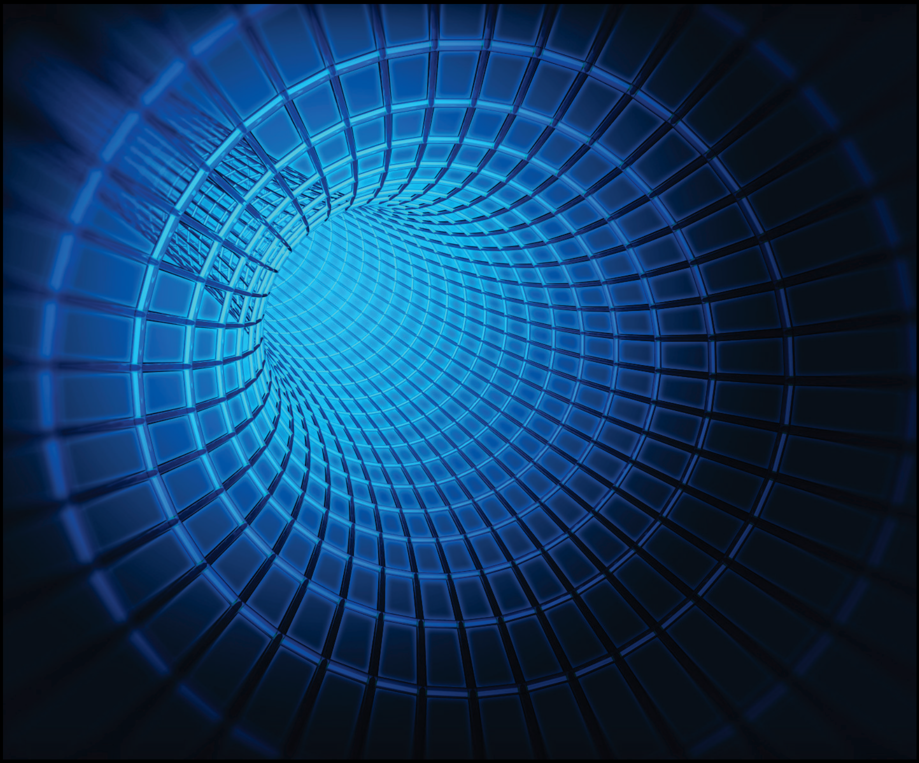
TIME and Consistent Relativity

Physical and Mathematical Fundamentals



Lyubomir T. Gruyitch, DSc

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ABOUT THE AUTHOR

Lyubomir T. Gruyitch

Lyubomir T. Gruyitch, DSc, was a Professor at the Ecole Nationale d'Ingénieurs, which integrated with the Institut Polytechnique de Sévenans at the University of Technology Belfort–Montbeliard, in France (1993-2007). He was also the AECI Professor of Control in the Department of Electrical Engineering at the University of Natal, Durban, South Africa (1992/1993), and a Professor of Automatic Control in the Faculty of Mechanical Engineering at the University of Belgrade, Serbia (1964–1992). He has also been a visiting professor at Ecole Centrale, Lille, France (1992); at Louisiana State University, Baton Rouge, Louisiana (1989/1990); and at the University of Notre Dame, Notre Dame, Indiana (1988/1989); as well as Research Associate at the University of Santa Clara, Santa Clara, California (1972). He has continued his research, lecturing, and consulting activity.

Dr. Gruyitch is the author of several published books and many scientific papers on dynamical systems, on control systems, and on time and its relativity. He has participated at many scientific conferences throughout the world. He has been honored with several awards and honors, including being honored with Doctor Honoris Causa by French Republic and the highest award presented by the Faculty of Mechanical Engineering, University of Belgrade, for teaching and scientific contributions to the faculty, 1964–1992, and an award from the Yugoslav Air Force Academy for teaching achievements in the undergraduate course Foundations of Automatic Control.

Dr. Gruyitch is a Certified Mechanical Engineer (Dipl. M. Eng.), Master of Electrical Engineering Sciences (M. E. E. Sc.), and Doctor of Engineering Sciences (DSc), all from the University of Belgrade, Serbia.

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Preface

0.1 Relevant citations

Fugit irreparabile tempus.
Irreparable time flees.
Le temps irréparable fuit.
Latin proverb

.... *the true and the beautiful are one and the same, as are likewise the false and the ugly.*
....
But once you have denied the principles of the sciences and have cast doubt upon the most evident things, everybody knows that you may prove whatever you will, and maintain any paradox.
Galileo GALILEI (1632) [191, pp. 131, 141]

Wherefore relative quantities are not the quantities themselves, whose names they bear, but those sensible measures of them (either accurate or inaccurate), which are commonly used instead of the measured quantities themselves. And if the meaning of words is to be determined by their use, then by the names time, space, place, and motion, their [sensible] measures are properly to be understood; and the expression will be unusual, and purely mathematical, if the measured quantities themselves are meant. On this account, those violate the accuracy of language, which ought to be kept precise, who interpret these words for the measured quantities. Nor do those less defile the purity of mathematical and philosophical truths, who confound real quantities with their relations and sensible measures.
Isaac NEWTON (1687) [360, p. 12]

Our whole progress up to this point may be described as a gradual development of the doctrine of relativity of all physical phenomena.
James Clerk MAXWELL (1877) [334, p. 80]

Peut-être suffirait-il de renoncer à cette définition, pour que la théorie de Lorentz *fût aussi complètement boulversée que l'a été le système de* Ptolémée *par l'intervention de* Copernic*. Si cela arrive un jour, cela ne prouvera pas que l'effort fait par* Lorentz *ait été inutile ; car* Ptolémée*, quoi qu'on pense, n'a pas été inutile à* Copernic*.*
Henri POINCARE (1905) [383]

English translation reads:

Perhaps the abandonment of this definition would suffice to overthrow the theory of Lorentz *as completely disrupted as the system of* Ptolemy *by the intervention of* Copernicus*. If this ever happens, it will not prove that the effort made by* Lorentz *has been useless; for* Ptolemy*, whatever we think, was not useless to* Copernicus*.*

The assumption which was in the pre-relativity physics of the absolute character of time (i.e. the independence of time of choice of the inertial system) does not follow at all from this definition. The theory of relativity is often criticized for giving, without justification, a central theoretical rôle to the propagation of light, in that it founds the concept of time upon the law of propagation of light. The situation, however, is somewhat as follows. In order to give physical significance to the concept of time, processes of some kind are required which enable relations to be established between different places. It is immaterial what kind of processes one chooses for such a definition of time. It is advantageous, however, for the theory, to choose only those processes concerning which we know something certain. This holds for the propagation of light in vacuo *in a higher degree than for any other process which could be considered, thanks to the investigations of Maxwell and H. A. Lorentz.*
Albert EINSTEIN (1950) [150, p. 27]

... le temps, dimension oubliée ...
Ilya PRIGOGINE (1982) [396, p. 1]

0.2 Why Consistent *Time* Relativity Theory?

We know, we are aware, that all physical processes, all movements and motions, take place in *time*. Our existence occurs in *time*. *Time* is one of the most important variables in physics, mathematics, biology, engineering, but also for economics, informatics, management, medicine, music, politics and philosophy. It has occupied human thoughts and studies from the most ancient epoch.

The book presents characteristic original thoughts and claims by scientists, researchers, physicists, philosophers and other thinkers on *time* and its properties. They, the common human knowledge and experience result in the definition and the characterization of *time* and its properties, which are proposed in the book.

The books [226], [231] expose in debt, exact, objective study of what is correct, inconsistent, paradoxical, even absurd, and wrong in Einstein‘s explanation of *time* and in Einstein‘s theory of *time* relativity. The substantial overall drawback and failure of Einstein‘s *time* relativity theory has opened the necessity for a consistent theory of *time* relativity. The new situation has taken place. What is *time*? What is its relativity? Many other questions and open problems have appeared, as well. Their list follows.

This book presents the fundamentals of the new theory of *time* relativity. Its physical basis are the proposed definition and characterization of *time,* as well as the solutions to the next problems. Its mathematical basis are various new linear coordinate transformations. They are different from Galilean - Newtonian coordinate transformations and from Lorentz transformations, which result from the new transformations as special or singular cases. This ensures the generality of **Consistent *Time* Relativity Theory** (**CTRT**), for short, **Consistent Relativity Theory** (**CRT**).

0.3 Problems to be solved

There have been various controversial interpretations of *time*. The book presents them in their original forms. They pose the following questions and rise the following related problems:

Problem 1 *Does time exist as a physical variable or it is an abstract (mathematical) variable, or even, only a parameter?*

Problem 2 *Are there several times or there is the unique time?*

Problem 3 *How is time explained, interpreted and exactly defined in science in general and in Einsteinian relativity theory established by Lorentz, Einstein and Poincaré in particular?*

The relativity theory founded by and based on the results of Lorentz [297] through [301], Einstein [114] through [157] and Poincaré [383], [386] will be called herein ***Einsteinian relativity theory***.

Problem 4 *Has Einsteinian relativity theory succeeded to explain time so that the explanation agrees with the physical reality and with the experience?*

Problem 5 *Is time really dependent variable (e.g. on space, on the light velocity), or it is an independent variable?*

Problem 6 *What are the properties of time and of the speed of its values flow?*

Problem 7 *Can velocity be defined before time has been explained and defined? If it can, then: how?*

Problem 8 *Can a speed numerical value (including the numerical value of the light speed) be determined without having specified a time scale and a time unit? If it can, then: how?*

Problem 9 *What is the influence of a time scale change on a speed (numerical) value [including the (numerical) value of the light speed], if any?*

Problem 10 *Is there any relationship between Einstein's interpretation of time relativity and Newton's explanation of relative time?*

Problem 11 *Is it impossible to interpret time in Newton's sense so that it satisfies Einstein's interpretation of time, or vice versa?*

Problem 12 *Is it impossible to characterize time so that it agrees with: physical reality, our experience and knowledge, Newton's explanation of time and Einstein's interpretation of time?*

Problem 13 *What is the relationship between properties of time and human (biological and/or psychological) feeling of time?*

Problem 14 *What are the meanings of the biological age and of the psychological age? What are the speeds of the biological aging and of the psychological aging?*

Problem 15 *Is the light speed invariant in vacuum?*

Problem 16 *Can the speed of an arbitrary material point (of a particle), or of a body, be greater than the light speed in some coordinate systems?*

Problem 17 *Do there exist coordinate transformations and from them deduced speed transformations relative to which an arbitrary speed is invariant in vacuum?*

Problem 18 *Under what necessary and sufficient conditions Lorentz transformations permit invariance of an arbitrary speed?*

Problem 19 *Under what necessary and sufficient conditions on the spatial transfer speed can Lorentz transformations obey Einstein‘s distance condition?*

Problem 20 *Do Galilei-Newton‘s transformations of coordinates satisfy Einstein‘s special condition for the distance preservation in the time-space environment?*

Problem 21 *Do Galilei-Newton‘s transformations of coordinates satisfy Einstein‘s generalized condition for the distance preservation in the time-space environment?*

Problem 22 *What are consequences of the proposed definition and characterization of time for the relativity theory from the point of view of the physical reality, the common human experience and knowledge?*

Problem 23 *Can such definition and characterization of time permit new time coordinate transformations, new space coordinate transformations and new velocity transformations?*

Problem 24 *Do there exist non-Lorentzian linear coordinate transformations that satisfy the general or Einstein's distance condition?*

Problem 25 *Do the accepted definition and characterization of time permit a relaxation of all a priory accepted assumptions and constraints in Einsteinian relativity theory?*

Problem 26 *What is the relationship between time and multiple time scale dynamical systems?*

Problem 27 *What are the meanings of partial and of complete compatibility of the transformations?*

Problem 28 *What is the meaning of consistency of the transformations?*

Problem 29 *What are the forms of partially compatible but consistent transformations that satisfy the general or Einstein's distance condition?*

Problem 30 *What are from them deduced velocity transformations like?*

Problem 31 *What are the forms of completely compatible and consistent transformations that satisfy the general or Einstein's distance condition?*

Problem 32 *What are from them deduced velocity and acceleration transformations like?*

Problem 33 *Do they lead to a new relativity theory, the basis of which are consistent transformations?*

Problem 34 *What is the relationship of the new results relative to the corresponding Galilei-Newton's and Einstein's results?*

Problem 35 *What are implications of the properties of time on: physical variables, dynamical systems and control?*

Problem 36 *What is the relationship between the properties of time and human (biological and/or psychological) feeling of time?*

The aim of the book is to explore these problems, to reply to the questions, to present the complete solutions to the problems and to contribute with other discoveries on the *time* relativity. The new results form the fundamentals of the new relativity theory called *Consistent Time Relativity Theory (CTRT), for short Consistent Relativity Theory (CRT).*

This work, which is theoretical with potential applications in diverse directions, is addressed (in alphabetical order) to biologists, chemists, economists, engineers, mathematicians, philosophers, physicists and psychologists; to lecturers and researchers; to everybody who has been excited by some of the above questions, or at least by some of the following general ones:

Problem 37

What is time?
What is relativity of time?
Is the light velocity invariant?
Is the light speed the limiting speed?
Is it possible to establish a consistent relativity theory?
Can the physical properties of time be the base of such theory?
Can the theory be relaxed of all Lorentz'-Einstein's assumptions?
Can such theory reflect the common human experience with time?

The book represents the refined, with several new results, joint version of the former books [226], [227], [228] and [231].

0.4 Acknowledgement

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Belgrade, March 12, June 1, 2014, April 10, 2015. Lyubomir T. Gruyitch

Chapter 1

Introduction

1.1 The goals of the book

The purpose of what follows is to clarify the phenomena of *time*, of its relativity and to present the new, consistent, physical and mathematical theory of *time* relativity based fully on the physical nature of the *time* properties. This determines several main goals of the book.

One goal of the book is to propose the definition and the characterization of *time* so that they express the physical reality, the common human experience with the *time* phenomenon, and the accumulated human understanding and knowledge about *time* and about the speed of *time* values flow, for short *the time speed.*

An additional goal is to present the physical and mathematical fundamentals of the new, consistent, *time* relativity theory that is fully relaxed of all (tacit) assumptions and restrictions of Einstein‘s relativity theory.

The goal is also to show how the new, consistent, relativity theory overcomes all inconsistencies, paradoxes, absurds and mistakes of Einstein‘s relativity theory (which were discovered and rigorously proved in [226], [231] and for which Einstein‘s *time* relativity theory is invalid in general).

Another goal of the book is to prove various consistent linear coordinate transformations, essentially different from Lorentz‘ ones, so that they obey the general, or generalized Einstein‘s, condition on the distance preservation.

Besides, the goal is to deduce from them the velocity / speed transformations and to test whether the light speed is generally invariant in vacuum, as well as to verify whether the light speed is the limiting speed.

Moreover, we will explore the conditions under which the spatial transfer speed can be invariant as used a priory in Einstein‘s relativity theory.

In order to achieve these goals Problems 1 through 37 will be solved herein.

1.2 Book composition

In addition to **Contents, List of figures, Preface** and this **Introduction**, the book is composed of six parts. The first four parts constitute the main body of the book. Chapters compose the parts. Some chapters are divided into sections that can have subsections composed eventually of subsubsections.

The first part, **Part I**, which is on "***TIME***", contains four chapters. Its first chapter entitled **"Interpretations of *Time*"** is devoted to various explanations and interpretations of *time.* It continues with the clarification of the sense, the meaning, the characteristics, the definition, the properties and the physical relativity of *time*. It concerns also the influence of *time* on velocity. It begins with a brief review of various characteristic interpretations of *time* from different points of view.

The second chapter entitled "**Newton and Einstein on *Time***" is devoted to the particular attention to Newton‘s explanation of *time*, to Einstein‘s meaning of *time*, and to their comparison that ends this chapter.

The third chapter is entitled "**Nature and Properties of *Time***". By referring to the common human experience and understanding of physical phenomena, the definition of *time* and the characterization of *time* represent the core of this chapter and are crucial for the whole further study in the book.

In order to avoid any vagueness, the notion, meaning and general features of *physical variable* are explained before concluding whether *time* is (not) a physical variable. It is important to distinguish among variable, its value and its numerical value, which is done at the beginning of this chapter. Further explanations concern spaces and relationships between spaces and physical variables.

In order to show the importance of *time*, we clarify the meaning in which the notions *energy*, *matter* (*substance*) and *space* will be used in the sequel. This enables us to explain what are *the primary constituents of the existence of everybody and everything.*

The general relationships among quantities, units and transformations, which are explained in this chapter, play an important role for understanding both the drawbacks of Einsteinian relativity theory and the need for a care about the use of the values and numerical values of all variables in coordinate transformations. The lack of such care in Einsteinian relativity theory has caused sever negative consequences, among which is the inconsistency of the theory.

The general clock principle explains the difference among the clock indications, the *time* values and the *time* numerical values, as well as their relationship. This is inherent to understand Newton‘s sense of *time* and Einstein‘s meaning of *time*, as well as their explanations of *time* relativity, to clarify the differences between them and to discover what is common for them.

The analysis of the well known Einsteinian example illustrates that *time* itself does not depend on the speed either of a moving body or of a moving spatial coordinate system.

This chapter ends with the importance of *time* for the human.

The fourth chapter entitled "**New Fundamentals**" 5 discovers common and general properties of physical variables. They are physical uniqueness and physical continuity of physical variables. These properties are essentially different from such mathematical properties of functions. They and the properties of *time* result in an important characteristic of the *time* variation of every physical variable value. Besides, this chapter discovers the new fundamental theorems on time speed and on the light speed noninvariance.

The properties of *time* concern also the relationship between *time* and speed, as well between *time* and space. A novel theorem is proved on the invariance of the *time* speed. These relationships and the novel theorem jointly show that *time* is independent of a choice of a coordinate system, i.e. *time* is independent of space. This confirms Newton's explanation of *time* and represents a crucial disagreement with Einstein's attitude cited above (that *time* itself depends on a choice of a coordinate system, hence, of space).

The new linear coordinate transformations and, from them deduce velocity / speed transformations prove the noninvariance of the light speed.

The end of this chapter is also the end of the first part, Part I. It presents the general modeling and relativity principles. Besides, it discovers the links among *time*, the principles and the dynamical systems.

The second part, **Part II,** is on "***TIME*** **FIELDS and RELATIVITY**".

The properties of *time* enable us to discover in this Part the existence of ***time fields***,. The definition, the explanation of the features of *time* fields and their descriptions begin the first chapter: "***Time*** **Fields and Transformations**". We explain also characteristics of various *time* fields and of the related generic coordinate transformations. They generate other transformations and lead to a reach diversity of directions in the novel mathematical relativity theory.

The second chapter explains "**Why not Einstein‘s Relativity Theory?**".

A presented brief account of the essential drawbacks of Einsteinian relativity theory, which are fully explained and proved in [226], [231], opens the necessity for a new theory on *time* and its relativity and helps us to establish guidelines for further developments of mathematical relativity theory in diverse directions. The new theory should be consistent. It should agree with the common human experience on *time* and on its relativity. Besides, it should be relaxed of all artificial assumptions, hypotheses and restrictions, either physical and/or mathematical. The third chapter is on **"Non-Einsteinian Approaches to Relativity"**. Various non-Lorentzian transformations studied in this chapter obey generalized Einstein's distance preservation condition, i.e. form the Poincaré group. This shows that Lorentz transformations, analyzed briefly, are not the exclusive linear coordinate transformations that form Poincaré group. This chapter discovers the link between dynamical systems and relativity. It ends the generalization of Galilean - Newtonian approach to relativity. The fourth chapter **"Conclusion on *Time* and *Time* Fields"** ends Part II.

The third part, **Part III,** is entitled "**PARTIALLY COMPATIBLE but CONSISTENT (PCC) RELATIVITY THEORY (RT)**". Four chapters compose this part. Its first chapter is on "**Partial Compatibility**". The chapter explains the meaning and presents the origin of the partial compatibility. It establishes generic forms of *time*-invariant nonuniform and uniform coordinate transformations and *time*-fields.

Note 38 *This book introduces various time and space coordinate transformations in which the time and space scaling factors, as well as every velocity, are all time-invariant (i.e., constant). Hence, the transformations are time-invariant. The books [227], [228] permit all time and space scaling factors, as well as every velocity, to be time-varying. Since the results represent the direct time-varying generalization of the corresponding results of this book, then time varying transformations are not treated in the sequel.*

The second chapter entitled "**Light Speed of the Arbitrary Point**" continues the preceding chapter with a series of various partially compatible, nonuniform or uniform, *time*-invariant transformations and related *time* fields by applying Lorentz - Einstein - Poincaré approach to the determination of the scaling coefficients exclusively for the light speed of an arbitrary point P.

Starting with the properties of *time*, we analyze the fundamentals of Einsteinian theory of *time* relativity. The accepted definition and characterization of *time* enables us to (re)prove exactly the basic formulae of Einsteinian relativity theory and to show that they represent, not just a special case, but the singular case. The proofs illustrate validity of the accepted definition and characterization of *time*, and reject Einstein's claims that *time* depends on spatial frames, i.e. on space, and that *time* itself is relative, which are incontestably accepted as fundamentals of Einsteinian theory of *time* relativity.

By using the properties of *time*, we relax largely, but not completely, the fundamentals of Einsteinian relativity theory from the constraints a priory imposed by Lorentz, Einstein and Poincaré. By following them, we still retain the spatial transfer speed also for the temporal transfer speed, and we set the generic speeds in the temporal coordinate transformations to be equal to the light speed relative to the corresponding integral space. Consequently, the resulting transformations are partially pairwise compatible but consistent.

The relaxation of the a priory accepted Einsteinian constraint to calculate all scaling coefficients strictly for the light speed of the arbitrary point P enables us to show in the third chapter "**Any Speed of the Arbitrary Point**" that the light speed is not invariant relative to inertial frames in vacuum in general. The results of this chapter establish the fundamentals for the new, non-Einsteinian, mathematical, *Partially Compatible* but fully *Consistent (PCC) Relativity Theory.* They incorporate the results of Einsteinian relativity theory as singular cases.

The mathematical results of this part involve those of Einsteinian relativity theory.

This Part is a bridge from Einsteinian relativity theory to the fundamentals of another new, mathematical, *Compatible* and *Consistent (CC) Relativity Theory.*

The fourth chapter entitled "**Conclusion on PCC Relativity Theory**" ends the third part.

The fourth part, **Part IV,** on "**COMPATIBLE and CONSISTENT (CC) RELATIVITY THEORY**" establishes the fundamentals for another novel mathematical relativity theory. It is fully and essentially different from

Einsteinian relativity theory. The basis is the set of completely compatible and consistent coordinate transformations. It begins with the chapter entitled "**Colinear Motions: Transformations**" and establishes the mathematical basis of the non-Einsteinian theory of *time* relativity. It is called *Compatible and Consistent Relativity Theory* (for short, *CC Relativity Theory*). The properties of *time*, which are characterized in the first part, form its basis. It permits the noninvariance of the light velocity. It deals with new transformations of the temporal coordinates and of the spatial coordinates. They yield new results essentially different from those of Einsteinian relativity theory on the coordinate transformations, as well as on the velocity transformations. The transformations are completely compatible. The values of all variables are in them consistently used relative to units, *time* axes and spatial frames.

The second chapter deals with "**Noncolinear Motions: Transformations**". It shows how the fundamentals of the CC relativity theory can be further mathematically developed to the cases when an arbitrary movement can have an arbitrary direction rather than fixed one tied with the direction of an arbitrarily accepted and then fixed constant unity vector **u**.

The chapter entitled **"Conclusion on CC Relativity Theory"** completes the fourth part.

The fifth part, **Part V,** presents **"GENERAL CONCLUSION".** Its first chapter **"Problem Solutions"** presents the solutions of the problems raised above. The second chapter entitled **"Summary on *time*"** summarizes the crucial properties of *time*.

The next part, Part **VI,** is named "**Subsidiary Parts**". Its chapters serve the first four parts of the book. The first one, **"Notational Details"**, presents a detailed list of notation with explanations. The next four chapters contain the proofs of the results of Part I through IV. It is followed by the chapter **"Used literature - Bibliography"** that contains the list of the consulted literature. The last chapter is the **"Index"** one composed of "Author Index" section and "Subject Index" section.

The topic of the book is extremely delicate. The discoveries disprove the fundamentals of Einsteinian relativity theory and reaffirm Galilean - Newtonian physics via its new developments and generalizations.

The book represents a refined and abbreviated version of author's former books [227] and [228] together with the summary of the results from [226] and [231] and with several new contributions.

1.3 On the notation and proofs

The stuff treatment requires exact, precise and clear proofs of theorems, as well as an adequate, fully precise, and as simple as possible, notation, and its consistent usage. In this connection it is to note that the following explanation:

$$(.),\ (..) \in \{i, j\},\ and$$
$$i, j \in \{-, 1, 2,\ ...\ ,\ s\},\ i \le j,\ are\ arbitrarily\ chosen\ and\ then\ fixed, \quad (1.1)$$

holds throughout the book if it is not stated otherwise. For the sake of the simplicity, this notational explanation, (1.1), will be omitted in the sequel. The minus sign, $-$, stands instead of the blank space in (1.1) to denote that the index (subscript or superscript) should be omitted.

The used mathematics is mainly linear algebra. Up to the chapter 15: "**Noncolinear Motions: Transformations**" of Part IV, the scaling coefficients are scalars, while in that chapter they are diagonal matrices.

We apply the infinitesimal calculus only in order to determine the formulae for the transformations of velocity / speed. Since all velocities are constant then the corresponding accelerations are equal to zero vector.

The formulae can appear more or less complicated due to the existence of the superscripts and subscripts, which sometimes carry their indexes.

The proofs show the causality among conditions of statements and the corresponding results. They permit verifications of the results, hence, a test of all the statements. They present methodologies used to arrive at the results and enable their further developments. The proofs of the new results are mainly presented in details due to the delicacy of the claims that largely disagree with Einsteinian relativity theory. It is simple to skip over the proofs, or just to avoid to open the corresponding appendix, if the reader is not interested in them. Most of the proofs are in the appendices. The proofs of theorems, which are obvious modifications of already elaborated proofs, are omitted.

The theorems present the conditions that are both necessary and sufficient for the claimed phenomenon to take place.

Part I

Time

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Chapter 2

Interpretations of *Time*

2.1 Introductory comment

Time, in and of itself, is an artist idealizing the world.
Jean-Marie GUYAU [346, p. 140]

Several original citations, rather than their interpretations or translations, illustrate importance and complexity of the nature, of the meaning, of the sense, and of the characteristics of *time*. They illustrate how long and from which particular points of view *time* has been studied, and why it has been a challenging notion and phenomenon to explain. They illustrate also the difficulty, practical impossibility, to collect, or at least to review, in one work all what was written about *time*. Hence, what follows does not pretend to be a review, but just to be an illustrative introduction.

2.2 *Time* as a topic

The problem of time has always baffled the human mind.
Hans REICHENBACH [406, p. 1]

Qu'est-ce donc le temps? Si personne ne me le demande, je le sais ; mais si on me le demande et que je veuille l'expliquer, je ne le sais plus.
What, then, is time? If no one asks me, I know what it is. If I wish to explain it to him who asks me, I do not know.
Saint AUGUSTIN [434, p. 264], [452, p. 58]

Le temps, c'est le grand mystère du monde.
Pierre JANET [252, p. 20]

Et je crois le même de plusieurs autres choses, qui sont fort simples et se connaissent naturellement, comme sont la figure, la grandeur, le mouvement, le lieu, le temps, etc., en sorte que, lorsqu'on veut définir ces choses, on les obscurcit et on s'embarasse[5].

René DESCARTES [96, p. 144]

Le concept de temps est beaucoup plus complexe que nous le pensons généralement.

Ilya PRIGOGINE [396, p. 7]

Attempts to grasp the nature of time are amongst the deepest and most puzzling challenges to the human mind.

Manfred EULER [172, p. 159]

*It is simply a gross misunderstanding to believe that **TIME** - the basic feature of all existence - can be deduced from or explained by science, as it is a fundamental presupposition of science; but neither should be consent to those forms of science which are determined to ignore time.*

Mogens WEGENER [477, p. 258]

It is quite impossible to make any physical statement at all without some implicit assumptions about the nature of space and time.

William Graham DIXON [101, p. 2]

In any attempt to bridge the domains of experience belonging to the spiritual and physical sides of our nature, Time occupies the key position.

Arthur EDDINGTON [110, p. 91]

Like other philosophic data, though very familiar to us, time has turned out to be most recalcitrant to careful description and analysis, hard to grasp in its manifold modes and phases, and even harder to explain.

....

Science simply takes it for granted.

John WILD [491, p. 540]

Pourquoi avoir choisi le thème du temps? C'est une banalité de dire que les hommes ont toujours tenté, sans grand succès, d'élaborer un discours cohérent sur le temps. Il n'y a qu'à voir la place énorme, et unique, qu'il occupe dans la littérature de toutes les époques. Il intervient dans de si nombreuses expressions langagières qu'on pourrait penser qu'il fait partie de nos concepts familiers. Chacun comprend de quoi on parle quand on parle de temps, ce qui devrait suffire à résoudre le problème qu'il pose une bonne fois pour toutes, de façon claire et distincte. Mais il faut prendre garde au fait que les concepts familiers sont souvent les plus mystérieux. Cela est particulièrement vrai du temps. Chacun *sent bien qu'il n'est pas une chose comme les autres, et qu'on n'en finira jamais de l'interroger. D'abord, il n'est une matière à aucun de nos cinq sens. Ensuite, il se présente à nous de façon toujours paradoxale. Il est à la fois familier et mystérieux, substantiel et fuyant, évident et indicible. Enfin, bien que sa direction soit "fléchée", comme disent les physiciens, il demeure un objet introuvable. Cette nature ambiguë du temps n'empêche pas que dans leurs interrogations sur l'univers et sur l'homme, les scientifiques sont sans cesse confrontés à ses lois, voire à ses caprices. Cette omniprésence du temps dans le champs des sciences n'est d'ailleurs pas sans soulever plusieurs questions : signifie-t-elle qu'il y a une universalité du temps ou bien reflète-t-elle une juxtaposition de status particuliers ? Et cette présence du temps en physique, n'est-elle pas incongrue ? La physique ne tend-elle pas plutôt à nier le temps en faisant appel aux "idéaux immobiles" que sont les lois universelles ? La question reste en effet posée de savoir si la physique a vocation à décrire l'immuable, ou bien si, au contraire, elle doit devenir la législation des métamorphoses.*

E. KLEIN et M. SPIRO [271, pp. 12, 13]

Understanding time in quantum mechanics is in fact intimately linked to understanding quantum mechanics itself, ...

J. Gonzalo MUGA, Rafael S. MAYATO and Iñigo L. EGUSQUIZA [355, p. 23]

Conclusion

Returning to the beginning question, 'What is time?', we have gone through all this to find that we cannot answer it. Time is too diverse a concept to be amenable to one answer. Time is many things, many processes, many types of experience. We cannot even answer the much simpler question, 'What is the experience of time?' - since we have seen that time experience is not a unitary 'sense'.

Robert E. ORNSTEIN [368, p. 109]

2.3 Arts and *time*

Passent les jours et passent les semaines
Ni temps passé
Ni les amours reviennent
Sous le pont Mirabeau coule la Seine
Vienne la nuit sonne l'heure
Les jours s'en vont je demeure
APPOLLINAIRE, *Alcools* [19, p. 156]

Le temps est un enfant.
HERACLITE, "Fragments" [308, p. 59]

Le temps, certes, fait mourir de vieillesse toutes chose ; cependant, s'il les détruit totalement, s'il dissout toute la matière qui les compose, d'où vient que Vénus peut reconduire aux rives de la lumière l'espèce des être vivants, d'où vient que la terre industrieuse peut aider ce retour à la vie, en fournissant à chaque espèce la nourriture qui lui convient? D'où viennent ces eaux dont sans cesse la mer se renouvelle - surgissement toujours jaillissant de ses sources intérieures, apport extérieur des fleuves qui lui reviennent de loin? D'où l'éther tire-t-il ce feu qui nourrit les astres [11] *? Toute substance mortelle devrait être consumée dès longtemps par l'infini du temps et l'écoulement innombrable des jours. Or si, en dépit de tout ce temps écoulé, il s'est trouvé des éléments pour permettre à ce monde de se renouveler et de se maintenir en vie, alors ces éléments sont à coup sûr dotés d'une nature immortelle - et il est impossible que rien retourne au néant.*

....

Ainsi le temps tire peu à peu de la nuit chaque découverte que la raison, ensuite, guide jusqu'aux rives de la limière.

LUCRECE [307, pp. 24, 25, 193]

Qu'est-ce que le temps?

L'ombre sur le cadran, la sonnerie de l'horloge, l'écoulement du sable, le jour et la nuit, l'été et l'hiver, les mois, les années, les siècles - tous, ils sont que des signes extérieurs et arbitraires, la mesure du temps mais non pas le temps lui-même. Le temps, c'est la Vie de l'Ame.

Henry Wadsworth LONGFELLOW [1, p. 38]

See more also in the books byBowra [50], Brelet [51], Kern [264], Meyerhoff [340], Poulet [389] through [391], Ricœur [411] through [413], and in the papers by Quinones [403], Rochberg [422] and Wiener [490].

2.4 Biology and *time*

From the perspective of the theoretical biologist, time is maybe ***the*** *most important concept that underlies evolution.*

George KAMPIS [258, p. 88]

Clocks of squirrel monkeys and Norway rats - the two most commonly used animals for these studies - keep time with great accuracy for months or even years. These biological clocks can be quite as accurate as our mechanical clocks.

....

Fig. 11 shows records of two congenitally blind rats whose clocks could not have been entrained by light since their optic nerves were entirely missing.

Our observations indicate that the 24-hour clock must have originated in the tropics where day and night have the same length, namely, 12 hours.

....

All of the evidence indicates that these two monkeys harbored inherent clocks that have the same length of period as that of the moon - but that they functioned quite independently of any influence of the moon.

This 29-30 -day clock must have been built into the nervous system of these animals far back in evolutionary eras when survival depended on ability of animals to adjust to the actual appearance of the full moon.

....

In two species of animals - both hybernators - ground squirrels and chipmunks - definite evidence was found for the existence of a yearly clock. In these animals the clock manifested itself not only in spontaneous running activity, but in daily food and water intake and body weight.

C. P. RICHTER [410, pp. 39, 46, 49]

.... the animal treats the past as if it were the present.

....

To perceive space, children as well as animals only need to open their eyes: it is there, present and intense. Time, in contrast, is a 'faded dream'.

Jean-Marie GUYAU [346, pp. 99, 100]

Le temps existe en biologie depuis l'apparition de la vie sur terre, il y a environ 4000 Millions d'années. Dès l'apparition des premières cellules, le temps a été marqué par des rythmes imposés par l'environnement et limité par leur disparition - à la suite d'une division de la cellule en deux, ou de son déclin et mort ou par prédation.

....

Le vieillissement biologique ne suit pas non plus l'horloge centrale de Greenwich. Nous possédons plusieurs marqueurs biologiques du vieillissement de l'organisme qui nous permettent de comparer le déroulement du temps biologique de chaque organe avec le temps du calendrier. Les rythmes de l'organisme changent aussi avec le vieillessement - un nouveau chapitre de la chronobiologie est représenté par sa rencontre avec la gérontologie (5).

Ladislas ROBERT [418, pp. 213, 214]

... les cellules des êtres dits procaryotes, et les cellules des eucaryotes*. Les premiers, tels que les bactéries, ne possèdent pas de noyau cellulaire organisé ; les deuxièmes ont un noyau cellulaire. L'essentiel du message génétique est contenu dans l'acide désoxyribonucléique, ou ADN. Celui-ci contient non-seulement le plan de construction des éléments de la cellule, mais aussi la séquence temporelle de la mise en route de ces constructions. Le déroulement de ce programme est la traduction d'un message unidimonsionnel contenu dans l'un des brins de la double hélice d'ADN, en une construction en trois dimensions et qui, en plus, évolue dans le temps. Les instructions contenues dans ce message comportent des directives pour la division des cellules, pour les différenciations et spécialisations,*

et aussi pour la construction d'un réseau macromoléculaire extracellulaire qui relie les cellules en tissus, en organes, et confère sa solidité et sa résistance à l'organisme. Chez les eucaryotes, ce message génétique est contenu dans le noyau cellulaire.

Ladislas ROBERT [420, pp. 35, 36]

LES HORLOGES BIOLOGIQUES

Ce terme a été longtemps monopolisé par les chronobiologistes (3), bien qu'il soit tout à fait légitime de l'utiliser pour désigner aussi les mécanismes qui déterminent la durée de vie de l'organisme (5). De nombreuses études ont été effectuées depuis le milieu de ce ciècle pour cerner ces mécanismes, à tous les niveaux accessibles à l'expérimentation, du niveau moléculaire cellulaire jusqu'au niveau des populations (épidémiologie) en passant par le niveau tissulaire et celui des organes et des organismes des unicellulaires jusqu'à l'homme.

Ladislas ROBERT [418, p. 221]

... la corrélation entre les deux types d'horloges, l'horloge cellulaire et celle de l'organisme entier, est établie.

....

... l'horloge de la cellule enregistre la durée de vie déja accomplie par l'individu au moment où le prélèvement a été effectué.

....

Dans ces conditions, les cellules restent au repos, ne se divisent pas, mais leurs métabolisme continue. Ce temps au repos n'a pas été enregistré par l'horloge cellulaire non plus.

Ladislas ROBERT [420, pp. 59, 60, 71]

En outre les modifications de notre organisme, devenues périodiques, constituent une véritable horloge physiologique, que l'homme - comme l'animal - utilise pour son orientation temporelle, surtout quand lui font défaut les repères fournis d'ordinaire par les changements de son environement (chap. I).

Paul FRAISSE [184, p. 12]

CONCLUSIONS

Les résultats expérimentaux succinctement résumés et les considérations théoriques qui en découlement montrent clairement la complexité du problème du temps en biologie. Sa prédominance en tant que paramètre essentiel des phénomènes biologiques est apparent à tous les niveaux d'études, du vieillissement cellulaires aux cycles biologiques, des organismes monocellulaires à l'homme. Certaines horloges peuvent être synchronisées, "entraînées" sur un rythme compatible avec la survie optimale de la cellule ou de l'organisme, d'autres phénomènes suivent un rythme fortement dépendent de facteurs intrinsèques....

Ladislas ROBERT [418, p. 235]

If two people meet twice they must have lived the same time between the two meetings, even if one of them has travelled to a distant part of the universe and back in interim.

....

Although we cannot try the experiment of sending a man to another part of the universe, we have enough scientific knowledge to compute the rates of atomic and other physical processes in a body at rest and a body travelling rapidly. We can say definitely that the bodily processes in the traveller occur more slowly than the corresponding processes in the man at rest (i.e. more slowly according to the Astronomer Royal's time). This is not particularly mysterious; it is well known both from theory and experiment that the mass or inertia of matter increases when the velocity increases. The retardation is a natural consequence of the greater inertia. Thus so far as bodily processes are concerned the fast-moving traveller lives more slowly. His cycle of digestion and fatigue; the rate of muscular response to stimulus; the development of his body from youth to age; the material processes in his brain which must more or less keep step with the passage of thoughts and emotions; the watch which ticks in his waistcoat pocket; all these must be slowed down in the same ratio. If the speed of travel is very great we may find that, whilst the stay-at-home individual has aged 70 years, the traveller has aged 1 year. He has only found appetite for 365 breakfasts, lunches etc.; his intellect, clogged by a slow-moving brain, has only traversed the amount of thought appropriate to one year of terrestrial life. His watch, which gives a more accurate and scientific reckoning, confirms this. Judging by the time which consciousness attempts to measure after its own rough fashion - and, I repeat, this is the only reckoning of time which we have a right to expect to be distinct from space - the two men have not ***lived*** *the same time between the two meetings.*

....

It might be useful for each individual to have a private time exactly proportioned to his time lived....

....

Thus in physical time (or Astronomer Royal's time) two people are deemed to have lived the same time between two meetings, whether or not that accords with their actual experience.

Arthur EDDINGTON [110, pp. 38 - 40]

L'âge (biologique ou chronologique) est aussi directement observable et pourrait donc être quantifiable. Selon Richardson et Rosen, le coefficient phénoménologique L_i*, de la thermodynamique, permet de proposer une échelle pour le temps.*

Ladislas ROBERT [419, p. 11]

These observations all reflect upon the fact that time is not a measure of age. (1) In all physiological respects, a 50-year-old man can be younger than a 40-year-old colleague. (2) A 30-month-old rat can in a meaningful way be *compared in age to an 80-year-old man. Even beyond interspecies comparison, the concept of age crosses levels of organization; it is assumed that examination of old cultured cells bears upon aging of the whole organism. (3) An old man is not altogether old. At the cellular level, he is in fact a mosaic, having gut epithelial cells with a turnover time of about five days coexisting with nerve cells in the CNS as old as his chronological age.*

I. W. RICHARDSON [408, pp. 746, 747]

It is conceivable that in some small grain of sand which we can hardly perceive there is hidden a whole world in which there is an immense number of living beings so small that they escape not only our perception, but also the perception of those tiny living beings which we hardly observe under a microscope. Is it not possible that there be a long series of such worlds, which, with respect to one another have the same relation as our single grain of sand has to the whole world?

....

Whatever the truth of matter, it really seems beyond doubt that what is for us a vanishing instant seems to be a very long time to those very tiny living beings.

R. J. BOSCOVICH [45, p. 225]

If subjects are asked to indicate the sequence of two sensory stimuli, temporal order tresholds in the domain of approximately 30 ms are observed independent of the sensory modality.

....

Order tresholds indicate directly that temporal processing is discontinuous. Different physical stimuli which are processed within a ***temporal window*** *of approximately 30 ms are treated as* ***co-temporal****, i.e. a temporal relationship with respect to the before-after dimension cannot be established for such distinct stimuli. Information gathered within a temporal window of 30 ms is treated as* ***a-temporal****, i.e. there is no temporal continuity defined and definable for stimuli that follow each other within such intervals.* ***Time*** *in the Newtonian sense (Runhau and Pöppel 1991) does not exist on an experiential level for intervals shorter than approximately 30 milliseconds.*

Ernst PÖPPEL [388, pp. 108, 109]

Let us postulate that a person's age is measured by the number of beats produced by their heart since the birth. Further, let us say that a standard person has a heart which always stays in perfect synchrony with a standard clock. People with a negatively rated heart would be fortunate, because they would be using life less quickly than a standard person. Negatively rated people could wear two watches, a standard watch to enable them to fit in with normal activities in the standard world and an appropriately slow running watch (with built-in calendar) for their life reckoning!

Barrie J. TONKINSON [466, p. 233]

.... la flèche biologique et psychologique du temps, telle que nous la connaissons, doit représenter une adaptation nécessaire de la vie et de la conscience aux conditions de l'univers quadridimensionnel.

....

Il doit donc y avoir quelque principe fondamentale interdisant d'associer la flèche du temps biologique ou psychologique avec une flèche anti-Carnot de l'univers matériel.

O. Costa de BEAUREGARD [29, p. 113]

To reverse the situation, if we take the duration of a biological phenomenon as the unit of time, for instance the duration of the healing process for one square centimeter of wound, sidereal changes of increasing size will correspond to this unit as we grow older and we will say that time is passing more quickly.

....

We are justified in thinking that biological time has some influence on our estimations of duration. In fact we saw earlier that in animals and humans these estimations are dependent on the temperature which activates or slows down biological exchanges.

Paul FRAISSE [185, p. 248][185, p. 248]

For more on biological feeling of *time* see also the books by Beauregard [29], Fraisse [185], Fraser and Lawrence [189], Guyau [235], Robert [418] through [421], Whitrow [487] and the papers by Boscovich [45], Euler [172], Kampis [258], and Richardson [408].

2.5 Economics and *time*

Today, in most industrialized societies, time is highly valued and considered to be more scarce than ever. Scarcity of time is on verge of replacing the scarcity of money, not only for those individuals who have enough money already, but for the economic system of whole societies. Economic activity is increasingly measured by the number of hours it takes to produce certain goods, and this serves as an apparently sound basis for comparing the economic standards of different countries. Every sense of crisis, so often connected with the fear of scarcity, also induces in us a feeling that "time is running out", as though it were limited, just as money used to be before inflation became rampant. If one inquires into the reasons why time has become so scarce and highly valued, the answer is this: time has become the medium in which the results of production may accumulate. Yet, a strange, complementary relationship seems to exist: the more one wants to produce, the more time becomes scarce. Those who produce most have the least of it. It is also no longer necessary to limit time as a medium of production to the production of economic goods alone. It has become a medium of production of all sorts of human, especially social, activities. We

want to get to know more and more people; we want to do more and more things. Until recently, we were led to believe that we live in an age of abundance economically-speaking as well as with regard to social activities. So the question arises: is it possible to increase the amount of time available in order to produce more? Is it possible to "produce" time?
H. NOWOTNY [366, p. 331]

2.6 Human and *time*

.... l'homme est très faible par rapport au temps.
Pierre JANET [252, p. 20]

The language in which we think and the concepts we employ all originate in time.
Stephen KERN [264, p. 45]

Since children have not yet developed the art of remembering, for them everything is in the present.
Jean-Marie GUYAU [346, p. 98]

Time is one of the continuing, compelling and universal experiences of our lives, one of the primary threads which combine in the weave of our experience. All our perceptual, intellectual and emotional experiences are interwined with time. We continually feel *time passing but where does it come from? We continually experience it but we cannot taste it, see it, smell it, hear it or touch it. How, then,* do *we experience time? What do we* use *to experience it?*
Robert E. ORNSTEIN [368, p. 15]

Any human being can be aware of the passage of time and can measure this without recourse to scientific instrumentation. Feel your pulse at the wrist of one arm with a finger of the hand of the other. Feel the steady rhythm of pulsation, a result of rhythmic flow of blood through the vessels of the wrist, propelled by contractions of your heart. Next, start counting the progression of your heartbeats. Soon you may begin to wonder how many more heartbeats there are for you. For all of us know that when our hearts stop beating, life ceases for us. Life will go on only as long as our heart is beating rhythmically. Thus by counting the number of heartbeats that occur during some event, for example, during its entire span, we can have a quantitative measure of the time this process is taking.

Generalizing from such concepts the average individual might easily conclude that time is unidirectional : it can only progress forward and never turn back to replay what has already happened once. The humanistic philosophies and religions, spiritual frameworks of man's existence, have also relied on the unidirectionality of time as a fundamental concept.

....

Our heartbeat is quite regular : for a normal adult it is 60-80 beats per minute. In a sense it is like a watch. The question arises : should we go by a mechanical clock, by our own heartbeat, or by still some other, more refined, perhaps biological timing mechanism?

Cornelius A. TOBIAS [465, pp. 268, 278]

We should, however, be aware of the Western mythology of time (the informal elements of time), for we attribute to time the properties of a material: time has a value for the man in the street - the average man. It has material properties. Time can be earned, time can be spent. It can be saved, it can be wasted. It is a commodity. Time is money. It can be translated into something measurable. From this point of view our modern mythological approach is not very different from that of the ancient myths.

An Arab whose American companion asked him to meet him in an hour answered with his own question: "What do you mean in an hour? Has the hour capacity, has it a volume, that we will meet in the hour?" He reminded the American that in Arabic you can only say before or after the hour, but not in the hour. It is only in the philosophy of the Western world that the hour has capacity, because we give it material properties.

Aharon KATCHALSKY-KATZIR [262, p. 292]

Ainsi le temps : lui non plus n'existe pas en soi. Ce sont les événements et eux seuls qui nous donnent le sentiment que quelque chose est passé, est imminent, va arriver. Non, nous devons le reconnaître, la perception du temps en soi, indépendamment du mouvement des choses ou de leur placide immobilité, personne ne l'a.

LUCRECE [307, pp. 31, 32]

Time is born from the very activity of the man who tries to reconstruct the changes in which he takes part. .... With the notion of time we arrive at the most complete adaptation of man to the successions which form the thread of his existence. Man thus has the impression that his conception of time is that of an absolute time best formulated by Newton.

Paul FRAISSE [185, p. 285]

A l'égard du temps, il est d'abord certain que nous n'en avons la notion que par la succession de nos idées ; il ne l'est pas moins que ce n'est pas la succession de nos idées qui fait le temps, puisque le temps a une mesure indépendante de nos idées, mesure que nous fournit le mouvement des corps.

....

De même s'il n'y avoit rien, il n'y auroit point de temps, parce que l'idée de temps est relative à des êtres qui existent successivement ; et il y en auroit un, parce que le tems ne seroit alors que la simple possibilité de succession dans des *êtres qui n'existeroient pas ; succession qui n'est rien de réel qu'autant qu'il y a réellement des êtres existants.*

Jean D'ALEMBERT [84, p. 356]

So long as we do not go outside the domain of consciousness, the notion of time is relatively clear. Not only do we distinguish without difficulty present sensation from the remembrance of past sensations or the anticipation of future sensations, but we know perfectly well what we mean when we say that of two conscious phenomena which we remember, one was anterior to the other; or that, of two foreseen conscious phenomena, one will be anterior to the other.

When we say that two conscious facts are simultaneous, we mean that they profoundly interpenetrate, so that analysis can not separate them without mutilating them.

The order in which we arrange conscious phenomena does not admit of any arbitrariness. It is imposed upon us and of it we can change nothing.

Henri POINCARE [385, pp. 35, 36], [386, p. 317]

D'abord, il n'est une matière à aucun de nos cinq sens.

E. KLEIN et M. SPIRO [271, p. 12]

Pour l'homme le temps ne se déroule pas d'une façon uniforme. Le temps "objectif", celui du calendrier, ne coïncide pas forcement avec le temps subjectif qui paraît plutôt s'accélérer avec l'age.

Ladislas ROBERT [418, p. 214]

We have intimate acquaintance with the nature of time and so it baffles our comprehension.

....

We are accustomed to think of a man apart from his duration. .... But to think of a man without his duration is just as abstract as to think of a man without his inside.

Arthur EDDINGTON [110, pp. 51 - 53]

L'enfant vit, et dépense son temps, comme s'il avait une réserve infinie du temps. Il n'a pas le sentiment que le temps lui est mesuré, compté, et la mort concerne les autres. Comme il vit - normalement - dans la dépendance et la confiance, le souci du lendemain n'est pas son souci. Il vit dans l'immédiateté, le présent, sans tisser encore sa propre vie. A l'adolescence, l'individu "réalise" qu'il sera adulte. Il essaie ses forces en tous domaines afin de voir de quoi il est capable. De là beaucoup de velléités, de projets, d'hésitations, et une instabilité normale à cet âge. La mort est lointaine, irréelle. L'adolescent a le sentiment qu'il a ***le temps d'être****. Pour l'heure, il a conscience de ses virtualité, de ses promesses, et il s'essaie. La maturité vient, ou s'annonce, lorsque l'homme "réalise" que le temps dont il dispose lui est mesuré, qu'il n'a, somme toute, que* *peu de temps à vivre, et décide, alors, de se déterminer : refoulant ses rêves, il se résout à* ***n'être que*** *..., et à entrer dans la vie active. Durant l'époque de sa belle maturité, l'adulte croit à sa mort, mais dans un monde réel, riche et consistant. La mort n'irréalise pas encore le monde et les êtres, et ne gêne pas son bonheur, qui est jubilation vitale, spontanéité, plénitude. Puis vient la fêlure, le* ***lapsus****, la chute de la vitalité : premier signe de la vieillesse. C'est le moment où la pensée et l'angoisse de la mort viennent altérer le bonheur, et bientôt le rendent impossible en sa perfection humaine. Tout bonheur ne sera plus qu'un bonheur mitigé. Car la mort, désormais, se mêle à tout, intervient toujours pour nous séparer de tout : du présent, de ce que nous avons, des autres et de nous même.*

....

Le temps en soi est indépendant de nous et de l'esprit. C'est le temps de la nature, la puissance destinale qui entraîne toutes choses, humaines ou non humaines, vers leur néant.

....

Les êtres purement naturels sont purement temporels. Ils passent et ne le savent pas. Nous passons comme tout ce qui est ***dans*** *le temps, mais nous savons que nous passons : nous sommes dans le temps, et le temps, par son concept, est en nous. Ce n'est pas nous, cependant, qui avons le dernier mot, mais le temps, car c'est indéfiniment que les heures s'ajoutent aux heures, les jours aux jours, tandis que nous n'avons à vivre qu'un nombre fini d'heures et de jours. Alors, notre dernière heure écoulée, disparaît la pensée du temps. Reste le Temps.*

Marcel CONCHE [74, pp. 151, 152, 154, 213]

Et si l'on a toutes les raisons d'admettre que l'homme est en bonne santé, ce sera la preuve que son âge physiologique réel n'est pas le même que son age légal, officiel.

....

On conçoit l'espace comme quelque chose qui nous entoure et le temps comme quelque chose qui s'écoule, à coté de nous et à travers nous.

....

L'individu évolue donc dans un temps personnel qui possède un début et une fin ; mais le temps de l'espace à laquelle il appartient, le temps "enveloppe", n'a pratiquement ni commencement ni fin.

....

Le seul temps qui compte pour l'homme, c'est le sien propre, celui qui se place entre un berceau et une tombe.

....

Et ce temps physiologique, nous pouvons l'atteindre directement au moyen des deux méthodes que nous avons décrites dans ce livre : celle basée sur l'étude de la cicatrisation des plaies et celle basée sur les cultures de tissus. Nous avons déjà vu que l'on obtenait ainsi une mesure de l'âge physiologique réel.

Lecomte du NOÜY [365, pp. 129, 187, 221]

We have no doubt that time goes one way.
T. GOLD [201, p. 403]

The present is only reality. While it slips away, we enter into a new present, thus always remaining in the eternal Now.

....

The flow of time is not under our control. We cannot stop it; we cannot turn it back; we have the feeling of being carried away by it, helplessly, like a piece of lumber in the current of a river.

....

The coming of death is the inescapable result of the irreversible flow of time. If we could stop time, we could escape death-the fact that we cannot makes us ultimately impotent, makes us equals of the piece of lumber drifting in the river current. The fear of death is thus transformed into a fear of time, the flow of time appearing as the expression of superhuman forces from which there is no escape. The phrase "passing away", by means of which we evasively speak of death without using its name, reveals our emotional identification of time flow with death.
Hans REICHENBACH [406, pp. 2 - 4]

Subjective Age. It is customary in the United States to take chronological age (CA) very seriously as a basis for classifying individuals, preparing statistical information, and making decisions of many types. Gerontologists recently have intensified their efforts to supplement CA with dimensions that bear more relationship to the individual's actual capacities. Functional age (FA) is a general concept that, in turn, has started to be fractioned into more specific concepts that are operationally defined.[20] *One can speak of biological age as one realm of FA, but, even more specifically, of retinal, epidermal, or bone-marrow age. In other words, alongside the familiar, uniform, concensual increments of CA, one may now introduce a variety of specific time-lines whose properties remain to be determined. A person may be "biologically young" and "psychologically old", for example.*

As part of this development, renewed interest is being shown in subjective age: how old the person is to himself. Even within the realm of subjective age, it appears that differentiations must be acknowledged.

....

Recognition that most people (young and old) seem to have their private estimations and interpretations of their own age can be helpful in avoiding overgeneralizations based only upon CA and externalistic measures.

....

..... There is not much advantage to listing all possible thematics of time. The value resides in learning how an individual continues to express his relationship to life through his use of time as he ages, and what new themes, if any, emerge.

....

1. The same unit of time is radically different when viewed as proportion-of-life-remaining.[30] *One day is a tiny fraction of (assumed) future time for a young adult, but a more substantial "piece" of life for the aged person.*

R. KASTENBAUM [261, pp. 29, 30, 33]

Normal man gives little indication of possessing the 24 -hour clock. That, however, he still does possess it but in a submerged state becomes apparent from the fact that under pathological conditions - for instance, after a severe blow on the head, a bout of lethargic encephalitis - the clock may appear quite as clearly as in animals. ....

Early man up to 800,000 years ago probably manifested a clock of the same type found in rats, monkeys and apes. It told him when to seek shelter at night to avoid his enemies; and when to wake up to avoid his enemies. The more accurate the clock, the better chances of survival.

....

Now under normal conditions man gives little indication of possessing a 24 -hour clock as seen in animals - but pathological conditions can still bring it out in full force.

The 29-30 -day clock still manifests itself in normal human females but, otherwise, it too has become submerged to appear only under pathological conditions, chiefly associated with various psychiatric disturbances.

Little is known about the existence of an inherent yearly clock in man - except for records of appearance of yearly cycles under pathological conditions in a few psychiatric patients.

C. P. RICHTER [410, pp. 39, 46, 49]

For further reading see the books by Alexander [3], [4], Gouguenheim [204], Guyau [235], Husserl [248], Poulet [389] through [391], Ricœur [411] through [413], and the paper by Kampis [258].

2.7 Information and *time*

Knowledge is based on the possibility of distinction.

....

Basic Principle 1 (Principle of Distinguishability)

Conceptual knowledge presupposes distinguishability. Thus, distinctions are possible as far as conceptual knowledge is possible.

....

Terminological Introduction 1 (Binarity)

The simplest distinction possible, i.e., minimal distinguishability, is called a binarity.

....

Terminological Introduction 4 (Information)

Information is a measure of distinguishability. Its unit is called a bit*. One bit is the amount of information of one binarity.*

....

Basic Principle 2 (Principle of Temporality)

Empirical knowledge presupposes temporality. The difference between the past and the future is a precondition of experience.

....

Proposition 5 (Distinguishability and Temporality Are Interwoven)

Distinguishability and temporality are interwoven. Any temporal transition can be looked upon as a change of distinguishabilities. Distinctions which are made in the past lead to distinguishabilities in the future.

On the one hand, temporality always represents itself as a change of distinguishabilities or, in other words, of information in time. On the other hand, there is a clear difference between distinguishabilities (distinctions which are possible in future) and distinctions (of the past). Moreover, there is a temporal quality as regards the semantic and the pragmatic aspects of information. The former is correlated to the pre-existence of bits, whereas the latter is correlated to their creation.

....

Terminological Introduction 9 (Potential Information)

Distinguishabilities of the future are called potential information.

Terminological Introduction 10 (Actual Information)

Distinctions of the past are called actual information.

Terminological Introduction 11 (Flow of Information)

The flow of information has to be regarded as the transition of potential to actual information.

Holger LYRE [311, pp. 83 - 86]

For more details see the paper by Lyre [311].

2.8 Mathematics and *time*

Mathematically, time is modelled as a (strictly) ordered set T. Referring to the elements of T as instants and interpreting their ordering as earlier-later relation leads to (physical) Time as a set of points in a linear - successive order. Furthermore, this set is considered to form a continuum - the continuum of the real numbers. This is based on the idea that between two Time points there is always a third Time point. However, in measuring temporal data we do not observe points of Time; what we observe are, for example, the factual positions of the hands of clocks. Therefore, taking the linear continuum as the mathematical model of time is adequate for the following definition of Time: Continuous Time is the abstract structure of unlimited observability.

....

To repeat my thesis: The substance/permanence based reduction of reality leads to the subject-object duality, and, furthermore, to a dualistic description of substantialized time as

- Time: the counting of successive (f)actual states,

- the Now as a unifying power.

Eva RUNHAU [427, pp. 55, 56, 63]

*The discussion of time has greatly suffered from the confusion of two concepts, from neglecting the distinction between **order** and **direction**.*

The points on a straight line, which is infinite on both sides, are arranged in a certain order; but the line does not possess a direction.

....

The distinction between past and future is intended to express the direction of time; ...

....

*DEFINITION. An event **A** is **causally connected** with an event **B** if **A** is a cause of **B**, or **B** is a cause of **A**, or there exists an event **C** which is a cause of **A** and of **B**.*

....

*... any two events that are connected by a causal chain and thus are **causally ordered** are also **temporally ordered**. If **A** lies at the beginning of the chain and **B** at the end, **A** is called **earlier** than **B**, and **B** is called **later** than **A**. This determination is unique.*

....

*The time order described is linear, but not yet directed. We can reverse the direction of causally ordered events; then **A** is called later than **B**.*

....

.... the theory of relativity has not contributed to the problem of time direction, but only to that of time order...

....

DEFINITION. The direction in which most thermodynamical processes in isolated systems occur is the direction of positive time.

Hans REICHENBACH [406, pp. 2 6, 27, 29, 39, 42, 127]

We shall regard an instant as a fundamental concept, which, for present purposes, it is unnecessary further to analyse, and shall consider the relations of order among the instants of which I am directly conscious.

Thus for such instants we find the following properties:

*(1) If an instant **B** be **after** an instant **A**, then the instant **A** is not **after** the instant **B**, and is said to be **before** it.*

*(2) If **A** be any instant, there is at least one instant which is **after A** and also at least one instant which is **before A**.*

*(3) If an instant **B** be **after** an instant **A**, there is at least one instant which is both **after A** and **before B**.*

(4) If an instant ***B*** *be* ***after*** *an instant* ***A*** *and an instant* ***C*** *be* ***after*** *the instant* ***B****, then the instant* ***C*** *is* ***after*** *the instant* ***A****.*

(5) If an instant ***A*** *be neither* ***before*** *nor* ***after*** *an instant* ***B****, the instant* ***A*** *is identical with the instant* ***B****.*

Alfred A. ROBB [416, p. 14]

Le Temps, dans la Physique et la Mécanique classique est, avec la longeur et la masse, une grandeur fondamentale.

Cependant, son aspect immatériel ne le rend accessible à des mesures, nécessaires à une formulation mathématique des lois physiques, que d'une façon indirecte et très particulière.

En fait, le Temps apparaît dans les équations comme un paramètre t, sans implication sur la nature de ce paramètre si ce n'est qu'il est lié à la notion confuse et intuitive de temps s'étendant de moins l'infini à plus infini.

Jacque MERLEAU-PONTY and Bruno MORANDO [339, p. 134]

Definition 3.1 *(**Time Space** ...) A metric space (T,ρ) is called a* ***time space*** *if*

i) *T is fully ordered with order "$\prec$";*

ii) *T has a minimal element $t_{\min} \in T$, i.e., for any $t \in T$ and $t \neq t_{\min}$, it is true that $t_{\min} \prec t$;*

iii) *for any t_1, $t_2 \in T$ such that $t_1 \prec t_2$, it is true that*

$$\lambda\rho(t_1, t_2) = \rho(t_2, t_{min}) - \rho(t_1, t_{min}) \leq \rho(t_2, t_1),$$

where $\lambda \in (0, 1]$ is a constant;

iv) T is unbounded from above, i.e., for any $M > 0$, there exists a $t \in T$ such that $\rho(t, t_{min}) > M$.

Anthony N. MICHEL and Ling HOU [343, p. 174: in "a)"]

The mathematical model we shall use for space-time, i.e. the collection of all events, is a pair $(\mathcal{M},\mathbf{g})$ where $\mathcal{M}$ is a connected four-dimensional Hausdorff C^∞ manifold and **g** *is a Lorentz metric (i.e. a metric of signature +2) on $\mathcal{M}$.*

....

Strictly speaking then, the model for space-time is not just one pair $(\mathcal{M},\mathbf{g})$ but a whole equivalence class of all pairs $(\mathcal{M}',\mathbf{g}')$ which are equivalent to $(\mathcal{M},\mathbf{g})$.

....

The metric **g** *enables the nonzero vectors at a point $p \in \mathcal{M}$ to be divided into three classes: a nonzero vector $\boldsymbol{X} \in T_p$ being said to be* timelike, spacelike *or* null *according to whether g($\boldsymbol{X}$,$\boldsymbol{X}$) is negative, positive or zero respectively (cf. figure 5).*

S. W. HAWKING and G. F. R. ELLIS [241, pp. 56, 57]

'Geometric Algebra' (GA) is the algebra of multi-dimensional symmetry ..... Space-Time Algebra (STA) is the branch of GA which specialises in 3d-space

+1d-time. Maxwell's equations emerge as essential features of multi-vector symmetry in STA .... These successes suggest that the GA of '3+3' space-time should be studied if 3-d time is to be taken seriously.
J. E. CARROLL [58, p. 55]

For more reading see also the books by Brown [52], Davies [88], Foster and Nightingale [183], Hawking and Ellis [241], Hawking and Penrose [243], Lorentz, Einstein and Minkowski [304], Merleau-Ponty and Morando [339], Nevanlinna [359], Robb [417], and the papers by Earman [106], Kull [281], Minkowski [348], [349], and von Weizsäcker [474].

2.9 Philosophy and *time*

Une seule parole demeure, celle du chemin 'est'. Sur celui-ci se trouvent de nombreux signes montrant que étant, il est inengendré et impérissable, entier, unique, sans frémissement et sans fin. Jamais il n'était ni ne sera, puisqu'il est maintenant, tout entier un, continu.

....

Il n'est pas non plus divisible puisqu'entier, il est homogène. Il n'y a pas quelque chose de plus qui l'empêcherait de se tenir uni, ni quelque chose de moins, il est plein d'étant, il est tout entier, continu, car étant jouxte étant. De plus, immobile dans les limites de liens énormes, il est sans commencement et sans fin, puisque genèse et destruction ont été repoussées très loin, la conviction vraie les a écartées.

....

Rien n'est ni ne sera en dehors de l'étant, car le destin l'a enchaîné pour qu'il soit entier et sans mouvement.

....

Je termine ici, pour toi, mon discours digne de créance et ma pensée sur la vérité. A partir de maintenant, apprends les opinions des mortels en écoutant l'arrengement trompeur de mes paroles.
PARMENIDE [373, pp. 17, 19, 21]

C'est de temps que le temps se nourrit.
M. **AUDIBERTI** [19, p. 211]

Time cures Sorrow.
St. AUGUSTINE [453, p. 169]

The concept of time is among the most fundamental elements of the set of philosophical concepts.
Frederick M. KRONZ [280, p. 7]

It is not, I believe, too much to say that all the vital problems of philosophy depend for their solution on the solution of the problem what Space and Time[1] *are and more particularly how they are related to each other.*

S. ALEXANDER [3, p. 35]

Le démiurge a donc l'idée [186] *de fabriquer une image mobile de l'éternité; et, tandis qu'il met le ciel en ordre, il fabrique de l'éternité qui reste dans l'unité une certaine image éternelle progressant suivant le nombre, celle-là même que précisément nous appelons le "temps"* [187]*.*

En effet, les jours, les nuits, les mois et [e] *les années n'existaient pas avant que le ciel fût né; c'est en même temps qu'il construisait le ciel, que le dieu s'arrangea pour qu'ils naquissent. Tout cela, ce sont des divisions du temps, et les expressions "il était", "il sera", ne sont que des modalités du temps, qui sont venues à l'être; et c'est évidemment sans réfléchir que nous les appliquons à l'être qui est éternel, de façon impropre. Certes, nous disons qu'"il était", qu'"il est" et qu'"il sera", mais, à parler vrai, seule l'expression "il est" s'applique à l'être qui est éternel* [38a]*. En revanche, les expressions "il était" et "il sera", c'est à ce qui devient en progressant dans le temps qu'il sied de les appliquer, car ces deux expressions désignent des mouvements.*

....

Le temps est donc né en même temps que le ciel afin que, engendrés en même temps, ils soient dissous en même temps, si jamais ils doivent connaître la dissolution [190]*; en outre, le temps a été engendré sur le modèle de la nature éternelle, pour qu'il entretienne avec elle la ressemblance la plus grande possible.*

PLATON [379, pp. 127, 128]

II. Le temps présent ne dépend point de celui qui l'a immédiatement précédé ;

René DESCARTES [96, p. 592]

L'espace ou le dessin fini, le temps ou le dessin qui est en train de se faire en un mouvement universel qui est le temps.

....

Tout ce qui est mouvement est temps et sert à l'indiquer. Le temps considéré comme continuité du mouvement, ou durée.

....

Le temps est le sens de la vie[22]*.*

(Sens : comme on dit le sens d'un cours d'eau, le sens d'une phrase, le sens d'une étoffe, le sens de l'odorat).

Paul CLAUDEL [67, p. 35, 36, 48]

Parmenides, fr. 8, 5 as quoted by Simplicius[1] *seems to proclaim the doctrine of the Eternal Now clearly and succinctly....*

John WHITTAKER [488, p. 16]

Tout ce que le temps suppose est rejeté par Parménide au profit de l'affirmation d'une entité qui réalise précisément la privation de ces mêmes implications.

L'étant est la privation pure du temps.

....

Le temps n'est pas à vaincre mais à créer.

Catherine COLLOBERT [71, p. 229, 279]

<§ 4

Metaphysical exposition of the concept of time>

Time is <1)> not an empirical concept that is somehow drawn from an experience. For simultaneity or succession would not themselves come into perception if the representation of time did not ground them a priori. *Only under its presupposition can one represent that several things exist at one and the same time (simultaneously) or in different times (successively).*

2) Time is a necessary representation that grounds all intuitions. In regard to appearances in general one cannot remove time, though one can very well take the appearances away from time. Time is therefore given a priori. *In it alone is all actuality of appearances possible. The latter could all disappear, but time itself (as the universal condition of their possibility)*[a] *cannot be removed.*

3) This a priori *necessity also grounds the possibility of apodictic principles of relations of time, or axioms of time in general. It has only one dimension: different times are not simultaneous, but successive (just as different spaces are not successive, but simultaneous). These principles could not be drawn from experience, for this would yield neither strict universality nor apodictic certainty. We would only be able to say: This is what common perception teaches, but not: This is how matters must stand. These principles are valid as rules under which alone experiences are possible at all, and instruct us prior to them, not through it.*[b]

4) Time is no discursive or, as one calls it, general concept, but a pure form of sensible intuition. Different times are only parts of one and the same time. That representation, however, which can only be given through a single object, is an intuition. Further, the proposition that different times cannot be simultaneous cannot be derived from a general concept. The proposition is synthetic, and cannot arise from concepts alone. It is therefore immediately contained in the intuition and representation of time.

5) The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation ***time*** *must therefore be given as unlimited. But where the parts themselves and every magnitude of an object can be determinately represented only through limitation, there the entire representation cannot be given through concepts, (<for they contain only partial representations)>,*[c] *but immediate intuition must ground them.*[d]

<§ 5

Transcendental exposition of the concept of time.

I can appeal to No. 3 where, in order to be brief, I have placed that which is *properly transcendental under the heading of the metaphysical exposition. Here I add further that the concept of alteration and, with it, the concept of motion (as alteration of place), is only possible through and in the representation of time - that if this representation were not* a priori *(inner) intuition, then no concept, whatever it might be, could make comprehensible the possibility of an alteration, i.e. of a combination of contradictorily opposed predicates (e.g. thing's being in a place and the not-being of the very same thing in the same place) in one and the same object.*[e] *Only in time can both contradictorily opposed determinations in one thing be encountered, namely* ***successively****. Our concept of time therefore explains the possibility of as much synthetic* ***a priori*** *cognition as is presented by the general theory of motion, which is no less fruitful.>*[3°]

Emmanuel KANT [259, pp. 162, 163, 178 - 180 in English, pp. 72, 73 in French.]

Supprimez le temps, il ne reste plus rien. L'antithèse se trouve dans Hérodote : "Qu'on prodigue le temps, tout le possible arrive",....

....

.... toute réalité est la conséquence directe de la conjugaison de l'espace et du temps.

Lecomte du NOÜY [365, pp. 199, 200]

(1) There is only ***one*** *Time, and all different times are parts of it.*

(2) Different times are not simultaneous but successive.

(3) Time cannot be thought away, but everything can be thought away from it.

(4) Time has three divisions, the past, the present and the future, which constitute two directions and a centre of indifference.

(5) Time is infinitely divisible.

(6) Time is homogeneous and a ***Continuum****, i.e. no one of its parts is different from the rest, nor separated from it by anything that is not time.*

(7) Time has no beginning and no end, but all beginning and end is in it.

(8) By reason of time we count.

(9) Rhythm is only in time.

(10) We know the laws of time ***a priori****.*

(11) Time can be perceived ***a priori****, although only in the form of line.*

(12) Time has no permanence, but passes away as soon as it is there.

(13) Time never rests.

(14) Everything that exists in time has duration.

(15) Time has no duration, but all duration is in it, and is the persistence of what is permanent in contrast with its restless course.

(16) All motion is only possible in time.

(17) Velocity is, in equal spaces, in inverse proportion to the time.

(18) Time is not measurable directly through itself, but only indirectly through motion, which is in space and time together: thus the motion of the sun and of the clock measure time.

(19) Time is omnipresent. Every part of time is everywhere, i.e. in all space, at once.

(20) In time taken by itself everything would be in succession.

(21) Time makes the change of accidents possible.

(22) Every part of time contains all parts of matter.

(23) Time is the ***principum individuationis****.*

(24) The now has no duration.

(25) Time in itself is empty and without properties.

(26) Every moment is conditioned by the preceding moment, and is only because the latter has ceased to be. (Principle of sufficient reason of existence in time. - See my essay on the principle of sufficient reason).

(27) Time makes arithmetic possible.

(28) The simple element in arithmetic is unity.

A. SCHOPENHAUER [439, pp. 227 - 230]

Puisse le temps, toujours comme aujourd'hui régler sa prospérité, le combler de biens et lui procurer l'oubli de ses fatigues.

PINDARE [71, p. 266]

Thus Space and Time depend each upon the other, but for different reasons. But in each case the ultimate reason of the presence of the other is found in the continuity which in fact belongs to each of them as we find them in fact. Without Space there would be no connection in Time. Without Time there would be no points to connect. It is the two different aspects of continuity which compel us in turn to see that each of the two, Space and Time, is vital to the existence of the other.

S. ALEXANDER [3, pp. 47, 48]

Or l'acte libre se produit dans le temps qui s'écoule, et non pas dans le temps écoulé.

Henri BERGSON [36, p. 166]

The reason why the cause is objectively in the effect† is that the cause's feeling cannot, as a feeling, be abstracted from its subject which is the cause. This passage of the cause into the effect is the cumulative character of time. The irreversibility of time depends on this character.

....

Simple physical feelings embody the reproductive character of nature, and also the objective immortality of the past. In virtue of these feelings time is the conformation of the immediate present to the past. Such feelings are 'conformal' feelings.

....

And yet there is always change; for time is cumulative as well as reproductive, and the cumulation of the many is not their reproduction as many.

....

This genetic passage from phase to phase is not in physical time: the exactly converse point of view expresses the relationship of concrescence to physical time. It can be put shortly by saying, that physical time expresses some features of the growth, but not *the growth of the features. The final complete feeling is 'satisfaction'.*

Physical time makes its appearance in the 'coordinate' analysis of the 'satisfaction'. The actual entity is the enjoyment of a certain quantum of physical time. But the genetic process is not the temporal succession: such a view is exactly what is denied by the epochal theory of time.

....

The quantum is that standpoint in the extensive continuum which is consonant with the subjective aim in its original derivation from God. Here 'God' is that actuality in the world, in virtue of which there is physical 'law'.

There is a spatial element in the quantum as well as a temporal element. Thus the quantum is an extensive region.

....

Physical time expresses the reflection of genetic divisibility into coordinate divisibility.

....

The ultimate evil in the temporal world is deeper than any specific evil. It lies in the fact that the past fades, that time is a 'perpetual perishing'.

Alfred North WHITEHEAD [483, pp. 237, 238, 283, 289, 340]

For further reading see also the books by Alexander [3], [4], Aristotle [11], [12], Beauregard [29], Bergson [36], Bohm [43], Claudel [67], D'Alembert [83], [84], Earman, Glymour, and Stachel [107], Eddington [108], [110], Fraser and Lawrence [189], Kant [259], [260], Leibniz [289], Mourelatos [352], Poincaré [381], Prigogine [395], [396], Prigogine and Stengers [397], [398], Reichenbach [404], [406], Ricœur [411] through [413], Russell [428], Whitrow [487], and the papers by Barrow [25], Dauer [87], Grünbaum [215], Kronz [280], Leibniz and Clarke [290], Locke [294], Mays [333], Russell [429], [430], Schopehauer [439], St. Augustine [451] and Voisé [473].

2.10 Physics and *time*

After what has been said, the next thing is to inquire into time.

....

Some of it has been and is not, some of it is to be and is not yet. From these both infinite time and any arbitrary time is composed.

....

Now since what changes changes from something to something, and every magnitude is continuous, the change follows the magnitude: it is because the magnitude is continuous that the change is too. And it is because the change is that the time is. (For the time always seems to have been of the same amount as the change).

Now the before and after is in place primarily; there, it is by convention. But since the before and after is in magnitude, it must also be in change, by analogy with what there is there. But in time, too, the before and after is present, because the one always follows the other of them.

....

But whenever [we do perceive] the before and after, then we speak of time.

For that is what time is: a number of change in respect of the before and after. So time is not change but in the way in which change has a number.

....

But number is [so called] in two ways: we call number both (a) that which is counted and countable, and (b) that by which we count.

....

The now is in a way the same, and in a way not the same...

....

It is manifest too that, if time were not, the now would not be either, and if the now were not, time would not be.

....

Time is the number of the motion, and the now is, as the moving thing is, like a unit number.

Moreover, time is both continuous, by virtue of the now, and divided at the now.

....

Not only do we measure change by time, but time by change also....

....

Time will measure what is changing and what is at rest....

....

And so all that neither changes nor is at rest is not in time; for to be in time is to be measured by time, and time is a measure of change and rest.

....

... time is always at a beginning and at an end. And for this reason it is thought always different.

....

It is in time that everything comes to be and ceases to be.

....

But the before is in time, for we use 'before' and 'after' according to the distance from the now, and the now is boundary of the past and the future.

....

.... time is everywhere the same...

ARISTOTLE [10, pp. 41, 43 - 46, 48 - 52]

For it is obvious that time is that in which the movement has occurred.

....

We must now enquire in what sense it is number of movement or measure[7] *- for it is better to call it measure of movement, since movement is continuous.*

....

But again, since time is, and is said to be, unbounded, how could it have a number?

PLOTINUS [380, pp. 175, 177]

Time it is, by which we measure the Motion of Bodies

....

Time therefore is not the motion of bodies.

St. AUGUSTINE [451, pp. 182, 183]

... while in the eternal nothing is flitting, but all is at once present, whereas no time is all at once present...

St. AUGUSTINE [454, p. 233]

Eh bien, je ne crois pas que l'on puisse trouver un seul acte primaire qui soit en rapport avec le temps. Il n'y a aucune excitation physique, aucune excitation matérielle qui détermine un acte en rapport avec le temps. Toute espèce d'excitation physique détermine en nous une réaction motrice qui est une adaptation à l'espace et ne contient pas d'adaptation au temps. Pour qu'il y ait adaptation au temps, il faut quelque chose de nouveau, de surajouté. Il existe alors ce que nous appelons les actes secondaires.

Pierre JANET [252, p. 53]

... several astronomers arrived at the conclusion that the rotation of the earth is gradually becoming slower and slower. They have found that in every millennium the last *stellar day is by one thousandth second longer than the* first *one.*

....

It would be, therefore, absurd to claim that two given intervals of time are equally long as they coincide with the same number of stellar days or stellar seconds.

C. NEUMANN [357, pp. 233, 234]

On ne peut comparer ensemble deux choses d'une nature différente, telles que l'espace et le temps : mais on peut comparer le rapport des parties du temps avec celui des parties de l'espace parcouru. Le temps par sa nature coule uniformément, ...

....

De-là il résulte que le mouvement uniforme est la mesure du temps la plus simple.

Jean D'ALEMBERT [84, pp. 125, 126]

Avant Galilée, le temps n'était pas une grandeur mesurable.

La vérité est que la bille roulante "chronométrée" par quelque sablier, ou (s'il faut en croire la légende) que le pendule pisan "chronométré" par le pouls de *l'auteur de la Dynamique, auront été, du premier coup, des réalisations variées du nouvel et universel étalon du temps : celui par la formule* $\boldsymbol{F} = m\boldsymbol{\gamma}$.

La formule de Galilée-Newton, en effet, introduit une relation universelle entre les quatre étalons de la longueur, de la force, de la masse et du temps. Des grandeurs correspondantes, les trois premières sont directement mesurables ; la formule nouvelle rend donc par définition le temps mesurable (indirectement).

....

En outre, une définition relative au temps : celle de l'unité de temps, qui se trouve directement rapportée à celle de l'espace par la loi d'inertie $v =$ *constante.*

....

... il y a une connexion très intime entre la définition du temps comme grandeur mesurable et l'énoncé du principe de relativité.

O. Costa de BEAUREGARD [30, pp. 35, 36]

An instant of time, without duration, is an imaginative logical construction.

Alfred North WHITEHEAD [482, p. 65]

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

H. MINKOWSKI [349, p. 339]

Physical time is, like space, a kind of frame in which we locate the events of the external world.

....

I think we might go so far as to say that time is more typical of physical reality than matter, ...

Arthur S. EDDINGTON [110, pp. 40, 275]

Every reference body (co-ordinate system) has its own particular time; unless we are told the reference-body to which the statement of time refers, there is no meaning in a statement of the time of an event.

Albert EINSTEIN [154, p. 26]

A relativistic view of time is adopted so that an instantaneous moment of time is nothing else than an instantaneous and simultaneous spread of the events of the universe. But in the concept of instantaneousness the concept of the passage of time has been lost.

....

A moment expresses the spread of nature as a configuration in an instantaneous three dimensional space. The flow of time means the succession of *moments, and this succession includes the whole of nature. Rest and motion are direct facts of observation concerning the relation of objects to the durations whose limits are the moments of this flow of time. By means of rest a permanent point is defined which is merely a track of event-particles with one event-particle in every moment.*

....

The properties of time and space express the basis of uniformity in nature which is essential for our knowledge of nature as a coherent system. The physical field expresses the unessential uniformities regulating the contingency of appearance. In a fuller consideration of experience they may exhibit themselves as essential; but if we limit ourselves to nature there is no essential reason for the particular nexus of appearance.

Thus times and spaces are uniform.

....

Time and Space. The homogeneity of time with space arises from their common share in the more fundamental quality of extension which is a quality belonging exclusively to events. By extension I mean that quality in virtue of which one event may be part of another or two events may have a common part. Nature is a continuum of events so that any two events are both parts of some larger event.

The heterogeneity of time from space arises from the difference in the character of passage in time from that of passage in space.

....

A 'spatial' route is a route which lies entirely in one instantaneous space. A 'historical' route is such that no two of its event-particles are simultaneous according to any time-system.

....

Thus the distinction of time from space, which I have just asserted, consists in the fact that passage along a spatial route has a different character from passage along a historical route.

Alfred North WHITEHEAD [481, pp. 7, 8, 67, 68]

Le Temps est de l'espace, et quelque chose de plus. Ce "quelque chose", c'est le progrès de l'instant qui, en avançant sur la ligne du Temps, l'enrichit d'une qualité supplémentaire, non spatiale, dont les équations différentielles ne rendent pas compte. Le Temps mathématique classique ne peut donc pas être considéré comme représentant le concret dans toute sa plénitude, mais seulement la forme spatiale du Temps des choses.

....

En premier lieu il est impossible de séparer physiquement le Temps de l'Espace. Cette union ne nous autorise pas à voir dans le temps une quatrième dimension de l'espace.....

....

En second lieu il est impossible de séparer physiquement l'Espace-Temps de la matière et de l'énergie.

....

En dernier lieu, il faut abandonner la croyance en un Temps universel, indépendant de tous les objets et de tous les phénomènes qui sont plangés dans son courant. Chaque objet – galaxie, planète, homme, molécule, proton – possède un temps propre, et ce Temps propre varie lorsque varie la vitesse de cet objet. Le temps d'une horloge en mouvement est plus lent que celui d'une horloge immobile.

François Le LIONNAIS [293]

There is only one Space and one Time, and though the mathematicians may deal with it by methods different from those of philosophy and common sense, it is still the same Space and Time which they all investigate each in his different way.

....

Space-Time therefore is neither in Time nor in Space ; but it ***is*** *Time and it* ***is*** *Space.*

S. ALEXANDER [3, p. 343]

The physical world of our experience is four dimensional. One of these dimensions is qualitatively distinct; it is called ***time****. The other three dimensions are called* ***space****, while the whole four dimensional structure is referred to as* ***space-time****.*

P. C. W. DAVIES [88, p. 10]

Ce "temps des physiciens", plutôt qu'un temps objectif, est un temps objectivé, qu'il faut éviter de réifier. Car le risque est grand de croire que le temps <u>*est*</u> *cet axe linéaire tracé sur tant de graphiques ... De fait, notre notion intuitive et concrète du temps est évidemment beaucoup plus riche que l'abstraction qu'en a tirée la physique théorique. Le temps tel que nous le vivons n'est évidemment pas uniforme, ni réversible ; la dissymétrie entre le passé et le futur en est même l'une des premières caractéristiques. Ce temps, le nôtre, n'est pas séparable en instants ponctuels ; sa structure est beaucoup plus floue : notre présent n'est pas un point sur cet axe abstrait, mais une petite zone temporelle (de quelques millisecondes ?) que fait "glisser" le cours du temps. La séparation entre le futur et le passé n'est pas cette coupure discontinue et sans extension que serait un présent sans épaisseur, mais une transition continue, qui transforme progressivement l'un en l'autre ; le présent est justement ce processus de transformation. Enfin, le temps n'a pas que cette "largeur", il a aussi une "épaisseur" : plutôt que par mince filament, il serait mieux décrit par l'image d'un cordage tressé.*

Jean-Marc LEVY-LEBLOND [291, p. 279]

Que le passé ne soit pas, puisqu'il n'est plus, j'en suis évidemment d'accord. .... Le temps passé ne revient pas, et c'est ce qu'on appelle le passé.

....

L'avenir n'est jamais donné (s'il l'était, il serait présent) : l'avenir est à venir, s'il vient, et c'est pourquoi il n'est pas.

....

... le présent ne m'a jamais fait défaut, je ne l'ai jamais vu cesser, jamais vu disparaître, mais seulement durer, toujours durer, avec des contenus certes différents, mais sans cesser pour autant de continuer et d'être présent.

....

... le temps n'est pas autre chose sans doute que cette présence à soi de l'espace ou de la matière.

André Comte SPONVILLE [73, p. 248, 254]

... le temps diffère continuellement de lui-même, et cella autrement que diffère d'elle-même la série des nombres entiers lorsqu'on ajoute chaque fois un. Car un est toujours un, alors qu'aucune différence temporelle n'est identique à aucune autre, ... Ce n'est donc pas d'une manière simplement quantitative, mais c'est qualitativement que le temps diffère continuellement de lui-même.

Marcel CONCHE [74, p. 211]

*Let us begin by distinguishing **quantitative** from **qualitative** properties of time. In measuring time by the help of clocks we make use of its quantitative, or **metrical**, properties. .... The theory of the metrical properties of time has been developed in great detail in modern physics-in particular, in Einstein's theory of relativity....*

*Our inquiry shall be focused on the qualitative, or **topological**, properties of time.*

....

Statement 1. Time goes from the past to the future.

....

Statement 2. The present, which divides the past from future, is now.

....

Statement 3. The past never comes back.

....

