

Foundations of Engineering Mechanics

M.Z. Kolovsky, A.N. Evgrafov, Yu. A. Semenov, A.V. Slousch
Advanced Theory of Mechanisms and Machines

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Advanced Theory of Mechanisms and Machines

Translated by L. Lilov

With 250 Figures



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Preface

This book is based on a lecture course delivered by the authors over a period of many years to the students in mechanics at the St. Petersburg State Technical University (the former Leningrad Polytechnic Institute). The material differs from numerous traditional text books on Theory of Machines and Mechanisms through a more profound elaboration of the methods of structural, geometric, kinematic and dynamic analysis of mechanisms and machines, consisting in both the development of well-known methods and the creation of new ones that take into account the needs of modern machine building and the potential of modern computers.

The structural analysis of mechanisms is based on a new definition of structural group which makes it possible to consider closed structures that cannot be reduced to linkages of Assur groups. The methods of geometric analysis are adapted to the analysis of planar and spatial mechanisms with closed structure and several degrees of movability. Considerable attention is devoted to the problems of configuration multiplicity of a mechanism with given input coordinates as well as to the problems of distinguishing and removing singular positions, which is of great importance for the design of robot systems. These problems are also reflected in the description of the methods of kinematic analysis employed for the investigation of both open („tree“-type) structures and closed mechanisms.

The methods of dynamic analysis were subject to the greatest extent of modification and development. In this connection, special attention is given to the choice of dynamic models of machines and mechanisms, and to the evaluation of their dynamic characteristics: internal and external vibration activity as well as frictional forces and energy losses due to friction at kinematic pairs. The dynamic analysis of machine assemblies is based on both models of „rigid“ mechanism and models that take into account the elasticity of links and kinematic pairs. Different engine characteristics are considered in the investigation of the dynamics of machine assemblies. Special attention is given to the dynamics of machines with feedback systems for motion control.

The limited volume of the text book did not allow the authors to include some traditional topics (the investigation of geometry of gearings, cam mechanisms, the parametric synthesis). The authors assume that these topics are presented to a satisfactory extent in the available text books.

The text book sets a large number of problems. Some of them are solved in details, the rest have only answers. The authors believe that the solution of the problems is necessary for the full understanding of the course.

In order to successfully master the material in the text book, the reader should possess a certain level of knowledge in the field of mathematics and theoretical mechanics. On the whole, the required level corresponds to the common programs taught in higher technical educational institutions.

The text book has been written by a team of authors and it is difficult to distinguish the participation of any one of them. The authors would like to note that the successful preparation of this new course was fostered with the great help of the lecturers of the Chair of Theory of Machines and Mechanisms (St. Petersburg State Technical University) and, most of all, with the continual support of Prof. G.A. Smirnov who was for many years the head of this chair. As it is known, the work on a text book is not finished with its publication. Coming out of press only signifies the beginning of this work. The authors will be genuinely grateful to the readers for any critical remarks on the material presented in this text book and for any suggestions for its improvement.

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Contents

1	Structure of Machines and Mechanisms	1
1.1	Machines and Their Role in Modern Production	1
1.2	Structure of a Machine and its Functional Parts	2
1.3	Mechanisms. Links and Kinematic Pairs	3
1.4	Kinematic Chains and Structural Groups. Generation of Mechanisms	10
1.5	Mechanisms with Excessive Constraints and Redundant Degrees of Movability	17
1.6	Planar Mechanisms	19
1.7	Mechanisms with Variable Structure. Structural Transformations of Mechanisms	24
1.8	Examples of Structural Analysis of Mechanisms	27
1.9	Problems	33
2	Geometric Analysis of Mechanisms	41
2.1	Problems of Geometric Analysis	41
2.2	Geometric Analysis of Open Kinematic Chains	44
2.3	Derivation of Equations of Geometric Analysis for Closed Kinematic Chains	52
2.4	Solution to the Equations of Geometric Analysis	58
2.5	The Inverse Problem of Geometric Analysis	66
2.6	Special Features of Geometric Analysis of Mechanisms with Higher Kinematic Pairs	70
2.7	Problems	72
3	Kinematic and Parametric Analysis of Mechanisms	79
3.1	Kinematic Analysis of Planar Mechanisms	79
3.2	Kinematic Analysis of Spatial Mechanisms	85
3.3	Kinematic Analysis of a Mechanism with a Higher Pair	90
3.4	Kinematics of Mechanisms with Linear Position Functions	93
3.5	Parametric Analysis of Mechanisms	103
3.6	Problems	108

4	Determination of Forces Acting in Mechanisms	121
4.1	Geometric Conditions for Transmission of Forces by Mechanisms...	121
4.2	Determination of Forces Acting in Mechanisms by the Graph-Analytic Method and the Method of Opening Kinematic Chains.....	128
4.3	Application of Equilibrium Equations of a Mechanism to its Kinematic and Parametric Analysis	133
4.4	General Formulation of the Force Analysis Problem	138
4.5	Equations of Kinetostatics. Determination of the Resultant Vector and of the Resultant Moment of Inertia Forces of Links	143
4.6	Solution of the Equations of Kinetostatics	147
4.7	Application of the General Equation of Dynamics for Force Analysis of Mechanisms	152
4.8	Force Analysis of Mechanisms with Higher Kinematic Pairs	157
4.9	Problems	158
5	Friction in Mechanisms	175
5.1	Friction in Kinematic Pairs	175
5.2	Models of Kinematic Pairs with Friction	178
5.3	Force Analysis of Mechanisms with Friction	185
5.4	Problems	194
6	Equations of Motion for a Mechanism with Rigid Links	211
6.1	Lagrange's Equations of the Second Kind for a Mechanism with a Single Degree of Movability	211
6.2	Lagrange's Equations of the Second Kind for Mechanisms with Several Degrees of Movability	216
6.3	An Example for Derivation of the Equations of Motion of a Mechanism	219
6.4	Problems	224
7	Dynamic Characteristics of Mechanisms with Rigid Links	235
7.1	Internal Vibration Activity of a Mechanism	235
7.2	Methods of Reduction of Perturbation Moments	237
7.3	External Vibration Activity of Mechanisms and Machines	239
7.4	External Vibration Activity of a Rotating Rotor and of a Rotor Machine	242

7.5	Balancing of Rotors	245
7.6	Vibration Activity of a Planar Mechanism	247
7.7	Loss of Energy due to Friction in a Cyclic Mechanism	252
7.8	Problems	254
8	Dynamics of Cycle Machines with Rigid Links	269
8.1	Mechanical Characteristics of Engines	269
8.2	Equations of Motion of a Machine. State of Motion	276
8.3	Determination of the Average Angular Velocity of a Steady-State Motion for a Cycle Machine	278
8.4	Determination of Dynamic Errors and of Dynamic Loads in a Steady-State Motion	280
8.5	Influence of the Engine Dynamic Characteristic on Steady-State Motions	286
8.6	Starting Acceleration of a Machine	289
8.7	Braking of a Machine	294
8.8	Problems	295
9	Dynamics of Mechanisms with Elastic Links	301
9.1	Mechanisms with Elastic Links and Their Dynamic Models	301
9.2	Reduction of Stiffness. Inlet and Outlet Stiffness and Flexibility of a Mechanism	305
9.3	Reduced Stiffness and Reduced Flexibility of a Mechanism with Several Degrees of Movability	308
9.4	Determination of Reduced Flexibilities with the Help of Equilibrium Equations of a Rigid Mechanism	311
9.5	Some Problems of Kinematic Analysis of Elastic Mechanisms ...	313
9.6	Dynamic Problems of Elastic Mechanisms	315
9.7	Free and Forced Vibration of Elastic Mechanisms	318
9.8	Problems	321
10	Vibration of Machines with Elastic Transmission Mechanisms	327
10.1	Dissipative Forces in Deformable Elements	327
10.2	Reduced Stiffness and Reduced Damping Coefficient	330
10.3	Steady-State Motion of a Machine with an Ideal Engine. Elastic Resonance	332
10.4	Influence of the Static Characteristic of an Engine on a Steady-State Motion	339

10.5 Transients in an Elastic Machine	342
10.6 Problems	349
 11 Vibration of a Machine on an Elastic Base.	
Vibration Isolation of Machines	361
11.1 Vibration of the Body of a Machine Mounted on an Elastic Base	361
11.2 Vibration of a Machine in the Resonance Zone. Sommerfeld Effect	364
11.3 Vibration Isolation of Machines	367
11.4 Problems	369
 12 Elements of Dynamics of Machines with Program Control	371
12.1 Basic Principles of Construction of Machines with Program Control	371
12.2 Determination of Program Control. Sources of Dynamic Errors ...	373
12.3 Closed Feedback Control Systems	378
12.4 Effectiveness and Stability of a Closed System	380
12.5 Problems	383
 References	387
 Index	389

1 Structure of Machines and Mechanisms

1.1

Machines and Their Role in Modern Production

Modern industrial production is reduced in the end to the execution of a great number of diverse *working processes*. Most processes are associated with treatment and transformation of initial raw materials into half- or fully finished products; such working processes are referred to as *technological*. Technological processes involve *transportation* of materials to the place of utilization as well as *energy* processes, i.e. generation and transformation of energy in forms most convenient for the respective process. Also, *information* processes, i.e. transmission and transformation of information are of great importance in modern production, ensuring execution of operations associated with control and organization of production.

The accomplishment of many working processes requires realization of certain *mechanical motions*. For instance, material processing on a lathe requires shifting the blank and the instrument; transportation of raw materials and of finished products is reduced to mechanical shifting; transformation of heat energy into electric energy requires rotations steam turbines and generators, and so on. The execution of working processes is also associated with the application of *forces* to materials in process in order to balance the weight of transported objects. A person is able to realize directly mechanical motions which allow him to carry out certain working processes manually. In modern production however the overwhelming majority of working processes associated with the realization of mechanical motions is carried out by machines.

We will call *machine* (or *machine aggregate*) a system designed to realize mechanical motions and force actions related to the execution of one or another working process. Machines are divided into *technological*, *transport*, *energy-converting* and *information* machines depending on the kind of working process.

In industrial production, in addition to machines, various *apparatuses* are used which are not directly associated with mechanical motion but with chemical, thermal and other processes or with transmission and transformation of information. Sometimes some of them are called machines, as well (e.g., electronic computing devices); however, the term “machine” will be used, in this course, only in the indicated sense.

1.2

Structure of a Machine and its Functional Parts

Modern machines are, as a rule, complex systems consisting of several subsystems. These subsystems are referred to as *functional parts* of machines. To the functional parts of a machine belong the engines, the mechanical system and the motion control system. The functional diagram of a very simple one-engine machine is represented in Fig. 1.1. E stands for the engine, MS for the mechanical system, PCS for the program control system, FCS for the feedback control system and WP for the working process performed by the machine.

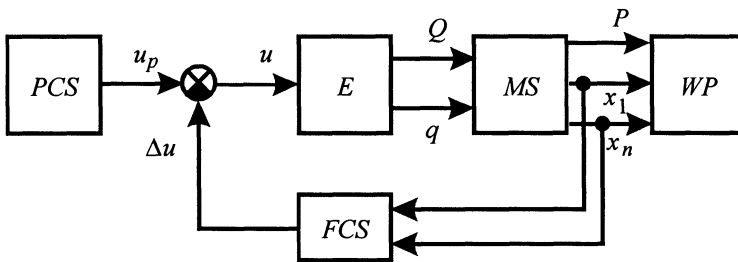


Fig. 1.1. Functional diagram of a one-engine machine

The completion of mechanical motions in a machine is always accompanied by transformation of some kind of energy into mechanical work. The *engine* is that part of the machine where such transformation actually takes place. *Electric, thermal, hydraulic, pneumatic* engines can be distinguished depending on the kind of the transformed energy. An *input engine parameter* u controls the energy transformation process. For electrical engines such a control parameter is the electrical voltage (for direct current engines) or the alternate current frequency; for internal-combustion engines control is achieved through change of the fuel quantity entering the combustion chamber; and so on. Each engine has an *output link*. This is a rigid body performing rotational (*rotary* engine) or reciprocating motion (*reciprocating* engine). The *output engine coordinate* is the generalized coordinate q determining the position of the link. The *generalized driving force* Q is generated in the engine acting on another functional part – the mechanical system connected with the engine; an equal and oppositely directed force $-Q$ acts on the output engine link in accordance with Newton's third law.

The *mechanical system* transforms the simplest motions created by engines into complex motions of the machine working organs, ensuring execution of working processes. Henceforth, such motions will be referred to as *machine program motions*. Output engine links are usually the inputs of a mechanical system. Therefore, the number of system inputs is equal to the number of engines. This number is referred to as *number of degrees of movability of a machine*. Fig. 1.2 shows the functional diagram of a machine with m degrees of movability. The

input parameters of its mechanical system are the coordinates $q_1, \dots, q_m$ of the output engine links and the output parameters are the coordinates $x_1, \dots, x_n$ of the machine working organs.

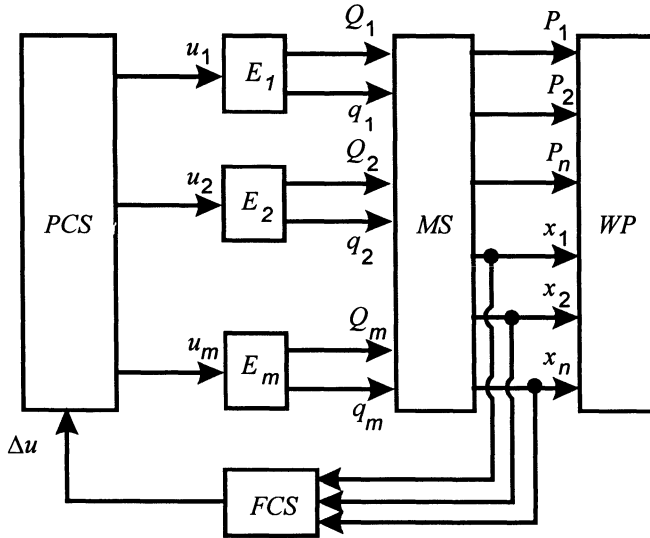


Fig. 1.2. Functional diagram of a multi-engine machine

The execution of a working process causes *workloads*, i.e. active forces P_s ($s=1, \dots, n$) acting on machine working organs. Mechanical systems of machines are in turn divided into simpler subsystems called mechanisms.

Systems for motion control are important functional parts of modern machines. *Systems of program control* produce *program control signals* u_p prescribing program motions of machines. Perturbation factors which will be considered in detail below cause *errors*, i.e. deviations of actual motions from the program motions. The correction of motion is achieved through a *feedback system*. It receives information about errors in positions, velocities or accelerations and forms *correcting controls* Δu which diminish these errors.

1.3

Mechanisms. Links and Kinematic Pairs

A connected system of bodies ensuring transmission and transformation of mechanical motions is called a *mechanism*. The bodies constituting a mechanism are referred to as *links*. Most often, the links of a mechanism are rigid bodies but mechanisms with liquid or elastic links exist, as well.

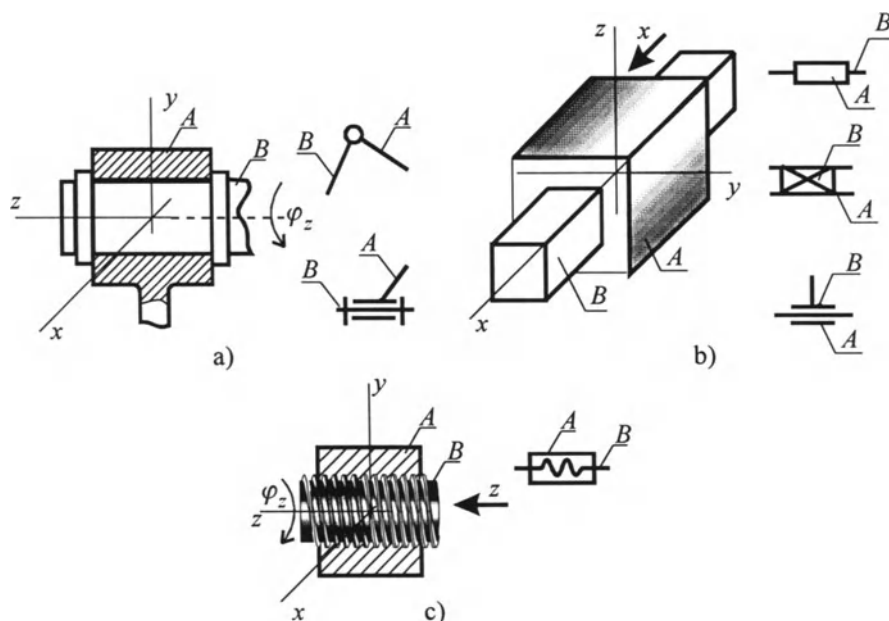


Fig. 1.3. Kinematic pairs of movability one: a) revolute, b) prismatic, c) screw

The constructive elements connecting links and imposing constraints on their motion are referred to as *kinematic pairs*. In mechanisms with links that are rigid bodies the kinematic pairs are realized in the form of cylindric (Fig. 1.3a) or spheric (Fig. 1.5a) joints, sliders and guides (Fig. 1.3b), screw couplings (Fig. 1.3c), contacting cylindric or planar surfaces (Fig. 1.6) and a lot of other constructive elements. Henceforth, only kinematic pairs constituted by rigid links will be considered.

Different *physical models* corresponding to different degrees of idealization of mechanism properties are used in the study of mechanisms. The choice of one or

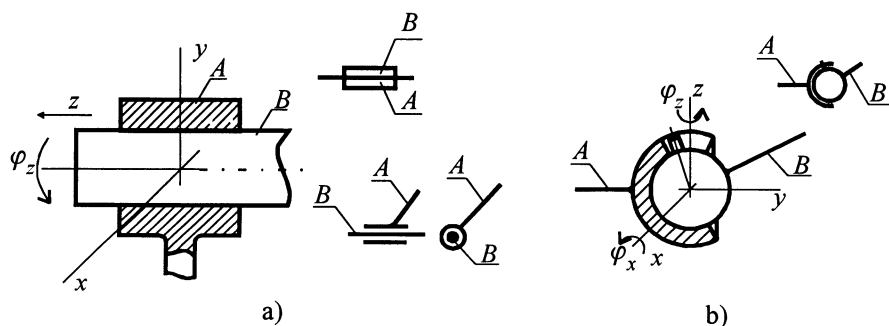


Fig. 1.4. Kinematic pairs of movability two: a) cylindric, b) spheric pair with a pin

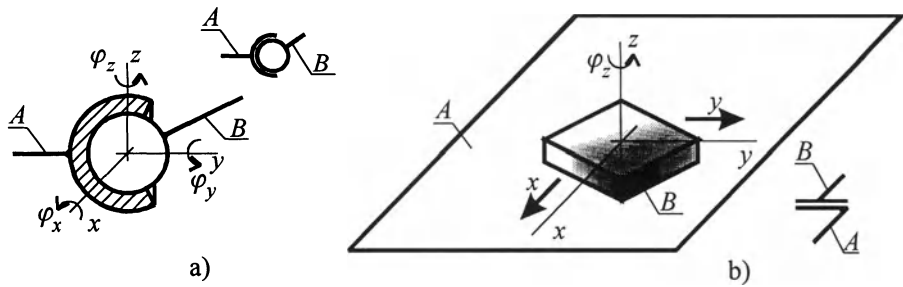


Fig. 1.5. Kinematic pairs of movability three: a) spheric, b) planar contact pair

another model depends primarily on the investigation goals and on what information about mechanism behaviour is needed in the analysis process. At different stages of a machine construction and investigation one and the same mechanism is described by different physical models. In the study of mechanism structure and kinematics one of the simplest physical models, referred to as a *mechanism with rigid links*, is usually used. The transition from a real mechanism to this model is based on the following assumptions:

1. All links and elements of kinematic pairs are considered nondeformable and rigid links are considered to be perfectly rigid bodies.
2. It is assumed that in a motion process no violation of the constraints imposed by kinematic pairs takes place and that these constraints themselves are *holonomic, stationary and bilateral*.

Henceforth, to make it short, mechanisms with rigid links will be referred to as *rigid mechanisms*, and the physical model of a machine consisting of only rigid mechanisms will be referred to as a *rigid machine*.

Like every physical model of a real system, the rigid mechanism model has limitations. For the solution of a large number of problems of statics and, particularly, of dynamics of mechanisms, one must use more complex models, taking into account deformations of links and of elements of kinematic pairs. Henceforth, such

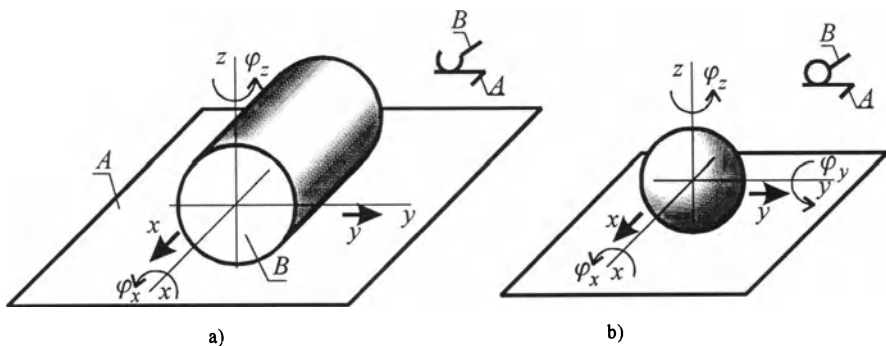


Fig. 1.6. Kinematic pairs of movability four and five: a) cylinder-plane, b) sphere-plane

models will be referred to as *mechanisms with elastic elements* or *elastic mechanisms*.

Let a mechanism with rigid links be composed of N movable links which are rigid bodies. Since the position of a free body in space is determined by six generalized coordinates, the position of all movable links is determined by $6N$ parameters. If kinematic pairs decrease the degrees of freedom by r , then the total mechanism, as a connected system of rigid bodies, possesses

$$w = 6N - r \quad (1.1)$$

degrees of freedom. For a mechanism with rigid links it is customary to call the number of its degrees of freedom *number of degrees of movability*. When we use more complex physical models (e.g., taking into account the deformability of links), then the number of degrees of freedom of a mechanism often turns out to be greater than the number of degrees of movability.

The division of a machine mechanical system into mechanisms is conditional and can be achieved in different ways. Usually, it is associated with separation of parts of the mechanical system which carry out specific functional tasks, or with representation of a complex system in the form of coupling of simpler systems. Inputs and outputs of mechanisms are accordingly distinguished.

The mechanism *inputs* are formed by those links on which generalized driving forces are directly applied. These are links connecting a mechanism with the output links of engines or links connecting it with previous mechanisms in the chain transmitting or transforming the motion. Internal and external inputs of a mechanism are distinguished.

In an *internal* input, generalized driving forces (equal and oppositely directed according to Newton's third law) are applied on two movable links of a given mechanism. These links are referred to as *input* links.

In an *external* input, a generalized driving force is applied only on one of the movable links of a given mechanism; an equal and oppositely directed force is applied either on the immovable link or on a link belonging to another mechanism.

The generalized coordinates determining the position of input links are referred to as *input mechanism coordinates*. When specifying the input coordinates one determines the mechanism *configuration*, i.e. the positions of all links. It follows that the number of independent input coordinates n has to be equal to the number of degrees of movability, i.e.

$$n = w. \quad (1.2)$$

Henceforth, a mechanism satisfying condition (1.2) will be referred to as a *normal* or *regular* mechanism. There are *singular* mechanisms which do not fulfill condition (1.2).

Mechanism *outputs* are formed by the links of machine working organs and by the links connected with input links of follow-up mechanisms. Such links are referred to as *output links* and the coordinates determining their position are referred to as *output mechanism coordinates*.

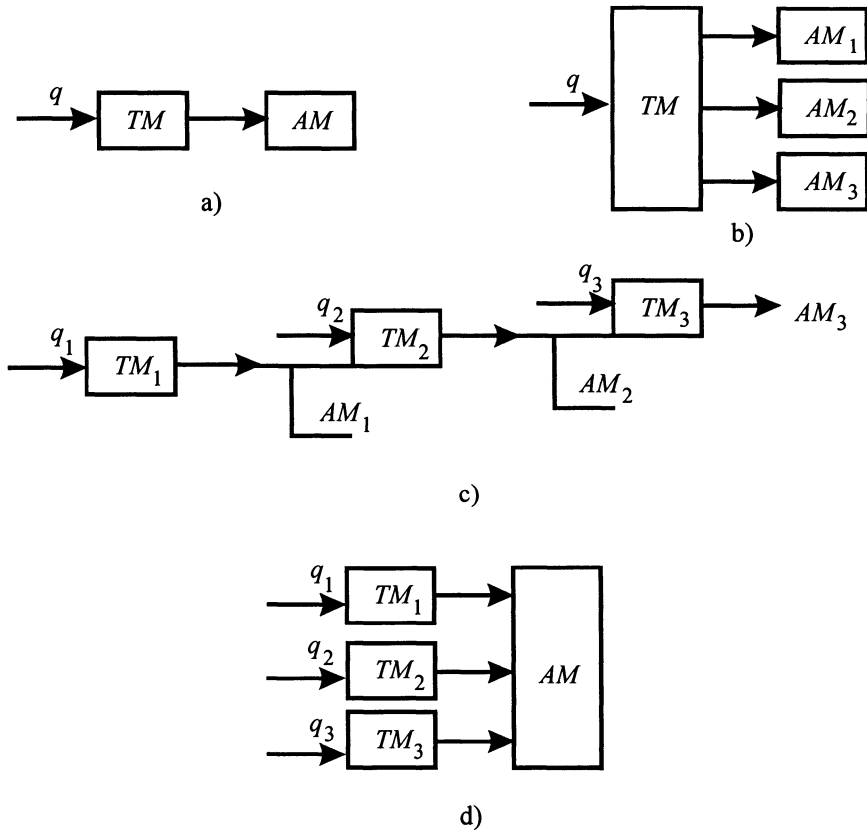


Fig. 1.7. Functional diagrams of a mechanical system: a) system of movability one formed by successive coupling of mechanisms, b) system of movability one formed by parallel coupling of mechanisms, c) system of multiple movability formed by successive coupling of mechanisms, d) system of multiple movability formed by parallel coupling of mechanisms

Different structures of a mechanical system arise depending on the method of decomposing the system into mechanisms. Some such structures are shown in Fig. 1.7.

Fig. 1.7a shows the functional diagram of a mechanical system constituted by *successive* coupling of two mechanisms of movability one. The first mechanism is directly connected with an engine and serves for changing the velocity of rotational or translational motion. Such a mechanism is called a *transmission* mechanism. The second mechanism transforms rotational or translational motions into program motions of working organs and is said to be an *actuating* mechanism. As a result of such coupling of mechanisms we obtain a mechanical system with one degree of movability.

Fig. 1.7b shows the functional diagram of a mechanical system with *parallel* coupling of actuating mechanisms. Such a structure is typical for *cycle* machines, in which coordinated motion of several mechanisms is needed. Here, the trans-

mission mechanism transmits motion to the actuating mechanisms the input links of which are connected to form the machine main shaft (main link) that performs one or several revolutions per cycle. Also this mechanical system has one degree of movability.

In the functional diagram shown in Fig. 1.7c, actuating mechanisms are coupled *successively* in such a way that the output links of the preceeding mechanisms are “frames” for the next ones. In this way, a system with several degrees of movability is obtained. Here, each of the mechanical system inputs is connected with the corresponding engine. Such a structure is used in some hoisting and handling machines, as well in robots and manipulators.

Another functional diagram of a mechanical system with several degrees of freedom is shown in Fig. 1.7d. Here the transmission mechanisms transmit motion to a common actuating mechanism (or link). A mechanical system with *parallel* structure is constituted. Such a structure appears in a number of robot constructions, walking or hoisting-and-handling machines, platform machines and others.

Classification of kinematic pairs. Kinematic pairs connecting mechanism links are classified according to the *number of degrees of freedom* in relative motion of the connected links. Let two links A and B , considered as perfectly rigid bodies, be connected by a kinematic pair. If link B would be a free rigid body, then it would have six degrees of freedom of the relative motion with respect to body A . A kinematic pair permitting s degrees of freedom of relative motion of links A and B is said to have *movability* s . It is obvious that s can take values from 1 to 5.

Fig. 1.3 shows kinematic pairs of *movability one*. A *revolute* pair (Fig. 1.3a) allows only rotation of link A relative to link B about the joint axis; the relative position of the links is defined by a single generalized coordinate φ_z . In a *prismatic* pair (Fig. 1.3b) the relative motion of links is reduced to translational displacement of slider B along a guide; the relative position is specified by the coordinate x . In a *screw* pair (Fig. 1.3c) a helical relative motion takes place, defined by the rotation angle φ_z of screw B relative to nut A and by the screw axial displacement z . Since $z = h\varphi_z/2\pi$ with h being the lead of the screw line, the relative displacement is determined by one parameter only and the pair has *movability one*.

Fig. 1.4 shows kinematic pairs of *movability two*: a *cylindric* pair (Fig. 1.4a) allowing relative rotation about the joint axis (coordinate φ_z) as well as axial displacement (coordinate z) and a *spheric pair with a pin* moving in a ring slot (Fig. 1.4b). Fig. 1.5 shows kinematic pairs of *movability three*: a *spheric* pair (Fig. 1.5a) and a *planar* pair (Fig. 1.5b); Fig. 1.6a shows a pair of *movability four*, constituted by a cylinder and a plane; Fig. 1.6b shows a pair of *movability five* (sphere–plane).

In Figs. 1.3–1.6 are also given the symbolic notations of the corresponding kinematic pairs used for representation in kinematic diagrams.

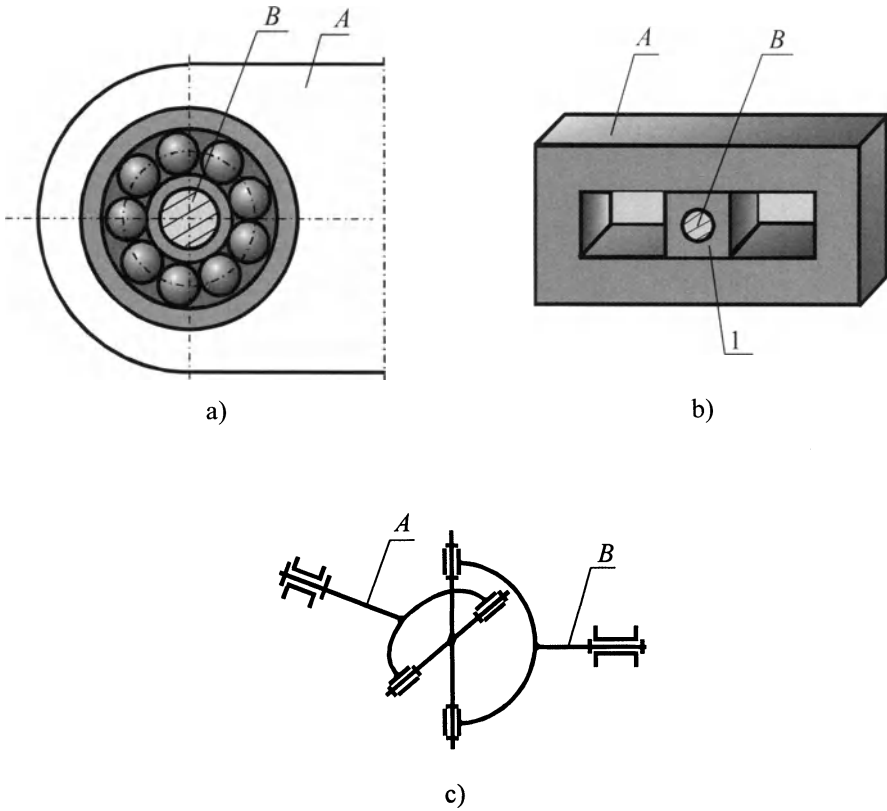


Fig. 1.8. Realization of kinematic pairs: a) revolute pair in the form of a ball bearing, b) cylinder-plane pair in the form of a connection of several planes, c) spheric pair in the form of three revolute pairs (Hooke joint)

We point out that in the literature another terminology is often used, according to which a pair belongs to the m -th class if its constraints take away m degrees of freedom of relative motion. Obviously, m -th class pair has movability $(6 - m)$.

It is necessary to point out that each kinematic pair is a physical model of a real construction of link coupling. It follows that, depending on the problem statement, one and the same coupling may be described by different kinematic pairs. For example, both radial and axial clearances exist in each real cylindric pair. In a number of problems, taking into account these clearances, a joint has to be considered either a revolute or a cylindric (or a spheric) pair.

We return again to the revolute pair (Fig. 1.3a). To bodies A and B we relate cylindrical surfaces of one and the same radius, and axes coinciding with the joint axis. It is obvious that for each value of tangle φ_z , i.e. for each relative position of the bodies A and B , these surfaces coincide. Analogously, in a prismatic pair the planes parallel to the relative displacement direction coincide, as well as the spheres belonging to bodies A and B in a spheric pair. Kinematic pairs with

common surfaces belonging to contiguous bodies and coinciding for each relative displacement are referred to as *lower pairs*.

There exist, however, kinematic pairs of still another kind: In each position bodies A and B have only common lines or points whose locations change during motion. Such kinematic pairs are referred to as *higher pairs*. Some of the simplest examples of higher pairs are shown in Fig. 1.6.

We point out that the notions “higher pair” and “lower pair” are related to physical models of real constructions and do not directly define their realization. For instance, the lower revolute pair shown in Fig. 1.3a may be built with the help of a roller bearing (Fig. 1.8a). In this case, bodies A and B do not have real elements with coinciding surfaces. However, if we think of them as cylindric surfaces with axes coinciding with the bearing axis and with identical radii, then such surfaces will coincide for any position of links and, therefore, this kinematic pair is a lower pair. On the other hand, the higher pair represented in Fig. 1.6a may be designed in the form shown in Fig. 1.8b. An additional slider 1 having common surfaces with bodies A and B is introduced here. In this case, the higher pair “cylinder–plane” is built with the help of two lower pairs. Analogously, Fig. 1.8c shows a construction of a spheric pair (Fig. 1.5a) with the help of revolute pairs. Such a construction of a spheric pair is called *Hooke’s joint*.

1.4

Kinematic Chains and Structural Groups. Generation of Mechanisms

Any assemblage of links connected by kinematic pairs constitutes a *kinematic chain*. A kinematic chain is said to be *open*, if it contains at least one link coupled with other links in only one pair (Fig. 1.9a); in the opposite case, the kinematic chain is said to be *closed* (Fig. 1.9b). An open kinematic chain has “*tree*”-structure if the chain is obtained through successive attachment of links in such a way that every following link is coupled with only one preceding link. A “*tree*” can be *simple* (Fig. 1.9c) or *branched* (Fig. 1.9d).

Let a kinematic chain contain N_c links and p_{sc} ($s = 1, \dots, 5$) kinematic pairs of movability s . If all constraints imposed on the motion of links are independent then such a kinematic chain has

$$w_c = 6N_c - \sum_{s=1}^5 (6-s)p_{sc} \quad (1.3)$$

degrees of movability, since each pair of movability s takes away $(6-s)$ degrees of freedom. For the kinematic chain shown in Fig. 1.10 we have $N_c = 5$, $p_{1c} = 2$, $p_{2c} = 3$, $p_{3c} = 2$, $p_{4c} = p_{5c} = 0$. Substituting these values into formula (1.3) we find $w_c = 2$. It has to be pointed out that, when we counted the degrees of movability of the kinematic chain, we took into account the degrees of freedom which were taken away by, both, the *external* kinematic pairs A, B, C (helping to

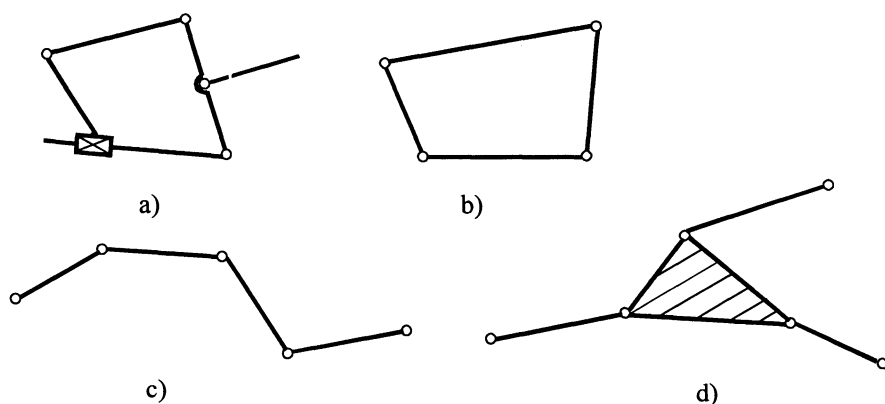


Fig. 1.9. Kinematic chains: a) open, b) closed, c) open chain with the structure of a simple tree, d) open chain with the structure of a branched tree

attach the considered kinematic chain to the fixed link or to another kinematic chain) and by the *internal* pairs D, E, F, G (coupling the chain links with each other).

The inputs of a kinematic chain are constituted by its *input links* (in Fig. 1.10 link 5) or by *input link pairs* (in Fig. 1.10 links 3 and 2). In the first case, an input coordinate q_1 is the coordinate which determines the position of the input link, belonging to the given chain, relative to a reference system connected with some other chain. Here the generalized driving force Q_1 is an external force for the chain

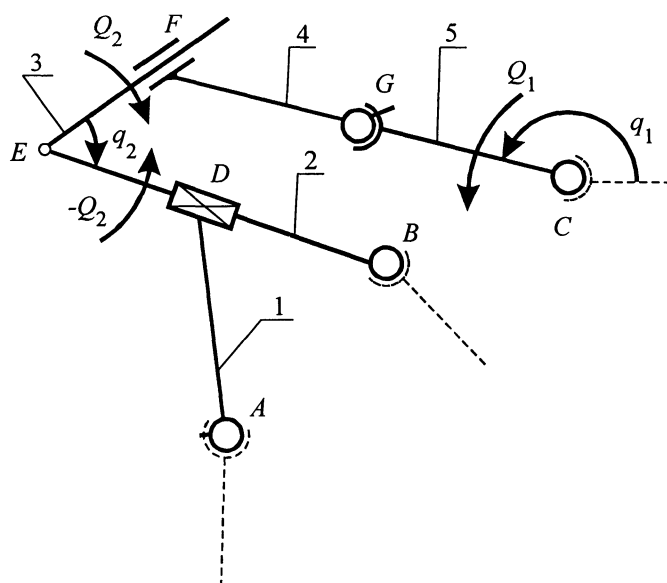


Fig. 1.10. Structural group of movability two

considered. In the second case, the input coordinate q_2 determines the relative position of two input links that belong to the given chain. The generalized driving forces Q_2 and $-Q_2$ applied to these links are internal forces. Inputs are said to be *independent*, if their corresponding input coordinates are independent.

A kinematic chain with given inputs is referred to as a *normal structural group of movability n* or simply a *structural group*, if the number of independent chain inputs n_c coincides with the number of degrees of movability w_c . As *simple structural group* is one that can not be splitted into several structural groups with smaller numbers of links. A simple structural group may have number zero of degrees of movability (and, therefore, number zero of inputs as well), i.e. $n_c = w_c = 0$. Such a structural group is called an *Assur group*.

The kinematic chain shown in Fig. 1.10 is a structural group; for this chain $n_c = w_c = 2$. It can be obtained by successive coupling of three simple structural groups. The first simple group is defined by links 1 and 2 constituting an Assur group ($N_c = 2$, $p_{1c} = 1$, $p_{2c} = 1$, $p_{3c} = 1$, $w_c = n_c = 0$). Link 3 forms the second group (with movability one); when attaching link 3 to the first group, the pair E of movability one is taken into account (external for this link; in the first group this kinematic pair is not taken into account since it belongs to the next group) and the input constituted by the input pair of links 2 and 3. Thus, for this group we have $w_c = 6 - 5 = 1$, $n_c = 1$. The third group (again with movability one) is constituted by links 4 and 5; they are attached to the previous group through kinematic pair F and to some external chain (or to the fixed link) through kinematic pair C . For this group we have $N_c = 2$, $p_{2c} = 2$, $p_{3c} = 1$, $w_c = 1$, $n_c = 1$.

Formula (1.3) can also be used to determine the number of degrees of movability of a mechanism. For this purpose, one of the chain links is assumed to be fixed; the motion of all other links is considered relative to the link which is referred to as *frame*. If N is the total number of movable links and if p_s is the number of pairs of movability s in this mechanism, then the number of its degrees of movability is determined by the expression

$$w = 6N - \sum_{s=1}^5 (6-s)p_s. \quad (1.4)$$

Generation principle of mechanisms. It is easy to show that every normal mechanism can be generated by the successive joining of simple structural groups to frame. This results from formula (1.4). By successive coupling of structural groups to the frame, we obtain a normal mechanism at each stage, since in every attached group the number of degrees of movability is equal to the number of inputs. It should only be kept in mind that the inputs constituted by links belonging to contiguous groups are taken into account only once: they are considered to belong to the attached group only.

The generation of a mechanism can be represented by its *structural diagram* that shows the coupling of structural groups with one another and that also

indicates the number of links in each group together with the number of degrees of movability.

The structure of a mechanism can be described by a *graph* whose *vertices* correspond to links and whose *edges* correspond to kinematic pairs. Moreover, the number of edges connecting contiguous vertices is equal to the movability of the corresponding kinematic pair. Bold lines display *root* edges that correspond to the kinematic pairs constituting the inputs of a mechanism.

Fig. 1.11 shows kinematic diagrams of some mechanisms and their structural diagrams and graphs. Fig. 1.11a shows a mechanism of movability one referred to as a *slider-crank mechanism*. It serves for transformation of the rotational motion of link 1, referred to as *crank*, into a translational motion of link 3 – the *slider*. Link 2, performing a plane-parallel motion, is referred to as *connecting rod*. The fourth link of this mechanism is the frame denoted by the label 0. The slider-crank mechanism consists of two structural groups: a one-bar group of movability one containing the revolute pair O and link 1, and a two-bar group containing the cylindric pair B , the spheric pair A and the prismatic pair together with links 2 and 3; here, the input coordinate is the rotation angle q of the crank. Fig 1.11b shows the same mechanism but in a different input position: the input coordinate in this mechanism is the angle q between the crank and the connecting rod. In this case, the structure of the mechanism is different: All movable links together constitute a simple structural group.

Fig. 1.11c shows a mechanism with three degrees of movability and with five movable links. Given the location of the inputs presented in the figure, the mechanism can be disintegrated into three simple groups of movability one: two one-bar groups (links 1 and 5) and a three-bar group (links 2, 3, 4). The one-bar groups form the first *structural layer* and the three-bar group constitutes the second layer. Changing disposition of inputs, we obtain the mechanism of different structure shown in Fig. 1.11d; there are two simple structural groups here: a group of movability one (link 1) and a group of movability two (links 2, 3, 4, 5).

In the mechanisms considered all links perform plane-parallel motion and the motion planes of all links are parallel. Such mechanisms are said to be *planar*. In Fig. 1.11e one more mechanism is represented; it is a six-bar mechanism (the number of links also includes the frame). It consists of a one-bar group of movability one (link 1) and of two two-bar Assur groups (links 2, 3 and 4, 5). Such two-bar groups are referred to as *dyads*. Some special features of such mechanisms will be considered later.

Mechanisms which are not planar are referred to as *spatial mechanisms*. Fig. 1.12a shows the spatial actuating mechanism of an industrial robot (robot effector) consisting of six links connected in sequence by kinematic pairs of movability one. All links are input links; the input coordinates $q_1 - q_6$ define the relative positions of links. Varying input coordinate values one can shift link 6 (the robot's gripper) and, thus, define the position of pole M and the orientation of the gripper. The structural diagram of the mechanism is shown in Fig. 1.12b.

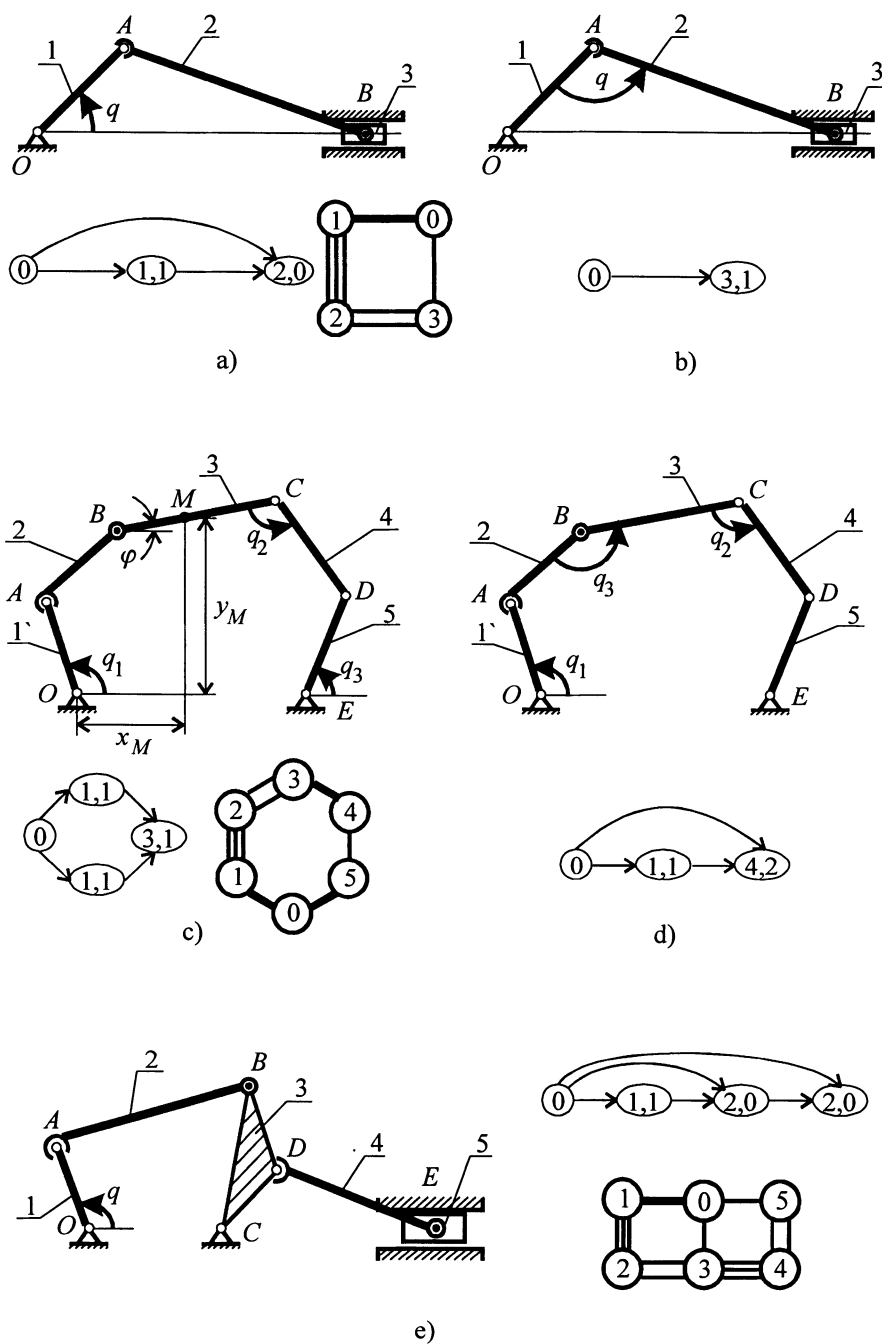


Fig. 1.11. Kinematic and structural diagrams of lever mechanisms: a) slider-crank mechanism with an external input, b) slider-crank mechanism with an internal input, c) and d) mechanism of movability three with different inputs, e) single six-bar mechanism

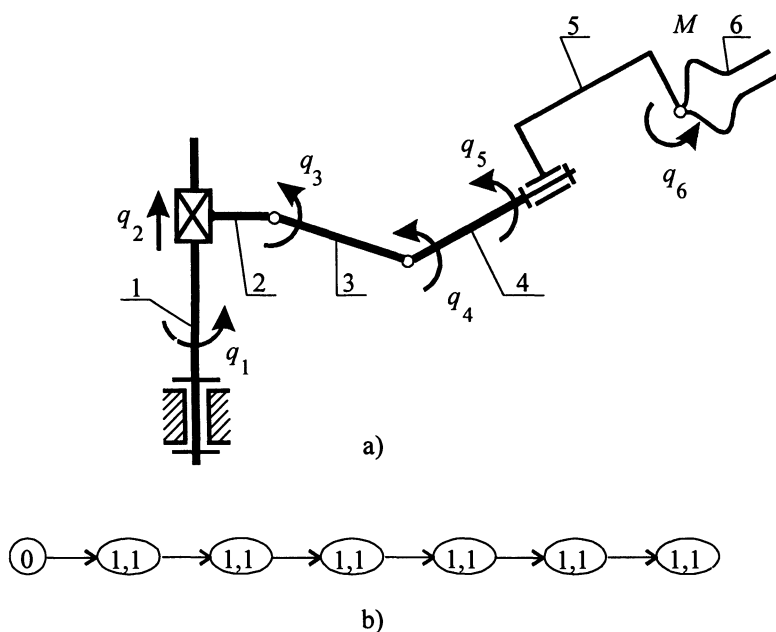


Fig. 1.12. Kinematic and structural diagram of an industrial robot

Fig. 1.13a shows the spatial mechanism of a light-wave guide used in laser welding. A laser ray is led by hinged mirrors from the radiation source to point C where the welding process is executed. The displacement and the orientation of link 6, carrying the welding head, is determined by a robot whose gripper is rigidly

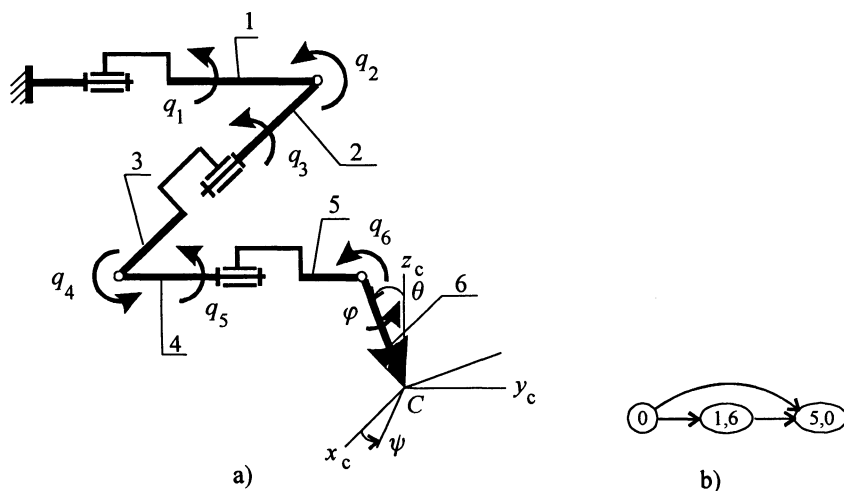
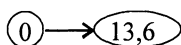
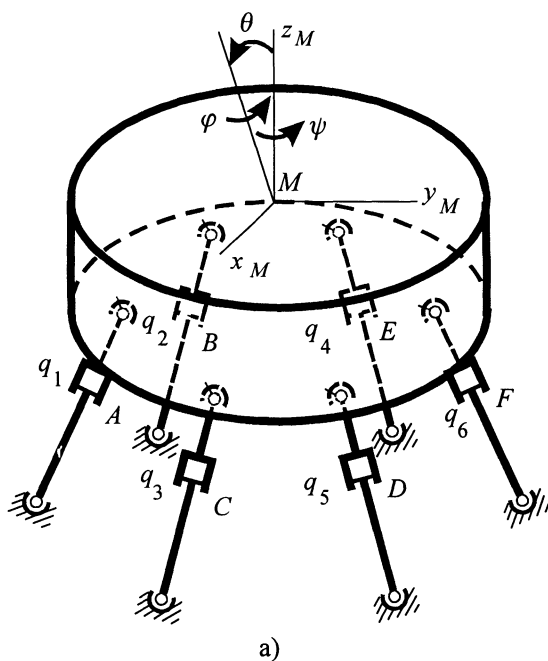


Fig. 1.13. Kinematic and structural diagram of the mechanism of a laser light waveguide

connected with link 6. In this way, the mechanism considered has six input coordinates, determining the position of link 6 relative to the frame. As input coordinates may be chosen, e.g., Cartesian coordinates x_c, y_c, z_c of pole C and the Euler angles φ, ϑ, ψ defining the head orientation relative to the frame. Here, link 6 has to be considered as a one-bar structural group of movability six which is conditionally “attached” to the frame through the specification of the six generalized coordinates (it can be said that this link is connected with the frame through a conditional “kinematic pair of movability six”). The rest of the movable links of the mechanism forms an Assur group with six revolute pairs of movability one. The structural diagram of the light-wave guide mechanism is shown in Fig.1.13b.

Fig. 1.14a shows the spatial mechanism of a Stewart platform possessing six degrees of movability. The input prismatic pairs A, B, C, D, E, F are built up by hydraulic cylinders and pistons which are set in motion by the pressure of working fluid entering the cylinders. The input coordinates $q_1 - q_6$ determine the position of pole M of the platform and its orientation relative to the frame. It can easily be shown that the investigated mechanism forms a simple structural group of thirteen links (Fig. 1.14b).



b)

Fig. 1.14. Kinematic and structural diagram of the mechanism of Stewart platform

1.5

Mechanisms with Excessive Constraints and Redundant Degrees of Movability

Formulae (1.3) and (1.4) have been derived under the assumption that the constraints, imposed by kinematic pairs on mechanism links, are independent. In some mechanisms there are constraints which repeat restrictions imposed by other constraints. Such constraints are said to be *excessive*.

Let us consider the four-bar mechanism shown in Fig. 1.15a. It has four revolute pairs of movability one. Determining the number of its degrees of movability according to formula (1.4), we obtain

$$w = 6 \cdot 3 - 4 \cdot 5 = -2.$$

In this way, the number of degrees of movability has turned out to be negative. This means that the four-bar linkage under consideration is not a mechanism but a stiff unchangeable system which, by the way, can be assembled only under specific geometric conditions. However, if all joints axes are strictly parallel, then, it is easy to understand that the considered system will prove to be a one degree of movability mechanism, performing a planar motion with a plane orthogonal to the axes of the joints. In this case, there will be excessive constraints among the constraints imposed on link motions by kinematic pairs, and their elimination will not affect the kinematics of the mechanism.

Elimination of excessive constraints is possible by increasing in the movability of some kinematic pairs. Let us replace joint *B* with a spheric pair and joint *C* with a cylindric pair; in this case, the mechanism shown in Fig. 1.15b is obtained. For it we find

$$w = 6 \cdot 3 - 4 \cdot 1 - 3 \cdot 1 - 2 \cdot 5 = 1.$$

This mechanism has one degree of movability and, moreover, the character of its link motions remains exactly the same as that of the mechanism shown in Fig. 1.15a.

The mechanism shown in Fig 1.11a does not have excessive constraints. They will only appear if the cylindric pair *B* and the spheric pair *A* are replaced by revolute pairs. If the joint axes *O*, *A* and *B* remain parallel, the plane motion of the mechanism links will be maintained. The same result is obtained if the cylindric and the spheric pairs in the mechanisms, shown in Fig. 1.11c and Fig. 1.11e, are replaced by revolute pairs.

The presence of excessive constraints in a mechanism has disadvantages as well as advantages. On the one hand, the presence of excessive constraints increases the requirements with respect to manufacturing accuracy of mechanism links and of elements of kinematic pairs. If, e.g., the joint axes in the mechanism shown in Fig. 1.15a are not exactly parallel, then the danger of blocking for this mechanism arises, since it can not move with nonparallel axes. On the other hand, it must be taken into account that the links of any mechanism are, in fact, not perfectly rigid bodies. Also the elements of kinematic pairs are deformable. Excessive

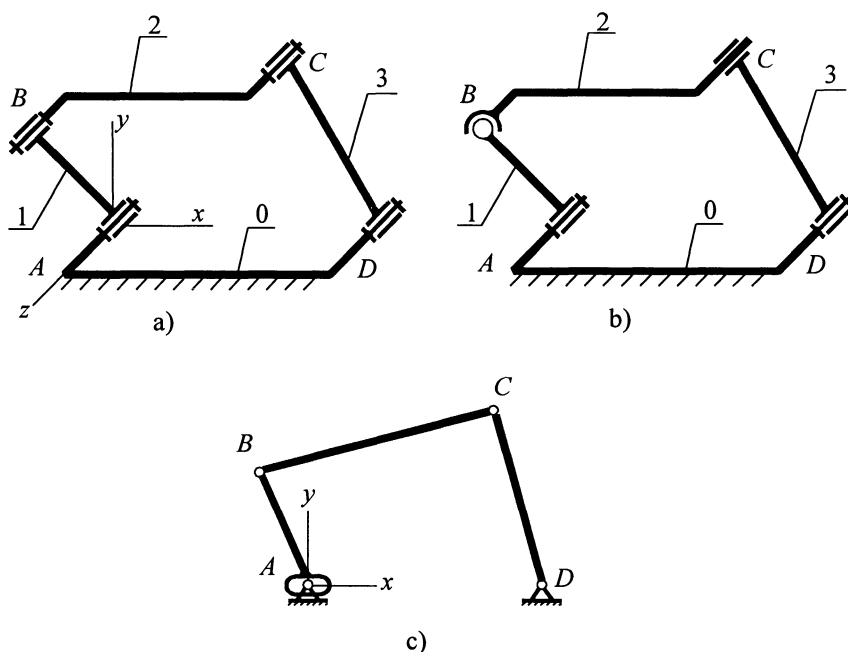


Fig. 1.15. Kinematic diagrams of a four-bar linkage: a) with excessive constraints, b) without excessive constraints, c) with a redundant degree of movability

constraints increase the stiffness of a mechanism and diminish deformations caused by transmitted forces.

Let us consider a mechanism with excessive constraints, e.g., the one shown in Fig. 1.15a. In every joint five constraints are imposed on link motions. For instance, at joint A constraints prevent shifting of link 1 along the rotation axis Az in the direction of axes Ax and Ay , as well as rotation about axis Az . This means that the transformation of the kinematic pair A into a cylindric pair will not increase the number of mechanism degrees of movability. A constraint of this kind is said to be *nonreleasing*. It is easy to see, that also the constraints preventing rotation about axes Ax and Ay are nonreleasing. On the other hand, if we remove the constraint which prevents shifting along axes Ax and Ay by introducing a slot in joint A in the corresponding direction, we will increase the number of mechanism degrees of movability by one (Fig. 1.15c). Constraints, whose elimination leads to an increase of the number of degrees of movability, are said to be *releasing*. In the mechanism, shown in Fig. 1.15a eight out of twenty constraints are releasing, and twelve are nonreleasing. We point out that in a mechanism with no excessive constraints all constraints are releasing.

Let us replace the cylindric pair C with a spheric pair in the mechanism shown in Fig. 1.15b. In this case, the mechanism will gain a *redundant* degree of movability. It is easy to see that this degree will not influence the transmission

of motion from link 1 to link 3: there arises only the possibility of an additional rotation of link 2 about the axis passing through the centers of spheres *B* and *C*. Such redundant degrees of movability are sometimes used for decreasing frictional forces or for achieving some other purposes.

1.6

Planar Mechanisms

In planar mechanisms, it is expedient to divide the constraints imposed on motion of links by kinematic pairs into two groups. The *first group* is related to constraints ensuring a plane motion of links; the *second group* restricts the relative displacement of links in the motion plane. Moreover, one and the same kinematic pair forms constraints of both kinds.

In the mechanism, shown in Fig. 1.15a, the first group is related to constraints which prevent relative shifting of links in *z*-direction and relative rotation of links about axes *x* and *y* at every of the four joints. It is clear from the foregoing exposition that all constraints of the first group are nonreleasing, while the constraints of the second group are releasing.

For planar mechanisms a separate consideration of movability in plane motion, permitted by constraints of the second group, turns out to be expedient. The number *w* of such degrees of movability of a planar mechanism can be determined in the following way. Let *N* be the number of movable links. Every link, being a rigid body, possesses three degrees of freedom in plane motion. Every kinematic pair can take away one or two degrees of freedom of the links in their plane motion, imposing one or two constraints from the second group. Any lower kinematic pair – prismatic, cylindric or spheric – takes away two degrees of freedom in plane motion; any higher pair, e.g. built from two cylindric surfaces with generating lines orthogonal to the motion plane, takes away one degree of freedom.

Let p_L be the number of lower pairs and p_H be the number of higher pairs; then

$$w = 3N - 2p_L - p_H. \quad (1.5)$$

Analogously, taking into account only the constraints from the second group, it is possible to investigate planar kinematic chains and planar structural groups. Fig. 1.16 shows some planar Assur groups for which the number of degrees of movability is equal to zero. Fig. 1.16a illustrates one-bar Assur groups containing one higher pair and one lower pair. Two-bar Assur groups are shown in Fig. 1.16b–d; the three-bar group (Fig. 1.16e) consists of three lower and one higher pair; the four-bar groups (Fig. 1.16f–g) consist of six lower pairs, and so on. Correspondingly, in Fig. 1.17 examples of planar structural groups of movability one (Fig. 1.17a–b), of movability two (Fig. 1.17c) and of movability three (Fig. 1.17d) are illustrated.

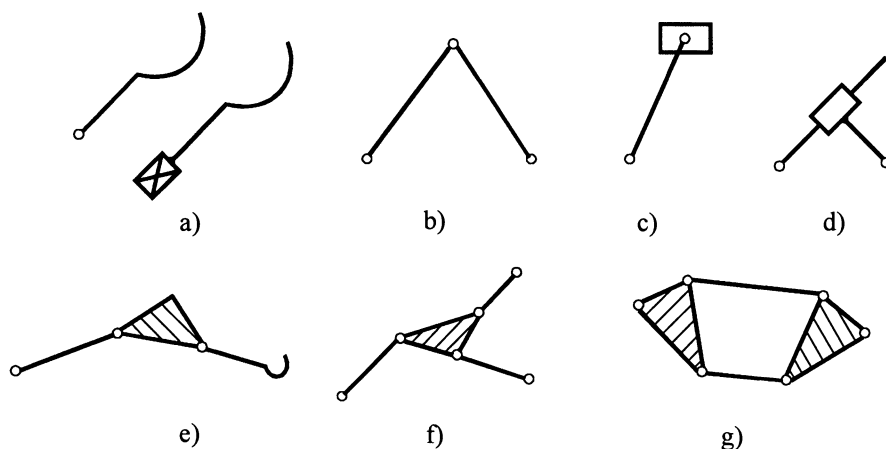


Fig. 1.16. Planar Assur groups

Planar mechanisms with one degree of movability are widely used in machines. Let us consider some most commonly used varieties of these mechanisms.

Planar linkages. A mechanism is referred to as a *linkage* if it has only lower kinematic pairs. In a planar linkage, spheric and cylindric pairs are equivalent in kinematical relation to revolute pairs (i.e. according to the character of permitted relative motion); because of that, henceforth we will consider only revolute and prismatic pairs. A linkage possessing revolute pairs exclusively is said to be *revolute*.

Planar linkages with one degree of movability can be generated through attaching a one-bar group of movability one with revolute or prismatic pair to the frame (Fig. 1.17a), and by attaching to this successively additional planar Assur groups. One of these mechanisms, the slider-crank mechanism, has been considered above. Other examples are given in Figs. 1.18 and 1.19. Fig. 1.18 shows fourbar mechanisms. The mechanism given in Fig. 1.18a is referred to as a *four-bar mechanism* or a *crank-and-rocker mechanism*, since link 3 performing reciprocating motion is referred to as a *rocker*. Fig. 1.18b displays a *slotted-link*

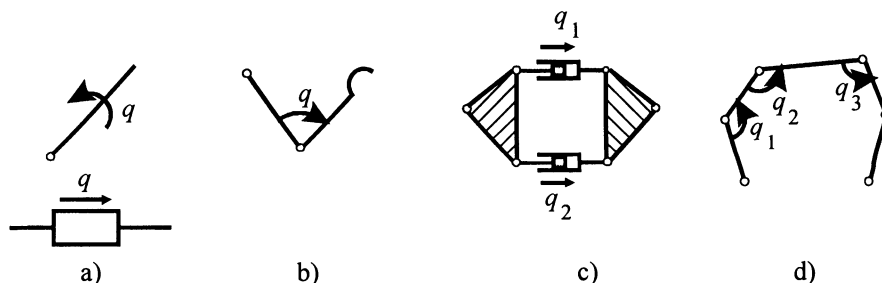


Fig. 1.17. Planar structural groups

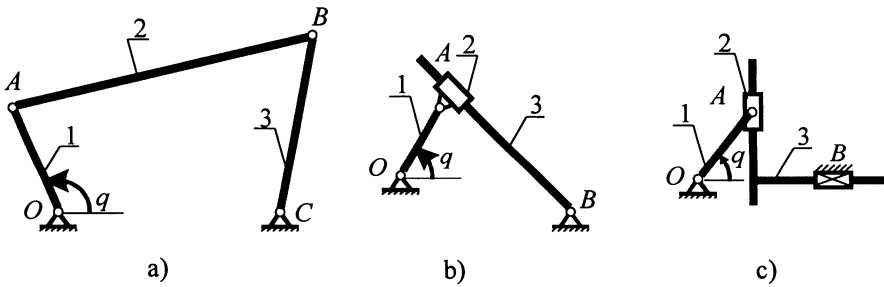


Fig. 1.18. Lever four-bar mechanisms: a) crank-and-rocker mechanism, b) slotted-link mechanism, c) scotch-yoke mechanism

mechanism consisting of a crank – link 1, a *slotted link* – link 3, which is a moving guide for the slide block 2 (*crosshead*). Here, the slotted link performs an incomplete turnover about an axis connected with the frame. In the *scotch-yoke* mechanism, shown in Fig. 1.18c, the slotted link 3 is performing a translational motion.

The mechanisms just shown have two-bar Assur groups; they are related to mechanisms of *second class*. In Fig 1.19a a mechanism is shown which is produced by attaching a four-bar Assur group (see Fig. 1.16f) to a one-bar group and to the frame. Such a group has three internal joints (B, C, D) and, because of that, it is called an Assur group of third class. Correspondingly, a mechanism possessing such a group is referred to as a mechanism of *third class*. Analogously, a group with four internal joints (B, C, D, E), which is shown in Fig. 1.16g, is related to the *fourth class*, as well as the mechanism including this group (Fig. 1.19b).

Planar cam mechanisms. In Fig. 1.20 are given diagrams of three-bar planar mechanisms with two lower pairs and one higher pair, constituted by *cam* 1 and *follower* 2. All of them consist of a one-bar group of movability one (link 1) and a one-bar Assur group, corresponding to Fig. 1.16a. Fig. 1.20a shows a mechanism with a translatable moving *pointed* follower, Fig. 1.20b shows a mechanism with a

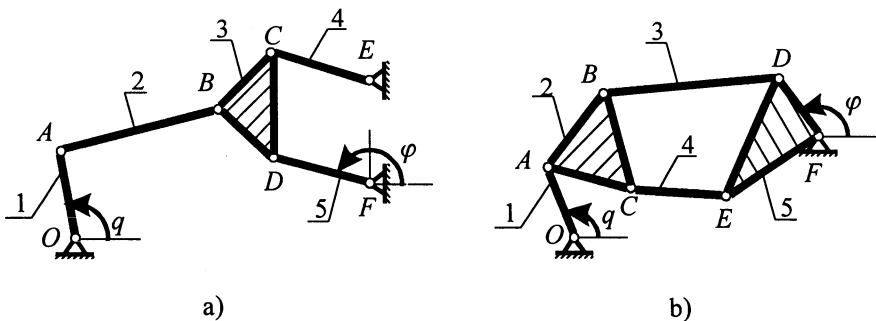


Fig. 1.19. Lever mechanisms of higher classes: a) third class, b) fourth class

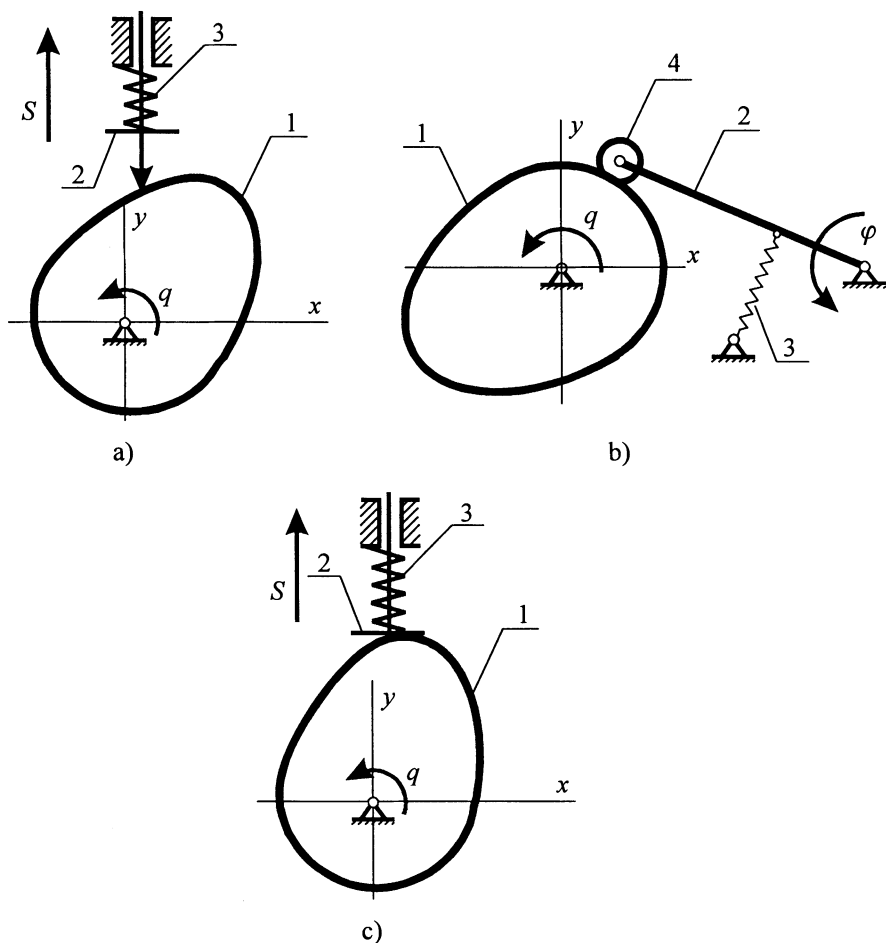


Fig. 1.20. Planar cam mechanisms: a) with a pointed follower, b) with a rocker follower, c) with a planar follower

rocker follower and Fig. 1.20c shows a mechanism with a *planar* follower. In all these mechanisms the higher pair creates a unilateral constraint: the position of the follower is constrained from only one side by the cam profile. A *geometric* or *force* closure of the mechanism is introduced with the help of spring 3 in order to guarantee permanent contact between the follower and the cam. The mechanism shown in Fig. 1.20b has a superfluous link – *roller* 4, allowing the replacement of sliding friction between the cam and the follower by rolling friction. This link and the superfluous kinematic pair (the kinematic pair between the follower and the cam) should not be taken into consideration in the analysis of the mechanism structure.

Planar transmission mechanisms. As mentioned before, mechanisms used for transmission of rotary motion from one link of a mechanical system to some other

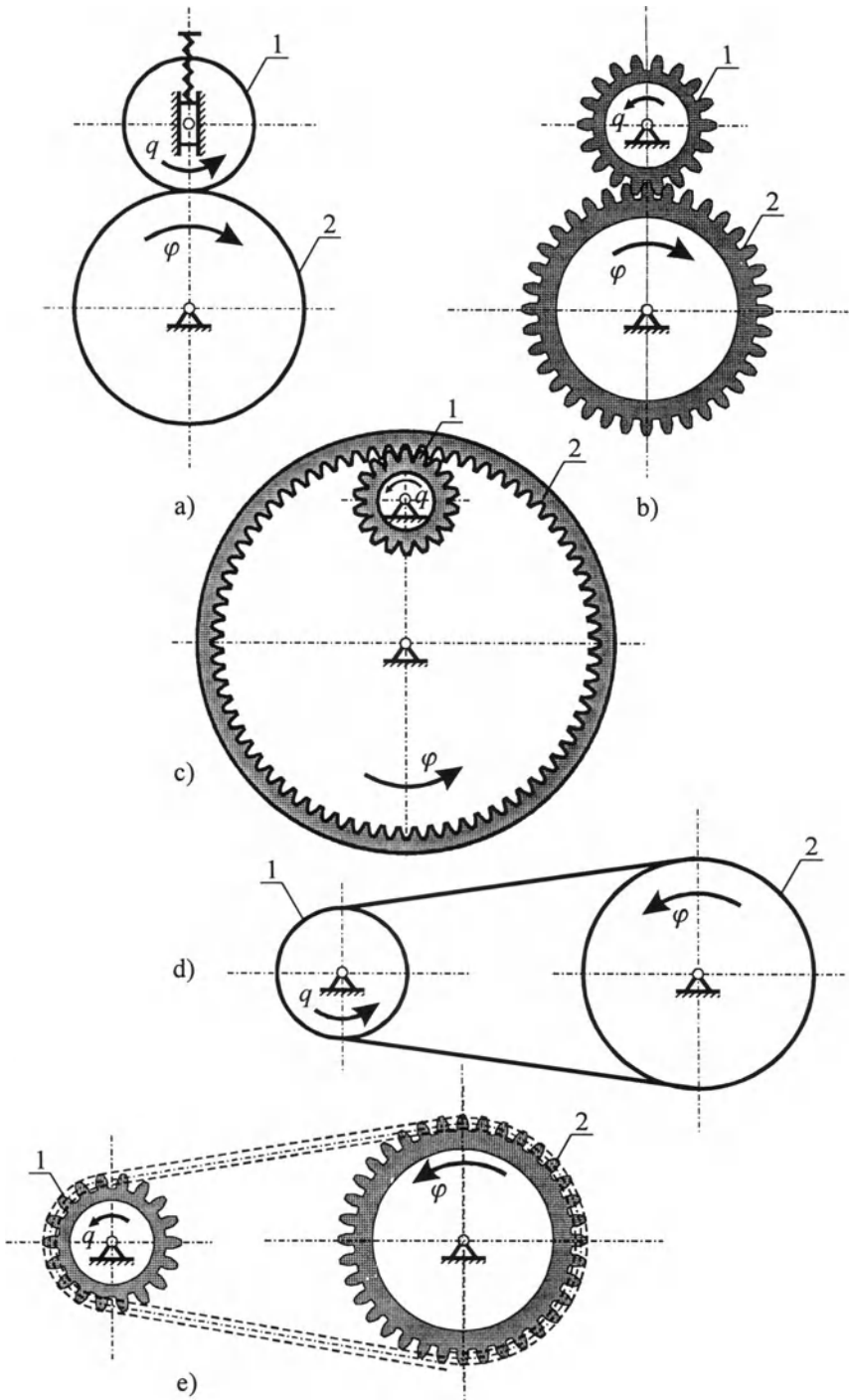


Fig. 1.21. Planar transmission mechanisms (transmissions): a) friction mechanism, b) external gearing, c) internal gearing, d) belt drive, e) chain transmission

links are transmission mechanisms. Fig. 1.21 shows planar transmission mechanisms: *friction* mechanism (Fig. 1.21a), *cylindric gearings* of external (Fig. 1.21b) and internal (Fig. 1.21c) meshing, *belt* drives (Fig. 1.21d), *chain* transmissions (Fig. 1.21e). All these mechanisms are designed to transmit rotary motion with a constant transmission ratio, i.e. with a constant ratio between the angular velocities of input 1 and output 2 link. In the friction mechanism the constant transmission ratio is ensured through pure rolling of cylinders and in the gearings it is achieved through engagement between the teeth of wheels. There exist also transmission mechanisms with variable transmission ratio. The transmission of rotation by means of elliptic gear wheels (Fig. 1.22) is an example.

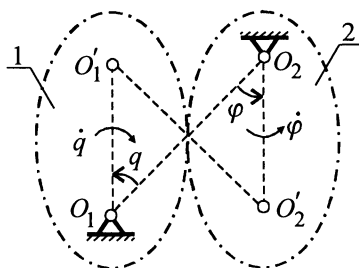


Fig. 1.22. Gearing with elliptic wheels

1.7

Mechanisms with Variable Structure. Structural Transformations of Mechanisms

It has been shown above that the structure of a mechanism depends on the location of its inputs. In the process of link motion, the points of application of generalized driving forces, i.e. the inputs of a mechanism, can vary.

Thus, e.g., in the four-piston internal combustion engine (Fig. 1.23), the slide blocks of the slider-crank mechanisms become both *driving* and *driven* ones during one cycle.

In the walking process, every leg of a walking mechanism (Fig. 1.24) is alternating between support phase and swing phase. This has the effect that the structure of the mechanism is time-varying.

Mechanisms of this kind are referred to as *mechanisms with variable structure*. The variation of mechanism structure allows to prevent links from accidental loads, to ensure required motions of input links, and so on. In mechanisms with alternative inputs, where, as in the examples above considered, the number of inputs exceeds the number of degrees of movability, the conditions for transmission of, both, motion and forces are improved. Such mechanisms will be considered in the following chapters.

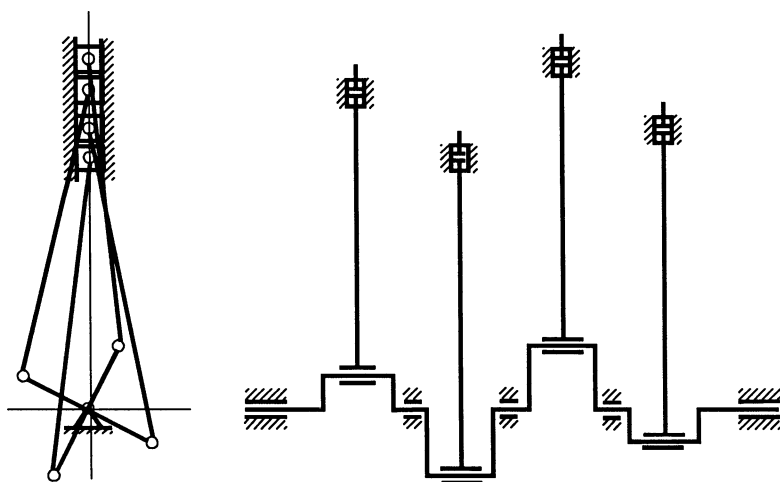


Fig. 1.23. Four-piston internal combustion engine

There are mechanisms with variable structure for which the number of degrees of movability exceeds the number of inputs (a car differential, a brake gear etc).

Sometimes a *structural transformation*, consisting in conditional displacement of inputs, is undertaken in order to simplify mechanism analysis.

Let us consider some examples. The mechanism shown in Fig. 1.11a, differs from the mechanism shown in Fig. 1.11b, only in the location of the input. A structural transformation, consisting in a displacement of the input, changes mechanism structure, making it more convenient for analysis through the substitution of one

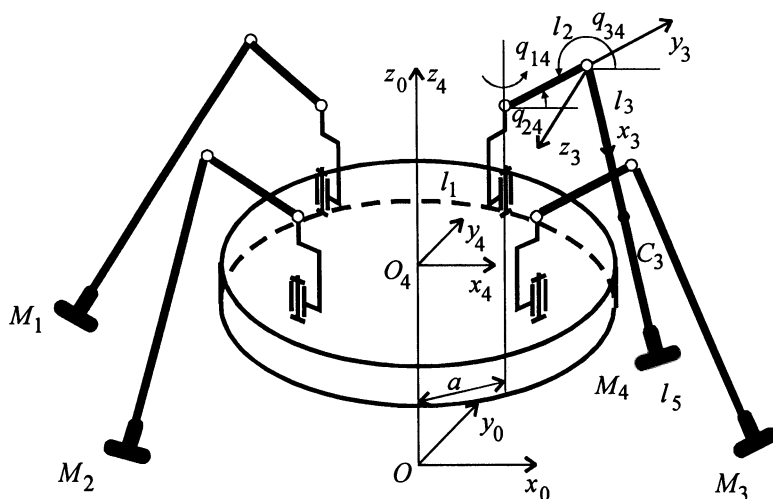


Fig. 1.24. Mechanism of a walking machine

three-bar group of movability one by two simpler groups: a one-bar group of movability one and a two-bar Assur group. An analogous simplification occurs in the transformation of the mechanism (see Fig. 1.11d) into another mechanism with a diagram depicted in Fig. 1.11c. In addition, the four-bar group of movability two is eliminated which complicates the analysis.

Let us consider the mechanism with diagram presented in Fig. 1.19a; it consists of a one-bar group of movability one (link 1) and a four-bar Assur group. Replacing the input coordinate q by the coordinate φ , we obtain a new structure: Two Assur groups (links 3, and 1, 2) are adjoined to a one-bar group of movability one (link 5). Changing the location of the input in the mechanism (Fig. 1.19b) we obtain a mechanism consisting of a one-bar group of movability one (link 5) and a four-bar Assur group of third class (links 4, 3, 2, 1).

In the investigation of mechanisms with several degrees of movability a structural transformation, in which inputs and outputs of a mechanism change their places, is often used. Henceforth, such a transformation is referred to as a *structural inversion*.

Let us give some examples of structural inversion of mechanisms. In the mechanism shown in Fig. 1.11c, the Cartesian coordinates of pole M of link 3 and the angle φ are output coordinates. Let us consider these coordinates as input coordinates and the coordinates q_1, q_2, q_3 as output ones. Then, the structure changes and it will be depicted by the structural diagram shown in Fig. 1.25. Here, link 3 is “connected” with the frame by a “pair of movability three”. This conditional connection means, in essence, that the position of the link – free in the plane motion of the body – is defined relative to the frame by means of three generalized coordinates. Assur groups consisting of links 1, 2 and 4, 5 are adjoined to link 3 and to the frame.

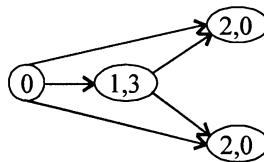


Fig. 1.25. Inversion structural diagram of a mechanism of movability three

In the mechanism of a light-wave guide (Fig. 1.13a) the coordinates of link 6 are input coordinates, while the rotation angles $q_1 - q_6$ of the links at joints are output coordinates. In the inversion transformation input and output coordinates change places and the mechanism obtains the structure shown in Fig. 1.26.



Fig. 1.26. Inversion structural diagram of a laser light-wave guide

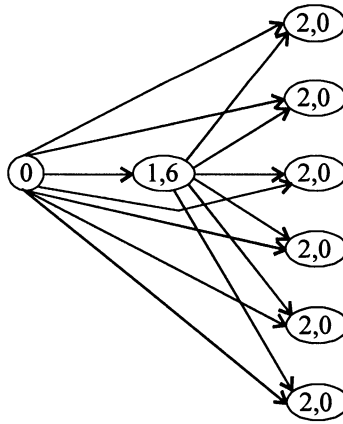


Fig. 1.27. Inversion structural diagram of Stewart platform

In the Stewart platform (Fig. 1.14) the inversion leads to replacement of the input coordinates $q_1 - q_6$ by the platform coordinates $x_M, y_M, z_M, \psi, \vartheta, \varphi$. As a result, the mechanism obtains the structure, shown in Fig. 1.27: one link of movability six, “connected” with the frame (i.e., in essence, considered as a free body, whose position is determined by six coordinates) and six two-bar Assur groups.

It has to be pointed out that the displacement of inputs results in a new mechanism not equivalent to the initial one. It turns out that the new structure is advantageous for solving some but not all problems of analysis; some problems have to be solved using the initial structure.

1.8

Examples of Structural Analysis of Mechanisms

Since any normal mechanism can be generated through successive attachment of structural groups to the frame, it is possible to divide a mechanism into separate structural groups. The division of mechanisms into groups, referred to as *structural analysis*, considerably simplifies its geometric, kinematic and dynamic investigation.

The structural analysis of a mechanism begins with the elaboration of the kinematic diagram. First the number N of movable links and the number

$P = \sum_{s=1}^5 p_s$ of kinematic pairs, belonging to the mechanism, are counted. Next, the

movability s of every kinematic pair and the total number $S = \sum_{s=1}^5 s p_s$ of

movability of all kinematic pairs of the mechanism are determined. With the numerical parameters N , P and S of the kinematic diagram thus obtained the

number w of mechanism degrees of movability is found. According to formula (1.4), the number of degrees of movability of a spatial mechanism is determined by the expression

$$w = 6N - \sum_{s=1}^5 (6-s)p_s = \sum_{s=1}^5 sp_s - 6 \left(\sum_{s=1}^5 p_s - N \right) = S - 6(P - N). \quad (1.6)$$

Analogously, for a planar mechanism we have

$$w = S - 3(P - N). \quad (1.7)$$

Let us consider some examples for determining the number w .

For the planar mechanism of a horizontal forging machine (Fig. 1.28a) we have $N = 9$, $P = S = 13$. Hence, $w = 1$. For the spatial mechanism of a platform (Fig. 1.28b) we have $N = 7$, $P = 9$, $S = 15$, therefore, $w = 3$.

The calculation of the degrees of movability from formulae (1.6) and (1.7) is correct only if there are no excessive constraints and no redundant degrees of movability in the mechanism structure. Their presence would violate the “normal” relations between the number of links, the number of constraints in kinematic pairs of a mechanism and the number of degrees of movability. Let us consider as an example the planar mechanism shown in Fig. 1.28c. This mechanism has one degree of movability, not three as it seems to be at first glance. The point is that at joint B , where three links 2, 3 and 4 come together, not one but two kinematic pairs are generated. Thus, with $N = 8$, $P = 12$, $S = 13$ we have $w = 1$. In this case, a single degree of movability of the mechanism can be realized, if the dimensions of its links satisfy the following conditions: $AO = BC$, $AD = BE$, $OC = AB = DE$. If these conditions are violated due to manufacturing or assembly errors, then the mechanism becomes a rigid invariable system (a truss). The constraints introduced by link 6 duplicate previously imposed constraints, and therefore, they are excessive. When calculating the number w , these constraints turn out to be “unnoticed” because of the excessive movability of roller 7 with respect to link 2.

It follows from the considered example, that the structural formula of a mechanism does not permit to determine separately the number γ of excessive constraints and the number η of redundant movabilities. It only allows to find the difference of these numbers:

$$\gamma - \eta = w - S + B(P - N). \quad (1.8)$$

Here, B is the number of independent parameters characterizing the position of a free body (in spatial motion $B = 6$, in plane motion $B = 3$).

Fig. 1.28d shows a planar mechanism of movability one possessing only prismatic pairs for which $B = 2$. Other mechanisms characterized by numbers $B = 1, 4$ and 5 also exist. In this section such mechanisms and mechanisms with excessive constraints will not be considered.

The next stage in the structural analysis is the division of the mechanism into structural groups, i.e. into kinematic chains satisfying the condition

$$n_c = w_c = S_c - B(P_c - N_c). \quad (1.9)$$

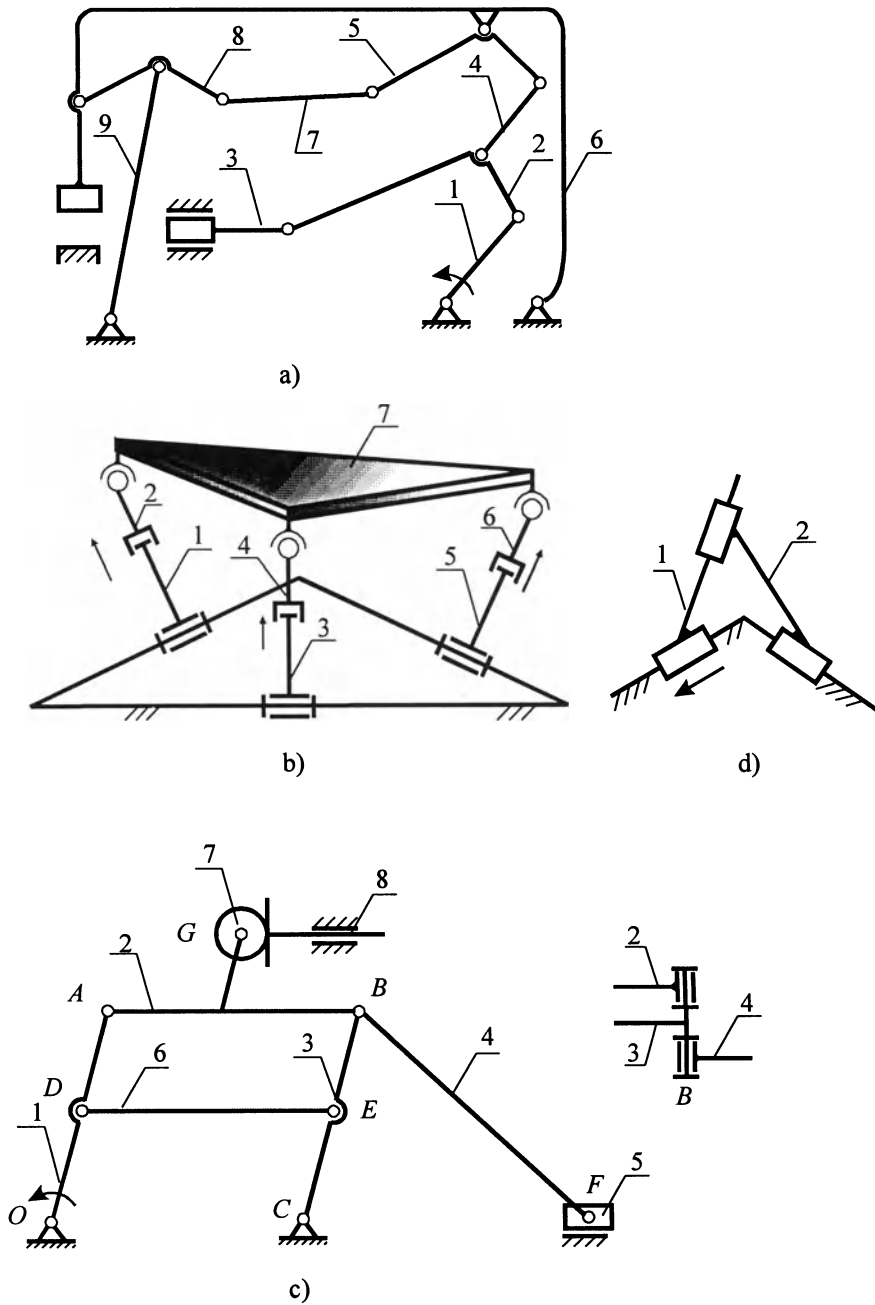


Fig. 1.28. Kinematic diagrams of lever mechanisms: a) horizontal forging machine, b) platform, c) mechanism with excessive constraints and with one redundant degree of movability, d) mechanism with prismatic pairs

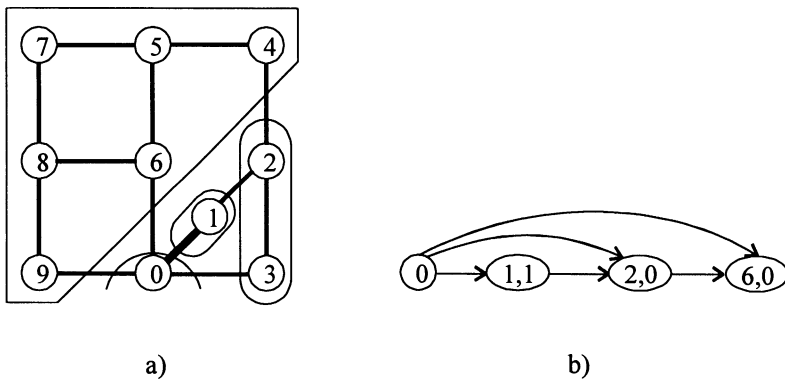


Fig. 1.29. Graph and structural diagram of a horizontal forging machine

The solution of this problem can be achieved with the help of graphs introduced in Sect. 1.4. Let us consider examples illustrating the process of division of mechanisms into structural groups.

Fig. 1.29a shows the graph of the mechanism of a horizontal forging machine (Fig. 1.28a). There are four independent loops in this graph: 0-1-2-3-0, 0-1-2-4-5-6-0, 6-5-7-8-6, 0-6-8-9-0, which differ in at least one vertex or edge. In graph theory, the number of independent loops, called cyclomatic number, is determined from the formula

$$K = P - N. \quad (1.10)$$

In the given case $K = 13 - 9 = 4$, i.e. there are no other independent loops in the graph.

If $K = 0$, then the graph describes an open kinematic chain which possesses "tree"-structure (see Fig. 1.12).

Let us point out a numerical relationship valid for a graph representing the structural group of a mechanism. From formula (1.9) it follows that

$$S_c - n_c = B(P_c - N_c) = BK_c, \quad (1.11)$$

i.e. the difference between the total number S_c of edges of the required graph and the number n_c of root edges (bold) is equal to the number of non-root edges (thin) and is a multiple of *three* for a planar mechanism (of *six* for a spatial mechanism). Obviously, in Assur groups $n_c = 0$.

If in independent loops there are vertices connected by root edges with the 0-vertex (frame) or with already defined structural units, then these graphs describe one-bar structural groups of movability one. In the graph of a mechanism of a horizontal forging machine considered here, vertex 1 and its incident root edge correspond to a group of this kind.

Vertices 2-3 together with the three thin edges in the first independent loop connected with vertex 0 characterize a two-bar Assur group. Condition (1.11) is not fulfilled for any of the remaining loops and for any of the pairs belonging to them. In total the three loops have nine thin edges and this means that these edges

and the vertices 4–9 describe a six-bar Assur group. Fig. 1.29b shows the structural diagram of the mechanism.

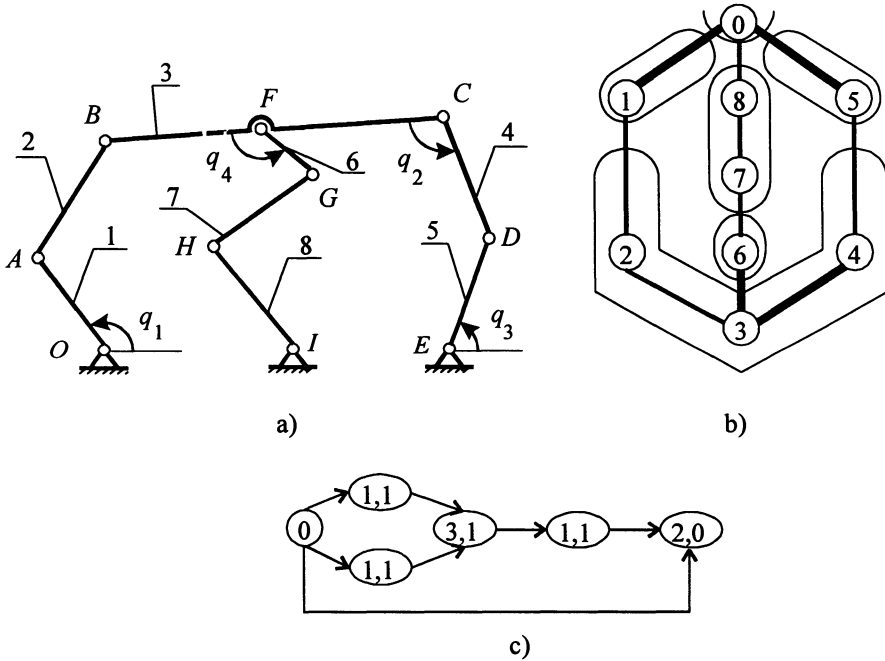


Fig. 1.30. Kinematic and structural diagram of a planar platform and its graph

The graph, shown in Fig. 1.30b, corresponds to the mechanism of a planar platform (Fig. 1.30a). For it $N = 8$, $P = S = 10$, $n = w = 4$. The graph contains three loops: 0-1-2-3-4-5-0, 0-1-2-3-6-7-8-0 and 0-8-7-6-3-4-5-0. Only two of these loops are independent ($K = 2$). The loop 0-1-2-3-4-5-0 contains three thin edges and this means that there is a structural group corresponding to it. This group of links disintegrates into three simple structural groups: Two one-bar groups of movability one (links 1, 5) and one three-bar group of movability one (links 2, 3, 4). Furthermore we distinguish between the one-bar group of movability one (link 6) and the Assur group (links 7 and 8). The structural diagram of the mechanism of the planar platform is depicted in Fig. 1.30c.

Fig. 1.31 shows the graph of a spatial platform (see Fig 1.28b) together with its structural diagram.

If in the planar mechanism depicted in Fig. 1.28c we remove links 6 and 7 which introduce excessive constraints and redundant movabilities we easily get the structural graph (Fig. 1.32a) and the structural diagram (Fig. 1.32b) of this mechanism. To the same mechanism containing the common joint B also the

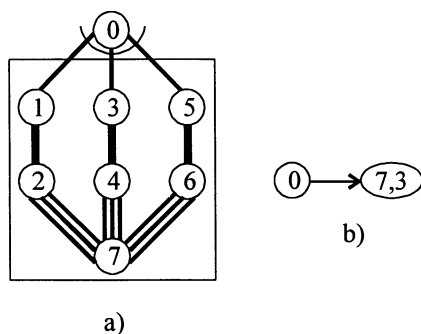


Fig. 1.31. Graph and structural diagram of a spatial platform

graph of Fig. 1.32c belongs. In this case, the structure of the mechanism is different (Fig. 1.32d).

It must be noted that a planar mechanism (Fig. 1.33a) may have a non-planar graph (Fig. 1.33b), for which some edges intersect in a plane. From now on, we will restrict ourselves to mechanisms with planar graphs.

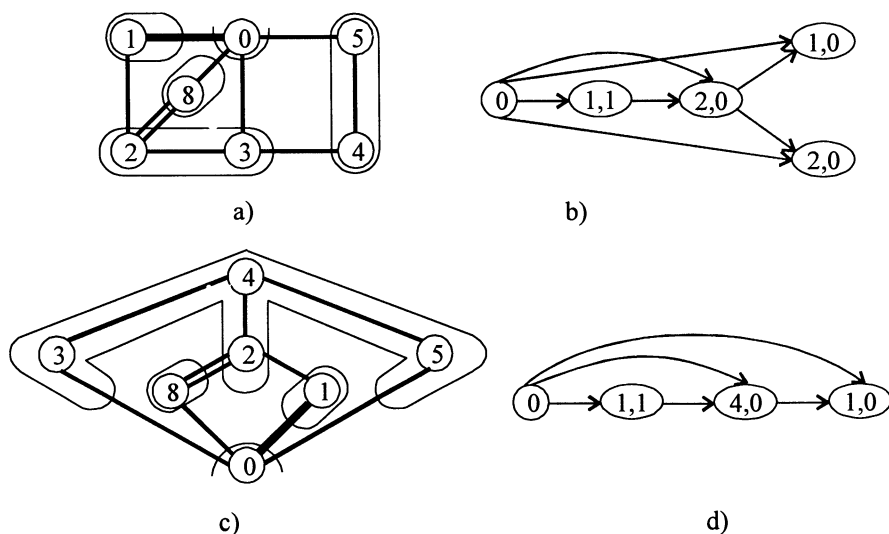


Fig. 1.32. Possible graphs and structural diagrams of a mechanism with common joint

The computer representation of a graph is either the adjacency $\{a_{ij}\}$ with the element a_{ij} being equal to the number of edges connecting vertices i and j or the incidence matrix $\{b_{ik}\}$ with $b_{ik} = 1$ if the i -th vertex and the k -th edge are incident and $b_{ik} = 0$ otherwise.

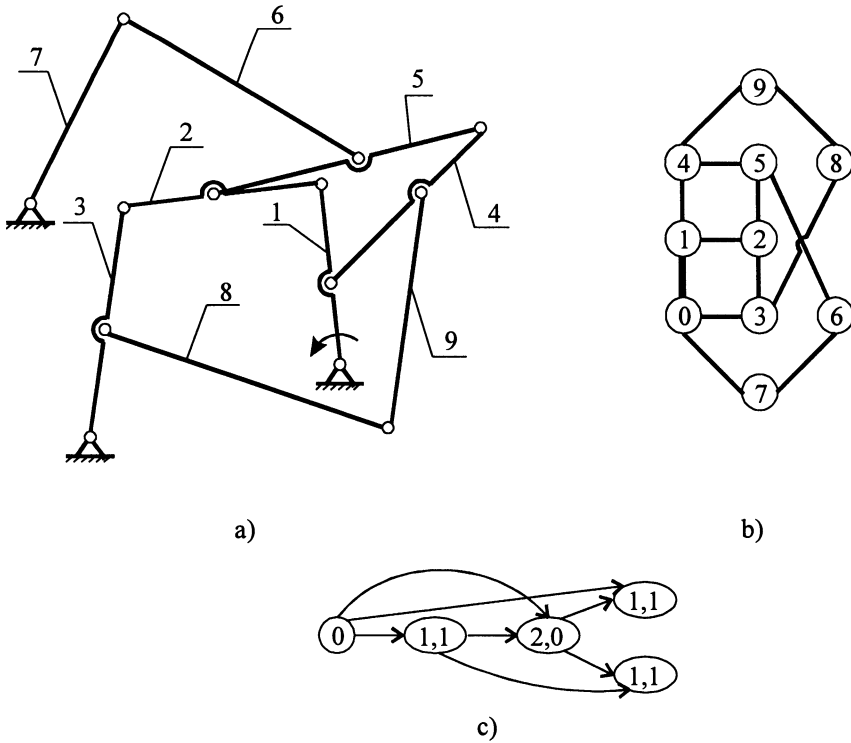


Fig. 1.33. Spatial graph and structural diagram of a planar mechanism

1.9 Problems

- 1.1. Determine the movability of the kinematic pairs shown in Fig. 1.34.

Answer: a) 3 ; b) 2 ; c) 2 ; d) 3 .

- 1.2. Are the kinematic pairs shown in Fig. 1.35 normal structural groups?

Answer: a) $n = 0$, $w = 1$, no; b) $n = w = 0$, yes; c) $n = w = 1$, yes; d) $n = 3$, $w = 4$, no.

- 1.3. Determine the number of degrees of movability and compare it with the number of the indicated inputs for the mechanism of a two-piston combustion engine (Fig. 1.36a), for the mechanism of a press (Fig. 1.36b), for the mechanism of engine valve-actuating drives (Fig. 1.36c) and for the slotted-link mechanism (Fig. 1.36d).

Answer: a) $w = 1$, $n = 2$, mechanism with variable structure;
 b) $w = 2$, $n = 1$, mechanism with variable structure; c) $w = 2$, $n = 1$, redundant degree of movability; d) $w = 1$, $n = 2$, excessive constraints.

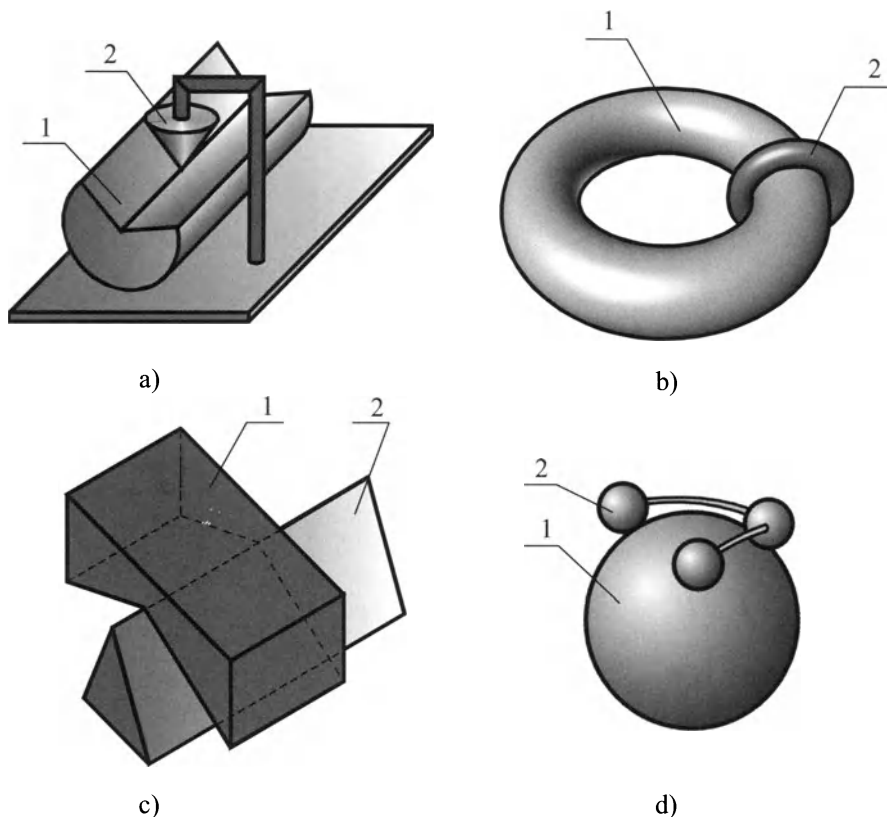


Fig. 1.34. Kinematic pairs

- 1.4. Prove that the degree of movability of the mechanism of a planar platform (Fig. 1.37) is equal to three, regardless of the number of supporting “legs”.
- 1.5. Determine the degree of movability of the mechanism of a washing machine (Fig. 1.38a); of a power shovel (Fig. 1.39a); of a micromanipulator for electronic microscopes (Fig. 1.40a) and do the structural analysis.

Answer: $w = 1, 3, 6$.

- 1.6 Construct the structural diagram of the winding machine (Fig. 1.41a) if generalized actuating forces are applied to links a) 1, 2, 3, b) 1, 11, 3.

Answer: a) Fig. 1.41b; b) Fig. 1.41c

- 1.7. Construct the structural diagram of the planar manipulator depicted in Fig. 1.42a and do the structural inversion.

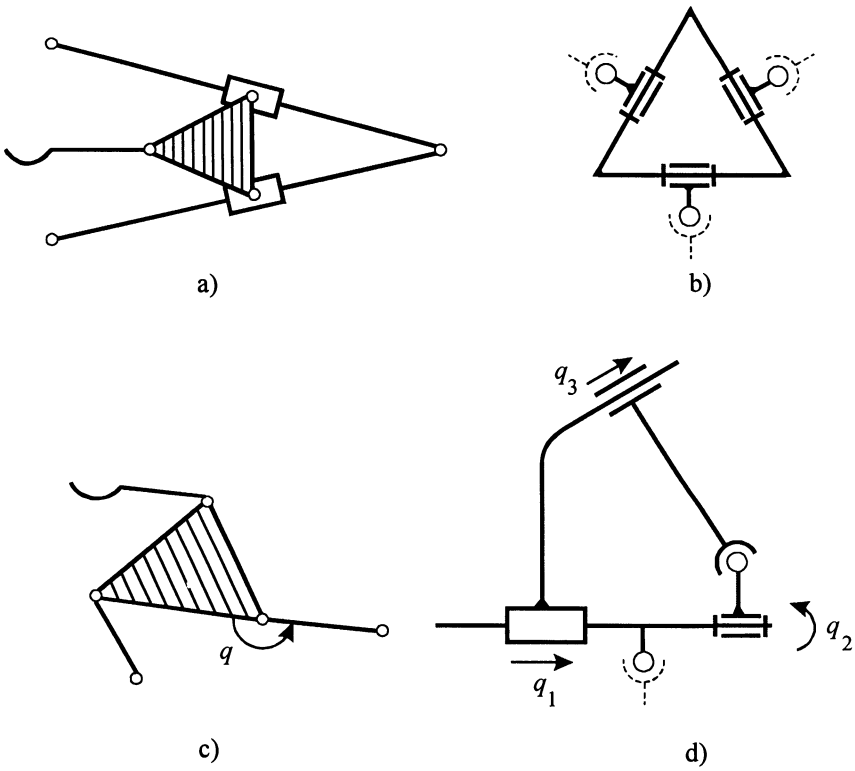


Fig. 1.35. Kinematic chains

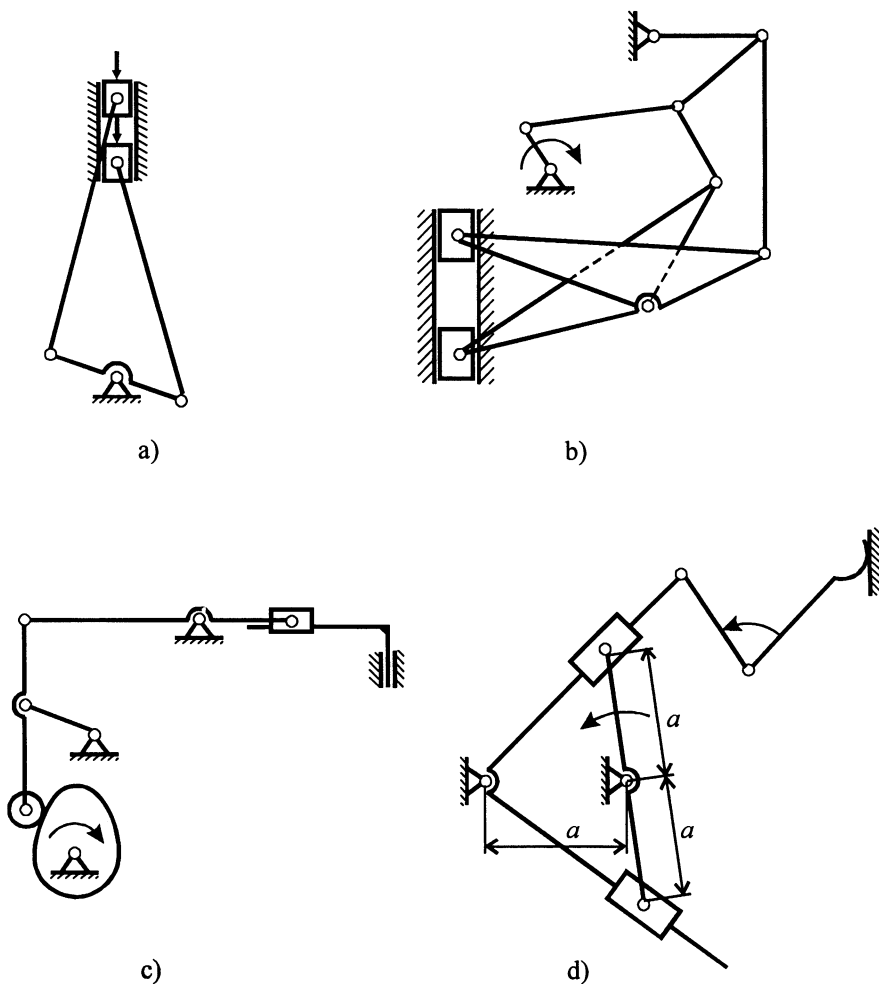


Fig. 1.36. Planar mechanisms: a) combustion engine, b) press, c) engine valve-actuating drives, d) slotted-link mechanism

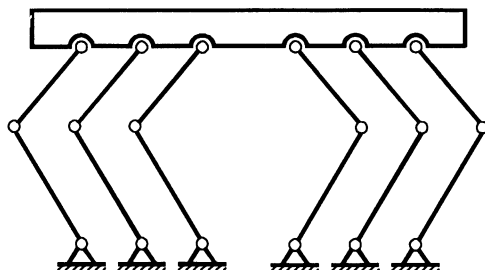
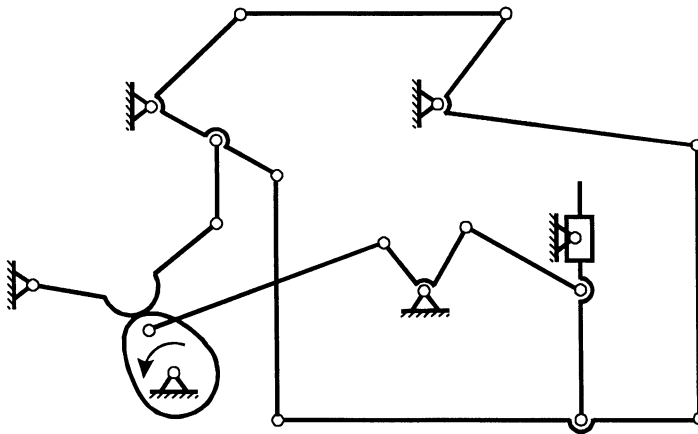
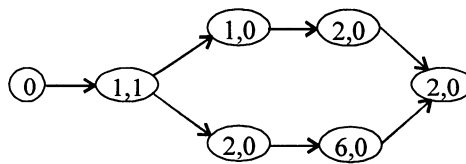


Fig. 1.37. Mechanism of a planar platform

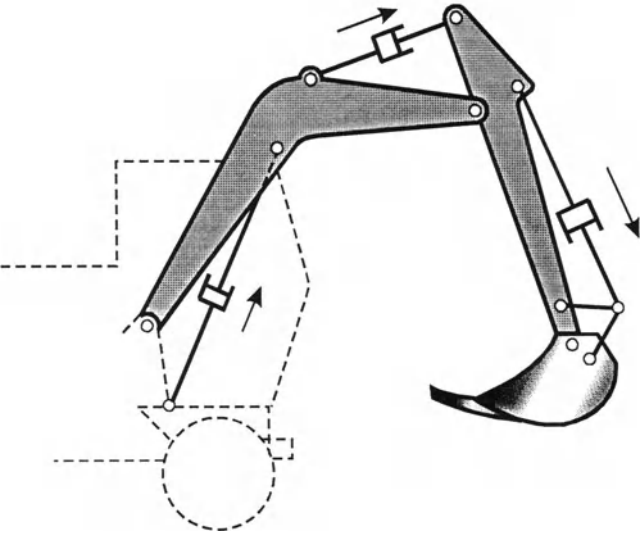


a)

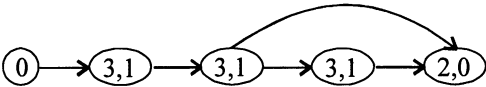


b)

Fig. 1.38. Mechanism of a washing machine and its structural diagram

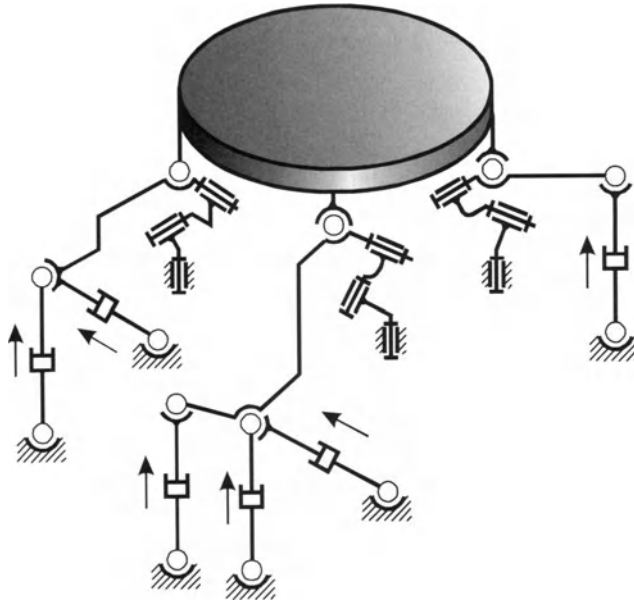


a)

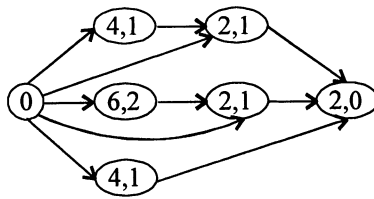


b)

Fig. 1.39. Mechanism of a power shovel and its structural diagram



a)



b)

Fig. 1.40. Mechanism of a micromanipulator for electronic microscopes and its structural diagram

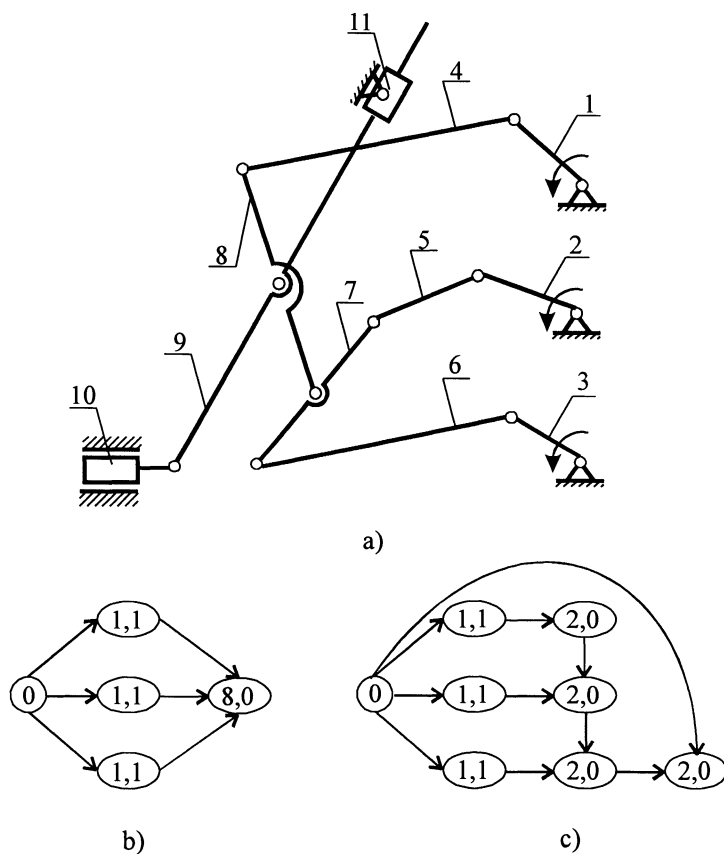


Fig. 1.41. The mechanism of a winding machine and its structural diagram.

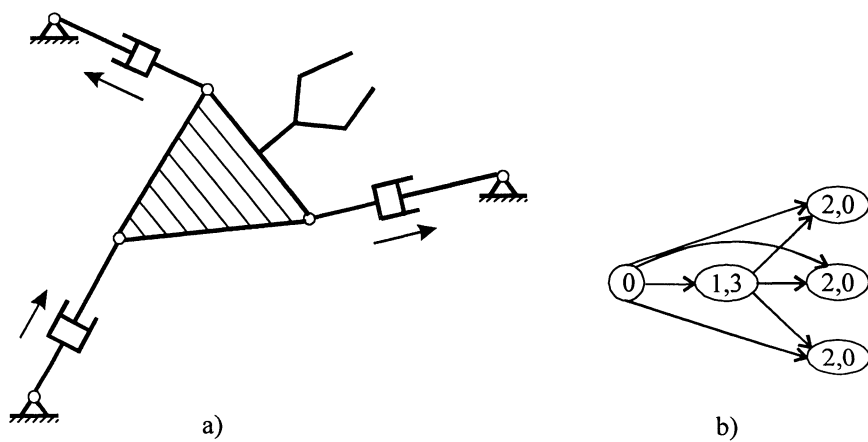


Fig. 1.42. The mechanism of a planar manipulator and its inversion structural diagram

2 Geometric Analysis of Mechanisms

2.1

Problems of Geometric Analysis

The subject of geometric analysis is to determine the *position functions* of a mechanism, i.e. relationships between output mechanism coordinates $x_1, x_2, \dots, x_m$ determining link positions and the input mechanism coordinates $q_1, q_2, \dots, q_n$. For mechanisms with n degrees of movability these functions are written in the form

$$x_s = \Pi_s(q_1, q_2, \dots, q_n) \quad (s = 1, 2, \dots, m). \quad (2.1)$$

The determination of these relationships constitutes the *direct problem of geometric analysis*. Having solved the direct problem it is possible to determine for given time functions of input variables $q_k(t)$ ($k = 1, 2, \dots, n$) the output coordinates as functions of time

$$x_s(t) = \Pi_s[q_1(t), q_2(t), \dots, q_n(t)] \quad (s = 1, 2, \dots, m). \quad (2.2)$$

For a mechanism with one degree of movability, the position functions are functions of a single variable q :

$$x_s = \Pi_s(q) \quad (s = 1, 2, \dots, m). \quad (2.3)$$

In simple cases these functions can be expressed in *explicit form*. For the mechanism shown in Fig. 2.1, for instance, the input coordinate is the rotation angle q of the crank while the angle ψ and the coordinate x_B can be considered as output coordinate.

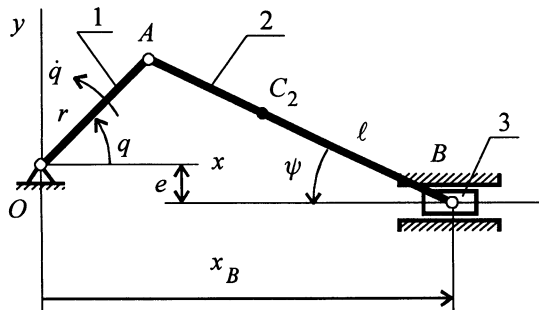


Fig. 2.1. Slider-crank mechanism

Projecting the polygonal line OAB onto axes Ox and Oy we obtain

$$\psi = \arcsin\left(\frac{r \sin q + e}{\ell}\right), \quad (2.4)$$

$$x_B = r \cos q + \sqrt{\ell^2 - (r \sin q + e)^2}, \quad (2.5)$$

where r is the radius of the crank, ℓ is the length of the connecting rod and e is the eccentricity.

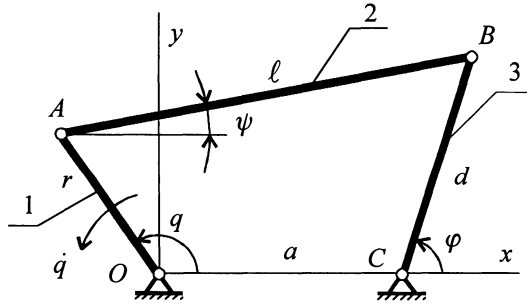


Fig. 2.2. Crank-and-rocker mechanism

In more complex cases, the position functions are determined in *implicit form*. Suppose that in the crank-and-rocker mechanism (Fig. 2.2) determination of the function $\varphi(q)$ is required. Projecting the polygonal line $OABC$ onto axes Ox and Oy and introducing the auxiliary angle ψ , we obtain the equations

$$\begin{aligned} F_1(\varphi, \psi, q) &= r \cos q + \ell \cos \psi - d \cos \varphi - a = 0, \\ F_2(\varphi, \psi, q) &= r \sin q + \ell \sin \psi - d \sin \varphi = 0. \end{aligned} \quad (2.6)$$

They determine in implicit form the dependence $\varphi(q)$. The auxiliary coordinate ψ can be eliminated from Eqs. (2.6). The result is the position function $\varphi(q)$ in the simpler but still implicit form

$$F(\varphi, q) = (r \cos q - d \cos \varphi - a)^2 + (r \sin q - d \sin \varphi)^2 - \ell^2 = 0. \quad (2.7)$$

Let us consider a planar mechanism with three degrees of movability (Fig. 2.3). Here, the input coordinates are the angles q_1, q_2, q_3 . The mechanism can be used for displacing the platform BC from a given position to another position. As output coordinates the coordinates x_M, y_M of some point M of the platform and the angle φ between the line BC and the axis Ox can be chosen. The position functions connecting input and output coordinates will be

$$x_M = \Pi_{x_M}(q_1, q_2, q_3), \quad y_M = \Pi_{y_M}(q_1, q_2, q_3), \quad \varphi = \Pi_{\varphi}(q_1, q_2, q_3). \quad (2.8)$$

Fig. 1.12a showed the open kinematic chain of a spatial robot. Input coordinates of the chain are the coordinates q_1-q_6 determining the relative positions of links. Output coordinates are the coordinates determining the spatial position of the executing device – gripper 6 considered as a rigid body. As output coordinates may be chosen, e.g., coordinates x_M, y_M, z_M of the gripper point M , selected to be a pole, and three Euler angles ψ, θ, φ determining the orientation of the gripper. In this way, the position functions can be represented in the form

$$\begin{aligned} x_M &= \Pi_{x_M}(q_1, \dots, q_6), & y_M &= \Pi_{y_M}(q_1, \dots, q_6), & z_M &= \Pi_{z_M}(q_1, \dots, q_6), \\ \psi &= \Pi_{\psi}(q_1, \dots, q_6), & \theta &= \Pi_{\theta}(q_1, \dots, q_6), & \varphi &= \Pi_{\varphi}(q_1, \dots, q_6). \end{aligned} \quad (2.9)$$

For mechanisms with higher kinematic pairs the position functions also depend on the form of contacting surfaces. This case will be considered separately.

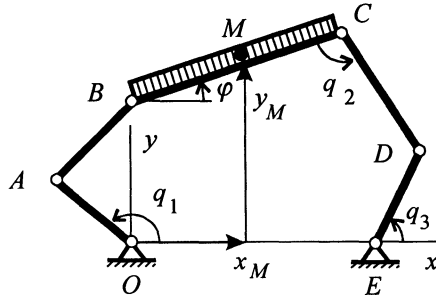


Fig. 2.3. Mechanism of movability three of a planar platform

In general, position functions can not be determined in simple analytical forms and the problem is to elaborate an algorithm for the calculation of output coordinates from input coordinates with the help of a computer. Because of geometric constraints, position functions exist only in a certain limited range of input coordinates. The determination of this area is also part of the direct problem of geometric analysis.

In mechanisms with several degrees of movability it is often necessary to solve the *inverse problem of geometric analysis*: Determine the values of input coordinates $q_1, q_2, \dots, q_n$ for given values of output coordinates $x_1, x_2, \dots, x_m$. In this form the problem of geometric analysis is formulated for *machines with program control*, where, e.g., one needs to determine those positions of engine output links for which the mechanism reaches a given position. Generally, the problem is to determine functions

$$q_k = \Phi_k(x_1, x_2, \dots, x_m) \quad (k = 1, 2, \dots, n). \quad (2.10)$$

Methods for solving the inverse problem will be considered separately.

2.2

Geometric Analysis of Open Kinematic Chains

We consider kinematic chains with “tree”-structure and demonstrate how position functions of such chains can be obtained, i.e. relationships between input and output coordinates.

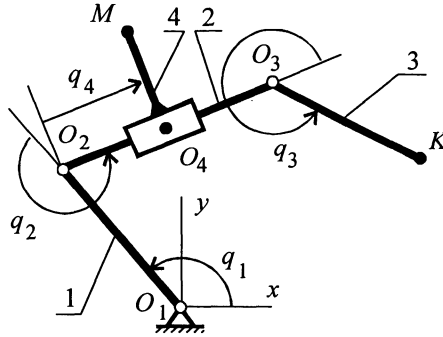


Fig. 2.4. Planar kinematic chain with a tree-structure

Fig. 2.4 shows a planar kinematic chain for which the number of degrees of movability is equal to four. Its input coordinates are the coordinates $q_1 - q_4$ determining the position of every link in the chain relative to the previous link. As output coordinates we choose coordinates $x_{O_s}, y_{O_s}, \varphi_s$ ($s = 1, 2, 3, 4$) specifying the positions of the links of the planar chain relative to the fixed reference system. We successively determine these coordinates for every link:

$$\begin{aligned}
 x_{O_1} &= \text{const}, \quad y_{O_1} = \text{const}, \quad \varphi_1 = q_1, \\
 x_{O_2} &= x_{O_1} + \ell_1 \cos q_1, \quad y_{O_2} = y_{O_1} + \ell_1 \sin q_1, \quad \varphi_2 = q_1 + q_2, \\
 x_{O_3} &= x_{O_2} + \ell_2 \cos(q_1 + q_2), \quad y_{O_3} = y_{O_2} + \ell_2 \sin(q_1 + q_2), \quad \varphi_3 = q_1 + q_2 + q_3, \\
 x_{O_4} &= x_{O_2} + \ell_4 \cos(q_1 + q_2), \quad y_{O_4} = y_{O_2} + \ell_4 \sin(q_1 + q_2), \quad \varphi_4 = q_1 + q_2.
 \end{aligned} \tag{2.11}$$

Henceforth, we will call *equations of geometric analysis* the equations relating output coordinates to input coordinates. In the above case, the equations have a single solution, i.e. the specification of input coordinates determines uniquely the configuration of kinematic chain with “tree”-structure. After specifying link coordinates it is easy to find the coordinates of poles M and K :

$$\begin{aligned}
 x_M &= x_{O_4} - \ell_4 \sin(q_1 + q_2), & y_M &= y_{O_4} + \ell_4 \cos(q_1 + q_2), \\
 x_K &= x_{O_3} + \ell_3 \cos(q_1 + q_2 + q_3), & y_K &= y_{O_3} + \ell_3 \sin(q_1 + q_2 + q_3).
 \end{aligned}$$

Similarly, the coordinates of any other point of the chain can be expressed.

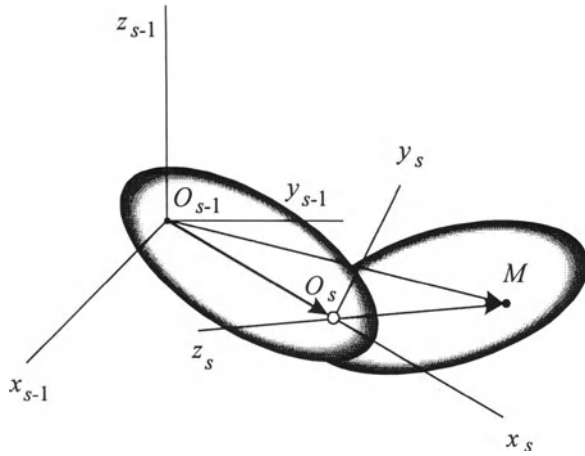


Fig. 2.5. Links of a spatial kinematic chain with a tree-structure

Let us consider the solution of the direct geometric problem for a spatial kinematic chain. Let two links of this chain be connected by a kinematic pair of movability one (Fig. 2.5). On the two links orthogonal reference systems $O_{s-1}x_{s-1}y_{s-1}z_{s-1}$ and $O_sx_sy_s z_s$, respectively are fixed. The position of the s -th coordinate system relative to the $(s-1)$ -st one is determined by the vector $\mathbf{O}_{s-1}\mathbf{O}_s$ and by the *direction cosine matrix*

$$A_{s-1,s} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}_{s-1,s}, \quad (2.12)$$

where

$$\begin{aligned} \alpha_{11} &= \cos(x_{s-1}, x_s), & \alpha_{12} &= \cos(x_{s-1}, y_s), & \alpha_{13} &= \cos(x_{s-1}, z_s), \\ \alpha_{21} &= \cos(y_{s-1}, x_s), & \alpha_{22} &= \cos(y_{s-1}, y_s), & \alpha_{23} &= \cos(y_{s-1}, z_s), \\ \alpha_{31} &= \cos(z_{s-1}, x_s), & \alpha_{32} &= \cos(z_{s-1}, y_s), & \alpha_{33} &= \cos(z_{s-1}, z_s). \end{aligned} \quad (2.13)$$

Because of the orthogonality of axes $O_{s-1}x_{s-1}y_{s-1}z_{s-1}$ and $O_sx_sy_s z_s$, the elements of the matrix $A_{s-1,s}$ satisfy six relations

$$\alpha_{k1}^2 + \alpha_{k2}^2 + \alpha_{k3}^2 = 1, \quad \alpha_{i1}\alpha_{k1} + \alpha_{i2}\alpha_{k2} + \alpha_{i3}\alpha_{k3} = 0, \quad (i, k = 1, 2, 3).$$

Therefore only three out of nine direction cosines are independent.

The vector $\mathbf{O}_{s-1}\mathbf{O}_s$ can be specified by its projections onto the axes of the $(s-1)$ -st coordinate system

$$\mathbf{r}_{O_s}^{(s-1)} = (x_{O_s}^{(s-1)}, y_{O_s}^{(s-1)}, z_{O_s}^{(s-1)})^T, \quad (2.14)$$

i.e. by the column containing the coordinates of point O_s in the system

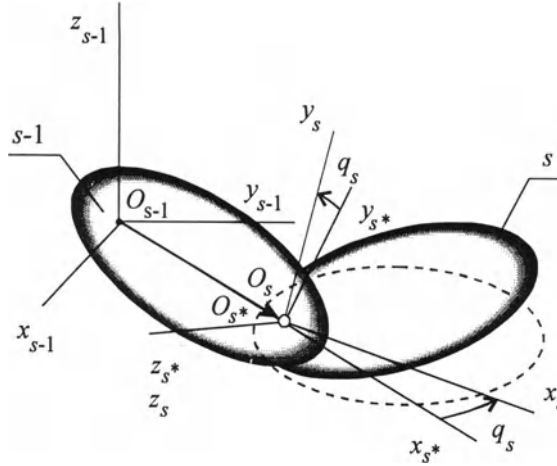


Fig. 2.6. Revolute kinematic pair of a spatial kinematic chain

$O_{s-1}x_{s-1}y_{s-1}z_{s-1}$ (the exponent T indicates transposition).

Let the links $(s-1)$ and s be interconnected by a *revolute* joint (Fig. 2.6). In this case, let the axis $O_s z_s$ be directed *along the rotation axis* of the joint in such a way that the input coordinate q_s is the angle of rotation of the link s relative to link $(s-1)$, measured from some initial position for which $q_s = 0$. The initial position of system the $O_s x_s y_s z_s$ is $O_{s*} x_{s*} y_{s*} z_{s*}$. Successively defining the direction cosines of the angles between the axes of systems $(s-1)$ and s , and then those between systems $(s-1)$ and s^* , we get the relation

$$A_{s-1,s} = A_{s-1,s*} A_{s*,s}. \quad (2.15)$$

This relation is true for any three coordinate systems. The matrix $A_{s-1,s*}$ is composed of direction cosines which remain constant when coordinate q changes:

$$A_{s-1,s*} = A_{s-1,s}(0) = \begin{pmatrix} \alpha_{11}(0) & \alpha_{12}(0) & \alpha_{13}(0) \\ \alpha_{21}(0) & \alpha_{22}(0) & \alpha_{23}(0) \\ \alpha_{31}(0) & \alpha_{32}(0) & \alpha_{33}(0) \end{pmatrix}. \quad (2.16)$$

The matrix $A_{s*,s}$ is a function of the rotation angle q_s . Since the rotation axis of the joint coincides with the axis $O_s z_s$, the elements of this matrix, according to relations (2.13), form a *rotation matrix* which, in what follows, will be denoted

$$P(q_s) = A_{s*,s} = \begin{pmatrix} \cos q_s & -\sin q_s & 0 \\ \sin q_s & \cos q_s & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.17)$$

Since the position of point O_s in system $(s-1)$ is invariable the column of projections of vector $\mathbf{O}_{s-1}\mathbf{O}_s$ is constant

$$r_{O_s}^{(s-1)} = r_{O_s}^{(s-1)}(0) = \text{const.} \quad (2.18)$$

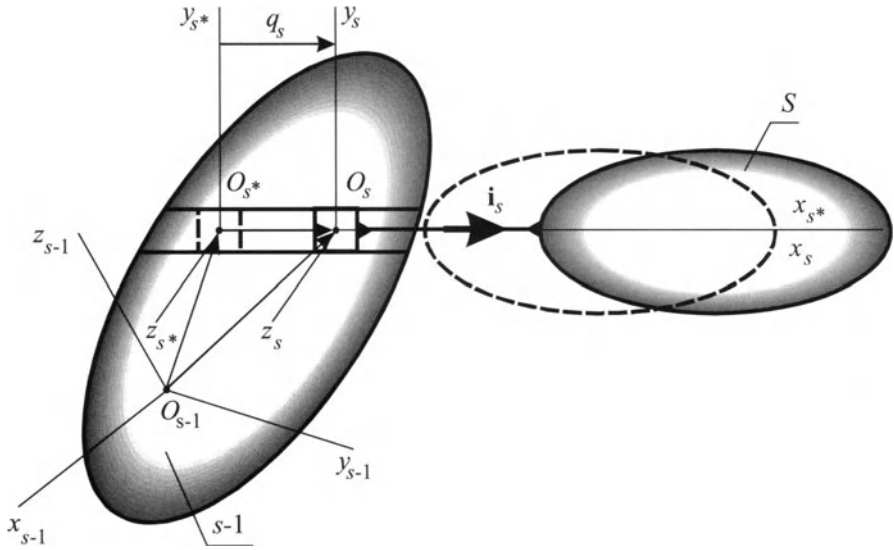


Fig. 2.7. Prismatic kinematic pair of a spatial kinematic chain

Let the links $(s-1)$ and s be connected by a *prismatic* joint (Fig. 2.7). Let the axis $O_s x_s$ be directed *along the line of translational motion* of link s relative to link $(s-1)$. We denote the relative displacement of the links q_s . At the initial position when $q_s = 0$ the system $O_s x_s y_s z_s$ is $O_{s*} x_{s*} y_{s*} z_{s*}$. Obviously, the translational motion of link s does not change the orientation of the axes. Therefore, we have

$$A_{s-1,s} = A_{s-1,s*} = A_{s-1,s}(0) = \text{const.} \quad (2.19)$$

From Fig. 2.7 it follows that

$$\mathbf{O}_{s-1}\mathbf{O}_s = \mathbf{O}_{s-1}\mathbf{O}_{s*} + \mathbf{O}_{s*}\mathbf{O}_s = \mathbf{O}_{s-1}\mathbf{O}_{s*} + q_s \mathbf{i}_s,$$

where $\mathbf{i}_s$ is the unit vector of axis $O_s x_s$. Projecting this equality onto the axes of the system $O_{s-1} x_{s-1} y_{s-1} z_{s-1}$ and taking into account that $\cos(x_{s-1}, x_{s*}) = \alpha_{11}(0)$, $\cos(y_{s-1}, x_{s*}) = \alpha_{21}(0)$, $\cos(z_{s-1}, x_{s*}) = \alpha_{31}(0)$, we obtain

$$r_{O_s}^{(s-1)} = r_{O_s}^{(s-1)}(0) + q_s (\alpha_{11}(0), \alpha_{21}(0), \alpha_{31}(0))^T. \quad (2.20)$$

Next, the equations of geometric analysis will be derived that will allow us to determine the coordinates of link s of the kinematic chain, shown in Fig. 2.5, and the Cartesian coordinates of some point M . Projecting the polygonal line $O_{s-1}O_sM$ onto the axes of the coordinate system $(s-1)$ we have

$$r_M^{(s-1)} = r_{O_s}^{(s-1)} + A_{s-1,s} r_M^{(s)}. \quad (2.21)$$

Here, it is taken into account that the transformation of the constant column of coordinates of point M from system s to system $(s-1)$ is reduced to the multiplication of this column by the matrix of direction cosines of the angles between the axes of these systems.

If the identity $1 \equiv 1$ is added to relation (2.21), then the connection between the input and the output coordinates of the kinematic chain can be as matrix

$$R_M^{(s-1)} = H_{s-1,s} R_M^{(s)}. \quad (2.22)$$

Four-dimensional column matrices of coordinates are

$$R_M^{(s-1)} = (x_M^{(s-1)}, y_M^{(s-1)}, z_M^{(s-1)}, 1)^T, \quad R_M^{(s)} = (x_M^{(s)}, y_M^{(s)}, z_M^{(s)}, 1)^T. \quad (2.23)$$

Furthermore a *transformation matrix* $H_{s-1,s}$ from system s to system $(s-1)$ is written in the block form

$$H_{s-1,s} = \left(\begin{array}{c|c} A_{s-1,s} & r_{O_s}^{(s-1)} \\ \hline (0,0,0) & 1 \end{array} \right). \quad (2.24)$$

Let us point out some properties of this transformation matrix and the direction cosine matrix $A_{s-1,s}$ figuring in it. The direction cosine matrix is an orthogonal matrix (the absolute value of its determinant is equal to unity). To invert such a matrix it is sufficient to transpose it:

$$A_{s-1,s}^{-1} = A_{s-1,s}^T = A_{s,s-1}. \quad (2.25)$$

The components of the inverse transformation matrix can be expressed by the blocks of the matrix $H_{s-1,s}$ itself:

$$H_{s-1,s}^{-1} = \left(\begin{array}{c|c} A_{s-1,s}^T & -A_{s-1,s}^T r_{O_s}^{(s-1)} \\ \hline (0,0,0) & 1 \end{array} \right) = H_{s,s-1}. \quad (2.26)$$

Because of relations (2.15)–(2.17) and (2.20) the transformation matrix is a function of the input coordinate q_s ; its form depends on the type of kinematic pair. For a revolute joint connecting link s with link $(s-1)$ we have

$$H_{s-1,s}^R = \left(\begin{array}{c|c} A_{s-1,s}(0)P(q_s) & r_{O_s}^{(s-1)}(0) \\ \hline (0,0,0) & 1 \end{array} \right). \quad (2.27)$$

For a prismatic joint we have

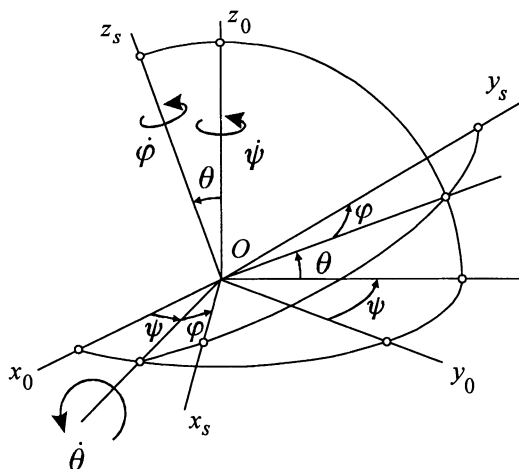


Fig. 2.8. Euler angles

$$A_{s-1,s}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (s=1,2,3), \quad A_{3,4}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then, with these matrices, we find

$$A_{0,1} = A_{0,1}(0)P(q_1) = \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{1,2} = A_{1,2}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

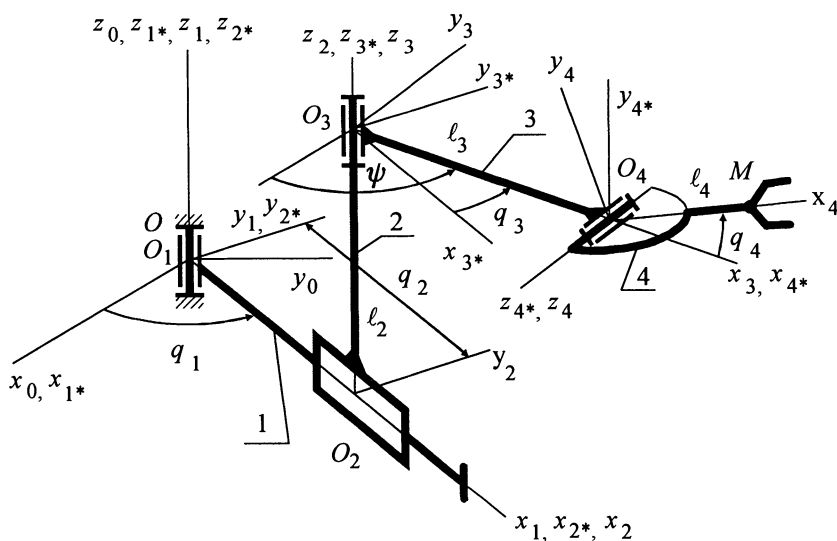


Fig. 2.9. Robot of movability four with definitions of coordinate systems

$$A_{2,3} = A_{2,3}(0)P(q_3) = \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{3,4} = A_{3,4}(0)P(q_4) = \begin{pmatrix} \cos q_4 & -\sin q_4 & 0 \\ 0 & 0 & -1 \\ \sin q_4 & \cos q_4 & 0 \end{pmatrix}.$$

We define the column marices of the coordinates of poles O_s in the s -th reference system according to formula (2.20):

$$r_{O_1}^{(0)} = (0,0,0)^T, \quad r_{O_2}^{(1)} = r_{O_2}^{(1)}(0) + q_2(\alpha_{11}(0), \alpha_{21}(0), \alpha_{31}(0))^T = (q_2, 0, 0)^T,$$

$$r_{O_3}^{(2)} = (0, 0, \ell_2)^T, \quad r_{O_4}^{(3)} = (\ell_3, 0, 0)^T.$$

Then, we write down the transformation matrices in the form

$$H_{0,1}(q_1) = \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_{1,2}(q_2) = \begin{pmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H_{2,3}(q_3) = \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 & 0 \\ \sin q_3 & \cos q_3 & 0 & 0 \\ 0 & 0 & 1 & \ell_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_{3,4}(q_4) = \begin{pmatrix} \cos q_4 & -\sin q_4 & 0 & \ell_3 \\ 0 & 0 & -1 & 0 \\ \sin q_4 & \cos q_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Pole M of the gripper, belonging to the fourth link, has coordinates $x_M^{(4)} = \ell_4$, $y_M^{(4)} = z_M^{(4)} = 0$. In accordance with formula (2.29) we have

$$R_M^{(0)} = (x_M^{(0)}, y_M^{(0)}, z_M^{(0)}, 1)^T = H_{0,1}(q_1)H_{1,2}(q_2)H_{2,3}(q_3)H_{3,4}(q_4)R_M^{(4)},$$

where $R_M^{(4)} = (\ell_4, 0, 0, 1)^T$. After multiplying the matrices we obtain the Cartesian coordinates of pole M of the gripper

$$\begin{aligned} x_M^{(0)} &= q_2 \cos q_1 + (\ell_4 \cos q_4 + \ell_3) \cos(q_1 + q_3), \\ y_M^{(0)} &= q_2 \sin q_1 + (\ell_4 \cos q_4 + \ell_3) \sin(q_1 + q_3), \\ z_M^{(0)} &= \ell_4 \sin q_4 + \ell_2. \end{aligned} \tag{2.33}$$

We define the angular orientation of the gripper with the help of the direction cosine matrix $A_{0,4} = A_{0,1}A_{1,2}A_{2,3}A_{3,4} =$

$$\begin{pmatrix} \cos q_4 \cos(q_1 + q_3) & -\sin q_4 \cos(q_1 + q_3) & \sin(q_1 + q_3) \\ \cos q_4 \sin(q_1 + q_3) & -\sin q_4 \sin(q_1 + q_3) & -\cos(q_1 + q_3) \\ \sin q_4 & \cos q_4 & 0 \end{pmatrix}.$$

In accordance with formulae (2.32) we find $\cos \theta = 0$, $\sin \theta = 1$, $\cos \psi = \cos(q_1 + q_3)$, $\sin \psi = \sin(q_1 + q_3)$, $\cos \varphi = \cos q_4$, $\sin \varphi = \sin q_4$. It is not difficult to determine the Euler angles from the above relations. In an analogous way the coordinates of the other robot links are determined.

For the description of geometry of open kinematic chains also other approaches are possible. E.g., a method based on the use of formulae for finite rotations of a rigid body is broadly applied when solving the direct problem of manipulative systems [1].

2.3

Derivation of Equations of Geometric Analysis for Closed Kinematic Chains

The method of geometric analysis of open kinematic chains considered above can be used to determine the position functions of mechanisms with closed kinematic chains. To this end, a closed kinematic chain is reduced to an open chain with "tree"-structure through opening of certain kinematic pairs. Following this we introduce coordinates determining the position of every link of the "tree" with respect to the previous link (relative coordinates) or with respect to the frame (absolute coordinates). Part of them can be input coordinates of a mechanism; at the same time *additional generalized coordinates* appear. The total number of these additional coordinates is equal to number of removed constraints, which number we will call the *degree of closure* of the mechanism. Closure conditions are derived for all removed constraints. These conditions are actually the equations of geometric analysis. With their help the position functions of mechanisms with closed chains are determined.

For *planar* mechanisms equations of geometric analysis are obtained, e.g., by projecting closed loops on the axes of a Cartesian coordinate system. As an example, let us derive the equations of geometric analysis for the planar mechanism of movability three shown in Fig.2.10. Its input coordinates are the angles q_1, q_2, q_3 . Let us open the closed chain at joints E and G , releasing the four constraints imposed by these joints on the plane motion of the mechanism. After opening the kinematic chain one has a structure of a branched „tree“. We define in a unique way the position of its links by angles $q_1 - q_3, \varphi_2, \varphi_3, \varphi_6, \varphi_7$. As mentioned earlier, the angles $\varphi_2, \varphi_3, \varphi_6, \varphi_7$ are additional coordinates. Projecting the polygonal lines $OABCDEO$ and $OABMFG$ on axes Ox and Oy , we obtain equations which express the closure conditions for the kinematic chain of the mechanism

$$\begin{aligned} x_0 + \ell_1 \cos q_1 + \ell_2 \cos \varphi_2 + \ell_3 \cos \varphi_3 + \ell_4 \cos(\varphi_3 + q_2 + \pi) - \ell_5 \cos q_3 &= x_E, \\ y_0 + \ell_1 \sin q_1 + \ell_2 \sin \varphi_2 + \ell_3 \sin \varphi_3 + \ell_4 \sin(\varphi_3 + q_2 + \pi) - \ell_5 \sin q_3 &= y_E, \\ x_0 + \ell_1 \cos q_1 + \ell_2 \cos \varphi_2 + BM \cos \varphi_3 + \ell_6 \cos \varphi_6 + \ell_7 \cos \varphi_7 &= x_G, \\ y_0 + \ell_1 \sin q_1 + \ell_2 \sin \varphi_2 + BM \sin \varphi_3 + \ell_6 \sin \varphi_6 + \ell_7 \sin \varphi_7 &= y_G. \end{aligned} \quad (2.34)$$

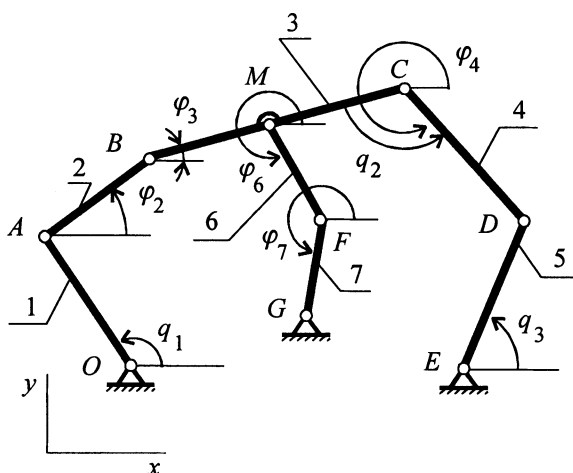


Fig. 2.10. Planar mechanism of movability three

It is convenient to compose the equations of geometrical analysis on the basis of structural groups. Such an approach allows for the whole system of equations to be divided into several independent groups of equations of lower order, which can successively be solved for each group, following the sequence of their attachment to the frame.

The *group-based method* for deriving equations is analogous to the method for deriving the equations of geometric analysis for a whole mechanism. Input and output coordinates are introduced for every structural group. As *input coordinates of a group* the coordinates that define the positions of those kinematic pairs with the help of which the considered group is attached to the previous one, and the mechanism input coordinates that fall into the given group are chosen. *Output coordinates of a group* are the coordinates determining the positions of those kinematic pairs to which the next groups are attached as well as the output coordinates of the mechanism. In every group an opening of some pairs is undertaken, thus reducing the group to one with a „tree“-structure, and additional generalized coordinates, which we will call *group coordinates*, are introduced. Further the equations of geometric analysis which relate the group coordinates to the input and the output ones are obtained. These equations are referred to as *group equations*.

Let us compose the group equations for the above considered mechanism of movability three. First, we consider the one-bar groups of movability one OA and ED , constituting the first structural layer. Input coordinates in group OA are coordinates x_0, y_0 of the frame attachment point O and the mechanism input coordinate q_1 . Correspondingly, in group ED input coordinates are x_E, y_E and angle q_3 . Since the considered groups are open chains, it is not necessary to introduce group coordinates here. The specification of q_1 and q_3 uniquely

defines the configuration of these groups. Output coordinates are the coordinates of points A and D at which the next group $ABCD$ is attached. The equations

$$\begin{aligned} x_A &= x_O + \ell_1 \cos q_1, & x_D &= x_E + \ell_5 \cos q_3, \\ y_A &= y_O + \ell_1 \sin q_1, & y_D &= y_E + \ell_5 \sin q_3 \end{aligned} \quad (2.35)$$

define relationships between the output and the input coordinates.

In the group $ABCD$ of the second layer, input coordinates are the coordinates of attachment points x_A, y_A, x_D, y_D as well as the mechanism input coordinate q_2 . Let us open this group at joint D by removing two constraints. The open chain obtained in this way has the structure of a simple „tree“ and it possesses three degrees of movability. With the given coordinates of point A the positions of links 2, 3, 4 of this chain are uniquely defined by the specification of angles $\varphi_2, \varphi_3, \varphi_4 = \varphi_3 + q_2 + \pi$ between links 2, 3, 4 and axis Ox . It is easy to see that the angles φ_2 and φ_3 are the group coordinates of group $ABCD$. The group equations can be obtained as closure conditions for the chain at point D . Projecting the polygonal line $ABCD$ on axes Ox and Oy , we have

$$\begin{aligned} \ell_2 \cos \varphi_2 + \ell_3 \cos \varphi_3 + \ell_4 \cos(\varphi_3 + q_2 + \pi) &= x_D - x_A, \\ \ell_2 \sin \varphi_2 + \ell_3 \sin \varphi_3 + \ell_4 \sin(\varphi_3 + q_2 + \pi) &= y_D - y_A. \end{aligned} \quad (2.36)$$

Group $ABCD$ can be opened at another joint, e.g. at joint C , dividing the chain into two open chains ABC and DC . The group equations

$$\begin{aligned} x_A + \ell_2 \cos \varphi_2 + \ell_3 \cos \varphi_3 &= x_D - \ell_4 \cos(\varphi_3 + q_2), \\ y_A + \ell_2 \sin \varphi_2 + \ell_3 \sin \varphi_3 &= y_D - \ell_4 \sin(\varphi_3 + q_2), \end{aligned} \quad (2.37)$$

derived from the closure conditions at point C , coincide identically with the trigonometric Eqs. (2.36). It is possible to choose as group coordinates the coordinates x_B, y_B of point B . Then, the closure conditions at point D would be written in the form of algebraic equations

$$\begin{aligned} (x_B - x_A)^2 + (y_B - y_A)^2 &= \ell_2^2, \\ (x_D - x_B)^2 + (y_D - y_B)^2 &= BD^2, \end{aligned} \quad (2.38)$$

where $BD^2 = \ell_3^2 + \ell_4^2 - 2\ell_3\ell_4 \cos q_2$. The output coordinates of group $ABCD$ are determined from the conditions

$$\begin{aligned} x_M &= x_A + \ell_2 \cos \varphi_2 + BM \cos \varphi_3, \\ y_M &= y_A + \ell_2 \sin \varphi_2 + BM \sin \varphi_3. \end{aligned} \quad (2.39)$$

In group MFG of the third structural layer, output coordinates are the coordinates x_M, y_M, x_G, y_G determining the positions of joints M and G . Opening the Assur group MFG at joint G , we obtain an open kinematic chain.

We define the positions of its links by means of group coordinates φ_6 and φ_7 . Then, the group equations are written in the form

$$\begin{aligned} \ell_6 \cos \varphi_6 + \ell_7 \cos \varphi_7 &= x_G - x_M, \\ \ell_6 \sin \varphi_6 + \ell_7 \sin \varphi_7 &= y_G - y_M. \end{aligned} \quad (2.40)$$

In *spatial* mechanisms the equations of geometric analysis are obtained through a multiplication of transformation matrices. These matrices have already been stated for prismatic and for revolute pairs. Let us show how to construct transformation matrices for other types of lower kinematic pairs.

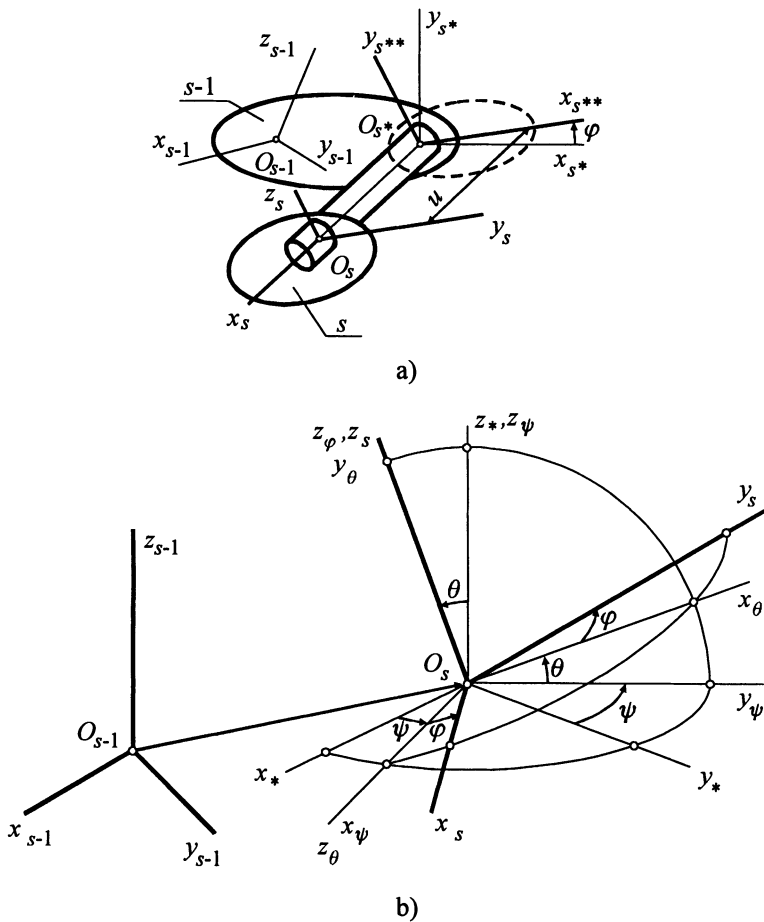


Fig. 2.11. Definition of the transformation matrices for kinematic pairs: a) for a cylindric pair, b) for a spheric pair

The *cylindric* pair can be considered as a series of a revolute and a prismatic pair (Fig. 2.11a). In accordance with the rules considered above, we introduce the

following coordinate systems: the system $O_{s-1}x_{s-1}y_{s-1}z_{s-1}$, related to link $(s-1)$; system $O_sx_sy_sz_s$ specifying the initial position of link s ; the auxiliary system $O_{s**}x_{s**}y_{s**}z_{s**}$ characterizing the rotation of the link s ; the system $O_sx_sy_sz_s$ related to the link s . We denote by φ the rotation angle and by u the translational displacement in the cylindric pair. With the direction cosine matrices

$$A_{s-1,s**} = A_{s-1,s}(0)P(\varphi), \quad A_{s**,s} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and with the column matrices of pole coordinates $r_{o_{s**}}^{(s-1)}$ and $r_{o_s}^{(s**)} = (0,0,u)^T$ we define the transformation matrix, characterizing rotational and translational motion

$$H_{s-1,s**}^R = \left(\begin{array}{c|c} A_{s-1,s}(0)P(\varphi) & r_{o_{s**}}^{(s-1)} \\ \hline (0,0,0) & 1 \end{array} \right), \quad H_{s**,s}^P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain the transformation matrix for a cylindric pair in the form of a matrix product

$$H_{s-1,s}^C(\varphi, u) = H_{s-1,s**}(\varphi)H_{s**,s}^P(u). \quad (2.41)$$

Analogously, a *spheric* pair can be considered as a series of three revolute pairs in every one of which a rotation, corresponding to one of the Euler angles ψ, θ, φ is carried out. We introduce the coordinate systems shown in Fig. 2.11b: system $O_{s-1}x_{s-1}y_{s-1}z_{s-1}$, related to link $(s-1)$; system $O_sx_sy_sz_s$ with axes parallel to axes $O_{s-1}x_{s-1}, O_{s-1}y_{s-1}, O_{s-1}z_{s-1}$ and with origin O_s ; system $O_sx_\psi y_\psi z_\psi$ obtained by rotation of system $O_sx_sy_sz_s$ through angle ψ about axis $O_s z_s$; system $O_sx_\theta y_\theta z_\theta$, obtained by rotation of system $O_sx_\psi y_\psi z_\psi$ through angle θ about axis $O_s z_\theta$ coinciding with the „nodal line“ x_ψ ; system $O_sx_s y_s z_s$ related to link s and obtained by rotation of system $O_sx_\theta y_\theta z_\theta$ through angle φ about axis $O_s z_s$, coinciding with axis $O_s y_\theta$. For the transformation matrix $H_{s-1,s}^C(\psi, \theta, \varphi)$ the expression (2.24) is valid. The direction cosine matrix is defined as a product of the rotation matrices $P(\psi), P(\theta), P(\varphi)$ and of the matrices

$$U_{\psi\theta} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_{\theta\varphi} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

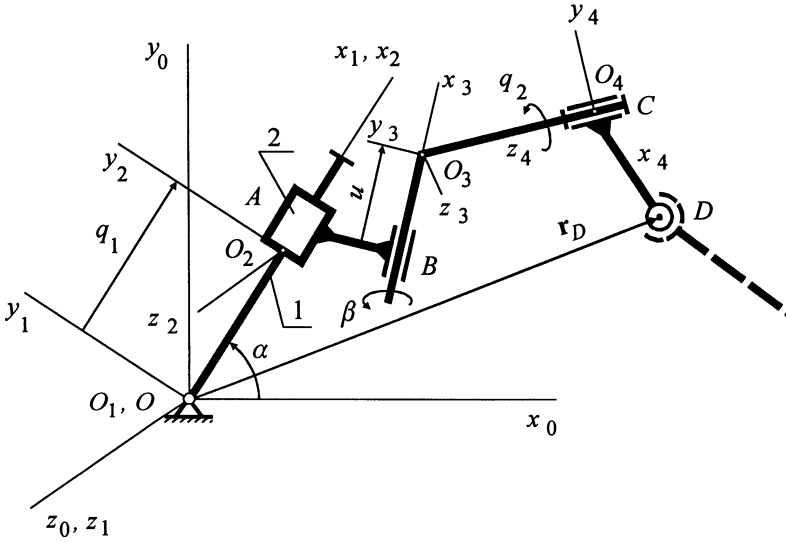


Fig. 2.12. Spatial closed kinematic chain

associated with the change of axis notations:

$$A_{s-1,s}(\psi, \theta, \varphi) = P(\psi)U_{\psi\theta}P(\theta)U_{\theta\varphi}P(\varphi). \quad (2.42)$$

The transformation matrix for a *spheric pair with a pin* of movability two can be obtained analogously.

Let us consider, as an example, a spatial kinematic chain (Fig. 2.12) which is a simple group of movability two. This chain has two external kinematic pairs – revolute O and spheric D as well as three internal pairs – prismatic A , cylindric B and revolute C . Following the general method presented above, we open the chain at joint D . This chain has the structure of a simple „tree“. Having opened joint D we release the three constraints imposed on the motion by the spheric pair; therefore, three group coordinates have to be introduced. Such coordinates are the angle α , defining the rotation of link 1 relative to frame 0; the angle β and the translational displacement u defining the motion of link 3 relative to link 2. The group equation has the form of a product of the transformation matrices and the column $R_D^{(4)}$ whose components are the constant quantities

$$R_{0,1}^R(\alpha)H_{1,2}^P(q_1)H_{2,3}^C(\beta, u)H_{3,4}^R(q_2)R_D^{(4)} = R_D^{(0)}. \quad (2.43)$$

The elements of the coordinate column $R_D^{(0)}$ are known from the geometric analysis of the previous groups. Developing expression (2.43) we obtain a system of three group equations with three unknowns α , β , u the determination of which leads to the solution of the direct problem of geometric analysis.

2.4

Solution to the Equations of Geometric Analysis

It is easy to see that because of their structure, expressions (2.11) and (2.33), derived for solving the direct problem of geometric analysis of open kinematic chains, have a unique solution, i.e. the specification of input coordinates uniquely defines the configuration of these chains.

The equations of geometric analysis, composed for closed kinematic chains, are trigonometric or nonlinear algebraic equations. If they have a solution, it is, as a rule, nonunique. We will study the problems arising in this connection illustrating it by the example of the mechanism, shown in Fig. 2.13. Here, with given input coordinate q the determination of the output coordinate φ may be obtained through a graphical method. Since the specification of q uniquely defines the position of point A , the solution is reduced to the location of the intersection point B of two circles with radii ℓ and d . It can be noticed from the figure that these circles have two intersection points B and B' and, correspondingly, two values φ and φ' for the output coordinate are obtained. In this way, with a given value of the input coordinate, the mechanism may have two different configurations and the choice of one of them can be made with the help of additional conditions.

If $r + a < \ell + d$ and $|r - a| > |\ell - d|$, then for a full revolution of the input link (crank) only one of the possible configurations is realized: the mechanism cannot move from position $OABC$ to position $OAB'C$ without being disassembled. In this case, it is said that the mechanism has two different *assemblies* and in the geometric analysis it is necessary to specify one of them.

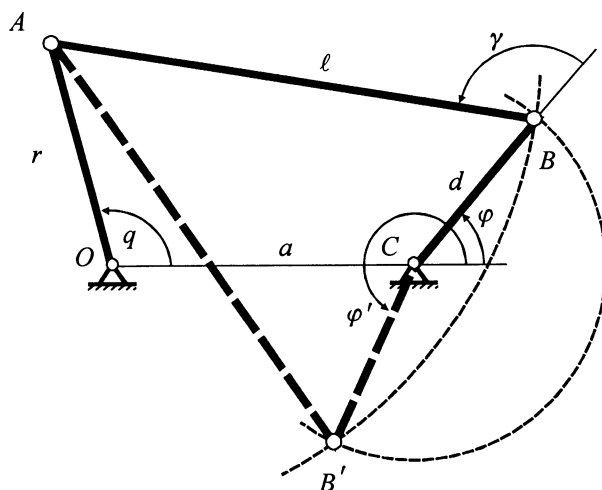


Fig. 2.13. Graphic determination of the position function for a crank-and-rocker mechanism

If $r + a \geq \ell + d$ and $|r - a| \leq |\ell - d|$, then the situation changes. The input link OA can no longer complete a full revolution and it becomes a rocker. This allows a transition from point B to point B' . In this transition there is a position in which points A , B and C are in line, point A takes one of the extreme positions and the two possible configurations for the given value of q coincide (Fig. 2.14). Such positions of a mechanism are called *singular*. Singular positions of a mechanism of movability one are characterized by the fact that either a displacement of the output link is impossible when driving forces are applied (the mechanism falls in a „dead“ position), or the displacement of the output link becomes undefined – a possibility arises for transition from one assembly to another. In the considered mechanism a motion branching is possible, if its parameters satisfy at least one of eight conditions $\pm r \pm \ell \pm d = a$ (parallel link mechanism).

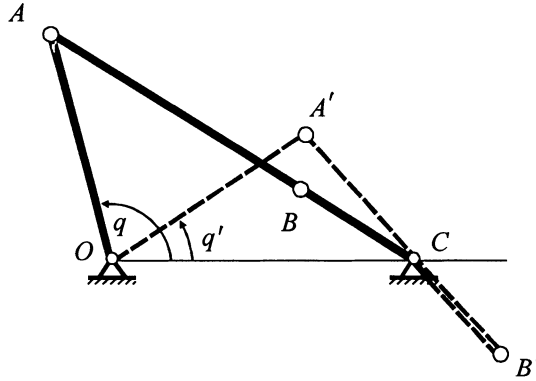


Fig. 2.14. Double-rocker mechanism in singular positions

The analytical solution of the problem of geometric analysis yields multiple roots corresponding to singular positions. In the considered case the equations of geometric analysis

$$\begin{aligned} d \cos \varphi + \ell \cos(\varphi + \gamma) &= r \cos q - a, \\ d \sin \varphi + \ell \sin(\varphi + \gamma) &= r \sin q \end{aligned} \quad (2.44)$$

determine the position functions $\gamma(q)$ and $\varphi(q)$. In order to obtain the relationship $\gamma(q)$ let us square the left and the right sides of Eqs. (2.44) and then add them together

$$d^2 + \ell^2 + 2\ell d \cos \gamma = (r \cos q - a)^2 + r^2 \sin^2 q, \quad (2.45)$$

whence with the help of the relationships

$$\cos \gamma = \frac{(r \cos q - a)^2 + r^2 \sin^2 q - \ell^2 - d^2}{2\ell d}, \quad \sin \gamma = \pm \sqrt{1 - \cos^2 \gamma}$$

we find the function $\gamma(q)$. Here, the double sign $\pm$ in front of the root shows that there are two configurations of the mechanism for a given input coordinate and fixed positions of the external kinematic pairs. For every configuration the function $\gamma(q)$ is unique. If link OA is a crank, then of the two solutions obtained from Eqs. (2.45) only one can be realized in a mechanism, while the second one is an accessory, practically nonrealizable solution. In Fig. 2.13 the basic solution $OABC$ is shown with solid lines, while the accessory one $OAB'C$ is shown with dashed lines. It is easy to see that Eqs. (2.44) have multiple roots when $\sin \gamma = 0$.

If rocker BC is an input link (Fig. 2.15), then both solutions of the system of equations of geometric analysis correspond to the two possible positions of one and the same assembly. In this case, the transition from one possible position to the other is accompanied by passing through the singular position of the mechanism, which, as a rule, is not desirable. The mechanism is pulled out from the singular position by inertia forces (with the help of flywheels) or by coupling of mechanisms identical with one another. The coupling of mechanisms is done in such a way that only one of them is in a singular position.

If, in the equations of geometric analysis (2.44), the inequality $|\cos \gamma| \leq 1$ is violated, then the links of the mechanism can not be brought together to form a closed kinematic chain (the *mechanism does not exist*). A break in the kinematic chain above considered will not occur, if $|\ell - d| < AC < \ell + d$, where

$$AC = \sqrt{a^2 + r^2 - 2ar \cos q}.$$

In order to define relationship $\varphi(q)$, it is necessary to write Eqs. (2.44) in the form

$$\begin{aligned} (d + \ell \cos \gamma) \cos \varphi - \ell \sin \gamma \sin \varphi &= r \cos q - a, \\ \ell \sin \gamma \cos \varphi + (d + \ell \cos \gamma) \sin \varphi &= r \sin q. \end{aligned}$$

It is not difficult to find from this system, using Cramer's rule,

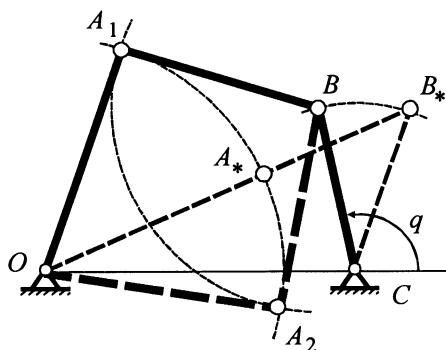


Fig. 2.15. Four-bar linkage in a singular and in two possible positions

$$\cos \varphi = \frac{(d + \ell \cos \gamma)(r \cos q - a) + r \ell \sin \gamma \sin q}{AC^2},$$

$$\sin \varphi = \frac{(d + \ell \cos \gamma)r \sin q - \ell \sin \gamma(r \cos q - a)}{AC^2},$$

where $AC^2 = d^2 + \ell^2 + 2d\ell \cos \gamma$.

From the same point of view, let us consider a mechanism of movability three (Fig. 2.16). We write down its group equations in an implicit form

$$\begin{aligned} F_1 &= x_0 + \ell_1 \cos q_1 - x_A = 0, & F_3 &= x_E + \ell_5 \cos q_3 - x_D = 0, \\ F_2 &= y_0 + \ell_1 \sin q_1 - y_A = 0, & F_4 &= y_E + \ell_5 \sin q_3 - y_D = 0, \end{aligned} \quad (2.46)$$

$$F_5 = \ell_2 \cos \varphi_2 + \ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2) - x_D + x_A = 0,$$

$$F_6 = \ell_2 \sin \varphi_2 + \ell_3 \sin \varphi_3 - \ell_4 \sin(\varphi_3 + q_2) - y_D + y_A = 0. \quad (2.47)$$

It is not difficult to understand that mechanism $OABCDE$ can move to position $OAB'C'DE$ without being disassembled. For this purpose, it is necessary at first, to decrease angle q_2 for the fixed values of angles q_1 and q_3 and to permit point B to go below line AC , and then, to restore the previous value of angle q_2 . In this way, both solutions of the group equations correspond to possible positions of the mechanism and the choice of the actual position must be made on the base of some additional conditions.

Suppose some position of the mechanism is given, i.e. one of the solutions $\varphi_2 = \varphi_{2*}$, $\varphi_3 = \varphi_{3*}$ of system (2.47) is chosen, corresponding to the given values $q_1 = q_{1*}$, $q_2 = q_{2*}$, $q_3 = q_{3*}$. We define the position of the mechanism as corresponding to $q_1 = q_{1*} + \Delta q_1$, $q_2 = q_{2*} + \Delta q_2$, $q_3 = q_{3*} + \Delta q_3$, where Δq_1 , Δq_2 , Δq_3 are small increments of the input coordinates. Simultaneously, we require that the new position of the mechanism, specified by values $\varphi_2 = \varphi_{2*} + \Delta \varphi_2$, $\varphi_3 =$

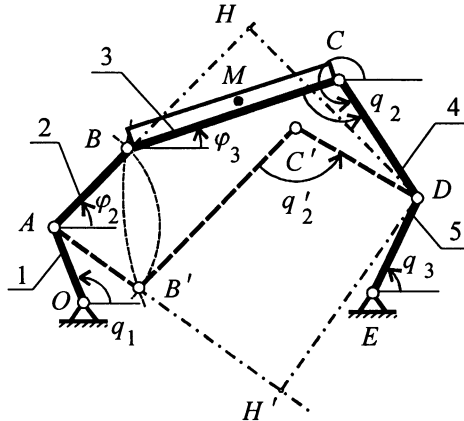


Fig. 2.16. Mechanism of movability three of a platform in two possible positions

$\varphi_{3*} + \Delta\varphi_3$, is close to the initial one, i.e. that $\Delta\varphi_2$, $\Delta\varphi_3$ are small quantities. Then, such a solution turns out to be unique, since the second position of the mechanism, corresponding to the same increments of coordinates q_1, q_2, q_3 will be close to $OAB'C'DE$ and, therefore, far from position $OABCDE$.

Let us determine the small increments $\Delta\varphi_2, \Delta\varphi_3$ from Eqs. (2.47). In this context, let us assume that increments $\Delta x_A, \Delta y_A, \Delta x_D, \Delta y_D$ have already been derived from the solution of equations of geometric analysis (2.46). Eqs. (2.47) can be written in matrix form

$$F(\varphi, \xi, q_2) = 0, \quad (2.48)$$

where the following coordinate columns $F = (F_5, F_6)^T$, $\varphi = (\varphi_2, \varphi_3)^T$, $\xi = (x_A, y_A, x_D, y_D)^T$ are introduced. In the given problem we define $\Delta\varphi$ from the equation

$$F(\varphi_* + \Delta\varphi, \xi_* + \Delta\xi, q_{2*} + \Delta q_2) = 0, \quad (2.49)$$

where $\Delta\xi, \Delta q_2$ are given small increments and φ_*, ξ_*, q_{2*} satisfy Eq. (2.48). In order to determine $\Delta\varphi$ one can use the iterative *Newton method* (or the *method of tangents* [2]). In accordance with this method the $(k+1)$ -st approximation of $\Delta\varphi$ is connected with the k -th approximation by the relationship

$$\Delta\varphi^{(k+1)} = \Delta\varphi^{(k)} - \left(\frac{\partial F}{\partial \varphi} \right)_k^{-1} F(\varphi_* + \Delta\varphi^{(k)}, \xi_* + \Delta\xi, q_{2*} + \Delta q_2) \quad (k = 1, 2, \dots). \quad (2.50)$$

It is proved that in a sufficiently small neighbourhood of the initial solution $\varphi = \varphi_*$ the sequence (2.50) converges, moreover a quadratic convergence is ensured. The expression

$$\left(\frac{\partial F}{\partial \varphi} \right)_k = \frac{\partial F}{\partial \varphi} (\varphi_* + \Delta\varphi^{(k)}, \xi_* + \Delta\xi, q_{2*} + \Delta q_2)$$

is the Jacobi matrix for system (2.48) which, in this case, has the following form

$$\frac{\partial F}{\partial \varphi} = \begin{pmatrix} -\ell_2 \sin \varphi_2 & -\ell_3 \sin \varphi_3 + \ell_4 \sin(\varphi_3 + q_2) \\ \ell_2 \cos \varphi_2 & \ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2) \end{pmatrix}. \quad (2.51)$$

The determinant of this matrix is the *Jacobian* of the initial system of Eqs. (2.48) and it is expressed by the formula

$$J = \det \left(\frac{\partial F}{\partial \varphi} \right) = \begin{vmatrix} -\ell_2 \sin \varphi_2 & -\ell_3 \sin \varphi_3 + \ell_4 \sin(\varphi_3 + q_2) \\ \ell_2 \cos \varphi_2 & \ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2) \end{vmatrix} = \ell_2 [\ell_3 \sin(\varphi_3 - \varphi_2) - \ell_4 \sin(\varphi_3 - \varphi_2 + q_2)]. \quad (2.52)$$

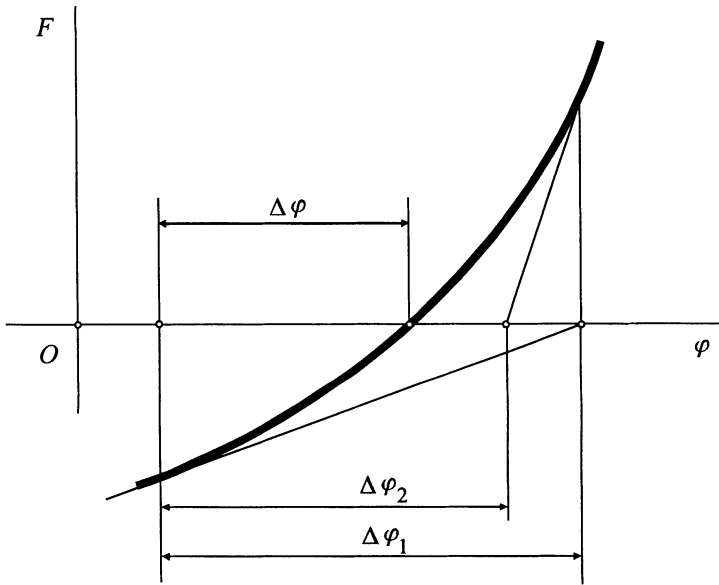


Fig. 2.17. Geometric interpretation of the Newton's method

The expression in square brackets is the projection of the polygonal line BCD onto the direction perpendicular to link AB . It is easy to convince oneself that in the two positions of the mechanism, corresponding to one and the same values q_1, q_2, q_3 (see Fig. 2.16), the values of the Jacobian are equal in magnitude and opposite in sign (line segments DH and DH' are equal).

In Fig. 2.17 a conditional geometric interpretation of Newton's method, related to the case when F and φ are one-dimensional is given. In order to avoid multiple calculations of the matrix $(\partial F / \partial \varphi)_k^{-1}$, inverse to the Jacobi matrix, a *modified Newton method* can be used, where a recurrent procedure is applied, corresponding to the formula

$$\Delta \varphi^{(k+1)} = \Delta \varphi^{(k)} - \left(\frac{\partial F}{\partial \varphi} \right)_*^{-1} F(\varphi_* + \Delta \varphi^{(k)}, \xi_* + \Delta \xi, q_{2*} + \Delta q_2) \quad k = 1, 2, \dots, \quad (2.53)$$

where $(\partial F / \partial \varphi)_* = \partial F / \partial \varphi(\varphi_*, \xi_*, q_{2*})$. The convergence of the successive approximations for the modified method turns out to be slower (linear). The Newton method can also be used to find values $\varphi_{2*}, \varphi_{3*}$ more precisely, if their approximate values $\tilde{\varphi}_{2*}, \tilde{\varphi}_{3*}$ are somehow found (e.g., through a geometric construction). In this case, the problem is reduced to the determination of $\Delta \varphi$ from the equation

$$F(\tilde{\varphi}_* + \Delta \varphi, \xi_*, q_{2*}) = 0.$$

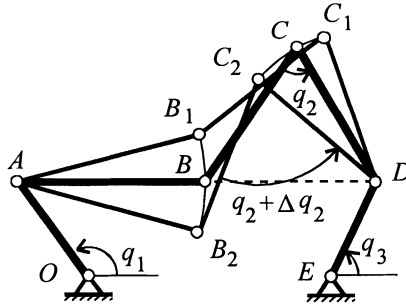


Fig. 2.18. Mechanism of movability three of a platform in singular position and in positions close to the singular one

The position of the mechanism, close to the initial position, cannot be obtained following the described method, if the Jacobian becomes zero. The determinant (2.52) can be written in the form

$$J = \begin{vmatrix} y_A - y_B & y_B - y_D \\ x_B - x_A & x_D - x_B \end{vmatrix}.$$

With the same formula $2S_{ABD}$ is determined that is the doubled surface of triangle ABD whose vertices are axes of passive joints (e.g., joints which are not input joints). The Jacobian becomes zero, if points A, B and D lie in line (Fig. 2.18). It is not difficult to see that in the position when q_1 and q_2 are fixed, it is impossible to give a negative small increment to angle q_3 ; when q_2 and q_3 are fixed, it is impossible to give a positive increment to angle q_1 ; when q_1 and q_3 are fixed, it is impossible to decrease angle q_2 . Moreover, when increasing q_2 by Δq_2 , the motion of the mechanism becomes indefinite (there are two positions OAB_1C_1DE and OAB_2C_2DE close together).

All these singularities can be explained by the fact that in the considered position

$$F(\varphi_*, \xi_*, q_{2*}) = 0, \quad \frac{\partial F}{\partial \varphi}(\varphi_*, \xi_*, q_{2*}) = 0,$$

which is an indication of root multiplicity and, therefore, of a singular position. In this position the two solutions of Eq. (2.48) fuse into one. It has to be noted that in mechanisms with several degrees of movability, the loss or gain of one degree of movability does not mean that the whole mechanism stops. However, the motions of input links can no longer be performed independently from one another, i.e. not all coordinates $q_1, q_2, \dots, q_n$ can be considered as generalized.

Naturally, in a neighbourhood of a singular position, both solutions of Eq. (2.48) are close to each other and the choice of one of them becomes difficult. The closer to zero the value of the Jacobian (2.52) is, the worse the successive approximations converge. Singular positions, as shown below, are also

undesirable from the viewpoint of force transmission. It follows from here that when assigning motion laws to a mechanism with several degrees of movability, the passages through singular positions must be avoided.

In the geometric analysis one can solve also a *direct problem of positioning* associated with the displacement of a mechanism from a given initial position q_{iI} to a given final position q_{iF} ($i = 1, 2, \dots, n$). Together with the initial position one of the several possible configurations is specified. In order to find whether this problem can be solved without passing through a singular position, it is necessary to define the value of the Jacobian in the initial and in the final positions. If the signs of the Jacobian in these positions are identical, then the problem can be solved in the following way. Divide the range $q_{iI} - q_{iF}$ into a finite number N of sufficiently small intervals; denote the division points by $q_i^{(1)}, q_i^{(2)}, \dots, q_i^{(N)} = q_{iF}$. Hence, we define $N + 1$ positions of the mechanism corresponding to $q_i^{(s)}$ ($s = 1, 2, \dots, N$). Since the initial position is given, we find, according to Newton's method, values $\varphi_r^{(1)}$ ($r = 1, 2, \dots, m$) in the first intermediate position ($s = 1$). Then, we analogously solve the problem of determining $\varphi_r^{(s)}$ for all remaining $s = 2, 3, \dots, N$. So, going ahead by small steps, we come to the final position. In essence, the solution of the positioning problem has coincided with the solution of a *path problem* – displacement of a mechanism along a prescribed „path“, which should be understood as the set of successive mechanism positions (configurations).

If the signs of the Jacobian in the initial and in the final positions are opposite, then the mechanism must pass through a singular position. To remove singular positions, excessive inputs, i.e. engines are introduced in mechanisms. For instance, in mechanism of movability three (Fig. 2.19) in addition to the basic inputs q_1, q_2, q_3 , an input q_4 at joint B has been introduced. In this case, inputs q_2 and q_4 work *alternatingly*: only one at a time. In the motion process the calculating device of the control system defines the values of the Jacobians $J = 2S_{ABD}$

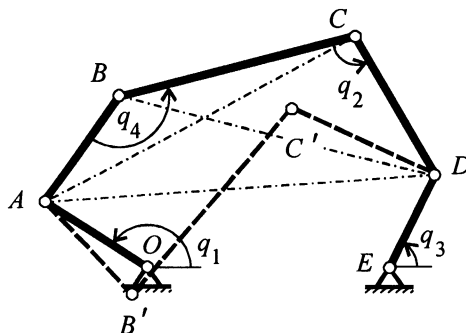


Fig. 2.19. Mechanism of movability three of a platform with excessive inputs

and $J^* = 2S_{ACD}$ in discrete points of the path. If $|J| > |J^*|$, the input q_2 is turned on and the input q_4 is turned off; if $|J| < |J^*|$, the input q_4 has to be turned off. Under such control of motion, passing of mechanisms through singular positions is excluded.

2.5

The Inverse Problem of Geometric Analysis

In solving the inverse problem of geometric analysis, the input coordinates q_k ($k=1,2,\dots,n$) of a mechanism with n degrees of movability are determined from given output coordinates x_s ($s=1,2,\dots,m$) of its links.

In order to solve the problem fulfilment of the condition $m = n$ is necessary. This condition, however, is not sufficient since, due to design restrictions, variants are possible, if the obtained solutions are not acceptable. If $m < n$, then $(n - m)$ input coordinates are arbitrarily assigned and the remaining m input coordinates are determined by solving the equations of geometric analysis. If $m > n$, there is in general no solution to the inverse problem. It is possible to obtain the solution by specifying n output coordinates of the links. We will illustrate the solution of the inverse problem with three examples.

As a first example, consider a planar kinematic chain with a „tree“-structure (Fig. 2.20a). The prescribed coordinates of point M are related to the unknown input coordinates q_1, q_2 by the following equations

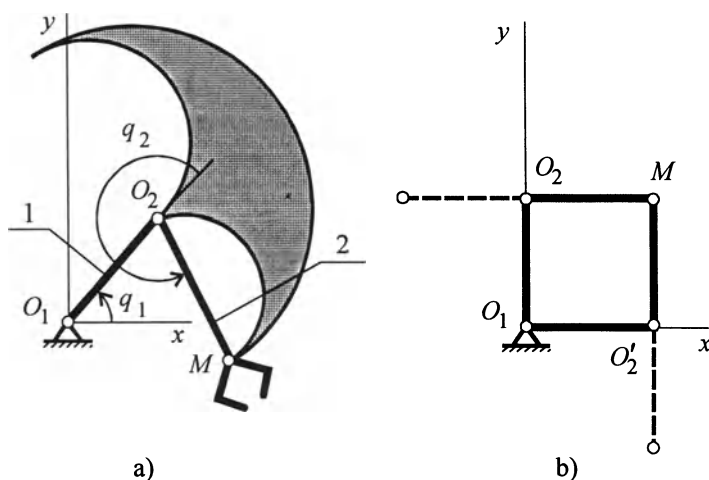


Fig. 2.20. Solution to the inverse problem of geometric analysis by using the example of a planar kinematic chain with tree-structure

$$\begin{aligned}x_M &= \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2), \\y_M &= \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2),\end{aligned}\quad (2.54)$$

where $\ell_1 = O_1O_2$, $\ell_2 = O_2M$. These equations can be written in the form

$$(x_M - \ell_1 \cos q_1)^2 + (y_M - \ell_1 \sin q_1)^2 = \ell_2^2, \quad (2.55)$$

$$x_M^2 + y_M^2 = \ell_1^2 + \ell_2^2 + 2\ell_1\ell_2 \sin q_2, \quad (2.56)$$

in which the unknowns q_1 and q_2 are separated. It is easy to notice that expressions (2.55) and (2.56) are equations of circles with centers O_2, O_1 and with radii O_2M, O_1M . They can be used to construct the performance zone of the output link of the chain, shown in Fig. 2.20a. From Eqs. (2.55) and (2.56) it is not a difficult task to find

$$\begin{aligned}q_1 &= \alpha \pm \arccos \frac{x_M^2 + y_M^2 + \ell_1^2 - \ell_2^2}{2\ell_1 \sqrt{x_M^2 + y_M^2}} + 2\pi k, \\q_2 &= \pm \arccos \frac{x_M^2 + y_M^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2} + 2\pi k \quad (k = 0, \pm 1, \pm 2, \dots),\end{aligned}$$

where the auxiliary angle α is defined from the relationships: $\cos \alpha = x_M / \sqrt{x_M^2 + y_M^2}$, $\sin \alpha = y_M / \sqrt{x_M^2 + y_M^2}$. When $x_M = y_M = \ell_1 = \ell_2$, we have $q_2 = \pm 90^\circ$, $\alpha = 45^\circ$, $q_1 = (0, 90^\circ)$. It is seen from Fig. 2.20b that there are two configurations ensuring the prescribed coordinates x_M, y_M . The choice of one or another position of the chain must be made on the basis of additional conditions. In Fig. 2.20b configurations, corresponding to solutions which are not roots of the initial Eqs. (2.55) and (2.56), are shown with dashed lines. If the prescribed point is outside the performance zone, then there is no solution to the inverse problem.

As a second example let us consider the spatial robot of movability four ($n = 4$) shown in Fig. 2.9. In solving this problem, no more than four of the six gripper coordinates can be prescribed. Consider the case when coordinates $x_M^{(0)}, y_M^{(0)}, z_M^{(0)}$ and ψ are given. The starting equations for the solution of this problem are four equations, obtained in Sect. 2.2:

$$x_M^{(0)} = q_2 \cos q_1 + (\ell_3 + \ell_4 \cos q_4) \cos(q_1 + q_3), \quad (2.57)$$

$$y_M^{(0)} = q_2 \sin q_1 + (\ell_3 + \ell_4 \cos q_4) \sin(q_1 + q_3), \quad (2.58)$$

$$z_M^{(0)} = \ell_2 + \ell_4 \sin q_4, \quad (2.59)$$

$$\psi = \begin{cases} q_1 + q_3 & \text{if } \varepsilon_4 = 1, \\ q_1 + q_3 - 180^\circ & \text{if } \varepsilon_4 = -1, \end{cases} \quad (2.60)$$

where $\varepsilon_4 = \text{sign}(\cos q_4)$. First, from Eqs. (2.59) we find

$$\sin q_4 = (z_M^{(0)} - \ell_2) / \ell_4. \quad (2.61)$$

A solution of the problem exists, if the condition

$$|\sin q_4| \leq 1 \quad (2.62)$$

is fulfilled, whence the inequalities $\ell_2 - \ell_4 \leq z_M^{(0)} \leq \ell_2 + \ell_4$ have to be satisfied. Then, we determine

$$\cos q_4 = \varepsilon_4 \sqrt{1 - \sin^2 q_4}. \quad (2.63)$$

With regard to $\cos \psi = \varepsilon_4 \cos(q_1 + q_3)$, $\sin \psi = \varepsilon_4 \sin(q_1 + q_3)$, Eqs. (2.57) and (2.58) are written in the form

$$\begin{aligned} x_M^{(0)} &= q_2 \cos q_1 + A_1 \varepsilon_4 \cos \psi, \\ y_M^{(0)} &= q_2 \sin q_1 + A_1 \varepsilon_4 \sin \psi, \end{aligned} \quad (2.64)$$

where

$$A_1 = \ell_3 + \ell_4 \cos q_4. \quad (2.65)$$

We find from system (2.64)

$$\tan q_1 = \frac{y_M^{(0)} - A_1 \varepsilon_4 \sin \psi}{x_M^{(0)} - A_1 \varepsilon_4 \cos \psi} = A_2, \quad (2.66)$$

$$q_1 = \arctan A_2 + \varepsilon_1 180^\circ, \quad \varepsilon_1 = 0 \text{ or } 1, \quad (2.67)$$

$$q_2 = (x_M^{(0)} - A_1 \varepsilon_1 \cos \psi) \cos q_1 + (y_M^{(0)} - A_1 \varepsilon_4 \sin \psi) \sin q_1. \quad (2.68)$$

Further, we obtain

$$q_3 = \begin{cases} \psi - q_1 & \text{if } \varepsilon_4 = 1, \\ \psi - q_1 + 180^\circ & \text{if } \varepsilon_4 = -1. \end{cases} \quad (2.69)$$

The solution of the inverse problem can be achieved in the following sequence: 1) compute $\sin q_4$ according to formula (2.61); 2) check condition (2.62) and if it is fulfilled, do the next step; 3) set $\varepsilon_4 = \pm 1$; 4) find $\cos q_4$ according to formula (2.63) and then find angle q_4 ; 5) compute A_1 according to formula (2.65); 6) compute A_2 according to formula (2.66); 7) set $\varepsilon_1 = 0$ or 1; 8) find q_1 according to formula (2.67); 9) compute q_2 according to formula (2.68); 10) compute q_3 according to formula (2.69). From the described algorithm it is seen that the problem has four solutions which differ in the values of ε_1 and ε_4 . This means that to fixed values of gripper coordinates $x_M^{(0)}, y_M^{(0)}, z_M^{(0)}, \psi$ there

correspond four different sets of values q_1, q_2, q_3, q_4 . It can be shown that in space to these four sets correspond not four but two different configurations of robot links. The reason is that the two solutions for which the value of ε_4 is one and the same but the values of ε_1 are different, provide one and the same configuration of robot links. That is why two out of the four solutions with $\varepsilon_1 = 1$ can be discarded. After this, two solutions remain which differ from one another in value ε_4 (for both solutions $\varepsilon_1 = 0$).

If $m = n$, then the inverse problem can be solved on the basis of the inverse structure of a mechanism. Consider a third example. Let the output coordinates x_M, y_M, φ_3 of the mechanism of movability three, shown in Fig. 2.16, be given. One needs to determine the input coordinates q_1, q_2, q_3 . In this case (see Fig. 1.25), the first structural group, if we start counting from the frame, is the one-bar group of movability three – link BC . The output coordinates of this open group – the coordinates of points B and C (to these points the following groups OAB and EDC are attached), are related with the input coordinates of the group x_M, y_M, φ_3 by the following relationships

$$\begin{aligned} x_B &= x_M - MC \cos \varphi_3, & x_C &= x_M + MC \cos \varphi_3, \\ y_B &= y_M - MC \sin \varphi_3, & y_C &= y_M + MC \sin \varphi_3. \end{aligned} \quad (2.70)$$

Further, we proceed to the analysis of the two-bar groups OAB and EDC . Opening them at points B and C , we obtain two simple „trees“. Let us introduce group coordinates φ_2 and φ_4 . Projecting the polygonal lines OAB and EDC on axes Ox and Oy , we obtain the conditions for chain closure

$$\begin{aligned} \ell_1 \cos q_1 + \ell_2 \cos \varphi_2 &= x_B - x_O, & \ell_5 \cos q_3 - \ell_4 \cos \varphi_4 &= x_C - x_E, \\ \ell_1 \sin q_1 + \ell_2 \sin \varphi_2 &= y_B - y_O, & \ell_5 \sin q_3 - \ell_4 \sin \varphi_4 &= y_C - y_E. \end{aligned} \quad (2.71)$$

Adding to these equations the relationship

$$\varphi_4 = \pi + \varphi_3 + q_2, \quad (2.72)$$

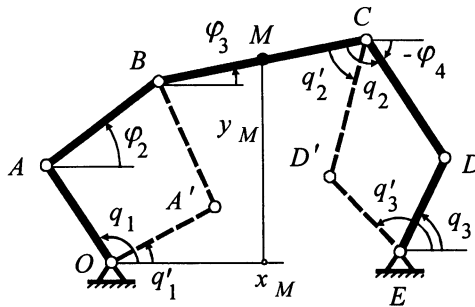


Fig. 2.21. Solution to the inverse problem of geometric analysis by using the example of a planar kinematic chain

we obtain a system of equations from which $\varphi_2, \varphi_4, q_1, q_2, q_3$ can be determined. The group equations (2.71) and (2.72) have four solutions, corresponding to the mechanism positions $OABCDE$, $OA'BCDE$, $OA'BCD'E$, $OABCD'E$, shown in Fig. 2.21. The Jacobian of system (2.71) and (2.72) has the following form

$$\det \left(\frac{\partial F}{\partial \psi} \right) = \begin{vmatrix} -\ell_1 \sin q_1 & -\ell_2 \sin \varphi_2 & 0 & 0 & 0 \\ \ell_1 \cos q_1 & \ell_2 \cos \varphi_2 & 0 & 0 & 0 \\ 0 & 0 & \ell_4 \sin \varphi_4 & -\ell_5 \sin q_3 & 0 \\ 0 & 0 & -\ell_4 \cos \varphi_4 & \ell_5 \cos q_3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \ell_1 \ell_2 \ell_4 \ell_5 \sin(q_1 - \varphi_2) \sin(\varphi_4 - q_3), \quad (2.73)$$

where $\psi = (\varphi_2, \varphi_4, q_1, q_2, q_3)^T$. Singular positions are those for which $\sin(q_1 - \varphi_2) = 0$ or $\sin(\varphi_4 - q_3) = 0$, i.e. when points O, A, B or C, D, E lie in line. We point out that the singular positions of the mechanism of inverse structure hide nothing dangerous for motion realization of the actual mechanism. They determine the extreme positions of the mechanism, lying on the boundary of its existence region in the space of coordinates x_M, y_M, φ_3 . As in the direct problem two problems are solved here. One is the *inverse problem of positioning* which is associated with the displacement of the mechanism output link from one given position $x_{MI}, y_{MI}, \varphi_{3I}$ to another position $x_{MF}, y_{MF}, \varphi_{3F}$. The other is the *inverse path problem*, ensuring displacement of the output links along a prescribed path. Most often, it is necessary to switch from the problem of positioning to the path problem of positioning, i.e. to divide the path into small sections, on which a solution is sought by the method of successive approximations:

$$q^{(k+1)} = q^{(k)} + \left(\frac{\partial \rho}{\partial q} \right)_k^{-1} \Delta \rho^{(k)} \quad (k = 1, 2, \dots), \quad (2.74)$$

where column matrices of coordinates $\rho = (x_M, y_M, \varphi_3)^T$, $q = (q_1, q_2, q_3)^T$ and the Jacobi matrices $(\partial \rho / \partial q)_k$ are introduced.

2.6

Special Features of Geometric Analysis of Mechanisms with a Higher Kinematic Pairs

In mechanisms with a higher kinematic pair, equations of surfaces mated at a kinematic pair are included in the equations of geometric analysis. Usually, a higher pair connects two one-bar structural groups. As examples of mechanisms with higher pairs serve the cam mechanisms shown in Fig. 1.20. Every one of them is

divided into a one-bar group of movability one (input link 1) and a Assur group of movability one (link 2).

The geometric analysis of a structural group with a higher kinematic pairs differs from the previous analysis in the way of constructing the group equations. Let us explain this difference with the example of the mechanism, shown in Fig. 2.22. Here, q is input coordinate and φ is output coordinate; they represent rotation angles of the mating links 1 and 2. In order to find the position function $\varphi(q)$, it is necessary to define the equations of the cylindric surfaces constituting the higher pair. Since the mechanism is planar, it is sufficient to define the equations of intersections of the surfaces with the motion plane. Let these equations be given in polar coordinates by functions $r_1(\theta_1)$ and $r_2(\theta_2)$, where θ_1 and θ_2 are polar angles counted from some lines O_1C_1 and O_2C_2 . At the tangency point A the mating profiles must have a common tangent the unit vector τ of which defines angles ψ_1 and ψ_2 against the radii r_1 and r_2 defined from the relationships

$$\tan \psi_1 = \frac{r_1}{dr_1/d\theta_1}, \quad \tan \psi_2 = \frac{r_2}{dr_2/d\theta_2}. \quad (2.75)$$

They are known from differential geometry of planar curves. Projecting the polygonal line O_1AO_2 on axes Ox and Oy , we have two equations

$$r_1 \cos(q + \theta_1) = a - r_2 \cos(\varphi + \theta_2), \quad (2.76)$$

$$r_1 \sin(q + \theta_1) = r_2 \sin(\varphi + \theta_2). \quad (2.77)$$

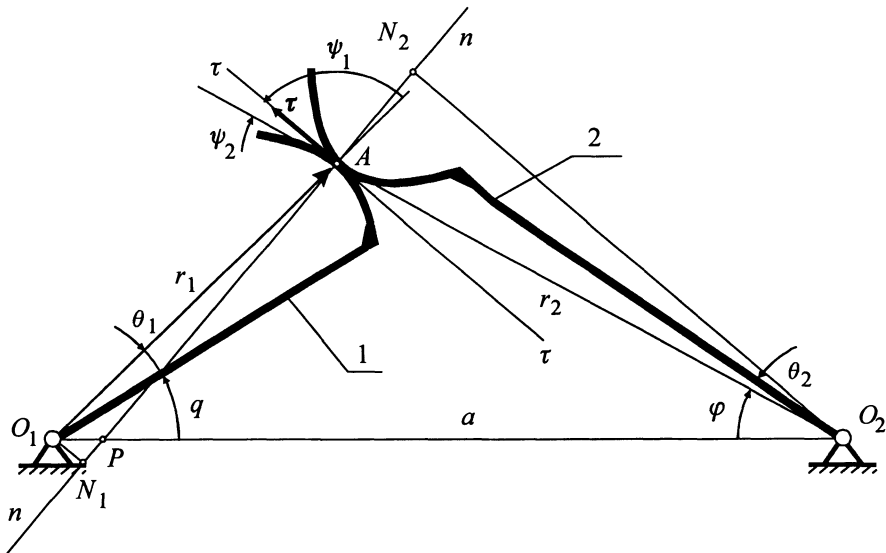


Fig. 2.22. Mechanism including a higher kinematic pair

We obtain a third equation using relationship (2.75):

$$\psi_1(\theta_1) + \psi_2(\theta_2) = \arctan \frac{r_1(\theta_1)}{dr_1/d\theta_1} + \arctan \frac{r_2(\theta_2)}{dr_2/d\theta_2} = \pi - q - \varphi - \theta_1 - \theta_2. \quad (2.78)$$

From Eqs. (2.76)–(2.78) it is possible to find the unknown polar angles θ_1 and θ_2 , defining the position of point A on every profile and the rotation angle of link 2. Here, angle q is the given input coordinate. Varying the value of q and determining the corresponding φ , one can obtain the position function $\varphi(q)$.

2.7 Problems

- 2.1. Determine the coordinates of the output link of the robot of movability three (Fig. 2.23) in the fixed coordinate system if $q_1 = 30^\circ$, $q_2 = 60^\circ$, $q_3 = 0.8$ m, $a = 1.6$ m, $b = 0.4$ m, $h = 0.2$ m.

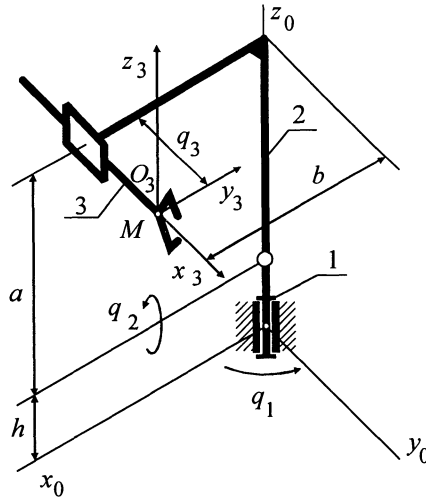


Fig. 2.23. Robot of movability three with tree-structure

Answer: $x_M^{(0)} = 0.2\sqrt{3} - 1$ (m), $y_M^{(0)} = 0.2 - 0.6\sqrt{3}$ (m), $z_M^{(0)} = 0.4\sqrt{3} + 1$ (m), $\theta = 60^\circ$, $\psi = -60^\circ$, $\varphi = 90^\circ$.

- 2.2. Derive the group equations with respect to angles ψ , φ , characterizing the positions of links 2 and 3 of the crank-and-rocker mechanism (see Fig. 2.2),

and solve them. Find the coordinates of point B and also determine the Jacobian of the equation system if $q = 60^\circ$, $r = 0.5$ m, $a = 1.0$ m. Solve the equations for the three cases: 1) $\ell = d = \sqrt{3}/2$ m, 2) $\ell = d = \sqrt{3}/4$ m, 3) $\ell = d = \sqrt{3}/5$ m. Prove that in the first case link OA is a crank.

Answer:

$$1) \psi = \begin{cases} 30^\circ \\ 270^\circ \end{cases}, \varphi = \begin{cases} 90^\circ \\ 210^\circ \end{cases}, x_B = \begin{cases} 1.0 \text{ m} \\ 0.25 \text{ m} \end{cases}, y_B = \begin{cases} \frac{\sqrt{3}}{2} \text{ m} \\ -\frac{\sqrt{3}}{4} \text{ m} \end{cases}, J = \begin{cases} \frac{-3\sqrt{3}}{8} \text{ m}^2 \\ \frac{3\sqrt{3}}{8} \text{ m}^2 \end{cases}.$$

$$2) \psi = 330^\circ, \varphi = 150^\circ, x_B = 0.625 \text{ m}, y_B = \sqrt{3}/8 \text{ m}, J = 0 \text{ (singular position)}.$$

3) no solution (break of the kinematic chain).

- 2.3. Define coordinates φ_2, φ_3 , characterizing the positions of links 2 and 3 of the manipulator of movability two (Fig. 2.24) and the coordinates of pole M for $OA = 0.7$ m, $AB = 0.8$ m, $BC = 1.0$ m, $DC = 0.4$ m, $BM = 1.2$ m, $x_O = y_O = 0$, $x_D = 0.4$ m, $y_M = -0.3$ m, $q_1 = 90^\circ$, $q_2 = 0$. Find these parameters for another assembly of links. Formulate the condition for a singular position.

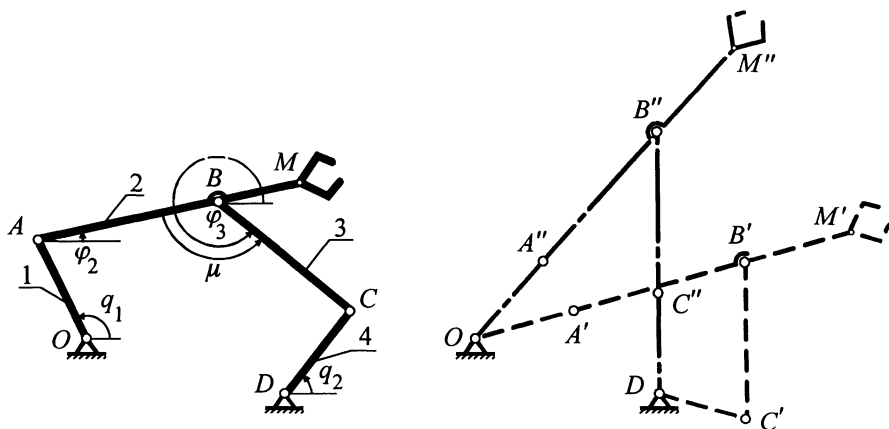


Fig. 2.24. Robot of movability two with a closed kinematic chain

Answer:

$$1) \varphi_2 = 0, \varphi_3 = 270^\circ, x_M = 2.0 \text{ m}, y_M = 0.7 \text{ m};$$

$$2) \varphi_2 = 257.32^\circ, \varphi_3 = 347.32^\circ, x_M = -0.439 \text{ m}, y_M = -1.25 \text{ m};$$

$$\sin(\varphi_3 - \varphi_2) = 0.$$

- 2.4. With the given parameters of the kinematic diagram of the mechanism of movability three (Fig. 2.25): $q_1 = 90^\circ$, $q_2 = 180^\circ$, $q_3 = 0$, $x_O = y_O = 0$, $x_C = 2 + \frac{\sqrt{3}}{2}$ m, $y_C = 1.5$ m, $x_H = -\sqrt{3}/2$ m, $y_H = 3.5$ m, $BK = KE$, $FE = BE = FB = \sqrt{3}$ m, $BM = 2.5$ m, $\gamma = 60^\circ$ determine the coordinates of links 2, 4, 6 and 7, as well as the Cartesian coordinates of pole M . Find the greatest possible number of mechanism assemblies.

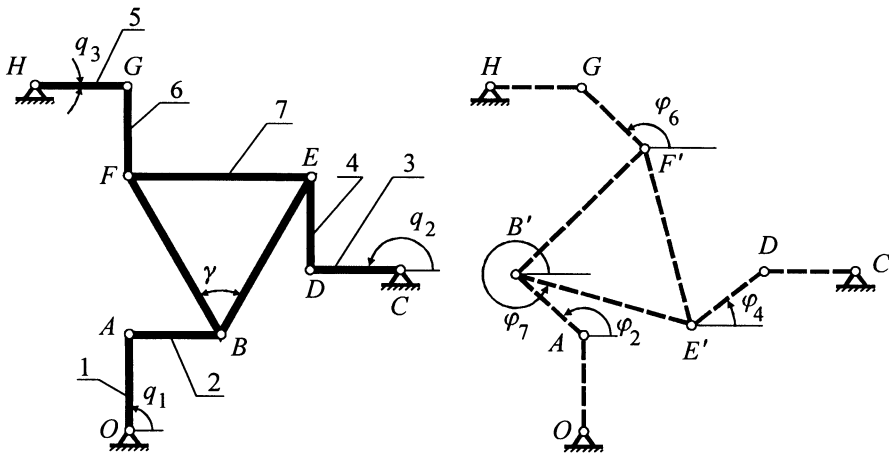


Fig. 2.25. Closed mechanism of movability three

Answer: $\varphi_2 = 0$, $\varphi_4 = 270^\circ$, $\varphi_6 = 90^\circ$, $\varphi_7 = 270^\circ$, $x_M = 1.0$ m, $y_M = 3.5$ m; apart from the six possible assemblies one more assembly is realized, shown with dashed line ($\varphi_2 = 126.47^\circ$, $\varphi_4 = 26.21^\circ$, $\varphi_6 = 134.36^\circ$, $\varphi_7 = 334.49^\circ$), $J = \ell_2 \ell_4 \ell_6 [FB \sin(\varphi_4 - \varphi_2) \sin(\varphi_7 + \gamma - \varphi_6) + BE \sin(\varphi_7 - \varphi_4) \sin(\varphi_2 - \varphi_6)] = -\sqrt{3}$ m⁴, $J = 0$ (lines FG , ED , AB intersect in one point).

- 2.5. Derive the group equations for the mechanism of movability three (Fig. 2.26); define the Jacobians related to the structure groups of the mechanism. Find the singular positions.

Answer: $J_1 = \ell_2 \cos(\varphi_3 + q_2) - \ell_3 \cos \varphi_3$; $J_2 = [\ell_5 \sin(\varphi_5 - \varphi_7) - \ell_6 \sin(\varphi_5 + q_3 - \varphi_7)] \ell_7$; $J_1 = 0$ (points A and C do not lie on the perpendicular to the guide of the slider); $J_2 = 0$ (points D, F, G lie on one line).

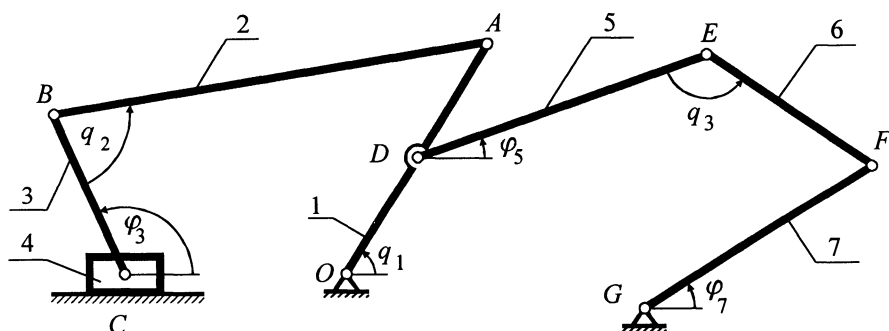


Fig. 2.26. Closed mechanism of movability three

- 2.6. Define the angular coordinates of the links of the robot of movability two (see Fig. 2.24), ensuring „hit“ of pole M at the point with coordinates $x_M = 2.0$ m, $y_M = 0.7$ m, if $OA = 0.7$ m, $AB = 0.8$ m, $BC = 1.0$ m, $DC = 0.4$ m, $BM = 1.2$ m, $x_D = 0.4$ m, $y_D = -0.3$ m. Show the positions of links when a loss of one or two degrees of movability occurs.

Answer:

- 1) $q_1 = 308.58^\circ$, $q_2 = 254.1^\circ$, $\varphi_2 = 38.6^\circ$, $\varphi_3 = 219.5^\circ$;
- 2) $q_1 = 90^\circ$, $q_2 = 0$, $\varphi_2 = 0$, $\varphi_3 = 270^\circ$;
- 3) $q_1 = 308.58^\circ$, $q_2 = 147.5^\circ$, $\varphi_2 = 38.6^\circ$, $\varphi_3 = 226.4^\circ$;
- 4) $q_1 = 90^\circ$, $q_2 = 136.4^\circ$, $\varphi_2 = 0$, $\varphi_3 = 226.4^\circ$.

- 2.7. Following the algorithm presented in Sect. 2.5, define the input coordinates of the robot of movability four shown in Fig. 2.9, if $\ell_2 = 1.0$ m, $\ell_3 = 2.0$ m, $\ell_4 = \sqrt{3}$ m, $\psi = 60^\circ$, $x_M^{(0)} = 3.75$ m, $y_M^{(0)} = 2 - 3.5\sqrt{3}$ m, $z_M^{(0)} = 1 + 0.5\sqrt{3}$ m.

Answer:

- 1) $q_1 = 45^\circ$, $q_2 = 2\sqrt{2}$ m, $q_3 = 15^\circ$, $q_4 = 30^\circ$;
- 2) $q_1 = 53.8^\circ$, $q_2 = 2\sqrt{8 + 2\sqrt{3}}$ m, $q_3 = 186.2^\circ$, $q_4 = 150^\circ$.

- 2.8. In Fig. 1.24 a four-legged walking machine is shown. Every leg consists of a thigh (links 1 and 2, connected with a revolute pair), a shin 3 and a foot 5. The thigh is connected with the body 4 of the machine and with the shin by revolute pairs. Define the coordinates of the foot point M_4 in the reference system, attached to the body of the machine, if $\ell_1 = a = 0.25$ m,

$$\ell_2 = 0.25\sqrt{3} \text{ m}, \quad \ell_3 = 0.55 \text{ m}, \quad \ell_5 = 0.05 \text{ m}, \quad q_{14} = 45^0, \quad q_{24} = 30^0, \\ q_{34} = 0.$$

$$\text{Answer: } x_{M4}^{(4)} = 0.25 + 0.3375\sqrt{2} \text{ m}, \quad y_{M4}^{(4)} = 0.3375\sqrt{2} \text{ m}, \quad z_{M4}^{(4)} = \\ 0.25 - 0.175\sqrt{3} \text{ m}.$$

- 2.9. Define the rotation angles of links 1, 2, and 3 of the fourth leg of the walking machine (see Fig. 1.24), if $\ell_1 = a = 0.25 \text{ m}$, $\ell_2 = 0.25\sqrt{3} \text{ m}$, $\ell_3 = 0.55 \text{ m}$, $\ell_5 = 0.05 \text{ m}$. Take the coordinates of the foot point M_4 from the previous problem.

$$\text{Answer: } q_{14} = \begin{cases} 45^0 \\ 225^0 \end{cases}, \quad q_{24} = \begin{cases} 30^0 \\ 150^0 \end{cases}, \quad q_{34} = \begin{cases} 0 \\ 180^0 \end{cases}, \quad J = \ell_2(\ell_3 +$$

$$\ell_5) \cos q_{34} [\ell_2 \cos q_{24} + (\ell_3 + \ell_5) \sin(q_{24} + q_{34})] = 0.10125\sqrt{3} \text{ m}^2.$$

In order to eliminate the vanishing of the Jacobian, it is necessary to avoid intersection of the foot paths with the rotation axis of the foot plane, as well as straightening of links 2 and 3 in one line.

- 2.10. Determine the position function of the cam mechanism (Fig. 2.27), representing the relationship between displacement s of the follower and the rotation angle q of the cam, if equation $r(\theta)$ of the cam profile is known in a polar coordinate system (r and θ are respectively the polar radius and the polar angle).

$$\text{Answer: } s(\theta) = \sqrt{r^2(\theta) - e^2} - \sqrt{r_0^2 - e^2}, \quad q = \theta + \arccos e/r(\theta) - \arccos e/r_0.$$

- 2.11. Determine the position function of the cylindric gearing (Fig. 2.28a), representing the relationship between the flat angle φ of the profile of the output link 2 and the angle q of the input link 1. The side-profile of the teeth is the involute curve (Fig. 2.28b) with the equation $\theta(\alpha) = \tan \alpha - \alpha$; $r(\alpha) = r_b / \cos \alpha$, with polar coordinates r_b (radius-vector of the base circle) and α (profile angle coinciding with the pressure angle for the present problem).

$$\text{Answer: } \varphi = \left(1 + \frac{r_{b1}}{r_{b2}} \right) \tan \alpha - \frac{r_{b1}}{r_{b2}} q.$$

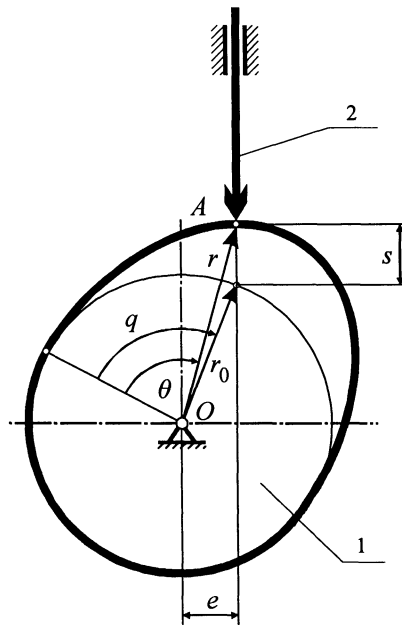


Fig. 2.27. Determination of the position function of a cam mechanism

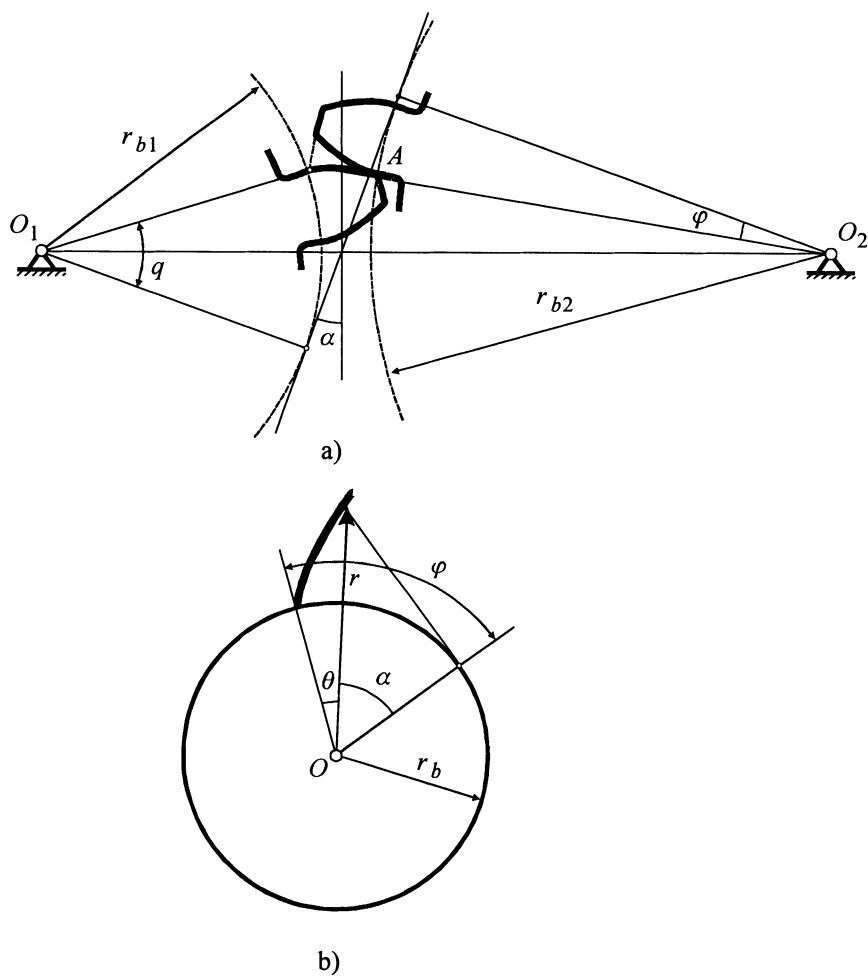


Fig. 2.28. Determination of the position function of a cylindric gearing

3 Kinematic and Parametric Analysis of Mechanisms

3.1 Kinematic Analysis of Planar Mechanisms

The goal of kinematic analysis is to determine velocities and accelerations of mechanism points, as well as angular velocities and accelerations of links. In addition, analogously to the geometric analysis, direct and inverse problems are solved.

The *direct* problem of kinematic analysis consists in determining first and second derivatives of position functions with respect to time

$$\dot{x}_s = \sum_{k=1}^n \frac{\partial \Pi_s}{\partial q_k} \dot{q}_k \quad (s = 1, 2, \dots, m), \quad (3.1)$$

$$\ddot{x}_s = \sum_{k=1}^n \frac{\partial \Pi_s}{\partial q_k} \ddot{q}_k + \sum_{k=1}^n \sum_{r=1}^n \frac{\partial^2 \Pi_s}{\partial q_k \partial q_r} \dot{q}_k \dot{q}_r \quad (3.2)$$

for given input coordinates q_k and their derivatives $\dot{q}_k, \ddot{q}_k$ ($k = 1, 2, \dots, n$).

The *inverse* problem is related to the determination of first and second time derivatives of input coordinates for given values of output coordinates and of their derivatives.

It is seen from formulae (3.1) and (3.2) that the problem of determining velocities and accelerations is reduced to finding first and second partial derivatives of positions functions with respect to input coordinates. These derivatives, depending only on geometric mechanism parameters, are called respectively *first* and *second geometric transfer functions* or *analogues of velocities* and *analogues of accelerations*.

Velocities and accelerations in a planar mechanism as well as first and second geometric transfer functions can be determined by analytic and by graph-analytic methods. Moreover, it is assumed that the direct problem of geometric analysis is already solved, i.e. the position functions of the mechanism are known. Consider some examples of analytical determination of first and second geometric transfer functions.

If in the slider-crank mechanism, shown in Fig. 2.1, crank 1 is an input link, then for the output link 3 the analogue of velocity and the analogue of acceleration are determined through a successive differentiation of the function $x_B(q)$, obtained in (2.5), with respect to the input coordinate q :

$$\frac{dx_B}{dq} = - \left[y_A + \frac{x_A(y_A + e)}{\sqrt{\ell^2 - (y_A + e)^2}} \right], \quad (3.3)$$

where $x_A = r \cos q$, $y_A = r \sin q$ and

$$\frac{d^2 x_B}{dq^2} = - \left\{ x_A + \frac{x_A^2(y_A + e)^2 + [x_A^2 - y_A(y_A + e)][\ell^2 - (y_A + e)^2]}{\sqrt{[\ell^2 - (y_A + e)^2]^3}} \right\}. \quad (3.4)$$

If the input link is slider 3, then for the output link 1 the analogue of velocity and the analogue of acceleration can be determined according to formulae:

$$\begin{aligned} \frac{dq}{dx_B} &= \left(\frac{dx_B}{dq} \right)^{-1}, \quad \frac{d^2 q}{dx_B^2} = \frac{d}{dx_B} \left(\frac{dq}{dx_B} \right) = \frac{dq}{dx_B} \frac{d}{dq} \left(\frac{dq}{dx_B} \right) = \\ &= \left(\frac{dx_B}{dq} \right)^{-1} \frac{d}{dq} \left(\left(\frac{dx_B}{dq} \right)^{-1} \right) = - \frac{d^2 x_B}{dq^2} \Big/ \left(\frac{dx_B}{dq} \right)^3. \end{aligned} \quad (3.5)$$

As another example, let us find the analogue of angular velocity of the output link 3 of the crank-and-rocker mechanism (see Fig. 2.2). To achieve this, let us differentiate with respect to the independent variable q in Eqs. (2.6) defining the position function $\varphi(q)$ in implicit form. After derivation we obtain the equation system

$$\begin{aligned} \frac{\partial F_1}{\partial q} + \frac{\partial F_1}{\partial \psi} \frac{d\psi}{dq} + \frac{\partial F_1}{\partial \varphi} \frac{d\varphi}{dq} &= 0, \\ \frac{\partial F_2}{\partial q} + \frac{\partial F_2}{\partial \psi} \frac{d\psi}{dq} + \frac{\partial F_2}{\partial \varphi} \frac{d\varphi}{dq} &= 0, \end{aligned} \quad (3.6)$$

from which we find

$$\frac{d\psi}{dq} = \frac{\frac{\partial F_1}{\partial \varphi} \frac{\partial F_2}{\partial q} - \frac{\partial F_1}{\partial q} \frac{\partial F_2}{\partial \varphi}}{\frac{\partial F_1}{\partial \psi} \frac{\partial F_2}{\partial \varphi} - \frac{\partial F_1}{\partial \varphi} \frac{\partial F_2}{\partial \psi}}, \quad \frac{d\varphi}{dq} = \frac{\frac{\partial F_1}{\partial \psi} \frac{\partial F_2}{\partial q} - \frac{\partial F_1}{\partial q} \frac{\partial F_2}{\partial \psi}}{\frac{\partial F_1}{\partial \psi} \frac{\partial F_2}{\partial \varphi} - \frac{\partial F_1}{\partial \varphi} \frac{\partial F_2}{\partial \psi}} = \frac{J_{\varphi'}}{J}, \quad (3.7)$$

where $J = \ell d \sin(\psi - \varphi)$ is the Jacobian of the initial equation system and $J_{\varphi'} = \ell r \sin(\psi - q)$. Equation system (3.6) does not have a solution, if $J = 0$, $J_{\varphi'} \neq 0$, i.e. in singular positions of the mechanism. If $J = 0$, $J_{\varphi'} = 0$, then one of the equations of system (3.6) is a corollary of the other and, because of this, the system has an infinite number of solutions. In this case, the motion of the

mechanism becomes undefined. At any nonsingular position, the system of linear Eqs. (3.6) has a unique solution. In particular, if $J \neq 0$, $J_{\varphi'} = 0$, equation system (3.6) has the solution $d\varphi/dq = 0$, which is a necessary condition for the extremum of function $\varphi(q)$. The two extrema of this function are reached at *extreme* positions of link 3 (points O, A and B lie in one straight line). Depending on the relationship between the lengths of links, the crank-and-rocker mechanism can become a double-crank or double-rocker mechanism. In the case of double-crank mechanism, the position function $\varphi(q)$ has no extremal values, and in the case of a double-rocker mechanism the function $\varphi(q)$ is continuous and finite with the exception of the boundary values, to which singular positions of the mechanism correspond.

The kinematic analysis of multi-link mechanisms can be realized with the use of the group-based method following the order of attachment of structural groups to the frame. Equations for the first derivatives of position functions can be obtained through direct derivation of group equations. E.g., differentiating Eqs. (2.35) and (2.36) with respect to q_1 , we obtain the equation system

$$\begin{aligned} \frac{\partial x_A}{\partial q_1} &= -\ell_1 \sin q_1, \quad \frac{\partial x_D}{\partial q_1} = 0, \quad \frac{\partial y_A}{\partial q_1} = \ell_1 \cos q_1, \quad \frac{\partial y_D}{\partial q_1} = 0, \\ -\ell_2 \sin \varphi_2 \frac{\partial \varphi_2}{\partial q_1} - [\ell_3 \sin \varphi_3 - \ell_4 \sin(\varphi_3 + q_2)] \frac{\partial \varphi_3}{\partial q_1} &= -\frac{\partial x_A}{\partial q_1}, \\ \ell_2 \cos \varphi_2 \frac{\partial \varphi_2}{\partial q_1} + [\ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2)] \frac{\partial \varphi_3}{\partial q_1} &= -\frac{\partial y_A}{\partial q_1}, \end{aligned} \quad (3.8)$$

from which we define the first derivatives

$$\begin{aligned} \frac{\partial \varphi_2}{\partial q_1} &= \frac{\ell_1 [\ell_3 \sin(q_1 - \varphi_3) - \ell_4 \sin(q_1 - \varphi_3 - q_2)]}{\ell_2 [\ell_3 \sin(\varphi_3 - \varphi_2) - \ell_4 \sin(\varphi_3 + q_2 - \varphi_2)]}, \\ \frac{\partial \varphi_3}{\partial q_1} &= \frac{\ell_1 \sin(\varphi_2 - q_1)}{\ell_3 \sin(\varphi_3 - \varphi_2) - \ell_4 \sin(\varphi_3 + q_2 - \varphi_2)}. \end{aligned} \quad (3.9)$$

The determinant of system (3.8) is the Jacobian of the group equations. Relationship (3.9) shows that in singular positions, when the Jacobian vanishes, the values of first derivatives become „infinitely large“. Close to singular positions, the first derivatives gain finite but rather large values. This means that at such positions small variations in input coordinates provide sharp, significant variations in output coordinates. When passing through singular positions, collisions occur which are extremely undesirable.

To determine the second partial derivatives of position functions, it is necessary to differentiate once more the group equations with respect to input coordinates. First, differentiating Eqs. (3.8) with respect to q_1 and then with respect to q_2 , we obtain the equation system

$$\begin{aligned}
\frac{\partial^2 x_A}{\partial q_1^2} &= -\ell_1 \cos q_1, \quad \frac{\partial^2 y_A}{\partial q_1^2} = -\ell_1 \sin q_1, \\
-\ell_2 \sin \varphi_2 \frac{\partial^2 \varphi_2}{\partial q_1^2} - [\ell_3 \sin \varphi_3 - \ell_4 \sin(\varphi_3 + q_2)] \frac{\partial^2 \varphi_3}{\partial q_1^2} = \\
-\frac{\partial^2 x_A}{\partial q_1^2} + \ell_2 \cos \varphi_2 \left(\frac{\partial \varphi_2}{\partial q_1} \right)^2 + [\ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2)] \left(\frac{\partial \varphi_3}{\partial q_1} \right)^2, \quad (3.10) \\
\ell_2 \cos \varphi_2 \frac{\partial^2 \varphi_2}{\partial q_1^2} + [\ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2)] \frac{\partial^2 \varphi_3}{\partial q_1^2} = \\
-\frac{\partial^2 y_A}{\partial q_1^2} + \ell_2 \sin \varphi_2 \left(\frac{\partial \varphi_2}{\partial q_1} \right)^2 + [\ell_3 \sin \varphi_3 - \ell_4 \sin(\varphi_3 + q_2)] \left(\frac{\partial \varphi_3}{\partial q_1} \right)^2
\end{aligned}$$

for determination of derivatives $\partial^2 \varphi_2 / \partial q_1^2, \partial^2 \varphi_3 / \partial q_1^2$, and one more equation system

$$\begin{aligned}
\frac{\partial^2 x_A}{\partial q_1 \partial q_2} &= \frac{\partial^2 y_A}{\partial q_1 \partial q_2} = 0, \\
-\ell_2 \sin \varphi_2 \frac{\partial^2 \varphi_2}{\partial q_1 \partial q_2} - [\ell_3 \sin \varphi_3 - \ell_4 \sin(\varphi_3 + q_2)] \frac{\partial^2 \varphi_3}{\partial q_1 \partial q_2} = \\
\ell_2 \cos \varphi_2 \frac{\partial \varphi_2}{\partial q_1} \frac{\partial \varphi_2}{\partial q_2} + [\ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2)] \frac{\partial \varphi_3}{\partial q_1} \frac{\partial \varphi_3}{\partial q_2} + \\
\ell_4 \cos(\varphi_3 + q_2) \frac{\partial \varphi_3}{\partial q_1}, \quad (3.11) \\
\ell_2 \cos \varphi_2 \frac{\partial^2 \varphi_2}{\partial q_1 \partial q_2} + [\ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2)] \frac{\partial^2 \varphi_3}{\partial q_1 \partial q_2} = \\
\ell_2 \sin \varphi_2 \frac{\partial \varphi_2}{\partial q_1} \frac{\partial \varphi_2}{\partial q_2} + [\ell_3 \sin \varphi_3 - \ell_4 \sin(\varphi_3 + q_2)] \frac{\partial \varphi_3}{\partial q_1} \frac{\partial \varphi_3}{\partial q_2} + \\
\ell_4 \sin(\varphi_3 + q_2) \frac{\partial \varphi_3}{\partial q_1},
\end{aligned}$$

from which the mixed second order derivatives are determined. The determinant of these linear equations again coincides with the Jacobian of the group equations. Therefore, a solution to these equation systems exists and it is unique for any nonsingular position of the mechanism.

The graph-analytical method for determination of the derivatives of position functions is based on the construction of *vector diagrams* of velocity analogues and acceleration analogues [3]. Let vector $\mathbf{r}$ be given, which is a function of a scalar parameter u (Fig. 3.1). We present it in the form

$$\mathbf{r} = r\mathbf{i}, \quad (3.12)$$

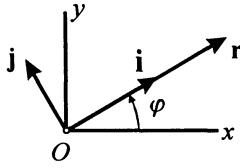


Fig. 3.1. Vector function of a scalar argument

where $\mathbf{i}$ is a unit vector. After differentiating (3.12) with respect to u , we obtain

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{i}}{\partial u} r + \mathbf{i} \frac{\partial r}{\partial u} = \mathbf{j} \frac{\partial \varphi}{\partial u} r + \mathbf{i} \frac{\partial r}{\partial u}, \quad (3.13)$$

where $\mathbf{j}$ is a unit vector orthogonal to vector $\mathbf{i}$, and φ is the angle between $\mathbf{r}$ and the fixed axis Ox . For unit vectors $\mathbf{i}$ and $\mathbf{j}$ the following relationships hold:

$$\frac{\partial \mathbf{i}}{\partial u} = \mathbf{j} \frac{\partial \varphi}{\partial u}, \quad \frac{\partial \mathbf{j}}{\partial u} = -\mathbf{i} \frac{\partial \varphi}{\partial u}. \quad (3.14)$$

Introducing the notations, shown in Fig. 3.2a, we write the equation of the closed vector loop $OABCD$ in the form

$$\ell_1 \mathbf{i}_1 + \ell_2 \mathbf{i}_2 + \ell_3 \mathbf{i}_3 + \ell_4 \mathbf{i}_4 = \mathbf{OE} + \ell_5 \mathbf{i}_5.$$

Differentiating with respect to q_1 , we obtain

$$\ell_1 \mathbf{j}_1 + \ell_2 \frac{\partial \varphi_2}{\partial q_1} \mathbf{j}_2 + \frac{\partial \varphi_3}{\partial q_1} \mathbf{BD}^* = 0, \quad (3.15)$$

where we have vector $\mathbf{BD}^* = \ell_3 \mathbf{j}_3 + \ell_4 \mathbf{j}_4$ and its modulus $BD^* = BD = \sqrt{\ell_3^2 + \ell_4^2 - 2\ell_3\ell_4 \cos q_2}$. The vector equation (3.15) contains two scalar quantities $\partial \varphi_2 / \partial q_1$ and $\partial \varphi_3 / \partial q_1$. It can be solved graphically by construction of the vector triangle $p_v ab$ (Fig. 3.2b). To find point c on the vector diagram of velocity analogue we make use of the property of similarity. According to this property, the vector diagram of velocity analogue for the relative velocities of link points is similar and analogously positioned to the figure connecting these points in the kinematic diagram of the mechanism. We leave the proof of this statement to the reader. Analogously, we determine point m , corresponding to pole M of the platform. Finding point m on the vector diagram of analogues and determining point projections on axes x and y , we obtain projections $\partial x_M / \partial q_1$, $\partial y_M / \partial q_1$.

Differentiating equation (3.15) with respect to q_1 , we obtain a vector equation which we write in the form

$$-\ell_1 \mathbf{i}_1 - \ell_2 \left(\frac{\partial \varphi_2}{\partial q_1} \right)^2 \mathbf{i}_2 + \ell_2 \frac{\partial^2 \varphi_2}{\partial q_1^2} \mathbf{j}_2 = \left(\frac{\partial \varphi_3}{\partial q_1} \right)^2 \mathbf{BD} - \frac{\partial^2 \varphi_3}{\partial q_1^2} \mathbf{BD}^*, \quad (3.16)$$

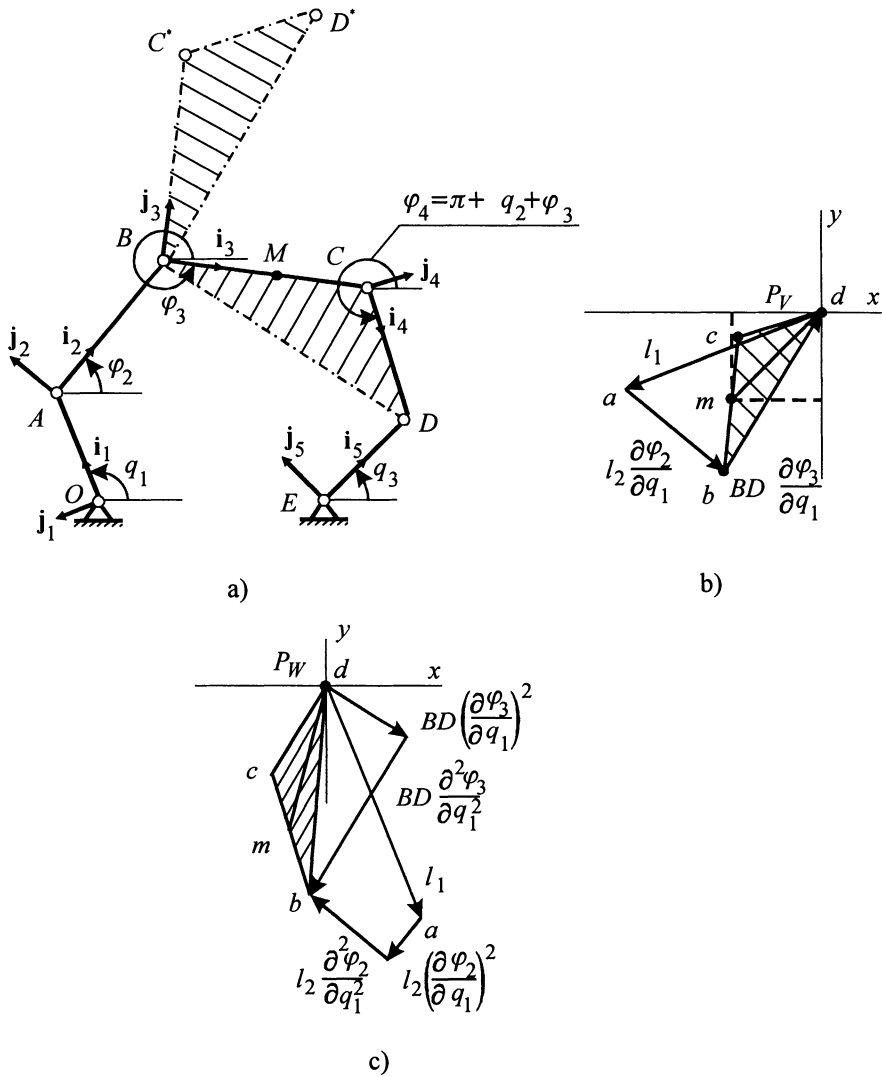


Fig. 3.2. Graph-analytical determination of the derivatives of position functions illustrated by the example of a mechanism of movability three of a platform

where $\mathbf{BD} = \ell_3 \mathbf{i}_3 + \ell_4 \mathbf{i}_4$. From the vector diagram (Fig. 3.2c), constructed on the basis of this equation, one can define $\ell_2 \cdot \partial^2 \varphi_2 / \partial q_1^2$, $\mathbf{BD} \cdot \partial^2 \varphi_3 / \partial q_1^2$, and hence, find the second derivatives.

If we differentiate Eq. (3.15) with respect to q_2 , we obtain the equation

$$\begin{aligned}
& -\ell_2 \frac{\partial \varphi_2}{\partial q_1} \frac{\partial \varphi_2}{\partial q_2} \mathbf{i}_2 + \ell_2 \frac{\partial^2 \varphi_2}{\partial q_1 \partial q_2} \mathbf{j}_2 - \ell_3 \frac{\partial \varphi_3}{\partial q_1} \frac{\partial \varphi_3}{\partial q_2} \mathbf{i}_3 + \frac{\partial^2 \varphi_3}{\partial q_1 \partial q_2} \mathbf{BD}^* - \\
& \ell_4 \frac{\partial \varphi_3}{\partial q_1} \left(\frac{\partial \varphi_3}{\partial q_2} + 1 \right) \mathbf{i}_4 = 0,
\end{aligned} \tag{3.17}$$

which allows us to determine the mixed partial derivatives.

3.2

Kinematic Analysis of Spatial Mechanisms

In kinematics of spatial mechanisms representing open kinematic chains absolute angular velocities and angular accelerations of links are determined as well as velocities and accelerations of individual points.

According to the rule of composition of rotations the *absolute angular velocity of link s* ($s = 1, 2, \dots, n$) is equal to the sum of relative angular velocities

$$\mathbf{\Omega}_s = \sum_{k=1}^s \boldsymbol{\omega}_k. \tag{3.18}$$

It can also be defined as the sum of the transport angular velocity and of the relative angular velocity. The role of the transport angular velocity is played by the absolute angular velocity $\mathbf{\Omega}_{s-1}$ of the previous link. Therefore,

$$\mathbf{\Omega}_s = \mathbf{\Omega}_{s-1} + \boldsymbol{\omega}_s. \tag{3.19}$$

Projecting this vector equality onto the axes of the coordinate system related to link s , we obtain

$$\Omega_s^{(s)} = (\Omega_{sx}^{(s)}, \Omega_{sy}^{(s)}, \Omega_{sz}^{(s)})^T = A_{s-1,s}^T \Omega_{s-1}^{(s-1)} + \omega_s^{(s)}. \tag{3.20}$$

In this, the column of coordinates of the relative angular velocity for a revolute pair is

$$\omega_s^{(s)} = (0, 0, \dot{q}_s)^T. \tag{3.21}$$

For a prismatic pair $\omega_s^{(s)} = (0, 0, 0)^T$.

For defining the *absolute angular acceleration of link s* it is necessary to differentiate with respect to time the vector Eq. (3.19). Taking into account the rule of differentiating vectors defined in a moving coordinate one gets

$$\frac{d\mathbf{\Omega}_s}{dt} = \frac{d\mathbf{\Omega}_{s-1}}{dt} + \mathbf{\Omega}_s \times \boldsymbol{\omega}_s + \frac{d\boldsymbol{\omega}_s}{dt}. \tag{3.22}$$

Denoting the absolute angular accelerations by the vectors $\mathbf{E}_s = d\mathbf{\Omega}_s / dt$ and $\mathbf{E}_{s-1} = d\mathbf{\Omega}_{s-1} / dt$ we write vector equality (3.22) in the form

$$\mathbf{E}_s = \mathbf{E}_{s-1} + \mathbf{\Omega}_s \times \mathbf{\omega}_s + \mathbf{\varepsilon}_s. \quad (3.23)$$

Projecting this vector equality onto the axes of the s -th coordinate system we obtain

$$\mathbf{E}_s^{(s)} = (\mathbf{E}_{sx}^{(s)}, \mathbf{E}_{sy}^{(s)}, \mathbf{E}_{sz}^{(s)})^T = \mathbf{A}_{s-1}^T \mathbf{E}_{s-1}^{(s-1)} + \tilde{\mathbf{\Omega}}_s^{(s)} \mathbf{\omega}_s^{(s)} + \mathbf{\varepsilon}_s^{(s)}, \quad (3.24)$$

where the skew-symmetric matrix

$$\tilde{\mathbf{\Omega}}_s^{(s)} = \begin{pmatrix} 0 & -\Omega_{sz}^{(s)} & \Omega_{sy}^{(s)} \\ \Omega_{sz}^{(s)} & 0 & -\Omega_{sx}^{(s)} \\ -\Omega_{sy}^{(s)} & \Omega_{sx}^{(s)} & 0 \end{pmatrix} \quad (3.25)$$

and the column of projections of the relative angular acceleration

$$\mathbf{\varepsilon}_s^{(s)} = (0, 0, \ddot{q}_s)^T \quad (3.26)$$

are introduced.

In accordance with a known theorem for the composition of velocities the *velocity of an arbitrary point K* , can be determined from the formula

$$\mathbf{v}_K = \mathbf{v}_O + \mathbf{\Omega} \times \mathbf{r}_K + d'\mathbf{r}_K / dt, \quad (3.27)$$

where $\mathbf{v}_O$ is the velocity of the origin of a moving coordinate system; $\mathbf{\Omega}$ is its absolute angular velocity; $\mathbf{r}_K$ is the vector, connecting the origin of the moving coordinate system with point K ; $d'\mathbf{r}_K / dt = \mathbf{v}_{rK}$ is the local derivative of the radius-vector of point K in the moving coordinate system, i.e. the relative velocity of point K . We make use of relationship (3.27) in order to define the velocity of the origin of the s -th coordinate system, i.e. of point O_s

$$\mathbf{v}_{O_s} = \mathbf{v}_{O_{s-1}} + \mathbf{\Omega}_{s-1} \times \mathbf{r}_{O_s} + \mathbf{v}_{rO_s}. \quad (3.28)$$

Proceeding to the columns of projections, we have

$$\mathbf{v}_{O_s}^{(s)} = \mathbf{A}_{s-1,s}^T (\mathbf{v}_{O_{s-1}}^{(s-1)} + \tilde{\mathbf{\Omega}}_{s-1}^{(s-1)} \mathbf{r}_{O_s}^{(s-1)} + \mathbf{v}_{rO_s}^{(s-1)}). \quad (3.29)$$

The column of coordinates of the relative velocity $\mathbf{v}_{rO_s}^{(s-1)}$ is zero, if links s and $(s-1)$ are connected by a revolute pair. If, however, these links are connected by a prismatic pair, then

$$\mathbf{v}_{rO_s}^{(s-1)} = \mathbf{A}_{s-1,s} (\dot{q}_s, 0, 0)^T = (\alpha_{11}, \alpha_{21}, \alpha_{31})^T \dot{q}_s. \quad (3.30)$$

For computing the velocity of an arbitrary point M belonging to link s we make use of the formula

$$\mathbf{v}_M = \mathbf{v}_{O_s} + \mathbf{\Omega}_s \times \mathbf{r}_M, \quad (3.31)$$

where $\mathbf{r}_M$ is the vector connecting points O_s and M . Here, it is taken into consideration that point M does not move relative to link s , i.e. $\mathbf{v}_{rM} = 0$.

The *acceleration of an arbitrary point K* is found by differentiating the vector relationship (3.27)

$$\mathbf{w}_K = \mathbf{w}_O + \mathbf{E} \times \mathbf{r}_K + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_K) + \mathbf{w}_{rK} + 2\mathbf{\Omega} \times \mathbf{v}_{rK}. \quad (3.32)$$

Here, the leading terms define the transport acceleration, the fourth is the relative acceleration and the last is the Coriolis acceleration of point K . Using relationship (3.32) we define the acceleration of point O_s which is the origin of the s -th coordinate system

$$\mathbf{w}_{O_s} = \mathbf{w}_{O_{s-1}} + \mathbf{E}_{s-1} \times \mathbf{r}_{O_s} + \mathbf{\Omega}_{s-1} \times (\mathbf{\Omega}_{s-1} \times \mathbf{r}_{O_s}) + \mathbf{w}_{rO_s} + 2\mathbf{\Omega}_{s-1} \times \mathbf{v}_{rO_s}.$$

Projecting this relationship onto the axes of the moving coordinate system, we obtain

$$\begin{aligned} w_{O_s}^{(s)} = A_{s-1,s}^T \Big(& w_{O_{s-1}}^{(s-1)} + \ddot{\mathbf{E}}_{s-1}^{(s-1)} \mathbf{r}_{O_s}^{(s-1)} + \ddot{\mathbf{\Omega}}_{s-1}^{(s-1)} \ddot{\mathbf{\Omega}}_{s-1}^{(s-1)} \mathbf{r}_{O_s}^{(s-1)} \\ & + w_{rO_s}^{(s-1)} + 2\ddot{\mathbf{\Omega}}_{s-1}^{(s-1)} \mathbf{v}_{rO_s}^{(s-1)} \Big). \end{aligned} \quad (3.33)$$

When links s and $(s-1)$ are connected by a revolute pair, then the last two terms in parentheses are equal to zero. When they are connected by a prismatic pair, then

$$w_{rO_s}^{(s-1)} = (\alpha_{11}, \alpha_{21}, \alpha_{31})^T \ddot{q}_s. \quad (3.34)$$

The relationships given above for the determination of velocities and of accelerations of mechanism points are recursive. In order to define the velocity and the acceleration of the output link, it is necessary to compute the velocities and the accelerations of the origins of all moving coordinate systems. Sometimes, the computation of the kinematic parameters of these points is not necessary. In such cases, the necessary velocities and accelerations can be defined through direct differentiation of the position functions of the corresponding points.

Let us define the velocity and the acceleration of point M on link s . According to formulae (2.29) and (2.30), we find the column of velocity projections of point M onto the axes of the fixed coordinate system

$$\mathbf{V}_M^{(0)} = (\dot{x}_M^{(0)}, \dot{y}_M^{(0)}, \dot{z}_M^{(0)}, 0)^T = \dot{H}_{0,s} \mathbf{R}_M^{(s)} + H_{0,s} \dot{\mathbf{R}}_M^{(s)}. \quad (3.35)$$

It is easy to see that the first term is the transport velocity and the second term is the relative velocity of this point. If point M is not moving relative to link s , then $\dot{\mathbf{R}}_M^{(s)} = (0, 0, 0, 0)^T$ and

$$\dot{H}_{0,s} = \sum_{\ell=1}^s H_{0,1}(q_1), \dots, \frac{\partial H_{\ell-1,\ell}(q_\ell)}{\partial q_\ell}, \dots, H_{s-1,s}(q_s) \dot{q}_\ell.$$

In a similar way one obtains the acceleration of point M

$$W_M^{(0)} = (\ddot{x}_M^{(0)}, \ddot{y}_M^{(0)}, \ddot{z}_M^{(0)}, 0)^T = \ddot{H}_{0,s} R_M^{(s)} + H_{0,s} \ddot{R}_M^{(0)} + 2\dot{H}_{0,s} \dot{R}_M^{(s)}. \quad (3.36)$$

In this expression, the first term is the transport component of acceleration, the second is the relative acceleration and the third is the Coriolis acceleration of point M . If point M is not moving relative to link s , then

$$\begin{aligned} W_M^{(0)} = & \sum_{\ell=1}^s H_{0,s}(q_s), \dots, \frac{\partial H_{\ell-1,\ell}(q_\ell)}{\partial q_\ell}, \dots, H_{s-1,s}(q_s) \ddot{q}_s R_M^{(s)} + \\ & \sum_{\ell=1}^s H_{0,1}(q_1), \dots, \frac{\partial^2 H_{\ell-1,\ell}(q_\ell)}{\partial q_\ell^2}, \dots, H_{s-1,s}(q_s) \dot{q}_\ell^2 R_M^{(s)} + \\ & \sum_{\ell=1}^s / \sum_{\ell=1}^s H_{0,1}(q_1), \dots, \frac{\partial H_{\ell-1,\ell}(q_\ell)}{\partial q_\ell}, \dots, \frac{\partial H_{m-1,m}(q_m)}{\partial q_m}, \dots, H_{s-1,s}(q_s) \dot{q}_\ell \dot{q}_m R_M^{(s)}. \end{aligned}$$

In the sum $\sum'$ the summation excludes $\ell = m$. The reader should himself determine the first and second derivatives of the matrices $H_{s-1,s}^R$ and $H_{s-1,s}^P$ with respect to q_s from formulae obtained in Sect. 2.2.

Columns (3.35) and (3.36) can be re-projected onto the axes of the moving coordinate system:

$$\begin{aligned} V_M^{(s)} &= H_{0,s}^{-1} \dot{H}_{0,s} R_M^{(s)} + \dot{R}_M^{(s)}, \\ W_M^{(s)} &= H_{0,s}^{-1} \ddot{H}_{0,s} R_M^{(s)} + \ddot{R}_M^{(s)} + 2H_{0,s}^{-1} \dot{H}_{0,s} \dot{R}_M^{(s)}. \end{aligned} \quad (3.37)$$

As an example, let us define the moduli of the absolute angular velocities and the absolute angular accelerations of the links of the robot whose kinematic diagram is shown on Fig. 3.3. The angular velocity of the first link is

$$\Omega_1^{(1)} = \omega_1^{(1)} = (0, 0, \dot{q}_1)^T, \quad \Omega_1 = \dot{q}_1.$$

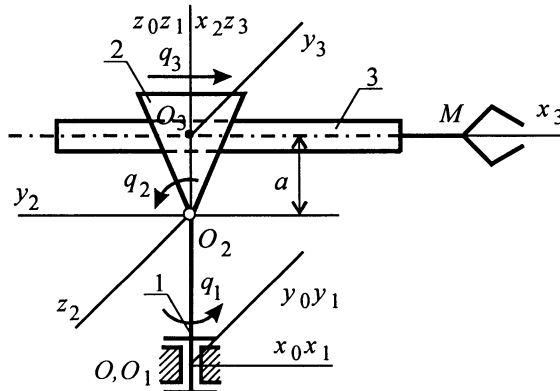


Fig. 3.3. Robot of movability three with tree-structure

For finding the projections of the remaining angular velocities onto the moving coordinate axes it is necessary to know the direction cosine matrix

$$A_{1,2} = \begin{pmatrix} -\sin q_2 & -\cos q_2 & 0 \\ 0 & 0 & -1 \\ \cos q_2 & -\sin q_2 & 0 \end{pmatrix}, \quad A_{2,3} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then,

$$\Omega_2^{(2)} = A_{1,2}^T \Omega_1^{(1)} + \omega_2^{(2)} = (\dot{q}_1 \cos q_2, -\dot{q}_1 \sin q_2, \dot{q}_2)^T,$$

$$\Omega_2 = \sqrt{\dot{q}_1^2 + \dot{q}_2^2}, \quad \omega_2^{(2)} = (0, 0, \dot{q}_2)^T,$$

$$\Omega_3^{(3)} = A_{2,3}^T \Omega_2^{(2)} + \omega_3^{(3)} = (\dot{q}_1 \sin q_2, -\dot{q}_2, \dot{q}_1 \cos q_2)^T, \quad \Omega_3 = \Omega_2, \quad \omega_3^{(3)} = (0, 0, 0)^T.$$

The angular accelerations of the links are

$$E_1^{(1)} = \varepsilon_1^{(1)} = (0, 0, \ddot{q}_1)^T, \quad E_1 = \ddot{q}_1,$$

$$E_2^{(2)} = A_{1,2}^T E_1^{(1)} + \ddot{\Omega}_2^{(2)} \omega_2^{(2)} + \varepsilon_2^{(2)} = (\ddot{q}_1 \cos q_2 - \dot{q}_1 \dot{q}_2 \sin q_2, -\ddot{q}_1 \sin q_2 - \dot{q}_1 \dot{q}_2 \cos q_2, \ddot{q}_2)^T,$$

$$E_2 = \sqrt{\ddot{q}_1^2 + \ddot{q}_2^2 + (\dot{q}_1 \dot{q}_2)^2},$$

where $\varepsilon_2^{(2)} = (0, 0, \ddot{q}_2)^T$, $E_3 = E_2$.

Let us also the absolute velocity of pole M of the gripper. First, we determine the velocity of pole O_3

$$v_{O_3}^{(3)} = A_{2,3}^T (v_{O_2}^{(2)} + \ddot{\Omega}_2^{(2)} r_{O_3}^{(2)} + v_{r_{O_3}}^{(2)}) = (\dot{q}_3 - a\dot{q}_2, \dot{q}_1(q_3 \cos q_2 - a \sin q_2), \dot{q}_2 q_3)^T,$$

where $v_{O_2}^{(2)} = (0, 0, 0)^T$, $r_{O_3}^{(2)} = (a, -q_3, 0)^T$, $v_{r_{O_3}}^{(2)} = (0, -\dot{q}_3, 0)^T$. Then, the velocity of point M is

$$v_M^{(3)} = v_{O_3}^{(3)} + \ddot{\Omega}_3^{(3)} r_M^{(3)} = (\dot{q}_3 - a\dot{q}_2, \dot{q}_1[(q_3 + \ell_3) \cos q_3 - a \sin q_2], \dot{q}_2(q_3 + \ell_3))^T,$$

where $r_M^{(3)} = (\ell_3, 0, 0)^T$.

The solution of the direct problem of kinematic analysis for closed structural groups is reduced to the determination of the first and the second derivatives of position functions. E.g., equations for the determination of the first partial derivatives $\partial \alpha / \partial q_1$, $\partial \beta / \partial q_1$, $\partial u / \partial q_1$ can be obtained by differentiation of relationships (2.43) with respect to q_1 :

$$\begin{aligned} & \frac{\partial H_{0,1}^R}{\partial \alpha} \frac{\partial \alpha}{\partial q_1} H_{1,2}^P(q_1) H_{2,3}^C(\beta, u) H_{3,4}^R(q_2) R_D^{(4)} + H_{0,1}^R(\alpha) \frac{\partial H_{1,2}^P}{\partial q_1} H_{2,3}^C(\beta, u) H_{3,4}^R(q_2) R_D^{(4)} + \\ & H_{0,1}^R(\alpha) H_{1,2}^P(q_1) \frac{\partial H_{2,3}^C}{\partial \beta} \frac{\partial \beta}{\partial q_1} H_{3,4}^R(q_2) R_D^{(4)} + H_{0,1}^R(\alpha) H_{1,2}^P(q_1) \frac{\partial H_{2,3}^C}{\partial u} \frac{\partial u}{\partial q_1} H_{3,4}^R(q_2) R_D^{(4)} = 0. \end{aligned}$$

Evaluating this matrix relationship we obtain three linear equations with three unknowns $\partial\alpha/\partial q_1, \partial\beta/\partial q_1, \partial u/\partial q_1$. If the velocity of any link of a chain has to be defined, it is necessary, first, to solve the equation system formulated above, then to define from an analogous system the partial derivatives $\partial\alpha/\partial q_2, \partial\beta/\partial q_2, \partial u/\partial q_2$, and finally, to compute the required velocity with the help of relationship (3.1).

3.3

Kinematic Analysis of a Mechanism with a Higher Pair

Let us solve the problem of kinematic analysis for the mechanism with a higher pair shown in Fig. 2.22. Differentiating Eqs. (2.75)–(2.77) with respect to q , and taking into account that the output coordinates θ_1, θ_2 and φ are functions of q , we obtain the following equations:

$$\begin{aligned}
 & r_1' \cos(q + \theta_1) \frac{d\theta_1}{dq} - r_1 \sin(q + \theta_1) \left(1 + \frac{d\theta_1}{dq} \right) + r_2' \cos(\varphi + \theta_2) \frac{d\theta_2}{dq} - \\
 & \quad r_2 \sin(\varphi + \theta_2) \left(\frac{d\theta_2}{dq} + \frac{d\varphi}{dq} \right) = 0, \\
 & r_1' \sin(q + \theta_1) \frac{d\theta_1}{dq} + r_1 \cos(q + \theta_1) \left(1 + \frac{d\theta_1}{dq} \right) + r_2' \sin(\varphi + \theta_2) \frac{d\theta_2}{dq} - \\
 & \quad r_2 \cos(\varphi + \theta_2) \left(\frac{d\theta_2}{dq} + \frac{d\varphi}{dq} \right) = 0, \tag{3.38} \\
 & \left[1 + \left(\frac{r_1}{r_1'} \right)^2 \right]^{-1} \frac{(r_1')^2 - r_1 r_1''}{(r_1')^2} \cdot \frac{d\theta_1}{dq} + \left[1 + \left(\frac{r_2}{r_2'} \right)^2 \right]^{-1} \frac{(r_2')^2 - r_2 r_2''}{(r_2')^2} \cdot \frac{d\theta_2}{dq} + \\
 & \quad 1 + \frac{d\theta_1}{dq} + \frac{d\theta_2}{dq} + \frac{d\varphi}{dq} = 0.
 \end{aligned}$$

Here, the derivatives of r_1 and r_2 with respect to θ_1 and to θ_2 are denoted by prime. As is well-known the curvature K of a curve defined in polar form $r(\theta)$ is

$$K = \frac{r^2 + 2(r')^2 - rr''}{[r^2 + (r')^2]^{3/2}}.$$

Taking into account this expression it is easy to transform Eqs. (3.38) into the following form

$$\begin{aligned}
& [r_1' \cos(q + \theta_1) - r_1 \sin(q + \theta_1)] \frac{d\theta_1}{dq} + [r_2' \cos(\varphi + \theta_2) - \\
& r_2 \sin(\varphi + \theta_2)] \frac{d\theta_2}{dq} - r_2 \sin(\varphi + \theta_2) \frac{d\varphi}{dq} = r_1 \sin(q + \theta_1), \\
& [r_1' \sin(q + \theta_1) + r_1 \cos(q + \theta_1)] \frac{d\theta_1}{dq} - [r_2' \sin(\varphi + \theta_2) - \\
& r_2 \cos(\varphi + \theta_2)] \frac{d\theta_2}{dq} - r_2 \cos(\varphi + \theta_2) \frac{d\varphi}{dq} = -r_1 \cos(q + \theta_1), \\
& K_1 \sqrt{r_1^2 + (r_1')^2} \frac{d\theta_1}{dq} + K_2 \sqrt{r_2^2 + (r_2')^2} \frac{d\theta_2}{dq} + \frac{d\varphi}{dq} = -1.
\end{aligned} \tag{3.39}$$

The determinant of this system of linear equations is also the Jacobian of system (2.75)–(2.77):

$$\begin{aligned}
J = & \begin{vmatrix} r_1' \cos(q + \theta_1) - r_1 \sin(q + \theta_1) & r_2' \cos(\varphi + \theta_2) - r_2 \sin(\varphi + \theta_2) & -r_2 \sin(\varphi + \theta_2) \\ r_1' \sin(q + \theta_1) + r_1 \cos(q + \theta_1) & -r_2' \sin(\varphi + \theta_2) - r_2 \cos(\varphi + \theta_2) & -r_2 \cos(\varphi + \theta_2) \\ K_1 \sqrt{r_1^2 + (r_1')^2} & K_2 \sqrt{r_2^2 + (r_2')^2} & 1 \end{vmatrix} = \\
& -K_1 \sqrt{r_1^2 + (r_1')^2} r_2 r_2' + K_2 \sqrt{r_2^2 + (r_2')^2} (r_2 r_1' \cos \alpha - r_1 r_2 \sin \alpha) + \\
& (r_1 r_2 - r_1' r_2') \sin \alpha - (r_1 r_2' + r_1' r_2) \cos \alpha,
\end{aligned} \tag{3.40}$$

where $\alpha = q + \theta_1 + \theta_2 + \varphi$.