Statement 4. We cannot change the past, but we can change the future.

....

Statement 5. We can have records of the past, but not of the future.

....

Statement 6. The past is determined, the future is undetermined.

Hans REICHENBACH [406, pp. 20 - 27]

It is not always appreciated that the direction of the flow of time is not just dependent upon the behaviour of the universe around us, but also upon the proper clock of the observer. We have seen that such a clock-observer cannot be contained within a point but requires a finite region of space-time. It is very real in the physical sense and cannot be assumed to exist where matter itself cannot exist, e.g. in an environment where there is free radiation but no rest mass.

It can, however, be identified with the presence of fundamental particles having rest mass and thus, for example, with the electron and proton. Such clocks can, of course, be annihilated. The fundamental observer together with his proper clock then ceases to exist, and proper time in that region is no longer valid. On the other hand, pair production can produce fundamental clock-observers whose proper time starts from the moment of formation.

Roger C. JENNISON [254, p. 79]

L'entropie devient ainsi un "indicateur d'évolution", et traduit l'existence en physique d'une "flèche du temps" : pour tout système isolé, le futur est la directon dans laquelle l'entropie augmente.

Ilya PRIGOGINE and Isabelle STENGERS [397, p. 189]

Setting aside the guidance of consciousness, we discover a signpost for time in the physical world itself. The signpost is a particular one, and I would not venture to say that the discovery of the signpost amounts to the same thing as the discovery of an objective 'going of on time' in the universe. But at any rate it provides a unique criterion for discriminating between past and future, whereas there is no corresponding absolute distinction between right and left. The signpost depends on a certain measurable physical quantity called entropy. Take an isolated system and measure its entropy at two instants t_1 and t_2: the rule is that the instant which corresponds to the greater entropy is the later. We can thus find out by purely physical measurements whether t_1 is before or after t_2 without trusting to the intuitive perception of the direction of progress of time in our consciousness. In mathematical form the rule is that the entropy S fulfils the law: dS/dt is always positive. This is the famous Second Law of Thermodynamics.

Arthur S. EDDINGTON [112, p. 463]

Time has no arrow.

....

Clausius' principle of entropy increase in every closed system was obviously not covariant under time reversal. Especially since the beginning of this century, however, the Ehrenfests [13] *and Smoluchowski* [14] *have offered a convincing analysis and refutation of Clausius' principle without affecting the practical applicability of the latter. To the best of my knowledge, their conceptual analysis has never been seriously challenged.*

To put the main result of this analysis in a nutshell: The entropy of a closed system may either decrease or increase, and its changes are governed, accordingly, by time-symmetrical laws.

....

Consequently, neither a phenomenologically nor a statistically defined entropy provides time with an arrow.

Henryk MEHLBERG [337, pp. 108, 114, 115]

The most important classes of phenomena characterizing a direction in time (which since Eddington are called ***arrows of time****) are:*

1. ***Radiation:*** *......*

2. ***Thermodynamics****: The Second Law* $dS/dt \geq 0$ *characterizes a direction in time and is conventionally also formulated as a law of nature ..... This arrow of time is clearly the most important one for our everyday life. It is expected by its applicability to human memory and other physiological processes to be responsible for our actual impression of a direction of time (the flow of time).*

3. ***Evolution:*** *....*

4. ***Quantum Mechanical Measurement:*** *....*

5. ***Exponential Decay:*** *....*

6. ***Gravity:*** *....*

....

...... thus arriving at the concept of a ***thermodynamico-mechanistic time****. The empirical basis of this concept is the observation that the thermodynamical arrow of time always and everywhere points in the same direction.*

....

However, statistics as a method of counting has nothing a priori to do with the physical concept of time and its direction. It is therefore not able to explain by itself the thermodynamical arrow of time.

....

.... the time coordinate is physically meaningless...

H.-Dieter ZEH [497, pp. 3, 4, 9, 31, 129]

Time and causality are not so samey as space. 'Before' is not the same as 'after', and causes are not the same as effects. Nevertheless, in other respects time and causality do manifest important sameness. Time is homogeneous – origin-indifferent – and appears to have no natural metrics; and it is characteristic of physical causes that they should be repeatable, that is, that the same cause should be followed by the same effect.

J. R. LUCAS and P. E. HODGSON [306, p. 29]

What we explicitly assume, as a postulate, is that there exists ***a universal time by means of which dynamical interactions are correlated****.*

L. P. HORWITZ [247, p. 111]

All change takes place in time. In order to determine this time, there must be a privileged, absolutely uniform movement which marks the duration of all other changes. Where is this primary clock to be found?

P. DUHEM [104, p. 185]

For, as it has been said, time exists by itself and in no way depends on motion; whatever characteristics it has, it has them all from itself and none from motion.

....

... time is the interval, duration and extent, not over which or through which, but ***in*** *which all motion and change occur. And motion is the measure of time: though not every motion, but only the motion of the celestial sphere which is truly one, continuous and uniform.*

B. TELESIO [461, pp. 187, 188]

Time which is the measure of motion is not in the heaven, but in the stars....

....

.... there are as many times in the universe as there are stars.

....

Motion is the measure of time rather than time being the measure of motion...

....

.... no time is the measure of motion unless previously some motion was the measure of time.

....

Nevertheless we say that time exists even if all things were at rest.

G. BRUNO [53, pp. 189 - 191]

Duration is fleeting extension.

....

Time is duration set out by measures.

....

A good measure of time must divide its whole duration into equal periods.

....

The revolutions of the sun and moon the properest measures of time.

....

No two parts of duration can be certainly known to be equal.

J. LOCKE [294, pp. 211, 215, 217]

Before describing the above methods we want to emphasize that time as such has no particular rhythm and time can adequately be expressed in terms of very different measures. All that we can say about time is that it flows into one direction only.

In the literature there exists a considerable confusion because the selection of a distinguished scale for the measure of time is confused with what is called a "definition of time".

L. JÁNOSSY [253, p. 95]

There are three ways in which our world appears to be unbounded and thus, perhaps, infinite. It seems that time cannot end. It seems that any interval of space or time can be divided and subdivided endlessly.

....

But recently it has become an established fact that the universe does have a beginning in time known as Big Bang. The Big Bang took place approximately

15 billion years ago. At that time our universe was the size of a point, and it has been expanding ever since. What happened before the Big Bang?

....

If the universe really does start as a point and eventually contract back to a point, is it really reasonable to say that there is no time except for the interval between these points? What comes before the beginning and after the end?

Rudy RUCKER [425, pp. 9 - 12]

Space as such is fixed. A point or a line does not flow out of existence. It is permanent and stable. But an instant no sooner comes into being than it passes away, not into another position but into non-existence, to be replaced by another. Memory and anticipation give us cognitive access to the past and future. Hence we may imagine a stretch of time as a world line. But closer analysis will show that this analogy with a spatial line breaks down, for all the parts of a line are coexistent. The parts of a duration, on the other hand, cannot coexist. They are necessarily successive. This temporal separation, or inability to be together, radically distinguishes a duration from a line.

In the second place, there is nothing in a single spatial dimension which corresponds to the present, past, and future, the so called modes of time. All the points on a given dimension exist together. Hence the selection of a single point is arbitrary. It may be here or there. But the instants of time do not coexist in this way, and the selection is not arbitrary. In fact, no selection of an instant in real time can ever be made, for I am restricted to the now that is given me now in the order of time. This order is inalterable.

....

The flow of time is irreversible, and in one direction only. It will not flow backwards, nor in a perpendicular direction.

John WILD [491, pp. 541, 542]

The scalings of time and space, the anticipatory spinbacks (and the resultant openings and closings of the spaces along a radial path) allow us to understand the propagation of the phenomenon of gravitation and the resulting distribution of masse.

Jean-Pierre GARNIER-MALET [193, p. 154]

The "doubling theory" considers that any space (dust, atom, cell, planet, star, galaxy, ...) is at the same time - but not in the same time flow - an horizon and a particle in another horizon. With a system of motions, called "fundamental motion", different stroboscopic time flows in different horizons allow any particle (or horizon) to become two doubled particles (or horizons) in temporal holes of its time flow.

***The anti-gravitation, the big bang, the missing masses**, the Universes (corresponding to three time flows of any particle or horizon) are explained by this cycle.*

....

Space and time scalings (Garnier-Malet, 1999, 2000) allow any particles to exchange their horizon during common temporal holes.

The end of our solar cycle allows this exchange in the final common temporal hole of the six embedded horizons of our three time flows (past, present, future).

The cycle of 25000 years is the time to transform an external initial virtual horizon to an internal virtual particle. This doubling is non-observable. It uses temporal holes in a virtual horizon which seems to be a real particle. But now, it is the end of our solar temporal hole and the beginning of a temporal window of the initial external horizon.

Some planetary disturbances could arise.

When the differences of the three time flows will be finished, the stars will seem to go down in the sky because C_0 of the particle becomes C_1 of the horizon when C_1 of the particle becomes C_2 of its horizon.

Jean-Pierre GARNIER-MALET [195, pp. 311, 312, 320]

The doubling theory [1] completes the basic principles of modern physics without throwing away existing laws.

....

This theory introduces a discontinuous flow of time which is defined by a succession of observation instants separated by non observation instants. .... we can define a stroboscopic time. This discontinuous time is apparently continuous for an observer which is moving in this time.

....

A flow of time can be defined by a periodical motion of a space into the horizon of the observer.

....

The exchange of radial and tangential is made with a time acceleration from 1 to 10.

....

- Invariance of inversion of the direction of time T : all the reactions between the elementary particles are also possible with a reversed time.

....

The fundamental doubling motion defines imperceptible times called "temporal openings" in which the time flow can be accelerated. Conversely, any time flow can be the consequence of "temporal openings" of decelerated time flow.

Jean-Pierre GARNIER-MALET and Philippe BOBOLA [196, pp. 123, 124, 132, 138, 139]

A fundamental doubling movement divides an initial time flow into several embedded time flows by using imperceptible "openings of time" in which the time flow is accelerated.

Jean-Pierre GARNIER-MALET and Philippe BOBOLA [197, p. 232]

Einstein time t', proper time (τ) and the Galilean invariant time T (same in both reference frames in relative motion) become numerically equal, showing again there is no time dilation.

....

We immediately see that there is no space contraction. *The postulate of space and time is respected. We are living in Galilean space and time!*

....

Moreover, there is only one Universe with one universal time. Time travel is absolutely physically impossible as it would entail the existence of an infinity of simultaneous past and future universes.

Adolph MARTIN [332, pp. 51, 52, 54]

The following authors wrote on *time* and physics in their books:

Alexander [3], Aristotle [11], [12], Beauregard [29], Cohen-Tannoudji and Spiro [70], D'Alembert [83], [84], Eddington [108], [110], Gouguenheim [204], Guyau [235], Hawking [240], [242], Kouznétsov [278], Lévy-Leblond [291], Lorentz, Einstein and Minkowski [304], Lucas and Hodgson [306], Lucrèce [307], Mehlberg [337], Merleau-Ponty and Morando [339], Nevanlinna [359], Nordenson [361], Ohanian [367], Poincaré [381], Prigogine [394], [395], [396], Prigogine and Stengers [397], Reichenbach [404], [406], Robb [417], Rucker [425], Wetzel [479], Whitrow [487], and Zeh [497].

Further on *time* and physics can be found in the following papers by:

Balian [19], Barrow [25], Bohm [42], Clarke [66], Duhem [104], Eddington [112], Gassendi [198], Gold [201], Grünbaum [215], Lindsay and Margenau [292], Neumann [357], Poincaré [386], Robb [417], Russell [429], [430], Schopenhauer [439], Telesio [461], Whitehead [485], Wiener [490] and Wild [491].

2.11 Psychology and *time*

Suppress the perception of differences and you suppress time.

....

Movement through space is what creates time in human consciousness. No movement, no time!

....

.... we measure time on the basis of the number of sensations and in no way on the basis of their pure duration; ....

Jean-Marie GUYAU [346, pp. 103, 116, 128]

Suprimez l'esprit, reste la durée sans limite : reste le présent.

André Comte SPONVILLE [73, p. 252]

Psychological time is doubtless conditioned by biological time, but one cannot be equated with the other, for the psychological processes are more complex,[15] *since they involve all the functions: the phenomena of contrast, automatic corrections, habit, etc.*

Paul FRAISSE [185, p. 248]

Le temps "objectif", celui du calendrier, ne coïncide pas forcement avec le temps subjectif qui paraît s'accélérer avec l'age.............. Pour notre organisme aussi il existe un déroulement du temps différentiel. Le déclin des fonctions suits des vitesses très différentes d'un organe ou tissu à l'autre.

Ladislas ROBERT [418, p. 214]

Le temps a besoin de l'âme, non pour être ce qu'il est (le temps présent), mais pour être ce qu'il n'est plus ou pas encore (la somme d'un passé et d'un avenir) : il a besoin de l'âme, non pour être le temps réel, le temps du monde ou de la nature, mais pour être, et c'est assez logique, le temps .... de l'âme!

André Comte SPONVILLE [73, p. 244]

Quant à la réversibilité du temps, dont parlent la mécanique quantique et la relativité, elle n'existe qu'au niveau des souvenirs évoqués grâce à l'évolution et à la complexité de notre système nerveux central.

Ladislas ROBERT [418, p. 235]

For it is not true that a day which in the state of hope appears long and in the state of fear short is either extended or contracted by the effect of such thoughts.

P. GASSENDI [198, p. 197]

It would be at once admitted that mental acts are related in time, they are either simultaneous or successive, but it would not universally or even commonly be admitted that they are spread out in space. Further, it is clear that the mental act stands in a temporal relation to its object ; whether of simultaneity or succession is not obvious from direct experience.

S. ALEXANDER [3, p. 27]

One of the innate dispositions of humans is an awareness of passing time and the ability to recollect or anticipate it as a concept which can be expressed. .... This awareness of time is the fifth psychological given.

....

For the self, time is entirely subjective, and subjective time may have little relation to external or clock time. Time does not exist as an independent philosophical or physical entity. It is a result of the reciprocal inter-relation of experience, maturation, learning and innate personality predispositions. The time which the self knows has quality as well as extent. As Whitrow has said, we do not experience time, per se, *but only what goes on in time, and the experience of time has both quality and quantity. For example, the child will experience* *as very long the time it takes to fill a pail at a well, or to wait and then run at a traffic stop light. This is time's extent. Habituation later changes this childhood perception of time by levelling certain aspects and sharpening other aspects, thus altering its quality. The self grows with these time changes. Furthermore, the self finds itself only in relation to time. For example, to ascertain whether one knew Harry before Dick, one compares oneself at various time periods: 'I knew Dick when I was still a student, but did I know Harry? Later on I took that job; yes, that's when I met Harry.'*

People who are disoriented in their former relation to time are maladjusted, for they have no past time. Amnesia for time cripples the self, causing it to feel empty and unable to function in the present. The confusion of schizophrenes concerning present time is a crucial aspect of their disorder.[9,10] *For example, schizophrenes sometimes think they are children when they are grownups, or otherwise forget themselves in past time. .... Returning soldiers, who have participated in a period of shocking war, report confusion as to who they really are,*[11,12] *because the nature of the self needs to be re-etsablished and related to its time.*

....

..... The self is inextricably bound to time because neither exists without the other. The self is the agent upon which time is recorded, but the agent cannot grow or act without the experience of time.

....

Time exists only when there is a psychological agent or self which perceives it, whether consciously or unconsciously. The self, in turn, is strung out along the thread of its time. It travels back and forth along this thread to search for memories or to project the future. But if the self is not found there, neither is time.

H. B. GREEN [207, pp. 2, 3, 14]

Nous avons en nous une machine à enregistrer le temps : c'est notre subconscience ; et une machine à concevoir le temps : c'est notre intelligence. Les deux mécanismes, bien que différents, sont basée sur la mémoire. Notre subconscience fournit à notre intelligence un renseignement brut : le temps semble s'écouler plus vite au fur et à mesure qu'on vieillit.

Lecomte du NOÜY [365, p. 237]

Roughly speaking, there are two levels of mind which I shall call R-mind and W-mind. Logic and systematical thinking belongs to W-mind, whereas speculation and intuition originate from R-mind. I suggest, W-mind tends to be in close contact with the external world, whereas R-mind feels itself more associated with inner reality. There are no sharp boundaries, however: mathematics as well as mathematical physics overlaps both R-mind and W-mind, and the link between mind and body is as close as that between knowledge and experience.

....

*Complex combinations of the multiple units of time-*cum*-space constitute* *the universe. The smallest units of time and their associated regions of space belong to R-mind, whereas the more coarse tempo-spatial regions are inquired into W-mind.*

S. C. TIWARI [463, p. 231]

Il devient dès lors évident qu'en dehors de toute représentation symbolique le temps ne prendra jamais pour notre conscience l'aspect d'un milieu homogène, où les termes d'une succession s'extériorisent les uns par rapport aux autres.

Henri BERGSON [36, p. 92]

For further reading see also the books by Beauregard [29], Bergson [36], Fraisse [185], Fraser and Lawrence [189], Guyau [235], Janet [252], Ornstein [368], and the papers by Comte-Sponville [73], Fraser and Lawrence [189], as well as by Gassendi [198].

2.12 Religion and *time*

The distinguishing mark between time and eternity is that the former does not exist without some movement and change, while in the latter there is no change at all. Obviously, then, there could have been no time had not a creature been made whose movement would effect some change. It is because the parts of this motion and change cannot be simultaneous, since one part must follow another, that, in these shorter or longer intervals of duration, time begins. Now, since God, in whose eternity there is absolutely no change, is the Creator and Ruler of time, I do not see how we can say that He created the world after a space of time had elapsed unless we admit, also, that previously some creature had existed whose movements would mark the course of time.

Again, sacred and infallible Scripture tells us that in the beginning God created heaven and earth in order. Now, unless this meant that nothing had been made before, it would have been stated that whatever else God had made before was created in the beginning. Undoubtedly, then, the world was made not in time but together with time. For, what is made in time is made after one period of time and before another, namely, after a past and before a future time. But, there could have been no past time, since there was nothing created by whose movements and change time could be measured.

The fact is that the world was made simultaneously with time, if, with creation, motion and change began.

Saint AUGUSTINE [433, pp. 211, 212]

ON NE PEUT CONCEVOIR UN TEMPS ANTERIEUR A L'EXISTENCE DU MONDE, CAR DIEU A CREE L'UN AVEC L'AUTRE

Si quelque esprit léger, vagabondant à travers les images des temps écoulés, s'étonne que vous, le Dieu toutpuissant, qui avez crée et conservé toutes choses, vous, l'ouvrier du ciel et de la terre, vous vous soyez abstenu, jusqu'aux jours *de la création, pendant des siècles innombrables, d'une telle œuvre, que celui-là s'éveille et prenne conscience de l'erreur attachée à son étonnement.*

....

Car ce temps même, c'est vous qui l'avez créé, et les temps n'ont pas pu s'écouler avant que vous fissiez les temps.

....

Tous les temps sont votre œuvre, vous êtes avant tous les temps et il ne se peut pas qu'il y eût un temps où le temps n'était pas.

....

LE TEMPS EST UNE DISTENSION DE L'AME

....

D'où il résulte pour moi que le temps n'est rien d'autre qu'une distension. Mais une distension de quoi, je ne sais au juste, probablement de l'âme elle-même.

Saint AUGUSTIN [434, pp. 262, 263, 275]

Il [s'en] suit de la Perfection Supreme de Dieu, qu'en produisant l'Univers il a choisi le meilleur Plan possible où il y ait la plus grande varieté [possible] avec le plus grand ordre [possible] ; le terrain, le lieu, le temps, les mieux menagés ; ....

LEIBNIZ [289, p. 49]

That the world and time had both one beginning, and the one did not anticipate the other

For if eternity and time are rightly distinguished by this, that time does not exist without some movement and transition, while in eternity there is no change, who does not see that there could have been no time had not some creature been made, which by some motion could give birth to change, - the various parts of which motion and change, as they cannot be simultaneous, succeed one another, - and thus, in these shorter or longer intervals of duration, time would begin? Since then, God, in whose eternity is no change at all, is the Creator and Ordainer of time....

..... then assuredly the world was made, not in time, but simultaneously with time.

Saint AUGUSTINE, [451, pp. 180, 181]

Thus, as it is proper for the created things to be only somewhere *with respect to place and* sometime *with respect to time, so it is proper for the Creator to be everywhere with regard to place and always with regard to time ...*

P. GASSENDI [198, p. 199]

For deeper analysis see also the books by and the papers by St. Augustine [451], Gassendi [198], Leibniz and Clarke [290].

2.13 Works on *time* in general

The structure of history, the uninterrupted forward movement of clocks, the procession of days, seasons, and years, and simple common sense tell us that time is irreversible and moves forward at a steady rate. Yet these features of traditional time were also challenged as artists and intellectuals envisioned times that reversed themselves, moved at irregular rhythms, and even came to a dead stop. In the fin de siècle*, time's arrow did not always fly straight and true.*
Stephen KERN [264]

The study of language, therefore, does reveal an evolution of the idea of time.
....
In my opinion, time is only one of the forms evolution takes; instead of producing evolution, time emerges from it. Time is, in fact, a consequence of the transition from the homogeneous to the heterogeneous; it is a differentiation instilled in things; it is the reproduction of similar effects in a different setting or of different effects in a similar setting. Instead of saying that time is the essential factor of change and, consequently, of progress, <it would be more appropriate to say that progress (evolution) constitutes a factor *and fundamental element of time: time is the abstract formula for describing change in the universe. In the completely homogeneous mass that, as a result of logical is sometimes thought to be at the origin of the present world, time does not yet exist.*
Jean-Marie GUYAU [346, pp. 97, 146]

I do not believe that there are objectively true meanings of space and time.
Mendel SACHS [432, p. 170]

The origin of the sun-dial. – *The first timekeeper or rather time indicator to be used was without doubt the sun-dial. Herodotus, a Greek historian of the fifth century B. C., states that the sun-dial came from Babylonia. There are other lines of evidence as well, so that it is safe to conclude that the sun-dial was in use in the valleys of the Tigris and the Euphrates at least as early as 2000 B. C.*

....
The origin of the clepsydra. – *In the previous chapter it was stated that the sun-dial was historically the first timekeeper to be used. The second in chronological order was the clepsydra, or the Greek water clock.*[1] *The statement is often made that the clepsydra was invented by Ctesibius at Alexandria in the second century B. C. Vitruvius says of him: "Methods of making water clocks have been investigated . . . first of all by Ctesibius the Alexandrian, who also discovered the natural pressure of the air and pneumatic principles. . . . Preëminent for natural ability and great industry, he is said to have amused himself with ingenious devices." The date is, however, altogether too late for the first introduction of the water clock. It must have been a special kind which Ctesibius invented.*

The water clock may have come from Babylonia along with the sun-dial, and some even claim for it an equally great antiquity. It may have originated in Egypt. It was used very early in India and China, but may have been independently discovered in these countries. It is stated by Pliny in his Historia Naturalis that Scipio Nasica brought it to Rome from Greece in 157 B. C.

Willis I. MILHAM [347, pp. 31, 48]

On mesure la vitesse d'un mouvement, ce qui implique que le temps lui aussi, est une grandeur.

Henri BERGSON [36, p. 80]

... we can assert that ***the very essence of time is its transience****, and that this is a fundamental concept that cannot be explained in terms of something still more fundamental.* ***Time is the mode of activity, and without activity there can be no time.***

G. J. WHITROW [487, p. 372]

We comprehend that even before there were any things time flowed...

....

... we understand that time flows in the same tenor as it flowed before; and if God reduced the whole universe to nothing, we comprehend that time will still flow; and we also understand that if God would wish to recreate the universe, time still would flow in the interval between its destruction and recreation.

P. GASSENDI [198, p. 195]

Il n'y a ni espace, ni temps à priori....

(There is neither space, nor time a priori .....)

Paul LANGEVIN [285, p. 32]

Time in itself is not a physical quantity; it is an abstract quantity: a number, the time value. Time makes physical sense only in relation to the counting of ideal periodic processes encapsulated in devices called clocks.

....

Since the dependence on velocity of the time exists, but, ..., this is only in connection with the graphical construction ...., any association of it with the running velocity of a clock, as long as no parameter characterizing the working machinery of the last enters into LT, is wrong. Nothing in physics denies the uniform elapse of time in empty space.

Alexandru Constantin CEAPA [60, pp. 4, 5]

The concept of time is an abstraction by induction from the observation of many different processes occurring in nature.

J. P. WESLEY [478, p. 262]

Time is the primeval cause of the universe. The universe ***is*** *time manifested as space. Time is everything. Even mind is temporal, conscious thought unfolding itself as a steady flow.*
S. C. TIWARI [463, p. 231]

The great thing about time is that it goes on.
Arthur EDDINGTON [110, p. 68]

Pour nous qui voulons la réforme du calendrier, le temps est une force universelle et toujours mouvante qui gouverne la vie de l'homme et toutes ses activités. Elle s'intègre à lui autant que son ombre, elle est avec nous toujours, mais toujours insaisissable. Nous savons que cette force du temps existe sans pouvoir la définir ou la capter. Son influence est totalement impersonnelle, universelle, autonome, qualités qui devront également caractériser tout système chronométrique efficace, qu'il s'agisse comme nous l'avons indiqué plus haut, de l'horloge ou bien du calendirer.
Elisabeth ACHELIS [1, pp. 38, 39]

En résumé, il convient de distinguer : 1° le temps pur; 2° le temps-nombre; 3° le temps-espace; 4° la durée vécue; 5° la temporalité. Le temps pur est le temps proprement dit, le temps primordial, dont le temps-quantité, la durée vécue et la temporalité sont, sous des formes différentes, la négation. Le temps ne se laisse pas saisir en lui-même : il ne se montre que nié.
Marcel CONCHE [74, p. 39]

· L'existence objective de la matière implique l'existence d'une flèche du temps.
· La flèche du temps n'est pas une propriété de l'espace-temps, c'est une propriété de la matière-espace-temps.
· C'est la matière qui, par son existence objective donne sa flèche au temps.
· La matière, c'est tout ce qui donne sa flèche au temps.
Gilles COHEN-TANNOUDJI [69, p. 130]

Une piste possible pour une première et modeste approche qui resterait dans le cadre de la physique théorique conventionnelle, serait de prendre comme point de départ, non les notions de temps et d'espace, mais celles d'espace et du mouvement. En d'autres termes, au lieu de considérer le mouvement comme une modification spatiale au cours du temps, il s'agirait de construire le temps comme un descripteur du mouvement dans l'espace, et de voir si une conceptualisation plus souple pourrait découler de ce point de vue.
Jean-Marc LEVY-LEBLOND [291, p. 281]

The time of our clocks and of history is a construct. It is used as an index to indicate and measure change and speed of change. In physics, it is a construct *in the form of a parameter, used to describe the sequential order of states and the speed of transitions from one state to another.*

Gerhard J. DALENOORT [85, p. 179]

Il est donc devenu impossible de concevoir le temps de l'univers matériel comme un front qui avance. Le temps de l'univers matériel est, au même titre que son espace, une étendu actuelle ; et le paramètre explorateur de cette " quatrième dimension " est, tout comme ceux qui paramètrent l'espace ordinaire, une coordonnée.

O. Costa de BEAUREGARD [29, p. 125]

Let x, y, z be rectangular co-ordinates for space, and let t denote time. The objects of our perception invariably include places and times in combination. Nobody has ever noticed a place except at a time, or a time except at a place.

H. MINKOWSKI [348], [446, p. 298]

We must conclude then that in order for the system to have become organized there was a unidirectional passage of time, and that the passage of time is an essential feature of self-organized systems and systems that are capable of transferring information from one to the other and of holding it.

Cornelius A. TOBIAS [465, p. 274]

For further reading see also the books by Bergson [36] and Janet [252], and the paper by Langevin [285].

2.14 Works on *time*: reviews

Reviews of the works on *time* can be found in the books by Andrillat [8], Atmanspacher and Dalenoort [14], Atmanspacher and Runhau [15], Attali [16], Bergson [36], Brelet [51], Collobert [71], Conche [74], Cullman [80], Čapek [81], de Broglie [93], de Grazia [95], Earman, Glymour, and Stachel [107], Fraisse [185], Franck [186], Fraser and Lawrence [189], Grünbaum [214], Hawking [240], Husserl [248], Hussey [249], Janet [252], Kern [264], Klein [270], Klein and Spiro [271], Le Lionnais [293], Lucas and Hodgson [306], Mainzer [314], Marder [319], Merleau-Ponty and Morando [339], Meyerhoff [340], Michon, Pouthas and Jackson [346], Muga, Mayato and Egusquiza [354], Nottale [364], Ornstein [368], Prigogine and Stengers [398], Prior [400], Saint Augustin [434], Selleri [444], von Franz [188], Wetzel [479], Whitrow [487], Whittaker [488], and in the papers by Ariotti [9], Bailey [20], Balian [19], Bergson [34], [35], Berkeley [37], Bitsakis [39], Bohm [41], Bowman [49], Bruno [53], Cornford [75], Craig [77], Čapek [82], Eddington [112], Ehlers [113], Frank [187], Friedman [190], Garnier-Malet and Bobola [197], Gödel [200], Goldschmidt [202], Grünbaum [215], Hiley and Fernandes [244], Horwitz [247], Jennison [254], Katchalsky-Katzir [262], Keswani [265], [266], Kiefer [268], Klose [272], Lachièze-Rey [282], Leibniz and Clarke

[290], Luminet [308], Martin [332], Maxwell [335], Mehlberg [337], Mercier [338], Meyerson [341], [342], Nesteruk [358], Paty [374], Plotinus [380], Pöppel [388], Reichenbach [405], Robb [417], Robert [418], Ruhnau [426], [427], Russell [429], [430], Sachs [432], Sambursky [435], [436], Selleri [443], Stone [455], [456], Tobias [465], Tonkinson [466], Wegener [477], Wesley [478], Whitehead [482], [484], and Whitrow [486].

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Chapter 3

Newton and Einstein on *Time*

3.1 Newton's explanation of *time*

.... we already have in Newtonian mechanics the
"notion of time [with] respect to a frame of reference."
Peter A. BOWMAN [49, p. 444]

Sir Isaac NEWTON (1642 - 1727) had written in Latin his capital work ***Philosophiae Naturalis Principia Mathematica*** (*Mathematical Principles of Natural Philosophy)*, [360]. He had completed this fundamental monograph on 8 May, 1686. It was published first by July 5, 1687. It became a founding column of the modern science, in which Newton wrote the following about *time*:

> *I. Absolute, true, and mathematical time, of itself, and*
> *from its own nature, flows equaly without relation to*
> *anything external, and by another name is called duration:*
> *relative, apparent, and common time, is some sensible and*
> *external (whether accurate or unequable) measure of*
> *duration by the means of motion, which is commonly*
> *used instead of true time; such as an hour, a day, a month,*
> *a year.*
> **Isaac NEWTON** [360, I of Scholium, p. 8]

This shows *Newton's* understanding that *time* is *a variable* and, as every variable, it can have different values. Moreover, this claims that the temporal value of the duration of a *time* interval can be measured with different units (*such as an hour, a day, a month, a year*). This means implicitly that *time* is a physical variable. We usually identify a variable (e.g. angle, *time*, voltage) with its value in a colloquial conversation ("angle is 90° " meaning "angle has the value of 90 degrees"; or "it is five PM" meaning "*time* has the value of five hours passed since noon"; or "voltage is 220V" meaning that the voltage value equals 220 Volts; for more details see Subsections "Numerical values of *time* and relativity" in 4.2, and "Values of physical variables" in 4.4). Numerical *time* value depends on *time* scale, on *time* unit (as well as on the accepted zero *time* value and the initial *time* value). Newton wrote in this sense about relative *time.* Above cited Bowman‘s comment should be understood in that sense.

At least physics, like mathematics, should clearly distinguish *variable* from *variable value* in a definition and/or in an explanation of the variable. *Newton* did distinguish *time* (which he called *absolute, true time*) from *time* value (which he called *relative, apparent, and common time)* in his explanation as cited above and in what follows.

Absolute time, in astronomy, is distinguished from relative, by the equation or correction of the apparent time. For the natural days are truly unequal, though they are commonly considered as equal, and used for a measure of time; astronomers correct this inequality that they may measure the celestial motions by a more accurate time. It may be, that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated and retarded, but the flowing of absolute time is not liable to any change. The duration of perseverance of the existence of things remains the same, whether the motions are swift or slow, or none at all: and therefore this duration ought to be distinguished from what are only sensible measures thereof; and from which we deduce it, by means of the astronomical equation. The necessity of this equation, for determining the times of a phenomenon, is evinced as well from the experiments of the pendulum clock, as by eclipses of the satellites of Jupiter.

Isaac NEWTON [360, pp. 9 -10]

Conclusion 39 ***Newton explained the absolute and the relative sense of time***

The above citations request us to note, at least for scientific fairness, that Newton distinguished well between

◦ *variable (quantity)*

and

◦ *its value (sensible measure),*

as well as he distinguished

◦ *absolute (true) time*

from

◦ *relative time*

and wrote about them both. In the modern literature, the reference to Newton's explanation of time is most often only to his explication of absolute sense of time, but usually without mentioning his clarification of relativity of time. Moreover, he is then often sharply criticized for the absolute sense of time!

Newton's understanding of absolute nature of time means that time is unique variable in the sense that there are not two or more different times as variables, i.e. that there are not several different time (i.e. temporal) variables. This is *the same as for all other physical variables (e.g. acceleration, angle, current, position, pressure, temperature, velocity, voltage). Moreover, he understood that time is independent variable, the value of which can be measured with imperfect accuracy and precision. This means that he considered (implicitly) time as a physical variable.*

Newton understood not only the meaning of absolute time, but also relativity of time, and explained its sense, as cited above. We will analyze, in what follows , where is a point for which Newton's essential meaning of time is not valid, if there is anyone. We will show that such a point does not exist, and that Einstein's interpretation of the numerical time value relativity is the same as Newton's.

3.2 Einstein's interpretation of *time*

Newton, accepte mes excuses!
Albert EINSTEIN [158, p. 34]

Albert EINSTEIN (1879 - 1955) explained the meaning of *time* as follows:

We are thus led also to a definition of "time" in physics. For this purpose we suppose that clocks of identical construction are placed at the points A, B, *and* C *of the railway line (co-ordinate system), and that they are set in such a manner that the positions of their pointers are simultaneously (in the above sense) the same. Under these conditions we understand by the "time" of an event the reading (position of the hands) of that one of these clocks which is in the immediate vicinity (in space) of the event. In this manner a time-value is associated with every event which is essentially capable of observation.*

Albert EINSTEIN [144, p.20], [154, pp. 23-40]

... In the pre-relativity physics this problem was solved by making unconcsciously two hypotheses :

I. The time is absolute ; the time of an event, t', relatively to K' is the same as the time relatively to K. If instantaneous signals could be sent to a distance, and if one knew that the state of motion of a clock had no influence on its rate, then this assumption would be physically established. For then clocks, similar to one another, and regulated alike, could be distributed over the systems K and K', at rest relatively to them, and their indications would be independent of the state of motion of the systems ; the time of an event would then be given by the clock in its immediate neighbourhood.

2. Length is absolute ; ....

Albert EINSTEIN [150, p. 24]

Before we draw any conclusions from these two principles we must first review the physical significance of the concepts 'time' and 'velocity'. It follows from what has gone before, that co-ordinates with respect to an inertial system are physically defined by means of measurements and constructions with the aid *of rigid bodies. In order to measure time, we have supposed a clock, U, present somewhere, at rest relatively to K. But we cannot fix the time, by means of this clock, of an event whose distance from the clock is not negligible; for there are no 'instantaneous signals' that we can use in order to compare the time of the event with that of the clock. In order to complete the definition of time we may employ the principle of the constancy of the velocity of light in a vacuum. Let us suppose that we place similar clocks at points of the system K, at rest relatively to it, and regulated according to the following scheme. A ray of light is sent out from one of the clocks, U_m, at the instant when it indicates the time t_m, and travels through a vacuum a distance r_{mn}, to the clock U_n; at this instant when this ray meets the clock U_n the latter is set to indicate the time $t_n = t_m + \frac{r_{mn}}{c}$.* The principle of the constancy of the velocity of light then states that this adjustment of the clocks will not lead to contradictions. With clocks so adjusted, we can assign the time to events which take place near any one of them. It is essential to note that this definition of time relates only to the inertial system K, since we have used a system of clocks at rest relatively to K. The assumption which was in the pre-relativity physics of the absolute character of time (i.e. the independence of time of the choice of the inertial system) does not follow at all from this definition.*

** Strictly speaking, it would be more correct to define simultaneity first, somewhat as follows: two events taking place at the points A and B of the system K are simultaneous if they appear at the same instant when observed from the middle point, M, of the interval AB. Time is then defined as the ensemble of the indications of similar clocks, at rest relatively to K, which register the same simultaneously.*

Albert EINSTEIN [150, pp. 26 - 27]

Remark 40 ***Clock indications and time***

*The clock hands indicate nothing else than a numerical time value relative to a chosen zero time value, a time unit and a time scale. They indicate it in terms of different accepted time units such as second, minute, hour. However, their positions (their indications) are **not** time. Let a moment representing the zero time value and a time scale be accepted and fixed. Then, a time unit used is determined by the speed of the clock hand and vice versa. For details see Section 4.8: "Clock principles" in 4. A change of the speed value of the clock hand means exclusively the change of a time unit, but it **does not** and **cannot** change time. Einstein identified time with the numerical time value.*

With this in mind we conclude as follows:

Comment 41 ***Einstein's crucial confusion***

Einstein did not distinguish time, is value and its numerical value.

As mathematics demands the clear understanding of the crucial difference between function and (numerical) function value, so physics demands the full understanding of the essential difference among (physical) variable, its value and its numerical value. Otherwise, the serious mistake results.

Conclusion 42 ***Einstein equalized time with its numerical value***

a) Einstein identified time with the numerical time value.

b) Einstein's meaning of the relativity of time is correct only in the sense of the relativity of the numerical value of time, but not in the sense of the relativity of time itself.

For more by Einstein about *time* see [114], [116], [131], [144], [150], [153], [154], [169], [305].

3.3 Einstein's versus Newton's explanation

Having understood Einstein's and Newton's explanations of *time* relativity we may summarize the following (see Section: : "Clock principles"):

° Newton had distinguished *time* (as a variable - temporal variable) from its value, and from its numerical value.

° Einstein identified *time* (as a variable) with its numerical value rather than to distinguish them.

° Newton had explained the relativity of *time* in the sense of the relativity of both *time* units and the numerical *time* value.

° Einstein's explanation of the relativity of *time* is correct only if it means the relativity of the numerical *time* value. It is incorrect if it means the relativity of *time* itself.

° There is not a collision between Newton's explication of the *time* relativity, and Einstein's explication if it is considered only in the sense of the relativity of the numerical *time* value. Otherwise, they are in a deep mutual opposition.

Conclusion 43 ***Einstein's versus Newton's explanation of time***

Einstein criticized Newton for the absolute meaning of time without mentioning anywhere Newton's explication of the relative sense of time.

If we refer to Newton for his explanation of time then we are obliged to refer to his complete explanation rather than only to its first part. By ignoring the second part of his explanation, its first part was used by all those scientists, who (have) followed Einstein's critique, as a reason to criticize Newton for considering time only in the absolute sense, but not in the relative sense. However, the second part of Newton's statement explains the relative sense of time.

Einstein treated only quantitative characteristics of time.

Newton explained both qualitative and quantitative properties of time.

If Einstein's explication of the relativity of time means the relativity of the numerical time value, then it is in Newton's sense. Consequently, Newton's explanation of the relative sense of time then incorporates Einstein's.

If Einstein's explication of the relativity of time means the relativity of time itself, then it opposes crucially Newton's.

Newton explained correctly the physical essence of the time relativity two and half centuries before Lorentz, Einstein and Poincaré wrote about this topic.

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Chapter 4

Nature and Properties of *Time*

4.1 Quantities, dimensions and units

A physical quantity is an absolute quantity,
defined as the product of the number of units,
multiplied by the size of the corresponding reference unit.
Paul MARMET

We will generalize, by referring to [226], [227], [228], the well known facts about units because Einstein ignored them, which was accepted by Lorentz, Poincaré and others.

An amount (the absolute value) $g \in R^+$ of a quantity g to be measured, the goal of the measurement, and the purpose of the quantity usage govern a choice of *a unit* 1_g for the quantity g, $unitg = 1_g$, i.e. $g \langle 1_g \rangle$. We use various amounts of the quantity g. We can utilize the quantity g for various purposes. Consequently, we can establish and apply various scales and units for the same quantity g, which we do often.

We should distinguish the unit used to measure the quantity g, which is denoted by $1_{(.)}$, from *the physical dimension* of the quantity g denoted by $phdimg$ and shown between the parentheses [] following the quantity g. The **physical dimension of the quantity** g is denoted, for example, by G, where G stands for "quantity g",

$$phdim(g) = \text{G}, \textit{ or equivalently}, \ g[\text{G}].$$

The physical dimension of distance d, of length l and of position ρ or r is L (L stands for "length"),

$$phdim(d) = phdim(l) = phdim(\rho) = phdim(r) = \text{L}.$$

Hence, d[L], l[L], ρ[L] and r[L]. Their mathematical dimensions, $dim(d)$, $dim(l)$, $dim(\rho)$, $dim(r)$, respectively, are equal to one,

$$dim(d) = dim(l) = dim(\rho) = dim(r) = 1.$$

The physical dimension of the velocity $\mathbf{v}$ and of the speed v of a moving body is LT^{-1},

$$phdim(\mathbf{v}) = phdim(v) = \mathrm{LT}^{-1},\ i.e.\ \mathbf{v}[\mathrm{LT}^{-1}],\ v[\mathrm{LT}^{-1}],$$

but their mathematical dimensions, $dim(\mathbf{v})$, $dim(v)$, are three and one, respectively, because $\mathbf{v} \in \mathfrak{V}^3$ and $v \in \mathfrak{V}^1$,

$$dim(\mathbf{v}) = 3,\ dim(v) = 1.$$

The notation between the parentheses $\langle . \rangle$ shows the units used to measure the value of the corresponding variable. We can use also the symbolic operator "$unit(.)$" in order to show the units used to measure the value of the variable. For example, if the value of the speed v of a moving body is measured in meters per second (ms^{-1}), then $unit(v) = ms^{-1}$, hence $v \left\langle ms^{-1} \right\rangle$.

Let 1_g be the accepted unit of the quantity g.

If the value of the quantity g is *time*-varying, then g is the variable g.

The numerical value of g will be denoted by $num\,(g)$. It expresses the number N_{1g} of the units 1_g contained in the value g of the variable g, which is measured with 1_g,

$$num\,(g) = N_{1g} = \frac{g\,\langle 1_g \rangle}{1_g},\ g\,\langle 1_g \rangle = N_{1g} 1_g = [num\,(g)]\, 1_g. \tag{4.1}$$

Let the quantity g be length, $g = L$. A rod of the length $L_m = 1m$ measured in meters has the same length L measured in millimeters, $L_{mm} = 1000mm$ or in kilometers, $L_{km} = 0.001km$, or in another length unit. Its numerical values $numL$ are different, $1_{L_m} = 1m \Longrightarrow numL_m = 1$, $L_{mm} = 1000mm \Longrightarrow numL_{mm} = 1000$, $1_{L_{km}} = 0.001km \Longrightarrow numL_{km} = 0.001$. The product of the numerical value of the length and of the corresponding length unit is constant, and invariant relative to a choice of the unit, $numL_m 1_{L_m} = 1\text{x}1m = 1m$, $numL_{mm} 1_{L_{mm}} = 1000\text{x}1mm = 1000\text{x}0.001m = 1m$, and $numL_{km} 1_{L_{km}} = 0.001\text{x}1km = 0.001\text{x}1000m = 1m$. This illustrates that

$$L_{mm}\,\langle mm \rangle = L_{km}\,\langle km \rangle = L_m\,\langle m \rangle = 1m.$$

The length size (value) is independent of the used units. However, the numerical value of the length size depends, not only on the length value (size), but also on the used units. Analogously, $3h = 180min = 10800s$, i.e. $3\text{x}1h = 180\text{x}1min = 10800\text{x}1s$. The value of the *time* interval (of three hours) is independent of the used *time* units (hour h, minute min, second s). Its numerical values depend on the used unit ($num(3h) = 3$, $num(180min) = 180$,

$num(10800s) = 10800 \notin \{3, 180\}$). These elementary, but fundamental, facts hold in general. Unfortunately, Einsteinian relativity theory has ignored them, which is the reason to recall them.

Let two scales and the corresponding units of the quantity g carry the subscripts i and j, 1_{g_i} and 1_{g_j}. Let the unit 1_{g_i} be κ_{ji} times bigger than 1_{g_j}, i.e. let the unit 1_{g_j} be κ_{ji} times smaller than 1_{g_i},

$$1\left\langle 1_{g_i}\right\rangle = \left(\kappa_{ji}\left\langle 1_{g_j}1_{g_i}^{-1}\right\rangle 1\left\langle 1_{g_i}\right\rangle\right)\left\langle 1_{g_j}\right\rangle = \left(\kappa_{ji}1_{g_j}\right)\left\langle 1_{g_j}\right\rangle,$$
$$1\left\langle 1_{g_j}\right\rangle = \left[\left(\kappa_{ji}\left\langle 1_{g_j}1_{g_i}^{-1}\right\rangle\right)^{-1} 1\left\langle 1_{g_j}\right\rangle\right]\left\langle 1_{g_i}\right\rangle = \left(\kappa_{ji}^{-1}1_{g_i}\right)\left\langle 1_{g_i}\right\rangle. \tag{4.2}$$

The amount of the quantity g measured in the scale (.) with the unit $1_{g_{(.)}}$ is denoted by $g_{(.)}$. Then, the following holds:

$$g_j\left\langle 1_{g_j}\right\rangle = \left[\kappa_{ji}\left\langle 1_{g_j}1_{g_i}^{-1}\right\rangle g_i\left\langle 1_{g_i}\right\rangle\right]\left\langle 1_{g_j}\right\rangle,$$
$$g_i\left\langle 1_{g_i}\right\rangle = \left[\left(\kappa_{ji}\left\langle 1_{g_j}1_{g_i}^{-1}\right\rangle\right)^{-1}\ g_j\left\langle 1_{g_j}\right\rangle\right]\left\langle 1_{g_i}\right\rangle, \tag{4.3}$$

or for short,

$$g_j = \kappa_{ji}g_i,\ g_i = \kappa_{ji}^{-1}\ g_j. \tag{4.4}$$

Law 44 ***General units law, and transformations***

◦ ***Units values***

Every unit $1_{g_{(.)}}$ of any quantity g should be positive real valued,

$$1_{g_{(.)}} \in R^+,\ (.) \in \{i, j\},\ \forall i, j \in \{-, 1, 2, ...,\},\ i \leq j. \tag{4.5}$$

◦ ***Units and quantity values***

The amount (size, absolute value) g of any quantity g is independent of a unit used. If it is measured with a unit $1_{g_{(.)}}$ then it equals the product of the numerical value, $\left[num(g_{(.)}\left\langle 1_{g_{(.)}}\right\rangle)\right]$, of the so measured quantity g, and the unit $1_{g_{(.)}}$,

$$g_i\left\langle 1_{g_i}\right\rangle = \left\{\left[num(g_i\left\langle 1_{g_i}\right\rangle)\right]1_{g_i}\right\}\left\langle 1_{g_i}\right\rangle \equiv$$
$$\equiv g_j\left\langle 1_{g_j}\right\rangle = \left\{\left[num(g_j\left\langle 1_{g_j}\right\rangle)\right]1_{g_j}\right\}\left\langle 1_{g_j}\right\rangle \equiv$$
$$\equiv g\left\langle 1_g\right\rangle = \left\{\left[num(g\left\langle 1_g\right\rangle)\right]1_g\right\}\left\langle 1_g\right\rangle \equiv const,$$
$$\forall i, j \in \{-, 1, 2, ...,\},\ i \leq j. \tag{4.6}$$

The bigger unit 1_{g_i}, the smaller number $N_{1_{g_i}}$ of the units 1_{g_i} in the amount (size, value) $g\left\langle 1_g\right\rangle$ of any quantity g, and vice versa. Their product is constant, and it equals the value of the quantity g, which is invariant relative to units,

$$1_{g_j} \neq 1_{g_i} \Longrightarrow N_{1_{g_j}} = num(g_j\left\langle 1_{g_j}\right\rangle) \neq N_{1_{g_i}} = num(g_i\left\langle 1_{g_i}\right\rangle),$$
$$1_g \notin \{1_{g_i}, 1_{g_j}\} \Longrightarrow N_{1_g} = num(g\left\langle 1_g\right\rangle) \notin \{N_{1_{g_i}}, N_{1_{g_j}}\},\ \textit{but, (4.6):}$$
$$g_i\left\langle 1_{g_i}\right\rangle = \left(N_{1_{g_i}}1_{g_i}\right)\left\langle 1_{g_i}\right\rangle \equiv g_j\left\langle 1_{g_j}\right\rangle = \left(N_{1_{g_j}}1_{g_j}\right)\left\langle 1_{g_j}\right\rangle \equiv$$
$$\equiv g\left\langle 1_g\right\rangle = \left(N_{1_g}1_g\right)\left\langle 1_g\right\rangle. \tag{4.7}$$

◦ ***Units and transformations***

Let two units 1_{g_i} and 1_{g_j} of a quantity g be related by the equations (4.2). Then, the values g_i and g_j of the same amount g of the quantity g measured in the scales i and j with the units 1_{g_i} and 1_{g_j} obey the transformations (4.3), or for short, (4.4).

The equations (4.4) and the third statement (”Units and transformations”) of the preceding *General units law, and transformations* 44 imply the following:

$$g_i \in R^+ \ and \left\{ \begin{array}{c} \kappa_{ji} \longrightarrow 0 \Longrightarrow g_j \longrightarrow 0 \Longrightarrow num\ g_j \longrightarrow 0, \\ \kappa_{ji} \longrightarrow \infty \Longrightarrow g_j \longrightarrow \infty \Longrightarrow num\ g_j \longrightarrow \infty \end{array} \right\},$$
$$g_j \in R^+ \ and \left\{ \begin{array}{c} \kappa_{ji} \longrightarrow 0 \Longrightarrow g_i \longrightarrow \infty \Longrightarrow num\ g_i \longrightarrow \infty, \\ \kappa_{ji} \longrightarrow \infty \Longrightarrow g_i \longrightarrow 0 \Longrightarrow num\ g_i \longrightarrow 0 \end{array} \right\}. \qquad (4.8)$$

These results and the first statement (”Units values”) of *General units law and transformations* 44 imply the following:

Corollary 45 ***Unacceptable units values***

Let the amount (size, absolute value) of an arbitrary quantity g be fixed. The smaller unit $1_{g_{(.)}}$ of the quantity g, the bigger number of the units $1_{g_{(.)}}$ in its same amount. If the unit $1_{g_{(.)}}$ were equal to zero, $1_{g_{(.)}} = 0$, then there would be infinitely many (zero) units $1_{g_{(.)}}$ in any nonzero amount of the quantity, i.e. the numerical amount of the quantity would be equal to infinity,

$$1_{g_i} \neq 0,\ \forall g_i \left\langle 1_{g_i} \right\rangle \in R^+,\ and\ 1_{g_j} = 0 \Longrightarrow num \left(g_j \left\langle 1_{g_j} \right\rangle \right) = \infty. \qquad (4.9)$$

The inverse analogy holds for the infinite unit 1_{g_k},

$$1_{g_i} \neq 0,\ \forall g_i \left\langle 1_{g_i} \right\rangle \in R^+,\ and\ 1_{g_k} = \infty \Longrightarrow num \left(g_k \left\langle 1_{g_k} \right\rangle \right) = 0. \qquad (4.10)$$

Any unit $1_{g_{(.)}}$ of any quantity g may not be equal either to zero or to infinity,

$$1_{g_{(.)}} \notin \{0, \infty\},\ (.) \in \{i, j\},\ i \leq j,\ i, j \in \{-, 1, 2, ...,\}. \qquad (4.11)$$

This holds for all the quantities, hence for all the physical variables including time and energy. This holds also for the mass of every particle and body.

It is proved in [226], [231], that Einsteinian relativity theory brakes *General units law, and transformations* 44, as well as its Corollary 45. The consequences are seriously negative. However, Galilean - Newtonian physics obeys both.

4.2 Definition and properties of *time*

What follows relies on [226], [227], [228], [229], [230], [231].

4.2.1 Principal Einstein's contradiction

Let us explain immediately a contradiction in the following Einstein's attitude cited above:

It is immaterial what kind of processes one chooses for such a definition of time. It is advantageous, however, for the theory, to choose only those processes concerning which we know something certain. This holds for the propagation of light in vacuo *in a higher degree than for any other process which could be considered ...*

Albert EINSTEIN [150, p. 27]

The condition under which the statement expressed in the first two sentences can hold is that "*those processes*" **should be explained, and can be afterwards applied to the clarification of *time*, only without referring to *time* itself**. If those processes can be clarified only by referring, explicitly or implicitly, to *time*, then we know less about them than about *time* itself. The propagation of light does not satisfy the preceding condition. One of the light main attributes is the light speed. Only the use of *time* can explain the notion and the sense of the speed in general, hence of the light speed in particular. This leads to the conclusion that *time* is crucial for the light propagation. We cannot explain the light propagation if we cannot explain *time*. This is a fundamental reason for which we cannot accept completely Einstein's cited claim. We present more detailed explanations in what follows.

4.2.2 Various claims on *time*

There are opinions and/or claims that:

- ◦ *time* does not exist,
- ◦ *time* is not a physical variable,
- ◦ *time* is only an abstract (mathematical) variable, or even a parameter only,
- ◦ *time* is a dependent variable whatever is its nature and it depends on space,
- ◦ *time* is an independent variable whatever is its nature,
- ◦ *time* is an independent physical variable.

These claims are contradictory. Which one is correct? The reply follows.

4.2.3 Definition of *time*

As every quantity, *time* has a value. Its value has been changing. Hence, it is a variable. It is an energy-matter free variable.

There is an attempt to define *time* t (mathematically rather than physically) from the integral of both the infinitesimal variation dg of a variable $g(.)$ and the speed υ of the variation of its value, by considering its speed υ either as dependent exclusively on its value g or as constant,

$$t - t_0 = \int_{g_0}^{g} \frac{dg}{\upsilon(g)},\ g_0 = g(\upsilon_0) = g\left[\upsilon(t_0)\right],\ g = g(\upsilon) = g\left[\upsilon(t)\right]. \tag{4.12}$$

Such approach meets the unsolvable problem of defining the speed without referring to *time*, which should be achieved because the approach is aimed to define *time* via the speed.

The speed (and the velocity) cannot be defined without referring to *time*.

Therefore, this approach becomes circulus vitiousus, hence inappropriate to define *time*.

We should first define, or at least we should first explain, *time*, and only afterwards we can define velocity and speed (and acceleration). Once we have completed their definitions, then, and only then, we can use the equation (4.12) with the full understanding in order to calculate or to measure the value of *time* relative to an accepted zero value of *time*, relative to an adopted its initial value, relative to an adopted its scale, and relative to an accepted its unit. In this regard see on Piaget's studies of children's feeling of *time* in [319, pp. 80 - 82, 201].

Let us repeat and conclude: the equation (4.12) cannot be used to explain and/or to define *time*.

We should first define *time* itself, and only afterwards we can define speed and acceleration.

Thousands of years lasted human effort to explain adequately *time* in terms of other known phenomena, processes, categories and/or variables. The accumulated knowledge, experience and understanding of physical categories, phenomena, processes and variables show clearly and doubtless that *time*, its nature, characteristics and properties, cannot be explained in terms of energy, matter and space, and/or in terms of another category, other phenomena and/or processes, and/or in terms of another variables. They show that *time* has its own, particular, especial, original nature - *the temporal nature*. They set us in the situation to face the fact that *time* is one of the elementary primary quantities, one of the ground-phenomena, one of the elementary substantial notions, which helps us to understand and to explain many others.

To be clear we state:

Definition 46 ***Definition of time***

TIME is an independent scalar variable (called also TEMPORAL VARIABLE)

- ***the value of which*** *determines* ***when*** *somebody or something started / ended, has started / has ended, starts / ends, or will start / will end to exist,*
- ***the values of which*** *determine* ***since when*** *and* ***until when*** *somebody or something existed / did not exist, has existed / has not existed, will exist / will not exist,*
- ***the values difference of which*** *determines* ***how long*** *somebody or something existed / did not exist, exists / does not exist, or will exist / will not exist,*
- ***the values of which*** *determine whether an event occurs* ***when*** *another event has not yet taken place, or* ***when*** *another event has already taken place, i.e.* ***the values of which*** *determine* ***the order*** *of events happening,*

- ***the value of which occupies (covers, encloses, imbues, impregnates, is over and in, penetrates) equally*** *everybody and everything (i.e. beings, objects, energy, matter, and space)* ***everywhere and always****,*

and

- ***the value of which has been, is, and will be permanently changing smoothly, strictly monotonously continuously, equally*** *in all spatial directions and their senses, in and around everybody and everything,* ***independently*** *of everybody and everything (i.e. independently of beings, objects, energy, matter, and space;* ***independently*** *of all other variables, independently of all happenings, movements and processes).*

Moment*, i.e.,* ***instant****, is the elementary time value.*

This book is aimed to explore the nature, sense, properties and characteristics of both *time* and its relativity.

4.2.4 *Time* value

We do use the equation (4.12) to measure the value of *time*, and to calculate its numerical value, but not to define *time*. We use it also as the mathematical basis for the functioning of a device (called *timekeeper*, or *clock* in general) that measures values of *time* and indicates the sizes of passed *time* intervals. This is explained in details in Section 4.8: "Clock principles".

We have emphasized this well known relationship, (4.12), between velocity (hence, speed) and *time* in order to explain the reason for which we disagree substantially with Einstein's attitude (cited at the beginning of the book) that the light velocity is so well defined (which should be then without any reference to *time*) that it can and should be used to define *time*. It is not possible because the speed as a change, as a variation, in *time*, is defined in terms of *time*.

Einstein himself recognized the fact that *time* has a value, which shows the following citation. Newton [360, I of Scholium, p. 8] had done the same, too.

In this manner a time-value is associated with every event which is essentially capable of observation.

Albert EINSTEIN [154, p. 24]

This citation expresses implicitly Einstein's attitude that *time* does exist as a variable. Unfortunately, he equalized *time* with its numerical value rather than to distinguish them clearly, which Newton had done.

There is an opinion that it does not matter how we define *time*, but it is important only how we measure its value, (e.g. [180, p. 5-2]). This would be correct if the definition could be arbitrary, which is inacceptable for *time*.

In this regard, the following questions on a measurement of a variable value appear in general:

What is the meaning of a measurement of something if we do neither know, nor we can explain, nor we can understand what is its sense?

Should scientists, physicists, engineers and physicians, be satisfied with a measurement if they cannot understand the sense, the meaning of that what is (to be) measured?

The replies to these questions are obviously negative. Therefore, we aim to present at first an objective and an adequate characterization of *time*. Afterwards, we can discuss how to measure its value. Such a discussion is out of the scope of this book.

The physical reality shows that it is the *time* value that passes (the *time* value changes), but not *time* itself. This is the same as for values changes of all (physical) variables, but the (physical) variables themselves do not change. For example, pressure cannot become temperature or voltage or *time*, and vice versa. We devote a particular attention to this notational and linguistic subtlety in order to be aware that Einstein did not distinguish *time* from the *time* value.

... we understand by the "time" of an event the reading (position of the hands) of that one of these clocks ...

Albert EINSTEIN [144], [154]

We will distinguish *time* being a variable from its value and from its numerical value, as Newton did essentially (see Section: "Newton's explanation of *time*" in Chapter 3: "Newton and Einstein on *Time*").

A fixed moment (instant) is usually denoted by t (or by τ) and by a subscript, e.g. t_2 (or τ_b). An arbitrary moment (instant) is denoted as *time* itself by t (or by τ). Its numerical value denoted by $numt$ is a unique real number, i.e. for every moment t there is exactly one ($\exists!$) real number x such that the numerical value of t equals x, and vice versa,

$$\forall t \in \mathfrak{T},\ \exists! x \in R \Longrightarrow x = numt;\ \forall x \in R,\ \exists! t \in \mathfrak{T} \Longrightarrow numt = x. \qquad (4.13)$$

For the sake of the simplicity, we usually denote simply, if it does not create a confusion, the numerical value $numt$ of the moment t by t, too.

Time does not have a total zero value $t_{ZeroTotal}$. Any moment can be conventionally accepted for **the relative zero moment**, **the relative zero instant** (**the relative numerical zero value of *time***) $t_{zero} = 0$. This gives a relative sense rather than an absolute sense to the zero moment. It is not a total zero instant, $t_{zero} \neq t_{ZeroTotal}$.

There are physical variables that do not have a total zero value, and others that do have a total zero value. Position does not have a total zero value. Temperature has the total zero value that is the zero Kelvin degree, $0K$. Celsius and Fahrenheit numerical zero values of temperature are its relative (and conventional) numerical zero values, but not total.

An **initial value of *time*** (**initial moment**, **initial instant**) denoted by t_0, $t_0 \in \mathfrak{T}$, can be also accepted conventionally, possibly different from its adopted zero value, so to be the instant of the beginning of the existence / of the nonexistence (of the appearance / of the disappearance) of somebody or of something: $t_0 \neq 0$ is permitted.

It is important to distinguish the total zero *time* value $t_{ZeroTotal} = 0$, which has not existed and will not happen, a conventionally accepted relative zero *time* value $t_{zero} = 0$, and an initial moment t_0 as the moment of the beginning of the existence / of the nonexistence of somebody or of something.

The supposed year of the birth of Jesus Christ has been largely accepted for the relative zero *time* value $t_{zeroChrist} = 0$ and for the initial moment $t_{0Christians}$ of both the existence of Christianity and the beginning of New Era, but it is not the total zero value of *time*,

$$t_{0Christians} = t_{zeroChrist} \neq t_{ZeroTotal}.$$

If the moment t_{BB} when Big Bang took place (or started) is the moment of the creation of our energy-matter universe (space, cosmos), then we can accept it for the initial moment t_0 and for the relative zero moment,

$$t_{BB} = t_0 = t_{zero} = 0.$$

It is then only the relative zero instant, but it is not and cannot be the total zero instant,

$$t_{BB} = t_0 = t_{zero} = 0 \neq t_{ZeroTotal}.$$

4.2.5 *Time* properties and characterization

Nobody has been able to verify experimentally whether there was the beginning of *time* (i.e. its minimal value). Nobody can now test experimentally whether there will be its end (i.e. its maximal value). We should rely on the physical reality, on the available experience, analyses, comprehension of, contemplations and knowledge about *time* in order to explain it, its characteristics and properties. We should not ignore the common sense without providing an uncontested proof for its incorrectness.

The physical reality, the performed analyses, the accumulated knowledge, the common sense, the comprehension and the experience permit us to state, and to accept herein, the following characterization of *time* in view of its Definition 46:

Axiom 47 ***Time: properties and characterization***

a) ***Time***

Time *(i.e.* ***the temporal variable****) denoted by* t *(or by* τ*) is an independent and unique scalar variable such that its value occupies (covers, encloses, imbues, impregnates, is over and in, penetrates) equally everybody and everything (i.e. beings, energy, matter, objects, and the position space) everywhere, and its value has been, is, and will be permanently changing smoothly, strictly monotonously continuously, equally in all directions and their senses, in and around everybody and everything, independently of everybody and everything (i.e. independently of beings, energy, matter, objects, and the position space; independently of all other variables, independently of all happenings, movements and processes).*

b) ***Time value***

The value of time t *(*τ*) is denoted also by* t *(or by* τ*), respectively. It is called* ***moment*** *or* ***instant****. It is an* ***instantaneous (momentous)*** *and* ***elementary time value****. It can happen exactly once and then it is the same everywhere* *for, and in, everybody and everything (i.e. for, and in, beings, energy, matter, objects, and the position space), for all other variables, for all happenings, for all movements and for all processes.*

There can be assigned exactly one real number to every moment (instant), and vice versa. The numerical value ***num*** *t of the moment t is a real number, $numt \in R$. The set of all moments is in the one-to-one correspondence with the set of all real numbers. The rule of the correspondence determines* ***an accepted zero numerical time value*** *t_{zero},* ***time scale*** *and* ***time unit*** *1_t (or 1_τ).*

A ***total zero moment*** *$t_{ZeroTotal}$ has not existed and will not occur. Any moment can be accepted for a* ***relative zero moment*** *t_{zero}.*

c) ***Time (temporal) order***

If a moment t_2 has not yet happened when a moment t_1 occurs, then the moment t_1 is ***before*** *the moment t_2, and the moment t_2 is* ***after*** *the moment t_1. Then, and only then, they are different moments. The moment t_1 happened* ***earlier*** *than t_2, while t_2 occurs* ***later*** *than t_1.*

Moments t_3 and t_4 are ***simultaneous*** *if, and only if the moment t_4 happens exactly then when the moment t_3 occurs. Then, and only then, they are* ***equal*** *and then they are the same moment, $t_3 = t_4$.*

Between any two moments there holds exactly one relationship, either before / earlier (i.e. after / later), or simultaneous. These relationships determine the unique ***time order*** *(i.e.* ***the temporal order****).*

Between any two different moments there is a third different moment. All instants (moments) form the temporally ordered set $\mathfrak{T}(t)$ called ***the time set*** *that is* ***continuum****.*

d) ***Time sets: past, present and future***

All instants that happened are the ***past instants****. They form the temporally ordered set $\mathfrak{T}_{pst}(t)$ of the past instants, which is* ***the past*** *and which does not have the first moment denoted by $t_{\inf}$,*

$$\inf \mathfrak{T}_{pst}(t) = t_{\inf} = -\infty.$$

A moment that is happening is ***the present (the current) moment****. It forms the singleton $\mathfrak{T}_{pzt}(t)$ of the present moment, which is* ***the present****. The present moment occurs after all past moments and we assign to it a real number bigger than the real numbers assigned to all past moments.*

All instants that have not yet happened are ***the future instants****. They form the temporally ordered set $\mathfrak{T}_{ftr}(t)$ of the future instants, which is* ***the future*** *and which does not have the last instant denoted by $t_{\sup}$,*

$$\sup \mathfrak{T}_{ftr}(t) = t_{\sup} = \infty.$$

The present instant is happening before all future instants and the real number assigned to it is smaller than the real numbers assigned to all future instants.

All three temporal sets, the past $\mathfrak{T}_{pst}(t)$, the present $\mathfrak{T}_{pzt}(t)$, and the future $\mathfrak{T}_{ftr}(t)$, are ***connected****. The past $\mathfrak{T}_{pst}(t)$, and the future $\mathfrak{T}_{ftr}(t)$ are also* ***dense****. They are all variable - they are all three (the past, the present and the future)*

time-varying *and* ***always disjoint in pairs****. Their union,* ***the time set*** $\mathfrak{T}(t)$*,*

$$\mathfrak{T}(t) = \mathfrak{T}_{pst}(t) \cup \mathfrak{T}_{pzt}(t) \cup \mathfrak{T}_{ftr}(t), \tag{4.14}$$

is also connected and dense, but it is time-invariant and permanent,

$$\mathfrak{T}(t) \equiv \mathfrak{T}. \tag{4.15}$$

e) ***Time flow, its orientation and duration***

The time flow *(i.e.* ***the temporal flow****) is an oriented (i.e. directed) variation of moments through a temporally ordered sequence of time values, from one, earlier, time value to another, later, time value. Its orientation is* ***the time or the temporal orientation (the temporal sense, the temporal direction, the temporal arrow)****. Its temporal length (temporal size) is* ***duration****.*

The temporal orientation (the temporal sense, the temporal direction, the temporal arrow) is the orientation (is the sense, is the direction, is the arrow oriented) from the past moments through the present moment towards the future moments, or equivalently, from the past through the present towards the future.

The duration of the instant (of the moment) is the minimal possible duration that is ***instantaneous (momentous)****, i.e. infinitesimal duration.*

f) ***Time interval***

The temporally both ordered and directed (oriented) set of all instants between two different time values is ***time interval****. The time interval* $[t_0, t_{trm}]$, $t_{trm} > t_0$*, reflects the duration from* ***the initial instant*** $t_0 \in \mathfrak{T}$ *to* ***the terminal instant*** $t_{trm} \in \mathfrak{T}$ *either of the existence or of the nonexistence of the related somebody or something (i.e. of the related being, or of the related form of energy, or of the related kind of matter, or of the related movement, object, process, or of the related rest).*

g) ***Age***

A time value difference $t - t_0 \geq 0$ *is* ***the age of somebody or something at the moment*** t ***relative to*** t_0 *(for short:* ***the age****), where* t_0 *is the initial moment, i.e. the moment of the beginning of the existence of the related somebody or something (i.e. of the related being or of the related form of energy, of the related kind of matter, of the related object, of the related movement, of the related process, or of the related rest).*

h) ***Time relativity***

Time (the temporal variable) itself ***is not relative.***

The relativity of time is exclusively the relativity of its numerical value with respect to the accepted

- *zero moment* $t_{(.)zero} \in \mathfrak{T}_{(.)}$,
- *initial instant* $t_{(.)0} \in \mathfrak{T}_{(.)}$,
- *time scale,*

and/or

- *time unit* $1_{t_{(.)}}$.

We will use this axiomatic characterization of *time* throughout the book as its keystone, and we will test it from the relativity point of view in the second and in the third part of the book.

Let us consider the properties and relativity of *time* in more details.

4.2.6 Existence of *time*

After a session on *time* at IEEE Conference on Systems, Man and Cybernetics in Beijing, China, October 14 - 16, 1996, a participant stated that there was a prophet in his country who had claimed that he had stopped *time* (meaning that he had stopped the *time* value evolution). Another participant then asked:

"For *how long* did the prophet stop *time*?"

The former participant did not reply.

As soon as there is a claim in the following sense:

"*Time* did not exist before certain event",

then some, or all, of the following questions rise naturally:

° Since which moment has not *time* existed?

° Until which moment did not *time* exist?

° How long did not *time* exist?

° During which *time* interval did not *time* exist?

° Why and how did *time* start to exist at the moment when the event occurred?

Trivially obvious replies to these questions illustrate that *time* had existed before the event happened and the *time* value was changing (increasing) in spite some processes or phenomena had been stopped, even if they had not existed before that moment, even if there had been a rest, and in spite the rest had possibly lasted during a long *time* interval before that instant.

Note 48 ***Speed, velocity and time***

The speed v_g *(the velocity* $\mathbf{v}_g$*) of a variation of the (vector) value of a variable* $g(.)$ $(\mathbf{g}(.))$ *is, respectively, a dependent (vector) variable. It is, by the well known definition, the quotient of the infinitesimal variation [i.e. of the differential]* dg $(d\mathbf{g})$ *of the value of the (vector) variable* $g(.)$ $(\mathbf{g}(.))$*, and of the infinitesimal value variation [i.e. of the differential]* dt *of time* t*, respectively,*

$$v_g = \frac{dg}{dt},\ \mathbf{v}_g = \frac{d\mathbf{g}}{dt},\ dt > 0. \tag{4.16}$$

If time had not existed or if it had not been a (physical) variable then we would not have been able to identify the speed as a (physical) variable, hence, the velocity as a vector (physical) variable. The existence of the speed (of the velocity) proves illustratively the existence of time and that it is a physical variable. Besides, it illustrates that we cannot clearly explain what is velocity or speed, that we cannot define precisely them, before we have explained what is time. Einstein's attitude (cited at the beginning of the book) that the speed c of light can be, and should be, used to define time is invalid and unacceptable.

We state now the axiom of the existence of *time*:

Axiom 49 ***Existence of time***
Time cannot be created or destroyed. It has ever existed and will exist for ever.

This Axiom agrees with the essence of Newton's characterization of absolute *time*.

In this regard we should clearly distinguish the existence of *time* from a possibility (from an impossibility) to measure its values.

Claims exist in a literature that *time* would cease to exist if there were not any change, motion, movement, transition, variation of any kind (except the flow of *time* values, which is ignored in that literature). In such a (hypothetical) case, there would not be any being, and any clock would not work. The values of *time* would not and could not be measured in such a situation. However, the existence of *time* does not depend on a possibility to measure its values and their flow. The flow of *time* values would continue regardless of such (im)possibility. It is independent of everybody and everything. Nobody and nothing can influence the *time* values flow. *Time* would continue to exist. It had existed (infinitely) long before the human became able to measure its value.

4.2.7 Uniqueness of *time*

By equalizing, i.e. by identifying, *time* with its value in our colloquial reference to *time*, we can make easily a confusion, or even a mistake. Everybody does make a fault when he/she uses relativity of the accepted zero instant, and/or of the adopted initial instant, and/or of the *time* scale and/or of the *time* unit in order to claim that *time* itself is relative and that there are several different "*times*".

Theorem 50 ***Uniqueness of time***
Time is unique (the temporal variable is unique). There are not two or more different "times" (different temporal variables). Time is not relative.

Appendix 20.1 contains various proofs of Theorem 50. Some of them prove that Lorentz transformations and the new transformations established in the sequel prove *time* uniqueness rather than to imply the *time* nonuniqueness.

However, there are infinitely many different possibilities for a correct, but relative, mathematical models (mathematical representations) of *time* values in the sense of the free choices of a relative zero *time* value, of a relative initial *time* value, of a *time* scale and of a *time* unit. Whatever choices are accepted, for a mathematical model (for a mathematical representation) of *time* to be physically correct it is necessary to ensure both its independence and continuous monotonous strict increasing of its value and of its numerical value, i.e. to ensure $dt > 0$.

Uniqueness and nonrelativeness of *time* agree with Newton's explanation of the absolute sense of *time*, [360, I of Scholium, p. 8].

4.2.8 Name for *time*

There are used many different terms for *time* in literature, e.g. "absolute time", "$\alpha i\omega\nu$" ("aion", "aions"), "apparent time", "astronomic time", "asymmetrical time", "atomic time", "autistic time", "biological time", "calendar time", "Christian time", "cinematic time", "circular time", "clock time", "closed time", "coherence time", "collective time", "common time", "coordinate time", "cosmic time", "cyclic time", "dial time", "differential time", "directed time", "dynamic(al) time", "earth time", "ecological time", "eigentime", "empty time", "ephemeris time", "external time", "filled time", "Greek time", "homogeneous time", "human time", "I-time", "imaginary time", "immanent time", "impulse time", "individual time", "instantaneous time", "intrinsic time", "irreversible time", "$\varkappa\alpha\iota\rho\acute{o}\zeta$" ("kairos", "kairoi"), "laboratory time", "local time", "local solar time", "mathematical time", "measurable time", "measured time", "microphysic time", "musical time", "natural time", "negative time", "objective time", "observable time", "ordinary time", "organic time", "parameter time", "parametric time", "perception time", "periodic time", "personal time", "physical time", "positive time", "pragmatic time", "prescientific time", "present time", "private time", "proper time", "psychological time", "public time", "pure time", "qualitative time", "quantitative time", "real time", "recorded time", "relative time", "relativistic time", "reversible time", "sacred time", "scientific time", "sensory-motor time", "sidereal time", "social time", "solar time", "space-time", "spacetime", "standard solar time", "stellar time", "subjective time", "theoretical time", "thermodynamic time", "time-at-a-distance", "time-number", "time of consciousness", "time of the being", "time of things", "time-space", "traditional time", "traveller's time", "true time", "uniform time", "universal time", "usual time", "venet time", "world time" [43], [51], [56], [66], [73], [74], [80], [86], [88], [101], [108], [110], [114], [119], [144], [150], [153], [154], [161], [169], [183], [184], [185], [186], [189], [190], [193], [247], [240] - [243], [248], [259], [264], [276], [278], [282], [283], [293], [312], [319], [334], [337], [339], [347], [351], [355], [359], [360], [374], [394] - [397], [404], [406], [415], [417], [418], [420], [425], [440], [445], [479], [480], [487], [491], [497]. These notions concern the same, unique, variable that is the temporal variable, i.e. *time*, but with associated, differently accepted, zero *time* value t_{zero}, and/or differently adopted initial *time* value t_0, and/or differently chosen *time* scale, and/or differently accepted *time* unit, and/or they express a different derived (mathematical) model [(mathematical) representation] of *time*, respectively.

4.2.9 Beginning, end, and *time*

Boundedness of all human abilities and of the duration of the human life appears as an obstacle to get a clear physical feeling and sense of spatial and temporal unboundedness, of infinity, such as unboudedness of the existence of *time*, i.e. the unboundedness of the *time* set $\mathfrak{T}$.

An arbitrary point P on a straight line is in infinity relative to the "beginning", and relative to the "end", of the line. Analogously, any human H is in the spatial and temporal infinity relative to the "beginning", and relative to the "end", of space and of the existence of *time*, i.e. relative to everybody and everything who and which are in the spatial and the temporal infinity with respect to the human H herself/himself. In this sense, each of us is in relative temporal and spatial infinity.

It seems that there is a confusion when the topics are the beginning of *time* and the date of the beginning of our energy-matter universe. Whichever date we accept for the date of the beginning of the existence of Earth (about $4{,}6 \times 10^9$ years ago [424, p. 11]), or of our solar system, or of our galaxy, or of our energy-matter universe (about 15×10^9 years ago [424, p. 11]), it is only the initial instant of the existence of Earth, or of our solar system, or of our galaxy, or of our physical universe, respectively. It is not the moment of the beginning of the existence of *time*. It is not the total zero value of *time*. Moreover, it is not the initial moment of the existence of matter and/or of energy. It can be accepted for a relative zero value of *time*. The initial instant has the relative zero value.

Saying today (in the XXI century) that the instant of the beginning of the existence of our energy-matter universe is the instant of the beginning of the existence of *time*, hence, the total zero value of *time*, is the mistake of the same rank as the wrong claim and the erroneous attitude of Ancient Age that Earth is the center of the universe.

Claiming that there did not exist anybody or anything, including matter and energy, including **nature**, before the beginning of the existence of our energy-matter universe is a total absurd. It means that nobody or nothing *could* create, was *able* to create by itself all, including matter and energy, including God and **nature**. It would mean that there existed an ability of nobody or of nothing to create, i.e. nobody or nothing possessed an ability and a power to create. The existence of such ability, of such power contradicts the nonexistence of anybody and anything. This shows the absurd of the claim that either anybody or anything did not exist, but that the ability, the power, of nobody or of nothing, did exist. If such absurd had been true, then it would have rejected completely the validity of the fundamental physical laws - the law of the preservation (of the conservation) of energy and matter (see Law 66 in Section 4.5) and the law of the energy - matter existence (see below Law 67 in Section 4.5).

Whatever are the dates of the beginnings of the existence of Earth, of our solar system, of our galaxy, of our energy-matter universe, we should understand that *time*, the position space, energy and matter had existed before those dates. Moreover, they will continue to exist after the disappearance of Earth, of our solar system, of our galaxy, and of our energy-matter universe if it happens in future. We should not repeat the egocentric mistakes of Ancient and Middle Age. We should follow the spirit of Jordano Bruno and the work of Galileo Galilei.

4.2.10 Existence, *time* value counting and measurement

The existence / the nonexistence of anybody and of anything created is said to be *in time* because *time* occupies (covers, encloses, imbues, impregnates, is over and in, penetrates) permanently and equally everybody and everything (beings, objects, energy, matter, happenings, movements, processes and the position space).

The existence of any created being in general, of any terrestrial being in particular, has its beginning and its end. The beginning happens before the end (or equivalently, the end occurs after the beginning).

It seems at first glance from a mathematical point of view that we can chose freely to count moments either from the instant of the beginning towards the instant of the end of the existence of somebody or of something, or from the instant of the end towards the instant of the beginning. It has appeared natural, and most often only possible, to reckon moments from the moment of the beginning towards the moment of the end of the existence of somebody or of something. The moment of the beginning of the existence is usually known, or at least available, or evaluable, but the moment of the end of the existence is rarely known. It is usually unpredictable, when the existence is in the course. Besides, such reckoning corresponds to the temporal orientation of the *time* value evolution. It is not possible to measure *time* values in the inverse temporal orientation, i.e. from the end towards the beginning of the existence of somebody or of something either before the existence, or during the existence or later. Therefore, we count naturally the moments from the moment of the beginning of the existence of somebody or of something towards the moment of the end of her/his, or of its, existence.

4.2.11 Numerical values of *time* and relativity

When we say that we measure a length, what we actually do is to measure ***the ratio of the lengths of two bodies****. In the same way as the absolute velocity of a body has no physical meaning, but only the relative velocity of one body with respect to another, as demonstrated by Galileo, the length of a body or the periods of a clock has no physical meaning, but only the ratio of the lengths of* ***two bodies*** *and the ratio of the periods of* ***two clocks****.*

....

One of the main characteristics of scale which points toward the need for a scale relativity theory is the nonexistence of an ***a priory*** *absolute scale.*

Laurent NOTTALE [362, pp. 21, 220]

We say: "It is eight forty five PM" in the sense: "The (relative) value of *time* is eight hours and forty five minutes this afternoon". If we say that it is twenty hours and forty five minutes, then we express another numerical *time* value (20:45) different from the preceding (8:45), but they both represent the same *time* value measured in two different *time* scales, $8h45minPM = 20h45min$, and relative to different numerical zero *time* values, noon and midnight, respectively. In this example, we use two different *time* units, which are hour and minute. The first *time* scale is PM scale (the subscript "PM"). It is not a complete daytime scale, while the second scale is a complete daytime scale (the subscript

"Compl"). The former has the numerical zero *time* value representing noon, $t_{zeroPM} = t_{noon}$. The latter has the numerical zero *time* value representing midnight, $t_{zeroCompl} = t_{midnight}$, which is also the numerical zero value of AM *time* scale (the subscript "AM"), $t_{zeroCompl} = t_{zeroAM}$. Hence, $t_{zeroCompl} \neq t_{zeroPM}$.

The numerical *time* value *numt* (the numerical value of the moment t, the numerical value of the instant t) depends on the moment (on the instant) accepted for the zero moment (for the zero instant) t_{zero}, on the *time* scale applied, and it depends on the adopted *time* unit. At different places we can accept them differently, what we really do. We can them adopt differently at the same place, but at different instants or during different *time* intervals. We do that as well. Even more. We use several different *time* scales in the same place at the same moment. Just a simple watch with three different hands for three different *time* scales with different *time* units is an example.

When we use the word *time* then it is not clear to everybody and always whether it means the temporal variable t - *time t,* or its value t, or its numerical value (*numt*) (compare Newton's explanation of *time* in Section 20.1.1 with Einstein's interpretation of *time* in Section 3.2).

When we refer to *time* in biology, chemistry, control science and control engineering, dynamical systems theory, econometrics, information science, mathematics, philosophy, physics, systems science and systems engineering, then it is indispensable to distinguish *time* (being a variable, the temporal variable) from its value and from its numerical value.

Conclusion 51 ***Time relativity: physical sense***

Once we clarify, understand and distinguish time (the temporal variable) from its value and from its numerical value, then we can accept to refer to time relativity exclusively in the sense of relativity of its numerical value numt with respect to the accepted zero moment $t_{(.)zero} \in \mathfrak{T}_{(.)}$, *to the used initial moment* $t_{(.)0} \in \mathfrak{T}_{(.)}$, *to the adopted time scale and/or to the used time unit* $1_{t_{(.)}}$ *[h) of Axiom 47].*

Analysis of *time* and of its relativity will show in the sequel why such precision is indispensable when we use the notion *time*.

Newton explained this well [360, p. 8, Scholium I] (see the Section 20.1.1). We might think that Einstein considered *time* relativity in the same sense [144], [150] through [169] (see Conclusion 42 in Section 3.2).

Nottale introduced *the principle of the scale relativity* and used it to establish *the theory of the scale relativity*, which are in Einsteinian sense [362, p. 196], [363, p. 122], [364, pp. 210, 219].

Claim 52 ***Notalle's principle of the scale relativity***

The laws of physics must be such that they apply to systems of reference whatever their state of scale.

Laurent NOTTALE *[362, p. 196]*

Remark 53 *Notalle's principle of the scale relativity imposes the condition on the laws of physics rather than on their mathematical models (mathematical descriptions, expressions).*

4.2.12 *Time* order

*A temporally ordered (*i.e. *a strictly increasing) time value sequence* determines that for any two moments t_1 and t_2 only one of the following relationships holds:

◦ t_1 and t_2 are different so that either t_1 is before (earlier than) t_2 [i.e. t_2 is after (later than) t_1], which is numerically expressed by $t_1 < t_2$ (i.e. $t_2 > t_1$), or t_1 is after (later than) t_2 [i.e. t_2 is before (earlier than) t_1], which is numerically expressed by $t_1 > t_2$ (i.e. $t_2 < t_1$), so that they are not simultaneous,

or

◦ t_1 and t_2 are equal so that they are simultaneous, which is numerically expressed by $t_1 = t_2$ (i.e. $t_2 = t_1$).

These relationships determine the *time* order (the temporal order).

The past moments (the past) occurred *before* the present moment (*before* the present) that is occurring *before* the future moments (*before* the future). Or equivalently, the future instants (the future) will occur *after* the present instant (*after* the present), which is occurring *after* the past instants (*after* the past).

The past and the future exist in the temporal continuum, i.e. in the *time* set $\mathfrak{T}$, but they are not temporally currently attainable for us. They are not being realized. The past occurred. It had been and it was. The present is available. The present is always being realized. It is *now.* Parts of the future will take place, but never the whole future. They will be. Some of them are or will be attainable. All three are always, permanently, *time*-varying. The duration of both past and future is infinite, but of present is instantaneous.

What occurred in the past is unchangeable. What is taking place now, in the present, is able of escaping a change. What could happen in future might provide a possibility to prepare its change when it is being realized in the present.

4.2.13 *Time* flow direction

The strict continuous monotonous variation of the *time* value (of moment, of instant) represents *the temporal flow of time values* (for short: *the temporal flow*). Its strictly increasing feature determines its sense (i.e. its arrow, its direction, its orientation): *time flow sense* (*arrow*, *direction, orientation*). Hence, there is *the temporal orientation* (*the temporal arrow*, *the temporal direction, the temporal sense) of the time value variation.* It is from the past moments (from the past) through the present (current) moment (through the present) towards the future moments (towards the future). Nobody and nothing can change either the temporal orientation or the speed of the *time* value evolution (or colloquially, "*time* speed"). The temporal orientation is in the literature known also as "*time*'s arrow" ("la flèche du temps" in French) [72], [336], [337], or "the direction of *time*" [406]. It is not a spatial orientation.

The evolution of *time* value cannot be reversed or stopped or decelerated or accelerated.

What is *now* a future moment, it can be *now* the present moment. What is *now* the present moment, it was a future moment, but it is already a past moment *now*. These four *now* are mutually temporally different.