The final goal of the kinematic analysis is the determination of the first geometric transfer function $d\varphi/dq$, which, for a mechanism transmitting rotational motion, is called the *transmission ratio* between output and input links. Solving the system equation (3.39), we obtain

$$\begin{aligned}
\frac{d\varphi}{dq} = & J^{-1} [K_1 \sqrt{r_1^2 + (r_1')^2} (-r_2' r_1 \cos \alpha + r_2 r_1 \sin \alpha) + \\
& K_2 \sqrt{r_2^2 + (r_2')^2} r_1 r_1' - (r_1 r_2 - r_1' r_2') \sin \alpha - (r_1 r_2' + r_1' r_2) \cos \alpha].
\end{aligned} \tag{3.41}$$

This expression can be simplified if one takes into account that (see Fig. 2.22)

$$\alpha = \pi - \psi_1 - \psi_2 = \pi - \arctan \frac{r_1}{r_1'} - \arctan \frac{r_2}{r_2'},$$

and, therefore,

$$\begin{aligned}
\tan \alpha = & -\frac{r_1 r_2' + r_2 r_1'}{r_1' r_2' - r_1 r_2}, \quad \cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}} = \frac{r_1 r_2}{\sqrt{[r_1^2 + (r_1')^2][r_2^2 + (r_2')^2]}}, \\
\sin \alpha = & \frac{r_1 r_2' + r_2 r_1'}{\sqrt{[r_1^2 + (r_1')^2][r_2^2 + (r_2')^2]}}.
\end{aligned}$$

From this we obtain

$$(r_1 r_2 - r_1' r_2') \sin \alpha - (r_1 r_2' + r_1' r_2) \cos \alpha = 0,$$

$$r_2 r_1 \sin \alpha - r_2' r_1 \cos \alpha = \frac{r_1 r_1' \sqrt{r_2^2 + (r_2')^2}}{\sqrt{r_1^2 + (r_1')^2}},$$

$$r_2 r_1' \cos \alpha - r_2 r_1 \sin \alpha = -\frac{r_2 r_2' \sqrt{r_1^2 + (r_1')^2}}{\sqrt{r_2^2 + (r_2')^2}}.$$

With this we find

$$\frac{d\varphi}{dq} = -\frac{K_1 \sqrt{r_2^2 + (r_2')^2} r_1 r_1' + K_2 \sqrt{r_2^2 + (r_2')^2} r_1 r_1'}{K_1 \sqrt{r_1^2 + (r_1')^2} r_2 r_2' + K_2 \sqrt{r_1^2 + (r_1')^2} r_2 r_2'} = -\frac{U_1(\theta_1)}{U_2(\theta_2)}, \quad (3.42)$$

where $U(\theta) = \frac{r(\theta)r'(\theta)}{\sqrt{r^2 + (r')^2}} = r(\theta)\cos\psi(\theta)$. Here, $\psi(\theta)$ is the angle between the tangent to a curve and the radius of point of tangency. In this way, $U(\theta)$ is the radius projection on direction of the common tangent $\tau - \tau$ to conjugated curves at point A , i.e. $U(\theta) = ON$. Then,

$$\frac{d\varphi}{dq} = \frac{\dot{\varphi}}{\dot{q}} = -\frac{O_1 N_1}{O_2 N_2}. \quad (3.43)$$

The minus sign in Eq. (3.43) means that the links rotate in opposite directions. From the similarity of triangles $O_1 P N_1$ and $O_2 P N_2$ it follows that

$$\frac{O_1 N_1}{O_2 N_2} = \frac{O_1 P}{O_2 P}, \quad (3.44)$$

where P is the instantaneous center of rotation of relative motion of the links. Putting the last two equalities together, we obtain

$$\frac{\dot{\varphi}}{\dot{q}} = -\frac{O_1 P}{O_2 P}. \quad (3.45)$$

Thus, the common normal $n - n$ at the tangency point A of the conjugated profiles passes through the instantaneous center of rotation P and divides the distance $O_1 O_2 = a$ between the axes into parts, inversely proportional to the angular velocities of the links. This is the modern interpretation of *Willis' theorem* for planar mechanisms (see [4]). In general, the *basic theorem* for a higher kinematic pair is formulated as follows: At every point of tangency between surfaces constituting a higher pair, the analogues of velocity vectors of relative motion with respect to generalized coordinates lie in the common tangent plane. In analytical form the conditions of the basic theorem [5] can be represented

in the following way

$$\mathbf{n} \cdot \mathbf{v}'_{21} = -\mathbf{n} \cdot \mathbf{v}'_{12} = 0, \quad (3.46)$$

where $\mathbf{n}$ is the unit vector of the common normal to the surfaces at the point of tangency, $\mathbf{v}'$ is the analogue of the velocity vector of the point of tangency of one surface relative to the other.

Constraint equations of the form (3.46) are also called *equations of meshing*, since they are found in solving problems of geometric and kinematic analysis of gear meshing. In the theory of meshing, the instantaneous center of rotation P is called a *pitch point*. When the transmission ratio varies, the pitch point moves along the distance between the axes; when the ratio is constant, the pitch point P is fixed.

3.4 Kinematics of Mechanisms with Linear Position Functions

Mechanisms transmitting motion with a constant transmission ratio are used in various machines and devices. Such mechanisms, as already mentioned, are called transmission mechanisms or briefly *transmissions*. Typical for transmission mechanisms are linear relationships between output and input coordinates, i.e. linear position functions

$$x_s = \Pi_s(q_1, q_2, \dots, q_n) = \sum_{k=1}^n i_{ks}^{-1} q_k \quad (s = 1, 2, \dots, m), \quad (3.47)$$

where $i_{ks} = (\partial \Pi_s / \partial q_k)^{-1}$ are the transmission ratios between the input link k and the output link s . Differentiating these functions twice with respect to time, we obtain an expression for velocities and accelerations

$$\dot{x}_s = \sum_{k=1}^n \frac{\partial \Pi_s}{\partial q_k} \dot{q}_k = \sum_{k=1}^n i_{ks}^{-1} \dot{q}_k, \quad (3.48)$$

$$\ddot{x}_s = \sum_{k=1}^n \frac{\partial \Pi_s}{\partial q_k} \ddot{q}_k + \sum_{k=1}^n \sum_{r=1}^n \frac{\partial^2 \Pi_s}{\partial q_k \partial q_r} \dot{q}_k \dot{q}_r = \sum_{k=1}^n i_{ks}^{-1} \ddot{q}_k. \quad (3.49)$$

From formulae (3.48) and (3.49) it can be noticed that with given $\dot{q}_k(t)$ and $\ddot{q}_k(t)$ the problem of determining the velocities and the accelerations is reduced to the determination of transmission ratios. Usually, these ratios are expressed by ratios of constant parameters in the kinematic diagram.

Examples of planar transmissions of movability one have already been considered in Sect. 1.6. The transmission ratio of the *friction transmission* with cylindric rollers (see Fig. 1.21a) without slippage is expressed by the ratios of the radii of rollers 2 and 1. This condition follows from the fact, that the circles of the rollers

are centrodes in the relative motion of links. In an analogous way the transmission ratio of the *belt drives* (see Fig. 1.21d) is determined.

For the cylindric gearing the transmission ratio is

$$i_{12} = \dot{q} / \dot{\varphi} = \pm r_2 / r_1, \quad (3.50)$$

where $r_1 = O_1P, r_2 = O_2P$ are the radii of the centrodes in the relative motion of links 1 and 2, called *initial circles*. Here the sign “minus” is related to an *external gearing* (Fig. 3.4), while the sign “plus” is related to an *internal gearing* (Fig. 3.5).

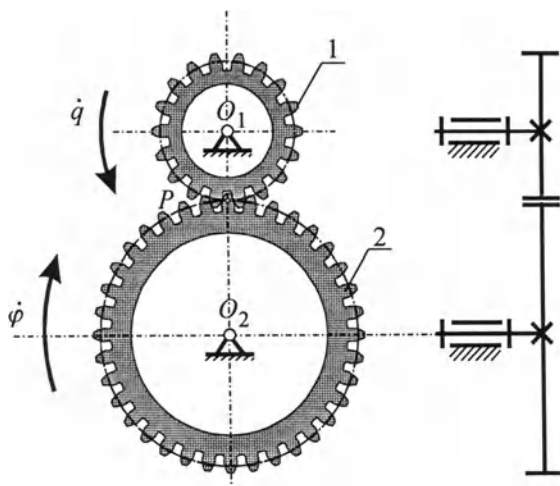


Fig. 3.4. External gearing

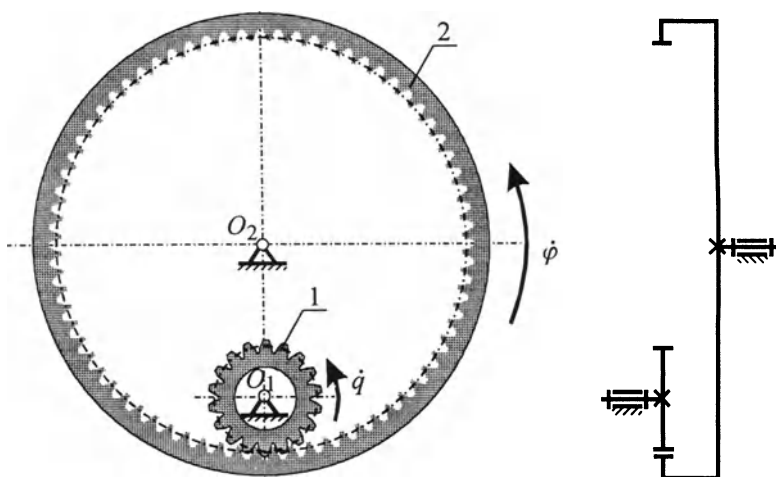


Fig. 3.5. Internal gearing

The transmission ratio of a gearing can also be expressed by the numbers z_1, z_2 of teeth. Teeth on wheel rings are ordered uniformly, that is why, a revolution of input link 1 through an angular step $\Delta q = 2\pi / z_1$ causes a revolution of output link 2 through an angular step $\Delta \varphi = 2\pi / z_2$. Then,

$$i_{12} = \dot{q} / \dot{\varphi} = \Delta q / \Delta \varphi = \pm z_2 / z_1. \quad (3.51)$$

In the *rack-and-pinion* transmission (Fig. 3.6), transforming rotational motion into translational and vice versa, the transmission ratio is

$$i_{12} = \dot{q} / \dot{x} = 1 / r, \quad (3.52)$$

where r is the radius of the initial circle of the pinion.

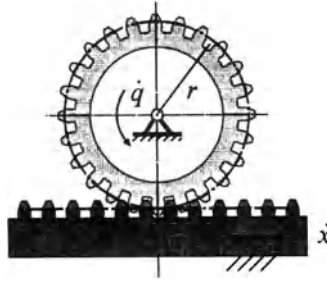


Fig. 3.6. Rack-and-pinion gearing

The transmission ratio that can be realized by a single pair of gears is small. For the realization of significant transmission ratios multi-stage gear mechanisms consisting of several pairs of gears are applied. Multi-stage mechanisms with fixed rotation axes are called *gear trains*. The transmission ratio of the gear train shown in Fig. 3.7 is defined in the following way

$$i_{1n} = \frac{\omega_1}{\omega_n} = \frac{\omega_1}{\omega_2} \frac{\omega_2}{\omega_3} \dots \frac{\omega_{n-1}}{\omega_n} = i_{12} i_{2'3} \dots i_{n-1'n} = (-1)^k \frac{z_2 z_3 \dots z_n}{z_1 z_2' \dots z_{n-1'}}, \quad (3.53)$$

where k is the number of external gearings. Therefore, the transmission ratio of a gear train is equal to the product of the transmission ratios of gear pairs.

For the transmission of rotational motion between two axes, located at considerable distance, *gear trains* with *intermediate gears* are applied. The transmission ratio of such a mechanism (Fig. 3.8) is:

$$i_{1n} = \frac{\omega_1}{\omega_n} = \frac{\omega_1}{\omega_2} \frac{\omega_2}{\omega_3} \dots \frac{\omega_{n-1}}{\omega_n} = i_{12} i_{23} \dots i_{n-1n} = (-1)^k \frac{z_n}{z_1}, \quad (3.54)$$

where k is the number of external gearings. From formula (3.54) it is seen that the numbers of teeth of the intermediate gears $2, 3, \dots, n-1$ do not influence the magnitude of the transmission ratio but they can change the sign of the trans-

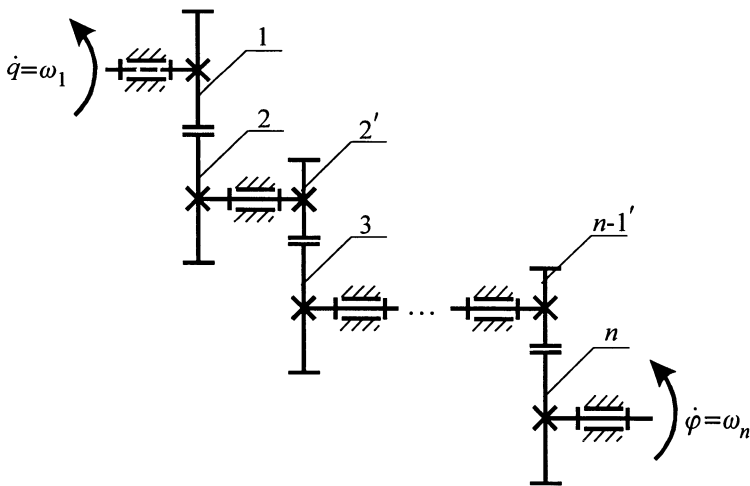


Fig. 3.7. Gear train

mission ratio. For this reason gears $2, 3, \dots, n-1$ are sometimes called *parasitic*.

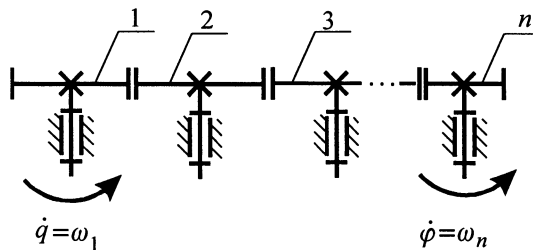


Fig. 3.8. Gear train with intermediate gears

We suggest that the reader defines, as an example, the number of teeth z_1, z_2 of gears in the clockwork (Fig. 3.9), if $z_{2'} = 15, z_3 = 45$. When solving this problem, it is necessary to take into consideration a number of conditions: 1) the condition of alignment, i.e. of the coincidence of the axes of gears 1, 2 and $2'$; 3) the condition of proportionality between the diameter of the *pitch* circle and the number of teeth

$$d = mz, \quad (3.55)$$

where m is the coefficient of proportionality, called *modulus of meshing*. In gearings, whose gears are cut without shifting, the pitch and the initial circles coincide. Henceforth, if not explicitly specified, the moduli of all gears are assumed to be identical. Also the pitch and the initial diameters are assumed identical.

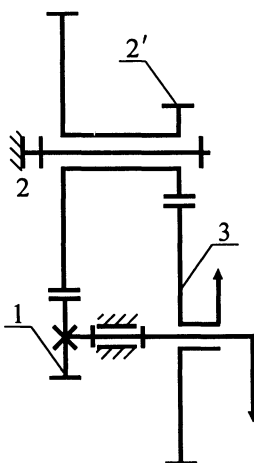


Fig. 3.9. Clockwork

Gear mechanisms with movable gear axes are referred to as *planetary* gears. If the number of degrees of movability of such a mechanism is greater than one, it is called a *differential* gear. One of the simplest planetary mechanisms is depicted in Fig. 3. 10. Wheel 2, whose axis moves in space, is called a *planet pinion* (*satellite gear*). Link H , carrying the movable axis O_2 , is called a *planet carrier*. The gear wheel 1, whose axis coincides with the rotation axis of the carrier, is referred to as *sun* or *central* gear. The specified group of links constitutes a *differential kinematic chain*. For every such group one can formulate equations of kinematic analysis. The derivation of these equations is simplified when using the *method of inversion of motion*. The essence of the method consists in reducing the problem of analysis of planetary mechanisms to the analysis of gear trains. Instead of absolute motions of links their relative motion with respect to the carrier is considered.

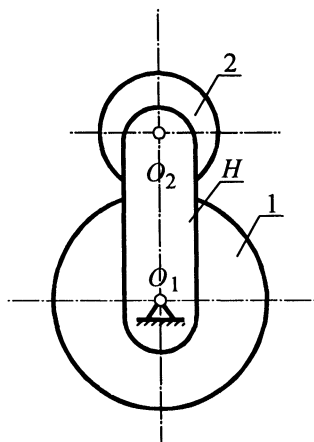


Fig. 3.10. The simplest planetary mechanism

As an example, let us define the transmission ratio i_{H1} of the planetary mechanism shown in Fig. 3.11a. In this mechanism, it is necessary to imagine that to all links an additional rotation with angular velocity $-\omega_H$ is introduced. As a result of this transformation, the planetary mechanism becomes a gear train and the carrier „stops“. Then, the ratio of angular velocities is written in the form of *Willis' formula*:

$$\frac{\omega_1 - \omega_H}{\omega_3 - \omega_H} = \left(-\frac{z_2}{z_1} \right) \left(-\frac{z_3}{z_{2'}} \right). \quad (3.56)$$

Taking into account, that wheel 3 is motionless ($\omega_3 = 0$), we obtain the ratio of the planetary mechanism

$$i_{H1} = \frac{\omega_H}{\omega_1} = \frac{z_1 z_{2'}}{z_1 z_{2'} - z_2 z_3} = \frac{z_1 z_{2'}}{(z_{2'} - z_2)(z_3 + z_{2'})}. \quad (3.57)$$

The formula shows that with the given mechanism large transmission ratios can be achieved since the denominator of formula (3.57) includes the difference of numbers $z_{2'}$ and z_2 . If one chooses wheels 2' and 2 nearly equal in size, then ω_1 will be rather small. In the case $z_{2'} = z_2$ the output link 1 will not move ($\omega_1 = 0$).

Kinematic equations can also be obtained by a *graph-analytical* method. To achieve this, it is necessary to associate a vector with the carrier and with every gear wheel. In the case of the carrier, the vector is drawn from its axis to the center of every planet pinion. For the wheels the vectors are drawn from the centers of the wheels to the poles of meshing. Then, the vector equations for closed chains are differentiated with respect to time. In the mechanism under consideration unit vectors $\mathbf{i}$ and $\mathbf{j}$ are associated with every link (Fig. 3.11b). The vector equations for the two closed loops read

$$\begin{aligned} r_H \mathbf{i}_H + r_2 \mathbf{i}_2 &= r_1 \mathbf{i}_1, \\ r_H \mathbf{i}_H + r_{2'} \mathbf{i}_{2'} &= r_3 \mathbf{i}_3, \end{aligned}$$

where $r_H = r_1 + r_2 = r_{2'} + r_3$. Differentiating these equations with respect to time we get

$$\begin{aligned} (r_1 + r_2) \omega_H \mathbf{j}_H + r_2 \omega_2 \mathbf{j}_2 &= r_1 \omega_1 \mathbf{j}_1, \\ (r_{2'} + r_3) \omega_H \mathbf{j}_H + r_{2'} \omega_{2'} \mathbf{j}_{2'} &= r_3 \omega_3 \mathbf{j}_3. \end{aligned}$$

Taking into account the directions of the unit vectors as well as the relationship $\omega_2 = \omega_{2'}$, $r = mz/2$, we obtain the kinematic equations

$$\begin{aligned} -(z_1 + z_2) \omega_H + z_2 \omega_2 &= -z_1 \omega_1, \\ -(z_{2'} + z_3) \omega_H + z_{2'} \omega_2 &= 0, \end{aligned}$$

from which we find the required transmission ratio i_{H1} .

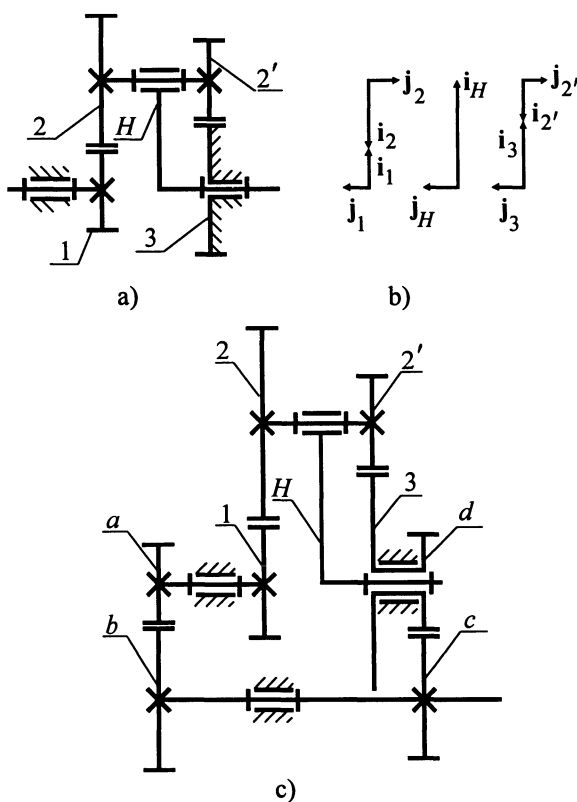


Fig. 3.11. Planetary mechanism with defined unit vectors

On the motion of certain links of a planetary mechanism additional constraints are imposed. In this case, constraint equations are formulated: a) if the r -th gear wheel is motionless, then $\omega_r = 0$; b) if the gear wheels and the carrier belong to one and the same link, then $\omega_f = \omega_e = \omega_H$, and so on; c) if the gear wheels constitute a gear train, then $\omega_\ell = i_{\ell m} \omega_m$. In order to define the transmission ratio of a planetary mechanism, it is necessary to solve the kinematic equations, obtained from the combination of Willis' formula with the constraint equations.

As an example, let us define the transmission ratio i_{H1} of the planetary mechanism shown in Fig. 3.11c. Here links 1 and 3 are closed by a gear train with wheels a, b, c, d . In this case, we have $\omega_3 = \omega_1 / i_{ad}$, where $i_{ad} = z_b z_d / z_a z_c$. On the basis of formula (3.56) we obtain

$$i_{H1} = \frac{1 - i_{13}^{(H)} \cdot i_{ad}^{-1}}{1 - i_{13}^{(H)}},$$

where $i_{13}^{(H)} = z_2 z_3 / z_1 z_2'$ is the transmission ratio between wheels 1 and 3 with stopped carrier (indicated by the upper index H).

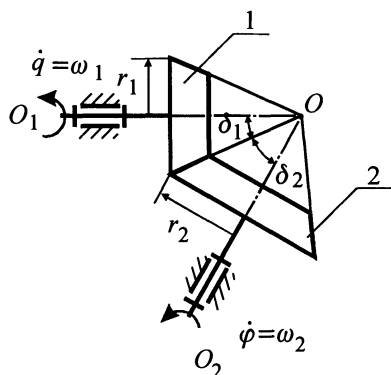


Fig. 3.12. Bevel gearing

Spatial gear mechanisms are used for transmission of rotational motion between intersecting and crossing axes. The transmission of motion between intersecting axes can be realized with the help of *bevel* gears (Fig. 3.12). Just as in cylindric transmissions there are initial cylinders (initial circles in planar transmissions), so in bevel gearings there are initial cones (initial circles in planar transmissions), so in bevel gearings there are initial cones 1 and 2, rolling on one another without slippage. In this case, the velocities of any two points coinciding in P , each belonging to cone and lying on the common generating line OP , are equal, i.e. $r_1\omega_1 = r_2\omega_2$, whence $i_{12} = \omega_1/\omega_2 = r_2/r_1$. Taking into account that $r_1 = OP \sin \delta_1$, $r_2 = OP \sin \delta_2$, we obtain

$$i_{12} = \sin \delta_2 / \sin \delta_1 = z_2 / z_1, \quad (3.58)$$

where δ_1, δ_2 are the angles between the wheel axes and the generating line OP of the initial cones. For the majority of bevel gears $\delta_1 + \delta_2 = 90^\circ$. Then, formula (3.58) takes the following form

$$i_{12} = \tan \delta_2 = \cot \delta_1. \quad (3.59)$$

Bevel gears are widely used in planetary mechanisms. Fig. 3.13a shows a *car differential*. Here the rotation is transmitted from the car engine by means of cone wheels 1 and 2 to the box H , serving as a carrier. In the box two identical planet pinions 3 and 3' rotate freely about their axes. The planet pinions are in meshing with two central wheels 4 and 5, which are fixed on the two separate half-axes of the drive wheels of the car. Let us determine the angular velocities of the central wheels, if the angular velocity ω_H of the carrier is known. In order to determine the angular velocities we will use the method of inversion of motions. For this purpose, we impart to all mechanism links a rotation with velocity $-\omega_H$. We obtain equations of kinematic analysis with the help of Willis' formula

$$i_{45}^{(H)} = \frac{\omega_4 - \omega_H}{\omega_5 - \omega_H}.$$

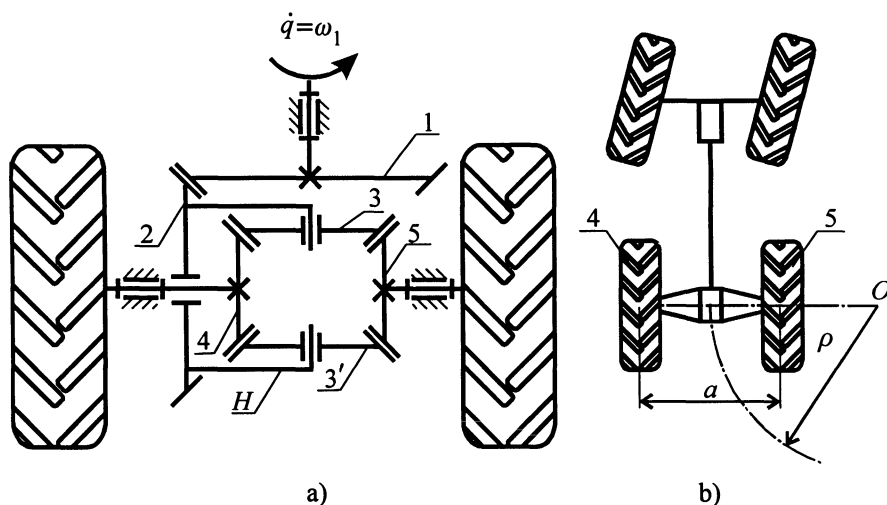


Fig. 3.13. Car differential

Since $z_4 = z_5$ and since the central wheels in the relative motion are rotating in opposite directions about one and the same geometric axis, we have $i_{45}^{(H)} = -1$, and therefore

$$\omega_4 + \omega_5 = 2\omega_H. \quad (3.60)$$

As a matter of fact, the car differential is a special mechanism in which, without taking into account the „excessive“ link 3', the number of inputs $n = 1$ does not coincide with the number of degrees of movability $w = 2$ (one kinematic Eq. (3.60) contains two unknown angular velocities ω_4 and ω_5). Indeed, if one lifts the car from the road with the help of a jack, then the motion of the car wheels will be undefined. The definiteness of wheel motions depends on the car motion on the road (on additional constraint conditions).

If the car is driven on a straight and smooth road, then the angular velocities of wheels 4 and 5 are identical and according to (3.60) they are equal to the angular velocity of the carrier $\omega_4 = \omega_5 = \omega_H$. If wheel 5 stops after meeting an obstacle ($\omega_5 = 0$), then the angular velocity of wheel 4 is equal to twice the velocity of the carrier, $\omega_4 = 2\omega_H$.

In turning, one of the car wheels, for example the wheel connected with the central wheel 4, covers a longer distance than the car wheel connected with the central wheel 5 (Fig. 3.13b). If the car makes a turn with radius ρ , then the angular velocities of wheels 4 and 5 are proportional to the radii of motion

$$\omega_4 / \omega_5 = \rho_4 / \rho_5, \quad (3.61)$$

where $\rho_4 = \rho + 0.5a$, $\rho_5 = \rho - 0.5a$, a is the wheel spacing. From relationships (3.60) and (3.61) we determine the angular velocities of wheels

$$\omega_4 = \frac{2\omega_H \rho_4}{\rho_4 + \rho_5} > \omega_5 = \frac{2\omega_H \rho_5}{\rho_4 + \rho_5}.$$

From these formulae it can be noticed that during a car turn the angular velocity of wheel 4 increases by $a\omega_H/2\rho$. The angular velocity of wheel 5 decreases by exactly the same quantity. Due to a difference in the angular velocities, there is no slippage of the car wheels and the tire wear decreases.

If the car is stopped with the help of a transmission brake, i.e. if an angular velocity $\omega_H = 0$ is imparted to the carrier, then, according to (3.60), the wheels will rotate with one and the same angular velocity but in opposite directions ($\omega_5 = -\omega_4$). This method is sometimes used by drivers for turning around a car on the spot without using the steering wheel.

Hyperboloidal gears whose initial surfaces are rotational hyperboloids, are used for transmission of rotational motion between crossing axes. Since the manufacture of hyperboloidal gears is complex, *hypoid* gears and *helical* gears are common practice. In the former the surfaces of hyperboloids are replaced by truncated cones, and in the latter the necks of hyperboloids are replaced by cylinders (Fig. 3.14a). In Fig. 3.14b the initial cylinders of two helical gears are shown. Let the peripheral velocities be $v_1 = \omega_1 r_1$ and $v_2 = \omega_2 r_2$. It is easy to notice that the normal components of the circumferential velocities of the initial cylinders must have equal magnitude $\omega_2 r_2 \cos \beta_2 = \omega_1 r_1 \cos \beta_1$, hence the transmission ratio is

$$i_{12} = \omega_1 / \omega_2 = r_2 \cos \beta_2 / r_1 \cos \beta_1 = z_2 / z_1. \quad (3.62)$$

With crossing angle $\beta_1 + \beta_2 = 90^\circ$ formula (3.62) takes the form

$$i_{12} = \omega_1 / \omega_2 = r_2 \tan \beta_1 / r_1 = r_2 / r_1 \tan \beta_2. \quad (3.63)$$

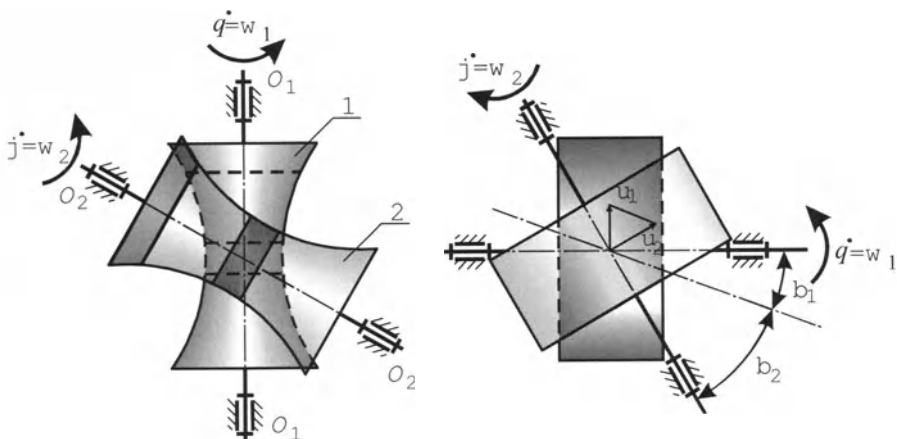


Fig. 3.14. Hyperboloidal gearings

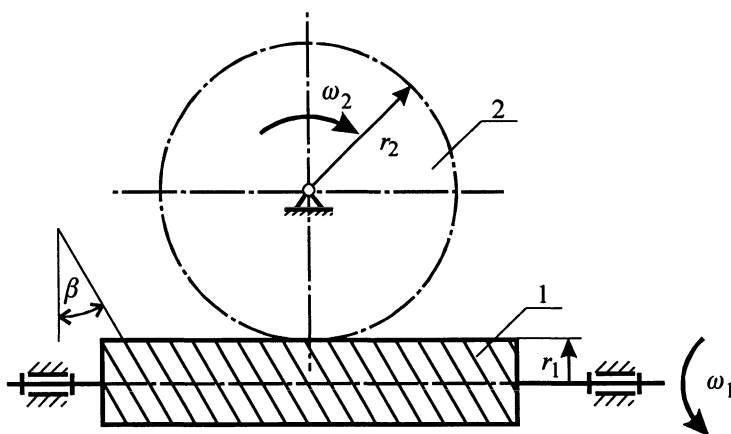


Fig. 3.15. Worm gearing

The formula for the transmission ratio of the *worm* gearing (Fig. 3.15) is analogous to formula (3.63), since this transmission is obtained as a particular case of the helical transmission. Here, link 1 has several full helical turns (threads). This link is most commonly made in the form of cylindric or globoidal *worm*. Link 2 is called a *worm wheel*. The transmission ratio for the worm gearing is

$$i_{12} = r_2 / r_1 \tan \beta = z_2 / z_1, \quad (3.64)$$

where β is the helix angle of the screw line of the worm, z_1 is the number of entries of the worm, z_2 is the number of teeth of the worm wheel.

3.5 Parametric Analysis of Mechanisms

As shown above, the kinematic analysis of mechanisms is reduced to the determination of partial derivatives of position functions. To the same problem also the parametric analysis of mechanisms is reduced. It is related to the determination of partial derivatives of position functions with respect to *constant mechanism parameters* $\alpha_1, \alpha_2, \dots, \alpha_l$ (lengths of links, coordinates of immovable joints etc.). Such partial derivatives are required for the accuracy analysis of a mechanism. A necessity of their determination may arise in the kinematic synthesis of a mechanism. Finally, the analysis of mechanisms with elastic links requires partial derivatives up to third order. We will acquaint ourselves with the methods for determining such partial derivatives by using as example the *accuracy* analysis of a mechanism.

Until now it has been assumed that the form and the dimensions of mechanism links are absolutely precise: There are no clearances at kinematic pairs and input pairs are executing program motions, i.e. only *ideal (theoretical)* mechanisms

have been considered. The *real (actual)* mechanisms differ from the ideal ones through the presence of errors.

We will call *errors* the deviations of the actual mechanism parameters from their design values. The deviations $\Delta\alpha_1, \Delta\alpha_2, \dots, \Delta\alpha_\ell$ of the constant geometric parameters of a mechanism from their nominal values $\alpha_1, \alpha_2, \dots, \alpha_\ell$ are referred to as *geometric errors*. The deviations $\Delta q_1, \Delta q_2, \dots, \Delta q_n$ of the input coordinates from their program values $q_1, q_2, \dots, q_n$ are referred to as *kinematic errors*. The geometric and kinematic errors are the primary sources of mechanism errors. For this reason they are called *primary errors*. Primary errors occur during manufacture and assembly of mechanisms (*technological* and *diagrammatic errors*) and as a result of usage of mechanisms (*static* and *dynamic errors*, *temperature* and *wear errors*).

Primary errors cause differences between positions of the output links of the real mechanism ($\sim$) on the one hand, and of the corresponding ideal mechanism, on the other. These differences are called *position errors*:

$$\Delta x_r = \tilde{x}_r - x_r = \tilde{\Pi}_r(q_1 + \Delta q_1, \dots, \alpha_\ell + \Delta\alpha_\ell) - \Pi_r(q_1, \dots, q_n) \quad (r = 1, 2, \dots, m). \quad (3.65)$$

Fig. 3.16 shows the position error $\Delta x_B = \tilde{x}_B - x_B$ of point B of the slider-crank mechanism, caused by errors Δr and Δl in the length of the crank and of the connecting rod, respectively, as well as because of a kinematic error Δq .

The difference of the displacements of output links of real and ideal mechanisms is called *displacement error*.

$$\begin{aligned} \Delta S_r &= \tilde{S}_r - S_r = (\tilde{x}_r^{(2)} - \tilde{x}_r^{(1)}) - (x_r^{(2)} - x_r^{(1)}) = \\ &= (\tilde{x}_r^{(2)} - x_r^{(2)}) - (\tilde{x}_r^{(1)} - x_r^{(1)}) = \Delta x_r^{(2)} - \Delta x_r^{(1)}. \end{aligned} \quad (3.66)$$

From this expression it is apparent that the displacement error of a mechanism can be defined as the difference between the position errors of the output link at the end (2) and at the beginning (1) of mechanism motion. The displacement errors cause *errors in velocity*

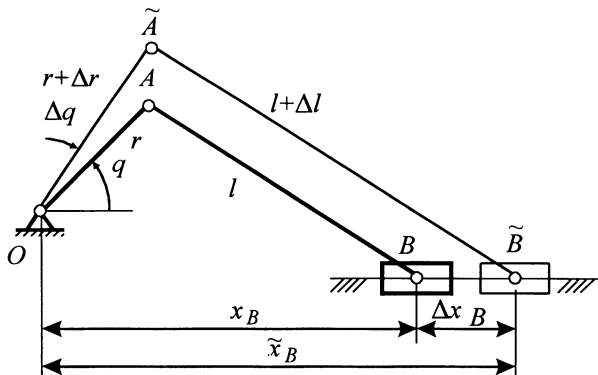


Fig. 3.16. Position error of the slider-crank mechanism

$$\Delta v_r = d(\Delta x_r) / dt \quad (3.67)$$

and errors in acceleration Δw_r .

In the accuracy analysis we will assume, both, the errors of mechanism and the primary errors causing them are small. This allows us to represent expressions (3.65) and (3.67), up to small quantities of second order, in the form

$$\Delta x_r = \sum_{k=1}^n \frac{\partial \Pi_r}{\partial q_k} \Delta q_k + \sum_{f=1}^{\ell} \frac{\partial \Pi_r}{\partial \alpha_f} \Delta \alpha_f \quad (r = 1, 2, \dots, m), \quad (3.68)$$

$$\Delta v_r = \sum_{k=1}^n \sum_{i=1}^n \frac{\partial^2 \Pi_r}{\partial q_k \partial q_i} \dot{q}_i \Delta q_k + \sum_{k=1}^n \frac{\partial \Pi_r}{\partial q_k} \Delta \dot{q}_k + \sum_{f=1}^{\ell} \sum_{i=1}^n \frac{\partial^2 \Pi_r}{\partial \alpha_f \partial q_i} \Delta \alpha_f \dot{q}_i. \quad (3.69)$$

When computing errors of a mechanism, it must be taken into account that the partial derivatives are defined for ideal mechanisms.

Two basic problems are solved in the error analysis. The *first* problem is the accuracy analysis of an existing mechanism with known primary errors. For this it is sufficient to find the partial derivatives and then to determine the mechanism errors with the help of formulae (3.68) and (3.69). The *second* problem is related to the synthesis and design of mechanisms satisfying given accuracy norms. We restrict ourselves to the consideration of the first problem only.

There are several methods for determining errors. Let us consider with two of them. The *graph-analytical* method for the determination of errors is based on the use of a closed vector loop and on the construction of an *error vector diagram*. As an example, let us define the position errors Δx_B , $\Delta \varphi$ of the slider-crank mechanism (Fig. 3.17), caused by primary errors Δq , Δr , $\Delta \ell$, Δe . To this end, we rigidly attach unit vectors $\mathbf{i}$, $\mathbf{j}$ to every mechanism link. For the absolute derivative of vector $\mathbf{u}$, defined in a moving coordinate system, the formula was obtained

$$\dot{\mathbf{u}} = \dot{u}\mathbf{i} + u\dot{\varphi} \cdot \mathbf{j}.$$

Analogously, one can define an increment of this vector, corresponding to the increments of modulus u and angle φ :

$$\Delta \mathbf{u} = \Delta u \mathbf{i} + u \Delta \varphi \cdot \mathbf{j}.$$

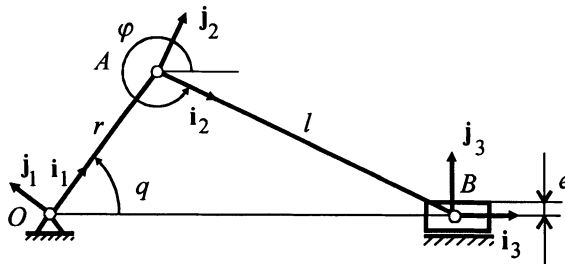


Fig. 3.17. Slider-crank mechanism with defined unit vectors

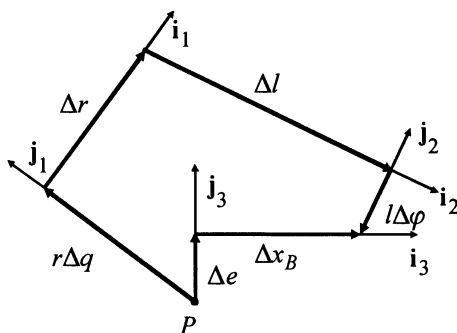


Fig. 3.18. Error vector diagram of the slider-crank mechanism

In the present case

$$r\mathbf{i}_1 + \ell\mathbf{i}_2 = x_B\mathbf{i}_3 + e\mathbf{j}_3,$$

whence follows

$$r\Delta q\mathbf{j}_1 + \Delta r\mathbf{i}_1 + \Delta\ell\mathbf{i}_2 + \ell\Delta\varphi \cdot \mathbf{j}_2 = \Delta e\mathbf{j}_3 + \Delta x_B\mathbf{i}_3.$$

In this equation, written in vector form, $\Delta\varphi$ and Δx_B are the unknowns. They are determined from the error vector diagram (Fig. 3.18) which corresponds to this equation.

The *analytical* method for the determination of mechanism errors is based on formulae (3.68) and (3.69). In order to determine the partial derivatives of group coordinates with respect to a certain parameter α_f , it is necessary to carry out a structural transformation of mechanism proposed by N. G. Bruevitch [6]. For this it is necessary: a) to fix the input coordinates ($q_k = \text{const}$, $k = 1, 2, \dots, n$); b) to introduce an additional link – here a slider with moving pivot that coincides with the direction of variation of the parameter α_f ($f = 1, 2, \dots, \ell$); c) to release the constraint which prevents a relative shift of the additional slider along the guide; d) to consider the parameter α_f as an input coordinate.

For determining the partial derivatives let us differentiate the equations of geometrical analysis derived earlier for the slider-crank mechanism

$$\begin{aligned} r \cos q + \ell \cos \varphi &= x_B, \\ r \sin q + \ell \sin \varphi &= e, \end{aligned}$$

correspondingly:

– with respect to q :

$$\begin{aligned} \frac{\partial x_B}{\partial q} + \ell \sin \varphi \frac{\partial \varphi}{\partial q} &= -r \sin q, \\ -\ell \cos \varphi \frac{\partial \varphi}{\partial q} &= r \cos q, \end{aligned}$$

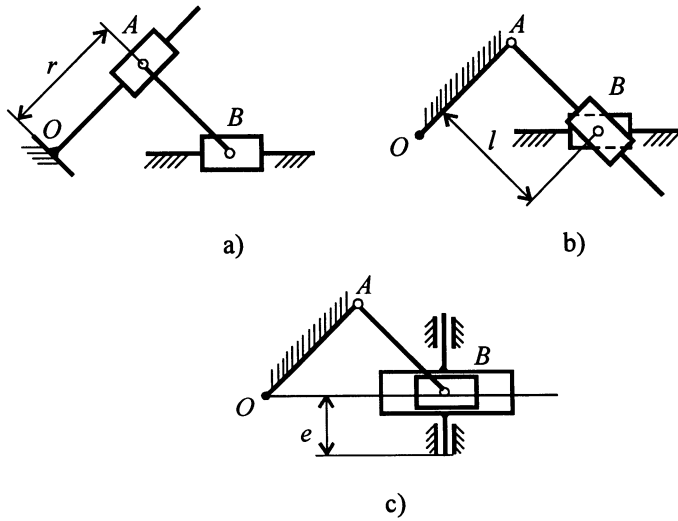


Fig. 3.19. Structural transformations of the slider-crank mechanism

– with respect to r (Fig. 3.19a):

$$\begin{aligned}\frac{\partial x_B}{\partial r} + l \sin \varphi \frac{\partial \varphi}{\partial r} &= \cos q, \\ -l \cos \varphi \frac{\partial \varphi}{\partial r} &= \sin q,\end{aligned}$$

– with respect to l (Fig. 3.19b):

$$\begin{aligned}\frac{\partial x_B}{\partial l} + l \sin \varphi \frac{\partial \varphi}{\partial l} &= \cos \varphi, \\ -l \cos \varphi \frac{\partial \varphi}{\partial l} &= \sin \varphi,\end{aligned}$$

– with respect to e (Fig. 3.19c):

$$\begin{aligned}\frac{\partial x_B}{\partial e} + l \sin \varphi \frac{\partial \varphi}{\partial e} &= 0, \\ -l \cos \varphi \frac{\partial \varphi}{\partial e} &= -1.\end{aligned}$$

According to Cramer rule, we find

$$\begin{aligned}\frac{\partial x_B}{\partial q} / \partial q &= r l \sin(\varphi - q) / J, & \frac{\partial \varphi}{\partial q} &= -r \cos q / J, \\ \frac{\partial x_B}{\partial r} / \partial r &= l \cos(q - \varphi) / J, & \frac{\partial \varphi}{\partial r} &= -\sin q / J, \\ \frac{\partial x_B}{\partial l} / \partial l &= l / J, & \frac{\partial \varphi}{\partial l} &= -\sin \varphi / J, \\ \frac{\partial x_B}{\partial e} / \partial e &= -l \sin \varphi / J, & \frac{\partial \varphi}{\partial e} &= 1 / J,\end{aligned}$$

where $J = \ell \cos \varphi$. With this the position errors of the mechanism are

$$\Delta \varphi = (-r \cos q \cdot \Delta q - \sin q \cdot \Delta r - \sin \varphi \cdot \Delta \ell + \Delta \ell) / J,$$

$$\Delta x_B = \ell [r \sin(\varphi - q) \cdot \Delta q + \cos(q - \varphi) \cdot \Delta r + \Delta \ell - \sin \varphi \cdot \Delta e] / J.$$

The velocity error of the slider is determined by the formula

$$\Delta \dot{x}_B = \frac{\partial x_B}{\partial q} \Delta \dot{q} + \left(\frac{\partial^2 x_B}{\partial q^2} \Delta q + \frac{\partial^2 x_B}{\partial r \partial q} \Delta r + \frac{\partial^2 x_B}{\partial \ell \partial q} \Delta \ell + \frac{\partial^2 x_B}{\partial e \partial q} \Delta e \right) \dot{q},$$

where the partial derivatives are:

$$\frac{\partial^2 \varphi}{\partial q^2} = \left[r \sin q + \ell \left(\frac{\partial \varphi}{\partial q} \right)^2 \sin \varphi \right] (\ell \cos \varphi)^{-1},$$

$$\frac{\partial^2 x_B}{\partial q^2} = -r \cos q - \ell \frac{\partial^2 \varphi}{\partial q^2} \sin \varphi - \ell \left(\frac{\partial \varphi}{\partial q} \right)^2 \cos \varphi,$$

$$\frac{\partial^2 x_B}{\partial r \partial q} = \frac{\partial}{\partial r} \left(\frac{\partial x_B}{\partial q} \right) = \frac{\sin(\varphi - q)}{\cos \varphi} - \frac{r \sin 2q}{2\ell \cos^3 \varphi},$$

$$\frac{\partial^2 x_B}{\partial \ell \partial q} = \frac{\partial}{\partial \ell} \left(\frac{\partial x_B}{\partial q} \right) = -\frac{r \sin \varphi \cos q}{\ell \cos^3 \varphi},$$

$$\frac{\partial^2 x_B}{\partial e \partial q} = \frac{\partial}{\partial e} \left(\frac{\partial x_B}{\partial q} \right) = \frac{r \cos q}{\ell \cos^3 \varphi}.$$

In this example, the mechanism errors were determined on the basis of given primary errors. In practice, most often, it is necessary to calculate errors not for a specific mechanism but of an entire group of mechanisms constructed from one and the same drawing. In this case, the estimation of accuracy is based on methods of probability theory which are not considered here.

3.6 Problems

- 3.1. Crank OA with length $r = 0.2$ m rotating uniformly with angular velocity $\dot{q} = 10$ 1/s sets in motion slider B and the connecting rod AB with length $\ell = 1.0$ m (see Fig. 2.1). Find the angular velocity and the angular acceleration of the connecting rod, as well as the velocity and the acceleration of the slider when the eccentricity is $e = 0.4\sqrt{2}$ m and $q = 45^\circ$. Determine also the acceleration of the mass center C_2 of the connecting rod when $AC_2 = C_2B$.

Answer: $\omega_{AB} = 2.0 \text{ 1/s}$, $\varepsilon_{AB} = 16.0 \text{ 1/s}^2$, $v_B = 2\sqrt{2} \text{ m/s}$, $w_B = 4\sqrt{2} \text{ m/s}^2$, $w_{C_2} = 2\sqrt{37} \text{ m/s}^2$.

- 3.2. Determine the angular velocities and the angular accelerations of the links of the four-bar linkage (see Fig. 2.2), as well as the velocity and the acceleration of point B , if $r = 0.4 \text{ m}$, $\ell = 0.8 \text{ m}$, $d = 0.6 \text{ m}$, $q = 150^\circ$, $\psi = 30^\circ$, $\varphi = 90^\circ$, $\dot{q} = 6.0 \text{ 1/s} = \text{const}$. Find the velocity of that point on the connecting rod AB for which the direction of the velocity coincides with the direction of the connecting rod.

Answer: $\omega_{AB} = 3.0 \text{ 1/s}$, $\omega_{BC} = 4.0 \text{ 1/s}$, $\varepsilon_{AB} = \sqrt{3} \text{ 1/s}^2$, $\varepsilon_{BC} = 16\sqrt{3}/3 \text{ 1/s}^2$, $v_B = 2.4 \text{ m/s}$, $w_B = 32\sqrt{3}/5 \text{ m/s}^2$, $v = r\dot{q}\sin(q - \varphi) = 1.2\sqrt{3} \text{ m/s}$.

- 3.3. In the crank-slotted link mechanism of a planing machine (Fig. 3.20), the length of the crank is $OA = r$, the length of the slotted link is $BC = \ell$, the distance is $OB = a$. The crank rotates with constant angular velocity $\dot{q} = \omega$. Find: 1) the group coordinates φ , u , x_M ; 2) the angular velocity of the slotted link $\dot{\varphi} = \omega_3$, the relative velocity of point A with respect to the slotted link $\dot{u} = v_r$, the velocity of the carriage M of the machine $\dot{x}_M = v_M$; 3) the angular acceleration of the slotted link $\ddot{\varphi} = \varepsilon_3$ and the acceleration of the carriage $\ddot{x}_M = w_M$; 4) the angle q for which $\omega_3 = 0$; 5) the condition under which the position function $\varphi(q)$ is monotonic; 6) the condition under which the mechanism has a singular position; 7) the maximum and minimum values of ω_3 ; 8) those positions of the crank, for which $\omega = \omega_3$.

Answer: 1) $u = \sqrt{a^2 + r^2 + 2ar \sin \omega t}$, $\varphi = \arccos(r \cos \omega t / u)$, $x_M = \ell r \cos \omega t u^{-1}$; 2) $\omega_3 = \omega r(r + a \sin \omega t)u^{-2}$, $v_r = \omega ar \cos \omega t u^{-1}$, $v_M = \omega \ell r(a + r \sin \omega t)(a \sin \omega t + r)u^{-3}$; 3) $\varepsilon_3 = \omega^2 ar \cos \omega t(a^2 - r^2)u^{-2}$, $w_M = \omega^2 \ell r [a(r^2 - a^2)(a + r \sin \omega t) - r^2(a \sin \omega t + r)^2]u^{-5} \cos \omega t$; 4) $\omega_3 = 0$ if $q = \arcsin(-r/a)$, which is possible for $r < a$ (oscillating slotted link); 5) for $r > a$ the position function $\varphi(q)$ has no extremal values (rotating slotted link); 6) for $r = a$ at $q = 270^\circ$ the motion of the slotted link is undetermined; 7) if $\cos q = 0$, then $\omega_{3\min} = r\omega/(r - a)$, $\omega_{3\max} = r\omega/(r + a)$ for $r < a$; $\omega_{3\min} = r\omega/(r + a)$, $\omega_{3\max} = r\omega/(r - a)$ for $r > a$; 8) $q = \arcsin(-a/r)$ for $r > a$; $q = \arcsin(2r^2 + a^2)/3ra$ for $0.5 \leq r < a$.

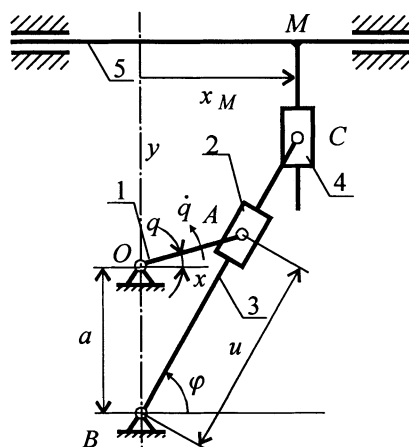


Fig. 3.20. Crank-slotted link mechanism of a planing machine

- 3.4. Find the angular velocity $\omega_4 = \dot{\varphi}_4$ of the output link of the mechanism of third class (Fig. 3.21) for $\ell_1 = OA = 0.1$ m, $\ell_2 = BA = 0.5$ m, $CD = 0.4$ m, $BC = BD = 0.4\sqrt{3}/3$ m, $\ell_4 = DF = 0.15$ m, $\ell_5 = CE = 0.15$ m, $y_F = y_E = 0.1 + 0.125\sqrt{3}$ m, $x_F = -0.175$ m, $x_E = -0.725$ m, $q = 60^\circ$, $\gamma_3 = 120^\circ$, $\dot{q} = 20$ 1/s.

Answer: $\omega_4 = J_4/J = -20$ 1/s where

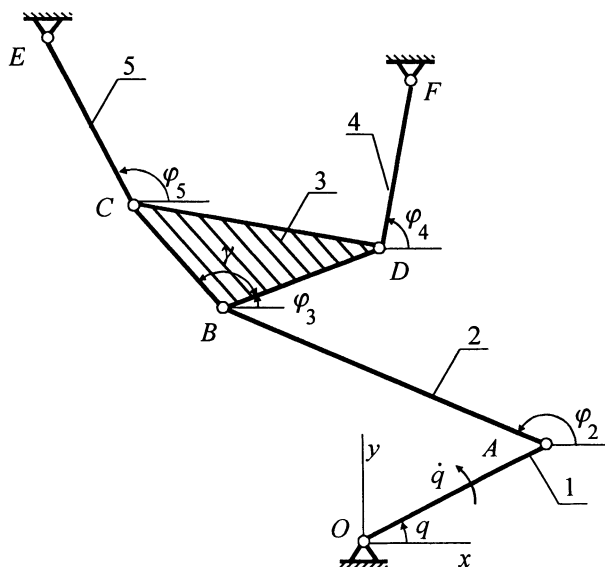


Fig. 3.21. Mechanism of third class

$$\begin{aligned}
 J_4 &= \ell_1 \ell_2 \ell_5 \dot{q} \{BC \sin(\varphi_3 + \gamma_3 - \varphi_5) \sin(\varphi_2 - q) + \\
 &\quad BD[\sin(\varphi_5 - \varphi_2) \sin(\varphi_3 - q) + \sin(\varphi_3 - \varphi_2) \sin(q - \varphi_5)]\} = 0,045 \text{ m}^4 / \text{s}, \\
 J &= \ell_2 \ell_4 \ell_5 [BC \sin(\varphi_3 + \gamma_3 - \varphi_5) \sin(\varphi_4 - \varphi_2) + \\
 &\quad BD \sin(\varphi_5 - \varphi_2) \sin(\varphi_4 - \varphi_3)] = -0,00225 \text{ m}^4, \\
 \varphi_2 &= 180^\circ, \varphi_3 = 30^\circ, \varphi_4 = 60^\circ, \varphi_5 = 120^\circ.
 \end{aligned}$$

- 3.5. Determine the analogues of angular velocities as well as the angular accelerations of links 1 and 3 of the one-cylinder mechanism (Fig. 3.22) when $\ell_1 = OA = 0.16 \text{ m}$, $\ell_2 = A'B = 0.32 \text{ m}$, $\ell_3 = BC = (0.48\sqrt{3} - 0.2) \text{ m}$, $x_O = y_O = 0$; $x_C = 0.96 \text{ m}$, $y_C = 0.4 \text{ m}$, input coordinate $q = 0.2\sqrt{3} \text{ m}$, group coordinates $\varphi_1 = 60^\circ$, $\varphi_3 = 330^\circ$. Find the angular velocities and the angular accelerations of links 1 and 3 for $\dot{q} = 0.5 \text{ m/s}$, $\ddot{q} = 0.1 \text{ m/s}^2$. Find the conditions for which $\omega_1 = 0$, $\omega_3 = 0$.

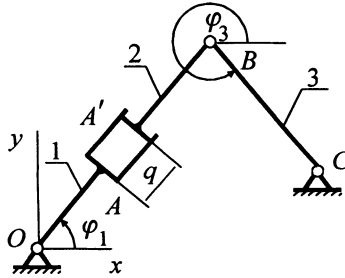


Fig. 3.22. One-cylinder mechanism

Answer:

$$\frac{d\varphi_1}{dq} = [(\ell_1 + q + \ell_2) \tan(\varphi_1 - \varphi_3)]^{-1} = 0,$$

$$\frac{d\varphi_3}{dq} = [\ell_3 \sin(\varphi_3 - \varphi_1)]^{-1} = -1.58 \text{ 1/m},$$

$$\frac{d^2\varphi_1}{dq^2} = \frac{F_1 \cos \varphi_3 + F_1 \sin \varphi_3}{(\ell_1 + q + \ell_2) \sin(\varphi_3 - \varphi_1)} = -1.92 \text{ 1/m},$$

$$\frac{d^2\varphi_3}{dq^2} = -\frac{F_1 \cos \varphi_1 + F_2 \sin \varphi_1}{\ell_3 \sin(\varphi_3 - \varphi_1)} = 0, \text{ where}$$

$$F_1 = 2 \sin \varphi_1 \frac{d\varphi_1}{dq} + (\ell_1 + q + \ell_2) \cos \varphi_1 \left(\frac{d\varphi_1}{dq} \right)^2 + \ell_3 \cos \varphi_3 \left(\frac{d\varphi_3}{dq} \right)^2,$$

$$F_2 = -2 \cos \varphi_1 \frac{d\varphi_1}{dq} + (\ell_1 + q + \ell_2) \sin \varphi_1 \left(\frac{d\varphi_1}{dq} \right)^2 + \ell_3 \sin \varphi_3 \left(\frac{d\varphi_3}{dq} \right)^2,$$

$$\omega_1 = 0, \quad \omega_3 = -0.792 \text{ 1/s}, \quad \varepsilon_1 = -0.479 \text{ 1/s}^2, \quad \varepsilon_3 = -0.158 \text{ 1/s}^2.$$

When $\cos(\varphi_1 - \varphi_3) = 0$, the angular velocity $\omega_1 = 0$, $\omega_3 \neq 0$ for all q .

- 3.6. Determine the group coordinates φ_3 and u of the links of the manipulator of movability two (Fig. 3.23), as well as the link velocities, if $x_0 = y_0 = 0$, $x_C = 0.6 \text{ m}$, $y_C = 0$, $\ell_3 = AB = 0.1\sqrt{3} \text{ m}$, $\ell_4 = BC = 0.5 \text{ m}$, $q_1 = 60^\circ$, $q_2 = 120^\circ$, $\dot{q}_1 = 2 \text{ 1/s}$, $\dot{q}_2 = 4 \text{ 1/s}$.

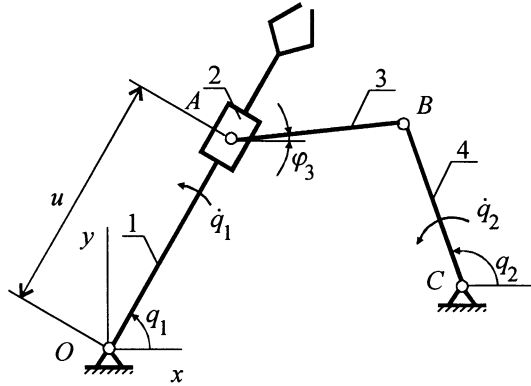


Fig. 3.23. Mechanism of movability two with a closed kinematic chain

$$\text{Answer: } u = 0.4 \text{ m}, \quad \varphi_3 = 30^\circ, \quad \dot{u} = -0.4\sqrt{3} \text{ m/s}, \quad \dot{\varphi}_3 = -12 \text{ 1/s}.$$

- 3.7. In the mechanism of movability two (Fig. 3.24) find velocities $\dot{\varphi}$, $\dot{x}_C$ and accelerations $\ddot{\varphi}$, $\ddot{x}_C$ of the output link when $\ell_1 = OA = 0.4 \text{ m}$, $\ell_2 = BA = 0.6 \text{ m}$, $\ell_3 = BC = 0.4\sqrt{3} \text{ m}$, $y_C = 0$, $q_1 = 60^\circ$, $q_2 = 30^\circ$, $\dot{q}_1 = 12 \text{ 1/s} = \text{const}$, $\dot{q}_2 = 2 \text{ 1/s} = \text{const}$, $\varphi = 180^\circ$, $x_C = -1.0 \text{ m}$.

$$\text{Answer: } \dot{\varphi} = 1 \text{ 1/s}, \quad \dot{x}_C = -1.8\sqrt{3} \text{ m/s}, \quad \ddot{\varphi} = -22.5\sqrt{3} \text{ 1/s}^2, \\ \ddot{x}_C = -36.3 \text{ m/s}^2.$$

- 3.8. Determine the angular velocities and accelerations of links 2 and 3, as well as the velocity and the acceleration of pole M of the gripper of the manipulator of movability two shown in Fig. 2.24 when $\ell_1 = OA = 0.7 \text{ m}$, $\ell_2 = AB = 0.8 \text{ m}$, $\ell_3 = BC = 1.0 \text{ m}$, $\ell_4 = DC = 0.4 \text{ m}$, $x_O = y_O = 0$,

$x_D = 0.4 \text{ m}$, $y_D = -0.3 \text{ m}$, $BM = 1.2 \text{ m}$, $q_1 = 90^\circ$, $q_2 = 0$, $\varphi_2 = 0$, $\varphi_3 = 270^\circ$, $\dot{q}_1 = 3 \text{ 1/s}$, $\dot{q}_2 = 2.8 \text{ 1/s}$, $\ddot{q}_1 = 12.24 \text{ 1/s}^2$, $\ddot{q}_2 = 1.275 \text{ 1/s}^2$. Obtain the necessary and sufficient conditions for an extremum of the function $\mu(q_1, q_2)$, where μ is the transmission angle.

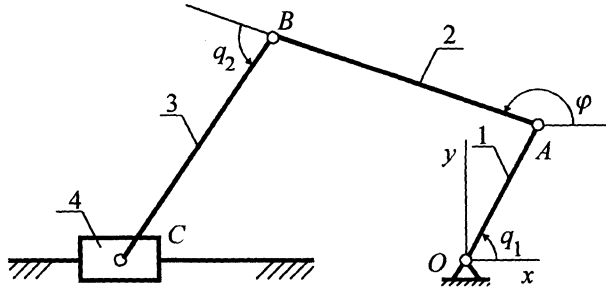


Fig. 3.24. Mechanism of movability two with a closed kinematic chain

Answer: $\dot{\varphi}_2 = 1.4 \text{ 1/s}$, $\dot{\varphi}_3 = 2.1 \text{ 1/s}$, $\dot{x}_M = -2.1 \text{ m/s}$, $\dot{y}_M = 2.8 \text{ m/s}$, $v_M = 3.5 \text{ m/s}$, $\ddot{\varphi}_2 = 3.0 \text{ 1/s}^2$, $\ddot{\varphi}_3 = 7.0 \text{ 1/s}^2$, $w_M = 12.49 \text{ m/s}^2$. The transmission angle μ takes extreme values for the maximum and the minimum values of $AC^2 = \ell_0^2 + \ell_1^2 + \ell_4^2 - 2\ell_1\ell_4 \cos(q_1 - q_2) - 2\ell_0\ell_1 \cos q_1 + 2\ell_0\ell_4 \cos q_2$, where $\ell_0 = OD$. The necessary conditions for extremum of functions AC^2 and μ : $\partial AC^2 / \partial q_1 = 0$, $\partial AC^2 / \partial q_2 = 0$ can be reduced to the form

$$\frac{\partial AC^2}{\partial q_1} + \frac{\partial AC^2}{\partial q_2} = 2\ell_0(y_A - y_C) = 0, \quad \frac{\partial AC^2}{\partial q_2} = 2\ell_1 \sin q_1 (x_C - x_A) = 0.$$

These conditions are fulfilled in the case when $\ell_2 = \ell_3$ and $|\ell_1 - \ell_4| < \ell_0 < \ell_1 + \ell_4$. For this reason, the transmission angle will take extremal values for $\sin q_1 = \sin q_2 = 0$. The sufficient conditions for maximum of functions AC^2 and μ

$$\frac{\partial^2 AC^2}{\partial q_1^2} = 2\ell_1 \cos q_1 (\ell_4 \cos q_2 + \ell_0) < 0,$$

$$\frac{\partial^2 AC^2}{\partial q_2^2} = 2\ell_4 \cos q_2 (\ell_1 \cos q_1 - \ell_0) < 0,$$

$$\frac{\partial^2 AC^2}{\partial q_1^2} \frac{\partial^2 AC^2}{\partial q_2^2} - \left(\frac{\partial^2 AC^2}{\partial q_1 \partial q_2} \right)^2 = 4\ell_0 \ell_1 \ell_4 (\ell_1 \cos q_1 - \ell_4 \cos q_2 - \ell_0) \cos q_1 \cos q_2 > 0$$

are fulfilled for $q_1 = 180^\circ$ and $q_2 = 0$ (Fig. 3.25a). Sufficient conditions for minimum of functions AC^2 and μ

$$\frac{\partial^2 AC^2}{\partial q_1^2} > 0, \quad \frac{\partial^2 AC^2}{\partial q_2^2} > 0, \quad \frac{\partial^2 AC^2}{\partial q_1^2} \frac{\partial^2 AC^2}{\partial q_2^2} - \left(\frac{\partial^2 AC^2}{\partial q_1 \partial q_2} \right)^2 > 0$$

are fulfilled for $q_1 = q_2 = 0$, if $\ell_1 > \ell_0 + \ell_4$ (Fig. 3.25b), as well as for $q_1 = 0, q_2 = 180^\circ$, if $\ell_0 > \ell_1 + \ell_4$. According to figures shown bellow, the conditions for the existence of the mechanism can be written in the form

$$|\ell_3 - \ell_2| < \ell_1 + \ell_0 + \ell_4 < \ell_3 + \ell_2, \quad |\ell_3 - \ell_2| < \ell_1 - \ell_0 - \ell_4 < \ell_3 + \ell_2.$$

These two conditions for the two cranks 1 and 4 can be reduced to the form

$$\ell_1 + \ell_0 + \ell_4 < \ell_3 + \ell_2, \quad |\ell_1 - \ell_0 - \ell_4| > |\ell_3 - \ell_2|.$$

Inequalities, analogous to Grashof's criterion were obtained for five-bar linkages (see [7]).

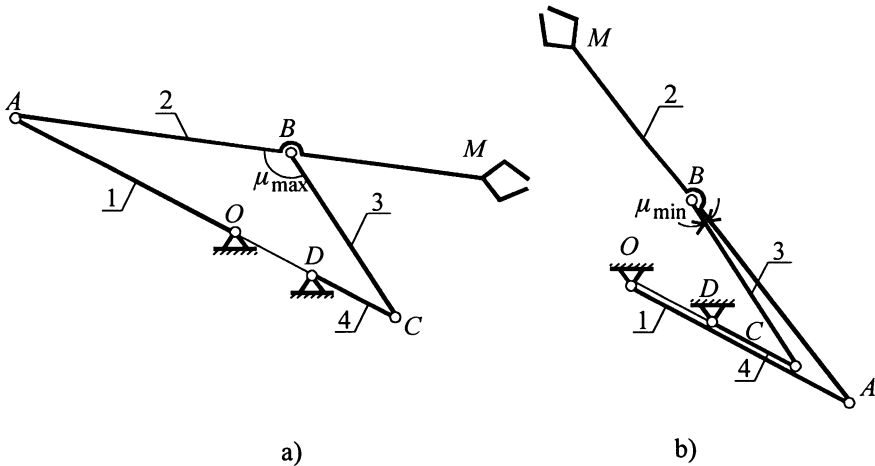


Fig. 3.25. Platform of movability two in the positions corresponding to extremal transmission angles

- 3.9. Determine the first partial derivatives of the group coordinates φ_2, φ_4 with respect to the input coordinates q_1, q_2, q_3 , as well as the velocity of point M of platform CD (Fig. 3.26), if $\ell_1 = \ell_2 = 0.4$ m, $\ell_3 = 0.6$ m, $\ell_4 = \ell_5 = 1.6$ m, $CM = 0.8$ m, $x_E = 1.2$ m, $y_E = 0$, $q_1 = 120^\circ$, $q_2 = 0.2$ m, $q_3 = 60^\circ$, $\dot{q}_1 = 20$ 1/s, $\dot{q}_2 = \sqrt{3}$ m/s, $\dot{q}_3 = 0$, $\varphi_2 = 60^\circ$, $\varphi_4 = 0$.

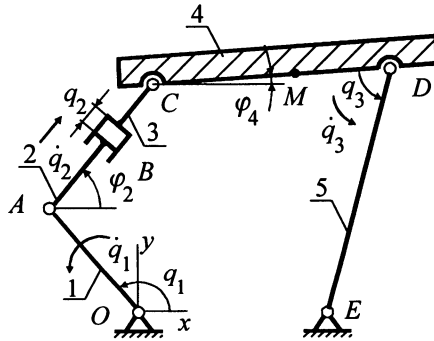


Fig. 3.26. Mechanism of movability three of a platform

Answer:

$$\begin{aligned}\partial\varphi_2/\partial q_1 &= \ell_1[\ell_4 \sin(q_1 - \varphi_4) + \ell_5 \sin(\varphi_4 + q_3 - q_1)]/J = 0, \\ \partial\varphi_4/\partial q_1 &= [\ell_1(\ell_2 + \ell_3 + q_2)\sin(\varphi_2 - q_1)]/J = 0.25, \\ \partial\varphi_2/\partial q_2 &= [\ell_5 \cos(\varphi_2 - \varphi_4 - q_3) - \ell_4 \cos(\varphi_4 - \varphi_2)]/J = -5/6\sqrt{3} \text{ 1/m}, \\ \partial\varphi_4/\partial q_2 &= (\ell_2 + \ell_3 + q_2)/J = -5/4\sqrt{3} \text{ 1/m}, \\ \partial\varphi_2/\partial q_3 &= -\ell_4 \ell_5 \sin q_3 / J = 4/3, \\ \partial\varphi_4/\partial q_3 &= [\ell_5(\ell_2 + \ell_3 + q_2)\sin(\varphi_4 + q_3 - \varphi_2)]/J = 0,\end{aligned}$$

where the Jacobian is

$$\begin{aligned}J &= (\ell_2 + \ell_3 + q_2)[\ell_4 \sin(\varphi_4 - \varphi_2) + \ell_5 \sin(\varphi_2 - \varphi_4 - q_3)] = -0.96\sqrt{3} \text{ m}^2, \\ v_M &= 3\sqrt{3} \text{ m/s}.\end{aligned}$$

- 3.10. Determine the link angular velocities relative to link 1 of the robot of movability three (Fig. 3.27), as well as the velocity of pole M of the gripper, if $BC = 0.3\sqrt{3} \text{ m}$, $\ell_2 = \ell_3 = \ell_5 = \ell_6 = 0.2 \text{ m}$, $CE = 0.6 \text{ m}$, $FC = GH = HM = 1.0 \text{ m}$, $FG = CH = 1.8 \text{ m}$, $y_A = 0.7 \text{ m}$, $z_A = 0$, $y_D = 0.6 \text{ m}$, $z_D = 0.1\sqrt{3} \text{ m}$, $y_C = 0$, $z_C = 0.1\sqrt{3} \text{ m}$, $q_1 = 30^\circ$, $q_2 = 0.1 \text{ m}$, $q_3 = 0.2 \text{ m}$, $\varphi_2 = 120^\circ$, $\varphi_4 = 240^\circ$, $\varphi_5 = \varphi_8 = 60^\circ$, $\varphi_7 = \varphi_9 = 330^\circ$, $\dot{q}_1 = 2 \text{ 1/s}$, $\dot{q}_2 = \dot{q}_3 = 3\sqrt{3} \text{ m/s}$. Find the singular positions of the robot mechanism.

Answer: $\dot{\varphi}_5 = -\dot{q}_2/(\ell_5 + q_2 + \ell_6)\tan(\varphi_7 - \varphi_5) = 0$, $\dot{\varphi}_7 = \dot{\varphi}_9 = \dot{q}_2/BC \sin(\varphi_7 - \varphi_5) = -10 \text{ 1/s}$, $\dot{\varphi}_2 = \dot{q}_3/(\ell_2 + q_3 + \ell_3)\tan(\varphi_4 - \varphi_2) = 5 \text{ 1/s}$, $\dot{\varphi}_4 = \dot{\varphi}_8 = \dot{q}_3/CE \sin(\varphi_4 - \varphi_2) = 10 \text{ 1/s}$. It is apparent from the derived formulae that the closed kinematic diagram of the robot ensures independence of the rotational motions.

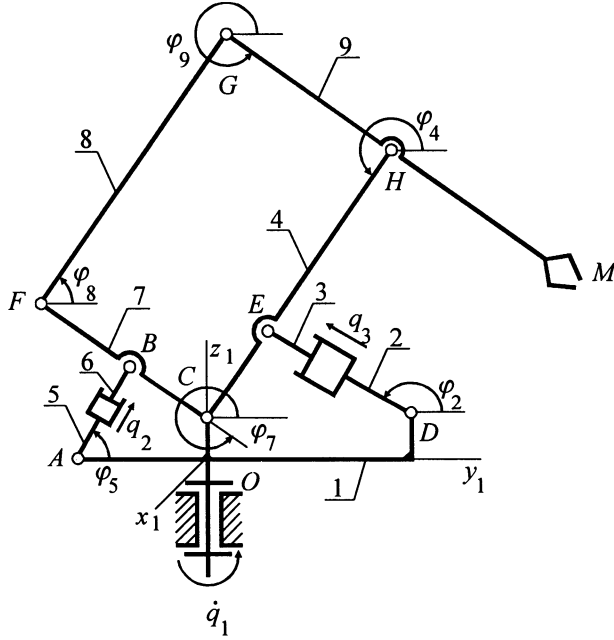


Fig. 3.27. Robot of movability three with a closed kinematic chain

The relation between the projections of the velocity of pole M and the velocities of the input links can be written in the form

$$\dot{\rho}_M = D\dot{q},$$

where $\dot{\rho}_M = (\dot{x}_M, \dot{y}_M, \dot{z}_M)^T$, $\dot{q} = (\dot{q}_1, \dot{q}_2, \dot{q}_3)^T = (2.0, 3\sqrt{3}, 3\sqrt{3})^T$; the Jacoby matrix is

$$D = \begin{pmatrix} -\frac{3(5+3\sqrt{3})}{20} & \frac{5}{6\sqrt{3}} & \frac{3}{2} \\ -\frac{5\sqrt{3}+9}{20} & -\frac{5}{6} & -\frac{3\sqrt{3}}{2} \\ 0 & -\frac{5}{3} & \sqrt{3} \end{pmatrix}.$$

Then, $v_M = \sqrt{(\dot{x}_M)^2 + (\dot{y}_M)^2 + (\dot{z}_M)^2} = 20.85$ m/s. The Jacobian of the system is

$$\det D = \frac{HM}{BC} \frac{CH}{CE} \frac{(CH \cos \varphi_4 - y_C - HM \cos \varphi_9) \sin(\varphi_9 - \varphi_4)}{\sin(\varphi_7 - \varphi_5) \sin(\varphi_4 - \varphi_2)}.$$

For the given mechanism, singular positions are those positions for which

$$x_M^{(1)} = 0, \quad y_M^{(1)} = -(CH \cos \varphi_4 - y_C - HM \cos \varphi_9) = 0, \quad \sin(\varphi_9 - \varphi_4) = 0,$$

i.e. when pole M lies on the rotation axis Oz or, when links 4(8) and 9(7) are in one line. Furthermore, the Jacoby matrix does not exist when $\sin(\varphi_7 - \varphi_5) = 0$ and $\sin(\varphi_4 - \varphi_2) = 0$.

- 3.11. Determine the generalized velocities $\dot{q}_1$, $\dot{q}_2$ of the mechanism of movability two (Fig. 3.28), if $\ell_1 = \ell_3 = 0.2$ m, $\ell_2 = 0.8$ m, $\ell_4 = 0.3$ m, $x_D = 2\sqrt{3}/15$ m, $y_D = 0$, $AM = MB$, $q_1 = 150^\circ$, $q_2 = (2\sqrt{3} - 3)/6$ m, $\varphi_2 = 30^\circ$, $\varphi_4 = 240^\circ$, $\dot{x}_M = 0.5$ m/s, $\dot{y}_M = -1.5\sqrt{3}$ m/s.

Answer: $\dot{q}_1 = 10$ 1/s, $\dot{q}_2 = 1.0$ m/s.

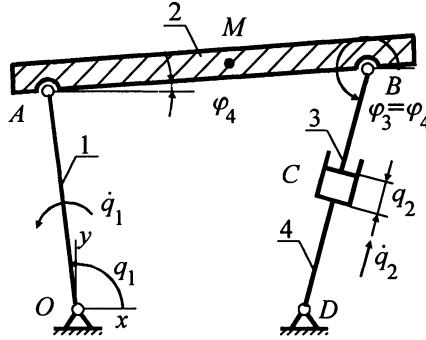


Fig. 3.28. Mechanism of movability two of a platform

- 3.12. Determine the projections of the angular velocity and of the angular acceleration of the output link onto the axes of the robot of movability three, shown on Fig. 2.23. Find also the projections of the velocity and of the acceleration of pole O_3 .

Answer:

$$\Omega_3^{(3)} = \begin{pmatrix} \dot{q}_1 \sin q_2 \\ -\dot{q}_2 \\ \dot{q}_1 \cos q_2 \end{pmatrix}, \quad E_3^{(3)} = \begin{pmatrix} \ddot{q}_1 \sin q_2 + \dot{q}_1 \dot{q}_2 \cos q_2 \\ -\ddot{q}_2 \\ \ddot{q}_1 \cos q_2 - \dot{q}_1 \dot{q}_2 \sin q_2 \end{pmatrix},$$

$$\dot{x}_{O_3}^{(3)} = \dot{q}_3 - a\dot{q}_2 + b\dot{q}_1 \cos q_2, \quad \dot{y}_{O_3}^{(3)} = q_3 \dot{q}_1 \cos q_2 - a\dot{q}_1 \sin q_2, \quad \dot{z}_{O_3}^{(3)} = q_3 \dot{q}_1 - b\dot{q}_1 \sin q_2,$$

$$\ddot{x}_{O_3}^{(3)} = -a(\ddot{q}_2 - \dot{q}_1^2 \sin q_2 \cos q_2) - q_3(\dot{q}_2^2 + \dot{q}_1^2 \cos^2 q_2) + b\ddot{q}_1 \cos q_2 + \ddot{q}_3,$$

$$\ddot{y}_{O_3}^{(3)} = -a(\ddot{q}_1 \sin q_2 + 2\dot{q}_1 \dot{q}_2 \cos q_2) + q_3(\ddot{q}_1 \cos q_2 - 2\dot{q}_1 \dot{q}_2 \sin q_2) + b\dot{q}_1^2 + 2\dot{q}_1 \dot{q}_2 \cos q_2,$$

$$\ddot{z}_{O_3}^{(3)} = -a(\dot{q}_2^2 + \dot{q}_1^2 \sin^2 q_2) + q_3(\ddot{q}_2 + \dot{q}_1^2 \sin q_2 \cos q_2) - b\ddot{q}_1 \sin q_2 + 2\dot{q}_2 \dot{q}_3.$$

- 3.13. Find the transmission ratio i_{13} of the planetary mechanism (Fig. 3.29), if the radii of the initial circles are: $r_1 = r_2 = r_{5'} = r_6 = 0.2$ m, $r_{3'} = r_8 = 0.5$ m, $r_4 = 0.8$ m, $r_5 = 0.4$ m, as well as the transmission ratio i_{1H_1} of the biplanetary mechanism (Fig. 3.30), if $r_1 = r_5 = r_6 = 0.2$ m, $r_2 = r_4 = 0.4$ m.

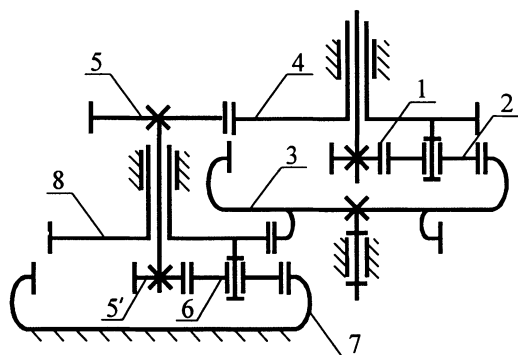


Fig. 3.29. Planetary gear mechanism

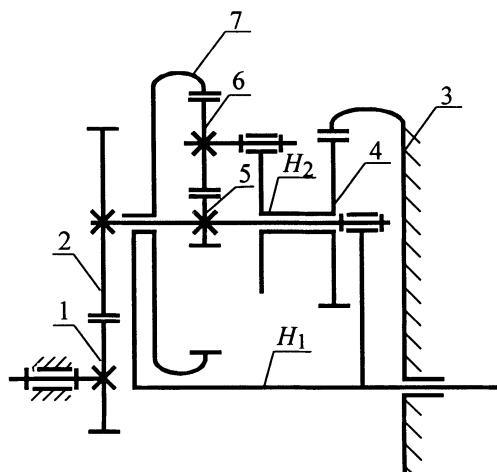


Fig. 3.30. Biplanetary gear mechanism

$$\text{Answer: } i_{13} = -\frac{r_3}{r_1} + \frac{r_5 r_{3'}}{r_8 r_4} \left(1 + \frac{r_7}{r_{5'}} \right) \left(1 + \frac{r_3}{r_1} \right) = 5, \text{ where } r_3 = r_1 + 2r_2 = 0.6 \text{ m, } r_7 = r_{5'} + 2r_6 = 0.6 \text{ m, } i_{1H_1} = 1 + \frac{r_3 r_2}{r_4 r_1} \left(1 + \frac{r_7}{r_5} \right) = 21, \text{ where } r_3 = r_1 + r_2 + r_4 = 1.0 \text{ m, } r_7 = r_5 + 2r_6 = 0.6 \text{ m.}$$

- 3.14. Determine the relative angular velocities $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$ of the wrist links of the robot, set in motion by differential bevel gear trains (Fig. 3.31), if the angular velocities $\omega_1, \omega_2, \omega_3$ of the rotor actuators and the numbers $z_1 - z_8$ of wheel teeth are known.

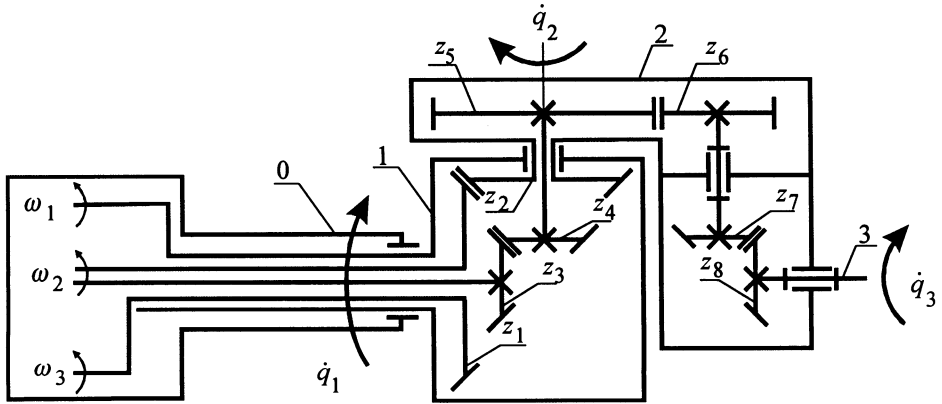


Fig. 3.31. Wrist of the robot

Answer:

$$\dot{q}_1 = \omega_1, \quad \dot{q}_2 = (\omega_2 - \omega_1)z_1 / z_2, \quad \dot{q}_3 = \frac{z_5 z_7}{z_6 z_8} \left[\left(\frac{z_3}{z_4} - \frac{z_1}{z_2} \right) \omega_1 + \frac{z_2}{z_1} \omega_2 - \frac{z_3}{z_4} \omega_3 \right].$$

For $z_1 = z_2, z_3 = z_4, z_5 = z_6, z_7 = z_8$ we have $\dot{q}_1 = \omega_1, \quad \dot{q}_2 = \omega_2 - \omega_1, \quad \dot{q}_3 = \omega_2 - \omega_3$.

- 3.15. Determine the transmission ratio i_{AB} between the input shaft A and the output shaft B of the differential bevel gear train (Fig. 3.32), if the number $z_1 - z_4$ of teeth and the corresponding half angles $\delta_1 - \delta_4$ of the initial cones are known.

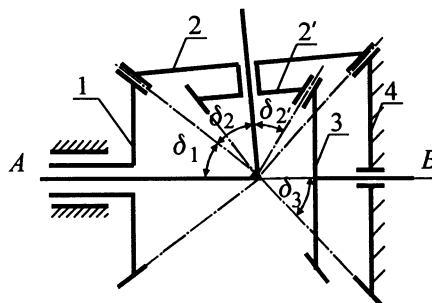


Fig. 3.32. Planetary bevel gear train

$$\text{Answer: } i_{AB} = i_{13}^{(4)} = \frac{1 + z_4 z_1^{-1}}{1 - (z_2' z_4 / z_3 z_2)} = \frac{\sin 2\delta_2 \sin \delta_3}{\sin \delta_1 \sin(\delta_2 - \delta_2')}.$$

- 3.16. Derive the position function for the mechanism with elliptic wheels shown in Fig. 1.22. Determine the transmission ratio i_{12} and find the angular acceleration of the output link when $\dot{q} = \text{const}$, when the semi-major axis of the ellipse is equal to a and the semi-minor axis is b , the distance between the axes is $O_1 O_2 = 2a$, the distance between the foci is $O_1 O_1' = O_2 O_2' = 2c$.

$$\text{Answer: } \varphi = 2 \arctan \left(\frac{a+c}{a-c} \tan \frac{q}{2} \right), \quad i_{12} = \frac{a^2 - 2ac \cos q + c^2}{a^2 - c^2};$$

$$\ddot{\varphi} = \frac{2ac(a^2 - c^2) \sin q}{(a^2 - 2ac \cos q + c^2)} \dot{q}^2.$$

- 3.17. Determine the position errors of platform BC (see Fig. 2.10) caused by the primary error $\Delta \ell_2$.

$$\text{Answer: } \Delta x_M = \frac{\partial x_M}{\partial \ell_2} \Delta \ell_2, \Delta y_M = \frac{\partial y_M}{\partial \ell_2} \Delta \ell_2, \Delta \varphi_3 = \frac{\partial \varphi_3}{\partial \ell_2} \Delta \ell_2, \text{ where}$$

$$\frac{\partial x_M}{\partial \ell_2} = \cos \varphi_2 - \ell_2 \sin \varphi_2 \frac{\partial \varphi_2}{\partial \ell_2} - BM \sin \varphi_3 \frac{\partial \varphi_3}{\partial \ell_2},$$

$$\frac{\partial y_M}{\partial \ell_2} = \sin \varphi_2 + \ell_2 \cos \varphi_2 \frac{\partial \varphi_2}{\partial \ell_2} + BM \cos \varphi_3 \frac{\partial \varphi_3}{\partial \ell_2},$$

$$\frac{\partial \varphi_2}{\partial \ell_2} = \frac{\ell_4 \cos(\varphi_3 + q_2 + \varphi_2) - \ell_3 \cos(\varphi_3 - \varphi_2)}{\ell_2 [\ell_3 \sin(\varphi_3 - \varphi_2) - \ell_4 \sin(\varphi_3 + q_2 - \varphi_2)]},$$

$$\frac{\partial \varphi_3}{\partial \ell_2} = \frac{1}{\ell_3 \sin(\varphi_3 - \varphi_2) - \ell_4 \sin(\varphi_3 + q_2 - \varphi_2)}.$$

4 Determination of Forces Acting in Mechanisms

4.1 Geometric Conditions for Transmission of Forces by Mechanisms

Mechanisms are used not only for producing program motions but also for transmitting forces, necessary, both, for performing working processes and for overcoming the inertia of moving links. For this reason the kinematic and geometric investigation must be complemented by the analysis of forces in the process of mechanism design.

A complete force analysis can be conducted only after the design of the mechanism construction, of its units and pieces. In order to carry out this analysis, it is necessary to know the inertia parameters of links (their masses and moments of inertia), as well as certain physical properties of constructive elements constituting links and kinematic pairs which can be known only after the construction is completed. In such a general form, the problem of force analysis will be considered in the following sections.

However, certain important mechanism characteristics, reflecting conditions of force transmission, can be investigated even during the phase of geometric and kinematic analysis, when the kinematic diagram of a mechanism is designed and its link dimensions are chosen. Such an investigation is necessary in order to avoid choosing geometric parameters which satisfy the kinematic requirements of a mechanism but which happen to be inefficient because they lead to inacceptably large reaction forces at kinematic pairs during working process, or because they require the generation of very large actuating forces.

The investigation of geometric conditions for the transmission of forces through a mechanism is conducted on the basis of a simplified physical model, which we will call a *static model*. With the help of the static model *equilibrium conditions of a mechanism* in different positions are essentially investigated. Moreover, it is assumed that in every position investigated only forces caused by workloads are taken into consideration whereas friction forces at kinematic pairs are neglected. Usually, the weight of links of transmission mechanisms is not taken into account, whereas the weight of executing organs of transporting machines including objects held by the executing organs is considered as workload.

The first problem of force analysis for the static model is to determine generalized forces to be applied to the input links of a mechanism in order to balance the action of workloads. In essence, these generalized forces are needed to

hold the mechanism in equilibrium; in the literature they are often called *balancing* forces. In low-speed mechanisms, in which inertia forces are negligible compared to workloads, the balancing forces are practically the driving forces.

The second problem of the force analysis static model is the determination of reaction forces at kinematic pairs of a mechanism.

We will acquaint ourselves with the force analysis based on the static model by examining a simple example. Let us consider the crank-slider mechanism in the position shown in Fig. 4.1. The workload P is applied to its output link – the slider. To be determined is the balancing generalized force Q (in the given case – a moment), which has to be applied to the input link – the crank, in order to keep the mechanism in equilibrium.

In the absence of friction forces the mechanism is a system with ideal constraints. Equilibrium conditions can be obtained with the help of the principle of virtual displacements according to which the total work done by active forces in any virtual displacement has to be equal to zero. The generalized coordinate q is given a virtual displacement δq ; as a result the output link will have a displacement δx_B , the magnitude of which, up to small quantities of second order, is determined in the following way:

$$\delta x_B = \frac{dx_B}{dq} \delta q = \Pi'_{xB}(q) \delta q, \quad (4.1)$$

where $\Pi_{xB}(q)$ is the position function expressing the coordinate x_B as function of the input coordinate q .

Setting equal to zero the work done by the active forces P and Q in a virtual displacement we obtain

$$Q \delta q + P \delta x_B = 0. \quad (4.2)$$

From (4.2) and (4.1) we find

$$Q = -P \frac{dx_B}{dq}. \quad (4.3)$$

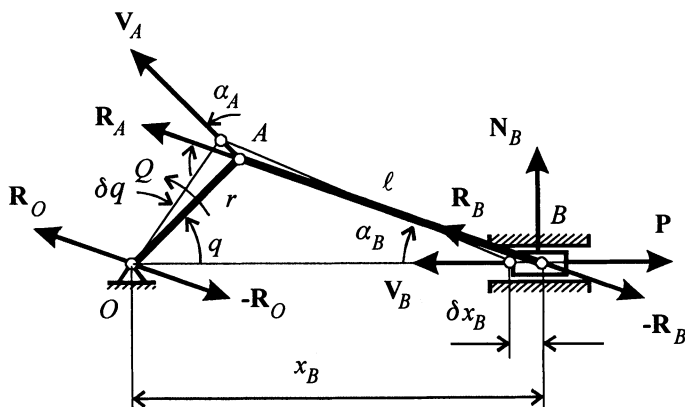


Fig. 4.1. Static model of the slider-crank mechanism

This shows that the generalized balancing force Q is equal to the workload P multiplied by the first derivative of the input link position function with respect to the generalized coordinate. Taking into account the relationship

$$\frac{dx_B}{dq} = \frac{\dot{x}}{\dot{q}},$$

one can write the expression (4.3) as an equation for powers:

$$Q\dot{q} = -P\dot{x}_B. \quad (4.4)$$

In essence, the relationships thus obtained express the well-known „golden rule of mechanics“: what is won in force, is lost in velocity. Apart from this, the relationships are also conditions for the transmission of forces from the mechanism input to its output. Formula (4.3) shows that in mechanism positions in which large workloads have to be overcome (P is a large quantity) the derivative dx_B/dq should be small quantity. Otherwise, a large balancing (driving) force would have to be generated. In the present example

$$\begin{aligned} x_B &= r \cos q + \sqrt{\ell^2 - r^2 \sin^2 q}, \\ \frac{dx_B}{dq} &= -r \sin q - \frac{r^2 \sin q \cos q}{\sqrt{\ell^2 - r^2 \sin^2 q}}, \end{aligned} \quad (4.5)$$

where r is the length of the crank and ℓ is the length of the connecting rod. In the mechanism positions close to $q = 0$ and $q = \pi$, the derivative dx_B/dq is small; therefore, in these positions it is possible to balance a large force P with a small moment Q .

Let us consider now a slider-crank mechanism, in which the slider is the input link, while the crank is the output link (Fig. 4.2). To the crank a workload – a moment M is applied. In this case, denoting by q the input coordinate ($q \equiv x_B$) and by φ the output coordinate, we find

$$Q = -M \frac{d\varphi}{dq}. \quad (4.6)$$

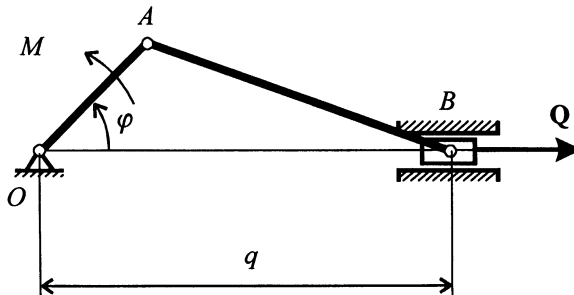


Fig. 4.2. Slider-crank mechanism with an input slider and output crank

As a result, according to (4.5), we obtain an expression for the derivative $d\varphi/dq$ in the implicit form:

$$\frac{d\varphi}{dq} = (-r \sin \varphi - \frac{r^2 \sin \varphi \cos \varphi}{\sqrt{\ell^2 - r^2 \sin^2 \varphi}})^{-1}. \quad (4.7)$$

It follows from (4.7) that in the end positions of the mechanism for $\varphi = 0$ and $\varphi = \pi$, the derivative $d\varphi/dq$ becomes infinitely large; this means that in these positions, it is impossible to overcome the moment M through the application of a force Q to the slider. Such positions of a mechanism are called *dead* positions; it is easily seen that they coincide with the singular positions considered in the previous chapters.

Let us proceed with the determination of reaction forces at kinematic pairs of the mechanism shown in Fig. 4.1. For determining the reactions at joints A and B one can use the principle of virtual displacements or either the condition of equal powers. In order to determine the reaction at joint B we release the system from the constraints imposed on the motion by this joint by replacing it by the reaction force $\mathbf{R}_B$. Then, we consider equilibrium conditions for of the slider. Since the connecting rod AB is in equilibrium under the action of two forces – reactions $\mathbf{R}_A$ and $-\mathbf{R}_B$, these forces have to be equal and opposite directed along the line AB . Using the condition of power balance, we obtain

$$\mathbf{R}_B \mathbf{V}_B = \mathbf{P} \mathbf{V}_B = -P \dot{x}_B, \quad (4.8)$$

where $\mathbf{V}_B$ is the velocity vector of point B . From (4.8) we find for the magnitude of the reaction $\mathbf{R}_B$:

$$|\mathbf{R}_B| = \frac{|P|}{\cos \alpha_B} = |\mathbf{R}_A|, \quad (4.9)$$

where α_B is the acute angle between vectors $\mathbf{V}_B$ and $\mathbf{R}_B$ at joints A and B known as *pressure angle* at joint B . It is apparent from expression (4.9), that the reactions $\mathbf{R}_A$ and $\mathbf{R}_B$ at joints A and B are determined by the force $\mathbf{P}$ and by the pressure angle α_B . For making the reactions at joints A and B small it is necessary to make the pressure angle α_B smaller. This can be achieved through a modification of mechanism geometry. Since in this example

$$\cos \alpha_B = \sqrt{1 - \frac{r^2}{\ell^2} \sin^2 q},$$

so that a small pressure angle α_B is achieved by a small ratio r/ℓ .

The pressure angle at joint A , i.e. the acute angle between vectors $\mathbf{V}_A$ and $\mathbf{R}_A$, affects the generalized balancing force Q . Using the condition of power balancing for crank OA , we obtain:

$$Q\dot{q} = \mathbf{R}_A \mathbf{V}_A, \quad (4.10)$$

where $\mathbf{V}_A$ is the velocity vector of point A . From (4.10) we find the balancing force Q :

$$Q = \frac{R_A V_A \cos \alpha_A}{\dot{q}} = \frac{|P| \dot{q} OA \cos \alpha_A}{\cos \alpha_B \dot{q}} = \frac{|P| OA \cos \alpha_A}{\cos \alpha_B}.$$

From this expression it is apparent that in order to decrease Q one must increase α_A . The maximum angle α_A , equal to $\pi/2$, and, therefore, the minimum force Q is achieved in the positions corresponding to the crank rotation angles $q = 0$ and $q = \pi$ i.e. in those end positions, where the derivative dx_B/dq becomes zero.

Reactions N_B at the prismatic pair and $\mathbf{R}_O$ at the revolute pair connecting the crank with the frame can be determined from equilibrium conditions for the slider and the crank. Considering equilibrium of the forces, applied to the slider (Fig. 4.3a), we find

$$N_B + R_B \sin \alpha_B = 0.$$

From here, we have

$$N_B = -R_B \sin \alpha_B, \quad |N_B| = |P| \tan \alpha_B.$$

The smaller the pressure angle α_B the smaller is the modulus of force N_B .

Considering the conditions for equilibrium of the forces, applied to the crank, we find (Fig. 4.3b)

$$\mathbf{R}_O = \mathbf{R}_A.$$

We note that, when implementing the static model, the balancing force as well as all reactions at kinematic pairs turn out to be proportional to the workload P . Therefore, when examining the conditions of force transmission from the view point of admissibility of a mechanism geometry, we can restrict ourselves to the determination of these forces for a unit workload.

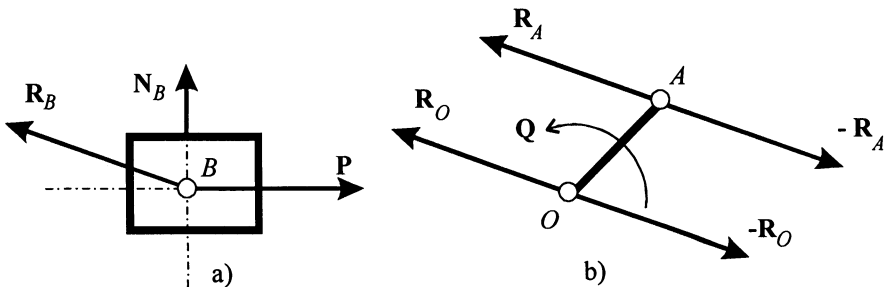


Fig. 4.3. Chart of the force action on the slider (a) and the crank (b)

Let us consider a mechanism with three degrees of movability (Fig. 4.4). The geometric and kinematic analysis of this mechanism has been carried out in Chaps. 2 and 3.

Let workloads be applied to platform BC in the plane motion. Having chosen point M of the platform ($BM = MC$) as reduction point, one can reduce the workloads to forces P_x and P_y and moment M_w . For the mechanism to be in equilibrium, it is necessary to apply to it generalized balancing forces Q_1 , Q_2 and Q_3 , corresponding to the generalized input coordinates q_1 , q_2 , q_3 . To determine the generalized balancing force Q_1 , let us freeze coordinates q_2 and q_3 and let us give a small increment δq_1 to coordinate q_1 . Expressing the work done by the forces P_x , P_y , Q_1 and moment M_w in the corresponding virtual displacement of the system, we obtain

$$Q_1 \delta q_1 + P_x \delta x_M + P_y \delta y_M + M_w \delta \varphi = 0, \quad (4.11)$$

where δx_M , δy_M , $\delta \varphi$ are the increments of coordinates x_M , y_M and the tilt angle φ of the platform, corresponding to δq_1 . Since

$$\delta x_M = \frac{\partial x_M}{\partial q_1} \delta q_1, \quad \delta y_M = \frac{\partial y_M}{\partial q_1} \delta q_1, \quad \delta \varphi = \frac{\partial \varphi}{\partial q_1} \delta q_1,$$

we obtain from (4.11) the relationship

$$Q_1 = -P_x \frac{\partial x_M}{\partial q_1} - P_y \frac{\partial y_M}{\partial q_1} - M_w \frac{\partial \varphi}{\partial q_1}. \quad (4.12)$$

Giving the mechanism a small displacement δq_2 for fixed q_1 and q_3 and then a small displacement δq_3 for fixed q_1 and q_2 , and deriving conditions analogous

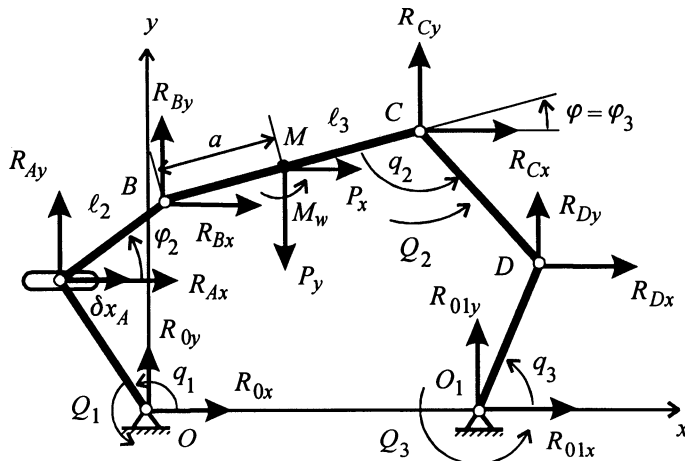


Fig. 4.4. Static model of a linkage of movability three

to (4.11), we obtain

$$Q_s = -P_x \frac{\partial x_M}{\partial q_s} - P_y \frac{\partial y_M}{\partial q_s} - M_w \frac{\partial \varphi}{\partial q_s} \quad (s = 2, 3). \quad (4.13)$$

In this way, for the determination of the generalized balancing forces it is necessary to know the first partial derivatives of position functions $x_M(q_1, q_2, q_3)$, $y_M(q_1, q_2, q_3)$, $\varphi(q_1, q_2, q_3)$ with respect to generalized coordinates. For determining these derivatives one can use the method considered in Chap.3, i.e. construct the vector diagrams of the analogues of mechanism velocities for a given position.

We note that the application of the principle of virtual displacements allows us to determine every single balancing force independently from the others. Let us show that the principle also allows us to determine individual components of reactions at kinematic pairs, if only the constraint corresponding to this component is a releasing constraint. In the mechanism shown in Fig. 4.4, all 12 components of reactions at the joints lying in the motion plane, correspond to releasing constraints. Therefore, all of them can be determined by the method presented below.

We eliminate the constraint at joint A which prevents the relative displacement of point A of link AB along the x -axis. Then, following the principle of constraint releasing, we consider R_{Ax} as an active force acting on mechanism $ABCD O_1$. It is easy to see that the release of this constraint leads to an additional degree of movability of the mechanism; the corresponding input coordinate is the displacement of point A , belonging to link AB in x -direction. Let us consider the equilibrium conditions for system $ABCD O_1$ under the action of forces P_x , P_y , moment M_w and balancing force R_{Ax} . To do this, let us fix coordinates q_2 and q_3 and then give a small increment δx_A to point A of link AB . From the equilibrium conditions we obtain an expression, analogous to (4.13)

$$R_{Ax} = -P_x \frac{\partial x_M}{\partial x_A} - P_y \frac{\partial y_M}{\partial x_A} - M_w \frac{\partial \varphi}{\partial x_A}. \quad (4.14)$$

In this way, successively eliminating the constraints at joints, it is possible to find all reaction forces. For determining the derivatives $\partial x_M / \partial x_A$, $\partial y_M / \partial x_A$, $\partial \varphi / \partial x_A$ one can use Eqs. (2.36). To the coordinate x_A , on the right side of the first equation (2.36) we give a small increment δx_A and we write the equations for the corresponding group coordinates φ_2 and φ_3 :

$$\begin{aligned} -\ell_2 \sin \varphi_2 \delta \varphi_2 - [\ell_3 \sin \varphi_3 - \ell_4 \sin(\varphi_3 + q_2)] \delta \varphi_3 &= -\delta x_A, \\ -\ell_2 \cos \varphi_2 \delta \varphi_2 - [\ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2)] \delta \varphi_3 &= 0. \end{aligned} \quad (4.15)$$

Taking into account that

$$\delta \varphi_2 = \frac{\partial \varphi_2}{\partial x_A} \delta x_A, \quad \delta \varphi_3 = \frac{\partial \varphi_3}{\partial x_A} \delta x_A,$$

we obtain

$$\begin{aligned}
 -\ell_2 \sin \varphi_2 \frac{\partial \varphi_2}{\partial x_A} + [-\ell_3 \sin \varphi_3 + \ell_4 \sin(\varphi_3 + q_2)] \frac{\partial \varphi_3}{\partial x_A} &= -1, \\
 \ell_2 \cos \varphi_2 \frac{\partial \varphi_2}{\partial x_A} + [\ell_3 \cos \varphi_3 - \ell_4 \cos(\varphi_3 + q_2)] \frac{\partial \varphi_3}{\partial x_A} &= 0.
 \end{aligned} \tag{4.16}$$

It is easy to determine $\partial \varphi_2 / \partial x_A$ and $\partial \varphi_3 / \partial x_A$ from (4.16). Differentiating further (2.37) with respect to x_A and taking into account that $\varphi \equiv \varphi_3$, we find

$$\begin{aligned}
 \frac{\partial x_M}{\partial x_A} &= 1 - \ell_2 \sin \varphi_2 \frac{\partial \varphi_2}{\partial x_A} - a \sin \varphi_3 \frac{\partial \varphi_3}{\partial x_A}, \\
 \frac{\partial y_M}{\partial x_A} &= \ell_2 \cos \varphi_2 \frac{\partial \varphi_2}{\partial x_A} + a \cos \varphi_3 \frac{\partial \varphi_3}{\partial x_A}, \\
 \frac{\partial \varphi}{\partial x_A} &= \frac{\partial \varphi_3}{\partial x_A}.
 \end{aligned} \tag{4.17}$$

The determinant of system (4.16) coincides with the Jacobian of the group equations (2.36). This means that in a neighbourhood of the singular positions the derivatives $\partial \varphi_2 / \partial x_A$ and $\partial \varphi_3 / \partial x_A$, and hence also the derivatives (4.17), become very large in magnitude. As a result, the generalized actuating forces (4.13) and the components (4.14) increase. This shows that the conditions of force transmission become disadvantageous in the neighbourhood of mechanism singular positions.

4.2

Determination of Forces Acting in Mechanisms by the Graph-Analytic Method and the Method of Opening Kinematic Chains

Forces acting in a planar mechanism that has no excessive constraints of the second group (see Sect. 1.5), can be determined by the graph-analytical method, which is reduced to a successive consideration of equilibrium conditions for structural groups. Moreover, the equilibrium equations are derived in a sequence, opposite to that of the geometric analysis, i.e. beginning from the group containing the output link (in general, beginning from the group of the last „structural layer“) and terminating with the groups directly attached to the frame.

As an example, let us consider the mechanism shown in Fig. 4.5. Let a force **P** be applied to the output link *BC* at point *M* ($BM = MC$) (Fig. 4.5a). Joint reactions will be denoted by a double index. As an example **R**₂₃ is the force exerted by link 2 on link 3 at joint *B*. The opposite force denoted by **R**₃₂ is exerted by link 3 on link 2 at the same joint. It is obvious that, in accordance with Newton's third law, **R**₂₃ = -**R**₃₂.

Let us consider the equilibrium conditions for the structural group *ABCD*. It is easy to see that to this group three external forces **R**₁₂, **R**₅₄ and **P** are applied.

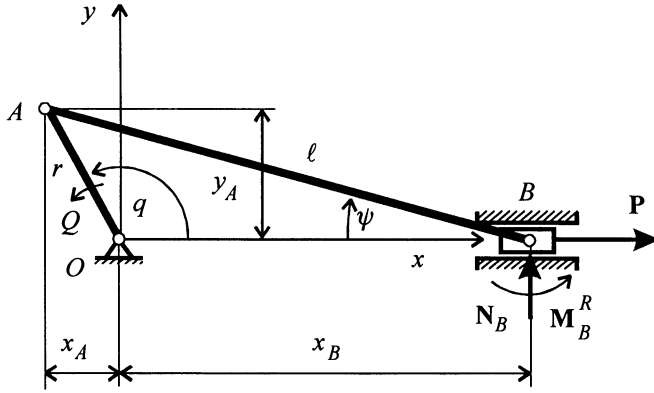


Fig. 4.6. Determination of forces in the slider-crank mechanism

In order to get acquainted with the method of determining forces by opening kinematic chains we will first study a simple example. Consider again the crank-slider mechanism (Fig. 4.6). After opening the prismatic pair we introduce the reactions appearing at this pair: moment M_B^R and force N_B which is orthogonal to the direction of the relative displacement at the pair. We write down the equilibrium conditions for the opened kinematic chain OAB . Let us, first, consider the moment equations for all external forces (i.e. for forces N_B , P and moment M_B^R) with respect to points B , A and O . Taking into account that joints A and B are „passive“, i.e. that at these joints no balancing forces are applied and that at joint O a balancing moment Q is applied, we obtain

$$\begin{aligned} \sum M_B = M_B^R = 0, \quad \sum M_A = N_B(x_B - x_A) + Py_A + M_B^R = 0, \\ \sum M_O = N_Bx_B + M_B^R + Py_0 + Q = 0. \end{aligned} \quad (4.18)$$

Solving these equations we find the unknown reactions N_B , M_B^R and the moment Q . Since

$$x_B = r \cos q + \sqrt{\ell^2 - r^2 \sin^2 q}, \quad x_A = r \cos q, \quad y_A = r \sin q, \quad y_0 = 0,$$

we obtain from (4.18):

$$\begin{aligned} M_B^R = 0, \quad N_B = -P \frac{y_A}{x_B - x_A} = -P \frac{r \sin q}{\sqrt{\ell^2 - r^2 \sin^2 q}}, \\ Q = -N_B x_B = P \frac{r \sin q}{\sqrt{\ell^2 - r^2 \sin^2 q}} (r \cos q + \sqrt{\ell^2 - r^2 \sin^2 q}) = \\ P \left(r \sin q + \frac{r^2 \sin q \cos q}{\sqrt{\ell^2 - r^2 \sin^2 q}} \right). \end{aligned} \quad (4.19)$$

It is easy to see that two expressions for Q obtained from formulae (4.3) and (4.9) are identical.

The method of opening kinematic chains can be applied successively to every single structural group of the mechanism. As an example let us consider the mechanism shown in Fig. 4.5. We open the mechanism at joint D and replace the action of the constraints by forces R_{Dx} and R_{Dy} . In order to determine these forces as well as the driving balancing moment Q_2 we write the equilibrium equations of the open kinematic chain $ABCD$ which require that the moments of all forces about axes A , B , C are equal to zero. Taking into account that joints A and B are passive, and that at joint C the moment Q_2 is applied, we obtain

$$\sum M_A = R_{Dy}(x_D - x_A) - R_{Dx}(y_D - y_A) + P_y(x_M - x_A) - P_x(y_M - y_A) = 0, \quad (4.20)$$

$$\sum M_B = R_{Dy}(x_D - x_B) - R_{Dx}(y_D - y_B) + P_y(x_M - x_B) - P_x(y_M - y_B) = 0, \quad (4.21)$$

$$\sum M_C = R_{Dy}(x_D - x_C) - R_{Dx}(y_D - y_C) + Q_2 = 0. \quad (4.22)$$

These equations can be solved in the following order. First, from the two leading equations we determine the unknown reactions R_{Dx} and R_{Dy} . In this case, a solution exists and it is unique if the determinant of the equation system (4.20) and (4.21) is not equal to zero. It is easily seen that the condition for this determinant to be zero,

$$\begin{vmatrix} x_D - x_A & -(y_D - y_A) \\ x_D - x_B & -(y_D - y_B) \end{vmatrix} = 0 \quad (4.23)$$

is fulfilled if and only if points A , B and D are in one straight line, i.e. if the mechanism is in a singular position (see Fig. 2.18). Having determined R_{Dx} and R_{Dy} , we find Q_2 from Eq. (4.22).

Following this, the remaining reactions at the joints of group $ABCD$ are determined without difficulty from the equilibrium conditions for the forces acting on these links. The balancing driving moments Q_1 and Q_3 are also easily determined from the equilibrium conditions for groups OA and O_1D .

Let us now formulate general rules for the determination of driving forces and reactions in arbitrary structural groups by the method of opening kinematic chains. For now, we will assume that the analysis is based on the static model. Suppose that we have the structural group shown in Fig. 2.12. It was shown in Chap. 2 that after opening the kinematic chain at joint D we obtain an open chain of „tree“-type in which group coordinates α , β and u are added in addition to the two generalized coordinates q_1 and q_2 . Let some active force $\mathbf{P}$ be applied to link CD at point M . We introduce the components of reaction $\mathbf{R}_D$ which are acting at the opened kinematic pair. Let them be denoted by $\mathbf{R}_{Dx}$, $\mathbf{R}_{Dy}$, $\mathbf{R}_{Dz}$. In order

to determine these forces together with the generalized driving forces Q_1 and Q_2 one may take advantage of two methods.

In the first method based on the the principle of virtual work we formulate the conditions that the total work done by all active forces for virtual displacements, $\delta q_2 = \delta\alpha = \delta\beta = \delta u = 0$, $\delta q_1 \neq 0$, $\delta q_1 = \delta\alpha = \delta\beta = \delta u = 0$, $\delta q_2 \neq 0$ etc. be equal to zero. As a result we obtain relationships analogous to (4.11). Moreover, taking into account that the generalized forces Q_α , Q_β , Q_u corresponding to coordinates α , β , u , are equal to zero (kinematic pairs are passive), we obtain the following equations:

$$\begin{aligned}
 Q_1 + P_x \frac{\partial x_M^{(0)}}{\partial q_1} + P_y \frac{\partial y_M^{(0)}}{\partial q_1} + P_z \frac{\partial z_M^{(0)}}{\partial q_1} + R_{Dx} \frac{\partial x_D^{(0)}}{\partial q_1} + \\
 R_{Dy} \frac{\partial y_D^{(0)}}{\partial q_1} + R_{Dz} \frac{\partial z_D^{(0)}}{\partial q_1} = 0, \\
 Q_2 + P_x \frac{\partial x_M^{(0)}}{\partial q_2} + P_y \frac{\partial y_M^{(0)}}{\partial q_2} + P_z \frac{\partial z_M^{(0)}}{\partial q_2} + R_{Dx} \frac{\partial x_D^{(0)}}{\partial q_2} + \\
 R_{Dy} \frac{\partial y_D^{(0)}}{\partial q_2} + R_{Dz} \frac{\partial z_D^{(0)}}{\partial q_2} = 0, \quad (4.24) \\
 P_x \frac{\partial x_M^{(0)}}{\partial \alpha} + P_y \frac{\partial y_M^{(0)}}{\partial \alpha} + P_z \frac{\partial z_M^{(0)}}{\partial \alpha} + R_{Dx} \frac{\partial x_D^{(0)}}{\partial \alpha} + R_{Dy} \frac{\partial y_D^{(0)}}{\partial \alpha} + R_{Dz} \frac{\partial z_D^{(0)}}{\partial \alpha} = 0, \\
 P_x \frac{\partial x_M^{(0)}}{\partial \beta} + P_y \frac{\partial y_M^{(0)}}{\partial \beta} + P_z \frac{\partial z_M^{(0)}}{\partial \beta} + R_{Dx} \frac{\partial x_D^{(0)}}{\partial \beta} + R_{Dy} \frac{\partial y_D^{(0)}}{\partial \beta} + R_{Dz} \frac{\partial z_D^{(0)}}{\partial \beta} = 0, \\
 P_x \frac{\partial x_M^{(0)}}{\partial u} + P_y \frac{\partial y_M^{(0)}}{\partial u} + P_z \frac{\partial z_M^{(0)}}{\partial u} + R_{Dx} \frac{\partial x_D^{(0)}}{\partial u} + R_{Dy} \frac{\partial y_D^{(0)}}{\partial u} + R_{Dz} \frac{\partial z_D^{(0)}}{\partial u} = 0.
 \end{aligned}$$

Here $x_M^{(0)}$, $y_M^{(0)}$, $z_M^{(0)}$, $x_D^{(0)}$, $y_D^{(0)}$, $z_D^{(0)}$ are the coordinates of points M and D in the system $Ox_0y_0z_0$. The system (4.24) can be solved in two steps. In the first step, R_{Dx} , R_{Dy} , R_{Dz} are determined from the last three Eqs. (4.24). (Such independent systems of equations equal in number to the degree of closure of the considered group, always appears when applying the method of opening). This system has a solution if the position is not singular, i.e. if

$$\begin{vmatrix}
 \frac{\partial x_D^{(0)}}{\partial \alpha} & \frac{\partial y_D^{(0)}}{\partial \alpha} & \frac{\partial z_D^{(0)}}{\partial \alpha} \\
 \frac{\partial x_D^{(0)}}{\partial \beta} & \frac{\partial y_D^{(0)}}{\partial \beta} & \frac{\partial z_D^{(0)}}{\partial \beta} \\
 \frac{\partial x_D^{(0)}}{\partial u} & \frac{\partial y_D^{(0)}}{\partial u} & \frac{\partial z_D^{(0)}}{\partial u}
 \end{vmatrix} \neq 0. \quad (4.25)$$

Subsequently, Q_1 and Q_2 are determined from the first two Eqs. (4.24).

In the second method we write the equilibrium equations in the form of equations for moments with respect to joint axes O, A, B, C and of equations for projections of forces onto the directions of joint axis B and axis x_1 . In addition, equilibrium equations are written for the external forces acting on the links to that side of the respective joint on which the opened pair D is located. In this way, we obtain the equations:

$$\begin{aligned}\sum M_{Cz4} &= M_{Cz4}(\mathbf{P}) + M_{Cz4}(\mathbf{R}_D) + Q_2 = 0, \\ \sum M_{Bx3} &= M_{Bx3}(\mathbf{P}) + M_{Bx3}(\mathbf{R}_D) = 0, \\ \sum M_{Oz1} &= M_{Oz1}(\mathbf{P}) + M_{Oz1}(\mathbf{R}_D) = 0, \\ \sum F_{O3x3} &= P \cos(O_3x_3, \mathbf{P}) + R_D \cos(O_3x_3, \mathbf{R}_D) = 0, \\ \sum F_{O1x1} &= P \cos(O_1x_1, \mathbf{P}) + R_D \cos(O_1x_1, \mathbf{R}_D) + Q_1 = 0.\end{aligned}\tag{4.26}$$

Here, the indices denote the origin of the coordinate system and the corresponding axis.

Usually, these equations are formulated using the apparatus of transformation matrices. With the help of a computer they are more easily formulated than the equivalent Eqs. (4.25). We note that in (4.26) the second, the third and the fourth equation constitute an independent group, from which it is possible to determine the components of vector $\mathbf{R}_D$.

4.3

Application of Equilibrium Equations of a Mechanism to its Kinematic and Parametric Analysis

The equilibrium equations of a mechanism can be used not only for the force analysis but also for the determination of the derivatives of position functions with respect to generalized input coordinates and to mechanism parameters, i.e. for the solution of problems of kinematic and parametric analysis. This approach has significant advantages in comparison to the methods considered in Chap. 3.

As an example let us consider the mechanism, shown in Fig. 4.7. Let us determine the first partial derivatives of position functions

$$y_k = \Pi_{yk}(q_1, q_2, q_3, \ell_1, \dots, \ell_5, d)\tag{4.27}$$

with respect to the input coordinates q_1, q_2, q_3 and to parameters $\ell_1, \dots, \ell_5, d$. We apply a force $\mathbf{P}_y = 1$ in y -direction at point K of link 3 and we determine the balancing moments Q_1, Q_2, Q_3 . This can be done, e.g., by constructing the force vector diagram shown in Fig. 4.5. The values of Q_1, Q_2, Q_3 are found from the equilibrium conditions for the structural groups $ABCD, OA$ and O_1D . On the other hand, from the principle of virtual displacements, it follows that

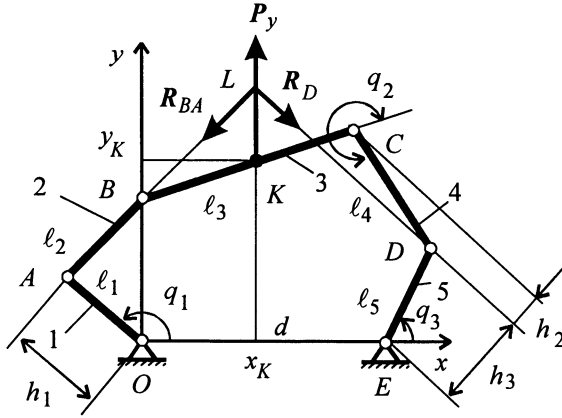


Fig. 4.7. Determination of the first derivatives of the position functions of a mechanism of movability three

$$Q_1 \delta q_1 + P_y \delta y_k^{(1)} = 0, \quad Q_2 \delta q_2 + P_y \delta y_k^{(2)} = 0, \quad Q_3 \delta q_3 + P_y \delta y_k^{(3)} = 0, \quad (4.28)$$

where $\delta y_k^{(i)}$ ($i = 1, 2, 3$) is the virtual displacement of point K in y -direction, caused by the variation of the i -th input coordinate, whereas the remaining input coordinates have zero variations. With $\delta y_k^{(i)} = \frac{\partial y_k}{\partial q_i} \delta q_i$ we obtain from Eqs. (4.28):

$$\frac{\partial y_k}{\partial q_1} = -\frac{Q_1}{P_y}, \quad \frac{\partial y_k}{\partial q_2} = -\frac{Q_2}{P_y}, \quad \frac{\partial y_k}{\partial q_3} = -\frac{Q_3}{P_y}. \quad (4.29)$$

This demonstrates how the first partial derivatives with respect to generalized coordinates are determined.

Now, let us remove constraint AB , replacing it by the reaction $\mathbf{R}_{BA}$, and let us write the equilibrium equation for the moments of forces $\mathbf{P}_y$ and $\mathbf{R}_{BA}$ relative to axis E . Apparently, this is a condition for the equilibrium of the open chain $BCDE$, since the other reactions forces at joints B, C, D, E are either internal forces or forces with zero moments with respect to point E . The condition for equilibrium of forces $\mathbf{P}_y$ and $\mathbf{R}_{BA}$ gives:

$$\mathbf{R}_{BA} \delta \ell_2 + \mathbf{P}_y \delta y_K^{(\ell_2)} = 0,$$

whence we obtain the derivative of y_k with respect to the parameter ℓ_2 :

$$\frac{\partial y_k}{\partial \ell_2} = -\frac{R_{BA}}{P_y}. \quad (4.30)$$

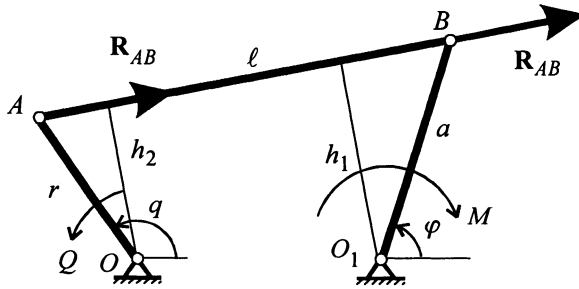


Fig. 4.8. Determination of the first derivatives of the position functions of the crank-and-rocker mechanism

In order to find the derivatives of the rotation angle φ of the platform with respect to coordinates q_1, q_2, q_3 , it is necessary to apply a unit moment M to link BC and to determine the balancing moments Q_1^M, Q_2^M, Q_3^M . Here we have:

$$\frac{\partial \varphi}{\partial q_1} = -\frac{Q_1}{M}, \quad \frac{\partial \varphi}{\partial q_2} = -\frac{Q_2}{M}, \quad \frac{\partial \varphi}{\partial q_3} = -\frac{Q_3}{M}. \quad (4.31)$$

Whereas the vector diagram of velocity analogues allows us to determine on one and the same graph the derivatives of different position functions with respect to one and the same coordinate (i.e., the input coordinate q) the method described here allows us to determine (under the application of one unit force) the derivatives of one position function with respect to different coordinates and parameters. Therefore, in a certain sense, both methods complement each other.

Consider one more example. For the four-bar crank-rocker mechanism (Fig. 4.8) let us determine the derivatives of the output coordinate φ with respect to the input coordinate q and to the parameter ℓ (the length of the connecting rod).

Having constructed the force vector diagram, and applying to the output a unit moment M , we find:

$$\frac{\partial \varphi}{\partial q} = -\frac{Q}{M} = -\frac{R_{AB} h_2}{R_{AB} h_1} = -\frac{h_2}{h_1},$$

where h_1 and h_2 are the shortest distances from points O and O_1 to the straight line AB . Analogously, we obtain:

$$\frac{\partial \varphi}{\partial \ell} = -\frac{R_{AB}}{M} = -\frac{1}{h_1}.$$

The application of equations for equilibrium of a mechanism turns out to be especially useful in the following cases:

- Determination of the derivatives with respect to parameters. A need for this arises in connection with the investigation of mechanism accuracy and with the determination of errors in position functions caused by errors in parameters.

The commonly known method for the determination of these derivatives, suggested by N.G. Bruevitch, is reduced to the construction of so-called „converted mechanisms“ [6]. In this case, the determination of derivatives of a given position function with respect to single parameter requires the construction of different converted mechanisms, in each of them the input coordinate is fixed and the investigated parameter becomes the input coordinate. When applying the equilibrium equations, it is sufficient to restrict oneself to the construction of a single force vector diagram only for any number of independent parameters.

- b) Determination of first derivatives of position functions for planar mechanisms of higher classes. In Fig. 4.9a a mechanism of fourth class is shown consisting of input link OA and of an Assur group of fourth class.

Let us determine the derivative of the angle φ with respect to the input coordinate q in Fig. 4.9. The construction of the diagram of velocity analogues is a sufficiently complex and labor-consuming problem. The solution is associated with the implementation of the method of false positions [8] or other equivalent graph-analytical methods. Using the equilibrium equations for this purpose, we apply a unit moment M to link 5. From the equilibrium conditions for link 2, to which three forces $\mathbf{R}_B$, $\mathbf{R}_C$ and $\mathbf{R}_A$ are applied, it follows that the action lines of these three forces must intersect in a single point (point L). From here, the line of action of force $\mathbf{R}_A$ is determined. Taking into account the fact that only the external force $\mathbf{R}_A$ and the moment M are applied to the structural group of fourth class, we obtain at once:

$$M\delta\varphi + R_A h_\varphi \delta\varphi = 0.$$

On the other hand, from the equilibrium condition for link 1 we have:

$$Q\delta q + R_A h_q \delta q = 0.$$

From here we obtain:

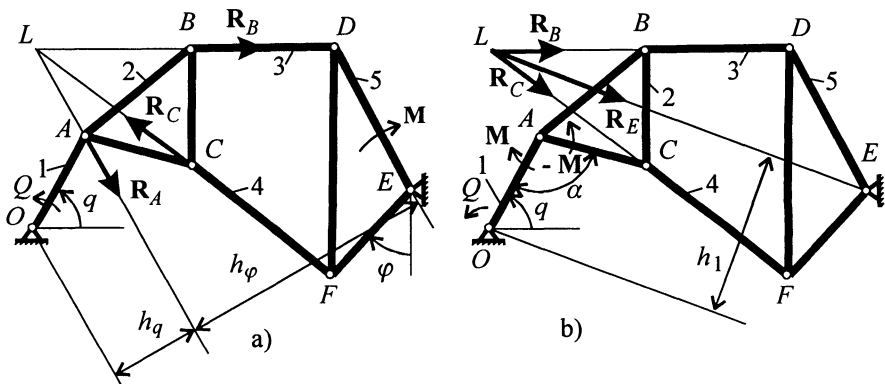


Fig. 4.9. Determination of the first derivatives of the position functions of a mechanism of fourth class

$$\frac{\partial \varphi}{\partial q} = \frac{Q}{M} = -\frac{h_q}{h_\varphi}.$$

In Fig. 4.9b constructions are shown, which are necessary for the determination of the derivative $\partial \alpha / \partial q$, where α is the angle between links 1 and 2.

From the force diagram we obtain $\partial \alpha / \partial q = R_E h_1 / M$.

- c) Determination of singular positions of complex linkages. For instance, it is apparent that the mechanism shown in Fig. 4.9a will be in a singular position, if the line of action of force $\mathbf{R}_A$ passes through point E since in this case the mechanism is in equilibrium for $M = 0$.

In Fig. 4.10 the structural group of movability one $ABCD$ with internal input q is shown. Opening it at point D and writing the equations for moments with respect to axes A , B and C , we obtain the equilibrium equations:

$$\begin{aligned} -R_{Dx}(y_D - y_A) + R_{Dy}(x_D - x_A) &= -P(x_M - x_A), \\ -R_{Dx}(y_D - y_B) + R_{Dy}(x_D - x_B) &= -P(x_M - x_B), \\ -R_{Dx}(y_D - y_C) + R_{Dy}(x_D - x_C) + Q &= 0. \end{aligned} \quad (4.32)$$

The determinant of this system

$$\Delta = \begin{vmatrix} -(y_D - y_A) & x_D - x_A & 0 \\ -(y_D - y_B) & x_D - x_B & 0 \\ -(y_D - y_C) & x_D - x_C & 1 \end{vmatrix} \quad (4.33)$$

$$= (x_D - x_A)(y_D - y_B) - (x_D - x_B)(y_D - y_A)$$

is equal to zero in singular positions. From here it follows that a singular position arises if points A , B and D lie in one straight line.

From the previous exposition it follows that unknown forces entering the equilibrium equation can be replaced by the corresponding derivatives of position functions. Thus, in Eqs. (4.32) substitutions can be made:

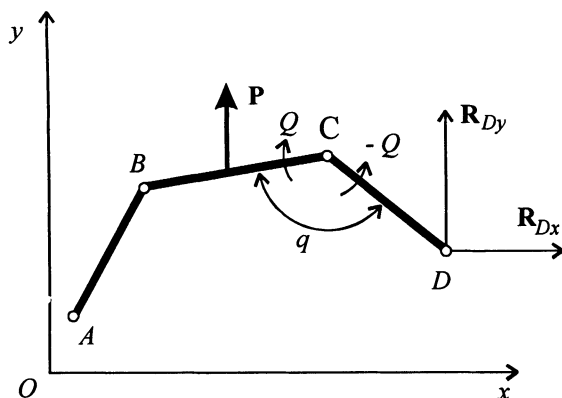


Fig. 4.10. Structural group of movability one $ABCD$ with an internal input

$$R_{Dx} = -P \frac{\partial y_M}{\partial x_D}, \quad R_{Dy} = P \frac{\partial y_M}{\partial y_D}, \quad Q = -P \frac{\partial y_M}{\partial q}. \quad (4.34)$$

In this case we obtain:

$$\begin{aligned} \frac{\partial y_M}{\partial x_D}(y_D - y_A) + \frac{\partial y_M}{\partial y_D}(x_D - x_A) &= -(x_M - x_A), \\ \frac{\partial y_M}{\partial x_D}(y_D - y_B) + \frac{\partial y_M}{\partial y_D}(x_D - x_B) &= -(x_M - x_B), \\ \frac{\partial y_M}{\partial x_D}(y_D - y_C) + \frac{\partial y_M}{\partial y_D}(x_D - x_C) + \frac{\partial y_M}{\partial q} &= 0. \end{aligned} \quad (4.35)$$

After differentiating these equations with respect to q , x_D , y_D , one can obtain equation systems for the second derivatives of position functions. So, differentiation with respect to q leads to the system:

$$\begin{aligned} \frac{\partial^2 y_M}{\partial x_D \partial q}(y_D - y_A) + \frac{\partial^2 y_M}{\partial y_D \partial q}(x_D - x_A) &= \\ -\frac{\partial y_M}{\partial x_D} \left(\frac{\partial y_D}{\partial q} - \frac{\partial y_A}{\partial q} \right) - \frac{\partial y_M}{\partial y_D} \left(\frac{\partial x_D}{\partial q} - \frac{\partial x_A}{\partial q} \right) - \frac{\partial x_M}{\partial q} - \frac{\partial x_A}{\partial q}, \\ \frac{\partial^2 y_M}{\partial x_D \partial q}(y_D - y_B) + \frac{\partial^2 y_M}{\partial y_D \partial q}(x_D - x_B) &= \\ -\frac{\partial y_M}{\partial x_D} \left(\frac{\partial y_D}{\partial q} - \frac{\partial y_B}{\partial q} \right) - \frac{\partial y_M}{\partial y_D} \left(\frac{\partial x_D}{\partial q} - \frac{\partial x_B}{\partial q} \right) - \frac{\partial x_M}{\partial q} - \frac{\partial x_B}{\partial q}, \\ \frac{\partial^2 y_M}{\partial x_D \partial q}(y_D - y_C) + \frac{\partial^2 y_M}{\partial y_D \partial q}(x_D - x_C) + \frac{\partial^2 y_M}{\partial q^2} &= \\ -\frac{\partial y_M}{\partial x_D} \left(\frac{\partial y_D}{\partial q} - \frac{\partial y_C}{\partial q} \right) - \frac{\partial y_M}{\partial y_D} \left(\frac{\partial x_D}{\partial q} - \frac{\partial x_C}{\partial q} \right). \end{aligned} \quad (4.36)$$

As soon as all first derivatives of position functions with respect to q have been determined, we find from (4.36) $\frac{\partial^2 y_M}{\partial x_D \partial q}$, $\frac{\partial^2 y_M}{\partial y_D \partial q}$, $\frac{\partial^2 y_M}{\partial q^2}$.

4.4

General Formulation of the Force Analysis Problem

In the force analysis based on the static method it was required that generalized driving forces applied to output links balance only the resistance forces caused by the working process. Actually, also other active forces arise in a mechanism: friction, gravity etc. Furthermore, during acceleration of links inertial forces occur

which must also be overcome by the driving forces. For this reason, in the majority of cases the analysis based on the static model can be considered only as a preliminary stage necessary for the estimation of the quality of a chosen kinematic diagram. A complete force analysis of a mechanism is done following the development of its construction when the structural features of links and of kinematic pairs are already defined, and when the dimensions of links as well as the materials from which the parts are made are known so that masses and moments of inertia can be computed.

In the force analysis the motion of a mechanism is considered as given. Usually, in the beginning it is assumed that this is a program motion, necessary for the performance of the working process. Only afterwards, the motion actually taking place can be determined more precisely from results of the dynamic analysis which will be considered in what follows.

In the complete force analysis of a mechanism, with the exception of the generalized driving forces, all active forces acting on mechanism links are considered as known. To the given forces belong:

- a) *Workloads*, i.e. forces acting on the working organs of a machine during the performance of the working process (e.g., cutting forces in a machine tool; forces of ground resistance acting on the shovel of an excavating machine; forces of air pressure acting on the piston of a compressor; moments of electromagnetic forces arising during rotation of the rotor of an electric generator etc. When we say that these forces are known, this means that the workloads are given as functions of motion parameters and of time. These functions are called *characteristics of workloads*. In general, they have the following form:

$$P_m = P_m(x_1, \dots, x_k, \dot{x}_1, \dots, \dot{x}_k, \ddot{x}_1, \dots, \ddot{x}_k, t) \quad (m = 1, \dots, \ell), \quad (4.37)$$

where ℓ is the total number of active forces; $x_1, \dots, x_k$ are coordinates defining the position of the working organs. In mechanisms with rigid links the coordinates x_s and their derivatives with respect to time can be expressed through generalized coordinates $q_1, \dots, q_w$, velocities $\dot{q}_1, \dots, \dot{q}_w$ and accelerations $\ddot{q}_1, \dots, \ddot{q}_w$ (w is the total number of degrees of movability of the mechanism):

$$\begin{aligned} x_s &= \Pi_s(q_1, \dots, q_w), \quad \dot{x}_s = \sum_{u=1}^w \frac{\partial \Pi_s}{\partial q_u} \dot{q}_u, \\ \ddot{x}_s &= \sum_{u=1}^w \left(\frac{\partial \Pi_s}{\partial q_u} \ddot{q}_u + \sum_{m=1}^w \frac{\partial^2 \Pi_s}{\partial q_u \partial q_m} \dot{q}_u \dot{q}_m \right), \end{aligned} \quad (4.38)$$

where Π_s are position functions.

Substituting (4.38) into (4.39), we obtain the workloads as functions of the generalized coordinates and of their derivatives:

$$P_m = P_m^*(q_1, \dots, q_w, \dot{q}_1, \dots, \dot{q}_w, \ddot{q}_1, \dots, \ddot{q}_w, t) \quad (m = 1, \dots, \ell). \quad (4.39)$$

For a given motion of a mechanism the quantities $q_u, \dot{q}_u, \ddot{q}_u$ ($u = 1, \dots, w$) are known at every point of time. Consequently, the P_m^* themselves can be determined for any given position of a mechanism, too.

Henceforth, it is assumed that workloads do not depend explicitly on time nor on accelerations $\ddot{q}_u$. Because of this, the expression (4.39) will be represented in the form

$$P_m = P_m^*(q_1, \dots, q_w, \dot{q}_1, \dots, \dot{q}_w). \quad (4.40)$$

- b) *Gravity forces of links.* These forces are constant but their points of application and, therefore, the moments of gravity forces with respect to s of reduction, change in the process of motion.
- c) In mechanisms springs are frequently used (Fig. 4.11) for ensuring force closure of kinematic chains. The *elastic forces* arising due to deformations of these springs are also active forces. Their magnitudes depend on the magnitudes of spring deformations, i.e. essentially on the coordinates of the spring attachment points (x_A, x_B, y_A, y_B in Fig. 4.11). For given motion laws these forces can be determined at any point in time or for any position of the mechanism.

The aim of the force analysis is to determine the generalized driving forces as well as the reactions at all kinematic pairs. The generalized driving forces are forces to be applied to the input links of a mechanism in order to achieve a given program motion during the execution of a working process. Having determined the driving forces it is possible to select driving motors for the machine.

Reactions at kinematic pairs are passive forces. As a rule, these are forces distributed on the contact surfaces of the constructive elements constituting a pair. However, in the framework of physical models of kinematic pairs considered in courses on theory of mechanisms and machines it is impossible, in general, to determine the distribution of reaction forces on element surfaces. Therefore, we restrict ourselves to the determination of resultant vectors and of resultant moments of reaction forces at every kinematic pair.

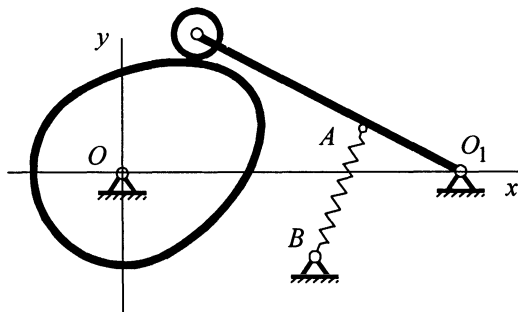


Fig. 4.11. Cam mechanism with oscillating follower

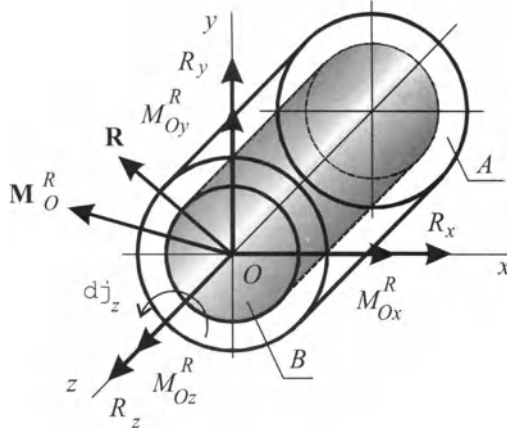


Fig. 4.12. The resultant vector $\mathbf{R}$ and the resultant moment $\mathbf{M}_O^R$ of the reaction forces at a kinematic pair

Let us consider a kinematic pair (Fig. 4.12); the resultant vector $\mathbf{R}$ of reaction forces and the resultant moment $\mathbf{M}_O^R$ are defined through the specification of six scalar quantities, namely their projections R_x , R_y , R_z , M_{Ox}^R , M_{Oy}^R , M_{Oz}^R onto the axes of the system $Oxyz$.

Next, we determine the total number of unknowns occurring in the force analysis. Let the mechanism have w degrees of movability and P kinematic pairs. Thus, the number of unknown generalized driving forces is equal to w and the number of the unknown components of reactions is $6P$. Thus, the overall number of unknowns is

$$n_u = w + 6P. \quad (4.41)$$

These unknowns can be determined by solving the motion equations of mechanism links. Let the number of the mobile links be equal to N . For every link it is possible to write two vector equations of motion on the basis of the theorems of momentum and of moment of momentum. If $\mathbf{K}_s$ is the momentum of link s and if $\mathbf{L}_{sO}$ is its moment of momentum, then the motion equations are written in the form:

$$\frac{d\mathbf{K}_s}{dt} = \sum_{k=1}^{g_s} \mathbf{F}_{sk}, \quad \frac{d\mathbf{L}_{sO}}{dt} = \sum_{k=1}^{g_s} \mathbf{r}_{sk} \times \mathbf{F}_{sk} \quad (s = 1, \dots, N), \quad (4.42)$$

where $\mathbf{F}_{sk}$ are the external forces, acting on link s ; $\mathbf{r}_{sk}$ are the radius-vectors of their points of application; g_s is the number of forces applied to link s . As is well known from theoretical mechanics, it is impossible to formulate additional independent equations of motion of a rigid body. The total number of vector

Eqs. (4.42) is equal to $2N$; projecting them on the coordinate axes we obtain $6N$ scalar equations.

Let a mechanism have no excessive constraints. Then, the formula holds

$$w = 6N - \sum_{s=1}^5 (6-s)p_s, \quad (4.43)$$

where p_s is the total number of pairs of movability s . Comparing the number of unknowns n_u and the number of equations $n_{eq} = 6N$, and taking into account

that $P = \sum_{s=1}^5 p_s$, we have

$$n_u - n_{eq} = 6P - 6\sum_{s=1}^5 p_s + \sum_{s=1}^5 sp_s = \sum_{s=1}^5 sp_s. \quad (4.44)$$

Thus, in such a formulation of the force analysis problem the number of unknowns is always larger than the number of equations, thus making this problem insoluble. It is worse, if there are excessive constraints in the mechanism, since in this case the number of unknown reactions is still larger whereas the number of equations remains unchanged.

In order to make the problem soluble, further refinements of the physical model and additional assumptions on the properties of kinematic pairs are necessary. One such refinement is the assumption that all kinematic pairs represent ideal constraints. In ideal constraints the work done by reaction forces at every kinematic pair is equal to zero in any virtual displacement, i.e.

$$\delta A = R_x \delta x + R_y \delta y + R_z \delta z + M_{Ox}^R \delta \varphi_x + M_{Oy}^R \delta \varphi_y + M_{Oz}^R \delta \varphi_z = 0. \quad (4.45)$$

Here δx , δy , δz are small virtual displacements along the coordinate axes and $\delta \varphi_x$, $\delta \varphi_y$, $\delta \varphi_z$ are small rotations about these axes. A revolute pair allows only a small rotation of link B (the journal) relative to link A (the sleeve) about an axis Oz . Thus, from (4.45) we have

$$\delta A = M_{Oz}^R \delta \varphi_z = 0.$$

For a rotation $\delta \varphi_z \neq 0$ we have $M_{Oz}^R = 0$. In this way one of the six components of reactions of the revolute pair has been determined and the number of unknowns has decreased by one which equals the degree of movability of the pair.

At a prismatic pair (Fig. 4.13a) any possible displacement of link B relative to link A is along an x -axis. Therefore, we have

$$\delta A = R_x \delta x = 0,$$

and, since $\delta x \neq 0$, we obtain $R_x = 0$, which also defines one of the unknown components of reactions.

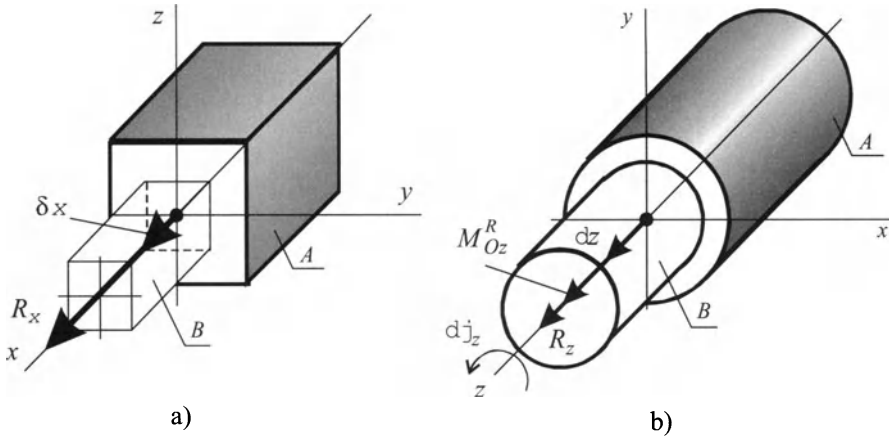


Fig. 4.13. Virtual displacements at a prismatic (a) and at a cylindric (b) kinematic pair

At a cylindric pair (Fig. 4.13b) the virtual displacements δz and $\delta\varphi_z$ are different from zero, so here

$$\delta A = R_z \delta z + M_{Oz}^R \delta\varphi_z = 0.$$

Due to the independence of virtual displacements δz and $\delta\varphi_z$ this condition has to be fulfilled for $\delta z \neq 0$, $\delta\varphi_z = 0$ as well as for $\delta z = 0$, $\delta\varphi_z \neq 0$. This leads to the conclusion that R_z and M_{Oz}^R must be equal to zero which determines two components of reactions for the cylindric pair of movability two.

In a similar way it can be shown that in any pair of movability s the condition of ideality leads to the appearance of s additional relationships for the components of reactions. As a result, $\sum_{s=1}^5 sp_s$ conditions appear for the mechanism as a whole, which makes the problem of force analysis soluble. In this case it is often said that the mechanism is *statically determinate*.

In cases, when friction must be taken into account, the statement of the problem must be substantially modified in a way which will be discussed in details below.

4.5

Equations of Kinetostatics. Determination of the Resultant Vector and of the Resultant Moment of Inertia Forces of Links

It is convenient to represent Eqs. (4.42) in a different form. Let us introduce the inertia forces of the material points of the link s

$$\Phi_{si} = -m_{si} \mathbf{w}_{si} \quad (i = 1, \dots, k_s; s = 1, \dots, N), \quad (4.46)$$

where m_{si} is the mass of the i -th material point, $\mathbf{w}_{si}$ is its acceleration. Let us recall that the „inertia force“ Φ_{si} is called a force only conditionally; actually, this is a *measure of motion* of a material point, like, e.g., momentum. Introducing inertia forces, it is possible to transform the left-hand sides of Eqs. (4.42); taking into account that $\mathbf{K}_s = \sum_i m_{si} \mathbf{v}_{si}$, $\mathbf{L}_{sO} = \sum_i \mathbf{r}_{si} \times m_{si} \mathbf{v}_{si}$, we obtain

$$\frac{d\mathbf{K}_s}{dt} = \frac{d}{dt} \sum_i m_{si} \mathbf{v}_{si} = \sum_i m_{si} \mathbf{w}_{si} = - \sum_i \Phi_{si} = -\Phi_s, \quad (4.47)$$

$$\begin{aligned} \frac{d\mathbf{L}_{sO}}{dt} &= \frac{d}{dt} \sum_i \mathbf{r}_{si} \times m_{si} \mathbf{v}_{si} = \sum_i (\mathbf{v}_{si} \times m_{si} \mathbf{v}_{si} + \mathbf{r}_{si} \times m_{si} \mathbf{w}_{si}) = \\ &= - \sum_i \mathbf{r}_{si} \times \Phi_{si} = -\mathbf{M}_{Os}^{(\Phi)}. \end{aligned} \quad (4.48)$$

Here Φ_s is the resultant vector of inertia forces of the s -th link and $\mathbf{M}_{Os}^{(\Phi)}$ is their resultant moment with respect to some arbitrarily chosen point O .

Let us collect the active forces $\mathbf{P}_{sk}$ and the reactions of kinematic pairs $\mathbf{R}_{sk}$ on the right-hand side of Eqs. (4.42):

$$\begin{aligned} \sum_k \mathbf{F}_{sk} &= \sum_k (\mathbf{P}_{sk} + \mathbf{R}_{sk}) = \mathbf{P}_s + \mathbf{R}_s, \\ \sum_k \mathbf{r}_{sk} \times \mathbf{F}_{sk} &= \sum_k \mathbf{r}_{sk} \times (\mathbf{P}_{sk} + \mathbf{R}_{sk}) = \mathbf{M}_{Os}^{(P)} + \mathbf{M}_{Os}^{(R)}, \end{aligned} \quad (4.49)$$

where $\mathbf{P}_s$ and $\mathbf{R}_s$ are the resultant vectors of active forces and of constraint reactions, acting on link s , $\mathbf{M}_{Os}^{(P)}$ and $\mathbf{M}_{Os}^{(R)}$ are the resultant moments with respect to point O . Substituting (4.49) into (4.42), we obtain the equations of motion in the following form:

$$\mathbf{P}_s + \mathbf{R}_s + \Phi_s = 0, \quad \mathbf{M}_{Os}^{(P)} + \mathbf{M}_{Os}^{(R)} + \mathbf{M}_{Os}^{(\Phi)} = 0 \quad (s = 1, \dots, N). \quad (4.50)$$

The equations of motion have taken the form of equilibrium equations. From this form it can be concluded that the active forces acting on every single mobile link of a mechanism together with the reactions of constraints and the inertia forces of links, form a balanced system. However, the artificiality of such a formulation should be kept in mind; in reality the inertia forces are not forces but measures of motion. Consequently, Eqs. (4.50) represent motion equations and not equilibrium equations. In order to stress this point Eqs. (4.50) are called *equations of kinetostatics* and the model of the force analysis of a mechanism based on them is called a *kinetostatic model*.

¹ One must distinguish the notations: $\mathbf{M}_O^R$ – the resultant moment of reaction forces at a kinematic pair; $\mathbf{M}_{Os}^{(R)}$ – the resultant moment of reaction forces, acting on the link s .

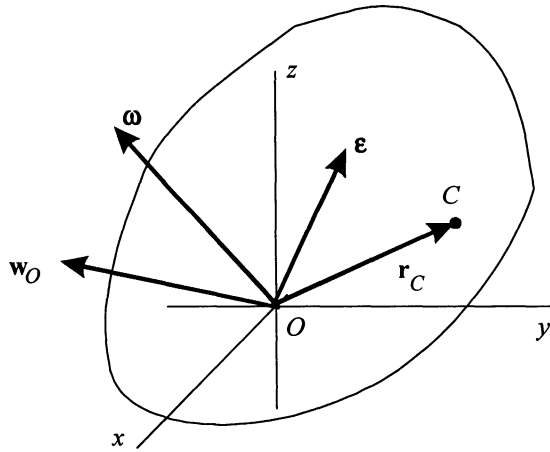


Fig. 4.14. The vectors of angular velocity ω and angular acceleration ϵ and the pole acceleration w_O of a link

For the derivation of equations in the form (4.50) we must know how to determine the resultant vector and the resultant moment of the inertia forces of a link if the motion is given. In the courses on analytical mechanics expressions for Φ_s and $M_{Os}^{(\Phi)}$ are developed in the general case of motion of a rigid body [9]. Let some point O (Fig. 4.14) be chosen as pole of a link and let r_C be a vector, defining the position of its mass center C .

If the pole acceleration w_O , the vector of angular velocity ω of a link and the vector of its angular acceleration ϵ are known (they are determined in the process of the kinematic analysis of a mechanism), then for the resultant vector of inertia forces Φ and for their resultant moment $M_O^{(\Phi)}$ with respect to point O the following expressions are valid [9]:

$$\Phi = -m w_C = -m(w_O + \epsilon \times r_C + \omega \times \omega \times r_C), \quad (4.51)$$

$$M_O^{(\Phi)} = -(I_O \epsilon + \omega \times I_O \omega + m r_C \times w_O). \quad (4.52)$$

Here m is the mass of the link, I_O is the inertia tensor at point O . If a coordinate system $Oxyz$ attached to the link is introduced, then the tensor I_O can be given by the matrix moments of inertia

$$I_O = \begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{xy} & J_y & -J_{yz} \\ -J_{xz} & -J_{yz} & J_z \end{pmatrix}, \quad (4.53)$$

where J_x , J_y , J_z are centroidal moments of inertia and J_{xy} , J_{yz} , J_{xz} are products of inertia. Evaluating expressions (4.51) and (4.52), we obtain

$$\begin{aligned}
\Phi_X &= -m \left[w_{Ox} + \varepsilon_y z_C - \varepsilon_z y_C + \omega_y (\omega_x y_C - \omega_y x_C) - \omega_z (\omega_z x_C - \omega_x z_C) \right], \\
\Phi_Y &= -m \left[w_{Oy} + \varepsilon_z x_C - \varepsilon_x z_C + \omega_z (\omega_y z_C - \omega_z y_C) - \omega_x (\omega_x y_C - \omega_y x_C) \right], \\
\Phi_Z &= -m \left[w_{Oz} + \varepsilon_x y_C - \varepsilon_y x_C + \omega_x (\omega_z x_C - \omega_x z_C) - \omega_y (\omega_y z_C - \omega_z y_C) \right], \\
M_{Ox}^{(\Phi)} &= -m (y_C w_{Oz} - z_C w_{Oy}) - J_x \varepsilon_x + J_{xy} \varepsilon_y + J_{xz} \varepsilon_z - J_{xy} \omega_z \omega_x - \\
&\quad J_{xz} \omega_x \omega_y - J_{yz} (\omega_z^2 - \omega_y^2) - \omega_z \omega_y (J_z - J_y), \\
M_{Oy}^{(\Phi)} &= -m (z_C w_{Ox} - x_C w_{Oz}) + J_{xy} \varepsilon_x - J_y \varepsilon_y + J_{yz} \varepsilon_z - J_{yz} \omega_x \omega_y - \\
&\quad J_{xy} \omega_y \omega_z - J_{xz} (\omega_x^2 - \omega_z^2) - \omega_x \omega_z (J_x - J_z), \\
M_{Oz}^{(\Phi)} &= -m (x_C w_{Oy} - y_C w_{Ox}) + J_{xz} \varepsilon_x + J_{yz} \varepsilon_y - J_z \varepsilon_z - J_{xz} \omega_y \omega_z - \\
&\quad J_{yz} \omega_x \omega_z - J_{xy} (\omega_y^2 - \omega_x^2) - \omega_x \omega_y (J_y - J_x).
\end{aligned} \tag{4.54}$$

From these relationships expressions for the projections of the resultant vector and the resultant moment of inertia forces can be obtained for special cases:

a) In the case of translation of a link $\omega = 0$, $\varepsilon = 0$; therefore

$$\begin{aligned}
\Phi_x &= -m w_{Ox}, & M_{Ox}^{(\Phi)} &= -m (y_C w_{Oz} - z_C w_{Oy}), \\
\Phi_y &= -m w_{Oy}, & M_{Oy}^{(\Phi)} &= -m (z_C w_{Ox} - x_C w_{Oz}), \\
\Phi_z &= -m w_{Oz}, & M_{Oz}^{(\Phi)} &= -m (x_C w_{Oy} - y_C w_{Ox});
\end{aligned} \tag{4.55}$$

b) For the rotation of a link about a fixed axis Oz we obtain (Fig. 4.15):

$$\omega_x = \omega_y = 0, \quad \omega_z = \omega, \quad \varepsilon_x = \varepsilon_y = 0, \quad \varepsilon_z = \varepsilon, \quad \mathbf{w}_O = \mathbf{0};$$

From (4.54) and (4.55) we find:

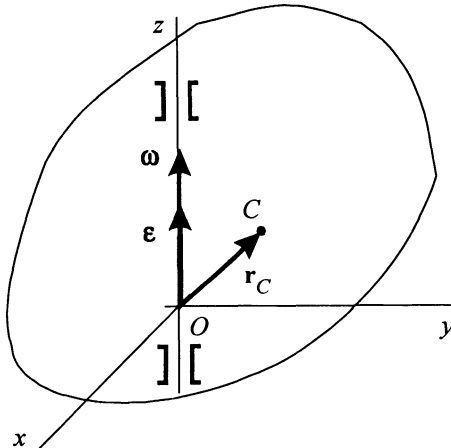


Fig. 4.15. The vectors of angular velocity ω and angular acceleration ε of a link rotating about a fixed axis Oz

$$\Phi_x = m(\varepsilon y_C + \omega^2 x_C), \quad \Phi_y = m(-\varepsilon x_C + \omega^2 y_C), \quad \Phi_z = 0, \quad (4.56)$$

$$M_{Ox}^{(\Phi)} = J_{xz}\varepsilon - J_{yz}\omega^2, \quad M_{Oy}^{(\Phi)} = J_{yz}\varepsilon + J_{xz}\omega^2, \quad M_{Oz}^{(\Phi)} = -J_z\varepsilon; \quad (4.57)$$

c) In the case of plane motion of a link in a plane, orthogonal to the z -axis, if we choose as pole the mass center of the link, we obtain

$$\begin{aligned} \Phi_x &= -mw_{Cx}, & M_{Cx}^{(\Phi)} &= J_{xz}^C\varepsilon - J_{yz}^C\omega^2, \\ \Phi_y &= -mw_{Cy}, & M_{Cy}^{(\Phi)} &= J_{yz}^C\varepsilon + J_{xz}^C\omega^2, \\ \Phi_z &= 0, & M_{Cz}^{(\Phi)} &= -J_z^C\varepsilon, \end{aligned} \quad (4.58)$$

where $J_{xz}^C, J_{yz}^C, J_z^C$ are the components of the inertia tensor I_C in the system $Cxyz$. It should be note that the resultant vector of inertia forces is assumed to be applied at the point relative to which the resultant moment is defined. In the case of rotation of a link about a fixed axis, this is a point lying on the rotation axis.

4.6 Solution of the Equations of Kinetostatics

For mechanisms with ideal constraints the equations of kinetostatics constitute a system of linear algebraic equations, which has a unique solution, if there are no excessive constraints in the system and if the mechanism is not in a singular position. For complex mechanisms with large numbers of mobile links, the system of equations of kinetostatics is of high order (for N links there are $6N$ equations). Its solution is substantially facilitated by the fact that it can be split into several independent systems, every one of which includes one generalized driving force together with the reactions of kinematic pairs acting on the links of a single structural group. Indeed, for every structural group without excessive constraints the following formula holds

$$w_G = 6N_G - \sum_{s=1}^5 (6-s)p_{sG}, \quad (4.59)$$

where w_G is the number of degrees of movability, N_G is the number of mobile links, p_{sG} is the number of kinematic pairs of movability s in the group. On the other hand, as it was shown above, the sum

$$n_u = w_G + \sum_{s=1}^5 (6-s)p_{sG} \quad (4.60)$$

represents the number of unknown driving forces and reactions at ideal constraints. Comparing expressions (4.59) and (4.60) we notice that $n_u = 6N_G$, i.e. the number of unknown forces and reactions is equal to the number of

equations of kinetostatics. For this reason, the equations of kinetostatics can be solved successively for every structural group. Moreover, the analysis should be done in sequence opposite to that of the geometric and kinematic analysis, i.e. it should start with the groups of the last layer. Then, the reactions at the external kinematic pairs of the groups of the m -th layer turn out to be known and they can be considered as given forces when analyzing the groups of the $(m-1)$ -st layer.

In a planar mechanism, the successive isolation of planar structural groups, makes it possible to determine for every group separately the components of reactions lying in the motion plane xOy (R_x, R_y, M_{Oz}^R) together with the generalized driving forces. The components of the reactions not lying in the motion plane appear in another group of equations. Often, due to the presence of excessive constraints, the determination of the reactions of this other group (R_z, M_{Ox}^R, M_{Oy}^R) becomes impossible; as is shown below, in this case it is necessary to restrict oneself to the determination of the reactions of releasing constraints.

Let us consider some examples.

- a) Fig. 4.16 shows a planar mechanism with one degree of movability, consisting of three structural groups: a group of movability one (link 1) and two groups of zero movability (links 2 and 3, 4 and 5). In the figure are shown the components of the reactions lying in the motion plane. Reactions with components R_x and R_y exist at the revolute pairs; reaction R_y and moment

M_{Oz}^R exist at the prismatic pairs. Every component of the reactions is denoted by two indices indicating the number of the acting link and the number of the link subject to this action. These notations were already used in Sect. 4.2.

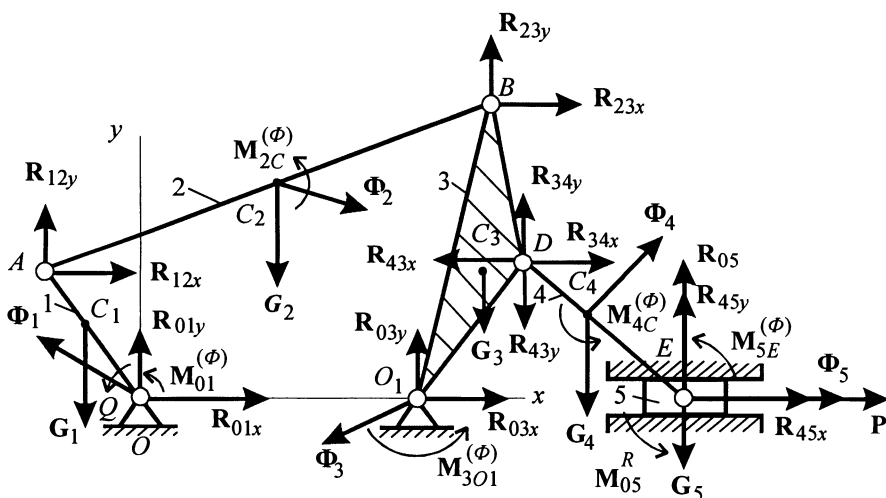


Fig. 4.16. Force diagram of a six-bar mechanism of movability one with three structural groups

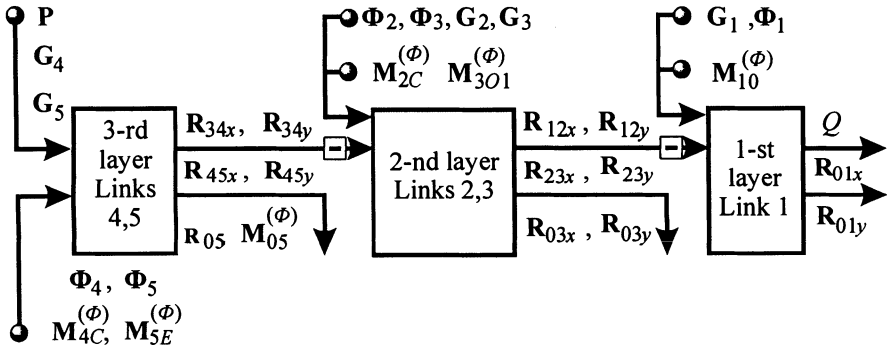


Fig. 4.17. Sequence of solution of the equations of kinetostatics for a mechanism with three structural groups

Also shown in the figure are the active forces – the given forces (P , G_1 to G_5), and the forces to be determined (the driving moment Q) – the inertia forces lying in the motion plane and the projections of the moments of inertia forces onto the z -axis orthogonal to the motion plane.

The sequence of solution of the equation system of kinetostatics is shown in the diagram of Fig. 4.17. First the 6 equations for links 4 and 5 belonging to the structural group of the third layer are solved. For every link two equations for x - and y -projections as well as equations for the moments about axes parallel to the z -axis are written. Reactions R_{34x} and R_{34y} are considered as given forces ($R_{43x} = -R_{34x}$ and $R_{43y} = -R_{34y}$) in the force analysis of the next group (links 2 and 3). Following this three equations for link 1 are solved, which is a group of the first layer. Moreover, the reactions R_{01x} , R_{01y} and the driving force Q are determined.

- c) Consider now a mechanism with three degrees of movability (Fig. 4.18) whose force analysis based on the static model has already been considered. Having defined the inertia forces and the moments of inertia forces of the links and having specified the values of the active forces P_{3x} and P_{3y} applied at point

C_3 and the value of moment M^w , we proceed to the force analysis of the last structural group $ABCD$. For the planar system of forces acting on the three links of this group we write 9 equations of kinetostatics, from which we determine the generalized driving force Q_2 and 8 reactions (R_{12x}, R_{12y} , $R_{23x}, R_{23y}, R_{34x}, R_{34y}, R_{54x}, R_{54y}$). Next, we proceed to the analysis of the one-bar groups OA and ED of the first layer. At this stage the forces $R_{21x} = -R_{12x}$, $R_{21y} = -R_{12y}$, $R_{45x} = -R_{54x}$, $R_{45y} = -R_{54y}$ are considered to be known already from the previous stage. From the equations of kinetostatics for link OA we determine R_{01x} , R_{01y} and the generalized driving force Q_1 ; from the equations for link ED we determine R_{05x} , R_{05y}

and the generalized driving force Q_3 . In Fig. 4.19 the algorithm for the force analysis of this mechanism is schematically presented. For every group the given („input“) forces and the forces found during the force analysis are indicated.

The force analysis based on the static model can be carried out not only by solving the equations of kinetostatics written individually for every link but also by other methods which were explained before. Let us consider how the three-bar group $ABCD$, shown in Fig. 4.18 could be analyzed employing the method of opening the kinematic diagram. Let us open the kinematic chain at joint D and let us introduce reactions R_{54x} and R_{54y} of releasing constraints. Let us write the equilibrium equations for the moments of all active forces, of all inertia forces and of the said constraint reactions about the axes of joints A, B, C . We obtain three equations:

$$\begin{aligned}
 &M_A(\Phi_2) + M_A(G_2) + M_{2C}^{(\Phi)} + M_A(P_{3x}) + M_A(P_{3y}) + M^w + M_A(\Phi_3) + \\
 &M_A(G_3) + M_{3C}^{(\Phi)} + M_A(\Phi_4) + M_A(G_4) + M_{4C}^{(\Phi)} + M_A(R_{54x}) + M_A(R_{54y}) = 0, \\
 &M_B(P_{3x}) + M_B(P_{3y}) + M^w + M_B(\Phi_3) + M_B(G_3) + \\
 &M_{3C}^{(\Phi)} + M_B(\Phi_4) + M_B(G_4) + M_{4C}^{(\Phi)} + M_B(R_{54x}) + M_B(R_{54y}) = 0, \\
 &M_C(\Phi_4) + M_C(G_4) + M_{4C}^{(\Phi)} + M_C(R_{54x}) + M_C(R_{54y}) + Q_2 = 0.
 \end{aligned}$$

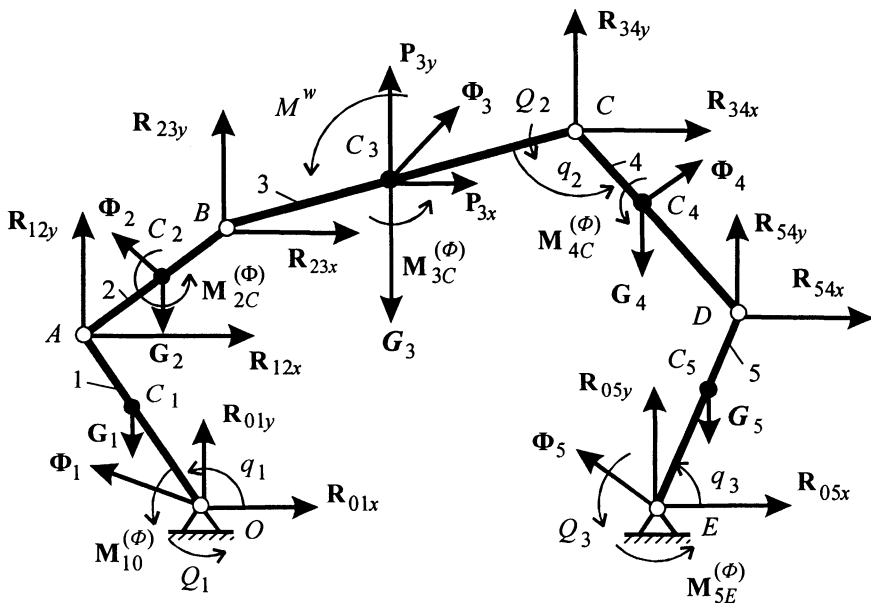


Fig. 4.18. Force diagram of a six-bar mechanism of movability three with three structural groups

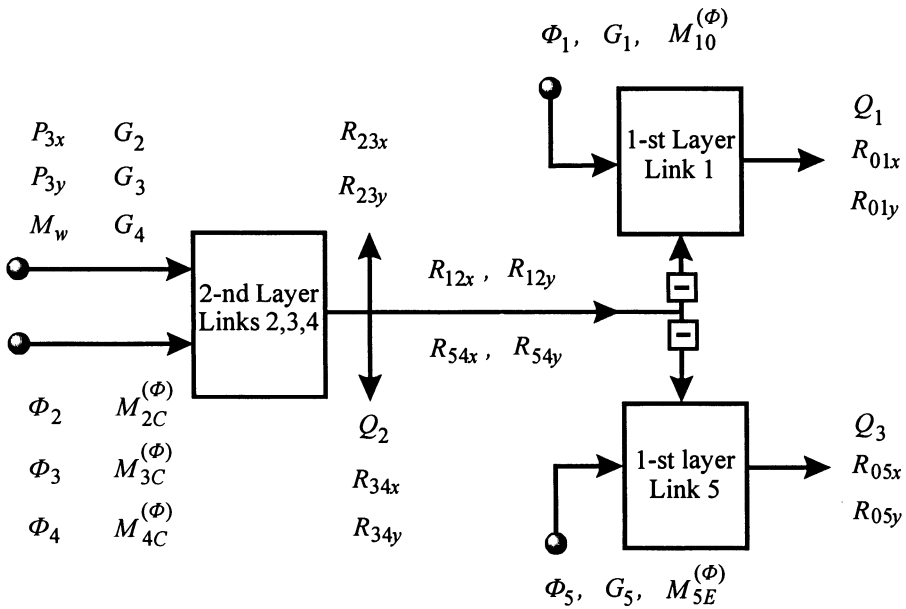


Fig. 4.19. Sequence of solution of the equations of kinetostatics for a mechanism of movability three with three structural groups

From these equations three unknown quantities are determined: R_{54x} , R_{54y} and Q_2 . After this, the determination of the reactions at joints A, B, C does not cause any difficulty; they can be found, for example, from the equations of x - and y -projections of the forces acting on links 2, 3, 4.

The solution of the equations of kinetostatics becomes complicated if there are excessive constraints in a mechanism. In this case, the system becomes statically undeterminate, since the number of unknown reactions and of driving forces exceeds the number of equations of kinetostatics. In a number of cases static indeterminacy can be avoided by increasing the movability of certain kinematic pairs (e.g., replacing revolute pairs by cylindric or spheric pairs). However, this method associated with a change of mechanism construction is often undesirable for technological reasons and also because the stiffness of the mechanical system is lowered. In certain special cases it turns out to be possible to increase conditionally the movability of kinematic pairs in the structural model by taking into account properties of the real construction. As an example, clearances in a revolute pair (between joint axis and sleeve) allows us to consider this pair as spheric pair with zero moments even in the presence of axes misalignment. In general, however, a complete force analysis of a mechanism with excessive constraints is possible only by rejecting the dynamic model of a rigid mechanism and by introducing a structural model that takes into account the elasticity of links and joints. The consideration of this complicated model is outside the scope of a course in theory of mechanisms and machines.

Moreover, it is always possible to conduct a particular force analysis of a mech-

anism, restricting oneself to the determination of generalized driving forces and reactions of all releasing constraints. (Remember that we call releasing a constraint the removal of which increases the number of degrees of movability of a mechanism by one (see Sect. 1.5)). Such a restricted analysis can be carried out by different methods; one of them is based on the use of the general equation of dynamics.

4.7

Application of the General Equation of Dynamics for Force Analysis of Mechanisms

For any system of material points with ideal constraints the total work done by all active forces and all inertia forces in any virtual displacement at any fixed moment of time is equal to zero. The analytical form of this statement

$$\sum_{K=1}^M (\mathbf{P}_K^* + \Phi_K^*) \delta \mathbf{r}_K = 0 \quad (4.61)$$

is called *equation of d'Alembert-Lagrange* or *general equation of dynamics*. In Eq. (4.61) $\mathbf{P}_K^*$ and Φ_K^* are the active force and the inertia force of the k -th material point, respectively, and $\delta \mathbf{r}_K$ is its virtual displacement, i.e. any elementary displacement compatible with the constraints imposed on the system at a given fixed moment of time (in contrast to a real elementary displacement during an infinitesimal increment of time Δt); M is the number of material points in the system.

Consider a mechanism link which is a rigid body. Let us introduce a coordinate system $Oxyz$ attached to this body (Fig. 4.20). For an arbitrary point of the link we have

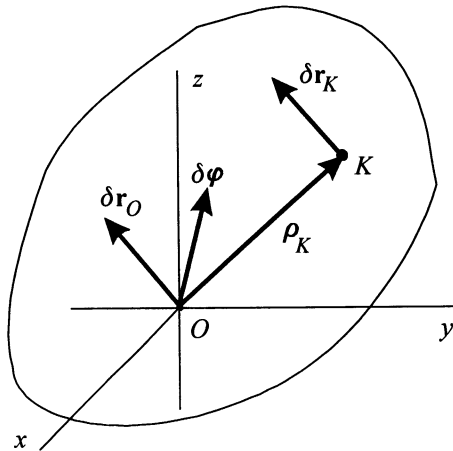


Fig. 4.20. A link and the vector of the virtual displacement $\delta \mathbf{r}_O$ of the pole, the vector of the infinitesimal rotation $\delta \varphi$ and the radius-vector ρ_K of the k -th point

$$\delta \mathbf{r}_K = \delta \mathbf{r}_O + \delta \boldsymbol{\varphi} \times \boldsymbol{\rho}_K, \quad (4.62)$$

where $\delta \mathbf{r}_O$ is a virtual displacement of pole O , $\delta \boldsymbol{\varphi}$ is a vector of infinitesimal rotation, $\boldsymbol{\rho}_K$ is the radius-vector of the k -th point. Substituting (4.62) into (4.61) we find

$$\begin{aligned} \sum_{K=1}^M (\mathbf{P}_K^* + \boldsymbol{\Phi}_K^*) (\delta \mathbf{r}_O + \delta \boldsymbol{\varphi} \times \boldsymbol{\rho}_K) &= \sum_{K=1}^M (\mathbf{P}_K^* + \boldsymbol{\Phi}_K^*) \delta \mathbf{r}_O + \\ &+ \sum_{K=1}^M \boldsymbol{\rho}_K \times (\mathbf{P}_K^* + \boldsymbol{\Phi}_K^*) \delta \boldsymbol{\varphi} = (\mathbf{P} + \boldsymbol{\Phi}) \delta \mathbf{r}_O + (\mathbf{M}_O^{(P)} + \mathbf{M}_O^{(\Phi)}) \delta \boldsymbol{\varphi}. \end{aligned} \quad (4.63)$$

Here, $\mathbf{P}$ and $\boldsymbol{\Phi}$ are the resultant vectors, and $\mathbf{M}_O^{(P)}$ and $\mathbf{M}_O^{(\Phi)}$ are the resultant moments of the active forces and of the inertia forces of a link, respectively. Summing the expressions (4.63) for all movable links we reduce Eq. (4.61) for a mechanism with rigid links and ideal kinematic pairs to the following form

$$\sum_{i=1}^N \left[(\mathbf{P}_i + \boldsymbol{\Phi}_i) \delta \mathbf{r}_{Oi} + (\mathbf{M}_{Oi}^{(P)} + \mathbf{M}_{Oi}^{(\Phi)}) \delta \boldsymbol{\varphi}_i \right] = 0, \quad (4.64)$$

where N is the number of movable links. It is necessary to point out that no single expression (4.63) is equal to zero, since the work done by the reactions acting on every individual link is not equal to zero.

If a mechanism has w degrees of freedom and if $q_1, \dots, q_w$ are its generalized coordinates, then

$$\delta \mathbf{r}_{Oi} = \sum_{s=1}^w \frac{\partial \mathbf{r}_{Oi}}{\partial q_s} \delta q_s, \quad \delta \boldsymbol{\varphi}_i = \sum_{s=1}^w \frac{\partial \boldsymbol{\varphi}_i}{\partial q_s} \delta q_s. \quad (4.65)$$

Substituting (4.65) into (4.64) and recognizing the independence of the variations δq_s of the generalized coordinates we obtain the following system of equations:

$$\sum_{i=1}^N \left[(\mathbf{P}_i + \boldsymbol{\Phi}_i) \frac{\partial \mathbf{r}_{Oi}}{\partial q_s} + (\mathbf{M}_{Oi}^{(P)} + \mathbf{M}_{Oi}^{(\Phi)}) \frac{\partial \boldsymbol{\varphi}_i}{\partial q_s} \right] = 0 \quad (s = 1, \dots, w). \quad (4.66)$$

We note that the expression $\partial \boldsymbol{\varphi}_i / \partial q_s$ should not be considered as the partial derivative of a position function $\boldsymbol{\varphi}(q_1, \dots, q_w)$, since, in general, a vector of rotation angle does not exist as position function but only as ratio of the infinitesimal rotation angle $\delta \boldsymbol{\varphi}_i$ and the infinitesimal partial virtual displacement δq_s . The expression $\partial \boldsymbol{\varphi}_i / \partial q_s$ can be considered also as ratio of the partial angular velocity $\boldsymbol{\omega}_i^S$ and the velocity $\dot{q}_s$ when $\dot{q}_k = 0$ for all $k \neq s$. The derivative $\partial \mathbf{r}_{Oi} / \partial q_s$ is the usual partial derivative of position function $\mathbf{r}_{Oi}(q_1, \dots, q_w)$ with respect to the coordinate q_s .

For a mechanism with a single degree of movability the system (4.66) is reduced to the single equation

$$\sum_{i=1}^N \left[(\mathbf{P}_i + \Phi_i) \frac{d\mathbf{r}_{Oi}}{dq} + (\mathbf{M}_{Oi}^{(P)} + \mathbf{M}_{Oi}^{(\Phi)}) \frac{d\varphi_i}{dq} \right] = 0. \quad (4.67)$$

Since in this case $d\mathbf{r}_{Oi}/dq = \mathbf{v}_{Oi}/\dot{q}$, $d\varphi_i/dq = \omega_i/\dot{q}$, where $\mathbf{v}_{Oi}$ is the velocity of point O_i , Eq. (4.67) is written also in the form

$$\sum_{i=1}^N \left[(\mathbf{P}_i + \Phi_i) \mathbf{v}_{Oi} + (\mathbf{M}_{Oi}^{(P)} + \mathbf{M}_{Oi}^{(\Phi)}) \omega_i \right] = 0. \quad (4.68)$$

From this it is seen that for a mechanism with a single degree of movability and with ideal kinematic pairs the total virtual power of all active forces and of all inertia forces at any given moment is equal to zero.

It is convenient to use the equation of d'Alembert–Lagrange in the form (4.64) for the determination of the generalized driving forces. Taking into account that the work done by a driving force Q_s along a virtual displacement δq_s is equal to $Q_s \delta q_s$, and distinguishing the generalized driving forces from other active forces, we have

$$\sum_{s=1}^w Q_s \delta q_s + \sum_{i=1}^N \left[(\mathbf{P}_{Ci} + \Phi_i) \delta \mathbf{r}_{Oi} + (\mathbf{M}_{Oi}^{(P_C)} + \mathbf{M}_{Oi}^{(\Phi)}) \delta \varphi_i \right] = 0, \quad (4.69)$$

where $\mathbf{P}_{Ci}$ is the resultant vector of all active forces on link i except driving forces, and $\mathbf{M}_{Oi}^{(P_C)}$ is the resultant moment of these forces. From (4.69) we obtain equations analogous to (4.66):

$$Q_s + \sum_{i=1}^N \left[(\mathbf{P}_{Ci} + \Phi_i) \frac{\partial \mathbf{r}_{Oi}}{\partial q_s} + (\mathbf{M}_{Oi}^{(P_C)} + \mathbf{M}_{Oi}^{(\Phi)}) \frac{\partial \varphi_i}{\partial q_s} \right] = 0 \quad (s = 1, \dots, w), \quad (4.70)$$

These equations can be used directly for the determination of the generalized driving forces Q_s . We note that the equations remain valid for mechanisms with any number of excessive ideal constraints. For a mechanism with a single degree of movability we find from (4.67):

$$Q = - \sum_{i=1}^N \left[(\mathbf{P}_{Ci} + \Phi_i) \frac{d\mathbf{r}_{Oi}}{dq} + (\mathbf{M}_{Oi}^{(P_C)} + \mathbf{M}_{Oi}^{(\Phi)}) \frac{d\varphi_i}{dq} \right]. \quad (4.71)$$

As an example let us consider the problem of determination of the driving force for the mechanism shown in Fig. 4.16. In a planar mechanism the vectors of virtual displacements of all points are parallel to the plane of motion and since the vectors of small rotations of links are orthogonal to this plane. Therefore, in order to derive the equations of d'Alembert–Lagrange, it suffices to determine the in-plane components of active forces and inertia forces and the orthogonal components of moments. The remaining components of forces and of moments are not doing work in a virtual displacement of a planar mechanism and, therefore, they do not affect the values of driving forces.

$\delta q = 0$, $\delta y_E \neq 0$ is equal to zero. It is easy to see that in this displacement only forces applied to links 4 and 5 do work. In this case we obtain from Eqs. (4.73)

$$R_{05} = -\mathbf{G}_4 \frac{\partial \mathbf{r}_{C4}}{\partial y_E} - \Phi_4 \frac{\partial \mathbf{r}_{C4}}{\partial y_E} - M_{4C}^{(\Phi)} \frac{\partial \psi_4}{\partial y_E} - \mathbf{P} \frac{\partial \mathbf{x}_E}{\partial y_E} - \Phi_5 \frac{\partial \mathbf{x}_E}{\partial y_E}. \quad (4.74)$$

For reasons of geometry we easily obtain (Fig. 4.21)

$$\begin{aligned} \frac{\partial \varphi_4}{\partial y_E} &= -\frac{1}{\ell_4 \cos \psi_4}, \quad \frac{\partial x_E}{\partial y_E} = \tan \psi_4, \\ \frac{\partial x_{C4}}{\partial y_E} &= \frac{DC_4}{\ell_4} \tan \psi_4, \quad \frac{\partial y_{C4}}{\partial y_E} = \frac{DC_4}{\ell_4}. \end{aligned} \quad (4.75)$$

Substituting (4.75) into (4.74), we find R_{05} in the given position. This same method can be used for the determination of reactions of arbitrary releasing constraints. In contrast to this, reactions corresponding to nonreleasing constraints cannot, in principle, be determined by this process of force analysis of a mechanism. As an example, let us consider the linkage shown in Fig. 4.22. Here, excessive constraints are imposed on the plane motion of the mechanism. Their presence is easily detected by applying the structural formula for a planar mechanism which gives:

$$w = 3N - 2p_\ell = 3 \cdot 4 - 2 \cdot 6 = 0.$$

In reality, the mechanism has a single degree of movability if the lengths of links satisfy the conditions $AB = ED = OO_1$, $OD = O_1E$, $AD = EB$. However, the components in x -direction of reactions at joints correspond to nonreleasing constraints. By releasing any one of them (e.g., by introducing a groove in the tie ED shown in the figure), we do not add a degree of movability to the mechanism. It follows that the reaction R_{Ex} (like the x -components of all other reactions) cannot be determined. It is not difficult to see that even in the absence of active forces and for the mechanism without motion, these reactions may be different from zero as a result of small differences in length of links AB and DE due to manufacturing inaccuracy.

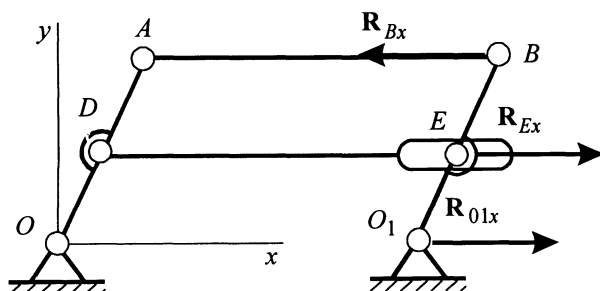


Fig. 4.22. Linkage with excessive constraints

4.8

Force Analysis of Mechanisms with Higher Kinematic Pairs

The surfaces of conjugated links are constructional elements forming a higher kinematic pair. In some cases these surfaces are point-tangent for every position of a mechanism (Fig. 4.23a); in other cases the tangency occurs along a certain line (Fig. 4.23b). When a point contact of rigid links is realized in the absence of frictional forces the reactions at a kinematic pair are reduced to a force $\mathbf{R}_n$ along the common normal of the surfaces in contact. Such a pair has movability five and only a single unknown component of reaction occurs. In the case of line contact, the interaction forces (in the absence of friction) are distributed along the contact line and directed along the common normal of the surfaces at every point.

In a planar cam mechanism (Fig. 4.24) the contact line is a straight line, the reaction forces lie in a plane. They are reduced to a resultant vector $\mathbf{R}_{12n}$ directed along the normal to the cam surface, and to a resultant moment $\mathbf{M}_{12A}^R$ which lies in the plane tangent to the profile. In this case, the higher kinematic pair has movability four.

An analogous picture exists in spur and helical involute cylindric gearings. Here, indeed, several pairs of teeth may simultaneously be meshing but all forces of contact interaction lie in the single plane which passes through the line of meshing and which is parallel to the rotation axes of wheels. In bevel, worm and hypoid gearings the contact line (if it exists) can be a spatial curve (Fig. 4.23b). In this case the interaction forces form a spatial system. When deriving force analysis equations additional components of reactions appear which usually are „excessive“ unknowns. In such cases a simplification of the model of the kinematic pair

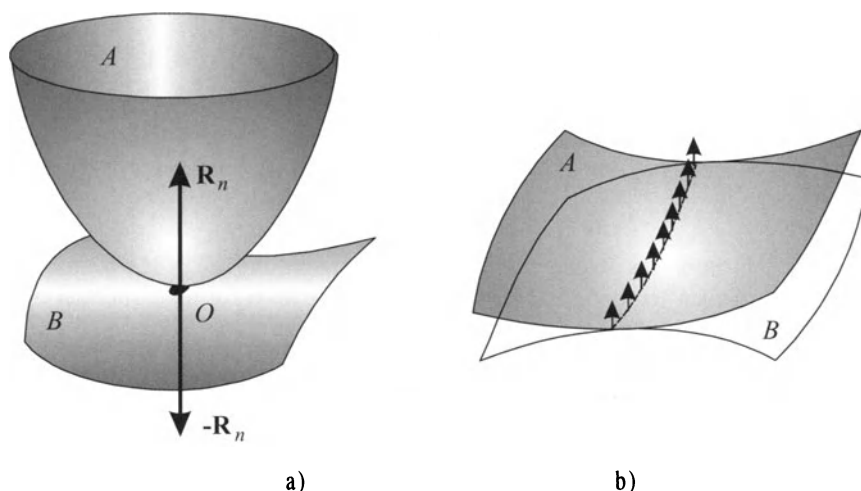


Fig. 4.23. Higher kinematic pair with a contact at a point (a) and along a line (b)

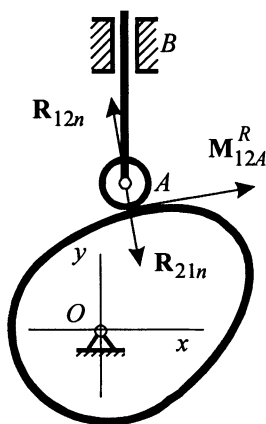


Fig. 4.24. The resultant vector $\mathbf{R}_{12}$ and the resultant moment $\mathbf{M}_{12}^R$ of the reaction forces at the higher kinematic pair of a planar cam mechanism

is achieved with a single unknown component of reaction $\mathbf{R}_n$ by proceeding to the conditional scheme of point interaction. It should be noted that in the force analysis of heavy loaded gearings the „rigid“ model of contact interaction without frictional forces gives only a very rough idea of force loads. Most commonly, the determination of these loads, which is associated with a stress analysis, relies on more complex models of force interaction, taking into account the elastic deformation of teeth, the influence of lubricant etc. However, the consideration of such models falls outside the scope of this course.

4.9 Problems

- 4.1. A load $\mathbf{P}=1000\text{ N}$ is applied to the slotted link of a scotch-yoke mechanism (Fig. 4.25). The length of the crank is $\ell_1 = 0.3\text{ m}$. Determine the reactions at kinematic pairs and the generalized balancing force Q for $q = 30^\circ$.

Solution: In the framework of the static model of the scotch-yoke mechanism let us derive the equilibrium equations for the structural Assur group (including links 2 and 3) for link 2 and for crank 1 (Fig. 4.26):

$$\begin{aligned}
 R_{12X} + P &= 0, & R_{32} + R_{12X} &= 0, & R_{01X} + R_{21X} &= 0, \\
 R_{12Y} + R_{03} &= 0, & R_{12Y} &= 0, & R_{01Y} + R_{21Y} &= 0, \\
 -R_{12X} \cdot AB + & & & & -R_{21X} \ell_1 \sin q + \\
 M_{03}^R - R_{12Y} \cdot BC &= 0, & M_{32}^R &= 0, & R_{21Y} \ell_1 \cos q + Q &= 0.
 \end{aligned}$$

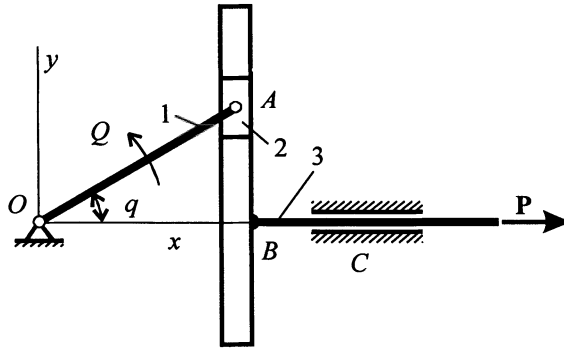


Fig. 4.25

From here we find: $R_{03}^R = 0$, $M_{03}^R = -500$ N, $R_{32} = 1000$ N, $M_{32}^R = 0$, $R_{12X} = -1000$ N, $R_{12Y} = 0$, $R_{01X} = -1000$ N, $R_{01Y} = 0$, $Q = 150$ N m.

In order to check Q we use the condition that the powers be equal:

$$Q\dot{q} = -P\dot{x}_C. \text{ From here we have } Q = -P \frac{\dot{x}_C}{\dot{q}} = -P \frac{dx_C}{dq}. \text{ Because of } x_C =$$

$$\ell_1 \cos q + BC \text{ one has } \frac{dx_C}{dq} = -\ell_1 \sin q, \text{ thus } Q = P\ell_1 \sin q = 150 \text{ N m.}$$

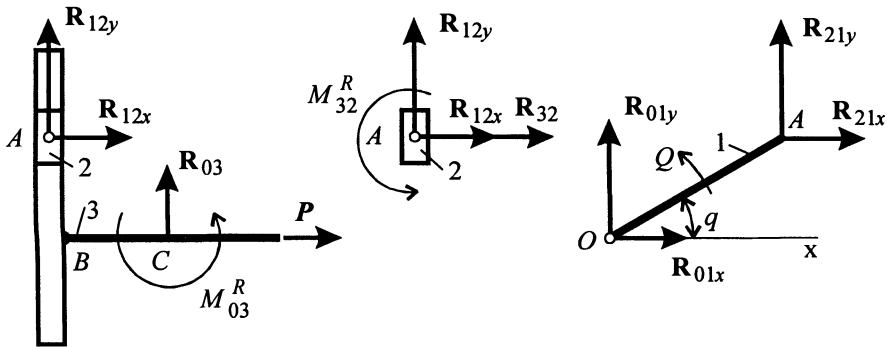


Fig. 4.26. Diagram of forces in problem 4.1

Answer: $R_{03}^R = 0$, $M_{03}^R = -500$ N, $R_{32} = 1000$ N, $M_{32}^R = 0$, $R_{12X} = -1000$ N, $R_{12Y} = 0$, $R_{01X} = -1000$ N, $R_{01Y} = 0$, $Q = 150$ N m.

- 4.2. In the turnover mechanism of a platform (Fig. 4.27) the output link 5 turns with the help of a hydraulic cylinder (links 2 and 3). Moreover, link 5 can make a turn up to 180° keeping the pressure angles at the kinematic pairs within allowable limits.

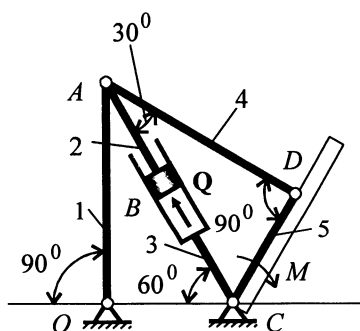


Fig. 4.27

For the given position of the mechanism find the reactions at the kinematic pairs, the pressure angles and the balancing force Q for

$$M = 200 \text{ N m} \text{ and } CD = \frac{\sqrt{3}}{2} \text{ m}.$$

Answer: $|R_{05}| = 400/\sqrt{3} \text{ N}$, $|R_{45}| = |R_{54}| = 400/\sqrt{3} \text{ N}$, $|R_{24}| = |R_{42}| = 400/\sqrt{3} \text{ N}$, $R_{12} = |R_{21}| = |R_{01}| = 400/\sqrt{3} \text{ N}$, $|R_{03}| = 400 \text{ N}$, $Q = 400 \text{ N}$. The pressure angles (angles $\langle \mathbf{R}_{45}, \mathbf{v}_D \rangle$, $\langle \mathbf{R}_{24}, \mathbf{v}_A \rangle$, $\langle \mathbf{Q}, \mathbf{v}_A \rangle$) and the directions of the reactions at the kinematic pairs are indicated in Fig. 4.28.

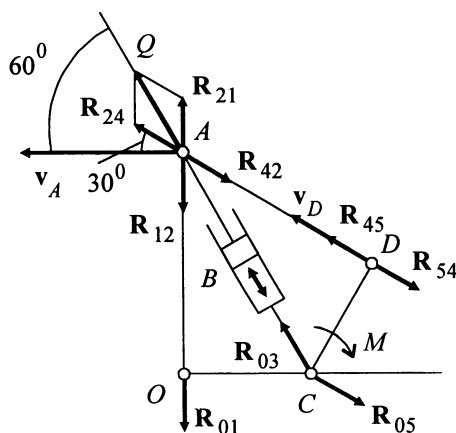


Fig. 4.28. Diagram of forces in problem 4.2

- 4.3. In Fig. 4.29 two diagrams of a power press are presented in positions when the workload $\mathbf{P}$ approaches the maximum value. The lengths of the corre-

spending links are equal. Find and compare the generalized balancing forces Q and Q^* for $\varphi = 5^\circ$; choose the preferable scheme.

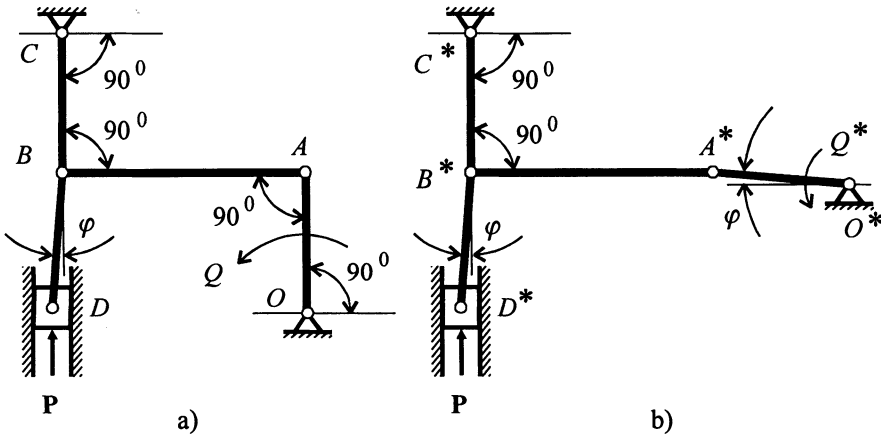


Fig. 4.29

Answer: $Q^* = Q \sin \varphi = 0.087Q$; the preferable scheme is the one in diagram b)

- 4.4. In the beam balance (Fig. 4.30) the mass of the measured load 1, lying on platform 2, is compared with the reference mass 3. The mass of body 1 is determined by the position of mass 3 on lever 4 when the mechanism of the beam balance is in equilibrium.

Find the relationship between arms O_1A , O_1B , O_2D and O_2C for which the moment of the weight of body 1 on lever EO_2 about pivot O_2 does not depend on the position of the body on platform 2.

Answer: $\frac{O_1A}{O_1B} = \frac{DO_2}{CO_2}$

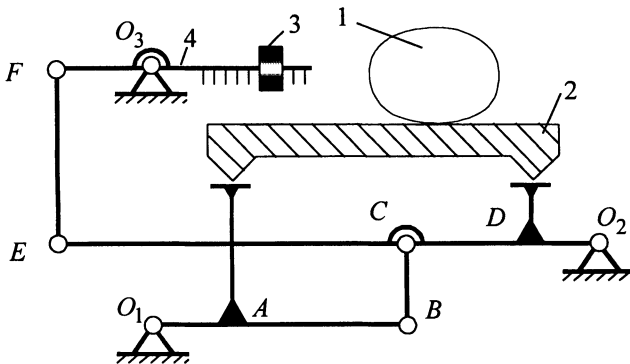


Fig. 4.30

- 4.5. In order to exclude the normal component of reaction acting on the slider, in some piston engines a design technique is used ensuring the presence of guaranteed side clearance during the displacement of the piston relative to the cylinder (Fig. 4.31).

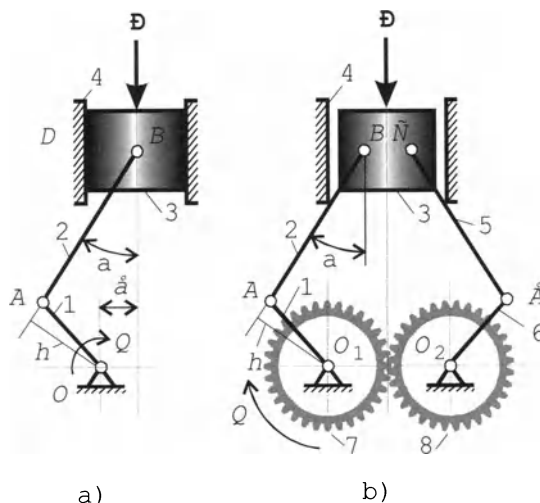


Fig. 4.31

Find and compare the reactions at the revolute and prismatic pairs as well as the balancing generalized forces Q in the mechanisms, designed according to diagrams a) and b) for equal workloads P . Consider the masses of the links to be equal to zero.

Answer:

For diagram a)

$$R_{43} = P \tan \alpha, \quad R_{12} = R_{23} = P / \cos \alpha, \quad Q = R_{21}h = R_{21}(x_B \sin \alpha - e \cos \alpha);$$

For diagram b)

$$R_{43} = 0, \quad R_{12} = R_{23} = P / (2 \cos \alpha), \quad Q = 2R_{21}h.$$

- 4.6. In the rear-dump truck shown in Fig. 4.32a the slotted-link mechanism of Fig. 4.32b is used. Body 3 is hinged to the support truck-slider 5 and to tractor 6 with the help of the tie-rod 4. During the turn-over of the body with the help of a hydraulic cylinder (links 1 and 2), truck 5 is pulled towards tractor 6 thus producing a large tilt angle of the body.

Find the generalized balancing force Q for the lifted body in the mechanism position shown in Fig. 4.32 if the vertical workload G is applied at point S of body 3 where $CS = BC = BD/3$.

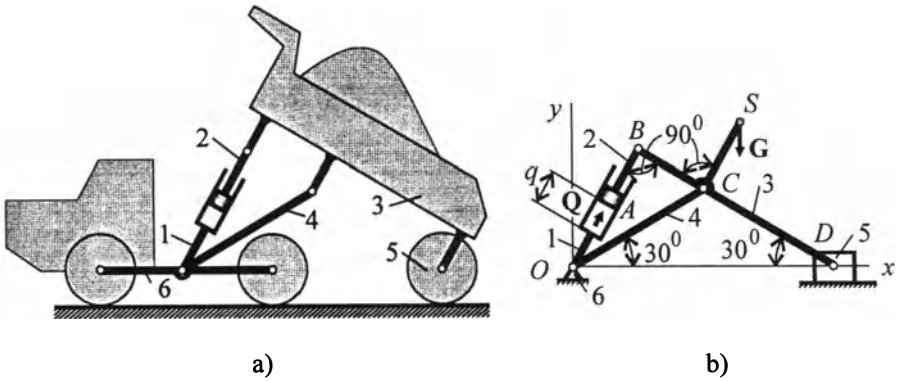


Fig. 4.32

Note: The relationship $\frac{dy_s}{dq}$ can be obtained from the velocity vector diagram shown in Fig. 4.33: $\frac{dy_s}{dq} = \frac{vs}{ab}$, where ab is the generalized velocity $\dot{q}$ and vs is the projection on the vertical axis of the velocity of point s (in the given mechanism position this velocity is directed vertically).

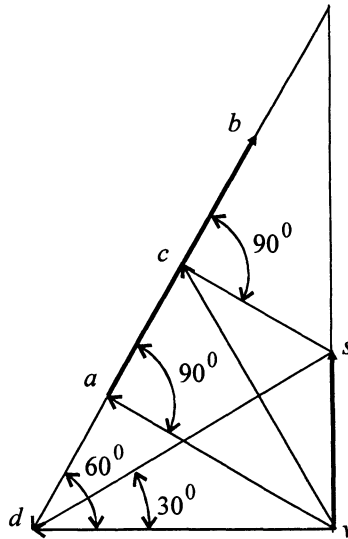


Fig. 4.33. Vector diagram of velocities for problem 4.6

$$\text{Answer: } Q = G \frac{dy_s}{dq} = G \tan 30^\circ = G \frac{1}{\sqrt{3}} \approx 0,577G.$$

- 4.7. Determine the reactions at the kinematic pairs of a six-bar linkage and the generalized balancing force Q , if the load moment $M = 400 \text{ N m}$, $OA = 0.1 \text{ m}$, $AB = 0.3 \text{ m}$, $CB = 0.4 \text{ m}$, $CD = 0.2 \text{ m}$, $DE = 0.4/\sqrt{3} \approx 0.23 \text{ m}$, $FE = 0.4 \text{ m}$, $q = 180^\circ$, $\varphi_2 = 30^\circ$, $\varphi_3 = 90^\circ$, $\varphi_4 = 300^\circ$, $\varphi_5 = 210^\circ$ (Fig. 4.34).

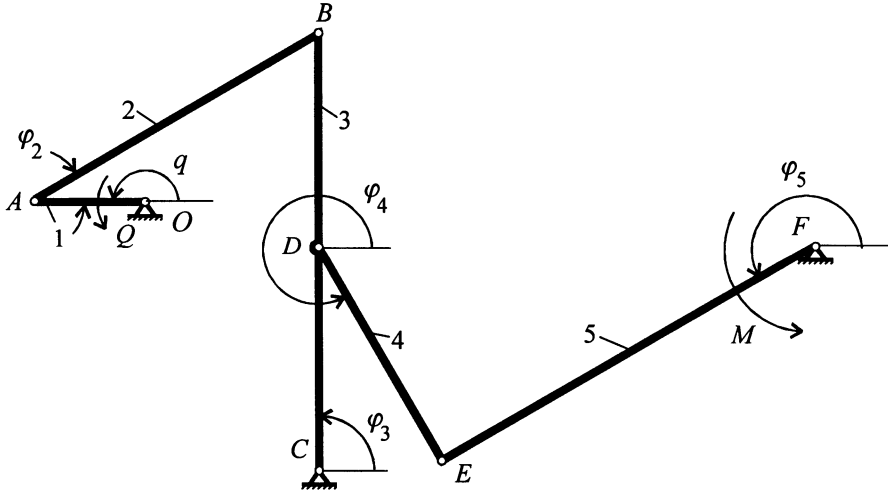


Fig. 4.34

Answer: $Q = -MJ_Q/J = 25/\sqrt{3} \approx 14.43 \text{ N m}$, $R_{01} = R_{12} = R_{23} = M|J_1/J| = 500\sqrt{3} \text{ N} \approx 866.03 \text{ N}$, $R_{34} = R_{45} = R_{50} = M|J_3/J| = 1000 \text{ N}$, $R_{03} = M\sqrt{J_1^2 + J_3^2 - 2J_1J_3\cos(\varphi_2 - \varphi_4)}/J = 500\sqrt{39}/3 \approx 1040.83 \text{ N}$, where

$$J = AB \cdot CB \cdot DE \cdot EF \cdot \sin(\varphi_2 - \varphi_3) \sin(\varphi_4 - \varphi_5),$$

$$J_Q = OA \cdot AB \cdot DE \cdot CD \cdot \sin(\varphi_2 - q) \sin(\varphi_4 - \varphi_3),$$

$$J_1 = AB \cdot DE \cdot CD \cdot \sin(\varphi_4 - \varphi_3), \quad J_3 = AB \cdot CB \cdot DE \cdot \sin(\varphi_2 - \varphi_3).$$

- 4.8. Find the generalized balancing forces Q_1 and Q_2 as well as the reactions at the kinematic pairs of a mechanism of movability two (Fig. 4.35), if $OA = 0.2 \text{ m}$, $AD = DM = MB$, $AB = 0.3 \text{ m}$, $BC = (0.5 + 0.1\sqrt{3}) \text{ m} \approx 0.673 \text{ m}$, $q_1 = 60^\circ$, $q_2 = 0$, $P = 100 \text{ N}$.

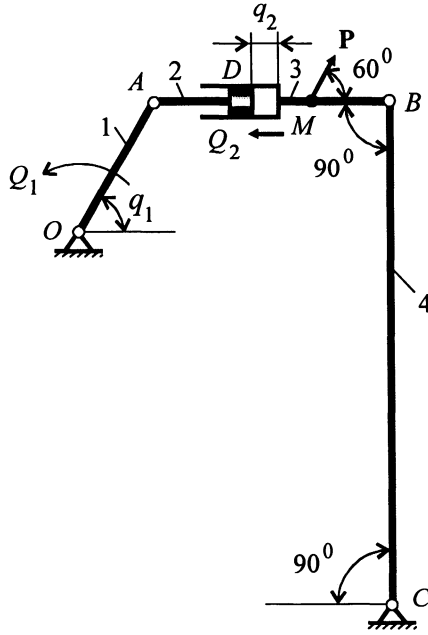


Fig. 4.35

$$\text{Answer: } Q_1 = \frac{10}{\sqrt{3}} \approx 5.77 \text{ N m}, \quad Q_2 = 50 \text{ N}, \quad R_{01} = R_{12} = R_{04} = \frac{100}{\sqrt{3}} \approx 57.74 \text{ N}, \quad R_{23} = \frac{50}{\sqrt{3}} \approx 28.87 \text{ N}, \quad M_{23}^R = \frac{5}{\sqrt{3}} \approx 2.89 \text{ N m}.$$

- 4.9. Determine the balancing forces Q_1 and Q_2 and the reactions at the kinematic pairs of a loader (Fig. 4.36). The workload is $P = 20 \text{ kN}$. The geometric parameters of the mechanism are: $AO = 0.8\sqrt{8} \text{ m} \approx 1.386 \text{ m}$, $BA = 1.2 \text{ m}$, $BE = \left(\frac{4.4\sqrt{3}}{3} - 1.6 \right) \text{ m} \approx 0.940 \text{ m}$, $CD = \left(\frac{124 - 3.2\sqrt{3}}{3} \right) \text{ m} \approx 2.286 \text{ m}$, $DE = 0.4\sqrt{3} \text{ m}$, $FG = 1.6 \text{ m}$, $GH = 0.8 \text{ m}$, $EH = \frac{0.8(6 - \sqrt{3})}{3} \text{ m} \approx 1.138 \text{ m}$, $EF = \frac{1.6\sqrt{3}}{3} \text{ m} \approx 0.924 \text{ m}$, $\gamma = 90^\circ$, $x_0 = y_0 = 0$, $x_B = 0.8\sqrt{3} \approx 1.386 \text{ m}$, $y_B = 1.2 \text{ m}$, $x_C = 0$, $y_C = \frac{0.8(2\sqrt{3} - 1)}{3} \text{ m} \approx 0.657 \text{ m}$, $x_M = 2.8\sqrt{3} \text{ m} \approx 4.84 \text{ m}$, $y_M = 2 \text{ m}$.

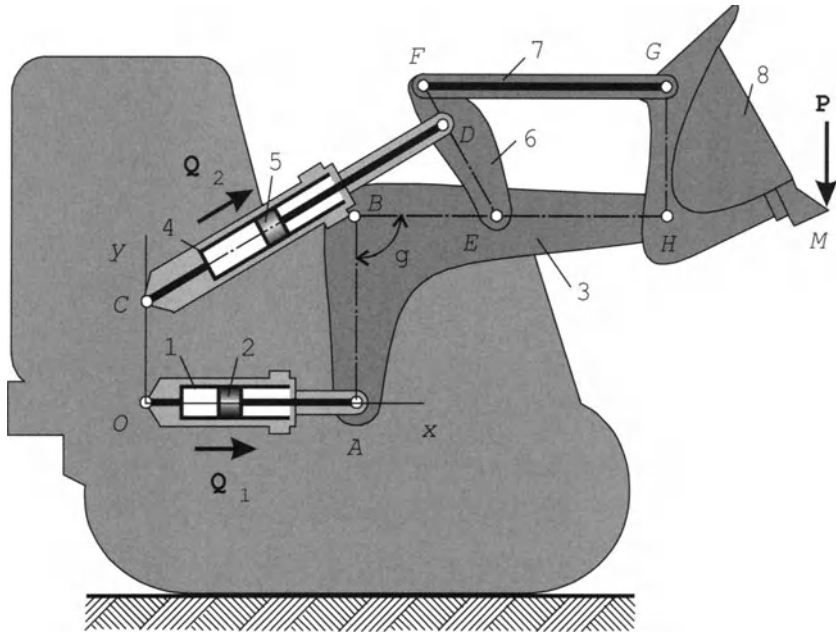


Fig. 4.36

Answer: $R_{78} = 20\sqrt{3} \text{ kN} \approx 34.6 \text{ kN}$, $R_{38} = R_{56} = R_{04} = Q_2 = 40 \text{ kN}$, $R_{36} = 20 \text{ kN}$, $R_{03} \approx 43 \text{ kN}$, $R_{01} = R_{23} = Q_1 \approx 50.3 \text{ kN}$.

- 4.10. In Fig. 4.37 the kinematic diagram of an earth-moving machine is shown. On shovel 4 the vertically directed earth resistance force $P = 40(\sqrt{3} - 1) \text{ kN} \approx 29.28 \text{ kN}$ is acting. Determine the reactions at the kinematic pairs and the generalized balancing forces Q_1 , Q_2 , Q_3 for the following output data: $OA = 0.56 \text{ m}$, $AB = 0.72 \text{ m}$, $BC = 0.8 \text{ m}$, $CI = 1.6 \text{ m}$, $CG = ED = 0.8(\sqrt{3} - 1) \text{ m} \approx 0.586 \text{ m}$, $BD = 0.8\sqrt{3} \text{ m} \approx 1.386 \text{ m}$, $EF = FG = 0.6 \text{ m}$, $x_0 = y_0 = x_I = 0$, $y_I = 0.8 \text{ m}$, $\gamma_1 = 120^\circ$, $\gamma_2 = 90^\circ$, $DK = 0.8\sqrt{3} \text{ m} \approx 1.386 \text{ m}$, $MK = 1.2 \text{ m}$, $q_1 = 0.32 \text{ m}$, $q_2 = 120^\circ$, $q_3 = 0.4 \text{ m}$.

Answer: $R_{34} = 40\sqrt{10 - 4\sqrt{3}} \text{ kN} \approx 70.11 \text{ kN}$, $Q_3 = -80 \text{ kN}$, $R_{54} = 80 \text{ kN}$, $R_{65} = 0$, $M_{65}^R = 0$, $R_{76} = 80 \text{ kN}$, $R_{07} = 40\sqrt{124 - 62\sqrt{3}} \text{ kN} \approx 163.04 \text{ kN}$, $Q_2 = 96(3 - \sqrt{3}) \text{ kN} \approx 121.72 \text{ kN}$, $Q_1 = 240(\sqrt{3} - 1) \text{ kN} \approx 175.69 \text{ kN}$.

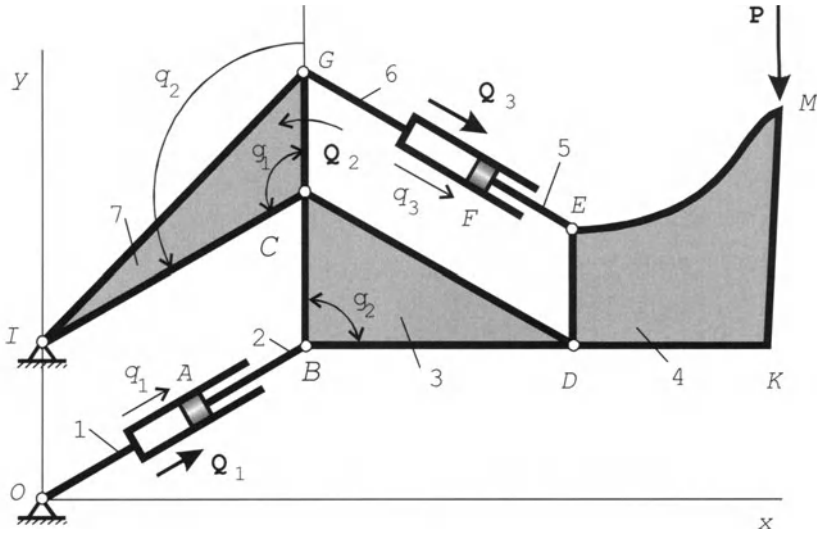


Fig. 4.37

- 4.11. Unbalance of a car wheel can cause deviation of the car from the straight line or vibration. For dynamic balancing of wheels balancing weights m are attached to the wheels (Fig. 4.38).

Determine the inertia forces arising in the wheel when a balancing weight of mass $m = 10 \text{ g}$ is lost, if the car is moving with constant velocity of 60 km/h without slipping. The wheel mass without the balancing weight is $M = 10 \text{ kg}$. The diameter of the rim to which the weight is attached is $d = 350 \text{ mm}$, the wheel diameter is $D = 650 \text{ mm}$, the wheel width is $b = 165 \text{ mm}$.

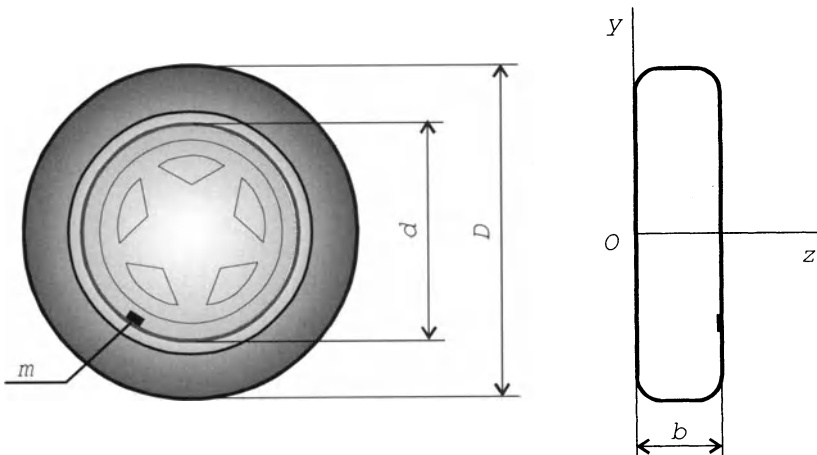


Fig. 4.38

Answer: The resultant vector of inertia forces is $|\Phi| = 4.899 \text{ N}$, the resultant moment of inertia forces is $M_0^\Phi = 0.808 \text{ N m}$.

- 4.12. The residual unbalance of a gyroscope rotor is estimated to be $me = 1 \text{ g cm}$ (Fig. 4.39). Determine the reactions at the shaft bearings of the gyroscope rotor as functions of its inertial load if the rotor rotates with a constant angular velocity of $\omega = 3000 \text{ s}^{-1}$.

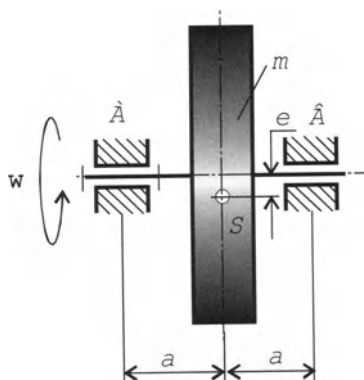


Fig. 4.39

Answer: $R_A = R_B = 45 \text{ N}$.

- 4.13. After installation of the faceplate 1 on the spindle 2, an end play of the faceplate is measured (Fig. 4.40). The pointer of indicator 3 installed at radius $R = 500 \text{ mm}$ deviates by 0.5 mm .

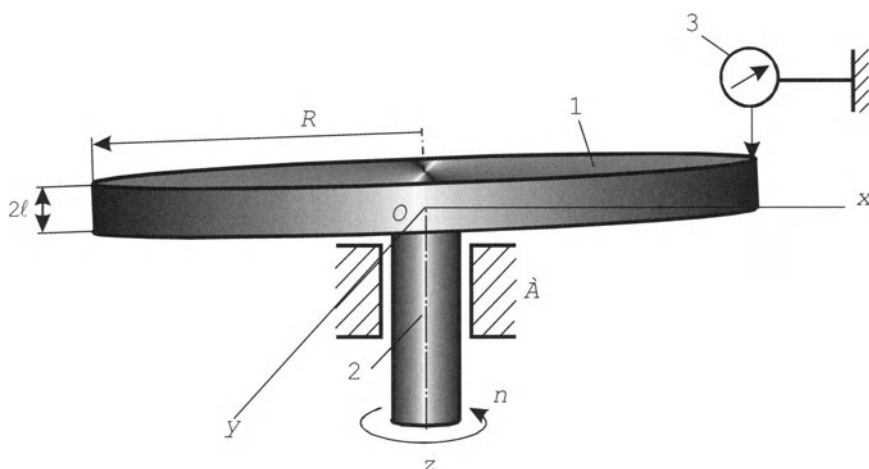


Fig. 4.40

Determine the moment of inertia forces acting on the faceplate if it rotates with a constant velocity $n = 500$ 1/min. The faceplate mass is $M = 100$ kg, the mass center is on the rotation axis, the thickness is $2l = 50$ mm.

Answer: $M_{Ox}^\phi = 0$, $M_{Oy}^\phi = 8.565$ N m, $M_{Oz}^\phi = 0$.

- 4.14. A planar cam mechanism consists of cam 1 and of a translationally moving follower 2 (Fig. 4.41). Derive the equations of kinetostatics for this mechanism.

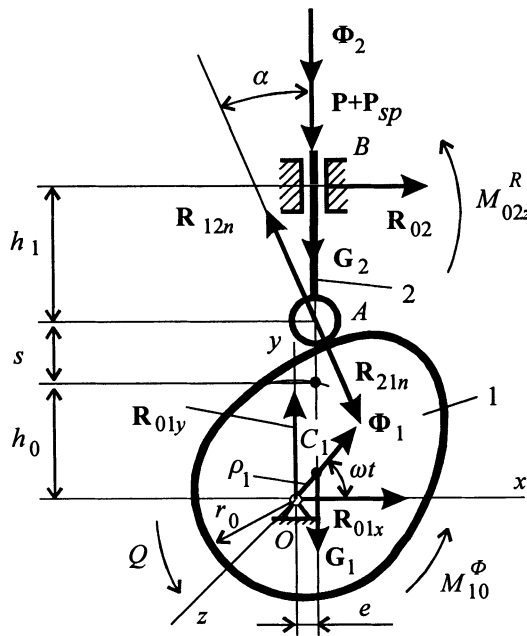


Fig. 4.41

Solution: The mechanism has two lower pairs and one higher pair. In the motion plane there are two unknown components of reactions at the revolute pair – R_{01x} and R_{01y} , at the prismatic pair – R_{02} and M_{02}^R , and at the higher kinematic pair – normal force $R_{12n} = -R_{21n}$. Together with the generalized force Q we have six unknowns. In order to find them we can write six equations of kinetostatics which under uniform rotation of the cam have the following form:

$$R_{01x} + \Phi_1 \cos \omega t + R_{12n} \sin \alpha = 0,$$

$$R_{01y} + \Phi_1 \sin \omega t - R_{12n} \cos \alpha - G_1 = 0,$$

$$Q - R_{12n} e \cos \alpha - R_{12n} \sin \alpha (h_0 + s) - G_1 \rho_1 \sin \omega t = 0,$$

$$R_{02} - R_{12n} \sin \alpha = 0,$$

$$R_{12n} \cos \alpha - P - P_{sp} - \Phi_2 - G_2 = 0,$$

$$-R_{12n} h_1 \sin \alpha + M_{02z}^R = 0.$$

Here ωt is the angle between radius OC_1 (C_1 is the mass center of the cam) and the x -axis, $\rho_1 = OC_1$, α is the pressure angle, Φ_1 and Φ_2 are inertia forces of the cam and the follower, G_1 and G_2 are gravity forces, P_{sp} is a force generated by the spring, pressing the follower against the cam.

- 4.15. For an involute spur cylindrical gearing determine the reactions lying in the motion plane (Fig. 4.42) and the generalized force Q .

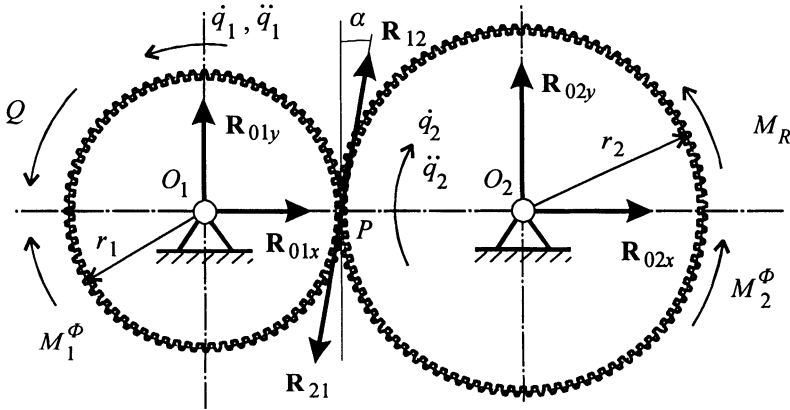


Fig. 4.42

Solution: Denote by $\dot{q}_1$ and $\dot{q}_2$ the angular velocities of gears with $\dot{q}_2 = \dot{q}_1/i$, where i is the transmission ratio; r_1 and r_2 are the radii of the initial circles; M_R is the moment of resistance forces; α is simultaneously the meshing angle and the pressure angle; $M_1^\phi = -J_{1z}\ddot{q}_1$ and $M_2^\phi = -J_{2z}\ddot{q}_2$ are the moments of inertia forces (it is assumed that the mass centers lie on the rotation axes); R_{12} is the normal reaction at the higher kinematic pair, applied to the second gear ($R_{21} = -R_{12}$). Deriving the equations of kinetostatics we have:

a) For wheel 1:

$$R_{01x} - R_{21} \sin \alpha = 0; \quad R_{01y} - R_{21} \cos \alpha = 0; \quad Q - R_{21} r_1 \cos \alpha - M_1^{\phi} = 0;$$

b) For wheel 2:

$$R_{02x} + R_{12} \sin \alpha = 0, \quad R_{02y} + R_{12} \cos \alpha = 0, \quad R_{12} r_2 \cos \alpha - M_2^{\phi} - M_R = 0.$$

Solving these equations we find (taking into account that $r_2 = r_1 i$, $\ddot{q}_2 = \ddot{q}_1 / i$): $Q = (J_{1z} + J_{2z} / i^2) \ddot{q}_1 + M_R / i$, $R_{12} = (J_{2z} \ddot{q}_1 / i + M_R) (i r_1 \cos \alpha)^{-1}$, $R_{01x} = R_{12} \sin \alpha$, $R_{01y} = R_{12} \cos \alpha$; $R_{02x} = -R_{12} \sin \alpha$, $R_{02y} = -R_{12} \cos \alpha$.

- 4.16. Slider 1 in a double-slider mechanism (Fig. 4.43) has coordinate $q = 0.4$ m, velocity $\dot{q} = \sqrt{2}$ m/s, acceleration $\ddot{q} = 10/9$ m/s². Determine the driving force Q , if the masses of the links are $m_1 = 9$ kg, $m_2 = 0$, $m_3 = 3$ kg. The length of the connecting rod is $AB = \ell = 0.5$ m. Do not take gravity into consideration.

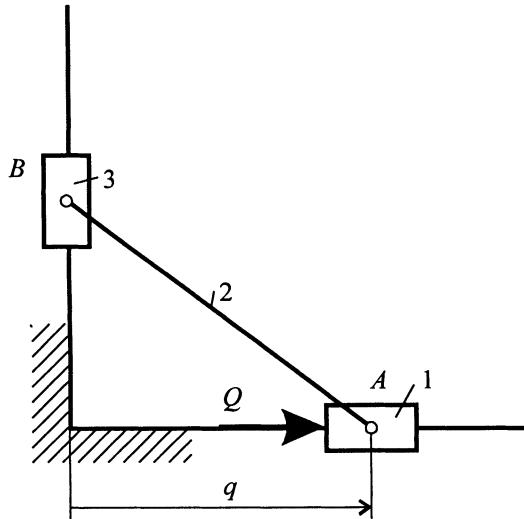


Fig. 4.43

$$\text{Answer: } Q = m_1 \ddot{q} + \frac{m_3 q}{(\ell^2 - q^2)} \left[(\ell \dot{q})^2 + q(\ell^2 - q^2) \ddot{q} \right] = 90 \text{ N.}$$

- 4.17. The translationally moving follower 2 of a central cam mechanism (Fig. 4.44) with mass $m_2 = 100$ g has a position function $S = 0.5 S_{\max} (1 - \cos \frac{\pi q}{q_y})$ along the stroke length, where q , q_y are, respectively, the current angle of cam rotation and its value corresponding to the end of the distance leg.

Define the cam rotation angle q^* for which the driving moment Q has maximum value. If the angular velocity of the cam is constant and equal to $\dot{q} = 100 \text{ 1/s}$, what is the driving moment? The maximum travel distance of the follower is $S_{\max} = 30 \text{ mm}$.

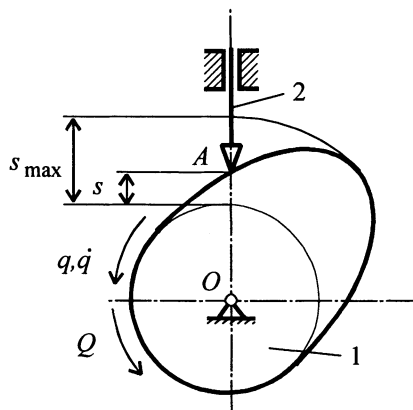


Fig. 4.44

$$\text{Answer: } Q_{\max} = \frac{\pi^3 m_2 \dot{q}^2 S_{\max}^2}{8 q_y^2} = 0.38 \text{ N m}, \quad q_{1*} = 30^\circ, \quad q_{2*} = 90^\circ.$$

- 4.18. Determine the generalized driving forces ensuring a program motion of a manipulator of movability two (Fig. 4.45). Compute these forces for the case when the second link is motionless, if $m_1 = 15 \text{ kg}$, $m_2 = 10 \text{ kg}$, $\ell = 0.5 \text{ m}$, $q_2 = 30^\circ$, $\ddot{q}_1 = 2 \text{ m/s}^2$. Neglect gravity.

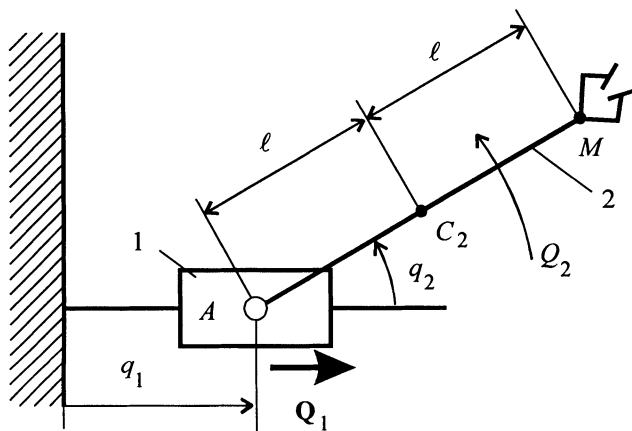


Fig. 4.45

Answer: $Q_1 = (m_1 + m_2)\ddot{q}_1 - m_2\ell\ddot{q}_2 \sin q_2 - m_2\ell\dot{q}_2^2 \cos q_2 = 50 \text{ N}$,
 $Q_2 = (m_2\ell^2 + J_{C2})\ddot{q}_2 - m_2\ell\ddot{q}_1 \sin q_2 = -5 \text{ N m}$.

- 4.19. Determine the driving moments Q_1 and Q_2 in a mechanism of movability two (Fig. 4.46), if $q_1 = 60^\circ$, $q_2 = 240^\circ$, $\dot{q}_1 = 0$, $\dot{q}_2 = \sqrt{\frac{\sqrt{3}}{2}} \text{ s}^{-1}$, $\ddot{q}_1 = 1 \text{ s}^{-2}$, $\ddot{q}_2 = 0.5 \text{ s}^{-2}$, $r = AB = 0.5 \text{ m}$, $\ell = BC = 1 \text{ m}$, the masses of the links are $m_1 = m_4 = 6 \text{ kg}$, $m_2 = m_3 = 10 \text{ kg}$, the axial moments of inertia are $J_2 = 1 \text{ kg m}^2$, $J_3 = 2 \text{ kg m}^2$. The mass center of link 2 coincides with point A , and the mass center of link 3 with point C . Neglect gravity.

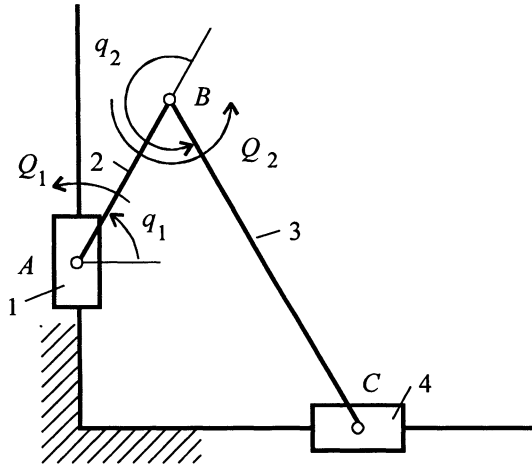


Fig. 4.46

Answer:

$$Q_1 = m(r^2 + \ell^2)\ddot{q}_1 + 2mr\ell \cos q_2 \ddot{q}_1 + m\ell^2\ddot{q}_2 + mr\ell \cos q_2 \ddot{q}_2 -$$

$$2mr\ell \sin q_2 \dot{q}_1 \dot{q}_2 - mr\ell \sin q_2 \dot{q}_2^2 + J_2\ddot{q}_1 + J_3(\ddot{q}_1 + \ddot{q}_2) = 16 \text{ N m},$$

$$Q_2 = m\ell^2\ddot{q}_1 + mr\ell \cos q_2 \ddot{q}_1 + m\ell^2\ddot{q}_2 + mr\ell \sin q_2 \dot{q}_1^2 + J_3(\ddot{q}_1 + \ddot{q}_2) = 23 \text{ N m},$$

where $m = m_1 + m_2 = m_3 + m_4$.

5 Friction in Mechanisms

5.1

Friction in Kinematic Pairs

At kinematic pairs of real mechanisms friction forces are acting. In many cases these forces have a major influence on the mechanism motion and must be taken into account in the force analysis.

Let S be the contact surface of the elements of a kinematic pair (Fig. 5.1). On this surface we select an elementary area dS in a neighbourhood of some point A . Let us consider the interaction forces generated in this neighbourhood and applied to one of the links of the kinematic pair. We resolve the resultant vector of these forces into components: dN – directed along the normal to surface S , and dF – lying in the tangent plane. We also resolve the resultant moment about point A into a normal component dM_p and a tangent component dM_r . Force dF is called *frictional force of sliding*, moment dM_r *frictional moment of rolling*, and moment dM_p *frictional moment of pivoting*. By their physical nature frictional forces are forces of resistance against motion. It follows that force dF is directed opposite to the vector of relative velocity v_{sl} (velocity of sliding) at point A , and vectors dM_r and dM_p are directed opposite to the tangent component ω_t and

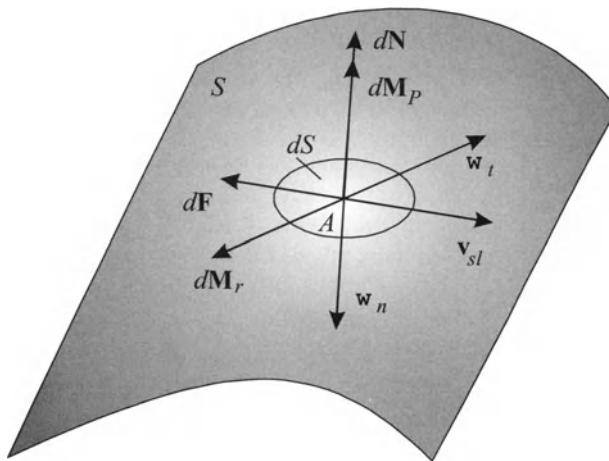


Fig. 5.1. Contact surface of the elements of a higher kinematic pair

to the normal component ω_n , respectively, of the vector of relative angular velocity.

Numerous experimental investigations have shown that the force analysis of mechanisms can be based, in most cases, on the law of dry friction, which is known in physics under the name *Amontons–Coulomb law*. In accordance with this law, the magnitudes of the frictional force dF and the moments dM_r and dM_p are assumed to be proportional to the modulus of the normal component of reaction dN :

$$|dF| = f|dN|, \quad |dM_r| = k|dN|, \quad |dM_p| = k_p|dN|, \quad (5.1)$$

where f is a dimensionless *coefficient of sliding friction*, while k and k_p are, respectively, *coefficients of rolling* and *of pivoting friction*, measured in centimeters.

From (5.1) and from the above assumptions about the directions of the forces and moments follow the vector relationships:

$$d\mathbf{F} = -f|dN|\frac{\mathbf{v}_{sl}}{|\mathbf{v}_{sl}|}, \quad d\mathbf{M}_r = -k|dN|\frac{\boldsymbol{\omega}_r}{|\boldsymbol{\omega}_r|}, \quad d\mathbf{M}_p = -k_p|dN|\frac{\boldsymbol{\omega}_n}{|\boldsymbol{\omega}_n|}. \quad (5.2)$$

Formulae (5.1) and (5.2) can directly be used for the determination of frictional forces at a higher kinematic pair with point contact. In the case of lower kinematic pairs and higher pairs with contact along a line, the resultant vector and the resultant moment of frictional forces are determined by integration of forces and moments, acting at elementary areas of the contact surface or along the contact line. So, e.g., the total frictional force at a lower kinematic pair can be determined according to the formula

$$\mathbf{F} = \int_S d\mathbf{F} = \int_S f|dN|\frac{\mathbf{v}_{sl}}{|\mathbf{v}_{sl}|}, \quad (5.3)$$

where S is the contact surface. For making use of this formula it is necessary to know the distribution law of the normal reactions over the surface S .

The friction coefficients of sliding, pivoting and rolling are experimentally determined; they depend on many factors: the properties of the material from which the contacting elements of kinematic pairs are made; the surface roughness; the presence of lubricant and its properties; and at last, the magnitude of the relative velocity and the relative angular velocity of links. In mechanics of machines these coefficients are supposed to be given and constant.

Formulae (5.1) and (5.2) are not applicable, if the sliding velocity at the contact point and the relative angular velocity are equal to zero, i.e., if the links constituting a kinematic pair are in a state of relative rest. In this case, the resultant forces and moments of frictional forces at a kinematic pair can be determined from the equilibrium conditions for links; moreover, they prove to be depending not on normal reactions but on directly applied external forces.

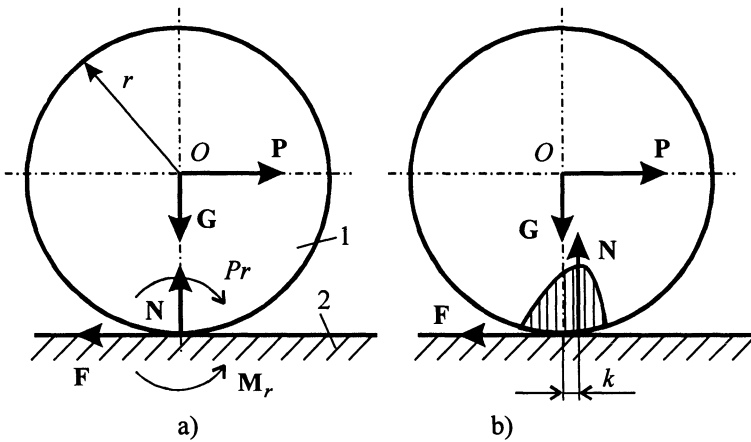


Fig. 5.2. Kinematic pair cylinder-plane

Let us explain by an example what has been said. In Fig. 5.2 a kinematic pair, constituted by cylinder 1 and plane 2 is shown. The gravity force G of the cylinder is balanced by the normal reaction N , which is the resultant of the elementary normal forces arising at the contact points lying on a cylinder element. Applying a horizontal external force P to the cylinder axis we find that when this force is sufficiently small the cylinder remains at rest. This means that force P is balanced by the horizontal component of reaction F , while moment Pr is balanced by moment M_r , whose vector is directed along an element of the cylinder. In this way,

$$F = P, \quad M_r = Pr. \quad (5.4)$$

Force F and moment M_r can occur only due to frictional forces with magnitudes that are determined by the magnitude of the force P alone, and that do not depend on N as is clear from formula (5.4). However, when increasing P , we find that for a certain value the state of rest will be violated. If the force P reaches a magnitude for which the condition

$$P \leq \frac{k}{r} N \quad (5.5)$$

is violated, where k is the coefficient of rolling friction, then the cylinder begins to roll over the plane without sliding. The sliding begins when the condition

$$P \leq f_s N \quad (5.6)$$

is violated, where f_s is the *coefficient of static friction* which usually somewhat exceeds the coefficient of sliding friction f . If $k/r < f_s$, then at first (with the increase of P) a rolling begins; if $k/r > f_s$, the opposite will be observed.

We also note that the occurrence of the moment M_r is related to the deformation of the cylinder and of the plane in the contact zone (see Fig. 5.2b) and to an unsymmetric distribution of normal forces, which causes a shift of their resultant N in the direction of the force vector P .

5.2

Models of Kinematic Pairs with Friction

The introduction of frictional forces leads to an increase in the number of unknown components of reactions at a kinematic pair, while the quantity of the equations of kinetostatics does not increase. For keeping the force analysis problem still solvable, it is necessary to introduce additional conditions equal in number to the number of additional unknowns. Most easily such conditions are introduced for a higher kinematic pair of first class (Fig. 5.3). Let the surfaces of the pair elements be deformed under the action of the normal force and let the surfaces touch each other in a small neighbourhood of point A . Furthermore, let the relative motion of links be defined by the specification of the sliding velocity $\mathbf{v}_{sl}$ and of the relative angular velocity ω . Let the z -axis be directed along the common normal to the surfaces at point A , and the x -axis along the line of action of vector $\mathbf{v}_{sl}$. Thus, all components of reactions are expressed through the magnitude of the normal force N . Using relationship (5.1), we find

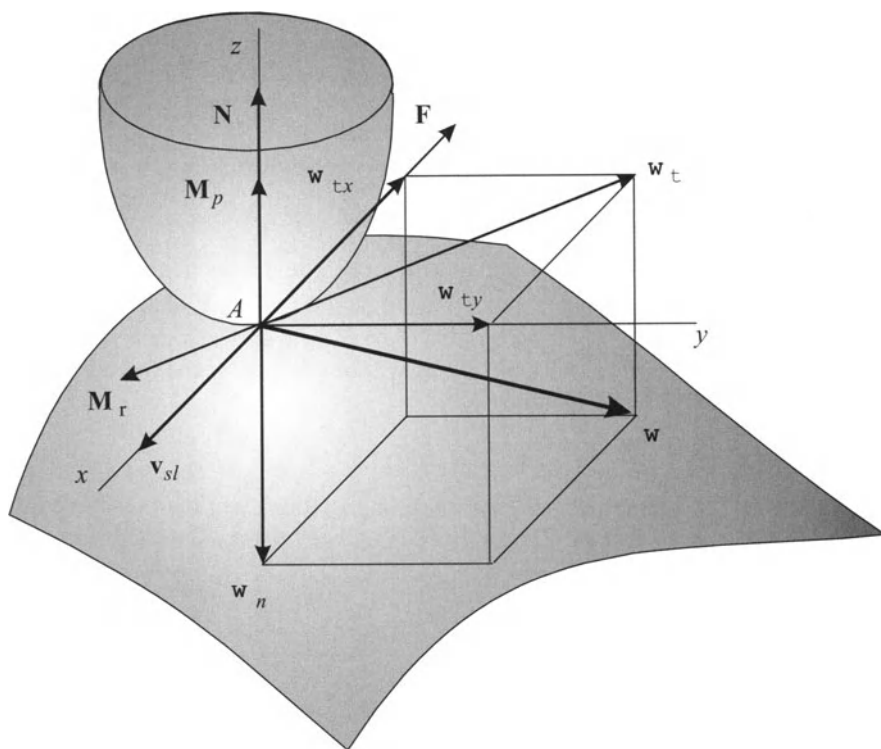


Fig. 5.3. Higher kinematic pair of first class

$$\begin{aligned}
 R_x = F_x &= -f|N|\frac{|\mathbf{v}_{sl}|}{|\mathbf{v}_{sl}|}, \quad R_y = 0, \quad R_z = N, \\
 M_{Ax}^R = M_{rx} &= -k|N|\frac{\omega_x}{|\omega_\tau|}, \quad M_{Ay}^R = M_{ry} = -k|N|\frac{\omega_y}{|\omega_\tau|}, \\
 M_{Az}^R = M_p &= -k_p|N|\frac{\omega_n}{|\omega_n|},
 \end{aligned} \quad (5.7)$$

where ω_τ is the component of the angular velocity lying in plane xAy , while ω_x , ω_y are its projections onto the x - and the y -axis, respectively. Formulae (5.7) express five components of reactions as functions of the sixth component.

The formulation of analogous relationships for pairs with fewer degrees of movability is a complex problem, since, in general, the law of distribution of normal reactions over the surface or the line of contact remains unknown. Usually, additional conditions are introduced which take into account design peculiarities of the elements of kinematic pairs. They allow to assume certain a priori hypotheses about the character of the distribution of normal reactions.

Let us consider some examples how to formulate such conditions. Essentially these examples show how to create dynamic models for the analysis of kinematic pairs with friction.

a) A prismatic pair in a planar mechanism. In the force analysis of a planar mechanism the problem of determining reactions lying in the motion plane is treated as follows. In the case of an ideal prismatic pair (Fig. 5.4), the unknown components of reactions are moment M_{Oz}^R and force R_y , orthogonal to the line of the slider motion. In the presence of friction one more component R_x occurs. In order for the problem of force analysis to keep soluble, this third component must be expressed in terms of the other two or, alternatively, all three components

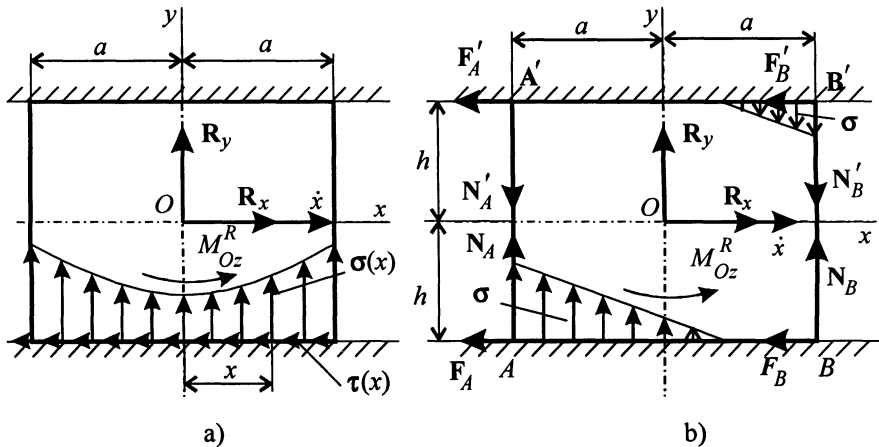


Fig. 5.4. Model of a planar prismatic kinematic pair with friction

must be expressed in terms of two new parameters. This can be achieved in different ways based on different assumptions about the character of the distribution of normal forces over the contact surface. Let us assume, first, that these forces are somehow distributed over only one of the two contacting planes, e.g. over the lower one (Fig. 5.4a). If $\sigma(x)$ is the normal force per unit length of the contact line at the point with coordinate x , then the specific frictional force τ , acting at this point is determined from the expressions (5.1) and (5.2):

$$\tau(x) = -f|\sigma(x)|\text{sign}\dot{x}. \quad (5.8)$$

Here, the function $\text{sign}\dot{x}$ indicates that the frictional forces acting on the slider are directed opposite to the slider velocity. From (5.8) we obtain

$$R_x = \int_{-a}^a \tau(x) dx = -f \int_{-a}^a |\sigma(x)|\text{sign}\dot{x} dx = -f|R_y|\text{sign}\dot{x}, \quad (5.9)$$

since in the given case

$$\int_{-a}^a |\sigma(x)| dx = |R_y|.$$

Thus, we have obtained an expression relating the reaction R_x to R_y thereby reducing the number of unknown components of reactions to only two. Note that when the slider touches the guide at the upper plane, the sign of R_y will be opposite, while the sign of R_x remains the same because of (5.9). In what follows, it is convenient to write the expression (5.9) in the form

$$R_x = -f R_y \text{sign} R_y \cdot \text{sign}\dot{x} \quad (5.10)$$

where we have used the identity $|R_y| \equiv R_y \text{sign} R_y$.

The assumption that the contact of the slider with the guide occurs in only one of the planes is not always acceptable. Often, it is necessary to take into account the skewing of the slider which leads to contact in both planes (Fig. 5.4b). In this case,

$$\int_{-a}^a |\sigma(x)| dx \neq |R_y|,$$

and the expression (5.9) appears incorrect.

In this case, it is possible to use a different model for the prismatic pair. Let us conditionally assume that the normal forces in the contact surfaces can be replaced by two concentrated forces N_A and N_B applied to the end points of the slider. Depending on the distribution of the normal forces $\sigma(x)$, the forces N_A and N_B are applied either to points A and B or with opposite directions to points A' and

B' . Let us now express all three components of reactions of the kinematic pair in terms of the two parameters N_A and N_B . From Fig. 5.4b we obtain

$$R_y = N_A + N_B, \quad (5.11)$$

$$\begin{aligned} R_x = -(F_A + F_B) &= -f(|N_A| + |N_B|)\text{sign}\dot{x} = \\ &= -f(N_A\text{sign}N_A + N_B\text{sign}N_B)\text{sign}\dot{x}. \end{aligned} \quad (5.12)$$

Taking into account that in the process of changing the contact point from one surface to the other, the direction of the frictional force remains unchanged whereas the direction of the moment of this force about point O is reversed, we obtain

$$M_{Oz}^R = (N_B - N_A)a - fh(N_B + N_A)\text{sign}\dot{x}. \quad (5.13)$$

Note that in the absence of friction ($f = 0$), reactions N_A and N_B will have the same signs, if $|R_y| < a^{-1}|M_{Oz}^R|$. This condition can be considered, generally speaking, as a criterion, whether to use the models described by Eqs. (5.11)–(5.13).

b) A revolute pair in a planar mechanism. In Fig. 5.5 the dynamic models of revolute pairs with friction are shown when only components of reactions lying in the plane orthogonal to the joint axis are taken into account. In the model shown in Fig. 5.5a, it is assumed that the forces of normal interaction are concentrated at point A and that at this point the frictional force F is applied. Projecting the frictional forces onto the coordinate axes and defining their moments, we find

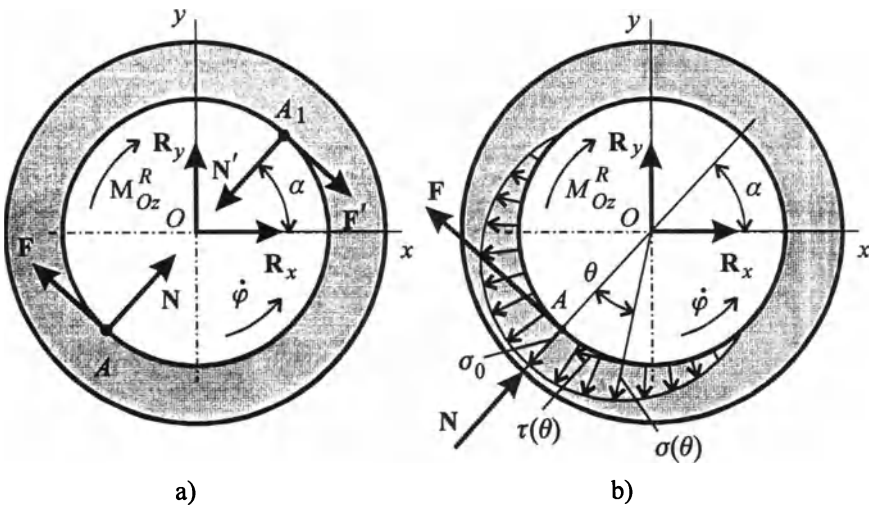


Fig. 5.5. Model of a planar revolute kinematic pair with friction

$$\begin{aligned}
 R_x &= N \cos \alpha - f N \sin \alpha, & R_y &= N \sin \alpha + f N \cos \alpha, \\
 M_{Oz}^R &= -fr|N|\text{sign}\dot{\phi} = -fr(1+f^2)^{-\frac{1}{2}}\sqrt{R_x^2 + R_y^2}\text{sign}\dot{\phi}.
 \end{aligned}
 \tag{5.14}$$

Here, α is the angle between the line of action of force N and the x -axis, and r is the radius of the journal. Formulae (5.14) take into account that a change of the sign of N leads to a change in the direction of force F , since its point of application moves to A_1 .

In the expression for the moment M_{Oz}^R the multiplier $-\text{sign}\dot{\phi}$ is introduced which indicates that the moment of the frictional forces at a revolute pair is directed opposite to the relative angular velocity. This expression also shows that the line of action of the resultant of the reaction forces at a revolute pair is tangent to the circle with the center point O and with radius $rf(1+f^2)^{-\frac{1}{2}}$.

In the model shown in Fig. 5.5b, it is assumed that the normal forces $\sigma(\theta)$ are distributed over a semi-circle and symmetrically about the point A . Usually, the distribution law is chosen in the form

$$\sigma(\theta) = \sigma_0 \cos \theta \quad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right).$$

The frictional forces are also distributed and

$$\tau(\theta) = f|\sigma(\theta)|.$$

Projecting the forces onto the radius AO and orthogonally to it and defining the force moment about point O , we find

$$\begin{aligned}
 N &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sigma(\theta) r d\theta \cos \theta = \sigma_0 r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta = \frac{\pi}{2} \sigma_0 r, \\
 F &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\sigma(\theta) r d\theta \cos \theta = \frac{\pi}{2} f\sigma_0 r, & M_{Oz}^R &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tau^2 d\theta = 2f\sigma_0 r^2.
 \end{aligned}$$

From this we obtain

$$\begin{aligned}
 R_x &= N \cos \alpha - fN \sin \alpha, & R_y &= N \sin \alpha + fN \cos \alpha, \\
 M_{Oz}^R &= -\frac{4}{\pi} fr|N|\text{sign}\dot{\phi}.
 \end{aligned}
 \tag{5.15}$$

Comparing these expressions with (5.14) we notice that they differ only by the factor $4/\pi$ of the moment of frictional forces. The first model is usually used for the analysis of kinematic pairs with significant clearances (e.g., worn out pairs).

c) Cylindric involute meshing (Fig. 5.6). Consider a gearing with involute meshing. Here, the normal force N_{12} is directed along the line of meshing MD under the angle α (angle of meshing) to the common tangent of the initial circles, adopted as y -axis. When the meshing is outside the pole P chosen as origin of the coordinates, then a frictional force F_{12} opposite to the relative velocity v_{21} is acting, i.e. orthogonally to MD . The moment of the rolling friction $M_r = k|N_{12}|$ is directed opposite to the relative angular velocity $\dot{q}_{21}$. It is combined with the moment of the force F_{12} about the origin of the coordinates. Projecting the forces onto the coordinate axes and adding the moments, we obtain the following equations for the model of a kinematic pair with friction:

$$\begin{aligned} R_x &= N_{12} \sin \alpha + f N_{12} \text{sign} N_{12} \cos \alpha \text{sign} \rho, \\ R_y &= N_{12} \cos \alpha - f N_{12} \text{sign} N_{12} \sin \alpha \text{sign} \rho, \\ M_{pz}^R &= -k N_{12} \text{sign} N_{12} \text{sign} \dot{q}_{21} + f N_{12} \text{sign} N_{12} \rho. \end{aligned} \quad (5.16)$$

Here, ρ is the length of the segment $\overline{PA}$ and $\text{sign} \rho = \text{sign}(\overline{MP} - \overline{MA})$. Eqs. (5.16) take into account that when the force N_{12} changes direction the force F_{12} retains its direction.

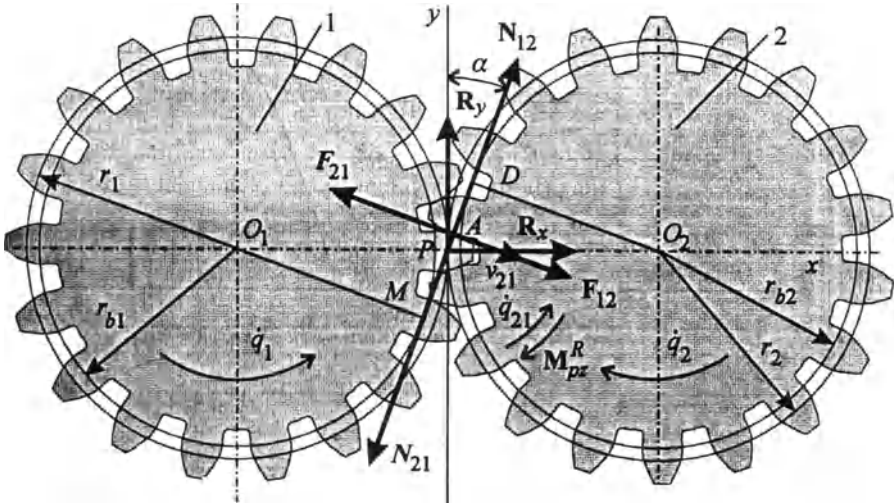


Fig. 5.6. Model of a cylindric involute meshing with friction

d) Worm meshing. In Fig. 5.7 the forces acting in a worm meshing are presented. It is assumed that the working surfaces of the worm turn and of the worm-wheel tooth are in contact at point B , i.e. it is conditionally assumed that a kinematic pair of movability five is formed by the meshing.

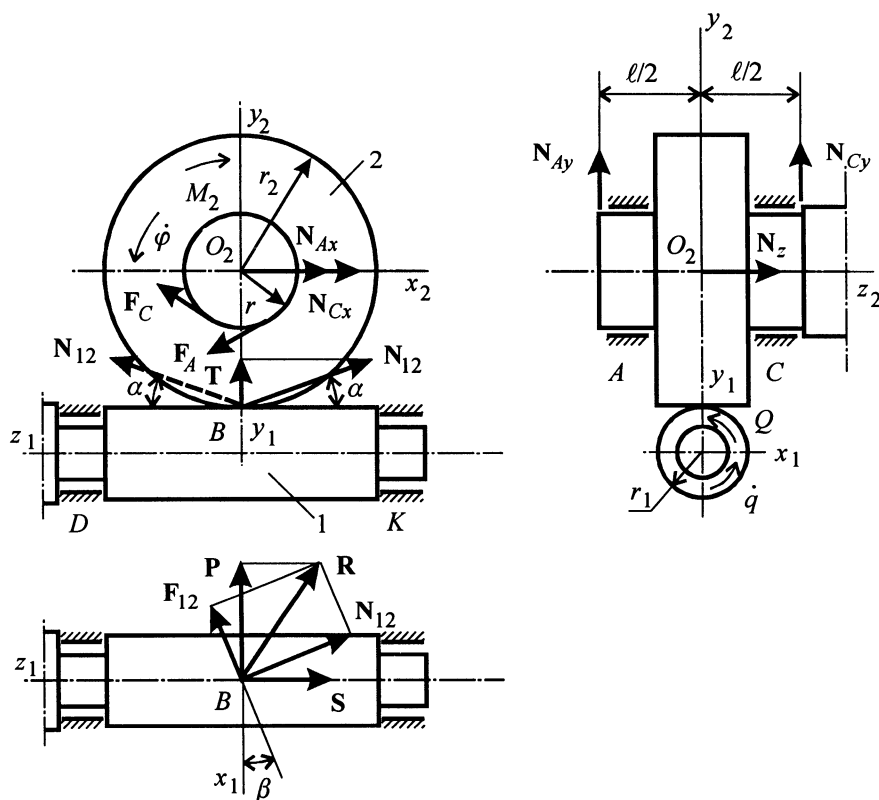


Fig. 5.7. Model of a worm gearing with friction at a higher kinematic pair

The normal force N_{12} (in the figure the force acting on the wheel is shown) makes an angle α with the plane parallel to the worm axis, and β is the helix angle of the turn on the worm. If the worm is leading, then force N_{12} is directed as shown in Fig. 5.7 by the solid line; when the wheel is leading (and the rotation of the worm is in the same direction), then the line of action of the force changes: It is shown by the dashed line in the figure. The force of the sliding friction F_{12} is directed opposite to the sliding velocity. We neglect the remaining frictional forces. Projecting the forces onto the coordinate axes we find the circle force S , the axial force P and the radial force T acting on the wheel:

$$\begin{aligned}
S = R_{x2} &= N_{12} \cos \alpha \cos \beta - F_{12} \sin \beta = \\
&N_{12} (\cos \alpha \cos \beta - f \operatorname{sign} N_{12} \sin \beta \operatorname{sign} \dot{q}), \\
P = -R_{z2} &= N_{12} \cos \alpha \sin \beta + F_{12} \cos \beta = \\
&N_{12} (\cos \alpha \sin \beta + f \operatorname{sign} N_{12} \cos \beta \operatorname{sign} \dot{q}), \\
T = R_{y2} &= N_{12} \sin \alpha.
\end{aligned} \tag{5.17}$$

When the moment M_2 , applied to the wheel, changes direction, then also the force N_{12} changes direction. The direction of the force F_{12} changes only with a sign change of $\dot{q}$.

5.3

Force Analysis of Mechanisms with Friction

The force analysis of mechanisms with friction reduces to solving the equations of kinetostatics including frictional forces in combination with the relationships (5.9)–(5.17) derived above for the mathematical models of kinematic pairs with friction. In the absence of excessive constraints the number of unknowns turns out to be equal to the number of equations. However, it is easy to see that the equations of the mathematical models of kinematic pairs with friction include nonlinear functions of reaction components, entering the equations (modulus, sign of reaction, and so on). Therefore, the complete system of equations of force analysis turns out to be nonlinear.

The nonlinearity of the equations causes various substantial complications of the solution procedure. In the first place, the procedure is more labour-consuming. As it is shown below, in a number of cases it is necessary to solve repeatedly systems of linear equations. In the second place, it may happen that for the investigated position of a mechanism with given kinematic parameters of motion and with given coefficients of friction, the force analysis equation system has no solution at all. From a physical point of view, this means that the investigated motion is inadmissible for the given mechanism with friction whatever the values of the driving forces are. In this case, we usually speak about *seizure* of the mechanism. A particular case of seizure is the effect of *self-braking*. It is impossible to bring the mechanism out of the state of rest no matter what force is applied to its input link. In such a mechanism an increase in the driving force causes an increase in the frictional forces, balancing its action. In the third place, the system of nonlinear equations may even have several solutions; in other words, under the same active forces the mechanism can execute a given motion with different driving forces and with different values of reactions. Usually, this occurs for such positions of the mechanism for which a self-braking is possible, but the active and the inertia forces have positive power, i.e. they „help“ the driving force, causing an effect of „*de-braking*“. Which of the solutions corresponds to the actual driving forces and reactions, cannot be decided in the framework of the original dynamic model of a rigid mechanism.

We will acquaint ourselves with the particularities of solving the problem of force analysis of mechanisms with friction by means of illustrative examples.