The moment that is now present moment and all past moments are temporally moving away from the (new) now present moment (from the present). They left the present into the past. Future moments are temporally approaching the present moment (will successively enter the present) from the future. Consequently, the element of the present has been continuously monotonously changing from an earlier moment to a later moment. This temporal propagation represents the temporal flow from the past *time* values through the present *time* value to future *time* values. It creates, expresses and preserves the *time* flow direction.

The present, the past and the future have ever existed and will exist for ever in the temporal continuum, in the *time* set $\mathfrak{T}$, but they have been, they are, and they will be permanently continuously changing. They are *time*-varying temporal sets. They are at every moment different from themselves at another moment. They are relative to the present instant - relative to the present. Although they are *time*-varying, the *time* set $\mathfrak{T}$ is *time*-invariant.

From a mathematical point of view, the present instant is the element of the intersection between the boundary (between the closure) of the set of the past instants and the boundary (the closure) of the set of the future instants, respectively. This intersection is the present. Since this intersection is singleton, then the present instant is the single and unique element of the intersection, hence, of the present. Although it is single and unique in this sense, it is permanently changing. The same holds for *now*.

We have accepted that the current *time* value (the present moment) is bigger than any past *time* value (it is after every past moment), and simultaneously it is smaller than any future *time* value (it is before every future moment). The same holds for their numerical values. Consequently, the strict monotonous temporal variation of the *time* value (from its past values through its present value towards its future values) is from its smaller (numerical) values towards its bigger (numerical) values. The strict monotonous temporal variation of the value of *time*, and of the numerical value of *time*, is *strict monotonous* ***increase*** *of its value, and of its numerical value.* Therefore,

$$dt > 0 \tag{4.17}$$

is only possible. Hence, $dt \leq 0$ is physically impossible and does not have a physical sense. In what follows, $dt > 0$ is only allowed and used.

4.2.14 Speed of the *time* value evolution (*time* speed)

Since the value and the numerical value of *time* are smoothly monotonously continuously strictly increasing then the incremental increase of the *time* value, i.e. its differential dt, is strictly positive, (4.17),

$$dt > 0, \tag{4.18}$$

which is from the physical point of view only meaningful. However, if we consider t as a mathematical description of *time* (*mathematical time* in Newton's terminology) and then treat it as a mathematical variable only, then $dt \leq 0$ can be accepted from a purely mathematical standpoint without having any physical sense. Such a mathematical property of the description of *time* is neither adequate to the nature of *time* nor physically justifiable. It is physically unacceptable.

In mathematical models of physical processes and of physical systems it is tacitly assumed that (4.18) holds.

We should clearly distinguish the velocity ($\mathbf{v}_t$ or $\mathbf{v}_\tau$) of the evolution of the *time* value (for short: *the time velocity*) from the velocity $\mathbf{v}_{(.)}$ of another variable value variation, of another motion and of another process.

The value (v_t or v_τ) of the velocity $\mathbf{v}_t$ or $\mathbf{v}_\tau$ is *the speed* v_t or v_τ of the evolution of the *time* value and of the numerical *time* value (for short: *the time speed*, or, *the temporal speed*).

There is a deep confusion in the literature on *time* in this connection. Speeds of variations of values of many variables, or of propagation of many processes, are mixed up with the *time* speed (v_t or v_τ). For example, the speed of the change of the biological state of an organism is referred to in certain literature as the *time* speed in spite it is not the *time* speed.

We are now going to show that the numerical value of both v_t and v_τ is the same and equals one ($numv_t = 1$ and $numv_\tau = 1$) in every *time* scale, hence in every integral space. It equals one relative to every *time* unit,

$$\mathbf{v}_t = \frac{d(t\mathbf{t}_u)}{dt} = (1\mathbf{t}_u)\,[\mathrm{TT}^{-1}]\left\langle 1_t 1_t^{-1}\right\rangle,$$
$$\mathbf{v}_\tau = \frac{d(\tau\boldsymbol{\tau}_u)}{d\tau} = (1\boldsymbol{\tau}_u)\,[\mathrm{TT}^{-1}]\left\langle 1_\tau 1_\tau^{-1}\right\rangle. \tag{4.19}$$

The following fundamental and universal *time* speed law is valid:

Theorem 54 ***Universal time speed law***

Time is the unique physical variable such that the speed v_t (v_τ) of the evolution (of the flow) of its value and of its numerical value:

a) is invariant relative to a choice of an initial moment, of a time scale and of a time unit, i.e. invariant relative to a choice of a time axis,

and

b) its value (its numerical value) equals one arbitrary time unit per the same time unit (equals one), respectively,

$$v_t = 1[TT^{-1}]\left\langle 1_t 1_t^{-1}\right\rangle = 1[TT^{-1}]\left\langle 1_\tau 1_\tau^{-1}\right\rangle = v_\tau,\ numv_t = numv_\tau = 1, \tag{4.20}$$

relative to arbitrary time axes T and T_τ, i.e. its numerical value equals 1 with respect to all time axes (with respect to all initial instants, all time scales and all time units).

Proof. Let $v_{xt}(.)$ be the speed of an arbitrary physical variable $x(.)$[X] $\langle 1_x \rangle$ measured relative to the *time* axis T,

$$v_{xt}(t;t_0) = \frac{dx(t;t_0)}{dt}[XT^{-1}]\left\langle 1_{x(t;t_0)}1_t^{-1}\right\rangle,$$

and $v_{x\tau}$ be its speed measured relative to a *time* axis T_τ,

$$v_{x\tau}(\tau;\tau_0) = \frac{dx(\tau;\tau_0)}{d\tau}[XT^{-1}]\left\langle 1_{x(\tau;\tau_0)}1_\tau^{-1}\right\rangle.$$

The initial moment ($t_0 \in \mathfrak{T}$ or $\tau_0 \in \mathfrak{T}_\tau$) is arbitrary and fixed.

Uniqueness and necessity. Let us accept at first for variable $x(.)$[X] $\left\langle 1_{x(.)}\right\rangle$ to be different from *time,*

$$\dim x(.) = X \neq T = \dim t \text{ and } unit\,(x(.)) = 1_{x(.)} \neq 1_{(.)} = unit(t). \tag{4.21}$$

Let its value obey (5.10),

$$v_{xt}(t;t_0) = 1[XT^{-1}]\left\langle 1_{x(t;t_0)}1_t^{-1}\right\rangle = v_{x\tau}(\tau;\tau_0) = 1[XT^{-1}]\left\langle 1_{x(\tau;\tau_0)}1_\tau^{-1}\right\rangle. \tag{4.22}$$

Let us change *time* axis so that

$$t_2 - t_1 = \mu_{t\tau}(\tau_2 - \tau_1),\ dt = \mu_{t\tau}d\tau,$$

where

$$\mu_{t\tau} \in R^+,\ \mu_{t\tau}\left\langle 1_t 1_\tau^{-1}\right\rangle,$$

is different from one,

$$\mu_{t\tau} \neq 1, \tag{4.23}$$

due to the change of the *time* axis. Since the variable *x(.)* is not *time*, then we do not change its scale and unit so that

$$1_{x(t;t_0)} \equiv 1_{x(\tau;\tau_0)} \Longrightarrow \eta_{xt\tau} = 1\left\langle 1_{x(t;t_0)}1_{x(\tau;\tau_0)}^{-1}\right\rangle. \tag{4.24}$$

Hence, the equations (5.12) imply

$$\begin{aligned}
1\left\langle 1_x 1_t^{-1}\right\rangle &\equiv v_{xt}(t;t_0)[XT^{-1}]\left\langle 1_x 1_t^{-1}\right\rangle = \frac{dx(t;t_0)}{dt}[XT^{-1}]\left\langle 1_{x(t;t_0)}1_t^{-1}\right\rangle = \\
&= \eta_{xt\tau}\left\langle 1_{x(t;t_0)}1_{x(\tau;\tau_0)}^{-1}\right\rangle \mu_{t\tau}^{-1}\left\langle 1_t^{-1}1_\tau\right\rangle \frac{dx(\tau;\tau_0)}{d\tau}[XT^{-1}]\left\langle 1_{x(\tau;\tau_0)}1_\tau^{-1}\right\rangle = \\
&= \mu_{t\tau}^{-1}v_{x\tau}(\tau;\tau_0)[XT^{-1}]\left\langle 1_x 1_\tau^{-1}\right\rangle \equiv \mu_{t\tau}^{-1}1\left\langle 1_x 1_\tau^{-1}\right\rangle.
\end{aligned} \tag{4.25}$$

Therefore, in order for (5.12), i.e. for (5.15),

$$1\left\langle 1_x 1_t^{-1}\right\rangle \equiv v_{xt}(t;t_0)\left\langle 1_{x(t;t_0)}1_t^{-1}\right\rangle \equiv v_{x\tau}(\tau;\tau_0)\left\langle 1_{x(\tau;\tau_0)}1_\tau^{-1}\right\rangle \equiv 1\left\langle 1_x 1_\tau^{-1}\right\rangle,$$

to hold it is necessary and sufficient that

$$1_t = 1_\tau.$$

This implies

$$\mu_{t\tau} = 1,$$

which results also from (5.15) and contradicts (5.13). The contradiction is a consequence of the assumption that the variable $x(.)[\mathrm{X}]\left\langle 1_{x(.)}\right\rangle$ is not *time.* Therefore, $x(.)[\mathrm{X}]\left\langle 1_{x(.)}\right\rangle$ is *time* $t[T]\langle 1_t\rangle$,

$$x(t;t_0)[X]\left\langle 1_{x(t;t_0)}\right\rangle \equiv (t-t_0)\,[T]\,\langle 1_t\rangle$$

and

$$x(\tau;\tau_0)[X]\left\langle 1_{x(\tau;\tau_0)}\right\rangle \equiv (\tau-\tau_0)\,[T]\,\langle 1_\tau\rangle\,.$$

Trivially, its unit changes if we change the *time* unit. Its value is measured with the *time* units 1_t and 1_τ. Its physical dimension is *time.* These conclusions and the last two equations prove necessity for the variable *x(.)* to be *time.*

Invariance and sufficiency. Let x be *time.* Let v_t be the *time* speed and its value be measured relative to the *time* axis T. Let v_τ be its speed, the value of which is measured relative to the *time* axis T_τ. We accept $1_t \neq 1_\tau$ so that $T \neq T_\tau$. In view of the definition of the speed of any variable applied to the speed of *time*, we may write the following:

$$v_t = \frac{dt}{dt} = 1[TT^{-1}]\left\langle 1_t 1_t^{-1}\right\rangle,\ and\ v_\tau = \frac{d\tau}{d\tau} = 1[TT^{-1}]\left\langle 1_\tau 1_\tau^{-1}\right\rangle.$$

These equations complete the proof. Q. E. D ■

The equations (5.9) and (5.10) verify the claim that both the *time* value and the numerical *time* value are strictly monotonously increasing, equally in all spatial directions since the directions of the unity vectors $\mathbf{t}_u$ and $\boldsymbol{\tau}_u$, and of their extensions $\mathbf{t}_{ue}$ and $\boldsymbol{\tau}_{ue}$, hence, the directions of the *time* axes $T_t = T$ and T_τ, are arbitrary relative to the space. The equations (5.9) and (5.10) show also that the *time* speed is independent of all beings, objects, processes, space and of all other variables.

Conclusion 55 ***Invariance of the time speed***

The speed of the (numerical) time value evolution [for short: ***the time speed****] is the same in all integral spaces. Its numerical value equals one. It is a universal constant. It is invariant relative to a choice of a time unit, of a time scale and of an integral space. It is independent of spatial coordinates, of movements of spatial frames and of the space. It is universal invariant.*

The time speed is evidently one arbitrary time unit per the same time unit, e.g. one second per second, one minute per minute, one hour per hour, one day per day, one year per year, one decade per decade, one century per century, ...

The numerical value of the time speed equals one, independently of everybody and everything. It is simple, but universal, constant and invariant numerical speed value. There is not another variable with such a property of the speed of its value variation. The speed of light (propagation) does not possess such properties.

This confirms the essence of Newton's attitude that "time, of itself, and from its own nature, flows equally without relation to anything external".

4.2.15 Continuous-*time* set and discrete-*time* set

The past $\mathfrak{T}_{pst}(t)$, the present $\mathfrak{T}_{pzt}(t)$, and the future $\mathfrak{T}_{ftr}(t)$ are *time*-varying temporal sets. They form the *time* set $\mathfrak{T}$, i.e. their union is the *time* set $\mathfrak{T}$, (4.14), which is independent of *time*, (4.15).

Let us summarize the properties of the *time* set $\mathfrak{T}$. There is not any moment that can simultaneously belong to both members of any pair composed of $\mathfrak{T}_{pst}(t)$, $\mathfrak{T}_{pzt}(t)$, and $\mathfrak{T}_{ftr}(t)$. Hence, these *time* sets are always disjoint in pairs.

Conclusion 56 ***Continuous-time set*** $\mathfrak{T}$

The time set *$\mathfrak{T}$ is the set of all moments (of all instants) t, or equivalently, it is the union of the past $\mathfrak{T}_{pst}(t)$, the present $\mathfrak{T}_{pzt}(t)$ and the future $\mathfrak{T}_{ftr}(t)$, (4.14). It is constant (time - invariant), (4.15), in the one-to-one correspondence with the set R of real numbers, and it is totally temporally ordered, connected, everywhere dense,*

a) with num inf $\mathfrak{T} = numt_{\inf} = -\infty$ and numsup $\mathfrak{T} = numt_{\sup} = \infty$,

b) with an instant conventionally accepted and fixed for the zero instant: $t_{zero} = 0$,

c) with an adopted time unit (second, minute, hour, day, ...),

d) with an accepted time scale (for seconds, minutes, hours, days, ...), and

e) with an arbitrary element denoted by t (or by τ), which takes place exactly ones and the value of which is strictly monotonously continuously increasing,

$$\mathfrak{T} = \{t : t[T] \langle s \rangle \, , \; numt \in R, \; dt > 0\}, \tag{4.26}$$

where $[T]$ denotes the physical dimension of time, and $\langle s \rangle$ specifies that the second s is the accepted unit of time.

f) Between any two different instants $t_1 \in \mathfrak{T}$ and $t_2 \in \mathfrak{T}$ there is a third instant $t_3 \in \mathfrak{T}$, either $t_1 < t_3 < t_2$ or $t_2 < t_3 < t_1$. The time set $\mathfrak{T}$ is ***continuum****. It is called also* ***the continuous-time set****.*

Time does not have either the minimal real value (the first instant $t_{\inf}$) or the maximal real value (the last instant $t_{\sup}$), which is expressed under a) by

$$numt_{\inf} = num \inf \mathfrak{T} = -\infty, \; numt_{\sup} = num \sup \mathfrak{T} = +\infty.$$

There was not the beginning of the *time* value evolution. There will not be its end.

If we should assign different *time* units and/or *time* scales to different beings, objects, processes, systems or subsets of space, then it appears useful to accept a reference *time* unit and a reference *time* scale. We should well define the reference *time* set $\mathfrak{T}$ and its corresponding mathematical model - the reference *time* axis T.

The *time* set $\mathfrak{T}$ is the *eternity.*

Discrete *time* denoted by t_i and discrete *time* set designated as $\mathfrak{T}_d$ represent discrete mathematical models of *time* t and of the *time* set $\mathfrak{T}$, respectively. They are used in computer science and computer engineering, in control science and control engineering, in econometrics, in information science, in mathematics, in systems science and systems engineering.

Definition 57 ***Discrete time set*** $\mathfrak{T}_d$

Discrete time set $\mathfrak{T}_d$ *is the set of all specially selected (discrete) moments* t_i*, the values of which increase discontinuously. It is in the one-to-one correspondence with the set of all integers. It is totally temporally ordered and denumerable set*

b) with an instant conventionally accepted and fixed for the zero moment: $t_{zero} = 0$*,*

c) with an adopted time unit (second, minute, hour, day, ...),

d) with an accepted time scale (for seconds, for minutes, for hours, for days, ...),

and

e) with an arbitrary element denoted by t_i *that takes place exactly ones and the value of which is strictly monotonously increasing:*

$$\mathfrak{T}_d = \{t_i : t_i \in \mathfrak{T},\ i \to |\infty| \Longrightarrow numt_i \to |\infty|\,,\ \infty > t_{i+1} - t_i > \delta_i \geq \delta > 0\}. \tag{4.27}$$

Discrete *time* t_i and discrete *time* set $\mathfrak{T}_d$ are very useful for mathematical (analytical, numerical) studies of events and of systems (as automata, digital computers, discrete event systems, microprocessors, sampled data systems, digitally controlled plants), which possess so fast transient processes that their duration can be neglected relative to the duration of other involved processes.

The discrete *time* set $\mathfrak{T}_d$ is a proper subset of the *time* set $\mathfrak{T}$, $\mathfrak{T}_d \subset \mathfrak{T}$.

4.3 *Time* scales, units and interval mappings

One second of time as judged from the sun is not equal to
one second of time as seen from the projected body.
Albert EINSTEIN [167, p. 5]

On the microscopic scale, as in elementary particle physics,
we have dimensions of the order of 10^{-22} *seconds and* 10^{-15}
centimeters. On the macroscopic scale, as in cosmology,
time can be of the order of 10^{10} *years (the age of the*
universe) and distance of the order 10^{28} *centimeters (the*
distance to the event horizon; i.e. the furthest distance
from which physical signals can be received).
Ilya PRIGOGINE [395, p. 1]

4.3.1 Dimensions and units of *time*

The **physical dimension of *time*** t is denoted by T, where T stands for "*time*",

$$phdim(t) = \mathrm{T},\ or\ equivalently,\ t[\mathrm{T}].$$

It cannot be expressed in terms of another variable. The physical dimension of a moment t_a and of a *time* interval $\mathcal{T}$ is also the dimension T of *time* t,

$$phdim(t_a) = phdim(t) = \mathrm{T},\ phdim(\mathcal{T}) = phdim(t) = \mathrm{T},$$

so that $t_a[\mathrm{T}]$ and $\mathcal{T}[\mathrm{T}]$. The mathematical dimension of *time* denoted by *dim*(t) equals one,

$$dim(t) = 1.$$

If the value of *time* t is measured in seconds (s), then $unit(t) = s$, hence: $t\left\langle s\right\rangle$.

The basic *time* unit 1_{tbasic} is *second, s,* $unit(t){=}1_{tbasic}{=}s$. It is defined as follows:

Definition 58 ***The definition of the second*** *[79, p. 19], [255, p. 921], [319, p. 39]*

The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of the Caesium-133 atom.

Comment 59 ***The meter being the basic length unit is defined in terms of the time unit***

The meter is the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second [255, p. 921].

The definitions of the second as the basic *time* unit, and of the meter as the basic length unit in terms of *time*, illustrate that *time* is independent of space, of velocity in general, and of the velocity of light in particular.

4.3.2 *Time* axes

The *time* set $\mathfrak{T}$, [see (4.15) in Axiom 47, and (4.26) in Conclusion 56], will be geometrically equivalently represented by a *time* axis denoted by T,

$$T = \left\{ \sigma : \begin{array}{c} \sigma \in R,\ d\sigma \in R^{+}, \\ \left(\begin{array}{c} \forall \sigma \in R \Longrightarrow \exists! t \in \mathfrak{T}, \\ numt = \sigma,\ numdt = d\sigma; \end{array} \right) \\ \left(\begin{array}{c} \forall t \in \mathfrak{T} \Longrightarrow \exists! \sigma \in R, \\ \sigma = numt,\ d\sigma = numdt \end{array} \right) \end{array} \right\}. \tag{4.28}$$

Their equivalence means that they are in the one-to-one correspondence. Both are temporally ordered. Every element of each of them can occur exactly once. The value of an arbitrary element of each of them is strictly monotonously continuously increasing. The *time* axis T possesses the features of the *time* set $\mathfrak{T}$ presented in Axiom 47 and in Conclusion 56. We write usually for short $t \in T$ in the sense $t \in \mathfrak{T}$ and $numt \in T$. We do this, for example, in all the graphical representations of the axis T, and whenever it does not create any confusion. In such cases t represents the nondimensional time variable $\overline{t}$,

$$\overline{t}\,[-] = \left(\frac{t\,[T]}{1_t\,[T]}\right)[-]\,,\ num\left(\overline{t}\right) \equiv num\,(t)\,. \tag{4.29}$$

4.3.3 *Time* scaling coefficients: definition

Time by itself, like other physical variables, does not determine a scale and a unit to measure its value. We accept them freely to suit our studies of phenomena. Therefore, we use various *time* scales and *time* units. They should be clearly distinguished from *time* itself. Their relativeness causes relativeness of numerical values of moments.

The existence and use of various *time* scales and *time* units imply the need for establishing relationships among them. *Time scaling coefficients* (*factors*) relate mutually different *time* scales and units.

An arbitrary element of both the accepted reference *time* set $\mathfrak{T}$ and the corresponding reference *time* axis T represents the corresponding arbitrary moment t. Their *time* unit 1_t is the basic *time* unit 1_{tbasic}. It is second, s,

$$1_t = 1_{tbasic} = s.$$

An arbitrary *time* value t is related to the basic *time* unit as follows:

$$t\ \langle 1_{tbasic}\rangle = t1_t\,\langle s\rangle = t\ \langle s\rangle\,,\ \forall t \in \mathfrak{T}.$$

Similarly, 1_τ is the *time* unit of the T_τ-axis and τ is an arbitrary instant measured with the *time* unit 1_τ. Let $1_\tau = s_\tau$ be the $\tau-$unit. It is called the $\tau-$second denoted by s_τ. Let it be equal to $\mathrm{c}^{-1}s = (2.99792458\text{x}10^8)^{-1}s$, for c=2.99792458x10^8 being the numerical value of the light speed c measured in ms^{-1}, num c =c=2.99792458x10^8. Then,

$$1_\tau\,\langle s_\tau\rangle = \left[(2.99792458\text{x}10^8)^{-1}\right]1_t\,\langle s\rangle\,, \tag{4.30}$$

i.e.

$$1_\tau\,\langle 1_\tau\rangle = (2.99792458\text{x}10^8)^{-1}1_t\,\langle 1_t\rangle\,, \tag{4.31}$$

or equivalently,

$$1_t\,\langle s\rangle = \left(2.99792458\text{x}10^8\right)1_\tau\,\langle s_\tau\rangle\,, \tag{4.32}$$

i.e.

$$1_t\,\langle 1_t\rangle = \left(2.99792458\text{x}10^8\right)1_\tau\,\langle 1_\tau\rangle\,. \tag{4.33}$$

The coefficient $\mu_{\tau t} = \mu_{s_\tau s} = (2.99792458\text{x}10^8)\left\langle s_\tau s^{-1}\right\rangle$ transforms the unit $1_t = s$ into $N_{s_\tau s} = 2.99792458\text{x}10^8$ units s_τ,

$$\mu_{\tau t}\left\langle s_\tau s^{-1}\right\rangle\,[-] = \mu_{s_\tau s}\left\langle s_\tau s^{-1}\right\rangle\,[-] = (2.99792458\text{x}10^8)\left\langle s_\tau s^{-1}\right\rangle\,[-]$$

and the coefficient $\mu_{t\tau} = \mu_{ss_\tau} = \left(2.99792458\text{x}10^8\right)^{-1} \left\langle ss_\tau^{-1} \right\rangle$ transforms the unit s_τ into $N_{ss_\tau} = \left(2.99792458\text{x}10^8\right)^{-1}$ units $1_t = s$.

Hence, $\mu_{\tau t}$ is *the time scaling coefficient* that transforms the unit $1_t = s$ of the T-axis into $N_{\tau 1_t} = 2.99792458\text{x}10^8$ units $1_\tau = s_\tau$ of the T_τ-axis:

$$1_t \langle 1_t \rangle = \left(N_{\tau 1_t} 1_\tau\right) \langle 1_\tau \rangle = \left(2.99792458\text{x}10^8 1_\tau\right) \langle 1_\tau \rangle \Longrightarrow$$
$$\mu_{\tau t} = \left(2.99792458\text{x}10^8\right) \left\langle 1_\tau 1_t^{-1} \right\rangle = \left(2.99792458\text{x}10^8\right) \left\langle s_\tau s^{-1} \right\rangle = \mu_{s_\tau s}.$$

The *time* scaling coefficient $\mu_{\tau t}$ transforms t seconds s into τ τ-units $1_\tau = s_\tau$ (τ-seconds):

$$\tau \langle s_\tau \rangle = \left\{ \left(\mu_{\tau t} \left\langle s_\tau s^{-1} \right\rangle\right) \left(t \langle s \rangle\right) \right\} 1_\tau \langle s_\tau \rangle = \left(\mu_{\tau t} t\right) \langle s_\tau \rangle = \left(2.99792458\text{x}10^8 t\right) \langle s_\tau \rangle .$$

It is now obvious that the numerical values $numN_{\tau 1_t} = N_{\tau 1_t}$ and $num\mu_{\tau t}$ of $N_{\tau 1_t}$ and of $\mu_{\tau t}$, respectively, are equal,

$$N_{\tau 1_t} \langle - \rangle = numN_{\tau 1_t} \langle - \rangle = num\mu_{\tau t} \langle - \rangle = 2.99792458\text{x}10^8 \langle - \rangle .$$

Analogously,

$$\mu_{t\tau} \left\langle ss_\tau^{-1} \right\rangle = \mu_{ss_\tau} \left\langle ss_\tau^{-1} \right\rangle = \left(2.99792458\text{x}10^8\right)^{-1} \left\langle ss_\tau^{-1} \right\rangle$$

is *the time scaling coefficient* that transforms the *time* unit $1_\tau = s_\tau$ of the T_τ-axis into $N_{t 1_\tau} = \left(2.99792458\text{x}10^8\right)^{-1}$ units $1_t = s$ of the T-axis. In general, τ units 1_τ transform into t units 1_t,

$$t \langle 1_t \rangle = \left[\left(2.99792458\text{x}10^8\right)^{-1} \left\langle 1_t 1_\tau^{-1} \right\rangle \tau \langle 1_\tau \rangle \right] \Longrightarrow$$
$$t \langle 1_t \rangle = \left[\mu_{t\tau} \left\langle 1_t 1_\tau^{-1} \right\rangle \tau \langle 1_\tau \rangle \right] = \left(\mu_{t\tau} \tau\right) \langle 1_t \rangle ,$$

or equivalently

$$t \langle s \rangle = \left(\mu_{t\tau} \tau\right) \langle s \rangle = \left[\left(2.99792458\text{x}10^8\right)^{-1} \tau \right] \langle s \rangle .$$

We may conclude that $N_{t 1_\tau}$ and $\mu_{t\tau}$ have the same numerical value,

$$N_{t 1_\tau} \langle - \rangle = numN_{t 1_\tau} \langle - \rangle = num\mu_{t\tau} \langle - \rangle = \left(2.99792458\text{x}10^8\right)^{-1} \langle - \rangle .$$

Altogether, we summarize that the *time* scaling coefficients $\mu_{\tau t}$ and $\mu_{t\tau}$ are mutually related by

$$\mu_{\tau t} \left\langle 1_\tau 1_t^{-1} \right\rangle = 2.99792458\text{x}10^8 \left\langle 1_\tau 1_t^{-1} \right\rangle = \mu_{t\tau}^{-1} \left\langle \left(1_t 1_\tau^{-1}\right)^{-1} \right\rangle .$$

We summarize now the preceding consideration by referring to (4.2) through (4.4).

Definition 60 ***Time scaling coefficients***

The basic time scaling coefficient *(for short,* ***the time scaling coefficient****)* μ_{ij} *transforms the time unit* $1_j = 1_{t_j}$ *of the* T_j*-axis into* N_{i1_j} *time units* $1_i{=}1_{t_i}$ *of the* T_i*-axis:*

$$1_j \left\langle 1_j \right\rangle = \left(N_{i1_j} 1_j \right) \left\langle 1_i \right\rangle = \left[\left(num\mu_{ij} \right) 1_j \right] \left\langle 1_i \right\rangle ,$$

so that $(t_j - t_{j0})$ *time units* 1_j *are transformed into* $t_i - t_{i0}$ *time units* 1_i *according to*

$$\left(t_i - t_{i0} \right) \left\langle 1_i \right\rangle = \mu_{ij} \left\langle 1_i 1_j^{-1} \right\rangle \left(t_j - t_{j0} \right) \left\langle 1_j \right\rangle = \left[\mu_{ij} \left(t_j - t_{j0} \right) \right] \left\langle 1_i \right\rangle ,$$

and

$$t_{i0} \left\langle 1_i \right\rangle = \mu_{ij} \left\langle 1_i 1_j^{-1} \right\rangle t_{j0} \left\langle 1_j \right\rangle = \left(\mu_{ij} t_{j0} \right) \left\langle 1_i \right\rangle ,$$

or, for short,

$$\left(t_i - t_{i0} \right) = \mu_{ij} \left(t_j - t_{j0} \right) , \ t_{i0} = \mu_{ij} t_{j0} . \tag{4.34}$$

Analogously,

$$\left(t_j - t_{j0} \right) = \mu_{ji} \left(t_i - t_{i0} \right) , \ t_{j0} = \mu_{ji} t_{i0} . \tag{4.35}$$

The equations (4.34), (4.35) result in:

$$\left(t_i - t_{i0} \right) \equiv \mu_{ij} \mu_{ji} \left(t_i - t_{i0} \right) ,$$

which implies

$$\mu_{ij} \mu_{ji} = 1, \ \mu_{ij} = \mu_{ji}^{-1} . \tag{4.36}$$

This agrees with (4.4).

The numerical values $numN_{i1_j} \left\langle - \right\rangle$ and $num\mu_{ij} \left\langle - \right\rangle$ are equal,

$$N_{i1_j} \left\langle - \right\rangle = numN_{i1_j} \left\langle - \right\rangle = num\mu_{ij} \left\langle - \right\rangle ,$$

but $N_{i1_j} \left\langle - \right\rangle$ and $\mu_{ij} \left\langle 1_i 1_j^{-1} \right\rangle$ are different,

$$N_{i1_j} \left\langle - \right\rangle \neq \mu_{ij} \left\langle 1_i 1_j^{-1} \right\rangle , \ \mu_{ij} \left\langle 1_i 1_j^{-1} \right\rangle = \left(N_{i1_j} \left\langle - \right\rangle \right) \left(1_i 1_j^{-1} \left\langle 1_i 1_j^{-1} \right\rangle \right) ,$$

due to different physical units used to measure their values.

Definition 60 implies also the following:

$$1_i \left\langle 1_i \right\rangle = \left[\left(N_{i1_j} \right)^{-1} 1_i \right] \left\langle 1_j \right\rangle , \ 1_j \left\langle 1_j \right\rangle = \left[\left(N_{j1_i} \right)^{-1} 1_j \right] \left\langle 1_i \right\rangle ,$$

due to

$$N_{j1_i} = \left(N_{i1_j} \right)^{-1} , \ \textit{i.e.}, \ N_{i1_j} = \left(N_{j1_i} \right)^{-1} .$$

The preceding equations mean that $(t_j - t_{j0})$ is measured with the *time* unit 1_j of the T_j-axis, and the result $(t_i - t_{i0})$ is measured with the *time* unit 1_i of the T_i-axis. This shows that μ_{ij} is measured with the *time* units 1_i and 1_j of the *time* axes T_i and T_j as follows:

$$unit \left(\mu_{ij} \right) = 1_i 1_j^{-1} , \ \mu_{ij} \left\langle 1_i 1_j^{-1} \right\rangle ,$$

and the physical dimension $phdim(\mu_{ij})$ of μ_{ij} is TT^{-1}, which is the reason to consider μ_{ij} dimensionless,

$$phdim(\mu_{ij}) = \mathrm{TT}^{-1}, \text{ i.e. } \mu_{ij}[\mathrm{TT}^{-1}] = \mu_{ij}[-].$$

We note that the *time* scaling coefficients are dimensionless,

$$phdim(\mu_{ij}) = -, \ \mu_{ij}[-],$$

because both *time* units 1_i and 1_j have the physical dimension T of *time*:

$$phdim\ t_{(.)} = phdim\ 1_{(.)} = \mathrm{T},\ (.) = i, j.$$

The initial moment carries the subscript "0" so that $t_{(.)0}$ is the accepted initial instant in the *time* set $\mathfrak{T}_{(.)}$ and in the *time* axis $T_{(.)}$. We accept $t_{(.)0} = 0$ because the choice of the initial moment does not have any influence on velocity, on speed and on characteristics of *time*-invariant transformations, and on properties and behavior of *time*-invariant processes and of *time*-invariant systems. We will treat only them. The results of this book are in the straight forward manner generalized to *time*-varying velocities, speeds and characteristics of coordinate, velocity and speed transformations in [227], [228].

The *time* scaling coefficient μ_{it}, which transforms $(t - t_0)$ *time* units $1_t = s$ into $(t_i - t_{i0})$ *time* units 1_i of the *time* axis T_i, will be simply denoted by μ_i,

$$\mu_{it} \equiv \mu_i, \ \mu_{ti} \equiv \mu_{it}^{-1}, \tag{4.37}$$

so that the equation (4.34) becomes

$$(t_i - t_{i0}) = \mu_i (t - t_0), \ t_{i0} = \mu_i t_0, \ \mu_i \in R^+, \tag{4.38}$$

by noting that $\mu_- = \mu = 1$. The *time* unit 1_i of the *time* axis T_i, and the *time* unit 1_t of the *time* axis T are interrelated by:

$$1_i \langle 1_i \rangle = [(num\mu_{ti})\, 1_i] \langle 1_t \rangle, \ 1_t \langle 1_t \rangle = [(num\mu_i)\, 1_t] \langle 1_i \rangle. \tag{4.39}$$

The *time* scaling coefficients have been introduced formally mathematically so far. However, they have a physical justification and sense in the framework of dynamical systems with multiple *time* scales. This will be explained in Subsection 5.3.2: "*Time* and dynamical systems with multiple *time* scales" of Section 5.3. They appear also natural from the point of view of biological and psychological processes. For this see in the sequel Subsection 4.10.1: "Aging, biological state and biological scales of *time*" of Section 4.10.

4.3.4 *Time* scaling coefficients: geometrical interpretation

Let T_i and T_j be two different *time* axes so that their *time* units mutually differ. If they are accepted parallel then interval mappings from one of them to another one can be by rays orthogonal to both of them, Fig. 4.1, and Fig. 4.3 under a).

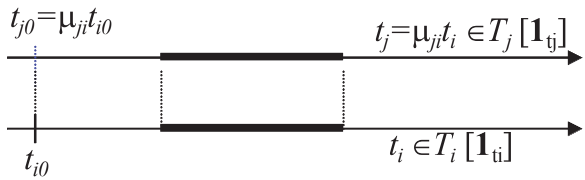

Figure 4.1: The classical geometric interpretation of the time scaling coefficients by orthogonal constant-*time* mappings.

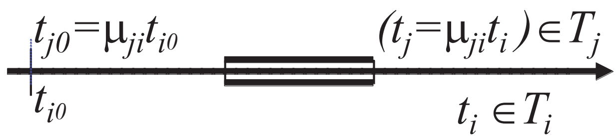

Figure 4.2: One axis is used for two *time* axes T_i and T_j with different *time* scales and *time* units.

The corresponding *time* scaling coefficients are μ_{ji} and μ_{ij}, respectively. Such a mapping is a classical *orthogonal constant-time mapping,* which is orthogonal to the *time* axes. It preserves the same length for the *time* units 1_{t_i} and 1_{t_j} of the *time* axes T_i and T_j, respectively. They are both orthogonal to space R^n. Therefore, we can use one porter axis for both *time* axes T_i and T_j without changing the length representing *time* units 1_{t_i} and 1_{t_j}, Fig. 4.2.

However, we can change the length representing a *time* unit, i.e. we may accept different lengths to represent different *time* units 1_{t_i} and 1_{t_j}, Fig. 4.3, b). These *time* mappings are also *constant-time mappings*, but they are not orthogonal to *time* axes, which makes them different from the orthogonal constant-*time* mappings, Fig. 4.1 and Fig. 4.3a).

4.3.5 *Time* axis transformation

The equations (4.34) and (4.38), repeated here as (4.40),

$$(t_j - t_{j0}) = \mu_{ji}\left(t_i - t_{i0}\right),\ t_{j0} = \mu_{ji}t_{i0}, \tag{4.40}$$

permit us to consider the (linear) *time* mapping of one *time* axis, say T_i, into another *time* axis, say T_j, in the temporal axes product $T_i \times T_j$, which determines a hyperplane *orthogonal to both space R^n and its representative axis $R^{(n)}$*, Fig. 4.4.

Since they are in the hyperplane orthogonal to space R^n and to its representative axis $R^{(n)}$, then we can rotate the axis T_j in the temporal hyperplane around the origin to become colinear with the axis T_i and then to translate the former to overlap with the *time* axes T_i. Then we can clearly visualize the unit transformation, Fig. 4.4. Finally, we may replace them both, T_i and T_j, by one

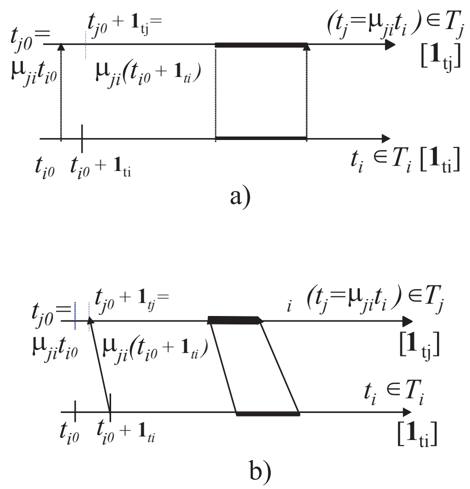

Figure 4.3: Constant-*time* mappings: a) which are orthogonal, b) which are not orthogonal.

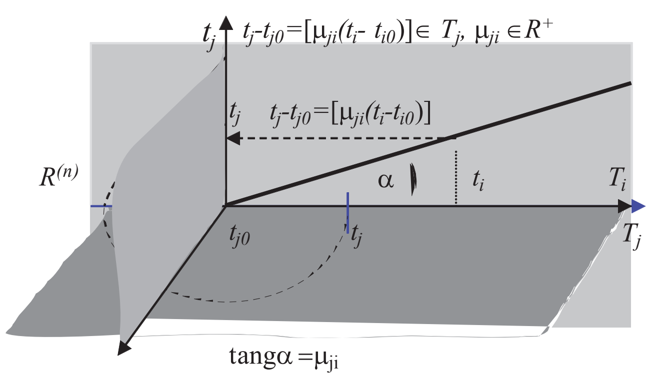

Figure 4.4: The (linear) *time* mapping of the *time* axis T_i into the *time* axis T_j in the temporal axes product $T_i \times T_j$.

porter axis that is gauged in their *time* units. The porter axis carries then two *time* scales, the *time* scales of the *time* axes T_i and T_j, Fig. 4.2.

4.4 Physical variables and spaces

4.4.1 Physical variables

A variable is a quantity, the value of which is changeable.

In order to consider the question whether *time* is a physical variable and in order to answer it, we should first clarify what we mean under *physical variable (*i.e. under *physical quantity, the value of which is changeable).*

Definition 61 ***Physical variable***

1) A scalar variable is a ***scalar physical variable,*** *for short a* ***physical variable****, if and only if both*

1a) it reflects and describes uniquely some directionally nonoriented physical phenomenon, hence, some directionally nonoriented external or internal physical situation of energy and/or of matter (including any material object and any being),

and

1b) its value reflects and characterizes uniquely the intensity (strength) of some physical phenomenon, hence, of some internal physical situation of energy and/or of matter (including any material object and any being), or the position of the place where the phenomenon occurs with respect to a reference point, or the moment when it happens and how long it lasts relative to an accepted reference (initial) moment.

2) The ***physical dimension*** *of a physical variable, which is denoted between the parentheses [.], is a functional set of all its component physical variables. It shows all the component physical variables and how are they functionally combined to compose the physical variable.*

A physical variable is ***an elementary (****also called:* ***a basic****)* ***physical variable*** *if and only if it has a single component physical variable that is the physical variable itself.*

3) A vector variable is a ***vector physical variable*** *if and only if*

3a) it reflects and describes uniquely some directionally oriented physical phenomenon, hence, some directionally oriented external or internal physical situation of energy and/or of matter (including any material object and any being) or the position of the place where the phenomenon occurs with respect to a reference point, or the moment when it happens and how long it lasts relative to an accepted reference (initial) moment,

3b) all its entries represent the same physical variable,

and

3c) its vector value expresses and characterizes uniquely the instantaneous vector intensity (i.e. the vector strength) and the instantaneous directional orientation (i.e. the direction and the sense) of the physical phenomenon, hence, of the instantaneous internal physical situation of energy and/or of matter, or *of position (of any material object and any being), or shows its instantaneous location relative to a reference point in the corresponding space, at every moment, or shows the vector composed elementwise of the moments when the phenomena occur.*

The preceding definition, Fundamental law of physics 66 and Law of energy - matter existence 67 imply the following characteristics of physical variables.

Note 62 ***Vector physical variable and vector of physical variables***
A vector variable is a ***vector physical variable*** *if, and only if, all its entries represent the same (scalar) physical variable.*

A vector variable is a ***vector of physical variables*** *if, and only if, some, or all, its entries contain different (scalar) physical variables.*

Claim 63 ***Uniqueness of physical variables***
Every physical phenomenon, hence, every physical situation of energy or of matter or of position (of any material object and any being) is characterized by the corresponding instantaneous value of the exactly one, unique, physical variable, or by a unique set of the values of the corresponding physical variables, and vice versa, at every moment.

Claim 64 ***The existence of a physical variable***
Every physical variable exists as long as the corresponding physical phenomenon exists.

Definition 61 of the physical variable emphasizes that, at least in engineering, in mathematics and in physics, we should clearly distinguish *variable (*i.e. *quantity, the value of which can vary)* from a *value of the variable* (a *variable value*), and from a *numerical value of the variable* (a *numerical variable value*) indicated relative to an accepted zero variable value, a variable scale and a variable unit.

4.4.2 Values of physical variables

Scalar physical variables are, for example, length, pressure and temperature. We often do not make a precise distinction between *a variable, its value* and its *numerical value* for the sake of simplicity.

We are used to say colloquially, "length" meaning *the value of length (the length size)*, i.e. we equalize *length* as a physical variable with its value (with its size) for the sake of simplicity. We do the same with other physical variables.

When we speak about *length* as a physical variable, then it is unique, but if we use "length" in the sense of the value (in the sense of the size) of the variable *length*, then it is not unique (there are infinitely many different values, sizes, of the variable *length*). If the length value is 40 m (meters), then its numerical value is 40 (in meters). If we say the length value is 4000 cm (centimeters) then its numerical value is 4000 (in centimeters), but its value is not changed as we know it very well, 40 m = 4000 cm. This warns that sometimes we should be strictly precise in distinguishing *physical variable* (e.g. *length*) from its value (length value, length size, "length", 40 m) and from its numerical value (40). Neither variable value (length size of 40 m = 4000 cm) nor numerical variable value (40 or 4000) is variable *(length)* itself. We used different length units (m and cm) in order to measure the same length value.

We say "Temperature is 30C" meaning that the numerical value of temperature is 30 in Celsius scale and degrees. If we say that temperature is 86 F (Fahrenheit degrees) or that temperature is 303,15 K (Kelvin degrees), then we express other numerical values (86 and 303,15) of temperature, which are different from the preceding (30). All three numerical values (30, 86 and 303,15) represent the same value of temperature θ, but measured in three different temperature scales (in Celsius, in Fahrenheit and in Kelvin scale, respectively), $30C = 86F = 303,15K$. Neither 30C nor 86F nor 303,15K is temperature. They represent the same temperature value. The numbers 30, 86 and 303,15 are different numerical temperature values in different temperature scales, which are expressed with respect to different units and relative to differently accepted numerical zero temperature value. In this example, the temperature unit $1_{\theta F}$ of Fahrenheit scale differs from those of Celsius scale and Kelvin scale, $1_{\theta C} = 1_{\theta K} \neq 1_{\theta F}$. All three scales have different numerical zero temperature value θ_{zero}. The numerical zero temperature value $\theta_{zeroK} = 0K$ in Kelvin scale is the zero value of temperature 0_θ, i.e. its *total* (usually called: *absolute*) *zero value* $0_{\theta total}$, $\theta_{zeroK} = 0K = 0_{\theta total} = 0_\theta$. However, the zero numerical temperature values θ_{zeroC} and θ_{zeroF} in Celsius scale and in Fahrenheit scale, respectively, are *relative numerical zero values* of temperature, $\theta_{zeroC} = 293,15K$, *and* $\theta_{zeroF} = 255,3722222...K$. They do not represent the same temperature value, $\theta_{zeroK} \neq \theta_{zeroC} \neq \theta_{zeroF}$, *and* $\theta_{zeroK} \neq \theta_{zeroF}$.

Speaking colloquially we say that we measure various variables (e.g. acceleration, pressure, speed, temperature, voltage). However, speaking strictly, rigorously, there is not a variable that is measured or can be measured, which is measurable. What is measurable and what we do measure is a *value* of a variable, which is *the instantaneous* and *local value* of the variable.

We should not be misled by the colloquial identification of a scalar or vector physical variable (e.g. acceleration, current, length, pressure, speed, temperature, velocity, voltage) with its scalar or vector value and with its numerical scalar or vector value, respectively.

There are not in the strict and the exact sense, for example, two or more different currents or pressures or temperatures or voltages. There are, in fact, different values and different numerical values, possibly also different forms, of current, of pressure, of temperature and of voltage. Alternating current / voltage and direct current / voltage are two forms of the same variable, which is, respectively, current / voltage. This holds for all variables, including all physical variables. They incorporate also *time*. There is not an exception. *Time* obeys this rule.

Variable, variable value and numerical variable value are all mutually essentially different. Neither variable nor its value can depend on its numerical value.

This analysis emphasizes that, at least in engineering, mathematics and physics, we should clearly distinguish (vector) variable from a (vector) value of the variable (a (vector) variable value), and from a numerical (vector) variable value, respectively.

Applied to *time* this means that we should carefully pay attention to the differences among *time* t, *time* value (moment, instant) t, and the numerical *time* value *numt* (see Conclusion 51 in Section 4.2).

4.4.3 Representation of a physical variable

We will treat the coordinate transformations in general, in the n-dimensional real vector space R^n (see Subsection "*Time* axes" in 4.3).

If an n-dimensional vector physical variable $\mathbf{g}(.)$ depends on $\mathbf{z} \in R^k$, then, from the mathematical point of view, $\mathbf{g}(.)$ is a mapping from R^k into R^n, $\mathbf{g}(.) : R^k \longrightarrow \mathfrak{G}^n \cup R^n$. The vector physical variable $\mathbf{g}(.)$ is composed of n scalar elements $\gamma_1(.)$, $\gamma_2(.)$, ..., $\gamma_n(.)$, $\gamma_i(.) : R^k \longrightarrow R$, $i = 1, 2, ..., n$, which represent the same (scalar) physical variable $g(.)$ with the values measured possibly with different units along different axes. The physical variable $\mathbf{g}(.)$ can be the position vector $\mathbf{r}_{(..)}$ of a point (.), $(..) \in \{G, L, P, P_R, P_{SU}\}$, or its velocity $\mathbf{v}_{(..)}$. The subscript "G" stands for *Generic*, "L" will be used for *Light*, "P" for an *arbitrary point* P, "R" for an arbitrarily accepted and then fixed *reference point* denoted by P_R, and "SU" designates *spatially uniform*.

Let the mutually orthogonal unity vectors $\mathbf{e}_{uk}$ form the unity basis $\{\mathbf{e}_{u1}, \mathbf{e}_{u2}, ..., \mathbf{e}_{un}\}$ of R^n, $\mathbf{e}_{ui}^T \mathbf{e}_{uj} = \delta_{ij}$, where δ_{ij} is *Kronecker delta*, $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$.

The vector value $\mathbf{g}(\mathbf{z})$ of the vector physical variable $\mathbf{g}(.)$ at $\mathbf{z}$ is represented relative to the unity basis $\{\mathbf{e}_{u1}, \mathbf{e}_{u2}, ..., \mathbf{e}_{un}\}$ as

$$\mathbf{g}(\mathbf{z}) = \gamma_1(\mathbf{z})\mathbf{e}_{u1} + \gamma_2(\mathbf{z})\mathbf{e}_{u2} + ... + \gamma_n(\mathbf{z})\mathbf{e}_{un} = (\gamma_1(\mathbf{z}) \ \ \gamma_2(\mathbf{z}) ... \ \gamma_n(\mathbf{z}))^T. \quad (4.41)$$

It has the *matrix intensity* (i.e. the *matrix strength*, the *matrix value*) $G(\mathbf{z})$ *(at* $\mathbf{z}$ *in general) relative to the unity vector* $\mathbf{u}$,

$$\mathbf{u} = u_1\mathbf{e}_{u1} + u_2\mathbf{e}_{u2} + ... + u_n\mathbf{e}_{un} = [u_1 I_n \ \ u_2 I_n ... \ u_n I_n] \begin{bmatrix} \mathbf{e}_{u1} \\ \mathbf{e}_{u2} \\ ... \\ \mathbf{e}_{un} \end{bmatrix} =$$

$$= (u_1 \ \ u_2 ... \ u_n)^T \in \mathfrak{R}^n, \ \|\mathbf{u}\| = \sum_{i=1}^{i=n} u_i^2 = 1, \ u_k > 0, \ k = 1, 2, .., n, \quad (4.42)$$

$$g_k(\mathbf{z}) = \frac{\gamma_k(\mathbf{z})}{u_k}, \ k = 1, 2, .., n, \ \mathbf{g}(\mathbf{z}) = G(\mathbf{z})\mathbf{u},$$
$$G(\mathbf{z}) = diag\{g_1(\mathbf{z}) \quad g_2(\mathbf{z}) \quad ... \quad g_n(\mathbf{z})\}. \quad (4.43)$$

If, and only if $\mathbf{g}(\mathbf{z})$ *is collinear with* $\mathbf{u}$ *(at* $\mathbf{z}$ in general*) then it* has the *intensity* (i.e. the *strength)* $\|\mathbf{g}(\mathbf{z})\|$ and the *algebraic* (i.e. *scalar*) *value* $g(\mathbf{z})$ (at

$\mathbf{z}$ in general) *relative to the unity vector* $\mathbf{u}$,

$$\mathbf{g}(\mathbf{z}) = g(\mathbf{z})\mathbf{u},\ g(\mathbf{z}) = \|\mathbf{g}(\mathbf{z})\|\ sign\left[\mathbf{g}^T(\mathbf{z})\mathbf{u}\right],\ \gamma_i(\mathbf{z}) \equiv g(\mathbf{z})u_i \Longrightarrow G(\mathbf{z}) = g(\mathbf{z})I.$$

where $signx = x/|x|^{-1}$ for $x \neq 0$, $sign0 = 0$. The intensity of $\mathbf{g}(\mathbf{z})$ is a freely accepted *norm* $\|\mathbf{g}(\mathbf{z})\|$ at $\mathbf{z}$, which is accepted herein to be its Euclidean norm on R^n spanned by the orthonormal vectors $\mathbf{e}_{uk}$, $\|\mathbf{g}(\mathbf{z})\| = \sqrt{\mathbf{g}^T(\mathbf{z})\mathbf{g}(\mathbf{z})}$, i.e.

$$\|\mathbf{g}(\mathbf{z})\| = \sqrt{\gamma_1^2(\mathbf{z}) + \gamma_2^2(\mathbf{z}) + \ ...\ + \gamma_n^2(\mathbf{z})}.$$

If the physical variable $\mathbf{g}(.)$ is the position vector $\mathbf{r}_{(..)}$ of a point $(.)$, $(..) \in \{G,\ L,\ P,\ P_R,\ P_{SU}\}$, then

$$\mathbf{r}_{(..)} = \rho_{(..)1}\mathbf{e}_{u1} + \rho_{(..)2}\mathbf{e}_{u2} + ... + \rho_{(..)n}\mathbf{e}_{un} = (\rho_{(..)1}\ \ \rho_{(..)2}\ \ \cdots\ \ \rho_{(..)n})^T \in R^n,$$
$$\mathbf{r}_{(..)} = r_{(..)}\mathbf{u},\ r_{(..)} = \left\|\mathbf{r}_{(..)}\right\| sign(\mathbf{r}_{(..)}^T\mathbf{u}),\ \left\|\mathbf{r}_{(..)}\right\| = \sqrt{\rho_{(..)1}^2 + \rho_{(..)2}^2 + \ ...\ + \rho_{(..)n}^2}. \tag{4.44}$$

If, and only if $\mathbf{r}_{(..)}$ and $\mathbf{u}$ are *colinear* then

$$\mathbf{r}_{(..)} = r_{(..)}\mathbf{u},\ r_{(..)} = \left\|\mathbf{r}_{(..)}\right\| sign(\mathbf{r}_{(..)}^T\mathbf{u}),\ \left\|\mathbf{r}_{(..)}\right\| = \rho_{(..)1}^2 + \rho_{(..)2}^2 + \ ...\ + \rho_{(..)n}^2.$$

We can represent the vector $\mathbf{r}_{(..)}$ also with respect to the unity vector $\mathbf{u}$. Such representation is in general determined by the matrix value $\mathrm{R}_{(..)}$ of $\mathbf{r}_{(..)}$ relative to $\mathbf{u}$,

$$\mathbf{r}_{(..)} = \mathrm{R}_{(..)}\mathbf{u},\ \mathrm{R}_{(..)} = diag\{r_{(..)1}\ \ r_{(..)2}\ \ \cdots\ \ r_{(..)n}\} \in R^{nxn},$$
$$r_{(..)k} = \frac{\rho_{(..)k}}{u_k},\ k = 1, 2, ..., n,\ (..) \in \{G, L, P, P_R, P_{SU}\}. \tag{4.45}$$

The algebraic (i.e. the scalar) value *g(*$\mathbf{z}$*)* of the vector variable $\mathbf{g}(.)$ at $\mathbf{z}$ relative to the unity vector $\mathbf{u}$ will be called for short its *value at* $\mathbf{z}$, or even simpler, its *value* if and only if $\mathbf{g}(.)$ is constant, $\mathbf{g}(\mathbf{z}) \equiv \mathbf{const}.$, and $\mathbf{u}$ is collinear with $\mathbf{g}(\mathbf{z})$. Then the unity vector $\mathbf{u}$ determines the direction of $\mathbf{g}(.)$ at $\mathbf{z}$. The value $g(\mathbf{z})$ determines the intensity and the sense of $\mathbf{g}(.)$ at $\mathbf{z}$ along $\mathbf{u}$.

4.4.4 *Time* and physical variables

Le physician admet quatre grandeurs fondamentales indépendantes: ***longueur, masse, temps et charge*****.*

La longueur est un concept premier et une notion acquise naturellement par tous; il est inutile d'essayer d'en donner une définition. Il en est de même du temps.

Marcelo ALONSO and Edward J. FINN, [5, p. 17]

English translation reads:

"The physicist admits four independent fundamental quantities: *length*, *mass*, *time* and *charge**.

The length is the primary concept and a notion acquired naturally by everybody; it is useless to give its definition. The same is valid for *time*."

We emphasize this because Alonso and Finn stated clearly that *time* is an independent and physical quantity ("grandeur"). This Galilean-Newtonian attitude, which has been accepted in the classical physics (e.g. also [424, p. 59]), disagrees with that of Einsteinian relativity theory. According to the latter *time* depends on spatial frames; hence, on space.

As we cannot see many physical variables (e.g. pressure, temperature, voltage), so we cannot see *time.* We can feel separately a value (an intensity) of temperature, of voltage, or of pressure at least by one of our senses. However, we do not feel a *time* value with only one of our senses; we do not have a particular sense for *time* [420, p. 23]. We do not feel a *time* value only with one particular our sense, but we feel it and its value evolution with our whole being, which we express colloquially by saying:

"*Time* passes (monotonously, or fast, or slowly)."

We feel *time* value and its increase, we feel its flow with our whole being because it imbues, impregnates, penetrates every our cell. It is in, and passes through, each of them [420, pp. 35 - 78].

Newton did not explicitly claim that *time* is a physical variable. He called it *true time* and he treated it equally as *mathematical time* (which was the reason for some authors to write that Newton considered *time* as a mathematical variable only, in spite he wrote: "*Absolute, true,* ***and*** *mathematical time*"). However, his explanation does not reject a physical nature of (*true*) *time* since he explained measures of a duration value [360, I of Scholium, p. 8] (see the second part of I of his Scholium, Section 20.1.1: "Newton's explanation of *time"*).

By defining *time* as that what clock hands indicate, Einstein in fact considered tacitly *time* as a physical variable (see Section 3.2: "Einstein's interpretation of *time*").

Eddington, Prigogine, Stengers and Zeh referred to Clausius second law of thermodynamics to put in evidence the following: the fact is that the increase of the value of entropy of a closed system and of an irreversible process happens if and only if the *time* value increases [112, p. 463], [396, pp. 18, 19], [397, p. 189], [497, pp. 3, 4, 9, 31, 129].

Prigogine [396, pp. 20, 101, 103, 168, 175, 180, 189, 199] emphasized the link of the stability theory of Lyapunov [310] with *time.*

Note 48, these examples, the physical phenomena, and the use of the properties of *time* (its independence of everybody and everything, the fixed temporal orientation of the flow of its values, the invariance and the constancy of the speed of the flow of its values) to construct its measuring device - *the clock*, show the indirect measurability of both *time* values and their temporal flow.

These facts, and the fact that *time* enables us to determine uniquely the instantaneous temporal situation of everybody and everything at any moment, its unavoidable inclusion in the definition and the determination of many physical variables (e.g. acceleration, speed, velocity), together with its uniqueness, imply the following statement:

Claim 65 ***Time and physical variables***

a) Time is a physical variable.

b) Time is its unique component, hence, it is an elementary primary physical variable; it is a basic physical variable.

c) Time value and its flow are (indirectly) measurable.

4.4.5 Spaces

La matière emplit - elle donc tout, prend - elle possession de l'espace?
Non : il y a du vide dans l'univers [15].
LUCRECE [307, p. 27]

The position (location) of everybody, and of everything composed of energy-matter can be determined by three mutually independent spatial coordinates relative to any point in the *position space* (or, equivalently, *the location space*).

As it is explained above, the position space is the largest, unbounded in all directions and all senses, three dimensional neighborhood of everybody and of everything created. It is our largest *spatial environment*.

Positions can take any real values. This causes the unboundedness of the position space in all directions and in all six (positive and negative) their senses.

The mathematical representation (the mathematical model) of the position space $\mathfrak{R}^3$ is the real vector space R^3, which is *the three dimensional geometric space*. We equalize them in the geometric sense, $\mathfrak{R}^3 = R^3$.

The energy - matter entity is real and it exists in the position space $\mathfrak{R}^3$. Its *energy - matter space* $\mathfrak{E}^3$ is in $\mathfrak{R}^3$, $\mathfrak{E}^3 \subseteq \mathfrak{R}^3$. It is a real space.

Variations of the positions of everybody and everything composed of energy-matter entity take place in the position space $\mathfrak{R}^3$, and occur in the course of *time t*.

Changes of orientations of material objects happen in the position space, and in the course of *time*, too. Additional three mutually independent angular variables determine orientation of every body in the position space. They compose *the orientation space*, or, equivalently, *the angle space* $\mathfrak{A}^3$. It is a real space because the angles are real variables. Its mathematical model is R^3.

The velocity space $\mathfrak{V}^6$ of a body is six dimensional ($n = 6$) because the body velocity is composed of the translational velocity of the center of its mass, and of the angular velocity of the body rotation about the center of mass. Each of these two subvelocities has three independent components, each of which is a real variable, represents one dimension and requires one independent axis for the representation of its values. It is a real space. Its mathematical model is the six dimensional real vector space R^6, $n = 6$ implies $R^n = R^6$. However, the realization of velocity is in the position space $\mathfrak{R}^3$.

The motion space $\mathfrak{M}^{12}$ of the (rigid) body is twelve dimensional; $n = 12$. It is a real space. Its mathematical model is the twelve dimensional real vector space $R^n = R^{12}$. Three independent position coordinates determine the body position (location), another three independent coordinates (angles) determine its orientation. These six coordinates and the six ones from the velocity space of the body compose its twelve dimensional motion space. The realization of a motion is in the motion space $\mathfrak{M}^n$ extended by the time set $\mathfrak{T}$, i.e. in the integral (motion) space $\mathfrak{I}^{n+1} = \mathfrak{T} \times \mathfrak{M}^n$. The projection of a motion into the motion space $\mathfrak{M}^n$ is its *trajectory.*

Although $\mathfrak{A}^3$, $\mathfrak{M}^{12}$ and $\mathfrak{V}^6$ are *real spaces*, they are **the instantaneous realization spaces** of angle, of motion, of velocity in the sense that their instantaneous values belong to these spaces, respectively. The Cartesian products of the instantaneous realization spaces with the time set $\mathfrak{T}$, i.e., $\mathfrak{I}^4 = \mathfrak{T} \times \mathfrak{A}^3$, $\mathfrak{I}^{13} = \mathfrak{T} \times \mathfrak{M}^{12}$, $\mathfrak{I}^7 = \mathfrak{T} \times \mathfrak{V}^6$, which are the corresponding integral space $\mathfrak{I}^4$, $\mathfrak{I}^{13}$, $\mathfrak{I}^7$, are the corresponding **realization spaces** of angle, of motion, of velocity, respectively. Having this in mind, we simplify the terminology by using the term *the realization space* also for *the instantaneous realization space.*

4.4.6 Spaces and physical variables

In fact, the theory supplies us with a simple connection[1]
between the space-expanse of the universe and
the average density of matter in it.
Albert EINSTEIN [154, p. 114], [155, p. 129]

The position coordinates (oriented lengths, oriented distances), the orientation coordinates (angles), the translational speeds (which compose the velocity of the center of mass of a body) and the angular speeds (which constitute the angular velocity of the body) are scalar physical variables. All values of each of them fill out the corresponding one dimensional space that is *the physical space of that scalar physical variable.*

Typical vector physical variables are *position vector variable* (for short, *position vector*, or just *position*) $\mathbf{r}(.)$, *velocity* (*vector*) *variable* $\mathbf{v}(.)$ (for short, *velocity*), and *acceleration* (*vector*) *variable* $\mathbf{a}(.)$ (for short, *acceleration*). They, and the vector of three independent orientation angles, span the corresponding three dimensional spaces.

Each of these spaces is *the space of the corresponding physical variable* - for short: *the physical variable space.* We will call them *physical spaces* in the sense that they are spanned by (one, or more, dimensional) vectors of the corresponding physical variables. They are all connected and unbounded.

If a physical variable is a vector of physical variables, then its physical space is *homogeneous* if, and only if both all the entries of the vector of physical variables are the same scalar physical variable (i.e. if, and only if the vector of physical variables is a vector physical variable), and its space contains only its vector values (i.e. its space does not comprise a value of another quantity). The physical space of every scalar physical variable is homogeneous if, and only if it does not contain a value of another quantity. The physical space of a vector of different physical variables is *heterogeneous.*

A physical space is *complete* (*full)* if, and only if there is not its energy-matter empty subset. Otherwise, it is *incomplete.*

The position space $\mathfrak{R}^3$ is *the physical space of the physical variable position* (for short: *the position physical space*, or simply, *the position space*). We equalized it with its mathematical model - the real vector space R^3, $\mathfrak{R}^3 = R^3$ by considering it as a homogeneous physical space of position only. However, it is in reality the heterogeneous physical space of the position vector variable $\mathbf{r}(.)$ despite its every entry is the same scalar physical variable - position denoted either by $\rho(.)$ or by $r(.)$, because there are values of other quantities (e.g. of temperature, of pressure) in the (real) position space.

Einsteinian relativity theory claims that the instantaneous diameter of our energy-matter space is bounded (between 6x10^5 and 10^{11} light years [110, p. 167], [154, p. 114], [155, p. 129], [424, p. 129], [445, pp. 109, 146, 211]) and has been expanding [155, p. 129], [183, p. 187], [204, pp. 28, 29], [240, p. 39], [425, p. 11]. This implies the existence of an energy-matter empty subspace of the position space. This means that the position space is an incomplete physical space.

Cited Einstein's claim [on the spatial expansion of our energy-matter space (universe), [154, p. 114], [155, p. 129]] suggests us to conclude that our energy-matter space is a proper subset of the (position) space.

In this connection we should note that the empty subspaces in the sense of astronomer Edwin Hubble, which he discovered among galaxies in our universe [240, p. 36], are not energy-matter empty subspaces. Signals propagate through them. The signals carry energy through such subspaces. They are not energy empty.

Whether the energy-matter empty subspace is the complement of our energy-matter universe to the whole position space, or there is another energy-matter universe (or, there are other energy-matter universes), it is an unanswered (and most probably, unanswerable) question.

All vector values of an n-vector physical variable $\mathbf{g}(.)$ form the *n-dimensional* **g**- *physical variable space* $\mathfrak{G}^n$. Its mathematical representation G^n is a subset of R^n, $G^n \subseteq R^n$.

Sometimes, for mathematical reasons, we form a vector $\mathbf{p}(.)$ composed of *p, p>1,* different scalar physical variables $p_i(.)$, $\mathbf{p}(.) = (p_1(.) \quad p_2(.) \quad ... \quad p_p(.))^T$. It is not a vector physical variable. It is a *vector of p-physical variables.* All its numerical vector values $\mathbf{p}$ then span *the p-dimensional* ***p****-physical variable space* $\mathfrak{P}^p$. Its mathematical representation P^p is a subset of R^p, $P^p \subseteq R^p$. It is *heterogeneous* **p**-physical space.

The general meaning of *space*, which will be used in the sequel if not stated otherwise, incorporates the meaning of the angle space $\mathfrak{A}^3$, of the energy-matter space $\mathfrak{E}^3$, of the motion space $\mathfrak{M}^{12}$, of the position space $\mathfrak{R}^3$, of the velocity space $\mathfrak{V}^3$, of the n-dimensional **g**- physical variable space $\mathfrak{G}^n$, of the p-dimensional ***p***-physical variable space $\mathfrak{P}^p$, of their mathematical representations R^3, E^3, R^{12}, G^n, and P^p, and of the n-dimensional real vector space R^n, where $E^3 \subseteq R^3$, $G^n \subseteq R^n$, $P^p \subseteq R^p$, and $\mathfrak{R}^3 = R^3$ in the geometrical sense.

In view of the need for the energy-matter entity in order to realize the existence and the transmission of any signal, and due to the (above) meaning of the energy-matter empty space, it is not possible to transmit any signal through an energy-matter empty subset or subspace of the position space. Consequently, there have not been and will not be possible any communication between our energy-matter universe and any other supposedly existing energy-matter universe as long as they have been, or they are, or they will be, disjoint.

If there were another energy-matter universe, which our energy-matter universe has been approaching and/or would be approaching, and vice versa, so that they would mutually join at some moment, then, until that moment we could not discover the existence of such an energy-matter universe. Since that moment, they would compose a new energy-matter universe. If this view were correct, then a further discussion on the existence of other energy-matter universes would escape the scientific framework and would rest in the world of imaginations and fantasy, in the framework of religions, in the joy of arts.

Energy and matter fill out the energy-matter space. Fields of many physical variables, such as density field, pressure field, temperature field, velocity field, imbue the energy-matter physical space. *Time*, too. But *time* imbues also the energy-matter empty space(s). It imbues the whole position space, while other physical variables do not if there is its energy-matter empty subset or subspace.

4.5 Physical constituents of the existence

4.5.1 Existence and physical constituents

Le verbe "être", dont on ne saurait se passer, entraine l'idée d'existence, et l'idée d'existence impose la notion de temps.
Lecomte du NOÜY [365, p. 187]

The verb "to be", without which we could not do, implies the idea of the existence, and the idea of the existence imposes the notion of time. If physics wants to use time, it first has to define it.
Albert EINSTEIN [167, p. 5]

Several notions are crucial for the presentation of the following stuff. We will briefly explain first their meanings accepted herein.

The existence of anybody or of anything is her/his or its *being*. It lasts *eternally* (without the beginning and the end), *semi-eternally* (with the beginning, but without the end, or, without the beginning and with the end), or *noneternally*, i.e. *finitely*, (with the beginning and with the end).

Analogously, *the nonexistence* of somebody or of something is her/his or its *nonbeing* that lasts eternally (without the beginning and the end), semi-eternally (with the beginning, but without the end, or, without the beginning and with the end), or noneternally (with the beginning and with the end).

Everybody who, and everything that has the beginning and the end was created in some manner. She/he, and it has existed finitely, noneternally. Who and what is created to exist can exist either semi-infinitely with the beginning and without the end, or finitely. Who and what can exist semi-infinitely without the beginning and with the end were not created, but they disappear, since they existed from ever until their end. Who and what are eternal, they were not created and will not disappear, since they have existed ever and will exist for ever.

A primary constituent of the existence of anybody and of anything created is that without which her/his and its existence is not possible. It is *elementary* if, and only if, it is its own unique constituent (component) so that it cannot be either transformed into another primary constituent, or extracted from, or obtained in any way from, or can be influenced by, another one. Otherwise, it is *complex (interconnected).*

Time, position space, *energy* and *matter* are the primary constituents *of the existence of everybody and everything created* (*of the existence*, for short in the sequel). *Time* and position space are *the elementary primary constituents of the existence.*

We refer to the definition of a neighborhood denoted by $\mathfrak{N}(P)$ of a point P as a set that contains a set $\mathfrak{N}_\alpha(P)$ of all points P_i such that $\left\|\mathbf{r}_{P_i}^O - \mathbf{r}_P^O\right\| < \alpha$ for any norm $\|.\|$ and for some $\alpha > 0$,

$$\mathfrak{N}_\alpha(\mathbf{x}) = \{\mathbf{x} : \ \|\mathbf{x}\| < \alpha\},$$
$$\mathfrak{N}_\alpha(P) = \left\{P_i \colon \ \left\|\mathbf{r}_{P_i}^O - \mathbf{r}_P^O\right\| < \alpha\right\},$$
$$\exists\ \alpha > 0 \Longrightarrow \mathfrak{N}_\alpha(P) \subseteq \mathfrak{N}(P),$$

where $\mathfrak{N}_\alpha(P)$ is *the* $\alpha-$*neighborhood of* P*.*

Position space (for short: *space* in its closest sense) denoted by $\mathfrak{R}^3$ is the three dimensional neighborhood, unbounded in all the directions and in every sense of each direction, of matter and of energy. It is a vector space with any accepted norm $\|.\|$. Hence, it is the three dimensional neighborhood, unbounded in all the directions and in every sense of each direction, of everybody and of everything, who and which was created, is created and/or will be created,

$$\mathfrak{R}^3 = \left\{ \begin{array}{c} \mathbf{x} : \dim\mathbf{x} = 3, \ \mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right], \ x_i \in \mathfrak{R}, \\ phdim x_i = L, \ i = 1, 2, 3, \ \ \|\mathbf{x}\| < \infty \end{array} \right\},$$

where *dim* $\mathbf{x}$ is *the mathematical dimension of the vector* $\mathbf{x}$, which is the minimal number ν of mutually independent vectors $\mathbf{e}_i$, $i = 1, 2, ..., \nu$, for which there exist real numbers α_i, $i = 1, 2, ..., \nu$, such that

$$\mathbf{x} = \sum_{i=1}^{i=\nu} \alpha_i \mathbf{e}_i, \ \mathbf{x} \in \mathfrak{R}^3 \Longrightarrow \nu = 3.$$

The notation *phdim* $x(.)$ stands for *the physical dimension of a variable* $x(.)$ and expresses its physical nature. For example, if the physical nature of $x(.)$ is length L then *phdim* $x(.) = \mathrm{L}$. The mathematical dimension $\dim \mathbf{v}$ of the velocity $\mathbf{v}$, $\mathbf{v} = [v_1 \;\; v_2 ... v_3]^T \in \mathfrak{V}^3 = \mathfrak{R}^3$, is equal to 3, $\dim \mathbf{v} = 3$. However its physical dimension, *phdim*$\mathbf{v}$, is equal to 1, *phdim*$\mathbf{v} = 1$.

Position space $\mathfrak{R}^3$ is the union of all three dimensional neighborhoods of every object (e.g. point, person) P, which existed, exists, and/or will exist,

$$\mathfrak{R}^3 = \bigcup_P \left[\bigcup_{\alpha > 0} \mathfrak{N}_\alpha(P), \; \dim \mathbf{r}_P^O = 3 \right].$$

It is connected and unique. There are not two or more different position spaces.

Position space involves both the absolute space defined by Newton [360, p. 8: II of Scholium] and the space that is a three dimensional continuum as Einstein explained [154, p. 55], [155, p. 61].

Time, *(*through its value), imbues, impregnates, penetrates, is throughout and over the position space. However, *time* is not a part of position space.

Time and position space are mutually independent. The existence of each is independent of another one. Neither (a part of, or a value of) *time* can be transformed into (a part of) position space, nor (a part of) position space can be transformed into (a part of, or a value of) *time*. They are quantitatively and qualitatively essentially different.

Energy and *matter (substance)* constitute everybody who, or everything that was created, hence, existed, and/or has existed, or will be created, hence, will exist noneternally. A spatially spread energy forms an *energy (energetic) field* (usually called simply *field*).

Energy and matter are the only primary constituents of the existence which are mutually interrelated. They are so linked that in principle each of them can influence another one. Mass is a characteristic of matter. Energy and mass can be either transformed in, or extracted from, or obtained in any way from, another one according to Einsteinian relativity theory. Therefore, energy and matter form *the energy-matter entity* that can be considered as one whole primary constituent, but not elementary one such as *time* and space. There is no way to extract, or to obtain, *time*, or energy-matter entity, from space, or vice versa, or either of them to influence another one in any way. The energy-matter constituent is interacted and interconnected (complex) primary constituent of the existence. Consequently, there are three completely mutually independent primary constituents of the existence: *time*, *position space*, and *energy-matter entity*. None of them can be transformed in, or extracted from, or obtained in any way from, or influenced by, anyone of other two. Their mutual independence is both qualitative and quantitative.

An energy-matter empty set is a nonempty (in the mathematical sense) part (subset) of position space, which is imbued, impregnated, penetrated by *time*, and which contains neither energy nor matter.

An energy-matter empty space (i.e., *energy-matter free space*) is a connected three dimensional part (subset) of position space, which has a nonempty interior in the mathematical sense, and which is imbued only by *time* (more precisely, by *time* value), hence, which contains neither energy nor matter, and which is such that any its extension in every direction and sense in which it is bounded, meets energy and/or matter. It is free of both energy and matter.

An energy-matter space denoted by $\mathfrak{E}^3$ is a connected three dimensional (proper or improper) part (subset) of position space $\mathfrak{R}^3$, which has a nonempty interior in the mathematical sense, and which is imbued not only by *time*, but which is filled out with energy and/or matter, and which is either the whole position space (its improper part), or which is its proper part such that any its extension in every direction and sense, in which it is bounded, meets an energy-matter empty set or an energy-matter empty space. It is also called *space, universe*, or *cosmos* in literature. It involves space in Einstein's sense [150, p. 3], and the universe in Einstein's sense [154, p. 114], [155, p. 129].

If an energy-matter space is not the whole position space, then it is disjoint from all other supposedly existing energy-matter spaces.

Either *time* or position space cannot be composed, created, obtained by combining, or by extracting from, and/or by transforming, other two primary constituents of the existence (energy and matter), and vice versa.

A fifth primary physical constituent of the existence of anybody and of anything created, has not been known. All other constituents of their existence are composed of, or contained in, the primary constituents, and/or represent transformations of the primary ones.

The intelligence, although being a fundamental constituent of the human existence, which enables us to think, to speak, to learn with understanding, to analyze, to conclude, even to create, is not a primary constituent of our existence. The human intelligence is based on energy and matter, and it is their qualitative transformation into an alive accumulator able for performing various thoughtful or fictitious processes, such as learning and memorizing; or such as artistic, engineering, religious or scientific creations; and managing. It is able to lead realizations of diverse productions and constructions.

The content, the sense, the characteristics and the properties of *time*, of energy and matter, and of position space, which express their natures, are different and cannot be mutually expressed. *Time* can be neither transformed into, nor expressed in terms of, position space, and vice versa. Neither energy nor matter can be transformed into, or expressed in terms of, anyone of another two primary constituents of the existence, i.e. in terms of *time* and/or position space, and vice versa.

4.5.2 Energy, matter, and fundamental laws of physics

Or, je l'ai montré, rien ne peut naître de rien,
et rien de ce qui a été créé ne peut retourner au néant.
LUCRECE [307, p. 34]

Die Energie der Welt ist konstant.
Die Entropie der Welt strebt einem Maximum zu.

Rudolf CLAUSIUS [397, p. 189]

Work *is the act of* producing a change of configuration in a system in opposition to a force which resists that change.*
Energy *is the capacity of doing work.*

....

The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible.

....

Energy cannot exist except in connexion with matter.
James Clerk MAXWELL [334, pp. 54, 55, 89]

Mass and energy in essence are identical.
Albert EINSTEIN [167, p. 6]

E=Mc² before Albert Einstein

When you read the original papers on relativity and physics, we find that $E{=}Mc^2$ is a relationship, which is much older than Einstein. This relationship was not included within the original Einstein paper but in a later paper. The relation between mass and energy began in 1881 with Thomsons electromagnetic mass and was modified in Heaviside formula in 1889. Poincarré in 1900 represented the mass by r and the energy density J of a fictitious fluid of radiation by the equation $J{=}r/c^2$. In 1904 Hasenorhl showed that the energy in a moving cavity would increase by $8E/(3c^3)$, which Abraham persuaded him to change for $4E/(2c^3)$. Also Soddy in 1904 suggested that the process of radioactive decay involves a conversion of mass into energy. But the idea of $E{=}mc^2$ as a potential energy for the potential of light is also explicit in Newtonian corpuscular theory. In Newton‘s Query 30, we read: "Are not gross Bodies and Light convertible into one another, . . . ". Books relating historical facts give the correct facts about science. About the History of Science, I suggest that you should read the Book "A Revolution Too Far" (1994) by Dr. Peter Rowland (PD Publications, 2 Ascot Park, Liverpool, L23 2XH).
Paul MARMET [321]

Energy is the physical source, the physical cause and the essential physical initiator or creator of actions, of movements, of motions, of processes in an energy-matter space, and the determinant of their physical strengths, of their physical intensities. It is the source of work. Energy can change conditions and the duration of the existence or of the nonexistence of a particular situation in the energy-matter space.

Matter (*substance*) is the physical carrier (i.e. the physical accumulator, the physical porter, and the physical transmitter) of energy.

If a quantity of matter (e.g. a body) is moving with a velocity $\mathbf{v}$, then it exhibits a *motion. The quantity of its motion* called *momentum* is denoted by $\mathbf{p}$. It is proportional to the velocity $\mathbf{v}$ of the matter (of the body), $\mathbf{p} = k\mathbf{v}$. The coefficient of proportionality k is called *mass of the matter (mass of the body)*, which is denoted by m, $k = m$. The mass of the matter (of the body) is the measure of the matter (of the body) resistance [i.e. of the matter (of the body) *inertia*] to a variation of the momentum. The mass of the matter (of the body) can be in principle transformed into energy, and vice versa, according to Einsteinian relativity theory, which is expressed by famous Einstein's formula $E = mc^2$, or by Marmet's (experimental) formula $E = Km$ in which $numK = numc^2$ [320, the equation 2.3 on p. 35]. These equations express mathematical proportionality (hence, the mathematical equivalence) between the energy of the matter (the body) and the mass of the matter. The matter mass (the body mass) is the matter (the body) characteristic.

This means that energy and matter (i.e. its mass) are inseparable.

Energy and matter form an entity: *the energy-matter entity.*

A particular energy reflects, influences and/or determines (internal and/or external) situation of (a part of) matter. Such situation of (that part of) matter is its *state.*

The transmitter and the executor of the energy influence on a variation of the velocity of a body is *force.* It is a vector variable denoted by $\mathbf{F}(.)$. The effect of a force that compels a body to pass a path $\mathbf{s}(t)$ is a scalar variable called *their work* $W(.)$ *over the path* $\mathbf{s}(t)$, which is mathematically determined as their (scalar) product, $W(t,..) = \mathbf{F}^T(t,..)\mathbf{s}(t)$. Their work variation $dW(t,..) = d\left[\mathbf{F}^T(t,..)\mathbf{s}(t)\right]$ done during the infinitesimal variation dt of the *time* value is another scalar variable - *their power* $P(.)$, $P(t,..) = d\left[\mathbf{F}^T(t,..)\mathbf{s}(t)\right]/dt$. If the force is constant, $\mathbf{F}(t,...) \equiv \mathbf{F} = \mathbf{const}$, then their power is the scalar product of the force and the body velocity $\mathbf{v}(t)$, $P(t,..) = \mathbf{F}^T(t,..)\mathbf{v}(t)$.

The energy-matter entity has the property of conservation. It is summarized in the well known fundamental law of physics:

Law 66 ***Fundamental Law of physics***

Energy and matter cannot be created from nothing or destroyed into nothing. They can only change (separately or jointly) their appearances, their forms, their phases, their kinds, their modes and/or their states.

Law 67 ***Law of the energy - matter existence***

a) Energy cannot exist there and then where and when matter does not exist.

b) Energy - matter entity has ever existed and will exist for ever.

These axiomatic laws express the knowledge on and the experience about the physical reality of the energy-matter entity.

If we thought of vacuum as of a strict energy-matter empty set, (or as of a strict energy-matter empty subspace), then there would not be energy in vacuum. Consequently, light could not propagate through the (strict) vacuum.

Einstein considered vacuum as a part of our energy-matter space filled out with a special material content called *ether* [145]. This sense of vacuum is not strict from the point of view of the energy-matter empty set (subspace).

In the sequel *vacuum* denotes a subset of our energy-matter space in which the speed of light (value) c is constant and equal to $2.99792458\text{x}10^5 Kms^{-1}$ (relative to the environment).

Axiom 49 agrees with Fundamental Law of physics (Law 66) and with Law of the energy - matter existence (Law 67). These laws show that *time* has ever existed and will exist for ever (Axiom 47). They show also that *time* cannot be created or destroyed.

4.6 *Time,* space and events. Simultaneity

4.6.1 *Time* axes and space

We associate a dimensionless unity vector $\mathbf{t}_u$ with the *time* axis T,

$$\mathbf{t}_u[-] = (1) \in R^1,$$

which has the same direction as the axis T. The positive sense of the *time* unity vector $\mathbf{t}_u$ coincides with the positive sense of the *time* axis T, as well as with the positive sense of the *time* vector $\mathbf{t} \in T^1 \cup R^1$, $\mathbf{t}\,[\text{T}]\,\langle s \rangle$,

$$\mathbf{t} = (t) = t\mathbf{t}_u \in T^1 \cup R^1 \ \textit{meaning}\ t \in \mathfrak{T},\ \textit{num}\mathbf{t} \in R^1.$$

They are oriented in the temporal sense, from the past instants towards the future instants (from the smaller towards the bigger numerical *time* values).

The *time* unity vector $\mathbf{t}_u = (1) \in R^1$ of the *time* axis T can be considered as the unity vector in R^1. Its extension in the vector space $R^{1+n} = R^1 \text{x} R^n$ is $\mathbf{t}_{ue}$,

$$\mathbf{t}_{ue}[-] = (1\ 0\ ...\ 0)^T \in R^{1+n}.$$

This permits us to define the extension $\mathbf{t}_e$ of the *time* vector $\mathbf{t}$ as its representation in $T \text{x} R^n$,

$$\mathbf{t}_e = (\sigma\ \ 0\ \ ...\ 0)^T = \sigma\mathbf{t}_{ue} \in T\text{x}R^n,\ \sigma = \textit{num}t,\ t \in \mathfrak{T}.$$

The extension T_e of T is the representation of the *time* axis T in R^{1+n},

$$T_e = \{\mathbf{t}_e :\ \mathbf{t}_e = \sigma\mathbf{t}_{ue} \in T\text{x}R^n,\ \sigma = \textit{num}t,\ t \in \mathfrak{T}\}.$$

We represent *the spatial unity vector* $\mathbf{u}$, (4.42), and *the position vector* $\mathbf{r}_{(.)} \in R^n$, (4.44), also in R^{1+n} and in $T\text{x}R^n$. Such their representations are, respectively, their extensions $\mathbf{u}_e$ and $\mathbf{r}_{(..)e}$ in both R^{1+n} and $T\text{x}R^n$,

$$\mathbf{u}_e = (0\ \ \mathbf{u}) = (0\ \ u_1\ u_2\ ...\ u_n)^T \in R^{1+n} \cup T\text{x}R^n,$$
$$\mathbf{r}_{(..)e} = (0\ \ \rho_1\ \ \rho_2\ \ ...\ \ \rho_n)^T = (0\ \ \mathbf{r}_{(..)}^T)^T \in R^{1+n} \cup T\text{x}R^n.$$

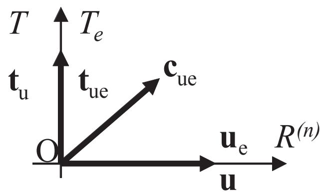

Figure 4.5: The space - *time* coordinate system $(R^{(n)}, T)$. The vectors $\mathbf{c}_{ue}$, $\mathbf{t}_u$, $\mathbf{t}_{ue}$, $\mathbf{u}$ and $\mathbf{u}_e$, are constant unity vectors. The vectors $\mathbf{c}_{ue}$, $\mathbf{t}_{ue}$ and $\mathbf{u}_e$ are, respectively, the representations of the unity vectors $\mathbf{c}_u \in R^n$, $\mathbf{t}_u \in R^1$ and $\mathbf{u} \in R^n$ in $R^n \text{x} T$.

The extended *time* unity vector $\mathbf{t}_{ue}$ is orthogonal to them, hence orthogonal to space R^n and to its symbolic representation $R^{(n)}$,

$$\mathbf{t}_{ue}^T \mathbf{u}_e = 0 \text{ and } \mathbf{t}_{ue}^T \mathbf{r}_{(..)e} = 0,\ \forall \mathbf{r}_{(..)e} = (0 \ \ \mathbf{r}_{(..)}^T)^T \in R^{1+n}, \text{ i.e. } \forall \mathbf{r}_{(..)} \in R^n. \tag{4.46}$$

These equations prove orthogonality between the *time* axis $T_{(.)}$ and space $R^n_{(.)}$. They express *time* independence of space.

The *time* axis $T_{(.)}$ is a symbolic geometric representation of the eternity.

4.6.2 *Time*, space and coordinate systems

Since the dimension of a physical variable space can be greater than three, then we will use the notion of space in the general sense that incorporates the n-dimensional real vector space R^n. Space will be arbitrary but fixed and assumed such that it enables the constant light speed. We accept to represent it symbolically, graphically and geometrically by an axis denoted by $R^{(n)}$ and by an arbitrarily chosen constant unity vector $\mathbf{u}$, (4.42). The axis $R^{(n)}$ is collinear with $\mathbf{u}$, Fig. 4.5.