Analysis of a planar linkage. Consider a crank-slider mechanism (Fig. 5.8a). Let us conduct the force analysis by taking into account frictional forces at the prismatic pair. After introducing the previously considered dynamic models of kinematic pairs with friction the equations of kinetostatics of every single structural group prove to be independent. Therefore, it is possible to consider first the structural group consisting of links 2 and 3. We suppose that the mass of link 2 is negligible. Thus, the reaction forces $\mathbf{R}_{12}$ and $\mathbf{R}_{32}$ applied at joints A and B must be directed along the line AB . Knowing the direction of force $\mathbf{R}_{23} = -\mathbf{R}_{32}$, it is possible to compose an independent system of equations of kinetostatics for the slider 3. Assuming that the velocity $\mathbf{v}_B$ is directed, as shown in the figure, i.e. $\dot{x} < 0$ and $\text{sign} \dot{x} = -1$, and employing the model of a prismatic pair with friction described by Eqs. (5.11)–(5.13), we obtain the following equation system of kinetostatics:

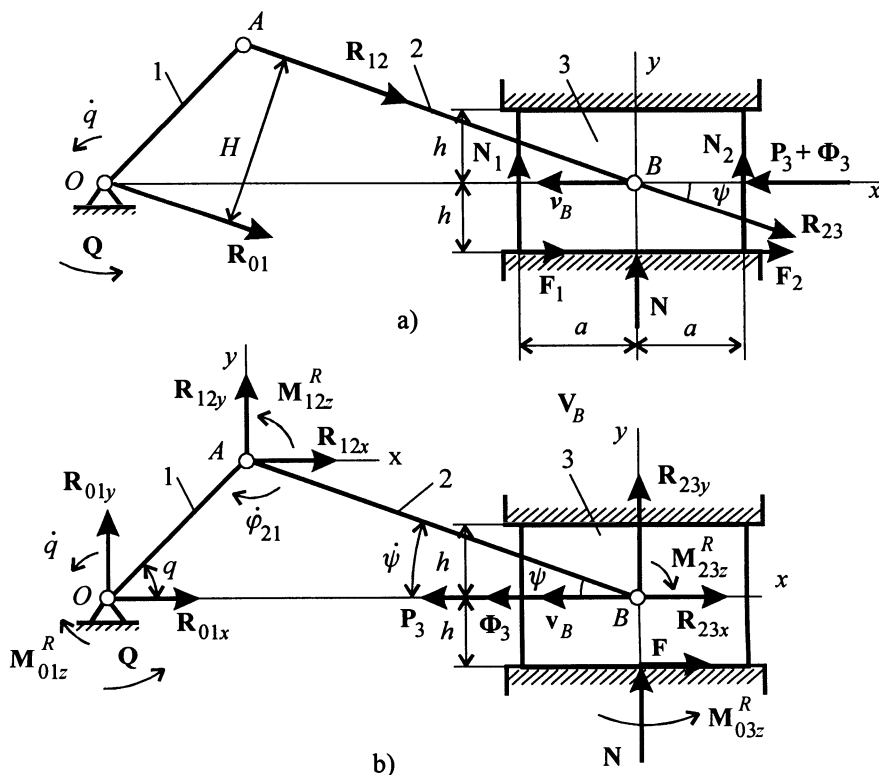


Fig. 5.8. Model of the slider-crank mechanism with friction

$$\begin{aligned}
N_1 + N_2 - R_{23} \sin \psi &= 0, \\
R_{23} \cos \psi + f(N_1 \text{sign} N_1 + N_2 \text{sign} N_2) - (P_3 + \Phi_3) &= 0, \\
(N_2 - N_1)a + fh(N_1 + N_2) &= 0.
\end{aligned} \tag{5.18}$$

Here, ψ is the angle between the direction of the reaction $\mathbf{R}_{23}$ and the x -axis; $\mathbf{P}_3 + \Phi_3$ is the sum of the active force and of the inertia force of link 3.

From the third equation in (5.18) we find that

$$N_2 = N_1 \frac{a - fh}{a + fh}.$$

Assuming that $a - fh > 0$ (which is practically always satisfied), we have

$$\text{sign} N_1 = \text{sign} N_2.$$

Thus, $N_1 \text{sign} N_1 + N_2 \text{sign} N_2 = (N_1 + N_2) \text{sign}(N_1 + N_2) = N \text{sign} N$, where $N = N_1 + N_2$. This allows re-writing the first and the second equation in (5.18) in a simpler form:

$$N - R_{23} \sin \psi = 0, \quad R_{23} \cos \psi + f N \text{sign} N = P_3 + \Phi_3.$$

Eliminating R_{23} from this system, we obtain an equation for the unknown force N :

$$N = \frac{P_3 + \Phi_3}{\cot \psi + f \text{sign} N}. \tag{5.19}$$

This equation is nonlinear, since function $\text{sign} N$ is nonlinear. Let us find the solution of this equation for different relationships between ψ and f :

1. Suppose that $\cot \psi > f$ (case of a „weak“ friction); then $\cot \psi + f \text{sign} N > 0$ for any $\text{sign} N$ and, therefore, $\text{sign} N = \text{sign}(P_3 + \Phi_3)$. If vector $(\mathbf{P}_3 + \Phi_3)$ is directed as shown in the figure, then $P_3 + \Phi_3 > 0$ and thus $\text{sign} N = 1$. From here

$$N = \frac{P_3 + \Phi_3}{\cot \psi + f} > 0.$$

In this case, the active and the inertia forces together „support“ the driving force. However, if vector $(\mathbf{P}_3 + \Phi_3)$ is directed opposite to $\mathbf{v}_B$ ($(\mathbf{P}_3 + \Phi_3) \cdot \mathbf{v}_B < 0$), then $\text{sign} N = -1$. Now, from Eq. (5.19) we obtain

$$N = \frac{P_3 + \Phi_3}{\cot \psi - f} < 0.$$

In this case, force $\mathbf{N}$ is directed downwards and force $\mathbf{R}_{23}$ from B towards A .

2. Suppose now that $\cot \psi < f$ (case of „strong“ friction or of „large“ pressure angle ψ). Let now vector $(\mathbf{P}_3 + \Phi_3)$ coincide in direction with vector $\mathbf{v}_B$. Then, Eq. (5.19) has two solutions. Indeed, assuming that $\text{sign} N = 1$, we have

$$N = \frac{P_3 + \Phi_3}{\cot \psi + f} > 0.$$

Having assumed that $\text{sign} N = -1$, we obtain a second solution

$$N = \frac{P_3 + \Phi_3}{\cot \psi - f} < 0.$$

In this case, we have to handle the *regime of de-braking*, when the sum of the active and the inertia force has the direction of velocity, i.e. when it plays the role of a driving force. Strictly speaking, in the framework of the model of a mechanism with rigid links, it is impossible to determine which of the solutions will actually be realized. It is only possible to show that some „physical“ reasons testify on behalf of the first solution. It is not difficult to understand that with increasing friction coefficient f one must expect an increase in the modulus of frictional force $|F|$, i.e. $d|F|/df > 0$. Investigating this solution, we obtain

$$|F| = \left| \frac{f(P_3 + \Phi_3)}{\cot \psi + f} \right| = (P_3 + \Phi_3) \frac{f}{\cot \psi + f},$$

since $P_3 + \Phi_3 > 0$. Hence,

$$\frac{d|F|}{df} = (P_3 + \Phi_3) \frac{\cot \psi + f - f}{(\cot \psi + f)^2} = (P_3 + \Phi_3) \frac{\cot \psi}{(\cot \psi + f)^2} > 0.$$

For the second solution we find

$$|F| = \left| \frac{f(P_3 + \Phi_3)}{\cot \psi - f} \right| = -(P_3 + \Phi_3) \frac{f}{\cot \psi - f},$$

since $\cot \psi - f < 0$. Hence, $\frac{d|F|}{df} = -(P_3 + \Phi_3) \frac{\cot \psi}{(\cot \psi - f)^2} < 0$. That is why the second solution is „questionable“ from a physical point of view. If $P_3 + \Phi_3 < 0$, i.e. if force $(P_3 + \Phi_3)$ plays the role of a resistance force, then for $\cot \psi < f$ Eq. (5.19) has no solutions. Assuming that $\text{sign} N = 1$, we obtain from (5.19)

$$N = \frac{P_3 + \Phi_3}{\cot \psi + f} < 0,$$

and for $\text{sign} N = -1$ we have

$$N = \frac{P_3 + \Phi_3}{\cot \psi - f} > 0.$$

The contradiction thus obtained shows that no solution exists. This case corresponds to the regime of self-braking, when in the investigated position of the mechanism and with the given direction of the force, motion becomes completely impossible.

Having determined the reaction forces acting on the slider, it is easy to find the remaining reactions in the mechanism. So, considering the equilibrium of link 2, we obtain ($m_2 \approx 0$):

$$\mathbf{R}_{12} = \mathbf{R}_{23},$$

and the equations of kinetostatics for link 1 give:

$$\mathbf{R}_{01} = \mathbf{R}_{12}, \quad Q = R_{12}H,$$

where H is the distance from point O to line AB .

Let us now conduct the force analysis of the crank-slider mechanism, taking into account frictional forces at the revolute pairs A , B and O . In this case, the direct solution of the nonlinear equations of kinetostatics is very complicated. It is necessary to use the method of successive approximations. As a model of friction at the prismatic pair we choose the relationship (5.9), assuming that the forces $\mathbf{N}_1$ and $\mathbf{N}_2$ have one and the same direction (Fig. 5.8b). For revolute pairs with friction coefficient f_1 we accept model (5.14). Let us introduce the notation

$$f_1 r (1 + f_1^2)^{-\frac{1}{2}} = r_f,$$

where the quantity r_f is called *radius of friction circle*. Assume that the radii r_f for the revolute pairs A , B and O are identical. We write the equations of kinetostatics for the structural group containing links 2 and 3. As in the case of ideal kinematic pairs these equations form an independent system. The equations of kinetostatics for slider 3 are:

$$\begin{aligned} R_{23x} &= P_3 + \Phi_3 - f N \text{sign} N, \quad R_{23y} + N = 0, \\ M_{03z}^R &= -f h N + r_f \sqrt{R_{23x}^2 + R_{23y}^2}. \end{aligned} \quad (5.20)$$

Here, P_3 and Φ_3 are the active force and the inertia force of the slider, respectively, f is the friction coefficient at the prismatic pair; R_{23x} and R_{23y} are the components of reaction at the prismatic pair. When defining the frictional force at the prismatic pair and the moment of frictional forces at the revolute pair, the direction of the velocity of the slider and the relative angular velocity at joint B are taken into account.

We derive the equations of kinetostatic for the connecting rod 2, assuming that its mass is negligible. In this case we obtain the following equations:

$$\begin{aligned} R_{12x} - R_{23x} &= 0, \quad R_{12y} - R_{23y} = 0 \\ (R_{12x} \sin \psi + R_{12y} \cos \psi) \ell &= r_f (\sqrt{R_{12x}^2 + R_{12y}^2} + \sqrt{R_{23x}^2 + R_{23y}^2}), \end{aligned} \quad (5.21)$$

where ψ is the angle between line AB and the x -axis, R_{12x} and R_{12y} are the components of reactions at joint A , ℓ is the length of the connecting rod.

We solve Eqs. (5.20)–(5.21) by the method of successive approximations, assuming that the frictional forces are sufficiently small to prevent a change in the sign of reaction N as well as seizure of the mechanism. First, we assume that there are no frictional forces and we find the reactions at the ideal kinematic pairs from the equations

$$\begin{aligned} R_{23x} &= P_3 + \Phi_3, \quad R_{23y} + N = 0, \quad M_{03z}^R = 0, \\ R_{12x} - R_{23x} &= 0, \quad R_{12y} - R_{23y} = 0, \quad R_{12y} \cos \psi + R_{12x} \sin \psi = 0, \end{aligned}$$

obtained from (5.20) and (5.21) for $f = 0$, $r_f = 0$. Solving these equations, we obtain the values of reactions in the „zero-order“ approximation:

$$\begin{aligned} (R_{12x})_0 &= (R_{23x})_0 = P_3 + \Phi_3, \quad (M_{03z}^R)_0 = 0; \\ (R_{12y})_0 &= (R_{23y})_0 = -(N)_0 = -(P_3 + \Phi_3) \tan \psi. \end{aligned}$$

Next, we substitute these reactions into the expression for the frictional forces in Eqs. (5.20) and (5.21). We then obtain the system of equations of the first-order approximation:

$$\begin{aligned} (R_{23x})_1 &= P_3 + \Phi_3 - f(N)_0 \operatorname{sign}(N)_0 = -(P_3 + \Phi_3)(1 - f \tan \psi), \\ (R_{23y})_1 + (N)_1 &= 0, \\ (M_{03z}^R)_1 &= -f h(N)_0 + r_f \sqrt{(R_{Bx})_0^2 + (R_{By})_0^2} = (P_3 + \Phi_3)(-f h \tan \psi + r_f \sec \psi), \\ (R_{12x})_1 &= (R_{23x})_1 = (P_3 + \Phi_3)(1 - f \tan \psi), \\ (R_{12y})_1 &= (R_{23y})_1, \\ \left[(R_{12x})_1 \sin \psi + (R_{12y})_1 \cos \psi \right] \ell &= \\ r_f \left(\sqrt{(R_{12x})_0^2 + (R_{12y})_0^2} + \sqrt{(R_{23x})_0^2 + (R_{23y})_0^2} \right) &= 2r_f (P_3 + \Phi_3) \sec \psi. \end{aligned}$$

Solving the last equation and taking into account the second and the fifth, we find

$$\begin{aligned} -(R_{23y})_1 &= -(R_{12y})_1 = (N)_1 = \\ (P_3 + \Phi_3) \left[\ell(1 - f \tan \psi) \sin \psi - 2r_f \sec \psi \right] (\ell \cos \psi)^{-1}. \end{aligned}$$

In this way, the values of all reactions have been determined in first-order approximation. Substituting them into the expressions for the frictional forces in Eqs. (5.20) and (5.21), it is possible to obtain the equations of the second approximation, and so on. It is easy to write an algorithm for executing the calculations by a computer. It can be shown that for real mechanisms this process converges sufficiently fast, so that we can practically restrict ourselves to the first-order approximation.

Let us obtain convergence conditions when $r_f = 0$, i.e. when there is friction at the prismatic pair only. In this case we find from (5.20) and (5.21)

$$\begin{aligned}
(R_{23x})_k &= (R_{12x})_k = \\
&\quad (P_3 + \Phi_3) \left[1 - f \tan \psi + f^2 \tan^2 \psi - \dots + (-1)^k f^k \tan^k \psi \right], \\
(R_{23y})_k &= (R_{12y})_k = (R_{23x})_k \tan \psi, \\
(N)_k &= -(R_{23y})_k = -(R_{23x})_k \tan \psi, \\
(M_{03z}^R)_k &= -f h (N)_{k-1} = f h (R_{23x})_{k-1} \tan \psi,
\end{aligned} \tag{5.22}$$

where k is the order of the approximation. It is obvious that the process of successive approximations converges as a geometric progression with the denominator $-f \tan \psi$ and that the convergence condition $|f \tan \psi| < 1$ is fulfilled for all real mechanisms. In practice, it can be assumed that $f \tan \psi < 0.1$, so that the second-order approximation already gives a correction of only 1%.

Next, we derive the equations of kinetostatics for the crank OA , taking into account that $R_{21x} = -R_{12x}$, $R_{21y} = -R_{12y}$, and assuming that the mass of the crank is negligible ($m_1 = 0$) and that the rotation of the crank is uniform ($\ddot{q} = 0$):

$$\begin{aligned}
R_{01x} + R_{21x} &= 0, \quad R_{01y} + R_{21y} = 0, \\
Q - R_{12x} \ell_1 \sin q + R_{12y} \ell_1 \cos q - r_f (\sqrt{R_{21x}^2 + R_{21y}^2} + \sqrt{R_{01x}^2 + R_{01y}^2}) &= 0,
\end{aligned}$$

where ℓ_1 is the length of crank OA . From here it is not difficult to determine the reactions R_{01x} and R_{01y} at the revolute pair O and the driving moment Q .

Force analysis of a worm transmission (see Fig. 5.7). The worm transmission has two revolute pairs and one higher kinematic pair of movability five, whose dynamic model is represented by Eqs. (5.17). The equations of kinetostatics for the worm and for the worm wheel turn out to be independent. First, the equations of kinetostatics for the worm wheel are derived: Projecting the forces acting on the wheel onto the coordinate axes $O_2x_2y_2z_2$ and writing the equations of the moments about these axes, we obtain (assuming that the coordinate axes are principal central axes of inertia of the wheel)

$$\begin{aligned}
R_{x2} + S &= 0, \quad R_{y2} + T = 0, \quad R_{z2} - P = 0, \\
M_{2x}^R - Pr_2 &= 0, \quad M_{2y}^R + Sr_2 = 0, \quad -J_{2O_2} \ddot{\phi} - M_2 + M_{2z}^R + Sr_2 = 0,
\end{aligned} \tag{5.23}$$

where J_{2O_2} is the inertia moment of the wheel about axis O_2z (point O_2 coincides with the mass center of the wheel). Using relation (5.17), we come to a system of six equations with six unknowns (N_{Ax} , N_{Ay} , N_{Cx} , N_{Cy} are forces reduced to the face planes A and C of the journal of the joint connecting the worm wheel with the support; N_{2z} is a force directed along the joint axis; N_{12} is the normal force exerted the wheel by the worm):

$$\begin{aligned}
& N_{Ax} + N_{Cx} - f_1(N_{Ay} + N_{Cy})\text{sign}\dot{\phi} + \\
& \quad N_{12}(\cos\alpha \cos\beta - f \text{sign } N_{12} \sin\beta \text{sign}\dot{q}) = 0, \\
& N_{Ay} + N_{Cy} + f_1(N_{Ax} + N_{Cx})\text{sign}\dot{\phi} + N_{12} \sin\alpha = 0, \\
& N_{2z} - N_{12}(\cos\alpha \sin\beta + f \text{sign } N_{12} \cos\beta \text{sign}\dot{q}) = 0, \\
& [N_{Cy} - N_{Ay} + f_1(N_{Cx} - N_{Ax})\text{sign}\dot{\phi}] 0.5\ell - \\
& \quad N_{12}(\cos\alpha \sin\beta + f \text{sign } N_{12} \cos\beta \text{sign}\dot{q})r_2 = 0, \\
& [N_{Cx} - N_{Ax} - f_1(N_{Cy} - N_{Ay})\text{sign}\dot{\phi}] 0.5\ell = 0, \\
& -J_{20_2}\ddot{\phi} - M_2 - \left[f_1 r \left(\sqrt{N_{Ax}^2 + N_{Ay}^2} + \sqrt{N_{Cx}^2 + N_{Cy}^2} \right) + k_p |N_{2z}| \right] \text{sign}\dot{\phi} + \\
& \quad N_{12}(\cos\alpha \cos\beta - f \text{sign } N_{12} \sin\beta \text{sign}\dot{q})r_2 = 0,
\end{aligned} \tag{5.24}$$

where f and f_1 are the friction coefficients at the higher pair and at the revolute pairs, respectively, k_p is the coefficient of pivoting friction at the revolute pair.

It is convenient to solve this system by the method of successive approximations. Assume (in „zero-order“ approximation) that $f_1 = 0$, $k_p = 0$ (the friction at a revolute pair is usually considerably less important than the friction in a worm meshing); we assume also, that $\dot{q} > 0$, $\dot{\phi} > 0$ (the directions of the angular velocities coincide with the directions shown in the figure). Thus, it is possible to define $(N_{12})_0$ directly from the last equation of (5.24):

$$(N_{12})_0 = \frac{M_2 + J_{20_2}\ddot{\phi}}{(\cos\alpha \cos\beta - f \sin\beta \text{sign}(N_{12})_0)r_2} = \frac{(M_2 + J_{20_2}\ddot{\phi})\cos\eta}{r_2 \cos\alpha \cos(\beta \pm \eta)}. \tag{5.25}$$

Here, the notation $\tan\eta = f \cos^{-1}\alpha$ is introduced; the sign „plus“ in the denominator corresponds to $(N_{12})_0 > 0$; this is the case, when the worm is the driving link. When $(N_{12})_0 < 0$, i.e. when the worm wheel is the driving link the sign „minus“ is used in (5.25). It is obvious, that expression (5.25) is equivalent to the two relationships:

$$\begin{aligned}
(N_{12})_0 &= \frac{(M_2 + J_{20_2}\ddot{\phi})\cos\eta}{r_2 \cos\alpha \cos(\beta + \eta)} > 0, \quad \text{for } M_2 + J_{20_2}\ddot{\phi} > 0; \\
(N_{12})_0 &= \frac{(M_2 + J_{20_2}\ddot{\phi})\cos\eta}{r_2 \cos\alpha \cos(\beta - \eta)} < 0, \quad \text{for } M_2 + J_{20_2}\ddot{\phi} < 0.
\end{aligned} \tag{5.26}$$

Having determined $(N_{12})_0$, it is also possible to find the remaining reactions in „zero-order“ approximation. Solving Eqs. (5.25), we obtain

$$(N_{Cx})_0 = (N_{Ax})_0 = -\frac{M_2 + J_{20_2}\ddot{\phi}}{r_2}, \quad (N_{2z})_0 = \frac{M_2 + J_{20_2}\ddot{\phi}}{r_2 \tan(\beta \pm \eta)},$$

$$(N_{Ay})_0 = -(M_2 + J_{20_2} \ddot{\varphi}) \left[\frac{1}{\ell \tan(\beta \pm \eta)} + \frac{\cos \eta}{2r_2 \cos(\beta \pm \eta)} \right],$$

$$(N_{Cy})_0 = (M_2 + J_{20_2} \ddot{\varphi}) \left[\frac{1}{\ell \tan(\beta \pm \eta)} - \frac{\cos \eta}{2r_2 \cos(\beta \pm \eta)} \right].$$

After this, it is possible to obtain even more precise values of the reactions by using the method of successive approximations. To this end, the values of reactions found above are substituted into Eqs. (5.24) with friction coefficients f_1 and k_p taken into account.

The equations of kinetostatics for the worm are solved analogously: From them the reactions of the support and the driving moment Q are determined. In this case, however, a particular situation exists when the motion turns out to be possible only in the regime of de-braking. Neglecting the friction at the revolute pair, let us derive the equations for the moments of the forces acting on the worm about the axis of its rotation. We obtain

$$Q - J_{10} \ddot{q} - Pr_1 = 0. \quad (5.27)$$

Here, J_{10} is the inertia moment of the worm with respect to the rotation axis; Eq. (5.27) takes into account that a force $-P$ is applied to the worm. Expressing $(P)_0$ by $(N_{12})_0$ and using (5.25), we have

$$(Q)_0 = J_{10} \ddot{q} + \frac{(M_2 + J_{20_2} \ddot{\varphi})(\cos \alpha \sin \beta + f \operatorname{sign} N_{12} \cos \beta) \cos \eta}{r_2 \cos \alpha \cos(\beta \pm \eta)} =$$

$$= J_{10} \ddot{q} + (M_2 + J_{20_2} \ddot{\varphi}) \frac{r_1}{r_2} \tan(\beta \pm \eta). \quad (5.28)$$

If $\eta < \beta$, then for $M_2 + J_{20_2} \ddot{\varphi} < 0$, i.e. when the worm wheel is the driving link, we have

$$(Q)_0 = J_{10} \ddot{q} - \left| M_2 + J_{20_2} \ddot{\varphi} \right| \frac{r_1}{r_2} |\tan(\beta - \eta)|. \quad (5.29)$$

In the regime of uniform rotation ($\ddot{q} = \ddot{\varphi} = 0$) we obtain $Q < 0$; in other words, in this case, the transmission can be turned even in the presence of a resisting moment acting on the worm provided a moment $-M_2$ is applied to the wheel. If $\eta > \beta$, then for $M_2 + J_{20_2} \ddot{\varphi} < 0$ we have

$$(Q)_0 = J_{10} \ddot{q} + \left| M_2 + J_{20_2} \ddot{\varphi} \right| \frac{r_1}{r_2} |\tan(\beta - \eta)|,$$

since, in this case, $\tan(\beta - \eta) < 0$. In this way, the rotation turns out to be possible only for $Q > 0$, i.e. when a driving moment corresponding to the regime of de-braking is applied to the worm. For $Q < 0$ the transmission of motion from the wheel to the worm becomes impossible because of self-braking.

5.4 Problems

5.1. A load moment $M = 100 \text{ N m}$ is applied to the movable slider (Fig. 5.9).

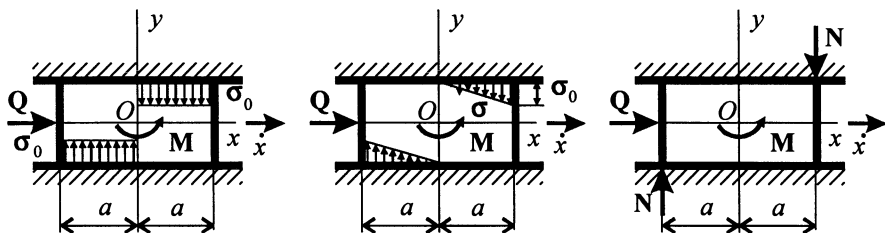


Fig. 5.9

Determine the driving force Q ensuring a uniform motion of the slider for:

- the normal forces per unit length of the contact line are uniformly distributed, i.e. $\sigma(x) = \sigma_0 = \text{const}$;
- the normal forces per unit length of the contact line are distributed according to the linear law $\sigma(x) = \frac{\sigma_0}{a}x$;
- the normal forces are concentrated at the end points of the slider.

The coefficient of sliding friction is $f = 0,2$, the length of the slider is $2a = 0.4 \text{ m}$.

Solution: The equilibrium equations are:

$$Q - 2f \int_0^a |\sigma(x)| dx = 0, \quad M - 2 \int_0^a \sigma(x)x dx = 0.$$

From here we find:

$$Q = 2f \int_0^a |\sigma(x)| dx, \quad M = 2 \int_0^a \sigma(x)x dx.$$

Case a):

$$M = 2\sigma_0 \int_0^a x dx = \sigma_0 a^2, \quad \text{i.e. } \sigma_0 = \frac{M}{a^2}.$$

The driving force is

$$Q = 2f \int_0^a \frac{M}{a^2} dx = \frac{2fM}{a} = 200 \text{ N}.$$

Case b):

$$M = 2 \frac{\sigma_0}{a} \int_0^a x^2 dx = \frac{2\sigma_0^2}{3}.$$

From here we have

$$\sigma_0 = 3M/2a^2, \quad Q = 2f\sigma_0 a^2/2a = f\sigma_0 a = 3fM/2a = 150 \text{ N}.$$

Case c):

$$M = N \cdot 2a, \quad Q = 2fN = \frac{fM}{a} = 100 \text{ N}.$$

Answer: a) $Q = 200 \text{ N}$; b) $Q = 150 \text{ N}$; c) $Q = 100 \text{ N}$.

- 5.2. To the slider (Fig. 5.10) a load moment $M = 100 \text{ N m}$ is applied together with a resistance force $P = 500 \text{ N}$ at an angle $\alpha = 30^\circ$ against the line of the slider. The coefficient of sliding friction between the slider and the guide is $f = 0.2$. The dimensions of the slider are: $a = 0.2 \text{ m}$, $h = 0.1 \text{ m}$. In the framework of the static model determine the normal forces N_A and N_B concentrated at the end points of the slider as well as the driving force Q if the slider moves with constant velocity.

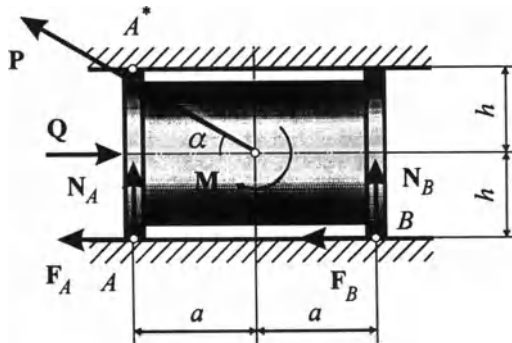


Fig. 5.10

Solution: The normal forces N_A and N_B and the driving force Q can be found from the equilibrium equations for the slider:

$$\begin{aligned}
 -f|N_A| - f|N_B| + Q - P \cos \alpha &= 0, \\
 N_A + N_B + P \sin \alpha &= 0, \\
 (N_B - N_A)a - fh(N_B + N_A) - M &= 0.
 \end{aligned}$$

From here:

$$N_A = \frac{(fh - a)P \sin \alpha - M}{2a} = -\frac{1}{2} \left(\frac{M}{a} + P \sin \alpha \right) + f \frac{Ph \sin \alpha}{2a} = -362.5 \text{ N}.$$

$$N_B = \frac{M - (fh + a)P \sin \alpha}{2a} = \frac{1}{2} \left(\frac{M}{a} - P \sin \alpha \right) - f \frac{Ph \sin \alpha}{2a} = 112.5 \text{ N}.$$

$$Q = P \cos \alpha + f|N_A| + f|N_B| = P \cos \alpha + \frac{f(M - fPh \sin \alpha)}{a} \approx 528 \text{ N}.$$

The sign „minus“ of the force N_A means, that it is applied at point A^* and directed downwards.

Answer: $N_A = -362.5 \text{ N}$, $N_B = 112.5 \text{ N}$, $Q \approx 528 \text{ N}$.

- 5.3. Jacks are used for lifting loads to a small height. In screw jacks (Fig. 5.11) the essential links are the screw 1 and the nut 2 forming a screw pair of movability one.

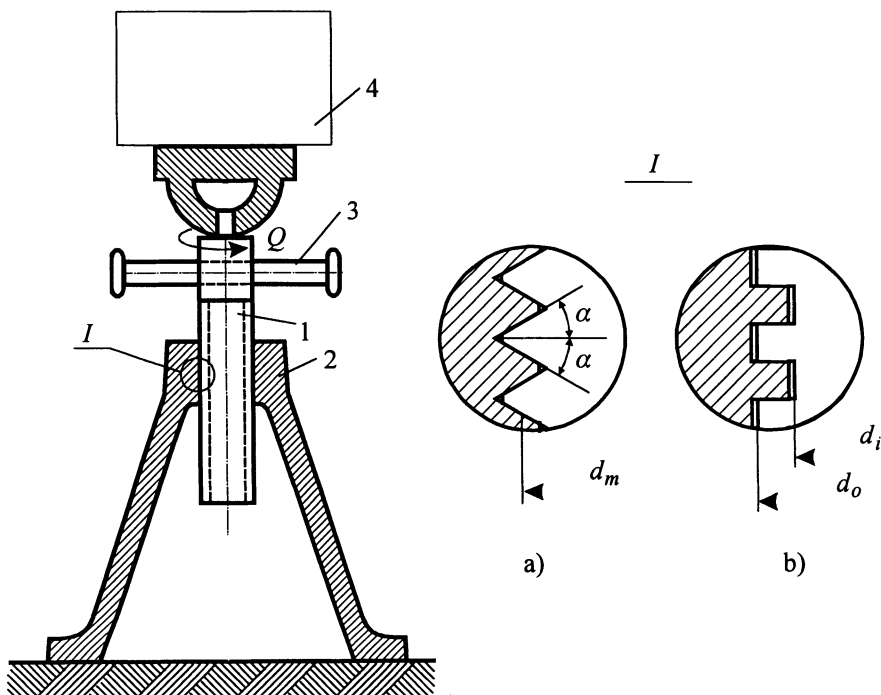


Fig. 5.11

Determine the moment Q to be applied to handle 3 in order to achieve a uniform lifting and lowering of load 4, assuming that the normal forces at the screw pair are uniformly distributed along the middle line of the thread. Take into account the friction at the screw pair only. The weight of the load is $G = 1000 \text{ N}$, the friction coefficient is $f = 0.1$.

Solve the problem for a triangular (a) and a square (b) thread. For the triangular thread the pitch diameter is $d_m = 22$ mm, the slope angle of the thread profile is $\alpha = 30^\circ$, the pitch of the thread is $h = 3$ mm. For the square thread the major diameter is $d_0 = 24$ mm and the minor diameter is $d_i = 21$ mm.

Solution: Let us write for uniform rotation the equation for the moments about the longitudinal axis of the forces acting on the screw of the jack. We obtain

$$Q - P \frac{d_m}{2} = 0,$$

where P is the circumferential force exerted to the screw by the nut, and d_m is the pitch diameter of the thread. Find the relation between the circumferential force P and the axial load of the screw.

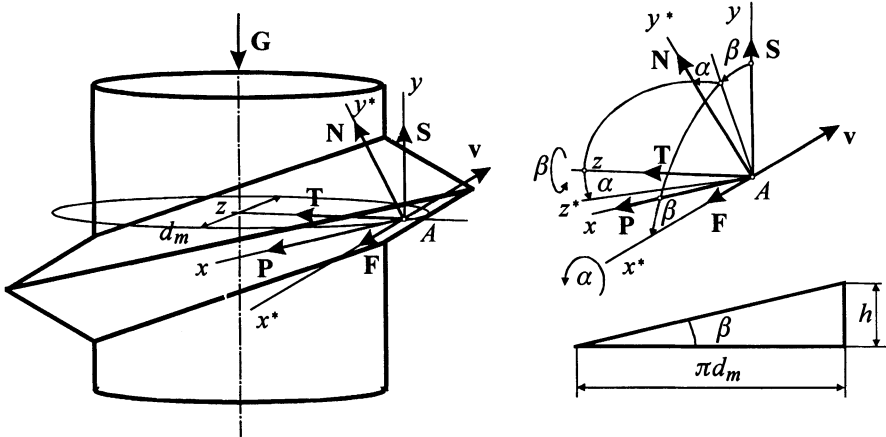


Fig. 5.12. Chart of forces for problem 5.3

Let us introduce two coordinate systems with origin at one of the contact points A (Fig. 5.12). In the first coordinate system $Axyz$ the y -axis is directed along the screw axis, the z -axis along the screw radius and the x -axis orthogonal to the radius. The axial force S is directed along the y -axis, the circumferential force P is in x -direction and the radial force T is in z -direction. The second coordinate system $Ax^*y^*z^*$ is drawn in such a

way that the y^* -axis is directed orthogonally to the screw surface, the x^* -axis along the tangent to the screw surface and the z^* -axis is directed such that a „right-handed“ coordinate system is obtained. Then, the normal component N of the reaction at the screw pair is directed along the y^* -axis and the tangent component (frictional force F) is along the x^* -axis.

The transition from one coordinate system to other can be represented as a sequence of a rotation of the coordinate system about the z -axis through the angle β (helix angle) and a rotation of the coordinate system about x^* -axis through angle α (slope angle of the thread profile). With the help of formulae given in Chap. 2, it is possible to project the forces N and F onto the axes of the coordinate system $Axyz$ and to express the forces S , P and T by the forces N and F . We obtain the following relationships (compare with expressions (5.17)):

$$\begin{aligned} S &= N(\cos \alpha \cos \beta - f \sin \beta \operatorname{sign} v), \\ P &= N(\cos \alpha \sin \beta + f \cos \beta \operatorname{sign} v), \\ T &= N \sin \alpha. \end{aligned}$$

Here, $\operatorname{sign} v$ is the sign of the sliding velocity of the screw relative to the nut; $\operatorname{sign} v = +1$ corresponds to the case of an axial motion of the screw opposite to the direction of action of the axial force S (lowering of load).

Consider the case $\operatorname{sign} v = 1$. Taking into account that $\alpha \neq 90^\circ$ and $\beta \neq 90^\circ$, we obtain:

$$\begin{aligned} P &= S \frac{\cos \alpha \sin \beta + f \cos \beta}{\cos \alpha \cos \beta - f \sin \beta} = S \frac{\cos \alpha \cos \beta (\tan \beta + \frac{f}{\cos \alpha})}{\cos \alpha \cos \beta (1 - \frac{f}{\cos \alpha} \tan \beta)} = \\ &= S \frac{\tan \beta + \tan \rho}{1 - \tan \rho \cdot \tan \beta} = S \tan(\rho + \beta), \end{aligned}$$

where $\tan \rho = \frac{f}{\cos \alpha}$. For the rectangular thread (in particular, the square thread) $\alpha = 0^\circ$ and then $\tan \rho = f$.

For the case $\operatorname{sign} v = -1$ we obtain:

$$P = S \frac{\cos \alpha \sin \beta - f \cos \beta}{\cos \alpha \cos \beta + f \sin \beta} = S \frac{\cos \alpha \cos \beta (\tan \beta - \frac{f}{\cos \alpha})}{\cos \alpha \cos \beta (1 + \frac{f}{\cos \alpha} \tan \beta)} =$$

$$= S \frac{\tan \beta - \tan \rho}{1 + \tan \rho \cdot \tan \beta} = S \cdot \tan(\beta - \rho),$$

i.e. in the process of lowering the load, the circumferential force P is somewhat smaller than in the process of lifting. Note that for $\beta < \rho$ self-braking occurs, i.e. the screw does not move under the action of the axial force.

During a uniform motion of the screw along the axis, the weight G acting on the screw is balanced by the axial component S of the nut reaction on the screw. For the triangular thread we obtain:

$$Q = G \frac{d_m}{2} \tan(\beta + \rho) = 1000 \cdot \frac{0.022}{2} \tan(2.485 + 6.587) = 1.756 \text{ N m},$$

$$\text{where } \beta = \arctan \frac{h}{\pi d_m} = \arctan \frac{3}{\pi 22} = 2.485^\circ, \quad \rho = \arctan \frac{f}{\cos \alpha} = 6.587^\circ.$$

In the process of lowering the weight we have:

$$Q = G \frac{d_m}{2} \tan(\beta - \rho) = 1000 \cdot \frac{0.022}{2} \tan(2.485 - 6.587) = -0.789 \text{ N m},$$

i.e. for lowering the weight, it is necessary to apply a smaller moment to the screw of the jack in opposite direction. For the square thread we have:

$$Q = G \frac{d_m}{2} \tan(\beta + \rho) = 1000 \cdot \frac{0.0225}{2} \tan(2.430 + 5.711) = 1.609 \text{ N m},$$

$$\text{where } \beta = \arctan \frac{h}{\pi d_m} = \arctan \frac{3 \cdot 2}{\pi(24 + 21)} = 2.430^\circ, \quad \rho = \arctan f = 5.711^\circ.$$

For lowering the weight we obtain:

$$Q = G \frac{d_m}{2} \tan(\beta - \rho) = 1000 \cdot \frac{0.0225}{2} \tan(2.430 - 5.711) = -0.645 \text{ N m}.$$

Answer: a) for the triangular thread: $Q = 1.756 \text{ N m}$, when lifting the load, and $Q = -0.789 \text{ N m}$, when lowering the load; b) for the rectangular thread: $Q = 1.606 \text{ N m}$, when lifting the load, and $Q = -0.645 \text{ N m}$, when lowering the load.

- 5.4. A *puller* is used for a nondestructive disassembly of elements, mounted in one another in an interference fit (Fig. 5.13). Body 1 leans against the motionless element 2. In the body are installed pawls 3, which are supplied to the surface of element 4 that must be removed. When turning handwheel 5 the pawls are moved in axial direction by means of a screw pair 6 (screw) – 7 (nut). Together with pawls 3, element 4 moves relative to the motionless element 2.

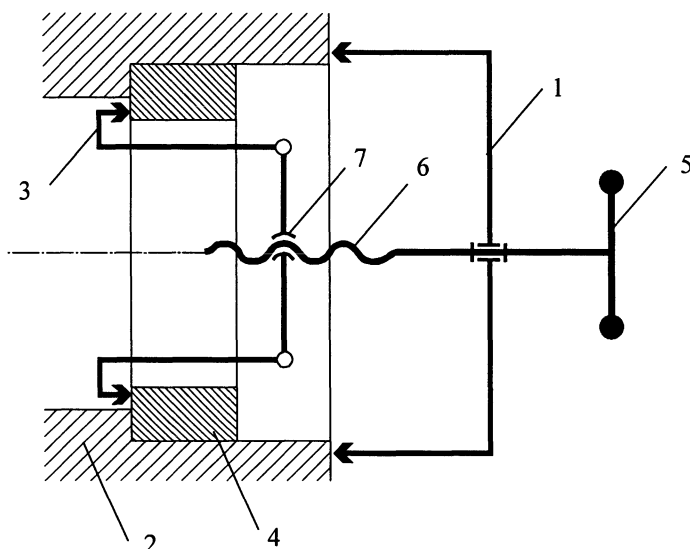


Fig. 5.13

In order to remove a bearing the moment $Q = 10 \text{ N m}$ is applied to screw 6. The screw has a triangular thread with slope angle $\alpha = 30^\circ$ of the thread. The pitch diameter of the thread is $d_m = 24 \text{ mm}$, the pitch of the thread is $h = 3 \text{ mm}$. The friction coefficient for the material of the screw and the nut is $f = 0.14$. Using the relationships obtained in the solution of problem 5.3, determine the axial force S of the resistance against the removal of the bearing. Take into account the friction in the thread only.

Answer: $S = 4105 \text{ N}$.

- 5.5. A turn-buckle is used for cable tightening (Fig. 5.14). In the nut 1 a screw 2 with a right-hand thread and a screw 3 with a left-hand thread are turned. The ends of the cables are fastened to screws 2 and 3. Turning nut 1, the screws come closer together, thus tightening the cable.

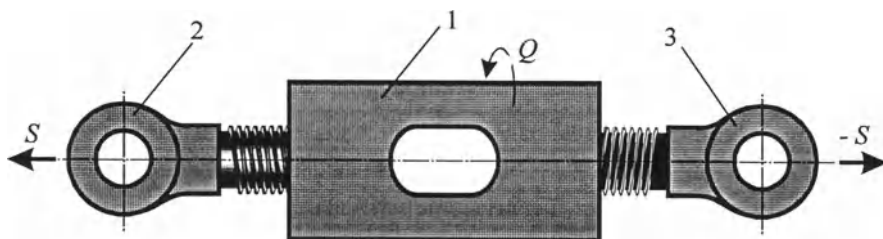


Fig. 5.14

Determine the moment Q required for tightening the cable to a normal force $S = 9000 \text{ N}$, assuming that the cable does not twist. The thread is square, the pitch of the thread is $h = 10 \text{ mm}$, the pitch diameter of the thread is $d_m = 40 \text{ mm}$. The friction coefficient between screw and nut is $f = 0.15$.

Answer: $Q = 41.82 \text{ N m}$.

- 5.6. A braking device (Fig. 5.15a) consists of a pulley 1 and a brake block 2. The pulley is stopped by pressing the brake block against it with force $P = 150 \text{ N}$. The dimensions of the block are: $a = 0.2 \text{ m}$, $l = 0.6 \text{ m}$, $b = 0.3 \text{ m}$. The friction coefficient between pulley 1 and block 2 is $f = 0.3$. Define the forces exerted to the block by the pulley for both directions of pulley rotation. Neglect the friction at the revolute kinematic pairs as well as the mass of the block.

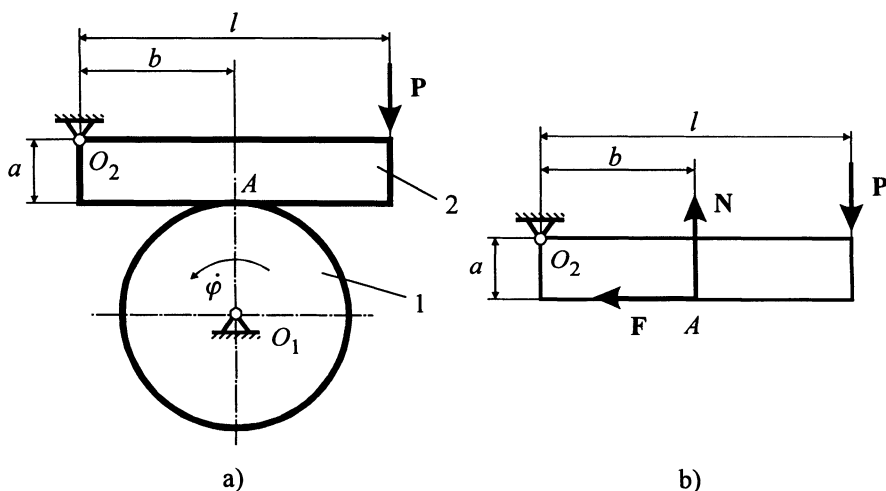


Fig. 5.15

Solution: Let us write the equation for the moments of the forces acting on the block with respect to axis O_2 (Fig. 5.15b):

$$-Pl + Nb - Fa = 0,$$

where F is the frictional force, $F = fN \text{sign} N \text{sign} \dot{\phi}$. When the rotation of the pulley is counter-clockwise, we have $\text{sign} \dot{\phi} = 1$, in the opposite case we have $\text{sign} \dot{\phi} = -1$. Let us express the normal component of reaction N :

$$N = \frac{Pl}{b - fa \cdot \text{sign} N \cdot \text{sign} \dot{\phi}}.$$

In the case when $\text{sign}\dot{\phi} = 1$, the solution for $\text{sign}N = 1$ is $N = 375 \text{ N}$, $F = 112.5 \text{ N}$ and there is no solution for $\text{sign}N = -1$. In the case when $\text{sign}\dot{\phi} = -1$, the solution for $\text{sign}N = 1$ is $N = 250 \text{ N}$, $F = 75 \text{ N}$ and there is no solution for $\text{sign}N = -1$.

Answer: In the case of counter-clockwise rotation of the pulley we have $N = 375 \text{ N}$, $F = 112.5 \text{ N}$; in the case of clockwise rotation of the pulley we have $N = 250 \text{ N}$, $F = 75 \text{ N}$.

- 5.7. In the crank-slider mechanism (Fig. 5.16a) $OA = l_1 = 0.4 \text{ m}$, $AB = l_2 = 0.5 \text{ m}$, $B_1B = B_2B$, $q = 90^\circ$, $\dot{q} = 9 \text{ s}^{-1}$, the mass of the slider is $m_3 = 10 \text{ kg}$, the coefficient of sliding friction for the materials of the slider and the frame is $f = 0.1$. Find the reaction at the kinematic pairs and the driving moment Q . Neglect the masses of the crank and of the connecting rod as well as the friction at the revolute pairs, $h = 0$.

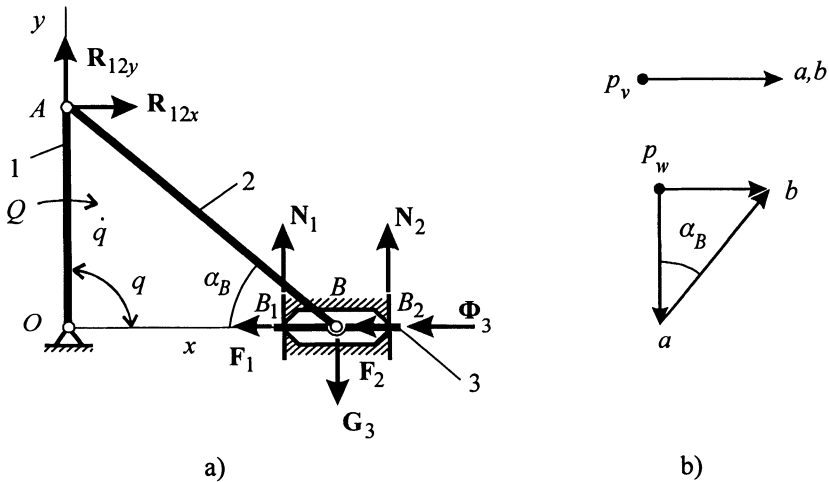


Fig. 5.16

Solution: Consider the structural Assur group consisting of links 2 and 3. Let us compose the equations of kinetostatics for this group:

$$\begin{aligned} R_{12x} - f |N_1| - f |N_2| - m_3 \ddot{x}_B &= 0, \\ R_{12y} + N_1 + N_2 - m_3 g &= 0, \\ -R_{12x} l_1 - R_{12y} x_B &= 0, \\ -N_1 \cdot B_1B + N_2 \cdot B_2B &= 0, \end{aligned}$$

where coordinate x_B of the slider is $x_B = \sqrt{l_2^2 - l_1^2} = 0.3$ m, its acceleration (Fig. 5.16b) is $\ddot{x}_B = l_1 \dot{q}^2 \tan \alpha_B = l_1 \dot{q}^2 \frac{l_1}{x_B} = 43.2$ m/s². From the last equation of kinetostatics it follows:

$$N_1 = N_2 = N.$$

Let us re-write the first three equations in the form:

$$R_{12x} - 2fN \cdot \text{sign}N = m_3 \ddot{x}_B,$$

$$R_{12y} + 2N = m_3 g,$$

$$4R_{12x} + 3R_{12y} = 0.$$

From here, we find N :

$$N = \frac{\begin{vmatrix} 1 & 0 & m_3 \ddot{x}_B \\ 0 & 1 & m_3 g \\ 4 & 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & -2f \text{sign}N \\ 0 & 1 & 2 \\ 4 & 3 & 0 \end{vmatrix}} = \frac{m_3(\ddot{x}_B + 3g)}{6 - 8f \text{sign}N} = \frac{m_3(4l_1 \dot{q}^2 + 9g)}{3(6 - 8f \text{sign}N)}.$$

There is no solution for $\text{sign}N = -1$. For $\text{sign}N = 1$ we have $N = 139.67$ N. Then, the reactions from the frame on the slider are $R_{03x} = -2fN = -31.83$ N, $R_{03y} = 2N = 318.32$ N; the reactions at joint A from the crank on the slider are $R_{12x} = 2fN + m_3 \ddot{x}_B = 463.83$ N, $R_{12y} = m_3 g - 2N = -220.22$ N; the reactions at joint B from the connecting rod on the slider are $R_{23x} = R_{12x} = 463.83$ N, $R_{23y} = R_{12y} = -220.22$ N; the reactions at joint O from the frame on the crank are $R_{01x} = R_{12x} = 463.83$ N, $R_{01y} = R_{12y} = -220.22$ N; the driving moment is $Q = -R_{12x}l_1 = -185.53$ N m (sign “minus” means that the moment is directed clockwise).

$$\text{Answer: } R_{01x} = R_{12x} = R_{23x} = 463.83 \text{ N, } R_{01y} = R_{12y} = R_{23y} = -220.22 \text{ N, } R_{03x} = -31.83 \text{ N, } R_{03y} = 318.32 \text{ N, } Q = -185.53 \text{ N m.}$$

- 5.8. Fig. 5.17 shows the design diagram of rear (a) and front (b) pneumatic brakes of a car. Under air pressure in cylinder 1, blocks 2 and 3 are pressed against the internal surface of the brake drum 4, providing braking. Compare the frictional forces in the brakes for the same air pressure and otherwise equal conditions.

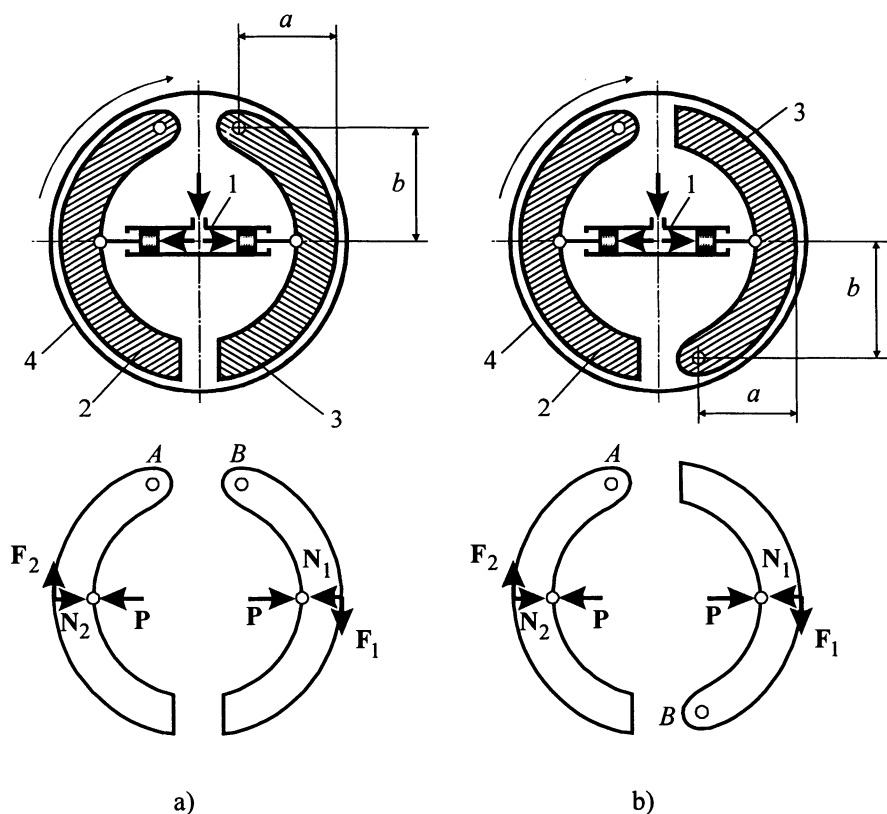


Fig. 5.17

Solution: Let us write the equations of the moments about axes A and B for the brake blocks according to diagram a). For block 2 we have

$$-(P - N_2)b - F_2a = 0$$

and for block 3:

$$(P - N_1)b - F_1a = 0.$$

Substituting the value of the frictional force $F = fN$, we find the normal reactions:

$$N_1 = \frac{Pb}{b + fa}, \quad N_2 = \frac{Pb}{b - fa}.$$

With this, the frictional forces are:

$$F_1 = fN_1 = \frac{fPb}{b + fa}, \quad F_2 = fN_2 = \frac{fPb}{b - fa}.$$

Let us write the equations of the moments for block 2 of the pneumatic

brake according to diagram b):

$$-(P - N_2)b - F_2a = 0,$$

and for block 3:

$$-(P - N_1)b - F_1a = 0.$$

For the normal reactions and for the frictional forces we obtain the expressions:

$$N_1 = \frac{Pb}{b - fa}, \quad N_2 = \frac{Pb}{b - fa}, \quad F_1 = \frac{fPb}{b - fa}, \quad F_2 = \frac{fPb}{b - fa}.$$

Comparing the sums $F_1 + F_2$ for both cases, we come to the conclusion that the frictional forces in the pneumatic brake, designed according to diagram b) are larger.

$$\text{Answer: For diagram a) } F_1 + F_2 = \frac{fPb}{b + fa} + \frac{fPb}{b - fa};$$

$$\text{For diagram b) } F_1 + F_2 = \frac{2fPb}{b - fa}.$$

- 5.9. The scotch-yoke mechanism (Fig. 5.18) consists of crank 1, slotted link 3, slotted-link crosshead 2 and the frame. The crank with length $l_1 = 0.2$ m rotates with constant angular velocity $\omega = 10 \text{ s}^{-1}$; the dimensions of the frame are $a = 0.1$ m, $b = 0.4$ m. The mass of the slotted link is $m_3 = 6$ kg, the remaining masses are neglected. On the slotted link the workload $P = 80$ N is acting. The coefficient of sliding friction between the slotted link and the frame is $f = 0.1$, and between the slotted link and the crosshead $f_1 = 0.1$. Neglecting frictional forces at the revolute pairs as well as gravity, find the reactions at the kinematic pairs and the driving force Q for the position $q = \omega t = 60^\circ$.

Solution: Let us write the equations of kinetostatics for the structural Assur group including links 2 and 3. For the prismatic pairs let us choose models with one-point contact (links 2 and 3) and with two-point contact (links 3 and 0).

$$\sum F_{2,3x} = 0 \Rightarrow P + \Phi_3 + R_{12x} + F_{03E} + F_{03D} = 0,$$

$$\sum F_{2,3y} = 0 \Rightarrow N_{03E} + N_{03D} + R_{12y} = 0,$$

$$\sum M_B = 0 \Rightarrow N_{03E}(b - x_A) + N_{03D}(b + a - x_A) - R_{12x}y_A = 0,$$

where $x_A = l_1 \cos \omega t$, $y_A = l_1 \sin \omega t$. For the second link:

$$\sum F_{2x} = 0 \Rightarrow R_{12x} + R_{32x} = 0, \quad \sum F_{2y} = 0 \Rightarrow R_{12y} + R_{32y} = 0.$$

From here it follows that

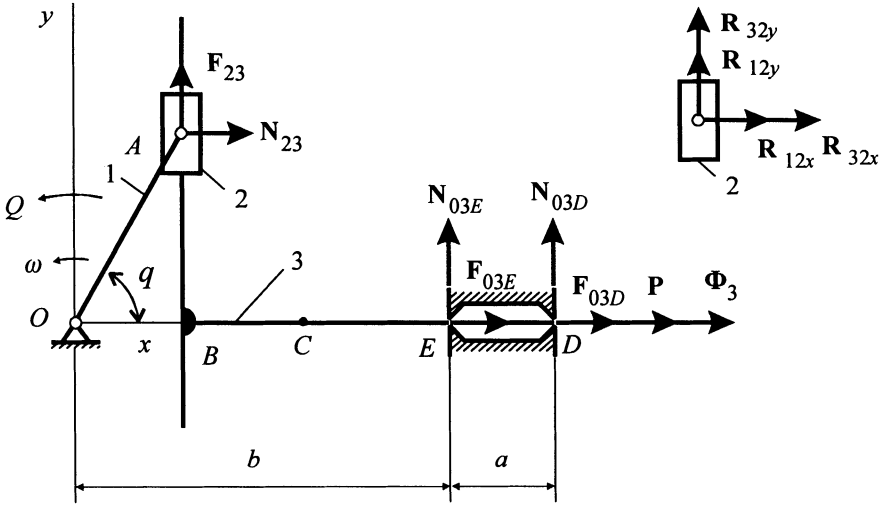


Fig. 5.18

$$R_{12x} = -R_{32x} = R_{23x} = N_{23}, \quad R_{12y} = -R_{32y} = R_{23y} = F_{23}.$$

Let us express the frictional forces through the normal components of reactions:

$$F_{03E} = f |N_{03E}| = f N_{03E} \text{sign} N_{03E},$$

$$F_{03D} = f |N_{03D}| = f N_{03D} \text{sign} N_{03D},$$

$$F_{23} = f_1 |N_{23}| = f_1 N_{23} \text{sign} N_{23}.$$

The inertia force of the third link is $\Phi_3 = -m_3 \ddot{x}_C = m_3 l_1 \omega^2 \cos \omega t$. Let us denote $U = P + m_3 l_1 \omega^2 \cos \omega t$, $\varepsilon_A = \text{sign} N_{23}$, $\varepsilon_D = \text{sign} N_{03D}$, $\varepsilon_E = \text{sign} N_{03E}$. Then, we rewrite the equations for the group of links 2 and 3 in the form:

$$\begin{cases} \varepsilon_E f N_{03E} + \varepsilon_D f N_{03D} + N_{23} = -U, \\ N_{03E} + N_{03D} + \varepsilon_A f_1 N_{23} = 0, \\ (b - l_1 \cos \omega t) N_{03E} + (b + a - l_1 \cos \omega t) N_{03D} - l_1 \sin \omega t N_{23} = 0. \end{cases}$$

We find the required forces according to Cramer's rule:

$$N_{03E} = \frac{\Delta_E}{\Delta}, \quad N_{03D} = \frac{\Delta_D}{\Delta}, \quad N_{23} = \frac{\Delta_A}{\Delta},$$

where the determinants of the equation systems are:

$$\begin{aligned} \Delta &= a - f(\varepsilon_E - \varepsilon_D)(\sin \omega t - f_1 \varepsilon_A \cos \omega t)l_1 + f f_1 \varepsilon_A [b \varepsilon_D - (b + a) \varepsilon_E], \\ \Delta_E &= U [(\sin \omega t - f_1 \varepsilon_A \cos \omega t)l_1 + f_1 \varepsilon_A (b + a)], \end{aligned}$$

$$\Delta_D = -U[(\sin \omega t - f_1 \varepsilon_A \cos \omega t)l_1 + f_1 \varepsilon_A b],$$

$$\Delta_A = -Ua.$$

Setting values $\varepsilon_A = \text{sign} N_{23} = \pm 1$, $\varepsilon_D = \text{sign} N_{03D} = \pm 1$, $\varepsilon_E = \text{sign} N_{03E} = \pm 1$, we obtain $2^3 = 8$ systems of equations. Solving them, we compare the signs obtained for the reactions N_{23} , N_{03D} , N_{03E} with the prescribed signs ε_A , ε_D , ε_E . In one of the cases these signs coincide: $\varepsilon_A = -1$, $\varepsilon_D = -1$, $\varepsilon_E = 1$. The solution of this system is the solution to the problem: $\Delta = 0.07236$ m, $\Delta_E = 18.649$ N m, $\Delta_D = -20.049$ N m, $\Delta_A = -14$ N m; $N_{23} = -193.48$ N, $N_{03D} = -277.07$ N, $N_{03E} = 257.72$ N. The driving moment is $Q = N_{32}l_1 \sin \omega t = -N_{23}l_1 \sin \omega t = 33.51$ N m.

Answer: $N_{23} = -193.48$ N, $N_{03D} = -277.07$ N, $N_{03E} = 257.72$ N, $Q = N_{32}l_1 \sin \omega t = -N_{23}l_1 \sin \omega t = 33.51$ N m.

- 5.10. In a planar cam mechanism (Fig. 5.19) cam 1 rotates with constant velocity $\dot{q}$. Follower 2 has dimensions: $L = 50$ mm, $a = 20$ mm, $h = 10$ mm. The journal radius of the cam shaft is $\rho = 5$ mm, the eccentricity of the cam mechanism is $e = 15$ mm. Conduct a force analysis of the mechanism in the position, for which the coordinate s of point K of the contact between cam and follower is $s = 70$ mm, the pressure angle is $\alpha = 30^\circ$, the workload P_2 and the inertia force Φ_2 of the follower add up to $P_2 + \Phi_2 = 100$ N. The friction coefficients at the revolute pair (f), at the higher pair (f_1) and at the prismatic pair (f_2) are equal to: $f = f_1 = f_2 = 0.1$. Neglect gravity.

Solution: The equations of kinetostatics derived for the follower and the cam are decoupled. They can be solved independently. Let us derive the equations of kinetostatics for the follower. The forces acted upon the follower by the cam are reduced to force N_{12} directed along the normal to the cam profile in the position considered, and the frictional force is F_{12} . Let us choose the model of a prismatic pair with two-point contact. The follower is subject to forces N_A and N_B and to corresponding frictional forces F_A and F_B exerted by the frame. Introduce the coordinate system $O_2x_2y_2$ for the follower; as a result we obtain the equations of kinetostatics in the following form:

$$N_{12}(\cos \alpha - f_1 \text{sign} N_{12} \sin \alpha) - f_2(N_A \text{sign} N_A + N_B \text{sign} N_B) = P_2 + \Phi_2,$$

$$N_{12}(\sin \alpha + f_1 \text{sign} N_{12} \cos \alpha) + N_A + N_B = 0,$$

$$-N_{12}(\sin \alpha + f_1 \text{sign} N_{12} \cos \alpha)L + (a - f_2 h)N_B - (a + f_2 h)N_A = 0.$$

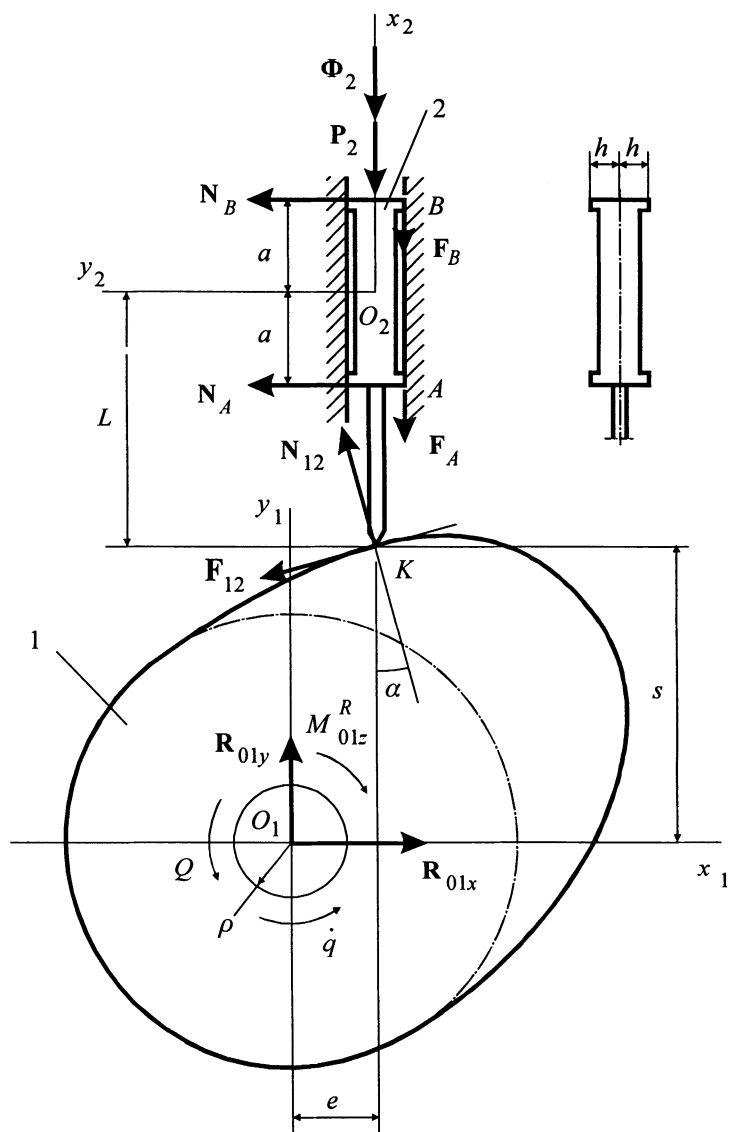


Fig. 5.19

Here, P_2 and Φ_2 are considered as positive if directed as shown in the picture.

The equation system is nonlinear because the functions $\text{sign}N_{12}$, $\text{sign}N_A$, $\text{sign}N_B$ have jumps for $N_{12} = 0$, $N_A = 0$, $N_B = 0$. The equations become linear, if the signs of the unknown reactions are given. The signs of N_{12} , N_A , N_B are chosen a-priorily to be what they would

be, in the same position of the mechanism, in the absence of frictional forces. Setting $f_1 = f_2 = 0$, we obtain:

$$\begin{aligned} N_{12} \cos \alpha &= P_2 + \Phi_2, \quad N_A = -0,5N_{12} \sin \alpha (L/a + 1), \\ N_B &= 0,5N_{12} \sin \alpha (L/a - 1). \end{aligned}$$

For the given direction of forces P_2 and Φ_2 we obtain $N_{12} > 0$, $N_A < 0$, $N_B > 0$ (if $L > a$). Substituting the values $\text{sign} N_{12} = 1$, $\text{sign} N_A = -1$, $\text{sign} N_B = 1$ into the equations of kinetostatics, we arrive at the system of linear equations

$$\begin{aligned} N_{12}(\cos \alpha - f_1 \sin \alpha) + f_2(N_A - N_B) &= P_2 + \Phi_2, \\ N_{12}(\sin \alpha + f_1 \cos \alpha) + N_A + N_B &= 0, \\ -N_{12}(\sin \alpha + f_1 \cos \alpha)L + (a - f_2h)N_B - (a + f_2h)N_A &= 0. \end{aligned}$$

From the last two equations we find N_A and N_B :

$$\begin{aligned} N_A &= -\frac{L + a - f_2h}{2a}(\sin \alpha + f_1 \cos \alpha)N_{12}, \\ N_B &= \frac{L - a - f_2h}{2a}(\sin \alpha + f_1 \cos \alpha)N_{12}. \end{aligned}$$

Substituting N_A and N_B into the first equation of the system we find N_{12} :

$$N_{12} = \frac{P_2 + \Phi_2}{\cos \alpha - f_1 \sin \alpha - f_2 \frac{L - f_2h}{a}(\sin \alpha + f_1 \cos \alpha)} = 148.74 \text{ N}.$$

Then, $N_A = -150.49 \text{ N}$, $N_B = 67.62 \text{ N}$. The corresponding frictional forces are $F_{12} = f_1 N_{12} = 14.87 \text{ N}$, $F_A = f_2 N_A = 15.05 \text{ N}$, $F_B = f_2 N_B = 6.76 \text{ N}$.

Let us write the equations of kinetostatics for the cam in the system $O_1x_1y_1$, using the model of friction at a revolute pair (5.14):

$$\begin{aligned} R_{01x} + N_{12} \sin \alpha + f_1 |N_{12}| \cos \alpha &= 0, \\ R_{01y} - N_{12} \cos \alpha + f_1 |N_{12}| \sin \alpha &= 0, \\ Q - N_{12}e(\cos \alpha - f_1 \sin \alpha) - N_{12}s(\sin \alpha + f_1 \cos \alpha) \\ &\quad - \frac{f\rho}{\sqrt{1+f^2}} \sqrt{R_{01x}^2 + R_{01y}^2} = 0. \end{aligned}$$

Solving the system, we find: $R_{01x} = -87.25 \text{ N}$, $R_{01y} = 121.38 \text{ N}$,

$$Q = 8 \text{ N m}, \quad M_{01z}^R = \frac{f\rho}{\sqrt{1+f^2}} \sqrt{R_{01x}^2 + R_{01y}^2} = 0.074 \text{ N m}.$$

Answer: $N_A = -150.49 \text{ N}$, $F_A = 15.05 \text{ N}$, $N_B = 67.62 \text{ N}$, $F_B = 6.76 \text{ N}$,
 $N_{12} = 148.74 \text{ N}$, $F_{12} = 14.87 \text{ N}$, $R_{01x} = -87.25 \text{ N}$, $R_{01y} = 121.38 \text{ N}$,
 $M_{01z}^R = 0.074 \text{ N m}$, $Q = 8.00 \text{ N m}$.

- 5.11. Solve problem 5.9 for a linear distribution law of normal forces at the prismatic pair connecting link 3 with the frame.

Answer: $N_{23} = 240.01 \text{ N}$, $N_{03D} = 512.02 \text{ N}$, $N_{03E} = 488.02 \text{ N}$.

- 5.12. Determine the reactions at the prismatic kinematic pairs of the scotch-yoke mechanism (see Fig. 5.18) for the following initial data: the mass of the slotted-link is $m_3 = 25 \text{ kg}$, the coefficient of sliding friction at the prismatic pairs is $f = 0.105$, the angular velocity of the crank is $\omega = 14 \text{ s}^{-1} = \text{const}$, the crank length is $l_1 = 0.6 \text{ m}$, $a = 0.1 \text{ m}$, $b = 0.7 \text{ m}$, the force acting on the slotted link is $P = -P_0 \text{sign} v_{30}$, where v_{30} is the relative slotted-link velocity, $P_0 = 430 \text{ N}$. Neglect the friction at the revolute pairs and the weight of the slotted link. Do the computations for $q = 90^\circ, 120^\circ$.

Answer:

q	$N_{03D}, \text{ N}$	$N_{03E}, \text{ N}$	$N_{23}, \text{ N}$
90°	no solution		
120°	-2260	2319	599.2
120° (lateral solution)	20470	-20130	-3223

6 Equations of Motion for a Mechanism with Rigid Links

6.1

Lagrange's Equations of the Second Kind for a Mechanism with a Single Degree of Movability

Until now, it was presumed that the motion law of a mechanism is known: it was assumed that the law is the program law necessary for the execution of a working process. In a real machine the actual motion differs from the program motion. This difference is primarily related to the properties of the engine, setting the machine in motion and generating driving forces applied to input links. The output velocity of the engine depends on the magnitude of the generalized driving force, and this must be taken into account when designing a machine assembly. It is necessary to integrate the system of differential equations of the mechanical system together with the engine characteristic. Usually, differential equations of motion for a mechanism are in the form of Lagrange's equations of the second kind.

For a mechanism with w degrees of movability, with rigid links and with ideal kinematic pairs, the Lagrange's equations of the second kind can be derived from the general equation of dynamics written in the form (4.53). The work which is done in a virtual displacement by inertia forces and which enters the equations can be expressed through the kinetic energy of the system. In courses on analytical mechanics it is proven that

$$\sum_{i=1}^N (\mathbf{\Phi}_i \delta \mathbf{r}_{Oi} + \mathbf{M}_{Oi}^{(\Phi)} \delta \boldsymbol{\varphi}_i) = \sum_{s=1}^w \left(\frac{\partial T}{\partial q_s} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} \right) \delta q_s, \quad (6.1)$$

where $T(q_1, \dots, q_w, \dot{q}_1, \dots, \dot{q}_w)$ is the kinetic energy of the system (in the present case a mechanism), represented as function of generalized coordinates and of their derivatives. As a result, the Eqs. (4.69) for the independent generalized coordinates are reduced to the form:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s + Q_{Rs} \quad (s = 1, \dots, w), \quad (6.2)$$

where

$$Q_{Rs} = \sum_{i=1}^N \left(\mathbf{P}_{Ri} \frac{\partial \mathbf{r}_{Oi}}{\partial q_s} + \mathbf{M}_{Oi}^{(P_0)} \frac{\partial \boldsymbol{\varphi}_i}{\partial q_s} \right) \quad (6.3)$$

are the *generalized forces of resistance* corresponding to all active forces except driving forces. Eqs. (6.2) are the *Lagrange equations of the second kind* for a mechanism.

The kinetic energy of a mechanism with a single degree of movability can always be represented in the form

$$T = \frac{1}{2} a(q) \dot{q}^2. \quad (6.4)$$

If q is a linear generalized coordinate, then the inertia coefficient $a(q) = m(q)$ is called *reduced mass* of a mechanism; for an angular generalized coordinate the quantity $a(q)$ has the dimension of moment of inertia and it is called a *reduced moment of inertia*. Henceforth, it is assumed that q is an angular coordinate and that expression (6.4) is written in the form

$$T = \frac{1}{2} J(q) \dot{q}^2, \quad (6.5)$$

where $J(q)$ is the reduced moment of inertia. Substituting (6.5) into (6.2) and taking into account that

$$\frac{\partial T}{\partial q} = \frac{1}{2} J'(q) \dot{q}^2, \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}} = \frac{d}{dt} [J(q) \cdot \dot{q}] = J'(q) \dot{q}^2 + J(q) \ddot{q},$$

we obtain

$$J(q) \ddot{q} + \frac{1}{2} J'(q) \dot{q}^2 = Q + Q_R. \quad (6.6)$$

Let us proceed to examples for deriving the equations of motion of a mechanism. Consider the slider-crank mechanism shown in Fig. 6.1.

We define the kinetic energy of the mechanism as a sum of the kinetic energies of its movable links. For the rotating link 1 we have

$$T_1 = \frac{1}{2} J_{10} \dot{q}^2,$$

where J_{10} is the moment of inertia of the link with respect to its axis. For the translationally moving slider 3 we obtain

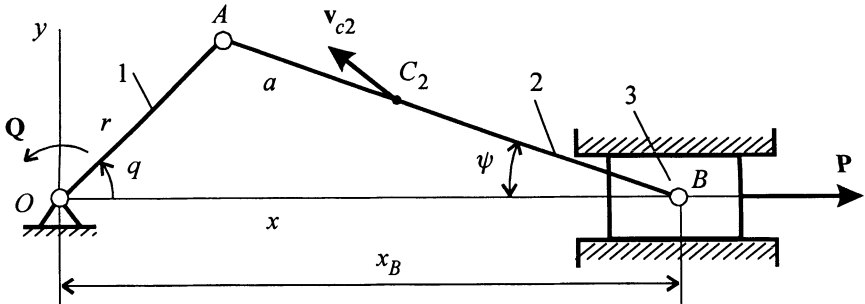


Fig. 6.1. Slider-crank mechanism

$$T_3 = \frac{1}{2} m_3 \dot{x}_B^2.$$

For link 2 executing a complex motion we find the kinetic energy by using the theorem of König, known from courses on theoretical mechanics:

$$T_2 = \frac{1}{2} (m_2 v_{C2}^2 + J_2^C \omega_2^2),$$

where m_2 is the link mass; J_2^C is the link moment of inertia with respect to the axis passing through the mass center C_2 and orthogonal to the motion plane; v_{C2} is the velocity of the mass center; ω_2 is the angular velocity. In this way, taking into account that

$$\dot{x}_B = \frac{dx_B}{dq} \dot{q}, \quad v_{C2}^2 = \left[\left(\frac{dx_{C2}}{dq} \right)^2 + \left(\frac{dy_{C2}}{dq} \right)^2 \right] \dot{q}^2, \quad \omega_2 = \dot{\psi} = \frac{d\psi}{dq} \dot{q},$$

where ψ is the rotation angle of link 2, we obtain

$$\begin{aligned} T &= T_1 + T_2 + T_3 = \\ &= \frac{1}{2} \left\{ J_{10} + m_2 \left[\left(\frac{dx_{C2}}{dq} \right)^2 + \left(\frac{dy_{C2}}{dq} \right)^2 \right] + J_2^C \left(\frac{d\psi}{dq} \right)^2 + m_3 \left(\frac{dx_B}{dq} \right)^2 \right\} \dot{q}^2 = \quad (6.7) \\ &= \frac{1}{2} J(q) \dot{q}^2. \end{aligned}$$

The expression in the braces is the reduced moment of inertia $J(q)$ of the mechanism. Employing the position functions $x_{C2}(q)$, $y_{C2}(q)$, $\psi(q)$, $x_B(q)$, one may represent $J(q)$ in an explicit form, but this analytic expression, even for such a comparatively simple linkage, proves to be rather cumbersome. Even more complex is the expression for $J'(q)$. For this reason, in practice, approximate representations of these functions based on Fourier series expansions are often used. It is easy to see that $J(q)$ is a periodic function with period of 2π and that it can be represented in the form of series:

$$J(q) = J_0 + \sum_{\ell=1}^{\infty} (J_{C\ell} \cos \ell q + J_{S\ell} \sin \ell q). \quad (6.8)$$

In order to determine the Fourier coefficients $J_0, J_{C1}, \dots, J_{S1}$, the values of $J(q)$ are computed for certain discrete values of q , e.g., for $q = 2\pi k / m$ ($k = 1, \dots, m$). To achieve this, the expression for $J(q)$ is used in the form (6.7), while the values of the derivatives of position functions are determined in the process of kinematic analysis of the mechanism. Then, familiar approximate relationships are used, expressing the Fourier coefficients through discrete values of a periodic function:

$$\begin{aligned}
J_0 &= \frac{1}{m} \sum_{k=1}^m J(2\pi k / m), \\
J_{C\ell} &= \frac{2}{m} \sum_{k=1}^m J(2\pi k / m) \cos(2\pi k \ell / m), \\
J_{S\ell} &= \frac{2}{m} \sum_{k=1}^m J(2\pi k / m) \sin(2\pi k \ell / m).
\end{aligned} \tag{6.9}$$

Furthermore, approximate representations of the functions $J(q)$ and $J'(q)$ are written

$$J(q) \approx J_0 + \sum_{\ell=1}^r (J_{C\ell} \cos \ell q + J_{S\ell} \sin \ell q), \tag{6.10}$$

$$J'(q) \approx \sum_{\ell=1}^r (-J_{C\ell} \sin \ell q + J_{S\ell} \cos \ell q) \ell. \tag{6.11}$$

A satisfactory approximation for the ℓ -th harmonic is obtained only under the condition $m \geq 4\ell$. It must also be recognized that the suppression of higher harmonics in the expression for the reduced moment of inertia restricts the application area of Eq. (6.6).

For deriving Lagrange equations, it is also necessary to determine the generalized force Q_R as a function of $q, \dot{q}, t$. Let us assume that the forces of gravity of mechanism links can be neglected and that the only active force of resistance during the execution of a working process is the force $\mathbf{P}$ depending on x_B and $\dot{x}_B$ (Fig. 6.1). Thus, we get from formula (6.3)

$$Q_R = P \frac{dx_B}{dq} = P \left[x_B(q), \frac{dx_B}{dq} \dot{q} \right] \frac{dx_B}{dq} = Q_R(q, \dot{q}). \tag{6.12}$$

The generalized force Q_R is often referred to as *reduced moment of resistance forces*. With respect to q the function $Q_R(q, \dot{q})$ is periodic with period 2π . If the input link is connected with the crank by means of a transmission mechanism with transmission ratio i , then the period is equal to $2\pi i$.

As a second example, let us consider the mechanism with a linear position function shown in Fig. 6.2. It consists of a two-stage transmission (wheels 1–4) and of rotor 5. Let $J_1 - J_5$ be the moments of inertia of the rotating masses with respect to their rotation axes; $z_1 - z_4$ are the numbers of teeth of the wheels; M_R is the moment of the resistance forces applied to the rotor. Composing the expression for the kinetic energy of the system, we have

$$\begin{aligned}
T &= \frac{1}{2} \left[J_1 \dot{q}^2 + (J_2 + J_3) \left(-\dot{q} \frac{z_1}{z_2} \right)^2 + (J_4 + J_5) \left(\dot{q} \frac{z_1 z_3}{z_2 z_4} \right)^2 \right] = \\
&\frac{1}{2} \left[J_1 + (J_2 + J_3) \left(\frac{z_1}{z_2} \right)^2 + (J_4 + J_5) \left(\frac{z_1 z_3}{z_2 z_4} \right)^2 \right] \dot{q}^2 = \frac{1}{2} J \dot{q}^2.
\end{aligned} \tag{6.13}$$

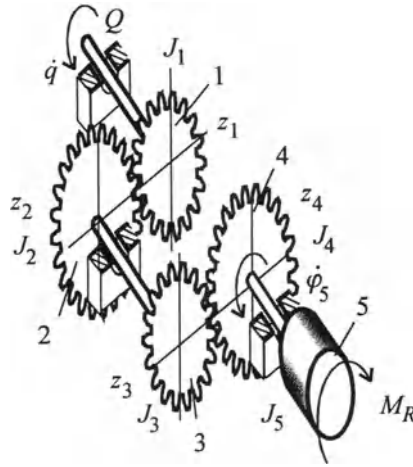


Fig. 6.2. Mechanism with a linear position function

In this case, the reduced moment of inertia does not depend on the coordinate q . The generalized force Q_R is determined from (6.3):

$$Q_R = -M_R \frac{z_1 z_3}{z_2 z_4}. \quad (6.14)$$

Substituting (6.13) and (6.14) into (6.6), we obtain the Lagrange equations

$$J \cdot \ddot{q} = Q - M_R \frac{z_1 z_3}{z_2 z_4}, \quad (6.15)$$

where

$$J = J_1 + (J_2 + J_3) \left(\frac{z_1}{z_2} \right)^2 + (J_4 + J_5) \left(\frac{z_1 z_3}{z_2 z_4} \right)^2 \quad (6.16)$$

is the reduced inertia moment of the mechanism. Note, that when reducing rotating masses the moment of inertia of every single one is divided by the square transmission ratio relating the mass to the input link.

Lagrange's equation of the second kind as well as the equation of d'Alembert–Lagrange can be used to determine the generalized driving force Q . For a given motion of the input link we find from Eq. (6.6):

$$Q(t) = J[q(t)]\ddot{q}(t) + 0.5J'[q(t)]\dot{q}^2 - Q_R[q(t), \dot{q}(t), t]. \quad (6.17)$$

However, the role of the Lagrange's equations in the dynamics of machines is not exhausted by this. As mentioned above, they are also used as differential equations of the motion of the mechanical part of a machine, from which the motion law $q(t)$ is determined. In both cases it is necessary to know the functions $J(q)$, $J'(q)$, $Q_R(q, \dot{q})$ in order to compose the Lagrange equations. The formulation of

these functions in analytical form usually requires rather cumbersome transformations for the composition of the expression of the kinetic energy and of its derivatives as well as for the determination of the work done by the active forces. Often, the equations of kinetostatics or d'Alembert–Lagrange equations are used to solve the first problem (determination of generalized driving force for a given motion law). Moreover, it is easy to show that these equations enable us to determine the quantities J and J' for a given value of q as well as the quantity Q_R for given q and $\dot{q}$.

In fact, let us assume that we have already defined, e.g., with the help of the equations of kinetostatics, the value of the generalized driving force Q under the following conditions:

$$q = q_*, \quad \dot{q} = 0, \quad \ddot{q} = 1 \text{ s}^{-2}, \quad P = 0,$$

i.e. for zero angular velocity of the input link, for unit angular acceleration and with no active forces (which is equivalent to $Q_R = 0$). Let us suppose that under these conditions we have $Q = Q_{1*}$ (Nm). Substituting the chosen values of q , $\dot{q}$, Q_R into Eq. (6.17), we find that

$$Q_{1*}(\text{Nm}) = J(q_*)(\text{Nm s}^2) \cdot 1(\text{s}^{-2}),$$

i.e. that the *numerical* value of Q_{1*} coincides with that of $J(q_*)$ expressed in Nm s^2 . Thus, in order to determine $J(q_*)$, it is not necessary to find the expression for the kinetic energy of the mechanism; the expression can be determined with the help of the equations of kinetostatics applied to some „conditional“ motion law. Giving to q different values and repeating this procedure, we find the values of $J(q)$ at a number of discrete points, which allows us to approximate this function, e.g., by a part of Fourier series (6.10). Let us find now $Q = Q_{2*}$ (Nm) for $q = q_*$, $\dot{q} = 1 \text{ s}^{-1}$, $\ddot{q} = 0$ and in the absence of frictional forces. Substituting these values into (6.17), we obtain $J'(q_*)(\text{Nm s}^2) \cdot 1 \text{ s}^{-2} = Q_{2*}$. Finally, having set $q = q_*$, $\dot{q} = \dot{q}_*$, $\ddot{q} = 0$, we find $Q = Q_{3*}$ for the given forces of resistance. Thus, from (6.17) we obtain

$$Q_R(q_*, \dot{q}_*) = 0.5 J'(q_*) \dot{q}_*^2 - Q_{3*}.$$

6.2

Lagrange's Equations of the Second Kind for Mechanisms with Several Degrees of Movability

As it is known from courses on analytic mechanics [9], the kinetic energy of a mechanical system with w degrees of freedom under holonomic-scleronic

constraints can be represented as a quadratic form of generalized velocities with coefficients depending on the generalized coordinates:

$$T = \frac{1}{2} \sum_{s,k=1}^w a_{sk}(q_1, \dots, q_w) \dot{q}_s \dot{q}_k. \quad (6.18)$$

Let us substitute this expression into the left-hand side of Eqs. (6.2). Differentiating (6.18) with respect to q_s and $\dot{q}_s$, we have

$$\frac{\partial T}{\partial \dot{q}_s} = \sum_{k=1}^w a_{sk} \dot{q}_k, \quad \frac{\partial T}{\partial q_s} = \frac{1}{2} \sum_{i,k=1}^w \frac{\partial a_{ik}}{\partial q_s} \dot{q}_i \dot{q}_k. \quad (6.19)$$

Furthermore, we find

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} = \sum_{k=1}^w \left(\sum_{i=1}^w \frac{\partial a_{sk}}{\partial q_i} \dot{q}_i \dot{q}_k + a_{sk} \ddot{q}_k \right). \quad (6.20)$$

Since

$$\sum_{k=1}^w \sum_{i=1}^w \frac{\partial a_{sk}}{\partial q_i} \dot{q}_i \dot{q}_k = \frac{1}{2} \sum_{i,k=1}^w \left(\frac{\partial a_{sk}}{\partial q_i} + \frac{\partial a_{si}}{\partial q_k} \right) \dot{q}_i \dot{q}_k,$$

we obtain from (6.19) and (6.20)

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = \frac{1}{2} \sum_{i,k=1}^w \left(\frac{\partial a_{sk}}{\partial q_i} + \frac{\partial a_{si}}{\partial q_k} - \frac{\partial a_{ik}}{\partial q_s} \right) \dot{q}_i \dot{q}_k + \sum_{k=1}^w a_{sk} \ddot{q}_k.$$

The expression

$$[i, k, s] = \frac{1}{2} \left(\frac{\partial a_{sk}}{\partial q_i} + \frac{\partial a_{si}}{\partial q_k} - \frac{\partial a_{ik}}{\partial q_s} \right)$$

is called *Christoffel symbol of first kind*. Using this expression, we write Eq. (6.2) in the following form:

$$\sum_{k=1}^w a_{sk} \ddot{q}_k + \sum_{i,k=1}^w [i, k, s] \dot{q}_i \dot{q}_k = Q_s + Q_{Rs} \quad (s = 1, \dots, w). \quad (6.21)$$

Eqs. (6.21) are used for the investigation of the motion of mechanisms with several degrees of movability. Moreover, a method analogous to the one considered in Sect. 6.1, is employed to obtain the values of coefficients a_{sk} , the Christoffel $[i, k, s]$ symbols as well as the values of the generalized resistance forces Q_{Rs} corresponding to certain given values of q_k and $\dot{q}_k$.

Suppose we have a computational algorithm enabling us to determine generalized driving forces Q_s for given q_k , $\dot{q}_k$, $\ddot{q}_k$. Let us prescribe specific values of coordinates $q_1^*, \dots, q_w^*$ defining the position of a mechanism. We first assume that all $Q_{Rs} = 0$ (no active forces in the system). Let us also assume that $\dot{q}_i = 0$ ($i = 1, \dots, w$) for $i \neq k$ and that $\ddot{q}_k$ is equal to unity (1 ms^{-2} or 1 s^{-2} , depending on whether coordinate q_k is linear or angular). With the help of the available

algorithm we calculate the values of the generalized driving forces Q_s ($s=1, \dots, w$) under the given conditions. Let them be denoted $Q_s = Q_{sk}^I$ (N or Nm). On the other hand, the chosen values of coordinates, velocities, accelerations and forces must satisfy Eqs. (6.21). Substituting them into these equations, we obtain

$$a_{sk}(q_1^*, \dots, q_w^*) \cdot 1 = Q_s = Q_{sk}^I. \quad (6.22)$$

From here it follows that the quantity $a_{sk}(q_1^*, \dots, q_w^*)$ numerically coincides with the value Q_{sk}^I . Set now $Q_{sk} = 0$ ($s=1, \dots, w$), $\ddot{q}_i = 0$ ($i=1, \dots, w$); $\dot{q}_i = 0$ for $i \neq k$, $\dot{q}_k = 1$ (ms^{-1} or s^{-1}). Then analogously, having determined the driving forces from the available computational algorithm and having obtained that $Q_{sk} = Q_{sk}^{II}$, we find after substituting into (6.21)

$$[k, k, s]^* \cdot 1^2 = Q_{sk}^{II}, \quad (6.23)$$

where $[k, k, s]$ is the value of the Christoffel symbol for $q_i = q_i^*$.

Next, we set $Q_{Rs} = 0$ ($s=1, \dots, w$); $\ddot{q}_i = 0$ ($i=1, \dots, w$), $\dot{q}_i = 0$ for $i \neq j, k$; $\dot{q}_j = 1$ (ms^{-1} or s^{-1}), $\dot{q}_k = 1$ (ms^{-1} or s^{-1}). After substituting the values of $\dot{q}_s$, $\ddot{q}_s$, Q_{Rs} into (6.21) we find:

$$2[j, k, s]^* \cdot 1^2 + [j, j, s]^* \cdot 1^2 + [k, k, s]^* \cdot 1^2 = Q_{sjk}. \quad (6.24)$$

From here we obtain

$$[j, k, s]^* = \frac{1}{2}(Q_{sjk} - Q_{sj}^{II} - Q_{sk}^{II}), \quad (6.25)$$

where Q_{sjk} is the s -th driving force, calculated for the given conditions. In this way, all Christoffel symbols are determined for the considered position of the mechanism.

Finally, suppose that for given $q_s = q_s^*$, $\dot{q}_s = \dot{q}_s^*$ and for a given value of t it is necessary to determine the values of the generalized resistance forces $Q_{Rs}(q_1^*, \dots, q_w^*, \dot{q}_1^*, \dots, \dot{q}_w^*)$. We assume $\ddot{q}_k = 0$ ($k=1, \dots, w$) and we determine the driving forces under the chosen conditions with the help of the available algorithm. Assume now that $Q_s = Q_s^{III}$. After substituting the given conditions into (6.21), we obtain

$$Q_{Rs}(q_1^*, \dots, q_w^*, \dot{q}_1^*, \dots, \dot{q}_w^*, t) = Q_s^{III} - \sum_{i,k=1}^w [i, k, s]^* \dot{q}_i^* \dot{q}_k^*, \quad (6.26)$$

where the $[i, k, s]^*$ have already been determined in the previous stages.

6.3

An Example for Derivation of the Equations of Motion of a Mechanism

As an example, illustrating the suggested method, let us consider the planar three-bar mechanism in the position shown in Fig. 6.3. The links of the mechanism are homogeneous rods of length 1 and mass m ; q_1 , q_2 , q_3 are the rotation angles at joints O , A , B . For the considered position we have $q_1 = 0$, $q_2 = \pi/2$, $q_3 = -\pi/2$. There are no resistance forces.

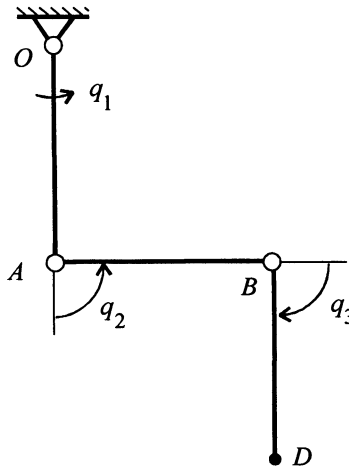


Fig. 6.3. Planar mechanism of movability three

First, define the coefficients a_{sk} and the Christoffel symbols using the expression for the kinetic energy, obtained in an analytical form. After determining the velocity components of the mass centers C_2 and C_3 of links AB and BD , we obtain (Fig. 6.4):

$$\begin{aligned}
 T = & \frac{1}{2} \left\{ \frac{1}{3} m \ell^2 \dot{q}_1^2 + \left[m v_{c2}^2 + \frac{1}{12} m \ell^2 (\dot{q}_1 + \dot{q}_2)^2 \right] + \right. \\
 & \left. \left[m v_{c3}^2 + \frac{1}{12} m \ell^2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \right] \right\} = \\
 & \frac{1}{2} m \ell^2 \left\{ \left[4 + 3 \cos q_2 + \cos(q_2 + q_3) + \cos q_3 \right] \dot{q}_1^2 + \left(\frac{5}{3} + \cos q_3 \right) \dot{q}_2^2 + \right. \\
 & \frac{1}{3} \dot{q}_3^2 + 2 \left[\frac{5}{3} + \frac{3}{2} \cos q_2 + \frac{1}{2} \cos(q_2 + q_3) + \cos q_3 \right] \dot{q}_1 \dot{q}_2 + \\
 & \left. 2 \left[\frac{1}{3} + \frac{1}{2} \cos(q_2 + q_3) + \frac{1}{2} \cos q_3 \right] \dot{q}_1 \dot{q}_3 + 2 \left(\frac{1}{3} + \frac{1}{2} \cos q_3 \right) \dot{q}_2 \dot{q}_3 \right\}. \quad (6.27)
 \end{aligned}$$

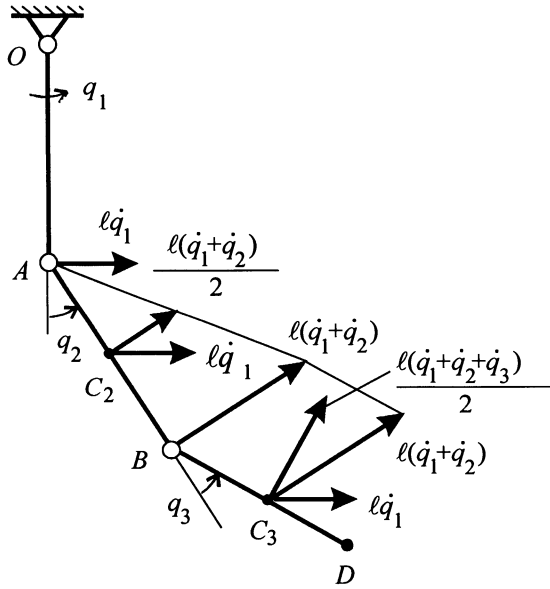


Fig. 6.4. The velocities of the mass centers C_2 and C_3 of a mechanism of movability three

Taking into account that in the considered position $\cos q_2 = 0$, $\cos q_3 = 0$, $\cos(q_2 + q_3) = 1$, we have

$$\begin{aligned} a_{11} &= 5m\ell^2, & a_{22} &= \frac{5}{3}m\ell^2, & a_{33} &= \frac{1}{3}m\ell^2, \\ a_{12} &= \frac{13}{6}m\ell^2, & a_{13} &= \frac{5}{6}m\ell^2, & a_{23} &= \frac{1}{3}m\ell^2. \end{aligned} \quad (6.28)$$

Substituting a_{ik} from (6.27) into the expressions for Christoffel symbols, we find:

$$\begin{aligned} [1,1,1] &= 0, & [1,1,2] &= \frac{3}{2}m\ell^2, & [1,1,3] &= -\frac{1}{2}m\ell^2, & [2,2,1] &= -\frac{3}{2}m\ell^2, \\ [2,2,3] &= -\frac{1}{2}m\ell^2, & [3,3,1] &= \frac{1}{2}m\ell^2, & [3,3,2] &= \frac{1}{2}m\ell^2, & [1,2,1] &= -\frac{3}{2}m\ell^2, \\ [1,2,3] &= -\frac{1}{2}m\ell^2, & [1,3,1] &= \frac{1}{2}m\ell^2, & [1,3,2] &= \frac{1}{2}m\ell^2, & [2,3,3] &= 0, \\ [2,3,1] &= \frac{1}{2}m\ell^2, & [2,3,2] &= \frac{1}{2}m\ell^2, & [2,2,2] &= [3,3,3] = [1,2,2] = [1,3,3] &= 0. \end{aligned} \quad (6.29)$$

Next, we show how the parameters (6.28) and (6.29) can be determined with the help of the method of kinetostatics.

1. Set $\ddot{q}_1 = 1 \text{ s}^{-2}$, $\ddot{q}_2 = \ddot{q}_3 = 0$, $\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$. Let us determine the inertia force components along the x - and y -axes as well as the moments of the inertia forces of the links (Fig. 6.5a):

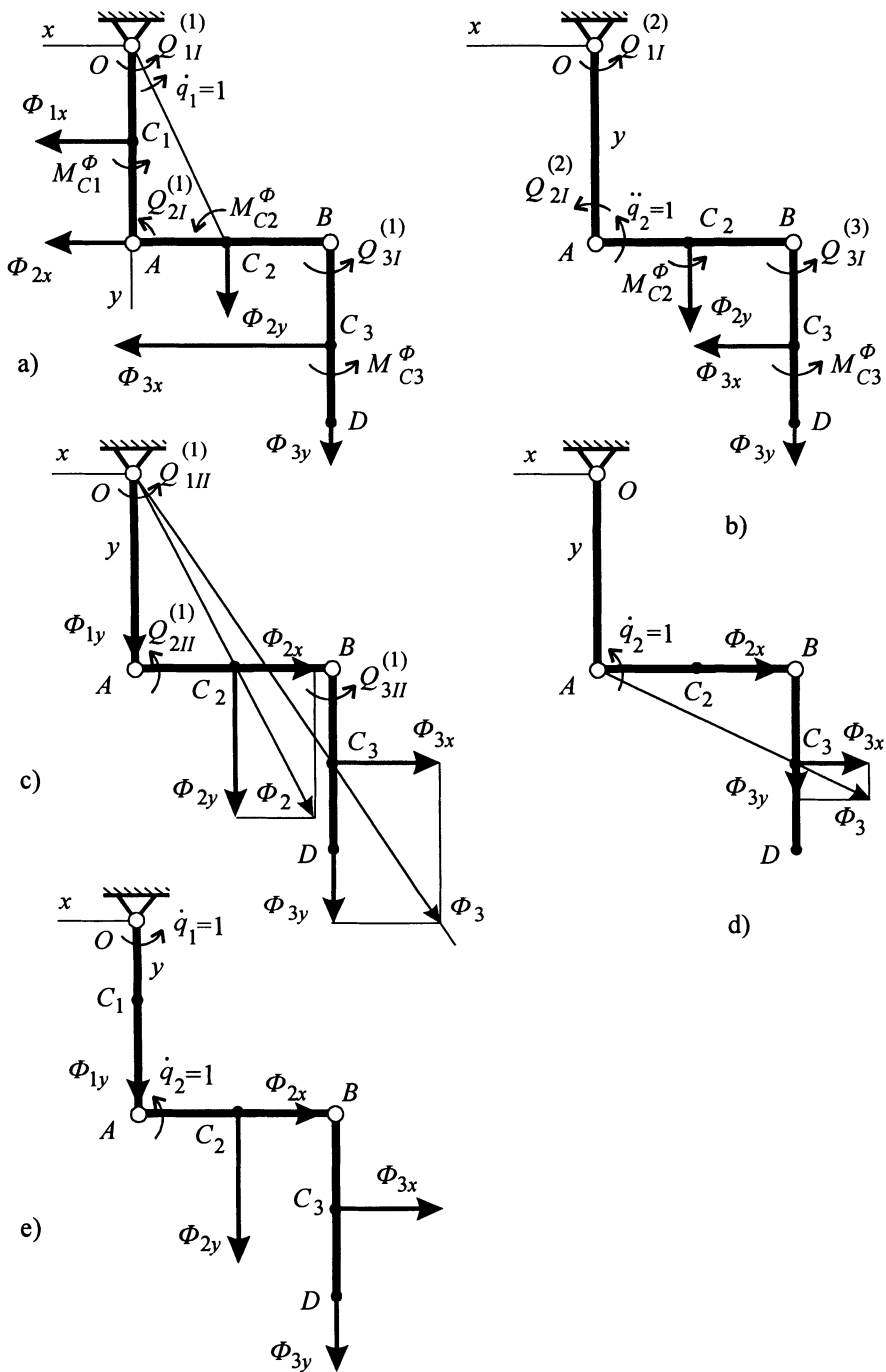


Fig. 6.5. Inertia forces and moments of inertia forces of a mechanism of movability three

$$\begin{aligned}\Phi_{1x} &= -mw_{C1x} = m\frac{\ell}{2}\ddot{q}_1 = \frac{1}{2}m\ell(N), \quad \Phi_{2x} = m\ell\ddot{q}_1 = m\ell(N), \\ \Phi_{2y} &= m\frac{\ell}{2}\ddot{q}_1 = \frac{1}{2}m\ell(N), \quad \Phi_{3x} = m\frac{3}{2}\ell\ddot{q}_1 = \frac{3}{2}m\ell(N), \quad \Phi_{3y} = m\ell\ddot{q}_1 = m\ell(N), \\ M_{C1}^\Phi &= M_{C3}^\Phi = -\frac{m\ell^2}{12}\dot{q}_1 = -\frac{1}{12}m\ell^2(N\cdot m).\end{aligned}$$

The driving moments to be applied at joints O , A , B are equal to the sum of the moments of the inertia forces of the corresponding links. From here, we obtain the coefficients a_{11}, a_{21}, a_{31} according to formulae (6.22):

$$\begin{aligned}a_{11} &= Q_{11}^I = m\ell^2\left(\frac{1}{2}\cdot\frac{1}{2} + 1\cdot 1 + \frac{1}{2}\cdot\frac{1}{2} + \frac{3}{2}\cdot\frac{3}{2} + 1\cdot 1 + 3\cdot\frac{1}{12}\right) = 5m\ell^2, \\ a_{21} &= Q_{21}^I = m\ell^2\left(\frac{1}{2}\cdot\frac{1}{2} + 1\cdot 1 + \frac{3}{2}\cdot\frac{1}{2} + 2\cdot\frac{1}{12}\right) = \frac{13}{6}m\ell^2, \\ a_{31} &= Q_{31}^I = m\ell^2\left(\frac{3}{2}\cdot\frac{1}{2} + \frac{1}{2}\right) = \frac{5}{6}m\ell^2.\end{aligned}$$

We assume that $\ddot{q}_1 = \ddot{q}_3 = 0$, $\ddot{q}_2 = 1\text{ s}^{-2}$, $\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$. Then (Fig. 6.5b):

$$\begin{aligned}\Phi_1 &= 0, \quad \Phi_{2x} = 0, \quad \Phi_{2y} = \frac{m\ell}{2}(N), \quad \Phi_{3x} = \frac{1}{12}m\ell(N), \\ \Phi_{3y} &= m\ell(N), \quad M_{c2}^\Phi = M_{c3}^\Phi = -\frac{1}{12}m\ell^2(N\cdot m), \quad M_{c1}^\Phi = 0.\end{aligned}$$

Having determined the sum of the moments of all these forces about the axes O , A , B , we find the driving forces Q_{12}^I , Q_{22}^I , Q_{32}^I and the numerically identical coefficients a_{12} , a_{22} , a_{32} :

$$a_{12} = Q_{12}^I = \frac{13}{6}m\ell^2, \quad a_{22} = Q_{22}^I = \frac{5}{3}m\ell^2, \quad a_{32} = Q_{32}^I = \frac{1}{3}m\ell^2.$$

In an analogous way we find a_{13} , a_{23} , a_{33} for the conditions $\ddot{q}_1 = \ddot{q}_2 = 0$, $\ddot{q}_3 = 1\text{ s}^{-2}$, $\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$ (naturally, $a_{13} = a_{31}$, $a_{23} = a_{32}$). It is obvious that the values of a_{ik} coincide with the values obtained above.

2. For determining the Christoffel symbols $[1,1,s]$ ($s=1,2,3$) we set the following conditions $\dot{q}_1 = 1\text{ s}^{-1}$, $\dot{q}_2 = \dot{q}_3 = 0$, $\ddot{q}_1 = \ddot{q}_2 = \ddot{q}_3 = 0$ and we determine the forces and the moments of inertia forces of the links (Fig. 6.5c):

$$\begin{aligned}\Phi_{1x} &= 0, \quad \Phi_{1y} = m\frac{\ell}{2}\dot{q}_1^2 = m\frac{\ell}{2}(N), \\ \Phi_{2x} &= -\frac{m\ell}{2}\dot{q}_1^2 = -\frac{1}{2}m\ell(N), \quad \Phi_{2y} = m\ell\dot{q}_1^2 = m\ell(N),\end{aligned}$$

$$\Phi_{3x} = -m\ell\dot{q}_1^2 = -m\ell(\text{N}), \quad \Phi_{3y} = \frac{3}{2}m\ell\dot{q}_1^2 = \frac{3}{2}m\ell(\text{N}), \quad M_{c1}^\Phi = M_{c2}^\Phi = M_{c3}^\Phi = 0.$$

From here, we find the driving moments Q_{k1}'' as sums of the moments of inertia forces with respect to the axes O , A , B and the corresponding Christoffel symbols (note that the lines of action of the inertia forces Φ_1 , Φ_2 , Φ_3 are passing through point O):

$$\begin{aligned} [1,1,1] &= Q_{11}'' = 0, \quad [1,1,2] = Q_{21}'' = \Phi_{2y}\frac{\ell}{2} + \Phi_{3y}\ell - \Phi_{3x}\frac{\ell}{2} = \frac{3}{2}m\ell^2(\text{Nm}), \\ [1,1,3] &= Q_{31}'' = \Phi_{3x}\frac{\ell}{2} = -\frac{1}{2}m\ell^2(\text{Nm}). \end{aligned}$$

3. The inertia forces corresponding to $\dot{q}_1 = \dot{q}_3 = 0$, $\dot{q}_2 = 1\text{ s}^{-1}$, $\ddot{q}_1 = \ddot{q}_2 = \ddot{q}_3 = 0$ are shown in Fig. 6.5d. Their moments about axes O , A , B are:

$$\begin{aligned} [2,2,1] &= Q_{12}'' = -\Phi_{2x}\ell - \Phi_{3x}\frac{3}{2}\ell + \Phi_{3y}\ell = -\frac{3}{2}m\ell^2(\text{Nm}), \\ [2,2,2] &= Q_{22}'' = 0, \quad [2,2,3] = Q_{32}'' = -\frac{1}{2}m\ell^2(\text{Nm}). \end{aligned}$$

4. Let us determine the Christoffel symbols $[1,2,s]$ ($s=1,2,3$). For this purpose we prescribe the conditions $\dot{q}_1 = 1\text{ s}^{-1}$, $\dot{q}_2 = 1\text{ s}^{-1}$, $\dot{q}_3 = 0$, $\ddot{q}_1 = \ddot{q}_2 = \ddot{q}_3 = 0$. Referring to Fig. 6.5e we find after simple calculations

$$\Phi_{1y} = \frac{1}{2}m\ell, \quad \Phi_{2x} = -2m\ell, \quad \Phi_{2y} = m\ell, \quad \Phi_{3x} = -4m\ell, \quad \Phi_{3y} = 3m\ell.$$

It follows from here

$$Q_{112} = -2m\ell \cdot \ell + m\ell \cdot \frac{\ell}{2} - 4m\ell \cdot \frac{3}{2}m\ell + 3m\ell \cdot \ell = -\frac{9}{2}m\ell^2.$$

Furthermore, we obtain

$$\begin{aligned} Q_{212} &= m\ell \cdot \frac{\ell}{2} + 3m\ell \cdot \ell - 4m\ell \cdot \frac{\ell}{2} = \frac{3}{2}m\ell^2, \\ [1,2,2] &= \frac{1}{2}\left(\frac{3}{2}m\ell^2 - \frac{3}{2}m\ell^2\right) = 0, \quad Q_{312} = -4m\ell \cdot \frac{\ell}{2} = -2m\ell^2, \\ [1,2,3] &= \frac{1}{2}\left(-2m\ell^2 + \frac{m\ell^2}{2} + \frac{m\ell^2}{2}\right) = -\frac{m\ell^2}{2}. \end{aligned}$$

The remaining Christoffel symbols can be determined analogously; all of them are, of course, identical with those obtained directly from the expression for the kinetic energy. It is natural that in this simple example the second method does not offer any obvious advantage. However, the advantages become highly apparent in

more complex cases in which the determination of generalized driving forces by a computer algorithm based on the method of kinetostatics, turns out to be much simpler than the analytic derivation of the Lagrange equations of second kind.

6.4 Problems

- 6.1. In the cargo winch (Fig. 6.6) a gear with number of teeth $z_1 = 30$ is mounted on the shaft of an electric motor. With this gear a second gear ($z_2 = 60$) is meshing on the shaft of which a drum with radius $r = 0.3$ m is mounted. A cable wound around the drum is fastened to a weight of mass $m = 100$ kg. The moment of inertia of the mass on the motor shaft is $I_1 = 0.05$ kg m² and the moment of inertia of the mass on the drum shaft is $I_2 = 0.09$ kg m². Formulate the equation of motion of the winch mechanism when it lifts the weight taking as generalized coordinate q the rotation angle of the motor shaft. Neglect the mass of the cable.

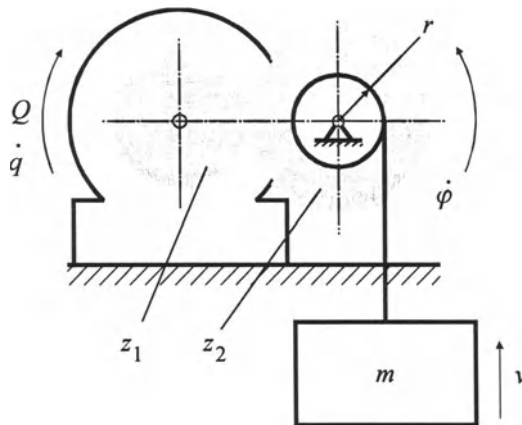


Fig. 6.6

Solution: We write the equilibrium equation in the form (6.6):

$$J(q)\ddot{q} + \frac{1}{2}J'(q)\dot{q}^2 = Q + Q_R.$$

For determining $J(q)$ we compose the expression for the kinetic energy:

$$T = \frac{1}{2}(J_1\dot{q}^2 + J_2\dot{\phi}^2 + m\dot{v}^2) = \frac{1}{2}J\dot{q}^2,$$

where $\dot{q}$ is the angular velocity of the motor shaft, $\dot{\phi}$ is the angular velocity of the drum shaft, m is the mass of the lifted weight, v is its velocity.

Taking into account the ratio of the velocities $|\dot{\phi}/\dot{q}| = z_1/z_2$, $|v/\dot{q}| = |\dot{\phi}r/\dot{q}| = z_1r/z_2$, we obtain the expression for the reduced moment of inertia J :

$$J = J_1 + J_2(z_1/z_2)^2 + mr^2(z_1/z_2)^2 = \text{const} = 2.3225 \text{ kg m}^2.$$

We determine the generalized force Q_R in accordance with (6.3):

$$Q_R = -mgr(z_1/z_2) = -147 \text{ N m}.$$

Thus, the motion equation takes the form:

$$2.3225\ddot{q} = Q - 147$$

$$\text{Answer: } 2.3225\ddot{q} = Q - 147.$$

- 6.2. A single-stage epicyclic-gear-train reducer (Fig. 6.7) consists of sun gear 1, planet pinion 2, planet carrier H and motionless internal gear 3. The numbers of wheel teeth are: $z_1 = 30$, $z_2 = 30$, $z_3 = 90$. A workload $M_H = 100 \text{ N m}$ is applied to the carrier H and a driving torque Q to the wheel shaft 1. The mass centers of the links are located on the axes of relative rotation, the moments of inertia of the links are $J_1 = 0.1 \text{ kg m}^2$, $J_2 = 0.1 \text{ kg m}^2$, $J_H = 0.16 \text{ kg m}^2$, the pinion mass is $m = 0.2 \text{ kg}$, the carrier radius is $r = 0.2 \text{ m}$. Formulate the equation of motion of the reducer.

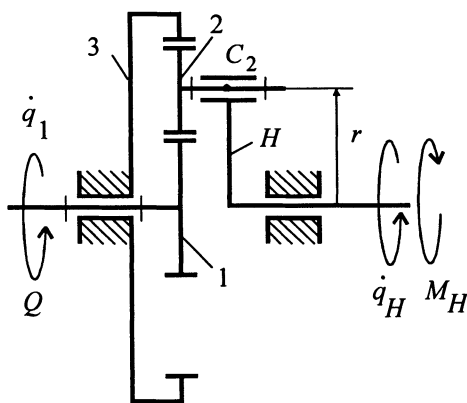


Fig. 6.7

Solution: We write the equation of motion in the form (6.6):

$$J(q_1)\ddot{q}_1 + \frac{1}{2}J'(q_1)\dot{q}_1^2 = Q + Q_R$$

where q_1 is the rotation angle of wheel 1, J the moment of inertia reduced to coordinate q_1 , Q_R is the reduced moment of inertia. For determining J we formulate the expression of the kinetic energy T :

$$T = \frac{1}{2} \left(J_1 \dot{q}_1^2 + J_2 \dot{q}_2^2 + m_2 v_{c2}^2 + J_H \dot{q}_H^2 \right) = \\ \frac{1}{2} \dot{q}_1^2 \left[J_1 + J_2 \left(\frac{\dot{q}_2}{\dot{q}_1} \right)^2 + m_2 \left(\frac{v_{c2}}{\dot{q}_1} \right)^2 + J_H \left(\frac{\dot{q}_H}{\dot{q}_1} \right)^2 \right].$$

From here

$$J = J_1 + J_2 \left(\frac{\dot{q}_2}{\dot{q}_1} \right)^2 + m_2 \left(\frac{v_{c2}}{\dot{q}_1} \right)^2 + J_H \left(\frac{\dot{q}_H}{\dot{q}_1} \right)^2,$$

where $\dot{q}_2$ is the angular velocity of the planet pinion, v_{c2} is the velocity of its mass center: $v_{c2} = \dot{q}_H r$. Let us find the angular velocity ratio $\frac{\dot{q}_H}{\dot{q}_1}$. For this purpose, we impart to all links of the reducer an angular velocity equal to $-\dot{q}_H$. Then, the carrier is motionless and we obtain a common gear train. For such a mechanism the relationship (Willis' formula) is valid:

$$\frac{\dot{q}_1 - \dot{q}_H}{\dot{q}_3 - \dot{q}_H} = \left(-\frac{z_2}{z_1} \right) \left(\frac{z_3}{z_2} \right).$$

Noticing that the third wheel is motionless ($\dot{q}_3 = 0$), we obtain:

$$\frac{\dot{q}_H}{\dot{q}_1} = \frac{1}{1 + \frac{z_3}{z_1}} = \frac{1}{4}.$$

An analogous expression can be obtained for the purpose of finding the ratio $\frac{\dot{q}_2}{\dot{q}_1}$:

$$\frac{\dot{q}_1 - \dot{q}_H}{\dot{q}_2 - \dot{q}_H} = \left(-\frac{z_2}{z_1} \right).$$

It is not difficult to find from here:

$$\frac{\dot{q}_2}{\dot{q}_1} = -\frac{1}{2}.$$

The reduced moment of inertia J is:

$$J = J_1 + J_2 \left(-\frac{1}{2} \right)^2 + m_2 \left(\frac{0.2 \cdot 1}{4} \right)^2 + J_H \left(\frac{1}{4} \right)^2 = 0.1355 \text{ kg m}^2,$$

i.e. J does not depend on the coordinate q_1 . For the reduced moment Q_R of the resistance forces we find:

$$Q_R = -M_H \frac{\dot{q}_H}{\dot{q}_1} = -100 \frac{1}{4} = -25 \text{ N m}.$$

Substituting J and Q_R into the equation of motion, we obtain:

$$0.1355\ddot{q} = Q - 25.$$

Answer: $0.1355\ddot{q} = Q - 25$.

- 6.3. Derive the equation of motion of the scotch-yoke mechanism (Fig. 6.8). The length of the crank is $l_1 = 0.3 \text{ m}$, its moment of inertia is $J_1 = 0.1 \text{ kg m}^2$ and the mass center is on the rotation axis O ; the link masses are $m_2 = 5 \text{ kg}$, $m_3 = 15 \text{ kg}$, the mass center of crosshead 2 of the slotted-link is at point A and the mass center of the slotted link 3 is at point B . A constant workload $P = 1000 \text{ N}$ acts on the slotted link and a driving moment Q is applied to the crank. Neglect gravity.

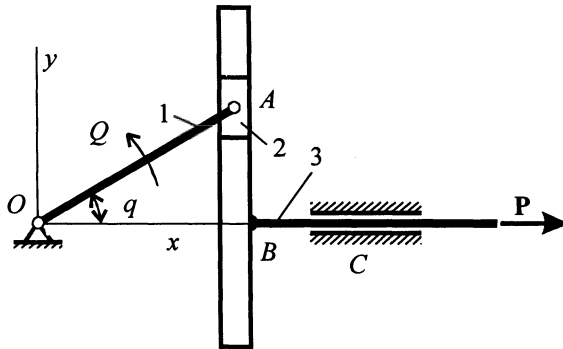


Fig. 6.8

Solution: For formulating the motion equation in the form (6.6), we down the expression of the kinetic energy T :

$$T = \frac{1}{2} (J_1 \dot{q}^2 + m_2 v_A^2 + m_3 v_B^2) = \frac{1}{2} \dot{q}^2 \left[J_1 + m_2 \left(\frac{v_A}{\dot{q}} \right)^2 + m_3 \left(\frac{v_B}{\dot{q}} \right)^2 \right].$$

This expression is taking into account that the crosshead 2 of the slotted link performs a curvilinear translational motion along a circle of radius l_1 and that the slotted link 3 performs a rectilinear reciprocating motion. With this, the reduced moment of inertia J is:

$$J(q) = J_1 + m_2 \left(\frac{v_A}{\dot{q}} \right)^2 + m_3 \left(\frac{v_B}{\dot{q}} \right)^2.$$

Because of $v_A = l_1 \dot{q}$ and $v_B = -l_1 \sin q \cdot \dot{q}$ one gets

$$J(q) = J_1 + m_2 l_1^2 + m_3 l_1^2 \sin^2 q,$$

$$J'(q) = 2m_3 l_1^2 \sin q \cos q.$$

The reduced moment of the resistance forces Q_R is

$$Q_R = P \frac{v_B}{\dot{q}} = -Pl_1 \sin q.$$

Substituting $J(q)$, $J'(q)$, Q_R into the expression (6.6), we obtain the equation of motion of the mechanism:

$$(J_1 + m_2 l_1^2 + m_3 l_1^2 \sin^2 q) \ddot{q} + m_3 l_1^2 \sin q \cos q \cdot \dot{q}^2 = Q - Pl_1 \sin q,$$

or

$$(0.55 + 1.35 \sin^2 q) \ddot{q} + 1.35 \sin q \cos q \cdot \dot{q}^2 = Q - 300 \sin q.$$

$$\text{Answer: } (0.55 + 1.35 \sin^2 q) \ddot{q} + 1.35 \sin q \cos q \cdot \dot{q}^2 = Q - 300 \sin q.$$

- 6.4. Derive the equations of motion of a mechanism of movability two (Fig. 6.9). Determine the driving moments Q_1 and Q_2 for $q_1 = 60^\circ$, $q_2 = 240^\circ$, $\dot{q}_1 = 0$, $\dot{q}_2 = 0.93 \text{ s}^{-1}$, $\ddot{q}_1 = 1 \text{ s}^{-2}$, $\ddot{q}_2 = 0.5 \text{ s}^{-2}$, $r = AB = 0.5 \text{ m}$, $l = BC = 1.0 \text{ m}$. The link masses are $m_1 = m_4 = 6 \text{ kg}$, $m_2 = m_3 = 10 \text{ kg}$. The axial moments of inertia are $J_2 = 1 \text{ kg m}^2$, $J_3 = 2 \text{ kg m}^2$. The mass center of link 2 coincides with point A and the mass center of link 3 coincides with point C . Neglect gravity.

Solution: We formulate the equations of motion in the form (6.2). From the fact that there is no workload and that gravity is neglected, we conclude that the generalized resistance forces Q_{R_s} are equal to zero. Thus:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s \quad (s = 1, 2).$$

The kinetic energy T of the mechanism is:

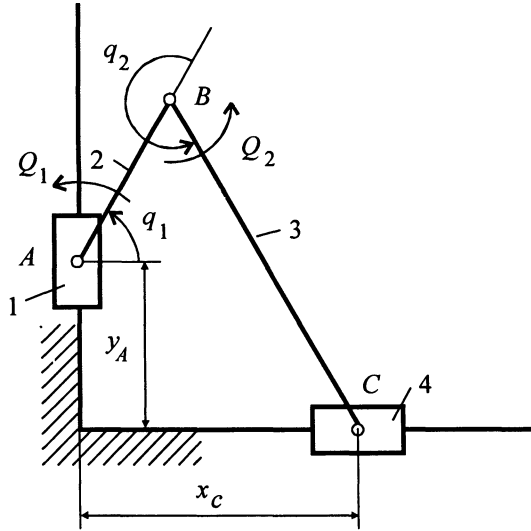


Fig. 6.9

$$T = T_1 + T_2 + T_3 + T_4 = \frac{1}{2} (m_1 \dot{y}_A^2 + m_2 \dot{y}_A^2 + J_2 \dot{q}_1^2 + m_3 \dot{x}_C^2 + m_4 \dot{x}_C^2 + J_3 (\dot{q}_1 + \dot{q}_2)^2).$$

Let us find the squares of the velocities entering the expression for T . To this end, we compose the group equations and differentiate them with respect to time:

$$\begin{cases} x_C = r \cos q_1 + l \cos(q_1 + q_2), \\ y_A = -r \sin q_1 - l \sin(q_1 + q_2), \\ \dot{x}_C = -r \dot{q}_1 \sin q_1 - l(\dot{q}_1 + \dot{q}_2) \sin(q_1 + q_2), \\ \dot{y}_A = -r \dot{q}_1 \cos q_1 - l(\dot{q}_1 + \dot{q}_2) \cos(q_1 + q_2). \end{cases}$$

Thus:

$$\begin{aligned} \dot{x}_C^2 &= r^2 \dot{q}_1^2 \sin^2 q_1 + l^2 (\dot{q}_1 + \dot{q}_2)^2 \sin^2(q_1 + q_2) + \\ &\quad 2rl \dot{q}_1 (\dot{q}_1 + \dot{q}_2) \sin q_1 \sin(q_1 + q_2), \\ \dot{y}_A^2 &= r^2 \dot{q}_1^2 \cos^2 q_1 + l^2 (\dot{q}_1 + \dot{q}_2)^2 \cos^2(q_1 + q_2) + \\ &\quad 2rl \dot{q}_1 (\dot{q}_1 + \dot{q}_2) \cos q_1 \cos(q_1 + q_2). \end{aligned}$$

We rewrite this expression in a more convenient form:

$$\begin{aligned} \dot{x}_C^2 &= [r \sin q_1 + l \sin(q_1 + q_2)]^2 \dot{q}_1^2 + l^2 \sin^2(q_1 + q_2) \dot{q}_2^2 + \\ &\quad 2l \sin(q_1 + q_2) [r \sin q_1 + l \sin(q_1 + q_2)] \dot{q}_1 \dot{q}_2, \\ \dot{y}_A^2 &= [r \cos q_1 + l \cos(q_1 + q_2)]^2 \dot{q}_1^2 + l^2 \cos^2(q_1 + q_2) \dot{q}_2^2 + \\ &\quad 2l \cos(q_1 + q_2) [r \cos q_1 + l \cos(q_1 + q_2)] \dot{q}_1 \dot{q}_2. \end{aligned}$$

Introducing the notation $m = m_1 + m_2 = m_3 + m_4$, we get:

$$T = \frac{1}{2} [J_2 + J_3 + m(r^2 + l^2) + 2mrl \cos q_2] \dot{q}_1^2 + \frac{1}{2} (J_3 + ml^2) \dot{q}_1^2 + (J_3 + ml^2 + mrl \cos q_2) \dot{q}_1 \dot{q}_2.$$

We determine the partial derivatives of the kinetic energy T with respect to the generalized velocities $\dot{q}_1$ and $\dot{q}_2$:

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_1} &= [J_2 + J_3 + m(r^2 + l^2) + 2mrl \cos q_2] \dot{q}_1 + (J_3 + ml^2 + mrl \cos q_2) \dot{q}_2, \\ \frac{\partial T}{\partial \dot{q}_2} &= (J_3 + ml^2) \dot{q}_2 + (J_3 + ml^2 + mrl \cos q_2) \dot{q}_1. \end{aligned}$$

We find the derivatives with respect to time:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1} \right) &= [J_2 + J_3 + m(r^2 + l^2) + 2mrl \cos q_2] \ddot{q}_1 - 2mrl \sin q_2 \cdot \dot{q}_1 \dot{q}_2 + \\ &\quad + (J_3 + ml^2 + mrl \cos q_2) \ddot{q}_2 - mrl \sin q_2 \cdot \dot{q}_2^2, \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2} \right) &= (J_3 + ml^2) \ddot{q}_2 + (J_3 + ml^2 + mrl \cos q_2) \ddot{q}_1 - mrl \sin q_2 \cdot \dot{q}_1 \dot{q}_2. \end{aligned}$$

We also define the partial derivatives of the kinetic energy T with respect to the generalized coordinates q_1 and q_2 :

$$\frac{\partial T}{\partial q_1} = 0, \quad \frac{\partial T}{\partial q_2} = -mrl \sin q_2 \cdot \dot{q}_1^2 - mrl \sin q_2 \cdot \dot{q}_1 \dot{q}_2.$$

Substituting there the found derivatives we obtain the equations of motion of the mechanism in the following form:

$$\begin{aligned} [J_2 + J_3 + m(r^2 + l^2) + 2mrl \cos q_2] \ddot{q}_1 + (J_3 + ml^2 + mrl \cos q_2) \ddot{q}_2 - \\ - 2mrl \sin q_2 \cdot \dot{q}_1 \dot{q}_2 - mrl \sin q_2 \dot{q}_2^2 &= Q_1, \\ (J_3 + ml^2 + mrl \cos q_2) \ddot{q}_1 + (J_3 + ml^2) \ddot{q}_2 + mrl \sin q_2 \dot{q}_1^2 &= Q_2. \end{aligned}$$

For the given values of parameters we obtain: $Q_1 = 16 \text{ N m}$, $Q_2 = 23 \text{ N m}$.

Answer:

$$\begin{aligned} [J_2 + J_3 + m(r^2 + l^2) + 2mrl \cos q_2] \ddot{q}_1 + (J_3 + ml^2 + mrl \cos q_2) \ddot{q}_2 - \\ - 2mrl \sin q_2 \cdot \dot{q}_1 \dot{q}_2 - mrl \sin q_2 \dot{q}_2^2 &= Q_1, \\ (J_3 + ml^2 + mrl \cos q_2) \ddot{q}_1 + (J_3 + ml^2) \ddot{q}_2 + mrl \sin q_2 \dot{q}_1^2 &= Q_2. \end{aligned}$$

$Q_1 = 16 \text{ N m}$, $Q_2 = 23 \text{ N m}$.

- 6.5. In a manipulator of movability three (Fig. 6.10) the link 1 is moving horizontally and link 2 turns about link 1. The link masses are m_1 and m_2 , respectively, the mass centers of links are S_1 and S_2 , the axial moment of inertia of link 2 is J_{S2} . Formulate the equation of motion of the manipulator. Neglect gravity.

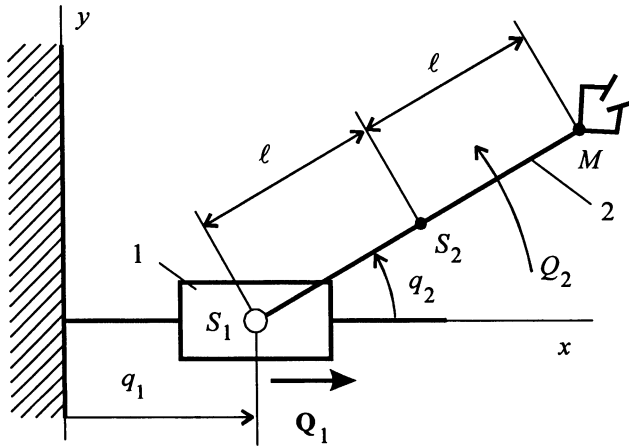


Fig. 6.10

Solution: We obtain the equations of motion from (6.2), assuming that workload and weights are equal to zero:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s \quad (s = 1, 2).$$

The kinetic energy T is:

$$T = \frac{1}{2} (m_1 v_{S1}^2 + m_2 v_{S2}^2 + J_{S2} \dot{q}_2^2),$$

where v_{S1} , v_{S2} are the velocities of the mass centers of links 1 and 2, respectively, $\dot{q}_2$ is the angular velocity of link 2. It is obvious that $v_{S1} = \dot{q}_1$. Let us find v_{S2} :

$$x_{S2} = q_1 + l \cos q_2, \quad y_{S2} = l \sin q_2,$$

$$\dot{x}_{S2} = \dot{q}_1 - l \dot{q}_2 \sin q_2, \quad \dot{y}_{S2} = l \dot{q}_2 \cos q_2,$$

$$v_{S2}^2 = \dot{x}_{S2}^2 + \dot{y}_{S2}^2 = (\dot{q}_1 - l \dot{q}_2 \sin q_2)^2 + (l \dot{q}_2 \cos q_2)^2.$$

Thus, the kinetic energy T takes the form:

$$T = \frac{(m_1 + m_2)\dot{q}_1^2}{2} + \frac{(m_2 l^2 + J_{S2})\dot{q}_2^2}{2} - m_2 l \dot{q}_1 \dot{q}_2 \sin q_2.$$

We find the derivatives of the kinetic energy:

$$\frac{\partial T}{\partial \dot{q}_1} = (m_1 + m_2)\dot{q}_1 - m_2 l \dot{q}_2 \sin q_2,$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1} \right) = (m_1 + m_2)\ddot{q}_1 - m_2 l \ddot{q}_2 \sin q_2 - m_2 l \dot{q}_2^2 \cos q_2,$$

$$\frac{\partial T}{\partial q_1} = 0,$$

$$\frac{\partial T}{\partial \dot{q}_2} = (m_2 l^2 + J_{S2})\dot{q}_2 - m_2 l \dot{q}_1 \sin q_2,$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2} \right) = (m_2 l^2 + J_{S2})\ddot{q}_2 - m_2 l \ddot{q}_1 \sin q_2 - m_2 l \dot{q}_1 \dot{q}_2 \cos q_2,$$

$$\frac{\partial T}{\partial q_2} = -m_2 l \dot{q}_1 \dot{q}_2 \cos q_2.$$

Substituting these derivatives, we obtain the equations of motion:

$$\begin{aligned} (m_1 + m_2)\ddot{q}_1 - m_2 l \ddot{q}_2 \sin q_2 - m_2 l \dot{q}_2^2 \cos q_2 &= Q_1, \\ (m_2 l^2 + J_{S2})\ddot{q}_2 - m_2 l \ddot{q}_1 \sin q_2 &= Q_2. \end{aligned}$$

Answer:

$$\begin{aligned} (m_1 + m_2)\ddot{q}_1 - m_2 l \ddot{q}_2 \sin q_2 - m_2 l \dot{q}_2^2 \cos q_2 &= Q_1, \\ (m_2 l^2 + J_{S2})\ddot{q}_2 - m_2 l \ddot{q}_1 \sin q_2 &= Q_2. \end{aligned}$$

- 6.6. In the robot of movability three (Fig. 6.11) column 1 rotates about its longitudinal axis. Along the column the cross-piece 2 is moving. From the cross-piece extends „arm“ 3, carrying the gripper 4. Formulate the equations of motion of the robot, if the masses of the links are m_1 , m_2 , m_3 ; J_1 is the moment of inertia of link 1 with respect to the axis Oz ; J_{2c2} , J_{3c3} are the moments of inertia of links 2 and 3 with respect to the axes parallel to Oz and passing through the mass centers of these links; Q_1 , Q_2 and Q_3 are the driving moments; G_2 and G_3 are the weights of the links.