The coordinate systems $(R^{(n)}, T)$, Fig. 4.5, and $(T, R^{(n)})$, Fig. 4.6, are orthogonal (Cartesean).

The $(R^{(n)}, T)$ - coordinate system, i.e. space - *time* coordinate system, is used in Einsteinian relativity theory, Fig. 4.5, in order to express the attitude that *time* depends on space, which contradicts Axiom 47. This attitude is fully accepted in Einsteinian relativity theory, consequently widely adopted in the modern physics.

The $(T, R^{(n)})$-frame, i.e. the *time* - space coordinate system presented in Fig. 4.6, expresses the *time* independence of space, and obeys Axiom 47. We will use it in the subsequent development. It is accepted in the classical (Galilean - Newtonian) both physics and mechanics, in mathematics, control theory, and systems theory. Galileiwas probably the first [191, p. 478] to use the abscissa and ordinate for two different variables, which were, respectively, *time* and speed [191, p. 199]. He was probably the first to use the (T, R^1)-coordinate system

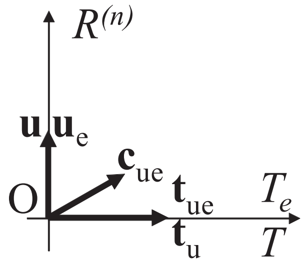

Figure 4.6: The *time* - space coordinate system $(T, R^{(n)})$ and the constant unity vectors $\mathbf{c}_{ue}$, $\mathbf{t}_{ue}$, and $\mathbf{u}_e$. They are, respectively, the representations of the vectors $\mathbf{c}_u \in R^n$, $\mathbf{t}_u \in R^1$ and $\mathbf{u} \in R^n$ in TxR^n.

[191, p. 478], which means that he was probably the first to consider happenings in the *time* - space coordinate system, more than two centuries before Einstein.

4.6.3 *Time* and space: integral space

The 1+n dimensional space product $\mathfrak{I}^{1+n} = \mathfrak{I} = \mathfrak{T}x\mathcal{R}^n$ represents Cartesean product of the *time* set $\mathfrak{T}$ and space $\mathcal{R}^n$,

$$\mathfrak{I}^{1+n} = \mathfrak{I} = \mathfrak{T}\text{x}\mathcal{R}^n = \{(t, \mathbf{x}) : t \in \mathfrak{T},\ \mathbf{x} \in \mathcal{R}^n\}. \tag{4.47}$$

It will be called **the** $(1+n)-$**dimensional integral space** $\mathfrak{I}^{1+n}$, for short: **the integral space** $\mathfrak{I}$, or ***time*-space** (**environment**) $\mathfrak{T}$x$\mathcal{R}^n$. Its mathematical representation is Cartesean product of the *time* axis T and the real vector space R^n,

$$I^{1+n} = I = T\text{x}R^n = \{(\sigma, \mathbf{x}) : \sigma \in T,\ \mathbf{x} \in R^n\}. \tag{4.48}$$

I will be also called *the integral space* for the sake of the simplicity. Motions of dynamical systems exist and propagate only in the temporal - spatial product $\mathfrak{I} = \mathfrak{T}$x $\mathcal{R}^n$. Integrals (i.e. solutions) of their n-th order mathematical models take place in the product set TxR^n. If a mathematical model of a system is in the form of a (vector) differential equation then system motions are the integrals of the mathematical model. They are called *integral curves* [90, p. 65]. These are some of the reasons to call space $\mathfrak{T}$x $\mathcal{R}^n$ (i.e. TxR^n) *the integral space* $\mathfrak{I}$ *(I)*, respectively. Another reason is the following: the temporal-spatial product $\mathfrak{T}$x$\mathcal{R}^n$ is *integral* in the sense that it provides integral (complete) information about *time* t and about a happening in $\mathcal{R}^n$ at any moment $t \in \mathfrak{T}$.

A pair $(t,\ x)\in \mathfrak{I}$ $[(\sigma,\ x)\in I]$ is called **an event (in** $\mathfrak{I}$**)** [***in*** I], respectively [183, p. 190], [257], [369, p. 9], [460, pp. 17-22]. It happens exactly once due to the properties of *time* (Axiom 47 in Subsection 4.2.5: "*Time* properties and characterization" of Section 4.2).

It is to understand the fundamental difference between the integral space $I = T$xR^n and the real vector space $R^{1+n} = R^1$xR^n in general. The nature of

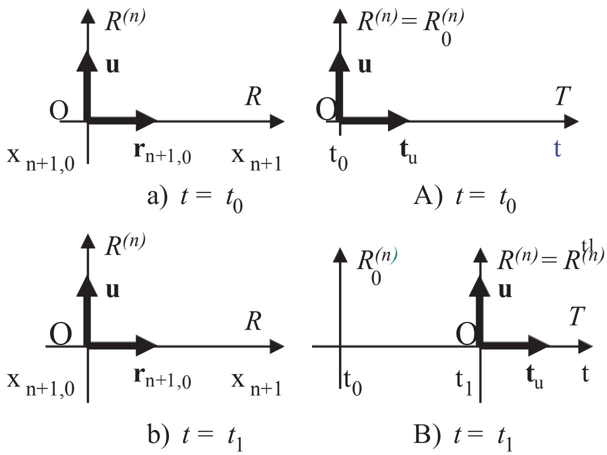

Figure 4.7: The positions of the $R^{(n)}$ axis: in the R^{1+n} vector space shown under a), b), and in the $I = T\mathrm{x}R^n$ integral space shown under A), B) at the initial moment t_0 represented under a), A), and at an arbitrary later moment t_1, $t_1 > t_0$, represented under b), B).

time causes the essential difference, which is illustrated in Fig. 4.7. The axis $R^{(n)}$, hence its origin O as well, preserves in R^{1+n} its position along the axis $R^1 = R$ at its initial position $x_{n+1} = x_{n+1,0}$ independently of the *time* value variation: at the initial moment $t = t_0$ and at an arbitrary moment $t = t_1$, $t_1 > t_0$ [a) and b) in Fig. 4.7]. However, the axis $R^{(n)}$, hence its origin O as well, changes its position in $I = T\mathrm{x}R^n$ along the *time* axis T, from its initial position $R_0^{(n)}$ at the initial moment $t = t_0$ to its position $R_{t1}^{(n)}$ at an arbitrary moment $t = t_1$, $t_1 > t_0$ [A) and B) in Fig. 4.7]. This is due to the independent *time* value variation (independent *time* value increase) along the *time* axis T (Axiom 47).

The axis $R^{(n)}$, hence its origin O as well, will be in its initial position in the $(1 + n)$-dimensional real vector space R^{1+n} at any moment after the initial moment t_0. Contrary to this, the axis $R^{(n)}$, hence its origin O as well, can be only in another position, along the *time* axis T, different from its initial position in the integral space I, after the initial moment t_0.

The *time* axis T, as well as *time* t, may not be treated just as another spatial axis, as another spatial coordinate, respectively. Such their treatment can lead to fatal mistakes. The same holds for the replacement of the *time* axis T by any artificial axis induced by an artificial variable $x = kt$, regardless of the value of the constant k, whether it is a real number ($k \in R$, $k \neq 0$, e.g. $k = \mathfrak{c}$ - the light speed numerical value), or a complex number (e.g. $k = \alpha + i\mathfrak{c} \in \mathfrak{C}$, $\alpha \in R$, $i = \sqrt{-1}$, $\mathfrak{C}$ is the set of all complex numbers), or an imaginary number (e.g. the imaginary unity value: $k = \sqrt{-1}$).

If we wish to show the initial position $R_0^{(n)}$ of the $R^{(n)}$-axis at any moment

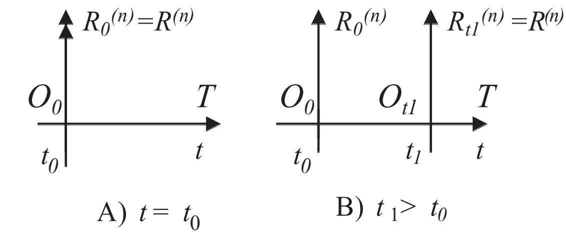

Figure 4.8: The initial and the instantaneous positions of $R^{(n)}$-axis in the $I^{n+1} = I$ integral space at the initial moment t_0 shown in A) and at an arbitrary later moment t, $t_0 < t$, shown in B).

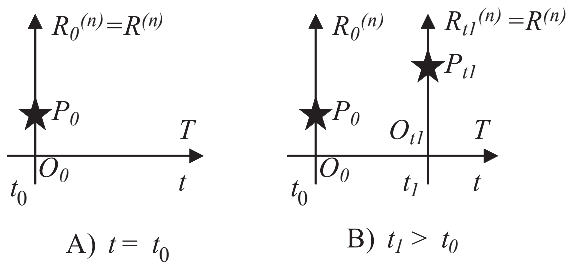

Figure 4.9: The initial position P_0 and the instantaneous position P_{t1} of an arbitrary point P in the $I^{n+1} = I$ integral space at the initial moment t_0 shown in A) and at an arbitrary later moment t_1, $t_0 < t_1$, shown in B).

$t_1 \in T$ in addition to its instantaneous position $R_{t1}^{(n)}$ at the same moment t_1, $t_1 > t_0$, then, Fig. 4.7 should be replaced by Fig. 4.8. Consequently, if we consider an initial position P_0 and an instantaneous position P_{t1} of an arbitrary point P, which, naturally, can move only in space R^n, then they are represented on the $R^{(n)}$-axis at every corresponding moment $t \in T$, Fig. 4.9. Therefore, the point P passed the path of the length l during the *time* interval $[t_0, t_1]$:

$$l = \overline{P_{0t1}P_{t1}} = \parallel \mathbf{r}_{Pt1} - \mathbf{r}_{P_{0t1}} \parallel, \tag{4.49}$$

and not the path of the length L:

$$L = \overline{P_0P_{t1}} = \left\| \begin{pmatrix} t_1 \\ \mathbf{r}_{Pt1} \end{pmatrix} - \begin{pmatrix} t_0 \\ \mathbf{r}_{P0} \end{pmatrix} \right\| \neq l, \tag{4.50}$$

as shown in Fig. 4.10.

We simplify usually the presentation. We do not show the instantaneous position $R_{t1}^{(n)}$ of the $R^{(n)}$-axis, but only the instantaneous position P_{t1} of the point P and of its projection P_{t10} on the initial position $R_0^{(n)}$ of the $R^{(n)}$-axis, on which we retain to present the initial position P_0 of the point P, Fig. 4.11. The point P passed the path of the length $\overline{P_0P_{t10}} = \overline{P_{0t1}P_{t1}} = l$, (4.49), during

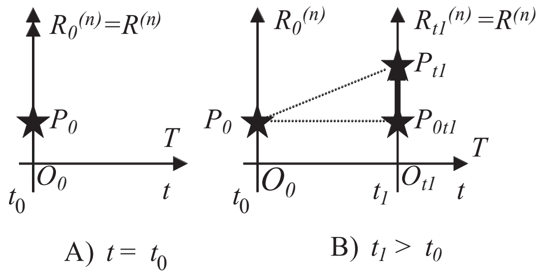

Figure 4.10: An arbitrary point P passed the path of the length $l = \overline{P_{0t1}P_{t1}}$ in the R^n-space during the *time* interval $[t_0, t_1]$, where P_{0t1} designates the initial position of the point P on $R_{t1}^{(n)}$ at the moment t_1.

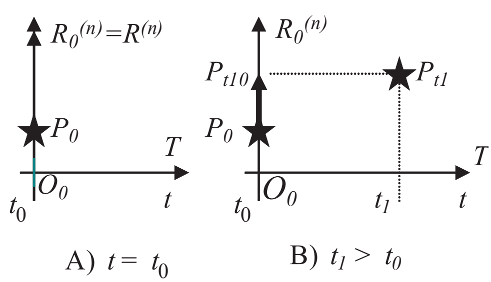

Figure 4.11: An arbitrary point P passed the path of the length $l = \overline{P_{0t1}P_{t1}} = \overline{P_0P_{t10}}$ in the R^n-space during the *time* interval $[t_0, t_1]$, where P_{t10} designates the projection of the instantaneous position of the point P at the moment t_1 on the $R^{(n)}$-axis in its initial position $R_0^{(n)}$.

the *time* interval $[t_0, t_1]$, but not the path of the length L, (4.50). An ignorance of these elementary facts can cause cardinal mistakes.

Notice that all the points in the hyperplane $H(t;T)$ in the integral space I,

$$H(t;T) = \{(\sigma, \mathbf{x}) : \ \sigma \in T, \ \sigma = numt, \ t \in \mathfrak{T}, \mathbf{x} \in R^n\}, \ H(t;T) \subset I, \quad (4.51)$$

correspond to the same moment t. We omit "$;T$" from the notation $H(t;T)$ if and only if the *time* axis T is known and fixed so that then $H(t;T) \equiv H(t)$. The hyperplane $H(t;T)$ is orthogonal to the *time* axis T (i.e. parallel with the $R^{(n)}$-axis) at every moment $t \in \mathfrak{T}$. Therefore, every ray in $H(t;T)$ is orthogonal to the *time* axis T and every point in the hyperplane $H(t;T)$ has the same temporal coordinate that equals t. The analogy holds for the hyperplane $H(\tau;T_\tau)$ relative to the *time* axis T_τ. This is a reason to carry out interval mappings from T_τ-axis to $\mathcal{L}$-axis, and vice versa, by rays orthogonal to the *time* axis T_τ. Such *time* mappings will be called *orthogonal constant time mappings in the integral space* $I = T \times R^n$. They should be distinguished from *orthogonal constant time mappings* from one *time* axis, say T_i, into another *time* axis, say T_j (Subsection "*Time* scaling coefficients: classical geometrical interpretation" in 4.3.4).

We transform all terms in equivalent ones so that all the transformed terms have the same physical dimensionality. The solutions are correct from the point of view of the homogeneity of the physical dimensionality of all terms. Such a transformation is mathematically correct. Its physical justification should be tested for every given case. For more on *time* mappings and dimensionality see [226], [227], [228] or [231].

4.6.4 Simultaneity of events

La simultanéité de deux événements, ou l'ordre de leur succession, l'égalité de deux durées, doivent être définis de telle sorte que l'énoncé des lois naturelles soit aussi simple que possible. En d'autres termes, toutes ces règles, toutes ces définitions ne sont que le fruit d'un opportunisme inconscient.

Henrie POINCARE [385, pp. 57, 58]

In English it reads:

Simultaneity of two events, or the order of their succession, the equality of the two durations, should be so defined that the statement of natural laws is as simple as possible. Otherwise, all those rules, all those definitions are only the fruit of an unconscientious opportunism.

We will first explain precisely what we understand under *event*, in order to explain simultaneity of two events.

Definition 68 ***Definition of the event***

An event E, the characteristics of which are called E-characteristics, is an ordered pair (t, X_E),

$$E = (t, X_E), \tag{4.52}$$

such that:

◦ *the moment $t \in \mathfrak{T}$, i.e. $numt \in T$,*

◦ *the set X_E is a nonempty subset of the R^n-space, $(X_E \neq \phi) \subseteq R^n$, and*

◦ *the E-characteristics (existence, features, attributes, ...) occupy the set X_E at the moment t.*

An event E occurs in the integral space $I = T \times R^n$ that can be also called *the event space*, $E \in I$. Every event E happens exactly ones. This means that the happening of the event E is uniquely determined by the pair (t, X_E) in the integral space I. The set X_E occupied by the characteristics (existence, features, attributes,...) of the event E at a moment t will be denoted more precisely by $X(t; E)$, or by $X_E(t)$,

$$X_E \equiv X_E(t) \equiv X(t; E).$$

Notice that $(t, x) \in H(t)$, $\forall x \in X(t; E)$, where $H(t)$ was defined in the equation (4.51) (Subsection "*Time* and space: integral space").

Definition 69 ***Simultaneity of events***

Two events E_1 and E_2 are ***simultaneous*** *(in the integral space $\mathfrak{I} = \mathfrak{T} \times \mathfrak{R}^n$, i.e. in $I = T \times R^n$) if, and only if, there are:*

◦ *moments t and t_i, $t \in \mathfrak{T}$, $t_i \in \mathfrak{T}$, i= 1, 2, and*

◦ *nonempty subsets X_i of the R^n-space occupied by the E_i-characteristics (existence, features, attributes,...) of the E_i-events at the moments t_i*

such that the pairs (t_i, x_i), i= 1, 2, belong to the hyperplane $H(t)$ for every $x_i \in X_i(t_i; E_i)$ at the moment t:

$$(t_i, x_i) \in H(t),\ \forall x_i \in X_i(t_i; E_i),\ i = 1, 2.$$

This definition and the definition of the hyperplane $H(t)$ imply:

$$t_1 = t_2 = t.$$

Two events E_1 and E_2 are simultaneous if, and only if, the moments t_1 and t_2 of their happenings are equal.

Friedman called the hyperplanes $H(t)$, $t \in \mathfrak{T}$, *the planes of absolute simultaneity* [190, p. 407].

Definition 69 determines *simultaneity of the events happening* that should be distinguished from *simultaneity of the events registration.* The latter involves the *time* interval needed for the transmission of information (of signals) about events happening from the places of the happenings to the measuring device(s). The *moment of the event registration* is the instant when information (the signal) about the event occurrence is received and registered. The information (the signal) transmission creates a delay relative to the moment of the event happening. We should be aware of this delay and we should determine it in order to get exact information about the moment of the event happening.

4.7 *Time*, velocity and light velocity

4.7.1 *Time*, relative velocities and their values

If there are several different *time* axes, then we may represent all the *time* axes as parallel axes. We can further replace them by one *porter time axis* (*carrier time axis*). It is accordingly gauged with the corresponding number of different *time* scales and of different *time* units.

Let $s+1$ different *time* scales be associated with T as follows: "an original *time* scale" T that is not indexed, (its subscript is considered zero and omitted), and the "i-th" *time* scale T_i for $i = 1, 2, \ldots, s$. In the sequel, "$i = -$" means that "i" should be omitted. An arbitrary moment measured in T-scale and in T_i-scale is designated, respectively, by t and t_i, $t \in T$ and $t_i \in T_i$. They are interrelated by (4.38).

The velocity $\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0}) \equiv \mathbf{v}_{(..)}^{O_i}(t_j;t_{j0})$ is the instantaneous velocity of a point (..) with respect to O_i measured with the length unit 1_{L_i} of R_i^n and with the *time* unit 1_j of T_j (for the introductory explication of the notation see Subsection 4.3.2: "*Time* axes" in Section 4.3: "*Time* scales, units and interval mappings").

The cases when $\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0})$ and $\mathbf{u}$ are not colinear are treated in [227], [228].

If, and only if, $\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0})$ and $\mathbf{u}$ are colinear then,

$$\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0}) = v_{(..)}^{O_i,j}(t_j;t_{j0})\mathbf{u} = \frac{d\mathbf{r}_{(..)}^{O_i,j}(t_j;t_{j0})}{dt_j} = \frac{dr_{(..)}^{O_i,j}(t_j;t_{j0})}{dt_j}\mathbf{u},\ dt_j > 0,$$

$$v_{(..)}^{O_i,j}(t_j;t_{j0}) = \|\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0})\| sign\left(\mathbf{u}^T\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0})\right),\ (..) \in \{G,\ L,\ P,\ R,\ SU\}.$$

Then the speed $v_{(.)}^{O_i,j}(t_j;t_{j0})$ is *the scalar (algebraic) value* of $\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0})$ relative to $\mathbf{u}$ at the moment t_j. It can be negative that happens when the sense of the velocity $\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0})$ is opposite to the sense of $\mathbf{u}$, i.e. $sign\left(\mathbf{u}^{\mathbf{T}}\mathbf{v}_{(..)}^{O_i,j}(t_j;t_{j0})\right) = -1$.

If, and only if $i = j$ then we can use the simplified notation by omitting the superscripts and arguments as follows:

$$\mathbf{r}_{(..)}^{O_i,i}(t_i;t_{i0}) \equiv \mathbf{r}_{(..)}(t_i;t_{i0}) \equiv r_{(..)}^{O_i,i}(t_i;t_{i0})\mathbf{u} \equiv r_{(..)}(t_i;t_{i0})\mathbf{u} \equiv r_{t(..)}^i\mathbf{u},$$
$$\mathbf{v}_{(..)}^{O_i,i}(t_i;t_{i0}) \equiv \mathbf{v}_{(..)}(t_i;t_{i0}) \equiv v_{(..)}^{O_i,i}(t_i;t_{i0})\mathbf{u} \equiv v_{(..)}(t_i;t_{i0})\mathbf{u} \equiv v_{t(..)}^i\mathbf{u},$$

in the special (colinearity) case, $(..) \in \{G, L, P, P_R, P_{SU}\}$.

If, and only if $\mathbf{v}_{(.)}(t_i;t_{i0})$ is constant then we write

$$\mathbf{v}_{(..)}(t_i;t_{i0}) \equiv \mathbf{v}_{(..)}^{O_i,i} = \mathbf{v}_{(..)}^i = \mathbf{const}. \neq \mathbf{0},\ (..) \in \{G, L, P, P_R, P_{SU}\}.$$

$$\mathbf{v}_{(..)}^i = \mathbf{v}_{(..)}^j \text{ implies } \mathbf{v}_{(..)}^i = \mathbf{v}_{(..)}^j = \mathbf{v}_{(..)}^{ij} = \mathbf{v}_{(..)}^{ji},\ (..) \in \{G, L, P, P_R, P_{SU}\}.$$

The velocity $\mathbf{v}_{O_j}^{i} \equiv v_{O_j}^{i}\mathbf{u} \equiv \mathbf{const}.$ in the colinearity case is the constant velocity of O_j with respect to O measured with the length unit 1_{L_i} of R_i^n and with the *time* unit 1_i of T_i. The zero value of $v_{O_j}^{i}$, $v_{O_j}^{i} = 0$, means that R_j^n is at rest relative to R^n.

We adopt also

$$0 \leq v_{O_i}^{(.)} \leq \; v_{O_j}^{(.)}.$$

The velocity $\mathbf{v}_{ji}^{(.)} \equiv v_{ji}^{(.)}\mathbf{u}$ in the colinearity case is the constant relative velocity of O_j and R_j^n with respect to O_i and R_i^n measured with the length unit $1_{L_{(.)}}$ of $R_{(.)}^n$ and with the *time* unit $1_{(.)}$ of $T_{(.)}$,

$$\mathbf{v}_{ji}^{(.)} \equiv \mathbf{v}_{O_j}^{(.)} - \mathbf{v}_{O_i}^{(.)} \equiv -\mathbf{v}_{ij}^{(.)},$$

i.e.

$$\mathbf{v}_{ji}^{(.)} \equiv v_{ji}^{(.)}\mathbf{u} \equiv \left(v_{O_j}^{(.)} - v_{O_i}^{(.)}\right)\mathbf{u} \equiv -\mathbf{v}_{ij}^{(.)},\; v_{ji}^{(.)} \in R_+,\, v_{ij}^{(.)} = -v_{ji}^{(.)},$$

in the colinearity case. It will denote the transfer velocity in the spatial coordinate transformations, *the spatial transfer velocity.* Notice that the zero superscript in

$$\mathbf{v}_{ji}^{0} \equiv v_{ji}^{0}\mathbf{u} \equiv \left(v_{O_j}^{O} - v_{O_i}^{O}\right)\mathbf{u} \equiv -\mathbf{v}_{ij}^{0},\; v_{ji}^{0} \in R_+,$$

denotes that the speed value is measured with the length unit 1_L of R^n and with the *time* unit 1_t of the *time* axis T. The speed $v_{ji}^{(.)}$ can (but need not) be also used as the temporal transfer speed in the temporal coordinate transformations, which holds for Lorentz transformations of the temporal coordinates.

If, and only if $\mathbf{v}_{ji}^{i} = \mathbf{v}_{ji}^{j}$ then we write $\mathbf{v}_{ji}$:

$$\mathbf{v}_{ji}^{i} = \mathbf{v}_{ji}^{j} \Longrightarrow \mathbf{v}_{ji}^{i} = \mathbf{v}_{ji}^{j} = \mathbf{v}_{ji} = -\mathbf{v}_{ij},$$
$$\mathbf{v}_{ji} = v_{ji}\mathbf{u} \textit{ in the colinearity case.}$$

Lorentz, Einstein and Poincaré accepted a priory for the spatial transfer speed $v_{ji}^{(.)}$ to be independent of a choice of a length unit and of a *time* unit. They denoted it by v in Lorentz transformations (7.20) through (7.23) (Section 7.2: "*Time* fields and Lorentz transformations"). They ignored completely the crucial influence of the units on the numerical values of speeds.

4.7.2 *Time*, light velocity, and light speed

La vitesse de la lumière - voisine de 300 000 km/s - était connue approximativement depuis 1676, grâce aux observations de Olaüs Roemer sur les satellites de Jupiter.

Jean-Pierre LUMINET [309, p. 24]

Time is indispensable and crucial for the definition and measurement of the speed (as well as of the acceleration, and of higher derivatives) of a value variation of every (biological, econometric, mathematical or physical) variable. The well known definition of the speed, (Note 48), holds also for the speed of light. It is not an exception. What is valid for the relativity of the speed value, (Note 48), it is valid also for the light speed value.

A light ray passes vector $d\mathbf{r}_L$ during the infinitesimal duration dt. They determine *the light velocity* $\mathbf{v}_L(t,\mathbf{x}) \equiv \mathbf{c}(t,\mathbf{x})$ in general, (4.16), [437, p. 177],

$$\mathbf{c}(t,\mathbf{x}) = \frac{d\mathbf{r}_L(t,\mathbf{x})}{dt},\ dt > 0,$$

in a place determined by the vector $\mathbf{x}$ *at a moment* t. The scalar value of the light velocity $\mathbf{c}(t,\mathbf{x})$ is the *light speed at* $(t,\mathbf{x}) \in \mathfrak{T}\text{x}\mathfrak{R}^n$ relative to the unity vector $\mathbf{u}$, which is denoted by $c(t,\mathbf{x})$ if $\mathbf{c}(t,\mathbf{x})$ is collinear with $\mathbf{u}$,

$$c(t,\mathbf{x}) = ||\mathbf{c}(t,\mathbf{x})||sign\left(\mathbf{c}^T(t,\mathbf{x})\mathbf{u}\right),\ \mathbf{c}(t,\mathbf{x}) = c(t,\mathbf{x})\mathbf{u}.$$

The light velocity and the light speed are constant in vacuum,

$$\mathbf{c}(t,\mathbf{x})\ \equiv \mathbf{c} = \mathbf{const.}\ \ and\ c(t,\mathbf{x}) \equiv c = const.,\ in\ vacuum,$$

[144, p. 15], [150, p. 26]. They are, respectively, *the light velocity in vacuum* and *the light speed in vacuum*, for short: *the light velocity* and *the light speed.* We use in this book the notion of vacuum in Einstein's sense, not in the sense of an energy-matter empty set or space (see Subsection "Energy, matter and fundamental laws of physics" in 4.5). In this regard see [263].

4.7.3 *Time*, light speed, units and coordinate systems

In order to measure the value of the speed of a variation of value of any variable, we should have well defined the *time* set $\mathfrak{T}_{(.)}$, hence the corresponding *time* axis $T_{(.)}$. We cannot determine the speed numerical value if we did not completely define the *time* set $\mathfrak{T}_{(.)}$, i.e. the corresponding *time* axis $T_{(.)}$. Besides, any change of a *time* unit or of a *time* scale can change, and most often changes, the numerical value of the speed of every variable except that of *time* (see Theorem 128 "Universal *time* speed law" in Subsection 5.4.1 "Fundamental theorem on *time* speed" of Section 5.4). The general rule holds also for the light speed that is not an exception. The unique exception is the *time* speed. It is invariant (it equals one) relative to all *time* units and relative to all *time* scales, i.e. relative to all time sets $\mathfrak{T}_{(.)}$ and time axes $T_{(.)}$.

Let us illustrate this by the following well known elementary fact that the light speed obeys the general rule.

The *time* axis T is the reference *time* axis with the *time* unit 1_t = second, s. The numerical (dimensionless) value c^t of the light speed c^t in vacuum, when it is measured relative to the length unit $1_L = 1\langle m\rangle$ of the spatial $R^{(n)}-$ axis and relative to the *time* unit $1_t = s$ of the T-axis, is $\mathrm{c}^t = 2.99792458 \text{ x } 10^8$.

° The value of the speed of light propagating in vacuum can be measured with $1_L1_t = ms^{-1}$ or with $1_L1_t = Kms^{-1}$,

$$c^t\!= 2.99792458\text{x}10^8 ms^{-1},\ or\ c^t\!= 2.99792458\text{x}10^5 Kms^{-1},$$

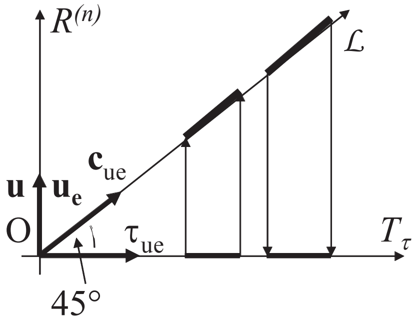

Figure 4.12: The *time* - space coordinate system $(T_\tau, R^{(n)})$. The light ray direction is the porter of the light axis $\mathcal{L}$. Interval mappings are represented by rays in the hyperplanes orthogonal to the *time* axis.

so that its numerical value c equals, respectively,

$$\mathrm{c}= 2.99792458\text{x}10^8, \; or \; \mathrm{c}= 2.99792458\text{x}10^5.$$

° We can measure it with $1_L 1_t = mmin^{-1}$ or with $1_L 1_t = Kmmin^{-1}$,

$$c= 1.798754748\text{x}10^{10} mmin^{-1}, \; or \; c= 1.798754748\text{x}10^7 Kmmin^{-1}.$$

Its numerical value $\mathfrak{c}$ then equals, respectively,

$$\mathrm{c}= 1.798754748\text{x}10^{10}, \; or \; \mathrm{c}= 1.798754748\text{x}10^7.$$

° If we measure the value of the speed of light propagating in vacuum with $1_L 1_t = mh^{-1}$ or with $1_L 1_t = Kmh^{-1}$,

$$c= 1.0792528488\text{x}10^{12} mh^{-1}, \; or \; c= 1.0792528488\text{x}10^9 Kmh^{-1},$$

then its corresponding numerical value is, respectively,

$$\mathrm{c}= 1.0792528488\text{x}10^{12}, \; or \; \mathrm{c}= 1.0792528488\text{x}10^9.$$

Let the third axis denoted by $\mathcal{L}$ be assigned with a light ray direction determined by the extension $\mathbf{c}_{ue}$ of the light speed unity vector $\mathbf{c}_u$ in space (T_τ, R^n), Fig. 4.12. The coordinate system $(\mathcal{L}, R^{(n)})$, Fig. 4.12, is not orthogonal because the extended unity vectors $\boldsymbol{\tau}_{ue}$, $\mathbf{c}_{ue}$ and $\mathbf{u}_e$, which are the hyper-planar vectors, form the angles of 45°,

$$\angle(\mathbf{c}_{ue}, \boldsymbol{\tau}_{ue}) = 45^\circ, \angle(\mathbf{c}_{ue}, \mathbf{u}_e) = 45^\circ.$$

The inclination angle of $\mathcal{L}$ to T_τ is taken 45° because the numerical value $\mathfrak{c}^\tau$ of the light speed in vacuum measured in meters per the *time* unit 1_τ of T_τ-axis equals 1:

$$c^t[\mathrm{LT}^{-1}]\left\langle ms^{-1}\right\rangle = 2.99792458\text{x}10^8\left\langle ms^{-1}\right\rangle\left\langle 1_\tau 1_\tau^{-1}\right\rangle =$$
$$= 2.99792458\text{x}10^8\left\langle 1_\tau s^{-1}\right\rangle\left\langle m1_\tau^{-1}\right\rangle =$$
$$= 2.99792458\text{x}10^8\left(2.99792458\text{x}10^8\right)^{-1}\left\langle ss^{-1}\right\rangle\left\langle m1_\tau^{-1}\right\rangle =$$
$$= 1\left\langle m1_\tau^{-1}\right\rangle = c^\tau\left\langle m1_\tau^{-1}\right\rangle, \text{ hence } \mathfrak{c}^\tau[-]\langle -\rangle = 1[-]\langle -\rangle .$$

This shows again, (for more details see in the sequel), the well known fact that a choice of the *time* unit influences the numerical value of the light speed. However, the choice of the units of length and of *time* does not and cannot influence either the light speed value or the light speed itself. This emphasizes once more the need to distinguish clearly variable, its value and its numerical value relative to the used units and axes.

The preceding examples illustrate also that different numerical values of a variable (speed in this case) can correspond to the same variable value. A change of the units does not change the value of the light speed, but it can change the numerical value of the light speed. In any calculation that involves the light speed we use its numerical value. It is crucially important for all calculations to use the correct numerical value of the light speed (as well as the correct numerical values of all other variables) relative to the accepted units. Otherwise, consequences can be serious, as they are in Einsteinian relativity theory (Section 7.3).

On the noninvariance of the light speed see Part III, Part IV, and in [206, p. 98], [332, pp. 53, 54] and [478, p. 261].

Let us consider the following statement from the book by D'Inverno [98].

Claim 70 ***Postulate by D'Inverno***

"Postulate II. Constancy of velocity of light:

The velocity of light is the same in all inertial systems.

Or stated another way: there is no overtaking of light by light in the empty space. The speed of light is conventionally denoted by c and has the exact numerical value 2.997 924 580 x 10^8 ms^{-1}*, but in this chapter we shall adopt* ***relativistic*** *units in which c is taken to be unity (i.e. c = 1)."*

R. D'INVERNO *[98, p. 20]*

Comment 71 *The acceptance of the light speed value for the unity speed value means that both the length unit and the time unit have been accordingly accepted and fixed. By changing any of these units, the numerical value of the light speed changes, in general, as shown above. The change of its numerical value need not mean any change either of the light speed value or of the light speed itself. It reflects the change of the unit.*

Note 72 *The fact that there is not overtaking of a light signal by a light signal in the empty space means that*

° *there is zero relative speed of every light signal with respect to another light signal moving in the same direction and in the same sense in vacuum,* *and*

° *if we tie and fix the origin of a frame with one of light signals, then the light speed of another one moving in parallel and in the same sense relative to that frame equals zero.*

Conclusion 73 *a) We should a priory allow that the light speed numerical value depends on the time unit of a chosen time axis* $T_{(.)}$ *and on the length unit of a chosen spatial frame* $R^n_{(.)}$.

It is erroneous to claim that the numerical value of the light speed is invariant relative to a choice of a time unit, i.e. relative to a choice of a time axis, in general.

b) The light speed numerical value is constant in vacuum, but it does not possess an invariance property relative to a selection of a time axis.

c) The numerical value of the light speed is not totally universal constant in the sense that it is not invariant. It depends in general on a choice of the time unit and on the length unit.

d) The fact that the numerical value of the light speed depends on the time unit of a chosen time axis has not been taken a priory into account in Lorentz transformations. Consequences are crucial, which will be explained in details in the next part, Part II: "Time Fields and Relativity".

e) The value of the time speed $v_{t(.)} = t^{(1)}_{(.)} = dt_{(.)}/dt_{(.)}$ *equals one arbitrary time unit per the same time unit independently of a choice of the time unit. It is invariant relative to time units, to time axes and to spatial frames (see Theorem 128).*

f) The value of the time speed is totally invariant.

g) The numerical value of the time speed $v_{t(.)} = t^{(1)}_{(.)}$ *equals* 1 *relative to all time units, to all time scales and to all spatial frames,*

$$numv_{t(.)} \equiv 1.$$

h) The numerical value of the time speed is the total universal constant and totally invariant.

The numerical values of speeds of variations of values of all other variables should be naturally permitted, in principle, a priory, to depend on the units of the accepted *time* axes.

4.7.4 Relative light velocities and their values

Neither ***matter****, nor* ***energy****,*
nor anything capable of being used as a ***signal***
can travel faster than 299,796 kilometers per second,
provided that the velocity is referred to one of the frames
of space and time considered in this chapter.
Arthur EDDINGTON [110, p. 57]

The relative light velocity $\mathbf{c}_j^i = c_j^i \mathbf{u}$ and the relative light speed c_j^i are measured with respect to R_j^n and its origin O_j, with the length unit 1_{L_j} and with the *time* unit 1_i, where $i = j$ is permitted. In this regard see [332, p. 53], [478, p. 261].

If, and only if, we use the same *time* axis for both R_i^n and R_j^n, $T_i = T_j$, i.e. $1_i = 1_j$ and $t_i = t_j$, then we will denote, respectively, the light speed relative to R_i^n and R_j^n by c_i and c_j,

$$t_i = t_j \Longrightarrow c_i^i = c_i^j = c_i \; and \; c_j^i = c_j^j = c_j.$$

If we accept R_i^n to be at rest (with respect to R^n) then we denote c_i^j as the light speed c^j measured with respect to the origins O and O_i of R^n and R_i^n with the length unit $1_{L_i} = 1_L$ of R_i^n and R^n, and with the *time* unit 1_j of T_j:

$$c_i^j = c^j \; if \; and \; only \; if \; v_{R_i^n}^{R^n} = v_{O_i}^{O} = 0.$$

Besides,

$$c^i = c^j \Longleftrightarrow c^i = c^j = c^{ij} = c^{ji}, \, c_i^i = c_j^j \Longleftrightarrow c_i^i = c_j^j = c_{ij} = c_{ji}.$$

In Einsteinian relativity theory, the light speed is considered a priory, by following Einstein [144, p. 29, pp. 101 - 107], [151, pp. 44 - 46, p.51], [153, pp. 129 - 135], [154, pp. 30 - 34], not only constant, but also invariant with respect to a choice of the units. Therefore, it is denoted simply by c in Lorentz transformations (7.20) through (7.23) (Section 7.2: "*Time* fields and Lorentz transformations").

4.8 Clock principles

However, one must understand that the change of time
between systems suggested by Einstein is only apparent
because clocks in different frames run at different rates.
This has erroneously been interpreted as time dilation
in the past, but we see now that it is nothing else than
clocks running at different rates in different frames.
Paul MARMET [320, p. 58]

A confusing feature in the theory of relativity is the use of time
and distance as parameters in explaining the constancy of the
velocity of light and the reduced frequencies of atomic clocks
in fast motion and in high gravitational field. ....
Instead of stating that the velocity of the signal were reduced
the theory of relativity explains that time close to mass centers
flows slower thus saving the basic assumption of the theory,
the constancy of the velocity of light.
Tuomo SUNTOLA [457, p. 477]

4.8.1 *Time* value measurement and clock

Time does not force any instrument to function. *Time* does not force or cause operation, functioning of any clock. Therefore, *time* value measurement, and a measurement of the value of a *time* interval duration have been indirect, rather than direct. They can be in terms of both a length passed during a *time* interval and the speed with which the length was passed. The value v of the measurement speed of a clock should be a positive constant because the speed of the *time* value evolution is constant and positive since it equals one (the equation (5.10) in Theorem 128, Conclusion 129). This permits us to apply the equation (4.12) for $v(g) \equiv v$ in the following simplified form:

$$t - t_0 = \int_{l_0}^{l} \frac{dl}{v} = \frac{l - l_0}{v},\ l_0 = l(t_0), l = l(t). \tag{4.53}$$

By analyzing this we arrive at the following:

Conclusion 74 ***Time, time value measurement and clock***

Let a clock measure and indicate the numerical time value exactly (i.e. accurately and precisely) relative to the accepted time axis (i.e. relative to the adopted $t_{zero} = 0$, the accepted initial instant t_0, the time scale and the time unit 1_t).

The accepted time unit 1_t is constant. The measurement speed v of the clock is (should be) therefore also constant.

What the clock shows, it is not time.

What it shows, it is just the measured number N_1 of the accepted time unit 1_t.

The product $N_1 1_t$ added to the initial time value t_0 is equal to the measured time value t relative to t_0,

$$t = t_0 + N_1 1_t, \tag{4.54}$$

and it is equal to the value of the duration $t - t_0$ of the time interval $[t_0, t]$, which is the measurement duration,

$$N_1 1_t = t - t_0. \tag{4.55}$$

In this regard Einstein stated that

....Under these conditions we understand by the "time" of an event the reading (position of the hands) of that one of these clocks ...

Albert EINSTEIN [144, p.20], [154, pp. 23-40]

Time is then defined as the ensemble of the indications of similar clocks, at rest relatively to K, which register the same simultaneously.

Albert EINSTEIN [150, pp. 26 - 27]

Note 75 *Conclusion 74 opposes crucially these Einstein's characterizations of both time and the clock. We will show the correctness of Conclusion 74 in the sequel.*

4.8.2 General clock principle

Since the *time* set $\mathfrak{T}_{(.)}$ is unbounded and since the length of the path of the pointer (of the hand) of every (e.g. mechanical, electromechanical or electronic - digital) clock is bounded, then the motion of the pointer cannot be translational. It should be periodic that is achieved by its rotational movement. The periodic motion is essentially also in the basis of a digital clock, although there is not a rotational pointer. What holds in the sequel for a mechanical and for an electromechanical clock, it holds in principle also for a digital clock and its basic periodic motion.

In order to present an illustrative analysis, we will refer to a clock that ensures constant rotational speed v of the top of its pointer along the scale that is gauged exactly (accurately and precisely) according to the accepted *time* set $\mathfrak{T}_{(.)}$, hence according to the accepted *time* axis $T_{(.)}$. An instantaneous angle of the pointer is φ . The pointer (the clock hand) angular speed ω is constant, so that, for the length size R of both the pointer radius and the scale radius,

$$v = R\omega,\ l = R\varphi + l_0,\ and\ l_0 = R\varphi_0, \tag{4.56}$$

where l_0 and φ_0 are an initial arc and an initial angle, respectively, of the pointer. By the definition, the angular speed ω is determined as the variation of the angle φ per *time* unit,

$$\omega = \frac{d\varphi}{dt}, \tag{4.57}$$

which, after integration, implies the well known relationship

$$t - t_0 = \frac{\varphi - \varphi_0}{\omega}, \tag{4.58}$$

or in another form,

$$\varphi - \varphi_0 = \omega(t - t_0). \tag{4.59}$$

The equations (4.56) and (4.58) can be combined into

$$t - t_0 = \frac{l - l_0}{v},\ l - l_0 = v(t - t_0). \tag{4.60}$$

The numerical value of the angular speed ω of the clock hand is determined by an accepted *time* unit, and vice versa. A change of the numerical value of the angular speed ω of the clock hand implies the corresponding change of the *time* unit, and vice versa. This means the following well known facts:

∘ If the angular speed ω is fixed, then, the bigger value of the angle difference $\varphi - \varphi_0$, (or equivalently, the bigger value of the length difference $l - l_0$), the bigger value of the *time* difference $t - t_0$, and vice versa, (4.59), (4.60).

∘ If the angle difference $\varphi - \varphi_0$ is fixed, e.g. if it is the clock indication read at the end of the measurement, then, the bigger value of the angular speed ω, the smaller value of the *time* difference $t - t_0$, and vice versa, (4.58).

∘ If the length difference $l-l_0$ is fixed, then, the bigger value of the speed v of the top of the clock hand, the smaller value of the *time* difference $t-t_0$, and vice versa, (4.60).

These very well known elementary facts emphasize that the clock hand does not show *time*. It shows directly the value of the length of the arc l passed by the top of its hand during the *time* interval $[t_0,\ t]$, hence the number N_1 of the *time* unit 1_t contained in the arc l. The clock hand shows indirectly the measured value of the duration $t-t_0$, hence the *time* value t relative to t_0 at the end of the measurement, in terms of the number N_1 and the *time* unit 1_t according to the equations (4.54) and (4.55).

A clock can have several different *time* scales such as those gauged in seconds (s), in minutes (min) and in hours (h). Each of the first two contains sixty equal arcs covering the whole circumferences of the scales. They correspond to seconds or to minutes, respectively. The third one has twelve equal arcs, each of which corresponds to one hour. They fill out the circumference of the hour scale. Every arc represents the corresponding *time* unit in the adequate *time* scale. Therefore, we will call them **the unity arcs**. Any unity arc is denoted by l_u. The unity arc l_u can be, but need not be, equal to the accepted length unit 1_L. All unity arcs of the same *time* scale determine the same angle that will be called **the unity angle** φ_u It can be, but need not be, equal to the accepted angle unit 1_φ. The unity angle φ_u represents also the same *time* unit 1_t as the unity arc l_u does. We will generalize this to clocks with multiple *time* scales.

The whole common human experience with, and understanding of, the physical phenomena establishes the following incontestable axiom:

Axiom 76 ***Time and clock***

Time does not have any influence on the clock, and vice versa. It is not time that forces the clock to work. The clock uses (a kind of) energy for its work, for its functioning. It is the energy that forces the clock to work.

The clock work, functioning, operation is fully determined by the crucial time properties, which are the invariance and constancy of the time flow speed (proved in Subsection 5.4.1) and time uniqueness (proved in Subsection 5.4.1 and in Section 20.1).

The angular speed of the clock hand rotation is, therefore, constant and invariantly fixed. In general, the speed of the clock indication variation is constant and invariantly fixed.

The human has been implicitly aware of this relationship between *time* and the clock since the most ancient epoch. It has been the primary basis for the clock design and construction, which leads to the following principle.

Principle 77 ***General clock principle***

The clock circumference is constant and equals $2\pi R$, where R is the clock radius. The clock possesses M different time scales.

R_i is the i-th scale radius and the radius of the i-th clock hand. The length of the i-th time scale equals $2\pi R_i$. It is the i-th scale circumference. The i-th scale

is gauged in N_{u_i} *equal arcs* l_{u_i} *of the i-th time scale, where* $i \in \{-, 1, 2, .., M\}$. *The number* N_{u_i} *is the number of the arcs* l_{u_i} *in the i-th scale circumference. Every arc* l_{u_i} *of the corresponding time scale is fixed and represents the same time unit* 1_{t_i} *of the i-th time scale. We will call* l_{u_i} ***the unity arc (of the i-th time scale)****. Its value can, but need not, be equal to the value of the length unit* 1_{L_i}. *The sum of all unity arcs of any fixed time scale equals the circumference* $(2\pi R_i)\langle 1_{L_i}\rangle$ *of the corresponding time scale,*

$$\sum_{j=1}^{j=N_{u_i}} l_{u_i}\langle 1_{L_i}\rangle = N_{u_i} l_{u_i}\langle 1_{L_i}\rangle = (2\pi R_i)\langle 1_{L_i}\rangle, \; i \in \{-, 1, 2, .., M\}. \tag{4.61}$$

a) The unity arc l_{u_i} *determines a constant angle* φ_{u_i} *that will be called* ***the unity angle (of the i-th time scale)****. Its numerical value can, but need not, be equal to the numerical value of the angle unit* 1_{φ_i}. *It corresponds to the time unit* 1_{t_i}. *The angle scaling coefficient* η_i *transforms 1 rad into* $N_{\varphi_i 1_{rad}}$ *angle units* 1_{φ_i},

$$\begin{aligned} 1rad\langle rad\rangle = \left(N_{\varphi_i 1_{rad}} 1_{\varphi_i}\right)\left\langle 1_{\varphi_i}\right\rangle = \left[\left(\eta_i\left\langle 1_{\varphi_i} rad^{-1}\right\rangle\right) 1rad\langle rad\rangle\right] = \\ = (\eta_i 1rad)\left\langle 1_{\varphi_i}\right\rangle, \\ N_{\varphi_i 1_{rad}} = numN_{\varphi_i 1_{rad}}\langle -\rangle = num\eta_i\langle -\rangle. \end{aligned} \tag{4.62}$$

From these equations we find easily another relationship between the units rad and 1_{φ_i} *as follows:*

$$1_{\varphi_i}\left\langle 1_{\varphi_i}\right\rangle = \left(N^{-1}_{\varphi_i 1_{rad}} 1rad\right)\langle rad\rangle = \left(\eta_i^{-1} 1rad\right)\langle rad\rangle. \tag{4.63}$$

The number N_{u_i} *is the number of the unity angles* φ_{u_i} *in the i-th time scale. Its product with the unity angle* φ_{u_i} *itself is constant, invariant and equals* $(2\pi\eta_i)\left\langle 1_{\varphi_i}\right\rangle$,

$$N_{u_i}\varphi_{u_i}\left\langle 1_{\varphi_i}\right\rangle = \eta_i\left\langle 1_{\varphi_i} rad^{-1}\right\rangle 2\pi\langle rad\rangle = (2\pi\eta_i)\left\langle 1_{\varphi_i}\right\rangle, i \in \{-, 1, 2, .., M\}, \tag{4.64}$$

or equivalently,

$$\eta_i^{-1}\left\langle 1^{-1}_{\varphi_i} rad\right\rangle N_{u_i}\varphi_{u_i}\left\langle 1_{\varphi_i}\right\rangle = \left(\eta_i^{-1} N_{u_i}\varphi_{u_i}\right)\langle rad\rangle = 2\pi\langle rad\rangle. \tag{4.65}$$

b) The product of a constant angular speed $\omega_i\left\langle 1_{\varphi_i} 1^{-1}_{t_i}\right\rangle$ *of the i-th clock hand, of the time unit* 1_{t_i} *and of the corresponding number* N_{u_i} *of the time units* 1_{t_i} *contained in one full hand rotation is constant, invariant relative to the value of* ω_i *and equals* $(2\pi\eta_i)\left\langle 1_{\varphi_i}\right\rangle$,

$$\begin{aligned} \omega_i\left\langle 1_{\varphi_i} 1^{-1}_{t_i}\right\rangle N_{u_i}\langle -\rangle 1\langle 1_{t_i}\rangle = (2\pi\eta_i)\left\langle 1_{\varphi_i}\right\rangle, \\ \forall(\omega_i = const.) \in R^+, \; i \in \{-, 1, 2, .., M\}, \end{aligned}$$

or equivalently,

$$\eta_i^{-1} \left\langle rad 1_{\varphi_i}^{-1} \right\rangle \omega_i \left\langle 1_{\varphi_i} 1_{t_i}^{-1} \right\rangle N_{u_i} \langle - \rangle 1 \langle 1_{t_i} \rangle = 2\pi \langle rad \rangle ,$$
$$\forall (\omega_i = const.) \in R^+, i \in \{-, 1, 2, .., M\} .$$

c) The bigger value of the angular speed ω_i *of the i-th clock hand, the smaller value of the time unit* 1_{t_i}*, and vice versa,*

$$1_{t_i} = \frac{\varphi_{u_i}}{\omega_i},\ \omega_i = \frac{\varphi_{u_i}}{1_{t_i}},\ i \in \{-, 1, 2, .., M\} .$$

d) The product of a constant speed $\upsilon_i \left\langle 1_{L_i} 1_{t_i}^{-1} \right\rangle$ *of the top of a clock hand and of the corresponding number* N_{u_i} *of the time units* 1_{t_i} *contained in one full hand rotation is constant, invariant relative to the speed* υ_i *and equals the i-th scale circumference* $2\pi R_i \langle 1_{L_i} \rangle$*,*

$$\upsilon_i \left\langle 1_{L_i} 1_{t_i}^{-1} \right\rangle N_{u_i} \langle - \rangle 1 \langle 1_{t_i} \rangle = 2\pi R_i \langle 1_{L_i} \rangle ,$$
$$\forall (\upsilon_i = const.) \in R^+, i \in \{-, 1, 2, .., M\} .$$

e) The bigger value of the speed υ_i *of the top of the clock hand, the smaller value of the time unit* 1_{t_i}*, and vice versa,*

$$1_{t_i} = \frac{l_{u_i}}{\upsilon_i},\ l_{u_i} = 2\pi N_{u_i}^{-1} R_i = const.,\ \upsilon_i = \frac{l_{u_i}}{1_{t_i}},\ i \in \{-, 1, 2, .., M\} .$$

f) If the i-th clock hand rotates with the angular speed ω_i *for an angle difference denoted by* $\varphi_{i_2} - \varphi_{i_1}$ *during a time interval* $[t_{i_1},\ t_{i_2}]$*, and if the number of the unity angles contained in* $\varphi_{i_2} - \varphi_{i_1}$ *is* N_i*,* $N_i = \left(\varphi_{i_2} - \varphi_{i_1}\right) / \varphi_{u_i}$*, then*

$$\frac{\varphi_{i_2} - \varphi_{i_1}}{\omega_i} = \frac{N_i \varphi_{u_i}}{\omega_i} = t_{i_2} - t_{i_1},\ i \in \{-, 1, 2, .., M\} . \tag{4.66}$$

g) If the top of the i-th clock hand rotates with the speed υ_i *along an arc difference denoted by* $l_{i_2} - l_{i_1}$ *during a time interval* $[t_{i_1},\ t_{i_2}]$*, and if the corresponding number of the unity arcs is* N_i*,* $N_i = (l_{i_2} - l_{i_1}) / l_{u_i}$*, then*

$$\frac{l_{i_2} - l_{i_1}}{\upsilon_i} = \frac{N_i l_{u_i}}{\upsilon_i} = t_{i_2} - t_{i_1},\ i \in \{-, 1, 2, .., M\} .$$

This well known and widely used ancient clock principle warns clearly that we should distinguish what the clock indicates, the value of the duration of the measurement and the *time* value, from *time* itself.

The clock hand indication is a chronometric model of *time*, but it is not *time*. Our reading of the clock hand indication is the numerical value of *time* relative to its initial value, both expressed in terms of the corresponding *time* unit. The reading is not *time* itself. It is only the relative numerical *time* value.

The clock hand *shows directly the number* of the *time* units contained in the corresponding *time* interval. It shows directly the numbers associated with the initial moment, with the current moment and with the final moment of the measured value of the *time* interval. Hence, it shows indirectly the value of the duration of the *time* interval. The clock hands indicate indirectly the measured value of *time*. They do *not* show *time*. We cannot see *time* on the clock or from the clock.

Any change of the value of the clock hand angular speed (i.e. any change of the value of the speed of the top of the clock hand) means exclusively the corresponding change of the associated *time* unit. It is not and cannot be a change of *time*.

Conclusion 78 ***Einstein‘s meaning of time is wrong***

a) Axiom 76 expresses the elementary physical fact that disproves Einstein‘s attitude that time depends on the clock speed. Einstein‘s claim that time depends on the clock speed is absurd and physical nonsense.

b) Einstein‘s explanation of time disagrees with the clock principle because it rejects Einstein’s equalization of time with its (numerical) value.

c) Einstein‘s explanation of time is fully wrong. It is another absurd and physical nonsense.

The clock principle is completely compatible with Newton’s explanation of *time* and with the characterization of *time* in Axiom 47 (Section 4.2 ”Definition and properties of *time*”). It rejects Einstein’s equalization of *time* with its (numerical) value.

4.8.3 Relativity theory based clock principle

General clock principle (Principle 77) does not consider whether the clock is moving or it is at rest. We will show that it holds in both cases. The *time* scaling coefficients defined by the equation (4.38) (Subsection 4.3.3: "*Time* scaling coefficients: definition" of Section 4.3: "*Time* scales, units and interval mappings") are crucial to show its validity when clocks move with constant velocities. Since we consider clocks in translational motions, then we should investigate how the *time* scaling coefficients depend on the speeds of the clocks themselves. If the clocks are at rest then the speeds of their translational motions are equal to zero, which is just a singular case of the translational motion.

Condition 79 ***Clock conditions***

Let two clocks be identical and let they work exactly (accurately and precisely) so that they operate perfectly equally when they are in the same conditions. One of them stays at rest. Another one is moving with a constant velocity $\mathbf{v}_m = v_m\mathbf{u}$, $0 < v_m < c$, *where c denotes both the light speed and its value when the light propagates through vacuum. The subscript ”m” designates ”moving”, and the subscript ”r” will denote ”at rest”.*

The reference time unit is second, $1_t = s$, *and the reference time axis is* T. *The reference length unit is meter,* $1_L = m$. *The reference angle unit is radian,* $1_\varphi = rad$. *The reference unit* 1_v *of the speeds* c *and* v_m *is meter/second,*

$1_v = 1_L 1_t^{-1} = ms^{-1}$, and the reference unit 1_ω of the angular speed ω is radian/second, $1_\omega = rad 1_t^{-1} = rads^{-1}$.

The angular speeds (of the hands) of both clocks are constant, and equal to ω_r at rest. The angular speed ω_m of the clock moving with the speed v_m is constant, but it can be different from the angular speed ω_r at rest due to its movement with the constant speed v_m.

The clock at rest shows the time values relative to the time axis T_r that is the reference time axis T, $T_r = T$ (i.e. relative to the time unit 1_{t_r} that is the reference time unit 1_t, $1_{t_r} = 1_t = s$). Its hand angle and the hand angular speed are, respectively, φ_r and ω_r. They are related by

$$\omega_r = \frac{\varphi_{r_2} - \varphi_{r_1}}{t_{r_2} - t_{r_1}} = \frac{N_r \varphi_{u_r}}{t_{r_2} - t_{r_1}},$$

where $\varphi_{r_2} - \varphi_{r_1}$ is the angle variation of the clock hand during the time interval $[t_{r_1}, t_{r_2}]$, and N_r is the number of the unity angles φ_{u_r} contained in $\varphi_{r_2} - \varphi_{r_1}$,

$$N_r = \frac{\varphi_{r_2} - \varphi_{r_1}}{\varphi_{u_r}}.$$

Analogously, the moving clock shows the time values relative to the time axis T_m (i.e. relative to the time unit 1_{t_m}). Its hand angle and the hand angular speed are, respectively, φ_m and ω_m. They are related by

$$\omega_m = \frac{\varphi_{m_2} - \varphi_{m_1}}{t_{m_2} - t_{m_1}} = \frac{N_m \varphi_{u_m}}{t_{m_2} - t_{m_1}},$$

where $\varphi_{m_2} - \varphi_{m_1}$ is the angle variation of the clock hand during the time interval $[t_{m_1}, t_{m_2}]$, and N_m is the number of the unity angles φ_{u_m} contained in $\varphi_{m_2} - \varphi_{m_1}$,

$$N_m = \frac{\varphi_{m_2} - \varphi_{m_1}}{\varphi_{u_m}}.$$

The preceding equations determine the following relationship between ω_m and ω_r:

$$\omega_m = \mu_{mr}^{-1} \frac{\varphi_{m_2} - \varphi_{m_1}}{\varphi_{r_2} - \varphi_{r_1}} \omega_r = \mu_{mr}^{-1} \frac{N_m \varphi_{u_m}}{N_r \varphi_{u_r}} \omega_r.$$

Let the origin $O_r = O$ of the frame at rest be fixed at the clock at rest. Let the origin O_m of the moving frame be tied with the moving clock so that the clock and the frame move with the same speed v_m.

Condition 80 *Let the time axes, the spatial axes and the positions of the clocks be interrelated by Lorentz transformations (7.20) through (7.23) presented by Einstein in [144, p. 28], [153, p. 36], [154, pp. 32, 33].*

The time scaling coefficient μ_{mr}, $\mu_{mr} \in R^+$, determines the relationship between the time unit 1_{t_r} and the time unit 1_{t_m} as follows in view of the equations

(4.34) through (4.39):

$$1_{t_m} = [(num\mu_{rm})\, 1_{t_m}] \langle 1_{t_r} \rangle \,,\ 1_{t_r} = [(num\mu_{mr})\, 1_{t_r}] \langle 1_{t_m} \rangle \,,$$
$$N_{1_{t_m}} 1_{t_m} \langle 1_{t_m} \rangle = \left[(num\mu_{mr})\, N_{1_{t_r}} 1_{t_r}\right] \langle 1_{t_m} \rangle \Longleftrightarrow$$
$$\Longleftrightarrow (t_m - t_{m_0}) \langle 1_{t_m} \rangle = [\mu_{mr}\,(t_r - t_{r_0})] \langle 1_{t_m} \rangle \,. \tag{4.67}$$

Let it be determined by

$$\mu_{mr} = \sqrt{\frac{1-\frac{v_m}{c}}{1+\frac{v_m}{c}}} = const.,\ 0 < v_m < c \Longrightarrow \mu_{mr} \in]0,\ 1[. \tag{4.68}$$

Note 81 *These conditions should be strictly verified, e.g. we should verify that the time axes, the spatial axes and the positions of the used clocks are really interrelated by Lorentz transformations presented by Einstein in [144, p. 28], [153, p. 36], [154, pp. 32, 33], if we intend to apply Lorentz transformations. The equations (4.67), (4.68) forbid* $v_m \in \{0, c\}$.

Theorem 82 ***Relativity theory based clock principle (general form)***

Let Clock conditions (Conditions 79 and 80) hold. The following product $\mu_{mr}^{-1}\frac{\varphi_m-\varphi_{m_0}}{\omega_m}$ *of the reciprocal value* μ_{mr}^{-1} *of the time scaling coefficient* μ_{mr} *and of the ratio* $\left(\varphi_m - \varphi_{m_0}\right)/\omega_m$ *of the angle difference* $\varphi_m - \varphi_{m_0}$*, which is measured with the angle unit* 1_{φ_m}*, and of the angular speed* ω_m*, which is measured with* $1_{\omega_m} = 1_{\varphi_m} 1_{t_m}^{-1}$*, is invariant relative to the units and to the angular speed* ω_m*, and it equals the time value difference* $t_r - t_{r_0} = t - t_0$ *measured with the reference time unit* $1_t = 1_{t_r}$ *of the clock at rest,*

$$\mu_{mr}^{-1}\frac{\varphi_m - \varphi_{m_0}}{\omega_m} = \mu_{mr}^{-1}\frac{N_m \varphi_{u_m}}{\omega_m} = t_r - t_{r_0} = t - t_0,\ \forall(\omega_m = const.) \in R^+. \tag{4.69}$$

The value of the time speed is invariant and equals one,

$$v_{t_r} = v_{t_m} = v_t = 1 \left\langle 1_{t(.)} 1_{t(.)}^{-1} \right\rangle,\ (.) \in \{m, r, -, 1, 2, ...\} \,. \tag{4.70}$$

The proof is in Appendix 20.2 "Proof of Theorem 82".

The statement of the theorem and its proof permit the following result. It enables us to calculate the value of the *time* difference $t_r - t_{r_0}$ indicated by the clock at rest directly from the value of the *time* difference $t_m - t_{m_0}$ indicated by the moving clock.

Corollary 83 ***Relativity theory based clock principle (special form)***

Let Clock conditions (Conditions 79 and 80) hold.

Let the moment when the movable clock starts to move with the constant speed v_m *be* t_{m_0} *in the time axis* T_m*, and* t_{r_0} *in the time axis* T_r*. Let the instant when the moving clock interrupts to move with the constant speed* v_m *be* t_m *in the time axis* T_m *and* t_r *in the time axis* T_r*. Then*

$$\left(N_{1_{t_m}} 1_{t_m}\right) \langle 1_{t_m} \rangle = \left[(num\mu_{mr})\, N_{1_{t_r}} 1_{t_r}\right] \langle 1_{t_m} \rangle \,,$$
$$1_{t_m} = [(num\mu_{rm})\, 1_{t_m}] \langle 1_{t_r} \rangle \Longrightarrow$$
$$(t_m - t_{m_0}) \langle 1_{t_m} \rangle = [\mu_{mr}\,(t_r - t_{r_0})] \langle 1_{t_m} \rangle = (t_r - t_{r_0}) \langle 1_{t_r} \rangle \,.$$

The time speed value v_t is invariant and equals one time unit per the same time unit whatever is the choice of the time unit, (4.70).

Theorem 82, this Corollary and the equations (4.66), (4.69), (4.70) agree completely with the equations (5.9), and with Universal *time* speed law, Theorem 128, (5.10). They show that there is not any "*time* dilation", but there is only the change of the *time* unit caused by the speed of the moving clock.

Let us fix the instant of our following observation as the instant of the end of the experiment and let us consider the difference $\varphi_m - \varphi_{m_0}$ that the hand of the moving clock indicates. The preceding proof of Relativity theory based clock principle in its general form (Theorem 82) and in its special form (Corollary 83) confirm the following well known relationships:

° The smaller variation of the angle φ_m of the hand of the moving clock was not caused by a variation of the *time* speed, which is invariant and its numerical value equals one, but it was then caused exclusively by the smaller value of its angular speed ω_m (of the hand of the moving clock).

° There was not any change (or, dilation) of *time*, but there was the change of the unity arc l_u from l_{u_r} to l_{u_m} so that the moving clock showed the arc difference $l_m - l_{m_0}$ instead of $l_r - l_{r_0}$, and there was the change of the unity angle φ_u from φ_{u_r} to φ_{u_m} so that the moving clock indicated the angle difference $\varphi_m - \varphi_{m_0}$ instead of $\varphi_r - \varphi_{r_0}$, as soon as the value of the angular speed changed from $\omega_r = const.$ to $\omega_m = const.$

° The deviation of the angle difference $\varphi_m - \varphi_{m_0}$ (relative to such difference $\varphi_r - \varphi_{r_0}$ shown by the clock at rest) is caused exclusively by the change of the value of the angular speed from $\omega_r = const.$ to $\omega_m = const.$

° It is only the change of the speed of the moving clock from zero to v_m, which caused the change of the angular speed of (the hand of) the clock from ω_r to ω_m.

° *Time* itself did not influence either the clock operation (the speed of the top of the clock hand and its angular speed) or the clock speed.

Let Clock conditions (Conditions 79 and 80) hold. If the clock itself moves and changes its own speed, from a very small value (e.g. starting from the zero speed value, being initially at rest, $v_r = 0$) to a very big value (say v_m =2x10^8 $\langle ms^{-1}\rangle$), then the angular speed of each of its hands changes (significantly), respectively, (in Lorentzian frame). This is expressed in the equation (4.68) by the dependence of the *time* scaling coefficient μ_{mr} on the speed v_m of the moving clock and by the relationship (20.16) (Appendix 20.2: "Proof of Theorem 82") between ω_m and ω_r. Each of these changes implies just the change of the *time* unit corresponding to the change of the clock hand speed. It does not, and cannot, mean that *time* itself changed.

We may now deduce the following:

Conclusion 84 ***Mistakes of Einsteinian relativity theory on time***

Einsteinian relativity theory does not, and cannot, prove either of the following statements because ***they are wrong:***

° *There are several different times.*

° *Time itself changes due to the change of the speed of a body (of a person, of a clock, ...).*

° *The speed of time varies with the variation of the speed of the body.*

The following is true:

° *Time is unique. There are not several different times.*

° *A change of the speed of a body (of a person, of a clock, ...) does not, and cannot, change either time, or time value, or the duration of the movement or its value. It can change only the time unit, hence the numerical values of time and of the duration, relative to the new time unit.*

° *The speed of time does not, and cannot, vary with the variation of the speed of the body. It is constant, invariant and its numerical value equals one.*

This Conclusion verifies the essence of Newton's explanation of *time* itself, and disproves Einstein's. However, it confirms both Newton's explication of "relative *time*" and Einstein's meaning of the relativity of *time* considered only in the sense of the relativity of the numerical *time* value. This is due to the fact that Newton's explanation of relative *time* incorporates Einstein's (for details see Section 3.3).

4.8.4 *Time* and the cause of the clock operation

Let us remind ourselves of the following well known obvious facts:

° There is not any clock that can function (can operate, can work) without using an energy. The cause of the operation of every clock is the corresponding kind of energy (energy exchange in general).

° *Time* does not cause operation of any clock.

° *Time* does not cause any variation of the speed of any clock hand of any clock.

° A change of the speed of a moving clock itself reflects a change of an energy exchanged between the clock and its environment.

° A change of the speed of a moving clock itself causes the corresponding changes of the speeds of the clock hands, hence of the speeds of the clock measurements, in Lorentzian frame.

° *Time* does not cause any change of the speed of a moving clock itself.

° The increase of the clock speed increases the kinetic energy of the clock hands, which causes the bigger resistance force to the clock hand rotation. Since the clock energy is fixed and the clock power is constant then the angular speed of the clock hands decreases, This implies the increase of the *time* unit of the clock. Evidently, there is neither any clock influence on *time* nor any *time* influence on the clock power, energy, the speed of the clock hand rotation, nor on the *time* unit.

Conclusion 85 ***Time and the cause of the clock operation***

Time does not influence operation of any clock.

It is a kind of energy that forces a clock to operate. Only the energy forces a clock hand to move.

Any variation of the speed of a clock hand, i.e. of the clock measurement speed, does not express any variation of time or of its value. It implies only the corresponding variation of the time scale and/or of the time unit. It implies only the corresponding variation of the numerical value of time relative to the corresponding variation of the time scale and/or of the time unit.

Any variation of the speed of the clock hand is caused by the corresponding variation of the power value. It expresses the power value variation. It does not, and cannot, mean a variation of time itself.

4.8.5 Energy and movement of clock itself

Marmet, by accepting that the mass varies with a speed variation and that the mass conversion coefficient $\gamma = 1/\sqrt{1-(v/c)^2}$, [320, the equation 2.2 on the page 31], explained, from the energy-mass point of view, how and why a change of the value of the speed of the clock itself causes variations of the values of the angular speeds of the clock hands (in this regard see the books [226], [231]). He explained that the phenomenon does not represent a change of *time*, but that it just causes variations of the corresponding *time* units.

We should distinguish two different phenomena:

◦ One phenomenon is the energy-mass cause of the changes of the values of the angular speeds of the clock hands due to a change of the value of the speed of the clock itself. This phenomenon does not reflect any variation or change of *time*.

◦ Another phenomenon is a change of the *time* unit caused by the change of the value of the angular speed of the clock hand. This phenomenon does not either reflect any variation or change of *time*.

The above explanations show again that Einstein's interpretation of *time* relativity in the sense of the relativity of its numerical value is just in Newton's sense (see Section 3.3). Both, Newton's and such Einstein's sense of the relativity of *time* mean the relativity of the numerical *time* values with respect to the zero instant, to initial moment, to *time* scale, and to *time* unit.

We can now analyze following Einstein's claim, in which "I" means one, I = 1 (=1_{t_m}), the superscript "prime" (') denotes "moving" (and corresponds to the subscript "*m*"), and notations without superscript and subscript correspond to "at rest" (correspond to those denoted in the above text by the subscript "*r*" or without subscript and superscript).

Claim 86 ***Einstein's fundamental claim on time relativity***

As judged from K, the clock is moving with the velocity v; as judged from this reference-body, the time which elapses between two strokes of the clock is not second, but $\frac{I}{\sqrt{1-\frac{v^2}{c^2}}}$ *seconds, i.e. a somewhat larger time. As a consequence of its motion the clock goes more slowly then when at rest."*

Albert EINSTEIN *[154, p. 37] (also [144, p. 32], [153, p. 41])*

This Einstein's claim is the fundamental of his theory of *time* relativity.

The *time* unit 1_{t_m} of the moving clock changed relative to the *time* unit 1_{t_r} of the clock at rest (and relative to the *time* unit of the moving clock itself when it is at rest). The numerical value of the *time* unit 1_{t_r} of the clock at rest is positive real number, $1_{t_r} \in R^+$. It corresponds to $\mu_{mr} = \left(1 - \frac{v_m}{c}\right)^{1/2} \left(1 + \frac{v_m}{c}\right)^{-1/2}$ *time* units 1_{t_m} of the moving clock, i.e. one *time* unit 1_{t_m} of the moving clock corresponds to $\mu_{rm} = \left(1 + \frac{v_m}{c}\right)^{1/2} \left(1 - \frac{v_m}{c}\right)^{-1/2}$ *time* units 1_{t_r} of the clock at rest, (4.67), (4.68). Since $\mu_{mr} \in]0,1[$, $\mu_{rm} \in]1,\infty[$, for $0 < v_m < c$, then the *time* unit 1_{t_r} is smaller (its duration is shorter) than (that of) the *time* unit 1_{t_m}. This confirms Einstein's conclusion that

the time which elapses between two strokes of the clock is not second,

and simultaneously rejects the immediate continuation of his claim 86:

but $\frac{I}{\sqrt{1-\frac{v^2}{c^2}}}$ *seconds,*

because $\mu_{mr} = \sqrt{\left(1 - \frac{v_m}{c}\right) / \left(1 + \frac{v_m}{c}\right)} \neq 1/\sqrt{1 - \frac{v^2}{c^2}}$. Besides, the further continuation (i.e. *a somewhat larger time*) of his conclusion does not hold, as well. If the clock were moving with the light speed, $v_m = c$, then the numerical value of the *time* unit 1_{t_m} of the moving clock would be infinitely big, $v_m = c \Longrightarrow num1_{t_m} = \infty$. Consequently, the clock hands would not move (see the books [226], [231]). Einsteinian interpretation would be that the *time* values flow was stopped, which is impossible. Let us explain this in another way.

Claim 87 ***Rebuttal to Einstein's fundamental claim on time relativity***
Let Clock conditions (Conditions 79 and 80) hold. Then, the following is true:

The time is the same, hence the speed of the time value evolution is the same and its numerical value equals one, for both the clock at rest and the moving clock.

Proof. We accepted Clock conditions 79 and 80 to hold because they were used by Einstein, [144, pp. 25 - 32], [153, pp. 33 - 37], [154, pp. 30 - 34]. Then the statements of Claim 87 follow directly from Relativity theory based clock principle in its general form (Theorem 82) and in its special form (Corollary 83). Q. E. D ■

Conclusion 88 ***What is wrong, what is correct in Einstein's fundamental claim, and in Newton's meaning***
Einstein's conclusion that the clock at rest indicates

"a somewhat larger time"

is wrong. This is the consequence of Einstein's equalization of time with "numerical value of time" (with the numerical time value), since he defined time as follows:

"Under these conditions we understand by the "time" of an event the reading (position of the hands) of that one of these clocks which is in the immediate vicinity (in space) of the event. In this manner a time-value is associated with every event which is essentially capable of observation." [144, p.20], [154, pp. 23-40]

"Time is then defined as the ensemble of the indications of similar clocks, at rest relatively to K, which register the same simultaneously." [150, pp. 26 - 27]

Einstein's conclusion that

"the time which elapses between two strokes of the clock is not second, but $\frac{I}{\sqrt{1-\frac{v^2}{c^2}}}$ seconds"

is partially true. It is correct provided only that we understand that it is the time interval that elapses between two strokes of the clock, but not time itself. However, the factor $1/\sqrt{1-(v/c)^2}$ does not relate the time unit of the moving clock to the time unit of the clock at rest. The equations (4.67), (4.68) determine exactly their relationship.

The equations 4.67, 4.68 imply the following:

° *for $v_m = 0$, $1_{t_m} = 1_{t_r} \in R^+$, which is trivially obvious,*

° *for $0 < v_m < c$, the time unit of the moving clock is bigger than that of the clock at rest, $1_{t_m} > 1_{t_r}$, so that the number of the units 1_{t_m} is smaller than the number of the units 1_{t_r} in the same time interval (see Law 44 in Section 4.1),*

° *for $v_m = c$, the time unit 1_{t_m} of the moving clock is infinitely times bigger than the time unit 1_{t_r} of the clock at rest, i.e. $1_{t_m} = \infty$ for $1_{t_r} \in R^+$, which forbids $v_m = c$. This is an extraordinary paradox of Einsteinian relativity theory. It is proved exclusively for the light speed of the arbitrary point, and it forbids simultaneously the arbitrary point to move with the light speed.*

If we accept, in view of Einstein's definitions of time, that Einstein's notion of time relativity is exclusively the notion of relativity of the time numerical value, then we should recognize the following:

° *Einstein did not explain what is time itself. He did not prove either the nonuniqueness of time (i.e. the existence of many temporal variables - many times) or relativity of time (of time itself),*

and

° *Einstein's notion of time relativity is incorporated in Newton's notion of relative time.*

Then, we cannot state that Einstein wrote about time relativity in a sense different from Newton's (see Conclusion 43 in Section 3.3: "Einstein's versus Newton's explanation"). Moreover, we cannot claim that Einstein's relativity theory disproves Newton's explanation of the absolute nature of time.

If somebody wished to try to reaffirm Einstein's claim in the sense that *time* itself is relative (Claim 86) by stating that the value of the angular speed ω_m of the hand of the moving clock is smaller because the *time* speed value was smaller for the moving clock (relative to the *time* speed value valid for the clock at rest), then it would be also completely wrong. There is not any variation of the *time* speed value. The decrease of the value of the clock hand angular speed is not a consequence of a variation of the *time* speed value because such variation have never existed and will never exist. The *time* speed value is constant, invariant and its numerical value equals one independently of motions of clocks and of their indications. It is just the speed v_m of the moving clock itself, which caused the decrease of the value of the clock hand angular speed. Einstein himself confirmed this as follows:

L'horloge en mouvement marche plus lentement qu'au repos. [144, p. 32]

Par suite de son mouvement, l'horloge marche plus lentement que lorsqu'elle est au repos. [153, p. 41]

As a consequence of its motion the clock goes more slowly than when at rest. [154, p. 37]

4.9 *Time* and movement

It is therefore unscientific to distinguish between rest and motion,
as between two different states of a body in itself,
since it is impossible to speak of a body being
at rest or in motion except with reference,
expressed or implied, to some other body.
....
Acceleration, like position and velocity,
is a relative term and
cannot be interpreted absolutely.*
James Clerk MAXWELL [334, pp. 22, 25]

Time is the temporal link between the light propagation and motions. Let us examine their relationships in the framework of motions of bodies.

Next example can be found in certain literature on Einsteinian relativity theory. They are used therein to justify the claim that the speed of a body influences the speed of the *time* value evolution, and that it causes a change of *time* itself, thus creating relativeness of *time* itself in the sense of the existence of several different *times* (several temporal variables).

We will consider the same example by analyzing *time* and the speeds of all motions relative to the same initial moment, the same *time* scale and the same *time* unit. We will show that the speed of the body does not and cannot influence the *time* speed (the speed of the *time* value increase).

For analysis of other examples used in Einsteinian relativity theory literature see [226], [231]. Its conclusions are the same as the conclusion of the following example analysis.

Example 89 *Two persons are denoted by P and P_i. They are represented by the points denoted as the persons, P and P_i, respectively, in Fig. 4.13 and Fig. 4.14. They carry two identical clocks that work perfectly equally under the same*

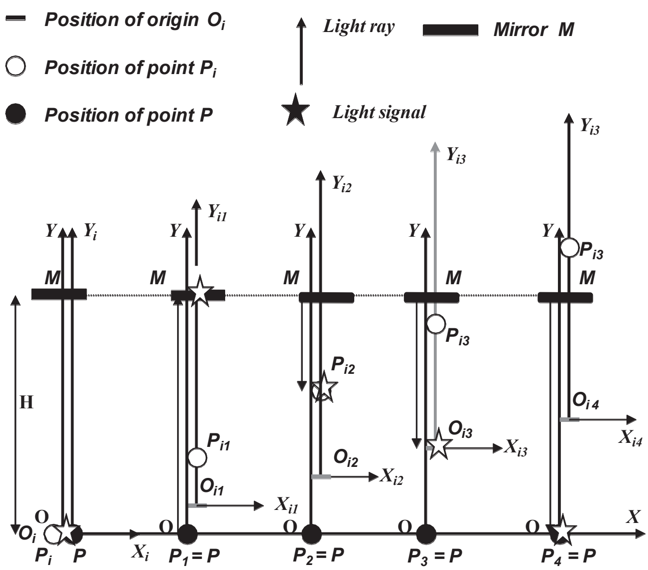

Figure 4.13: Different positions of the origin O_i, of the point P_i, and of the light ray in the spaces represented by the $X_{(.)}Y_{(.)}$-planes.

conditions (e.g. when they are at rest, where "at rest" means the rest relative to the Earth, with which the spatial frame R^n is tied).

The initial positions of the persons do not carry any numerical subscript. The person P was staying unmovable on the ground at the position O all the time during the experiment. The person P_i was moving translationally with a constant velocity $\mathbf{v}_{P_i}^O = v_{P_i}^O \mathbf{u}$ with respect to the origin O of R^2. Her/his relative velocity with respect to the origin O_i of R_i^2 was $\mathbf{v}_{P_i}^{O_i} = v_{P_i}^{O_i} \mathbf{u}$. The initial moment $t_0 = 0$ was the moment when the origin O_i of R_i^2 and the person P_i were in the position O and when a light source simultaneously emitted a light signal. The origin O_i and the coordinate system R_i^2 were moving with a constant velocity $\mathbf{v}_{O_i}^O = v_{O_i}^O \mathbf{u}$ with respect to O and R^2, $v_{O_i}^O \leq v_{P_i}^O$ so that $v_{P_i}^{O_i} \in R_+$. (By the way, note that such an experiment is unrealizable because the speed cannot instantaneously, hence discontinuously, change its value from zero to $v_{O_i}^O \in R^+$). The light ray reached the mirror M in its constant position M at the moment t_1, Fig. 4.13 and Fig. 4.14. It passed the length $H = \overline{OM}$ with the speed c relative to $I = T \times R^2$ during the time interval $[0, t_1] \subset T$:

$$\overline{OM} = ct_1 \Longrightarrow t_1 = \frac{\overline{OM}}{c}. \tag{4.71}$$

The reflected light signal flew in the reverse sense. The light signal reached the person P_i at her/his position P_{i2} at the moment t_2. The moment t_2 was evidently later than the moment t_1, $t_2 > t_1$, because the light ray had to pass the longer

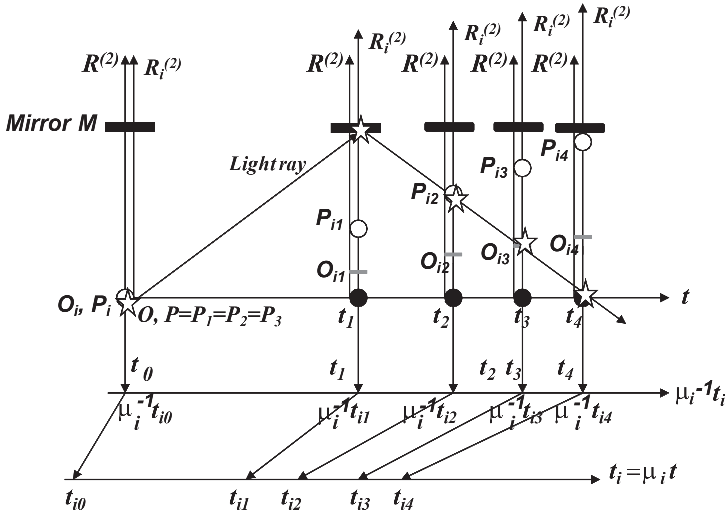

Figure 4.14: Different positions of the origin O_i, of the point P_i, and of the light ray in the integral spaces.

distance OMP_{i2} to reach the person P_i than the distance OM needed to reach the mirror M. The origin O_i was in its position O_{i2} at the moment t_2. Hence:

$$\overline{OMP_{i2}} = \overline{OM} + \overline{MP_{i2}},\ \textit{and}\ \overline{OM} = \overline{OP_{i2}} + \overline{MP_{i2}} \Longrightarrow$$

$$\Longrightarrow ct_1 = v^O_{P_i} t_2 + c(t_2 - t_1) \Longrightarrow t_2 = \frac{2}{1 + \frac{v^O_{P_i}}{c}} \frac{\overline{OM}}{c} = \frac{2}{1 + \frac{v^O_{P_i}}{c}} t_1. \tag{4.72}$$

If we consider two extreme situations and the corresponding values of the speed $v^O_{P_i}$ of the person P_i with respect to O, then:

a) $t_2 = 2t_1$ *if the person P_{i2} had not moved, i.e. if $\mathbf{v}^O_{P_i} = \mathbf{0}$,*

b) $t_2 = t_1$ *if the person P_{i2} had been moving (hypothetically) with the speed of light, i.e. if $\mathbf{v}^O_{P_i} = c\mathbf{u}$.*

These particular results verify the correctness of the last equation in (4.72). The light arrived at the origin O_i of R^2_i in its position O_{i3} at the moment denoted as t_3. The origin O_i was moving with the speed $v^O_{O_i}$ with respect to the origin O. The light signal passed the distance $MP_{i2}O_{i3}$ during the time interval $[t_3 - t_1]$. Altogether:

$$\overline{OM} = \overline{OO_{i3}} + \overline{MO_{i3}} \Longrightarrow ct_1 = v^O_{O_i} t_3 + c(t_3 - t_1) \Longrightarrow$$

$$t_3 = \frac{2}{1 + \frac{v^O_{O_i}}{c}} \frac{\overline{OM}}{c} = \frac{2}{1 + \frac{v^O_{O_i}}{c}} t_1. \tag{4.73}$$

For the two extreme situations and the corresponding values of the speed $v^O_{O_i}$ of the origin O_i with respect to the origin O we find:

a) $t_3 = 2t_1$ *if the coordinate system* R_i^2 *and its origin* O_i *did not move, i.e. if* $\mathbf{v}_{O_i}^O = \mathbf{0}$,

b) $t_3 = t_1$ *if the coordinate system* R_i^2 *and its origin* O_i *were moving with the speed of light* $\mathbf{v}_{O_i}^O = c\mathbf{u}$.

These specific results are correct and verify the last equation in (4.73). The light ray reached the person P *later, at the moment* t_4. *The light signal passed the longest path* $OMP_{i2}O_{i3}P$ *from its initial position* O *to the person* P: $t_4 > t_3 > t_2 > t_1$, *Fig. 4.13 and Fig. 4.14. Therefore, it was natural that the clocks at the persons* P_i *and* P *showed different moments,* t_2 *and* t_4, *if they were identical and synchronized, and if they were working with the same scales and units (of T-axis), when the light signal arrived at the persons. Evidently:*

$$\overline{OM} = \overline{MO} \Longrightarrow ct_1 = c(t_4 - t_1) \Longrightarrow t_4 = 2t_1. \tag{4.74}$$

If, additionally, the moving clock in the position P_{i2} *showed some other moment, say* t_{i2}, *different from* t_2, *due to the high speed* $v_{P_i}^O$ *of its own motion, then it would mean that the clock used another time unit, say* 1_i, *different from the time unit* 1_t *of the time axis* T *[see Principle 77 "General clock principle" and Theorem 82 "Relativity theory based clock principle (general form)" in Section 4.8: "Clock principles"]. A moment* t_i *of the time axis* T_i *is related to the corresponding moment* t *of the time axis* T *by the time scaling coefficient* μ_i : $t_i = \mu_i t$. *This means that*

$$t_{i1} = \mu_i t_1, \tag{4.75}$$

which yields:

$$t_1 = \mu_i^{-1} t_{i1}. \tag{4.76}$$

Since we did not change the length unit, $1_{L_i} = 1_L$, *and since the change of the time unit and of the time axis do not change the length, then*

$$\mathbf{r}_{P_i}^{O,i}(t_i) \equiv \mathbf{r}_{P_i}^{O}(t). \tag{4.77}$$

The speed unit 1_v, $1_v = 1_L 1_t^{-1}$, *changed to* 1_{v_i}, *exclusively due to the change of the time unit from* 1_t *to* 1_i. *These facts caused the change of the numerical values of the speeds as follows:*

$$v_{O_i}^{O,i} = \frac{dr_{O_i}^{O,i}(t_i)}{dt_i} = \frac{dr_{O_i}^{O}(t)}{d(\mu_i t)} = \mu_i^{-1}\frac{dr_{O_i}^{O}(t)}{dt} = \mu_i^{-1} v_{O_i}^{O},$$

$$v_{P_i}^{O,i} = \frac{dr_{P_i}^{O,i}(t_i)}{dt_i} = \frac{d\left[r_{O_i}^{O,i}(t_i) + r_{P_i}^{O_i,i}(t_i)\right]}{dt_i} = v_{O_i}^{O,i} + v_{P_i}^{O_i,i} = \frac{dr_{P_i}^{O}(t)}{dt_i} =$$

$$= \frac{d\left[r_{O_i}^{O}(t) + r_{P_i}^{O_i}(t)\right]}{d(\mu_i t)} = \mu_i^{-1}\left(v_{O_i}^{O} + v_{P_i}^{O_i}\right) = \mu_i^{-1} v_{P_i}^{O}. \tag{4.78}$$

Altogether, $v^i = \mu_i^{-1} v$ *in general. Hence,* $c^i = \mu_i^{-1} c$, $v_{O_i}^{O,i} = \mu_i^{-1} v_{O_i}^{O}$, $v_{P_i}^{O,i} = \mu_i^{-1} v_{P_i}^{O}$. *These equations yield*

$$\frac{v_{O_i}^{O,i}}{c^i} = \frac{v_{O_i}^{O}}{c} \text{ and } \frac{v_{P_i}^{O,i}}{c^i} = \frac{v_{P_i}^{O}}{c}. \tag{4.79}$$

Let us determine the relationships among the moments t_{i1}, t_{i2}, t_{i3} and t_{i4}. We will refer again to Fig. 4.13 and Fig. 4.14, to the equations (4.75) through (4.79), and we will use the values $v_{P_i}^{O,i}$, $v_{P_i}^{O_i,i}$, $v_{O_i}^{O,i}$ and c^i of the speeds of P_i, O_i and of the light signal if the time value is measured in terms of the unit 1_i of T_i:

$$\overline{OM} = ct_1 = c^i t_{i1} \Longrightarrow t_{i1} = \frac{\overline{OM}}{c^i} = \mu_i \frac{\overline{OM}}{c} = \mu_i t_1. \tag{4.80}$$

$$\overline{OMP_{i2}} = \overline{OM} + \overline{MP_{i2}}, \text{ or } \overline{OM} = \overline{OP_{i2}} + \overline{MP_{i2}}$$
$$\Longrightarrow c^i t_{i1} = v_{P_i}^{O,i} t_{i2} + c^i (t_{i2} - t_{i1}) \Longrightarrow$$
$$t_{i2} = \mu_i t_2 = \frac{2}{1 + \frac{v_{P_i}^{O,i}}{c^i}} t_{i1} = \frac{2}{1 + \frac{v_{P_i}^{O}}{c}} \mu_i t_1. \tag{4.81}$$

This is evidently the equation (4.72) multiplied by μ_i.

$$\overline{OM} = \overline{OO_{i3}} + \overline{MO_{i3}} \Longrightarrow c^i t_{i1} = v_{O_i}^{O,i} t_{i3} + c^i (t_{i3} - t_{i1}) \Longrightarrow$$
$$t_{i3} = \mu_i t_3 = \frac{2}{1 + \frac{v_{O_i}^{O,i}}{c^i}} t_{i1} = \frac{2}{1 + \frac{v_{O_i}^{O}}{c}} \mu_i t_1. \tag{4.82}$$

This is evidently the equation (4.73) multiplied by μ_i. Further, $\overline{OM} = \overline{MO}$ implies

$$c^i t_{i1} = c^i (t_{i4} - t_{i1}) \Longrightarrow t_{i4} = \mu_i t_4 = 2t_{i1} = 2\mu_i t_1. \tag{4.83}$$

This is the equation (4.74) multiplied by μ_i. The equations (4.80) through (4.83) are the equations (4.71) through (4.74) multiplied by μ_i. They show that there was only change of the time unit and the induced change of the speed unit, but there was not any change of time itself. The moving clock showed the same time values as the clock at rest, but measured with the time unit 1_i rather than with the time unit 1_t of the clock at rest. There might be a significant change of the time unit, hence of the speed unit, if the clock was moving with a (very) high speed. In this regard see [332, p. 51].

Notice that Fig. 4.13, Fig. 4.14 and (4.77) imply

$$\mathbf{r}_{P_i}^{O}(t) = \mathbf{r}_{O_i}^{O}(t) + \mathbf{r}_{P_i}^{O_i}(t) = \mathbf{r}_{P_i}^{O_i,i}(t_i) + \mathbf{v}_{O_i}^{O,i} t_i. \tag{4.84}$$

If we set this in Lorentz form (7.22),

$$\mathbf{r}_{P_i}^{O}(t) = \lambda \left[\mathbf{r}_{P_i}^{O_i,i}(t_i) + \mathbf{v}_{O_i}^{O,i} t_i \right], \tag{4.85}$$

then Lorentz space scaling factor

$$\lambda = 1. \tag{4.86}$$

Lorentz transformations (7.20) through (7.23) are not applicable to the direct modeling the considered system.

Note 90 *Notice that if there were a change of the length unit so that* $\mathbf{r}_{P_i}^{O_i,i}(t_i) = \eta_i \mathbf{r}_{P_i}^{O}(t)$ *then (4.78) and (4.83) should be replaced by (4.87) and (4.88),*

$$v_{P_i}^{O,i} = \frac{dr_{P_i}^{O,i}(t_i)}{dt_i} = \frac{d\left[r_{O_i}^{O,i}(t_i) + r_{P_i}^{O_i,i}(t_i)\right]}{dt_i} = v_{O_i}^{O,i} + v_{P_i}^{O_i,i} = \frac{d\left[\eta_i r_{P_i}^{O}(t)\right]}{dt_i} =$$
$$= \frac{d\left[\eta_i r_{O_i}^{O}(t) + \eta_i r_{P_i}^{O_i}(t)\right]}{d(\mu_i t)} = \eta_i \mu_i^{-1}\left(v_{O_i}^{O} + v_{P_i}^{O_i}\right) = \eta_i \mu_i^{-1} v_{P_i}^{O}. \tag{4.87}$$

In general, $v_i^i = \eta_i \mu_i^{-1} v$*. Hence,* $c_i^i = \eta_i \mu_i^{-1} c$, $v_{O_i}^{O,i} = \eta_i \mu_i^{-1} v_{O_i}^{O}$, $v_{P_i}^{O,i} = \eta_i \mu_i^{-1} v_{P_i}^{O}$ *so that*

$$\frac{v_{O_i}^{O,i}}{c_i^i} = \frac{v_{O_i}^{O}}{c} \text{ and } \frac{v_{P_i}^{O,i}}{c_i^i} = \frac{v_{P_i}^{O}}{c}. \tag{4.88}$$

The equations (4.83) and (4.88) show that the ratios of the speeds are invariant relative to the changes of the time unit and of the length unit, which is important for this example. Since the characteristic instants depend only on the ratios of the values of the speeds (but not on the separated values of the speeds), then the changes of the time unit and/or of the length unit, hence the change of the speed unit, do not influence the values of the instants. This justifies the condition in the above example that there is not a change of the length unit. It enables us to analyze the influence of the change of the time unit only.

Conclusion 91 ***Lorentz transformations do not enable direct modeling***

The preceding example, and other examples presented in [226], [231], illustrates that Galilean - Newtonian approach to the analysis of the relationships among time, speed and motions (of light and of bodies) provides correct and accurate results.

Besides, it shows that time was the same for all those participating in the experiments, including the moving clocks that changed time units (relative to the time unit of the clock at rest) due to their own speeds.

Lorentz transformations, consequently Einsteinian approach, is inapplicable in the physical sense to them, [226], [231].

Conclusion 92 ***Relative speed between two parallel light signals***

Let two light signals move in parallel in the same sense in vacuum. Each of them can be taken for the reference axis - inertial axis.

If we accepted Einsteinian attitude that the light speed were invariant relative to the speed of inertial frames, then the speed of each of the two light signals relative to another one would be equal to their speed relative to a reference frame that is at rest. This would mean that each light ray would move with the speed of 2.99792458 x 10$^5 \left\langle Kms^{-1}\right\rangle$ *relative to another one, which is impossible, hence meaningless.*

The light signals move with equal speeds with respect to the reference frame. This implies zero relative speed of each of them with respect to another one. Therefore, we should allow a priory that the light speed value depends on the *integral space with respect to which it is measured. This does not contradict Einstein's postulate on the constancy of the light speed value in vacuum, but it discredits his assertion that the light speed is the same relative to all inertial frames, i.e. that it is invariant relative to a choice of the integral space. This is crucial for the interpretation of the constancy of the light speed in vacuum. The speed of light propagating in vacuum is constant, but its value and numerical value depend in general on a choice of the integral space with respect to which it is measured, rather than to be invariant relative to this choice. It is not invariant. This will be confirmed in different ways in the sequel (see also [226], [231]).*

Conclusion 93 ***Noninvariance of the light speed***

The numerical value of the light speed is not a universal invariant, while the numerical value of the speed of the time value propagation is a universal constant (equal to one). Moreover, the latter is also a universal invariant (see Conclusion 129 and [226], [231]).

For more examples see [226], [231].

4.10 Human and *time*

4.10.1 Aging, biological state and biological scales of *time*

The biologist has no operational definition of aging.
I. W. RICHARDSON [408, pp. 752]

The (*temporal*, sometimes called *chronometric*) aging process is irreversible. This is due to the temporal orientation of the flow of *time* values, which is expressed by the strict continuous increase of the *time* value. Nothing and nobody can stop this process and in this sense its evolution continues independently of the beings, of the space, and of all other variables. The speed of the (temporal) aging process is the speed of the *time* value evolution, which is universally constant. It is invariant. Its numerical value equals one relative to all *time* axes, hence relative to all *time* scales and relative to all *time* units (see Theorem 128 or Conclusion 129 in Subsection 5.4.1: "Speed of the *time* value evolution" of Section 4.2: "Definition and properties of *time* "). There is not another variable with such properties of the speed of its value variation. *Time* is the unique variable with such speed features. We have just repeated this once more in order to emphasize what follows.

We should clearly distinguish the speed of the (temporal) aging process, the numerical value of which is invariant and equals one, from the speeds of the evolutions of all other processes, including all processes in beings, hence in human, and from the speeds of the propagations of all other phenomena. There is a deep confusion, even in the scientific literature, due to the equalization of the *time* speed with the speeds of other processes and/or with the speeds of the propagations of other phenomena. Such equalization is a fundamental mistake. It led to claims that there are various, different *times*. Unfortunately, it has been occupying almost all branches of human activity.

The speeds of aging processes and phenomena can be variable and can imply different *time* scales and different *time* units as natural for those processes and phenomena. We do assign different *time* scales and/or different *time* units to different processes or phenomena. We do accommodate *time* scales and/or *time* units to the speeds of the evolutions of the corresponding processes or to the speeds of the propagations of the corresponding phenomena.

Persons P_A and P_B can have the same (temporal) age, but their appearance, their biological states (physical and psychological state, and their health) can reflect different states of their organisms, of themselves. We may speak about their "biological aging" in the sense of the speeds of their biological processes. We should clearly distinguish, respectively, between their "biological aging" (between their "biological age") and their temporal aging (and their temporal age). We will call, for short, *temporal age* and *temporal aging* just *age* and *aging*, respectively. The speed of aging is constant, invariant and has the numerical value 1 (one) relative to all *time* units, *time* scales, initial moments, zero instants, and relative to all biological processes in any being including the human..

Biological state is an adequate expression, rather than "biological age" or even only "age", to determine the biological situation of any being, which the term "biological age" is aimed at. The biological state and its evolution have different nature from the nature of *time,* from age and from aging.

If we wish to use the term "biological age of a person" in the temporal sense, then we should assume that the most adequate (temporal) age is well determined, is well defined (according to the appropriate criteria), for every possible biological state of the human organism.

Definition 94 ***Age, biological state and biological age***

Temporal age, for short: age, of a person at a moment $t \in \mathfrak{T}$, *who was born at a moment* $t_0 \in \mathfrak{T}$, *is the temporal length* $t - t_0$ *of the time interval* $[t_0, t]$.

The biological state, for short: state, of a person at a moment $t \in \mathfrak{T}$ *is her/his overall, complete, biological situation of at the same moment* $t \in \mathfrak{T}$.

The biological age of a person at a moment $t \in \mathfrak{T}$ *is the (temporal) age at the same moment* $t \in \mathfrak{T}$, *which corresponds most adequately to the real (to the actual) (biological) state of person's organism at the moment* $t \in \mathfrak{T}$.

Let us consider the hypothetical example, which is popular in Einsteinian relativity theory, of the identical twins of the same sex, (two brothers or two sisters) in order to satisfy the condition for their full equality. One rested on Earth. Another one flew enormous round trip distance with an unbelievable high speed and returned after 30 years. The state of the unmovable brother/sister was such that the he/she seemed older for 30 years than the brother/sister who flew. This does not mean that there ages were different, that there aging speeds were different. This means only that the speed of the variation of the biological state of the brother/sister, who stayed on the earth, was much bigger than the speed of the variation of the biological state of the brother/sister who flew. The biological state of the brother/sister on the ground was changing much faster than the biological state of the brother/sister who flew with the very big speed. However, *time* interval passed during the journey was the same for both. The *time* speed was equal for both. Only the speeds of variations of their biological states were different. These speeds were not the *time* speed. Their biological states are not *time* values or *time* itself. The biological age of the flying brother/sister reflected his/her biological state that was adequate for a person younger for 30 years than his/her brother/sister who rested on Earth.

4.10.2 Psychological feeling of *time*

3. One sometimes hears this reversibility of motion
referred to as the reversibility of time.
I shall not comment further on this abuse of terminology.
D. PARK [372, p. 266]

Newton's explanation of the relative sense of *time* has already enabled us to understand why different persons think that *time* has different speeds for them; i.e. why each of them thinks that she/he has her/his own "personal *time*". In fact, they think of the speeds of their personal processes or of the speeds of other processes. For example, if a theater performance was dynamic and interesting then its evolution appeared "fast" and a person says usually: "*Time* passed fast during the performance"; or, if another performance was almost static and uninteresting then the person would say: "*Time* passed slowly" in spite the performances lasted equally.

Every person accepts, at least unconsciously, a *time* scale that naturally corresponds to some mean speed, or to a representative speed, of a process. Or even more, a person can accept, can use, also unconsciously, different *time* units that naturally correspond to speeds of different biological, psychological, electrochemical and/or other physical processes in her/his body. This explains briefly biological and psychological personal feeling of *time* and of the *time* speed, which attracted much interest, [3], [29], [36], [45], [74], [109], [110], [172], [185], [198], [214], [235], [245], [258], [270], [271], [262], [346], [365], [385], [386], [397], [404], [406], [408], [416], [418] - [421], [463], [465] and [487].

Processes in different organs of higher developed beings propagate with different speeds. Their speeds are not and cannot be the *time* speed. We should not coincide speeds of various processes with the *time* speed.

However, their speeds determine natural *time* scales for the corresponding organs. Such beings are biological (in the widest sense including psychological) systems with multiple *time* scales. The speeds of the organic processes can change with age and/or with a variation of the biological state of the organ and/or of the whole organism. Consequently, the natural *time* scales (themselves) of the organs, hence of the organism, are *time*-varying in general (more about *time*-varying velocities/speeds and accelerations, as well as about *time*-varying coordinate, velocity/speed and acceleration transformations see [227], [228]).

Chapter 5

New Fundamentals

5.1 Physical variables, *time* and new principles

5.1.1 Introduction

*Beginning with the slowest motion, it will never acquire any degree of speed (*velocità*)† without first having passed through all the gradations of lesser speed - or should I say of greater slowness?*

....

*I did not say, nor dare I, that it was impossible for nature or for God to confer immediately that velocity which you speak of. I do indeed say that **de facto** nature does not do so - that the doing of this would be something outside the course of nature, and therefore miraculous.*

....

... I tell you that the movable body does pass through the said gradations, but without pausing in any of them. So that even if the passage requires but a single instant of time, still, since every small time contains infinite instants, we shall not lack a sufficiency of them to assign to each its own part of the infinite degrees of slowness, though the time be as short as you please.

Galileo GALILEI [191, pp. 20 - 22]

Comme un Corps ne peut occuper plusieurs lieux à la fois, il ne peut arriver d'un lieu à un autre dans le même instant : le mouvement ne peut donc se faire que durant un certain tems.

Jean D'ALEMBERT [83, p. 2: III]

The motion of a material particle which has continuous existence in time and space is the type and exemplar of every form of continuity.

James Clark MAXWELL [334, p. 18]

Physical variables can be scalar variables (e.g. density, pressure, speed, temperature) or vector variables (e.g. acceleration, velocity). However, we may formally consider also scalar physical variables as entries of a vector or of a matrix for mathematical reasons (e.g. in order to simplify a mathematical treatment of a problem).

A mathematical description of a system can contain scalar, vector and/or matrix variables. We will call all of them *system variables.* This enables us to present in brief the new physical principles in four forms: in scalar, vector, matrix and system form [227], [228].

5.1.2 Nonlinearities: continuity and discontinuity

Let us consider four different nonlinearities (nonlinear mappings) $y_i(.) : R_+ \to R_+$, $i = 1, 2, 3, 4$. The first one is single valued, well defined and mathematically continuous everywhere on R_+ except at $x = \alpha$, where it is discontinuous and double-valued, a) of Fig. 5.1:

$$y_1(x) \left\{ \begin{array}{c} = 0, \ x \in [0, \alpha] \\ = M, \ x \in [\alpha, \infty[. \end{array} \right\}$$

Let, for the sake of simplicity and clarity, the input variable x depend linearly on *time* t for $t \geq t_0 = 0$, b) of Fig. 5.1. The moment when $x(t)$ becomes equal to α is denoted by τ. The mathematical response $y_1[x(t)]$ of the nonlinearity is mathematically continuous, well defined and single-valued everywhere on $\mathfrak{T}_0$ except at the moment $t = \tau$ when it is discontinuous, c) of Fig. 5.1. However, the output variable y_1 of such a physically realized nonlinearity, if the exact realization were possible, should be a physical variable and would vary continuously in *time* since $t = 0$ on, d) of Fig. 5.1. The physical response would be continuous, well defined and single-valued since $t = 0$ on, d) of Fig. 5.1.

The second nonlinearity, Fig. 5.2, is single-valued everywhere, well defined and mathematically continuous everywhere on R_+ except at $x = \alpha$, where it is discontinuous, a) of Fig. 5.2:

$$y_2(x) \left\{ \begin{array}{c} = 0, \ x \in [0, \alpha] \\ = M, \ x \in]\alpha, \infty[. \end{array} \right\}$$

The input variable x depends linearly on *time* t for $t \geq 0$, b) of Fig. 5.2. The mathematical response $y_2[x(t)]$ of the nonlinearity is single-valued, well defined and mathematically continuous everywhere on $\mathfrak{T}_0$ except at the moment $t = \tau$ when it is discontinuous, c) of Fig. 5.2. However, the output variable y_2 of such a physically realized nonlinearity, if the exact realization were possible, would be a physical variable and would vary continuously in *time* since $t = 0$ on, d) of Fig. 5.2. The physical response would be single-valued, well defined and continuous since $t = 0$ on, d) of Fig. 5.2.

The third nonlinearity, Fig. 5.3, is single-valued, well defined and mathematically continuous everywhere on R_+ except at $x = \alpha$, where it is discontinuous

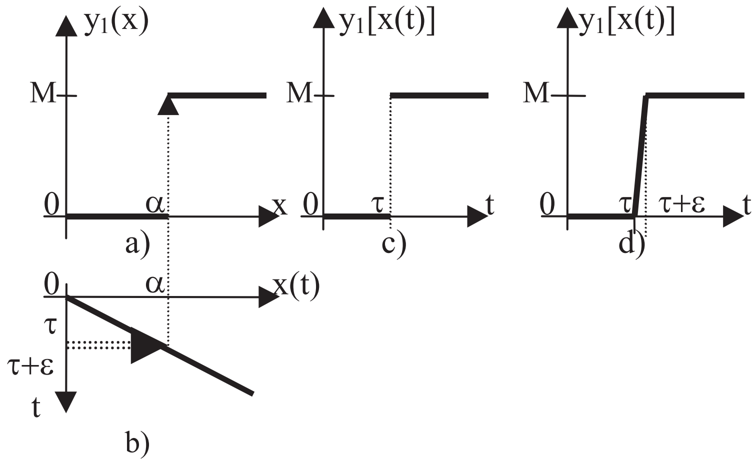

Figure 5.1: Mathematically discontinuous nonlinearity. It is well defined and single valued everywhere on R_+ except for $x = \alpha$ where it is double valued: 0 and M.

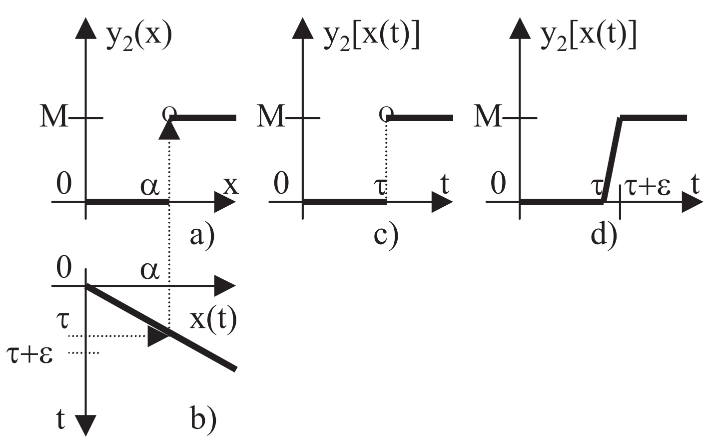

Figure 5.2: Mathematically discontinuous nonlinearity. It is well defined and single valued everywhere on R_+, and continuous on R_+ except for $x = \alpha$.

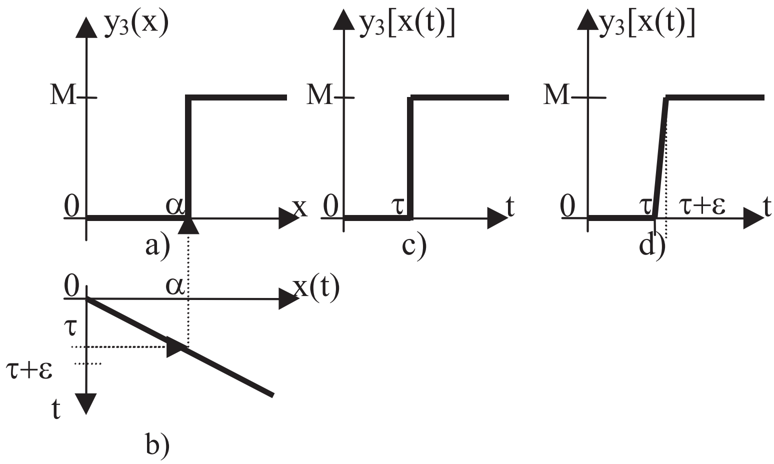

Figure 5.3: Mathematically discontinuous nonlinearity. It is well defined, single valued and continuous everywhere on R_+ except at $x = \alpha$ where it is discontinuous, multivalued and can take any value from the interval $[0, M]$.

and multi-valued, a) of Fig. 5.3:

$$y_3(x) \left\{ \begin{array}{c} = 0, \ x \in [0, \alpha[, \\ \in [0, M], \ x = \alpha, \\ = M, \ x \in]\alpha, \infty[. \end{array} \right\}$$

It is an asymmetric relay type nonlinearity. The input variable x depends linearly on *time* t for $t \geq 0$, b) of 5.3. The mathematical response $y_3[x(t)]$ of the nonlinearity is single-valued, well defined and mathematically continuous everywhere on $\mathfrak{T}_0$ except at the moment $t = \tau$ when it is discontinuous, c) of 5.3. However, the output variable y_3 of such a physically realized nonlinearity, if the realization were possible, should be a physical variable and would vary continuously in *time* since $t = 0$ on, d) of 5.3. The physical response would be single-valued, well defined and mathematically continuous since $t = 0$ on, d) of 5.3.

The fourth nonlinearity, Fig. 5.4, is single-valued, well defined and mathematically continuous everywhere on R_+, a) of Fig. 5.4:

$$y_4(x) \left\{ \begin{array}{c} = 0, \ x \in [0, \alpha], \\ = \frac{M}{\beta}(x - \alpha), \ x \in [\alpha, \beta] \\ = M, \ x \in]\beta, \infty[. \end{array} \right\}$$

It is an asymmetric saturation type nonlinearity with a linear part. The input variable x depends linearly on *time* t, b) of Fig. 5.4. The mathematical response $y_4[x(t)]$ of the nonlinearity is single-valued, well defined and mathematically continuous everywhere on $\mathfrak{T}_0$, c) of Fig. 5.4. The output variable y_4 of such

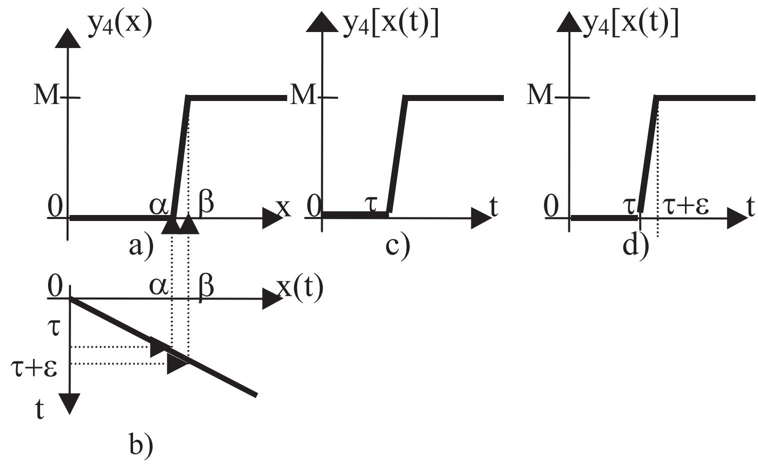

Figure 5.4: Mathematically continuous nonlinearity. It is well defined and continuous everywhere on R_+.

a physically realizable and realized nonlinearity, which is a physical variable, varies also continuously in *time* since $t = 0$ on, d) of Fig. 5.4. The physical response is also single-valued, well defined and mathematically continuous since $t = 0$ on, d) of Fig. 5.4.

This simple analysis illustrates an essential difference among the values of mathematical and physical variables from the point of view of their continuity and uniqueness. The former can be multi-valued, but the latter are always and everywhere single-valued. This illustrates also a crucial difference among variations of values of mathematical and of physical variables. The former can be discontinuous in *time*, but the latter are always and everywhere continuous in *time*.

This consideration results from the very nature of mathematical variables, which can be abstract, and physical variables that are real. Such properties of physical variables and strict monotonous continuous evolution of *time* value (always in the sense of increasing *time* values) led to the recently established principles that will be presented in the sequel (see [227], [228]). The preceding examples illustrate the principles.

5.1.3 Physical Continuity Principle (PCP)

Scalar form

Principle 95 ***Physical Continuity Principle (PCP): scalar form***

A scalar physical variable can change its value from one value to another one only by passing through every intermediate value.

Any scalar physical variable y_i (e.g. angle, current, density, position, speed, temperature, voltage) of the preceding examples can change its value from its physically realizable value M_1 to its physically realizable value M_2 only by passing through, by taking on, every value from the interval $[M_1, M_2]$. It cannot avoid, it cannot jump over, any value from this interval. It cannot take any value out of that interval during its variation only from the value M_1 to the value M_2, and vice versa.

If the physical variable y_i depends on another variable x_i, $y_i = f_i(x_i)$, then such a dependence is physically continuous. This means that for the mathematical description $f_i(.)$ of the dependence of y_i on x_i to be adequate to the real physical dependence it is *necessary* (but not sufficient) that the function $f_i(.)$ is continuous.

Matrix and vector form

A matrix (vector) variable, the entries of which are different scalar physical variables, is called, respectively, a *matrix (vector) of physical variables.* If all entries of a vector variable represent the same scalar physical variable then and only then it is *a vector physical variable* (see for details Definition 61 "Physical variable").

Principle 96 ***Physical Continuity Principle (PCP): matrix and vector form***

A vector physical variable or a matrix (vector) of physical variables can change, respectively, its value from one vector or matrix (vector) value to another one only by passing elementwise through every intermediate vector or matrix (vector) value.

If a vector physical variable $\mathbf{y}$ (e.g. acceleration, force, torque, velocity) or a matrix Y (or, a vector $\mathbf{w}$) of physical variables (e.g. composed of angle, current, density, energy, position, temperature and/or of voltage) depends on another variable x, respectively, $\mathbf{y} = \mathbf{f}(x)$ or $Y = F(x)$ (or, $\mathbf{w} = \mathbf{g}(x)$), then such a dependence is physically continuous. This means that for the mathematical description $\mathbf{f}(.)$ or $F(.)$ (or, $\mathbf{g}(.)$) of the dependence of $\mathbf{y}$ or of Y (or of $\mathbf{w}$) on x to be adequate to the real physical dependence it is *necessary* (but not sufficient) that the vector function $\mathbf{f}(.)$ or the matrix function $F(.)$ (or, the vector function $\mathbf{g}(.)$) is continuous.

System form

The term system physical variables denotes all physical scalar and vector variables, as well as all matrix (vector) variables, the elements of which are physical variables, and which are related to the system.

Principle 97 ***Physical Continuity Principle (PCP): system form***

The system physical variables can change, respectively, their (scalar or vector or matrix) values from one (scalar or vector or matrix) value to another one only by passing (elementwise) through every their intermediate (scalar or vector or matrix) values.

Notice that physical continuity of physical variables is not related to the *time* value evolution. Physical continuity is the property of every physical variable.

Corollary 98 ***Mathematical model of a physical variable, mathematical model of a physical system and PCP***

a) For a mathematical (scalar or vector) variable to be, respectively, an adequate description of a (scalar or vector) physical variable it is necessary that it obeys Physical Continuity Principle.

b) For a mathematical model of a physical system to be an adequate description of the physical system it is necessary that its system variables obey Physical Continuity Principle; that is that the mathematical model obeys Physical Continuity Principle.

This corollary means that the mathematical variables y_1 and y_2 from the preceding examples, Fig. 5.1 through Fig. 5.2, cannot be adequate representations of any physical variable. Consequently, the mathematical nonlinearities $y_1(x)$ and $y_2(x)$ are not strictly exactly physically realizable (this should not be mixed with their digital simulation on the display). The same holds for the signum nonlinearity defined by $y(x) = sign\ x = x\left|x\right|^{-1}$ for $x \neq 0$ and $sign$ $0 = 0$, as well as for the binary nonlinearity defined by $y(x) = -1$ for $x < 0$, and $y(x) = 1$ for $x \geq 0$. They do not permit variable value to pass through the intermediate values between -1 and 0, and between 0 and 1. However, the relay nonlinearity defined by $y_3(x) = rel\ x = sign\ x$ for $x \neq 0$, and $rel\ 0 \in [-1, 1]$, satisfies PCP. It permits variable value to pass through all the intermediate values between -1 and 1. Is it physically exactly realizable? Let us consider another property of physical variables.

5.1.4 Physical Uniqueness Principle (PUP)

Scalar form

Principle 99 ***Physical Uniqueness Principle (PUP): scalar form***

A scalar physical variable possesses a unique local instantaneous real value in any place (in any being or in any object) at any moment.

Any scalar physical variable y_i (e.g. angle, current, density, energy, position, speed, temperature, voltage) can take on exactly one instantaneous value anywhere. It cannot either be without a well determined value (the zero value is well determined value), or be with several different values in one place at the same moment.

For the relay nonlinearity, Fig. 5.3, to satisfy also PUP it is necessary and sufficient that its output variable y_3 takes exactly one value from the interval $[-1, 1]$ for $x = 0$ at any moment. However, it is mathematically permitted that it takes all values from the interval $[-1, 1]$ for $x = 0$. It does not satisfy PUP.

PUP implies that any physical variable cannot change instantaneously its value. In this regard see [355, p. 1].

Matrix and vector form

Principle 100 ***Physical Uniqueness Principle (PUP): matrix and vector form***

A vector physical variable or a matrix (vector) of physical variables possesses, respectively, a unique local instantaneous real vector or matrix (vector) value in any place (in any being or in any object) at any moment.

Any physical vector variable or a matrix (vector) of physical variables can take on exactly one instantaneous, respectively, vector or matrix (vector) value anywhere at any instant. It cannot either be without a well determined matrix (vector) value (the zero matrix and the zero vector are well determined matrix value and vector value), or be with several different matrix (vector) values in one place at the same moment.

Let a vector physical variable $\mathbf{y}$ (e.g. acceleration, force, torque, velocity) or a matrix Y (or, a vector $\mathbf{w}$) of physical variables (e.g. composed of angle, current, density, energy, position, temperature and/or of voltage) depend on another variable x, respectively, $\mathbf{y} = \mathbf{f}(x)$ or $Y = F(x)$ (or, $\mathbf{w} = \mathbf{g}(x)$). Such a dependence is everywhere uniquely defined, i.e. $\mathbf{y} = \mathbf{f}(x)$ or $Y = F(x)$ (or, $\mathbf{w} = \mathbf{g}(x)$) has a unique (scalar or matrix or vector) value for every x. This means that for the mathematical description $\mathbf{f}(.)$ or $F(.)$ (or, $\mathbf{g}(.)$) of the dependence of $\mathbf{y}$ or of Y (or of $\mathbf{w}$) on x to be adequate to the real physical dependence it is *necessary* that the vector function $\mathbf{f}(.)$ or the matrix function $F(.)$ (or, the vector function $\mathbf{g}(.)$) is everywhere well defined as single valued.

System form

Principle 101 ***Physical Uniqueness Principle (PUP): system form***

The system physical variables possess unique local instantaneous real (scalar or vector or matrix) values in any place (in any being or in any object) at every moment.

PUP is important for an adequate mathematical modeling of physical systems and physical processes.

Corollary 102 ***Mathematical model of a physical variable, mathematical model of a physical system and PUP***

a) For a mathematical (scalar or vector) variable to be, respectively, an adequate description of a physical (scalar or vector) variable it is necessary that it obeys Physical Uniqueness Principle.

b) For a mathematical model of a physical system to be an adequate description of a physical system it is necessary that its system variables obey Physical Uniqueness Principle; that is that the mathematical model obeys Physical Uniqueness Principle.

Like PCP, PUP is necessary, but not sufficient for an adequate mathematical modeling a physical system.

5.1.5 Physical Continuity and Uniqueness Principle (for short: PCUP)

Physical Continuity and Uniqueness Principle (for short: *PCUP*) is composed of Physical Continuity Principle (PCP) and Physical Uniqueness Principle (PUP). It has also several forms.

Scalar form

Principle 103 ***Physical Continuity and Uniqueness Principle (shortly: PCUP): scalar form***

A scalar physical variable can change its value from one value to another one only by passing through every intermediate value and it possesses a unique local instantaneous real value in any place (in any being or in any object) at any moment.

The asymmetric saturation nonlinearity with the linear part, Fig. 5.4, is the only one among the nonlinearities depicted in Fig. 5.1 through Fig. 5.4 which obeys PCUP.

Matrix and vector form

Principle 104 ***Physical Continuity and Uniqueness Principle (shortly: PCUP): matrix and vector form***

A vector physical variable or a matrix (vector) of physical variables can change, respectively, its vector or matrix (vector) value from one vector or matrix (vector) value to another one only by passing elementwise through every intermediate vector or matrix (vector) value and it possesses a unique local instantaneous real vector or matrix (vector) value in any place (in any being or in any object) at any moment.

Let a vector physical variable $\mathbf{y}$ (e.g. acceleration, force, torque, velocity), or a matrix Y (or, a vector $\mathbf{w}$) of physical variables (e.g. composed of angle, current, density, energy, position, temperature and/or of voltage), depend on another variable x, respectively, $\mathbf{y} = \mathbf{f}(x)$ or $Y = F(x)$ (or, $\mathbf{w} = \mathbf{g}(x)$). Such a dependence is everywhere uniquely defined, i.e. $\mathbf{y} = \mathbf{f}(x)$ or $Y = F(x)$ (or, $\mathbf{w} = \mathbf{g}(x)$) has a unique (scalar or matrix or vector) value for every x. Besides, they can change their values only by passing through all intermediate values. This means that for the mathematical description $\mathbf{f}(.)$ or $F(.)$ (or, $\mathbf{g}(.)$) of the dependence of $\mathbf{y}$ or of Y (or of $\mathbf{w}$) on x to be adequate to the real physical dependence it is *necessary* that the vector function $\mathbf{f}(.)$ or the matrix function $F(.)$ (or, the vector function $\mathbf{g}(.)$) is everywhere uniquely defined and that it can change its value only by passing (elementwise) through all intermediate (scalar or matrix or vector) values.

System form

Principle 105 ***Physical Continuity and Uniqueness Principle (shortly: PCUP): system form***

The system physical variables (including those their derivatives or integrals, which are also physical variables related to the system), can change, respectively, their (scalar or vector or matrix) values from one (scalar or vector or matrix) value to another one only by passing through every intermediate (scalar or vector or matrix) value, and they possess unique local instantaneous real (scalar or vector or matrix) values in any place at any moment.

PCUP appears important for an accurate modeling physical systems.

Corollary 106 ***Mathematical model of a physical variable, mathematical model of a physical system and PCUP***

a) For a mathematical (scalar or vector) variable to be, respectively, an adequate description of a physical (scalar or vector) variable it is necessary that it obeys Physical Continuity and Uniqueness Principle.

b) For a mathematical model of a physical system to be an adequate description of a physical system it is necessary that its system variables obey Physical Continuity and Uniqueness Principle; i.e. that the mathematical model obeys Physical Continuity and Uniqueness Principle.

From a mathematical modeling point of view, this poses the next question, i.e. opens the following mathematical problem left unsolved for future investigations:

Problem 107 ***Open mathematical problem***

What are necessary and sufficient conditions for a mathematical model to obey PCUP?

5.1.6 *Time* Continuity and Uniqueness Principle (shortly: TCUP)

Principles 103 through 105 and Axiom 47 imply *time* continuity of every physical variable:

Principle 108 ***Time Continuity and Uniqueness Principle (TCUP)***

Any (scalar or vector) physical variable (any vector / matrix of physical variables) can change, respectively, its (scalar / vector / matrix) value from one (scalar / vector / matrix) value to another one only continuously in time by passing (elementwise) through every intermediate (scalar / vector / matrix) value and it possesses a unique local instantaneous real (scalar / vector / matrix) value in any place (in any being or in any object) at any moment.

Definition 109 *The system form of TCUP means that all system variables satisfy TCUP.*

TCUP appeared very useful for stability study of nonlinear dynamical systems and for control synthesis for such systems [227], [228], [230].

Corollary 110 ***Mathematical representation of a physical variable and mathematical model of a physical system linked with TCUP***

a) For a mathematical (scalar or vector) variable to be, respectively, an adequate description of a physical (scalar or vector) variable it is necessary that it obeys Time Continuity and Uniqueness Principle.

b) For a mathematical model of a physical system to be an adequate description of the physical system it is necessary that its system variables obey Time Continuity and Uniqueness Principle; or equivalently, that the mathematical model obeys Time Continuity and Uniqueness Principle.

c) For a mathematical model of a physical system to be an adequate description of the physical system it is necessary that its solutions are unique and continuous in time.

This corollary shows that, in the control framework, we should look for such control algorithms that ensure uniqueness and *time* continuity to solutions of mathematical models of control systems.

The principles of physical continuity and of temporal continuity (*PCP, PCUP, TCUP*), and of physical uniqueness (*PUP, PCUP, TCUP*) of physical variables, and their corollaries, show the relationships between physical variables and pure mathematical variables, between physical systems and their mathematical models, between physics and mathematics. These relationships appear crucial for adequate mathematical modeling physical processes and systems.

5.2 Modelling and relativity principles

5.2.1 Modeling principles

Principle 111 ***Natural laws are independent of their modeling***

Natural phenomena, natural laws, natural relationships, hence the laws of physics, determine essentially their mathematical models, and are independent of their own modeling and models (of their mathematical descriptions, expressions).

The former govern the latter, but vice versa does not hold.

Principle 112 ***Choices of scales and units of coordinates are free***

The choices of scales and of units of coordinates of integral spaces cannot, therefore, do not, influence natural phenomena, natural laws, hence, they cannot influence laws of physics. They influence the forms of the resulting mathematical models of natural phenomena, of natural laws, hence of laws of physics.

Note 113 *Once, at the beginning, the scales and units were freely chosen. Afterwards they should be consistently used throughout the subsequent considerations and calculations.*

Principle 114 ***Selections of transformations of coordinates used for the modeling are free***

A new form of a transformed mathematical model cannot, hence does not, change the physical reality. It cannot, hence does not, change either natural laws or natural relationships, which govern natural phenomena and physical properties of physical processes and systems, and which are used for their mathematical modeling.

Principle 115 ***Transformations of coordinates and their inverses are different in general.***

The equality of a transformation and its inverse is a singular case that is trivial. They are different in general.

Note 116 *The time scaling coefficients and the space scaling coefficients should be a priory permitted mutually all different. This is not satisfied in Einsteinian relativity theory, in which the scaling coefficients are a priory accepted the same in pairs. This resulted in the equality of all the scaling coefficients. Einsteinian relativity theory does not fulfil this Principle. (See Section 7.3)*

Principle 117 ***The form of the transformed mathematical model can be different from the form of the original mathematical model.***

The form of the transformed mathematical model depends, not only on natural phenomena, natural laws and physical properties of processes and systems, but also on the applied transformations. What must be satisfied is the equivalence between the transformed and the original mathematical model.

A transformed mathematical model and an original mathematical model are equivalent if, and only if the application of the inverse transformation to the transformed mathematical model results into the original mathematical model.

Principle 118 ***Equivalence of mathematical models***

A transformed mathematical model is equivalent to the original mathematical model if, and only if, the application of the inverse transformation to the transformed mathematical model results in the original mathematical model.

5.2.2 Principle of relativity of values of variables

Principle 119 ***The (numerical) values of all variables should be consistently measured and consistently expressed relative to the accepted and fixed integral spaces.***

The values and the numerical values of all the variables should be consistently measured and consistently expressed with respect to the accepted and fixed scales and units of the relevant coordinates. This holds for both time and velocity (speed) including the time speed and the light velocity (speed). However, the time speed value is invariant with respect to all time axes. It equals one time unit per the same time unit whatever is the choice of the time unit.

Note 120 *Einsteinian relativity theory does not satisfy this principle (see Section 7.3). It accepts a priory the values of the light speed and of the spatial transfer speed to be invariant relative to all integral spaces with inertial frames. However, it adopts the value of the speed of the arbitrary point P to be dependent on the integral spaces.*

5.2.3 Principle of mathematical models relativity

Principle 121 ***Mathematical models are relative to integral spaces.***

Mathematical models (mathematical descriptions, expressions) of natural phenomena, of natural laws, of natural relationships, hence of laws of physics, are determined with respect to integral spaces whatever are the scales and unites of their coordinates.

Remark 122 *The light speed (value and numerical value) is (are) constant, but not invariant, in vacuum in general. By accepting tacitly, but a priory, the invariance of both the light speed and the spatial transfer speed, Einsteinian relativity theory violates both Principle of relativity of values of variables and Principle of mathematical models relativity in general.(See Section 7.3)*

Note 123 *Scales and units cannot, therefore do not, influence natural phenomena, natural laws, hence the laws of physics. The former can influence only mathematical models of the latter. Therefore, Principle of mathematical models relativity concerns mathematical descriptions of the laws of physics rather than the laws themselves. This is a refinement of Nottale's principle of the scale relativity (Claim 52). In this connection see Remark 53.*

5.3 *Time,* principles and dynamical systems

5.3.1 *Time* and motions of dynamical systems

A mathematical model of the internal dynamics of a dynamical physical system can have *Cauchy form (5.1)* (also called: *the normal form* and *the state form*),

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[t, \mathbf{x}(t)],\ dt > 0,\ \mathbf{x} \in R^n,\ \mathbf{f}(.) : \mathfrak{T}\text{x}R^n \to R^n. \tag{5.1}$$

An initial instant $t_0 \in \mathfrak{T}$ determines the unbounded *time* subset $\mathfrak{T}_0$ of the *time* set $\mathfrak{T}$, and the corresponding unbounded *time* subinterval T_0 of the *time* axis T, $\mathfrak{T}_0 = \{t : t \in \mathfrak{T},\ t \geq t_0\}$, $\mathfrak{T}_0 \subset \mathfrak{T}$, $T_0 = \{\sigma : \sigma \in T,\ \sigma = numt \geq numt_0,\ t \in \mathfrak{T}\}$, $T_0 \subset T$. A solution to *(*5.1*)*, which starts from an initial state $\mathbf{x}_0$ at the accepted initial moment $t_0 \in \mathfrak{T}$, is denoted by $\boldsymbol{\chi}(.; t_0, \mathbf{x}_0)$. Its instantaneous vector value $\boldsymbol{\chi}(t; t_0, \mathbf{x}_0)$ at a moment $t \in \mathfrak{T}_0$ represents the instantaneous state $\mathbf{x}(t)$ of the system at the same moment t, $\boldsymbol{\chi}(t; t_0, \mathbf{x}_0) \equiv \mathbf{x}(t)$, so that it obeys every initial condition: $\boldsymbol{\chi}(t_0; t_0, \mathbf{x}_0) \equiv \mathbf{x}(t_0) = \mathbf{x}_0$, and, by the definition of the system motion (solution), it satisfies identically the equation (5.1),

$$\frac{d\boldsymbol{\chi}(t; t_0, \mathbf{x}_0)}{dt} \equiv \mathbf{f}[t, \boldsymbol{\chi}(t; t_0, \mathbf{x}_0)],\ dt > 0. \tag{5.2}$$

For the mathematical model (5.1) to be an adequate description of the internal dynamics of the corresponding dynamical physical system it is necessary to obey the system form of Physical Continuity and Uniqueness Principle (Corollary 106), or equivalently, *Time* Continuity and Uniqueness Principle (Corollary 110). This permits the existence and uniqueness of the solutions of the model *(*5.1*)* that will be referred to as *the dynamical system (5.1).*

Theorem 124 ***Physical principles and dynamical system motions***
Let the dynamical system (5.1) be an adequate mathematical model of a physical system. Then, for the dynamical system (5.1) to have well defined, continuous, forward-time continuously differentiable and unique motions it is necessary (but not sufficient) that it satisfies the system form of the Physical Continuity and Uniqueness Principle, or equivalently, the system form of Time Continuity and Uniqueness Principle.

Proof. Let the dynamical system (5.1) be an adequate mathematical model of a physical system.

Necessity. Let motions of the dynamical system (5.1) exist and be well defined, continuous, forward-*time* continuously differentiable and unique. Then the system obeys the system form of both PCUP and TCUP in view of Principle 105, the Principle 108 and Definition 109. Q.E.D ■

This permits, but does not guarantee, to non-Lipschitzian dynamical systems, i.e. to dynamical systems of the form (5.1) with non-Lipschitzian $\mathbf{f}(.)$, to have unique motions that are (only) forward-*time* continuously differentiable. The following example illustrates this claim.

Example 125 *Let*

$$\frac{dx(t)}{dt} = -2k\sqrt{x(t)},\ \ (k>0)\in R,\ x\in R_+,\ dt>0.$$

The function $f(x)=k\sqrt{x}$ is non-Lipschitzian on any neighborhood of $x=0$ in R_+, (i.e. for $x\geq 0$). It obeys the PCUP on R_+. The system has the unique motions for $x_0\in R_+$,

$$\chi(t;t_0,x_0)\left\{\begin{array}{l} =\left(\begin{array}{c}\left[\sqrt{x_0}-k(t-t_0)\right]^2, \\ t\in\left[t_0,t_0+k^{-1}\sqrt{x_0}\right]\end{array}\right) \\ \\ =0,\quad t\in[t_0+k^{-1}\sqrt{x_0},\infty[\end{array}\right\},\ x_0\in R_+.$$

They are continuously forward-time differentiable.

A non-Lipschitzian mathematical system can have unique motions in spite it does not obey the PCUP, (which means that it is not physically realizable, i.e. it is not an adequate mathematical description of a physical system from the point of view of physics). This and the above theorem mean that the PCUP is not necessary condition for uniqueness of motions of pure mathematical systems (in spite it is a necessary condition for uniqueness of motions of physical systems, and in spite it is a necessary condition for a mathematical model to be adequate to the corresponding physical system). Let us illustrate this by the following example [227], [228].

Example 126 *The pure mathematical system*

$$\frac{dx(t)}{dt} = -ksignx(t), \ \ (k > 0) \in R, \ x \in R, \ dt > 0, \tag{5.3}$$

is non-Lipschitzian and it does not obey PCUP because the function sign(.) does not satisfy PCUP. Its value belongs to $\{-1, 0, 1\}$. *It changes its value from -1 to 0 and from 0 to 1 without passing through all intermediate values. This explains why it does not satisfy PCP, hence PCUP. However, the mathematical system (5.3) has the unique solutions defined by*

$$\chi(t;t_0,x_0)\left\{ \begin{array}{c} \left(\begin{array}{c} = [|x_0| - k(t-t_0)]\, signx_0, \\ t \in \left[t_0, t_0 + k^{-1}|x_0|\right] \end{array} \right) \\ = 0, \qquad t \in [t_0 + k^{-1}|x_0|, \infty[\end{array} \right\}, \\ \forall\, (x_0, t, t_0) \in R_+ x \mathfrak{T}_0 x \mathfrak{T}.$$

They are forward time differentiable. Moreover, they are forward-time continuously differentiable at every $(t \neq t_0 + k^{-1}|x_0|) \in T_0$. *But, their derivatives do not satisfy PCP, PCUP and TCUP because the derivatives are discontinuous at* $t = t_0 + k^{-1}|x_0|$. *The motions are not forward-time continuously differentiable at every* $t \in \mathfrak{T}_0$. *The mathematical system (5.3) is not an adequate mathematical description of a physical system, i.e. it is not exactly physically realizable.*

5.3.2 *Time* and dynamical systems with multiple *time* scales

.... any given dynamics will generate its own intrinsic time scale.
I. W. RICHARDSON and Robert ROSEN [409, p. 423]

There are physical systems in which processes flow with essentially different evolutionary speeds. Examples are control systems of thermal processes controlled by electronic controllers. The values of the speeds of thermal processes are much smaller than those of electrical processes in electronic controllers. Other examples are power systems. Their mechanical and electrical subsystems have essentially different speeds and durations of transient processes. In general, the same holds for all thermal processes and mechanical plants controlled by electrical and/or electronic controllers.

The existence of processes that flow with inherently different speeds implies their different natural *time* scales and *time* units. Our acceptance of different *time* scales and *time* units for different process means only that such scales and units correspond adequately to the speeds of the evolutions of the processes. It does not, and cannot, mean that the processes take place in different *times*. They occur in the same *time*.

The internal dynamics of a large class of such systems is mathematically modeled by (5.4) and (5.5), [227], [228],

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}_1[t, \mathbf{x}(t), \mathbf{y}(t), M],\ \mathbf{x} \in R^{n_x},\ \mathbf{y} \in R^s,\ M \in R^{sxs},\ dt > 0,$$
$$\mathbf{f}_1(.) : \mathfrak{T} \mathrm{x} R^{n_x} \mathrm{x} R^s \mathrm{x} R^{sxs} \rightarrow R^{n_x}, \tag{5.4}$$

$$M\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}_2[t, \mathbf{x}(t), \mathbf{y}(t), M],\ \mathbf{f}_2(.) : \mathfrak{T} \mathrm{x} R^{n_x} \mathrm{x} R^s \mathrm{x} R^{sxs} \rightarrow R^s,\ dt > 0. \tag{5.5}$$

Matrix $M = diag\{\mu_1\ \mu_2\ ...\ \mu_s\}$ contains (possibly, but not necessarily, small) *time* scaling factors (coefficients) $\mu_{(.)}$, $\mu_{(.)} \in R^+$, $R^+ =]0, \infty[$. They enable the introduction of s-different *time* scales and/or units defined by (4.38) (see Subsection: "*Time* scaling coefficients: definition" in 4.3.3).

The *time* scaling factors $\mu_{(.)}$ determine the temporal coordinate transformation (4.38). They enable us to link the relativity theory with the theory of dynamical systems with multiple *time* scales.

We will examine from the relativity theory point of view under what conditions the system can possess multiple *time* scales.

Comment 127 ***Time Continuity and Uniqueness Principle and uniqueness of system motions***

Let the dynamical system (5.1) satisfy the system form of Physical Continuity and Uniqueness Principle (PCUP), or equivalently, the system form of Time Continuity and Uniqueness Principle (TCUP), where the state acceleration is also included in the system variables. Axiom 47 and PCUP, or equivalently TCUP, guarantee the existence of motions of the system so that they and their first two derivatives obey the principles (Principle 105, Principle 108 and Definition 109). This implies the existence, continuity and twice forward-time continuous differentiability of the system motions due to $dt > 0$*. Let us verify their uniqueness. If they were not unique, then there would exist* $(t_0^*, \mathbf{x}_0^*) \in \mathfrak{T}xR^n$ *such that the system would have at least two different motions* $\boldsymbol{\chi}_1(.; t_0^*, \mathbf{x}_0^*)$ *and* $\boldsymbol{\chi}_2(.; t_0^*, \mathbf{x}_0^*)$ *passing through* $\mathbf{x}_0^*$ *at* t_0^**. They obey identically (5.1) by the definition of the system motions, (5.2):*

$$\frac{d\boldsymbol{\chi}_k(t; t_0^*, \mathbf{x}_0^*)}{dt} \equiv \mathbf{f}[t, \boldsymbol{\chi}_k(t; t_0^*, \mathbf{x}_0^*)],\ dt > 0,\ k = 1, 2.$$

Their initial speeds are equal since $\mathbf{f}(.)$ *obeys PCUP,*

$$\frac{d\boldsymbol{\chi}_k(t; t_0^*, \mathbf{x}_0^*)}{dt}\ |_{t=t_0} \equiv \mathbf{f}(t_0^*, \mathbf{x}_0^*),\ k = 1, 2.$$

Hence, in order for the motions $\boldsymbol{\chi}_1(.; t_0^*, \mathbf{x}_0^*)$ *and* $\boldsymbol{\chi}_2(.; t_0^*, \mathbf{x}_0^*)$ *to become different it is necessary that their forward derivatives become first different. Let* $\tau^* \in [t_0^*, \infty[$ *be the first moment when*

$$\frac{d\boldsymbol{\chi}_1(t; t_0^*, \mathbf{x}_0^*)}{dt}\ |_{t=\tau^*} \neq \frac{d\boldsymbol{\chi}_2(t; t_0^*, \mathbf{x}_0^*)}{dt}\ |_{t=\tau^*}\ . \tag{5.6a}$$

This means the double value of the state acceleration at the moment τ^**, which is impossible since it obeys PCUP, which disproves (5.6a). We will show this in another way. The definition of* τ^* *and continuity of the motions imply*

$$\boldsymbol{\chi}_1(\tau^*; t_0^*, \mathbf{x}_0^*) = \boldsymbol{\chi}_2(\tau^*; t_0^*, \mathbf{x}_0^*).$$

Hence

$$\mathbf{f}\left[\tau^*, \boldsymbol{\chi}_1(\tau^*; t_0^*, \mathbf{x}_0^*)\right] = \mathbf{f}\left[\tau^*, \boldsymbol{\chi}_2(\tau^*; t_0^*, \mathbf{x}_0^*)\right] \tag{5.7}$$

because the function $\mathbf{f}(.)$ *fulfills PCUP. The equations (5.7) and (5.1) imply*

$$\frac{d\boldsymbol{\chi}_1(t; t_0^*, \mathbf{x}_0^*)}{dt}\, |_{t=\tau^*} = \frac{d\boldsymbol{\chi}_2(t; t_0^*, \mathbf{x}_0^*)}{dt}\, |_{t=\tau^*}\ , \tag{5.8}$$

which contradicts (5.6a). Since (5.7) is correct then (5.6a) is incorrect, which is the consequence of the supposed existence of t_0^*, $\mathbf{x}_0^*$ *and* τ^**. Consequently, such* t_0^*, $\mathbf{x}_0^*$ *and* τ^* *do not exist, which implies*

$$\boldsymbol{\chi}_1(t; t_0, \mathbf{x}_0) \equiv \boldsymbol{\chi}_2(t; t_0, \mathbf{x}_0).$$

5.4 New fundamental theorems

The following new fundamental theorems prove the complete failure of Einstein's both postulates on *time* an on the light speed. They discover the physical nonsense of Einstein's postulates.

5.4.1 Fundamental theorem on *time* speed

We should clearly distinguish the velocity ($\mathbf{v}_t$ or $\mathbf{v}_\tau$) of the evolution of the *time* value t or τ (for short: *the time velocity*) from the velocity $\mathbf{v}_{(.)}$ of another variable value variation, of another motion and of another process. The time velocity $\mathbf{v}_t$ (or $\mathbf{v}_\tau$) has only one element - the *time* speed v_t (or v_τ),

$$\mathbf{v}_t = [v_t] \in \mathfrak{T},\ \mathbf{v}_\tau = [v_\tau] \in \mathfrak{T}.$$

They are temporal vectors. If we wished to consider $\mathbf{v}_t$ (or $\mathbf{v}_\tau$) in space then it spans uniformly the whole space. It is the same everywhere in the space. It is fully independent of space. We prove this claim in what follows.

The value (v_t or v_τ) of the velocity $\mathbf{v}_t$ or $\mathbf{v}_\tau$ is *the speed* v_t or v_τ of the evolution of the *time* value t or τ and of the numerical *time* value (for short: *the time speed*, or, *the temporal speed*).

There is a deep confusion in the literature on *time* in this connection. Speeds of variations of values of many variables, or of propagation of many processes, are mixed up with the *time* speed (v_t or v_τ). For example, the speed of the change of the biological state of an organism is referred to in certain literature as the *time* speed in spite it is not the *time* speed.

We are now going to show that the numerical value of both v_t and v_τ is the same and equals one ($num(v_t) = 1$ and $num(v_\tau) = 1$) in every *time* scale, hence in every integral space. It equals 1 (one) relative to every *time* unit,

$$\mathbf{v}_t = \frac{d(t\mathbf{t}_u)}{dt} = (1\mathbf{t}_u)\,[\mathrm{TT}^{-1}]\left\langle 1_t 1_t^{-1}\right\rangle, \ \mathbf{v}_\tau = \frac{d(\tau\boldsymbol{\tau}_u)}{d\tau} = (1\boldsymbol{\tau}_u)\,[\mathrm{TT}^{-1}]\left\langle 1_\tau 1_\tau^{-1}\right\rangle. \tag{5.9}$$

The following universal *time* speed law is the fundamental theorem on *time* speed:

Theorem 128 ***Universal time speed law***

Time is the unique physical variable such that the speed v_t (v_τ) of the evolution (i.e., of the flow) of the time values and of its numerical values:

a) is invariant relative to everybody and everything, hence invariant relative to a choice of an initial moment, of a time scale and of a time unit, i.e. invariant relative to a choice of a time axis, invariant relative to spatial coordinates,

and

b) its value (its numerical value) equals one arbitrary time unit per the same time unit (equals one), respectively,

$$v_t = 1[TT^{-1}]\left\langle 1_t 1_t^{-1}\right\rangle = 1[TT^{-1}]\left\langle 1_\tau 1_\tau^{-1}\right\rangle = v_\tau,\ num\,(v_t) = num\,(v_\tau) = 1, \tag{5.10}$$

relative to arbitrary time axes T and T_τ, i.e. its numerical value equals 1 with respect to all time axes (with respect to all initial instants, all time scales and all time units).

Proof. Let us show the proof that nobody and nothing can influence either *time* or *time* speed or *time* flow [227], [228], [231]. Let $v_{xt}(.)$ be the speed of an arbitrary physical variable $x(.)[\mathrm{X}]\left\langle 1_x\right\rangle$ measured relative to the *time* axis T,

$$v_{xt}(t;t_0) = \frac{dx(t;t_0)}{dt}[XT^{-1}]\left\langle 1_{x(t;t_0)}1_t^{-1}\right\rangle,$$

and $v_{x\tau}$ be its speed measured relative to a *time* axis T_τ,

$$v_{x\tau}(\tau;\tau_0) = \frac{dx(\tau;\tau_0)}{d\tau}[XT^{-1}]\left\langle 1_{x(\tau;\tau_0)}1_\tau^{-1}\right\rangle.$$

The initial moment ($t_0 \in \mathfrak{T}$ or $\tau_0 \in \mathfrak{T}_\tau$) is arbitrary and fixed.

Uniqueness and necessity. Let us accept at first for variable $x(.)[\mathrm{X}]\left\langle 1_{x(.)}\right\rangle$ to be different from *time*,

$$\dim x(.) = X \neq T = \dim t \text{ and } unit\,(x(.)) = 1_{x(.)} \neq 1_{(.)} = unit(t). \tag{5.11}$$

Let its value obey (5.10),

$$v_{xt}(t;t_0) = 1[XT^{-1}]\left\langle 1_{x(t;t_0)}1_t^{-1}\right\rangle = \ v_{x\tau}(\tau;\tau_0) = 1[XT^{-1}]\left\langle 1_{x(\tau;\tau_0)}1_\tau^{-1}\right\rangle. \tag{5.12}$$

Let us change *time* axis so that

$$t_2 - t_1 = \mu_{t\tau}(\tau_2 - \tau_1),\ dt = \mu_{t\tau} d\tau,$$

where

$$\mu_{t\tau} \in R^+,\ \mu_{t\tau}\left\langle 1_t 1_\tau^{-1}\right\rangle,$$

is different from one,

$$\mu_{t\tau} \neq 1, \tag{5.13}$$

due to the change of the *time* axis. Since the variable *x(.)* is not *time*, then we do not change its scale and unit so that

$$1_{x(t;t_0)} \equiv 1_{x(\tau;\tau_0)} \Longrightarrow \eta_{xt\tau} = 1\left\langle 1_{x(t;t_0)} 1_{x(\tau;\tau_0)}^{-1}\right\rangle. \tag{5.14}$$

Hence, the equations (5.12) imply

$$1\left\langle 1_x 1_t^{-1}\right\rangle \equiv v_{xt}(t;t_0)[XT^{-1}]\left\langle 1_x 1_t^{-1}\right\rangle = \frac{dx(t;t_0)}{dt}[XT^{-1}]\left\langle 1_{x(t;t_0)} 1_t^{-1}\right\rangle =$$
$$= \eta_{xt\tau}\left\langle 1_{x(t;t_0)} 1_{x(\tau;\tau_0)}^{-1}\right\rangle \mu_{t\tau}^{-1}\left\langle 1_t^{-1} 1_\tau\right\rangle \frac{dx(\tau;\tau_0)}{d\tau}[XT^{-1}]\left\langle 1_{x(\tau;\tau_0)} 1_\tau^{-1}\right\rangle =$$
$$= \mu_{t\tau}^{-1} v_{x\tau}(\tau;\tau_0)[XT^{-1}]\left\langle 1_x 1_\tau^{-1}\right\rangle \equiv 1\left\langle 1_x 1_\tau^{-1}\right\rangle. \tag{5.15}$$

Therefore, in order for (5.12), i.e. for (5.15),

$$1\left\langle 1_x 1_t^{-1}\right\rangle \equiv v_{xt}(t;t_0)\left\langle 1_{x(t;t_0)} 1_t^{-1}\right\rangle \equiv v_{x\tau}(\tau;\tau_0)\left\langle 1_{x(\tau;\tau_0)} 1_\tau^{-1}\right\rangle \equiv 1\left\langle 1_x 1_\tau^{-1}\right\rangle,$$

to hold it is necessary and sufficient that

$$1_t = 1_\tau.$$

This implies

$$\mu_{t\tau} = 1,$$

which results also from (5.15) and contradicts (5.13). The contradiction is a consequence of the assumption that the variable $x(.)[\mathrm{X}]\left\langle 1_{x(.)}\right\rangle$ is not *time.* Therefore, $x(.)[\mathrm{X}]\left\langle 1_{x(.)}\right\rangle$ is *time* $t[T]\left\langle 1_t\right\rangle$,

$$x(t;t_0)[X]\left\langle 1_{x(t;t_0)}\right\rangle \equiv (t - t_0)[T]\left\langle 1_t\right\rangle$$

and

$$x(\tau;\tau_0)[X]\left\langle 1_{x(\tau;\tau_0)}\right\rangle \equiv (\tau - \tau_0)[T]\left\langle 1_\tau\right\rangle.$$

Trivially, its unit changes if we change the *time* unit. Its value is measured with the *time* units 1_t and 1_τ. Its physical dimension is *time*. These conclusions and the last two equations prove necessity for the variable *x(.)* to be *time*.

Invariance and sufficiency. Let x be *time*. Let v_t be the *time* speed and its value be measured relative to the *time* axis T. Let v_τ be its speed, the value of which is measured relative to the *time* axis T_τ. We accept $1_t \neq 1_\tau$ so that

$T \neq T_\tau$. In view of the definition of the speed of any variable applied to the speed of *time*, we may write the following:

$$v_t = \frac{dt}{dt} = 1[TT^{-1}]\left\langle 1_t 1_t^{-1} \right\rangle, \; v_\tau = \frac{d\tau}{d\tau} = 1[TT^{-1}]\left\langle 1_\tau 1_\tau^{-1} \right\rangle \Longrightarrow v_t = v_\tau.$$

These equations complete the proof. Q. E. D ■

The equations (5.9) and (5.10) verify the claim that both the *time* value and the numerical *time* value are strictly monotonously increasing, equally in all spatial directions since the directions of the unity vectors $\mathbf{t}_u$ and $\boldsymbol{\tau}_u$, and of their extensions $\mathbf{t}_{ue}$ and $\boldsymbol{\tau}_{ue}$, hence, the directions of the *time* axes $T_t = T$ and T_τ, are arbitrary relative to space. The equations (5.9) and (5.10) show also that the *time* speed is independent of all beings, objects, processes, space and of all other variables.

Conclusion 129 ***Invariance of the time speed***

The speed of the (numerical) time value evolution [for short: ***the time speed****] is the same in all integral spaces. Its numerical value equals one. It is a universal constant. It is invariant relative to a choice of a time unit, of a time scale and of an integral space. It is independent of spatial coordinates, of movements of spatial frames and of space. It is universal invariant.*

The time speed is evidently one arbitrary time unit per the same time unit, e.g. one second per second, one minute per minute, one hour per hour, one day per day, one year per year, one decade per decade, one century per century, ...

The numerical value of the time speed equals one, independently of everybody and everything. It is simple, but universal, constant and invariant numerical speed value. There is not another variable with such a property of the speed of its value variation. The speed of light (propagation) does not possess such properties.

Note 130 *The invariance of the time speed (numerical) value implies time uniqueness. If there were several times (temporal variables) then the speeds of variations of their (numerical) values would be different, which is impossible due to Theorem 128. Since the time speed is invariant, then the initial moment and the zero moment can be freely chosen so that they can be mutually different. Their choices do not influence either time or its speed. The latter holds only for time. The former is the expression of the general rule that a choice of a variable scale and unit cannot influence either the variable itself or its value. For example, a choice of a temperature scale and temperature unit does not influence either temperature of a body or its value; a selection of pressure scale and unit does not have any impact either on gas pressure or on its value. Einstein's different "times" might represent, therefore, only the same time with possibly associated different initial moments and different zero moments, the values of which can be measured with different time units and in different time scales. Their speeds would be mutually equal and equal to 1 (one) time unit per the same time unit independently of the chosen time units.*

Conclusion 131 ***Independence and uniqueness of the time speed and of time***

Theorem 128 shows that the time speed is fully independent of everybody and everything. Nobody and nothing can influence either time or the speed of the flow of time values. Time and its speed are unique. There do not exist two or more times. There are not two or more different time speeds.

Mathematical proofs based on (various generalized) Lorentz transformations in [227, Section 23.1: *Time* Uniqueness, pp. 613-616], [228, Section 18.1: *Time* Uniqueness, pp. 571 - 574] verify Conclusion 129 and Conclusion 131. They verify the above conclusion and the results of Section 20.1.

Conclusion 132 ***Einstein's attitude on time and on its speed fails***

Einstein's attitude that there exist different time and different time speeds (e.g., [144, p.20], [150, pp. 24, 26, 27], [154, pp. 23-40] [167, p. 5], for the relevant citations of Einstein's attitude see Section 3.2) is wrong.

Conclusion 133 ***Newton's attitude is correct***

Newton's explanation of time [360, Scholium, p. 8] (for the relevant citation of Newton's explanation see Section 3.3) is substantially correct.

Note 134 *Uniqueness of time is the general, common, property of all physical variables. Pressure, temperature, voltage, ,.. are unique. The sets of their values are infinite.*

Comment 135 *Theorem 128 verifies mathematically common human physical experience on and with time. They led to the definition of time in [229, Definition 2, p. 3], [230, Definition 2, p. 3].*

Comment 136 *Since the value and the numerical value of time are smoothly monotonously continuously strictly increasing then the incremental increase of the time value, i.e. its differential dt, is strictly positive,*

$$dt > 0, \tag{5.16}$$

which is from the physical point of view only meaningful. However, if we consider t as a mathematical description of time (mathematical time in Newton's terminology) and then treat it as a mathematical variable only, then $dt \leq 0$ can be also accepted from a purely mathematical standpoint without having any physical sense. Such a mathematical property of the description of time is neither adequate to the nature of time nor physically justifiable. It is physically unacceptable.

In mathematical models of physical processes and of physical systems it is tacitly assumed that (5.16) holds.

Comment 137 ***Time speed and clock***

The fact (Theorem 128) that the time speed numerical value is invariantly equal to 1 (one) determines that the clock pointer should move (rotate) with *a constant positive speed. Since it is not possible to realize it with a pointer translational movement, then the periodic rotational movement is accepted with a constant angular speed. The human has understood this since the most ancient epoch.*

Conclusion 138 ***Failure of Einstein's fundamental postulate on time***

Einstein's postulate on the existence of various times, of various time speeds, on time dependence on space, on spatial coordinates, and that time is that what the clock hand indicates, is not only completely mathematically and physically wrong, but it is a physical nonsense.

5.4.2 Fundamental theorem on the light speed noninvariance

Let w denote an arbitrary nonzero speed and its value, $w \in R$, $w \neq 0$. The value w is accepted to be the reference speed value v_{SU} in the formula 12.36 (Corollary 474, Theorem 475) for the transformation of the velocity $\mathbf{v}_P(.)$ of the arbitrary point P from an integral space $I_j = T_j \text{x} R_j^n$ into an integral space $I_i = T_i \text{x} R_i^n$ (the sign plus: +), and vice versa (the sign minus: -), where the value of the speed of R_j^n relative to R^n is bigger than the value of the speed of R_i^n relative to R^n, i.e. $v \in R^+$,

$$\mathbf{v}_P(t_i;t_{i0}) = \frac{\mathbf{v}_P(t_j;t_{j0}) + \mathbf{v}}{1 + \frac{v v_P(t_j;t_{j0})}{(w^j)^2}}, \text{ and } \mathbf{v}_P(t_j;t_{j0}) = \frac{\mathbf{v}_P(t_i;t_{i0}) - \mathbf{v}}{1 - \frac{v v_P(t_i;t_{i0})}{(w^i)^2}}. \tag{5.17}$$

If we adopt the light speed (value) c to be a special (value for the) speed w, $w^j = w^i = c$, then the equations (5.17) reduce to Einstein's formulae for the transformation of velocities expressed by (7.43) and (7.45), i.e.

$$\mathbf{v}_P(t_i;t_{i0}) = \frac{\mathbf{v}_P(t_j;t_{j0}) + \mathbf{v}}{1 + \frac{v v_P(t_j;t_{j0})}{c^2}}, \text{ and } \mathbf{v}_P(t_j;t_{j0}) = \frac{\mathbf{v}_P(t_i;t_{i0}) - \mathbf{v}}{1 - \frac{v v_P(t_i;t_{i0})}{c^2}}. \tag{5.18}$$

Hence, the formulae (5.17) are more general then Einstein's (7.43) and (7.45), equivalently (5.18). The former incorporate the latter as a special case.

Theorem 139 ***The reference speed w is invariant, but not the light speed, in general*** *[226], [231]*

Let $w^{(\cdot)}$ be an arbitrary reference nonzero constant speed (value), $w^{(\cdot)} \in R$, $w^{(\cdot)} \neq 0$ in (5.17).

a) The reference speed $w^{(\cdot)}$ is invariant relative to all integral spaces over which the transformations (5.17) are true, which is the consequence of the formulae (5.17) themselves, and it is not a feature of the reference speed $w^{(\cdot)}$ itself.

b) The light speed $c_{(\cdot)}^{(\cdot)}$ is not invariant in general. For it to be invariant due to (5.17) it is necessary and sufficient to be the reference speed $w^{(\cdot)}$, $c_{(\cdot)}^{(\cdot)} = w^{(\cdot)} \equiv w$, i.e. it is necessary and sufficient for the formulae (5.17) to reduce to Einstein's *law of the composition of velocities (5.18). The corresponding linear coordinate transformations are Lorentz transformations (7.20) - (7.23).*

Proof. Notice that (5.17) results from (12.28) through (12.31) which imply (12.36). Let $w^{(\cdot)}$ be an arbitrary reference non-zero constant speed (value, $w^{(\cdot)} \in R$, $w^{(\cdot)} \neq 0$), in (5.17).

a) Let in the equations (5.17) the speed value $v_P(t_{(.)}; t_{(.)0})$ of the speed of the arbitrary point P be equal to $w^{(\cdot)}$, $v_P(t_{(.)}; t_{(.)0}) \equiv w^{(\cdot)}$, so that $\mathbf{v}_P(t_{(.)}; t_{(.)0}) \equiv w^{(\cdot)}\mathbf{u}$, $(.) = i, j$. The equations take the following forms:

$$w^i\mathbf{u} = \mathbf{v}_P(t_i; t_{i0}) = \frac{w^j\mathbf{u} + v\mathbf{u}}{1 + \frac{vw^j}{(w^j)^2}} = \frac{1 + \frac{v}{w^j}}{1 + \frac{v}{w^j}} w^j\mathbf{u} = w^j\mathbf{u} = \mathbf{v}_P(t_j; t_{j0}),$$

$$w^j\mathbf{u} = \mathbf{v}_P(t_j; t_{j0}) = \frac{w^i\mathbf{u} - \mathbf{v}}{1 - \frac{vw^i}{(w^i)^2}} = \frac{1 - \frac{v}{w^i}}{1 - \frac{v}{w^i}} w^i\mathbf{u} = w^i\mathbf{u} = \mathbf{v}_P(t_i; t_{i0}). \qquad (5.19)$$

These results prove the invariance of an arbitrary reference velocity, $\mathbf{w}^{(\cdot)} \equiv w^{(\cdot)}\mathbf{u}$. Since this holds for every non-zero reference speed w, then the invariance is not a property of a particular reference velocity $w^{(\cdot)}$. It is the property of the formulae (5.17).

b) Let the arbitrary point P move with the light velocity $\mathbf{c}_{(\cdot)}^{(\cdot)}$, $\mathbf{v}_P(t_{(.)}; t_{(.)0}) \equiv \mathbf{c}_{(\cdot)}^{(\cdot)} \equiv c_{(\cdot)}^{(\cdot)}\mathbf{u}$. The formulae (5.17) yield

$$\mathbf{v}_P(t_i; t_{i0}) \equiv \mathbf{c}_i^i = \frac{c_j^j\mathbf{u} + v\mathbf{u}}{1 + \frac{vc_j^j}{(w^j)^2}}, \ \mathbf{v}_P(t_j; t_{j0}) \equiv \mathbf{c}_j^j = \frac{c_i^i\mathbf{u} - \mathbf{v}}{1 - \frac{vc_i^i}{(w^i)^2}}.$$

For the light speed to be invariant it is necessary and sufficient that $c_j^j \equiv c_i^i$, i.e. it is necessary and sufficient that

$$\mathbf{v}_P(t_i; t_{i0}) \equiv \mathbf{c}_i^i = \frac{c_j^j \pm v}{1 \pm \frac{vc_j^j}{(w^j)^2}}\mathbf{u} \equiv c_j^j\mathbf{u} = \mathbf{c}_j^j \equiv \mathbf{v}_P(t_j; t_{j0}),$$

$$\mathbf{v}_P(t_j; t_{j0}) \equiv \mathbf{c}_j^j = \frac{c_i^i - v}{1 + \frac{vc_i^i}{(w^i)^2}}\mathbf{u} \equiv c_i^i\mathbf{u} = \mathbf{c}_i^i \equiv \mathbf{v}_P(t_i; t_{i0}).$$

For these identities to hold it is necessary and sufficient that

$$c_{(\cdot)}^{(\cdot)} = w^{(\cdot)} = w.$$

This result reduces the formulae (5.17) to Einstein's law of the composition of velocities (5.18) and $w^{(\cdot)} \equiv c$, i.e. the acceptance of the invariant light speed for the reference speed. For $w^{(\cdot)} \equiv c$ the transformation (12.28) through (12.31) become Lorentz transformations (7.20) - (7.23) Q. E. D ■

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Part II

Time Fields and Relativity

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Chapter 6

Time Fields and Transformations

6.1 *Time* field: definition and properties

Thus instead of regarding ourselves as, so to speak, swimming along in an ocean of space (as we usually do), we are to think of ourselves
rather as somehow pursuing a course in an ocean of time; ....
Alfred A. ROBB [417, p. 19]

Au fond, ne butons-nous pas sur la trop forte prégnance d'une représentation traditionnelle du temps qui, en définitive, l'assimile à l'espace ? Cet "axe du temps", c'est, après tout, une droite spatiale, dont nous ne questionnons même plus la pertinence, et que nous traçons sur ces désormais coutumiers diagrammes, horaires ferroviaires ou lignes d'univers einsteiniennes. Or ces schémas, conceptuellement spatio-temporels, sont matériellement spatio-spatiaux, dessinés qu'ils sont sur le plan du tableau ou de la feuille.
Jean-Marc LEVY-LEBLOND [291, pp. 280, 281]

6.1.1 *Time* axis, temporal environment and space

A geometrical representation of the *time* set $\mathfrak{T}$, or of $\mathfrak{T}_{(.)}$, by the corresponding *time* axis (denoted by T, or by $T_{(.)}$) requires that the nature of *time* is well expressed and preserved as much as possible. The *time* set $\mathfrak{T}$, or $\mathfrak{T}_{(.)}$, is totally ordered and everywhere dense set in view of Axiom 47 (Section 4.2 "Characterization of *Time*"). The temporal orientation (the temporal direction) and the temporal flow are its crucial and characteristic features.

We say "as much as possible" because it is not possible without a simulation to show exactly geometrically and graphically the temporal flow, i.e. the permanent strict continuous monotonous increase of the *time* value. Hence, the geometrical representation of *time* and of the *time* set $\mathfrak{T}$ is only partial. In order to avoid a great loss of information about the properties of *time*, a choice of the *time* axis T can be arbitrary under the condition that it expresses the independent nature of *time* and the temporal flow of *time* values. The first requirement can be achieved by accepting the *time* axis T to be orthogonal to space R^n, hence to be orthogonal to the spatial axis $R^{(n)}$, and to be the abscissa axis of an accepted coordinate system. The second requirement can be achieved descriptively by presenting the condition $dt > 0$, which has been accepted to hold without any exception in this book. Any axis that is a straight line abscissa and orthogonal to space R^n (or equivalently, to the spatial axis $R^{(n)}$) may be taken for the *time* axis T (or $T_{(.)}$).

All *time* axes $T_{(.)}$ with

1° the same zero instant $t_{(.)zero} = 0$,

2° the same *time* unit $1_{(.)}$,

3° with the same *time* scale,

4° which are orthogonal to space R^n, hence to the spatial axis $R^{(n)}$, i.e. which obey $\mathbf{t}_{u(.)e}^T \mathbf{u}_e = \mathbf{0}$,

and

5° which are abscissa axes in the corresponding coordinate systems,

coincide in the temporal sense. They characterize the unique *time* axis with such characteristics. They represent the same *time* axis in the temporal sense (relative to space). If at least one of the first four attributes is different for two *time* axes then, and only then, they are different (in the temporal sense). If an axis does not satisfy the fifth condition, then it cannot be an adequate *time* axis. Even if it obeys the first four conditions, it cannot then fulfill the requirement to express *time* independence of space. It should not be accepted for a *time* axis. We emphasize this because this opposes essentially Einsteinian relativity theory that uses the coordinate systems in which the *time* axis is the ordinate axis representing *time* as dependent variable of space.

If a single *time* axis T is chosen to hold at every point $\mathbf{x}$ in R^n then it *occupies* (*encloses, imbues, impregnates, is over and in, holds across and through, penetrates*) the whole R^n-space, Fig. 6.1. The chosen *time* axis T and the R^n-space determine then the unique integral space I, which we accept at this stage of the consideration.

A *time* axis valid at a point $\mathbf{x}$ at a moment t is $T_m(t;\mathbf{x})$, $m \in \{-,1,2,...\}$. A *time* axis is valid over a set $S \subseteq R^n$ at a moment $t \in T$ if and only if it is valid at every point $\mathbf{x} \in S$ at the moment t. It is then, and only then, denoted by $T_m(t;S)$.

The *time* axis $T_{(.)}$, an arbitrarily chosen and then fixed n-vector $\mathbf{a}$ and a real number κ, $\mathbf{a} \in R^n$, $\kappa \in R$, determine a $\mathbf{T}_{(.)}$ - hyperplane orthogonal to the spatial axis $R^{(n)}$, hence orthogonal to space R^n, Fig. 6.1 (Subsection "*Time* axes and space" in 4.6), as follows:

$$\mathbf{T}_{(.)}(\mathbf{a},\kappa) = \left\{T_j(t;\mathbf{x}) : \ T_j(t;\mathbf{x}) = T_{(.)}, \, \mathbf{x} \in R^n, \, \mathbf{a}^T\mathbf{x} = \kappa\right\}. \qquad (6.1)$$

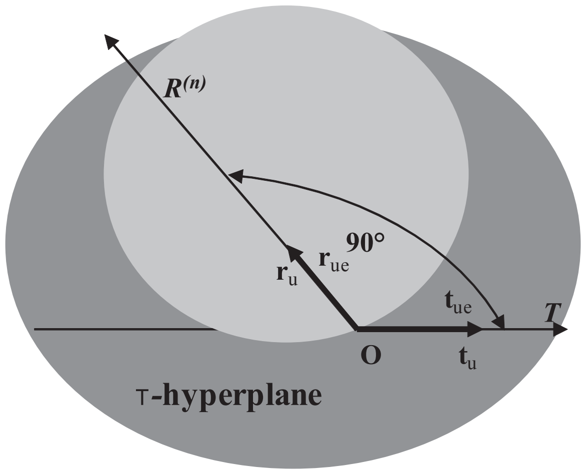

Figure 6.1: A geometrical representation of the *time* axis T and of the corresponding temporal hyperplane $\mathbf{T}$ relative to the R^n-space.

When both $\mathbf{a}$ and κ are known and fixed then, for short, $\mathbf{T}_{(.)}(\mathbf{a},\kappa) \equiv \mathbf{T}_{(.)}$. All *time* axes in the $\mathbf{T}_{(.)}$ - hyperplane are equal to the *time* axis $T_{(.)}$,

$$T_i \in \mathbf{T}_{(.)} \Longleftrightarrow T_i = T_{(.)}.$$

The equality of all the *time* axes in the $\mathbf{T}$-hyperplane is in the temporal sense, that is that they carry the same zero *time* point, the same *time* unit, the same *time* scale, they are orthogonal to R^n and to $R^{(n)}$, and they are abscissa axes in the corresponding coordinate systems. If the first four attributes hold for all *time* axes from two hyperplanes $\mathbf{T}_j$ and $\mathbf{T}_k$ then, and only then, they are equal in the temporal sense,

$$(T_i \in \mathbf{T}_j \iff T_i \in \mathbf{T}_k) \Longleftrightarrow \mathbf{T}_j = \mathbf{T}_k.$$

The T_j- *environment* (*the temporal environment* T_j) is the set of all $\mathbf{T}_i$-hyperplanes that are temporally equal to the $\mathbf{T}_j$-hyperplane,

$$\mathsf{T}_j = \{\mathbf{T}_i : \mathbf{T}_i = \mathbf{T}_j,\ i \in \{-, 1, 2, ...\}\},$$

or equivalently, the T_j-*environment* is composed of all *time* axes T_i that are temporally equal to the T_j-axis, Fig. 6.2,

$$\mathsf{T}_j = \{T_i :\ T_i = T_j,\ i \in \{-, 1, 2, ...\}\}.$$

The T_j-*environment* occupies (fulfills, imbues, impregnates, is over and in, is valid on and in, holds across and through, penetrates) the R^n-space.

We have tacitly assumed so far that every *time* axis that is valid at a point in R^n is simultaneously valid everywhere in R^n. However, there can be different

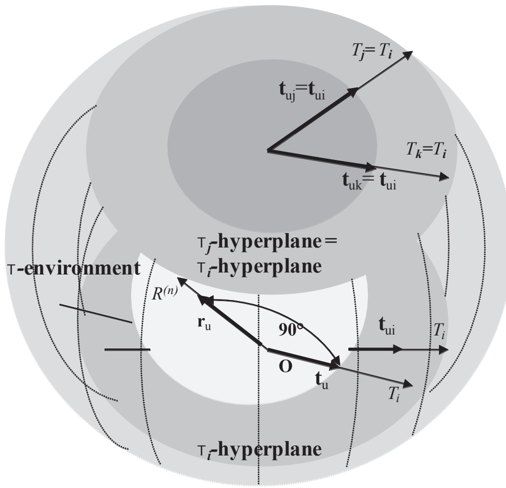

Figure 6.2: A symbolic geometrical representation of temporally equal *time* axes T_i, T_j and T_k, of the corresponding temporal hyperplanes $\mathbf{T}_i$ and $\mathbf{T}_j$, and of the *time* T-environment relative to the R^n-space.

subsets of the R^n-space, e.g. S_i^n and S_j^n, such that there are local temporal environments T_i and T_j, which occupy, respectively, S_i^n and S_j^n, Fig. 6.3, Fig. 6.4.

All the axes in the T_j-environment are mutually temporally equal, e.g. the *time* axes T_j and T_k are temporally equal, in Fig. 6.4. They belong to the T_j-environment, but not to the T_i-environment. The latter contains the *time* axis T_i that is not temporally equal to the *time* axes T_j and T_k. By following the above explanation of the temporal equality of all the *time* axes from the same $\mathsf{T}_{(.)}$-environment, we can represent symbolically the T_i-environment and the T_j-environment by arbitrary *time* axes T_i and T_j belonging to them, respectively, Fig. 6.4.