Answer:

$$\begin{aligned} (J_1 + m_3 q_3^2)\ddot{q}_1 - m_3 e \ddot{q}_3 + 2m_3 q_3 \dot{q}_1 \dot{q}_3 &= Q_1, \\ (m_2 + m_3)\ddot{q}_2 &= Q_2 - (G_2 + G_3), \\ m_3 \ddot{q}_3 - m_3 e \ddot{q}_1 - m_3 q_3 \dot{q}_1^2 &= Q_3, \end{aligned}$$

where $J_0 = J_1 + J_{2c2} + J_{3c3} + m_2 a^2 + m_3 e^2$.

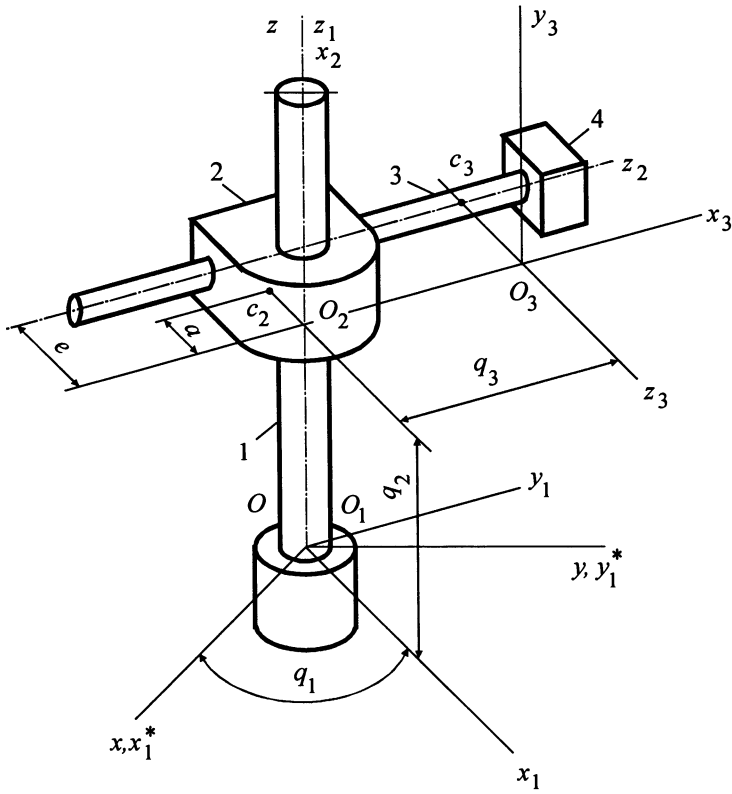


Fig. 6.11

7 Dynamic Characteristics of Mechanisms with Rigid Links

For an assessment of the quality of a mechanism, it is necessary to proceed from the acting forces and from the constraint reactions obtained in the force analysis, to some general dynamic criteria which reflect the most important properties of a mechanism in the most typical dynamic regimes. In this chapter we will consider methods of defining dynamics characteristics as well as methods of improving the qualities of a mechanism through a modification of its parameters and through the introduction of certain complementary devices.

7.1 Internal Vibration Activity of a Mechanism

Let us consider a cyclic mechanism with rigid links and with ideal kinematic pairs (Fig. 7.1), which is a combination of a transmission mechanism with transmission ratio i ($\varphi = q/i$) with an actuating mechanism with a nonlinear position function.

Assuming that the generalized resistance force can be represented in the form (6.12), we write the equation of motion of the mechanism in the form (6.6):

$$J(q)\ddot{q} + \frac{1}{2}J'(q)\dot{q}^2 = Q + Q_R(q, \dot{q}). \quad (7.1)$$

The reduced moment of inertia $J(q)$ can be represented in the form (6.8); in what follows, this series expansion is abbreviated

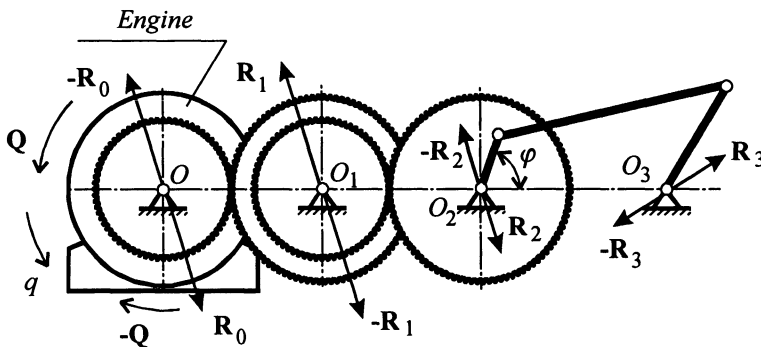


Fig. 7.1. Mechanism with rigid links and with ideal kinematic pairs

$$J(q) = J_0 + \tilde{J}(q) = \frac{1}{2\pi i} \int_0^{2\pi i} J(q) dq + \tilde{J}(q), \quad (7.2)$$

where the variable part of the reduced moment of inertia of the mechanism is $\tilde{J}(q)$ which, in the present case, has period $2\pi i$. The reduced moment of the resistance forces can be represented analogously, also being a periodic function of q with period $2\pi i$:

$$Q_R(q, \dot{q}) = Q_{R0}(\dot{q}) + \tilde{Q}_R(q, \dot{q}), \quad (7.3)$$

where the leading term

$$Q_{R0}(\dot{q}) = \frac{1}{2\pi i} \int_0^{2\pi i} Q_{R0}(q, \dot{q}) dq \quad (7.4)$$

is the average value of the reduced moment of resistance forces and $\tilde{Q}_R = Q_R(q, \dot{q}) - Q_{R0}(\dot{q})$.

One of the most typical performances of a cyclic mechanism is the steady-state motion when the angular velocity of the input link is close to some constant value $\dot{q} = \omega_0$. Let us consider a mechanism characteristic which reflects the dynamic properties of the mechanism during steady-state motion. We assume that the input link rotates with constant angular velocity ω_0 . Let us find the generalized driving force (moment) to be applied to the input link in order to achieve such a motion. Substituting $\dot{q} = \omega_0$, $q = \omega_0 t$, $\ddot{q} = 0$ into (7.1), we have

$$Q(t) = \frac{1}{2} J'(\omega_0 t) \omega_0^2 - Q_R(\omega_0 t, \omega_0) = -Q_{R0}(\omega_0) + \tilde{Q}(t), \quad (7.5)$$

where $\tilde{Q}(t)$ is the variable part of the driving moment. The opposite moment

$$L(t) = -\tilde{Q}(t), \quad (7.6)$$

acted upon the engine by the mechanical system, is called *perturbation moment*. The ability of the mechanism to generate a variable perturbation moment during a uniform rotation of the input link reflects its *internal vibration activity*. From the expression (7.5) it is seen that the internal vibration activity of the system is caused by the variability of the reduced moment of inertia of the mechanism and by the explicit dependence of the reduced moment of resistance forces on the coordinate q .

The perturbation moment is a periodic function of t with period $T = 2\pi i / \omega_0 = 2\pi / \nu$, where ν is the angular velocity of the input link of the actuating mechanism. This moment does not have a constant term. It can be represented in the form of a Fourier series:

$$L(t) = \sum_{k=1}^{\infty} L_k \cos(k\nu t + \alpha_k). \quad (7.7)$$

The internal vibration activity of a mechanism is causing a number of undesirable dynamical phenomena typical for cyclic machines. Some of these are considered further below. At this point we only note that the internal vibration activity causes variable dynamic loads at higher kinematic pairs of a transmission mechanism. If during uniform rotation of the input link the condition

$$Q(t) = -Q_{R0}(\omega_0) + \tilde{Q}(t) < 0$$

is fulfilled for certain intervals of time, i.e. if the driving moment is alternating in sign, then also the moment in the transmission mechanism changes its sign. Moreover, a „rearrangement“ of clearances is occurring in gearings, when the driving wheels become driven wheels. This is an undesirable effect contributing to increased wear of gearings. In order to avoid this, the constant term $Q_{R0}(\omega_0)$ of the resistance moment is sometimes artificially increased, e.g., by special damping devices mounted on the output shaft of the transmission mechanism which generate additional resistance forces.

7.2 Methods of Reduction of Perturbation Moments

A reduction of the perturbation moment (7.5) is achieved by reducing the variable component $\tilde{J}(q)$ of the reduced moment of inertia and of the variable component $\tilde{Q}_R(q, \dot{q})$ of the reduced moment of resistance forces. For reducing $\tilde{J}(q)$ one must try to reduce the masses of those movable links of the actuating mechanism whose coordinates are related to the coordinate q through nonlinear position functions. In cycle machines a reduction of $\tilde{Q}_R$ is sometimes achieved by a shift in the cycle of synchro-working mechanisms in such a way that the magnitude of $Q_R(q, \dot{q})$ becomes balanced. However, all these design methods can only be used under certain conditions; their potential is usually rather limited. However, there are a few reduction general methods of mechanism vibration activity which are based on the introduction of special devices that reduce $L(t)$.

Unloaders. Unloaders are complementary devices introduced in mechanisms for reducing the perturbation moments caused by the mechanisms. Fig. 7.2 shows a cam mechanism on the input shaft of an actuating mechanism designed for reducing the perturbation moment generated by this mechanism at a given angular velocity $\nu = \omega_0 / i$ of the input shaft. If we chose the cam profile in such a way that the unload moment $M_u = h_u P_u$ applied by the cam to the input shaft is as close to $-L(t)$ as possible we can reduce the total perturbation moment

ism by a cam unloader is principally possible for only one particular value of the angular velocity. When ω_0 changes, also the perturbation moment changes, namely the term caused by the inertia forces ($-0.5\tilde{J}'(\omega_0 t)\omega_0^2$) so that the condition of unloading is no longer fulfilled. This is especially noticeable in transient motion (during acceleration and deceleration of a machine), when the unloader substantially increases the perturbation moment.

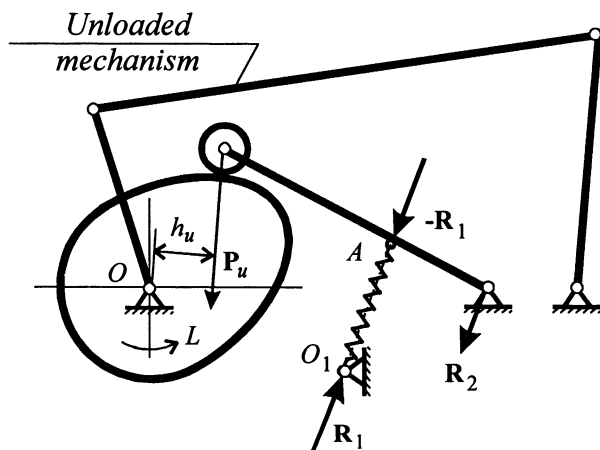


Fig. 7.2. Cam unloader

Fig. 7.3 shows a spring unloader designed to unload the inertia force generated by the translationally moving slotted link of a scotch-yoke mechanism. In this case, the elastic force R_1 of the spring connecting the slotted link with the frame compensates the inertia force of the massive link 3, thus decreasing the reactions at the kinematic pairs A and B and the variable component of the driving moment acting on crank 1.

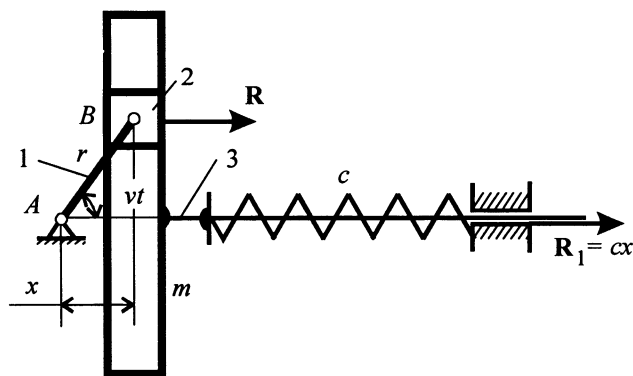


Fig. 7.3. Spring unloader

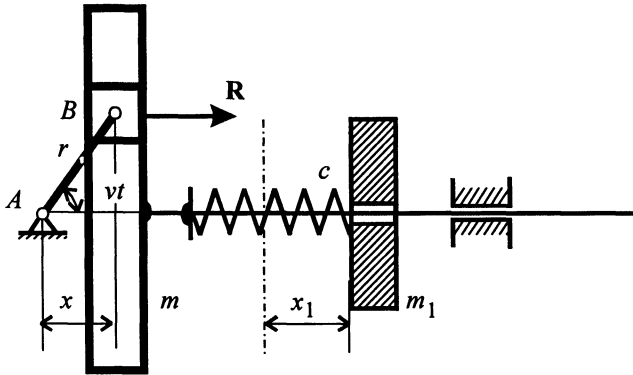


Fig. 7.4. Dynamic absorber

Absorbers. It is not difficult to recognize that, while diminishing the perturbation moment, unloaders simultaneously generate variable forces acting on the body of a machine (forces R_1 and R_2 in Fig. 7.2, force R_1 in Fig. 7.3). This can be avoided through the application of dynamic absorbers. Fig. 7.4 shows a dynamic absorber attached to the slotted link of the mechanism, considered in the previous example. The absorber consists of the mass m_1 and of a spring with stiffness c connecting this mass with the slotted link. In this case, the inertia force generated by the moving slotted link is compensated by the dynamic absorber inertia force transmitted through the spring. The effect of the dynamic dampening is reached when

$$c = mm_1(m + m_1)^{-1}v^2.$$

It should be noted, that for two reasons the mass m_1 must not be too small. First, because the displacement of a small mass is very large ($x_{1\max} \approx rm/m_1$, where r is the crank radius); second, because of friction which for a small mass can essentially reduce the effect of the absorber.

7.3

External Vibration Activity of Mechanisms and Machines

During motion of mechanisms time-varying forces are acting on the body of a machine. Such forces occur, first of all, at kinematic pairs connecting movable links of a mechanism to the body. So, e.g., the mechanism shown in Fig. 7.1 acts on the body with forces $-\mathbf{R}_0$, $-\mathbf{R}_1$, $-\mathbf{R}_2$, $-\mathbf{R}_3$ at joints O , O_1 , O_2 , O_3 , respectively. Equal and oppositely directed forces $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ are applied from the body to the mechanism links; henceforth, these latter ones will be called *external reactions*. They will be denoted $\mathbf{R}_k^{(e)}$, where k is the number of the link to which the force is applied.

The variable forces acting on the body of a machine can cause a number of phenomena of vibrational character (vibration of the body of a machine as a rigid body with respect to the base, resulting from the elasticity of the supports; elastic vibration of the body; vibration of the building, where a machine is installed etc.). In this context, the ability of a mechanism to excite variable forces acting on the body is called *external vibration activity*. The reduction of the external vibration activity of individual mechanisms and of machines in general is one of the most important problems in modern machine-building, since vibration often result in losses of quality of machine performance of strength, reliability and durability. The reduction of the external vibration activity of machines has recently become a major social task. Machine vibration influence on people working with machines. They disturb their work capacity and they become hazardous for human health in the case of continuous exposure.

Balancing of mechanisms and machines. One method of reduction of machine vibration activity is balancing of mechanisms. A mechanism is called *balanced*, if its time-varying external reactions form a balanced system of forces¹ independent of the motion law. Let us consider a mechanism with N movable links with given motion laws. For every single link we formulate the equations of kinetostatics

$$\begin{aligned} \mathbf{P}_k^{(e)} + \mathbf{P}_k^{(i)} + \Phi_k + \mathbf{R}_k^{(e)} + \mathbf{R}_k^{(i)} &= 0, \\ \mathbf{M}_{Ok}^{(Pe)} + \mathbf{M}_{Ok}^{(Pi)} + \mathbf{M}_{Ok}^{(\Phi)} + \mathbf{M}_{Ok}^{(Re)} + \mathbf{M}_{Ok}^{(Ri)} &= 0. \end{aligned} \quad (7.8)$$

Here, $\mathbf{P}_k^{(e)}$ is the sum of the external active forces applied to the k -th link; $\mathbf{P}_k^{(i)}$ is the sum of the internal active forces, i.e. interaction forces between links; Φ_k is the resultant vector of inertia forces of the link; $\mathbf{R}_k^{(e)}$ is the sum of the forces acted upon the link by the frame; $\mathbf{R}_k^{(i)}$ is the sum of the internal constraint reactions, i.e. the sum of the forces acted upon the k -th link by the other movable links; $\mathbf{M}_{Ok}^{(Pe)}$, $\mathbf{M}_{Ok}^{(Pi)}$, $\mathbf{M}_{Ok}^{(\Phi)}$, $\mathbf{M}_{Ok}^{(Re)}$, $\mathbf{M}_{Ok}^{(Ri)}$ are the resultant moments of the corresponding forces with respect to some center O . We take the sum of the Eqs. (7.8) over $k = 1, \dots, N$. According to Newton's third law we have

$$\sum_{k=1}^N \mathbf{P}_k^{(i)} = 0, \quad \sum_{k=1}^N \mathbf{R}_k^{(i)} = 0, \quad \sum_{k=1}^N \mathbf{M}_{Ok}^{(Pi)} = 0, \quad \sum_{k=1}^N \mathbf{M}_{Ok}^{(Ri)} = 0, \quad (7.9)$$

since the interaction forces of every pair of movable links are equal in magnitude and opposite in direction. As a result, we obtain

$$\mathbf{P}^{(e)} + \Phi + \mathbf{R}^{(e)} = 0, \quad \mathbf{M}_O^{(Pe)} + \mathbf{M}_O^{(\Phi)} + \mathbf{M}_O^{(Re)} = 0, \quad (7.10)$$

¹ A balanced mechanism can act on the frame with constant forces. Such time invariant reactions can be caused, e.g., by the weight of links.

where $\mathbf{P}^{(e)}$, Φ , $\mathbf{R}^{(e)}$ are the resultant vectors, while $\mathbf{M}_O^{(Pe)}$, $\mathbf{M}_O^{(\Phi)}$, $\mathbf{M}_O^{(Re)}$ are the resultant moments. For balancing a mechanism, according the adopted definition, it is necessary and sufficient to satisfy the conditions

$$\mathbf{R}^{(e)} = 0, \quad \mathbf{M}_O^{(Re)} = 0. \quad (7.11)$$

Hence, (7.10) requires that the conditions

$$\mathbf{P}^{(e)} + \Phi = 0, \quad \mathbf{M}_O^{(Pe)} + \mathbf{M}_O^{(\Phi)} = 0 \quad (7.12)$$

be fulfilled, i.e. the external active forces and the inertia forces of the mechanism links must constitute a balanced system of forces.

External reactions of mechanisms acting on the body of a machine are transmitted to the foundation on which the machine is installed. The external vibration activity of a machine is, therefore, conditioned by the external vibration activity of its mechanisms. However, when formulating balancing conditions of a machine it is necessary to recognize that in many cases active forces applied to mechanism links constitute internal forces with respect to the machine as a whole. Consider, e.g., a generalized driving force Q applied to the input link of the mechanism shown in Fig. 7.1. According to Newton's third law the equal and oppositely directed force $-Q$ acts on the stator of the motor and, if the motor is installed on the body of the machine, on this body itself. Also on the body the moment $-M_R$ equal and opposite to the moment of resistance forces is acting, if the source of these forces is connected with the body of the machine. An internal force for the machine as a whole will also be the elastic force $\mathbf{R}_1$ shown in Fig. 7.2, since the opposite force $-\mathbf{R}_1$ is applied to the body at point O_1 . Internal forces are the forces of gas pressure in a compressor or in an internal-combustion engine (Fig. 7.5): The force $\mathbf{P}$ is applied to the piston, while the force $-\mathbf{P}$ is applied to the body by the cylinder.

If all external active forces applied to the links of a mechanism are internal forces for the machine as a whole, then the balance of the machine will be ensured under the fulfillment of the conditions

$$\Phi = 0, \quad \mathbf{M}_O^{(\Phi)} = 0, \quad (7.13)$$

i.e. when the inertia forces are balanced.

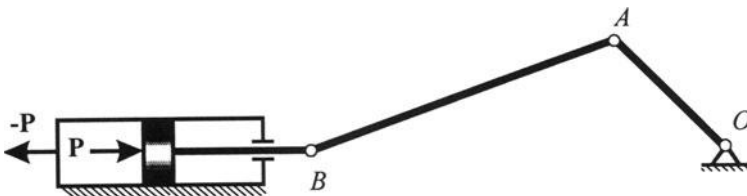


Fig. 7.5. Slider-crank mechanism with internal force $\mathbf{P}$

It is natural, to adopt as measure of unbalance of a mechanism the resultant vector $\mathbf{R}^{(e)}$ and the resultant moment $\mathbf{M}_O^{(Re)}$ of its external reactions. It must be recognized, however, that in the case $\mathbf{R}^{(e)} \neq 0$ the quantity $\mathbf{M}_O^{(Re)}$ depends on the choice of the reduction center O . For this reason $\mathbf{M}_O^{(Re)}$ can be considered as measure of unbalance only for a fixed position of point O . It is known that the scalar product $\mathbf{R}^{(e)} \cdot \mathbf{M}_O^{(Re)}$ does not depend on the reduction center.

In modern high-speed machines the balancing of mechanisms and machines as a whole does not solve entirely the problem of eliminating external vibration activity. Balanced forces applied to the body at different points can cause deformations which result in intensive vibration. Besides, it is incorrect to think that the internal active forces not presented in condition (7.13) do not affect, in general, the level of the external vibration activity. It should be kept in mind that these forces affect the motion law of a mechanism and, therefore, the magnitude of the inertia forces. E.g., the internal forces $\mathbf{P}$ and $-\mathbf{P}$ shown in Fig. 7.5, can cause nonuniformity of the rotation of crank OA , which, generally speaking, results in a change of the action of the machine on the base.

7.4

External Vibration Activity of a Rotating Rotor and of a Rotor Machine

There are many machines in which the only movable link is a rotor in rotational motion. Rotors electric motors, pumps, centrifugal compressors, turbines and centrifuges belong to this group. Let us investigate the external vibration activity of a rotor (Fig. 7.6), considered as a rigid body. A driving moment Q and a resistance moment M_R are applied to the rotor; it is rotating with angular velocity ω and with angular acceleration ε . We introduce a coordinate system $Oxyz$ attached to the rotor. Projecting Eqs. (7.10) onto these axes, taking into account that in this case $\mathbf{P}^{(e)} = 0$, $M_{Ox}^{(Pe)} = M_{Oy}^{(Pe)} = 0$, $M_{Oz}^{(Pe)} = Q + M_R$, and using expressions (4.95) and (4.96), we find

$$R_x^{(e)} = -\Phi_x = -m(\omega^2 x_c + \varepsilon y_c), \quad R_y^{(e)} = -\Phi_y = -m(\omega^2 y_c - \varepsilon x_c), \quad (7.14)$$

$$R_z^{(e)} = -\Phi_z = 0,$$

$$M_{Ox}^{(Re)} = -M_{Ox}^{(\Phi)} = J_{yz}\omega^2 - J_{xz}\varepsilon, \quad M_{Oy}^{(Re)} = -M_{Oy}^{(\Phi)} = -J_{xz}\omega^2 - J_{yz}\varepsilon, \quad (7.15)$$

$$M_{Oz}^{(Re)} = -M_{Oz}^{(Pe)} - M_{Oz}^{(\Phi)} = -Q - M_R + J_z\varepsilon. \quad (7.16)$$

Here, x_c and y_c are coordinates of the mass center C of the rotor. The gravity force of the rotor does not enter expression 7.14. It is generating reactions of constant magnitude which can be determined from equations of statics. They must

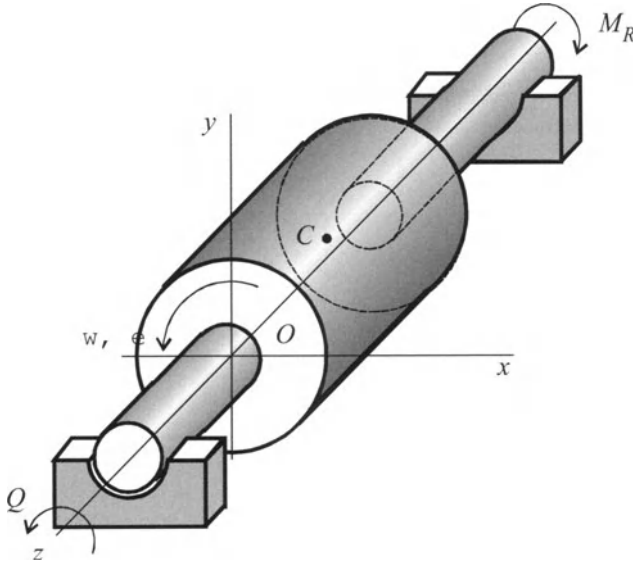


Fig. 7.6. Rotating rotor

be added to the dynamic reactions². From relationship (7.14) it follows that the first balancing condition $\mathbf{R}^{(e)} = 0$ is fulfilled for any ω and ε , if and only if

$$x_c = y_c = 0, \quad (7.17)$$

i.e. if the mass center of the rotor lies on the rotation axis. When this condition is fulfilled the rotor is called *statically balanced*. From the expressions (7.15) it follows that a rigid rotor does not generate dynamic moments with respect to the axes Ox and Oy for any ω and ε if and only if

$$J_{xz} = J_{yz} = 0, \quad (7.18)$$

i.e. if the z -axis is principal axis of inertia of the rotor. If the conditions (7.17) and (7.18) are fulfilled, i.e. if the rotation axis is a principal central axis of inertia, then the rotor is called *dynamically balanced*.

Comparing (7.16) with the motion equation of the rotating rotor

$$J_z \varepsilon = Q + M_R,$$

it is easy to see that $M_{Oz}^{(Re)} = 0$ for any motion law. Hence, the reactions at the supports of the rotating rotor cannot generate a moment about the rotation axis. This conclusion is correct only under the assumption that friction forces at the rotor supports are considered as active forces.

² The moment of the gravity force about the axis x is constant; the moment about the z -axis can be considered as part of the moment M_R of the resistance forces entering Eq. (7.16)

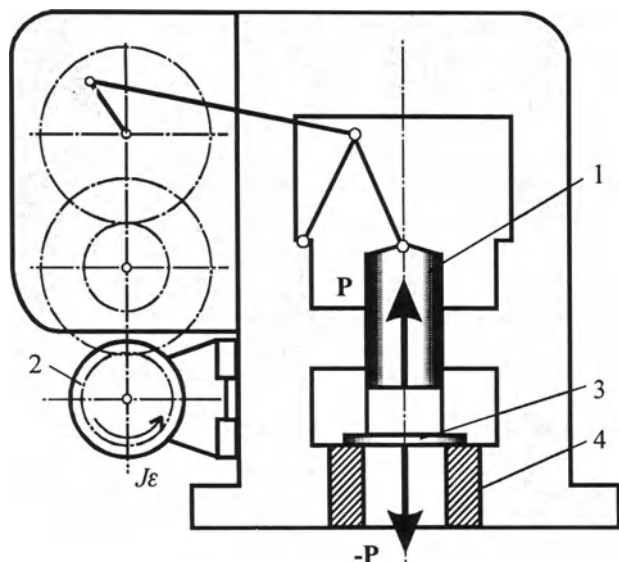


Fig. 7.7. Diagram of a blanking press

Let us assume, next, that the driving moment as well as the resistance force which generates a moment M_R , are internal forces for the machine as a whole (note that to internal forces also belong friction forces at the supports). When investigating the vibration activity of a rotor machine under these conditions active forces must be excluded from expression (7.16). As a result we obtain:

$$M_{Oz}^{(Re)} = J_z \varepsilon. \quad (7.19)$$

This moment is the only action on the base of a rotor machine with a dynamically balanced rotor and with internal active forces. The action is equal to zero for uniform rotation of the rotor. A moment about the rotation axis is acting only during run-up and run-out of the machine and for nonuniform rotation of the rotor. It will be shown below that a nonuniform rotation of the motor rotor occurs, in particular, during steady-state motion of a cyclic machine. This is caused by internal vibration activity of the mechanical system. In this case, the action of the machine on the base has vibratory character. It may cause severe vibration in, both, the base of the machine and the building in which it is installed.

Let us consider a typical example which illustrates what was said above. Fig. 7.7 shows the diagram of a blanking press. The motor 2, mounted on the body, drives the punch 1 by means of a two-stage gear-train reducer and a linkage. At the moment of punching a piece from blank 3 an impact occurs with very large forces P and $-P$ acting on the punch and on the die 4, respectively. Since these forces are internal forces for the machine as a whole they do not act on the base. However, at the moment of impact a sharp decrease in the velocity of the punch occurs and, as a result, a decrease in the velocities of all links of the rigid mechanism, including the motor rotor that usually possesses a large moment of inertia J .

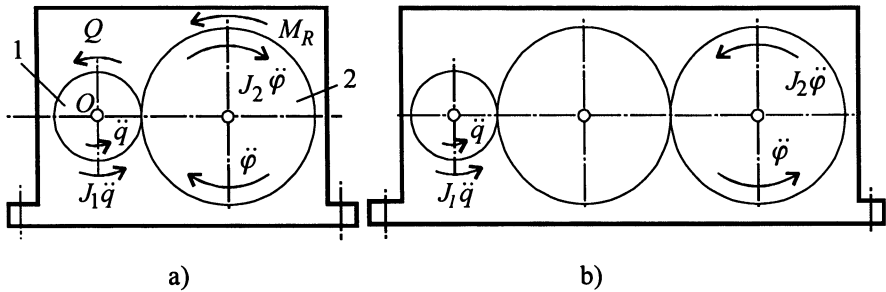


Fig. 7.8. Diagram of a two-rotor machine

A time-varying moment of inertia $J\varepsilon$ acts on the base of the press. The frequency of this process is determined by the number of cycles of the machine per unit of time.

Fig. 7.8a shows the diagram of a two-rotor machine. If both rotors are dynamically balanced and if the moments Q and M_R are internal forces for the machine, then the impact of the machine on the base (external vibration activity) is reduced to the moment $M_{Oz}^{(Re)} = J_1\ddot{q} - J_2\ddot{\varphi} = (J_1 - J_2i^{-1})\ddot{q}$, where i is the transmission ratio of the transmission connecting the two rotors. For $J_1 = J_2i^{-1}$ the machine is completely balanced. Once again we point out that the balance is violated if only one of the active generalized forces is an external force. So, e.g., if rotor 2 in Fig. 7.8a is the rotor of a fan and if the moment M_R is generated by aerodynamic forces, then the body of the machine is subject to the moment $-M_R$. In the two-rotor machine with the diagram shown in Fig. 7.8b, both rotors are rotating in the same direction. Therefore, in this case the inertial moments $J_1\ddot{q}$ and $J_2\ddot{\varphi}$ are added and balance cannot be achieved during accelerating motions.

7.5 Balancing of Rotors

In modern machines the angular velocities of rotors reach $10\,000\text{ s}^{-1}$ and more, and velocities on the order of $300 - 600\text{ s}^{-1}$ are common. For such velocities any rotor unbalance, either static or dynamic is causing very large reactions at supports. If, e.g., the mass center of a rotor is at distance of 1 mm from the rotation axis, then an angular velocity of 1000 s^{-1} causes a dynamic load on the base exceeding 100 times the weight of the rotor. It is natural that the problem of rotor balancing is one of the cardinal problems in machine building.

The approach to the problem of balancing rotors depends on the choice of the dynamic model used for the rotor. To be distinguished are: *stiff* rotors which can be considered as perfectly rigid; *elastic* rotors whose deformability must be taken

into account in balancing; *flexible* rotors whose velocity of rotation exceeds some *critical* value. The choice of dynamic model depends on the velocity of rotation and on the desired accuracy of balancing. The operation of achieving balance is often called *balancing* and the devices on which the balancing is carried out are called *balancing stands*.

Balancing of a stiff rotor. The aim of static balancing is to satisfy the conditions (7.17) for the rotor. To this end, the rotor is mounted on supports in the form of prismatic rules (Fig. 7.9). By placing a balancing weight m_b , the mass center is shifted to the rotation axis in order to ensure that when the rotor is released from an arbitrary initial position without initial velocity, it will not roll on the prismatic supports.

The accuracy of static balancing depends on the coefficient k of the rolling friction of the rotor journals over the prisms. Let m be the mass of the rotor and let e be the distance from the mass center to the rotation axis. Then, the largest driving moment forcing the rotor to roll over the prisms, is equal to mge . Rolling occurs if this moment is greater than the moment mgk of rolling friction. Unbalance will not be detected if $e \leq k$; the residual unbalance of the rotor is determined by the moment of the mass: $me \leq mk$.

The dynamic balancing of a rotor with the goal to satisfy conditions (7.17) and (7.18), can be achieved by two balancing masses placed in two arbitrarily chosen planes orthogonal to the rotation axis, the so-called correction planes.

Let z_1 and z_2 be the coordinates of the correction planes (Fig. 7.10); m_1 and m_2 are the masses of the balancing weights; x_1, y_1, x_2, y_2 are their coordinates in the correction planes (the system $Oxyz$ is attached to the rotor); m is the mass of the rotor; x_c, y_c are the coordinates of its mass center. Then conditions (7.17) will be fulfilled, if

$$\begin{aligned} m_1 x_1 + m_2 x_2 + m x_c &= 0, \\ m_1 y_1 + m_2 y_2 + m y_c &= 0. \end{aligned} \quad (7.20)$$

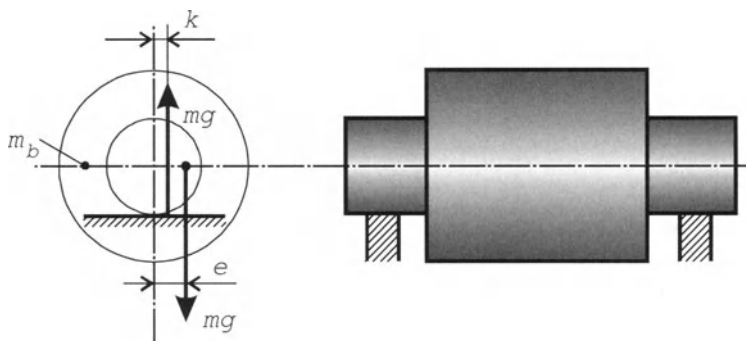


Fig. 7.9. Rotor on prismatic rules

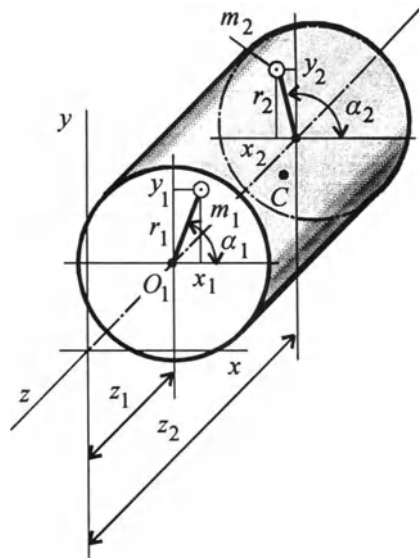


Fig. 7.10. Diagram of the dynamic balancing of a rotor

Conditions (7.20) requires that the mass center of the system consisting of the rotor and of the balancing weights lies on the rotation axis. We formulate expressions for the products of inertia J_{xz}^* and J_{yz}^* of the system and equate them to zero:

$$\begin{aligned} J_{xz}^* &= J_{xz} + m_1 x_1 z_1 + m_2 x_2 z_2 = 0, \\ J_{yz}^* &= J_{yz} + m_1 y_1 z_1 + m_2 y_2 z_2 = 0. \end{aligned} \quad (7.21)$$

Fulfilling these conditions, the already balanced rotor will be dynamically balanced. In this way, it is necessary to define from Eqs. (7.20) and (7.21) masses m_1 and m_2 of the weights and their coordinates x_1, y_1, x_2, y_2 . Since the number of unknowns exceeds the number of equations, it is necessary to set two additional conditions, by which it is possible to choose the values of the radii $r_1 = \sqrt{x_1^2 + y_1^2}$ and $r_2 = \sqrt{x_2^2 + y_2^2}$, and to seek angles α_1, α_2 (Fig.7.10) and values m_1, m_2 .

7.6

Vibration Activity of a Planar Mechanism

The analysis of the external vibration activity of a planar mechanism is often restricted to the determination of the components of the resultant vector and of the resultant moment of the external reactions lying in the motion plane. If axes Ox and Oy are defined in this plane, then the problem is to determine components

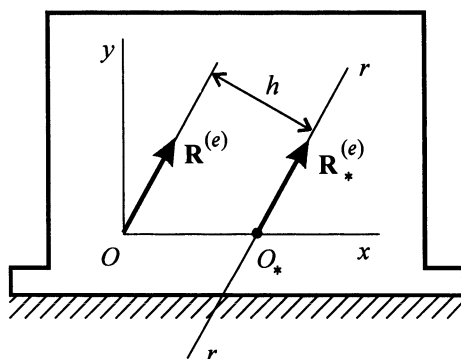


Fig. 7.11. The resultant of the external reactions of a mechanism

$R_x^{(e)}$, $R_y^{(e)}$, $M_{Oz}^{(Re)}$. For every position of the mechanism a straight line $r-r$ parallel to the vector $\mathbf{R}^{(e)}$ can be found as line of action of the resultant of all external reactions $\mathbf{R}_*^{(e)}$ (Fig. 7.11). Its position is determined from the condition

$$M_{O_*r}^{(Re)} = R^{(e)}h - M_{Oz}^{(Re)} = 0, \quad (7.22)$$

where O_* is a point on the straight line $r-r$ (e.g., the intersection point of the x -axis with this line). During motion of mechanism, both, magnitude and direction of the vector $\mathbf{R}^{(e)}$ change, and the position of the line $r-r$ changes, as well.

Balancing planar mechanism. Assume that all active forces (except forces of gravity, which will not be taken into account here) are internal forces for the machine as a whole. Then,

$$\mathbf{R}^{(e)} = -\Phi = m\mathbf{w}_c, \quad (7.23)$$

where $\mathbf{w}_c$ is the vector of the absolute acceleration of the mechanism mass center. Since for planar motion the vector $\mathbf{w}_c$ lies in the motion plane, also the vector $\mathbf{R}^{(e)}$ lies in this plane. From (7.23) it follows that the first balancing condition (7.12) $\mathbf{R}^{(e)} \equiv 0$ is fulfilled, if $\mathbf{w}_c \equiv 0$, i.e. if $\mathbf{v}_c = \text{const}$. However, for a stationary machine a constant velocity $\mathbf{v}_c$ can only be zero. Thus, for fulfilling the condition $\mathbf{R}^{(e)} = 0$, it is necessary that $\mathbf{v}_c = 0$, i.e. the mass center of the whole mechanism must remain fixed during the motion. In principle, it is possible to fulfil this condition by attaching additional masses as counterweights to the mechanism links. Fig. 7.12 shows a method how to balance a slider-crank mechanism by means of two counterweights which are attached to the rocker and to the crank. Let C_1 and C_2 be the mass centers of the crank and the rocker, respectively; K_1 and K_2 are the mass centers of the counterweights; B is the mass center of the slider; m_1 , m_2 , m_3 are the masses of these links; $OA = r$,

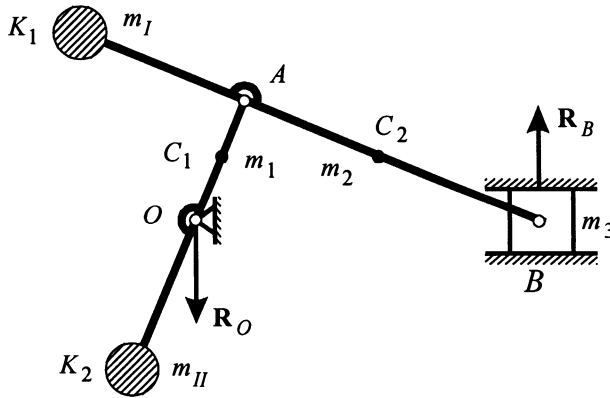


Fig. 7.12. The attaching of counterweights to links of a mechanism

$AB = \ell$, $AC_2 = a_2$, $AK_1 = a_I$, $OK_2 = a_{II}$, $OC_1 = a_1$. By choosing the mass m_I of the first counterweight from the condition

$$m_I a_I = m_2 a_2 + m_3 \ell,$$

we shift the mass center of the system rocker–slider to the point A . The next task must be to bring the mass center of the system to the point O . For this, the condition

$$m_I a_I = m_1 a_1 + (m_1 + m_2 + m_3) r$$

must be fulfilled. The disadvantage of this balancing method is the very large total mass of the counterweights. Indeed, e.g., for $a_1 = a_2 = \ell/2$ we obtain from the first condition $m_I = m_2 + 2m_3$. If $a_{II} = a_I = r/2$, then the second condition gives

$$m_{II} = m_1 + 4m_2 + 6m_3.$$

In this way, the total mass of the counterweights must exceed several times the mass of all movable links of the mechanism. As a result its internal vibration activity increases substantially and the dynamic properties become worse.

Besides this, the balancing condition $M_{Oz}^{(Re)} = 0$ will no longer be satisfied. A moment $M_{Oz}^{(Re)}$ will result from the external reactions $\mathbf{R}_O$ and $\mathbf{R}_B$ (see Fig. 7.12).

Balancing of the first harmonics of inertia forces. Let us consider an arbitrary planar mechanism (Fig. 7.13) consisting of a transmission and of an actuating mechanism. Let the input link of the actuating mechanism (the crank) rotate with constant angular velocity v . Assuming that the active forces are internal forces for the machine, we have

$$R_x^{(e)} = -\Phi_x = m\ddot{x}_c = mv^2 x_c''(\varphi),$$

$$R_y^{(e)} = -\Phi_y = m\ddot{y}_c = mv^2 y_c''(\varphi).$$

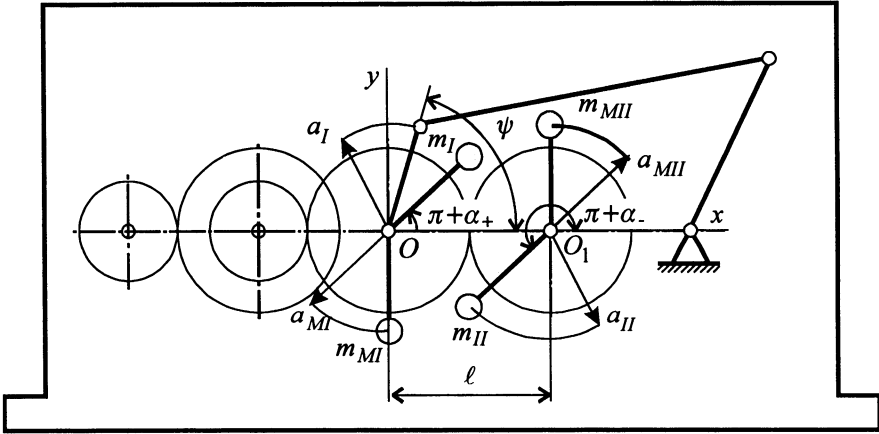


Fig. 7.13. The balancing of the first harmonic of the resultant vector of the inertia forces

Here, $x_c(\varphi)$ and $y_c(\varphi)$ are the coordinates of the mass center of the mechanism. They are periodic functions of the rotation angle φ of the crank; $x_c''(\varphi)$ and $y_c''(\varphi)$ are second-order derivatives with respect to φ , themselves periodic functions. Let us expand the functions $x_c''(\varphi)$ and $y_c''(\varphi)$ into Fourier series. Retaining only the first harmonics of this series we obtain

$$\begin{aligned} R_x^{(e)} &= m v^2 (a_x \cos \varphi + b_x \sin \varphi) + \dots, \\ R_y^{(e)} &= m v^2 (a_y \cos \varphi + b_y \sin \varphi) + \dots \end{aligned} \quad (7.24)$$

For arbitrary coefficients a_x, b_x, a_y, b_y the vector $\mathbf{R}^{(e)}$ with components (7.24) can be represented in the form of a sum of two vectors $R_+^{(e)}$ and $R_-^{(e)}$ in the following way:

$$\begin{aligned} R_x^{(e)} &= R_{x+}^{(e)} + R_{x-}^{(e)} = m v^2 \left[(0.5(a_x + b_y) \cos \varphi - 0.5(a_y - b_x) \sin \varphi) + \right. \\ &\quad \left. (0.5(a_x - b_y) \cos \varphi + 0.5(a_y + b_x) \sin \varphi) \right] = 0.5 m v^2 \sqrt{(a_x + b_y)^2 + (a_y - b_x)^2} \times \\ &\quad \cos(\alpha_+ + \varphi) + 0.5 m v^2 \sqrt{(a_x - b_y)^2 + (a_y + b_x)^2} \cos(\alpha_- - \varphi), \end{aligned}$$

$$\begin{aligned} R_y^{(e)} &= R_{y+}^{(e)} + R_{y-}^{(e)} = m v^2 \left[(0.5(a_y - b_x) \cos \varphi + 0.5(a_x + b_y) \sin \varphi) + \right. \\ &\quad \left. (0.5(a_y + b_x) \cos \varphi - 0.5(a_x - b_y) \sin \varphi) \right] = 0.5 m v^2 \sqrt{(a_x + b_y)^2 + (a_y - b_x)^2} \times \\ &\quad \sin(\alpha_+ + \varphi) + 0.5 m v^2 \sqrt{(a_x - b_y)^2 + (a_y + b_x)^2} \sin(\alpha_- - \varphi), \end{aligned}$$

$$\tan \alpha_+ = \frac{a_y - b_x}{a_x + b_y}, \quad \tan \alpha_- = \frac{a_y + b_x}{a_x - b_y}.$$

It is easy to see that the vector $\mathbf{R}_+^{(e)}$ has a constant modulus

$$|\mathbf{R}_+^{(e)}| = 0.5m\nu^2 \sqrt{(a_x + b_y)^2 + (a_y - b_x)^2}$$

and that it is rotating in the same direction as the crank, when φ varies from 0 to 2π . Vector $\mathbf{R}_-^{(e)}$ with modulus

$$|\mathbf{R}_-^{(e)}| = 0.5m\nu^2 \sqrt{(a_x - b_y)^2 + (a_y + b_x)^2}$$

is rotating in the opposite direction with the same angular velocity. It is possible to balance every one of these vectors through the centrifugal forces of two counterweights, one of which is attached to the crank and the other is mounted on an additional shaft which is connected with the crank through an external gear train with $i = -1$. The initial positioning of the counterweights (in the position of the mechanism corresponding to $\varphi = 0$) is determined by angles $\pi + \alpha_+$ and $\pi + \alpha_-$ (see Fig. 7.13), while the masses m_I and m_{II} , and the radii a_I and a_{II} are determined from the relationships

$$m_I a_I = |\mathbf{R}_+^{(e)}| \nu^{-2}, \quad m_{II} a_{II} = |\mathbf{R}_-^{(e)}| \nu^{-2}.$$

In this case, the counterweights are less bulky than in the case of shifting the mass center of the system to a fixed point. On the other hand, in this case an additional gearing is required.

Most often the balancing is restricted to the installation of a single counterweight reducing the first harmonic of the unbalanced force, but not ensuring its complete removal. It is possible, e.g., to choose the mass of the counterweight, the radius and the initial angle of its position on the crank in such a way that the maximum value of the modulus of $\mathbf{R}^{(e)}$ is minimized.

In principle, it is possible to balance also the first harmonic of the moment $M_{Oz}^{(Re)}$, which is a periodic function of the rotation angle φ . For this purpose, again, two counterweights can be placed on the axes O and O_I (see Fig. 7.13) rotating in opposite directions with angular velocity ν (counterweights m_{MI} and m_{MII}). The masses of the counterweights and their distances from the rotation centers are determined from the formula

$$m_{MI} a_{MI} = m_{MII} a_{MII} = \frac{(M_{Oz}^{(Re)})_1}{\ell \nu^2},$$

where ℓ is the distance OO_I , $(M_{Oz}^{(Re)})_1$ is the amplitude of the first harmonic of the moment $(M_{Oz}^{(Re)})$. In an analogous way, by means of counterweights rotating with angular velocity $k\nu$ on additional rotation axes, it is possible to balance $k - e$ harmonics of the dynamic reactions. Because of the complexity of this construction such balancing is applied only in exceptional cases.

7.7

Loss of Energy due to Friction in a Cyclic Mechanism

The motion of a cyclic mechanism is accompanied not only by the generation of variable forces resulting in internal and external vibration activity; in a moving mechanism also processes related to energy transformation occur.

The basic energy process taking place in a mechanism is the transformation of the work done by driving forces into work done by the forces of useful resistance which act when executing a working process. Part of the work done by the driving forces in a mechanism is converted into kinetic energy of moving links, into potential energy of the potential forces acting on mechanism links (forces of gravity, forces of elasticity in elastic elements) and into work of friction forces at kinematic pairs. In the steady-state motion of a cyclic mechanism, the variations of kinetic and of potential energy over one cycle are equal to zero, since in the beginning and at the end of the cycle the coordinates and the velocities of all material points are one and the same. For this reason, for a mechanism the balance of the works per cycle can be written in the following form:

$$A_{DF} = A_{UR} + A_{FR}, \quad (7.25)$$

where A_{DF} is the work of driving forces, A_{UR} is the work of forces of useful resistance, A_{FR} is the work of friction forces characterizing the loss of energy in a mechanism. The work of friction forces, which is transformed into heat energy has the effect of heating the contacting elements of kinematic pairs. This causes an increase in the intensity of wear and a decrease in the durability of the mechanism. The loss of energy due to friction worsens the economic indicators of machine performance.

Indeed, the loss due to friction occurring in a mechanism during the execution of a working process together with the external and the internal vibration activity has to be considered one of the important dynamic characteristics determining the quality of the mechanical system of a machine.

After the kinematic and the force analysis including the determination of friction forces at kinematic pairs, the computation of the energy loss in a mechanism is not difficult. Let us consider as example the slider-crank mechanism shown in Fig. 7.14. Let us assume that in some position of the mechanism, defined by the generalized coordinate q , the friction force F at the prismatic pair and the moments $M_{Oz}^R, M_{Az}^R, M_{Bz}^R$ of friction forces at the revolute pairs have already been found. Then, knowing the velocity $\dot{x}_B$ of the slider motion and the relative angular velocities $\dot{q}, \dot{\phi}, \dot{\psi}$ at the revolute pairs O, A, B , it is possible to determine the power of the friction forces

$$N_{FR} = |M_{Oz}^R \dot{q}| + |M_{Az}^R \dot{\phi}| + |M_{Bz}^R \dot{\psi}| + |F \dot{x}_B|. \quad (7.26)$$

The work of the friction forces per cycle during uniform rotation of the input link with angular velocity $\dot{q} = \omega_0$ is determined by the integration of this expression

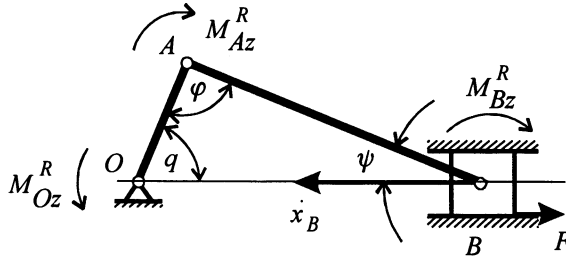


Fig. 7.14. The friction force and the moments of friction forces at the kinematic pairs of a slider-crank mechanism

$$A_{FR} = \int_0^{\frac{2\pi}{\omega_0}} N_{FR} dt = \int_0^{\frac{2\pi}{\omega_0}} \left(|M_{Oz}^R \dot{q}| + |M_{Az}^R \dot{\varphi}| + |M_{Bz}^R \dot{\psi}| + |F \dot{x}_B| \right) dt.$$

Taking into account that

$$\begin{aligned} |M_{Oz}^R \dot{q}| dt &= |M_{Oz}^R| dq, \quad |M_{Az}^R \dot{\varphi}| dt = \left| M_{Az}^R \frac{d\varphi}{dq} \right| dq, \\ |M_{Bz}^R \dot{\psi}| dt &= \left| M_{Bz}^R \frac{d\psi}{dq} \right| dq, \quad |F \dot{x}_B| dt = \left| F \frac{dx_B}{dq} \right| dq, \end{aligned}$$

we obtain

$$A_{FR} = \int_0^{2\pi} \left(|M_{Oz}^R| + \left| M_{Az}^R \frac{d\varphi}{dq} \right| + \left| M_{Bz}^R \frac{d\psi}{dq} \right| + \left| F \frac{dx_B}{dq} \right| \right) dq. \quad (7.27)$$

For an approximate computation of this integral we determine for k discrete positions $q = 2\pi s/k$ ($s = 0, \dots, k-1$) the values of the friction forces and moments and also the geometric transfer functions $d\varphi/dq$, $d\psi/dq$, dx_B/dq of the mechanism. Then, the approximate value is calculated from the formula

$$A_{FR} \cong \frac{2\pi}{k} \sum_{s=0}^{k-1} \left(|M_{Oz}^R|_s + \left| M_{Az}^R \frac{d\varphi}{dq} \right|_s + \left| M_{Bz}^R \frac{d\psi}{dq} \right|_s + \left| F \frac{dx_B}{dq} \right|_s \right). \quad (7.28)$$

The quality of a mechanism can also be characterized by a parameter such as *efficiency*. The efficiency of a cycle mechanism in steady-state motion is the ratio of the work done per cycle by forces of useful resistance and the work done per cycle by the driving forces:

$$\eta = \frac{A_{UR}}{A_{DF}}. \quad (7.29)$$

The efficiency of a transmission mechanism with a linear position function can be defined as the ratio of the power of forces of useful resistance and the power of the driving forces. The quantity

$$\psi = 1 - \eta \quad (7.30)$$

is called *loss factor*. Because of the expression (7.25) it is the ratio of the losses due to friction and the work done by the driving forces.

Let us recall that the reactions at kinematic pairs and, hence, also the friction forces depend on, both, active forces of useful resistance and inertia forces. Therefore, the efficiency and the loss factor depend not only on the quality of the mechanism, on properties of its kinematic pairs and on friction coefficients, but also on the working regime, on the program motion and on the workload. If, e.g., there is no useful load at all ($A_{UF} = 0$), then the inertia forces of the mechanism links will still cause reactions at the kinematic pairs and, therefore, friction forces. In such a regime always $\eta = 0$, $\psi = 1$. With the increase in useful load, for a given motion law of the input link, the efficiency of a mechanism will increase, since the loss due to friction will increase more slowly than the work done by the forces of useful resistance.

In order to exclude the influence of the inertia forces on the efficiency, a conditional design model of a mechanism can be used which takes into account only driving forces and forces of useful resistance. In this model it is assumed that the masses of all links are equal to zero.

The friction forces computed on the basis of such a model will be proportional to the useful load in every position of the mechanism and the efficiency will be characterized by the properties of the kinematic pairs only.

In the design of mechanisms different methods are used for increasing the efficiency and for reducing losses due to friction. The most efficient method is the reduction of the friction coefficients at kinematic pairs. This is achieved by using rolling supports instead of sliding supports, by lubricating kinematic pairs etc.

7.8 Problems

- 7.1. Fig. 7.15 shows a vibration exciter – a device designed for excitation of mechanical vibration. The vibration exciter is used in vibrohammers, vibratory pile drivers, shakers etc. The motor is turning the unbalanced link 1 (unbalanced mass) which is connected through the gearing 3–4 with the unbalanced mass 2. The unbalanced masses are $m_1 = m_2 = 1$ kg, the radii of the mass centers are $r_1 = r_2 = 0.1$ m, the angular velocities are $\omega_1 = \omega_2 = 10 \text{ s}^{-1}$. Find the resultant vector $\mathbf{R}^{(e)}$ and the resultant moment $\mathbf{M}_O^{(Re)}$ of the external reactions.

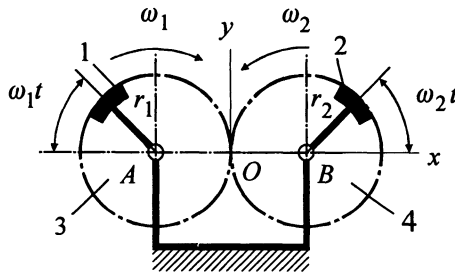


Fig. 7.15

Answer:

$$R_x^{(e)} = 0, R_y^{(e)} = 20 \sin \omega t, R_z^{(e)} = 0, M_{Ox}^{(Re)} = 0, M_{Oy}^{(Re)} = 0, M_{Oz}^{(Re)} = 0.$$

- 7.2. A machine consists of in itself a motor, a one-stage gear train reducer and an actuating mechanism (Fig. 7.16). All rotating links are statically and dynamically balanced. The moment of inertia of the shaft connecting the motor rotor to the gear z_1 is $J_1 = 3 \text{ kg m}^2$. The moment of inertia of the shaft connecting the second gear z_2 and the actuating mechanism is $J_2 = 10 \text{ kg m}^2$. Find the moment of inertia J_f of a flywheel that can balance the machine.

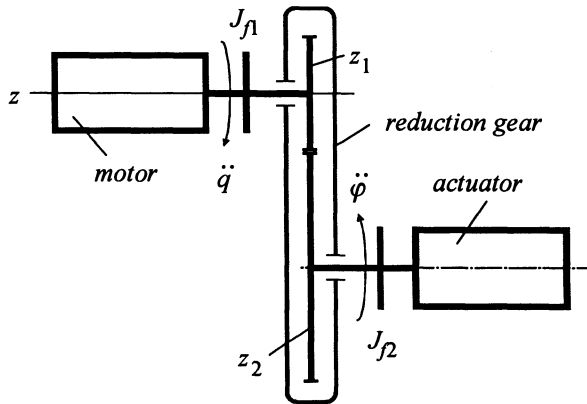


Fig. 7.16

Solution: If the rotating links are dynamically balanced, then the condition for the machine to have no external vibration activity is $M_{Oz}^{(Re)} = 0$.

Case 1: The flywheel is mounted on the shaft of the motor rotor. Then

$$M_{Oz}^{(Re)} = (J_1 + J_{f1})\ddot{q} - J_2\ddot{\phi} = 0.$$

Taking into account that $|\ddot{\varphi}/\ddot{q}| = |z_1/z_2|$, we find the moment of inertia of the flywheel J_{f1} : $J_{f1} = J_2 \frac{z_1}{z_2} - J_1 = 2 \text{ kg m}^2$.

Case 2: The flywheel is mounted on the shaft of the actuating mechanism. Then:

$$M_{Oz}^{(Re)} = J_1 \ddot{q} - (J_2 + J_{f2}) \ddot{\varphi} = 0.$$

From here $J_{f2} = J_1 \frac{z_2}{z_1} - J_2 = -4 \text{ kg m}^2$. The minus sign indicates that the shaft of the actuating mechanism is overloaded compared with the motor shaft, so that the flywheel with J_{f2} cannot balance the machine.

Answer: $J_{f1} = 2 \text{ kg m}^2$; it is impossible to balance the machine with flywheel J_{f2} .

- 7.3. The cam unloader (Fig. 7.17a) consists of cam 1, follower 2 and closing spring 3. The cam is an eccentric, i.e. disc with geometric center O_1 at a distance $e = 50 \text{ mm}$ from the rotation axis O . A perturbation moment $L = 30 \cos \omega_0 t - 10 \sin 2\omega_0 t$ is applied to the shaft of the eccentric where ω_0 is the average angular velocity of the shaft. Find the stiffness c of the closing spring and the initial deformation S_0 of the spring. Do not take into account the masses of the links.

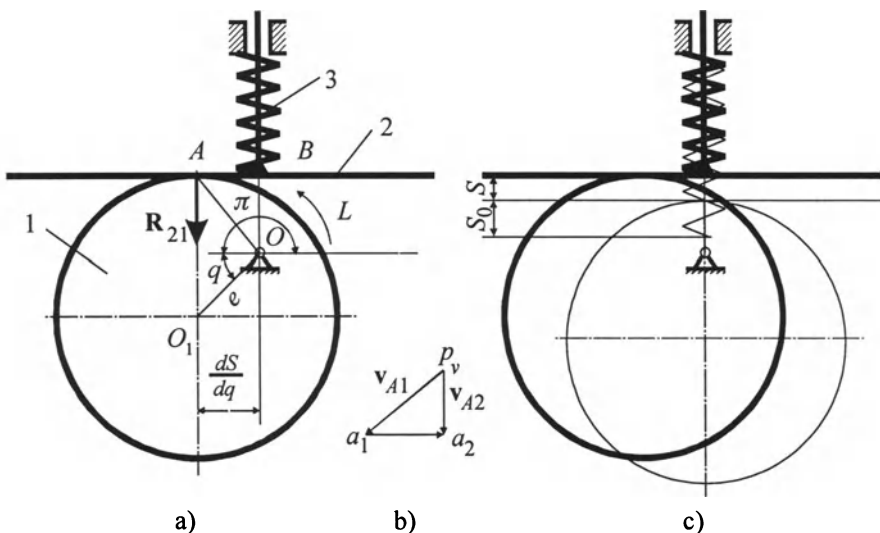


Fig. 7.17

Solution: We choose the parameters of the cam-unloader such that the condition $M_u + L = 0$ is fulfilled, where M_u is the moment of the unloader. This is the moment of the reaction forces from the follower on the cam: $M_u = h_u R_{21}$, where $h_u = AB$ is the lever of force R_{21} . Comparing the triangle OAB and the triangle of the velocity vector diagram of point A (Fig. 7.17b), it is possible to show that $h_u = \frac{dS}{dq}$, where $\frac{dS}{dq}$ is the analogue of the velocity of the point A on the follower. If one does not take into account the mass of the follower nor the friction forces, then the reaction R_{21} is equal to the elastic force of the spring: $R_{21} = c(S_0 + S)$ where c is the stiffness of the spring, S_0 is the initial deformation of the spring, S is the displacement of the follower relative to the lowest position (Fig. 7.17c). It follows that

$$c(S_0 + S) \frac{dS}{dq} = -L.$$

Since the displacement of the follower is $S = e + e \sin(q + \pi) = e(1 - \sin q)$, we have $dS/dq = -e \cos q$. Therefore:

$$(-cS_0 e - ce^2) \cos q + \frac{ce^2}{2} \sin 2q = -a \cos \omega_0 t - b \sin 2\omega_0 t,$$

where $a = 30 \text{ N m}$ and $b = -10 \text{ N m}$. For a constant cam angular velocity ω_0 we have $q = \omega_0 t$ and comparing the coefficients of sines and cosines

we obtain: $c = -\frac{2b}{e^2} = 8000 \frac{\text{N}}{\text{m}}$, $S_0 = \frac{a - ce^2}{ce} = 0.025 \text{ m}$.

$$\text{Answer: } c = 8000 \frac{\text{N}}{\text{m}}, S_0 = 0.025 \text{ m}.$$

7.4. In a washing machine 1 (Fig. 7.18) washing and spinning of the laundry is done in drum 2 with horizontal rotation axis.

1. Determine the horizontal motion $x_1(t)$ of the machine in the spinning cycle. The wet laundry has mass $m_2 = 5 \text{ kg}$, its mass center is at the radius $r = 8.5 \text{ cm}$ from the drum axis, the spinning frequency of the drum is $n = 380 \text{ min}^{-1}$ and the mass of the machine itself is $m_1 = 80 \text{ kg}$. Neglect friction at the axes of the wheels and also the wheel masses. The mass center of the machine coincides with the drum axis.
2. Determine the maximum force exerted by the machine on the floor in the spinning cycle.

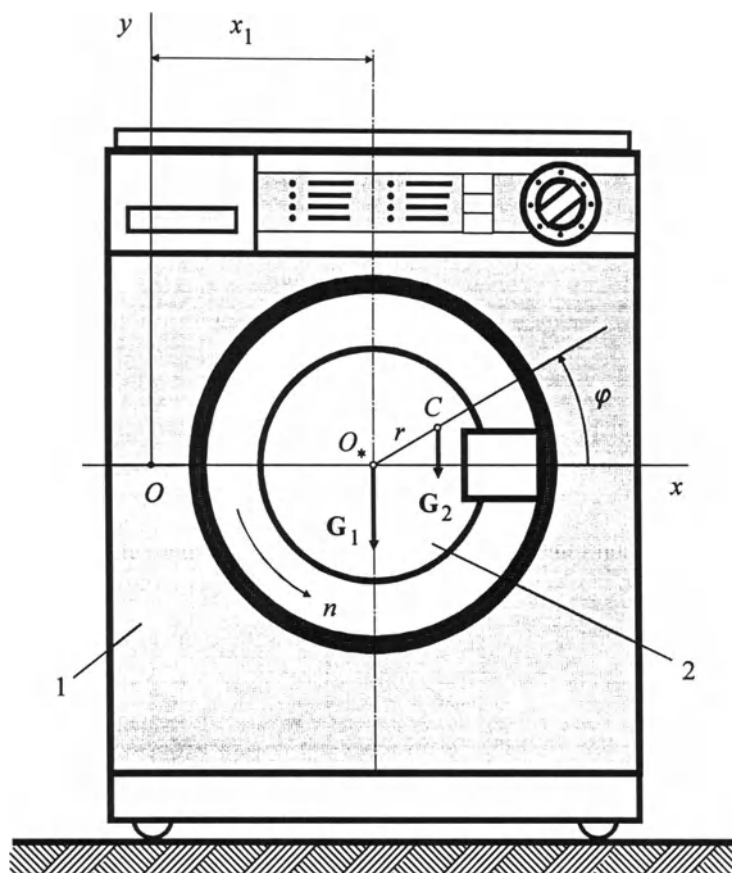


Fig. 7.18

3. Determine the minimum drum angular velocity required for vertical motions of the machine (the machine starts to „jump“).

Solution: Equations of motion are obtained from the theorem for the motion of center of mass:

$$(m_1 + m_2)\ddot{\mathbf{r}}_S = \mathbf{P}^{(e)} + \mathbf{R}^{(e)},$$

where $\mathbf{r}_S$ is the radius-vector of the mass center of the system „machine plus laundry“. For the coordinates x_C and y_C of the mass center of the system we find:

$$x_C = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}, \quad y_C = \frac{y_1 m_1 + y_2 m_2}{m_1 + m_2},$$

where m_1 is the mass of the machine (without laundry); m_2 is the laundry mass; x_1, y_1 are the coordinates of the mass center of the machine; x_2, y_2 are the coordinates of the mass center of the laundry:

$$y_1 = 0, \quad x_2 = x_1 + r \cos \varphi = x_1 + r \cos \omega t,$$

$$y_2 = r \sin \varphi = r \sin \omega t, \quad \omega = \frac{2\pi n}{60} = 39.8 \text{ s}^{-1}.$$

Substituting y_1 , x_2 and y_2 we obtain:

$$x_C = x_1 + \frac{m_2}{m_1 + m_2} r \cos \omega t, \quad y_C = \frac{m_2}{m_1 + m_2} r \sin \omega t.$$

The external forces applied to the system are the weight G_1 of the machine, the weight G_2 of the laundry, the reaction R_{0M} of the support (floor). The differential equations for the motion of the system center of mass have the form ($G_1 = m_1 g$, $G_2 = m_2 g$, $R_{M0} = R_{0M}$):

$$(m_1 + m_2)(\ddot{x}_1 - r\omega^2 \frac{m_2}{m_1 + m_2} \cos \omega t) = 0,$$

$$-(m_1 + m_2) \frac{m_2}{m_1 + m_2} r\omega^2 \sin \omega t = -R_{M0} - m_1 g - m_2 g.$$

From the first equation the horizontal motion x_1 of the machine can be found and from the second the force R_{M0} acting on the floor.

1. The horizontal motion of the machine is:

$$\ddot{x}_1 = \frac{m_2}{m_1 + m_2} r\omega^2 \cos \omega t, \quad \dot{x}_1 = \frac{m_2}{m_1 + m_2} r\omega \sin \omega t + C_1,$$

$$x_1 = -\frac{m_2}{m_1 + m_2} r \cos \omega t + C_1 t + C_2.$$

Assuming that initially the machine is motionless, i.e. $\dot{x}_1(0) = 0$, we have $C_1 = 0$. When the origin of the coordinate system is in the initial position of the system center of mass $x_C(0) = C_2 = 0$ we obtain:

$$x_1 = -\frac{m_2}{m_1 + m_2} r \cos \omega t = -0.005 \cos 39.8 t.$$

2. The force acting on the floor is:

$$R_{M0} = -m_1 g - m_2 g + m_2 r\omega^2 \sin \omega t = -833.9 + 673.2 \sin 39.8 t.$$

For $\sin \omega t = -1$ we have $R_{M0} = -1507.1 \text{ N}$ (the minus sign indicates that the force is directed downwards).

3. The machine does not move vertically if the condition: $R_{M0} \leq 0$ is satisfied, i.e.

$$\omega^2 \leq \frac{m_1 + m_2}{m_2} \frac{g}{r}, \quad \omega \leq 44.27 \text{ s}^{-1}.$$

Answer: $x_1 = 0.005 \sin 39.8t \text{ m}$, $|R_{M0}|_{\max} = 1507.1 \text{ N}$, $\omega_{\max} = 44.27 \text{ s}^{-1}$.

- 7.5. The rotor of an electric motor (Fig. 7.19) with mass $m = 400 \text{ kg}$ rotating with angular velocity 3000 1/min . The upper limit for the dynamic load on the bearing is $R = 400 \text{ N}$. Determine the admissible deviation e of the principal central axis of inertia of the rotor from the rotation axis.

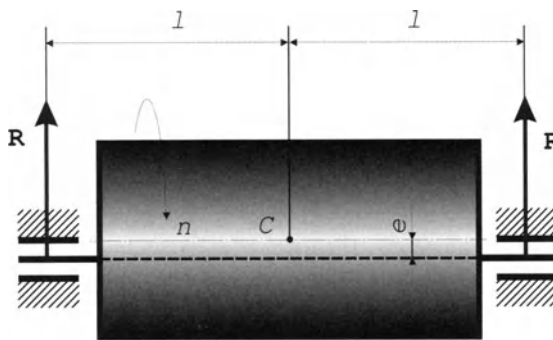


Fig. 7.19

Answer: $e = 0.0203 \text{ mm}$.

- 7.6. The crankshaft of a single one-piston engine (Fig. 7.20) consists of a bearing journal 1 and of a crank 2, which includes brake shoes 3 and crankpin 4. On the shaft carries are mounted two identical flywheels 5 and 6 with radius $r = 0.5 \text{ m}$. Flywheels 5 and 6 and the bearing journal 1 are statically and dynamically balanced. The mass center of crank 2 is at the point C . Find the masses m_A and m_B to be fixed on the wheel rims in order to balance the system. The mass of the crank is $m_2 = 21 \text{ kg}$, $h = 0.2 \text{ m}$, $b = 1.4 \text{ m}$.

Answer: $m_A = 4.8 \text{ kg}$, $m_B = 3.6 \text{ kg}$.

- 7.7. In a planar six-bar mechanism (Fig. 7.21) the dimensions of the links are as follows: $l_1 = OA = 0.1 \text{ m}$, $l_2 = AB = 0.3 \text{ m}$, $l_3 = BE = 0.4 \text{ m}$, $l_4 = DF = 0.4 \text{ m}$, $DE = 0.3 \text{ m}$. The mass centers of the links are at points C_1 , C_2 , C_3 , C_4 and C_5 respectively. Furthermore, $OC_1 = 0$, $AC_2 = C_2B$, $BC_3 =$

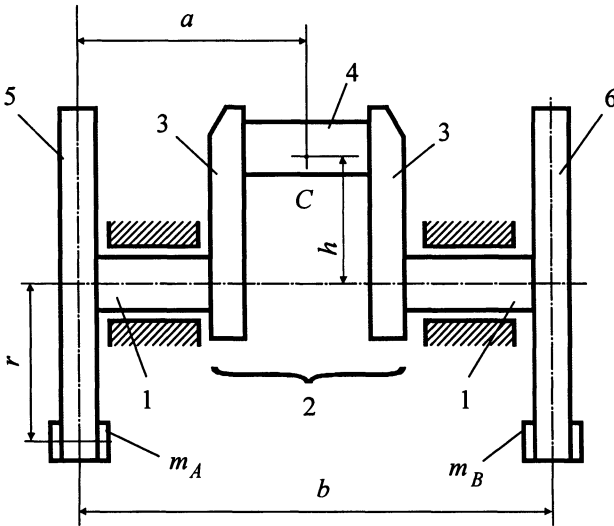


Fig. 7.20

C_3E , $DC_4 = C_4F$, $FC_5 = 0$. The masses of the links are: $m_1 = 6$ kg, $m_2 = 9$ kg, $m_3 = 12$ kg, $m_4 = 12$ kg, $m_5 = 12$ kg. Select the masses m_I , m_{III} and m_{IV} of counterweights attached to links 1, 3 and 4, respectively, in such a way that the mass center of the mechanism remains motionless. With K_1 , K_3 and K_4 being the mass centers of the counterweights assume that $OK_1 = 0.1$ m, $EK_3 = 0.4$ m, $DK_4 = 0.1$ m.

Answer: $m_I = 4.5$ kg, $m_{III} = 82.5$ kg, $m_{IV} = 72$ kg.

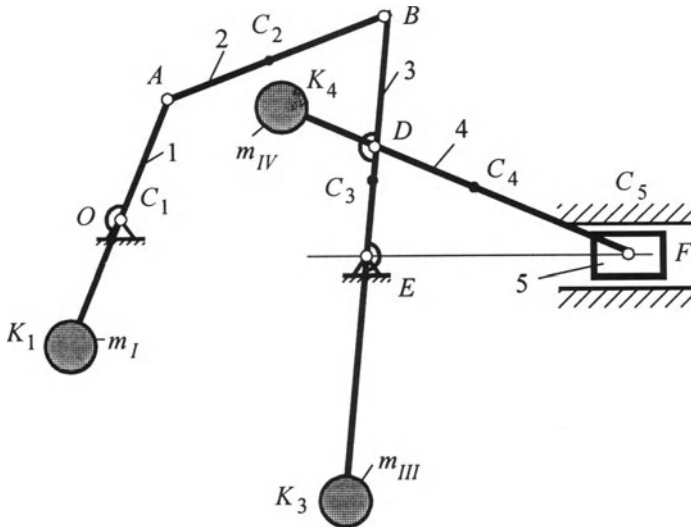


Fig. 7.21

- 7.8. A compressor is a machine for compressing air or gas. Components of the piston compressor (Fig. 7.22) are crank 1, rocker 2, piston 3, cylinder 4, inlet valve 5 and discharge valve 6. The length of the crank is $l_1 = OA = 0.1$ m, the length of the rocker is $l_2 = AB = 0.4$ m. The mass center of the crank is on the rotation axis O , the mass center of the rocker is at point C_2 with $AC_2 = 0.1$ m; the mass center of the piston is at point B . The mass of the rocker is $m_2 = 4$ kg, the mass of the piston is $m_3 = 5$ kg. Compute the masses m_I and m_{II} of counterweights which balance the first harmonic of the inertia forces. Assume that the counterweights are placed at the radius $a_I = a_{II} = 0.1$ m.

Hint: The acceleration w_B of the mass center of the slider can be assumed to be

$$w_B = -l_1 v^2 \left(\cos vt + \frac{l_1}{l_2} \cos 2vt \right),$$

where v is the constant angular velocity of the crank.

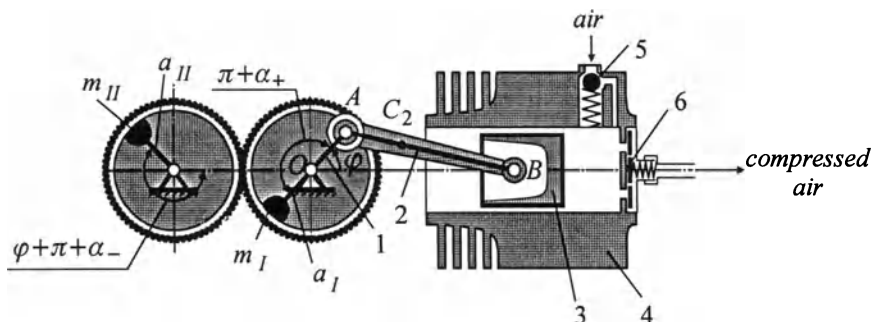


Fig. 7.22

Solution: Imagine that the mass of the second link consists of two point masses m_{2A} and m_{2B} concentrated at points A and B , respectively, in such a way that the mass center of the second link remains at point C_2 :

$$m_{2A} + m_{2B} = m_2, \quad m_{2A} AC_2 = m_{2B} C_2 B.$$

From this follows

$$m_{2A} = m_2 \frac{C_2 B}{AB} = 3 \text{ kg}, \quad m_{2B} = m_2 \frac{AC_2}{AB} = 1 \text{ kg}.$$

Under these conditions the masses m_{2B} and m_3 will be concentrated at point B , i.e. $m_B = m_{2B} + m_3 = 6$ kg, whereas at point A only a part of the mass of link 2 is concentrated: $m_A = m_{2A} = 3$ kg. Taking into account

that the mass of the crank is on the rotation axis and restricting ourselves to the first harmonic we obtain:

$$\begin{aligned} R_x^{(e)} &= -\Phi_x = -l_1 v^2 (m_A \cos vt + m_B \cos vt), \\ R_y^{(e)} &= -\Phi_y = -l_1 v^2 (m_A \sin vt). \end{aligned}$$

Comparing this with expression (7.24) and assuming that $\varphi = vt$, we find coefficients a_x , b_x , a_y , b_y :

$$ma_x = -l_1(m_A + m_B), \quad mb_x = 0, \quad ma_y = 0, \quad mb_y = -l_1 m_A.$$

With this, the moduli of the vectors of circular harmonics $R_+^{(e)}$ and $R_-^{(e)}$ are:

$$\begin{aligned} |R_+^{(e)}| &= 0.5v^2 \sqrt{(ma_x + mb_y)^2 + (ma_y - mb_x)^2} = l_1 v^2 (m_A + \frac{m_B}{2}), \\ |R_-^{(e)}| &= 0.5v^2 \sqrt{(ma_x - mb_y)^2 + (ma_y + mb_x)^2} = l_1 v^2 \frac{m_B}{2}. \end{aligned}$$

The associated angles α_+ and α_- are:

$$\begin{aligned} \tan \alpha_+ &= \frac{ma_y - mb_x}{ma_x + mb_y} = 0, \quad \alpha_+ = 0^0, \\ \tan \alpha_- &= \frac{ma_y + mb_x}{ma_x - mb_y} = 0, \quad \alpha_- = 0^0. \end{aligned}$$

The masses of the counterweights are:

$$\begin{aligned} m_I &= \frac{|R_+^{(e)}|}{a_I v^2} = m_A + \frac{m_B}{2} = 6 \text{ kg}, \\ m_{II} &= \frac{|R_-^{(e)}|}{a_{II} v^2} = \frac{m_B}{2} = 3 \text{ kg}. \end{aligned}$$

$$\text{Answer: } m_I = 6 \text{ kg}, \quad m_{II} = 3 \text{ kg}, \quad \alpha_+ = 0^0, \quad \alpha_- = 0^0.$$

- 7.9. A four-cylinder in-line engine (Fig. 7.23) is a series of identical crank-slider mechanisms (1, 2, 3, 4) with a common crankshaft. All cylinder axes are parallel to the axis Ox . The dominant forces acting on the frame are the first and the second harmonics Φ_{1k} and Φ_{2k} of the inertia forces of the pistons (sliders). They are given by the formulae:

$$\begin{aligned} \Phi_{1x} &= mr v^2 [\cos \varphi + \cos(\varphi + \alpha_{21}) + \cos(\varphi + \alpha_{31}) + \cos(\varphi + \alpha_{41})], \\ \Phi_{2x} &= mr v^2 \lambda [\cos 2\varphi + \cos 2(\varphi + \alpha_{21}) + \cos 2(\varphi + \alpha_{31}) + \cos 2(\varphi + \alpha_{41})], \end{aligned}$$

where v is the angular velocity of the crank; $\lambda = \frac{r}{l}$; r is the length of the crank, l is the length of the rocker; α_{21} , α_{31} , α_{41} are the angles of the 2-nd, 3-rd and 4-th crank, respectively, against the first; φ is the crank rotation angle of the 1-st mechanism.

Determine to within two harmonics the action of the engine on the base for $\varphi = 0^\circ$ and for the crank angular velocity is $v = 20 \text{ s}^{-1}$. The angles are: $\alpha_{21} = 180^\circ$, $\alpha_{31} = 180^\circ$, $\alpha_{41} = 0^\circ$. The dimensions of the engine are: $r = 0.1 \text{ m}$, $l = 0.4 \text{ m}$, $a = 0.4 \text{ m}$, $b = 0.8 \text{ m}$. Every slider has mass $m = 10 \text{ kg}$.

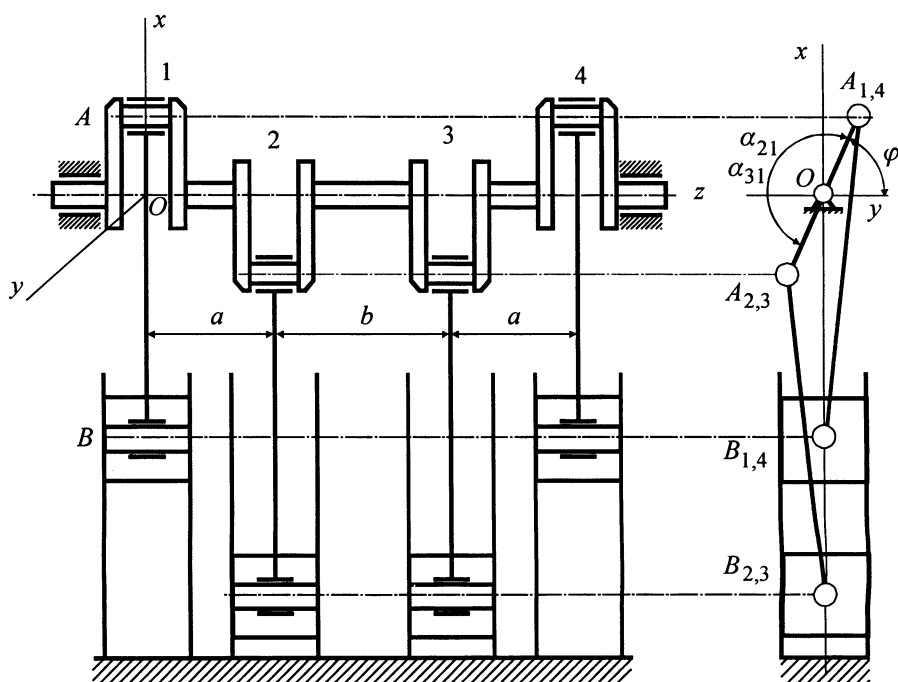


Fig. 7.23

Answer: $\Phi_{1x} = 0$, $\Phi_{2x} = 400 \text{ N}$,

$$M_{0y}^{(\Phi_1)} = mr v^2 \left[a \cos(q + 180^\circ) + (a + b) \cos(q + 180^\circ) + (2a + b) \cos q \right] = 0,$$

$$M_{0y}^{(\Phi_2)} = mr v^2 \lambda \left[a \cos(2q + 360^\circ) + (a + b) \cos(2q + 360^\circ) + (2a + b) \cos 2q \right] = 320 \text{ N.m.}$$

- 7.10. In a double-slider mechanism (Fig. 7.24) sliders 1 and 3 are interconnected by the rocker 2. Slider 1 is the input link. The length of link 2 is $l = AB = 0.5$ m. A workload P is applied to the output link 3. The coefficient of sliding friction at the prismatic pairs is $f = 0.1$. Find the relationship between the efficiency of the mechanism and the generalized coordinate q . Neglect friction at the revolute pairs and do not take gravity into account.

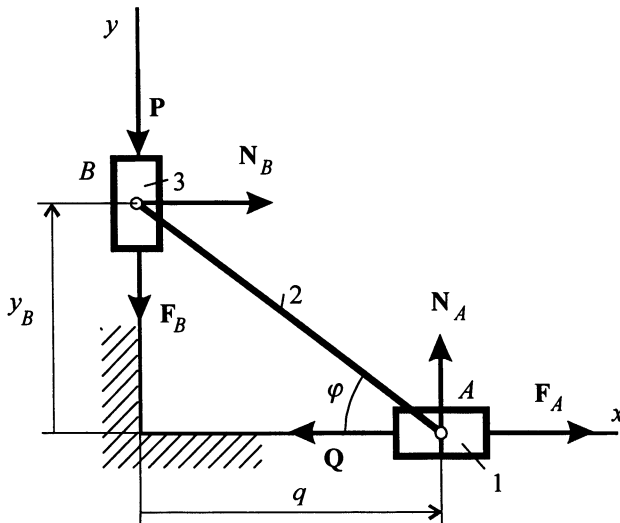


Fig. 7.24

Solution: We find the instantaneous efficiency η of the mechanism as ratio of the power of the workload to the power of the driving force:

$$\eta = \left| \frac{N_{UR}}{N_{DF}} \right| = \left| \frac{P\dot{y}_B}{Q\dot{q}} \right|.$$

We formulate the equilibrium equations of the mechanism ($F_A = fN_A$, $F_B = fN_B$):

$$\begin{cases} fN_A + N_B - Q = 0, \\ N_A - (fN_B + P) = 0, \\ -qN_A - y_B fN_A + y_B Q = 0, \end{cases}$$

where $y_B = \sqrt{l^2 - q^2}$. It is obvious that $\dot{y}_B = -\frac{q}{\sqrt{l^2 - q^2}}\dot{q} = -\frac{q}{y_B}\dot{q}$.

The driving force Q is:

$$Q = \frac{\begin{vmatrix} f & 1 & 0 \\ 1 & -f & P \\ -(q + fy_B) & 0 & 0 \end{vmatrix}}{\begin{vmatrix} f & 1 & -1 \\ 1 & -f & 0 \\ -(q + fy_B) & 0 & y_B \end{vmatrix}} = P \frac{q + fy_B}{y_B - fq}.$$

Hence,

$$\eta = \frac{q(y_B - fq)}{y_B(q + fy_B)} = \frac{1 - f \frac{q}{y_B}}{1 + f \frac{y_B}{q}} = \frac{\left(1 - \frac{fq}{\sqrt{l^2 - q^2}}\right)q}{q + f\sqrt{l^2 - q^2}}.$$

From this expression it follows that $\eta = 0$ for $q = 0$ and for

$$q = q^* = \frac{l}{\sqrt{f^2 + 1}} = 0.4975 \text{ m};$$

η has the maximum value for

$$q = q^{**} = \frac{l}{\sqrt{(f + \sqrt{f^2 + 1})^2 + 1}} = 0.3355.$$

This maximum is

$$\eta_{\max} = \frac{1 - f(f + \sqrt{f^2 + 1})^{-1}}{1 + f(f + \sqrt{f^2 + 1})} = 0.819.$$

The graph of the function $\eta(q)$ is shown in Fig. 7.25. Seizure of the mechanism occurs if $q > q^*$.

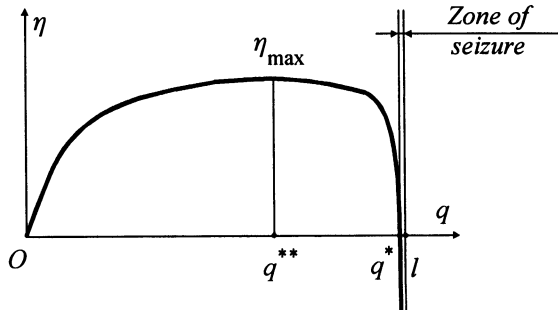


Fig. 7.25. The graph of the function $\eta(q)$ in problem 7.10

$$\text{Answer: } \eta = \frac{\left(1 - \frac{fq}{\sqrt{l^2 - q^2}}\right)q}{q + f\sqrt{l^2 - q^2}}.$$

8 Dynamics of Cycle Machines with Rigid Links

8.1

Mechanical Characteristics of Engines

For the solution of problems of machine dynamics we usually use the simplest dynamic engine models displaying the dependencies among the following time laws: The law $u(t)$ of the input parameter (control) of the engine, the law $q(t)$ of the generalized coordinate and the law $Q(t)$ of the generalized driving force (see Sect. 1.2). The mathematical relationships describing these dependencies are called *mechanical characteristics* of engines. Comparatively rarely, it is necessary to refer to more complex models which take into account the dynamics of internal physical processes in engines. In the present course such models will not be considered.

Characteristics of a direct-current electric motor with separate excitation. In order to get acquainted with the basic varieties of mechanical characteristics of engines we consider as example a direct-current electric motor with separate excitation. The general diagram is shown in Fig. 8.1. The rotation of the output link (the rotor) of the motor results from the interaction between the current in the rotor winding and the magnetic field generated by the excitation winding. According to the law of electromagnetic induction, the rotation of the rotor induces an electromotive force E (e.m.f.) in the rotor winding which is proportional to the magnitude of the magnetic flux Φ of the excitation winding and to the rotor angular velocity $\dot{q}$:

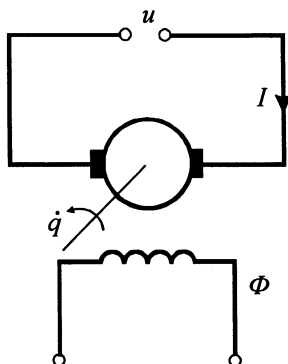


Fig. 8.1. The general diagram of an electric motor

$$E = k\Phi\dot{q},$$

where k is the coefficient of proportionality. When a current I is flowing in the rotor circuit voltage losses occur because of the resistance R and the inductance L . In terms of these losses we write the equation of the circuit in the form

$$u = E + RI + L\dot{I} = k\Phi\dot{q} + RI + L\dot{I}. \quad (8.1)$$

On the other hand, according to Ampere law, the driving moment Q is related to the current I through the equation

$$Q = k\Phi I. \quad (8.2)$$

Eliminating I from Eqs. (8.1) and (8.2), we obtain

$$u = k\Phi\dot{q} + \frac{R}{k\Phi}Q + \frac{L}{k\Phi}\dot{Q}. \quad (8.3)$$

Using the notation $\frac{L}{R} = \tau$, $\frac{k\Phi}{R} = r$, $\frac{k^2\Phi^2}{R} = s$, it is easy to give this expression the form

$$\tau\dot{Q} + Q = ru - s\dot{q}. \quad (8.4)$$

The expression (8.4) relating the input (u) to the output parameters ($Q, \dot{q}$) of the motor is called *dynamic characteristic*. The parameter τ is called *electromagnetic time constant*. It characterizes the inertia of the electromagnetic processes occurring in a motor. Its magnitude is usually in range of 0.02 s to 0.1 s. The parameter s is called *steepness of the motor characteristic*. The larger the steepness s , the smaller is the influence of load variations on the magnitude of the rotor angular velocity.

The characteristic (8.4) is widely used in the analysis of dynamic processes in machines that are driven by direct-current electric motors with separate excitation. When a static process with $Q = \text{const}$ is investigated, the expression (8.4) is reduced to a *static* characteristic of the motor:

$$Q = ru - s\dot{q}. \quad (8.5)$$

The static characteristic can also be used for the investigation of dynamic processes if $\tau\dot{Q} \ll Q$, i.e. in cases when either the time constant τ or the derivative $\dot{Q}$ is a small quantity.

In Fig. 8.2 two families of static characteristics are constructed: in Fig. 8.2a *working characteristics* expressing functions $Q(\dot{q})$ for different constant values of u are depicted; in Fig. 8.2b *control characteristics* $\dot{q}(u)$ for different constant values of Q are presented. In this case all these characteristics are *linear*.

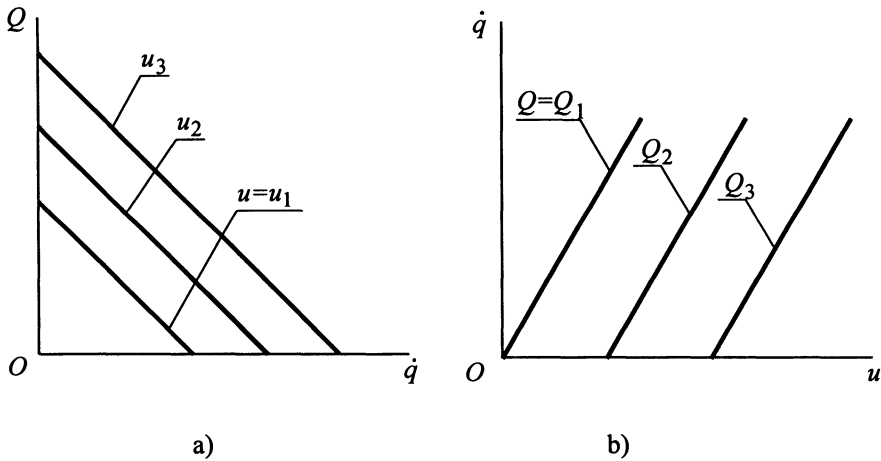


Fig. 8.2. Static characteristics of a direct-current electric motor with separate excitation: a) working characteristics, b) control characteristics

The control characteristic corresponding to $Q = 0$ (i.e. the determining function $\dot{q}(u)$ in the absence of load on the motor) is called *characteristic of idling*. Under certain conditions this characteristic can also be considered as an approximation of characteristic for $Q \neq 0$. This happens in cases, when the motor characteristic is *stiff* enough, i.e. when the influence of the load on the velocity can be neglected in first approximation. The characteristic obtained under this assumption is called an *ideal characteristic*; it can be reduced to the form

$$\dot{q} = \frac{r}{s} u. \quad (8.6)$$

According to this characteristic, the angular velocity of the rotor is completely determined by the value of the input parameter of the motor; when this characteristic is used, the motor behaves like a „velocity source“.

An important characteristic of a motor in steady state is its *efficiency* (eff.), which is the ratio of the outlet power to the power transmitted to the motor by the energy source. Multiplying the expression (8.3) by $I = (k\Phi)^{-1} Q$ and assuming that $\dot{I} = 0$, we obtain:

$$uI = \dot{q}Q + R(k\Phi)^{-2} Q^2 = \dot{q}Q + s^{-1} Q^2.$$

The product uI is the consumed power, $\dot{q}Q$ is the outlet power. Dividing the equation by $\dot{q}Q$, we obtain

$$\frac{uI}{\dot{q}Q} = \frac{1}{\eta} = 1 + \frac{Q}{s\dot{q}}, \quad (8.7)$$

where η is the efficiency of the motor. In a steady state

$$\dot{q} = \frac{r}{s}u - \frac{Q}{s} = \omega_{id} - \frac{Q}{s},$$

where ω_{id} is the angular velocity of idling corresponding to a given value of u . With this the relationship (8.7) yields

$$\eta = \frac{\dot{q}}{\omega_{id}}. \quad (8.8)$$

Let us compute the average value of the efficiency of a motor in a dynamic state in which $u = \text{const}$, whereas $Q(t)$ and $\dot{q}(t)$ are periodic functions of time. For this purpose we multiply equation (8.3) by I and take the average values of both sides over one period T . We obtain

$$\frac{1}{T} \int_0^T uI dt = \frac{1}{T} \int_0^T \dot{q}Q dt + \frac{1}{T} \int_0^T RI^2 dt + \frac{1}{T} \int_0^T LI\dot{I} dt.$$

The quantities

$$\frac{1}{T} \int_0^T uI dt = N_e, \quad \frac{1}{T} \int_0^T \dot{q}Q dt = N_M$$

are the average values of the power provided by the mains and of the outlet power, respectively. Furthermore, because of the periodicity of $I(t)$

$$\frac{1}{T} \int_0^T LI\dot{I} dt = \frac{LI^2}{2} \Big|_0^T = 0.$$

Consequently,

$$\eta = \frac{N_M}{N_e} = \left(1 + \frac{\overline{Q^2}}{sN_M} \right)^{-1}, \quad (8.9)$$

where $\overline{Q^2}$ is the average value of the squared moment over one period.

Characteristics of an electric induction motor. The functional principle of the induction motor is the generation of a magnetic field with a certain vector of intensity that rotates with an angular velocity equal to the frequency ν of the alternating current feeding the stator winding divided by the number p_P of pairs of poles of its magnetic system. In the absence of a load (in idling) the rotor being entrained by this field is rotating with this angular velocity

$$\dot{q} = \frac{\nu}{p_P}. \quad (8.10)$$

By varying frequency ν one can control the velocity of the motor; such controlled induction motors have recently found wide-spread application because of the availability of efficient and cheap frequency converters.

Expression (8.10) can be considered as an ideal induction motor characteristic in which the frequency ν is a control parameter. When a load is applied, the rotor starts to fall behind the magnetic field; the result is the so-called rotor slip σ defined by the formula

$$\sigma = 1 - p_P \dot{q} \nu^{-1}. \quad (8.11)$$

The dependence of the slip on the load is rather complex. It is described, generally speaking, by a system of differential equations relating currents in the windings of the motor. However, for steady state one can use approximate dependencies; the most widely used approximation is the so-called *specified formula of Kloss*

$$Q = \frac{2Q_c(1 + a\sigma_c)}{\sigma_c\sigma^{-1} + \sigma\sigma_c^{-1} + 2a\sigma_c}. \quad (8.12)$$

Here, the parameter a is the ratio of the resistance of the stator circuit to the resistance of the rotor circuit; the parameters Q_c and σ_c are called *critical moment* and *critical slip*, respectively. All three parameters are given in catalogues of induction motors.

Expression (8.12) is a static characteristic having the form shown in Fig. 8.3. The maximum value of the resistance moment Q is Q_c , and σ_c is the slip corresponding to this load. Usually, $\sigma_c = 0.1 \div 0.2$, while Q_c exceeds $2 \div 2.5$ times the *nominal motor moment* (corresponding to the nominal power for the nominal angular velocity: $Q_n = N_n \dot{q}_n^{-1}$). It is obvious from Fig. 8.3 that in this case the static characteristic is nonlinear. Actually, in working regimes as a rule $Q < 0.5Q_c$; to these values of Q corresponds the section Ob of the static characteristic, where it is close to linear:

$$Q \approx \frac{2Q_c}{\nu\sigma_c} (\nu - p_P \dot{q}). \quad (8.13)$$

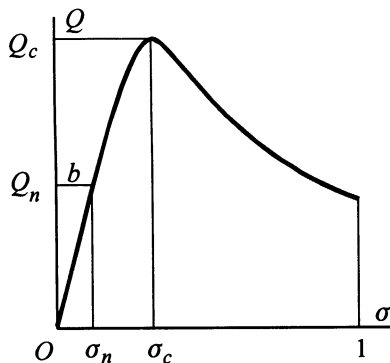


Fig. 8.3. Static characteristic of an induction motor

If in a dynamic state the quantities Q and σ are not outside this section, then it is possible to use a dynamic characteristic which is analogous to the characteristic (8.4):

$$\tau \dot{Q} + Q \approx \frac{2Q_c}{v\sigma_c} (v - p_P \dot{q}), \quad (8.14)$$

where the electric time constant can be determined by the formula

$$\tau = (v\sigma_c)^{-1}.$$

General forms of mechanical characteristics of engines. In general, the mechanical characteristics of engines of different kinds (heat, hydraulic, pneumatic) can be represented in forms analogous to those obtained above. During idling, for $Q = 0$, the engine behaviour is characterized by an *ideal kinematic* characteristic

$$\dot{q} = f(u), \quad (8.15)$$

which, in general, is nonlinear. Such a characteristic approximately describes an engine when the state velocity depends only slightly on the load. Not only the electrical motors above considered but also hydraulic motors with positive-displacement control and with throttle control possess such properties. In combustion engines and in pneumatic motors the opposite case is observed: the value of the input parameter u predetermines to a large degree the value of the generalized driving force Q . The steady state of such engines can approximately be described by the *ideal force* characteristic

$$Q = F(u). \quad (8.16)$$

In general, in investigations of steady states static characteristics of the form

$$\dot{q} = f_s(u, Q). \quad (8.17)$$

are used. Solving for Q they can be represented in the form:

$$Q = F_s(u, \dot{q}). \quad (8.18)$$

These characteristics take into account the influence of the load on the generalized velocity. This influence is apparent to a greater or smaller degree in all real engines. Control characteristics obtained from the relationship (8.17) for $Q = \text{const}$, and working characteristics obtained from the relationship (8.18) for $u = \text{const}$ are, generally speaking, nonlinear. As a rule, with the increasing load the generalized velocity $\dot{q}$ becomes smaller and the working characteristics are „falling“. The value of the derivative $\partial Q / \partial \dot{q}$ taken with the opposite sign,

$$s = -\frac{\partial Q}{\partial \dot{q}}, \quad (8.19)$$

is called *steepness* of the static characteristic at a given point. In the case of a „falling“ characteristic $s > 0$. If the generalized velocity depends only slightly on the load, the static engine characteristic is called *stiff*; if velocity variations influence the value of the moment only slightly, then the characteristic is called *soft*.

In some problems of machine dynamics the values of the parameter u and those of the generalized velocity $\dot{q}$ can be considered to be close to some average values u_0 and ω_0 . In these cases, the nonlinear static characteristic can be linearized in the vicinity of the point (u_0, ω_0) :

$$Q = F_s(u_0, \omega_0) + \frac{\partial F_s}{\partial u}(u_0, \omega_0)(u - u_0) + \frac{\partial F_s}{\partial \dot{q}}(u_0, \omega_0)(\dot{q} - \omega_0). \quad (8.20)$$

In some classes of cycle engines, e.g., in combustion engines with pistons in which the coordinate of the output link (crankshaft) is related to the input coordinate by a nonlinear position function the generalized driving force depends not only on the generalized velocity $\dot{q}$, but also on the generalized coordinate q . In this case, the static characteristic is presented in the form

$$Q = Q_s(u, \dot{q}, q), \quad (8.21)$$

where Q_s is a periodic function of q . In a piston engine the period of this function is equal to $2\pi/m$, where m is some integer determined by the engine design, in particular, by the number of cylinders.

The static characteristics adequately reflect the properties of real engines only in static regimes of machine performance, i.e. in cases when the quantities u , $\dot{q}$ and Q are constant, or when they are varying insignificantly and slowly. In more general cases, it is necessary to take into account the inertia of the physical processes taking place in an engine. In a direct-current electric motor such inertia is related to the inductance of the armature circuit; due to this inertia, a variation in the voltage does not result in a sudden variation of the current in the rotor circuit. A *transient* occurs whose duration depends on the time constant τ .

The inertia has a different nature in other types of engines. In a hydraulic motor, e.g., it is caused by the compressibility of the fluid. In any case, however the inertia has the effect that for a fixed value of the input parameter the generalized velocity of the output link depends not only on the load but also on its derivatives with respect to time. In a first approximation this can be taken into account by introducing in the static characteristic (8.17) the first derivative of $Q(t)$ and by representing it in the form

$$\dot{q} = f_s(u, \tau \dot{Q} + Q), \quad (8.22)$$

or in the form, analogous to the relationship (8.18):

$$\tau \dot{Q} + Q = F_s(u, \dot{q}). \quad (8.23)$$

In this general case the parameter τ is called *fundamental time constant* of the engine and the expressions (8.22) and (8.23) are its *dynamic characteristics*. It must be noted that for some classes of engines, characteristics of the kind (8.22) and (8.23) can be used only when the quantity $\dot{q}$ is comparatively narrowly bounded. In investigations of other dynamic states one should use more complex dynamic models which are not considered in this book.

8.2

Equations of Motion of a Machine. State of Motion

For a machine assembly with a single degree of movability, with a mechanical system consisting of mechanisms with rigid links and with ideal kinematic pairs, the complete system of equations includes an equation of motion of the mechanical system, e.g., in the form of Lagrange's equation of the second kind, and also engine characteristics. If, furthermore, the ideal kinematic characteristic of the form (8.15) is chosen, then for a given law of variation of the input parameter $u(t)$, the law of variation of the the motor angular velocity and, therefore, its motion law are determined by the characteristic:

$$\dot{q}(t) = f[u(t)], \quad q(t) = \int_0^t f[u(t)] dt. \quad (8.24)$$

For the determination of the driving moment the Lagrange equation can be used:

$$Q(t) = J[q(t)]\ddot{q} + \frac{1}{2} J'[q(t)]\dot{q}^2 - Q_R[q(t), \dot{q}(t)]. \quad (8.25)$$

In other words, this is the problem of investigating the dynamics of the mechanical system of a machine for a given motion. This problem has been investigated in detail in previous chapters. Such formulation of the problem of dynamics of a machine assembly is acceptable, if the engine possesses a stiff characteristic. For an engine with a soft characteristic, an approximate investigation of the machine can be based on the ideal force characteristic (8.16). In this case, the generalized driving force is determined by the engine characteristic

$$Q(t) = F[u(t)] \quad (8.26)$$

and the motion law of the machine can be found through an integration of the equation

$$J(q)\ddot{q} + \frac{1}{2} J'(q)\dot{q}^2 = F[u(t)] + Q_R(q, \dot{q}), \quad (8.27)$$

which is obtained by substituting the characteristic (8.26) into the Lagrange equation.

As a rule, the use of ideal characteristics is acceptable in the early stages of design. In a more precise dynamics analysis the dependency of the motion law on the load, which is expressed by static and dynamic characteristics, must be taken into account. Using a static characteristic we get the motion equation by substituting the static characteristic (8.18) into the Lagrange equation

$$J(q)\ddot{q} + \frac{1}{2} J'(q)\dot{q}^2 = F_s[u(t), \dot{q}] + Q_R(q, \dot{q}). \quad (8.28)$$

This is a nonlinear second-order differential equation which can be integrated for a given law $u(t)$ and for a given initial or boundary conditions.

When a dynamic engine characteristic is used, the problem is reduced to the integration of a system of two differential equations with two unknowns $q(t)$ and $Q(t)$:

$$J(q)\ddot{q} + \frac{1}{2}J'(q)\dot{q}^2 = Q + Q_R(q, \dot{q}),$$

$$\tau\dot{Q} + Q = F_s[u(t), \dot{q}].$$

State of motion of a machine. Usually, the dynamics investigation of a machine assembly is limited to the determination and analysis of certain special solutions of the differential equations of motion which describe the most typical states of motion.

- a) *Steady-state motion.* Steady-state motion is typical for machines working under constant load and for cycle machines performing a periodically repeating process. Usually, in a steady-state regime the input parameter is a constant quantity

$$u = u_0 = \text{const}. \quad (8.29)$$

In a machine with a rotating engine a periodic motion is developing with an angular velocity close to some average value ω_0 :

$$\dot{q} = \omega_0 + \dot{\psi}, \quad (8.30)$$

with $|\dot{\psi}|_{\max} \ll |\omega_0|$. In what follows a regime satisfying the conditions (8.29) and (8.30) will be called a *steady-state motion* of machine.

- b) *Transients.* Transients occur during *starting acceleration* and *run-out* of a machine and as a result of load variations. To the *process of starting acceleration* corresponds a special solution $\dot{q}(t)$ satisfying the initial conditions $t = 0, \dot{q} = 0$; during the starting acceleration a transition of the machine from a state of rest to a steady-state motion occurs. The starting acceleration is called *non-controllable*, if $u = u_0 = \text{const}$; during a *controllable* starting acceleration, $u(t)$ is time-varying. Normally, $u(t)$ increases smoothly from zero to $u = u_0$.

Run-out of a machine is the transient from a steady-state motion to the state of rest. In a *free run-out* the engine is turned off and the deceleration of the machine is caused by resistance forces. In the state of *braking*, after turning-off the engine, an active braking moment is generated that accelerates the process of run-out. During a *dynamic braking*, the kinetic energy of a machine is recuperated, i.e. returned, in one way or another, to the energy source.

Often, due to changing characteristics of the working process, a transition from one steady-state regime of the machine to another occurs. In this case, the transient is the result of load variation.

8.3

Determination of the Average Angular Velocity of a Steady-State Motion for a Cycle Machine

We investigate a steady-state motion of a machine taking into account the static characteristic of the engine. We obtain the motion equation by substituting $u = u_0$ into equation (8.28):

$$J(q)\ddot{q} + \frac{1}{2}J'(q)\dot{q}^2 = F_s(u_0, \dot{q}) + Q_R(q, \dot{q}). \quad (8.31)$$

We try to find a periodic solution of this equation in the form of Eq. (8.30):

$$q = \omega_0 t + \psi(t). \quad (8.32)$$

Using relationships (7.2) and (7.3) we write equation (8.31) in the form

$$J_0\ddot{q} - F_s(u_0, \dot{q}) - Q_{R0}(q, \dot{q}) = -\tilde{J}(q)\ddot{q} - \frac{1}{2}\tilde{J}'(q)\dot{q}^2 + \tilde{Q}_R(q, \dot{q}). \quad (8.33)$$

The influence of the static characteristic of the engine becomes apparent in two ways. First, the average angular velocity of the rotor is no longer determined by the value of the input parameter only, but it depends also on the load. Second, the variable forces occurring in the mechanical system which determine the inner vibration activity of a mechanism result in oscillations of the angular velocity of the rotor. It is easy to see that the vibration processes are excited by right-hand side terms in equation (8.33) which depend explicitly on q . Assuming that the oscillations of the angular velocity are small ($|\dot{\psi}| \ll \omega_0$), it is possible to find a solution of equation (8.33) by the method of successive approximation, where as an initial „zero“ approximation a solution in the form

$$q^{(0)}(t) = \omega_0 t \quad (8.34)$$

is chosen which satisfies the equation

$$J_0\ddot{q}^{(0)} - F_s(u_0, \dot{q}^{(0)}) - Q_{R0}(q^{(0)}, \dot{q}^{(0)}) = 0. \quad (8.35)$$

Substituting expression (8.34) into equation (8.35) we obtain an equation for ω_0 :

$$F_s(u_0, \omega_0) + Q_{R0}(\omega_0) = 0. \quad (8.36)$$

This equation has a simple physical interpretation. It means that in the system an angular velocity of the engine rotor is established such that the average moment of the driving forces $F_s(u_0, \omega_0)$ is equal to the average moment of the resistance forces $-Q_{R0}(\omega_0)$. Equation (8.36) can be solved graphically by determining the intersection points of the graphs of $F_s(u_0, \dot{q})$ and $-Q_{R0}(\dot{q})$ (Fig. 8.4). As we can see from the figure, there may be several intersection points which correspond to different solutions of Eq. (8.36). Only stable solutions of Eq. (8.35) correspond

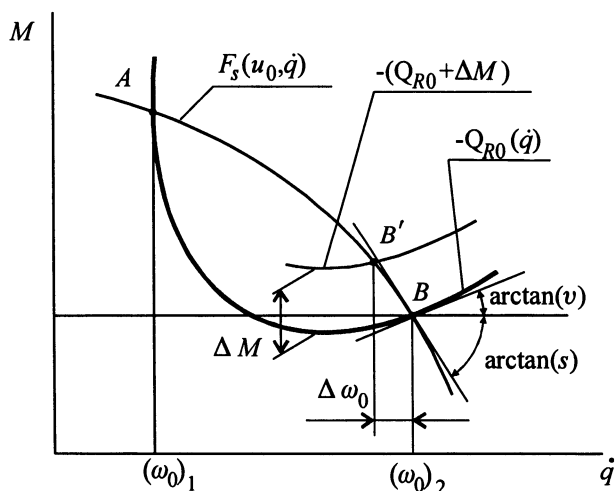


Fig. 8.4. Graphical determination of the average angular velocity

to realizable steady-state motions. In order to detect such solutions we formulate an equation for variations. We set $\dot{q}^{(0)} = \omega_0 + \nu$ and substitute this expression into Eq. (8.35). Considering ν as a small quantity, we expand F_s and Q_{R0} into Taylor series in the vicinity of $\nu = 0$, retaining the terms up to the first power of ν and of its time derivatives:

$$J_0 \dot{\nu} - F_s(u_0, \omega_0) - \frac{\partial F_s}{\partial \dot{q}}(u_0, \omega_0) \nu - Q_{R0}(\omega_0) - \frac{dQ_{R0}}{d\dot{q}}(\omega_0) \nu = 0. \quad (8.37)$$

According to relationship (8.19) $\frac{\partial F_s}{\partial \dot{q}}(u_0, \omega_0) = -s$, where s is the steepness of the static engine characteristic. By analogy we introduce the quantity

$$v = -\frac{dQ_{R0}}{d\dot{q}}(\omega_0), \quad (8.38)$$

which we will call *steepness of the average moment of resistance forces*. Taking into account Eq. (8.36), we obtain from Eq. (8.37)

$$J_0 \dot{\nu} + (s + v) \nu = 0.$$

The investigated steady-state motion is stable, if the general solution of this equation $\nu = Ce^{-(s+v)t/J_0}$ tends to zero when t goes to infinity.

From this, we find the stability condition for the investigated solution:

$$s + v > 0. \quad (8.39)$$

Using this condition, it is easy to find out that the solution corresponding to point A in Fig. 8.4 is unstable, whereas the solution corresponding to point B is stable.

Let us suppose that as a result of changes in the parameters of the working process or of the resistance forces the average moment $-Q_{R0}$ is changed by some increment ΔM (see Fig. 8.4). Moreover, point B of the intersection of the graphs moves to B' ; thus, the average angular velocity decrease equal to $\omega_0 - \Delta\omega$. In the vicinity of point B the graphs of F_s and $-Q_{R0}$ can be replaced to within small quantities by their tangents. Moreover, we have

$$\Delta\omega_0 = -\frac{\Delta M}{s + v}.$$

We will call the quantity

$$\alpha = -\frac{\Delta\omega_0}{\Delta M} = (s + v)^{-1} \quad (8.40)$$

coefficient of sensitivity or simply *sensitivity* of the investigated state with respect to the variation in the load.

The sensitivity is an important characteristic of the regime of steady-state motion. If the sensitivity is high, then the average angular velocity changes significantly even for slight variations in the load, which are inevitable under real operating conditions of machines. In such cases it is necessary to undertake special measures in order to reduce the sensitivity. One such measure is the introduction of a governor ensuring an increase of the input parameter for an increase of the load, i.e. the governor transfers the engine to a new working characteristic for which the previous magnitude of the angular velocity corresponds to the increased load.

The governor is controlling the machine motion by the principle of feedback. For this purpose most often a transducer is installed on the engine rotor for measuring its average angular velocity $\dot{q}_{av}$; if it is below the normal value ω_0 , then a signal Δu is generated in the governor which is proportional to the difference $\dot{q}_{av} - \omega_0$, and which increases the value of the input parameter; a negative signal Δu is formed for $\dot{q}_{av} - \omega_0 > 0$. Such a system for stabilization of the angular velocity is called a *negative tachometric feedback*.

Another method for reducing the sensitivity is the use of a transmission mechanism with a variable transmission ratio (speed gear-box, variable-speed drive). With an increase of the load in the actuating mechanism, the transmission ratio is increased (either by an operator working with the machine or automatically). This causes the average moment of the resistance forces, reduced to the engine shaft, to regain its previous magnitude.

8.4

Determination of Dynamic Errors and of Dynamic Loads in a Steady-State Motion

Let us now find the dynamic error $\psi(t)$ of the motion law of an engine output link. Following the method of successive approximations, we substitute the solution $q^{(0)}(t)$ found above into the right-hand side of Eq. (8.33). We obtain

$$-\tilde{J}(\omega_0 t)0 - \frac{1}{2}\tilde{J}'(\omega_0 t)\omega_0^2 + \tilde{Q}_R(\omega_0 t, \omega_0) = \tilde{L}(t), \quad (8.41)$$

where $\tilde{L}(t)$ is the perturbation moment (7.6), introduced earlier as a characteristic of the inner vibration activity of a mechanism. Thus, the cause for the nonuniformity of rotation of the engine rotor in a steady-state regime is the inner vibration activity conditioned by the explicit dependence of the reduced moment of inertia and of the reduced moment of resistance forces on the generalized coordinate q .

For finding an approximate solution of Eq. (8.33), we replace the left-hand side moments F_s and Q_{R0} by their linearized expressions:

$$\begin{aligned} F_s(u_0, \dot{q}) &\approx F_s(u_0, \omega_0) - s(\dot{q} - \omega_0) = M_0 - s\dot{\psi}, \\ Q_{R0}(\dot{q}) &\approx Q_{R0}(\omega_0) - v(\dot{q} - \omega_0) = -M_0 - v\dot{\psi}, \end{aligned} \quad (8.42)$$

where M_0 is the moment corresponding to point B in Fig. 8.4. Substituting relationships (8.41) and (8.42) into Eq. (8.33), we obtain a linear differential equation of second order

$$J_0\ddot{\psi} + (s + v)\dot{\psi} = \tilde{L}(t). \quad (8.43)$$

Here, $\ddot{\psi} \equiv \ddot{q}$ is taken into account. Representing $\tilde{L}$ in the form of a Fourier series (7.7), we obtain from Eq. (8.43)

$$J_0\ddot{\psi} + (s + v)\dot{\psi} = \sum_{l=1}^{\infty} L_l \cos(lvt + \alpha_l), \quad (8.44)$$

where $v = \omega_0/i$ is the angular velocity of the input link of the actuating mechanism. To the steady-state motion of the system corresponds a periodic solution of Eq. (8.44), which can be obtained by methods known from the theory of linear differential equations in the following form:

$$\psi = \sum_{l=1}^{\infty} \frac{L_l \cos(lvt + \alpha_l + \delta_l)}{lv \sqrt{J_0^2 l^2 v^2 + (s + v)^2}}, \quad (8.45)$$

$$\dot{\psi} = -\sum_{l=1}^{\infty} \frac{L_l \sin(lvt + \alpha_l + \delta_l)}{\sqrt{J_0^2 l^2 v^2 + (s + v)^2}}, \quad (8.46)$$

where

$$\delta_l = -\arg[-J_0 lv + (s + v)j], \quad j^2 = -1, \quad (8.47)$$

or

$$\cos \delta_l = \frac{-J_0 lv}{\sqrt{J_0^2 l^2 v^2 + (s + v)^2}}, \quad \sin \delta_l = \frac{-(s + v)}{\sqrt{J_0^2 l^2 v^2 + (s + v)^2}}.$$

Expressions (8.45) and (8.46) determine the dynamic error of the motion law in the first approximation. For higher precision one should substitute $q = \omega_0 t + \psi(t)$ into the right-hand side of Eq. (8.33) and solve for the next approximation. However, as a rule, the accuracy of the first approximation proves to be entirely sufficient for practical calculations.

From formulae (8.45) and (8.46) it follows that the harmonic of the perturbation moment with frequency $\omega = l\nu$ causes the appearance of harmonics with the same frequency in the dynamic errors in angle and in velocity. Moreover, the amplitudes a_ψ and $a_{\dot{\psi}}$ of these harmonics are related to the amplitude of the perturbation moment through the relations

$$a_\psi(\omega) = \frac{L_l}{\omega(s+\nu)\sqrt{1+\tau_M^2\omega^2}}, \quad a_{\dot{\psi}}(\omega) = \frac{L_l}{(s+\nu)\sqrt{1+\tau_M^2\omega^2}}. \quad (8.48)$$

Here, $\tau_M = J_0(s+\nu)^{-1}$ is a system parameter of dimension time called *mechanical time constant* for machine assembly. Formulae (8.48) show that with increasing frequency ω the ratios a_ψ/L_l and $a_{\dot{\psi}}/L_l$ decrease. Usually, this has the effect that in the spectrum of dynamic errors the low-frequency components are predominant.

For this reason, the Fourier series (8.45) and (8.46) are usually converging fast. Often, it is possible to restrict oneself to only two or three leading harmonics altogether. It must be noted that the representation of the solution in the form of Fourier series may prove unacceptable in cases when in a cycle machine in steady-state motion jumps of the perturbation moment occur as a result of impact processes (e.g., in presses) or jumps of the second derivative of the position function (in cam mechanisms). In such machines, free vibrations caused by the perturbation jumps are superimposed on the relatively smooth machine motion described by solutions of the form (8.44)–(8.46). Such vibrations are usually called *accompanying* vibrations. However, we should keep in mind that for adequate investigations of steady-state motions in a system with jumps in perturbations the transition to an elastic model of the mechanism is required.

The nonuniformity of rotation of engine rotor is usually characterized by the *coefficient of nonuniformity*

$$\delta = \frac{\dot{\psi}_{\max} - \dot{\psi}_{\min}}{\omega_0}.$$

In the literature (see, e.g., [8]) one finds admissible values of the coefficient of nonuniformity for machines of various kinds. We note that, as a rule, the nonuniformity of rotation by itself does not effect the quality of the working process. In most cases it is dangerous since it causes additional energy losses in the engine and increased dynamic loads in transmission mechanisms. Moreover, the nonuniformity of rotation of engine rotors with usually large moments of inertia causes dynamic actions of the machine on the base.

Influence of the nonuniformity of rotation on the energy losses in engines. Substituting the dynamic velocity error (8.46) into the linearized engine characteristic (8.42), we obtain

$$Q(t) = M_0 + s \sum_{l=1}^{\infty} \frac{L_l \sin(lvt + \alpha_l + \delta_l)}{\sqrt{J_0^2 l^2 v^2 + (s + v)^2}} = M_0 + \tilde{Q}, \quad (8.49)$$

where $\tilde{Q}$ is the variable component of the driving moment. Let us determine the energy losses caused by this variable component in a direct-current electric motor with separate excitation. From relationship (8.9) it follows that the average power loss in such a motor is determined by the expression

$$N_{lost} = \tilde{Q}^2 s^{-1}.$$

Substituting the relationship (8.49) into this expression and averaging the with respect to time over the cycle period $2\pi/v$, we obtain

$$N_{lost} = \frac{v}{2\pi} \int_0^{2\pi/v} \frac{Q^2}{s} dt = \frac{1}{s} \left[M_0^2 + \frac{M_0 v}{\pi} \int_0^{2\pi/v} \tilde{Q}(t) dt + \frac{v}{2\pi} \int_0^{2\pi/v} \tilde{Q}^2 dt \right].$$

The first term defines the power lost upon a constant driving moment, the second term is equal to zero and the third term determines the additional losses caused by the variable moment. With (8.49) we find for this last term the expression:

$$\tilde{N}_{lost} = s \sum_{l=1}^{\infty} \frac{L_l^2}{2 [J_0^2 l^2 + (s + v)^2]}. \quad (8.50)$$

On the other hand, the average power taken by the motor from the mains is:

$$N_e = u_0 I_{av} = u_0 \frac{v}{2\pi} \int_0^{2\pi/v} I dt = u_0 \frac{v}{2\pi} \int_0^{2\pi/v} \frac{1}{k\Phi} (M_0 + \tilde{Q}) dt = M_0 \omega_{id},$$

where ω_{id} is the angular velocity of idling. From here we find the loss coefficient

$$\zeta = \tilde{N}_{lost} / N_e = \tilde{N}_{lost} / M_0 \omega_{id}. \quad (8.51)$$

Computations show that for large coefficients of nonuniformity of rotation ($\delta > 0.2$) the loss coefficient can reach 0.02–0.03.

Dynamic loads in a transmission mechanism. The nonuniformity of rotation has the effect that the dynamic (variable) moment reduced to the output link in a transmission mechanism differs from the variable part of the driving moment. This

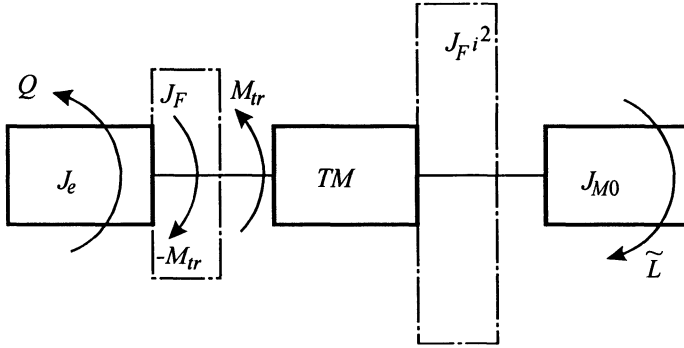


Fig. 8.5. Block diagram of the mechanical system of a machine

difference is determined by the inertia force moment of the engine rotor. In Fig. 8.5 a diagram is shown of the mechanical system of a machine assembly consisting of an engine rotor, a transmission and of actuating mechanisms. A moment Q is applied to the engine rotor which possesses a constant moment of inertia J_e ; M_{tr} is a moment occurring in the transmission mechanism. We formulate the motion equation of the rotor

$$J_e \ddot{q} = Q - M_{tr}.$$

Taking into account that $\ddot{q} = \ddot{\psi}$ and assuming, in accordance with expression (8.42), that $Q = M_0 - s\dot{\psi}$, we have

$$J_e \ddot{\psi} = M_0 - s\dot{\psi} - M_{tr}.$$

From here we get

$$M_{tr} = M_0 - s\dot{\psi} - J_e \ddot{\psi} = M_0 + \tilde{M}_{tr},$$

where

$$\tilde{M}_{tr} = -s\dot{\psi} - J_e \ddot{\psi} \quad (8.52)$$

is the variable part of the moment in the transmission mechanism. Differentiating relationship (8.46) we find $\ddot{\psi}$; substituting $\dot{\psi}$ and $\ddot{\psi}$ into expression (8.52) we find after some simple transformations

$$\tilde{M}_{tr} = \sum_{l=1}^{\infty} \frac{\sqrt{s^2 + J_e^2 l^2 v^2}}{\sqrt{J_0^2 l^2 v^2 + (s + v)^2}} L_l \cos(lvt + \alpha_l + \delta_l + \xi_l), \quad (8.53)$$

where $\xi_l = \arg(s + J_e l v j)$.

The variable moment acting in a transmission mechanism has negative effects on the performance of a machine: it causes elastic vibrations which reduce the functional accuracy of mechanisms and which lead to increased wear of the transmission. The energy losses caused by elastic vibrations lead to an additional decrease in the efficiency of the transmission mechanism. It should be noted,

however, that every single amplitude M_{trl} of the harmonics of moment $\tilde{M}_{tr}$ is smaller than the amplitude of the corresponding harmonic L_l of the perturbation moment $\tilde{L}$, since for $v > 0$

$$\left(\frac{M_{trl}}{L_l} \right)^2 = \frac{s^2 + J_e^2 l^2 v^2}{J_0^2 l^2 v^2 + (s+v)^2} < 1 \quad (8.54)$$

because $s+v > s$, $J_0 > J_e$. The softer the engine characteristic and the smaller the moment of inertia of the engine rotor, the smaller is the fraction in expression (8.54).

Methods for reducing dynamic errors and dynamic loads during steady-state motion of a machine. From the expressions (8.46), (8.49) and (8.53) it is easily seen that a reduction of the nonuniformity of rotation, of the variable part of the driving moment as well as of the moment in the transmission mechanism can be achieved by reducing the amplitudes of the harmonics of the perturbation moment. Thus, a reduction of the inner vibration activity of the mechanical system of a machine has an improving effect on all dynamic quality indicators of a steady-state motion. These indicators also depend on such parameters of a machine assembly as J_0 , J_e , s , v . It is easy to notice that an increase in J_0 , s or v leads to a reduction of the dynamic error in velocity $\dot{\psi}$ (in expression (8.46) these quantities appear in the denominator so that an increase leads to a reduction of the amplitudes of all harmonics). Note J_0 can be increased in two ways: Either by increasing J_e , i.e. by attaching an additional mass to the engine rotor, or by increasing the average value of the reduced moment of inertia J_{M0} of the actuating mechanism (see Fig. 8.5), i.e. by installing an additional mass on the output shaft of the transmission mechanism. Such an additional mass intended for reducing the nonuniformity of rotation is called a *flywheel*. It is obvious that for achieving a prescribed value of the reduced moment of inertia of the flywheel, a flywheel mounted at the output of the transmission mechanism must have a moment of inertia i^2 times larger than the flywheel mounted at the input of the mechanism; here i is the transmission ratio. From this point of view it is preferable to mount the flywheel at the input of the transmission mechanism. For such an installation of the flywheel the variable part of the driving moment (8.49) also diminishes, whereas the dynamic loads in the transmission mechanism, generally speaking, increase. In fact, when increasing J_0 and J_e by one and the same quantity, i.e. when installing the flywheel next to the engine, both, the numerators and the denominators of the fractions (8.54) are increased. Since the numerators are smaller than the denominators, the magnitudes of the fractions, generally speaking, increase.

An improvement of all dynamic quality criteria of a steady-state motion is achieved by increasing the steepness v of the characteristic of the average moment of resistance forces. In addition, the velocity errors and the variable moments $\tilde{Q}$ and $\tilde{M}_{tr}$ are reduced. An increase of the steepness s of the engine

characteristic also reduces the nonuniformity of rotation. But this causes, in general, an increase of the quantities $\tilde{Q}$ and $\tilde{M}_{tr}$.

All these results have a simple physical meaning. The source of perturbation $\tilde{L}(t)$ is the actuating mechanism. By installing a flywheel on the shaft of the engine rotor or by increasing the steepness of its static characteristic, we introduce an inertia or an additional driving moment that somehow compensates this perturbation and thereby reduces the nonuniformity in rotation. However, the application of two opposite actions on both ends of a kinematic chain formed by a transmission mechanism and, in the case of increase of steepness s , including the engine rotor, leads to a loading of this chain through the applied actions.

Let us note that in the cases when the perturbation source is the engine (this occurs, e.g., in machines with combustion engines), the situation is different. The installation of a flywheel on the output engine has the effect of unloading the transmission mechanism.