Since all the *time* axes belonging to the same temporal environment are temporally equal, then the whole $\mathsf{T}_{(.)}$-environment can be symbolically represented by a single *time* axis $T_{(.)}$, Fig. 6.5.

If there are several different temporal environments, then each of them can be symbolically represented by one *time* axis, Fig. 6.6.

The preceding analysis explains why the integral space $I = T\text{x}R^n$ has the dimension $1 + n$ in spite it is symbolically represented by only two (orthogonal) axes: by the *time* axis T as the abscissa axis and by the symbolic space axis $R^{(n)}$ as the ordinate axis, Fig. 6.5 and Fig. 6.6.

Possibility of an arbitrary choice of *time* axes $T_{(.)}$ in the corresponding T_i - environment permits us to select them in different $\mathsf{T}_{(.)}$-environments so to be all parallel, Fig. 6.7.

All parallel *time* axes can be represented by a single carrier *time* axis that carries the corresponding *time* scales and the corresponding *time* units of all

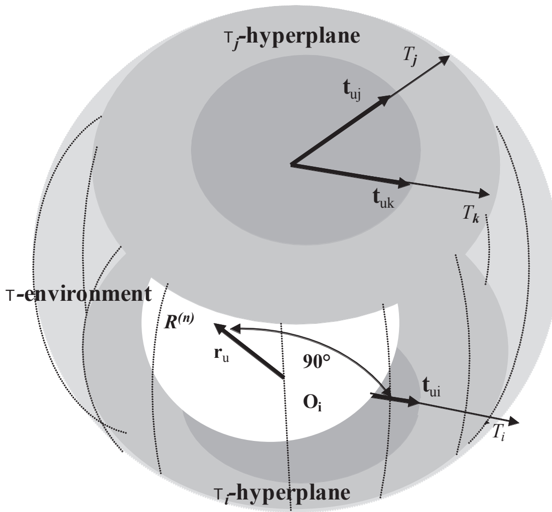

Figure 6.3: A symbolic geometrical representation of temporally equal *time* axes T_j and T_k, of the corresponding temporal hyperplanes $\mathbf{T}_j$ and $\mathbf{T}_k$, and of the *time* T-environment relative to the R^n-space. The *time* axis T_i is not temporally equal to the *time* axes T_j and T_k, and it does not belong to the T-environment.

parallel *time* axes (see Subsubsection "*Time* axis transformation" in 4.3).

6.1.2 Definition and properties of *time* fields

A fast electrical process, for which a *time* set $\mathfrak{T}_i$, equivalently *time* axis T_i, is adequate, can occur in a point $\mathbf{x}$ of a nonempty subset S of the R^n-space at a moment t, that is that

$$\mathfrak{T}_i, \ i.e.\ T_i\ , \ is\ valid\ in\ \mathbf{x} \in S\ at\ the\ moment\ t,$$

which is denoted by $\mathfrak{T}_i(t;\mathbf{x})$, i.e. by $T_i(t;\mathbf{x})$,

$$\mathfrak{T}_i = \mathfrak{T}_i(t;\mathbf{x}),\ T_i = T_i(t;\mathbf{x}).$$

Another *time* set $\mathfrak{T}_{(.)}$, i.e. *time* axis $T_{(.)}$, can occupy the whole subset S at the same moment t, that is that

$$\forall \mathbf{x} \in S,\ \mathfrak{T}_{(.)},\ \text{i.e. } T_{(.)},\ is\ valid\ in\ \mathbf{x}\ at\ the\ moment\ t,$$

which is denoted by $\mathfrak{T}_{(.)}(t;S)$, i.e. $T_{(.)}(t;S)$, or simply $\mathfrak{T}_{(.)}(t)$, i.e. $T_{(.)}(t)$, for the known and fixed S,

$$\mathfrak{T}_{(.)}(t) = \mathfrak{T}_{(.)}(t;S) \Longleftrightarrow \mathfrak{T}_{(.)}(t) = \mathfrak{T}_{(.)}(t;\mathbf{x}),\ \forall \mathbf{x} \in S,$$
$$T_{(.)}(t) = T_{(.)}(t;S) \Longleftrightarrow T_{(.)}(t) = T_{(.)}(t;\mathbf{x}),\ \forall \mathbf{x} \in S.$$

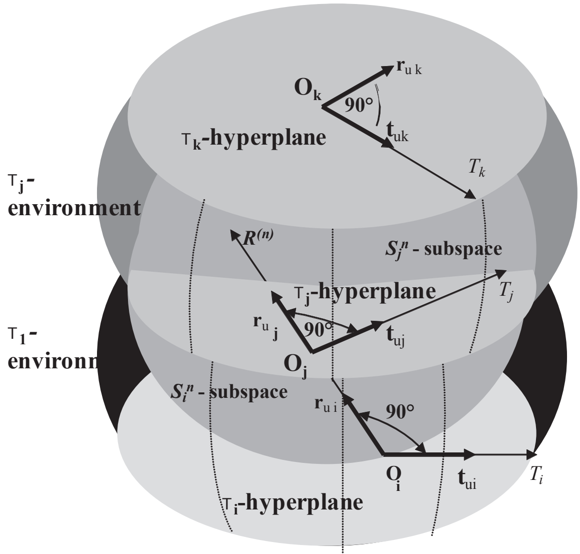

Figure 6.4: A symbolic geometrical representation of temporally equal *time* axes T_j and T_k of the corresponding temporal hyperplanes $\mathbf{T}_j$ and $\mathbf{T}_k$ in the T_j-temporal environment, and of *time* axis T_i of the temporal T_i-environment. A symbolic geometrical representation of the temporal environments relative to the S_i^n-subspace and the S_j^n-subspace, respectively. The *time* axis T_i is not temporally equal to the *time* axes T_j and T_k.

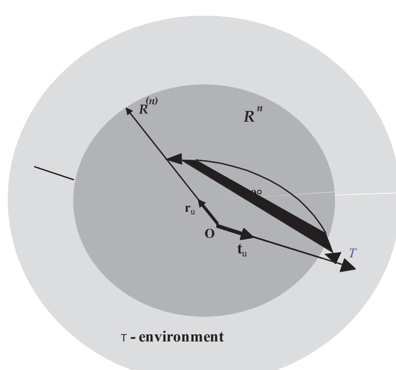

Figure 6.5: The whole temporal T-environment is symbolically represented by one *time* axis T.

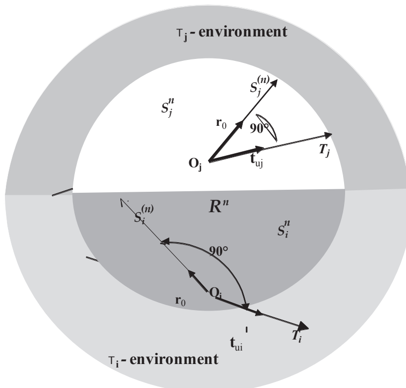

Figure 6.6: Every temporal environment is symbolically represented by one *time* axis.

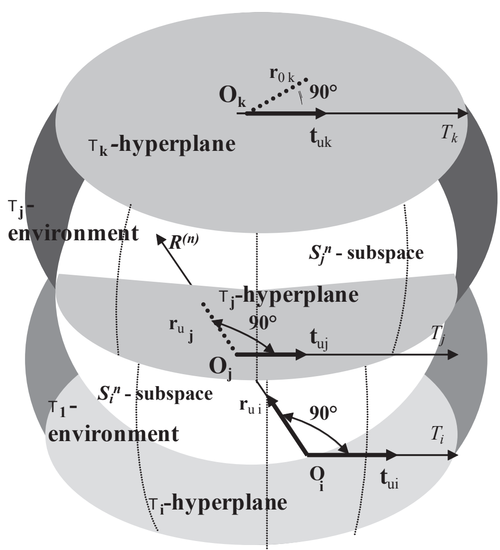

Figure 6.7: All *time* axes can be selected parallel.

Definition 140 *Time* ***(temporal) environment*** *[227], [228]*

$\mathsf{T}_{(.)}$, *which is* ***the*** *time* ***(temporal) environment of the set*** S ***at a moment*** t, ***is the set of all the*** *time* ***sets*** $\mathfrak{T}_i(t;S)$, *or* ***equivalently of all the*** *time* ***axes*** $T_i(t;S)$, ***which are equal, respectively, to*** $\mathfrak{T}_{(.)}(t;S)$, *or* ***to*** $T_{(.)}(t;S)$,

$$\mathsf{T}_{(.)}(t;S) = \left\{\mathfrak{T}_i(t;S) : \ \mathfrak{T}_i(t;S) = \mathfrak{T}_{(.)}(t;S),\ i \in \{-,1,2,...\}\right\},$$
$$\mathsf{T}_{(.)}(t;S) = \left\{T_i(t;S) : \ T_i(t;S) = T_{(.)}(t;S),\ i \in \{-,1,2,...\}\right\}.$$

Property 141 *Time* ***(temporal) environment*** *[227], [228]*

The time (temporal) environment of the set S *at a moment* t *occupies (encloses, imbues, impregnates, is over, is on and in, holds across and through, penetrates) the set* S *at the moment* t, $\mathsf{T}_{(.)}(t;S) \equiv \mathsf{T}_{(.)}(t)$.

In a special case, the set S *can be singleton,* $S = \{\mathbf{x}\}$*; i.e. it can represent a [known] single point* $\mathbf{x}$. *In such a case the* T*-environment* $\mathsf{T}_{(.)}(t;S)\mid_{S=\{\mathbf{x}\}} \equiv \mathsf{T}_{(.)}(t;\mathbf{x})$ *occupies the point* $\mathbf{x}$ *at the moment* t*, respectively,*

$$\mathsf{T}_{(.)}(t;S)\mid_{S=\{\mathbf{x}\}} = \mathsf{T}_{(.)}(t;\mathbf{x}).$$

A slow thermal process, for which a *time* axis T_j is appropriate, can occupy another subset S_k of the R^n-space at the same moment t. Then, the T_j-environment occupies the set S_k at the moment t, $\mathsf{T}_j(t) = \mathsf{T}_j(t;S_k)$.

These examples illustrate the possibility for the existence of different *time* scales over different subsets of the R^n-space at the same moment. Moreover, we use q different *time* scales T_k, $k = 1,2,...,q$, over the same set S at the same moment t. They determine q different T-environments T_k of S, $k = 1,2,...,q$,

$$\mathsf{T}_k(t;S) = \{\mathfrak{T}_j(t;S) : \ \mathfrak{T}_j(t;S) = \mathfrak{T}_k(t;S),\ j \in \{-,1,2,...\}\},\ k \in \{1,2,...,q\},$$

equivalently,

$$\mathsf{T}_k(t;S) = \{T_j(t;S) : \ T_j(t;S) = T_k(t;S),\ j \in \{-,1,2,...\}\},\ k \in \{1,2,...,q\}.$$

Every clock with several hands rotating with different angular speeds contains several different *time* scales and *time* units in the same place at every moment. A clock K fulfills the set S_K. Each of its *time* scales occupies S_K.

Let us observe two persons P_k and P_m. They occupy subsets S_k and S_m of the R^n-space, $n = 3$, at the same moment, or during the same *time* interval. We accept at first that the persons are at rest during that *time* interval. Speeds of their divers biological processes can be (very) different. It is a reason why each of them thinks of (and occasionally speaks about) "her / his own *time*". In fact, they accept different *time* scales corresponding to the speeds of their biological processes. Thinking of their own different *times*, they in fact think of their different *time* scales and different *time* units, and/or of different speeds of their biological processes. Their psycho-biological *clocks operate with different speeds* that induce *different time scales and different time units.* Therefore, different *time* axes T_{k_i}, $i = 1,2,...,n_k$, and T_{m_j}, $j = 1,2,...,n_m$, can be associated with the persons P_k and P_m, where n_k and n_m are the numbers of different *time* scales and units, hence of different *time* axes, of the persons P_k and P_m, respectively. Several different *time* (temporal) environments $\mathsf{T}_{l(.)}(t; S_l)$ occupy each of the persons,

$$\mathsf{T}_{l(.)}(t; S_l) = \mathsf{T}_{l(.)}(t; P_l),\ (.) = 1, 2, ..., n_l, l \in \{k, m\}.$$

This illustrates also the reason for the existence of different *time* units over different subsets of the R^n-space at the same moment, or during the same *time* interval. After the *time* interval has elapsed, the persons move by carrying their *time* scales and their *time* units with themselves.

These examples illustrate the following phenomena:

◦ Several different temporal environments can occupy the same subset of the R^n-space at the same moment, or during the same *time* intervals.

◦ A temporal environment can be (but need not be only one) over exactly one subset of the R^n-space at any moment, or during a *time* interval.

◦ A temporal environment can enclose different (possibly, but not necessarily, disjoint) subsets of the R^n-space at the same moment, or during the same *time* intervals.

◦ A temporal environment can be (but need not be) *time*-invariant. There are both *time*-invariant and *time*-varying temporal environments.

The collection of all temporal environments (of all T-environments), or equivalently, of all *time* axes, which are associated with all subsets S_i of the R^n-space, *occupies* (*covers, encloses, imbues, impregnates, is over and in, holds across and through, penetrates*) space R^n with a particular *temporal structure*. The temporal structure of space is composed of the *time* scales and of the *time* units at every moment. More precisely, it is composed of *time* axes.

The spatial distribution of *time* axes determines spatial domains of the validity of every *time* axis, i.e. of every temporal environment. This leads us to the notion of *the time field over a set* $\mathcal{A}$**,** $\mathcal{A} \subseteq R^n$**,** in a local sense, or *over the* R^n*-space* in the global sense. The empty set will be denoted by ϕ.

Definition 142 ***Time field***

The instantaneous temporal field, *for short:* ***the time field, of*** *(i.e.* ***over and in, across and through****)* ***a set*** $\mathcal{A}$**,** $\mathcal{A} \subseteq R^n$**, [*****of*** *(i.e.* ***over and in, across and through****)* ***the*** R^n***-space if and only if*** $\mathcal{A} = R^n$***] at a moment*** *t is a family of all instantaneous temporal environments* $\mathsf{T}_i(t; S; \mathcal{A})$ *[*$\mathsf{T}_i(t; S)$*] such that each of them occupies (covers, encloses, impregnates, is over and in, is across and through, penetrates) at least one nonempty subset S of the set* $\mathcal{A}$ *[of the* R^n*-space] at the moment t, respectively. The instantaneous time field over the set* $\mathcal{A}$ *[over the* R^n*-space] at the moment t is denoted, respectively, by*

$\mathbb{T}$*(t;$\mathcal{A}$) [*$\mathbb{T}(t;R^n)\equiv\mathbb{T}(t)$*],*

$$\mathbb{T}_{(.)}(t_{(.)};A)=\left\{\begin{array}{c}\mathsf{T}(t_{(.)},\mathbf{x}):\ \exists\,(S\neq\phi)\subseteq A,\ \exists k\in\{-,1,2,...\}\Longrightarrow\\ \exists\mathsf{T}_k(t_{(.)};S;\mathcal{A})\neq\phi\ \mathit{and}\ \mathsf{T}(t_{(.)},\mathbf{x})=\mathsf{T}_k(t_{(.)};S;\mathcal{A}),\ \forall\mathbf{x}\in S\end{array}\right\},$$

$$\mathbb{T}_{(.)}(t_{(.)};R^n)=\left\{\begin{array}{c}\mathsf{T}(t_{(.)},\mathbf{x}):\ \exists\,(S\neq\phi)\subseteq R^n,\ \exists k\in\{-,1,2,...\}\Longrightarrow\\ \exists\mathsf{T}_k(t_{(.)};S)\neq\phi\ \mathit{and}\ \mathsf{T}(t_{(.)},\mathbf{x})=\mathsf{T}_k(t_{(.)};S),\ \forall\mathbf{x}\in S\end{array}\right\}\equiv$$
$$\equiv\mathbb{T}_{(.)}(t_{(.)}).\tag{6.2}$$

The instantaneous *time* field of the position space $\mathfrak{R}^3$ at any moment expresses the instantaneous temporal structure of both the position space itself and of everybody and everything existing in the position space, relative to *time* axes at that moment.

Temporal equality of two *time* axes $T_i(t;S;\mathcal{A})$ and $T_j(t;S;\mathcal{A})$, $S\subseteq\mathcal{A}$, which belong to the same temporal environment $\mathsf{T}_m(t;S;\mathcal{A})$, $T_i(t;S;\mathcal{A})$ and $T_j(t;S;\mathcal{A})\in\mathsf{T}_m(t;S;\mathcal{A})$, and Definition 142, imply the following:

Proposition 143 *The instantaneous time field $\mathbb{T}(t;\mathcal{A})$ of a set $\mathcal{A}$, $\mathcal{A}\subseteq R^n$, [$\mathbb{T}(t)$ of the R^n-space] at a moment t is a family of all time-sets $\mathfrak{T}_i(t;S;\mathcal{A})$ [$\mathfrak{T}_i(t;S)$], i.e. of all T-axes $T_i=T_i(t;S;\mathcal{A})$ [$T_i=T_i(t;S)$], which occupy at least one nonempty subset S of the set $\mathcal{A}$ [of the R^n-space] at the moment t,*

$$\mathbb{T}(t;\mathcal{A})=\left\{\begin{array}{c}\mathfrak{T}(t,\mathbf{x}):\exists\,(S\neq\phi)\subseteq\mathcal{A},\ \exists i\in\{-,1,2,..\}\Longrightarrow\\ \exists\mathfrak{T}_i(t;S;\mathcal{A})\neq\phi\ \mathit{and}\ \mathfrak{T}(t,\mathbf{x})=\mathfrak{T}_i(t;S;\mathcal{A}),\ \forall\mathbf{x}\in S\end{array}\right\},$$
$$\mathbb{T}(t)=\left\{\begin{array}{c}\mathfrak{T}(t,\mathbf{x}):\exists\,(S\neq\phi)\subseteq R^n,\ \exists i\in\{-,1,2,..\}\Longrightarrow\\ \exists\mathfrak{T}_i(t;S)\neq\phi\ \mathit{and}\ \mathfrak{T}(t,\mathbf{x})=\mathfrak{T}_i(t;S),\ \forall\mathbf{x}\in S\end{array}\right\},$$

equivalently,

$$\mathbb{T}(t;\mathcal{A})=\left\{\begin{array}{c}T(t,\mathbf{x}):\exists\,(S\neq\phi)\subseteq\mathcal{A},\ \exists i\in\{-,1,2,..\}\Longrightarrow\\ \exists T_i(t;S;\mathcal{A})\neq\phi\ \mathit{and}\ T(t,\mathbf{x})=T_i(t;S;\mathcal{A}),\ \forall\mathbf{x}\in S\end{array}\right\},$$
$$\mathbb{T}(t)=\left\{\begin{array}{c}T(t,\mathbf{x}):\exists\,(S\neq\phi)\subseteq R^n,\ \exists i\in\{-,1,2,..\}\Longrightarrow\\ \exists T_i(t;S)\neq\phi\ \mathit{and}\ T(t,\mathbf{x})=T_i(t;S),\ \forall\mathbf{x}\in S\end{array}\right\}.$$

A *time* field $\mathbb{T}(t;\mathcal{A})$ over a set $\mathcal{A}$ is *time*-varying in general. If and only if the *time* field is *time*-invariant then

$$\mathbb{T}(t;\mathcal{A})\equiv\mathbb{T}(\mathcal{A}),\ \mathbb{T}(t;R^n)\equiv\mathbb{T}(R^n)=\mathbb{T}.\tag{6.3}$$

Notice that a movement of a *time* axis $T_{(.)}$ relative to the R^n-space does not mean its movement from one T- environment into another T- environment. For example, a person carries a watch with three different and fixed *time* scales and *time* units (hence, with three different but fixed *time* axes) from one place to another one without changing the *time* scales and the *time* units, hence, without changing the *time* axes. However, such a movement of the *time* axes can change, and usually does change, an instantaneous *time* field of a set $\mathcal{A}$.

Definition 144 ***Homogeneous and heterogeneous time fields***

a) The time field $\mathbb{T}(t)$ *is* ***spatially homogeneous*** *(for short:* ***homogeneous) at a moment t*** *if and only if there is a time set* $\mathfrak{T}$*, equivalently, a time axis* T*, which is unique over the whole* R^n*-space and over any nonempty subset of the* R^n*-space at the moment* t*:*

$$[\mathfrak{T}(t;S) = \mathfrak{T}(t;R^n) \ \ \forall\,(S \neq \phi) \subseteq R^n] \Longleftrightarrow \mathbb{T}(t) = \{\mathfrak{T}(t;R^n)\}\,,$$

or equivalently,

$$[T(t,S) = T(t,R^n) \ \ \forall\,(S \neq \phi) \subseteq R^n] \Longleftrightarrow \mathbb{T}(t) = \{T(t,R^n)\}\,.$$

Otherwise, the time field $\mathbb{T}(t)$ *is* ***spatially heterogeneous*** *(for short:* ***heterogeneous) at the moment t****.*

b) The time field $\mathbb{T}(t)$ *is* ***homogeneous (heterogeneous)*** *if and only if it is, respectively, homogeneous (heterogeneous) at every moment* $t \in \mathfrak{T}$*.*

A *time* field can be homogeneous at one instant and heterogeneous at another instant. This expresses the *time*-varying nature of *time* fields.

We may conclude that the *time* field $\mathbb{T}(t, \mathfrak{E}^3)$ over our energy-matter space $\mathfrak{E}^3$ is heterogeneous (heterogeneous at every moment $t \in \mathfrak{T}$).

Time fields can be single-layer or multi-layer in the following sense:

Definition 145 ***Multi-layer and single-layer time fields***

a) A time field $\mathbb{T}(t)$ *is* ***an instantaneous multi-layer time field at a moment t*** *if and only if for every partition of space* R^n *into disjoint nonempty subsets* S_i*,*

$$\cup S_i = R^n,\ S_i \cap S_k = \phi,\ \forall i \neq k,\ k = 1, 2, ...,$$

there is a nonempty set S_j*,* $S_j \subseteq R^n$*, at the moment* t *over which there are* k_j*,* $k_j = k(t, S_j) \in \{2, 3, ...\}$*, different temporal environments* $\mathsf{T}_{jm_j}(t, S_j)$*,* $m_j = 1$*, 2, ...,* k_j*, at the moment* t*.*

A time field $\mathbb{T}(t)$ *is* ***an instantaneous single-layer time field at a moment t*** *if, and only if there is such a partition of space* R^n *into disjoint nonempty subsets* S_i *that over every* S_i *there is a unique temporal environment* $\mathsf{T}_i(t, S_i)$*,* $\forall i = 1, 2, ...,$ *at the moment* t*.*

b) A time field $\mathbb{T}(t)$ *is* ***multi-layer (single-layer) time field*** *if and only if, respectively, it is the instantaneous multi-layer (single-layer) time field at every moment* $t \in \mathfrak{T}$*.*

Every homogeneous *time* field (at a moment t) is a single-layer *time* field (at the moment t), respectively.

A heterogeneous *time* field (at a moment t) can be a single-layer *time* field (at the moment t), or a multi-layer *time* field (at the moment t).

A single-layer *time* field can be a homogeneous *time* field (at a moment t) or a heterogeneous *time* field (at the moment t).

Every multi-layer *time* field (at a moment t) is a heterogeneous *time* field (at the moment t).

∘ A *time* field $\mathbb{T}(t)$ can be multi-layer at one moment t_1 (or, during a *time* interval $[t_1, t_{f1}[$) and single-layer at another moment t_2, $t_2 \neq t_1$, (or, during a *time* interval $[t_2, t_{f2}[$, $[t_1, t_{f1}[\cap[t_2, t_{f2}[= \phi$). This expresses *time*-varying nature of the *time* field $\mathbb{T}(t)$.

∘ A *time* field $\mathbb{T}(t)$ can be *partially multi-layer at a moment* t, or *completely multi-layer at a moment* t. In the former case, for every partition of space R^n into disjoint nonempty subsets S_i there are at least two nonempty subsets S_i and S_j of R^n, $S_k \subseteq R^n$, $k = i, j$, $S_i \neq S_j$, such that there is a unique temporal environment (i.e. a unique *time* set, a unique *time* axis) over S_i at the moment t, and there are at least two different temporal environments (i.e. two different *time* sets, two different *time* axes) over S_j at the same moment. In the latter case, for every partition of space R^n into disjoint nonempty subsets S_i there are at least two different temporal environments (i.e. two different *time* sets, two different *time* axes) over every subset S_i of R^n.

∘ A *time* field $\mathbb{T}(t)$ can be partially multi-layer at one moment t_1 (or, during a *time* interval $[t_1, t_{f1}[$) and completely multi-layer at another moment t_2, $t_2 \neq t_1$, (or, during a *time* interval $[t_2, t_{f2}[$, $[t_1, t_{f1}[\cap[t_2, t_{f2}[= \phi$). This expresses the *time*-varying nature of the *time* field $\mathbb{T}(t)$.

∘ A *time* field $\mathbb{T}(t)$ is *partially multi-layer* if and only if it is partially multi-layer at every moment $t \in T$.

∘ A *time* field $\mathbb{T}(t)$ is *completely multi-layer* if and only if it is completely multi-layer at every moment $t \in T$.

Time axes, temporal environments and *time* fields result from our conventions on the choice of the zero moment, the *time* scale and the *time* unit, i.e. they result from our choices of *time* axes. Therefore, they have a relative sense rather than an absolute sense. Evidently, they are **not** *time*. We use them to measure *time* value and the values of *time* intervals.

Time axes, temporal environments and *time* fields help us to model *time*, to represent some features of *time*, and to study phenomena, events, motions, processes, the existence of somebody or of something.

We can use mathematical descriptions, operations and relationships in order to establish links among different forms of *time* fields and/or to transform *time* fields from given forms into new forms. Any such transformation **does not and cannot** influence *time*, hence, it **does not and cannot** change *time* itself.

By saying "they occur in *time*" we express the physical fact that *time* (value) occupies (covers, encloses, imbues, impregnates, is over and in, holds across and through, penetrates) everybody, everything and space in the sense that its value "enters" temporally everybody and everything including space, "stays" temporally in everybody and everything including space, and "leaves" temporally everybody and everything including space; i.e. the *time* (value) "passes" temporally through everybody and everything, including space, permanently, instantaneously, independently, simultaneously, smoothly, strictly monotonously continuously increasing its values (itself), uniquely and equally in all spatial directions.

We will study heterogeneous *time* fields in what follows in order to investigate temporal coordinate transformations. We will treat them jointly with spatial coordinate transformations.

6.1.3 Temporal environment

A (probably fictitious) being living only in one dimensional space (along a straight line) could not sense the existence of other two spatial dimensions around itself. If it had a reach imagination, it could imagine two, or three, dimensional, or even an m-dimensional, space. The same holds for a being living in a two dimensional space (in a plane) relative to three and more dimensional spaces. The analogy would be true for us with respect to four and more dimensional spaces, if they really existed as the realization spaces.

We are aware of the existence of n mutually independent physical variables, which led to the introduction of the n-dimensional physical space. Starting with Galilei [191, Fig.11 on the page 199 in the English edition], as Strauss remarked according to Drake [102] [191, Drake's comment on p. 478 in English edition], if not earlier, the spatial coordinate systems have been extended by the *time* axis. Einstein and Minkowski enforced this by introducing the *time* coordinate as the fourth spatial coordinate to extend the three dimensional (position) space to the four dimensional space-*time* environment treated too formally, really wrongly, as the four dimensional position space.

In view of the essentially different nature of *time* from that of the position space, their product as an entire new position space is so artificial that it is physically unjustifiable and meaningless.

Is it reasonable and justifiable to speak of an s-dimensional *time* environment *(*space) rather than to think only of the one dimensional physical *time* environment *(*space) ?

This means essentially whether an s-dimensional *time* vector has any sense.

The fact is that there exist complex systems (e.g. control systems of mechanical plants, of power systems, of robots; factory, market, society, being, human), with multiple natural different *time* scales and units, i.e. with multiple different *time* axes that are mutually independent.

The multiple different *time* axes are induced by different speeds of evolutions of the processes in the complex systems. There are mutually independent s different *time* axes T_i, $i = 1, 2, 3, \ldots, s$, $s \in \{2, 3, \ldots\}$. Each of them is characterized by its unity *time* vector $\mathbf{t}_{ui}^s$, $\mathbf{t}_{ui}^s = (\delta_{i1}\ \delta_{i2}\ \ldots\ \delta_{ij}\ \ldots\ \delta_{is})^T \in R^s$, and by its $(n+s)$-dimensional extension $\mathbf{t}_{uie}^s = \begin{pmatrix}\mathbf{t}_{ui}^{s^T} & 0\ 0\ \ldots\ 0\end{pmatrix}^T \in R^{n+s}$, $i = 1, 2, 3, \ldots, s$, $s \in \{2, 3, \ldots\}$.

This inspires us to imagine an s-dimensional *temporal* (or, *time*) *space* $\mathfrak{T}_M^s$ spanned by s independent unity *time* vectors $\mathbf{t}_{ui}^s$, i.e. by s independent *time* axes T_i,

$$\mathfrak{T}_M^s = \{\mathbf{t}_M^s : \ \mathbf{t}_M^s = M\ \mathbf{t}_{\mathbf{1}}^s = tM\mathbf{1}^s,\ t \in \mathfrak{T}\},\ \mathfrak{T}_I^s = \mathfrak{T}^s,\ where$$
$$M = diag\{\mu_1\ \mu_2\ \ldots\ \mu_s\},\ \mathbf{t}_{\mathbf{1}}^s = t\mathbf{1}^s \in \mathfrak{T}^s,\ \mathbf{1}^s = (1\ 1\ \ldots\ 1)^T \in R^s, \qquad (6.4)$$

or equivalently,

$$\mathfrak{T}_M^s = \left\{ \mathbf{t}_M^s : \ \mathbf{t}_M^s = (\mu_1 t \ \mu_2 t \ . \ . \ . \ \mu_s t)^T, \ t \in \mathfrak{T} \right\}. \tag{6.5}$$

The *time* vector $\mathbf{t}_M^s$ was used in the framework of high performance control of dynamical systems (see [227], [228], [232], [233]).

The *time* space $\mathfrak{T}_M^s$ is the physical space of the *time* vector variable $\mathbf{t}_M^s$, but it is not a realization space. It is space originated and spanned by s different *time* axes. It is useful for vector matrix transformatins related to noncolinear velocities, hence for noncolinear motions [227], [228].

If, and only if the *time* axes are fixed (constant), then the corresponding *time* space is also fixed (constant, *time*-invariant). Otherwise it itself is *time*-varying.

The *time* space $\mathfrak{T}_M^s$ is not the *time* field $\mathbb{T}(A)$, $\mathfrak{T}_M^s \neq \mathbb{T}(A)$, $A \subseteq R^n$. The former is spanned by s mutually different *time* axes, while the latter is a collection (a family) of *time* axes over a subset A of R^n.

6.2 *Time* fields. Generic transformations

6.2.1 Speed of a generic point G

G denotes **the generic point**. It is the point the data of which appear in the *time* coordinate transformations. It can be **the arbitrary point** P that is not fixed, or a **fixed point** (e.g., **the light signal** L). The generic point G is the arbitrary point P in Lorentz transformations (7.20) through (7.23).

The reference speeds denoted by q and w in general are the speeds, the values (or the squared value) of which exist in the *time* coordinate transformations. The reference speed is single in Lorentz transformations (7.20) through (7.23): $q = w$. It is the light speed c in Lorentz transformations (7.20) through (7.23): $q = w = c$.

If velocity $\mathbf{v}_G(t_{(.)}; t_{(.)0})$ of a freely chosen and then fixed *generic point* G is *time*-varying then it is not sufficient to determine uniquely the position vector $\mathbf{r}_G(t_{(.)}; t_{(.)0})$ of the point G even if we know its initial position vector $\mathbf{r}_{G(.)0}$,

$$\mathbf{v}_G(t_{(.)}; t_{(.)0}) \neq \mathbf{const.} \Longrightarrow \mathbf{r}_G(t_{(.)}; t_{(.)0}) \neq \mathbf{r}_{G(.)0} + \mathbf{v}_G(t_{(.)}; t_{(.)0})(t_{(.)} - t_{(.)0}). \tag{6.6}$$

Without losing in generality and for the sake of the simplicity we accept for the initial position vector $\mathbf{r}_{G(.)0}$ to be the zero vector if not stated otherwise,

$$\mathbf{r}_{G(.)0} \equiv \mathbf{r}_G^{O_{(.)}}(t_{(.)0}; t_{(.)0}) \equiv \mathbf{0}, \ G \in \{L, P, P_R, P_{SU}\}.$$

The generic point G can be either

- ◦ a light signal L, or
- ◦ an arbitrary point P (which represents an arbitrary material point, i.e. an arbitrary particle, or an arbitrary object, or an arbitrary being, or an arbitrary human), or
- ◦ a freely selected and then fixed *reference point* P_R or P_{SU}.

If the velocity $\mathbf{v}_G(t_{(.)};t_{(.)0})$ of the generic point G relative to $R^n_{(.)}$ and to its origin $O_{(.)}$ is *time*-varying then we will use its *instantaneous* [*at a moment* $t_{(.)}$] *average vector value* $\widetilde{\mathbf{v}}_G(t_{(.)};t_{(.)0})$, and its *instantaneous average algebraic value* $\widetilde{v}_G(t_{(.)};t_{(.)0})$, which is called *the instantaneous average speed,* of the generic point G, *over the time interval* $[t_{(.)0},t_{(.)}]$ *at a moment* $t_{(.)}$. They are defined by

$$\widetilde{\mathbf{v}}_G(t_{(.)};t_{(.)0}) = \left\{ \begin{array}{c} \mathbf{v}_G(t_{(.)0};t_{(.)0}) = \mathbf{v}_{G(.)0},\ t_{(.)} = t_{(.)0}, \\ \frac{\mathbf{r}_G(t_{(.)};t_{(.)0})-\mathbf{r}_{G(.)0}}{t_{(.)}-t_{(.)0}} \,|_{\mathbf{r}_{G(.)0}=\,\mathbf{0}},\ t_{(.)} \in]t_{(.)0},\infty[\end{array} \right\}, \tag{6.7}$$

or equivalently by,

$$\widetilde{\mathbf{v}}_G(t_{(.)};t_{(.)0}) = \left\{ \begin{array}{c} \mathbf{v}_G(t_{(.)0};t_{(.)0}) = \mathbf{v}_{G(.)0},\ t_{(.)} = t_{(.)0}, \\ \frac{1}{t_{(.)}-t_{(.)0}} \int_{t_{(.)0}}^{t_{(.)}} \mathbf{v}_G(\sigma;t_{(.)0})d\sigma,\ t_{(.)} \in]t_{(.)0},\infty[\end{array} \right\}, \tag{6.8}$$

$$\widetilde{\mathbf{v}}_G(t_{(.)};t_{(.)0}) \equiv \widetilde{\mathbf{v}}_{Gt}^{(.)} \equiv \widetilde{v}_G(t_{(.)};t_{(.)0})\mathbf{u} \equiv \widetilde{v}_{Gt}\mathbf{u} \ \textit{if}\ \widetilde{\mathbf{v}}_G(t_{(.)};t_{(.)0}) \textit{ and } \mathbf{u} \textit{ are colinear}. \tag{6.9}$$

The use of the average velocity (i.e., average speed) in the framework of *time*-varying fields and transformations enables us to extend to them all results obtained for *time*-invariant fields and transformations [227], [227].

The average velocity $\widetilde{\mathbf{v}}_G(t_{(.)};t_{(.)0})$ is constant if, and only if, the instantaneous velocity $\mathbf{v}_G(t_{(.)};t_{(.)0})$ is constant,

$$\mathbf{v}_G(t_{(.)};t_{(.)0}) \equiv \widetilde{\mathbf{v}}_G(t_{(.)};t_{(.)0}) \equiv \mathbf{v}_G^{(.)} = \mathbf{const.},\ G \in \{L, P, P_R, P_{SU}\}. \tag{6.10}$$

This greatly simplifies (6.6) and (6.7) in the obvious manner.

We will treat only the cases with *time*-invariant (constant) velocity of the generic point G, which is expressed by (6.10), $G \in \{L, P, P_R, P_{SU}\}$. This enables us to represent the position vector $\mathbf{r}_G(t_{(.)};t_{(.)0})$ as the homogeneous linear function of $(t_{(.)} - t_{(.)0})$ with *time*-invariant vector gain $\mathbf{v}_G$ that is the velocity of the generic point G,

$$\mathbf{r}_G(t_{(.)};t_{(.)0}) = \mathbf{v}_G(t_{(.)} - t_{(.)0}),\ G \in \{L, P, P_R, P_{SU}\}. \tag{6.11}$$

We will use (6.10) and (6.11) in order to determine the scaling coefficient functions in the transformations of the temporal coordinates and of the spatial coordinates.

6.2.2 *Time*, velocity and generic transformations

There are in reality several different *time* sets, i.e. several different *time* axes, and several different *time* environments, over subspaces of our energy-matter space $\mathfrak{E}^3$, and over subspaces of space R^n in general. The real *time* field is heterogeneous and multi-layer. It contains s different *time* scales, which induce s different *time* hyperplanes and s different *time* environments. Their number s can vary in *time*. Constant value of s will be assumed in the sequel for the sake of the simplicity.

In what follows we will consider generic forms of the *time*-invariant coordinate transformations in the integral space with a heterogeneous *time* field containing s different *time* axes and *time* environments. The results will be valid for single-layer heterogeneous *time* fields and for multi-layer (hence, also heterogeneous) *time* fields. *Time*-invariant *time* fields are the topic of this book. *Time*-varying *time* fields are studied in the books [227], [228]. The former are linked with *time*-invariant coordinate transformations, and the latter are associated with *time*-varying coordinate transformations.

The *time* scaling coefficient function $\mu_i(.)$ depends in general on the position ρ_{Gt} of the generic point G at a moment t,

$$\mu_i(.) : R^n \rightarrow R^+,\ t_i - t_{i0} = \mu_i(t, \mathbf{r}_{Gt})(t - t_0),\ t_{i0} = \mu_i(\rho_{G0})t_0. \tag{6.12}$$

The *time* scaling coefficient functions $\alpha_j^i(.)$ and $\alpha_i^j(.)$, $\alpha_j^i(.)$ and $\alpha_i^j(.)$: $R^n \text{x} R^n \text{x} R_+ \text{x} R \text{x} R^+ \text{x} R^+ \rightarrow R^+$, and space scaling coefficient functions $\lambda_j^i(.)$ and $\lambda_i^j(.)$, $\lambda_j^i(.)$ and $\lambda_i^j(.)$: $R^n \text{x} R^n \text{x} R_+ \text{x} R \text{x} R^+ \text{x} R^+ \rightarrow R^+$, are in general real valued (scaling coefficient) functions of

○ of the position vector $\mathbf{r}_{Gt}^{(.)}$ at a moment t of the generic point G with respect to the origin $O_{(.)}$ of $R_{(.)}^n$,

○ of the velocity $\mathbf{v}_G^{(.)}$, i.e. of the speed $v_G^{(.)}$, of the generic point G with respect to the origin $O_{(.)}$ of $R_{(.)}^n$,

○ of an arbitrarily accepted and then fixed generic, or reference, positive *time*-invariant (constant) velocities $\mathbf{q}^{(.)}$ and $\mathbf{w}^{(.)}$, i.e., speeds $q^{(.)}$ and $w^{(.)}$,

$$\mathbf{q}^{(.)} \equiv q^{(.)}\mathbf{u},\ \mathbf{w}^{(.)} \equiv w^{(.)}\mathbf{u},$$

and

○ of the *time*-invariant (constant) velocity $\mathbf{v}_P^{(.)}$ i.e., of the speed $v_P^{(.)}$, of the arbitrary point P with respect to the origin $O_{(.)}$ of $R_{(.)}^n$ when the value of *time* is measured with the *time* unit $1_{t_{(.)}}$ of the *time* axis $T_{(.)}$ and the value of length (of position, of distance) is measured with the length unit $1_{L_{(.)}}$ of $R_{(.)}^n$.

The values of the *time* and space scaling coefficient functions are determined also by

○ an arbitrarily accepted and then fixed constant temporal transfer generic velocity $\boldsymbol{\vartheta}^{(.)}$, i.e. by its speed value $\vartheta^{(.)}$, $\vartheta^{(.)} \in R_+$,

and by

○ an arbitrarily accepted and then fixed constant spatial transfer velocity $\mathbf{v}_{ji}^{(.)}$, i.e. by its speed value $v_{ji}^{(.)} \in R_+$, which is the value of the speed of the origin O_j of R_j^n with respect to the origin O_i of R_i^n.

The following equations, (6.13) through (6.17), represent **the basic general generic scalar forms** of the *time*-invariant coordinate transformations:

$$(t_i - t_{i0}) = \alpha_j^i\left(\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^j, v_G^j, q^j, w^j\right)\left[(t_j - t_{j0}) + \frac{\vartheta^j}{qw} r_G(t_j; t_{j0})\right],$$

$$\alpha_j^i(\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^j, v_G^j, q^j, w^j) \in R^+,$$

$$\forall(\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^j, v_G^j, q^j, w^j) \in R^n \text{x} R^n \text{x} R_+ \text{x} R \text{x} R^+ \text{x} R^+, \tag{6.13}$$

$$(t_j - t_{j0}) = \alpha_i^j \left(\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^i, v_G^i, q^i, w^i\right) \left[(t_i - t_{i0}) - \frac{\vartheta^i}{qw} r_G(t_i; t_{i0})\right],$$
$$\alpha_i^j (\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^i, v_G^i, q^i, w^i) \in R^+,$$
$$\forall (\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^i, v_G^i, q^i, w^i) \in R^n \mathsf{x} R^n \mathsf{x} R_+ \mathsf{x} R \mathsf{x} R^+ \mathsf{x} R^+, \tag{6.14}$$

$$\mathbf{r}_P(t_i; t_{i0}) = \lambda_j^i \left(\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^j, v_G^j, q^j, w^j\right) \left[\mathbf{r}_P(t_j; t_{j0}) + v_{ji}^j (t_j - t_{j0}) \mathbf{u}\right],$$
$$\lambda_j^i \left(\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^j, v_G^j, q^j, w^j\right) \in R^+,$$
$$\forall (\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^j, v_G^j, q^j, w^j) \in R^n \mathsf{x} R^n \mathsf{x} R_+ \mathsf{x} R \mathsf{x} R^+ \mathsf{x} R^+, \tag{6.15}$$

$$\mathbf{r}_P(t_j; t_{j0}) = \lambda_i^j \left(\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^i, v_G^i, q^i, w^i\right) \left[\mathbf{r}_P(t_i; t_{i0}) - v_{ji}^i (t_i - t_{i0}) \mathbf{u}\right],$$
$$\lambda_i^j (\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^i, v_G^i, q^i, w^i) \in R^+,$$
$$\forall (\mathbf{r}_G^i, \mathbf{r}_G^j, \vartheta^i, v_G^i, q^i, w^i) \in R^n \mathsf{x} R^n \mathsf{x} R_+ \mathsf{x} R \mathsf{x} R^+ \mathsf{x} R^+, \tag{6.16}$$

where

$$G \in \{L, P, P_R, P_{SU}\} \ \textit{is permitted in special cases},$$
$$q^{(\cdot)} \in R^+,\ w^{(\cdot)} \in R^+,\ q^{(\cdot)} \textit{ and } w^{(\cdot)} \textit{ are constant}. \tag{6.17}$$

These transformations are generic when all movements, velocities and accelerations are mutually parallel, i.e., colinear. If they are not parallel then we need to use matrices, which is explained in Section 15.1

Note 146 *Since the choice of the initial instant t_0 does not influence the time-invariant transformations and the related time fields, we accept throughout the book $t_0 = 0$. It is trivial to replace t_0 by* 0 *(zero) in the transformations (6.13) through (6.16) that then take the slightly simpler forms.*

6.3 Compatibility. Consistency

6.3.1 Compatibility of the transformations

We distinguish different types of compatibility of the transformations.

Definition 147 ***Compatibility of the transformations***

*a) The temporal coordinate transformations (6.13) and (6.14) are **compatible** if, and only if they yield an identity as soon as one temporal coordinate and the corresponding (with the same subscript) spatial coordinate are eliminated from them [without using the spatial coordinate transformations (6.15) and (6.16)]. Otherwise, they are **incompatible**.*

Analogously, the spatial coordinate transformations (6.15) and (6.16) are ***compatible*** *if, and only if they yield an identity as soon as one spatial coordinate and the corresponding (with the same subscript) temporal coordinate are eliminated from them [without using the temporal coordinate transformations (6.13) and (6.14)]. Otherwise, they are* ***incompatible****.*

b) The transformations (6.13) through (6.16) are ***pairwise compatible*** *if and only if both the temporal coordinate transformations (6.13) and (6.14) are compatible, and the spatial coordinate transformations (6.15) and (6.16) are compatible. Otherwise, they are* ***pairwise incompatible****.*

c) The transformations (6.13) through (6.16) are ***entirely compatible*** *if and only if both 1) and 2) hold:*

1) The temporal coordinate transformations (6.13) and (6.14) yield, by means of the spatial coordinate transformations (6.15) and (6.16), an identity as soon as temporal and spatial coordinates with the same subscripts are eliminated from them.

2) The spatial coordinate transformations (6.15) and (6.16) yield, by means of the temporal coordinate transformations (6.13) and (6.14), an identity as soon as temporal and spatial coordinates with the same subscripts are eliminated from them.

Otherwise they are ***entirely incompatible****.*

d) The transformations (6.13) through (6.16) are ***partially*** *(i.e.* ***restrictively****) [****pairwise, entirely****]* ***compatible*** *if and only if they are [pairwise, entirely] compatible, respectively, exclusively either when the arbitrary point P moves with the speed restricted to be equal to a specific speed (e.g. to be equal to the light speed), or when the generic point G, hence the reference point P_R or P_{SU}, moves with the speed restricted to be equal to a specific speed, or the product value qw of the generic speeds q and w should be equal to a squared specific speed value.*

e) The transformations (6.13) through (6.16) are ***completely (pairwise, entirely) compatible*** *if and only if they are, respectively, (pairwise, entirely) compatible for any nonzero speed of the arbitrary point P and for any nonnegative speed of the generic point G, hence, of the reference point P_R or P_{SU}, and for any positive value of the product qw of the generic speeds q and w; hence, which are not restricted to be equal to specific speed values.*

Note 148 *Pairwise compatibility of the equations (6.13) through (6.16) is necessary and sufficient for their validity and applicability in separate pairs [(6.13), (6.14)], and [(6.15), (6.16)]. Pairwise incompatibility expresses their invalidity in separate pairs and it prevents their separate applications in pairs.*

Complete compatibility of the time coordinate transformations (6.13), (6.14) is important because it reflects clearly the time independence property (the Axiom 47). Their partial compatibility reflects clearly this time property only in the case of a specific speed (e.g. the light speed) of the arbitrary point P, or in the case of the specific (e.g. light) speed of the generic point G, or for a specific value of the product qw.

We will treat both partially and completely (entirely, pairwise) compatible transformations.

6.3.2 Consistency of values and of transformations

The numerical value of every quantity depends on the unit used to measure the value of the quantity.

The value of every variable should be consistently measured and used in terms of the units of the corresponding integral space (for details see Subsections 4.2 and 4.3).

Einsteinian relativity theory rejected a priory to ensure such consistency to the values of the light speed and of the spatial transfer speed in Lorentz transformations (7.20) through (7.23), hence in all formulae resulting from Lorentz transformations. This is the consequence of the a priory accepted invariance of these speeds relative to all inertial frames and relative to all *time* axes. We will release the transformations of such constraints.

Once we accept the units for all variables related to a fixed integral space, then we should use them consistently throughout the calculations. This leads to the following:

Definition 149 ***Consistency of values and of transformations***

The numerical values of any variable are consistent in the transformations *[e.g. in (6.13) through (6.16)] if, and only if the same scale (e.g. the same time scale for time, the same length scale for length) and the same unit (e.g. the same time unit for time, the same length unit for length) are applied to (measure) all the values of the variable, which are related to the same integral space in the transformations. Otherwise they are* ***inconsistent****.*

The transformations *[e.g. (6.13) through (6.16)]* ***are consistent*** *if, and only if the numerical values of all variables in them are consistent. Otherwise they are* ***inconsistent****.*

It will be shown in Part III that Lorentz transformations are inconsistent. The same is true for the velocity transformations known as Einstein's law of the velocity composition. These facts open the problem of consistency of the transformations, i.e. of finding consistent transformations.

6.4 Basic mathematical problem

We should solve the following problem in the framework of time-invariant coordinate and velocity transformations:

Problem 150 ***Essence of the basic mathematical problem***

The essence of the basic mathematical problem of the theory of time relativity, for the given (or to be determined) time scaling scalar coefficient function $\mu(.)$ in the basic temporal coordinate transformation (6.12), is the determination of the scalar scaling functions $\alpha_j^i(.)$, $\alpha_i^j(.)$, $\lambda_j^i(.)$ and $\lambda_i^j(.)$ so that

1° they obey the consistent coordinate transformations (6.13) through (6.17), and that

2° the consistent coordinate transformations (6.12), (6.13) through (6.17) ensure the preservation of the distance (of the length), which means that they transform

$$[\mathbf{r}_P^T(t_i) \quad t_i\mathbf{v}_G^{i^T}]D[\mathbf{r}_P^T(t_i) \quad t_i\mathbf{v}_G^{i^T}]^T, \tag{6.18}$$

into

$$[\mathbf{r}_P^T(t_j) \quad t_j\mathbf{v}_G^{j^T}]D[\mathbf{r}_P^T(t_j) \quad t_j\mathbf{v}_G^{j^T}]^T, \tag{6.19}$$

and vice versa, where

° *the arbitrary point P in $\mathbf{r}_P(t_i)$ and in $\mathbf{r}_P(t_j)$ can be either*

– *the generic point G in general,*

– *the light signal L in a special case, or*

– *the reference point P_R or P_{SU} in other special cases,*

i.e. $P \in \{G, L, P_R, P_{SU}\}$ is permitted, and

° *the velocity $\mathbf{v}_G^{(.)}$ of the generic point can be either*

– *the relative velocity $\mathbf{c}_{(.)}^{(.)}$ of the light signal L in a special case, or*

– *the relative velocity $\mathbf{v}_P^{(.)}$ of the arbitrary point P, or*

– *the relative velocity $\mathbf{v}_R^{(.)}$ ($\mathbf{v}_{SU}^{(.)}$) of the reference point P_R (P_{SU}), respectively, in another special case, i.e. $G \in \{L, P, P_R, P_{SU}\}$ is permitted.*

Besides, the compatibility properties of the transformations should be determined.

We will present solutions for different cases of this problem in the framework of the partially compatible transformations and in the framework of the completely compatible ones. In both cases the use of the numerical values of speeds will be consistent (Note 154).

Note 151 *The matrix D is a block diagonal matrix,*

$$D = \begin{pmatrix} A & O \\ O & -B \end{pmatrix}.$$

Such its structure enables the separation of the temporal coordinates from the spatial coordinates. Its structure and $A \neq B$ reflect the complete time independence of space

$$D = blockdiag\{A \quad -B\},\ A \neq B.$$

The matrices A and B can, but need not, be different.

If the matrices A and B are equal then they do not express jointly time independence of space. They can be different from the identity matrix I, which is a special case relative to their choice,

$$A = B \neq I. \tag{6.20}$$

A more special case (i.e. the singular case) occurs when both A and B are the identity matrix I,

$$A = B = I. \tag{6.21}$$

The distance relationship (6.22) expresses then the preservation of the Euclidean distance relationship.

Condition 152 ***General condition for the length preservation in integral spaces***

The application of the coordinate transformations (6.12), (6.13) through (6.17) to (6.18) should ensure the preservation of the distance (of the length), which is expressed by the following general identity in the framework of time-invariant transformations:

$$\begin{gathered}[\mathbf{r}_P^T(t_i) \quad t_i\mathbf{v}_G^{i^T}]D[\mathbf{r}_P^T(t_i) \quad t_i\mathbf{v}_G^{i^T}]^T \equiv [\mathbf{r}_P^T(t_j) \quad t_j\mathbf{v}_G^{j^T}]D[\mathbf{r}_P^T(t_j) \quad t_j\mathbf{v}_G^{j^T}]^T, \\ D = blockdiag\,\{A \quad -B\} \in R^{2nx2n}, \\ A \in R^{nxn} \text{ and } B \in R^{nxn} \text{ are any positive definite matrices,} \end{gathered} \tag{6.22}$$

This condition expresses the preservation of the distance (the length). Its meaning is that its satisfaction ensures the fulfillment of the condition 2° of the essence of the basic mathematical problem 150. We emphasize for the following reason:

Note 153 *Both sides of the identity are equal to zero in the case $G = P$ and $A = B$, so that the identity is then trivially satisfied: $0 = 0$. This is due to $\mathbf{r}_P(t_{(.)}) = t_{(.)}\mathbf{v}_P^{(.)}$:*

$$\begin{gathered}[\mathbf{r}_P^T(t_{(.)}) \quad t_{(.)}\mathbf{v}_G^{(.)^T}]D[\mathbf{r}_P^T(t_{(.)}) \quad t_{(.)}\mathbf{v}_G^{(.)^T}]^T \equiv \\ \equiv \alpha\mathbf{r}_P^T(t_{(.)})[I \quad I] \begin{matrix} A & O \\ O & -A \end{matrix} [I \quad I]^T\mathbf{r}_P(t_{(.)}) \equiv \\ \equiv \mathbf{r}_P^T(t_{(.)})\,[A \quad -A]\,[\begin{bmatrix} I \\ I \end{bmatrix}\mathbf{r}_P(t_{(.)}) \equiv \mathbf{r}_P^T(t_{(.)})\,(A-A)\,\mathbf{r}_P(t_{(.)}) \equiv 0.\end{gathered}$$

Regardless of this triviality: $0 = 0$, we should show, in order to avoid any suspicion, that application of 6.12), (6.13) through (6.17) to the left-hand side of the identity (6.22) transforms it into the right-hand side of the identity.

The triviality: $0 = 0$ does not exist if A $\neq$ B, the general Condition 152.

Note 154 *The identity (6.22), and the quadratic forms 6.18, 6.19, show the consistent use of the numerical values of all the variables relative to the corresponding integral spaces.*

Remark 155 *The position vector $\mathbf{r}_P(t_{(.)};t_{(.)0})$ and the velocity $\mathbf{v}_G^{(.)}$ are in general related formally to different points in (6.22) by following Einstein, [114] through [154], and Minkowski, [348]. However, there is not a convincible physical justification to accept the position vector of one point (e.g. of the arbitrary point P) and the velocity of another point (e.g. of the reference point) in (6.22). It is more natural to use the position vector $\mathbf{r}_{(..)}(t_{(.)};t_{(.)0})$ and the velocity $\mathbf{v}_{(..)}^{(.)}$ of the same point (..), which can be the generic point G, or the light signal L, or the arbitrary point P or the reference point P_R or P_{SU}.*

"Formally" means above that $\mathbf{r}_P(t_{(.)}; t_{(.)0})$ and $\mathbf{v}_G^{(.)}$ correspond to P and G, respectively, for $P \neq G$ in (6.22), but they are really and essentially taken both equal to a light signal L, $G = P = L$, in all proofs of (6.22) in Einsteinian relativity theory. They reduced (6.22) to This is sufficient to understand its severe restrictiveness, i.e., its full invalidity except in that singular case.

Conclusion 156 ***The condition for the preservation of the distance relationship***

The condition for the preservation of the distance relationship will be used in the synthetic form (6.22). Special characteristic cases of the choice of the matrices A and B are the following:

- *The matrices A and B are equal but different from the identity matrix I, (6.20).*
- *The matrices A and B are the identity matrix I, (6.21).*

These special cases characterize Einstein's relativity theory.

6.5 General, special and singular case

The scaling coefficient functions $\alpha_j^i(.)$, $\alpha_i^j(.)$, $\lambda_j^i(.)$ and $\lambda_i^j(.)$ can be a priory left mutually different or accepted pairwise equal, or even all equal, in the general generic coordinate transformations (6.13) through (6.17). From this point of view we distinguish the following different cases:

Case 157 ***General case of the scaling coefficient functions***

Either the time scaling coefficient functions $\alpha_j^i(.)$ and $\alpha_i^j(.)$ are mutually different, or space scaling coefficient functions $\lambda_j^i(.)$ and $\lambda_i^j(.)$ are mutually different, or both, in general:

$$\alpha_j^i(.) \neq \alpha_i^j(.) \text{ and/or } \lambda_j^i(.) \neq \lambda_i^j(.). \tag{6.23}$$

This condition expresses the usual qualitative relationship between a transformation and its inverse. A transformation and its inverse are mutually different in general. This case is beyond Einsteinian relativity theory.

Case 158 ***Special case of the scaling coefficient functions***

Both the time scaling coefficient functions $\alpha_j^i(.)$ and $\alpha_i^j(.)$ are equal and space scaling coefficient functions $\lambda_j^i(.)$ and $\lambda_i^j(.)$ are equal, but space scaling coefficient functions can be different from the time scaling coefficient functions:

$$\begin{gathered}\alpha_j^i(.) = \alpha_i^j(.) = \alpha_{ji}(.) \text{ and } \lambda_j^i(.) = \lambda_i^j(.) = \lambda_{ji}(.),\\ \alpha_{ji}(.) = \alpha_{ij}(.) \neq \lambda_{ji}(.) = \lambda_{ij}(.) \text{ is possible }.\end{gathered} \tag{6.24}$$

This case is also beyond Einsteinian relativity theory.

All the scaling coefficient functions are mutually equal only in a very special case, i.e. in a singular case.

Case 159 ***Singular case of the scaling coefficient functions***
All the scaling coefficient functions $\alpha_j^i(.)$, $\alpha_i^j(.)$, $\lambda_j^i(.)$ *and* $\lambda_i^j(.)$ *are equal, (6.25),*

$$\alpha_j^i(.) = \alpha_i^j(.) = \alpha = \lambda_j^i(.) = \lambda_i^j(.) = \lambda \textit{ is possible.} \tag{6.25}$$

Einsteinian relativity theory has treated only this case.
This book deals with all three cases.

The temporal coordinate transformations (6.13), (6.14) are expressed in general in terms of the position $r_G(t_{(.)}; t_{(.)0})$ of the generic point G with respect to the origin $O_{(.)}$ of $R^n_{(.)}$. Relative to the meaning of the point G we distinguish the following characteristic cases. The generic point G can be either

- ° a light signal L, or
- ° the arbitrary point P, or
- ° the temporal reference point P_R, or
- ° the spatial reference point P_{SU}, i.e.

$$G \in \{L, P, P_R, P_{SU}\}.$$

This implies

$$\mathbf{r}_G(t_{(.)}; t_{(.)0}) \in \left\{\mathbf{r}_L(t_{(.)}; t_{(.)0}),\ \mathbf{r}_P(t_{(.)}; t_{(.)0}),\ \mathbf{r}_R(t_{(.)}; t_{(.)0}),\ \mathbf{r}_{SU}(t_{(.)}; t_{(.)0})\right\},$$
$$r_G(t_{(.)}; t_{(.)0}) \in \left\{r_L(t_{(.)}; t_{(.)0}),\ r_P(t_{(.)}; t_{(.)0}),\ r_R(t_{(.)}; t_{(.)0}),\ r_{SU}(t_{(.)}; t_{(.)0})\right\},$$
$$\mathbf{v}_G \in \left\{\mathbf{c}^{(.)}_{(.)}, \mathbf{v}_P, \mathbf{v}_R, \mathbf{v}_{SU}\right\},\ v_G \in \left\{c^{(.)}_{(.)}, v_P, v_R, v_{SU}\right\}.$$

The temporal transfer speed $\vartheta^{(.)}$ is in general different from and independent of the light speed $c^{(.)}$, of the speed $v_P^{(.)}$ of the arbitrary point P, of the speed $v_R^{(.)}$ of the temporal reference point P_R, of the speed $v_{SU}^{(.)}$ of the spatial reference point P_{SU}, and of the spatial transfer speed $v_{ji}^{(.)}$,

$$\textit{in general } \vartheta^{(.)} \notin \left\{c^{(.)}, v_P^{(.)}, v_R^{(.)}, v_{SU}^{(.)}, v_{ji}^{(.)}\right\}. \tag{6.26}$$

This reflects the *time* independence of space.

The speed $v_R^{(.)}$ of the temporal reference point P_R can be accepted for the temporal transfer speed $\vartheta^{(.)}$ in a special case,

$$\vartheta^{(.)} \equiv v_R^{(.)} \textit{ is permitted.} \tag{6.27}$$

The speed $v_P^{(.)}$ of the arbitrary point P can be adopted for the temporal transfer speed $\vartheta^{(.)}$ in another special case,

$$\vartheta^{(.)} \equiv v_P^{(.)} \textit{ is permitted.} \tag{6.28}$$

The preceding cases are beyond Einsteinian relativity theory.

In a more special case, and in the singular case that characterizes Einsteinian relativity theory, the spatial transfer speed $v_{ji}^{(.)}$ is simultaneously the temporal transfer speed $\vartheta^{(.)}$,

$$\vartheta^{(.)} = v_{ji}^{(.)} \text{ is permitted.} \tag{6.29}$$

It is denoted simply by v in Einsteinian relativity theory,

$$\vartheta^{(.)} \equiv v. \tag{6.30}$$

Case 160 ***Time independence and the transformations.***
Time independence of space, (Axiom 47), is completely expressed in the temporal coordinate transformations (6.13), (6.14) if and only if

◦ *the generic point G is not an arbitrary point P, $G \neq P$,*

◦ *the temporal transfer speed $\vartheta^{(.)}$ is neither the speed $v_P^{(.)}$ of the arbitrary point P nor the spatial transfer speed $v_{ji}^{(.)}$, $\vartheta^{(.)} \notin \left\{ v_{ji}^{(.)}, v_P^{(.)} \right\}$, and*

◦ *the generic speeds q and w are neither the speed v_P of the arbitrary point P nor the spatial transfer speed $v_{ji}^{(.)}$, where $q^{(.)}$ and $w^{(.)} \notin \{v_P^{(.)}, v_{ji}^{(.)}\}$.*

Case 161 ***Time-varying and time-invariant transformations.***
The transformations (6.12), (6.13) through (6.17) are time-varying coordinate transformations if and only if

◦ *at least one scaling coefficient function among the scaling coefficient functions $\mu_i(.)$, $\alpha_j^i(.)$, $\alpha_i^j(.)$, $\lambda_j^i(.)$ and $\lambda_i^j(.)$ depends explicitly or implicitly (through its arguments) on time t, or*

◦ *a choice of the initial moment t_0 influences the transformations (either through a scaling coefficient function, or through a position coordinate of a point, or through a speed value),*

or both.

Otherwise, they are time-invariant.

Note 162 *The transformations (6.13) through (6.17) will be used to determine the corresponding speed transformations. This will incorporate derivatives of the scaling coefficient functions. If they do not depend explicitly on time t, but their arguments depend on time t, then their derivatives will not be identically equal to zero in general. Their derivatives can then influence the velocity transformations and the acceleration transformations. Therefore, the transformations are considered as time-varying in such a case. However, in what follows, all velocities and speeds are time-invariant (constant). Therefore, all coordinate transformations are time-invariant and all accelerations are equal to the zero vector.*

For the cases when some velocity or speed is time varying, which implies nonzero acceleration see [227], [228]. These books contain the new acceleration transformations.

Case 163 ***Uniformity and nonuniformity of the temporal coordinate transformations.***

The temporal coordinate transformations (6.12), (6.13), (6.14) are ***spatially uniform*** *if and only if*

$\circ$ *all the time scaling coefficient functions,* $\mu(.)$, $\alpha_j^i(.)$ *and* $\alpha_i^j(.)$, *do not depend on the position and on the speed of the arbitrary point* P,

$\circ$ *the generic point* G *is not the arbitrary point* P, $G \neq P$,

$\circ$ *the speed* $v_P^{(.)}$ *of the arbitrary point* P *is neither the temporal transfer speed* $\vartheta^{(.)}$ *nor the generic speed* $q^{(.)}$ *nor* $w^{(.)}$; $\vartheta^{(.)}$, $q_G^{(.)}$, $q^{(.)}$, $w^{(.)} \notin \left\{v_P^{(.)}\right\}$.

This is uniformity of the transformations over space. It is their ***spatial uniformity****.*

If, additionally, the time scaling function $\mu_i(.)$ *depends on the (initial and/or the instantaneous) position vector of the reference point* P_R *then* ***the (spatial) uniformity*** *of the transformations is incomplete - it is* ***weak****.*

If the time scaling function $\mu_i(.)$ *does not depend on the (instantaneous and initial) position vector either of the reference point* P_R *or of the arbitrary point* P *then* ***the (spatial) uniformity*** *of the transformations is* ***complete****.*

The transformations are nonuniform *if and only if either*

$\circ$ *at least one of the time scaling coefficient functions,* $\mu(.)$, $\alpha_j^i(.)$ *and/or* $\alpha_i^j(.)$, *depends on the position and/or on the speed of the arbitrary point* P, *or*

$\circ$ *the generic point* G *is the arbitrary point* P, $G = P$, *or*

$\circ$ *the speed* $v_{Pt}^{(.)}$ *of the arbitrary point* P *is the temporal transfer speed* $\vartheta^{(.)}$, $v_P^{(.)} \equiv \vartheta^{(.)}$, *or*

$\circ$ *the speed* $v_{Pt}^{(.)}$ *of the arbitrary point* P *is at least one of the generic speeds* $q^{(.)}$ *or* $w^{(.)}$, $v_P^{(.)} \in \left\{q^{(.)}, w^{(.)}\right\}$.

This is ***the spatial nonuniformity*** *of the transformations.*

The generic transformations take particular forms determined by their arguments.

The temporal transfer speed ϑ can be adopted independently of other variables, but need not.

Case 164 ***Independent choice of the temporal transfer speed*** ϑ

The temporal transfer speed ϑ *is independent of the spatial transfer speed* v_{ji}, $\vartheta^{(.)} \neq v_{ji}^{(.)}$. *This is the general case. It reflects time independence of space.*

Case 165 ***Dependent choices of the temporal transfer speed*** ϑ

a) The speed $v_R^{(.)}$ *of the reference point* P_R *is also the temporal transfer speed* ϑ, $v_R^{(.)} = \vartheta^{(.)}$. *Both are constant. This choice reflects the time independence of space.*

b) The speed $v_P^{(.)}$ *of the arbitrary point* P *is also the temporal transfer speed* ϑ, $\vartheta^{(.)} = v_P^{(.)}$. *Both are constant. This choice does not reflect the time independence of space.*

c) The spatial transfer speed $v_{ji}^{(.)}$ *is also the temporal transfer speed* $\vartheta^{(.)}$, $v_{ji}^{(.)} = \vartheta^{(.)}$. *Both are constant. This is a special case. It does not reflect the time independence of space.*

The generic speeds q and w can be differently chosen. Their selection can be free, independent of other variables, or can depend on them. In this concern there are the following possibilities:

Case 166 ***Free choice of the generic speeds***

The generic speeds q and w are freely chosen, independently of all other variables, in the temporal coordinate transformations (6.13), (6.14). This is the general case of their selection. It reflects the time independence property and enables uniformity of the temporal coordinate transformations over space.

Case 167 ***Special choices of the generic speeds***

a) The light speed is chosen for both generic speeds, $q^{(\cdot)}$ and $w^{(\cdot)}$, $q^{(\cdot)} = w^{(\cdot)} = c_{(\cdot)}^{(\cdot)}$. This reflects the time independence of space and permits uniformity of the temporal coordinate transformations over space.

b) The speed of the arbitrary point P is chosen for at least one of the generic speeds $q^{(\cdot)}$ and $w^{(\cdot)}$, $v_P^{(\cdot)} \in \left\{q^{(\cdot)}, w^{(\cdot)}\right\}$. This does not reflect the time independence property and causes nonuniformity of the temporal coordinate transformations over space.

If the light signal L is accepted for the reference point P_R in the coordinate transformations (6.13) through (6.17), $L = P_R$, then the light speed numerical value can be assumed dependent on a choice of the integral space, or independent of such a choice. From this point of view we distinguish two different cases.

Case 168 ***General case of the light speed numerical value***

The numerical value of the light speed in vacuum depends on the accepted integral space

$$c_i^i \neq c_j^j \ . \tag{6.31}$$

This is not acceptable in Einsteinian relativity theory.

This condition expresses the general law. It reflects the natural dependence of a speed numerical value on the accepted integral space. It holds also for the light speed in general (Conclusion 309).

Case 169 ***Special cases of the light speed numerical value***

a) The numerical value of the light speed in vacuum depends on the accepted time axis only,

$$c_i^i = c^i \neq c_j^j = c^j \ . \tag{6.32}$$

b) The numerical value of the light speed in vacuum depends on the accepted frame only,

$$c_i^i = c_i \neq c_j^j = c_j . \tag{6.33}$$

These cases are inacceptable in Einsteinian relativity theory.

The general law allows independence of the speed numerical value of choices of a *time* unit and of a length unit only in a singular case.

Case 170 ***Singular case of the light speed numerical value***

The numerical value of the light speed in vacuum is invariant relative to integral spaces,

$$c_i^i = c_j^j = c_{ij} = c_{ji} = c. \tag{6.34}$$

Einsteinian relativity theory accepts this case a priory.

We will treat all the preceding cases.

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Chapter 7

Why not Einstein‘s Relativity Theory?

What follows is the summary of the discoveries on Lorentz- Einstein‘s mistakes, inconsistencies, paradoxes or absurds proved in [226] [227], [228], [231]. They explain why Einstein!s theory of *time* relativity (for short: *Einstein‘s relativity theory*) is invalid, i.e., why it fails.

7.1 Einstein’s condition and transformations

A homogeneous *time* field contains a single *time* axes only. If it should rest homogeneous with the original *time* axis, then this demand prevents any transformation of the temporal coordinate. If we do not impose such a demand then the *time* field can rest homogeneous only with the new *time* axis instead of the original one. Otherwise, it will be transformed into a heterogeneous *time* field with the two (or more) *time* axes.

Heterogeneous *time* fields can be single-layer or multi-layer heterogeneous *time* fields. A single-layer *time* field can also contain several different *time* axes, but space can be so partitioned that there is not overlapping among the *time* axes over any subspace or subset. A multi-layer *time* field contains several different *time* axes and whatever is the space partition into nonempty disjoint subspaces and/or subsets there are (for every partition) at least two different *time* axes that hold over the same subspace or subset. The corresponding temporal coordinate transformations establish links among the *time* axes that can belong to the same layer or to different layers regardless of the multiplicity of the layers.

Einstein imposed the following condition as crucial for the validity of the coordinate transformations in general, hence for the validity of Lorentz transformations, as well, [144] through [154]:

Condition 171 ***Einstein‘s generalized distance condition***

For the validity of the coordinate transformations from the integral space $\mathfrak{I}_i = \mathfrak{T}_i \times \mathcal{R}_i^n$ *into the integral space* $\mathfrak{I}_j = \mathfrak{T}_j \times \mathcal{R}_j^n$ *they should preserve the distance (the length) expressed by the following condition:*

$$\begin{bmatrix} \mathbf{r}_P(t_i;t_{i0}) \\ (t_i - t_{i0})\mathbf{c} \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(t_i;t_{i0}) \\ (t_i - t_{i0})\mathbf{c} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{r}_P(t_j;t_{j0}) \\ (t_j - t_{j0})\mathbf{c} \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(t_j;t_{j0}) \\ (t_j - t_{j0})\mathbf{c} \end{bmatrix},$$
$$D = blockdiag\{A \quad -A\} \in R^{2nx2n},\ A \text{ is any positive definite matrix.} \tag{7.1}$$

This condition is a special case of the general distance preservation condition (6.22) introduced in [227], [228]. Since we consider *time*-invariant transformations and velocities / speeds then we may accept $t_0 = 0$ without losing in generality.

The left-hand side of (7.1) is a quadratic form in $\mathbf{r}_P(t_i;t_{i0})$ and $t_i\mathbf{c}$,

$$\begin{bmatrix} \mathbf{r}_P(t_i) \\ t_i\mathbf{c} \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(t_i) \\ t_i\mathbf{c} \end{bmatrix}. \tag{7.2}$$

Definition 172 ***Poincaré group*** *(e.g. [88, p. 11], [489])*

The coordinate transformations that obey (7.1) form Poincaré group. The coordinate transformations obeying the condition (7.1) are Gaussean.

The condition (7.1) expresses:

...the fundamental idea of the general principle of relativity: All Gaussean coordinate systems are essentially equivalent for the formulation of the general laws of nature.

Albert EINSTEIN [144, p. 85], [153, p. 54], [154]

Einstein accepted $B = A$ to hold in (7.1). Such a choice of the matrix B, $B = A$, does not reflect *time* independence of the space. However, it does not prove either *time* dependence on the space.

The matrix A is different from the identity matrix I in the general relativity theory, $B = A \neq I$.

In the special relativity theory, Einstein used the identity matrix I for the matrices A and B, $B = A = I$. Then the condition (7.1) is expressed compactly as follows, [144, p. 77], [150, p. 61], [153, p. 97], [154],

$$[\mathbf{r}_P^T(t_i) \quad t_i\mathbf{c}_i^{i^T}]I[\mathbf{r}_P^T(t_i) \quad t_i\mathbf{c}_i^{i^T}]^T$$

or equivalently in the form of Euclidean norm,

$$\left\| \mathbf{r}_P^T(t_i) \quad t_i\mathbf{c}_i^{i^T} \right\|^2.$$

This induces the condition for Euclidean distance preservation in integral spaces in Euclidean form,

$$\left\| \mathbf{r}_P^T(t_i) \quad t_i\mathbf{c}_i^{i^T} \right\|^2 \equiv \left\| \mathbf{r}_P^T(t_j) \quad t_j\mathbf{c}_i^{i^T} \right\|^2.$$

Lorentz and Einstein used the position vector $\mathbf{r}_P(t_i)$ of the point P to represent a light signal L, $\mathbf{r}_P(t_i) \equiv \mathbf{r}_L(t_i)$, in order to verify the preservation of the distance, i.e. of the length. They did not allow the point P to be arbitrary because they used explicitly the light velocity $\mathbf{c}$ for the velocity of the arbitrary point P in their derivation of Lorentz transformations.

Note 173 *Notice that*

$$\mathbf{r}_P(t_{(.)}) = \mathbf{r}_L(t_{(.)}) = t_{(.)}\mathbf{c}, \text{ and } D = blockdiag\{A \quad -A\}$$

imply

$$[\mathbf{r}_P^T(t_{(.)}) \qquad t_{(.)}\mathbf{c}^T] \begin{bmatrix} A & O \\ O & -A \end{bmatrix} \begin{bmatrix} \mathbf{r}_P(t_{(.)}) \\ t_{(.)}\mathbf{c} \end{bmatrix} \equiv$$

$$\equiv \mathbf{r}_L^T(t_{(.)})A\mathbf{r}_L(t_{(.)}) - \ t_{(.)}\mathbf{c}^T A(t_{(.)}\mathbf{c} \equiv 0 \equiv \mathbf{r}_L^T(t_{(.)})A\mathbf{r}_L(t_{(.)}) - \mathbf{r}_L^T(t_{(.)})A\mathbf{r}_L(t_{(.)}).$$

This reduces Einstein's generalized condition (7.1) into a trivial identity $0 \equiv 0$*:*

$$[\mathbf{r}_P^T(t_i) \quad t_i\mathbf{c}^T] \begin{bmatrix} A & O \\ O & -A \end{bmatrix} \begin{bmatrix} \mathbf{r}_P(t_i) \\ t_i\mathbf{c} \end{bmatrix} \equiv 0 \equiv$$

$$\equiv [\mathbf{r}_L^T(t_j) \quad \mathbf{r}_L^T(t_j)] \begin{bmatrix} A & O \\ O & -A \end{bmatrix} \begin{bmatrix} \mathbf{r}_L(t_j) \\ \mathbf{r}_L(t_j) \end{bmatrix} \equiv 0. \tag{7.3}$$

Such a trivial case, which was treated by Einstein [144, p. 77], [153, p. 97], [150, p. 61], [154, p. 88], does not occur in the general Condition 152 due to A ≠ B.

This Note is a special case of Note 153 in Section 6.4.

Comment 174 *Einstein himself replaced, since he considered the case* $A = B$*, [150, p. 28: the equations (22) and (22a)], the condition (7.1) by the condition that the identity*

$$\mathbf{r}_P^T(t_i)A\mathbf{r}_P(t_i) \equiv t_i^2\mathbf{c}^T A\mathbf{c} \Longleftrightarrow P = L, \tag{7.4}$$

implies the identity

$$\mathbf{r}_P^T(t_j)A\mathbf{r}_P(t_j) \equiv t_j^2\mathbf{c}^T A\mathbf{c} \Longleftrightarrow P = L, \tag{7.5}$$

and vice versa, i.e. that (7.4) and (7.5) are equivalent. It is obvious, as Einstein wrote, [150, p. 28: the equations (22) and (22a)], that these identities can be set in the form (7.3), due to $\mathbf{r}_L(t_{(.)}) \equiv t_{(.)}\mathbf{c}$ *:*

$$\mathbf{r}_L^T(t_i)A\mathbf{r}_L(t_i) + t_i^2\mathbf{c}^T(-A)\mathbf{c}_i^i \equiv 0 \Longleftrightarrow P = L,$$

$$\mathbf{r}_L^T(t_j)A\mathbf{r}_L(t_j) + t_j^2\mathbf{c}^T(-A)\mathbf{c}_j^j \equiv 0 \Longleftrightarrow P = L.$$

This explains why the condition (7.1) in the case $A = B$ *means the trivial equivalence between (7.4) and (7.5), which is expressed by* $0 = 0$.

Paradox 175 Triviality of Einstein‘s relativity theory

Note 153, Note 173 and Comment 174 discover triviality of Einstein‘s relativity theory because its basis is the trivial identity $0 \equiv 0$.

Minkowski [348] confirmed the use of the position vector $\mathbf{r}_P(t_{(.)})$ of the arbitrary point P (not necessarily of the light signal L) in general in (7.1). The characteristics of two different points are then used in (7.1), the instantaneous position vector $\mathbf{r}_P(t_{(.)})$ of the arbitrary point P and the velocity $\mathbf{c}$ of the light signal L. There was not shown a convincing physical reason for such their combination. It is natural that the position vector and its speed characterize the same point in (7.1). A possible mathematical reason seems to come from the form of Lorentz transformations of the temporal coordinates. In them, the position $r_P(t_{(.)})$ is the instantaneous position of the arbitrary point P, and the generic speeds q and w are both equal to the light speed c. However, Lorentz transformations were determined exclusively for the light signal L as the arbitrary point P, $L = P$. We will retain the condition (7.1) in the framework of Einsteinian relativity theory and its development since Einstein claimed that Lorentz transformations satisfy this generalized Einstein‘s condition. We will confirm this in the sequel by starting with Axiom 47, i.e. with the *time* independence property, rather than to accept Einstein’s attitude that *time* depends on the space.

Einstein used the light speed as invariant relative to integral spaces, [144, p. 29, pp. 101 - 107], [151, pp. 44 - 46, p.51], [153, pp. 129 - 135], [154, pp. 30 - 34], that is that $c_i^i = c_j^j = c$. This has been unquestionably accepted in Einsteinian relativity theory.

In general, any change of a *time* unit changes the numerical value of speed if we do not change accordingly and simultaneously the length unit (see the Remark 302 and Conclusion 304 in Section 8.2).

7.2 *Time* Fields and Lorentz transformations

7.2.1 Lorentz transformations

Origin of Lorentz transformations

H. A. Lorentz wrote fairly the following, published in French:

Ce furent ces considérations publiées par moi en 1904 qui donnèrent lieu à Poincaré d’écrire son mémoire sur la Dynamique de l’électron, dans lequel il a attaché mon nom à la transformation dont je viens de parler. Je dois remarquer à ce propos que la même transformation se trouve déjà dans un article de M. VOIGT publié en 1887 et que je n’ai pas tiré de cet artifice tout le parti possible. En effet, pour certaines des grandeurs physiques qui entrent dans les formules, je n’ai pas indiqué la transformation qui convient le mieux. Cela a été fait par POINCARE et ensuite par M. EINSTEIN et MINKOWSKI.

H. A. LORENTZ [301, p. 295]

This text translated into English reads:

These were those considerations published by me 1904, which influenced Poincaré to write his memoir on the Dynamics of electron, in which he attached my name to the transformation about which I have spoken. I must remark in this concern that the same transformation is there in the paper by Mr. VOIGT published 1887 and that I did not take from this artifice all possible part. In fact, for certain physical quantities that appear in the formulae, I did not indicate a transformation that is the most convenient. This was done by POINCARE and afterwards by Mr. EINSTEIN and MINKOWSKI.

By following **Woldemar Voigt** (1887), [472], **Hendrik Antoon Lorentz**, as explained honestly by himself in the cited text, used partially Voigt transformation of both temporal and spatial coordinates, [297] through [304].

Lorentz stated also that the transformations were attributed to him and used under his name at the early development of Einsteinian relativity theory at first by Poincaré (1905), , and later by Einstein (1907), [116], [144] through [154], and Minkowski (1908) [348].

Einstein claimed in his famous paper [114] that he proved several new results among which new coordinate transformations. They are the same as those by Lorentz. However, Einstein did not refer either in [114] (1905) to the transformations as introduced by Voigt (1887) [472, published 1887], or as used first by Lorentz (1904), or to the corresponding paper by Lorentz [298, published 1904]. Einstein referred to the transformations under the name *Lorentz transformations* for the first time 1907 in [116].

We will follow the commonly accepted name "Lorentz transformations" by having in mind that they are due essentially to Voigt.

For more details see [226], [231].

Slightly generalized basic Lorentz transformations

Axiom 47 suggests the independence between the *time* scaling coefficient functions and the space scaling coefficient functions in general. Therefore, we will present at first a slightly generalized form of the transformations in order to explain the approach by Lorentz, Einstein and Poincaré, the conditions under which they used the transformations and how these conditions simplified the transformations. The subsequent analysis will justify such a slight generalization.

Definition 176 ***The slightly generalized basic Lorentz transformations*** *There are positive real numbers α_j^i, α_i^j, λ_j^i and λ_i^j such that the following coordinate transformations hold:*

$$(t_i - t_{i0}) = \alpha_j^i \left[(t_j - t_{j0}) + \frac{v_{ji}^j}{\left(c_j^j\right)^2} r_P(t_j; t_{j0}) \right], \tag{7.6}$$

$$(t_j - t_{j0}) = \alpha_i^j \left[(t_i - t_{i0}) - \frac{v_{ji}^i}{\left(c_i^i\right)^2} r_P(t_i; t_{i0}) \right], \tag{7.7}$$

$$\mathbf{r}_P(t_i;t_{i0}) = \lambda_j^i[\mathbf{r}_P(t_j;t_{j0}) + v_{ji}^j(t_j - t_{j0})\mathbf{u}], \tag{7.8}$$

$$\mathbf{r}_P(t_j;t_{j0}) = \lambda_i^j[\mathbf{r}_P(t_i;t_{i0}) - v_{ji}^i(t_i - t_{i0})\mathbf{u}], \tag{7.9}$$

and that they obey the condition (7.1). These transformations will be called ***the slightly generalized basic Lorentz transformations****.*

Remark 177 ***Time-invariance of the scaling coefficients***
The scaling coefficients are accepted a priory time-invariant in order for the transformations to correspond to time - invariant units.

Note 178 *The fully generalized time-invariant basic Lorentz transformations are determined in Definition 315. See also Remarks 316, 317, and 404.*

The temporal coordinate transformations (7.6), (7.7) are expressed in general in terms of the position coordinate $r_P(t_{(.)};t_{(.)0})$ of the arbitrary point P with respect to the origin $O_{(.)}$ of $R^n_{(.)}$.

Remark 179 ***Lorentz - Einstein - Poincaré approach***
The original Lorentz - Einstein - Poincaré (for short: ***Einsteinian****) approach to the coordinate transformations has two different stages:*

◦ ***The stage of the determination of both the scaling coefficients and the final form of the transformations****. The characteristic of this stage is that the scaling coefficients are determined for the arbitrary point P representing a light signal, hence moving exclusively with the light velocity:*

$$\mathbf{r}_P^{(.)}(..) \equiv \mathbf{r}_L^{(.)}(..),\ \mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)},\ \textit{i.e. } v_P^{(.)} \equiv c_{(.)}^{(.)},\ \textit{at this, the first, stage.}$$

◦ ***The stage of an application of the scaling coefficients and of the transformations****. The characteristic of this stage is that the scaling coefficients determined in the first stage, and the so resulting coordinate transformations, are used in applications of the transformations for an arbitrary position and for an arbitrary velocity of the arbitrary point P (hence, not necessarily moving exclusively with the light velocity),*

$$\mathbf{r}_P^{(.)}(..) \neq \mathbf{r}_L^{(.)}(..),\ \mathbf{v}_P^{(.)} \neq \mathbf{c}_{(.)}^{(.)},\ \textit{i.e. } v_P^{(.)} \neq c_{(.)}^{(.)},\ \textit{permitted at the second stage.}$$

Paradox 180 ***Inconsistency of the stages of Einstein's relativity theory***
By comparing the two stages we can conclude immediately that they are mutually inconsistent from the point of view of the meaning of the arbitrary point P, hence of its velocity $\mathbf{v}_P^{(.)}$*.*

Einsteinian conditions

The original Lorentz transformations were determined under the following a priory accepted conditions, which were also a priory adopted by Einstein and Poincaré, and in the whole Einsteinian relativity theory:

Condition 181 *All the time scaling coefficients (α_i^j and α_j^i) are equal:*

$$\alpha_i^j \equiv \alpha_j^i \equiv \alpha. \tag{7.10}$$

In order for this condition to be satisfied it is necessary and sufficient that the arbitrary point P moves exclusively with the light speed. Otherwise. Einstein's theory of time relativity disappears (for details see Paradox 230 and Paradox 231).

Condition 182 *All the space scaling coefficients (λ_i^j and λ_j^i) are equal:*

$$\lambda_i^j \equiv \lambda_j^i \equiv \lambda. \tag{7.11}$$

In order for this condition to be satisfied it is necessary and sufficient that the arbitrary point P moves exclusively with the light speed. Otherwise. Einstein's theory of time relativity disappears (for details see Paradox 230 and Paradox 231).

Condition 183 *The arbitrary point P moves with the velocity of light:*

$$\mathbf{v}_P^{(.)}(t_{(.)}) \equiv \mathbf{c}_{(.)}^{(.)} \equiv c_{(.)}^{(.)}\mathbf{u}, \tag{7.12}$$

The generic speeds q and w are fixed to the light speed c: $q = w = c$.