8.5

Influence of the Engine Dynamic Characteristic on Steady-State Motions

The dynamic characteristic (8.23) of an engine differs from the static one by the term $\tau\dot{Q}$ in the left-hand side. In the case of a steady-state motion it can be represented in the form

$$\tau\dot{Q} + Q = F_s(u_0, \dot{q}). \quad (8.55)$$

Let us consider what it means to take into account the dynamic characteristic of an engine in the investigation of a steady-state motion of a machine assembly. In this case, the problem is reduced to the determination of a periodic solution of the system of differential equations (8.31) and (8.35). We write these equations in the form

$$J_0\ddot{q} - Q - Q_{R0} = -\tilde{J}(q)\ddot{q} - \frac{1}{2}\tilde{J}'(q)\dot{q}^2 + \tilde{Q}_R(q, \dot{q}), \quad (8.56)$$

$$\tau\dot{Q} + Q - F_s(u_0, \dot{q}) = 0. \quad (8.57)$$

In the absence of perturbations which are characterized by the terms on the right-hand side of Eq. (8.56) the system under consideration would have a stationary solution of the form

$$q = q^{(0)} = \omega_0 t, \quad Q = Q_0 = const, \quad (8.58)$$

corresponding to uniform rotation if the driving moment were constant. As before, we will assume that in the presence of perturbations the steady-state motion remains

close to the state of uniform rotation ($|\dot{\psi}| \ll |\omega_0|$), and that the driving moment differs only slightly from a constant value. Then, in order to solve Eqs. (8.56) and (8.57), we can apply the method of successive approximations in a form analogous to the one considered above. At first, let us find the solution of the system of equations

$$\begin{aligned} J_0 \ddot{q}^{(0)} - Q^{(0)} - Q_{R0}(\dot{q}^{(0)}) &= 0, \\ \tau \dot{Q}^{(0)} + Q^{(0)} - F_s(u_0, \dot{q}^{(0)}) &= 0. \end{aligned}$$

Setting $\dot{q}^{(0)} = \omega_0$, $\ddot{q}^{(0)} = 0$, $Q^{(0)} = M_0$, $\dot{Q}^{(0)} = 0$, we find

$$-M_0 - Q_{R0} = 0, \quad M_0 = F_s(u_0, \omega_0).$$

In this way, an equation for the determination of the average angular velocity of the engine rotor is obtained which has the form (8.36). This means that the consideration of the engine dynamic characteristic does not influence the first-order approximation of the magnitude of the average angular velocity ω_0 . Having determined ω_0 from Eq. (8.36), and having substituted $q_0 = \omega_0 t$ into the right-hand side of Eq. (8.56), we obtain an equation system for the determination of $q(t)$ and $Q(t)$ in first-order approximation:

$$\begin{aligned} J_0 \ddot{q} - Q - Q_{R0}(\dot{q}) &= -\frac{1}{2} \tilde{J}'(\omega_0 t) \omega_0^2 + \tilde{Q}_R(\omega_0 t, \omega) = \tilde{L}(t), \\ \tau \dot{Q} + Q - F_s(u_0, \dot{q}) &= 0. \end{aligned} \quad (8.59)$$

We will seek a solution of system (8.59) in the form

$$q = \omega_0 t + \psi(t), \quad Q = M_0 + \mu(t).$$

Using relationship (8.42), we obtain

$$J_0 \ddot{\psi} + v \dot{\psi} - \mu = \tilde{L}(t), \quad \tau \dot{\mu} + \mu + s \dot{\psi} = 0. \quad (8.60)$$

From the first equation we find

$$\mu = J_0 \ddot{\psi} + v \dot{\psi} - \tilde{L}(t),$$

whence follows

$$\dot{\mu} = J_0 \ddot{\psi} + v \ddot{\psi} - \dot{\tilde{L}}(t).$$

Substituting μ and $\dot{\mu}$ into the second equation (8.60), we obtain a third-order differential equation for ψ :

$$\tau J_0 \ddot{\psi} + (J_0 + \tau v) \ddot{\psi} + (s + v) \dot{\psi} = \tau \dot{\tilde{L}}(t) + \tilde{L}(t). \quad (8.61)$$

In most cases, in real machines $\tau v \ll J_0$ which allows us to remove the second term in the coefficient of $\ddot{\psi}$. Dividing all terms of Eq. (8.61) by $s + v$ we obtain

$$\tau\tau_M\ddot{\psi} + \tau_M\dot{\psi} + \psi = (s+v)^{-1} \left[\tau\dot{\tilde{L}}(t) + \tilde{L}(t) \right], \quad (8.62)$$

where τ_M is the mechanical time constant introduced above. The dynamic error in velocity in a steady-state motion is defined as a partial periodic solution of this equation. Using for $\tilde{L}(t)$ the expansion (7.7), we find $\dot{\psi}$ in the following form

$$\dot{\psi} = \sum_{l=1}^{\infty} \frac{\sqrt{1+\tau^2 l^2 v^2} L_l \cos(lv t + \zeta_l)}{(s+v) \sqrt{(1-\tau\tau_M l^2 v^2)^2 + \tau_M^2 l^2 v^2}}, \quad (8.63)$$

where $\zeta_l = \arg(1 + j\tau l v) - \arg(1 - \tau\tau_M l^2 v^2 + j\tau_M l v)$, $j^2 = -1$. The nonuniformity of rotation is characterized, first of all, by the amplitudes of the harmonics of the series (8.63). The amplitude of the l -th harmonic is found to be a product of the coefficient $L_l (s+v)^{-1}$ by the function

$$A(\omega) = \frac{\sqrt{1+\tau^2 \omega^2}}{\sqrt{(1-\tau\tau_M \omega^2)^2 + \tau_M^2 \omega^2}}, \quad (8.64)$$

where $\omega = lv$. Fig. 8.6 shows the graphs of functions (8.64) constructed for different values of the ratio τ/τ_M . For $\tau/\tau_M < \sqrt{2} - 1$ the form of the curves differs only slightly from the one obtained for $\tau = 0$. For $\tau/\tau_M > \sqrt{2} - 1$ an additional maximum of the function $A(\omega)$ occurs. The analysis of expression (8.64) shows that this maximum occurs at

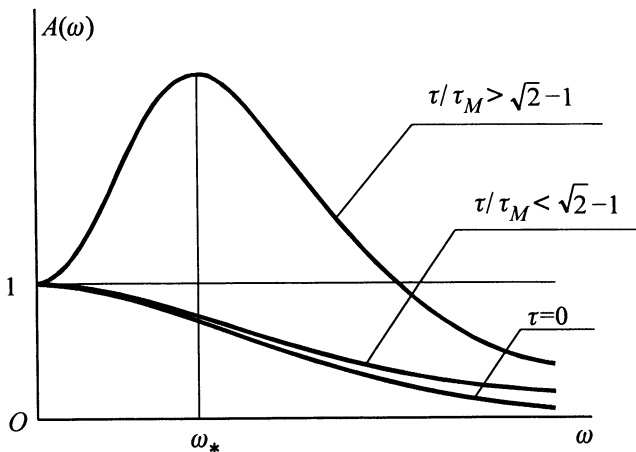


Fig. 8.6. The amplitude-frequency characteristic of the dynamic error in velocity

$$\omega = \omega_* = \frac{1}{\tau} \sqrt{2 \frac{\tau}{\tau_M} + \frac{\tau^2}{\tau_M^2} - 1} . \quad (8.65)$$

The magnitude of the maximum value of $A(\omega)$ depends itself on the magnitude of the ratio τ/τ_M . For $\tau/\tau_M = 2$ it reaches 2.5 and for $\tau/\tau_M = 4$ it grows to 4.5. The increase in coefficient $A(\omega)$ for $\omega = \omega_*$ means an increase in the amplitude of that harmonic of $\dot{\psi}$ which frequency $l\nu$ is closed to ω_* . Respectively, the nonuniformity of rotation increases. This phenomenon is called a *motive resonance* of a machine. For fixed τ , the function $A(\omega)$ for a given ω depends on τ_M . It can be shown that this dependency is not monotonic: $A(\omega)$ reaches a maximum for

$$\tau_M = \tau_M^* = \frac{\tau}{1 + \tau^2 \omega^2} . \quad (8.66)$$

If $\tau_M < \tau_M^*$, then an increase in this parameter may lead to an increase in $A(\omega)$. However, τ_M is proportional to J_0 ; that is why, an increase in the average value of J_0 , e.g. adding a flywheel, may lead to an increase in the nonuniformity of rotation. We note that for a number of reasons the technical progress in machine building is marked by an increase of the ratio τ/τ_M for real machine assemblies. Moreover, the above mentioned singularities in machine behaviour in a steady-state regime are increasingly apparent which makes it necessary to take into consideration the engine dynamic characteristic.

8.6 Starting Acceleration of a Machine

We begin the study of transients with an investigation of a noncontrollable starting acceleration of a machine. Let us at first assume that the static characteristic of the engine has been chosen; since the starting acceleration is noncontrollable, $u(t) = u_0 = \text{const}$. We also assume that the reduced moment of inertia is constant and that the reduced moment of resistance forces does not depend on the coordinate q ; thus, the motion equation (8.28) takes the following form:

$$J_0 \ddot{q} = F_s(u_0, \dot{q}) + Q_{R0}(\dot{q}) . \quad (8.67)$$

The omission of the variable components $\tilde{J}(q)$ and $\tilde{Q}_R(q, \dot{q})$ is usually admissible in the investigation of transients.

The solution of Eq. (8.67) for initial conditions $t = 0$, $\dot{q} = 0$ corresponds to the starting acceleration of a machine. Denoting $\dot{q} = \omega$, we obtain a differential equation with separated variables

$$J_0 \frac{d\omega}{dt} = F_s(u_0, \omega) + Q_{R0}(\omega). \quad (8.68)$$

Solving it, we find

$$t = J_0 \int_0^{\omega} \frac{d\omega}{F_s(u_0, \omega) + Q_{R0}(\omega)} = t(\omega). \quad (8.69)$$

Inverting the function (8.69), we obtain the function $\omega(t)$. The time duration of the starting acceleration can be determined as

$$t_a = J_0 \int_0^{\omega_0} \frac{d\omega}{F_s(u_0, \omega) + Q_{R0}(\omega)} = t(\omega_0). \quad (8.70)$$

It is easy to show that this integral diverges. In fact, for $\omega = \omega_0$ the denominator under the integral vanishes (since ω_0 is the steady-state angular velocity determined from Eq. (8.36)). That is why, the integral is an improper one. It diverges, if

$$\frac{\partial F_s}{\partial \omega}(u_0, \omega) + \frac{dQ_{R0}}{d\omega}(\omega_0) = -(s + v) \neq 0, \quad (8.71)$$

which is a stability condition for the regime of a steady-state motion. Thus, the time duration of the starting acceleration is, theoretically, infinitely large. For this reason as time duration of the starting acceleration the time interval is taken which is needed to reach an angular velocity close but smaller than ω_0 . Most often, it is accepted that

$$t_a = J_0 \int_0^{0.95\omega_0} \frac{d\omega}{F_s(u_0, \omega) + Q_{R0}(\omega)}. \quad (8.72)$$

From this formula it can be seen that the time duration of the starting acceleration is proportional to J_0 ; that is why, the diminution of the machine moment of inertia is one of the most effective methods for reducing the transient time.

Starting acceleration upon linear characteristics of machine and engine. Let

$$F_s(u_0, \omega) = M_0 - s(\omega - \omega_0), \quad Q_{R0} = -M_0 - v(\omega - \omega_0), \quad (8.73)$$

where $M_0 = F_s(u_0, \omega_0) = -Q_{R0}(\omega_0)$. Substituting (8.73) into (8.67), we obtain

$$J_0 \dot{\omega} + (s + v)(\omega - \omega_0) = 0.$$

Dividing by $s + v$ and taking into account that $J_0(s + v)^{-1} = \tau_M$, we have

$$\tau_M \dot{\omega} + \omega = \omega_0. \quad (8.74)$$

The general solution of this equation is written in the form

$$\omega = \omega_0 + C \exp(-t/\tau_M).$$

From the initial condition $\omega(0) = 0$ we find that $C = -\omega_0$; from here

$$\omega = \omega_0 [1 - \exp(-t/\tau_M)]. \quad (8.75)$$

Setting $\omega = 0.95 \omega_0$, $t = t_a$, we obtain

$$t_a = -\tau_M \ln 0.05 \approx 3\tau_M.$$

This shows that the duration of the starting acceleration is proportional to τ_M .

Determination of the moment in a transmission mechanism. Let us find the moment M_{tr} acting in a transmission mechanism during the starting acceleration. Composing the motion equation of the engine rotor, we have

$$I_e \dot{\omega} = Q - M_{tr} = M_0 - s(\omega - \omega_0) - M_{tr},$$

where J_e is the moment of inertia of the rotor. Since

$$\dot{\omega} = \omega_0 \tau_M^{-1} \exp(-t/\tau_M), \quad \omega - \omega_0 = -\omega_0 \exp(-t/\tau_M), \quad \tau_M = J_0 (s + v)^{-1},$$

we obtain

$$\begin{aligned} M_{tr} &= M_0 + (s\omega_0 - J_e \omega_0 \tau_M^{-1}) \exp(-t/\tau_M) = \\ &= M_0 + \omega_0 \exp(-t/\tau_M) (J_M s - J_e v) J_0^{-1}, \end{aligned} \quad (8.76)$$

where $J_M = J_0 - J_e$. In Fig. 8.7 possible forms of the function $M_{tr}(t)$ during a starting acceleration are shown. It is obvious that for $J_M s > J_e v$ the moment in the transmission mechanism acting during starting acceleration exceeds the moment in a steady-state regime. The condition $J_M s < J_e v$ is more preferable when the moment M_{tr} in the transmission does not exceed M_0 during the whole transient.

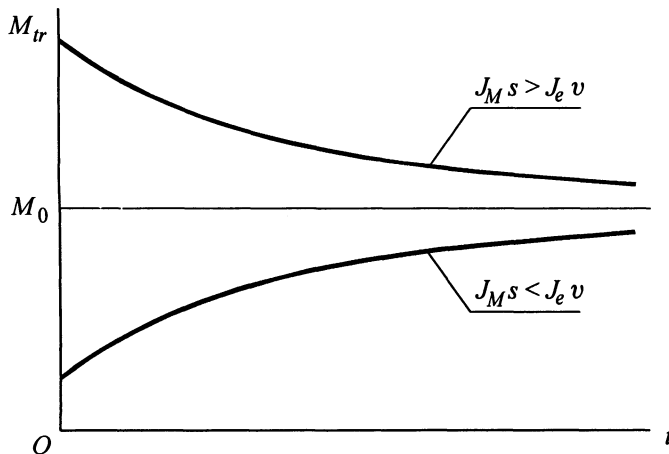


Fig. 8.7. The driving moment during starting acceleration

Influence of the engine dynamic characteristic on the starting acceleration. Restricting ourselves to the consideration of a system with linear characteristics (8.73), we write the motion equations of a machine in the form

$$\begin{aligned} J_0 \dot{\omega} &= Q + Q_{R0}(\dot{q}) = Q - M_0 - v(\omega - \omega_0) , \\ \tau \dot{Q} + Q &= M_0 - s(\omega - \omega_0) . \end{aligned} \quad (8.77)$$

From the first equation we determine the driving moment Q

$$Q = J_0 \dot{\omega} + M_0 + v(\omega - \omega_0) .$$

Substituting this equation into the second one of Eqs. (8.77), we obtain

$$\tau(J_0 \ddot{\omega} + v \dot{\omega}) + J_0 \dot{\omega} + M_0 + v(\omega - \omega_0) = M_0 - s(\omega - \omega_0)$$

or, after simplifications,

$$\tau \tau_M \ddot{\omega} + \left(\tau_M + \frac{\tau v}{s + v} \right) \dot{\omega} + \omega = \omega_0 .$$

Henceforth, we will assume that $\tau v(s + v)^{-1} \ll \tau_M$ and that we can neglect the corresponding term in the coefficient at $\dot{\omega}$.

We finally obtain

$$\tau \tau_M \ddot{\omega} + \tau_M \dot{\omega} + \omega = \omega_0 . \quad (8.78)$$

The starting acceleration is described by a partial solution of Eq. (8.78) corresponding to certain initial conditions. One of these conditions is obvious:

$$t = 0, \quad \omega = 0. \quad (8.79)$$

The second initial condition needs more detailed explanation. The point is that at the moment when the engine is switched on, the driving moment is equal to zero, whereas the moment of resistance forces is $-Q_{R0} = M_0 - v\omega_0 > 0$ (see Fig. 8.8).

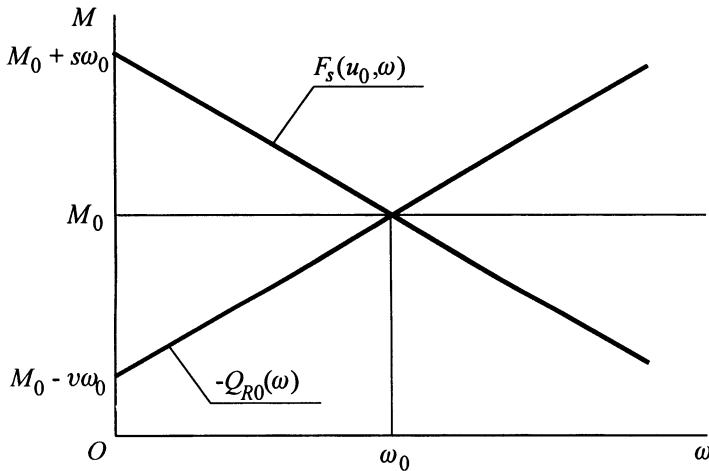


Fig. 8.8. Static characteristics of an engine and the moment of resistance forces

Therefore, this is not the starting instant. With a motionless rotor the driving moment starts to increase in accordance with the engine dynamic characteristic where one should set $\omega = 0$:

$$\tau \dot{Q} + Q = M_0 + s\omega_0. \quad (8.80)$$

The start occurs at that moment when the partial solution of Eq. (8.80) corresponding to condition $Q(0) = 0$ reaches a value equal to $-Q_{R0}(0)$. If the time duration of the starting acceleration is counted from this moment on, then as a second initial condition one should adopt

$$t = 0, \quad \dot{\omega} = 0. \quad (8.81)$$

Seeking the general solution of the nonhomogeneous differential equation (8.78), let us first find the roots of its characteristic equation

$$\tau \tau_M \lambda^2 + \tau_M \lambda + 1 = 0.$$

Solving this equation, we find

$$\lambda_{1,2} = -\frac{1}{2\tau} \pm \frac{\sqrt{\tau_M^2 - 4\tau\tau_M}}{2\tau\tau_M}. \quad (8.82)$$

In what follows, it is necessary to consider two cases:

- a) If $\tau_M > 4\tau$, then the roots of the quadratic equation are real and negative. The general solution of Eq. (8.78) is presented in the form

$$\omega = \omega_0 + C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

The initial conditions (8.79) and (8.81) allow us to determine the constants C_1 and C_2 :

$$C_1 = \lambda_2 \omega_0 (\lambda_1 - \lambda_2)^{-1}, \quad C_2 = -\lambda_1 \omega_0 (\lambda_1 - \lambda_2)^{-1}.$$

In this case, the starting acceleration is an aperiodic process in which

$$\omega = \omega_0 \left(1 + \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t} \right). \quad (8.83)$$

An approximate form of the graph of function $\omega(t)$ is shown in Fig. 8.9a. The angular velocity ω increases monotonically tending toward ω_0 . It can be shown that in this case for all values of t we have $\omega < \omega_0$.

- b) If $\tau_M < 4\tau$, then the roots of the quadratic equation (8.82) are complex-conjugate:

$$\lambda_{1,2} = -\frac{1}{2\tau} \pm j \frac{\sqrt{4\tau\tau_M - \tau_M^2}}{2\tau\tau_M} = n \pm jk. \quad (8.84)$$

Using the initial conditions, we find

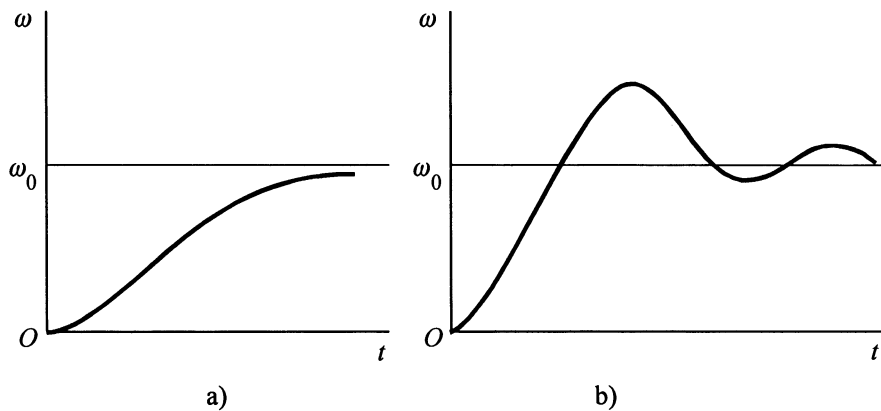


Fig. 8.9. The angular velocity of the main shaft during starting acceleration: a) case $\tau_M > 4\tau$, b) case $\tau_M < 4\tau$

$$\omega = \omega_0 \left[1 - e^{-nt} \left(nk^{-1} \sin kt + \cos kt \right) \right]. \quad (8.85)$$

In this case, the starting acceleration proves to be a damped oscillation process (Fig. 8.9b). The maximum value of the angular velocity is reached for $t = \pi/k$. In this case, during the starting acceleration the angular velocity reaches values exceeding ω_0 which is often undesirable.

8.7 Braking of a Machine

Let us consider the braking process of a machine when the engine is turned off and the brake is turned on, thus generating an additional moment of resistance $-M_b$ which we will consider to be constant in magnitude. In this case, the motion equation of a rigid machine is written in the form

$$J_0 \dot{\omega} = Q_{R0}(\omega) - M_b. \quad (8.86)$$

For the linear characteristic $Q_{R0}(\omega) = -M_0 - v(\omega - \omega_0)$ this equation takes the form

$$J_0 \dot{\omega} = -M_0 - v(\omega - \omega_0) - M_b$$

or

$$\tau_b \dot{\omega} + \omega = \omega_0 - (M_0 + M_b) v^{-1}, \quad (8.87)$$

where $\tau_b = J_0 v^{-1}$ is the time constant during braking. Solving Eq. (8.87) for the initial condition $\omega(0) = \omega_0$, we find

$$\omega = \omega_0 - (M_0 + M_b) v^{-1} \left[1 - \exp(-t/\tau_b) \right]. \quad (8.88)$$

From conditions $t = t_b$, $\omega = 0$ we determine the time duration of braking

$$t_b = -\tau_b \ln \left[1 - \omega_0 v (M_0 + M_b)^{-1} \right] . \quad (8.89)$$

Let J_e be the inertia moment of the engine rotor and let the braking moment be directly applied to the rotor. Writing the motion equation of the rotor in the form

$$J_e \dot{\omega} = M_{tr} - M_b ,$$

where M_{tr} is the moment in the transmission mechanism, we obtain

$$M_{tr} = M_b - J_e (v \tau_b)^{-1} (M_0 + M_b) \exp(-t/\tau_b) . \quad (8.90)$$

For $t = 0$ moment M_{tr} takes maximum value equal to M_b . Usually, one tries to achieve that the moment $(M_{tr})_{\max}$ does not exceed moment M_0 ; from this condition, it is possible to choose the value of the braking moment.

8.8 Problems

- 8.1. Determine at which constant angular velocity ω of the rotor, a direct-current electric motor with separate excitation generates maximum power N if its static characteristic is linear.

Solution: Assuming that in a static regime the driving moment on the engine shaft is defined by the relationship

$$Q = s(\omega_{id} - \omega),$$

where s is the steepness of the engine static characteristic and ω_{id} is the angular velocity of idling, we have

$$N = Q\omega = s(\omega_{id}\omega - \omega^2).$$

This expression reaches a maximum for $\omega = \omega_{id}/2$. The maximum power itself is $N_{\max} = s \omega_{id}^2 / 4$.

- 8.2. In Fig. 8.10 the kinematic diagram of a pump is shown. The length of the crank is $l_{OA} = r$, the moment of inertia of the crank with respect to the rotation axis is J_{O1} , the masses of the second and third links are m_2 and m_3 , respectively, the resistance force P_R is constant in magnitude and it is always directed opposite to the velocity of point B : $P_R = -P_{R0} \text{sign} \dot{X}_B$. Assuming that the driving moment is constant and equal to $Q = 2r P_{R0} / \pi$,

and that the angular velocity for $q = 0$ is equal to ω_0 , find the angular velocity for $q = \pi$. Neglect the influence of gravity.

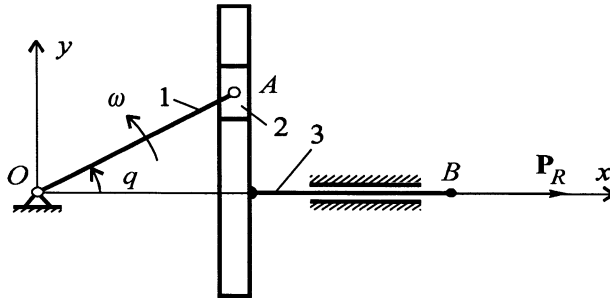


Fig. 8.10

Solution: The variation in the kinetic energy of the mechanism is equal to the total work of the external forces applied on its links (driving forces and resistance forces):

$$\Delta T = \Delta A_e + \Delta A_R.$$

The work of the driving forces in the interval of variation $\Delta q = 180^\circ = \pi$ of the generalized coordinate is determined by the expression

$$\Delta A_e = \pi \frac{2rP_{R0}}{\pi} = 2rP_{R0}.$$

The displacement of the slider during the change of the generalized coordinate from 0 to π is $\Delta X_B = 2r$, so that the work of the resistance forces over this displacement is

$$\Delta A_R = -P_{R0}\Delta X_B = -2rP_{R0}.$$

For this reason, in all considered intervals the variation in the kinetic energy is equal to zero.

The kinetic energy of the mechanism is equal to

$$T = \frac{1}{2} (J_{O1} + m_2 r^2 + m_3 r^2 \sin^2 q) \omega^2.$$

For $q = 0$ and for $q = \pi$, $\sin q = 0$ and, in this way, for $q = \pi$ $\omega = \omega_0$.

- 8.3. The mechanism of a double-acting pump (see Fig. 8.11) connected to a reducer with transmission ratio $i = 10$ is set into motion by a direct-current electric motor with separate excitation for which the angular velocity of idling is $\omega_{id} = 150 \text{ s}^{-1}$, and the steepness of the characteristic is

$s = 0.1 \text{ N m s}$. The resistance force applied to the piston of the pump is given by the expression $P_R = -P_0 \text{sign} \dot{X}_B$, where $\dot{X}_B$ is the projection of the velocity of point B onto the x -axis.

Setting the length of the crank $r = 0.1 \text{ m}$, $P_0 = 1570.8 \text{ N}$, find the average value of the angular velocity in a steady-state regime. The static characteristic of the motor is assumed to be linear.

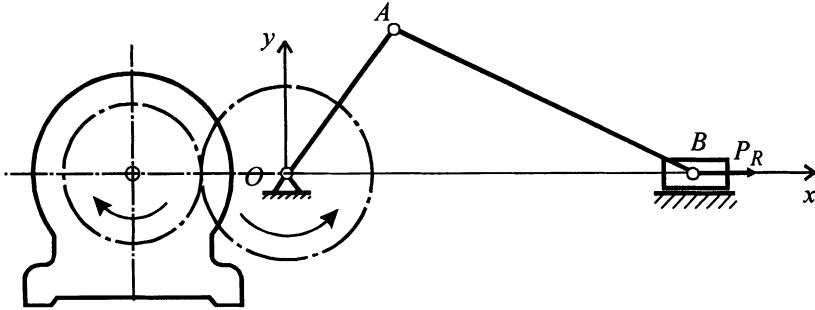


Fig. 8.11

Solution: The average value of the generalized resistance force Q_{R0} for a single revolution of the crank is determined by the expression

$$Q_{R0}(\dot{q}) = \frac{1}{2\pi} \int_0^{2\pi} Q_R(q, \dot{q}) dq,$$

where the generalized coordinate q is the rotation angle of the crank. The integral itself represents the work of the resistance forces during one revolution of the crank. This work can easily be calculated directly:

$$\int_0^{2\pi} Q_R(q, \dot{q}) dq = -2P_0 r \approx -314.16 \text{ N m}.$$

Hence, $Q_{R0} = -50 \text{ N m}$ independent of the angular velocity of the crank. On the other hand, the moment of the driving forces (the generalized driving force) for a average angular velocity ω_0 of the crank, is

$$Q(\omega_0) = F_s(u_0, \omega_0) = is(\omega_{id} - i\omega_0) = 150 - 10\omega_0 \text{ (N m)}.$$

From the condition $F_s(u_0, \omega_0) = -Q_{R0}(\omega_0)$, we obtain $\omega_0 = 10 \text{ s}^{-1}$.

- 8.4. An emery wheel is set in motion by an induction motor with a static characteristic approximated by the expression $F_s(u_0, \omega_0) = 625 -$

$(20 - \dot{q})^2 \text{ Nm}$. The moment of the resistance forces applied to the wheel is constant: $Q_{R0} = -400 \text{ Nm}$. Determine the angular velocity of the wheel in a steady-state regime.

Solution: In a steady-state motion the angular velocity is defined by expression (8.36):

$$F_s(u_0, \omega_0) + Q_{R0}(\omega_0) = 0,$$

from which follows the equality

$$625 - (20 - \dot{q})^2 - 400 = 0.$$

Solving this quadratic equation, we obtain two roots

$$\dot{q}_{1,2} = 20 \pm \sqrt{400 - 175} = 20 \pm 15 \text{ (s}^{-1}\text{)}.$$

Thus, $\omega_{01} = 5 \text{ s}^{-1}$, $\omega_{02} = 35 \text{ s}^{-1}$. Actually, a state is realized for which $s + v > 0$, where $v = -dQ_{R0}/d\omega_0 = 0$, $s = -\partial F_s/\partial\omega_0$. As is easily seen, $s = -30 \text{ Nm s}$ for $\omega_0 = 5 \text{ s}^{-1}$ and $s = 30 \text{ Nm s}$ for $\omega_0 = 35 \text{ s}^{-1}$. It is clear that it is possible to realize only the steady-state regime with $\omega_0 = \omega_{02} = 35 \text{ s}^{-1}$.

- 8.5. All parts of a machine consisting of an engine, a reducer and a scotch-yoke mechanism (Fig. 8.12) execute motions in a horizontal plane. The direct-current electric motor with separate excitation has the following parameters: the nominal power is $N = 2.1 \text{ kW}$; the nominal angular velocity of the motor rotor is $\omega_n = 140 \text{ s}^{-1}$; the angular velocity of idling of the motor is $\omega_{id} = 150 \text{ s}^{-1}$; the moment of inertia of the rotor together with the flywheel and the reducer, reduced to the motor shaft, is $J_e = 0.43 \text{ kg m}^2$; the electrodynamic time constant of the motor is negligibly small. The transmission ratio of the reducer is $i = 10$; the length of the crank is $l_{O_1A} = 0.6 \text{ m}$; the moment of inertia of the crank about the rotation axis is $J_{O_1} = 0.7 \text{ kg m}^2$; the mass of the crosshead is $m_2 = 5 \text{ kg}$; the mass of the slotted link is $m_3 = 25 \text{ kg}$. The work load P_{wl} – the force applied to the slotted link – changes according to the harmonic law $P_{wl} = P_0 \sin \varphi$, where $P_0 = 500 \text{ N}$.

Determine:

- 1) The law $\varphi(t)$ of the steady-state motion of the crank in first approximation;
- 2) The coefficient of nonuniformity δ of rotation of the crank;
- 3) The variable part $\tilde{Q}$ of the driving moment;

- 4) The variable part $\tilde{M}_{tr}$ of the moment in the reducer.

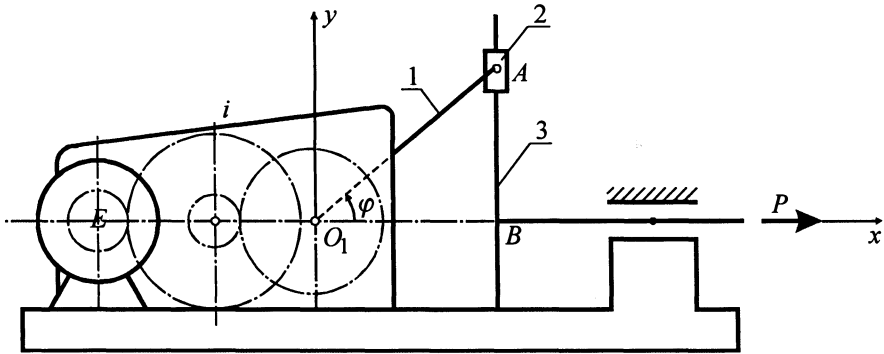


Fig. 8.12

Answer:

- 1) $\varphi(t) \approx 14t + 0.0227 \cos(28t + 4.65) \text{ rad},$
 $\dot{\varphi}(t) \approx 14 + 0.635 \cos(28t - 0.0617) \text{ s}^{-1};$
- 2) $\delta \approx \frac{\dot{\varphi}_{\max} - \dot{\varphi}_{\min}}{\omega_0} = 0.09;$
- 3) $\tilde{Q} \approx 95.3 \cos(28t + 3.08) \text{ N m};$
- 4) $\tilde{M}_{tr} \approx 771 \cos(28t + 1.385) \text{ N m}.$

- 8.6. A load with mass 20 kg is hoisted by a winch consisting of a belt transmission and a drum with radius $r = 0.2 \text{ m}$ (Fig. 8.13) on which a rope is reeled. To drive the pulley 1 from the motor a moment $Q = M_0 - s\omega_1$ is applied where $M_0 = 45 \text{ N m}$ is the starting moment, $s = 3 \text{ N m s}$ is the steepness of the static characteristic of the motor, ω_1 is the angular velocity of pulley 1 and of the rotor of the motor. The moment of inertia of the drive pulley together with the motor rotor with respect to the rotation axis O_1 is $J_{O1} = 0.05 \text{ kg m}^2$. The moment of inertia of the driven pulley 2, together with the drum, with respect to the rotation axis O_2 is $J_{O2} = 0.2 \text{ kg m}^2$. The radii of the pulleys are $r_1 = 0.3 \text{ m}$, $r_2 = 0.4 \text{ m}$, respectively. Determine the variation law of the angular velocity $\omega_1(t)$ of the pulley and the steady value of the hoisting velocity V , if at the initial moment the system is at rest. Neglect the masses of the belt and the rope.

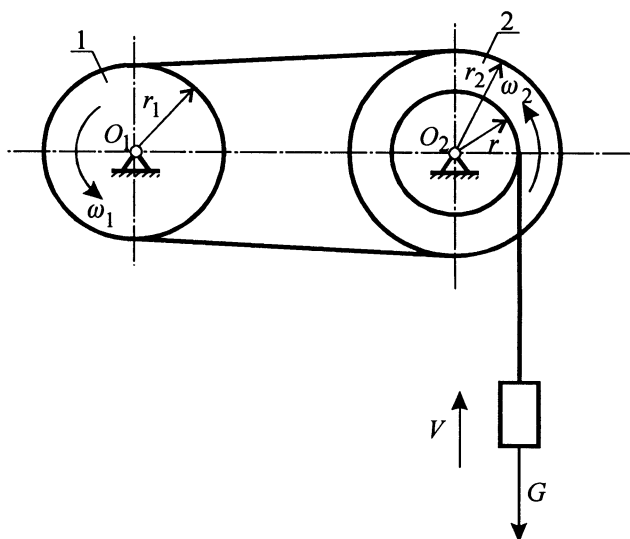


Fig. 8.13

Answer: $\omega_1(t) = \omega_0(1 - e^{-t/\tau})$, where $\omega_0 = 5.2 \text{ s}^{-1}$, $\tau_M = 0.204 \text{ s}$; $V = 0.78 \text{ m/s}$.

- 8.7. After turning off the motor, a fan is decelerated by the moment of aerodynamic resistance forces proportional to the squared angular velocity; $M_R = -0.5\omega^2 \text{ Nm}$, and by the constant brake moment $M_{br} = -2 \text{ Nm}$. Determine the time t_{br} , necessary to stop the fan as well as the angle φ_{br} during the time of braking, if the initial angular velocity of the rotor is $\omega_0 = 20 \text{ s}^{-1}$ and the moment of inertia of the rotor is $J_e = 0.05 \text{ kg m}^2$.

Answer: $t_{br} \approx 0.0736 \text{ s}$, $\varphi_{br} \approx 0.231 \text{ rad} \approx 13.2^\circ$.

9 Dynamics of Mechanisms with Elastic Links

9.1

Mechanisms with Elastic Links and Their Dynamic Models

We will call *mechanism with elastic links* or simply *elastic mechanism* a dynamic model of a real mechanism obtained under the assumption that some constructive elements of its links and kinematic pairs are deformable. From this definition it follows that mechanisms with elastic links can be obtained from the corresponding rigid mechanisms, if in the latter some rigid constraints are replaced by deformable ones. Let us illustrate this with concrete examples.

In Fig. 9.1a is shown a lever rocker mechanism with rigid links. Assuming that the connecting rod AB can be deformed in longitudinal direction, and introducing in the mechanism model the corresponding elastic element, we obtain a mechanism with elastic links, whose model is shown in Fig. 9.1b. The assumption about the elasticity of joint A (along two mutually orthogonal directions), or of joint D (along the x -axis) leads to the models shown in Fig. 9.1c,d.

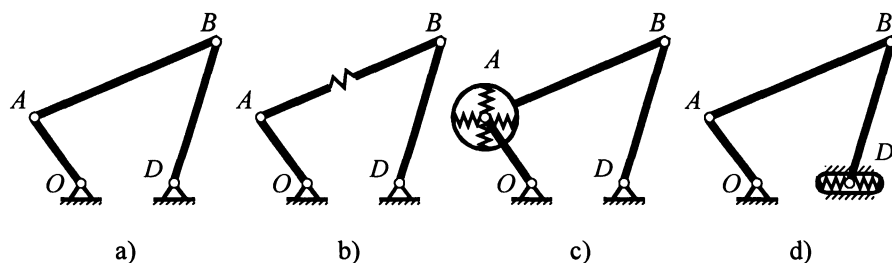


Fig. 9.1. Models of a rigid and of an elastic mechanism

In Fig. 9.2a a model of a rigid mechanism is shown transmitting rotation from engine D to the actuating link – rotor R . Assuming that shaft 0–1, connecting the engine with gear 1 deforms (twists), we obtain the model of an elastic mechanism, shown in Fig. 9.2b. If one considers as deformable shafts 0–1 and 2–3 as well as the gears at the contact point of the teeth, one obtains a more complex model, shown in Fig. 9.2c. Finally, taking into account the deformability of the shaft supports, it is possible to complicate even further the model of the mechanism (Fig. 9.2d).

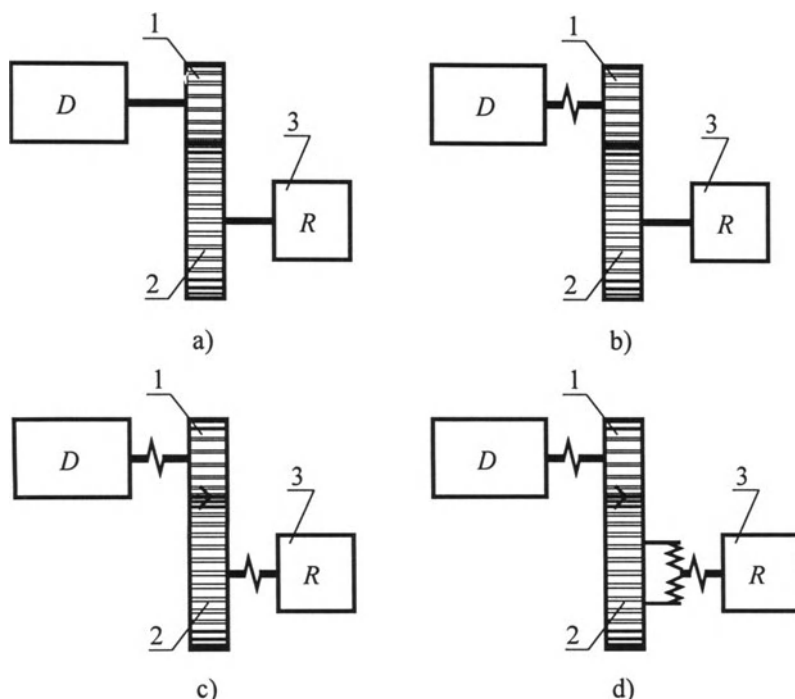


Fig. 9.2. Models of a transmission mechanism

The adequate choice of one or another model of a mechanism with elastic links with respect to the investigated dynamic processes is a rather complex problem solution of which is based on, both, the comparison of the elastic characteristics of different mechanism links and the consideration of the physical nature of the processes being analyzed.

When replacing the rigid constraints by deformable elements we should take as a base the following assumptions:

- a) The introduced elements will be considered as *one-dimensional*, i.e. their deformation is determined by the assignment of one scalar parameter (longitudinal tension or compression, twist angle etc.). Moreover, the real, stiff constraint can be replaced through one or several one-dimensional elements. So, in Figs. 9.1b and 9.1d the rigid constraint is replaced by one element and in Fig. 9.1c the flexibility of the joint is reflected by the introduction of two one-dimensional elements.
- b) It will be assumed that the considered mechanism, in principle, has to be rigid in order to realize a program motion, and that small deformations of the elastic elements lead to deviations of its motion from the program motion, i.e. to dynamic errors. Hence, we will not consider as elastic a mechanism, whose elastic elements serve for the realization of the functions of a rigid mechanism, e.g., for the force closure of the follower in the cam mechanism shown in Fig. 9.3a. In the latter case a mechanism with elastic elements is obtained if the elasticity of the follower rod is taken into account (Fig. 9.3b).

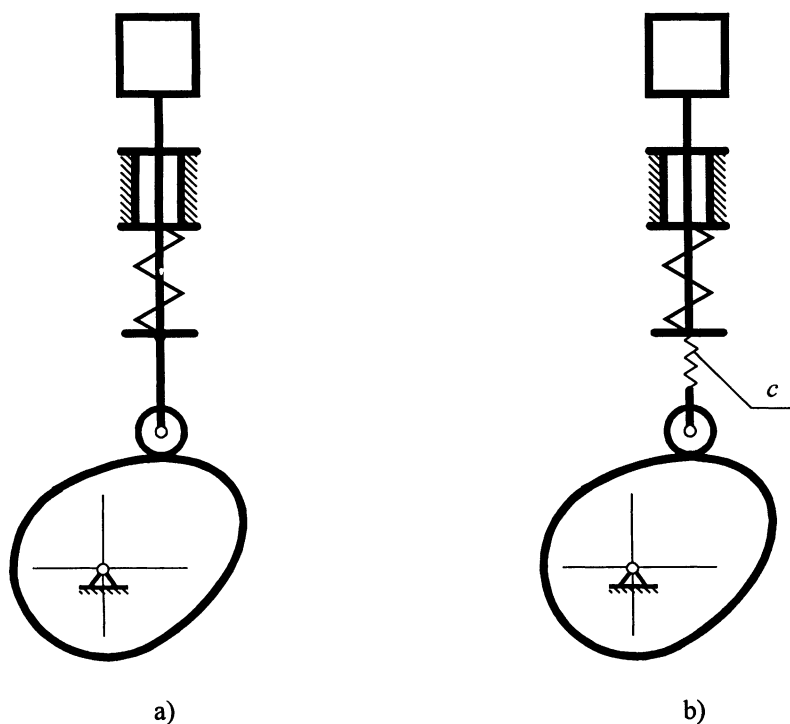


Fig. 9.3. Models of a rigid and an elastic cam mechanism

- c) A mechanism with elastic links is realized only if the introduced elastic elements increase the number of degrees of freedom of the mechanical system.

To achieve this the releasing constraints must become elastic ones. The rigid mechanism, shown in Fig. 9.4a has an excessive constraint but does not become an elastic mechanism when we introduce the elastic element shown in Fig. 9.4b, since, in this case, the number of degrees of freedom does not increase.

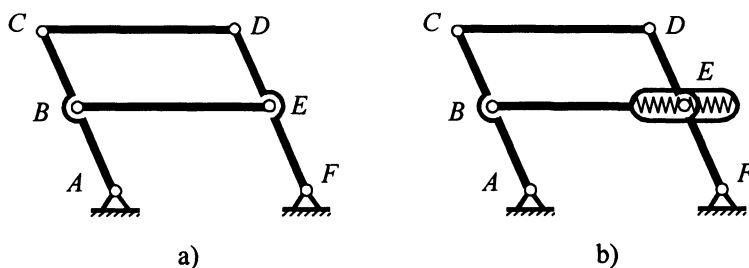


Fig. 9.4. Mechanism with a nonreleasing constraint

d) Henceforth, it is assumed that within the limits of small deformations of the elastic elements, these elements have linear elastic characteristics and that they obey Hooke's law. A basic characteristic of an elastic element is its *stiffness* c or *flexibility* $e = 1/c$.

Stiffness of a linear elastic element is called the ratio of the generalized force applied to the element, and the deformation caused by this force.

The linear stiffness is

$$c = P / \Delta \text{ N/m}, \quad (9.1)$$

where P is the force in newtons and Δ is the deformation in meters.

The angular stiffness is

$$c = M / \varphi \text{ N m}, \quad (9.2)$$

where M is the moment in newton-meters and φ is the angular deformation in radians. (The notation of linear and angular stiffness with one and the same letter is convenient for a number of reasons). Analogously we define the linear and the angular flexibility:

$$e = \Delta / P \text{ m/N or } e = \varphi / M \text{ 1/N m}. \quad (9.3)$$

In simple cases the stiffness and the flexibility can be defined according to formulae that are known from courses on strength of materials. For an extended or compressed homogeneous rod we have:

$$c = EF / L, \quad e = L / (EF), \quad (9.4)$$

where E is Young's modulus of the material, F is the cross sectional area and L is the length of the rod. For a rod deformed by a torsional moment we have

$$c = GI_p / L, \quad e = L / (GI_p), \quad (9.5)$$

where G is the shear modulus, I_p is the polar moment of inertia of the cross section. Stiffnesses of some more complex elements are given in reference books.

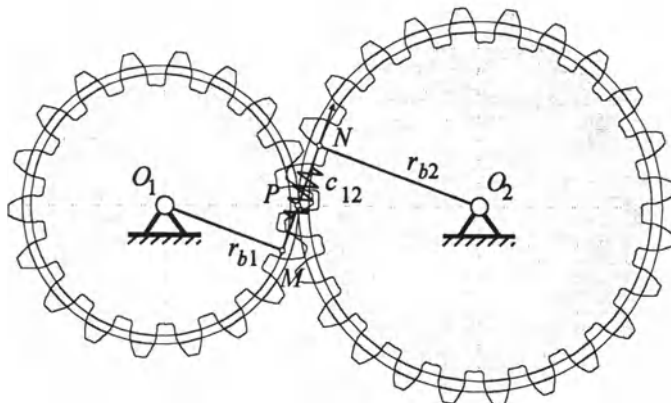


Fig. 9.5. Elastic gearing

Expressions for the stiffness of gearings require particular considerations. In an involute meshing the contact stiffness of a spur gearing is defined as the ratio of the force, acting at the contact point and directed along the common normal of the teeth profiles, to the total deformation of the teeth, arising at the contact point (Fig. 9.5). For gearings with steel wheels the stiffness can be determined according to the formula [10, 11]

$$c_{12} = kb, \quad (9.6)$$

where b is the width of the gear rim and $k = 15 \times 10^9 \text{ N/m}^2$.

9.2

Reduction of Stiffness. Inlet and Outlet Stiffness and Flexibility of a Mechanism

Let us consider a mechanism (Fig. 9.6) with an elastic element with stiffness equal to c . We fasten the input link OA , thus fixing the value of coordinate q , and apply at point K a force P directed along the axis Kx . Then, a balancing force F directed from B to A , and a deformation θ occurs at the elastic element. From the principle of virtual work we find that

$$F\delta\theta + P\delta x = 0, \quad (9.7)$$

where $\delta\theta$ and δx are virtual displacements. From here we obtain

$$F = -P(\delta x / \delta\theta) = -P(dx / d\theta). \quad (9.8)$$

Force F causes a small increment of the elastic element, opposite in direction to this force:

$$\Delta\theta = -F / c. \quad (9.9)$$

As a result of the deformation of link AB a displacement of point K occurs:

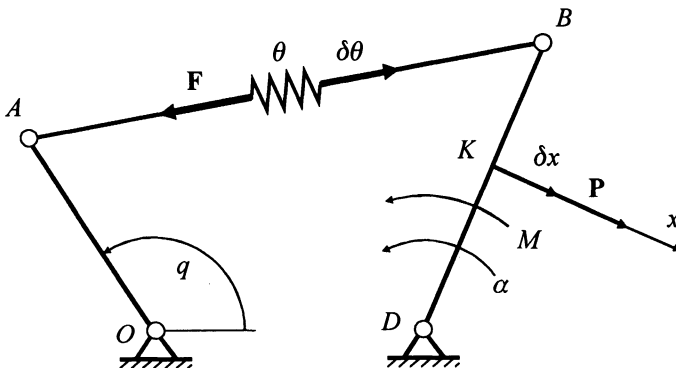


Fig. 9.6. Determination of the reduced stiffness of a mechanism

$$\Delta x = (dx/d\theta)\Delta\theta = -(dx/d\theta)Fc^{-1} = (dx/d\theta)^2 Pc^{-1}. \quad (9.10)$$

The ratio

$$c_{Kx} = P/\Delta x = c(dx/d\theta)^{-2} \quad (9.11)$$

is called the stiffness of the elastic element *reduced to the x -axis at point K* . Analogously the reduced flexibility is defined:

$$e_{Kx} = \Delta x/P = e(dx/d\theta)^2. \quad (9.12)$$

In the same way a reduced angular stiffness of an elastic element with respect to some axis is defined. In order to determine the angular stiffness of an elastic element (Fig. 9.6) with respect to axis D we apply a moment M to link BD , and let the link turn through an angle $\Delta\alpha$. From the relationship

$$M\Delta\alpha + F\Delta\theta = 0$$

we find

$$F = -M(d\alpha/d\theta), \quad \Delta\theta = -Fc^{-1} = Mc^{-1}(d\alpha/d\theta), \quad \Delta\alpha = \Delta\theta(d\alpha/d\theta).$$

From here we obtain the reduced stiffness c_D :

$$c_D = M/\Delta\alpha = c(d\theta/d\alpha)^2, \quad e_D = c^{-1}(d\alpha/d\theta)^2. \quad (9.13)$$

Let us consider a mechanism with two elastic elements arranged in series (Fig. 9.7). In this case we have

$$F_1 = -P(\partial x/\partial\theta_1), \quad F_2 = -P(\partial x/\partial\theta_2),$$

where θ_1 and θ_2 are deformations of the elastic elements. Defining the total displacement of point K in the direction Kx , we obtain

$$\Delta x = (\partial x/\partial\theta_1)\Delta\theta_1 + (\partial x/\partial\theta_2)\Delta\theta_2 = P\left(c_1^{-1}(\partial x/\partial\theta_1)^2 + c_2^{-1}(\partial x/\partial\theta_2)^2\right).$$

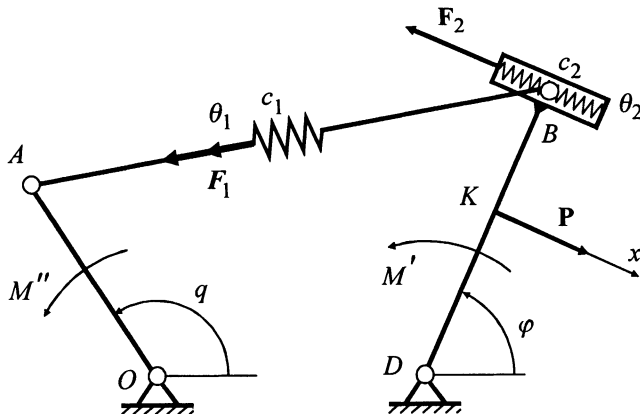


Fig. 9.7. Determination of the stiffness of a mechanism

Hence,

$$e_{Kx} = \Delta x / P = e_1 (\partial x / \partial \theta_1)^2 + e_2 (\partial x / \partial \theta_2)^2 = e_{Kx1} + e_{Kx2}. \quad (9.14)$$

Thus, the reduced stiffness of two elastic elements arranged in series is equal to the sum of the reduced stiffnesses of the individual elements.

Let the angle φ be the output coordinate of the considered mechanism. We will call *outlet stiffness* c_φ the ratio of the moment M' applied at the output to the displacement $\Delta\varphi$ of the output link of the mechanism, provided the input is fixed. Analogously to (9.14) we obtain

$$c_\varphi = e_\varphi^{-1} = \left[e_1 (\partial \varphi / \partial \theta_1)^2 + e_2 (\partial \varphi / \partial \theta_2)^2 \right]^{-1}. \quad (9.15)$$

Let us now fasten the output link, fixing coordinate φ , and let us apply moment M'' to the input link. We will call *inlet stiffness* of the mechanism the ratio of this moment to the angular displacement Δq of the input link caused by this moment. Analogously to (9.15) we have:

$$c_q = e_q^{-1} = \left[e_1 (\partial q / \partial \theta_1)^2 + e_2 (\partial q / \partial \theta_2)^2 \right]^{-1}. \quad (9.16)$$

Let us consider one more example which is important for the solution of many practical problems. Let us find the flexibility reduced to the output of a transmission mechanism (Fig. 9.8). To this end, we fix the input link – the motor rotor 0, and we apply a unit moment M at the output. We want to find the flexibility of the elastic elements, reduced to the mechanism output. In order to achieve this, it is necessary to multiply the flexibility of each element by the squared ratio of the variation of the output coordinate and the magnitude of deformation of the element. We obtain:

$$e_{01\varphi} = e_{01} (i_{12})^{-2}, \quad e_{12\varphi} = e_{12} (i_{12} r_{b1})^{-2} = e_{12} (r_{b2})^{-2}, \quad e_{23\varphi} = e_{23}, \quad (9.17)$$

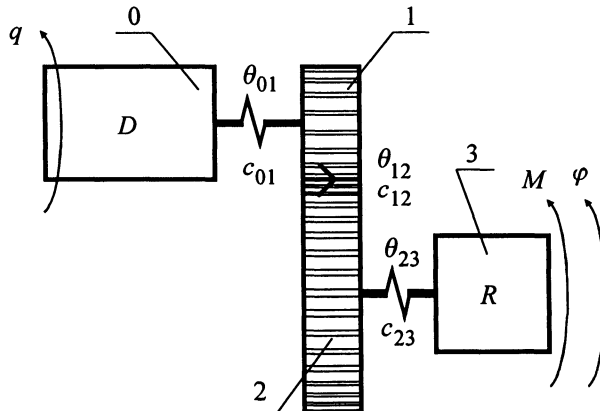


Fig. 9.8. Determination of the stiffness of a transmission mechanism

where i_{12} is the transmission ratio of the gearing, r_{b1} is the radius of the base circle of wheel 1 (see Fig. 9.5). Adding together the flexibilities (9.17), we find the outlet flexibility of the mechanism:

$$e_\varphi = e_{01}(i_{12})^{-2} + e_{12}(i_{12}r_{b1})^{-2} + e_{23}. \quad (9.18)$$

Thus, when making a reduction to the output of a transmission mechanism, the flexibility of an elastic element is divided by the squared transmission ratio, which relates the deformation of this element to the output coordinate.

In an analogous way the flexibility of a transmission mechanism, reduced to its input is defined. For the mechanism shown in Fig. 9.8 we obtain:

$$e_q = e_{01} + e_{12}(r_{b1})^{-2} + e_{23}(i_{12})^2. \quad (9.19)$$

It is obvious that

$$e_q = e_\varphi (i_{12})^2. \quad (9.20)$$

9.3

Reduced Stiffness and Reduced Flexibility of a Mechanism with Several Degrees of Movability

Let us consider a mechanism with several degrees of movability. An example of a mechanism with three degrees of movability ensuring the positioning of a planar platform BD is shown in Fig. 9.9. Here motors 1, 2 and 3 are installed at joints O , D and F , respectively; their stators are mounted on the frame and on the platform, and the rotors are connected with links OA , BD and FE through coaxial, elastic transmission mechanisms. Input coordinates of these mechanisms are the rotation angles of the rotors q_1, q_2, q_3 , and output coordinates are the rotation angles $\varphi_1, \varphi_2, \varphi_3$. We denote by c_1, c_2, c_3 the stiffnesses of the transmission mechanisms reduced to their outputs, by i_1, i_2, i_3 the transmission ratios and by $\theta_1, \theta_2, \theta_3$ the deformations of the mechanisms reduced to their outputs. The coordinates of platform BD are x_K, y_K and φ ; moreover, $BK = KD$.

We apply forces $\mathbf{P}_x, \mathbf{P}_y$ to link BD at point K as well as a moment M , supposing that the inputs of the mechanisms are fixed, i.e. with braked rotors of the motors. It is obvious that in this case $\theta_i = \Delta\varphi_i$. Moreover, as a result of the deformation of the elastic elements, link BD will experience a displacement defined by the variations $\Delta x_K, \Delta y_K, \Delta\varphi$ of the coordinates. We introduce vectors

$$\Delta\mathbf{p} = (\Delta x_K, \Delta y_K, \Delta\varphi)^T, \quad \mathbf{V} = (P_x, P_y, M)^T, \quad (9.21)$$

where ^{T} is denoting transposition of a vector or of a matrix.

We will call *matrix of reduced flexibilities* for the outputs of the elastic mechanisms the matrix $E_{\rho\nu}$, relating vectors (9.21) in such a way that

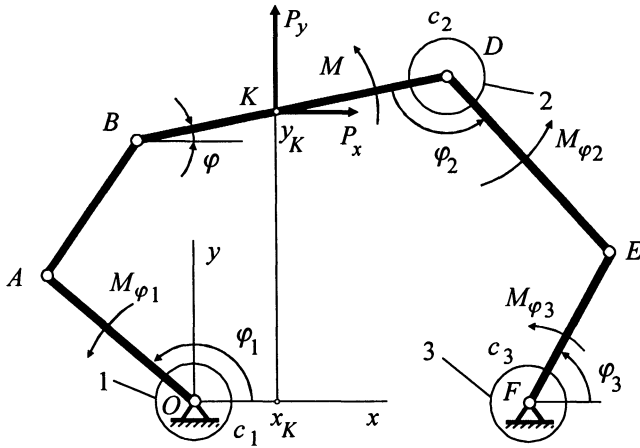


Fig. 9.9. Mechanism with three elastic drives

$$\Delta \mathbf{p} = E_{\rho v} \mathbf{V}. \quad (9.22)$$

The elements e_{ik} ($i, k = 1, 2, 3$) forming this matrix will be called *reduced flexibilities*.

Let us find an expression for the matrix $E_{\rho v}$. In accordance with the principle of virtual work, we set equal to zero the virtual work of all active forces; we obtain

$$\mathbf{M}_\varphi^T \delta \boldsymbol{\varphi} + \mathbf{V}^T \delta \mathbf{p} = 0, \quad (9.23)$$

where $\mathbf{M}_\varphi^T = (M_{\varphi 1}, M_{\varphi 2}, M_{\varphi 3})$ is the vector of the balancing moments arising at the outputs of the transmission mechanisms; $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^T$ is the vector of their output coordinates. We further have

$$\delta \mathbf{p} = (\partial \mathbf{p} / \partial \boldsymbol{\varphi})_0 \delta \boldsymbol{\varphi}. \quad (9.24)$$

Here $(\partial \mathbf{p} / \partial \boldsymbol{\varphi})_0$ is a matrix composed by the first derivatives of the position functions ρ_1, ρ_2, ρ_3 with respect to the output coordinates of the transmission mechanisms that are in equilibrium. From (9.23) and (9.24) it follows that

$$\mathbf{M}_\varphi^T = -\mathbf{V}^T (\partial \mathbf{p} / \partial \boldsymbol{\varphi})_0 \text{ or } \mathbf{M}_\varphi = -(\partial \mathbf{p} / \partial \boldsymbol{\varphi})_0^T \mathbf{V}. \quad (9.25)$$

On the other hand, the deformation $\Delta \varphi_i$ of the i -th elastic transmission mechanism, reduced to its output, is equal to the ratio of moment $M_{\varphi i}$ and stiffness c_i . Introducing a diagonal flexibility matrix

$$E_d = \text{diag}(c_1^{-1}, c_2^{-1}, c_3^{-1}), \quad (9.26)$$

and taking into account that the moment of the forces of elasticity is directed opposite to the deformation, we obtain

$$\Delta\varphi = -E_d \mathbf{M}_\varphi. \quad (9.27)$$

Therefore,

$$\Delta\mathbf{p} = (\partial\mathbf{p}/\partial\varphi)_0 \Delta\varphi = (\partial\mathbf{p}/\partial\varphi)_0 E_d (\partial\mathbf{p}/\partial\varphi)_0^T \mathbf{V}. \quad (9.28)$$

Comparing (9.27) with (9.22) we obtain

$$E_{\rho v} = (\partial\mathbf{p}/\partial\varphi)_0 E_d (\partial\mathbf{p}/\partial\varphi)_0^T. \quad (9.29)$$

This result can be extended to the more general case of an arbitrary mechanism. Let in arbitrary mechanism the inputs be fixed and let forces P_i be applied to the mechanism at points K_i in the directions of the axes x_i ($i = 1, \dots, m_1$) and moments M_i on the i -th links about axes z_i ($i = m_1 + 1, \dots, m$). The forces and moments are ordered in an m -dimensional vector $\mathbf{V}$. Let us denote by $\mathbf{p}$ the m -dimensional vector, whose components are respectively the displacements of points K_i in x_i -directions and the rotations of the links about axes z_i due to the presence of s elastic elements with stiffnesses $c_1, \dots, c_s$. We denote the vector of deformations of these elements by $\boldsymbol{\theta}$. Then, from the principle of virtual work we obtain

$$\mathbf{F}^T \delta\boldsymbol{\theta} + \mathbf{V} \delta\mathbf{p} = 0, \quad (9.30)$$

where $\mathbf{F}$ is the vector of the forces and the moments arising at the elastic elements. From (9.30) we find

$$\mathbf{F} = -(\partial\mathbf{p}/\partial\boldsymbol{\theta})^T \mathbf{V}. \quad (9.31)$$

Furthermore, we define the vector of deformations $\boldsymbol{\theta}$:

$$\boldsymbol{\theta} = -E_d \mathbf{F}, \quad (9.32)$$

where $E_d = \text{diag}(c_1^{-1}, \dots, c_s^{-1})$. From here we have

$$\Delta\mathbf{p} = (\partial\mathbf{p}/\partial\boldsymbol{\theta}) \Delta\boldsymbol{\theta} = (\partial\mathbf{p}/\partial\boldsymbol{\theta})_0 E_d (\partial\mathbf{p}/\partial\boldsymbol{\theta})_0^T \mathbf{V}. \quad (9.33)$$

In this way, the matrix of the reduced flexibilities has the following form:

$$E_{\rho v} = (\partial\mathbf{p}/\partial\boldsymbol{\theta})_0 E_d (\partial\mathbf{p}/\partial\boldsymbol{\theta})_0^T. \quad (9.34)$$

Here $(\partial\mathbf{p}/\partial\boldsymbol{\theta})$ is the matrix of the derivatives $\partial p_k / \partial \theta_i$ ($k = 1, \dots, m$; $i = 1, \dots, s$), evaluated in an equilibrium position of the mechanism. We note that matrices (9.29) and (9.34) are always symmetric.

9.4

Determination of Reduced Flexibilities with the Help of Equilibrium Equations of a Rigid Mechanism

The relationships obtained above show that, in essence, the determination of the reduced flexibilities of an elastic mechanism is reduced to the determination of the first derivatives of the output coordinates with respect to the deformations of the elastic elements. Since the deformations are small variations of some parameters (lengths of links, angles between axes etc.) this problem is reduced to the problem of parametric analysis considered in Chaps. 3 and 4. A convenient way to solve it in the considered case is to employ the equilibrium equations, using the method explained in Sect. 4.3. It must be emphasized that when applying this method we employ the equilibrium equations for a rigid non-deformed mechanism.

As an example we consider an elastic mechanism with the diagram shown in Fig. 9.10. It differs from the mechanism considered in Sect. 9.3 in that links AB and DE are assumed to be flexible in longitudinal direction; we denote their stiffnesses by c_4 and c_5 and their deformations with θ_4 and θ_5 .

Having fixed the inputs (the rotors of the motors), we determine the reduced flexibilities for the output link BD . They form a matrix E_{vp} that relates vectors

$$\mathbf{V} = (P_x, P_y, M)^T \text{ and } \Delta \mathbf{p} = (\Delta x_K, \Delta y_K, \Delta \varphi)^T.$$

Next, we introduce the vector of the forces of elasticity $\mathbf{F} = (M_{\varphi 1}, M_{\varphi 2}, M_{\varphi 3}, F_4, F_5)^T$, where $M_{\varphi i}$ ($i=1,2,3$) are the outlet moments of the transmission mechanisms; F_4, F_5 are the elastic forces in rods AB and DE . We denote by U the matrix relating vectors $\mathbf{F}$ and $\mathbf{V}$. In this way,

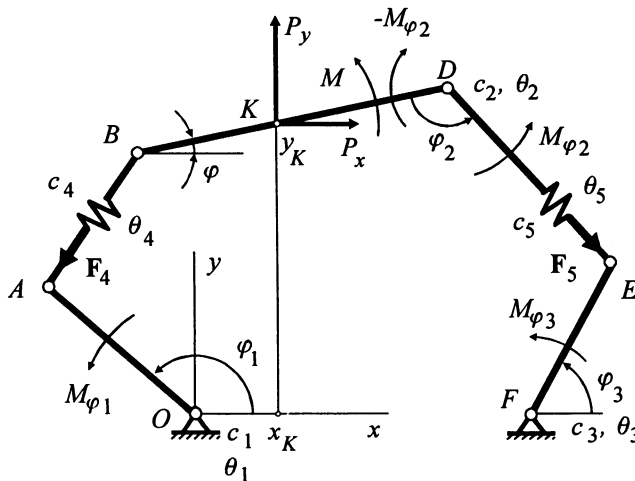


Fig. 9.10. Mechanism with five elastic elements

$$\mathbf{F} = U\mathbf{V}. \quad (9.35)$$

In order to determine the matrix U that consists of five rows and three columns, we can proceed in the following way. We apply to link BD force $P_x = 1$, assuming that $P_y = 0$ and $M = 0$, and we determine, in one way or another (see Chap. 4), the reactions which constitute the vector $\mathbf{F}^x$ (assuming that the mechanism is rigid). We denote these reactions by $M_{\varphi 1}^x$, $M_{\varphi 2}^x$, $M_{\varphi 3}^x$, F_4^x , F_5^x . Since in this case $\mathbf{F} = (0, 0, 1)^T$, it follows from (9.35) that the values found for the reactions form the first column of matrix U . Analogously, setting $\mathbf{V} = (0, 1, 0)^T$ and $\mathbf{V} = (0, 0, 1)^T$, we respectively determine the second and the third column. In this way we find:

$$U = (\mathbf{F}^x, \mathbf{F}^y, \mathbf{F}^z). \quad (9.36)$$

On the other hand, from the principle of virtual work we have:

$$\mathbf{F}^T \delta \boldsymbol{\theta} + \mathbf{V}^T \delta \mathbf{p} = 0, \quad (9.37)$$

where $\boldsymbol{\theta}$ is the vector of the deformations of elastic elements. From here

$$\mathbf{F} = -(\partial \mathbf{p} / \partial \boldsymbol{\theta})_0^T \mathbf{V}. \quad (9.38)$$

Comparing (9.35) and (9.38), we obtain

$$U \equiv -(\partial \mathbf{p} / \partial \boldsymbol{\theta})_0^T. \quad (9.39)$$

In this way, the determination of the matrix $(\partial \mathbf{p} / \partial \boldsymbol{\theta})_0$ can be replaced by the determination of the matrix U . This reduce the problem to solving three times the equilibrium equations of the rigid mechanism. From (9.34) and (9.39) we obtain:

$$E_{pv} = U^T E_d U, \quad (9.40)$$

where $E_d = \text{diag}(c_1^{-1}, \dots, c_5^{-1})$.

The determination of the reduced stiffnesses and reduced flexibilities of an elastic mechanism is directly related to the solution of problems of statics of mechanisms through the determination of static deformations. Problems of this kind arise, in particular, in investigations of the positioning accuracy of robot-technical systems and in the determination of *static errors*. For the above considered planar platform these errors are characterized by the components of the vector $\Delta \mathbf{p}$.

9.5

Some Problems of Kinematic Analysis of Elastic Mechanisms

The introduction of elastic elements leads to an increase in the number of degrees of freedom of a mechanism, while the number of degrees of movability remains the same. The position functions of an elastic mechanism, together with the generalized input coordinates $q_1, \dots, q_n$, include as arguments the components of the vector of deformations $\theta = (\theta_1, \dots, \theta_s)^T$. In the kinematic analysis, the deformations as well as their derivatives with respect to time, $d\theta_k/dt \equiv \dot{\theta}_k$, are considered to be small quantities. It follows that in the expressions for derivatives of position functions containing these quantities only first-order terms must be kept.

Let us consider an elastic mechanism with a single degree of movability. Let x be one of the output coordinates. The corresponding position function is represented in the form:

$$x = \Pi_x(q, \theta_1, \dots, \theta_s) = \Pi_x(x, \theta). \quad (9.41)$$

Differentiating (9.41) with respect to time, we obtain the expression for the first derivative of the position function:

$$\dot{x} = (\partial \Pi_x / \partial q)(q, \theta) \dot{q} + (\partial \Pi_x / \partial \theta) \dot{\theta}. \quad (9.42)$$

In this expression the derivatives $\partial \Pi_x / \partial q$ and $\partial \Pi_x / \partial \theta$ must be determined for the deformed mechanism, which considerably complicates the problem of kinematic analysis. In order to avoid this, we expand these derivatives into series of powers of θ and retain only terms of first order. We obtain:

$$\dot{x} = (\partial \Pi_x / \partial q)_0 \dot{q} + (\partial \Pi_x / \partial \theta)_0 \dot{\theta} + (\partial^2 \Pi_x / \partial q \partial \theta)_0 \dot{q} \theta. \quad (9.43)$$

Here, the index 0 indicates that the derivatives of the position functions are determined for $\theta = 0$.

If a reference system attached to the rigid mechanism is introduced, then the first term in expression (9.43) represents the transport velocity and the second represents the relative velocity. The third term appears because the derivatives of the position functions are determined for the non-deformed mechanism. It reflects an „additional“ velocity, the formulation of which requires the second derivative $(\partial^2 \Pi_x / \partial q \partial \theta)_0$.

Differentiating the expression (9.43) with respect to time, and retaining only the terms of first order, we obtain:

$$\begin{aligned} \ddot{x} = & (\partial^2 \Pi_x / \partial q^2)_0 (\dot{q})^2 + (\partial \Pi_x / \partial q)_0 \ddot{q} + (\partial \Pi_x / \partial \theta)_0 \ddot{\theta} + \\ & 2(\partial^2 \Pi_x / \partial q \partial \theta)_0 \dot{q} \dot{\theta} + \left[(\partial^2 \Pi_x / \partial q \partial \theta)_0 \ddot{q} + (\partial^3 \Pi_x / \partial q^2 \partial \theta)_0 (\dot{q})^2 \right] \theta. \end{aligned} \quad (9.44)$$

The first two terms in Eq. (9.44) characterize the transport acceleration defined for the rigid mechanism, the third term corresponds to the relative acceleration, and the fourth summand corresponds to the Coriolis acceleration. The last term defines an „additional“ acceleration which is caused by the fact that the partial derivatives are computed for $\theta = 0$.

The obtained expressions show that for the determination of angular velocities of links of an elastic mechanism and of the linear velocities of points, in addition to first derivatives of position functions also the mixed second derivatives must be determined, whereas for the determination of accelerations in addition to first and second derivatives the mixed third derivative $\partial^3 \Pi_x / \partial q^2 \partial \theta$ must be determined.

In some cases, often met in practice, the kinematic analysis is simplified. Let us consider a transmission mechanism with linear position functions (Fig. 9.11). Here, the output coordinate φ is related to the input coordinate q and to the deformations $\theta_1, \theta_2, \theta_3$ by the relationships

$$\varphi = q(i_{12}i_{34})^{-1} + \theta_1(i_{12}i_{34})^{-1} + \theta_2 i_{34}^{-1} + \theta_3 = aq + \mathbf{b}^T \boldsymbol{\theta}, \quad (9.45)$$

where $a = (i_{12}i_{34})^{-1}$ and $\mathbf{b} = ((i_{12}i_{34})^{-1}, i_{34}^{-1}, 1)$. Differentiating (9.45) we obtain:

$$\dot{\varphi} = a\dot{q} + \mathbf{b}^T \dot{\boldsymbol{\theta}}, \quad \ddot{\varphi} = a\ddot{q} + \mathbf{b}^T \ddot{\boldsymbol{\theta}}. \quad (9.46)$$

The kinematic analysis is also simplified in cases when the motion of the elastic mechanism consists in small vibrations taking place about some equilibrium position $q = q_0$ of the rigid mechanism. Such a situation occurs, e.g., in a machine when abruptly finishes a positioning process with a jump of acceleration. In this case, when the engine stops, vibrations caused by the deformations of the elastic elements occur in the system. Setting in (9.43) and (9.44) $q = q_0$, $dq/dt = 0$,

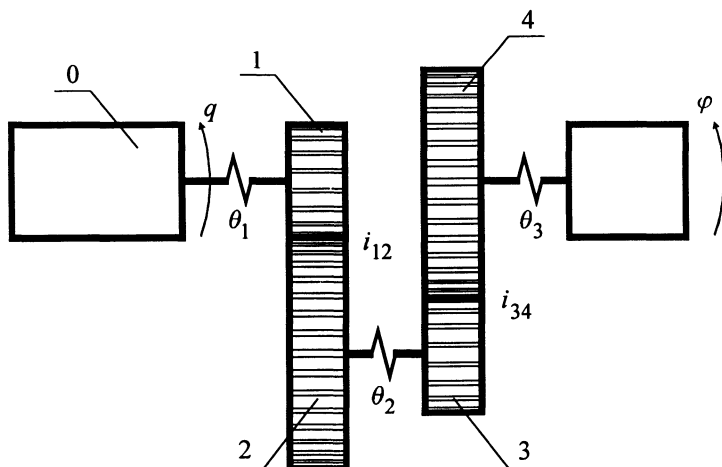


Fig. 9.11. Elastic transmission mechanism

$d^2q/dt^2 = 0$, we obtain:

$$\dot{x} = (\partial \Pi_x / \partial \dot{\theta})_0 \dot{\theta}, \quad \ddot{x} = (\partial \Pi_x / \partial \ddot{\theta})_0 \ddot{\theta}. \quad (9.47)$$

In a cyclic machine with an elastic transmission mechanism and with a rigid actuating mechanism (Fig. 9.12) the output coordinate x is expressed through the input coordinate q and the vector of deformations θ in the following way (see (9.45)):

$$x = \Pi_x(\varphi) = \Pi_x(aq + b^T \theta) \quad (9.48)$$

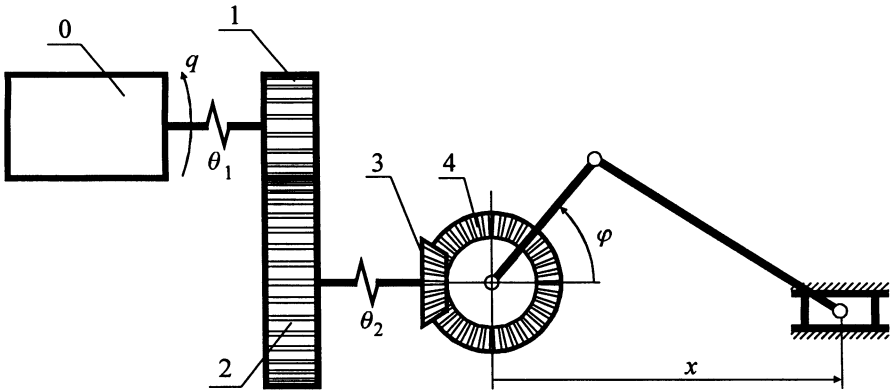


Fig. 9.12. Elastic transmission mechanism and rigid actuating mechanism

Differentiating (9.48) with respect to time and retaining the terms of first order, we obtain:

$$\dot{x} = (\partial \Pi_x / \partial \varphi)_0 (a\dot{q} + b^T \dot{\theta}) + (\partial^2 \Pi_x / \partial \varphi^2)_0 a\dot{q} b^T \theta. \quad (9.49)$$

Expression (9.49) is simplified, if only small vibrations are considered in the vicinity of the equilibrium position of the rigid mechanism. Setting $q = q_0$,

$dq/dt = 0$, $d^2q/dt^2 = 0$, we have:

$$\dot{x} = (\partial \Pi_x / \partial \dot{\theta})_0 \dot{\theta}, \quad (9.50)$$

$$\ddot{x} = (\partial \Pi_x / \partial \ddot{\theta})_0 \ddot{\theta}. \quad (9.51)$$

9.6

Dynamic Problems of Elastic Mechanisms

The dynamic analysis of elastic mechanisms must be based on dynamic models that take into account their kinematic parameters: masses and moments of inertia of the links. In most cases, it turns out that most appropriate models are those

presenting a mechanism as a system of perfectly rigid bodies connected by inertialess (i.e. „massless“) elastic elements. In these cases, the method of kinetostatics may be used for the derivation of motion equations. To this end, the resultant vectors and the resultant moments of the inertia forces of the rigid bodies are determined as functions of the kinematic parameters of motion. These forces and moments are added to the active forces, acting on the links, in order to determine the deformations of the elastic elements caused by the combined action of active forces and inertia forces.

A basic problem of the dynamics of elastic mechanisms is the investigation of vibrations in the vicinity of an equilibrium position or of a program motion. We will acquaint ourselves with methods how to formulate this problem by considering concrete examples.

Let us formulate the equations of small vibrations for the mechanism shown in Fig. 9.10 in the vicinity of the equilibrium position given by the values of the input coordinates (rotation angles of the motors) q_{10}, q_{20}, q_{30} . In this position the motors are braked and the mechanism can accomplish only small vibrations due to deformations $\theta_1, \theta_2, \theta_3$ of the transmission mechanisms and deformations of the rods AB (θ_4) and DE (θ_5). Let us suppose that platform BD , the mass centre of which is at point K , and links OA and EF possess substantial masses. The inertia forces of the platform performing a plane motion are reduced to the resultant vector Φ with projections $\Phi_x = -mw_{Kx}$ and $\Phi_y = -mw_{Ky}$ ¹ onto the coordinate axes, where m is the mass of the platform, and to the resultant moment $M_K^\Phi = -J_K \varepsilon$, where J_K is the moment of inertia of the platform with respect to axis Kz , and ε is its angular acceleration. The inertia forces of links OA and EF are reduced to moments $M_1^\Phi = -J_1 \varepsilon_1$ and $M_3^\Phi = -J_3 \varepsilon_3$, where J_1 and J_3 are the link moments of inertia with respect to the rotation axes, ε_1 and ε_3 are the angular accelerations. Let us introduce the vector

$$\mathbf{r}^T = (\Delta x_K, \Delta y_K, \Delta \varphi, \theta_1, \theta_3), \quad (9.52)$$

defining the deviations of the system from the equilibrium position, the vector of inertia forces

$$\Phi^T = (\Phi_x, \Phi_y, M_K^\Phi, M_1^\Phi, M_3^\Phi) \quad (9.53)$$

and the diagonal matrix of kinetic parameters

$$A = \text{diag}(m, m, J_K, J_1, J_3). \quad (9.54)$$

Then, the vector of inertia forces and of moments can be represented in the following form:

$$\Phi = -A\ddot{\mathbf{r}} = -A(\partial \mathbf{r} / \partial \theta)_0 \ddot{\theta}, \quad (9.55)$$

¹ It is assumed that the mass centre K is on line BD .

where $\theta^T = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ is the vector of deformations of the elastic elements.

Let us, first, suppose that there are no active external forces. Thus, the deformations are caused only by the action of inertia forces. During the process of deformations in the elastic elements, elastic moments $M_{\varphi 1}, M_{\varphi 2}, M_{\varphi 3}$ (on the outputs of the elastic transmission mechanisms) and elastic forces F_4, F_5 (in rods AB and DE) are acting. Let us once more introduce the vector of elastic forces (see Sect. 9.4)

$$\mathbf{F}^T = (M_{\varphi 1}, M_{\varphi 2}, M_{\varphi 3}, F_4, F_5). \quad (9.56)$$

Every single elastic force is equal to the product of the stiffness of the corresponding element and its deformation, and it is directed opposite to the deformation. Hence,

$$\mathbf{F} = -\text{diag}(c_1, \dots, c_5) \theta = -C\theta. \quad (9.57)$$

Here, $C = (E_d)^{-1}$ is the diagonal stiffness matrix. In accordance with the principle of virtual work we have:

$$\mathbf{F}^T \delta \theta + \Phi^T \delta \mathbf{r} = 0 \quad \text{or} \quad \mathbf{F} = -(\partial \mathbf{r} / \partial \theta)_0^T \Phi. \quad (9.58)$$

Substituting (9.55) and (9.57) into (9.58), we obtain the differential equation of *free vibration* of the system:

$$(\partial \mathbf{r} / \partial \theta)_0^T A (\partial \mathbf{r} / \partial \theta)_0 \ddot{\theta} + C\theta = 0. \quad (9.59)$$

The matrix $(\partial \mathbf{r} / \partial \theta)_0$ is evaluated in the equilibrium position of the rigid mechanism. In this case, relationship (9.39) can be used and the matrix is determined with the help of the equilibrium equations.

Let us now assume that active forces are applied to the system, which are reduced to a moment M and to forces P_x and P_y , applied to platform BD at point K . These may, e.g., be forces associated with a process of technological treatment of a workpiece mounted on the platform. According to (9.52) the work of the active forces δW in a virtual displacement $\delta \mathbf{r}$, can be represented in the form:

$$\delta W = \mathbf{V}_a^T \delta \mathbf{r} = (P_x, P_y, M, 0, 0) (\delta x_K, \delta y_K, \delta \varphi, \delta \theta_1, \delta \theta_3)^T, \quad (9.60)$$

where $\mathbf{V}_a$ is the vector of the active forces associated with the components of vector $\Delta \mathbf{r}$. With regard to the active forces, the equation of small vibrations in the vicinity of the equilibrium position takes the following form:

$$(\partial \mathbf{r} / \partial \theta)_0^T A (\partial \mathbf{r} / \partial \theta)_0 \ddot{\theta} + C\theta = (\partial \mathbf{r} / \partial \theta)_0^T \mathbf{V}_a. \quad (9.61)$$

This equation describes *forced vibration* of the mechanism which occur in the system if the active forces have vibratory character.

Let us return to the equation of free vibrations (9.59). Writing

$$A_* = (\partial \mathbf{r} / \partial \boldsymbol{\theta})_0^T A (\partial \mathbf{r} / \partial \boldsymbol{\theta})_0, \quad (9.62)$$

we represent this equation in the form

$$A_* \ddot{\boldsymbol{\theta}} + C \boldsymbol{\theta} = 0. \quad (9.63)$$

Analogously, substituting (9.62) into (9.61), we obtain

$$A_* \ddot{\boldsymbol{\theta}} + C \boldsymbol{\theta} = (\partial \mathbf{r} / \partial \boldsymbol{\theta})_0^T \mathbf{V}_a. \quad (9.64)$$

General methods for the investigation of free and forced vibrations of mechanical systems are considered in detail in courses on theory of vibrations. Here, the influence of dissipative forces arising during the deformations of elastic elements is usually taken into account. In the investigation of free and forced vibrations of machine units the mechanical characteristics of the engines and the characteristics of the working processes are also taken into account. One of the most actual problems of this kind is considered in Chap. 10.

9.7

Free and Forced Vibration of Elastic Mechanisms

Here we restrict ourselves to the presentation of only a few general aspects of problems in the vibration analysis of mechanisms with elastic links.

Equation (9.63) is the vector form of a system of linear differential equations of order $2s$, where s is the order of vector $\boldsymbol{\theta}$. It can be proven that this equation has always s linearly independent particular solutions of the kind

$$\boldsymbol{\theta}_m = \boldsymbol{\theta}_{m0} \cos(k_m t + \alpha_m) \quad (m = 1, \dots, s). \quad (9.65)$$

For the considered mechanism every one of these solutions corresponds to small vibration in the vicinity of the given equilibrium position, for which all deformations θ_m vary according to a harmonic law, in synchronism and in phase. Such free vibration is called *natural vibration* of the system and the frequencies k_m are called *natural frequencies*. Let us substitute (9.65) into equation (9.63). We obtain

$$-k_m^2 A_* \boldsymbol{\theta}_{m0} \cos(k_m t + \alpha_m) + C \boldsymbol{\theta}_{m0} \cos(k_m t + \alpha_m) = 0.$$

After division by the common factor $\cos(k_m t + \alpha_m)$, we obtain a linear vectorial algebraic equation (i.e. a system of s scalar equations) for the vector of the amplitudes of deformations $\boldsymbol{\theta}_{m0}$:

$$(C - A_* k_m^2) \boldsymbol{\theta}_{m0} = 0. \quad (9.66)$$

As it is known from linear algebra, this equation has a nonzero solution, if and only if the determinant of the coefficient matrix is equal to zero. This condition, written in the form

$$\det(C - A_* k_m^2) = 0, \quad (9.67)$$

is the *frequency equation* because its roots $k_1, k_2, \dots, k_s$ are namely the natural frequencies. It can be proven that the frequency equation has always s real positive roots, among them possibly multiple roots. Henceforth, we will assume that $k_r^2 \geq k_m^2$, if $r > m$.

It can be proved that to every simple root k_m^2 corresponds a solution of equation (9.66) and, moreover, since the determinant of this system of algebraic equations is equal to zero, the vector θ_{m0} is determined up to an arbitrary scalar factor. The components of this vector determine the amplitudes of deformations, when the system is in free vibration with the m -th natural frequency. From what has been said it is clear that one of the amplitudes can be given arbitrarily, while the components of the vectors θ_{m0} define the ratios of the amplitudes. The vectors θ_{m0} are called *natural modes* of the system. It can be shown that the natural modes, corresponding to different natural frequencies, satisfy the conditions

$$(A_* \theta_{m0})^T \theta_{r0} = 0, \quad (C \theta_{m0})^T \theta_{r0} = 0 \quad (r \neq m).$$

Vectors satisfying these conditions are called *orthogonal in the metrics A_* and C* .

It can also be proven that to every natural frequency of multiplicity d are associated d natural modes which are independent solutions of equation (9.66). As a result the total number of natural modes is always equal to s .

We point out that the total number of natural frequencies and of natural modes of an elastic mechanism is not a property intrinsic to the mechanism itself. This property is determined by the choice of physical model, more precisely, by the choice of elastic elements. The larger the number of elastic elements the larger is the number of degrees of freedom of the system and with it the dimension of the vector θ . The number of natural frequencies increases and so does the *width of the spectrum*, i.e. the difference $k_{\max} - k_{\min}$. When choosing a physical model for a mechanism, it is necessary to know what the width of the spectrum of natural frequencies should be, in order to properly describe the vibration processes arising in the mechanism during its performance. One of the most important criteria is the width of the frequency spectrum of exciting forces causing forced vibration of the system.

Free vibration arises, if an elastic mechanism is deflected from its equilibrium position and if then, in the absence of exciting forces, the loads which have caused the initial deformations are removed. Such a situation arises, e.g., during the action of short-term impact loads or after finishing a positioning process. In the absence of dissipative forces in the system free vibration would represent a *polyharmonic* process – a sum of harmonic vibrations with natural frequencies. In a real mechanical system there are always dissipative forces causing damping of the free vibration; the study of these dissipative forces allows us to estimate the duration of this damping process.

The most important characteristics of an elastic mechanical system are revealed by the investigation of the free vibration. Its natural frequencies and natural

modes, the determination of which is necessary for the solution of many other problems of dynamics.

Let us now proceed with the analysis of forced vibration described by equation (9.64). We assume that the exciting force on the right-hand side of this equation is a harmonic function of time with frequency ω . Then, equation (9.64) takes the form:

$$A_* \ddot{\theta} + C\theta = \left(\partial \mathbf{r} / \partial \theta\right)_0^T \mathbf{V}_{a0} \cos \omega t = \Theta_0 \cos \omega t, \quad (9.68)$$

where Θ_0 is the vector of the amplitudes of the exciting forces. The general solution of this equation is given, as is known, by the general solution of equation (9.63) and by a particular solution of equation (9.68). In a real system the first term corresponding to free vibration is, as has already been noted, a damped process. That is why after a transient, usually of very short duration, the system is undergoing forced harmonic vibration corresponding to the particular solution of equation (9.68). Following well known results of the theory of linear differential equations, this solution should be sought in the form:

$$\theta = \theta_0 \cos \omega t. \quad (9.69)$$

Substituting (9.69) into (9.68), and deviding the left-hand and the right-hand side by the common factor $\cos \omega t$, which is not identically equal to zero, we obtain

$$(C - A_* \omega^2) \theta_0 = \Theta_0 \quad \text{or} \quad \theta_0 = (C - A_* \omega^2)^{-1} \Theta_0. \quad (9.70)$$

Developing this vector expression we find:

$$\theta_{0p} = \left[\det(C - A_* \omega^2) \right]^{-1} \sum_{m=1}^s D_{pm}(\omega^2) \theta_{0m} \quad (p = 1, \dots, s). \quad (9.71)$$

Here, $D_{pm}(\omega^2)$ is the cofactor of the element in the p -th row and the m -th column of the matrix $D(\omega^2) = (C - A_* \omega^2)$. Comparing (9.71) with the frequency equation (9.67), one can easily verify that for $\omega = k_m$ ($m = 1, \dots, s$), i.e. identity of the frequency of the exciting force with one of the natural frequencies of the system, the expressions (9.71) cease to exist, since then $\det D = 0$. For values ω close to k_m the amplitudes of the deformations of elastic elements become very large. Naturally, in this case, also the amplitudes of the corresponding elastic forces increase which can turn out to be dangerous for the performance of the mechanism. Vibration with large amplitude that occurs when the frequency of the exciting force approaches one of the natural frequencies, is called *resonance* vibration. Dissipative forces which are considered in detail in the next chapter somewhat reduce vibration amplitudes in the resonance zone; also the values of the amplitudes for $\omega = k_m$ become limited. Nevertheless, the risk of resonances usually persists.

Often, the exciting force in a machine is polyharmonic. So, e.g., in a cyclic machine perturbation forces acting in a steady-state motion of the machine (see Chap. 8) are periodic processes and can be represented as a sum of harmonics

forming Fourier series. By virtue of the superposition principle valid for linear systems the particular solution of equation (9.68) is also represented by a Fourier series including harmonics of the same frequencies. Moreover, resonances may arise, if the frequency of any harmonic of the perturbation turns out to be equal to one of the natural frequencies of the elastic mechanism.

A small vibration of elastic mechanisms also arises in the process of execution of program motions. The investigation of these vibrations leads to the analysis of systems of linear differential equations with variable coefficients, the consideration of which falls outside the scope of this course.

9.8 Problems

- 9.1. Determine the outlet stiffness c_x of a slider-crank mechanism (Fig. 9.13), if the revolute pairs O , A and B are elastic pairs with stiffnesses c_O , c_A , c_B not depending on the directions of the forces acting at the corresponding pair. Find the mechanism positions for which c_x takes maximum and minimum values.

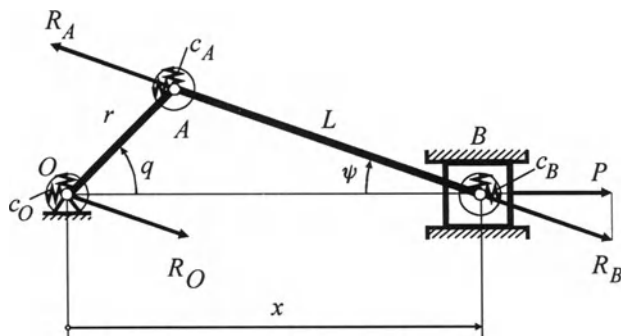


Fig. 9.13. Determination of the stiffness of a mechanism with elastic joints

Solution: Let us apply at the output a force P directed along the x -axis. From force diagrams for the kinematic pairs we find the reactions R_b , R_a and R_o

$$R_b = R_a = R_o = P/\cos\psi.$$

It follows that $(\partial x/\partial \theta_B) = (\partial x/\partial \theta_A) = (\partial x/\partial \theta_O) = -1/\cos\psi$, where θ_A , θ_B , θ_O are the deformations of the joints. Adding together the output-reduced flexibilities of the series-connected elastic elements, we find from (9.15):

$$c_x = e_x^{-1} = \left[c_B^{-1} (\partial x / \partial \theta_B)^2 + c_A^{-1} (\partial x / \partial \theta_A)^2 + c_O^{-1} (\partial x / \partial \theta_O)^2 \right]^{-1} = \\ (c_B^{-1} + c_A^{-1} + c_O^{-1})^{-1} \cos^2 \psi.$$

The stiffness is maximal for $q = 0$ and $q = \pi$, and minimal for $\tan q = L / r$.

- 9.2. Determine the inlet stiffness c_q of an epicyclic gear train (Fig. 9.14), if the elastic elements are the contact elements of the involute gears with stiffnesses c_{12} and c_{23} , respectively, along the common normals to the teeth surfaces at the contact points. The moduli of all wheels are identical.

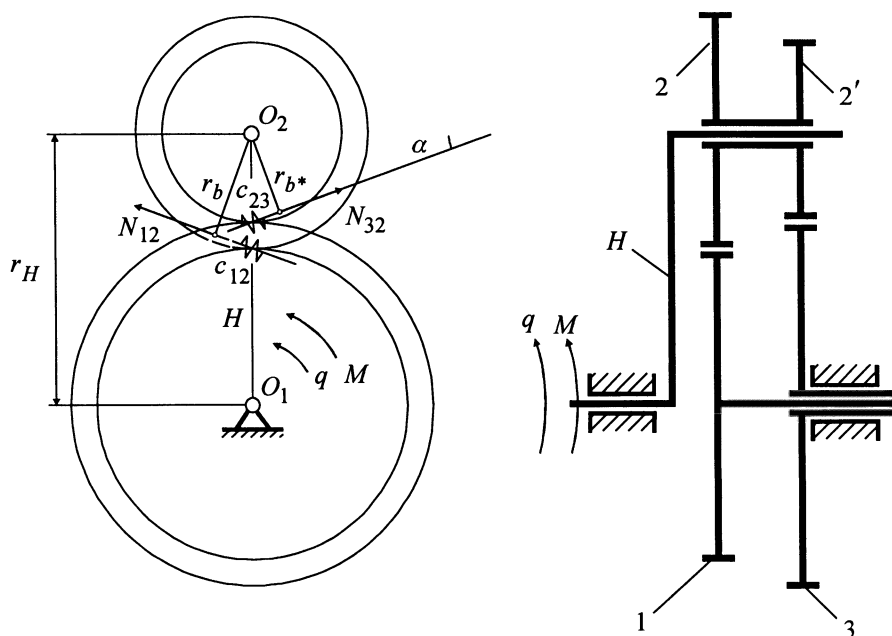


Fig. 9.14. Determination of the stiffness of an epicyclic gear train

Answer:
$$c_q^{-1} = [r_H \cos \alpha (z_2 - z_{2*})]^{-2} \left[(z_2)^2 / c_{23} + (z_{2*})^2 / c_{12} \right],$$

where r_H is the radius of the planet carrier, z_2 and z_{2*} are the teeth numbers of the planet pinion, α is the meshing angle.

- 9.3. Determine the matrix of flexibilities at point M (gripper pole) of the robot actuating mechanism in the position shown in Fig. 9.15 in directions of the axes Ox , Oy , Oz . The drive mechanisms of the actuating links are

elastic. Their outlet stiffnesses are c_1 , c_2 , c_3 , respectively. The dimensions a_1 , a_2 , a_3 of the links are shown in the figure.

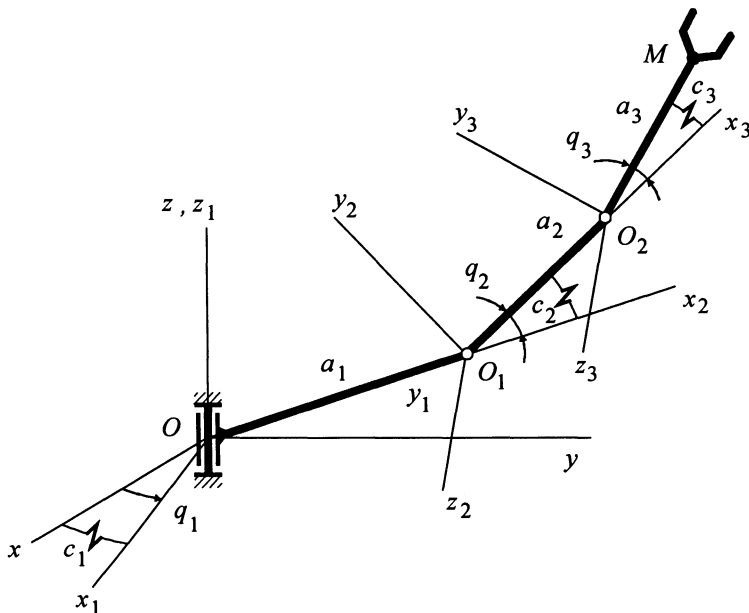


Fig. 9.15. Elastic mechanism of a robot

Solution: By the methods presented in Chap. 2 we determine the position functions $x_M(q_1, q_2, q_3)$, $y_M(q_1, q_2, q_3)$, $z_M(q_1, q_2, q_3)$. Differentiating them, we obtain the elements of the matrix $\partial \mathbf{r}_M / \partial \mathbf{q}$, where $\mathbf{r}_M^T = (x_M, y_M, z_M)$, $\mathbf{q}^T = (q_1, q_2, q_3)$:

$$\begin{aligned} \partial x_M / \partial q_1 &= a_1 \cos q_1 + a_2 \cos q_1 \cos q_2 + a_3 \cos q_1 \cos(q_2 + q_3), \\ \partial x_M / \partial q_2 &= a_2 \sin q_1 \sin q_2 + a_3 \sin q_1 \sin(q_2 + q_3), \\ \partial x_M / \partial q_3 &= a_3 \sin q_1 \sin(q_2 + q_3), \\ \partial y_M / \partial q_1 &= -a_1 \sin q_1 - a_2 \sin q_1 \cos q_2 - a_3 \sin q_1 \cos(q_2 + q_3), \\ \partial y_M / \partial q_2 &= -a_2 \sin q_2 \cos q_1 - a_3 \cos q_1 \sin(q_2 + q_3), \\ \partial y_M / \partial q_3 &= -a_3 \cos q_1 \sin(q_2 + q_3), \\ \partial z_M / \partial q_1 &= 0, \quad \partial z_M / \partial q_2 = a_2 \cos q_2 + a_3 \cos(q_2 + q_3), \\ \partial z_M / \partial q_3 &= a_3 \cos(q_2 + q_3). \end{aligned}$$

The flexibility matrix is determined by formula (9.34). Moreover, the matrix E_d is diagonal:

$$E_d = \text{diag}(c_1^{-1}, c_2^{-1}, c_3^{-1}).$$

As a result, the elements of the flexibility matrix are represented in the following form:

$$\begin{aligned} e_{xx} &= c_1^{-1}(\partial x_M / \partial q_1)^2 + c_2^{-1}(\partial x_M / \partial q_2)^2 + c_3^{-1}(\partial x_M / \partial q_3)^2, \\ e_{xy} &= c_1^{-1}(\partial x_M / \partial q_1)(\partial y_M / \partial q_1) + c_2^{-1}(\partial x_M / \partial q_2)(\partial y_M / \partial q_2) + \\ &\quad c_3^{-1}(\partial x_M / \partial q_3)(\partial y_M / \partial q_3), \\ e_{yy} &= c_1^{-1}(\partial y_M / \partial q_1)^2 + c_2^{-1}(\partial y_M / \partial q_2)^2 + c_3^{-1}(\partial y_M / \partial q_3)^2, \\ e_{xz} &= c_1^{-1}(\partial x_M / \partial q_1)(\partial z_M / \partial q_1) + c_2^{-1}(\partial x_M / \partial q_2)(\partial z_M / \partial q_2) + \\ &\quad c_3^{-1}(\partial x_M / \partial q_3)(\partial z_M / \partial q_3), \\ e_{yz} &= c_1^{-1}(\partial y_M / \partial q_1)(\partial z_M / \partial q_1) + c_2^{-1}(\partial y_M / \partial q_2)(\partial z_M / \partial q_2) + \\ &\quad c_3^{-1}(\partial y_M / \partial q_3)(\partial z_M / \partial q_3), \\ e_{zz} &= c_1^{-1}(\partial z_M / \partial q_1)^2 + c_2^{-1}(\partial z_M / \partial q_2)^2 + c_3^{-1}(\partial z_M / \partial q_3)^2. \end{aligned}$$

The flexibility matrix of type (9.34) is symmetric. That is why, $e_{yx} = e_{xy}$,

$$e_{zx} = e_{xz}, \quad e_{zy} = e_{yz}.$$

- 9.4. Determine the natural frequencies of the slider-crank mechanism in the equilibrium position shown in Fig. 9.16. The inertia moment of crank 1 with respect to the rotation axis O is J , the mass of slider 3 is m , the mass of the connecting rod 2 is negligibly small. Elastic elements are: the transmission connecting the braked rotor of the motor with the crank and the connecting rod which can deform in longitudinal direction. The stiffness of the transmission, reduced to the rotation angle of the crank, is c_1 , the stiffness of the connecting rod is c_2 . The radius of the crank is a , the length of the connecting rod is L .

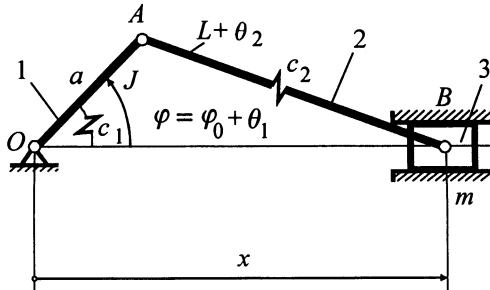


Fig. 9.16. To the determination of the natural frequencies of an elastic mechanism

Solution:

1) We determine the matrix $(\partial \mathbf{r} / \partial \theta)_0$, where $\mathbf{r} = (x, \varphi)^T$ is the vector of „output“ coordinates, $\theta = (\theta_1, \theta_2)^T$ is the vector of the deformations of the elastic elements. We obtain:

$$\begin{aligned} x &= a \cos \varphi + (L^2 - a^2 \sin^2 \varphi)^{1/2}, \quad \varphi = \varphi_0 + \theta_1, \\ (\partial x / \partial \theta_1)_0 &= \partial x / \partial \varphi = -a \sin \varphi - (L^2 - a^2 \sin^2 \varphi)^{1/2} a^2 \sin \varphi \cos \varphi = u(\varphi), \\ (\partial x / \partial \theta_2)_0 &= (\partial x / \partial L) = L^{-1} (L^2 - a^2 \sin^2 \varphi)^{1/2} = v(\varphi), \\ \partial \varphi / \partial \theta_1 &= 1, \quad \partial \varphi / \partial \theta_2 = 0. \end{aligned}$$

2) We determine the matrix A_* according to formula (9.62):

$$A_* = \begin{pmatrix} u(\varphi) & 1 \\ v(\varphi) & 0 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} u(\varphi) & v(\varphi) \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} mu^2(\varphi) + J & mu(\varphi)v(\varphi) \\ mu(\varphi)v(\varphi) & mv^2(\varphi) \end{pmatrix}.$$

3) We derive the frequency equation in form (9.67):

$$\begin{vmatrix} c_1 - k^2(mu^2(\varphi) + J) & -k^2 mu(\varphi)v(\varphi) \\ -k^2 mu(\varphi)v(\varphi) & c_2 - k^2 mv^2(\varphi) \end{vmatrix} = 0.$$

Developing the determinant and solving the resulting quadratic equation for k^2 , we find

$$\begin{aligned} k_{1,2}^2 &= \left(c_1 mv^2(\varphi) + c_2 mu^2(\varphi) + c_2 J \pm \left\{ [c_1 mv^2(\varphi) + c_2 mu^2(\varphi) + c_2 J]^2 - \right. \right. \\ &\quad \left. \left. 4Jmv^2(\varphi)c_1c_2 \right\}^{1/2} \right) [2Jmv^2(\varphi)]^{-1} \quad (\text{for } \varphi = \varphi_0). \end{aligned}$$

10 Vibration of Machines with Elastic Transmission Mechanisms

10.1

Dissipative Forces in Deformable Elements

In mechanisms with rigid links motion is accompanied, as is well-known, by the generation of dissipative forces acting at kinematic pairs. Earlier the influence of forces of Coulomb friction on the motion of rigid mechanisms and on the efficiency mechanism has been investigated. In mechanisms with elastic links dissipative forces are also caused by deformations of elastic elements

Depending on the physical nature of dissipative forces two varieties are distinguished: Forces of *internal friction* and forces of *construction damping*. Forces of internal friction appear in the material of deformable elements of a mechanism. They are caused by friction at the boundaries of grains of structural materials and are associated with the presence of material imperfections called *dislocations*. Forces of construction damping are caused by frictional forces acting at contacting surfaces of elements constituting so-called „immovable“ joints. During the deformation of inter-connected elements relative slipping occurs which is accompanied by friction on contacting surfaces.

The resistance forces represent *dissipative* forces, i.e. their action is accompanied by dissipation of mechanical energy and by its conversion into thermal energy.

The general analytical expression for dissipative forces has the following form:

$$F_{diss} = -F_d(\theta, |\dot{\theta}|) \text{sign} \dot{\theta}, \quad (10.1)$$

where θ is the deformation of an elastic element and $\dot{\theta}$ is the velocity of deformation. If θ is an angular deformation, then the analytical expression

$$M_{diss} = -M_d(\theta, |\dot{\theta}|) \text{sign} \dot{\theta}, \quad (10.2)$$

defines the dependency of the moment on deformation and on deformation velocity. The dissipative force is called *positional*, if it can be represented in the form

$$F_{diss} = -F_d(\theta) \text{sign} \dot{\theta}, \text{ or } M_{diss} = -M_d(\theta) \text{sign} \dot{\theta}. \quad (10.3)$$

Usually, the forces of internal friction can be considered as positional.

Dissipative forces of *viscous* friction for which

$$F_{diss} = -F_d(|\dot{\theta}|) \text{sign} \dot{\theta}, \text{ or } M_{diss} = -M_d(|\dot{\theta}|) \text{sign} \dot{\theta}. \quad (10.4)$$

are particularly distinguished. Forces of viscous friction arise in bearings with fluid lubricant. A special case of viscous friction is the *linear* friction for which

$$F_{diss} = -b|\dot{\theta}|\text{sign } \dot{\theta} = -b\dot{\theta}, \text{ or } M_{diss} = -b|\dot{\theta}|\text{sign } \dot{\theta} = -b\dot{\theta}. \quad (10.5)$$

The coefficient of proportionality b is called *damping coefficient*.

Linear friction is the simplest form of analytical description of dissipative forces. That is why, there is often a strive to represent dissipative forces in the form of linear friction with damping coefficient b depending on motion parameters. Such a representation is called an *equivalent linearization* of dissipative forces. Moreover, the condition of equality of the energy dissipated for a cycle of the vibrating process in the original elastic-dissipative element and in the „linearized“ one is usually chosen as a criterion of equivalence.

The quantity of energy, dissipated for a cycle of vibration in a dissipative element may be determined after building the so called „hysteresis loop“ (Fig. 10.1) expressing the force (the moment) as a function of deformation during a cyclic loading. In the process of loading the representative point of the graph describes a closed curve. For a linear elastic force and a positional dissipative force the area limited by this curve

$$S = \oint M d\theta = \oint c\theta d\theta + \oint M_d(\theta)\text{sign } \dot{\theta} d\theta \quad (10.6)$$

is equal to the work done by moment M and defines the energy dissipated for a cycle, since the first summand in expression (10.6) is equal to zero.

Let deformation θ changes according to a harmonic law

$$\theta = \theta_0 \sin \omega t. \quad (10.7)$$

Substituting (10.7) into (10.6) we have

$$S = \int_0^{2\pi/\omega} M_d(\theta_0 \sin \omega t) |\cos \omega t| \omega dt = \int_0^{2\pi} M_d(\theta_0 \sin \xi) |\cos \xi| d\xi. \quad (10.8)$$

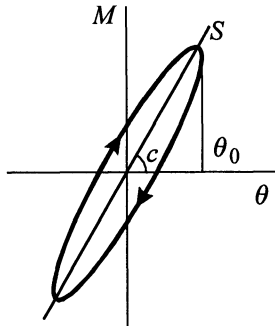


Fig. 10.1. The force and the moment in an elastic element as a function of the deformation

It is obvious that S depends only on the amplitude of vibrations and does not depend on frequency ω of the vibration process; this is an important feature of positional dissipative forces. On the other hand, in the case of linear friction (10.5) we have

$$S = \int_0^{2\pi/\omega} b\omega\theta_0 \cos \omega t \omega\theta_0 \cos \omega t dt = \pi b \omega \theta_0^2. \quad (10.9)$$

Comparing expressions (10.8) and (10.9) we find that for harmonic vibration the coefficient of equivalent linearization b , obtained from the condition of equality of energy losses for a cycle must be defined in the following way:

$$b = \frac{S(\theta_0)}{\pi \omega \theta_0^2}. \quad (10.10)$$

The dissipation of energy is often characterized by the *coefficient of absorption* ψ equal to the ratio of the area of hysteresis loop to the maximum value of potential energy of elastic deformation. For a linear element with stiffness c we have

$$\Pi_{\max} = \int_0^{\theta_0} c\theta d\theta = \frac{c\theta_0^2}{2}, \quad (10.11)$$

where $\Pi_{\max}$ is the maximum value of the potential energy; θ_0 is the amplitude of deformation. Then, taking into account (10.9), the coefficient of absorption may be represented in the form:

$$\psi = \frac{S}{\Pi_{\max}} = \frac{\pi b \theta_0^2 \omega}{0.5 c \theta_0^2} = \frac{2\pi b \omega}{c}. \quad (10.12)$$

Hence, the equivalent damping coefficient b is associated with the stiffness of the element by the following relationship:

$$b = \frac{\psi c}{2\pi \omega}. \quad (10.13)$$

This formula is used later on when choosing values of the coefficients of resistance of elastic-dissipative elements. It is obvious that it has sense only in the case of harmonic law of variation of θ . However, in most cases, it is necessary to take into account dissipative forces only when considering resonance vibrations which prove to be close to harmonic ones. In more complex cases, when polyharmonic vibrations are excited, other methods of equivalent linearization of dissipative forces are applied which will not be considered here.

Let us note that for most structural materials the value of the coefficient of absorption ψ lies in the boundaries

$$0.2 < \psi < 0.6. \quad (10.14)$$

10.2

Reduced Stiffness and Reduced Damping Coefficient

Let us consider the dynamic model of the machine assembly presented in Fig 10.2. Here, 0 is the engine, 1–4 are gears, 5 is an actuating mechanism depicted in the form of a turning rotor. Those elements that are regarded as elastic are marked in the figure; $c_1, \dots, c_5$ are their stiffnesses. The stiffnesses of shafts 0–1, 2–3, 4–5 are determined from formula (9.14) and the stiffnesses of gear meshings 1–2 and 3–4 are determined from formula (9.15). The damping coefficients $b_1, \dots, b_5$ reflect the influence of dissipative forces. Assuming that the vibration in the system is close to harmonic, we will consider the expressions (10.13) to be valid. The parameters $\theta_1, \dots, \theta_5$ represent the deformations of the elastic elements.

In accordance with the method exposed in Sect. 9.2 we determine the flexibility of the transmission mechanism reduced to the input. To this end, it is necessary to multiply the flexibility of every elastic element by the squared transmission ratio relating this element to the mechanism output and to add together all values thus obtained. We find:

$$\begin{aligned} e_{1re} &= c_1^{-1}, \quad e_{2re} = c_2^{-1} \frac{1}{r_1^2}, \quad e_{3re} = c_3^{-1} i_{12}^2, \\ e_{4re} &= c_4^{-1} \frac{i_{12}^2}{r_3^2}, \quad e_{5re} = c_5^{-1} i_{12}^2 i_{34}^2, \end{aligned} \quad (10.15)$$

where r_1 and r_3 are the radii of the basic circles of the involute gear wheels 1 and 3. Adding together relationships (10.15) we obtain the flexibility of the mechanism reduced to its output. The inverse of this sum represents the reduced stiffness c of the mechanism:

$$c = \left(\frac{1}{c_1} + \frac{1}{c_2 r_1^2} + \frac{i_{12}^2}{c_3} + \frac{i_{12}^2}{c_4 r_3^2} + \frac{i_{12}^2 i_{34}^2}{c_5} \right)^{-1}. \quad (10.16)$$

Let us note that in expression (10.16) the flexibility of every elastic element is multiplied by the squared transmission ratio relating this element to the input of the mechanism. That is why, the greatest contribution to the total flexibility have those elastic elements for which this transmission ratio is maximal. For a transmission mechanism working in the regime of a reducer such elastic elements prove to be the output shaft and, in general, the elements located closest to the output. For increasing the stiffness of the transmission it is necessary, above all, to try to increase the stiffness just of these elements.

Let us now suppose that, upon a fixed mechanism output, the input link is vibrating according to the harmonic law $\theta = \theta_0 \sin \omega t$. Then, all elastic elements of the mechanism will be deformed according to a harmonic law and their dissipative properties can be specified by the damping coefficients $b_1, \dots, b_5$ which can be determined according to formula (10.13). Moreover, at the mechanism input there will be a moment of nonelastic dissipative forces. For this moment the coefficient

of equivalent linearization b can be determined according to a formula analogous to (10.16)

$$b = \left(\frac{1}{b_1} + \frac{1}{b_2 r_1^2} + \frac{i_{12}^2}{b_3} + \frac{i_{12}^2}{b_4 r_3^2} + \frac{i_{12}^2 i_{34}^2}{b_5} \right)^{-1}. \quad (10.17)$$

If we assume that for all elements the coefficient of dissipation has one and the same value, then the reduced damping coefficient (10.17) will be related to the reduced stiffness (10.16) by relationship (10.13).

Derivation of motion equations. The dynamic model shown in Fig. 10.2 is a system with six degrees of freedom whose generalized coordinates are the rotation angles of the rotor (q_0), the gear wheels (q_1, q_2, q_3, q_4) and the input link of the actuating mechanism (φ). In many cases, however, the moments of inertia of gear wheels are small compared with the moment of inertia J_0 of the engine rotor and with the reduced moment of inertia J_M of the actuating mechanism. Then, neglecting the masses of wheels, we obtain a system with two degrees of freedom with generalized coordinates q_0 and φ . Let us derive the motion equations of such a „two-mass“ system. The kinetic energy is written in the form

$$T = \frac{1}{2} J_0 \dot{q}_0^2 + \frac{1}{2} J_M (\varphi) \dot{\varphi}^2. \quad (10.18)$$

We introduce the new variable $q_1 = i\varphi$, where $i = i_{12}i_{34}$ is the transmission ratio of the mechanism. Then,

$$T = \frac{1}{2} \left[J_0 \dot{q}_0^2 + J_M \left(\frac{q_1}{i} \right) \frac{1}{i^2} \dot{q}_1^2 \right] = \frac{1}{2} \left[J_0 \dot{q}_0^2 + J_1(q_1) \dot{q}_1^2 \right], \quad (10.19)$$

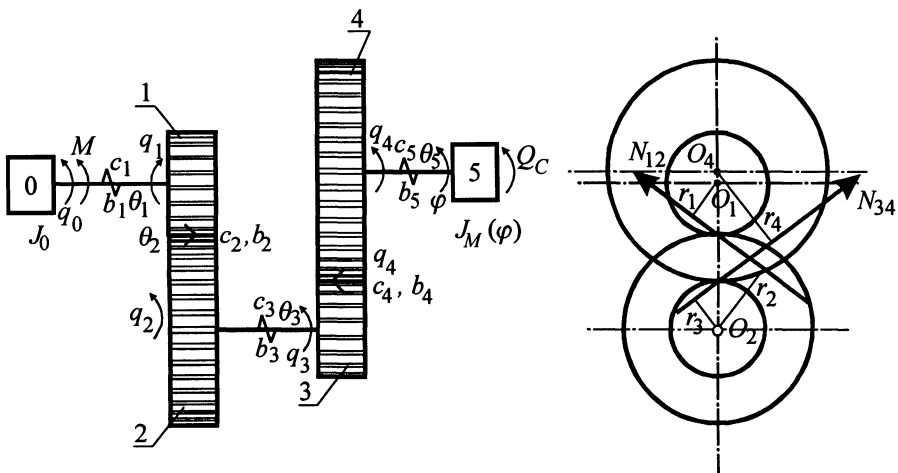


Fig. 10.2. Dynamic model of a machine aggregate

where $J_1(q_1) = i^{-2} J_M(q_1 i^{-1})$. For a cycle mechanism $J_M(\varphi)$ is a periodic function of φ with period 2π ; moreover, $J_1(q_1)$ is a $2\pi i$ periodic function. The moment acting at the transmission mechanism and reduced to its input is:

$$M_{tr} = b\dot{\theta} + c\theta = b(\dot{q}_0 - \dot{q}_1) + c(q_0 - q_1), \quad (10.20)$$

where c and b are respectively the reduced stiffness and the coefficient of resistance, while θ is the deformation reduced to the input. The reduced moment of resistance forces, applied to the actuating mechanism, is defined in the same way as for a rigid mechanism:

$$Q_{Rr} = Q_{Rr}(\varphi, \dot{\varphi}) = Q'_R(q_1, \dot{q}_1).$$

Reducing this moment to the input of the transmission mechanism we obtain

$$Q_R(q_1, \dot{q}_1) = \frac{1}{i} Q'_R(q_1, \dot{q}_1). \quad (10.21)$$

In the case of a cycle mechanism Q_R is a periodic function of q_1 with period $2\pi i$. We now write the motion equation of the system in the form of Lagrange's equation of the second kind. We have:

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_0} &= J_0 \dot{q}_0, \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_0} = J_0 \ddot{q}_0, \quad \frac{\partial T}{\partial q_0} = 0, \\ \frac{\partial T}{\partial \dot{q}_1} &= J_1(q_1) \dot{q}_1, \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} = J_1(q_1) \ddot{q}_1 + \frac{dJ_1}{dq_1} \dot{q}_1^2, \quad \frac{\partial T}{\partial q_1} = \frac{1}{2} \frac{dJ_1}{dq_1} \dot{q}_1^2. \end{aligned}$$

Substituting these expressions into the Lagrange equation we obtain:

$$J_0 \ddot{q}_0 = -M_{tr} + Q = Q - b(\dot{q}_0 - \dot{q}_1) - c(q_0 - q_1), \quad (10.22)$$

$$J_1(q_1) \ddot{q}_1 + \frac{1}{2} J_1' \dot{q}_1^2 = M_{tr} + Q_R = b(\dot{q}_0 - \dot{q}_1) + c(q_0 - q_1) + Q_R(q_1, \dot{q}_1). \quad (10.23)$$

It is necessary to add to these equations the characteristic of the engine, either the ideal (8.15), the static (8.18) or the dynamic (8.23). As a result a system of three equations with three unknowns q_0 , q_1 and Q is obtained.

10.3

Steady-State Motion of a Machine with an Ideal Engine. Elastic Resonance

Let us suppose that the ideal characteristic (8.15) is used for an engine. Then, for $u = u_0 = \text{const}$ the output link of the engine rotates uniformly with angular velocity

$$\dot{q}_0 = f(u_0) = \omega_0. \quad (10.24)$$

Here, $q_0 = \omega_0 t$, $\ddot{q}_0 = 0$. In this case, the rotation angle q_0 can be considered as a given function of time, and the investigation of machine motion is reduced to the determination of $q_1(t)$ by integration of Eq. (10.23). Following this, it is possible to determine from Eq. (10.22) the variation law of driving moment Q :

$$Q(t) = b(\omega_0 - \dot{q}_1) + c(\omega_0 t - q_1). \quad (10.25)$$

The periodic functions $J_1(q_1)$ and $Q_R(q_1, \dot{q}_1)$ can be represented in a form analogous to (7.2) and (7.3):

$$J_1(q_1) = J_{10} + \tilde{J}_1(q_1), \quad Q_R = Q_{R0}(\dot{q}_1) + \tilde{Q}_R(q_1, \dot{q}_1), \quad (10.26)$$

where J_{10} and $Q_{R0}(\dot{q}_1)$ are the average values of the periodic functions $J_1(q_1)$ and $Q_R(q_1, \dot{q}_1)$:

$$J_{10} = \frac{1}{2\pi i} \int_0^{2\pi i} J_1(q_1) dq_1, \quad Q_{R0}(\dot{q}_1) = \frac{1}{2\pi i} \int_0^{2\pi i} Q_R(q_1, \dot{q}_1) dq_1. \quad (10.27)$$

Substituting relationships (10.26) into Eqs. (10.23) and taking into account equation (10.24) we obtain

$$\begin{aligned} J_{10}\ddot{q}_1 + b(\dot{q}_1 - \omega_0) + c(q_1 - \omega_0 t) - Q_{R0}(\dot{q}_1) = \\ - \tilde{J}_1(q_1)\ddot{q}_1 - \frac{1}{2}\tilde{J}_1'(q_1)\dot{q}_1^2 + \tilde{Q}_R(q_1, \dot{q}_1). \end{aligned} \quad (10.28)$$

A periodic solution corresponding to a steady-state motion of the machine can be found for this equation by the method of successive approximations based on the assumption of proximity of the steady-state motion to the state of uniform rotation. As in the case of a rigid machine (see Sects. 8.3 and 8.4) the deviation from the state of uniform rotation is caused by moments on the right-hand side of Eq. (10.28). These moments explicitly depend on the coordinate q_1 . In order to determine the initial approximation of the desired solution, we equate to zero the right-hand side of this equation. We obtain the equation

$$J_{10}\dot{q}_1^0 + b(\dot{q}_1^0 - \omega_0) + c(q_1^0 - \omega_0 t) - Q_{R0}(\dot{q}_1^0) = 0, \quad (10.29)$$

which has a solution

$$q_1^0 = \omega_0 t - \Delta, \quad (10.30)$$

describing a state of uniform rotation of the output shaft of the transmission mechanism. The static deformation $\theta_{st} = \Delta$ is given by the equation

$$c\Delta + Q_{R0}(\omega_0) = 0, \quad (10.31)$$

which is obtained through substitution of the solution (10.30) into Eq. (10.29).

We obtain the next approximation to the solution by substituting solution (10.30) into the right-hand side of Eq. (10.28). On the left-hand side of this equation we linearize the function $Q_{R0}(\dot{q}_1)$ in the vicinity of ω_0 :

$$Q_{R0}(\dot{q}_1) \cong Q_{R0}(\omega_0) + \frac{dQ_{R0}}{d\dot{q}_1}(\omega_0)(\dot{q}_1 - \omega_0) = Q_{R0}(\omega_0) - v(\dot{q}_1 - \omega_0). \quad (10.32)$$

As result we obtain a linear differential equation:

$$J_{10}\ddot{q}_1 + (b + v)(\dot{q}_1 - \omega_0) + c(q_1 - \omega_0 t) = -\frac{1}{2}\tilde{J}'_1(\omega_0 t - \Delta)\omega_0^2 + \tilde{Q}_R(\omega_0 t - \Delta, \omega_0). \quad (10.33)$$

Comparing the right-hand side of this expression with the expression for the perturbation moment $\tilde{L}(t)$ (7.5), we notice that

$$-\frac{1}{2}\tilde{J}'_1(\omega_0 t - \Delta)\omega_0^2 + \tilde{Q}_R(\omega_0 t - \Delta, \omega_0) = \tilde{L}\left(t - \frac{\Delta}{\omega_0}\right), \quad (10.34)$$

Hence, on the right-hand side of Eq. (10.33) appears a perturbation moment shifted in time by $-\Delta/\omega_0$. Decomposing relationship (10.34) in Fourier series, we obtain

$$\tilde{L}\left(t - \frac{\Delta}{\omega_0}\right) = \sum_{l=1}^{\infty} L_l \cos(l\nu t + \beta_l) \quad (\nu = \omega_0/i), \quad (10.35)$$

in which the obtained amplitudes L_l of harmonics are the same as those in the case of a rigid machine. We seek a solution of Eq. (10.33) in the form

$$q_1(t) = \omega_0 t - \Delta + \tilde{\theta}(t), \quad (10.36)$$

where $\tilde{\theta}(t)$ is the variable part of the deformation of an elastic mechanism. Substituting equality (10.36) into Eq. (10.33) and taking into account relationships (10.34), (10.35) and (10.31) we arrive at an equation for the determination of $\tilde{\theta}$:

$$J_{10}\ddot{\tilde{\theta}} + (b + v)\dot{\tilde{\theta}} + c\tilde{\theta} = \sum_{l=1}^{\infty} L_l \cos(l\nu t + \beta_l). \quad (10.37)$$

Let us find the periodic solution of this differential equation possessing frequency ν and corresponding to the steady-state motion of a machine in first approximation. The solution has the following form:

$$\tilde{\theta} = \sum_{l=1}^{\infty} \frac{L_l \cos(l\nu t + \beta_l + \gamma_l)}{\sqrt{(c - J_{10}l^2\nu^2)^2 + (b + v)^2 l^2\nu^2}}, \quad (10.38)$$

where

$$\gamma_l = \arg\left[c - J_{10}l^2\nu^2 + (b + v)lvj\right]. \quad (10.39)$$

Expression (10.38) defines the dynamic error related to the elasticity of the transmission mechanism. It shows that the source of this error is the internal vibration activity of the mechanical system, which is characterized by the magnitude of the perturbation moment. The smaller the perturbation moment $\tilde{L}(t)$, the smaller are the amplitudes of the harmonics of deformation $\tilde{\theta}$.

We consider a special case. Let the average value of the moment of resistance forces does not depend on the angular velocity, thus

$$Q_{R0}(\dot{q}_1) = -M_0 = \text{const}, \quad v = 0. \quad (10.40)$$

We also suppose that one harmonic component predominates in the perturbation moment:

$$\tilde{L}(t) = L_0 \cos \omega t. \quad (10.41)$$

Then, for the amplitude of deformation (10.38) we obtain the expression

$$|\tilde{\theta}|_{\max} = \frac{L_0}{\sqrt{(c - J_{10} \omega^2)^2 + b^2 \omega^2}}. \quad (10.42)$$

Employing relationship (10.13), we obtain

$$|\tilde{\theta}|_{\max} = \frac{L_0}{c} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{k^2}\right)^2 + \frac{\psi^2}{4\pi^2}}}, \quad (10.42)$$

where $k^2 = c/J_{10}$ is the squared frequency of the free torsional vibration of the system shown in Fig. 10.3a; ψ is the coefficient of absorption corresponding to the reduced damping coefficient. In Fig. 10.4 the graph of the *magnification factor* is constructed

$$D(\omega) = |\tilde{\theta}|_{\max} \frac{c}{L_0} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{k^2}\right)^2 + \frac{\psi^2}{4\pi^2}}} \quad (10.44)$$

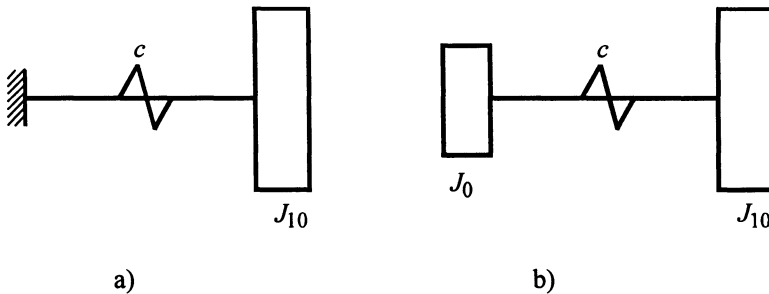


Fig. 10.3. Dynamic models of a mechanical system with an elastic transmission mechanism

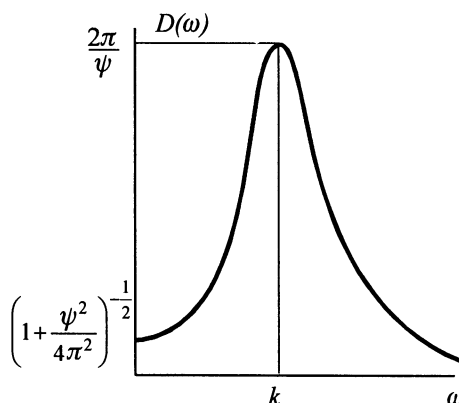


Fig. 10.4. The magnification factor as a function of the perturbation frequency

as a function of the frequency of perturbation ω . Taking into account the feasible values of ψ (see (10.14)), we obtain

$$10 \leq D_{\max} \leq 30.$$

In this way, for a frequency of vibration equal to the natural frequency of an elastic mechanism, an abrupt increase in the amplitude of vibration occurs. This phenomenon is called *elastic resonance* of a machine. In the considered case, the amplitude of resonance vibration is completely determined by the level of dissipative forces characterized by the magnitude of the coefficient of absorption ψ . If dissipative forces depend on velocity so that $\nu > 0$, then the magnitudes of vibration decrease in all frequencies including the resonance as well. Nevertheless, the magnitudes remain high enough and for this reason the appearance of resonance vibration during a steady-state motion in a normal working state is considered to be inadmissible.

The quantity $\psi^2/4\pi^2$ usually does not exceed 0.01. For this reason the influence of dissipative forces is essential in a very narrow range of frequencies only that are close to the resonance frequency. In non-resonance states these forces need not be taken into account at all.

Pre-resonance and post-resonance state of machine performance. In most cases, the frequency of the harmonic which predominates in $\tilde{L}(t)$ coincides with the frequency of rotation ν_M of the output shaft of the transmission mechanism which is equal to ω_0/i . Resonance occurs if $\nu_M = k$. If $\nu_M < k$ then the state is called a *pre-resonance* state; taking into account the danger of a resonance state and possible accidental deviations of quantities ν_M and k , one usually assumes that in a pre-resonance state it is necessary that

$$\nu_M \leq \frac{k}{\sqrt{2}}. \quad (10.45)$$

Under this condition we obtain from relationship (10.44) $D(\nu_M) \leq 2$. When working in a pre-resonance state, there still exists the danger of excitation of resonance vibration if frequencies of higher harmonics of the perturbation moment $\tilde{L}(t)$ are equal to $m\nu_M$, where m is an integer. In cycle machines the second harmonic with frequency $2\nu_M$ is especially dangerous. It is usually associated with the perturbation moments related to inertia effects. In high-speed machines the fulfilment of condition (10.45) often proves to be impossible. For fulfilling (10.45) the frequency k and, therefore, the stiffness c must be too large. Practically one succeeds in setting a pre-resonance state with $\nu_M \leq 150 - 200 \text{ s}^{-1}$. Taking this into account, in many cases a post-resonance state of the machine performance is ensured for which

$$\nu_M \geq \sqrt{\frac{3}{2}}k, \quad (10.46)$$

while at the same time the quantity $D(\nu_M)$ does not exceed 2, and monotonically decreases with increasing ν_M . During a post-resonance state the danger of resonance vibration occurs during the starting acceleration of a machine, when the angular velocity ν_M which grows from zero to the working value, necessarily is equal to k at some moment. In order to avoid that the amplitudes of resonance vibration reach dangerous values, one strives to increase the rate of starting acceleration by increasing the engine power.

Nonuniformity of rotation of an actuating mechanism. Differentiating equality (10.38) we find

$$\dot{\theta} = -\sum_{l=1}^{\infty} \frac{L_l l \nu \sin(l\nu t + \beta_l + \gamma_l)}{\sqrt{(c - J_{10} l^2 \nu^2)^2 + (b + \nu)^2 l^2 \nu^2}}. \quad (10.47)$$

Under the fulfilment of conditions (10.40) and (10.41) we have

$$\left| \dot{\theta} \right|_{\max} = \frac{L_0 \omega}{\sqrt{(c - J_{10} \omega^2)^2 + b^2 \omega^2}} = \frac{L_0}{c} \omega D(\omega). \quad (10.48)$$

It is obvious that in resonance the dynamic error in velocity sharply increases. For the coefficient of nonuniformity of shaft rotation we obtain the following expression from equality (10.48) at $\omega = \nu_M$:

$$\delta = \frac{2 \left| \dot{\theta} \right|_{\max}}{\nu_M} = \frac{2L_0}{c} D(\nu_M). \quad (10.49)$$

We note that the increase in J_{10} (e.g., through the mounting of an additional fly-wheel at the output of the transmission mechanism) may lead to an increase in the

coefficient of nonuniformity because approaches the natural frequency k to frequency ν_M .

Dynamic moment in a transmission mechanism. According to formula (10.20) we determine the moment transmitted by a mechanism:

$$M_{tr} = b\ddot{\theta} + c(\Delta + \tilde{\theta}) = c\Delta + c\tilde{\theta} + b\ddot{\theta}. \quad (10.50)$$

The first term $c\Delta = -Q_{R0}(\omega_0) = M_0$ represents the static component of the moment; the dynamic component is

$$\tilde{M}_{tr} = c\tilde{\theta} + b\ddot{\theta}. \quad (10.51)$$

Substituting (10.38) and (10.47) into (10.51), we obtain

$$\tilde{M}_{tr} = \sum_{l=1}^{\infty} \frac{L_l \sqrt{c^2 + b^2 l^2 \nu^2} \cos(l\nu t + \beta_l + \gamma_l + \zeta_l)}{\sqrt{(c - J_{10} l^2 \nu^2)^2 + (b + \nu)^2 l^2 \nu^2}}, \quad (10.52)$$

$$\zeta_l = \arg(c + b l \nu j) .$$

For harmonic perturbation and $\nu = 0$

$$\left| \tilde{M}_{tr} \right|_{\max} = \frac{L_0 \sqrt{1 + \frac{\psi^2}{4\pi^2}}}{\sqrt{\left(1 - \frac{\omega^2}{k^2}\right)^2 + \frac{\psi^2}{4\pi^2}}} \cong L_0 D(\omega), \quad (10.53)$$

since $\psi^2/4\pi^2 \ll 1$. The resonance value of $\left| \tilde{M}_{tr} \right|_{\max}$ exceeds 10–30 times L_0 .

In a post-resonance state the quantity $\left| \tilde{M}_{tr} \right|_{\max}$ decreases when the ratio ω/k increases. This circumstance is used for the protection of a transmission mechanism from dynamic loading. If we series-connect to a transmission mechanism a link with sufficiently small stiffness, provided that $k \ll \nu_M$ is fulfilled, then the dynamic moment proves to be essentially smaller than the perturbation moment. Such an elastic link, ensuring the protection of the transmission mechanism from dynamic loading, is a flexible coupling. The physical explanation of this effect is that in a post-resonance state the perturbation moment acting on the actuating mechanism is partially balanced by the moment J_{10} of inertia forces of a vibrating mass so that the transmission mechanism is unloaded from dynamic loading.

10.4

Influence of the Static Characteristic of an Engine on a Steady-State Motion

Let us turn back to Eqs. (10.22) and (10.23) describing the motion of a machine with an elastic transmission mechanism. Assuming that the moment Q entering Eq. (10.22), is determined by the static engine characteristic (8.18), we arrive at a system of equations for determination of $q_0(t)$ and $q_1(t)$ in a steady-state for $u = u_0 = \text{const}$:

$$J_0 \ddot{q}_0 + b(\dot{q}_0 - \dot{q}_1) + c(q_0 - q_1) - F_s(u_0, \dot{q}_0) = 0, \quad (10.54)$$

$$J_{10} \ddot{q}_1 + b(\dot{q}_1 - \dot{q}_0) + c(q_1 - q_0) - Q_{R0}(\dot{q}_1) = -\tilde{J}_1(q_1) \ddot{q}_1 - \frac{1}{2} \tilde{J}'_1(q_1) \dot{q}_1^2 + \tilde{Q}_R(q_1, \dot{q}_1). \quad (10.55)$$

Relationships (10.26) and (10.27) are taken into account in these equations. On the right-hand side of Eq. (10.55) there are terms explicitly depending on q_1 , which are perturbation actions causing a nonuniformity of rotation of, both, the output and the input shaft of a transmission mechanism. Following the method of successive approximation we equate these terms to zero and seek a solution of the system of equations

$$J_0 \ddot{q}_0^0 + b(\dot{q}_0^0 - \dot{q}_1^0) + c(q_0^0 - q_1^0) - F_s(u_0, \dot{q}_0^0) = 0, \quad (10.56)$$

$$J_{10} \ddot{q}_1^0 + b(\dot{q}_1^0 - \dot{q}_0^0) + c(q_1^0 - q_0^0) - Q_{R0}(\dot{q}_1^0) = 0, \quad (10.57)$$

corresponding to a uniform engine rotation. Substituting into Eqs. (10.56) and (10.57)

$$q_0^0 = \omega_0 t, \quad q_1^0 = \omega_0 t - \Delta, \quad (10.58)$$

we obtain equations for the determination of the average angular velocity ω_0 and of the static deformation Δ of the transmission mechanism:

$$F_s(u_0, \omega_0) = -Q_{R0}(\omega_0) = c\Delta. \quad (10.59)$$

Comparing these results with Eq. (8.36) we find that in this (first) approximation the magnitude of the average angular velocity is the same as that in the system with a rigid transmission mechanism. Eqs. (10.59) for ω_0 can have several solutions (see Fig. 8.4); for the following analysis it is necessary to identify the stable solutions. We will obtain stability conditions after formulating equations for variations of the solutions (10.58). To this end, we set

$$q_0^0 = \omega_0 t + \xi_0, \quad q_1^0 = \omega_0 t - \Delta + \xi_1, \quad (10.60)$$

where ξ_0 and ξ_1 are small variations of the solutions. Substituting (10.60) into (10.56), taking into account (10.59) and preserving terms which are linear with respect to ξ_0 , ξ_1 and to their derivatives, we have:

$$J_0 \ddot{\xi}_0 + b(\dot{\xi}_0 - \dot{\xi}_1) + c(\xi_0 - \xi_1) + s\dot{\xi}_0 = 0, \quad (10.61)$$

$$J_{10} \ddot{\xi}_1 + b(\dot{\xi}_1 - \dot{\xi}_0) + c(\xi_1 - \xi_0) + v\dot{\xi}_1 = 0. \quad (10.62)$$

We write the characteristic equation of this system:

$$D(\lambda) = \begin{vmatrix} J_0 \lambda^2 + (b+s)\lambda + c & -(b\lambda + c) \\ -(b\lambda + c) & J_{10} \lambda^2 + (b+v)\lambda + c \end{vmatrix} = 0. \quad (10.63)$$

Developing this determinant we obtain:

$$J_0 J_{10} \lambda^4 + [(b+v)J_0 + (b+s)J_{10}] \lambda^3 + [(J_0 + J_{10})c + b(s+v) + sv] \lambda^2 + (s+v)c\lambda = 0. \quad (10.64)$$

This equation has a single zero root indicating the possibility of rotation of the whole system as a rigid unit. The stability of solution (10.58) is insured in the case when the remaining roots of Eq. (10.64) are either negative or have negative real parts. To achieve this, first all coefficients of this equation must be positive, which is always fulfilled, if $s > 0$ and $v > 0$. Second, the condition

$$[(b+v)J_0 + (b+s)J_{10}] [(J_0 + J_{10})c + b(s+v) + sv] > J_0 J_{10} (s+v)c$$

must be satisfied, which is also fulfilled, if $s > 0$ and $v > 0$. We recall that in a rigid machine for the stability of a steady-state motion it is sufficient that $(s+v)$ be a positive quantity.

Determination of dynamic errors. We seek the solution of Eqs. (10.54) and (10.55) in first approximation in the form

$$q_0 = \omega_0 t + \psi_0, \quad q_1 = \omega_0 t - \Delta + \psi_1, \quad (10.65)$$

where ψ_0 and ψ_1 are unknown functions of time, and at the same time $\dot{\psi}_0$ and $\dot{\psi}_1$ are small in comparison to ω_0 . In accordance with the method of successive approximations, we substitute into the left-hand sides of Eqs. (10.54) and (10.55) expressions (10.65), and into the right-hand side of (10.55) expressions (10.58). Linearizing the characteristics $F_s(u_0, \dot{q}_0)$ and Q_{R0} in the vicinity of $\dot{q}_0 = \dot{q}_1 = \omega_0$, and taking into account (10.34), we obtain the system of differential equations

$$\begin{aligned} J_0 \ddot{\psi}_0 + (b+s)\dot{\psi}_0 + c\psi_0 - b\dot{\psi}_1 - c\psi_1 &= 0, \\ J_{10} \ddot{\psi}_1 + (b+v)\dot{\psi}_1 + c\psi_1 - b\dot{\psi}_0 - c\psi_0 &= \tilde{L}\left(t - \frac{\Delta}{\omega_0}\right). \end{aligned} \quad (10.66)$$

It is convenient to seek the solution of this system, i. e. the dynamic errors ψ_0 and ψ_1 , in an operator form. Rewriting Eqs. (10.66) in the operator form

$$\begin{aligned} [J_0 p^2 + (b+s)p + c] \psi_0 - (bp+c) \psi_1 &= 0, \\ -(bp+c) \psi_0 + [J_{10} p^2 + (b+v)p + c] \psi_1 &= \tilde{L} \left(t - \frac{\Delta}{\omega_0} \right), \end{aligned} \quad (10.67)$$

we find the transfer functions relating ψ_0 and ψ_1 to $\tilde{L}$:

$$\psi_0 = -w_0(p)\tilde{L} = \frac{bp+c}{D(p)}\tilde{L}, \quad \psi_1 = w_1(p)\tilde{L} = \frac{J_{10}p^2 + (b+s)p + c}{D(p)}\tilde{L}, \quad (10.68)$$

where $D(p)$ is given by expression (10.63) for $\lambda = p$.

Let us determine the amplitude of the dynamic error in the angular velocity of the output link of a transmission mechanism ψ_1 , if

$$\tilde{L} = L_0 \cos \omega t, \quad b = v = 0.$$

From expressions (10.68) and (10.63) we have:

$$\dot{\psi}_1 = p w_1(p)(L_0 \cos \omega t), \quad D(p) = (J_0 p^2 + sp + c)(J_{10} p^2 + c) - c^2.$$

From here

$$|\dot{\psi}_1|_{\max} = \frac{\sqrt{(c - J_0 \omega^2)^2 + s^2 \omega^2}}{\sqrt{[c(J_0 + J_{10}) - J_0 J_{10} \omega^2]^2 \omega^2 + (c - J_{10} \omega^2)^2 s^2}} L_0. \quad (10.69)$$

One can see from this expression that the dynamic error $\dot{\psi}_1$ can take relatively large values in those cases when one of the terms under the square root of the denominator of (10.69) vanishes. This can occur, if

$$\omega^2 = \frac{c(J_0 + J_{10})}{J_0 J_{10}}, \quad (10.70)$$

or if

$$\omega^2 = \frac{c}{J_{10}}. \quad (10.71)$$

In the first case, the frequency ω coincides with the natural frequency of the two-mass system shown in Fig. 10.3b. In the second case, it coincides with the natural frequency k of the system shown in Fig. 10.3a. In both cases resonance may not occur if the second term is sufficiently large. The danger of resonance exists when the frequencies (10.70) and (10.71) turn out to be close in magnitude. This takes place for $J_0 \gg J_{10}$; besides

$$c \frac{J_0 + J_{10}}{J_0 J_{10}} \cong c \frac{J_0}{J_0 J_{10}} = \frac{c}{J_{10}}.$$

In this way, an engine possessing sufficiently large steepness s of the static characteristic is able to decrease the vibration of a machine only if the moment of inertia of the rotor is not too large in comparison with J_{10} .

10.5

Transients in an Elastic Machine

Let us consider a process of starting acceleration in a machine with an elastic transmission mechanism and with an ideal engine. We assume (as in the case of investigation of the starting acceleration of a rigid machine) that the reduced moment of inertia is constant ($\tilde{J}_1(q_1) \cong 0$) and that the moment of resistance forces does not depend on the coordinate q_1 ($\tilde{Q}_R(q_1, \dot{q}_1) \cong 0$). We also assume that the moment $Q_{R0}(\dot{q}_1)$ does not depend on the velocity $\dot{q}_1$. Thus,

$$J_1(\dot{q}_1) = J_{10} = \text{const}, \quad Q_R(q_1, \dot{q}_1) = -M_0 = \text{const}. \quad (10.72)$$

For an ideal engine the motion law $q_0(t)$ of its rotor can be considered as given and as identical with the program motion law. Besides, $\ddot{q}_0(t) = \ddot{q}_p(t)$. Taking also into account that $q_1 = q_0 + \theta$, we represent Eq. (10.23) in the form

$$J_{10}(\ddot{q}_0 + \ddot{\theta}) + b\dot{\theta} + c\theta = -M_0 \quad (10.73)$$

or

$$\ddot{\theta} + 2n\dot{\theta} + k^2\theta = -\ddot{q}_p - \frac{M_0}{J_{10}}, \quad (10.74)$$

where $2n = b/J_{10}$, $k^2 = c/J_{10}$. During the starting acceleration the machine passes from a state of rest $q_0 = 0$, $\dot{q}_0 = 0$ to a steady-state motion for which $\dot{q}_0 = \omega_0$.

Constant starting acceleration. Let us suppose at first that the starting acceleration takes place with a constant angular acceleration of the engine rotor

$$\ddot{q}_p = \varepsilon = \frac{\omega_0}{t_a}, \quad (10.75)$$

where t_a is the time duration of acceleration. The law of variation of $\ddot{q}_p(t)$ is graphically represented in Fig. 10.5a. It can analitically be represented in the form

$$\begin{aligned} \ddot{q}_p(t) &= \varepsilon & \text{for } 0 < t \leq t_a, \\ \ddot{q}_p(t) &= 0 & \text{for } t > t_a. \end{aligned} \quad (10.76)$$

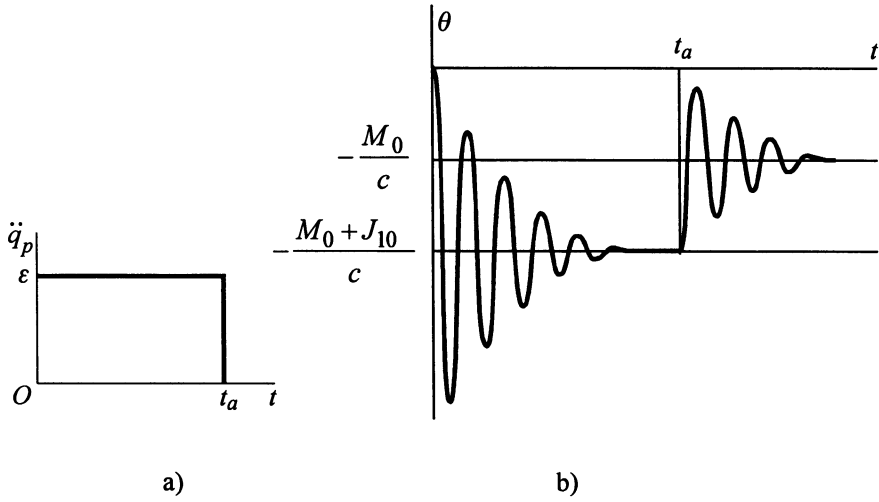


Fig. 10.5. a) Program acceleration and b) deformation of an elastic element for a rectangular variation law of the program acceleration

Using the „unit function“ $\eta(t)$ that is equal to zero for $t < 0$ and to unity for $t \geq 0$, one can bring Eq. (10.74) to the form

$$\ddot{\theta} + 2n\dot{\theta} + k^2\theta = -\left(\frac{M_0}{J_{10}} + \varepsilon\right)\eta(t) + \varepsilon\eta(t - t_a) = -\mu(t). \quad (10.77)$$

To the law of variation of deformation $\theta(t)$ corresponds a particular solution of this equation with initial conditions $t = 0$, $\theta = 0$, $\dot{\theta} = 0$. As is known, this solution can be represented in the form of a Duhamel integral. By integration we obtain

$$\begin{aligned} \theta &= \int_0^t k_1^{-1} e^{-n(t-\tau)} \sin k_1(t-\tau) \mu(\tau) d\tau = \\ &= -\left(\frac{M_0}{J_{10}} + \varepsilon\right) \int_0^t k_1^{-1} e^{-n(t-\tau)} \sin k_1(t-\tau) d\tau + \varepsilon \int_{t_a}^t k_1^{-1} e^{-n(t-\tau)} \sin k_1(t-\tau) d\tau = \\ &= -\left[1 - e^{-nt} \left(\cos k_1 t + \frac{n}{k_1} \sin k_1 t\right)\right] k^{-2} \left(\frac{M_0}{J_{10}} + \varepsilon\right) + \\ &\quad \left\{1 - e^{-n(t-t_a)} \left[\cos k_1(t-t_a) + \frac{n}{k_1} \sin k_1(t-t_a)\right]\right\} k^2 \varepsilon, \end{aligned} \quad (10.78)$$

where $k_1 = \sqrt{k^2 - n^2}$. Usually, the time duration of the starting acceleration t_a exceeds at least 5–10 times the „period“ of the free damped vibration equal to

$2\pi/k_1$. In this case the function $\theta(t)$ has the form shown in Fig. 10.5b. Jumps in the program acceleration at times $t=0$ and $t=t_a$ cause an excitation in the system of damped free vibrations. The amplitudes at these moments reach values $(M_0/J_{10} + \varepsilon)k_1^{-2}$ and εk_1^{-2} respectively. This vibration causes a dynamic loading in the transmission mechanism. The maximal absolute value of the moment is approximately equal to $2(M_0 + J_{10}\varepsilon)$. The damping of free vibration is usually characterized by the logarithmic decrement δ which is the logarithm of the ratio of two successive maxima of $\theta(t)$. One can see from expression (10.78) that this logarithm is equal to $2\pi n/k_1$. Since

$$\delta = \frac{2\pi n}{k_1} = \frac{2\pi n}{\sqrt{k^2 - n^2}} \cong \frac{2\pi n}{k}$$

and

$$\frac{n}{k} = \frac{b}{2J_{10}k} = \frac{\psi}{2\pi} \cdot \frac{c}{2J_{10}k^2} = \frac{\psi}{4\pi},$$

we obtain

$$\delta \cong \frac{2\pi\psi}{4\pi} = \frac{\psi}{2}. \quad (10.79)$$

If the value of ψ is in the range (10.14), then $0.1 \leq \delta \leq 0.3$. This means that the maximal deviation $\theta_{\max}$ decreases by the factor 1.1–1.35 during one period of vibration. For ten periods the amplitude of vibration decreases by the factor 2.6–20.

Free damped vibration caused by perturbation forces is called *accompanying vibration* [11]. In order to suppress the accompanying vibration, one usually tries to avoid jumps in the program acceleration during the starting acceleration. For this purpose smoother laws of variation of the program accelerations are employed.

Starting acceleration of a machine for a program acceleration that changes according to a sine law. Let us suppose now that the program acceleration of the engine rotor $\ddot{q}_p(t)$ changes according to the law

$$\begin{aligned} \ddot{q}_p &= \varepsilon_m \sin \frac{\pi}{t_a} t & \text{for } 0 < t \leq t_a, \\ \ddot{q}_p &= 0 & \text{for } t > t_a. \end{aligned} \quad (10.80)$$

This law of acceleration variation is graphically shown in Fig. 10.6a. Since the relation

$$\int_0^{t_a} \ddot{q}_p(t) dt = \int_0^{t_a} \varepsilon_m \sin \frac{\pi}{t_a} t dt = \omega_0,$$

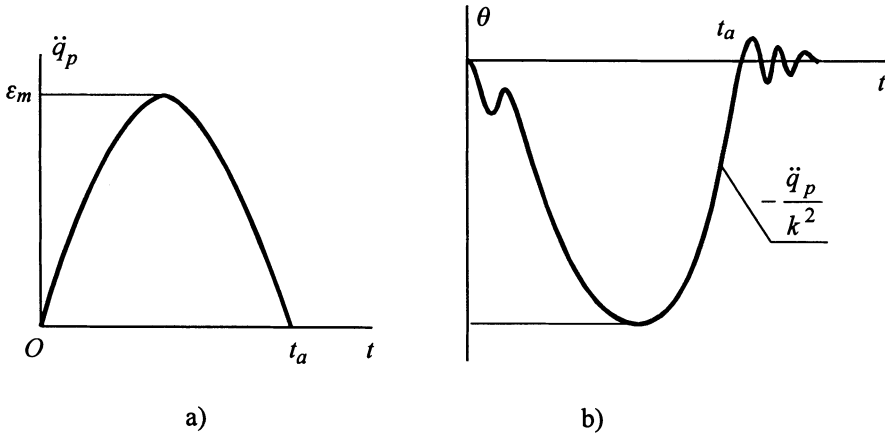


Fig. 10.6. a) Program acceleration and b) deformation of an elastic element for a sinusoidal variation law of the program acceleration

must be fulfilled, we obtain that

$$\varepsilon_m = \frac{\pi\omega_0}{2t_a}. \quad (10.81)$$

Hence, in this case the maximal value of the program acceleration and, therefore, the maximal value of the transport inertial force is $\pi/2$ times larger than in the previous case. Let us also assume that during the starting acceleration $M_0 = 0$. It is convenient to seek the solution of the motion equation

$$\ddot{\theta} + 2n\dot{\theta} + k^2\theta = -\ddot{q}_p(t) \quad (10.82)$$

in the form

$$\theta = -\frac{\ddot{q}_p(t)}{k^2} + \zeta. \quad (10.83)$$

Here the first term corresponds to the „quasistatic“ deformation equal to the ratio of the inertia force $-J_{10}\ddot{q}_p$ to the mechanism stiffness while the second summand reflects the free vibration accompanying the process of starting acceleration. With the acceleration (10.80) we have

$$\theta = -\frac{\varepsilon_m}{k^2} \sin \frac{\pi}{t_p} t + \zeta. \quad (10.84)$$

Substituting (10.84) into (10.82) we obtain an equation for ζ :

$$\ddot{\zeta} + 2n\dot{\zeta} + k^2\zeta = -\frac{q_p^{IV}}{k^2} + 2n\frac{\ddot{q}_p}{k^2} = \frac{\varepsilon_m\pi^2}{k^2 t_a^2} \sin \frac{\pi}{t_a} t + \frac{2n\pi}{k^2 t_a} \varepsilon_m \cos \frac{\pi}{t_a} t. \quad (10.85)$$

This equation must be solved with initial conditions corresponding to $\theta(0) = 0$, $\dot{\theta}(0) = 0$:

$$t = 0, \quad \zeta = \frac{\ddot{q}_p(0)}{k^2} = 0, \quad \dot{\zeta} = \frac{\ddot{q}_p(0)}{k^2} = \frac{\varepsilon_m \pi}{k^2 t_a}. \quad (10.86)$$

Since we usually have $2\pi/k < 0.1t_a$ and $n/k = \psi/4\pi$, the perturbations on the right-hand side of Eq. (10.85) are small:

$$\varepsilon_m \frac{\pi^2}{k^2 t_a^2} < 0.0025 \varepsilon_m, \quad \varepsilon_m \frac{2\pi n}{k^2 t_a} < \frac{0.1\psi}{2\pi} \varepsilon_m.$$

Therefore, the accompanying vibration has a small amplitude, approximately 20–30 times less than that for the acceleration law (10.76). A graph of $\theta(t)$ illustrating this case is shown in Fig. 10.6b.

The considered examples allow to draw some conclusions:

- For a machine with an ideal engine, it is desirable to ensure smooth starting acceleration free of jumps in the program acceleration.
- It is necessary that the time duration of starting acceleration does not exceed significantly the period of free vibration of the one-mass system shown in Fig. 10.3a ($kt_a > (10 - 20)\pi$).

The fulfilment of these conditions guarantees small amplitudes of accompanying vibration.

Braking of a machine. Let us consider the process of braking of a machine with elastic transmission mechanism, supposing that in the process of braking a braking moment M_b directed opposite to the velocity is applied to the engine rotor (with moment of inertia J_0) while a moment $-M_0$ is applied to the actuating mechanism (with a constant reduced moment of inertia J_1). The system of motion equations is written in the form

$$J_0 \ddot{q}_0 = b(\dot{q}_1 - \dot{q}_0) + c(q_1 - q_0) - M_b, \quad (10.87)$$

$$J_1 \ddot{q}_1 = -b(\dot{q}_1 - \dot{q}_0) - c(q_1 - q_0) - M_0, \quad (10.88)$$

After dividing all terms of Eqs. (10.87) and (10.88) by J_0 and J_1 , respectively, we subtract the first equation from the second. Taking into account that $q_1 - q_0 = \theta$, we obtain

$$\ddot{\theta} = -b(J_0^{-1} + J_1^{-1})\dot{\theta} - c(J_0^{-1} + J_1^{-1})\theta - M_0 J_1^{-1} + M_b J_0^{-1}. \quad (10.89)$$

We denote

$$J_* = (J_0^{-1} + J_1^{-1})^{-1} = \frac{J_0 J_1}{J_0 + J_1}, \quad bJ_*^{-1} = 2n_0, \quad cJ_*^{-1} = k_0^2. \quad (10.90)$$

Here, k_0 is the natural frequency of the two-mass system shown in Fig. 10.3b ($J_{10} = J_1$). Substituting (10.90) into (10.89), we bring the motion equations to this form:

$$\ddot{\theta} + 2n_0\dot{\theta} + k_0^2\theta = -M_0J_1^{-1} + M_bJ_0^{-1}. \quad (10.91)$$

At the initial moment the elastic mechanism is deformed by the load $-M_0$ and the velocities of both masses are equal. Therefore, a solution of Eq. (10.91) should be sought for the initial conditions

$$t = 0, \quad \theta = -\frac{M_0}{c}, \quad \dot{\theta} = 0. \quad (10.92)$$

The general solution of Eq. (10.91) is sought in the form

$$\theta = Ae^{-n_0t} \sin(k_1t + \alpha) + \theta_*, \quad (10.93)$$

where $k_1 = \sqrt{k_0^2 - n_0^2}$, θ_* is a particular solution of the equation, A and α are constants determined from the initial conditions. The particular solution θ_* can be shown to have the form of a constant

$$\theta_* = \frac{1}{k_0^2} (M_bJ_0^{-1} - M_0J_1^{-1}). \quad (10.94)$$

Substituting solution (10.93) into the initial conditions (10.92), we have

$$\theta(0) = A \sin \alpha + \theta_* = \frac{M_0}{c}, \quad \dot{\theta}(0) = A(k_1 \cos \alpha - n_0 \sin \alpha) = 0. \quad (10.95)$$

From this we find after simple transformations

$$A = -\frac{M_0 + M_b}{k_0k_1J_1}, \quad \sin \alpha = \frac{k_1}{k_0}. \quad (10.96)$$

Substituting expressions (10.96) and (10.94) into the solution (10.93) we obtain

$$\theta = (k_0)^{-2} (M_bJ_0^{-1} - M_0J_1^{-1}) - (k_0k_1J_1)^{-1} (M_0 + M_b) e^{-n_0t} \sin(k_1t + \alpha). \quad (10.97)$$

If the braking moment is chosen so that

$$M_bJ_0^{-1} = M_0J_1^{-1}, \quad (10.98)$$

then at the end of the braking process, after the damping of the free vibration described by the second term on the right-hand side of expression (10.97), it turns out that the mechanism is in an undeformed state ($\theta = 0$).

In order to determine the time duration of braking, we add together Eqs. (10.87) and (10.88). We obtain

$$J_0\ddot{q}_0 + J_1\ddot{q}_1 = -(M_b + M_0). \quad (10.99)$$

We introduce new generalized coordinate

$$q_* = \frac{J_0 q_0 + J_1 q_1}{J_0 + J_1}. \quad (10.100)$$

Then, Eq. (10.99) is written in the form

$$(J_0 + J_1) \ddot{q}_* = -(M_b + M_0). \quad (10.101)$$

The initial coordinates q_0 and q_1 are expressed through the coordinates q_* and θ in the following way:

$$q_0 = q_* - \frac{J_1}{J_0 + J_1} \theta, \quad q_1 = q_* - \frac{J_0}{J_0 + J_1} \theta. \quad (10.102)$$

Thus, the transition from the initial coordinates q_0 and q_1 to coordinates q_* and θ is reduced to a linear transformation, which has the consequence that Eqs. (10.87) and (10.88) take the form of Eqs. (10.91) and (10.101), where the new coordinates q_* and θ prove to be decoupled. Such coordinates are called *normal modes* of the system. Solving Eq. (10.101) we find

$$q_* = -\frac{M_b + M_0}{J_0 + J_1} t^2 + \dot{q}_*(0)t + q_*(0). \quad (10.103)$$

At the initial moment both masses rotate with the same angular velocity ω_0 ; the value of coordinate q_* can be assumed to be zero. Then,

$$q_*(t) = -\frac{M_b + M_0}{J_0 + J_1} t^2 + \omega_0 t. \quad (10.104)$$

Substituting solutions (10.104) and (10.97) into formulae (10.102), it is possible to obtain the final expressions for q_0 and q_1 . The braking time t_b can be defined as the time it takes for the velocity $\dot{q}_*$ to become zero. From relationships (10.104) we find

$$\dot{q}_*(t_b) = -2 \frac{M_b + M_0}{J_0 + J_1} t_b + \omega_0 = 0,$$

whence

$$t_b = \frac{(J_0 + J_1) \omega_0}{2(M_b + M_0)}. \quad (10.105)$$

If condition (10.98) is fulfilled and if by the time $t = t_b$ the free vibration has damped, then at this moment the system comes to a state of rest with zero deformation, which is the desirable state. The damping time of vibration is usually defined to be the time necessary for decreasing the amplitude by the factor twenty. For a coefficient of absorption $\psi = 0.6$ this leads to the condition

$$t_b > 20 \frac{\pi}{k_0}. \quad (10.106)$$

This indicates that the braking moment M_b should not be chosen too large. Its admissible magnitude obtained from expressions (10.105) and (10.106) depends on the natural frequency k_0 :

$$(M_b)_{\max} \leq \frac{(J_0 + J_1)\omega_0 k_0}{40\pi} - M_0. \quad (10.107)$$

10.6 Problems

10.1. In Fig. 10.7 a diagram of the mechanical system of a machine is represented in which the rotor engine 0 is connected with the rotatory actuating mechanism 3 by a gearing consisting of gear wheels 1 and 2 and of shafts 0–1 and 2–3. The lengths of the shafts are $l_{0-1} = l_{2-3} = 50$ mm, the diameters of the shafts are $d_{0-1} = 8$ mm, $d_{2-3} = 28$ mm, the numbers of teeth of the gears are $Z_1 = 19$, $Z_2 = 65$, the modul of the gearing is $m = 2$ mm, the thickness of the gear wheels is $B = 10$ mm, the moment of inertia of the rotor of the actuating machine is $J_3 = 0.45 \text{ kg m}^2$. Assuming that all elements of the machine's mechanical system are made of steel, determine:

- the stiffnesses of the shafts;
- the stiffness of the gearing, conditioned by the deformation of the wheel teeth;
- the stiffness of the transmission mechanism as a whole (reduced to shaft 0–1);
- the moment of inertia of the mechanism (reduced to the same shaft) not taking into account the moment of inertia of the engine rotor.

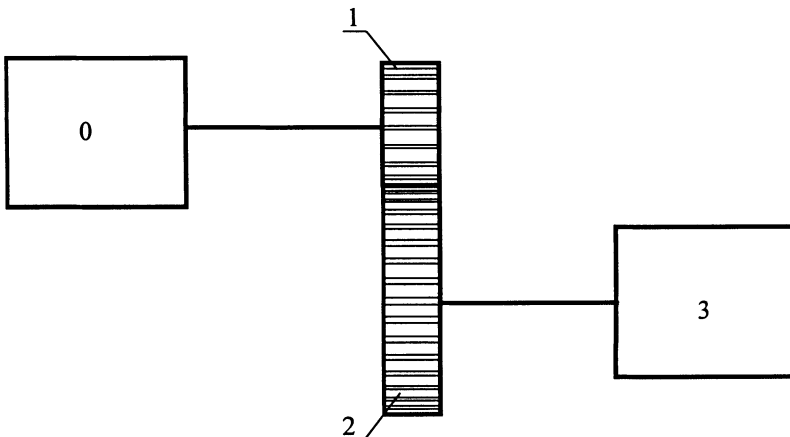


Fig. 10.7

Solution:

1) The transmission ratio of the transmission mechanism is $i = 65/19 = 3.421$, $i^2 = 10.70$.

2) The masses and the moments of inertia of the gear wheels are: $M_j = \pi r_j^2 B \rho$, $J_j = M_j r_j^2 / 2$, where $r_j = m Z_j / 2$ is the radius of the pitch circle of the j -th wheel, $\rho = 7800 \text{ kg/m}^3$ is the density of steel, $j = 1, 2$.

We have $M_1 = 0.0885 \text{ kg}$, $M_2 = 1.035 \text{ kg}$, $J_1 = 1.869 \times 10^{-5} \text{ kg m}^2$, $J_2 = 2.187 \times 10^{-3} \text{ kg m}^2$.

3) The moments of inertia of wheel 2, of rotor 3 and of the actuating mechanism (reduced to shaft 0–1) are:

$$J_2^* = J_2 / i^2 = 1.869 \times 10^{-4} \text{ kg m}^2, \quad J_3^* = J_3 / i^2 = 3.846 \times 10^{-2} \text{ kg m}^2.$$

4) The reduced moments of inertia J_{im} of the transmission mechanism and J of the mechanical system are:

$$J_{im} = J_1 + J_2^* = 2.028 \times 10^{-4} \text{ kg m}^2,$$

$$J = J_{im} + J_3^* = 3.866 \times 10^{-2} \text{ kg m}^2.$$

5) The stiffnesses of the shafts and the stiffness of the gearing are:

$$C_{01} = GJ_{p01} / l_{01}, \quad C_{23} = GJ_{p23} / l_{23}, \quad C_{12} = kBr_{b1}^2,$$

where J_{p01} and J_{p23} are the polar moments of inertia of the shaft cross-sections: $J_p = \pi d^4 / 32$, d is the diameter of a shaft; G is a shear modulus, for steel $G = 8 \times 10^{10} \text{ N/m}^2$; k is a numerical coefficient; for steel wheels $k = 15 \times 10^9 \text{ N/m}^2$; $r_{b1} = r_1 \cos 20^\circ$ is the radius of the base circle of the first wheel. As a result we have $C_{01} = 643.4 \text{ N/m}$, $C_{23} = 96550 \text{ N/m}$, $C_{12} = 47820 \text{ N/m}$.

6) The reduced stiffnesses of the second shaft and the transmission mechanism as a whole are:

$$C_{23}^* = \frac{C_{23}}{i^2} = 8250 \text{ N/m}, \quad C_r = \frac{1}{1/C_{01} + 1/C_{12} + 1/C_{23}^*} = 589.5 \text{ N/m}.$$

- 10.2. In Fig. 10.8 three dynamic models of the machine considered in the foregoing problem are depicted. The first model (Fig. 10.8a) takes into account the deformation of shaft 0–1 only; the second model (Fig. 10.8b) takes into account the deformations of both shafts 0–1 and 2–3; the third model takes into account, both, the elastic deformations of the shafts and the deformation of the teeth of the transmission. Derive the equations of free vibration and determine the natural frequencies and modes for all three models.

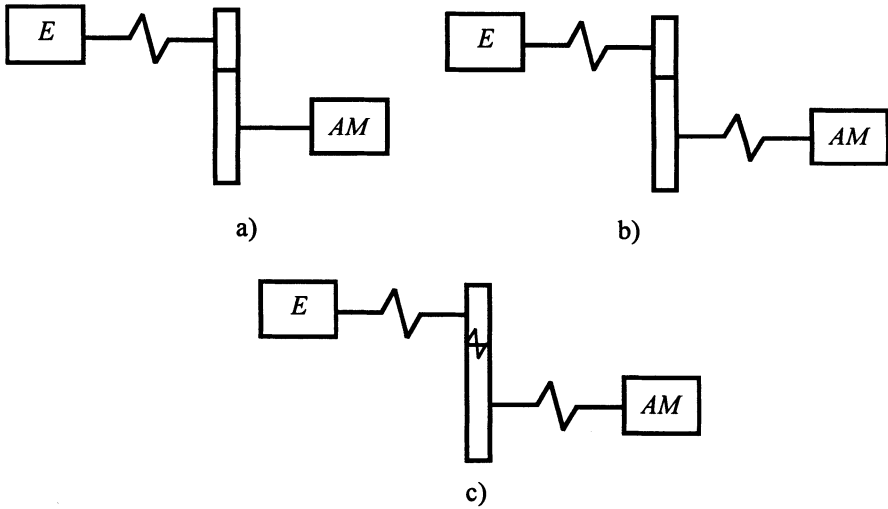


Fig. 10.8

Solution:

1) When considering the first dynamic model we assume that the stiffness of the transmission is equal to the stiffness of shaft 0–1: $C_{0-1} = 643.4 \text{ N m}$ and that the reduced moment of inertia of the mechanism is $J = 3.866 \times 10^{-2} \text{ kg m}^2$.

We introduce variables q_0 (rotation angle of the engine shaft) and φ (rotation angle of the rotor of the actuating mechanism). $q_1 = i\varphi$ is the reduced rotation angle of the actuating mechanism. Then, the deformation of the transmission mechanism is $\theta = q_1 - q_0$ and the equation of the free vibration of the rotor of the actuating mechanism takes the form

$$J\ddot{q}_1 = C_{01}(q_0 - q_1).$$

Assuming that the motion of the rotor is uniform, i.e. $\ddot{q}_0 = 0$, we obtain

$$J\ddot{\theta} + C_{01}\theta = 0,$$

whence follows the natural frequency $k = \sqrt{C_{01}/J} = 129.0 \text{ s}^{-1}$.

Remark: when employing the quantity C_r instead of C_{01} as stiffness of the transmission mechanism, the value of the natural frequency is $k = \sqrt{C_r/J} = 123.5 \text{ s}^{-1}$.

2) For the analysis of the second model we introduce variables q_0 , q_1 and φ which are respectively the rotation angles of the engine, of wheel 1 and of the rotor of the actuating mechanism. Besides, we introduce quantities $\theta_1 = q_1 - q_0$ and $\theta_2 = i\varphi - q_0$. Setting the moment of inertia of the trans-

mission mechanism $J_{tm} = 2.028 \times 10^{-4} \text{ kg m}^2$, the moment of inertia of the rotor of the actuating mechanism $J_3 = 3.846 \times 10^{-2} \text{ kg m}^2$ and the stiffnesses of the shafts $C_{01} = 643.4 \text{ N m}$ and $C_{23} = 8250 \text{ N m}$, we write the equations of free vibration of the mechanism in the form:

$$\begin{aligned} J_{tm}\ddot{\theta}_1 + C_{01}\theta_1 + C_{23}(\theta_1 - \theta_2) &= 0, \\ J_3\ddot{\theta}_2 + C_{23}(\theta_2 - \theta_1) &= 0. \end{aligned}$$

The frequency equation has the form:

$$\begin{vmatrix} C_{01} + C_{23} - J_{tm}k^2 & -C_{23} \\ -C_{23} & C_{23} - J_3k^2 \end{vmatrix} = 0.$$

Developing the determinant we obtain

$$J_{tm}J_3k^4 - [(C_{01} + C_{23})J_3 + C_{23}J_{tm}]k^2 + C_{01}C_{23} = 0$$

and numerically

$$7800 \times 10^{-6} k^4 - 343.71 k^2 + 5.308 \times 10^6 = 0.$$

We obtain $k_1 = 124.3 \text{ s}^{-1}$, $k_2 = 6637 \text{ s}^{-1}$. The natural modes corresponding to these frequencies are:

$$\begin{aligned} h_1^{(1)} &= 1, & h_2^{(1)} &= 1.078, \\ h_1^{(2)} &= 1, & h_2^{(2)} &= -1.82 \times 10^{-4}. \end{aligned}$$

3) For the investigation of the third model we introduce variables q_0 , q_1 , q_2 and φ which are respectively the rotation angles of the engine, gear wheel 1, gear wheel 2 and the rotor of the actuating mechanism. Defining deformations $\theta_1 = q_1 - q_0$, $\theta_2 = q_2 - q_0$ and $\theta_3 = i\varphi - q_0$, we obtain the equations of free vibration in the form

$$\begin{aligned} J_1\ddot{\theta}_1 + C_{01}\theta_1 + C_{12}(\theta_1 - \theta_2) &= 0, \\ J_2\ddot{\theta}_2 + C_{12}(\theta_2 - \theta_1) + C_{23}(\theta_2 - \theta_3) &= 0, \\ J_3\ddot{\theta}_3 + C_{23}(\theta_3 - \theta_2) &= 0. \end{aligned}$$

The frequency equation is:

$$\begin{vmatrix} C_{01} + C_{23} - J_1k^2 & -C_{23} & 0 \\ -C_{12} & C_{12} + C_{23} - J_2k^2 & -C_{23} \\ 0 & -C_{23} & C_{23} - J_3k^2 \end{vmatrix} = 0.$$

Developing the determinant and substituting $C_{01} = 643.4 \text{ N m}$, $C_{12} = 47820 \text{ N m}$, $C_{23} = 8250 \text{ N m}$, $J_1 = 1.597 \times 10^{-5} \text{ kg m}^2$, $J_2 = 1.869 \times 10^{-4} \text{ kg m}^2$ and $J_3 = 3.846 \times 10^{-5} \text{ kg m}^2$, we obtain

$$-1.148 \times 10^{-10} k^6 + 0.3828 k^4 - 1.664 \times 10^7 k^2 + 2.538 \times 10^{11} = 0.$$

The natural frequencies and modes are:

$$\begin{aligned} k_1 &= 123.5 \text{ s}^{-1}, & h_1^{(1)} &= 1, & h_2^{(1)} &= 1.013, & h_3^{(1)} &= 1.091, \\ k_2 &= 6636 \text{ s}^{-1}, & h_1^{(2)} &= 1, & h_2^{(2)} &= 0.999, & h_3^{(2)} &= -0.0049, \\ k_3 &= 5.736 \times 10^4 \text{ s}^{-1}, & h_1^{(3)} &= 1, & h_2^{(3)} &= -0.0855, & h_3^{(3)} &= 5.6 \times 10^{-6}. \end{aligned}$$

- 10.3. A load moment $M = M_0 \sin \omega t$ is applied to the rotor of the actuating machine with an ideal engine, considered in the foregoing problems. Define the steady-state motion law of the rotor of the actuating mechanism, if $M_0 = 10 \text{ N m}$, $\omega = 100 \text{ s}^{-1}$. Compare the results obtained for the three different models suggested in problem 10.2. It is assumed that there is a slight dissipation in the system which does not affect the forced motion but which ensures the damping of the free vibration.

Solution:

- 1) When employing the first dynamic model, the equation of vibration of the rotor of the actuating mechanism has the form

$$J\ddot{\theta} + C_{01}\dot{\theta} = \frac{M_0}{i} \sin \omega t.$$

The forced component of the solution is

$$\theta = A \sin \omega t, \text{ where } A = \frac{M_0}{i} \frac{1}{J(k^2 - \omega^2)} \approx 1.14 \times 10^{-2}.$$

The actual motion law of the actuating mechanism is

$$\varphi(t) = \frac{\omega_0 t + A \sin \omega t}{i},$$

where ω_0 is the angular velocity of the engine rotor.

Remark: when employing the quantity C_r instead of C_{01} as stiffness of the transmission mechanism, and when taking into account the change of the natural frequency, the amplitude of the forced vibration is $A = 1.44 \times 10^{-2}$.

- 2) When employing the second dynamic model, the equations of the vibration of the rotor of the actuating mechanism are

$$J_{im}\ddot{\theta}_1 + C_{01}\theta_1 + C_{23}(\theta_1 - \theta_2) = 0,$$

$$J_3\ddot{\theta}_2 + C_{23}(\theta_2 - \theta_1) = \frac{M_0}{i} \sin \omega t.$$

We seek the forced vibration in the form

$$\theta_j(t) = A_j \sin \omega t \quad (j = 1, 2).$$

We obtain for A_j a system of linear equations

$$(C_{01} + C_{23} - J_{im}\omega^2)A_1 - C_{23}A_2 = 0, \quad -C_{23}A_1 + (C_{23} - J_3\omega^2)A_2 = \frac{M_0}{i}.$$

Solving this system we find $A_2 = 1.39 \times 10^{-2}$, and the motion of the actuating mechanism is described by the relationship

$$\varphi(t) = \frac{\omega_0 t + A_2 \sin \omega t}{i}.$$

3) When employing the third dynamic model, the equations of the vibration of the rotor of the actuating mechanism are represented in the form:

$$J_1\ddot{\theta}_1 + C_{01}\theta_1 + C_{12}(\theta_1 - \theta_2) = 0,$$

$$J_2\ddot{\theta}_2 + C_{12}(\theta_2 - \theta_1) + C_{23}(\theta_2 - \theta_3) = 0,$$

$$J_3\ddot{\theta}_3 + C_{23}(\theta_3 - \theta_2) = \frac{M_0}{i} \sin \omega t.$$

From the system of linear equations

$$(C_{01} + C_{12} - J_1\omega^2)A_1 - C_{12}A_2 = 0,$$

$$-C_{23}A_1 + (C_{12} + C_{23} - J_2\omega^2)A_2 - C_{23}A_3 = 0,$$

$$-C_{23}A_2 + (C_{23} - J_3\omega^2)A_3 = \frac{M_0}{i}$$

we find the amplitude of vibration of the actuating mechanism $A_3 = 1.44 \times 10^{-2}$. The motion law of this mechanism is determined by the relationship $\varphi(t) = (\omega_0 t + A_3 \sin \omega t)/i$.

- 10.4. A machine consists of an engine, an elastic transmission mechanism and a rotatory actuating mechanism. The engine has an ideal characteristic, the stiffness of the transmission mechanism reduced to the engine shaft is $C = 500 \text{ N/m}$, the transmission ratio is $i = 10$, the moment of inertia of the rotor of the actuating mechanism reduced to the engine shaft is $J = 0.2 \text{ kg m}^2$, the coefficient of damping of the system reduced to the engine shaft is $b = 2 \text{ N m s}$. A perturbation moment $L = L_0 \sin \omega t$, $L_0 = 10 \text{ N m}$ is applied to the rotor of the actuating mechanism. Find the

amplitude of the steady vibration of the rotor of the actuating mechanism for $\omega = 30 \text{ s}^{-1}$; find the frequency ω for which this amplitude is maximal and determine this maximum amplitude.

Solution: Let q_0 and φ be the rotation angles of the engine and of the rotor of the actuating mechanism. Then, $\theta = i\varphi - q_0$ is the reduced deformation of the transmission mechanism, and the equation of the vibration of the rotor of the actuating mechanism is written in the form

$$J\ddot{\theta} + b\dot{\theta} + C\theta = \frac{L_0}{i} \sin \omega t.$$

Substituting $\theta = A \sin(\omega t + \beta)$ into the equation we obtain

$$A(C - J\omega^2) \sin(\omega t + \beta) + Ab\omega \cos(\omega t + \beta) = \frac{L_0}{i} \sin[(\omega t + \beta) - \beta].$$

Comparing the coefficients of $\cos(\omega t + \beta)$ and of $\sin(\omega t + \beta)$ we obtain

$$A(C - J\omega^2) = \frac{L_0}{i} \cos \beta, \quad Ab\omega = -\frac{L_0}{i} \sin \beta.$$

From this we get

$$A = \frac{L_0}{iJ\sqrt{(k^2 - \omega^2)^2 + 4n^2\omega^2}}, \quad \beta = \arctan \frac{-2n\omega}{k^2 - \omega^2},$$

where $k^2 = C/J = 2500 \text{ s}^{-1}$, $n = b/2J = 5 \text{ s}^{-1}$. For $\omega = 30 \text{ s}^{-1}$ the amplitude of steady vibration is $A = 3.07 \times 10^{-3}$. This amplitude reaches a maximal value at $\omega = \omega^* = \sqrt{k^2 - 2n^2}$; $\omega^* = 49.497 \text{ s}^{-1}$ and $A_{\max} = 0.01$.

- 10.5. The machine considered in the foregoing problem accelerates from a state of rest. The engine rotor moves with a constant angular acceleration ε . Find the motion law of the rotor of the actuating mechanism.

Solution:

1) Let q_0 and φ be respectively the rotation angles of the rotors of the engine and of the actuating mechanism, $q_1 = i\varphi$ and $\theta = q_1 - q_0$. The motion equation of the rotor of the actuating mechanism is:

$$J\ddot{q}_1 + b(\dot{q}_1 - \dot{q}_0) + C(q_1 - q_0) = 0.$$

Substituting $\ddot{q}_1 = \ddot{q}_0 + \ddot{\theta} = \varepsilon + \ddot{\theta}$, we find

$$J\ddot{\theta} + b\dot{\theta} + C\theta = -J\varepsilon.$$

2) The general motion of this equation is written in the form

$$\theta = Ae^{-nt} \cos(k_1 t - \delta) - \frac{\varepsilon}{k^2}, \quad k = \sqrt{\frac{C}{J}}, \quad n = \frac{b}{2J}, \quad k_1 = \sqrt{k^2 - n^2}.$$

We have $\theta = 0, \dot{\theta} = 0$ for $t = 0$, therefore $A \cos \delta - \varepsilon/k^2 = 0$, $-A(n \cos \delta - k_1 \sin \delta) = 0$. In this way, $\cos \delta = k_1/k$, $\sin \delta = n/k$.

3) Numerical results:

$$k^2 = 2500 \text{ s}^{-2}, \quad n = 5 \text{ s}^{-1}, \quad k_1 = 49.749 \text{ s}^{-1},$$

$$\sin \delta = 0.1, \quad \cos \delta = 0.995, \quad \delta = 0.1002,$$

$$\varphi = \varepsilon \left[\frac{t^2}{20} + 4.02 \times 10^{-5} e^{-5t} \cos(49.749t + 0.1002) - 4.00 \times 10^{-5} \right].$$

- 10.6. The machine shown in Fig. 10.9 consists of an engine, an elastic transmission mechanism and a scotch-yoke actuating mechanism. The system of motion control ensures with high precision a rotation velocity of the engine $\omega_e = 140 \text{ s}^{-1}$. The transmission ratio of the reducer is $i = 10$, the length of the crank is $l_{OA} = r = 0.6 \text{ m}$, the moment of inertia of the crank about its own rotation axis is $J_O = 0.7 \text{ kg m}^2$, the mass of the cross-head is $m_2 = 5 \text{ kg}$, the mass of the slotted link is $m_3 = 25 \text{ kg}$, the mass center of the crank is on the rotation axis. To the slotted link a load along the x -axis is applied: $P_x = P_0 \sin \varphi$, where φ is the rotation angle of the crank, $P_0 = 500 \text{ N}$. The stiffness of the transmission mechanism is $C = 1500 \text{ N m}$. Neglecting the mass of the transmission mechanism and considering that the mechanism links perform planar motions in a horizontal plane, determine the motion law of the crank and the moment at the drive.

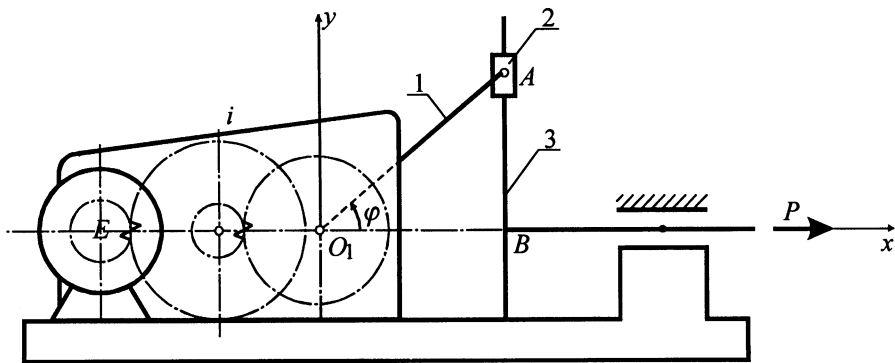


Fig. 10.9

Solution:

1) Let q_0 be the rotation angle of the engine rotor, φ the rotation angle of the crank shaft, $\theta = i\varphi - q_0$ the deformation of the transmission mechanism, $q_1 = i\varphi$. The motion equation of the actuating mechanism is:

$$J_{M0}\ddot{q}_1 + C(q_1 - q_0) - Q_{R0} = \tilde{Q}_R - \frac{1}{2} \frac{dJ}{dq_1} \dot{q}_1^2 - \tilde{J}_M \ddot{q}_1.$$

Here, J_M is the moment of inertia of the scotch-yoke mechanism reduced to the engine shaft, J_{M0} is its average value and $\tilde{J}_M$ is its variable part; Q_R is the moment of resistance forces reduced to the engine shaft, Q_{R0} is its average value and $\tilde{Q}_R$ is its variable part.

2) The kinetic energy and the reduced moment of inertia of the mechanism are

$$\begin{aligned} T &= \frac{1}{2} (J_O \dot{\varphi}^2 + m_2 V_A^2 + m_3 V_B^2), \\ \dot{\varphi} &= \frac{\dot{q}_1}{i}, \quad V_A = \frac{r \dot{q}_1}{i}, \quad V_B = \frac{r \dot{q}_1}{i \sin(q_1/i)}, \\ T &= \frac{1}{2} \frac{1}{i^2} \left(J_O + m_2 r^2 + m_3 r^2 \sin^2 \frac{q_1}{i} \right) \dot{q}_1^2, \\ J_M &= \frac{1}{i^2} \left(J_O + m_2 r^2 + m_3 r^2 \sin^2 \frac{q_1}{i} \right) = \frac{1}{i^2} \left[J_O + m_2 r^2 + \frac{1}{2} \left(1 - \cos \frac{2q_1}{i} \right) \right] = \\ &= 0.07 - 0.045 \cos(0.2q_1) \text{ kg m}^2, \\ \frac{1}{2} \frac{dJ_M}{dq_1} &= \frac{m_3 r^2}{2i^3} \sin \frac{2q_1}{i} = 0.0045 \sin(0.2q_1) \text{ kg m}^2. \end{aligned}$$

3) The reduced moment of resistance forces is:

$$\begin{aligned} Q_R &= P_x \frac{dx_B}{dq_1} = -\frac{P_x}{i} r \sin \varphi = -\frac{P_0}{i} r \sin^2 \varphi = -\frac{P_0}{2i} r (1 - \cos 2\varphi) = \\ &= -15 + 15 \cos(0.2q_1) \text{ N m}. \end{aligned}$$

4) The equation of zero-order approximation (the generating equation) is:

$$J_{M0}\ddot{q}_1 + C(q_1^0 - q_0) - Q_{R0} = 0,$$

here,

$$J_{M0} = \frac{1}{i^2} \left(J_O + m_2 r^2 + \frac{1}{2} m_3 r^2 \right) = 0.07 \text{ kg m}^2,$$

$$Q_{R0} = -\frac{P_0}{2i} r = -15 \text{ N m}.$$

$\Delta = q_1^0 - q_0 = \frac{Q_{R0}}{C} = -0.01$ is the constant component of the deformation of the transmission mechanism.

5) On the left-hand side of the second motion equation we substitute $q_1 = q_1^0 + \theta = q_0 + \Delta + \theta$ and on the right-hand side of this same equation $q_1 = q_1^0$, respectively. We obtain:

$$J_{M0}\ddot{\theta} + C\theta = L,$$

$$L = -\frac{1}{2} \frac{dJ_M}{dq_1} (q_1^0) \omega_0^2 + \tilde{Q}_R = -0.0045 \times 140^2 \sin(28t - 0.002) + 15 \cos(28t - 0.002) = 89.466 \cos(28t + 1.4) \text{ N m}.$$

As a result we obtain the equation of first-order approximation:

$$0.07\ddot{\theta} + 1500\theta = 89.466 \cos(28t + 1.4).$$

Assuming weak dissipation leading to damping of free vibration but not affecting the forced motion, we seek the solution in the form:

$$\theta = A \cos(28t + 1.4).$$

Substituting it into the equation of the first approximation, we obtain

$A = 6.19 \times 10^{-2}$, whence

$$q_1(t) = 140t - 0.01 + 0.0619 \cos(28t + 1.4),$$

$$M_d = C(\Delta + \theta) = -15 + 92.86 \cos(28t + 1.4) \text{ (Nm)}.$$

- 10.7. The machine considered in the foregoing problem is equipped with a direct-current electric motor with separate excitation whose power rating is $N_{er} = 2.1 \text{ kW}$, the rated angular velocity is $\omega_{er} = 140 \text{ s}^{-1}$, the angular velocity of idling is $\omega_{id} = 150 \text{ s}^{-1}$ and the axial moment of inertia of the rotor together with the flywheel is $J_r = 0.43 \text{ kg m}^2$. Supposing that the static characteristic adequately describes the engine, determine the motion law of the crank and the moment at the drive of the machine.

Solution:

1) The equation of the static characteristic of the motor can be written in the form:

$$Q = s(\omega_{id} - \omega).$$

Here, the steepness s of the static characteristic is determined by the relationship

$$s = \frac{N_{er}}{\omega_{er}(\omega_{id} - \omega_{er})} = 1.5 \text{ N m s}.$$

2) The motion equations of the machine together with the static characteristic of the motor are:

$$\begin{aligned} J_r \ddot{q}_0 + C(q_0 - q_1) &= Q, \\ J_{M0} \ddot{q}_1 + C(q_1 - q_0) - Q_{R0} &= \tilde{Q}_R - \frac{1}{2} \frac{dJ}{dq_1} \dot{q}_1^2 - \tilde{J}_M \ddot{q}_1, \\ Q &= s(\omega_{id} - \omega). \end{aligned}$$

The reduced moment of inertia and the reduced moment of resistance forces are the same as in the previous problem.

3) The equations of zero-order approximation are:

$$\begin{aligned} J_r \ddot{q}_0^0 + C(q_0^0 - q_1^0) &= Q^0, \\ J_{M0} \ddot{q}_1^0 + C(q_1^0 - q_0^0) - Q_{R0} &= 0, \\ Q^0 &= s(\omega_{id} - \dot{q}_0^0). \end{aligned}$$

Solving this system, we obtain $q_0^0 = \omega_0 t$, $q_1^0 = \omega_0 t + \Delta$, where

$$\Delta = \frac{Q_{R0}}{C} = -0.01, \quad Q^0 = -C\Delta = 15 \text{ N m}, \quad \omega_0 = (s\omega_{id} - Q^0)/s = 140 \text{ s}^{-1}.$$

4) The perturbation moment is

$$\begin{aligned} L &= -\frac{1}{2} \frac{dJ_M}{dq_1} \omega_0^2 + \tilde{Q}_R = L_0 \cos(28t + \alpha), \\ L_0 &= 89.466, \quad \alpha = 1.4. \end{aligned}$$

5) Substituting $q_0 = q_0^0 + \psi$, $q_1 = q_1^0 + \theta$ into the first equation and into the left-hand side of the second of the motion equations, we obtain the equations of first-order approximation

$$\begin{aligned} J_r \ddot{\psi} + s\dot{\psi} + C(\psi - \theta) &= 0, \\ J_{M0} \ddot{\theta} + s\dot{\theta} + C(\theta - \psi) &= L, \end{aligned}$$

or in an operator form and numerically

$$\begin{aligned} (0.43p^2 + 1.5p + 1500)\psi - 1500\theta &= 0, \\ (0.07p^2 + 1500)\theta - 1500\psi &= L. \end{aligned}$$

To solve this system we compute its determinant

$$D(p) = (0.43p^2 + 1.5p + 1500)(0.07p^2 + 1500) - 1500^2$$

and the transfer functions

$$w_1(p) = \frac{1500}{D(p)}, \quad w_2(p) = \frac{0.43p^2 + 1.5p + 1500}{D(p)}.$$

The steady values of the variables are determined from the conditions:

$$\begin{aligned}\psi(t) &= L_0 |w_1(28j)| \cos(28t + \alpha + \beta), \quad \beta = \arg(w_1(28j)), \\ \theta(t) &= L_0 |w_2(28j)| \cos(28t + \alpha + \gamma), \quad \gamma = \arg(w_2(28j)).\end{aligned}$$

Carrying out the indicated computations, we find

$$\psi(t) = 0.2343 \cos(28t - 1.635), \quad \theta(t) = 0.1818 \cos(28t - 1.599).$$

6) The moment in the drive is determined by the expression

$$M_d = C\Delta + C(\psi - \theta) = M_{d0} + M_{\max} \cos(28t + \nu).$$

Here,

$$\begin{aligned}M_{\max} &= 79.604 \text{ N m}, \quad \nu = 1.382, \\ M_d &= -15 + 79.604 \cos(28t + 1.382) \text{ N m}.\end{aligned}$$

11 Vibration of a Machine on an Elastic Base. Vibration Isolation of Machines

11.1

Vibration of the Body of a Machine Mounted on an Elastic Base

No base on which a machine is being mounted is perfectly rigid. Time-varying forces caused by unbalanced mechanisms which act on the machine body lead to deformations of the base or of elements connecting the machine body with the base. Also, vibrations of the machine body itself occur which may lead to perturbations of the working process; so, e.g., the vibration of the body of a metal-cutting machine tool may deteriorate the accuracy of the blank processing.

Let us consider a one-rotor machine (Fig. 11.1); we suppose that the rotor with mass m_r , rotating with angular velocity $\dot{q} = \omega$, possesses a static unbalance caused by a displacement e of its mass center relative to the rotation axis. In this case, the action of the rotor on the body is reduced to a centrifugal force Φ with components $\Phi_x = m_r \omega^2 e \cos \omega t$ and $\Phi_y = m_r \omega^2 e \sin \omega t$. Let us suppose that the stiffness of the base in x -direction is large enough, so that its deformation in this direction can be ignored, while in y -direction the properties of the elastic base are

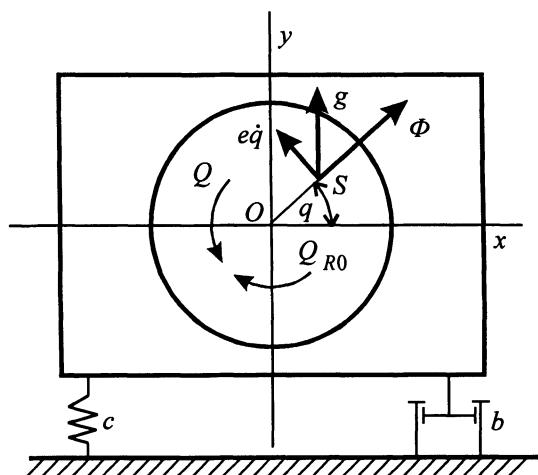


Fig. 11.1. Dynamic model of a one-rotor machine on an elastic base

characterized by a stiffness c and a damping coefficient b . Then, the vibration of the machine, considered as a rigid body with mass m , are described in y -direction by the equation

$$m\ddot{y} = -G - c(y - \Delta_{st}) - b\dot{y} + m_r\omega^2 e \sin \omega t, \quad (11.1)$$

where y is the displacement of the body measured from the position of static equilibrium; $G = mg$ is the weight of the machine; Δ_{st} is the static deformation of the elastic base. Since $G = c\Delta_{st}$, Eq. (11.1) is brought to the form

$$m\ddot{y} + b\dot{y} + cy = m_r\omega^2 e \sin \omega t. \quad (11.2)$$

A particular solution of this equation, which can be sought in the form

$$y = a \sin(\omega t + \varphi), \quad (11.3)$$

corresponds to the steady-state vibration of the body. Substituting solution (11.3) into Eq. (11.2), we obtain

$$\begin{aligned} (c - m\omega^2)a \sin(\omega t + \varphi) + ba\omega \cos(\omega t + \varphi) &= m_r\omega^2 e \sin[(\omega t + \varphi) - \varphi] = \\ &= m_r\omega^2 e [\sin(\omega t + \varphi)\cos\varphi - \cos(\omega t + \varphi)\sin\varphi]. \end{aligned}$$

Equating the coefficients of $\sin(\omega t + \varphi)$ and of $\cos(\omega t + \varphi)$ on the left- and in the right-hand side of this equation, we find

$$a(c - m\omega^2) = m_r\omega^2 e \cos\varphi, \quad (11.4)$$

$$ba\omega = -m_r\omega^2 e \sin\varphi. \quad (11.5)$$

From here we obtain the following expressions for the amplitude a of vibration and for the phase shift φ :

$$a = \frac{m_r\omega^2 e}{\sqrt{(c - m\omega^2)^2 + b^2\omega^2}}, \quad \tan\varphi = -\frac{b\omega}{c - m\omega^2}. \quad (11.6)$$

In Fig. 11.2 graphs are constructed of the amplitude a of vibration as a function of the frequency of the rotor angular velocity ω corresponding to different values of the damping coefficient b . When ω approaches the natural frequency $k = \sqrt{c/m}$ of the system the amplitude of vibration sharply increases and a resonance vibration occurs which, as a rule, is inadmissible. The graphs show that the most preferable working state of the machine is a pre-resonance state with $\omega \ll k$. Usually, a state is considered as pre-resonance if $\omega < k/2$; besides, $a < m_r e / 3m$. In order to ensure the work of the machine in a pre-resonance state the stiffness c of the base must be large enough. From the condition $\omega < k/2$, we obtain

$$c = mk^2 > 4m\omega^2.$$

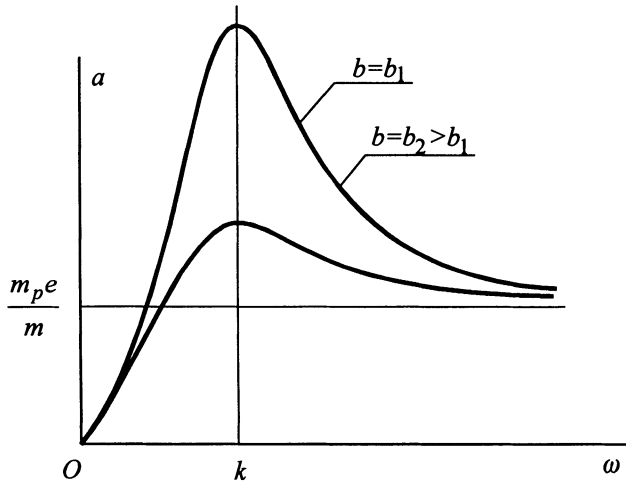


Fig. 11.2. The amplitude of vibration as a function of the rotor angular velocity

For the static deformation caused by the weight this yields

$$\Delta_{st} = \frac{mg}{c} < \frac{g}{4\omega^2}.$$

Usually, fulfilment of this condition for high-speed machines is technically impossible. Actually, for $\omega = 500 \text{ s}^{-1}$ we obtain $\Delta_{st} < 10^{-2} \text{ mm}$. As a rule, such a stiff connection of the machine with the base cannot be materialized. That is why a post-resonance state with $\omega > 2k$ is usually acceptable for rotors with angular velocities exceeding $200\text{--}300 \text{ s}^{-1}$. For a machine to work in a post-resonance state, it is necessary that the stiffness of the base satisfies the condition

$$c < \frac{m\omega^2}{4}. \quad (11.7)$$

Fulfilment of this condition is ensured by the introduction of special deformable elements set between the body and the base. Such elements called *bushings* have other purposes as well which will be discussed below.

For $\omega > 2k$ the amplitude of body vibration is in the range $1.33 m_r e m^{-1} > a > m_r e m^{-1}$. Making it smaller can be done either by reducing the static rotor unbalance, characterized by the quantity $m_r e$, or by increasing the total mass of the machine. Usually, the latter is achieved by attaching a massive foundation to the machine body.

We note that in pre-resonance and in post-resonance states the influence of the dissipative forces (coefficient b) on the amplitude of vibration is, as a rule, unessential and negligible. For the determination of the amplitude we use the formula

$$a \cong \frac{m_r \omega^2 e}{|c - m \omega^2|}, \quad (11.8)$$

obtained from relationship (11.4) for $b = 0$.

11.2

Vibration of a Machine in the Resonance Zone. Sommerfeld Effect

If a machine is in a post-resonance state, it must go through the resonance zone during the starting acceleration and during run-out. This can cause a sharp increase of the amplitude of vibration.

When investigating resonance vibration of a machine, one can no longer neglect dissipative forces acting in the supports. Another possibly essential phenomenon is the influence of body vibrations, in combination with non-ideal engine characteristics, on the rotor angular velocity. In order to investigate this influence let us consider a machine with an unbalanced rotor as a system with two degrees of freedom. As generalized coordinates we choose the rotation angle q of the rotor and the displacement y of the body (Fig. 11.1), measured from the static equilibrium position. Let us determine the kinetic energy of the system. We find the kinetic energy of the body as kinetic energy of a translational motion

$$T_1 = \frac{1}{2}(m - m_r)\dot{y}^2. \quad (11.9)$$

The kinetic energy of the rotor is given by the formula of König

$$T_2 = \frac{1}{2}m_r v_s^2 + \frac{1}{2}J_r^s \dot{q}^2, \quad (11.10)$$

where v_s is the velocity of the rotor mass center composed by the velocity $\dot{y}$ of the translational displacement and the relative velocity $e\dot{q}$; J_r^s is the moment of inertia about the axis passing through the mass center. Projecting the transport and the relative velocities onto the coordinate axes, we find

$$\begin{aligned} v_s^2 &= v_{sx}^2 + v_{sy}^2 = (e\dot{q} \sin q)^2 + (\dot{y} + e\dot{q} \cos q)^2 = \\ &= e^2 \dot{q}^2 + \dot{y}^2 + 2e\dot{q}\dot{y} \cos q. \end{aligned} \quad (11.11)$$

Substituting (11.11) into (11.10) and adding together T_2 and T_1 , we finally obtain

$$T = \frac{1}{2}(m\dot{y}^2 + J_r^s \dot{q}^2 + 2m_r e\dot{q}\dot{y} \cos q), \quad (11.12)$$

where $J_r^0 = J_r^s + m_r e^2$ is the moment of inertia of the rotor about the rotation axis. The generalized force Q_y corresponding to the coordinate y is composed of the elastic force $-cy$ and of the dissipative force $-b\dot{y}$, while the generalized force Q_q is composed of the driving moment Q and the moment $Q_{R0}(\dot{q})$ of resistance forces. In this way we have

$$Q_y = -b\dot{y} - cy, \quad Q_q = Q + Q_{R0}(\dot{q}). \quad (11.13)$$

Let us compose the motion equations of the system in the form of Lagrange equations of the second kind. From (11.12) we obtain

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} &= m\ddot{y} + m_r e (\ddot{q} \cos q - \dot{q}^2 \sin q), \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} &= J_r^0 \ddot{q} + m_r e \dot{y} \cos q. \end{aligned}$$

Taking into account (11.13) and using the static characteristic of the engine $Q = F_s(u_0, \dot{q})$, we obtain the motion equations in the following form

$$m\ddot{y} + b\dot{y} + cy = m_r e (-\ddot{q} \cos q + \dot{q}^2 \sin q), \quad (11.14)$$

$$J_r^0 \ddot{q} - F_s(u_0, \dot{q}) - Q_{R0}(\dot{q}) = -m_r e \dot{y} \cos q. \quad (11.15)$$

We will solve the system of Eqs. (11.14) and (11.15) by the method of successive approximations based on the assumption of the proximity of the rotor motion law to a uniform rotation. Moreover, the right-hand side of Eq. (11.15) reflecting the influence of the body vibration on the rotor revolution can, in the first-order approximation assumed to be zero. Then, Eq. (11.15) takes form (8.35) and the angular velocity $\dot{q} = \omega$ can be found from Eq. (8.36). Substituting $\dot{q} = \omega$ into the right-hand side of Eq. (11.14) we arrived at an equation identical with Eq. (11.2), from which we can obtain in first-order approximation the variation law of $y(t)$ in the form (11.3). Substituting this into the right-hand side of Eq. (11.15) we have

$$\begin{aligned} J_r^0 \ddot{q} - F_s(u_0, \dot{q}) - Q_{R0}(\dot{q}) &= m_r e \omega^2 a \sin(\omega t + \varphi) \cos \omega t = \\ &= \frac{1}{2} m_r e \omega^2 a [\sin \varphi + \sin(2\omega t + \varphi)], \end{aligned} \quad (11.16)$$

where we find a and φ according to formulae (11.6).

From this equation an even more accurate variation law of the rotor angular velocity $\dot{q}(t)$ should be found. It is obvious that the average value of this velocity will differ from ω , since a constant moment

$$M_v = \frac{1}{2} m_r e \omega^2 a \sin \varphi \quad (11.17)$$

has appeared in the right-hand side of Eq. (11.16). This moment is referred to as *vibrational* moment. In order to clarify the physical meaning of this moment let us

find the work of the dissipative forces acting in the deformable elements of the elastic supports for one period of body vibration. We have

$$A = \int_0^{\frac{2\pi}{\omega}} b\dot{y}\dot{y} dt = \int_0^{\frac{2\pi}{\omega}} ba^2\omega^2 \cos^2(\omega t + \varphi) dt = \pi a^2 b \omega. \quad (11.18)$$

Taking into account equality (11.5), we obtain

$$A = -\pi a m_r e \omega^2 \sin \varphi. \quad (11.19)$$

Comparing (11.17) and (11.19), we obtain

$$M_v = -\frac{A}{2\pi}. \quad (11.20)$$

Thus, the vibrational moment is the ratio of the work of dissipative forces in the rotor supports and the rotation angle of the rotor for one period. This clarifies the physical meaning of the vibrational moment. It reflects the additional load on the engine caused by the engine energy consumption necessary to overcome the dissipative forces in the supports during vibration. Using expressions (11.18) and (11.20), we obtain

$$M_v = -\frac{a^2 b \omega}{2} = -\frac{bm_r^2 e^2 \omega^5}{(c - m\omega^2)^2 + b^2 \omega^2} = M_v(\omega). \quad (11.21)$$

The average rotor angular velocity ω should now be determined from Eq. (11.16) retaining in its right-hand side the vibrational moment and setting $\ddot{q} = 0$:

$$-F_s(u_0, \dot{q}) - Q_{R0}(\dot{q}_{av}) = M_v(\dot{q}_{av}).$$

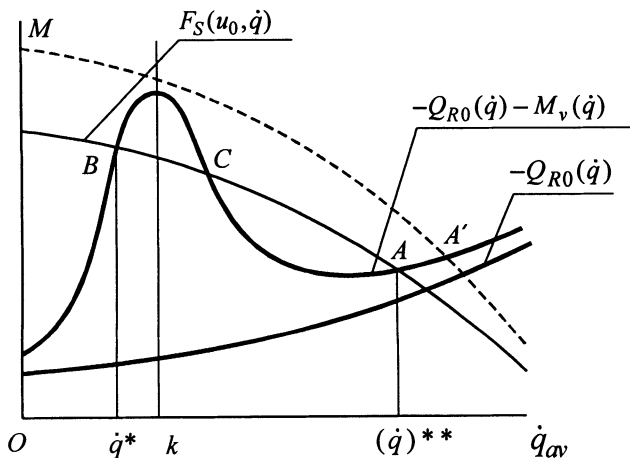


Fig. 11.3. Sommerfeld effect

In Fig. 11.3 the graphical solution of this equation is shown. In the resonance zone the characteristic of the total moment $M_v + Q_{R0}$ of resistance forces makes a sharp jump caused by the large amplitudes of body vibration and – as a consequence – large values of the vibrational moment. As a result the graphs of the moments $F_s(u_0, \dot{q}_{av})$ and $M_v + Q_{R0}$ can have intersection points with abscissae close to the resonance frequency (points B and C in Fig. 11.3). The state of rotor revolution with average angular velocity $\dot{q}^*$ is stable. This means that having reached the corresponding angular velocity during the starting acceleration, a machine may „stick“ to this velocity. In this case, the starting acceleration ceases because the engine is unable to overcome the resonance zone and to reach a steady-state motion corresponding to the angular velocity $\dot{q}^{**}$. This phenomenon is called *Sommerfeld effect* (after the name of the scholar who first described it). In order to exclude the possibility of this phenomenon to occur one must increase the engine power. The engine characteristic shown in Fig. 11.3 with dashed line ensures that the machine reaches the steady-state motion corresponding to point A' . Let us note that the Sommerfeld effect can also arise in a machine with an elastic transmission mechanism when the latter passes the zone of elastic resonance during the starting acceleration.

11.3 Vibration Isolation of Machines

By mounting a machine on elastic-damping vibration isolators we can solve another important problem, as well, namely, the reduction of the dynamic action of a machine on the base, i.e. the reduction of its external vibration activity.

We will consider the problem of vibration isolation in a more general form: We assume that a harmonic force $F = F_0 \sin \omega t$ (Fig. 11.4) is applied to a machine without specifying the nature of this force (it can be either a result of an active external effect or an inertia force). As a result, the body will perform harmonic vibration in accordance with equality (11.3); the amplitude a and the initial phase φ of this vibration are determined by formula (11.6) for $m_r \omega^2 e = F_0$. The force R transmitted to the base is determined from the relationship

$$R = b\dot{y} + cy = ba\omega \cos(\omega t + \varphi) + ca \sin(\omega t + \varphi) = \frac{\sqrt{c^2 + b^2 \omega^2} F_0}{\sqrt{(c - m\omega^2)^2 + b^2 \omega^2}} \sin(\omega t + \varphi + \eta), \quad (11.22)$$

where $\tan \eta = b\omega / c$.

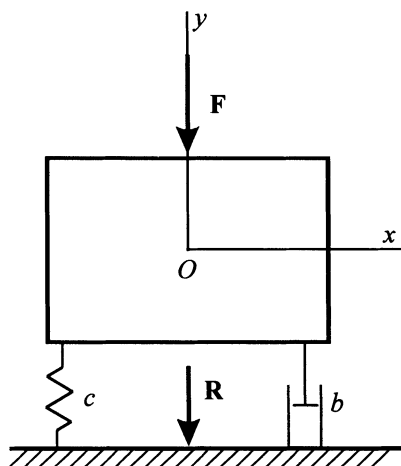


Fig. 11.4. Mounting of a machine on elastic-damping vibration isolator

Let us take into consideration the coefficient of vibration isolation

$$K = \frac{R_{\max}}{F_0} = \sqrt{\frac{c^2 + b^2 \omega^2}{(c - m\omega^2)^2 + b^2 \omega^2}}. \quad (11.23)$$

Provided we have a rigid connection of the machine body with the base, the maximal force transmitted to the base equals F_0 . Thus, the

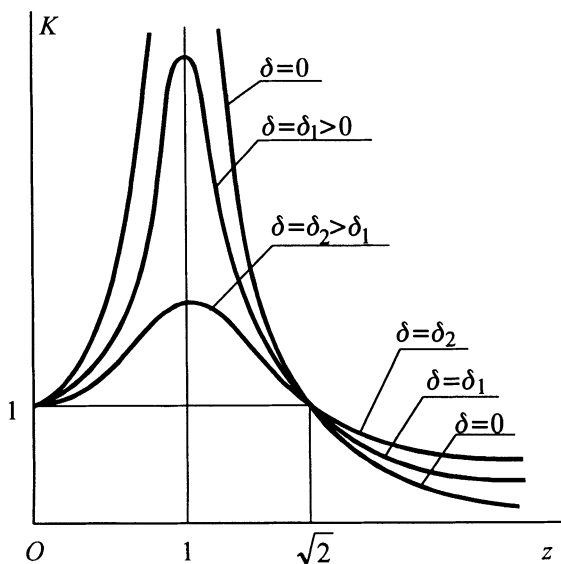


Fig. 11.5. The coefficient of vibration isolation as a function of the dimensionless damping coefficient

relationship (11.23) characterizes the degree of reduction of the maximal force, achieved by the application of a vibration isolator. Fig. 11.5 shows graphs of the coefficient of vibration isolation as function of the ratio $z = \omega/k$ ($k = \sqrt{c/m}$) for different values of the dimensionless damping coefficient $\delta = b/(2mk)$. The domain $z > \sqrt{2}$ is the domain of efficient vibration isolation ($K < 1$). In this domain an increase in the magnitude of δ has the effect of deteriorating the efficiency. For this reason vibration isolators usually have small damping coefficients, so that in the domain of efficiency the influence of dissipative forces can be neglected which allows the assumption

$$K = \frac{c}{|c - m\omega^2|} = \frac{1}{|1 - z^2|}. \quad (11.24)$$

11.4 Problems

- 11.1. A rotor machine with a rotor frequency $n = 1200$ rpm is elastically mounted on the base in such a way that horizontal displacements and rotations of the body can be considered unessential. The static deformation of the support caused by the weight of the machine is 3 mm. Determine whether the machine works in a post- or in a pre-resonance state.

Solution: The magnitude of static deformation is given by the expression $\Delta_{st} = mg/C$, where mg is the weight of the machine and C is the stiffness of the supports in vertical direction; the squared of the natural frequency of vibration of the machine as a rigid unit on the base is $k^2 = C/m = g/\Delta_{st} = 9.81/0.003 = 3270 \text{ s}^{-2}$. Hence, $k = 57.18 \text{ s}^{-1}$ and the frequency of perturbation is $\omega = n\pi/30 = 125.6 \text{ s}^{-1}$. Since $\omega > k\sqrt{2}$, the working state of the machine is a post-resonance state.

- 11.2. A device with mass $m = 2$ kg is mounted on a movable base with the help of shock-absorbers with stiffness $C = 10000$ N/m and with a damping coefficient $b = 4$ kg/s. The base is undergoing harmonic vibration in vertical direction with amplitude $A = 0.01$ m and frequency $\omega = 140 \text{ s}^{-1}$. Determine the amplitude of the acceleration of the device in a steady-state motion.

Solution:

- 1) Let x be the displacement of the mass center and y the displacement of the base. Then, the motion equation of the device is

$$m\ddot{x} + b(\dot{x} - \dot{y}) + C(x - y) = -mg,$$

where $y = A \cos \omega t$. After simple transformations we obtain

$$\ddot{x} + 2n\dot{x} + k^2x = F \cos(\omega t + \varphi) - g,$$

where

$$k^2 = \frac{C}{m} = 5000 \text{ s}^{-2}, \quad n = \frac{b}{2m} = 1 \text{ s}^{-1}, \quad F = A\sqrt{k^4 + 4n^2\omega^2} = 50.078 \text{ m s}^{-2}.$$

2) We seek a steady solution of this equation in the form

$$x = X \cos(\omega t + \varphi - \beta) - \Delta_{st},$$

where $\Delta_{st} = mg / C = g / k^2 = 1.96 \text{ mm}$ is the static deformation of the shock-absorber.

$$X = \frac{F}{\sqrt{(k^2 - \omega^2)^2 + 4n^2\omega^2}} = 3.429 \text{ mm}$$

is the vibration amplitude of the device and $\omega^2 X = 67.2 \text{ m s}^{-2}$ is the acceleration amplitude of the device.

12 Elements of Dynamics of Machines with Program Control

12.1

Basic Principles of Construction of Machines with Program Control

In modern machine building two methods are applied for designing program motions of working organs of a machine necessary for the fulfillment of specified working processes. The first method which is widely used for single-engine cycle machines consists in feeding the engine input with an input signal ensuring a nearly uniform rotation at the output of a rotor engine or a periodically reciprocating motion (in a linear engine). The transformation of these simplest motions into desired program motions of working organs is carried out by actuating mechanisms (lever-, cam-, gear- and other ones) which have, as a rule, nonlinear position functions. One might say that *programming mechanisms* are applied in such machines.

In the second method program motions of working organs are produced by the choice of suitable variation laws of engine input parameters $u_k(t)$ which together represent the *program control*. Here, mechanisms with either linear or nonlinear position functions can be employed for the transmission of motion from the engine to the working organs. In recent years program control is applied more and more for technological as well as for transport machines. It is often carried out by a human operator; cars, airplanes, cranes are controlled in such a way. At the same time, automatic machines with program control, particularly industrial robots, are used more and more.

Program control is especially useful in those cases when, depending on certain conditions, the working organs of a machine must produce several different program motions (e.g. to move a load along different paths). The transition from one program motion to another is then reduced to a change of program control which is carried out rather easily in modern control systems. When applying the first method, a change of program motion is associated with a change either of the structure or of parameters of the programming mechanism which at least requires new adjustment of the mechanism or replacement of certain links of the mechanism. The advantages of the principle of program control ensure broad application in machines used in flexible automated production, in which a fast reorganizing of the system for every new working process is necessary.

At the same time, there are some intrinsic disadvantages for machines with program control. In Fig. 12.1 diagrams of machine assemblies are depicted, in which working organs carry out reciprocating motions. In the diagram shown in

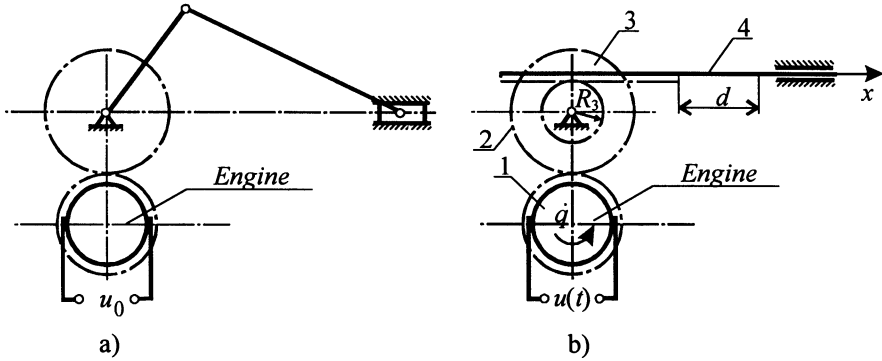


Fig. 12.1. Methods for derivation of a reciprocating motion of the working organ of a machine

Fig. 12.1a the engine rotor rotates with constant angular velocity; the conversion of this rotation into a reciprocating motion of the working organ (slider) is carried out by a programming slider-crank mechanism. In the diagram shown in Fig. 12.1b, the input of a direct-current electric motor with separate excitation is fed by the alternating input voltage $u(t)$ producing a forward-backward rotational motion of the rotor, which by means of the gearing and the rack-and-pinion mechanism is transformed into a reciprocating motion of the output link – the gear rack; in this way, this diagram uses the principle of program control.

Comparing these diagrams, the advantage of the system with program control is obvious in the case, when, e.g., the length of the stroke of the working organ must be changed. In the first diagram this stroke is equal to twice length of the crank and in order to change the stroke, it is necessary to change this length, which requires a new adjustment of the mechanism. Let us consider how this change of stroke is achieved in the second diagram. We assume that for the engine the ideal characteristic (8.6) can be chosen and that the input voltage changes according to the harmonic law

$$u = u_0 \cos \omega t. \quad (12.1)$$

Substituting (12.1) into characteristic (8.6), we obtain

$$\dot{q} = \frac{r}{s} u = \frac{r}{s} u_0 \cos \omega t.$$

Integrating this expression under the assumption that for $t = 0$, $q = 0$, we find

$$q = \frac{r}{s\omega} u_0 \sin \omega t. \quad (12.2)$$

From here, it is easy to determine the motion law of the output link:

$$x = \frac{q}{i_{12}} R_3 = \frac{r u_0 R_3}{s \omega i_{12}} \sin \omega t, \quad (12.3)$$

where i_{12} is the transmission ratio of the gearing and R_3 is the radius of the initial circle of gear wheel 3. From expression (12.3) one can see that the stroke of the rack is

$$d = 2 \frac{ru_0 R_3}{s\omega i_{12}}. \quad (12.4)$$

For changing it, it is sufficient to change the voltage u_0 .

On the other hand, it is obvious that in the first diagram the deviation of the stroke from the nominal value depends only on how accurately the crank length is manufactured and that it does not depend on any other parameters of the system. In the second diagram, the magnitude of the stroke depends on many parameters, in part mechanical (R_3, i_{12}) and in part associated with the engine characteristic (r, s, u_0). For this reason, ensuring the accuracy of the output displacement is more difficult in this case.

Comparing both systems we also notice that in the first case only the working organ of the machine is executing a reversing motion, while in the second case all links of the mechanical system including the engine rotor do. It is natural that those dynamic loads which primarily depend on inertia forces will be more significant in a system with program control. This demonstrates the great importance of dynamic factors and, therefore, the increasing importance of dynamics analysis in the design of machines with program control.

12.2

Design of Program Control. Sources of Dynamic Errors

One of the main problems in the design of machines with program control is the design of the program control ensuring the fulfilment of a given program motion. Here, in contrast to machines with programming mechanisms the solution of the kinematic problem – the problem of obtaining a required motion law – is closely interlaced with the problem of dynamic analysis. In order to illustrate how to solve the problem of determining a program motion, we consider as example the system with the diagram depicted in Fig. 12.1b.

We formulate the motion equations of this system in the form of Lagrange's equations of the second kind. If J_e, J_1, J_2, J_3 are respectively the moments of inertia of the engine rotor and of the gear wheels 1, 2, 3, and if m is the mass of rack 4, then the kinetic energy of the mechanism is represented in the form

$$T = \frac{1}{2} \left[(J_e + J_1) \dot{q}^2 + (J_2 + J_3) i_{12}^{-2} \dot{q}^2 + m R_3^2 i_{12}^{-2} \dot{q}^2 \right] = \frac{1}{2} J_r \dot{q}^2, \quad (12.5)$$

where $J_r = J_e + J_1 + (J_2 + J_3) i_{12}^{-2} + m R_3^2 i_{12}^{-2}$ is the reduced moment of inertia and $\dot{q}$ is the angular velocity of the engine rotor. Supposing that the resistance

forces are negligibly small, we obtain the motion equation of the mechanical system in the form

$$J_r \ddot{q} = Q, \quad (12.6)$$

where Q is the driving moment. Due to the fact that in systems with program control large time-varying inertia forces occur which cause significant variations in the driving moment, one must employ the dynamic engine characteristic. Supposing that a direct-current motor with separate excitation is used, we assume this characteristic to be of the following form (see (8.4)):

$$\tau \dot{Q} + Q = ru - s\dot{q}, \quad (12.7)$$

where τ , r , s are parameters of the electric motor (see Sect. 8.1).

Let $x_p(t)$ be a given program motion law of the output link. It is easy to determine the program law for the variation of $q(t)$ from the kinematic relationships:

$$q_p = \frac{x_p \dot{i}_{12}}{R_3}. \quad (12.8)$$

Substituting q_p into Eq. (12.6) we define the variation law of the driving moment $Q(t)$ during the program motion:

$$Q_p = J_p \ddot{q}_p(t). \quad (12.9)$$

Next, we determine the program control u_p from Eq. (12.7):

$$u_p = r^{-1}(\tau \dot{Q}_p + Q_p + s\dot{q}_p) = r^{-1}(\tau J_r \ddot{q}_p + J_r \ddot{q}_p + s\dot{q}_p). \quad (12.10)$$

Introducing the mechanical time constant $\tau_M = J_r/s$, we bring the expression (12.10) into the form

$$u_p = sr^{-1}(\tau \tau_M \ddot{q}_p + \tau_M \ddot{q}_p + s\dot{q}_p). \quad (12.11)$$

It might seem that the problem of determination of the program control is solved: By feeding the engine input with the voltage u_p given by the relationship (12.11), we should obtain the required program motion law. Actually, however, a number of circumstances occur which cause significant deviations of the actual motion law from the program law, and in a number of cases this makes it impossible to realize the program motion.

The Problem of realizability of a program motion. Systems with program control often solve the problem of displacing of a working organ from one position to another according to a given motion law. Let us suppose that the rack in Fig. 12.1b must be displaced by distance d with an acceleration varying according to law whose graph is shown in Fig. 12.2. Here, t_p is the time duration for the program displacement. The rack must pass the first half of the way with a constant acceleration w_0 and the second half with a constant deceleration $-w_0$. If d is

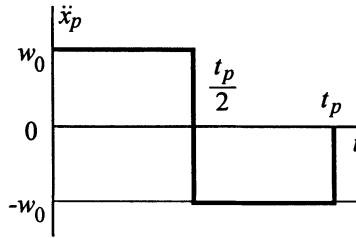


Fig. 12.2. The simplest variation law of the program acceleration

the magnitude of the required displacement and the initial velocity is equal to zero, then from the condition of acceleration uniformity we have:

$$d = w_0 t_p^2 / 4, \quad w_0 = 4d / t_p^2. \quad (12.12)$$

However, it is impossible to realize such a motion. In fact, at the initial moment the acceleration must change from 0 to w_0 with a jump. For this, the engine angular acceleration must change with a jump, i.e. at this moment the quantity $\ddot{q}_p$ must take an „infinitely large value“. Then, the voltage $u(t)$ must also be infinitely large, which, of course, is impossible. Earlier (see Sect. 10.4) it was shown that during the realization of motion laws with acceleration jumps, vibrations are excited in the mechanical system as a result of deformations of elastic links. We will now show that the realization of jumps is, generally speaking, impossible, because of the inertia of the motor electromagnetic system. Let us now suppose that a periodic reciprocating motion of a working organ must be realized according to the law

$$x_p = a \sin \omega t, \quad (12.13)$$

where a and ω are the given amplitude and frequency. Substituting equality (12.13) into expression (12.8), we find:

$$q_p = \frac{a i_{12}}{R_3} \sin \omega t. \quad (12.14)$$

Substituting (12.14) into (12.11), we find the program control:

$$u_p = s i_{12} (r R_3)^{-1} a \omega \left[(1 - \tau \tau_M \omega^2) \cos \omega t - \tau_M \omega \sin \omega t \right]. \quad (12.15)$$

In this way the input voltage must have an amplitude

$$(u_p)_{\max} = s i_{12} (r R_3)^{-1} a \omega \sqrt{(1 - \tau \tau_M \omega^2)^2 + \tau_M^2 \omega^2}. \quad (12.16)$$

For a given value of a the amplitude $(u_p)_{\max}$ increases with increasing of ω ; for large values of ω it becomes approximately proportional to ω^3 . Since the am-

plitudes $u(t)$ are bounded, difficulties arise in the real system, when trying to realize high-frequency vibration of working organs of a machine with the help of a system with program control. For this reason, usually only comparatively low-frequency program motions are realized in such systems.

Influence of initial conditions. Substituting Q from motion Eq. (12.6) into the dynamic engine characteristic (12.7), we obtain the motion equation of the engine rotor in the form

$$\tau J_r \ddot{q} + J_r \dot{q} + s \dot{q} = ru(t) \quad (12.17)$$

or after division by s

$$\tau \tau_M \ddot{q} + \tau_M \dot{q} + \dot{q} = rs^{-1}u(t). \quad (12.18)$$

The program motion $q_p(t)$ is a particular solution of this equation for $u(t) = u_p(t)$ for specified initial conditions. The general solution of the linear nonhomogeneous Eq. (12.18) for $\dot{q}(t)$ is written in the form

$$\dot{q} = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \dot{q}_p(t), \quad (12.19)$$

where C_1 and C_2 are constants determined from the initial conditions; λ_1 and λ_2 are the roots of the characteristic equation

$$\tau \tau_M \lambda^2 + \tau_M \lambda + 1 = 0,$$

whence follows

$$\lambda_{1,2} = \frac{-\tau_M \pm \sqrt{\tau_M^2 - 4\tau \tau_M}}{2\tau \tau_M}. \quad (12.20)$$

It is easy to verify that the roots of Eq. (12.20) are always real and negative (for $\tau_M > 4\tau$) or complex-valued with a negative real part (for $\tau_M < 4\tau$). It follows that the first two terms in the expression (12.19) tend to zero and, therefore,

$$\dot{q} \rightarrow \dot{q}_p \text{ when } t \rightarrow \infty.$$

This means that the program motion in the system is not established at once but only after finishing a transient. For the initial conditions $t = 0$, $\dot{q} = 0$, $\ddot{q} = 0$, i.e. for the motion of the system from a state of rest, we obtain from (12.17)

$$\dot{q}(0) = C_1 + C_2 + \dot{q}_p(0), \quad \ddot{q}(0) = C_1 \lambda_1 + C_2 \lambda_2 + \ddot{q}_p(0) = 0.$$

In the case of program motion (12.14) we obtain

$$C_1 + C_2 = -\dot{q}_p(0) = -ai_{12}R_3^{-1}\omega, \quad C_1 \lambda_1 + C_2 \lambda_2 = -\ddot{q}_p = 0.$$

From these equations we find

$$C_1 = \frac{\lambda_2}{\lambda_1 - \lambda_2} a i_{12} R_3^{-1} \omega, \quad C_2 = \frac{\lambda_1}{\lambda_2 - \lambda_1} a i_{12} R_3^{-1} \omega. \quad (12.21)$$

Therefore, the velocity of the working organ $\dot{x}(t)$ will change according to the law

$$\dot{x}(t) = \left(\frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t} \right) a \omega + \dot{x}_p(t). \quad (12.22)$$

The motion of the working organ will correspond to the program motion only after the damping of the transient described by the first term on the right-hand side of expression (12.22).

Inadequacy of the system dynamic model. When determining the program control, we started from the system's dynamic model described by Eqs. (12.6) and (12.7). In reality, these equations always correspond to the real system only approximately. They do not take into account the elasticity of the real links of a mechanical system nor deviations of the true values of parameters τ, r, s, J_p from their nominal values etc. All this leads to deviations of real motions from the nominal motions, i. e. to dynamic errors. Let us suppose that in the above considered example as a dynamic model of the engine is chosen its ideal characteristic

$$\dot{q} = r s^{-1} u. \quad (12.23)$$

Let us estimate which dynamic errors will be induced by such a simplification of the dynamic model. In accordance with the characteristic (12.23), we substitute into the right-hand side of motion equation (12.18) the following expression

$$u(t) = u_p(t) = s r^{-1} \dot{q}_p(t).$$

As a result we obtain

$$\tau \tau_M \ddot{q} + \tau_M \dot{q} + \dot{q} = \dot{q}_p. \quad (12.24)$$

The solution of this equation $\dot{q}(t)$ will determine the „real“ variation law of the rotor angular velocity (if we consider the dynamic engine characteristic to be the real one), while $\dot{\psi} = \dot{q}(t) - \dot{q}_p(t)$ will determine the dynamic velocity error. Replacing in (12.24) $\dot{q}$ by $\dot{q}_p + \dot{\psi}$, we obtain an equation for the dynamic error

$$\tau \tau_M \ddot{\psi} + \tau_M \dot{\psi} + \dot{\psi} = -\tau \tau_M \ddot{q}_p - \tau_M \dot{q}_p. \quad (12.25)$$

The general solution of this equation is the sum of the general solution of the homogeneous equation

$$\dot{\psi}^0(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \quad (12.26)$$

where λ_1 and λ_2 are the roots of the characteristic equation (12.20), and of the particular solution corresponding to the terminal dynamic error, which is established in the system after the end of transient (12.26). In order to determine the terminal error, we write Eq. (12.25) in the operator form

$$(\tau\tau_M p^2 + \tau_M p + 1)\dot{\psi} = (-\tau\tau_M p^2 - \tau_M p)\dot{q}_p. \quad (12.27)$$

From here we obtain the transfer function $w(p)$ relating the dynamic angular velocity error to the program angular velocity:

$$w(p) = -\frac{\tau\tau_M p^2 + \tau_M p}{\tau\tau_M p^2 + \tau_M p + 1},$$

so that in operator form

$$\dot{\psi} = w(p)\dot{q}_p. \quad (12.28)$$

Provided we have a program motion according to the harmonic law (12.4), it is possible to use for the determination of the dynamic error the frequency characteristic

$$w(j\omega) = -\frac{\tau\tau_M (j\omega)^2 + \tau_M (j\omega)}{\tau\tau_M (j\omega)^2 + \tau_M (j\omega) + 1}. \quad (12.29)$$

In accordance with the known operator relationships, we have

$$\dot{\psi} = |w(j\omega)|a\omega i_{12}R_3^{-1} \cos[\omega t + \arg(w(j\omega))], \quad (12.30)$$

where the modulus of the frequency characteristic is given by the formula

$$|w(j\omega)| = \frac{\sqrt{1 + \tau^2 \omega^2} \tau_M \omega}{\sqrt{(1 - \tau\tau_M \omega^2)^2 + \tau_M^2 \omega^2}}. \quad (12.31)$$

Thus, the dynamic error (12.30) turns out to be a harmonic process and the ratio of its amplitude to the amplitude of the program velocity for a given frequency ω is determined by the magnitude of the amplitude-frequency characteristic (12.31).

It is easy to notice that the disregard of dynamic engine properties, which is associated with the use of the ideal engine characteristic, may lead to very large dynamic errors. So, e.g., for $\tau = 0.03$ s, $\tau_M = 0.1$ s⁻¹, $\omega = 10$ s⁻¹ (these values are typical for real systems with program control) we obtain $|w(j\omega)| = 0.85$, i.e. the amplitude of the error is almost equal to the amplitude of the program velocity.

12.3

Closed Feedback Control Systems

Feedback is used for the purpose of increasing the accuracy of systems with program control. The structural diagram of a feedback control system is shown in Fig. 12.3. Here, measuring devices (sensors) measuring the rotation angle $q(t)$ and the angular velocity $\dot{q}(t)$ of the rotor and comparing them with the program

values are installed at the engine output (the rotor shaft). Differences $\psi = q(t) - q_p(t)$ and $\dot{\psi} = \dot{q}(t) - \dot{q}_p(t)$ are errors in rotor coordinate and rotor angular velocity. Signals ψ and $\dot{\psi}$ are fed to the input of the feedback control system (FCS), which is a controller – a device forming a signal Δu which is added to the signal u_0 of the program control that feeds the engine input. The control law relating the feedback signal Δu to errors ψ and $\dot{\psi}$ is usually chosen in the form

$$\Delta u = -\kappa\psi - \kappa_1\dot{\psi}, \quad (12.32)$$

where κ and κ_1 are positive coefficients called *coordinate gain* and *velocity gain*, respectively. From formula (12.32) it can be seen that the sign of the correcting signal Δu is opposite to the signs of the errors. This has the consequence that for positive errors, i.e. for $q > q_p$ and $\dot{q} > \dot{q}_p$, the correcting signal diminishes the magnitude of the input parameter and consequently diminishes the engine velocity, and hence the magnitude of the errors, as well. In this way, the formation of the control law according to formula (12.32) is aiming, generally speaking, to the diminution of dynamic errors and, therefore, to an increase of the accuracy of the system response to the program motion. A feedback built according to such a principle is called *negative*. A system provided with a feedback connecting its output and input is called *closed*.

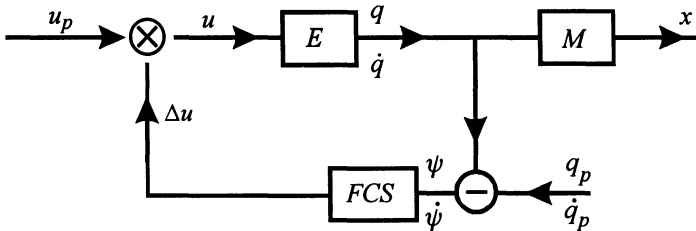


Fig. 12.3. Structural diagram of a feedback control system

The closed system depicted in Fig. 12.3 remains efficient even in the case when the program signal u_0 does not feed the input. In this case, the signal at the engine input is formed as a reaction of the FCS to the mismatch between the motion law $q(t)$, measured at the engine output, and the program motion law q_p , introduced at the input of the feedback. In principle, in the absence of errors ($\psi = 0$, $\dot{\psi} = 0$) the engine is motionless but this will immediately lead to the appearance of a negative error that causes a positive signal at the engine input. A system that is built according to such a principle is called *tracking*.

The system shown in Fig. 12.3 measures the error at the engine output and, therefore, it does not react to errors arising in the mechanical system. In modern machines systems are applied which directly measure the motion law $x(t)$ of the

working organ and compare it with $x_p(t)$. At the same time, the signal Δu is formed in accordance with errors $\psi_1(t) = x(t) - x_p(t)$ and $\dot{\psi}_1(t) = \dot{x}(t) - \dot{x}_p(t)$. Such feedback systems will not be considered here.

12.4

Effectiveness and Stability of a Closed System

Let us return to the consideration of the system depicted in Fig. 12.1b; we suppose that in this system, the motion of which is described by Eq. (12.18), feedback (12.32) is introduced. Substituting into Eq. (12.18)

$$u(t) = u_p(t) + \Delta u = u_p(t) - \kappa\psi - \kappa_1\dot{\psi}, \quad q = q_p + \psi$$

and assuming that $u_p = rs^{-1}\dot{q}_p$, we have

$$\tau\tau_M(\ddot{q}_p + \ddot{\psi}) + \tau_M(\ddot{q}_p + \ddot{\psi}) + \dot{q}_p + \dot{\psi} = rs^{-1}(u_p - \kappa\psi - \kappa_1\dot{\psi}). \quad (12.33)$$

After elementary transformations we obtain the following equation for the dynamic error:

$$\tau\tau_M\ddot{\psi} + \tau_M\ddot{\psi} + (1 + rs^{-1}\kappa_1)\dot{\psi} + rs^{-1}\kappa\psi = -\tau\tau_M\ddot{q}_p - \tau_M\dot{q}_p. \quad (12.34)$$

Let us determine the terminal error for the program motion (12.14). The transfer function of system (12.34) relating the error $\dot{\psi}$ to the program angular velocity $\dot{q}_p$ is

$$w_1 = -\frac{\tau_M p^2 (\tau p + 1)}{\tau\tau_M p^3 + \tau_M p^2 + (1 + rs^{-1}\kappa_1)p + rs^{-1}\kappa}. \quad (12.35)$$

Employing the frequency characteristic $w_1(j\omega)$, we obtain an expression for the amplitude of the harmonic error in velocity:

$$(\dot{\psi})_{\max} = \alpha_0 |w_1(j\omega)| = \frac{\tau_M \omega^2 \sqrt{1 + \tau^2 \omega^2}}{\sqrt{(1 + rs^{-1}\kappa_1 - \tau\tau_M \omega^2)^2 \omega^2 + (rs^{-1}\kappa - \tau_M \omega^2)^2}} \alpha_0, \quad (12.36)$$

where $\alpha_0 = ai_{12}\omega R_3^{-1}$. In the absence of feedback, i.e. for $\kappa = 0, \kappa_1 = 0$ the magnitude of the amplitude of the dynamic velocity error is given by the expression

$$(\dot{\psi})_{\max}^0 = \frac{\tau_M \omega \sqrt{1 + \tau^2 \omega^2}}{\sqrt{(1 - \tau\tau_M \omega^2)^2 + \tau_M^2 \omega^2}} \alpha_0. \quad (12.37)$$

The effectiveness of introducing feedback can be characterized by the ratio between the amplitudes of errors in the closed and in the open systems. Dividing (12.36) by (12.37), we obtain

$$K_e = \frac{(\dot{\psi})_{\max}}{(\dot{\psi})_{\max}^0} = \frac{\omega \sqrt{(1 - \tau \tau_M \omega^2)^2 + \tau_M^2 \omega^2}}{\sqrt{(1 + rs^{-1} \kappa_1 - \tau \tau_M \omega^2)^2 \omega^2 + (rs^{-1} \kappa - \tau_M \omega^2)^2}}. \quad (12.38)$$

The smaller the coefficient of effectiveness K_e , the more effective is the introduction of feedback. It is easy to see that the first turns under the square roots in the numerator and in the denominator in (12.38) satisfy the inequality

$$|1 + rs^{-1} \kappa_1 - \tau \tau_M \omega^2| > |1 - \tau \tau_M \omega^2| \quad (12.39)$$

for arbitrary κ_1 , if $\tau \tau_M \omega^2 < 1$; if $\tau \tau_M \omega^2 > 1$, then the inequality (12.39) is satisfied for

$$\kappa_1 > 2sr^{-1}(\tau \tau_M \omega^2 - 1). \quad (12.40)$$

The second terms under the square roots satisfy the inequality

$$|rks^{-1} - \tau_M \omega^2| > \tau_M \omega^2, \quad (12.41)$$

if

$$\kappa > 2sr^{-1} \tau_M \omega^2. \quad (12.42)$$

It follows that both inequalities (12.39) and (12.41) are fulfilled for sufficiently large values of κ and κ_1 satisfying conditions (12.42) and (12.40), and that at the same time the absolute value of the numerator in (12.38) will certainly be smaller than the absolute value of the denominator, i.e. the condition for the effectiveness of the control ($K_e < 1$) will be fulfilled. Moreover, with a further increase in the gain coefficients, the quantity K_e will monotonically decrease tending to zero; at the same time the amplitude of the dynamic velocity error will tend to zero.

Stability conditions of a closed system. One should think that by increasing the gain coefficient of the feedback, it is possible to ensure arbitrarily high accuracy of execution of the program motion. In reality the possibilities of improving the accuracy are restricted for a number of reasons, the most essential one being the necessity of ensuring the stability of the closed system. For the investigation of the stability of the considered system let us return again to Eq. (12.34). For the asymptotic stability of the system it is necessary and sufficient that all roots of the characteristic equation

$$\tau \tau_M \lambda^3 + \tau_M \lambda^2 + (1 + rs^{-1} \kappa_1) \lambda + rs^{-1} \kappa = 0 \quad (12.43)$$

have negative real parts. That is why all coefficients of this equation must be positive which is fulfilled if $\kappa > 0$, $\kappa_1 > -sr^{-1}$, and if, in addition, for the third-order

equation (12.43) the condition of Hurwitz must be fulfilled which in the present case takes the form

$$\tau \tau_M r s^{-1} \kappa < \tau_M (1 + r s^{-1} \kappa_1). \quad (12.44)$$

Denoting $\kappa r s^{-1} = \alpha$, $\kappa_1 r s^{-1} = \beta$, we bring the condition (12.44) to the form

$$\tau \alpha < 1 + \beta.$$

Fig. 12.4 represents the areas of stability of the system in the plane of the parameters α , β corresponding to different values of τ . The larger the engine time constant τ , the smaller is the domain of admissible values α , β , and, hence, the smaller gain coefficients κ and κ_1 .

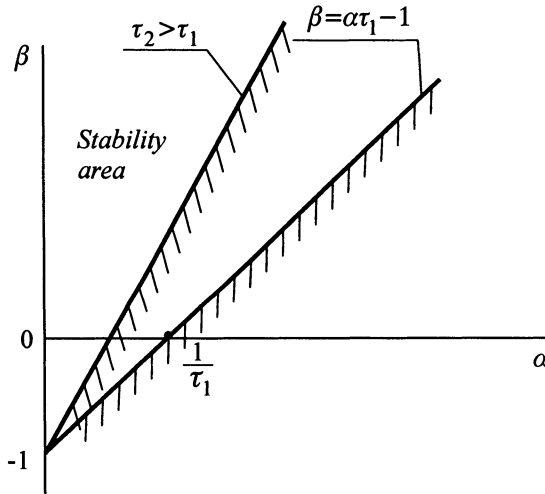


Fig. 12.4. Area of stability of the closed system

Thus, an increase in the gain coefficients of a feedback system can lead to instability of the closed system. A negative feedback, which by the principle of action should induce a diminution of the dynamic error, in reality, turns out to be the cause of its unlimited increase. Without recourse to a detailed description of all processes occurring in a closed system, let us only point out that instability is, in essence, caused by the engine inertia, a characteristic of which is the engine time constant τ . This inertia leads to a phase shift of the engine vibration moment with respect to that vibration component of the transient which must be damped by the moment. As a result, the engine moment provided by the feedback signal becomes an excitation moment instead of a damping moment. The larger the value of τ , the stronger this circumstance manifests itself.

It must be noted that several other elements of the control system possess inertia, as well. E.g., the signal at the controller output Δu is related to the dynamic

error ψ through a more complex function than the one described by expression (12.32). In first approximation the dynamic processes carried out in a controller are described by an equation of the kind

$$\tau_c \dot{\Delta u} + \Delta u = -\kappa \psi - \kappa_1 \dot{\psi}, \quad (12.45)$$

where τ_c is the controller time constant. Usually the „delay“ at the controller is small ($\tau_c \ll \tau, \tau_M$), so that it can be neglected for small gain coefficients. However, with the increase in κ and κ_1 the influence of the small time constant τ_c on the stability of the system becomes essential.

In general, the larger the gain coefficients of a feedback loop are, the more precise the dynamic model of the system must be. In particular, this concerns the analysis of the elasticity of the links of a mechanical system. This analysis appears necessary in systems for motion control of precision machines, in which the program motions must be carried out with high accuracy.

12.5 Problems

- 12.1. A vertical column rotating about its axis and with a gripper attached to it is considered as automatic operator. The moment of inertia with respect to the rotation axis of the column including the gripper and the payload is $J_c = 1.5 \text{ kg m}^2$. The system of program control must ensure the rotation of the column from the initial position $\varphi(0) = 0$ to the end position $\varphi(T) = \varphi_T = \pi$ for the positioning time $T = 1 \text{ s}$. The required variation law of the column angular acceleration is

$$\ddot{\varphi}_p = \varphi_T \frac{2\pi}{T^2} \sin \Omega t, \quad \Omega = \frac{2\pi}{T}.$$

The column is set in motion by a direct-current electric motor with separate excitation whose dynamic characteristic

$$\tau \dot{Q} + Q = ru - s\omega$$

has the following parameters:

$$\tau = 0.01 \text{ s}, \quad r = 0.232 \text{ N m / V}, \quad s = 0.486 \text{ N m s}.$$

Here, u is the control voltage and ω is the angular velocity of the motor shaft.

The moment of inertia of the motor rotor together with the transmission mechanism is $J_r = 0.01 \text{ kg m}^2$; the transmission ratio is $i = 50$, so that $\omega = i\dot{\varphi}$.

Formulate the program variation law of the control voltage and find the dynamic errors during its execution.

Solution:

1) The program control, as a rule, is defined on the basis of the ideal engine characteristic, i.e.

$$u_p = \frac{s}{r} \omega_p, \quad \omega_p = i \int_0^t \ddot{\varphi}_p(t) dt = i \frac{\varphi_T}{T} (1 - \cos \Omega t).$$

Hence,

$$u_p = \frac{si}{r} \frac{\varphi_T}{T} (1 - \cos \Omega t).$$

2) The motion equation of the column is solved together with the equation of the engine dynamic characteristic

$$J\dot{\omega} = Q, \quad \tau\dot{Q} + Q = ru_p - s\omega.$$

Here, $J = J_p + J_c / i^2 = 0.0106 \text{ kg m}^2$. After the usual transformations we obtain

$$\tau\tau_M\ddot{\omega} + \tau_M\dot{\omega} + \omega = i \frac{\varphi_T}{T} (1 - \cos \Omega t),$$

where $\tau_M = J / s = 0.0218 \text{ s}$.

The general solution of this equation is

$$\omega = De^{-nt} \cos(k_1 t + \beta) + \tilde{\omega}.$$

Here, D and β are arbitrary constants and $\tilde{\omega}$ is a particular solution,

$$n = 1/2\tau = 50 \text{ s}^{-1}, \quad k = \sqrt{1/\tau\tau_M} = 67.7 \text{ s}^{-1}, \quad k_1 = \sqrt{k^2 - n^2} = 45.7 \text{ s}^{-1}.$$

We note that the first term is rapidly damping: Within 0.1 s the amplitude of vibrations diminishes by the factor 148.

We seek the second term in the form $\tilde{\omega} = A + B \cos(\Omega t - \delta)$. Substituting this expression into the equation and solving the resulting system we find

$$A = i \frac{\varphi_T}{T}, \quad B = \frac{k^2 i \varphi_T / T}{\sqrt{(k^2 - \Omega^2)^2 + 4n^2 \Omega^2}} = 0.9991 \frac{\varphi_T}{T},$$

$$\sin \delta = \frac{2n\Omega}{\sqrt{(k^2 - \Omega^2)^2 + 4n^2 \Omega^2}} = 0.137,$$

$$\cos \delta = \frac{k^2 - \Omega^2}{\sqrt{(k^2 - \Omega^2)^2 + 4n^2 \Omega^2}} = 0.991.$$

Thus, the angular velocity of the column after the transient is

$$\omega = -\frac{\varphi_T}{T} [1 - 0.9991 \cos(\Omega t - \delta)]$$

and the dynamic velocity error is

$$\dot{\psi} = i \frac{\varphi_T}{T} [\cos \Omega t - 0.9991 \cos(\Omega t - \delta)] = i \frac{\varphi_T}{T} (0.009 \cos \Omega t - 0.137 \sin \Omega t).$$

The dynamic error of the angle is determined by the expression

$$\psi = i \frac{\varphi_T}{T\Omega} [\sin \Omega t - 0.9991 \sin(\Omega t - \delta)] = i \frac{\varphi_T}{T\Omega} (0.009 \sin \Omega t + 0.137 \cos \Omega t).$$

- 12.2. The column of the machine described in the foregoing problem is accelerated; the required program acceleration is $\ddot{\phi}_p = \varepsilon = 0.1 \text{ s}^{-2}$. Find the dynamic position error, applying a program control and employing a control system with position feedback with a gain coefficient $\kappa = 50 \text{ V}$ in the feedback loop. Find the maximal admissible value of this coefficient.

Solution:

1) We define the program control on the basis of the engine ideal characteristic:

$$u_p = \frac{s}{r} \dot{q}_p = i \frac{s}{r} \dot{\phi}_p = i \frac{s}{r} \varepsilon t,$$

where, the parameters of the engine dynamic characteristic are those of the previous problem: $s = 0.486 \text{ N m s}$, $r = 0.232 \text{ N m / V}$, $\tau = 0.01 \text{ s}$, while the mechanical time constant is $\tau_M = 0.0218 \text{ s}$.

The motion equation of the controlled system is:

$$\tau \tau_M \ddot{q} + \tau_M \dot{q} + q = \frac{r}{s} u.$$

2) In the case of program control we have $u = u_p$

$$\tau \tau_M \ddot{q} + \tau_M \dot{q} + q = i \varepsilon t.$$

We determine only the forced motion since the free motion vanishes rapidly: $\dot{q} = i \varepsilon (t - \tau_M)$. Moreover, the dynamic velocity error is $\dot{\psi}^0 = -i \varepsilon \tau_M = -0.109 \text{ s}^{-1}$, while the dynamic angle error is $\psi^0 = -i \varepsilon \tau_M t$.

3) In the case of a feedback control system we have $u = u_p + \kappa(q_p - q)$ with $q_p = i \varepsilon t^2 / 2$. We obtain

$$\tau\tau_M\ddot{q} + \tau_M\ddot{q} + \dot{q} = i\varepsilon t + \frac{r}{s}\kappa i\varepsilon t^2/2 - \frac{r}{s}\kappa q.$$

Assuming again that the free motion is rapidly damping, we determine only the forced component of the solution $q = i\varepsilon t^2/2 - s\tau_M i\varepsilon/(r\kappa)$. The dynamic position error is $\psi = -s\tau_M i\varepsilon/(r\kappa) \approx -0.0457 \approx 2.62^\circ$. We note that the calculated dynamic position errors are those of the engine shaft. The errors of the column position are $i = 50$ times less.

4) The limit value of the gain coefficient in the feedback loop is determined by the inequality

$$\kappa < \frac{s}{r\tau} = 209.5 \text{ V}.$$

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Index

- absolute angular
 - acceleration of a link 85
 - velocity of a link 85
- acceleration of an arbitrary point 86
- accuracy analysis
 - , first problem of 105
 - , second problem of 105
- Amontons–Coulomb law 176
- angle
 - of meshing 183
 - , pressure 124, 170
- apparatus 1
- assembly of a mechanism 58
- balancing 246
 - a planar mechanism 248
 - of a rotor
 - , dynamic 246
 - , static 246
 - of a stiff rotor 246
 - of mechanisms and machines 240
 - of the first harmonics of inertia forces 249
 - stands 246
- basic theorem for a higher kinematic pair 92
- break in a kinematic chain 60
- braking, dynamic 277
- bushing 363
- car differential 100
- cam 21
- characteristic
 - of engine
 - , control 270
 - , dynamic 270
 - , ideal 271
 - , ideal force 274
 - , ideal kinematic 274
 - , idling 271
 - , mechanical 269
 - , mild 275
 - , static 270
 - , stiff 274
 - , working 270
 - of workloads 139
- circle of a gear
 - , base 76
 - , initial 95
 - , pitch 96
- class of a mechanism 9, 21
- classification of kinematic pairs according to
 - the character of contacting of the pair elements 10
 - the number of degrees of movability 8
 - the number of imposed constraints 9
- closed system 379
- closure of a mechanism 22
 - , force 22
 - , geometric 22
- coefficient
 - of absorption 329
 - of nonuniformity 282
 - of pivoting friction 176, 192
 - of rolling friction 176, 246
 - of sensitivity 280
 - of sliding friction 176, 202, 210
 - of static friction 177
 - , damping 328
- connecting rod 13
- configuration of a mechanism 6
- constraints 5
 - , bilateral 5
 - , excessive 17
 - , holonomic 5
 - , nonreleasing 18
 - , releasing 18
 - , stationary 5
- construction damping 327
- control
 - signal, program 3
 - , correcting 3
 - , program 371

- control system
 - , closed 379
 - , feedback 3, 379
 - , program 3
- coordinate
 - of a mechanism
 - –, input 6
 - –, output 6
 - of a structural group 53
 - –, input 53
 - –, output 53
- coordinates
 - , group 53
- correction planes 246
- coupling of mechanisms 7
 - , parallel 7
 - , successive 7
- Christoffel symbols of first kind 217
- crosshead 21
- damping, construction 327
- de-braking 185, 188, 193
- degree of closure of a mechanism 52
- device 1
- dislocations 327
- dynamic braking 277
- dynamic absorber 239
- effect, Sommerfeld 364
- effectiveness of a closed system 380
- efficiency 253, 265, 271
- engine 2
 - link, output 2
 - parameter, input 2
 - , electric 2
 - , thermal 2
 - , hydraulic 2
 - , pneumatic 2
 - , reciprocating 2
 - , rotary 2
- equation
 - of d'Alembert–Lagrange 152, 154
 - of dynamics, general 152, 155
 - of geometric analysis 44
 - of kinetostatics 144
 - of meshing 93
 - of second kind, Lagrange 212, 216
- equilibrium conditions of a mechanism 121
- equivalent linearization 328
- error 3, 104
 - in acceleration 105
 - in velocity 104
 - , diagrammatic 104
 - , displacement 104
 - , dynamic 104, 280
 - , geometric 104
 - , kinematic 104
 - , position 104
 - , primary 104
 - , static 104
 - , technological 104
 - , temperature 104
 - , wear 104
- Euler angles 49
- external reactions 239, 249
- feedback, negative 280, 379
- feedback system 3, 379
- flexibility 304
- flywheel 285
- follower 21
 - , planar 22
 - , pointed 21
 - , rocker 22
- force
 - of a link, gravity 140
 - of sliding, frictional 175
 - vector diagram 129
 - , balancing 122
 - , dissipative 327
 - –, positional 327
 - , elastic 140
 - , generalized driving 2
 - , inertia 122, 139, 143, 144, 149, 167, 170
- forces 1
 - of resistance, generalized 212, 228, 235
- formula
 - , specified, of Kloss 273
 - , Willis' 98, 226
- frame 12
- friction
 - , coefficient of pivoting 176, 192
 - , coefficient of rolling 176, 246
 - , coefficient of sliding 176, 202, 210
 - , coefficient of static 177
 - , internal 327
 - , linear 328
 - , viscous 327
- frictional

- force of sliding 175
- moment of pivoting 175
- moment of rolling 175, 183
- functional part of a machine 2
- gain 379
 - , coordinate 379
 - , velocity 379
- gear train 95
 - with intermediate gears 95
- gearing
 - , bevel 100
 - , cylindric 94
 - –, external 94
 - –, internal 94
 - , hyperboloidal 102
 - –, helical 102
 - –, hypoid 102
 - , worm 103
- general equation of dynamics 152, 155
- generalized driving force 2
- generalized forces of resistance 212, 228, 235
- generation principle of mechanisms 12
- graph of a mechanism 13
 - , edges of 13
 - –, root 13
 - , vertices of 13
- gravity force of a link 140
- group-based method for deriving equations 53
- group coordinates 53
- inertia force 122, 139, 143, 144, 149, 167, 170
- inertia tensor 145
- input engine parameter 2
- Jacobian 62
- kinematic chain 10
 - , break in 60
 - , closed 10
 - , differential 97
 - , open 10
 - – with tree-structure 10
 - –, simple 10
 - –, branched 10
 - , number of degrees of movability of 10
- kinematic pair 4
 - , basic theorem for a higher 92
 - , cylinder-plane 8
 - , cylindric 8, 55
 - , higher 10
 - , input 11
 - , lower 10
 - , planar 8
 - , prismatic 8
 - , reaction at 140
 - , revolute 8
 - , screw 8
 - , sphere-plane 8
 - , spheric 8, 56
 - – with a pin 8, 57
- kinetostatic model 144
- Lagrange equations of second kind 212, 216
- linearization, equivalent 328
- link 3
 - , absolute angular acceleration of 86
 - , absolute angular velocity of 86
 - , driven 24
 - , driving 24
 - , input 6, 11
 - , output 6
- linkage 20
- loss factor 254
- machine 1
 - aggregate 1
 - with program control 43
 - , cycle 7
 - , energy-converting 1
 - , functional part of 2
 - , information 1
 - , number of degrees of movability of 2
 - , technological 1
 - , transport 1
- magnification factor 335
- matrix
 - of inertia moments 145
 - of reduced flexibilities 308
- , direction cosine 45
- , Jacobi 62
- , rotation 46
- , transformation 48
- mechanical
 - motions 1
 - system 2

- mechanism 3
 - with excessive constraints 17
 - with redundant degree of mobility 18
 - with variable structure 24
 - , actuating 7
 - , assembly of a 58
 - , balanced 240
 - , cam 21
 - , class of 9, 21
 - , closure of 22
 - –, force 22
 - –, geometric 22
 - , configuration of 6
 - , converted 136
 - , coordinate of 6
 - –, input 6
 - –, output 6
 - , crank-and-rocker (,four-bar) 20
 - , degree of closure of 52
 - , elastic 6, 301
 - , equilibrium conditions of 121
 - , graph of 13
 - –, edges of, 13
 - – –, root 13
 - –, vertices of 13
 - , ideal (theoretical) 103
 - , input of 6
 - –, external 6
 - –, internal 6
 - , normal (regular) 6
 - , number of degrees of movability of 6, 12
 - , output of 6
 - , planar 13, 19
 - , position of 59
 - –, singular 59
 - – (–)dead 59
 - – (–)undefined 59, 81
 - , programming 371
 - , real (actual) 104
 - , rigid 5, 301
 - , scotch-yoke 21
 - , singular 6
 - , slider-crank 13
 - , slotted-link 20
 - , spatial 13, 55
 - , structural analysis of 27
 - , structural diagram of 12
 - , structural inversion of 26
 - , structural transformation of 25
 - , transmission 7
 - –, belt 24
 - –, friction 24
 - –, gear 24
 - – –, differential 97
 - – –, planetary 97
 - – –, rack-and-pinion 95
- method
 - of inversion of motion 97
- , Newton 62
- –, modified 63
- modes, normal 348
- modulus of meshing 96
- moment
 - , of inertia, reduced 212, 235
 - of inertia forces, resultant 144, 145
 - of pivoting, frictional 175
 - of resistance forces, reduced 214, 237
 - of rolling, frictional 175, 183
- , critical 273
- , nominal 273
- , perturbation 236
- , vibrational 365
- negative feedback 280
- Newton method 62
- , modified 63
- normal modes 348
- number of degrees of movability
 - of a kinematic chain 10
 - of a machine 2
 - of a mechanism 6, 12
- output link of an engine 2
- output of a mechanism 6
- physical model 4
 - of elastic mechanism 6
 - of rigid mechanism 5
- pitch point 93
- planet carrier 97
- planet pinion (satellite gear) 97
- , reduced 306
- pressure angle 124, 170
- position function 41
 - , explicit representation of 41
 - , implicit representation of 42
- position of a mechanism 59
- , singular 59
- (–)dead 59, 124

- (–)undefined 59, 81
- problem of accuracy analysis
 - , first 105
 - , second 105
- problem of geometric analysis 41
 - , direct 41
 - , inverse 43, 66
 - , path 65
 - , positioning 65
- problem of kinematic analysis 79
 - , direct 79
 - , inverse 79
- program control signal 3
- program control system 3

- rack-and-pinion transmission 95
- radius of friction circle 189
- reaction at a kinematic pair 140
- reactions, external 239, 249
- „rearrangement“ of clearances 237
- reduced
 - flexibility 306
 - mass 212
 - moment of inertia 212, 235
 - moment of resistance forces 214, 326, 237
 - stiffness 306
- resonance
 - , elastic 332
 - , motive 289
- resultant moment of inertia forces 144, 145
- resultant vector of inertia forces 144, 145
- rocker 20
- roller 22
- rotor
 - , balancing of 245
 - –, dynamic 246
 - –, static 246
 - , dynamically balanced 243, 247
 - , elastic 245
 - , flexible 246
 - , statically balanced 243
 - , stiff 245
- run-out 277
 - , free 277

- seizure 185
- self-braking 185, 188
- sensitivity 280
- slider 13

- slip 273
 - , critical 273
- slotted link 21
- Sommerfeld effect 364
- stability 380
 - of a closed system 380
- starting acceleration 277
 - , controllable 277
 - , non-controllable 277
- state of performance
 - , post-resonance 336
 - , pre-resonance 336
- static balancing of a rotor 246
- static model 121
- statical determination 143
- steady-state motion 236, 277
- steepness
 - of the engine static characteristic 274
 - of the average moment of resistance forces 279
- stiffness 304
 - , angular 304
 - , inlet 305
 - , linear 304
 - , outlet 305
 - , reduced 306
- structural
 - analysis 27
 - diagram 12
 - group 12
 - –, Assur 12
 - –, two-bar (dyad) 13
 - –, simple 12
 - inversion 26
 - layer 13
 - transformation 25
- sun gear 97
- system
 - of program control 3
 - , closed 379
 - , tracking 379

- tensor of inertia 145
- time constant
 - , electromagnetic 270
 - , fundamental 275
 - , mechanical 282
- tracking system 379
- transfer function 79
 - , first geometric 79

–, second geometric 79

transient 275

transmission ratio 91

unloaders 237

vector diagram

– of velocity analogues 82

– of acceleration analogues 82

–, force 129

velocity of an arbitrary point of a link 86

velocity, critical 246

vibration activity

–, external 240, 241, 242

–, internal 236

vibration exciter 254

vibration of elastic mechanisms

–, forced 318

–, free 318

–, resonance 320

vibration, accompanying 282, 346

Willis' formula 98, 226

working process 1

–, information 1

–, energy 1

–, technological 1

–, transportation 1

workload 3, 139

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