Condition 184 *The numerical values $c_{(.)}^{(.)}$ and $v_{(.)}^{(.)}$ of the light speed and of the spatial transfer speed, respectively, are invariant relative to time axes:*

$$c_{(..)}^{(.)} \equiv c_{(..)},\ v_{ji}^{(.)} \equiv v_{ji}, \tag{7.13}$$

and the values of the light speed and of the spatial transfer speed, respectively, hence their numerical values as well, are also invariant with respect to inertial spatial coordinate systems:

$$c_{(..)} \equiv c,\ v_{ji} \equiv v. \tag{7.14}$$

[144, p. 29, pp. 101 - 107], [151, pp. 44 - 46, p.51], [153, pp. 129 - 135], [154, pp. 30 - 34].

Condition 185 *The position $\mathbf{r}_P(t_{(.)}; t_{(.)0})$ of the arbitrary point P is the position $\mathbf{r}_L(t_{(.)}; t_{(.)0})$ of the corresponding light signal:*

$$\mathbf{r}_P(t_{(.)}; t_{(.)0}) \equiv \mathbf{r}_L(t_{(.)}; t_{(.)0}) \equiv r_L(t_{(.)}; t_{(.)0})\mathbf{u} \equiv c(t_{(.)} - t_{(.)0})\mathbf{u}, \\ r_L(t_{(.)}; t_{(.)0}) \in R_+,\ \forall t_{(.)} \in T_{(.)}. \tag{7.15}$$

Presentation of Lorentz transformations

The preceding analysis permits us to discover how restrictive are conditions under which Lorentz transformations and the whole Einsteinian relativity theory claimed to hold. Consequently, they determine the singular case.

Case 186 ***Singular case: the basic Lorentz transformations***
The conditions 181 through 185 determine the following ***singular case*** *of the joint temporal and spatial coordinate transformations (7.6) through (7.9),*

$$(t_i - t_{i0}) = \alpha\left[(t_j - t_{j0}) + \frac{v}{c^2} r_P(t_j; t_{j0})\right], \tag{7.16}$$

$$(t_j - t_{j0}) = \alpha\left[(t_i - t_{i0}) - \frac{v}{c^2} r_P(t_i; t_{i0})\right], \tag{7.17}$$

$$\mathbf{r}_P(t_i; t_{i0}) = \lambda[\mathbf{r}_P(t_j; t_{j0}) + v(t_j - t_{j0})\mathbf{u}], \tag{7.18}$$

$$\mathbf{r}_P(t_j; t_{j0}) = \lambda[\mathbf{r}_P(t_i; t_{i0}) - v(t_i - t_{i0})\mathbf{u}]. \tag{7.19}$$

The transformations (7.16) through (7.19) represent ***the basic Lorentz transformations****.*

Problem 187 ***Determination of the scaling coefficients***
The mathematical part of Einsteinian relativity theory starts with the determination of the scaling coefficients α and λ so that they obey both the basic Lorentz transformations (7.16) through (7.19) and Einstein's trivial distance condition (7.1).

Solution 188 ***Lorentz transformations***
The final forms of the basic Lorentz transformations (7.16) through (7.19) are ***Lorentz transformations*** *(7.20) through (7.23), [297] through [301]:*

$$(t_i - t_{i0}) = \frac{(t_j - t_{j0}) + \frac{v}{c^2} r_P(t_j; t_{j0})}{\sqrt{1 - \frac{v^2}{c^2}}}, \ \alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{7.20}$$

$$(t_j - t_{j0}) = \frac{(t_i - t_{i0}) - \frac{v}{c^2} r_P(t_i; t_{i0})}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{7.21}$$

$$\mathbf{r}_P(t_i; t_{i0}) = \frac{\mathbf{r}_P(t_j; t_{j0}) + v(t_j - t_{j0})\mathbf{u}}{\sqrt{1 - \frac{v^2}{c^2}}}, \ \lambda = \alpha, \tag{7.22}$$

$$\mathbf{r}_P(t_j; t_{j0}) = \frac{\mathbf{r}_P(t_i; t_{i0}) - v(t_i - t_{i0})\mathbf{u}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{7.23}$$

Note 189 *Lorentz transformations (7.20) through (7.23) restrict the value v of the (spatial) transfer speed v to be less than the value c of the light speed c. They are undefined for $v = c$, i.e. they, and all the results deduced from them, may not be used for $v = c$.*

However, they do not restrict the value $v_P^{(.)}$ of the speed $v_P^{(.)}$ of the arbitrary point P, provided it is different from v, $v_P^{(.)} \neq v$. They permit bigger values of the speed of the arbitrary point than the light speed value. They permit faster movements of bodies than the light propagation.

Note 190 *The geometrical interpretations and explanations of the physical origin of Lorentz transformations can be found in the books [226], [231].*

The books [226], [231] contain also the detailed analysis of the features of Lorentz transformations, which shows the following.

Theorem 191 ***Lorentz transformations obey Einstein's condition*** *[226], [231]*

Lorentz transformations (7.20) through (7.23) satisfy the condition (7.1), i.e. (7.2), for an arbitrary position and for an arbitrary velocity $\mathbf{v}_P^{(.)}$ of the arbitrary point P.

Conclusion 192 ***Singularity of Lorentz transformations relative to the scaling coefficients*** *[226], [231]*

The time scaling coefficients $\alpha_{(.)}^{(..)}$, as well as the space scaling coefficients $\lambda_{(.)}^{(..)}$, are accepted in Einsteinian relativity theory a priory equal and invariant relative to the coordinate transformations (Condition 181 and Condition 182), $\alpha_{(.)}^{(..)} \equiv \alpha$ and $\lambda_{(.)}^{(..)} \equiv \lambda$. As a consequence of these conditions, they are all mutually equal, hence $\alpha = \lambda$.

Conclusion 193 ***Singularity of Lorentz transformations relative to the position and the speed of the arbitrary point P*** *[226], [231]*

Lorentz, Einstein and Poincaré determined the scaling coefficients $\alpha = \lambda$ in (7.20) through (7.23) for the arbitrary point P moving exclusively with the speed of light [Condition 183, (7.12) and Condition 185, (7.15)].

Conclusion 194 ***Singularity of Lorentz transformations relative to the light speed*** *[226], [231]*

The value of the light speed was assumed a priory invariant relative to the transformations of both the temporal and the spatial coordinates,

$$c_i^i = c_j^j = c_{ij} = c_{ji} = c,$$

[Condition 184, (7.13), (7.14)]. This expresses an a priory principal inconsistency of Lorentz transformations because such invariance does not apply a priory to an arbitrary speed (Theorem 139, Subsection 5.4.2).

Conclusion 195 ***Singularity of Lorentz transformations relative to the spatial transfer speed***

The value of the spatial transfer speed was assumed a priory invariant relative to the transformations of both the temporal and the spatial coordinates,

$$v_{ji}^{i} = v_{ji}^{j} = v_{ji} = v,$$

[Condition 184, (7.13), (7.14)]. This expresses an a priory principal inconsistency of Lorentz transformations because this does not apply a priory to an arbitrary speed.

Conclusion 196 ***Strict Lorentz transformations***

The transformations (7.20) through (7.23) are the ***strict Lorentz transformations*** *if and only if the arbitrary point P moves with the light speed [Condition 183, i.e. (7.12), and Condition 185, (7.15)], that is that* ***the strict Lorentz transformations are defined by (7.12), (7.15), (7.20) through (7.23).*** *This is due to the conditions under which the scaling coefficients and the transformations were determined.*

In the strict Lorentz transformations (7.12), (7.15), (7.20) through (7.23) the position vector of the arbitrary point P is the position vector of the light signal L, $\mathbf{r}_P(t_{(.)};t_{(.)0}) \equiv \mathbf{r}_L(t_{(.)};t_{(.)0})$*, in view of (7.15). [226], [231]*

Conclusion 197 ***Slightly generalized basic Lorentz transformations***

If

° *the conditions 181 through 185 do not hold, i.e. if the scaling coefficients may be different,*

° *the arbitrary point P can move with a speed different from the light speed, and/or if*

° *the values of both the light speed and the spatial transfer speed are not a priory accepted invariant relative to time axes and to the inertial frames,*

then the basic Lorentz transformations (7.16) through (7.19) should be replaced by the ***slightly generalized basic Lorentz transformations*** *(7.6) through (7.9). [226], [231]*

Conclusion 198 ***Determination versus applications of the scaling coefficients in Lorentz transformations***

The scaling factors α *and* λ *were determined for the singular case when both the arbitrary point P moves exclusively with the light speed and the light speed numerical value is invariant relative to the time unit and to the length unit. In spite of these facts, the so determined scaling factors* α *and* λ *were used in applications of Lorentz transformations (7.20) through (7.23) also for the general case when the arbitrary point P moves with an arbitrary speed, hence not necessarily with the light speed. This is an inconsistency between the two stages of Lorentz transformations. [226], [231]*

Theorem 199 ***Insufficiency of the scaling factors for the separate validity of the pairs of Lorentz transformations in general***

The value of the scaling coefficients α *and* λ, $\alpha = \lambda = 1/\sqrt{1-(v/c)^2}$, *is not sufficient either for the separate validity of the pair (7.20), (7.21) of Lorentz transformations of the temporal coordinates, or for the separate validity of the pair (7.22), (7.23) of Lorentz transformations of the spatial coordinates when the arbitrary point* P *moves with an arbitrary constant velocity* $\mathbf{v}_P^{(.)}$ *with respect to the origin* $O_{(.)}$ *of* $R^n_{(.)}$,

$$\mathbf{v}_P^{(.)} \neq \mathbf{c}, \tag{7.24}$$

[226], [231].

Theorem 200 ***Sufficiency of the scaling factors for the joint validity of Lorentz transformations in general*** *[226], [231]*

The value $1/\sqrt{1-(v/c)^2}$ *of the constant scaling coefficients* α *and* λ, $\alpha = \lambda = 1/\sqrt{1-(v/c)^2}$, *is sufficient for the joint validity of the quadruple (7.20) through (7.23) of Lorentz transformations of the temporal and spatial coordinates when the arbitrary point* P *moves with an arbitrary constant velocity* $\mathbf{v}_P^{(.)}$ *with respect to the origin* $O_{(.)}$ *of* $R^n_{(.)}$.

Time **fields, Lorentz transformations and Einstein's principles**

Time **fields and Lorentz transformations** Lorentz equations (7.20) and (7.21) define mutual relationships between two *time*-invariant either homogeneous or heterogeneous *time* fields $\mathbb{T}_i(t, A_i)$ and $\mathbb{T}_j(t, A_j)$, or between two temporal environments $\mathsf{T}_i(S_i)$ and $\mathsf{T}_j(S_j)$ of the same heterogeneous *time* field $\mathbb{T}(A)$, $(S_k \neq \phi) \subseteq A \subseteq R^n$, $k = i, j$, or from a homogeneous *time* field $\mathbb{T}_i(t, B_i)$ and a heterogeneous *time* field $\mathbb{T}_j(t, B_j)$, $(B_k \neq \phi) \subseteq S_k \subseteq A \subseteq R^n$, $k = i, j$.

What were the reasons to introduce and use such transformations?

Einstein's principles The principles of the special and general relativity theory by Einstein, [144, pp. 11, 51, 52], [153, pp. 15, 108], [154], [169, pp. 130], which are cited in what follows, enable us to reply partially to the question. Other replies are given in the works by Lorentz, [297] through [301], and by Poincaré, [383], [386]. For more details see Subsection 7.3.3: "Einstein's principles and misusage".

Principle 201 ***The special relativity principle***

If, relative to R^n_i, R^n_j *is a uniformly moving co-ordinate system devoid of rotation, then natural phenomena run their course with respect to* R^n_j *according to exactly the same general laws as with respect to* R^n_i. *This statement is called the* ***principle of relativity*** *(in the restricted sense).*

Albert EINSTEIN *[144, p. 11], [153, p. 15], [154, p.13], [155, p. 16]*

Comment 202 ***Natural laws and frames***

The essence of Einstein's special relativity principle is that the natural phenomena and laws do not depend on uniformly moving frames. It does not concern mathematical models of natural phenomena and laws.

Principle 203 ***Principe de la relativité généralisé***

Au contraire, nous entendrons par "principe de la relativité généralisé" l'affirmation suivante : Quels que soient leurs mouvements, tous les systèmes de référence K, K' sont équivalents au point de vue de l'expression des lois de la nature.

Albert EINSTEIN *[144, p. 52], [153, p. 108]*

English translation reads:

Principle 204 ***The general relativity principle***

On the contrary, we shall understand by "the generalized relativity principle" the following statement: Whatever are their movements, all reference systems K, K' are equivalent from the point of view of the expression of natural laws.

Conclusion 205 ***Einstein's statement of his general relativity principle supports Galilean - Newtonian physics***

Galilean - Newtonian physics is valid in the framework of Galilean - Newtonian coordinate systems. Let any of them be denoted as K. Einsteinian relativity theory holds in the framework of Lorentzian coordinate systems. Let any Lorentzian coordinate system be K'. Einstein's above statement (Claims 203, 204) of his general relativity principle implies that they, K and K', are equivalent from the point of view of the expression of natural laws. Consequently, Galilean - Newtonian physics is valid, i.e. its expressions of the natural laws on velocity, acceleration, mass, force and energy can be only equivalent to those of Einsteinian relativity theory, but not wrong.

7.2.2 Homogenous forms of Lorentz transformations

Variable velocity of the arbitrary point P

The coordinate transformations should hold not only for a constant but also for a variable velocity $\mathbf{v}_P(t_{(.)}; t_{(.)0})$ of the arbitrary point P relative to an integral space $I_{(.)} = T_{(.)} \times R^n_{(.)}$.

We can express the value of the instantaneous position

$$\mathbf{r}_P(t_{(.)}; t_{(.)0}) = \int_{t_{(.)0}}^{t_{(.)}} \mathbf{v}_P(\sigma; t_{(.)0}) d\sigma \tag{7.25}$$

of the arbitrary point P relative to the origin $O_{(.)}$ in terms of $(t_{(.)} - t_{(.)0})$ by using the instantaneous average velocity $\widetilde{\mathbf{v}}_P(t_{(.)}; t_{(.)0})$, (6.7) through (6.9), together with (7.25):

$$\mathbf{r}_P(t_{(.)}; t_{(.)0}) = \widetilde{\mathbf{v}}_P(t_{(.)}; t_{(.)0})(t_{(.)} - t_{(.)0}). \tag{7.26}$$

This transforms (7.20) through (7.23) into their homogeneous forms (7.27) through (7.30):

$$(t_i - t_{i0}) = \mu_{ij}[\widetilde{v}_P(t_j; t_{j0})](t_j - t_{j0}), \ \mu_{ij}[\widetilde{v}_P(t_j; t_{j0})] = \frac{1 + \frac{v\widetilde{v}_P(t_j; t_{j0})}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{7.27}$$

$$(t_j - t_{j0}) = \mu_{ji}[\widetilde{v}_P(t_i;t_{i0})](t_i - t_{i0}),\ \mu_{ji}[\widetilde{v}_P(t_i;t_{i0})] = \frac{1 - \frac{v\widetilde{v}_P(t_i;t_{i0})}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}. \quad (7.28)$$

$$\mathbf{r}_P(t_i;t_{i0}) = \eta_{ij}[\widetilde{v}_P(t_j;t_{j0})]\mathbf{r}_P(t_j;t_{j0}),\ \eta_{ij}[\widetilde{v}_P(t_j;t_{j0})] = \frac{1 + \frac{v}{\widetilde{v}_P(t_j;t_{j0})}}{\sqrt{1-\frac{v^2}{c^2}}}, \quad (7.29)$$

$$\mathbf{r}_P(t_j;t_{j0}) = \eta_{ji}[\widetilde{v}_P(t_i;t_{i0})]\mathbf{r}_P(t_i;t_{i0}),\ \eta_{ji}[\widetilde{v}_P(t_i;t_{i0})] = \frac{1 - \frac{v}{\widetilde{v}_P(t_i;t_{i0})}}{\sqrt{1-\frac{v^2}{c^2}}}. \quad (7.30)$$

The preceding equations, (7.27) through (7.30), show that Lorentz transformations are *time*-varying homogeneous linear transformations as soon as the arbitrary point P moves with a variable speed. The coefficients $\mu_{(.)}$ and $\eta_{(.)}$ depend on the instantaneous average value of the speed of the point P. They depend on a choice of the arbitrary point P, which is a reason to call Lorentz transformations (7.27) through (7.30) **non-uniform** (**over** $\mathrm{R}^n_{(.)}$) in general.

Constant velocity of the arbitrary point P

Note 206 *We accepted that the velocity* $\mathbf{v}_P$ *of the arbitrary point* P *is constant,*

$$\mathbf{v}_P^{(.)} = v_P^{(.)}\mathbf{u} = \mathbf{const.}, \quad (7.31)$$

so that the average velocity is also constant:

$$\mathbf{v}_P^{(.)} = \mathbf{const.} \Longrightarrow \widetilde{\mathbf{v}}_P(t_{(.)};t_{(.)0}) \equiv \mathbf{v}_P^{(.)}. \quad (7.32)$$

We can express the value of the instantaneous position of the arbitrary point P relative to the origin $\mathrm{O}_{(.)}$ in terms of $(t_{(.)} - t_{(.)0})$ by using the constancy of the velocity $\mathbf{v}_P$, together with (7.25) and (7.26):

$$\mathbf{r}_P(t_{(.)};t_{(.)0}) = r_P(t_{(.)};t_{(.)0})\mathbf{u} = \mathbf{v}_P^{(.)}(t_{(.)} - t_{(.)0}) = v_P^{(.)}(t_{(.)} - t_{(.)0})\mathbf{u}. \quad (7.33)$$

This transforms the equations (7.20) through (7.23) into their homogeneous forms (7.34) through (7.37):

$$(t_i - t_{i0}) = \mu_{ij}(t_j - t_{j0}),\ \mu_{ij} = \frac{1 + \frac{vv_P^j}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = const., \quad (7.34)$$

$$(t_j - t_{j0}) = \mu_{ji}(t_i - t_{i0}),\ \mu_{ji} = \frac{1 - \frac{vv_P^i}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = const., \quad (7.35)$$

$$\mathbf{r}_P(t_i;t_{i0}) = \eta_{ij}\mathbf{r}_P(t_j;t_{j0}),\ \eta_{ij} = \frac{1 + \frac{v}{v_P^j}}{\sqrt{1-\frac{v^2}{c^2}}} = const., \quad (7.36)$$

$$\mathbf{r}_P(t_j;t_{j0}) = \eta_{ji}\mathbf{r}_P(t_i;t_{i0}),\ \eta_{ji} = \frac{1-\frac{v}{v_P^i}}{\sqrt{1-\frac{v^2}{c^2}}} = const. \tag{7.37}$$

The equations, (7.34) through (7.37), show that Lorentz transformations (7.20) - (7.23) are ordinary *time*-invariant homogeneous linear transformations (hence, with constant gains) as soon as the arbitrary point P moves with a constant speed. The scaling coefficients $\mu_{(.)}$ and $\eta_{(.)}$ are constant but their values depend on a choice of the point P via its speed. They are independent of the current moment $t_{(.)}$. The homogeneous forms (7.34) through (7.37) of Lorentz transformations show that they are nonuniform (over $R^n_{(.)}$) in general since they depend on a choice of the arbitrary point P via its speed.

The arbitrary point P moves with the light velocity

Note 207 *If we restrict the choice of the arbitrary point P only to points that move with the speed of light, i.e. we demand for the speed $v_P^{(.)}$ of the point P to be equal to the speed of light $c_{(.)}^{(.)}$ and that the light speed is invariant relative to integral spaces, $v_P^{(.)} \equiv c_{(.)}^{(.)} \equiv c$. This is the singular case because we require that every arbitrary point moves exclusively with the speed of light and that the light speed value is invariant relative to integral spaces. This means that we then deal with the strict Lorentz transformations (7.12), (7.15), (7.20) through (7.23).*

If the velocity $\mathbf{v}_P(t_{(.)};t_{(.)0})$ of the arbitrary point P is the velocity $\mathbf{c}$ of light,

$$\mathbf{v}_P(t_{(.)};t_{(.)0}) \equiv \mathbf{c} = c\mathbf{u},$$

then we can express the instantaneous position $\mathbf{r}_P(t_{(.)};t_{(.)0})$, (7.25), of the point P relative to the origin $O_{(.)}$ in terms of $(t_{(.)} - t_{(.)0})$ by using the light velocity $\mathbf{c}$:

$$\mathbf{r}_P(t_{(.)};t_{(.)0}) = \mathbf{r}_L(t_{(.)};t_{(.)0}) = \mathbf{c}(t_{(.)} - t_{(.)0}).$$

This transforms the equations (7.20) through (7.23) into their homogeneous forms (7.38) through (7.41):

$$(t_i - t_{i0}) = \mu_{ij}(t_j - t_{j0}),\ \mu_{ij} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = const., \tag{7.38}$$

$$(t_j - t_{j0}) = \mu_{ji}(t_i - t_{i0}),\ \mu_{ji} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = const., \tag{7.39}$$

$$\mathbf{r}_P(t_i;t_{i0}) = \mathbf{r}_L(t_i;t_{i0}) = \eta_{ij}\mathbf{r}_P(t_j;t_{j0}),\ \eta_{ij} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = const., \tag{7.40}$$

$$\mathbf{r}_P(t_j;t_{j0}) = \mathbf{r}_L(t_j;t_{j0}) = \eta_{ji}\mathbf{r}_P(t_i;t_{i0}),\ \eta_{ji} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = const. \tag{7.41}$$

The equations, (7.38) through (7.41), prove that the strict Lorentz transformations (7.12), (7.15), (7.20) through (7.23), are ordinary *time*-invariant homogeneous linear transformations (thus, with constant gains). The coefficients $\mu_{(.)}$ and $\eta_{(.)}$ are constant. Their values do not depend on the choice of the point P. They are independent also of the current moment $t_{(.)}$. The strict Lorentz transformations (7.12), (7.15), (7.38) and (7.39) of the temporal coordinates are **uniform** (**over** $R^n_{(.)}$) since they are independent of a choice of the arbitrary point P, i.e. they hold uniformly over the space.

7.2.3 Lorentz transformations and velocity

Nonzero variable velocity

Lorentz transformations led to new formulae on velocity, [144] through [154], which differ from those known in Galilean - Newtonian mechanics. Let us discuss them for the three possible cases of the velocity of the movement of the arbitrary point P [by ignoring the fact that we know that the transformations were determined under the conditions 181 through 185, i.e. (7.10) through (7.15)].

Let us consider *time*-varying velocity / speed of the arbitrary point P rather than only *time*-invariant. The velocity $\mathbf{v}_P(.)$ of the arbitrary point P relative to the integral space $I_{(.)} = T_{(.)} \times R^n_{(.)}$, i.e. with respect to the origin $\mathrm{O}_{(.)}$ of $R^n_{(.)}$, is defined by (7.42),

$$\mathbf{v}_P^{O_{(.)}}(t_{(.)};t_{(.)0}) = \mathbf{v}_P(t_{(.)};t_{(.)0}) = \frac{d\mathbf{r}_P(t_{(.)};t_{(.)0})}{dt_{(.)}}. \tag{7.42}$$

We consider at first a variable nonzero velocity of the point P and we use (7.42), (7.20) and (7.22) in order to determine $\mathbf{v}_P^{O_i}(t_i;t_{i0}) = \mathbf{v}_P(t_i;t_{i0})$ in terms of $\mathbf{v}_P^{O_j}(t_j;t_{j0}) = \mathbf{v}_P(t_j;t_{j0})$:

$$\mathbf{v}_P(t_i;t_{i0}) = \frac{\mathbf{v}_P(t_j;t_{j0}) + \mathbf{v}}{1+\frac{v v_P(t_j;t_{j0})}{c^2}} = \frac{1+\frac{v}{v_P(t_j;t_{j0})}}{1+\frac{v v_P(t_j;t_{j0})}{c^2}} \mathbf{v}_P(t_j;t_{j0}). \tag{7.43}$$

The homogeneous form results now directly from the preceding equations:

$$\mathbf{v}_P(t_i;t_{i0}) = \gamma_{ij}[v_P(t_j;t_{j0})]\mathbf{v}_P(t_j;t_{j0}),\ \gamma_{ij}[v_P(t_j;t_{j0})] = \frac{1+\frac{v}{v_P(t_j;t_{j0})}}{1+\frac{v v_P(t_j;t_{j0})}{c^2}}. \tag{7.44}$$

Similarly, (7.42), (7.21) and (7.23) determine $\mathbf{v}_P(t_j;t_{j0})$ in terms of $\mathbf{v}_P(t_i;t_{i0})$:

$$\mathbf{v}_P(t_j;t_{j0}) = \frac{\mathbf{v}_P(t_i;t_{i0}) - \mathbf{v}}{1-\frac{v v_P(t_i;t_{i0})}{c^2}} = \frac{1-\frac{v}{v_P(t_i;t_{i0})}}{1-\frac{v v_P(t_i;t_{i0})}{c^2}} \mathbf{v}_P(t_i;t_{i0}), \tag{7.45}$$

so that the homogeneous form is found as:

$$\mathbf{v}_P(t_j;t_{j0}) = \gamma_{ji}[v_P(t_i;t_{i0})]\mathbf{v}_P(t_i;t_{i0}),\ \gamma_{ji}[\mathbf{v}_P(t_i;t_{i0})] = \frac{1-\frac{v}{v_P(t_i;t_{i0})}}{1-\frac{v v_P(t_i;t_{i0})}{c^2}}. \tag{7.46}$$

The equations (7.44) and (7.46) show the homogeneous quasi linear forms of the nonlinear equations (7.43) and (7.45) between $\mathbf{v}_P(t_i;t_{i0})$ and $\mathbf{v}_P(t_j;t_{j0})$, in which the gains $\gamma_{ij}(.)$ and $\gamma_{ji}(.)$ depend on the current moment $t_{(.)}$ via the instantaneous vector value $\mathbf{v}_P(t_{(.)};t_{(.)0})$ of the velocity of the arbitrary point P.

Notice that in this case the velocity of the arbitrary point P is qualitatively different from the light velocity because the former is variable and the latter is constant.

Remark 208 *The first equation (7.43) is modified in Einsteinian relativity theory into the following form to express the composition of the velocities* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *of two arbitrary points:* P_1 *and* P_2 *via their values* $\overline{\mathbf{v}_{12}}$, $\overline{\mathbf{v}_1}$ *and* $\overline{\mathbf{v}_2}$ *normalized relative to the light speed c,*

$$\overline{\mathbf{v}_{12}} = \frac{\overline{\mathbf{v}_1} + \overline{\mathbf{v}_2}}{1 + \overline{v_1 v_2}}, \ \overline{\mathbf{v}_{12}} = \frac{\mathbf{v}_{12}}{c}, \ \overline{\mathbf{v}_1} = \frac{\mathbf{v}_1}{c}, \ \overline{\mathbf{v}_2} = \frac{\mathbf{v}_2}{c}, \tag{7.47}$$

which is true if, and only if $\mathbf{v}_1 = \mathbf{v}_P(t_j;t_{j0})$, $\mathbf{v}_2 = \mathbf{v}$ *and* $\mathbf{v}_{12} = \mathbf{v}_P(t_i;t_{i0})$. *This means that the first equation (7.47) is correctly deduced from Lorentz transformations (7.20) through (7.23) if, and only if* $\mathbf{v}_1$ *and* $\mathbf{v}_{12}$ *represent the velocity of the same point (P) relative to two different integral spaces mutually related by Lorentz transformations (7.20) through (7.23), and* $\mathbf{v}_2$ *is the spatial transfer velocity* $\mathbf{v}$. *Consequently, the equation (7.47) does not follow from Lorentz transformations (7.20) through (7.23) for arbitrary velocities* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *(Paradox 232).*

The equation (7.47) expresses Einstein's ***addition theorem of velocities*** *[150, p. 35], also called* ***the law of the composition of velocities*** *in the literature on Einsteinian relativity theory. Since such modification does not follow from (7.43) through (7.46) for (arbitrary) velocities of arbitrary two points 1 and 2, then Einstein tried in [150, pp. 31, 32, 35] to prove (7.47) by using Minkowski's approach [86, p. 49], [98, p. 30], [150, p. 30], [154, pp. 121, 122, 150 - 153], [348, p. 88], [349, p. 350], [464, p. 271].*

Constant nonzero velocity of the arbitrary point *P*

We get directly from the preceding results (7.43), (7.44), (7.45) and (7.46) the formulae in the case the velocity of the arbitrary point is constant. In the singular case it can be the light velocity. For details see [226], [231].

7.2.4 Lorentz transformations and acceleration: paradox

We consider a variable velocity of the point P, which permits a non-trivial acceleration. The obtained acceleration transformations read, [226] - [228], [231], which is easy to verify,

$$\mathbf{a}_P(t_i;t_{i0}) = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left[1 + \frac{v v_P(t_j)}{c^2}\right]^3} \mathbf{a}_P(t_j;t_{j0}), \ \mathbf{a}_P(t_j;t_{j0}) = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left[1 - \frac{v v_P(t_i;t_{i0})}{c^2}\right]^3} \mathbf{a}_P(t_i;t_{i0}). \tag{7.48}$$

Their homogeneous forms result now directly from the preceding equations:

$$\mathbf{a}_P(t_i;t_{i0}) = \xi_{ij}[v_P(t_j;t_{j0})]\mathbf{a}_P(t_j;t_{j0}),\ \mathbf{a}_P(t_j;t_{j0}) = \xi_{ji}[v_P(t_i;t_{i0})]\mathbf{a}_P(t_i;t_{i0}),$$

$$\xi_{ij}[v_P(t_j;t_{j0})] = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left[1+\frac{vv_P(t_j;t_{j0})}{c^2}\right]^3},\ \xi_{ji}[v_P(t_i;t_{i0})] = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left[1-\frac{vv_P(t_i;t_{i0})}{c^2}\right]^3}. \quad (7.49)$$

7.2.5 Compatibility problem in Einsteinian relativity theory

Results on compatibility properties of Lorentz transformations

Theorem 209 ***Partial compatibility of the strict Lorentz transformations***

The strict Lorentz transformations (7.12), (7.15), (7.20) through (7.23) can be only partially compatible.

Theorem 210 ***Partial pairwise compatibility of Lorentz transformations***

Lorentz transformations (7.20) through (7.23) are partially pairwise compatible. They are not completely pairwise compatible.

Theorem 211 ***Partial entire compatibility of Lorentz transformations***

a) Lorentz transformations (7.20) through (7.23) are partially entirely compatible.

b) The strict Lorentz transformations (7.12), (7.15), (7.20) through (7.23) are restrictively entirely compatible.

Results on compatibility properties of the velocity transformations

Definition 212 ***Compatibility of the velocity transformations***

The velocity transformations (7.43) and (7.45), are ***compatible*** *if, and only if they yield an identity as soon as one velocity variable is eliminated from them. Otherwise, they are* ***incompatible****.*

Since the general case incorporates the special case, we will consider compatibility of the velocity transformations by permitting *time* varying velocity of the arbitrary point P.

Theorem 213 ***Velocity transformations compatibility***

The velocity transformations (7.43) and (7.45), which result from Lorentz transformations (7.20) through (7.23), are partially compatible.

Results on compatibility properties of the acceleration transformations

Definition 214 ***Compatibility of the acceleration transformations***
The acceleration transformations (7.48) are **compatible** *if and only if they yield the identity as soon as all the variables with the same subscripts are eliminated from them. Otherwise, they are* **incompatible**.

Theorem 215 ***Acceleration transformations compatibility***
The acceleration transformations (7.48), which result from the speed transformations (7.43) and (7.45), hence from Lorentz transformations (7.20) through (7.23), are partially compatible.

7.3 Failure of Einstein‘s Relativity Theory

7.3.1 Inapplicability of Lorentz transformations

Lorentz and Einstein exploited the transformations of the temporal coordinates. In spite of this they used the light speed as invariant not only relative to length units but also relative to *time* units. We follow their approach in the sequel by considering various cases.

Case 216 ***The spatial frames are mutually at rest***
If the frames R^n, R_i^n and R_j^n are all mutually at rest: $v = 0$ then Lorentz transformations vanish so that there is not any nonidentity coordinate transformation. Lorentz transformations of the time coordinate reduce to $t_j = t_i = t$.

Case 217 ***Different time units. Spatial frames are at mutual rest. The same length unit.***
We accept in this case that all spatial frames are mutually at rest and carry the same length unit. The time unit is only changed. Lorentz transformations vanish. However, it is shown (Subsection 8.2 "Dynamical systems based approach to the relativity") that the transformations fulfill both the condition (6.22) and Einstein's condition (7.1

Conclusion 218 ***Spatial frames at rest prevent temporal transformations***
All spatial frames at mutual rest prevent any change of the temporal coordinates in the framework of Einsteinian relativity theory. Then Lorentz transformations reduce to the identity transformation.

Case 219 ***Same time axes and inertial frames in a mutual relative movement***
If both the frames are mutually moving, $v \in R^+$, and the temporal coordinate transformations are the identity transformation,

$$(t_i - t_{i0}) = (t_j - t_{j0}) = (t - t_0), \tag{7.50}$$

then the basic Lorentz transformations (7.16) through (7.19) and Lorentz transformations (7.20) through (7.23) are not valid.

This is the case of Galilean-Newtonian spatial coordinate transformations. They satisfy (Theorem 291) the general distance preservation condition (6.22) (Condition 152 in Section "Basic mathematical problem" 6.4). This means that they satisfy also Einstein's generalized condition (7.1) even if the light velocity $\mathbf{c}$ *is in it replaced by the velocity* $\mathbf{v}_{Pt}^{(.)}$ *of the arbitrary point P, so that the identity in (7.1) becomes for* $t_i = t_j = t$ *:*

$$\begin{bmatrix}\mathbf{r}_i^T(t) & t\mathbf{v}_{Pt}^{i^T}\end{bmatrix} D \begin{bmatrix}\mathbf{r}_i^T(t) & t\mathbf{v}_{Pt}^{i^T}\end{bmatrix}^T \equiv \begin{bmatrix}\mathbf{r}_j^T(t) & t\mathbf{v}_{Pt}^{j^T}\end{bmatrix} D \begin{bmatrix}\mathbf{r}_j^T(t) & t\mathbf{v}_{Pt}^{j^T}\end{bmatrix}^T,$$

or, if the notation for the arbitrary point P in the position vectors $\mathbf{r}_P(t)$ *is replaced by the notation* $\mathbf{r}_L(t)$ *for the light signal L, then the identity in (7.1) is set into the following form in this case:*

$$\begin{bmatrix}\mathbf{r}_L^T(t) & t\mathbf{c}^T\end{bmatrix} D \begin{bmatrix}\mathbf{r}_L^T(t) & t\mathbf{c}^T\end{bmatrix}^T \equiv 0 \equiv \begin{bmatrix}\mathbf{r}_L^T(t) & t\mathbf{c}^T\end{bmatrix} D \begin{bmatrix}\mathbf{r}_L^T(t) & t\mathbf{c}^T\end{bmatrix}^T.$$

Conclusion 220 ***Same time axes and inertial spatial frames reject Lorentz transformations***

Lorentz transformations are inapplicable if both the time axes are equal (i.e. there is not a transformation of the temporal coordinate and of time unit) and the inertial spatial frames are in relative mutual movements.

Galilean-Newtonian transformations and Lorentz transformations are essentially different. The former concern the spatial coordinates only and satisfy Einstein's condition (7.1) in the cases explained above. The latter transform simultaneously, in the same way and in the same ratio, both the temporal and the spatial coordinates.

Case 221 ***Simple physical systems and Lorentz transformations***

Lorentz transformations are inapplicable in the physical sense to the real systems considered in the Example 89 because all movements occur in the same, unique time, and the same time axis is valid for all of them, Conclusion 91 (for other examples see the books [226], [231]).

Case 222 ***Different time units. Spatial frames are in relative movements. The same length unit.***

We change only the time unit, but we retain the same length unit. The spatial frames are in a mutual relative movement. Both the temporal and the spatial coordinates are transformed. The coordinate transformations obey the condition (6.22) (Section 8.3 "Generalized Galilean - Newtonian approach"). Hence, they satisfy Einstein's condition (7.1) as soon as either the light velocity $\mathbf{c}$ *is in it replaced in general by the velocity* $\mathbf{v}_{Pt}^{(.)}$ *of the arbitrary point P, or if the position vectors* $\mathbf{r}_P(t_i; t_{i0})$ *and* $\mathbf{r}_P(t_j; t_{j0})$ *represent the position vectors of the light signal L, i.e. P=L and* $\mathbf{r}_P(t_{(.)}; t_{(.)0}) \equiv \mathbf{r}_L(t_{(.)}; t_{(.)0})$.

Note 223 *The transformations in the preceding case apply to both temporal and spatial coordinates. They are inherently different from Lorentz transformations, because there was not a change of the length unit.*

Note 224 *If we wished to apply transformations of both temporal and spatial coordinates to the mathematical models of the physical (mechanical) systems, then we could apply any transformations presented herein. They need not be Lorentz transformations (7.20) through (7.23). The transformations of mathematical models cannot influence either the physical reality or the basic, original, mathematical models.*

Note 225 *The preceding cases show that Lorentz transformations are not any exceptional transformations that obey Einstein's condition (7.1).*

Case 226 ***Different time units and different time axes. Spatial frames are at relative movements. Different length units and spatial frames.***

This case is the topic of the next parts of the book.

Conclusion 227 ***Inapplicability of Lorentz transformations***

The above results show that Lorentz transformations (7.20) through (7.23) are not applicable as soon as:

° *there is not a change of the spatial coordinates (requiring* $v = 0$*),*
and/or

° *the time axes are the same, i.e. there is not a change of the temporal coordinate, (7.51) (valid only for* $v = 0$*).*

Lorentz transformations are inapplicable in the physical sense to the Example 89 (for other examples see the book [226]).

Conclusion 228 ***Inapplicability of Einstein's law of velocity composition***

Einstein's formulae (7.43) and (7.45) of the transformation of velocities are inapplicable under (7.51),

$$t_i = t_j, \tag{7.51}$$

i.e. they are invalid when there is not a change of the time axis (hence, of the time unit).

Note 229 *All the above results are valid also for time-varying transformations and velocities / speeds [227], [228].*

7.3.2 Paradoxes of Lorentz transformations

So you see that the power of truth is such that when you try to attack it, your very assaults reinforce and validate it.

Galileo GALILEI [191, p. 203]

All statements made by Einstein with regard to events and processes in relation to a system in which the observer is at rest are necessarily

based on the classical time concept.

....

The result is that Einstein's Theory of Relativity is based on the indiscriminate use of the word 'time' in two different meanings

which makes his Theory untenable from a logical point of view.
Harald NORDENSON [361, p. 120]

Paradox 230 ***Lorentz-Einstein‘s general paradox on the scaling coefficients: time-varying speeds***
Equations (7.27) and (7.28) imply for $v > 0$, $\widetilde{v}_P(t_j; t_{j0}) \neq c$ *and* $\widetilde{v}_P(t_i; t_{i0}) \neq c$:

$$\mu_{ij}[\widetilde{v}_P(t_j; t_{j0})]\mu_{ji}[\widetilde{v}_P(t_i; t_{i0})] = \frac{1+\frac{v\widetilde{v}_P(t_j;t_{j0})}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1-\frac{v\widetilde{v}_P(t_i;t_{i0})}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \neq 1. \tag{7.52}$$

This violates both the general time scaling coefficients rule (4.4) and the special time scaling coefficients rule (4.36), (4.37). In order for the rules to be satisfied, i.e., for:

$$\mu_{ij}[\widetilde{v}_P(t_j; t_{j0})]\mu_{ji}[\widetilde{v}_P(t_i; t_{i0})] = \frac{1+\frac{v\widetilde{v}_P(t_j;t_{j0})}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1-\frac{v\widetilde{v}_P(t_i;t_{i0})}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = 1 \tag{7.53}$$

to hold it is necessary and sufficient that the temporal transfer speed vanishes: $v = 0$. *However, for* $v = 0$ *Lorentz temporal transformations (7.20) and (7.21) reduce to the identities:*

$$t_i - t_{i0} = t_j - t_{j0},\ t_j - t_{j0} = t_i - t_{i0},$$

which means that there are not transformations of the time coordinates. The whole Einsteinian theory of time relativity fails to exist by reducing to a part of Galilean-Newtonian physics.

Equations (7.29) and (7.30) imply for $v > 0$, $\widetilde{v}_P(t_j; t_{j0}) \neq c$ *and* $\widetilde{v}_P(t_i; t_{i0}) \neq c$:

$$\eta_{ij}[\widetilde{v}_P(t_j; t_{j0})]\eta_{ji}[\widetilde{v}_P(t_i; t_{i0})] = \frac{1+\frac{v}{\widetilde{v}_P(t_j;t_{j0})}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1-\frac{v}{\widetilde{v}_P(t_i;t_{i0})}}{\sqrt{1-\frac{v^2}{c^2}}} \neq 1.$$

This violates the general scaling coefficients rule (4.4). In order for the rule to be fulfilled, i.e., for:

$$\eta_{ij}[\widetilde{v}_P(t_j; t_{j0})]\eta_{ji}[\widetilde{v}_P(t_i; t_{i0})] = \frac{1+\frac{v}{\widetilde{v}_P(t_j;t_{j0})}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1-\frac{v}{\widetilde{v}_P(t_i;t_{i0})}}{\sqrt{1-\frac{v^2}{c^2}}} = 1$$

to hold it is necessary and sufficient that the spatial transfer speed vanishes: $v = 0$. *However, for* $v = 0$ *Lorentz spatial transformations (7.22) and (7.23) become the identities:*

$$\mathbf{r}_P(t_i; t_{i0}) = \mathbf{r}_P(t_j; t_{j0}), \mathbf{r}_P(t_j; t_{j0}) = \mathbf{r}_P(t_i; t_{i0}),$$

which signifies that there are not transformations of the space coordinates. The whole Einsteinian theory of time relativity fails to exist by becoming only a part of Galilean-Newtonian physics.

This paradox is the direct consequence of Einstein‘s stringent conditions (7.10) and (7.11) on the scaling coefficients.

Paradox 231 ***Lorentz-Einstein‘s special paradox on the scaling coefficients: constant speeds***

Equations (7.34) and (7.35) imply for $v > 0$, $v_P^i \neq c$ *and* $v_P^j \neq c$:

$$\mu_{ij}\mu_{ji} = \frac{1+\frac{vv_P^j}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1-\frac{vv_P^i}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \neq 1. \tag{7.54}$$

This violates both the general time scaling coefficients rule (4.4) and the special time scaling coefficients rule (4.36), (4.37). In order for the rules to be satisfied, i.e., for:

$$\mu_{ij}\mu_{ji} = \frac{1+\frac{vv_P^j}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1-\frac{vv_P^i}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = 1$$

to hold it is necessary and sufficient that the temporal transfer speed vanishes: $v = 0$. *However, for* $v = 0$ *Lorentz temporal transformations (7.20) and (7.21) reduce to the identities:*

$$t_i - t_{i0} = t_j - t_{j0},\ t_j - t_{j0} = t_i - t_{i0},$$

which means that there are not transformations of the time coordinates. The whole Einsteinian theory of time relativity fails to exist by reducing to a part of Galilean-Newtonian physics.

Equations (7.36) and (7.37) imply for $v > 0$, $v_P^i \neq c$ *and* $v_P^j \neq c$:

$$\eta_{ij}\eta_{ji} = \frac{1+\frac{v}{v_P^j}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1-\frac{v}{v_P^i}}{\sqrt{1-\frac{v^2}{c^2}}} \neq 1. \tag{7.55}$$

This violates the general scaling coefficients rule (4.4). In order for the rule to be fulfilled, i.e., for:

$$\eta_{ij}\eta_{ji} = \frac{1+\frac{v}{v_P^j}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1-\frac{v}{v_P^i}}{\sqrt{1-\frac{v^2}{c^2}}} = 1$$

to hold it is necessary and sufficient that the spatial transfer speed vanishes: $v = 0$. *However, for* $v = 0$ *Lorentz spatial transformations (7.22) and (7.23) become the identities:*

$$\mathbf{r}_P(t_i; t_{i0}) = \mathbf{r}_P(t_j; t_{j0}), \mathbf{r}_P(t_j; t_{j0}) = \mathbf{r}_P(t_i; t_{i0}),$$

which signifies that there are not transformations of the space coordinates. The whole Einsteinian theory of time relativity fails to exist by becoming only a part of Galilean-Newtonian physics.

This paradox is the direct consequence of Einstein‘s stringent conditions (7.10) and (7.11) on the scaling coefficients.

Paradox 232 ***Einstein's velocity paradox***

The equations (7.44) and (7.46) imply

$$\gamma_{ij}[v_P(t_j;t_{j0})]\gamma_{ji}[\mathbf{v}_P(t_i;t_{i0})] = \frac{1+\frac{v}{v_P(t_j;t_{j0})}}{1+\frac{vv_P(t_j;t_{j0})}{c^2}}\frac{1-\frac{v}{v_P(t_i;t_{i0})}}{1-\frac{vv_P(t_i;t_{i0})}{c^2}} \neq 1.$$

This violates the general scaling coefficients rule (4.4). In order for the rule to be fulfilled, i.e., for:

$$\gamma_{ij}[v_P(t_j;t_{j0})]\gamma_{ji}[\mathbf{v}_P(t_i;t_{i0})] = \frac{1+\frac{v}{v_P(t_j;t_{j0})}}{1+\frac{vv_P(t_j;t_{j0})}{c^2}}\frac{1-\frac{v}{v_P(t_i;t_{i0})}}{1-\frac{vv_P(t_i;t_{i0})}{c^2}} = 1$$

to hold it is necessary and sufficient that the speed $v_P(t_{(.)};t_{(.)0})$ *of the arbitrary point* P *is the light speed:*

$$v_P(t_i;t_{i0}) \equiv v_P(t_j;t_{j0}) \equiv c.$$

Einstein's law of the velocity composition holds only for the light speed of the arbitrary point P*. However, Einstein's relativity theory forbids to the arbitrary point* P *to move with speed of light.*

This paradox is the direct consequence of Einstein's stringent conditions (7.10) through (7.14) on the scaling coefficients.

Paradox 233 ***Einstein's acceleration paradox***

The equations (7.49) imply

$$\xi_{ij}[v_P(t_j;t_{j0})]\xi_{ji}[v_P(t_i;t_{i0})] = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left[1+\frac{vv_P(t_j;t_{j0})}{c^2}\right]^3}\frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left[1-\frac{vv_P(t_i;t_{i0})}{c^2}\right]^3} \neq 1.$$

This violates the general scaling coefficients rule (4.4). In order for the rule to be fulfilled, i.e., for:

$$\xi_{ij}[v_P(t_j;t_{j0})]\xi_{ji}[v_P(t_i;t_{i0})] = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left[1+\frac{vv_P(t_j;t_{j0})}{c^2}\right]^3}\frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left[1-\frac{vv_P(t_i;t_{i0})}{c^2}\right]^3} = 1$$

to hold it is necessary and sufficient that the speed $v_P(t_{(.)};t_{(.)0})$ *of the arbitrary point* P *is the light speed:*

$$v_P(t_i;t_{i0}) \equiv v_P(t_j;t_{j0}) \equiv c.$$

Einstein's law of the acceleration composition holds only for the light speed of the arbitrary point P*. However, if the arbitrary point* P *moves with speed of light then its acceleration is equal to zero. What a paradox!*

This paradox is the direct consequence of Einstein's stringent conditions (7.10) through (7.14) on the scaling coefficients.

The analysis of Lorentz coordinate transformations, and from them resulting velocity transformations raised the following questions

Problem 234 ***Questions on invariance of the transfer velocity v***

° *What is the justification for the spatial transfer velocity* $\mathbf{v}_{ji}$, *hence for the spatial transfer speed* v_{ji}, *to be invariant,* $(\mathbf{v}_{ji} \equiv \mathbf{v},\ v_{ji} \equiv v)$, *relative to integral spaces related by Lorentz transformations (7.20) through (7.23), since*

- *v must be less than the light speed, hence different from the latter, and*
- *the speed value less than the light speed value is not invariant under Lorentz transformations?*

° *What are consequences if we permit a priory for the spatial transfer velocity* $\mathbf{v}$, *hence for the spatial transfer speed* v, *to depend on integral spaces in the basic Lorentz transformations (7.16) through (7.19), i.e. to be non-invariant relative to them?*

The replies to these questions are presented in what follows, in the next Summary 236 , and in the second and the third part of the book.

Theorem 235 ***Necessity of the invariance of the spatial transfer velocity* v *for the validity of the generalized Lorentz transformations***

In order for the transformations (7.56) through (7.59),

$$(t_i - t_{i0}) = \widehat{\alpha}[(t_j - t_{j0}) + \frac{v_{ji}^j}{c^2} r_P(t_j; t_{j0})], \tag{7.56}$$

$$(t_j - t_{j0}) = \widehat{\alpha}[(t_i - t_{i0}) - \frac{v_{ji}^i}{c^2} r_P(t_i; t_{i0})], \tag{7.57}$$

$$\mathbf{r}_P(t_i; t_{i0}) = \widehat{\lambda}[\mathbf{r}_P(t_j; t_{j0}) + v_{ji}^j (t_j - t_{j0})\mathbf{u}], \tag{7.58}$$

$$\mathbf{r}_P(t_j; t_{j0}) = \widehat{\lambda}[\mathbf{r}_P(t_i; t_{i0}) - v_{ji}^i (t_i - t_{i0})\mathbf{u}]. \tag{7.59}$$

together with (4.38) to obey (7.1) it is necessary and sufficient that (7.60) holds,

$$v_{ji}^j \equiv v_{ji}^i \equiv v_{ji} \equiv v. \tag{7.60}$$

Summary 236 ***Paradoxes of Lorentz transformations***

It was shown that Lorentz transformations guarantee the invariance of the velocity (of the speed) if and only if it is the light velocity (the light speed) (Theorem 139). They do not allow simultaneously for the transfer speed to be equal to the light speed (otherwise, the scaling coefficients become infinite) but they hold only for the invariant transfer speed. What a paradox!

Moreover, Lorentz transformations do not imply the invariance of the spatial transfer speed v.

If we permit the a priory noninvariance of the transfer speed in the basic Lorentz transformations (7.16) through (7.19) then they become (7.56) through

(7.59), which yield (7.61) for the light speed (the subscript c) of the point P,

$$\widehat{\alpha}_c = \frac{1}{\sqrt{\left(1+\frac{v_{ji}^j}{c}\right)\left(1-\frac{v_{ji}^i}{c}\right)}} = \widehat{\lambda}_c. \tag{7.61}$$

Then they can satisfy Einstein's length preservation condition (7.1) if, and only if the transfer velocity is invariant relative to integral spaces, i.e. if, and only if (7.60) holds. This is an exception from the general rule and a paradox of Lorentz transformations, hence a paradox of Einsteinian relativity theory.

Besides, the paradox shows an inconsistency of Lorentz transformations. They allow for the speed of the arbitrary point P to depend on the integral space, but they are founded on the a priory invariance of the transfer speed v.

Einsteinian relativity theory does not explain what makes the transfer speed so exceptional. Does this happen because v, together with the light speed c, completes the ratio (v/c) that intervenes in the scaling factors α and λ in Lorentz transformations (7.20) through (7.23)? If it were so, then there would be another paradox: if v represented the speed of the arbitrary point P, then it would obey the general rule, i.e. its value and its numerical value would depend on the integral space (then it would be noninvariant); but if it represented the transfer speed, then it would not obey the rule (then it would be invariant).

Theorem 237 ***Velocity/speed invariance relative to integral spaces interrelated by Lorentz transformations***

In order for a velocity $\mathbf{v}_P(t_{(.)};t_{(.)0})$ */speed* $v_P(t_{(.)};t_{(.)0})$, $\mathbf{v}_P(t_{(.)};t_{(.)0}) = v_P(t_{(.)};t_{(.)0})\mathbf{u}$, *to be invariant relative to integral spaces interrelated by Lorentz transformations and from them deduced velocity/speed transformations it is necessary and sufficient that* $\mathbf{v}_P(t_{(.)};t_{(.)0})$ *is the light velocity:* $\mathbf{v}_P(t_{(.)};t_{(.)0}) \equiv \mathbf{c}$, *i.e. thatv is the light speed:* $v_P(t_{(.)};t_{(.)0}) \equiv c$.

Proof. Let Lorentz transformations (7.20) - (7.23) hold. They induce the velocity/speed transformations (7.20) - (7.23). Let the integral spaces $\mathfrak{I} = \mathfrak{T}\times\mathfrak{R}^n$, $\mathfrak{I}_{(.)} = \mathfrak{T}_{(.)}\times\mathfrak{R}^n_{(.)}$, $(.) = i, j$, be interrelated by Lorentz transformations. They induce the velocity/speed transformations (7.43).

Necessity. Let an arbitrary velocity $\mathbf{v}_P(t_{(.)};t_{(.)0})$ /speed $v_P(t_{(.)};t_{(.)0})$,

$$\mathbf{v}_P(t_{(.)};t_{(.)0}) = v_P(t_{(.)};t_{(.)0})\mathbf{u},$$

be invariant relative to the integral spaces $\mathfrak{I}_{(.)} = \mathfrak{T}_{(.)}\times\mathfrak{R}^n_{(.)}$, $(.) = -, i, j$, interrelated by Lorentz transformations (7.20) - (7.23). In order for Lorentz transformation to be nontrivial it is necessary and sufficient that the transfer speed $v \neq 0$. We will present two approaches to the necessity part of the proof.

Approach 1: The invariance of $\mathbf{v}_P(t_{(.)};t_{(.)0})$ and the velocity/speed transformations (7.43) mean that

$$\mathbf{v}_P(t_i;t_{i0}) = \frac{\mathbf{v}_P(t_j;t_{j0}) + \mathbf{v}}{1+\frac{vv_P(t_j;t_{j0})}{c^2}} = \mathbf{v}_P(t_j;t_{j0}),$$

which implies

$$[v_P(t_j;t_{j0})+v]\,\mathbf{u}=\left[v_P(t_j;t_{j0})+\frac{vv_P^2(t_j;t_{j0})}{c^2}\right]\mathbf{u},\ i.e.,$$

$$for\ v\neq 0:\ 1=\frac{v_P^2(t_j;t_{j0})}{c^2}\Longrightarrow v_P(t_j;t_{j0})\equiv c.$$

Approach 2:

$$\mathbf{v}_P(t_j;t_{j0})=\mathbf{v}_P(t_i;t_{i0})=\frac{\mathbf{v}_P(t_j;t_{j0})+\mathbf{v}}{1+\frac{vv_P(t_j;t_{j0})}{c^2}}=\frac{1+\frac{v}{v_P(t_i;t_{i0})}}{1+\frac{vv_P(t_i;t_{i0})}{c^2}}\mathbf{v}_P(t_i;t_{i0})\Longleftrightarrow$$

$$\frac{v}{v_P(t_i;t_{i0})}=\frac{vv_P(t_i;t_{i0})}{c^2}\Longleftrightarrow v_P(t_i;t_{i0})\equiv c.$$

Sufficiency. Let $v_P(t_{(.)};t_{(.)0})\equiv c$, i.e., $\mathbf{v}_P(t_{(.)};t_{(.)0})\equiv \mathbf{c}$. The velocity/speed transformations (7.43) become:

$$\mathbf{v}_P(t_i;t_{i0})=\mathbf{c}=\frac{\mathbf{v}_P(t_j;t_{j0})+\mathbf{v}}{1+\frac{vv_P(t_j;t_{j0})}{c^{j2}}}=\frac{v_P(t_j;t_{j0})+v}{1+\frac{vv_P(t_j;t_{j0})}{c^{j2}}}\mathbf{u}=$$

$$=\frac{c+v}{1+\frac{vc}{c^2}}\mathbf{u}=c\frac{1+\frac{v}{c}}{1+\frac{v}{c}}\mathbf{u}=c\mathbf{u}=\mathbf{v}_P(t_j;t_{j0}).$$

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This Theorem confirms Theorem 139. Since Theorem 237 holds for any velocity/speed, then it holds also for the spatial transfer speed $v_{ji}^{(.)}$:

Corollary 238 ***Invariance of the spatial transfer speed***

In order for the spatial transfer speed $v_{ji}^{(.)}$ to be invariant relative to the integral spaces interrelated by Lorentz transformations (7.20) - (7.23) it is necessary and sufficient that $v_{ji}^{(.)}$ is the light speed: $v_{ji}^{(.)}\equiv c$.

Conclusion 239 ***The spatial transfer velocity is not invariant if it is different from the light speed.***

Paradox 240 ***Paradox of the invariance of the transfer speed in Lorentz transformations***

The spatial transfer speed $v_{ji}^{(.)}$ is a priory accepted invariant relative to Lorentz transformations (7.20) - (7.23), which is possible if, and only if, it is the light speed c (Corollary 238). Unfortunately, for $v_{ji}^{(.)}\equiv v\equiv c$ (7.20) - (7.23) become meaningless because then

$$\alpha=\lambda=\left(1-v^2/c^2\right)^{-1/2}=\left(1-c^2/c^2\right)^{-1/2}=1/0=\infty.$$

This paradox leads to the following:

Absurd 241 ***Einsteinian relativity theory is absurd: its debacle***
Lorentz transformations (7.20) - (7.23) are the basis of Einsteinian relativity theory. They hold exclusively for the invariant transfer speed $v_{ji}^{(.)}$ *(Condition 184):* $v_{ji}^{(.)} \equiv v$*, which is possible if, and only if, it is the light speed:* $v_{ji}^{(.)} \equiv v = c$*. Lorentz transformations (7.20) - (7.23) become undefined, meaningless, absurd, for* $v = c$*, implying the absurdity of Einsteinian relativity theory because it strictly forbids* $v = c$*.*

This summary answers the questions in the Problem 234 on the invariance of the transfer velocity. Although Absurd 241 shows the complete debacle of Einsteinian relativity theory, we continue exploring it from other points of view.

7.3.3 Einstein's paradoxes, mistakes and absurd

Paradox of the imposed light speed limitation

Lorentz' - Einstein's attitude, that the value of the light speed is invariant, *holds formally for the (numerical) value of the light speed with respect to integral spaces that are mutually related by Lorentz transformations (7.20) through (7.23).* This is the *formal invariance of the light velocity (Theorem 139).* We call the formal invariance of the light velocity ***Lorentz-Einstein invariance of the light velocity***.

Conclusion 242 ***Lorentz-Einstein invariance of the light velocity is formal and an exception***
For every speed w *there exist linear coordinate transformations between two integral spaces, which form Poincaré group, and from deduced speed transformations so that they ensure invariance of the chosen speed* w *relative to the considered integral spaces.*

Lorentz-Einstein invariance of the light velocity results from the choice of Lorentz transformation for the coordinate transformations, from the choice of the light speed to be the reference speed and from the structure of Einstein's formula for the speed transformation (Theorem 139).

The light speed is not invariant relative to arbitrary integral spaces.

Speeds bigger than the light speed are possible (Theorem 139). Einstein's attitude that the light speed is the limiting speed, is wrong.

All the results of Einsteinian relativity theory are obtained under the assumption that the arbitrary point P moves with the invariant light speed c (Condition 183, the equation 7.12). However, the same results forbid to everybody and to everything any movement with the spatial transfer speed v not smaller than the light speed c, in spite the basic results were obtained just for the light speed c of the arbitrary point P. This is an extravagant paradox of Einsteinian relativity theory (see Conclusion 88 and Absurd 241).

The analysis of Einstein's postulate that the light speed is the same relative to all time axes and in all inertial frames, i.e. the same in all integral spaces with inertial frames, follows from other points of view.

Einstein's principles and their misusage

Original statements of the principles Several Einstein's statements of his principles follow. The principles are the theoretical fundamentals of the whole Einsteinian relativity theory.

Principle 243 ***Einstein's principles***

"The following reflexions are based on the principle of relativity and on the principle of the constancy of the velocity of light. These two principles we define as follows :

1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.

2. Any ray of light moves in the "stationary" system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body. Hence

$$velocity = \frac{light\ path}{time\ interval}$$

where time interval is to be taken in the sense of the definition in § 1."

Albert EINSTEIN *[114, pp. 895, 898], [119, p. 123], [144, pp. 11, 51, 52], [151, pp. 41, 44], [156, pp. 8, 9, 14], [160, pp. 143, 147], [161, pp. 123, 124, 130], [169, pp. 173, 174], [305, p. 41]*

Remark 244 ***Natural laws are independent of inertial frames***

Einstein's above statement under the item 1. is equivalent to his statement cited in Principle 201. Therefore, Comment 202 applies to it, i.e. a choice of an inertial coordinate system cannot influence natural phenomena and natural laws. Consequently, mathematical models obtained by using coordinates of such frames cannot influence natural phenomena and natural laws. The latter are independent of the former (Principle 111).

Einstein and Infeld called both principles *nouvelles suppositions* with which *La théorie de la relativité commence* (*The relativity theory begins*)[169, pp. 173, 174].

Einstein called the second principle - *the principle of the constancy of the velocity of light* [150, p. 28], [154, p. 150], [160, pp. 255 - 257], [161, pp. 124, 130], [305, p. 41]. He referred to it also as *the law of the constancy of the velocity of light in vacuo* or *the law of propagation of light* [154, pp. 17, 19, 20].

Einstein stated that the second principle agrees with the first one [153, p. 23], [154, pp. 19, 20], [161, p. 124].

The first principle cited above is the *special relativity principle.* Einstein expressed it also in the form cited in Principle 201. He generalized it in Principles 203 and 204, or in another form as follows:

Principle 245 ***Einstein‘s principle on coordinate systems equivalency***

All Gaussian co-ordinate systems are essentially equivalent for the formulation of the general laws of nature.

Albert EINSTEIN *[153, p. 108], [154, p. 97]*

Conclusion 246 ***Galilean - Newtonian frames are Gaussean coordinate systems***

Galilean - Newtonian coordinate systems are Gaussean, as proved in Theorem 291 and summarized in Conclusion 292.

Einstein's above statement confirms that Galilean - Newtonian coordinate systems are equivalent to all Gaussean coordinate systems.

Einstein wrote the following on the importance of the principle of the constancy of the light speed for the principle of relativity:

If we want to preserve the principle of relativity, we must assume that the principle of the constancy of the velocity of light holds for any arbitrary system not in accelerated motion.

Albert EINSTEIN [161, p. 124]

Remark 247 *This Einstein's statement refers only to the constancy of the light speed, but not to its invariance.*

Einstein's confusion: the principle of the constancy or of the invariance of the light speed? Einstein equalized the total light speed with the relative light speed with respect to integral spaces containing inertial spatial frames, in vacuum.

However, the relative light speed with respect to different *time* axes and/or different inertial frames is not equal to the total light speed in general. It and its numerical value depend in principle on the accepted *time* unit and length unit, as well as on the speed of the corresponding inertial frame. Changing anyone of them, or all, the value and the numerical value of the relative light speed change in principle. The light speed and its value rest constant relative to the reference stationary frame, but they are not invariant relative either to *time* axes or inertial frames.

Einstein's claims illustrate and imply the following:

Mistake 248 ***Einstein equalized the constancy of the light speed with its invariance***

Einstein treated and used the principle of the constancy of the light speed as it were the principle of the invariance of the light speed. This has been accepted as the fundamental postulate of Einsteinian relativity theory (Condition 184). This is not only a sever confusion but essential mistake

The principle of the light speed constancy states that the light speed is not *time*-varying, i.e. that it is *time*-invariant, in vacuum, and that it is independent of the speed of its source. This means clearly that the speed of the moving source does not influence the speed of light emitted from the source. The speed of light is the same with respect to the stationary frame whatever is the speed of the source that emitted light. However, this does not, and cannot, mean that the speed of light is the same relative to all bodies (including the light source itself) uniformly moving with (arbitrary) different constant speeds. The principle does not, and cannot, state that the speed of light is the same in vacuum relative to all inertial frames whatever are their constant speeds.

Besides, Einstein's second principle itself does not claim that the numerical value of the light speed is independent of a choice of a *time* unit and/or of a length unit. Its total value with respect to the stationary frame in vacuum is $c = 2.99792458 \text{x} 10^5 Kms^{-1}$. The principle does not, and cannot, claim that the light speed has this (numerical) value in vacuum with respect to all *time* axes $T_{(.)}$ and with respect to all inertial frames $R^n_{(.)}$, i.e. with respect to all integral spaces $I_{(.)} = T_{(.)} \text{x} R^n_{(.)}$. The light speed value is constant in vacuum with respect to every integral space $I_{(.)} = T_{(.)} \text{x} R^n_{(.)}$ with the inertial frame $R^n_{(.)}$, but its constant value is relative to the integral space rather than invariant (see Subsection "*Time*, light velocity, and light speed" in 4.7.2).

Note 249 *The relative speeds of light signal with respect to a fast moving train and with respect to another light signal, all moving in parallel in the same sense, are both mutually different, and different from the light speed with respect to the railway station. This agrees with Galilean - Newtonian physics, and rejects Lorentz - Einstein invariance of the light speed as valid in general.*

The light speed independence of the speed of the light source and of the speed of an observer concerns the light speed with respect to a frame and/or environment treated as stationary (from the terrestrial point of view). Such independence is crucially different from the relativeness of the light speed with respect to the moving source itself and to the moving observer. These two phenomena (independence, i.e. invariance, of the speed, and the dependence, i.e. relativeness, of the speed) should not be a priory equalized either mutually or with the third phenomenon - the constancy of the (light) speed.

This analysis warns us to distinguish the constancy of the light speed from its invariance.

Claim 250 ***Constancy of the light speed is not its invariance***

The light speed independency of its source speed and of an observer speed, and the constancy of the light speed with respect to the stationary frame in vacuum do not, and cannot, either mean or imply the light speed invariance relative to all inertial frames and/or relative to all time axes.

The light speed in vacuum and its numerical value are constant, but they are not universal constants, they are not invariant, relative to all integral spaces that incorporate inertial spatial frames.

Either Einstein or anybody else did not prove that Lorentz transformations (7.20) through (7.23) hold for two parallel light signals moving in the same sense. The relative speed of each of them with respect to another one equals zero (Galilean - Newtonian rule), not the light speed (Einsteinian rule that the light speed is invariant). More detailed study in [226], [231] resulted into the following:

Conclusion 251 ***Application of Einsteinian results to light signals allows their overtaking***

The application of Lorentz transformations, hence of Einstein's law of the velocity composition, to light signals permits their overtaking that is impossible. In fact, Lorentz transformations, hence Einstein's law of the velocity composition, may not be applied to light signals, hence to the light speed.

The applicability either of Lorentz transformations or of Einstein's law of the velocity composition to light signals has not been, and cannot be, proved.

Consequently, Einsteinian demonstration, based on Einstein's law of the velocity composition, of the light speed invariance is not a valid proof.

Claim 252 ***Light signals obey Galilean - Newtonian law rather than Einsteinian***

Light signals do not obey Lorentz transformations and Einstein's law of the velocity composition because they permit the overtaking. Their velocities obey Galilean - Newtonian law of the velocity composition. [226]

This claim opposes crucially the basic postulate of Einsteinian relativity theory on the light speed invariance. It agrees with the results by Martin [332, pp. 53, 54] and Wesley [478, pp. 261]. It agrees with Einstein's below claim 257 on the noninvariance of the light speed relative to moving frames (e.g. when there is not a change of any unit, i.e. when Galilean - Newtonian transformations hold):

Noninvariance of the light speed and the principles of relativity Einstein claimed that the noninvariance of the light speed in vacuum violates the principles of relativity [154, pp. 18, 19], *[155, pp. 22, 23]*

The following is proved in [226], [231]:

° The validity of Galilean - Newtonian velocity transformation does not contradict the principle of relativity in Einstein's example in *[154, pp. 18, 19], [155, pp. 22, 23]*. Einstein did not succeed to disprove Galilean - Newtonian velocity law.

° The relative light speed values are constant with respect to both the rails and the carriage, but they are mutually different for the nonzero constant speed v of the carriage with respect to the rails. The constancy of the light speed is not the invariance of the light speed. (See the Note 249).

° Einstein's law of the composition of velocities does not, and cannot, disprove Galilean - Newtonian law. In this regard see also [332, p. 53] and [478, p. 261].

Theorem 253 ***Galilean - Newtonian law applied to the light speed relative to Galilean inertial frames agrees with the relativity principle***

a) The light speed obeys Galilean - Newtonian law (8.6) of the relative speed with respect to all inertial frames, including the stationary frame, for which the same time axis and the same length unit and scale are valid.

b) The form of the law is the same relative to all inertial frames, and it agrees with the relativity principle.

For an example illustrating this theorem and its proof see [226], [231].

Other direct consequences follow.

Corollary 254 ***The light speed noninvariance obeys the relativity principle***

The noninvariance of the light speed does not violate the relativity principle.

These results verify once more Claim 252 and Claim 250. They agree with Conclusion 259.

Claim 255 ***The light speed is not invariant relative to all inertial frames***

The light speed is invariant only relative to Lorentzian inertial frames, but not relative to other frames.

Theorem 253, the above claims and Corollary 254 to Theorem 253 acknowledge the recent studies by Martin [332, pp. 47, 48, 53] and Wesley [478, p. 261]. More clearly, Martin and Wesley verified, respectively, that the light velocity $\mathbf{C}'$, i.e. the light speed "c*", relative to an inertial frame moving with the velocity $\mathbf{V}$, i.e. with the speed v, satisfies Galilean - Newtonian law, respectively, $\mathbf{C}' = \mathbf{C} \pm \mathbf{V}$, depending on the senses of the vectors $\mathbf{C}$ and $\mathbf{V}$, i.e. "c*$= c - v$". These results confirm the preceding theorem and claims.

In this concern see below Conclusion 270.

We can now summarize the above analyses and results.

The principle of the constancy of the light speed claims only that the light speed is constant and that the speed of its source or of an observer cannot change the light speed in vacuum. The principle does not, and cannot, state that the light speed (value) relative to the light source, or relative to the observer, considered as unmovable (stationary), is the same as relative to a moving source, or relative to a moving observer, in general.

The principle does not claim that the light speed is invariant with respect to all integral spaces $I_{(.)} = T_{(.)} \times R^n_{(.)}$ with inertial frames $R^n_{(.)}$.

Conclusion 256 ***Einstein's principle of the light speed constancy does not either mean or imply the light speed invariance***

Claim 257 ***Einstein's claim on the relative velocity of light***

But the ray moves relatively to the initial point of k, when measured in the stationary system, with the velocity $c - v$"...

Albert EINSTEIN *[114, p. 900], [151, p. 45], [156, p. 15], [160, pp. 148], [269].*

Martin noted Einstein's inconsistency as follows:

Time $t = 50$ of reception of light by m is calculated by Einstein and all others by:

$$t = \frac{AM}{c_0 - v} = \frac{10}{1 - 0.8} = 50, \quad x = c_0 t = 50. \tag{1}$$

....

Einsteinians use equation (1) to calculate t, which is very different from Einstein's own formula for addition of velocities—a glaring inconsistency!

Adolphe MARTIN [332]

It disproves the statement that the light velocity is invariant relative to all inertial frames. It agrees with the common sense and the experience.

Claim 258 ***Absurd of Lorentz-Einstein invariance of the light velocity***

Lorentz-Einstein invariance of the light velocity (Condition 184) is wrong. It represents a physical absurd stating that the light speed is the same relative to a stationary body, relative to a slowly moving body, relative to a high speed moving vehicle.

Conclusion 259 ***The light velocity is relative***

The light velocity with respect to inertial frames is in general noninvariant, hence relative rather than total (absolute).

Let us note immediately that these conclusions are valid for the Doppler effect as shown recently by Laski [286, p. 351] and Martin [332, pp. 53, 54] (see Comment 299).

The above results agree with the analysis of Subsection 4.9.

The noninvariance of the light speed does not violate the principle of relativity.

Galilean - Newtonian law of the velocity composition for frames, signals or bodies in mutual translational motions obeys the principle of relativity.

Conclusion 260 ***Einstein's principle of relativity does not disprove Galilean - Newtonian law of the velocity composition, neither vice versa holds. They are in mutual agreement*** *[226], [231].*

The books [226], [231] discover and prove various paradoxes or mistakes in Einstein's proofs. For example:

Conclusion 261 ***Einstein's simple mistake disproves the basic proof of his famous paper*** *[114]****.***

In this concern see also the analyses by Ceapa [60] and Keswani [265, pp. 138, 139].

Famous Fizeau's experimental result disagrees with Einstein's, which is contrary to Einstein's claim, [226], [231].

Conclusion 262 ***Fizeau's experimental result does not verify Einstein's law of the velocity composition.*** *[226], [231]*

This conclusion verifies Conclusion 259.

On Einstein's interpretation of the experiments by Michelson and Morley

Michelson's experiment suggested the assumption that, relative to a coordinate system moving along with Earth, and, more generally, relative to any system in nonaccelerated motion, all phenomena proceed according to exactly identical laws. Henceforth, we will call this assumption in brief "the principle of relativity".

Albert EINSTEIN [160, p. 383]

The law of the constancy of the speed of light, corroborated through the development of electrodynamics and optics, combined with Michelson's famous experiment that decisively demonstrated the equality of all inertial systems (principle of special relativity), relativized the concept of time, where every inertial system had to be given its own special time.

Albert EINSTEIN [168, p. 238]

The most important of these experiments are those of Michelson and Morley, ...

Albert EINSTEIN [150, p. 25]

The famous Michelson's [344] and Michelson's - Morley's [345] experiments have been considered as the doubtless experimental verifications of Einsteinian relativity theory [28, p. 258], [40, pp. 2 - 4], [41, pp. 14 - 16, 25, 107], [86, p. 59], [101, pp. 10, 23], [111, pp. 19 - 21], [116, pp. 253, 257], [119, pp. 120, 121], [150, p. 25], [153, pp. 58, 59], [154, pp. 53, 147], [160, pp. 253, 257], [161, pp. 121, 540, 545], [164, pp. 460, 526, 527], [166, p. 238], [169, p. 170], [240, p. 20 (English edition)], [242, pp. 57, 65 (French edition)], [269, pp. 6 - 9, 91, 119, 121, 139, 140, 231], [284, pp. 11 - 17], [305, pp. 3 - 6], [351, pp. 8 - 13, 15, 81, 82, 161, 384], [361, p. 100], [369, pp. 32 - 37], [404, pp. 170, 195, 201, 202, 260, 261], [415, pp. 9 - 11], [416, pp. 9, 11], [444, pp. 10, 94, 95, 133, 257, 261], [460, p. 14], [482, pp. 115, 118], [487, pp. 231 - 234, 254, 255], [493, pp. 51 - 53].

Michelson - Morley experiment conforms with this postulate but does not itself constitute definitive proof, because it is not possible to verify that velocity of light is the same in both directions, only that the total time taken for the round trip is invariant with respect to orientation of the instrument.

A. F. KRACKLAUER [279, p. 333]

Marmet [331] has recently discovered the overlooked phenomena in Michelson - Morley experiment, [345], and concluded as follows:

6 - Analysis of the new results

We have shown here that, in Michelson - Morley experiment, using classical physics, the time for light travel between any pair of mirrors, in any di-

rection, is always the same, independently of the direction of the moving frame and also independently of having light moving either parallel or transverse to the frame velocity.

Paul MARMET [331, p. 47]

In this regard see also the works by Ceapa [60], [62, p. 83].

The analyses and results of Ceapa [60], [62, p. 83], Kracklauer [279, p. 333], Marmet [331], Martin [332] and Wesley [478], and the above results, open the problem of a rigorous repetition, the exact investigation and the adequate interpretation of Michelson - Morley experiments, as well as of Trouton - Noble experiment [351, pp. 13, 14, 161 - 163] and of Barashenkov - Kapuscik - Lablin experiment [21, pp. 196 - 198]. The following analysis confirms this comment.

Michelson and Morley wrote:

If, therefore, an apparatus is so constructed as to permit two pencils of light, which have traveled over paths at right angles to each other, to interfere, the pencil which has traveled in the direction of Earth's motion, will in reality travel $\frac{4}{100}$ *of a wavelength farther than it would have done, were Earth at rest.*

....

The conditions for producing interference of two pencils of light which had traversed paths at right angles to each other were realized in the following simple manner.

Albert A. MICHELSON [344, p. 93]

If then the paths ab *and* ac *are equal, the two rays interfere along* ad.

Albert A. MICHELSON and Edward W. MORLEY [345, p. 335]

The analysis of the experiments by Michelson and Morley in [226], [231] shows that the light signals could not meet at the glass plate in any position of the interferometer. They could not arrive in any position of the glass plate at the same moment. Therefore, their interference was not possible in principle due to the following absurds: [226], [231],

Absurd 263 ***The speed v of Earth vanishes***

In order for the two light signals to interfer in Michelson - Morley experiments it is necessary and sufficient that the speed v of Earth is equal to zero: v=0, i.e. that Earth is at rest.

Absurd 264 ***The distance absurd***

Since the speed v of Earth is positive: v>0, then for the two light signals to interfer in Michelson - Morley experiments it is necessary and sufficient that

$$d = (v/c)\, D/\, [1 - (v/c)] = (v/c)\, D/\, [1 + (v/c)] = d_1, \qquad (7.62)$$

which is impossible due to v>0.

Neither Einstein himself nor the whole Einsteinian relativity theory claimed the above absurd. They both avoided the absurd by ignoring their own basic postulate. They followed and accepted Einstein's attitude to use the principle of the constancy of the light speed as it were the principle of the invariance of the light speed, and by it the induced above analysis. This itself is a sufficient proof to reject Einstein‘s attitude on the invariance of the light speed.

The absurd is physically clear: the light propagated with the speed $V = c$ relative to the stationary xy-frame and with the following relative speeds with respect to Earth and the interferometer: $V - v = c - v$ and $V + v = c + v$ in the forward and the backward direction, respectively. This was used not only by Michelson and Morley, but has been acknowledged by Einstein and by the whole Einsteinian relativity theory in concern with the experiments by Michelson and Morley. This itself is another sufficient proof to reject Einstein's attitude that the light speed is invariant.

Remark 265 ***The interference of two light pencils was not possible in Michelson's and Michelson's - Morley's experiments.****[226], [231]*

The theoretical basis of the experiments by Michelson and Morley

Michelson and Morley used Galilean - Newtonian relative speeds ($V - v = c - v$ and $V + v = c + v$) of light with respect to Earth (i.e. with respect to the moving interferometer tied with Earth) in the above analysis. This shows evidently that in their consideration the speed of light is not invariant relative to inertially moving frames and bodies. They did not claim anywhere the invariance of the light speed.

Note 266 ***Michelson's and Morley's approach reduces Einstein's law to Galilean - Newtonian law.*** *[226], [231]*

Conclusion 267 ***Galilean - Newtonian velocity law is the basis for the experiments of Michelson and Morley***

Michelson and Morley used Galilean - Newtonian law of the velocity composition, hence, the noninvariance of the light speed in the theoretical basis for their experiments. [226], [231]

How is it possible that, by following Einstein himself [116], [119], [150], [153], [154, pp. 53, 147], [160, pp. 253, 257], [161, pp. 121, 540, 545], [164, pp. 460, 526, 527], [168, p. 238], the whole Einsteinian relativity theory claims that Michelson's and Michelson's - Morley's experiments prove the light speed invariance, since their analysis is based on Galilean - Newtonian relative light speed, hence on its noninvariance?

Conclusion 268 ***Galilean - Newtonian physics is the theoretical fundamental for the experiments of Michelson and Morley***

Michelson and Morley used Galilean-Newtonian theory as the theoretical basis for their experiments, which has been accepted in Einsteinian relativity theory. [226], [231] Einsteinian relativity theory should have applied its own postulates and approach rather than Galilean - Newtonian one in the theoretical treatment of Michelson - Morley's experiments.

We should immediately note the following fact in view of Conclusion 268, and the preceding analysis:

Remark 269 ***Einsteinian relativity theory ignores itself***

Einsteinian relativity theory ignores its own postulates and approach in treating the theoretical background of the experiments by Michelson and Morley. [226], [231]

The study of [226], [231] gives a sufficient proof that the experiments and the analyses by Michelson and Morley, which have been considered as the unquestionable experimental proofs of the light speed invariance, imply that the light speed is not invariant relative to inertial frames and relative to bodies moving with constant velocities. Their theoretical analyses show just contrary to Einstein's attitude of the light speed invariance.

The light speed is relative with respect to inertial frames and to bodies moving with constant velocities in general. (Formal exceptions are the integral spaces over which Lorentz relationships hold among temporal and spatial coordinates.) Based on such analyses Michelson and Morley deduced their interpretations of the experiments and conclusions. They did not conclude that the speed of light is invariant (in vacuum).

Since the Galilean - Newtonian physics is the theoretical basis of Michelson and Morley experiment, then Einstein's reference to their experiment as the proof of invalidity of Galilean - Newtonian physics is absurd.

Conclusions

Conclusion 270 ***The light speed is neither invariant relative to the interferometer moving together with Earth, nor relative to Earth.****[226], [231]*

This conclusion agrees with Claim 250 and Conclusion 256. It verifies Claim 255. They are confirmed in the sequel: Theorem 473, Corollary 474, Theorem 475, the Remarks 477 through 480, Corollary 491, Theorem 495, the Remark 496 and the Remark 499.

Note 271 *We should note that nobody, neither Michelson nor Morley, nor Einstein, nor anybody else, tried to associate different time axes (i.e. different time units) to different light signals, or to the same light signal moving in different directions (forward, backward). Everybody applied naturally the same time axis, regardless of its choice, for all light signals and for their movements in any direction. This characterizes Galilean - Newtonian physics and its correct application, and rejects simultaneously the application of Lorentz transformations, i.e. of Einsteinian relativity theory, to the experiments by Michelson and Morley.*

The preceding analyses enable us to summarize them in the form of the following conclusion.

Conclusion 272 ***On the experiments of Michelson and Morley***

The experiments by Michelson and Morley

1. are theoretically based on Galilean - Newtonian physics,

2. do not disprove the uniqueness of time,

3. do not prove the invariance of the light speed,

3. do not disprove Galilean - Newtonian law of the velocity composition,

4. do not permit an application of Einsteinian relativity theory to be their theoretical fundamental.

5. Michelson and Morley did not claim either that time is not unique, but they used the same time axes for the two light signals, or that the light speed is invariant.[226]

The preceding conclusions agree with

° Conclusion 91 through Conclusion 93, Conclusion 294, Conclusion 298, Conclusion 304, Conclusion 305 and Conclusion 309, and

° with the recent results by Martin [332, pp. 47, 48, 53] and Wesley [478, p. 261] that the light speed obeys Galilean - Newtonian law of the velocity composition.

7.3.4 Concluding rebuttals to Einstein's postulates

Rebuttal to Einstein's postulate on *time*

Theorem 50 result in the following:

Rebuttal 273 ***Rebuttal to Einstein's postulate on time nonuniqueness***

Time is unique (Theorem 50, Theorem 128). The clock hands do not show time. The clock hand indicates only the relative numerical value of time (Axiom 76, Principle 77, Theorem 82). The numerical value of time, indicated by the clock hand, is relative to the accepted initial moment, time unit and time scale. Einstein's postulate that time depends on spatial frames and on the speed of the moving body (of the clock) means that there exist several different times each being "the reading (position of the hands)" of the corresponding clock. This is a substanital mistake and physical nonsense.

Rebuttal 274 ***Rebuttal to Einstein's postulate on time relativeness***

The uniqueness of time (Theorem 50) together wit the constancy and invariance of the time speed (Theorem 128) prove that time itself is not relative, which holds for every physical variable. Time and its speed are independent of everybody and everything. Einstein's postulate that time is relative is wrong and physically unacceptable.

Rebutall to Einstein‘s postulate on the light speed invariance

Theorem 139 implies directly the following:

Corollary 275 ***Lorentz -Einstein invariance of the light speed is not a property of the light speed itself*** *[226], [231]*

Lorentz - Einstein invariance of the light speed, which results formally from Einstein's formulae (7.43) and (7.45), equivalently from (5.18), is the consequence of both the property of Einstein's formulae themselves and the acceptance of the light speed c *for the reference speed* w*,* $w = c$*. It is not a property of the light speed itself.*

In this connection see Theorem 475 and Claim 500.

Remark 276 ***Rejection of the invariance of the light speed in view of Einstein's formula*** *[226], [231]*

Corollary 275 disproves the claim that Einstein's formulae (7.43), (7.45), equivalently (5.18), prove the invariance of the light speed as the light speed property. In other words, the formulae and from them deduced invariance of the light speed do not imply the light speed invariance relative to every inertial frame.

Conclusion 277 ***Failure of Einstein‘s fundamental postulate on the light speed invariance***

The a priory accepted invariance of the light speed is the primary fundamental postulate of Einsteinian relativity theory. Each, Theorem 139, Corollary 275, and Remark 276, disproves the validity of the claim that the light speed is invariant in general, i.e. with respect to every inertial frame. Consequently, the a priory acceptance of the light speed invariance is the a priory restriction that limits severely the validity of Einsteinian relativity theory.

Theorem 139, Corollary 275, and Remark 276, agree with Conclusions 92 and 93, and Conclusion 259. These conclusions are confirmed by the results of the next Part: Theorem 473, Corollary 474, Theorem 475, Remarks 477 through 480, Corollary 491, Theorem 495, Remark 496, Theorem 475 and Remark 499.

7.4 Conclusion on Einstein's Theory

The theory of relativity does not topple Newton's and Maxwell' theories .... They will have to accept some modifications of their laws but thereby gain their security.

Albert EINSTEIN [167, p. 5]

Many mistakes, inconsistencies, paradoxes and absurds of Einstein‘s relativity theory result in its failure. Let us repeat only some of them.

Einstein‘s relativity theory accepts a priory the following assumptions for its postulates:

Claim 278 ***Einstein‘s postulate on time***

Time and its speed depend on spatial frames, on space, on the speed of a moving body (of a clock), on the speed of biological processes of the human.

Rebuttal 279 ***Rebuttal to Einstein‘s postulate on time***

Einstein‘s postulate on time is physical nonsense. Time and its speed are independent of everybody and everything. Nobody and nothing can influence them, and vice versa. Time and its speed are unique. There do not exist versus different times or time speeds. The time speed is invariant.

Claim 280 ***Einstein‘s postulate on the light speed***

The light speed is the same in all inertial frames.

Rebuttal 281 ***Rebuttal to Einstein‘s postulate on the light speed***

The light speed is not invariant.

For every speed there are coordinate transformations among integral spaces such that the chosen speed is invariant relative to them. The light speed is not exceptional. Such transformations for the light speed are Lorentz transformations.

Theorem 282 ***Lorentz transformations and the distance preservation condition***

Claim 283 *The necessary and sufficient condition for Lorentz transformations to obey the condition for the distance preservation is the invariance of the temporal transfer speed:* $v_{ji}^{(.)} \equiv v$.

Property 284 ***Lorentz transformations restriction***

Lorentz transformations forbid to the spatial transfer speed to be equal to, or greater than, the light speed, but the spatial transfer speed is to be less than the light speed: $v < c$.

Otherwise, Einstein‘s relativity theory vanishes.

Paradox 285 ***The spatial transfer speed invariance and Einstein‘s relativity theory***

For the spatial transfer speed $v_{ji}^{(.)}$ *to be invariant it is necessary and sufficient that it is equal to the light speed:* $v = c$.

Mistake 286 ***Einstein‘s time relaativity theory and Galilean-Newtonian physics***

Einstein‘s time relativity theory contains wrong modifications of, extensions of, rebuttals to, Galilean-Newtonian physics.

For the proofs of all mistakes, wrong interpretations, inconsistencies, paradoxes, and absurds of Einstein‘s relativity theory, which result in its debacle, see the books [226], [231].

Chapter 8

Non-Einsteinian Approaches to Relativity

8.1 Galilean - Newtonian approach

Claim 287 ***Galilei's discovery and description of the principle of relativity***

For a final indication of the nullity of the experiments brought forth, this seems to me the place to show you a way to test them all very easily. Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs

many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other.
Galileo GALILEI *(1564 - 1642) [191, pp. 186, 187]*

Contrary to common knowledge,
the theory of relativity did not originate with Einstein.
The idea that all motion is relative was first put forward by Galileo,
Einstein simply expended upon basic Galilean relativity
in developing his special theory of relativity
first published one hundred years ago in 1905.
Michael J. Kelly [263, p.78]

Newton expressed Galilei's above description of the principle of relativity in the compact form, clear style and precise statement as follows:

Claim 288 ***Newton's statement of the principle of relativity***

COROLLARY V

The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion.
Isaac NEWTON *[360, p. 19]*

Galilean viewpoint eliminates Einsteinian concepts of space contraction and time delation. Furthermore, it removes the twin paradox.
Adolph MARTIN [332, p. 54]

Let us investigate whether the distance preservation condition (6.22) (Condition 152) holds under Galilean coordinate transformations, which are characterized as follows.

Case 289 ***The same time unit. Spatial frames are in relative movements. The same length unit.***
In this case the spatial frames are in a mutual relative movement,

$$v^{O}_{O_j} \neq 0, \text{ and } v^{0}_{ji} \in R^{+}. \tag{8.1}$$

We accept the same time unit for all the time axes and the same length unit for all spatial frames,

$$1_{t_i} = 1_{t_j} = 1_t,\ 1_{L_i} = 1_{L_j} = 1_L,\ but\ \mathbf{r}_P^{O_i}(t;t_0) \neq \mathbf{r}_P^{O_j}(t;t_0).$$

Hence, we do not change either the time scale or the time unit, i.e.

$$t_i = t_j = t. \tag{8.2}$$

The position vectors $\mathbf{r}_P^{O_i}(t;t_0)$ and $\mathbf{r}_P^{O_j}(t;t_0)$ of the arbitrary point P relative to the origins O_i and O_j of the movable frames R_i^n and R_j^n, respectively, are interrelated by the spatial transfer velocity $\mathbf{v}_{ji}^0$ measured with the time unit 1_t of the reference time axis T and with the length unit 1_L of R^n as follows:

$$\mathbf{r}_P^{O_i}(t;t_0) = \mathbf{r}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0(t - t_0). \tag{8.3}$$

This is the spatial coordinate transformation related to Galilean - Newtonian principle of relativity (the Claim 287 and the Claim 288).

Galilean-Newtonian transformation *(8.3) implies the following, also well known,* ***Galilean-Newtonian velocity transformations*** *if the velocity is time-varying:*

$$\mathbf{v}_P^{O_i}(t;t_0) = \frac{d\mathbf{r}_P^{O_i}(t;t_0)}{dt} = \frac{d\left[\mathbf{r}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0(t-t_0)\right]}{dt} = \mathbf{v}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0 \Longrightarrow$$
$$\mathbf{v}_P^{O_i}(t_0;t_0) = \mathbf{v}_P^{O_j}(t_0;t_0) + \mathbf{v}_{ji}^0,\ i.e.\ \mathbf{v}_{P0}^{O_i} = \mathbf{v}_{P0}^{O_j} + \mathbf{v}_{ji}^0, \tag{8.4}$$

$$\widetilde{\mathbf{v}}_P^{O_i}(t;t_0)\mid_{t\ =t_0} = \widetilde{\mathbf{v}}_P^{O_i}(t_0;t_0) = \mathbf{v}_P^{O_i}(t;t_0) = \mathbf{v}_{P0}^{O_i} = \mathbf{v}_{P0}^{O_j} + \mathbf{v}_{ji}^0,$$
$$\widetilde{\mathbf{v}}_P^{O_i}(t;t_0)\mid_{t\ >t_0} = \frac{\int_{t_0}^t \mathbf{v}_P^{O_i}(t;t_0)dt}{(t-t_0)}\mid_{t\ >t_0} = \frac{\mathbf{r}_P^{O_i}(t;t_0)}{(t-t_0)}\mid_{t\ >t_0} =$$
$$= \frac{\mathbf{r}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0(t-t_0)}{(t-t_0)}\mid_{t\ >t_0} = \widetilde{\mathbf{v}}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0, \tag{8.5}$$

and the following if the velocity is constant:

$$\mathbf{v}_P^{O_i}(t;t_0) = \frac{d\mathbf{r}_P^{O_i}(t;t_0)}{dt} = \frac{d\left[\mathbf{r}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0(t-t_0)\right]}{dt} = \mathbf{v}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0 \Longrightarrow$$
$$\mathbf{v}_P^{O_i}(t_0;t_0) = \mathbf{v}_P^{O_j}(t_0;t_0) + \mathbf{v}_{ji}^0,\ i.e.\ \mathbf{v}_{P0}^{O_i} = \mathbf{v}_{P0}^{O_j} + \mathbf{v}_{ji}^0, \tag{8.6}$$

There is not a change of the speed unit. This is the general Galilean - Newtonian rule. It holds also for the light velocity,

$$\mathbf{c}^{O_i} = \mathbf{c}_i^0 = \mathbf{c}^{O_j} + \mathbf{v}_{ji}^0 = \mathbf{c}_j^0 + \mathbf{v}_{ji}^0, \tag{8.7}$$

relative to the integral spaces with the same time unit and with the same length unit. Let us test whether Galilean - Newtonian transformations (8.3) through (8.7) satisfy the condition (6.22).

Note 290 ***Einsteinian relativity theory does not disprove the validity of Galilean - Newtonian velocity transformations***

Einsteinian relativity theory claims that the above Galilean - Newtonian velocity transformations hold only for the small values of the speeds, but not for their big values, without restricting its claim only to the integral spaces interrelated by Lorentz transformations (7.20) through (7.23).

Einsteinian relativity theory has not shown what is wrong in the above results relative to the integral spaces for which the results are established. It has neglected the fact that the above results hold in the Galilean - Newtonian framework, i.e. for Galilean - Newtonian integral spaces, and not for the Lorentzian ones. It has not shown what is wrong in the Galilean - Newtonian modeling physical phenomena and physical systems.

Theorem 291 ***Galilean - Newtonian transformations obey generalized Einstein's distance condition, i.e. they form the Poincaré group***

Let the spatial frames be at mutual relative movement that is expressed by (8.1). Let the same time unit and the same length unit hold for all integral spaces, i.e. let (8.2) be valid. Let $B = A$ *in the matrix* D *and* $\mathbf{v}_G^{(.)} \equiv \mathbf{v}_P^{O_{(.)}}$. *Then the transformations (8.3), (8.6), satisfy the general distamce preservation condition (6.22) for an arbitrary velocity* $\mathbf{v}_P^{O_{(.)}}$ *of the arbitrary point* P *(by permitting the light velocity* $\mathbf{c}_{(.)}$ *if and only if* $P = L$*, the velocity* $\mathbf{v}_R^{O_{(.)}}$ *of the reference point* P_R *if and only if* $P = P_R$*, and the velocity* $\mathbf{v}_{SU}^{O_{(.)}}$ *of the reference point* P_{SU} *if and only if* $P = P_{SU}$*),*

$$\mathbf{v}_P^{O_{(.)}} \equiv \mathbf{c}_{(.)} \iff P = L, \mathbf{v}_P^{O_{(.)}} \equiv \mathbf{v}_R^{O_{(.)}} \iff P = P_R,$$
$$\mathbf{v}_P^{O_{(.)}} \equiv \mathbf{v}_{SU}^{O_{(.)}} \iff P = P_{SU}. \tag{8.8}$$

The proof is given in Appendix 21.1 "Proof of Theorem 291". This theorem holds also when the velocity $\mathbf{v}_{Pt}^{O_{(.)}}$ of the arbitrary point P is time-varying [228].

Conclusion 292 ***Galilean - Newtonian relativity***

Galilean - Newtonian spatial coordinate transformations determined by (8.3) form the Poincaré group.

Galilean - Newtonian coordinate systems are Gaussean.

Galilean - Newtonian spatial coordinate transformation (8.3) and Galilean - Newtonian velocity transformation (8.6) represent the mathematical basis for the principle of ***Galilean - Newtonian relativity*** *(the Claim 287 and the Claim 288), which is also expressed by Galilei's "statement that* ***the laws of mechanics are valid in all inertial frames****" [255, p. 692], or equivalently, the equations of motion do not change their forms under the Galilean - Newtonian transformation (8.3).*

Conclusion 293 ***The complete failure of Einstein's claim on Galilean - Newtonian transformations***

It is important to understand that the light velocity obeys simultaneously Galilean - Newtonian rule of the velocity transformation (8.6) and the condition

(6.22). This is not only a crucial opposition to the a priory accepted restriction in Einsteinian relativity theory, which is its primary keystone, that the light speed is invariant, hence, that Galilean - Newtonian transformation (8.3) is invalid for the light velocity. It shows also the complete failure of Einstein‘s claim that Galilean-Newtonian coordinate transformations do not obey the condition (6.22) for the length preservation.

Conclusion 294 ***Galilean - Newtonian velocity transformation (8.6) holds for the light velocity as shown in 8.7.***

Comment 295 *Galilean - Newtonian spatial coordinate transformation (8.3) can be set into the following form of a linear function of the position vector of the point P, where the gain is a function of the average speed of the point P rather than a constant in general,*

$$\mathbf{r}_P^{O_i}(t;t_0) = \eta\left(\widetilde{v}_P^{O_j}\right)\mathbf{r}_P^{O_j}(t;t_0),\ \eta\left(\widetilde{v}_P^{O_j}\right) = 1 + \frac{v_{ji}^0}{\widetilde{v}_P^{O_j}(t;t_0)}.$$

If the speed of the arbitrary point P is constant, then its average speed is also constant. They imply a constant value of the coefficient function $\eta(.)$,

$$v_P^{O_j}(t;t_0) \equiv v_P^{O_j} \Longrightarrow \widetilde{v}_P^{O_j}(t;t_0) \equiv v_P^{O_j} \Longrightarrow \eta\left(\widetilde{v}_P^{O_j}\right) \equiv \eta = const.$$

Then, the position vector $\mathbf{r}_P^{O_i}(t;t_0)$ of the point P is the ordinary linear function of $\mathbf{r}_P^{O_j}(t;t_0)$ (with the constant gain),

$$\mathbf{r}_P^{O_i}(t;t_0) = \eta\mathbf{r}_P^{O_j}(t;t_0),\ \eta = 1 + \frac{v_{ji}^0}{v_P^{O_j}} = const.$$

Another important property of the following Galilean - Newtonian coordinate transformations (8.9), (8.10), which results from (8.3), is their complete compatibility,

$$\mathbf{r}_P^{O_i}(t;t_0) = \mathbf{r}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0(t-t_0), \tag{8.9}$$

$$\mathbf{r}_P^{O_j}(t;t_0) = \mathbf{r}_P^{O_i}(t;t_0) - \mathbf{v}_{ji}^0(t-t_0). \tag{8.10}$$

Claim 296 ***Galilean - Newtonian coordinate transformations (8.9), (8.10) are completely compatible***

Proof. Galilean - Newtonian spatial coordinate transformations (8.9), (8.10) yield the following, after eliminating, for example $\mathbf{r}_P^{O_i}(t;t_0)$, from them:

$$\mathbf{r}_P^{O_j}(t;t_0) \equiv \mathbf{r}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0(t-t_0) - \mathbf{v}_{ji}^0(t-t_0) \equiv \mathbf{r}_P^{O_j}(t;t_0).$$

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We summarize Theorem 291 and Claim 296 as follows.

Conclusion 297 ***Galilean - Newtonian transformations (8.9), (8.10) form the Poincaré group and they are completely compatible.***

Conclusion 298 ***Inapplicability of Lorentz transformations***

The above transformations concern only the spatial coordinates. The temporal coordinates are not transformed. Lorentz transformations and Einsteinian relativity theory are inapplicable to this case.

Comment 299 *Laski has recently showed [286, p. 351] correctness of Galilean - Newtonian velocity transformation for Doppler effect, rather than Einstein's law of the composition of velocities, which results from Lorentz transformations.*

Note 300 *The transformations (14.21), (14.22) represent the full generalization of Galilean - Newtonian (spatial coordinate) transformations (8.9), (8.10). See also the Claim 508.*

8.2 Dynamical systems approach to relativity

The spatial coordinate systems are all assumed to be at relative rest in the framework of dynamical systems with multiple *time* scales. Let us analyze whether the condition (6.22) holds in such a case.

Case 301 ***Different time units. Frames are at mutual rest. The same length unit.***

We accept in this case that all the spatial frames are mutually at relative rest and that they carry the same length unit 1_L. We accept that their origins coincide. This enables us to analyze the influence of a change of the time unit only, hence of the time axis only:

$$\mathbf{v}_{O_i}^{O,(.)} \equiv \mathbf{v}_{O_j}^{O,(.)} \Longrightarrow \mathbf{v}_{ji}^{(.)} \equiv \mathbf{0}. \tag{8.11}$$

The former means that

$$1_{L_i} = 1_{L_j} = 1_L \text{ and } \mathbf{r}_P^{O_i}(t_i;t_{i0}) \equiv \mathbf{r}_P^{O_j}(t_j;t_{j0}). \tag{8.12}$$

We change the time units as follows:

$$1_{t_{(.)}} \left\langle 1_{t_{(.)}} \right\rangle = \left[\left(num\mu_{(.)} \right)^{-1} 1_t \right] \left\langle 1_t \right\rangle,\ 1_t \left\langle 1_t \right\rangle = \left(num\mu_{(.)} 1_{t_{(.)}} \right) \left\langle 1_{t_{(.)}} \right\rangle,$$

so that

$$t_{(.)} - t_{(.)0} = \mu_{(.)}(t - t_0),\ t_{(.)0} = \mu_{(.)} t_0. \tag{8.13}$$

This implies a change of the speed unit as follows:

$$1_{v^{O_i,i}} \left\langle 1_{v^{O_i,i}} \right\rangle = 1_{L_i} 1_{t_i}^{-1} \left\langle 1_{L_i} 1_{t_i}^{-1} \right\rangle = 1_{L_i} \left\langle 1_{L_i} \right\rangle \left[(num\mu_i)^{-1} 1_t \left\langle 1_t \right\rangle \right]^{-1} =$$

$$= 1_{L_j} \left\langle 1_{L_j} \right\rangle \left[(num\mu_i)^{-1} \left(num\mu_j \right) 1_{t_j} \left\langle 1_{t_j} \right\rangle \right]^{-1} = \frac{num\mu_i}{num\mu_j} 1_{v^{O_j,j}} \left\langle 1_{v^{O_j,j}} \right\rangle,$$

which implies

$$\mathbf{v}_{(..)}^{O_i,i} = \frac{d\mathbf{r}_{(..)}(t_i;t_{i0})}{dt_i} = \frac{d\mathbf{r}_{(..)}(t;t_0)}{\mu_i dt} = \frac{d\mathbf{r}_{(..)}(t_j;t_{j0})}{\mu_i \mu_j^{-1} dt_j} =$$
$$= \frac{\mu_j}{\mu_i}\mathbf{v}_{(..)}^{O_j,j},\ \mathbf{v}_{(..)} \equiv \mathbf{v}_{(..)}^{(.)} \in \left\{\mathbf{c}_{(.)}^{(.)}, \mathbf{v}_G^{(.)}, \mathbf{v}_P^{(.)}, \mathbf{v}_R^{(.)}\right\}, \tag{8.14}$$

and

$$\widetilde{\mathbf{v}}_{(..)}^{i}(t_i;t_{i0}) = \frac{\mathbf{r}_{(..)}(t_i;t_{i0})}{t_i - t_{i0}} = \frac{\mathbf{r}_{(..)}(t_j;t_{j0})}{\mu_i\mu_j^{-1}(t_j - t_{j0})} = \frac{\mu_j}{\mu_i}\widetilde{\mathbf{v}}_{(..)}^{j}(t_j;t_{j0}). \tag{8.15}$$

These are the general rules. They are well known in Galilean - Newtonian physics. They express the influence of the change of the time unit only, i.e. of the time axis only, (without any change of the spatial coordinates), on the velocity. There is not any exception. They hold also for the light velocity for which $\widetilde{\mathbf{c}}_{(.)}^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$ *because it is constant,*

$$\mathbf{c}^{O_i,i} = \mathbf{c}^i = \frac{d\mathbf{r}_L(t_i;t_{i0})}{dt_i} = \frac{d\mathbf{r}_L(t_i;t_{i0})}{\mu_i dt} = \frac{d\mathbf{r}_L(t_j;t_{j0})}{\mu_i\mu_j^{-1}dt_j} = \frac{\mu_j}{\mu_i}\mathbf{c}^{O_j,j} = \frac{\mu_j}{\mu_i}\mathbf{c}^j. \tag{8.16}$$

Remark 302 ***The numerical value of the light speed depends on time unit***

The equations (8.16) show the dependence of the numerical value of the light speed on a time unit, which is expressed in terms of its dependence on the time scaling coefficients μ_i *and* μ_j*.*

Let us test whether the preceding transformations caused by the change of the *time* axis obey the condition (6.22), i.e. whether (8.12) through (8.16) satisfy this condition.

Theorem 303 ***Transformations (8.14) through (8.16) form Poincaré group***

Let the spatial frames be at a mutual relative rest that is expressed by (8.11). Let their origins coincide. Let the transformations of the time units of different time axes obey (8.13). Then they, together with the velocity transformations (8.14) through (8.16), satisfy the condition (6.22) for an arbitrary velocity $\mathbf{v}_{Gt}^{(.)}$ *(including the light velocity* $\mathbf{c}^{(.)}$*, the velocity* $\mathbf{v}_{Pt}^{(.)}$ *of the arbitrary point P, the velocity* $\mathbf{v}_{Rt}^{(.)}$ *of the reference point* P_R *and the velocity* $\mathbf{v}_{SUt}^{(.)}$ *of the reference point* P_{SU}*).*

Proof. Let the velocity $\mathbf{v}_{Gt}^{(.)}$ be an arbitrary velocity by permitting for it to be the velocity $\mathbf{c}^{(.)}$ of the light signal L, or the velocity $\mathbf{v}_{Pt}^{(.)}$ of the arbitrary point P, or the velocity $\mathbf{v}_{Rt}^{(.)}$ of the reference point P_R, or the velocity $\mathbf{v}_{SUt}^{(.)}$ of the reference point P_{SU}. We transform the left-hand side of (6.22) by applying (8.12) through (8.15):

$$\begin{bmatrix} \mathbf{r}_P(t_i;t_{i0}) \\ (t_i - t_{i0})\widetilde{\mathbf{v}}^i_{Gt}(t_i;t_{i0}) \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(t_i;t_{i0}) \\ (t_i - t_{i0})\widetilde{\mathbf{v}}^i_{Gt}(t_i;t_{i0}) \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} \mathbf{r}_P(t_j;t_{j0}) \\ \frac{\mu_i}{\mu_j}(t_j - t_{j0})\widetilde{\mathbf{v}}^i_{Gt}(t_i;t_{i0}) \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(t_j;t_{j0}) \\ \frac{\mu_i}{\mu_j}(t_j - t_{j0})\widetilde{\mathbf{v}}^i_{Gt}(t_i;t_{i0}) \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} \mathbf{r}_P(t_j;t_{j0}) \\ (t_j - t_{j0})\widetilde{\mathbf{v}}^j_{Gt}(t_j;t_{j0}) \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(t_j;t_{j0}) \\ (t_j - t_{j0})\widetilde{\mathbf{v}}^j_{Gt}(t_j;t_{j0}) \end{bmatrix}.$$

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Conclusion 304 ***The numerical value of the light speed depends on a time unit***

The light speed does not represent any exception of the general rule that the numerical value of speed depends on a time unit. By obeying this rule the light speed obeys also the condition (6.22).

Conclusion 305 ***Inapplicability of Einsteinian relativity theory***

Notice that the above transformation concerned only the temporal coordinates. The spatial coordinates were not transformed. Lorentz transformations and Einsteinian relativity theory are inapplicable to this case.

The above theorem establishes a new link between the theory of dynamical systems with multiple *time* scales and the relativity theory in general. It is completely beyond Einsteinian relativity theory.

8.3 Generalized Galilean - Newtonian approach

Time independence of the space permits us to change only the *time* unit also in the case when the frames are in a relative mutual movement.

Case 306 ***Different time units. Frames are in relative mutual movements. The same length unit.***

We change only the time unit as defined by (8.13) and we accept in this case that the frames are in a mutual relative movement, but that they retain the same length unit, i.e.

$$v^{(.)}_{ji} = \left[v^{O,(.)}_{O_j} - v^{O,(.)}_{O_i}\right] \in R^+, \tag{8.17}$$

$$1_{L_i} = 1_{L_j} = 1_L \implies \mathbf{r}^{O_k}_P(t_i;t_{i0}) \equiv \mathbf{r}^{O_k}_P(t_j;t_{j0}),\ k \in \{-,i,j\}. \tag{8.18}$$

Since a change of a time unit cannot change a distance or a length then the position vectors $\mathbf{r}^{O_i}_P(t;t_0)$ and $\mathbf{r}^{O_j}_P(t;t_0)$ of the arbitrary point P relative to the *origins O_i and O_j of R_i^n and R_j^n, respectively, are interrelated by the transfer velocity $\mathbf{v}_{ji}^0$ as follows:*

$$\mathbf{r}_P^{O_i}(t;t_0) = \mathbf{r}_P^{O_j}(t;t_0) + \mathbf{v}_{ji}^0(t-t_0), \tag{8.19}$$

when it is measured with the time unit 1_t of the reference time axis T and with the length unit 1_L of R^n. If we change (only) the time unit, and if we use 1_{t_i} instead of 1_t then

$$\mathbf{r}_P^{O_i}(t_i;t_{i0}) = \mathbf{r}_P^{O_j}(t_i;t_{i0}) + \mathbf{v}_{ji}^i(t_i-t_{i0}), \tag{8.20}$$

$$\mathbf{r}_P^{O_j}(t_i;t_{i0}) = \mathbf{r}_P^{O_i}(t_i;t_{i0}) - \mathbf{v}_{ji}^i(t_i-t_{i0}), \tag{8.21}$$

Or, if we replace 1_t by 1_{t_j},

$$\mathbf{r}_P^{O_i}(t_j;t_{j0}) = \mathbf{r}_P^{O_j}(t_j;t_{j0}) + \mathbf{v}_{ji}^j(t_j-t_{j0}), \tag{8.22}$$

$$\mathbf{r}_P^{O_j}(t_j;t_{j0}) = \mathbf{r}_P^{O_i}(t_j;t_{j0}) - \mathbf{v}_{ji}^j(t_j-t_{j0}). \tag{8.23}$$

These spatial coordinate transformations represent ***the slightly generalized Galilean - Newtonian spatial coordinate transformations*** *(see Note 300 and Claim 508). They together with (8.13) imply the following velocity transformations from the frame R_i^n into the frame R_j^n without changing the time axis, i.e. the same time axis is over both R_i^n and R_j^n,*

$$\mathbf{v}_P^{O_i,i}(t_i;t_{i0}) = \frac{d\mathbf{r}_P^{O_i}(t_i;t_{i0})}{dt_i} = \mathbf{v}_P^{O_j,i}(t_i;t_{i0}) + \mathbf{v}_{ji}^i \tag{8.24}$$

$$\mathbf{v}_P^{O_i,j}(t_j;t_{j0}) = \frac{d\mathbf{r}_P^{O_i}(t_j;t_{j0})}{dt_j} = \mathbf{v}_P^{O_j,j}(t_j;t_{j0}) + \mathbf{v}_{ji}^j \tag{8.25}$$

and the following change of the speed unit:

$$1_{v^{O_i,i}}\left\langle 1_{v^{O_i,i}}\right\rangle = 1_{L_i}1_{t_i}^{-1}\left\langle 1_{L_i}1_{t_i}^{-1}\right\rangle = 1_{L_i}\left\langle 1_{L_i}\right\rangle\left[(num\mu_i)^{-1}\,1_t\left\langle 1_t\right\rangle\right]^{-1} =$$
$$= 1_{L_i}\left\langle 1_{L_i}\right\rangle\left[(num\mu_i)^{-1}\,num\mu_j 1_{t_j}\left\langle 1_{t_j}\right\rangle\right]^{-1} = \frac{num\mu_i}{num\mu_j}1_{v^{O_j,j}}\left\langle 1_{v^{O_j,j}}\right\rangle. \tag{8.26}$$

The equations (8.24) and (8.25) are ***the slightly generalized Galilean - Newtonian velocity transformations.***

If we transform the velocity from the time axis T_i to the time axis T_j, both valid over R_i^n, i.e. without changing the frame (R_i^n), then

$$\mathbf{v}_P^{O_i,i}(t_i;t_{i0}) = \frac{d\mathbf{r}_P^{O_i}(t_i;t_{i0})}{dt_i} =$$
$$= \frac{d\mathbf{r}_P^{O_i}(t_j;t_{j0})}{\mu_i dt} = \frac{d\mathbf{r}_P^{O_i}(t_j;t_{j0})}{\mu_i\mu_j^{-1}dt_j} = \frac{\mu_j}{\mu_i}\mathbf{v}_P^{O_i,j}(t_j;t_{j0}). \tag{8.27}$$

This holds for the arbitrary point P and its arbitrary velocity. Hence, (8.27) holds for the transfer velocity,

$$\mathbf{v}_{ji}^{i} = \frac{\mu_j}{\mu_i}\mathbf{v}_{ji}^{j}. \tag{8.28}$$

because the definition of $\mathbf{v}_{ji}^{(.)} = v_{ji}^{(.)}\mathbf{u}$, (8.17), and (8.27) furnish

$$\mathbf{v}_{ji}^{i} = \mathbf{v}_{O_j}^{O,i} - \mathbf{v}_{O_i}^{O,i} = \frac{\mu_j}{\mu_i}\mathbf{v}_{O_j}^{O,j} - \frac{\mu_j}{\mu_i}\mathbf{v}_{O_i}^{O,j} = \frac{\mu_j}{\mu_i}\mathbf{v}_{ji}^{j}.$$

The equations (8.24), (8.27) and (8.28) imply the following velocity transformation from the integral space $I_i = T_i \times R_i^n$ into the integral space $I_j = T_j \times R_j^n$:

$$\mathbf{v}_P^{O_i,i}(t_i;t_{i0}) = \frac{\mu_j}{\mu_i}\left[\mathbf{v}_P^{O_j,j}(t_j;t_{j0}) + \mathbf{v}_{ji}^{j}\right]. \tag{8.29}$$

Analogously, (8.25), (8.27) and (8.28) yield

$$\mathbf{v}_P^{O_j,j}(t_j;t_{j0}) = \frac{\mu_j}{\mu_i}\left[\mathbf{v}_P^{O_i,i}(t_i;t_{i0}) - \mathbf{v}_{ji}^{i}\right]. \tag{8.30}$$

The equations (8.29) and (8.30) represent ***the generalized Galilean - Newtonian velocity transformations****.*

Moreover, the formulae hold also for the average speed,

$$\widetilde{\mathbf{v}}_P^{O_i,i}(t_i;t_{i0}) = \frac{\mathbf{r}_P^{O_i}(t_i;t_{i0})}{t_i - t_{i0}} = \frac{\mathbf{r}_P^{O_i}(t_j;t_{j0})}{\mu_i\mu_j^{-1}(t_j - t_{j0})} = \frac{\mu_j}{\mu_i}\widetilde{\mathbf{v}}_P^{O_i,j}(t_j;t_{j0}), \tag{8.31}$$

which express the influence of the change of the time unit only, and, due to (8.13) and (8.18),

$$\widetilde{\mathbf{v}}_P^{O_i,i}(t_i;t_{i0}) = \frac{\mathbf{r}_P^{O_i}(t_i;t_{i0})}{(t_i - t_{i0})} = \frac{\mathbf{r}_P^{O_i}(t_j;t_{j0})}{\mu_i\mu_j^{-1}(t_j - t_{j0})} =$$

$$= \frac{\mathbf{r}_P^{O_j}(t_j;t_{j0}) + \mathbf{v}_{ji}^{j}(t_j - t_{j0})}{\mu_i\mu_j^{-1}(t_j - t_{j0})} = \frac{\mu_j}{\mu_i}\left[\widetilde{\mathbf{v}}_P^{O_j,j}(t_j;t_{j0}) + \mathbf{v}_{ji}^{j}\right], \tag{8.32}$$

which express the change of both the time unit and the frame. These are the general rules.

The general equations (8.27) are valid for the light velocity, which we verify as follows due to (8.13) and (8.18):

$$\mathbf{c}^{O_i,i} = \frac{d\mathbf{r}_L(t_i;t_{i0})}{dt_i} = \frac{d\mathbf{r}_L(t_j;t_{j0})}{\mu_i dt} = \frac{d\mathbf{r}_L(t_j;t_{j0})}{\mu_i\mu_j^{-1}dt_j} = \frac{\mu_j}{\mu_i}\mathbf{c}^{O_i,j}. \tag{8.33}$$

Besides, the light velocity obeys the general equation (8.29),

$$\mathbf{c}^{O_i,j} = \mathbf{c}_i^{j} = \mathbf{c}^{O_j,j} + \mathbf{v}_{ji}^{j} = \mathbf{c}_j^{j} + \mathbf{v}_{ji}^{j} \Longrightarrow$$

$$\mathbf{c}^{O_i,i} = \mathbf{c}_i^{i} = \frac{\mu_j}{\mu_i}\mathbf{c}^{O_i,j} = \frac{\mu_j}{\mu_i}\mathbf{c}_i^{j} = \frac{\mu_j}{\mu_i}\left(\mathbf{c}_j^{j} + \mathbf{v}_{ji}^{j}\right). \tag{8.34}$$

Remark 307 ***Einsteinian relativity theory does not disprove the validity of Galilean - Newtonian velocity transformations***

Einsteinian relativity theory rejects the general validity of the above (slightly) generalized Galilean - Newtonian velocity transformations.

Einsteinian relativity theory has not shown what is wrong in the above results relative to the integral spaces for which they are established. It cannot disprove the general validity of the above velocity transformations in Galilean - Newtonian integral spaces. It is not possible to disprove them because they are correct relative to Galilean - Newtonian integral spaces.

Einsteinian relativity theory has ignored the fact that the above results hold in Galilean - Newtonian framework, i.e. for Galilean - Newtonian integral spaces, and not for Lorentzian ones.

We will show in Part III and in Part IV that Lorentz transformations (7.20) through (7.23) are invalid over Galilean - Newtonian and other non-Lorentzian integral spaces.

The light velocity is not any exception when it is considered in the (slightly) generalized Galilean - Newtonian integral spaces.

Let us test whether the transformations (8.19) through (8.34) satisfy the condition (6.22).

Theorem 308 *Let the spatial frames be in a mutual relative movement that is expressed by (8.17). Let the transformations of the time units of different time axes, and of different temporal coordinates obey (8.13). Then they, together with the transformations (8.19) through (8.32), satisfy the condition (6.22) for $A = B$ in D, for $G = P$ and for an arbitrary velocity of the arbitrary point P.*

Appendix 21.2 exposes the proof.

This theorem is beyond Einsteinian relativity theory. It confirms the validity of the transformations (8.13) through (8.16), (8.19) through (8.25), (8.29) through (8.34), from the point of view of the preservation of the generalized distance in integral spaces.

Conclusion 309 ***The light velocity and the light speed obey the general velocity rule***

The light velocity and the light speed obey the general velocity and speed transformation rule. They are not any exception of the rule. The numerical vector value of the light velocity and the numerical value of the light speed are relative to time axes and spatial frames. They are not invariant, although they are constant in vacuum.

8.4 Guideline

The preceding analyses and conclusions point out that we should a priory take into account and allow in coordinate transformations:

∘ the relative light velocity $\mathbf{c}_{(.)} = c_{(.)}\mathbf{u}$ with respect to the corresponding moving inertial frame $R^n_{(.)}$ if there is not a change either of the *time* unit or of the length unit, as established by Galilei (Section 8.1 "Galilean - Newtonian transformations") and confirmed by Einstein himself in his cited statement (Claim 257), or

∘ the relative light velocity $\mathbf{c}^{(.)} = c^{(.)}\mathbf{u}$ with respect to the corresponding *time* axis $T_{(.)}$ if we change only the *time* unit, hence, if we transform only the *time* axis (Section 8.2 "Dynamical systems approach to relativity"), and all spatial frames are mutually at rest, or

∘ the relative light velocity $\mathbf{c}^{(.)}_{(.)} = c^{(.)}_{(.)}\mathbf{u}$ with respect to the corresponding moving inertial frame $R^n_{(.)}$ if there is a change of the *time* unit only, i.e. if there is only a transformation of the *time* axis (Section 8.3 "Non-Einsteinian approach to relativity"), or

∘ the relative light velocity $\mathbf{c}^{(.)}_{(.)} = c^{(.)}_{(.)}\mathbf{u}$ with respect to the corresponding integral space $I_{(.)} = T_{(.)} \times R^n_{(.)}$ if there are possible both the change of the *time* unit, i.e. a transformation of the *time* axis, and the change of the length unit relative to the moving inertial frame $R^n_{(.)}$, which justifies the slight generalization of Lorentz transformations and their complete replacement by new general linear transformations that will be established in the sequel,

∘ different temporal and spatial transfer speeds,

∘ noninvariance of both the temporal and the spatial transfer speeds,

∘ different *time* scaling coefficients,

∘ different space scaling coefficients,

∘ a free choice of the generic speeds in the transformations of the temporal coordinates (which are replaced by the invariant light speed value in Lorentz transformations of the temporal coordinates), and

∘ a free choice of the reference speed in the transformations of the temporal coordinates, and

∘ a free choice of the spatial transfer in the transformations of the spatial coordinates.

The subsequent study will follow this guideline.

Chapter 9

Conclusion on *Time* and *Time* Fields

The word "time" has been used not only colloquially but also in the scientific literature in several crucially different meanings, such as:

° temporal variable - *time*, which is its basic, essential meaning,
° value of the temporal variable,
° instantaneous value of the temporal variable,
° numerical value of the temporal variable,
° temporal interval,
° duration.

The multiple use of the word "time" has created enormous and essential confusion even in science.

The existence, the nature, the sense, the meaning, the properties, and the (non)uniqueness of *time* have been attracting human thoughts and efforts to explain them since the most ancient epoch. It seems that there is not an area of human activity where *time* has not attracted an attention. This is natural because *time*, by its always changing temporal value, imbues, impregnates, is over and in, passes through every cell of the body of everybody; because all processes and motions propagate in *time*, and all actions occur in *time*. We do not have a particular, separate sense to feel and/or to measure *time* values and their flow, possibly because we do that with every our cell. This could explain why every human has its own feeling, and consequently, understanding of *time*.

The views on *time* have varied in the widest diapason, from claims that *time* does not exist, that it is not a physical variable, that it is our fiction, that it is a result of our imagination, an expression of our subjective feeling, that it is just a mathematical parameter, that it is dependent on space, a kind of a spatial coordinate, that there are infinitely many different *times* with variable speeds, up to the claims that it has existed, either with the beginning once or have existed for ever, that it is a physical variable with well measurable value, that it is an independent physical variable, that it is absolute, that it is the unique physical variable with the special properties and with invariant constant speed.

All those thoughts, all analyses, experiments and studies, our understanding of physical reality and our knowledge, show that *time* possesses such unique characteristic nature, sense and properties, that they cannot be expressed or explained in terms of other variables, or in terms of nontemporal phenomena, processes and/or categories. They clarify that *time* is its own component with the self-contained nature, meaning, characteristics, and properties. This led herein to introduce the definition of *time* and to explain and describe its properties, i.e., to present its characteristics, in the form of an axiom

Definition 46 and Axiom 47 are the keystones of this book. They present are the synthetic summary that describes and explains the nature, the properties, the characteristics and the sense of *time*. By relying on them we will be able to reply to solve Problems 1 through 37, and to establish new physical principles and to develop fundamentals of the new physical and mathematical relativity theory called *Consistent Relativity Theory.*

The unbounded variety of speeds of evolutions of (artistic, biological, chemical, economical, physical, political, social, technical) processes and by them induced the unlimited diversity of the choices of *time* axes (of relative zero moments, of initial moments, of *time* scales, and of *time* units), and our everyday use of some of them, and not always the same of them, and not everywhere the same, enabled us to discover the existence of *time fields.*

Time fields are governed by *time* axes so that they can be homogeneous or heterogeneous, single-layer or multi-layer *time* fields, *time* - invariant or *time* - varying ones. They express a reach temporal structure of space.

Among various coordinate transformations and *time* fields there are Galilean - Newtonian coordinate transformations and *time* fields, and others that are different from them, from Lorentz transformations, and from Lorentzian *time* fields. It is shown herein that all of them form Poincaré group of the transformations. Consequently, we cannot reject Galilean - Newtonian transformations by claiming wrongly that they do not obey Einstein's generalized distance preservation condition, i.e. that they do not form Poincaré group.

There is a simple, but a crucial mathematical link, which has a physical meaning, between the theory of dynamical systems with multiple *time* scales and the theory of relativity of *time.* The *time* scaling coefficients are this link. They will play important role in the temporal coordinate transformations studied in the sequel. They are missing in Einsteinian relativity theory.

There is also an important interconnection between the general, common properties of all physical variables, which are expressed in the various forms of *Physical Continuity and Uniqueness Principle (PCUP)*, and the continuous flow of *time* values. The latter is expressed in the form of *Time Continuity and Uniqueness Principle (TCUP).* These principles appear crucial for an adequate mathematical modeling physical phenomena, processes and systems. Besides, they are significant for synthesis and implementation of fine controls of dynamical systems.

Part III

Partially Compatible but Consistent Relativity Theory (PCC RT)

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Chapter 10

Partial Compatibility

10.1 Origin of partial compatibility

Lorentz, Einstein and Poincaré considered the speed of the arbitrary point P to be the light speed when they determined the scaling factors α and λ. Consequently, the strict Lorentz transformations (7.12), (7.15), (7.20) through (7.23), and Lorentz transformations (7.20) through (7.23) are partially (rather than completely) pairwise compatible, i.e. they are pairwise compatible if and only if the arbitrary point P moves with the speed of light. This is the origin of the partial compatibility.

We will follow in this Part such Lorentz - Einstein - Poincaré approach to the determination of the scaling functions $\alpha_{(.)}^{(.)}(.)$ and $\lambda_{(.)}^{(.)}(.)$ in the various Galilean-Newtonian generalizations of Lorentz transformations. This means that we will determine in this part of the book the scaling coefficients on the basis of Definition 46 and Axiom 47 and by considering arbitrary points moving with the speed of light.

We will also relax the conditions on the light speed and on the transfer speeds to allow to be noninvariant, and we will relax the constraints on the scaling coefficients to permit to be mutually different.

Furthermore, we will verify (pairwise, entire) compatibility of the obtained transformations and we will test them for their complete compatibility. We wish to explore whether the relaxations and the generalizations of Einsteinian approach can result in complete (pairwise, entire) compatibility.

The new *time*-invariant coordinate transformations will be exploited to get new velocity transformations for constant velocities. The invariance of the light speed and of the spatial transfer speed will be then tested.

The proofs will contain, in addition to the necessity part and the sufficiency part, also the test and/or the proof of the aimed (pairwise, entire) compatibility.

10.2 *Time*-invariant nonuniformity

10.2.1 On nonuniformity

The existence of stationary, unmovable, clocks shows the existence of several stationary *time*-axes at one place. Relationships among them are *time*-invariant.

In the framework of the *time*-invariant both transformations and *time* fields we can allow only a constant velocity of the arbitrary point P :

$$\mathbf{v}_P^{O_{(.)},(.)}(t_{(.)};t_{(.)0}) \equiv \mathbf{v}_P^{(.)} = v_P^{(.)}\mathbf{u} = \mathbf{const}. \tag{10.1}$$

Therefore, there is not acceleration of the point P.

Time-invariance of the transformations and of speeds permits an arbitrary choice of the initial instant t_0. We accept $t_0 = 0$ in this framework. Hence,

$$\mathbf{r}_{(..)}^{O_{(.)},(.)}(t_{(.)};t_{(.)0}) \equiv \mathbf{r}_{(..)}^{O_{(.)},(.)}(t_{(.)}) \equiv \mathbf{r}_{(..)}(t_{(.)}),\ (..) \in \{G, L, P, P_R, P_{SU}\}.$$

The constancy of the speed value, (10.1), permits the following relationship among the position vector $\mathbf{r}_{(..)}(t_{(.)})$ of a point (..), its velocity $\mathbf{v}_{(..)}^{(.)}$ and *time* $t_{(.)}$, which are measured with the length unit $1_{L_{(.)}}$ and the *time* unit $1_{(.)}$ of the integral space $I_{(.)} = T_{(.)}\text{x}R_{(.)}^n$:

$$\mathbf{r}_{(..)}(t_{(.)}) \equiv \mathbf{v}_{(..)}^{(.)}t_{(.)},\ (..) \in \{G, L, P, P_R, P_{SU}\}. \tag{10.2}$$

The speed and velocity unit $1_{v_{(.)}}$ is then determined by both the *time* unit $1_{(.)}$ and the length unit $1_{L_{(.)}}$, $1_{v_{(.)}} = 1_{L_{(.)}}1_{(.)}^{-1}$.

Nonuniformity of the transformations can be due to a dependence of anyone of them on the arbitrary point P, or due to a dependence of any (*time* or space) scaling coefficient on a characteristic (position, speed / velocity) of the arbitrary point P. If this is true only for the *time* scaling coefficients then the nonuniformity is in the temporal domain, and the uniformity is over space. If only space scaling coefficients depend on a characteristic of the arbitrary point P then the nonuniformity is only over space, but the uniformity is in the temporal domain.

10.2.2 Weak nonuniformity

The basic *time* scaling coefficients are constant in this framework,

$$\mu_i(.) \equiv \mu_i = const. \in R^+ \Longrightarrow t_i = \mu_i t. \tag{10.3}$$

The values of other scaling coefficients are also constant, but they are determined by constant positive values of the generic (i.e., reference) speeds $q^{(\cdot)}$ and $w^{(\cdot)}$ of freely accepted and then fixed generic (i.e., reference) velocities $\mathbf{q}^{(\cdot)} = q^{(\cdot)}\mathbf{u}$ and $\mathbf{w}^{(\cdot)} = w^{(\cdot)}\mathbf{u}$, by the constant speed $v_P^{(.)}$ of the arbitrary point P and by the constant nonnegative spatial transfer speed $v_{ji}^{(.)}$,

$$\alpha_j^i(q^j, v_{ji}^j, v_P^j, w^j) \equiv \alpha_j^i = const. \in R^+,\ \alpha_i^j(q^i, v_{ji}^i, v_P^j, w^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(q^j, v_{ji}^j, v_P^j, w^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(q^j, v_{ji}^j, v_P^j, w^j) \equiv \lambda_i^j = const. \in R^+.$$

Either $q^{(.)}$ or $w^{(.)}$ can be equal in this framework to the speed $v_P^{(.)}$ of the arbitrary point P, but their product should be different from $\left[v_P^{(.)}\right]^2$.

The basic general generic transformations (6.13) through (6.17) become the following:

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)\right], \tag{10.4}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i)\right], \tag{10.5}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u}\right], \tag{10.6}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - \ v_{ji}^i t_i \mathbf{u}\right], \tag{10.7}$$

where $q^{(.)}$ and $w^{(.)}$ are constant and

$$q^{(.)}, w^{(.)} \in R^+,\ q^{(.)} w^{(.)} \notin \left\{\left[c_{(.)}^{(.)}\right]^2, \left[v_P^{(.)}\right]^2\right\},\ v_{ji}^{(.)} \in R_+. \tag{10.8}$$

Claim 310 ***Time independence and the transformations.***

The coordinates $r_P(t_j)$ and $r_P(t_i)$ represent in (10.4), (10.5) the position coordinates of the arbitrary point P with respect to the origins O_j and O_i of the spatial frames R_j^n and R_i^n, respectively. The temporal coordinate transformations (10.4), (10.5) depend on $r_P(t_j)$ and $r_P(t_i)$ and on the values v_{ji}^i and v_{ji}^j of the spatial transfer speed. The transformations do not reflect the independence property of time, (Axiom 47).

Claim 311 ***Time invariance of the transformations.***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j are constant, hence, they do not depend either explicitly or implicitly on time t. A choice of the initial moment t_0 does not influence the transformations. The transformations (10.3) through (10.8) are time-invariant.

Claim 312 ***Nonuniformity of the transformations.***

The position coordinate $r_P(t_{(.)})$ and the position vector $\mathbf{r}_P(t_{(.)})$ represent, respectively, in general in (10.4) through (10.7) the instantaneous position coordinate and the instantaneous position vector of the arbitrary point P relative to $O_{(.)}$ in $R_{(.)}^n$,

$$\mathbf{r}_P^{O_{(.)}}(t_{(.)}) = \mathbf{r}_P(t_{(.)}) = r_P(t_{(.)})\mathbf{u}. \tag{10.9}$$

The temporal coordinate transformations are nonuniform over space because they depend on a choice of the arbitrary point P.

Claim 313 ***Weak nonuniformity of the transformations.***

Nonuniformity of the transformations (10.4), (10.5) is weak due to $q^{(.)}w^{(.)} \neq \left[v_P^{(.)}\right]^2$.

Definition 314 ***Weak nonuniformity of the time field***

The transformations (10.3) through (10.8) determine a ***time-invariant weakly nonuniform time field*** *if and only if the transformations hold for every pair of the time axes including those from different layers in the case the time field is multi-layer and time-invariant.*

10.2.3 Nonuniformity

The scaling coefficients are constant. Their values are determined by constant values of the speed $v_P^{(.)} \in R^+$ of the arbitrary point P, and of the spatial transfer speed $v_{ji}^{(.)}$:

$$\alpha_j^i(v_{ji}^j, v_P^j) \equiv \alpha_j^i = const. \in R^+,\ \alpha_i^j(v_{ji}^i, v_P^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_{ji}^j, v_P^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_{ji}^i, v_P^i) \equiv \lambda_i^j = const. \in R^+,$$

and the basic *time* scaling coefficient is also constant,

$$\mu_i(.) \equiv \mu_i = const. \in R^+. \tag{10.10}$$

In this framework, both $q^{(.)}$ and $w^{(.)}$ are replaced by the value $v_P^{(.)}$ of the constant velocity $\mathbf{v}_P^{(.)}$ of the arbitrary point P so that the basic general generic transformations (6.13) through (6.17) take the form of the equations (10.11) through (10.14):

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{\left(v_P^j\right)^2} r_P(t_j) \right],\ v_{ji}^j \in R_+,\ v_P^j \in R^+, \tag{10.11}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{\left(v_P^i\right)^2} r_P(t_i) \right],\ v_{ji}^i \in R_+,\ v_P^i \in R^+, \tag{10.12}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u} \right], \tag{10.13}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right]. \tag{10.14}$$

Definition 315 ***The time-invariant generalized basic Lorentz transformations***

The coordinate transformations (10.11) through (10.14), in which the time and space scaling coefficients α_j^i, α_i^j, λ_j^i *and* λ_i^j *are positive real numbers, are called herein* ***the time-invariant generalized basic Lorentz transformations.***

Note 316 *The invariant value c of the light speed in the basic Lorentz transformations (7.16), (7.17), hence in their final forms that are Lorentz transformations (7.20), (7.21) of the temporal coordinates (Solution 188), is replaced by the noninvariant value $v_P^{(.)}$ of an arbitrary nonzero speed of the arbitrary point P in (10.11), (10.12). The invariant value v of the spatial transfer speed in the basic Lorentz transformations (7.18), (7.19), hence in their final forms that are Lorentz transformations (7.22), (7.23) of the spatial coordinates (Solution 188), is replaced by its noninvariant value $v_{ji}^{(.)}$ in (10.11) through (10.14). Besides all the scaling factors are permitted a priory to be mutually different in (10.11) through (10.14), while $\alpha_j^i \equiv \alpha_i^j \equiv \alpha$ and $\lambda_j^i \equiv \lambda_i^j \equiv \lambda$ are a priory accepted in the basic Lorentz transformations (7.16) through (7.19), hence in Lorentz transformations (7.20) through (7.23), i.e. in Einsteinian relativity theory, which resulted in $\alpha = \lambda$.*

Note 317 *The time-invariant generalized basic Lorentz transformations (10.11) through (10.14) incorporate the slightly generalized basic Lorentz transformations (7.6) through (7.9). The former become the latter by setting $v_P^{(.)} \equiv c_{(.)}^{(.)}$ in (10.11) and (10.12).*

If, additionally, we set $\alpha_j^i \equiv \alpha_i^j \equiv \alpha$ in (10.11) and (10.12), $\lambda_j^i \equiv \lambda_i^j \equiv \lambda$ in (10.13) and (10.14), $v_P^{(.)} \equiv c_{(.)}^{(.)} \equiv c$ and $v_{ji}^{(.)} \equiv v$ in (10.11) through (10.14) then they become the basic Lorentz transformations (7.16) through (7.19).

The transformations (11.52) through (11.55) represent the final form of the time-invariant generalized basic Lorentz transformations (10.11) through (10.14) in the general case (see Conclusion 371).

Claim 318 ***Time independence and the transformations.***

The position coordinates $r_P(t_j)$ and $r_P(t_i)$ represent the position coordinates of the arbitrary point P with respect to the origins O_j and O_i of the spatial frames R_j^n and R_i^n, respectively, (10.9). The transformations (10.11), (10.12) contain also the values of the spatial transfer speed $v_{ji}^{(.)}$. They do not express the independence property of time, (Axiom 47).

Claim 319 ***Time invariance of the transformations.***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j are constant. A choice of the initial moment t_0 does not influence the transformations. Hence, the transformations (10.10), (10.11) through (10.14) are time-independent.

Claim 320 ***Nonuniformity of the transformations.***

The values of the scaling coefficients α_j^i, α_i^j, λ_j^i and λ_i^j are determined by the speed $v_P^{(.)}$ of the arbitrary point P. The instantaneous position coordinate $r_P(t_{(.)})$ of the arbitrary point P, and its speed $v_P^{(.)}$, relative to $O_{(.)}$ and to $R_{(.)}^n$ intervene in (10.11), (10.12). The temporal coordinate transformations (10.11), (10.12) are nonuniform over space.

Definition 321 ***Nonuniformity of the time field***

The transformations (10.11) through (10.14) determine a ***time-invariant nonuniform time field*** *if and only if the transformations hold for every pair of the time axes including those from different layers in the case the time field is multi-layer and time-invariant.*

10.2.4 General nonuniformity

The scaling coefficients are constant. Their values can be determined by the initial position $\mathbf{r}_{P0}^{(.)} = \mathbf{r}_P(t_{(.)0})$ of the arbitrary point P, by its speed $v_P^{(.)}$ and by the speeds $q^{(.)}$, $v_{ji}^{(.)}$ and $w^{(.)}$,

$$\begin{aligned}
\alpha_j^i(\mathbf{r}_{P0}^i, \mathbf{r}_{P0}^j, q^j, v_{ji}^j, v_P^j, w^j) &\equiv \alpha_j^i = const. \in R^+, \\
\alpha_i^j(\mathbf{r}_{P0}^i, \mathbf{r}_{P0}^j, q^i, v_{ji}^i, v_P^i, w^i) &\equiv \alpha_i^j = const. \in R^+, \\
\lambda_j^i(\mathbf{r}_{P0}^i, \mathbf{r}_{P0}^j, q^j, v_{ji}^j, v_P^j, w^j) &\equiv \lambda_j^i = const. \in R^+, \\
\lambda_i^j(\mathbf{r}_{P0}^i, \mathbf{r}_{P0}^j, q^i, v_{ji}^i, v_P^i, w^i) &\equiv \lambda_i^j = const. \in R^+.
\end{aligned}$$

We accept that $\mathbf{r}_{P0}^{(.)} = \mathbf{0}$ is fixed so that both

$$\mu_i(\mathbf{0}) \equiv \mu_i = const. \in R^+ \Longrightarrow t_i = \mu_i t, \tag{10.15}$$

and the basic general generic transformations (6.13) through (6.17) take the following forms:

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j) \right], \tag{10.16}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i) \right], \tag{10.17}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u} \right], \tag{10.18}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right], \tag{10.19}$$

where

$$q^{(.)}, w^{(.)} \in R^+,\ q^{(.)} w^{(.)} \neq \left[c_{(.)}^{(.)} \right]^2,\ v_{ji}^{(.)} \in R_+,$$

$$q^{(.)} w^{(.)} = \left[v_P^{(.)} \right]^2 \ \textit{is permitted but not required.} \tag{10.20}$$

Claim 322 ***Time independence and the transformations.***

The equations (10.16), (10.17) do not express the independence property of time, (Axiom 47), because $r_P(t_i)$ and $r_P(t_j)$ represent in them the position of the arbitrary point P relative to the origins O_i and O_j of the spatial frames R_i^n and R_j^n, respectively. Another reason is the dependence of the temporal coordinate transformations (10.16), (10.17) on the spatial transfer speed $v_{ji}^{(.)}$.

Claim 323 ***Time invariance of the transformations.***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i *and* μ_j *do not depend on time* t. *A choice of the initial moment* t_0 *does not influence the transformations. The equations (10.15) through (10.20) determine time-invariant coordinate transformations.*

Claim 324 ***Nonuniformity of the transformations.***

The values of the scaling coefficients α_j^i, α_i^j, λ_j^i *and* λ_i^j *are determined by the speed* $v_P^{(.)}$ *of the arbitrary point* P. *The position coordinate* $r_P(t_{(.)})$ *and the position vector* $\mathbf{r}_P(t_{(.)})$ *represent in (10.16) through (10.19) the instantaneous position coordinate and the instantaneous position vector of the arbitrary point* P *relative to* $O_{(.)}$ *in* $R^n_{(.)}$, *(10.9). The temporal coordinate transformations (10.16), (10.17) are nonuniform over space because they depend on the position of the arbitrary point* P.

Claim 325 ***Nonuniformity of the transformations is general***

Nonuniformity of the transformations (10.15) through (10.20) is general because

$$q^{(.)}w^{(.)} = \left[v_P^{(.)}\right]^2,$$

is permitted but not required in (10.16) through (10.19).

Definition 326 ***General nonuniformity of the time field***

The transformations (10.15) through (10.20) determine a ***time-invariant generally nonuniform time field*** *if and only if the transformations hold for every pair of the time axes including those from different layers in the case the time field is multi-layer and time-invariant.*

Note 327 ***Links among the transformations***

When we compare the time-invariant nonuniform transformations (10.11) - (10.14) with the general ones, (10.16) through (10.20), we see that the former result from the latter as a special case in which $q^{(.)}w^{(.)} = \left(v_P^{(.)}\right)^2$.

The only difference between the time-invariant general nonuniform transformations (10.16) through (10.20) and the time-invariant weakly nonuniform transformations (10.4) through (10.8) is in the condition (10.8) imposed on $q^{(.)}w^{(.)}$ *in the latter case, which is not demanded in the former case as shown in (10.20).*

In view of these links among the transformations we will treat the general transformations in details. The obtained results will be valued for the other two particular cases under specific modifications that will be explained.

10.3 *Time*-invariant uniformity

10.3.1 On uniformity

Uniformity of the coordinate transformations means that the temporal coordinate transformations hold uniformly over space. This is *the uniformity of the* *temporal coordinate transformations over space* (see Case 163 in Subsection 6.5: "General, special and singular cases"). Their forms are invariant relative to a choice of the arbitrary point P and relative to its characterizations (its position, speed and/or its acceleration). When any particular instantaneous value of *time* (moment, instant) takes place, then it is the same unique *time* value over space (hence, to emphasize once more, it is independent of a choice of the arbitrary point P and of its characterizations). However, the numerical *time* value depends on a free choice of the *time* axis, but it is explicitly independent of space and of the chosen spatial frame.

The uniform temporal coordinate transformations are expressed in terms of the position and of the speed of an arbitrarily chosen and then fixed reference point P_R. It can, but need not, be the fixed light signal L. Hence, it can, but need not, move with the light speed. The *temporal transfer speed* is a freely chosen and then fixed constant speed $\vartheta^{(\cdot)}$. The generic speeds $q^{(\cdot)}$ and $w^{(\cdot)}$ are also freely selected and then fixed positive valued constant speeds. A selection of $\vartheta^{(\cdot)}$, P_R, $q^{(\cdot)}$ and $w^{(\cdot)}$ determines a type of uniformity of the transformations.

We distinguish the following three types of uniformity relative to the choice of $\vartheta^{(\cdot)}$: a) *relative uniformity* if, and only if, the spatial transfer speed $v_{ij}^{(\cdot)}$ is simultaneously the temporal transfer speed $\vartheta^{(\cdot)}$, $\vartheta^{(\cdot)} = v_{ij}^{(\cdot)}$, b) *weak uniformity* if, and only if, the speed $v_R^{(\cdot)}$ of the reference point P_R different from the light signal is simultaneously the temporal transfer speed $\vartheta^{(\cdot)}$, $\vartheta^{(\cdot)} = v_R^{(\cdot)} \neq c_{(\cdot)}^{(\cdot)}$, and c) *uniformity* if, and only if, the temporal transfer speed $\vartheta^{(\cdot)}$ is a freely selected and then fixed positive valued constant speed different from the light speed and from the reference point speed, $\vartheta^{(\cdot)} \notin \left\{c_{(\cdot)}^{(\cdot)}, v_R^{(\cdot)}\right\}$. These basic types of uniformity of the temporal coordinate transformations can be each 1) *special*, 2) *ordinary*, and 3) *general*. They are *special* if, and only if, both the light speed is generic speed, $c_{(\cdot)}^{(\cdot)} = q^{(\cdot)} = w^{(\cdot)}$, and the light signal L is the reference point P_R, $L = P_R$. They are *ordinary*, if, and only if, the generic speeds $q^{(\cdot)}$ and $w^{(\cdot)}$ can be different from the light speed, i.e. $q^{(\cdot)} \neq c_{(\cdot)}^{(\cdot)}$ and $w^{(\cdot)} \neq c_{(\cdot)}^{(\cdot)}$ are permitted, but the light signal L is the reference point P_R, $L = P_R$. We will omit the adjective *ordinary* from the expression *ordinary (relative, weak) uniformity*. They are *general*, if, and only if, the generic speeds $q^{(\cdot)}$ and $w^{(\cdot)}$ are freely selected and then they are fixed positive valued constant speeds, and the light signal L is not the reference point P_R, $P_R \neq L$, in general.

Time independence of space is clearly expressed by uniformity of the temporal coordinate transformations. They are independent of the characteristics (the position, the speed and the acceleration) of the arbitrary point P. Therefore, they can be completely entirely compatible in the temporal domain, but only partially entirely compatible in the spatial domain.

A *time field is (relatively, weakly) uniform* if and only if the temporal coordinate transformation between any two *time* axes (including those from different layers if the *time* field is multi-layer) is (relatively, weakly) uniform, respectively.

The relatively uniform transformations and *time* fields are partially uniform because the temporal transfer speed $\vartheta^{(.)}$ depends on a choice of the spatial transfer speed $v_{ji}^{(.)}$. They are uniform relative to a chosen $v_{ji}^{(.)}$.

In the weakly uniform transformations and *time* fields the temporal transfer speed $\vartheta^{(.)}$ is determined by a choice of the reference point P_R and its speed $v_R^{(.)}$.

The temporal transfer speed $\vartheta^{(.)}$ is independent of both $v_{ji}^{(.)}$ and $v_R^{(.)}$ in uniform coordinate transformations and *time* fields.

We retain the (relative) zero moment $t_{zero} = 0$ to be the initial moment t_0, $t_0 = 0$, because it does not influence *time*-invariant transformations and *time* fields.

Besides, we can be interested also in *uniformity of the spatial coordinate transformations over space*. Since they, by their definition, contain the position of the arbitrary point P, then their uniformity over space is meaningful only in the sense that space scaling coefficients are uniform over space, hence independent of the characteristics of the arbitrary point P.

If, and only if, both the temporal coordinate transformations and the spatial coordinate transformations are uniform over space then *the coordinate transformations are completely uniform*.

10.3.2 Special relative uniformity

In order to ensure uniformity of the temporal coordinate transformations, hence of a *time* field, the transformations should be independent of the choice of the arbitrary point P. Therefore, the light signal L can be, and in this framework it will be, accepted as the moving generic point G, i.e. as the moving reference point P_R for the temporal coordinate transformations, $L = G = P_R$.

In order to ensure *time* invariance of the transformations, the scaling coefficients should be independent of *time*. Altogether, the scaling coefficient functions reduce to positive real numbers because the relative light speed $c_{(.)}^{(.)}$, the speed $v_P^{(.)}$ of the arbitrary point P and the spatial transfer speed $v_{ji}^{(.)}$ are all constant,

$$\alpha_j^i(c_j^j, v_{ji}^j) \equiv \alpha_j^i = const. \in R^+,\ \alpha_i^j(c_i^i, v_{ji}^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_P^i, v_{ji}^i) \equiv \lambda_i^j = const. \in R^+,$$

and the basic *time* scaling coefficient is also constant,

$$\mu_i \equiv const. \in R^+ \Longrightarrow\ t_i = \mu_i t. \tag{10.21}$$

The temporal coordinate transformations depend on the position coordinate $r_L(t_{(.)})$ of the light signal relative to the origin $O_{(.)}$ of $R_{(.)}^n$ rather than on the position coordinate $r_P(t_{(.)})$ of the arbitrary point P relative to the origin. Since $r_L(t_{(.)})$ determines the instantaneous position of the light signal with respect to the origin $O_{(.)}$ of $R_{(.)}^n$, then the light speed value $c_{(.)}^{(.)}$ relative to the integral space $I_{(.)}$ will be used. The basic general generic equations (6.13) through (6.17) are replaced in this setting by the basic generic equations (10.22) through (10.25),

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{(c_j^j)^2} r_L(t_j) \right],\ v_{ji}^j \in R_+,\ \alpha_j^i \in R^+, \tag{10.22}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{(c_i^i)^2} r_L(t_i) \right],\ v_{ji}^i \in R_+,\ \alpha_i^j \in R^+, \tag{10.23}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u} \right],\ \lambda_j^i \in R^+, \tag{10.24}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - \ v_{ji}^i t_i \mathbf{u} \right],\ \lambda_i^j \in R^+. \tag{10.25}$$

Claim 328 ***Time independence and the transformations***

The independence property of time, (Axiom 47), is expressed by the fact that $r_L(t_{(.)})$ in the equations (10.22), (10.23) represents the position of the light signal and not the position of the arbitrary point P. However, the time independence property is not completely reflected in the temporal coordinate transformations (10.22), (10.23) because the spatial transfer speed $v_{ji}^{(.)}$ is also the temporal transfer speed.

Claim 329 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend (either explicitly or implicitly) on time t. A choice of the initial instant t_0 does not influence the transformations. The transformations (10.21) through (10.25) are time-invariant.

Claim 330 ***Uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i and λ_i^j do not depend either on the position (vector) or on the speed of the arbitrary point P. The position coordinate $r_L(t_{(.)})$ represents in (10.22), (10.23) the position coordinate of the light signal with respect to the origin $O_{(.)}$ in $R_{(.)}^n$. The temporal coordinate transformations (10.22), (10.23) contain the relative light speed value $c_{(.)}^{(.)}$, (instead of the value c of the light speed with respect to the origin O of R^n at rest), which is uniform over $R_{(.)}^n$.

The temporal coordinate transformations (10.21) through (10.23) are uniform over space because they do not depend on characteristics (the position, the speed, the acceleration) of the arbitrary point P. They depend on the position of the fixed light signal L.

Claim 331 ***Special relative uniformity of the transformations***

The uniformity of the transformations is special because the temporal coordinate transformations depend on the characteristics of the light signal (on its position and on its speed) due to $q^{(.)} \equiv c_{(.)}^{(.)}$ and $w^{(.)} \equiv c_{(.)}^{(.)}$.

The uniformity is relative because the spatial transfer speed $v_{ji}^{(.)}$ is also the temporal transfer speed.

Definition 332 ***Transformations and a time field***

A time-invariant time field is ***a time-invariant special relatively uniform time field*** *if and only if the transformations (10.21) through (10.25) hold for every pair of time axes including those from different layers if the time field is multi-layer.*

10.3.3 Relative uniformity

The temporal coordinate transformations, hence a *time* field, should depend on the position of the fixed light signal, $P_R = L$, and the spatial transfer speed $v_{ji}^{(.)}$ should be the temporal transfer speed. The relative uniformity of the temporal coordinate transformations incorporates their special relative uniformity.

The scaling coefficients are positive real numbers,

$$\alpha_j^i(q^j, v_R^j, v_{ji}^j, w^j) \equiv \alpha_j^i = const. \in R^+,$$
$$\alpha_i^j(q^i, v_R^i, v_{ji}^i, w^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_P^i, v_{ji}^i) \equiv \lambda_i^j = const. \in R^+,$$

and the basic *time* scaling coefficient is also constant,

$$\mu_i = const. \in R^+ \Longrightarrow\ t_i = \mu_i t. \tag{10.26}$$

The basic general generic equations (6.13) through (6.17) become now the equations (10.27) through (10.30):

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_L(t_j) \right], \tag{10.27}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i} r_L(t_i) \right], \tag{10.28}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) +\ v_{ji}^j t_j \mathbf{u} \right], \tag{10.29}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right], \tag{10.30}$$

where

$$q^{(.)}, w^{(.)} \in R^+,\ q^{(.)} \neq v_P^{(.)},\ w^{(.)} \neq v_P^{(.)},\ q^{(.)} w^{(.)} \neq \left[v_P^{(.)} \right]^2,\ v_{ji}^{(.)} \in R_+,$$

$$q^{(.)}, v_P^{(.)}, w^{(.)} = c_{(.)}^{(.)} \textit{ are permitted but not demanded.} \tag{10.31}$$

The temporal coordinate transformations (10.27) and (10.28) are independent of the position $r_P(t_{(.)})$ of the arbitrary point P and of its speed. This enables uniformity of the transformations over space.

Claim 333 ***Time independence and the transformations***

The temporal coordinate transformations (10.27), (10.28) express the independence property of time, (Axiom 47), because they depend explicitly only on the distance, and implicitly on the speed, of the fixed light signal L in view of

$$\mathbf{r}_L(t_{(.)}) = r_L(t_{(.)})\mathbf{u} = c_{(.)}^{(.)} t_{(.)} = c_{(.)}^{(.)} t_{(.)}\mathbf{u}.$$

However, they do not reflect fully the time independence of space since the spatial transfer speed $v_{ji}^{(.)}$ is also the temporal transfer speed.

Claim 334 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend (either explicitly or implicitly) on time t. The initial instant t_0 does not intervene in the transformations (10.26) through (10.30). They are time-invariant.

Claim 335 ***Relative uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i and λ_i^j do not depend on the position or on the speed of the arbitrary point P. The coordinate $r_L(t_{(.)})$ is in (10.27), (10.28) the position coordinate of the fixed light signal L relative to the origin $O_{(.)}$ in $R_{(.)}^n$. The temporal coordinate transformations (10.27), (10.28) do not depend on the light speed value in principle. Therefore, their uniformity is not special.

The temporal coordinate transformations are uniform over space because the light signal L is invariant relative to a choice of the arbitrary point P. Their uniformity is relative to the spatial transfer speed $v_{ji}^{(.)}$ because it is also the temporal transfer speed in (10.27), (10.28).

Definition 336 ***The transformations and a time field***

A time-invariant time field is ***a time-invariant relatively uniform time field*** *if and only if the transformations (10.26) through (10.31) are valid for every pair of the time axes including those from different layers if the time field is multi-layer.*

10.3.4 General relative uniformity

The generic speeds $q^{(.)}$ and $w^{(.)}$ can, but need not, be equal to the light speed $c_{(.)}^{(.)}$ and/or to the speed $v_R^{(.)}$ of the reference point P_R. General relative uniformity of the temporal coordinate transformations incorporates their relative uniformity and their special relative uniformity.

The basic *time* scaling coefficient is positive real number,

$$\mu_i = const. \in R^+ \Longrightarrow \; t_i = \mu_i t. \tag{10.32}$$

The other scaling coefficients are also positive real numbers,

$$\alpha_j^i(q^j, v_R^j, \vartheta^j, w^j) \equiv \alpha_j^i = const. \in R^+, \; \alpha_i^j(q^i, v_R^i, \vartheta^i, w^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) \equiv \lambda_j^i = const. \in R^+, \; \lambda_i^j(v_P^i, v_{ji}^i) \equiv \lambda_i^j = const. \in R^+.$$

The basic general generic equations (6.13) through (6.17) take the form of the equations (10.33) through (10.36),

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_R(t_j) \right], \tag{10.33}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i} r_R(t_i) \right], \tag{10.34}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u} \right], \tag{10.35}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right], \tag{10.36}$$

where

$$q^{(\cdot)}, w^{(\cdot)} \in R^+,\ v_{ji}^{(\cdot)} \in R_+,\ q^{(\cdot)}, v_P^{(\cdot)}, w^{(\cdot)} \in \{c_{(\cdot)}^{(\cdot)}, v_R^{(\cdot)}\} \text{ are permitted.} \tag{10.37}$$

The temporal coordinate transformations (10.33) and (10.34) are independent of the position $r_P(t_{(\cdot)})$ of the arbitrary point P and of its speed. This implies uniformity of the transformations over space.

Claim 337 ***Time independence and the transformations***

The independence property of time, (Axiom 47), is expressed partially due to the facts that the temporal coordinate transformations (10.33), (10.34) depend on the spatial transfer speed $v_{ji}^{(\cdot)}$, but are independent of the positions and speeds of the arbitrary point P.

Claim 338 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend (either explicitly or implicitly) on time t. The initial instant t_0 does not appear in the transformations (10.32) through (10.37). They are time-invariant.

Claim 339 ***General relative uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend on the positions and on the speeds of the arbitrary point P and of the light signal L. The coordinate $r_R(t_{(\cdot)})$ represents in the temporal coordinate transformations (10.33) and (10.34) the position coordinate of the fixed reference point P_R relative to the origin $O_{(\cdot)}$ in $R_{(\cdot)}^n$.

The temporal coordinate transformations (10.33), (10.34) are uniform over space because the reference point P_R is invariant with respect to a choice of the arbitrary point P. Their uniformity is relative to the spatial transfer speed $v_{ji}^{(\cdot)}$, which is also the temporal transfer speed. Their uniformity is general because the generic speeds $q^{(\cdot)}$ and $w^{(\cdot)}$ can be, but need not be, equal to the light speed $c_{(\cdot)}^{(\cdot)}$ and/or to the speed $v_R^{(\cdot)}$ of the reference point P_R.

Definition 340 ***The transformations and a time field***

A time-invariant time field is ***a time-invariant general relatively uniform time field*** *if and only if the transformations (10.32) through (10.37) are valid for every pair of the time axes including those from different layers if the time field is multi-layer.*

10.3.5 Special weak uniformity

The special weak uniformity is characterized by the choice of the light speed $c_{(.)}^{(.)}$ for the generic speed, $q^{(.)} \equiv w^{(.)} \equiv c_{(.)}^{(.)}$, and by the selection of the light signal position for the generic position, $r_G(t_j) = r_L(t_j)$. A light signal L is the moving generic point G, $L = G$. The speed $v_R^{(.)}$ of the fixed reference point P_R is the temporal transfer speed instead of $v_{ji}^{(.)}$.

The scaling coefficients are positive real numbers. Their values are determined by the values of the relative light speed $c_{(.)}^{(.)}$, the speed $v_R^{(.)}$ of the fixed reference point P_R, the speed $v_P^{(.)}$ of the arbitrary point P and the spatial transfer speed $v_{ji}^{(.)}$, which are all constant,

$$\alpha_j^i(c_j^j, v_R^j) \equiv \alpha_j^i = const. \in R^+,\ \alpha_i^j(c_i^i, v_R^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_P^i, v_{ji}^i) \equiv \lambda_i^j = const. \in R^+.$$

The basic *time* scaling coefficient is also constant,

$$\mu_i \equiv const. \in R^+ \Longrightarrow\ t_i = \mu_i t. \tag{10.38}$$

The temporal coordinate transformations depend on the position $r_L(t_{(.)})$ of the light signal relative to the origin $O_{(.)}$ of $R_{(.)}^n$ rather than on the position $r_P(t_{(.)})$ of the arbitrary point P relative to the origin. The basic general generic equations (6.13) through (6.17) are replaced in this setting by the basic generic equations (10.39) through (10.42),

$$t_i = \alpha_j^i \left[t_j + \frac{v_R^j}{(c_j^j)^2} r_L(t_j) \right],\ v_R^j \in R_+, \tag{10.39}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_R^i}{(c_i^i)^2} r_L(t_i) \right],\ v_R^i \in R_+, \tag{10.40}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) +\ v_{ji}^j t_j \mathbf{u} \right], \tag{10.41}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) -\ v_{ji}^i t_i \mathbf{u} \right]. \tag{10.42}$$

Claim 341 ***Time independence and the transformations***

The independence property of time, (Axiom 47), is expressed by the fact that $r_L(t_{(.)})$ in the equations (10.39) and (10.40) is the position of the light signal rather than the position of the arbitrary point P. The time independence property is completely reflected in the temporal coordinate transformations (10.38) through (10.40) because the spatial transfer speed $v_{ji}^{(.)}$ is not the temporal transfer speed.

Claim 342 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i *and* μ_j *do not depend (either explicitly or implicitly) on time* t. *A choice of the initial instant* t_0 *does not influence the transformations. The transformations (10.38) through (10.42) are time-invariant.*

Claim 343 ***Uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i *and* λ_i^j *do not depend on any characteristic (position, speed, acceleration) of the arbitrary point* P. *The coordinate* $r_L(t_{(.)})$ *represents in (10.39), (10.40) the position coordinate of the light signal with respect to the origin* $O_{(.)}$ *in* $R^n_{(.)}$. *The temporal coordinate transformations (10.39), (10.40) contain the relative values of the light speed* $c^{(.)}_{(.)}$, *(instead of the value* c *of the light speed with respect to the origin* O *of* R^n *at rest) and of the speed* $v_R^{(.)}$ *of the reference point* P_R. *Besides, the temporal coordinate transformations (10.38) through (10.40) do not depend on the characteristics of the arbitrary point* P. *Therefore, they are uniform over* $R^n_{(.)}$.

Claim 344 ***Special weak uniformity of the transformations***

The uniformity of the transformations is special and weak because the temporal coordinate transformations depend on the characteristics of the light signal (on its position and on its speed) and on the speed of the reference point P_R.

Definition 345 ***The transformations and a time field***

The transformations (10.38) through (10.42) determine ***a time-invariant specially weakly uniform time field*** *if and only if they hold for every pair of time axes including those from different layers if the time field is multi-layer.*

10.3.6 Weak uniformity

In order to guarantee weak uniformity of the temporal coordinate transformations, and of a *time* field, the transformations should depend on the position of the fixed light signal, $P_R = L$, and the speed $v_R^{(.)}$ of the reference point P_R should be the temporal transfer speed. However, the generic speeds $q^{(\cdot)}$ and $w^{(\cdot)}$ can, but need not, be different from the light speed.

The basic *time* scaling coefficient is defined by

$$\mu_i = const. \in R^+ \Longrightarrow \ t_i = \mu_i t. \tag{10.43}$$

The scaling coefficients are positive real numbers,

$$\alpha_j^i(q^j, v_R^j, w^j) \equiv \alpha_j^i = const. \in R^+,\ \alpha_i^j(q^i, v_R^i, w^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_P^i, v_{ji}^i) \equiv \lambda_i^j = const. \in R^+.$$

The basic general generic equations (6.13) through (6.17) become now the equations (10.44) through (10.47):

$$t_i = \alpha_j^i \left[t_j + \frac{v_R^j}{q^j w^j} r_L(t_j) \right], \tag{10.44}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_R^i}{q^i w^i} r_L(t_i) \right], \tag{10.45}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u} \right], \tag{10.46}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right], \tag{10.47}$$

where

$$q^{(.)}, w^{(.)} \in R^+, \; q^{(.)} \neq v_P^{(.)}, \; w^{(.)} \neq v_P^{(.)}, \; q^{(.)} w^{(.)} \notin \{ c_{(.)}^{(.)^2}, v_P^2 \}, \; v_{ji}^{(.)} \in R_+,$$

$$q^{(.)}, v_P^{(.)}, w^{(.)} \in \left\{ c_{(.)}^{(.)}, v_R^{(.)} \right\} \text{ are permitted but not demanded.} \tag{10.48}$$

Claim 346 ***Time independence and the transformations***

The temporal coordinate transformations (10.44), (10.45) depend explicitly only on the fixed speeds $q^{(.)}$, $v_R^{(.)}$, and $w^{(.)}$, and on the position of the fixed light signal L. They depend implicitly on the speed of the light signal L. They express the independence property of time, (Axiom 47).

Claim 347 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend (either explicitly or implicitly) on time t. The initial instant t_0 does not influence the transformations (10.43) through (10.47). They are time-invariant.

Claim 348 ***Weak uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i and λ_i^j do not depend on the position and on the speed of the arbitrary point P. The position $r_L(t_{(.)})$ represents in (10.44), (10.45) the position of the fixed light signal L relative to the origin $O_{(.)}$ in $R_{(.)}^n$. The temporal coordinate transformations (10.44), (10.45) do not depend explicitly on the light speed in principle. They depend explicitly on the speed $v_R^{(.)}$ of the reference point P_R. They are weakly uniform over space also because the light signal L is invariant relative to a choice of the arbitrary point P. Their uniformity is not relative to the spatial transfer speed $v_{ji}^{(.)}$ that is not the temporal transfer speed in (10.44), (10.45).

Definition 349 ***The transformations and a time field***

A time-invariant time field is ***a time-invariant weakly uniform time field*** *if and only if the transformations (10.43) through (10.48) are valid for every pair of the time axes including those from different layers if the time field is multi-layer.*

10.3.7 General weak uniformity

We allow for the generic speeds $q^{(.)}$ and $w^{(.)}$ to be equal to the light speed $c_{(.)}^{(.)}$ and/or to the speed $v_R^{(.)}$ of the temporal reference point P_R that is neither the light signal, $P_R \neq L$, nor moves with the light speed, $v_R^{(.)} \neq c_{(.)}^{(.)}$. The general weak uniformity of the temporal coordinate transformations incorporates their special weak uniformity and their weak uniformity.

The scaling coefficients are positive real numbers,

$$\alpha_j^i(q^j, v_R^j, w^j) \equiv \alpha_j^i = const. \in R^+,\ \alpha_i^j(q^i, v_R^i, w^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_P^i, v_{ji}^i) \equiv \lambda_i^j = const. \in R^+.$$

The basic *time* scaling coefficient is also a positive real number,

$$\mu_i = const. \in R^+ \Longrightarrow\ t_i = \mu_i t. \tag{10.49}$$

The basic general generic equations (6.13) through (6.17) take now the form of the equations (10.50) through (10.53):

$$t_i = \alpha_j^i \left[t_j + \frac{v_R^j}{q^j w^j} r_R(t_j) \right], \tag{10.50}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_R^i}{q^i w^i} r_R(t_i) \right], \tag{10.51}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) +\ v_{ji}^j t_j \mathbf{u} \right], \tag{10.52}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right], \tag{10.53}$$

where

$$q^{(.)}, w^{(.)} \in R^+,\ v_{ji}^{(.)}, v_R^{(.)} \in R_+,$$

$$q^{(.)}, v_P^{(.)}, w^{(.)} \in \{c_{(.)}^{(.)}, v_R^{(.)}\}\ \textit{are permitted but not demanded.} \tag{10.54}$$

Claim 350 ***Time independence and the transformations***

The transformations reflect completely the independence property of time, (Axiom 47), due to the facts that the temporal coordinate transformations (10.50), (10.51) do not depend on the spatial transfer speed $v_{ji}^{(.)}$, on the position or on the speed of the arbitrary point P.

Claim 351 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend (either explicitly or implicitly) on time t. The transformations (10.49) through (10.53) do not depend on the initial instant t_0. They are time-invariant.

Claim 352 ***General weak uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend on the positions and on the speeds of the arbitrary point P and of the light signal L. The coordinate $r_R(t_{(.)})$ represents in the temporal coordinate transformations (10.50) and (10.51) the position coordinate of the fixed reference point P_R relative to the origin $O_{(.)}$ in $R_{(.)}^n$, which is not the light signal, $P_R \neq L$.

The temporal coordinate transformations (10.50), (10.51) are uniform over space because the reference point P_R is invariant with respect to a choice of the *arbitrary point P. Their uniformity is weak because the speed of the reference point P_R is the temporal transfer speed. Their weak uniformity is general because the generic speeds $q^{(.)}$ and $w^{(.)}$ can be, but need not be, equal to the light speed $c_{(.)}^{(.)}$ and/or to the speed $v_R^{(.)}$ of the reference point P_R.*

Definition 353 ***The transformations and a time field***

A time-invariant time field is ***a time-invariant general weakly uniform time field*** *if and only if the transformations (10.49) through (10.54) are valid for every pair of the time axes including those from different layers if the time field is multi-layer.*

10.3.8 Special uniformity

In order to ensure special uniformity of the temporal coordinate transformations, which is neither weak nor relative, the transformations should be independent of both the choice of the arbitrary point P and the spatial transfer speed $v_{ji}^{(.)}$, and the basic temporal transformation should be independent of the initial position of the reference point P_R,

$$\mu_i(.) = \mu_i = const. \in R^+ \Longrightarrow t_i = \mu_i t. \tag{10.55}$$

Besides, the temporal transfer speed is $\vartheta^{(.)}$, which can be different from the light speed $c_{(.)}^{(.)}$, and from the speed $v_R^{(.)}$ of P_R. A light signal L will be accepted as the generic point G for the temporal coordinate transformations, $L = G$, and $q^{(.)} \equiv w^{(.)} \equiv c_{(.)}^{(.)}$.

The *time* and space scaling coefficients are independent of *time* in order to ensure *time* invariance of the transformations. They are positive real numbers, the values of which are determined by the constant values $c_{(.)}^{(.)}$ of the speed of light, $\vartheta^{(.)}$ of the generic speed as the temporal transfer speed and $v_{ji}^{(.)}$ of the spatial transfer speed,

$$\alpha_j^i(c_j^j, \vartheta^j) = \alpha_j^i = const. \in R^+,\ \alpha_i^j(c_i^i, \vartheta^i) = \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) = \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_P^i, v_{ji}^i) = \lambda_i^j = const. \in R^+,$$

The temporal coordinate transformations depend on the position $r_L(t_{(.)})$ of the fixed light signal L relative to the origin $O_{(.)}$ of $R_{(.)}^n$ and on the temporal speed $\vartheta^{(.)}$. The basic general generic equations (6.13) through (6.17) are replaced in this setting by the equations (10.56) through (10.59):

$$t_i = \alpha_j^i \left[t_j + \frac{\vartheta^j}{(c_j^j)^2} r_L(t_j) \right],\ \vartheta^j \in R_+, \tag{10.56}$$

$$t_j = \alpha_i^j \left[t_i - \frac{\vartheta^i}{(c_i^i)^2} r_L(t_i) \right],\ \vartheta^i \in R_+, \tag{10.57}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u}\right], \ v_{ji}^j \in R_+, \tag{10.58}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - \ v_{ji}^i t_i \mathbf{u}\right], \ v_{ji}^i \in R_+. \tag{10.59}$$

Claim 354 ***Time independence and the transformations***

The equations (10.56), (10.57) reflect fully the time independence, (Axiom 47). This is due to the facts that the position coordinate $r_L(t_{(.)})$ is the position coordinate of the light signal, not of the arbitrary point P, and the spatial transfer speed $v_{ji}^{(.)}$ does not influence the temporal coordinate transformations.

Claim 355 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend (either explicitly or implicitly) on time t. The transformations (10.55) through (10.59) do not contain the initial moment t_0. They are time-invariant.

Claim 356 ***Uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend either on the position or on the speed of the arbitrary point P. The coordinate $r_L(t_{(.)})$ represents in (10.56), (10.57) the position coordinate of the light signal with respect to the origin $O_{(.)}$ in $R_{(.)}^n$.

The temporal coordinate transformations (10.55) through (10.57) are uniform over space because they do not depend on the characteristics of the arbitrary point P. Their uniformity is not relative because they do not depend on the spatial transfer speed $v_{ji}^{(.)}$. It is not weak since the temporal transfer speed need not be the speed $v_R^{(.)}$ of the reference point P_R.

Claim 357 ***Special uniformity of the transformations***

The uniformity of the transformations is special because the temporal coordinate transformations (10.56), (10.57) depend on the speed of the light signal.

Definition 358 ***The transformations and a time field***

The transformations (10.55) through (10.59) determine ***a time-invariant specially uniform time field*** *if and only if they hold for every pair of time axes including those from different layers if the time field is multi-layer and time-invariant.*

10.3.9 Uniformity

In order to ensure uniformity of the temporal coordinate transformations, which is neither relative nor weak nor special, the temporal coordinate transformations should be independent of the characteristics of the arbitrary point P, of the spatial transfer speed $v_{ji}^{(.)}$, and the generic speeds $q^{(\cdot)}$ and $w^{(\cdot)}$ should be different from the light speed $c_{(.)}^{(.)}$. The reference point P_R will be accepted as the fixed light signal L, $P_R = L$, and the speed $\vartheta^{(\cdot)}$ independent of the speed $v_R^{(.)}$ of the reference point will be the temporal transfer speed.

The basic *time* scaling factors are positive real numbers,

$$\mu_i(.) \equiv \mu_i = const. \in R^+ \Longrightarrow t_i = \mu_i t. \tag{10.60}$$

The scaling coefficients are constant in order to ensure the *time* independence of the transformations. Their values are positive real numbers, which are determined by the constant values of the speeds $q^{(.)}$, $v_{ji}^{(.)}$, $v_P^{(.)}$, $\vartheta^{(.)}$ and $w^{(.)}$,

$$\alpha_j^i(q^j, \vartheta^j, w^j) \equiv \alpha_j^i = const. \in R^+,\ \alpha_i^j(q^i, \vartheta^i, w^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_P^i, v_{ji}^i) \equiv \lambda_i^j = const. \in R^+.$$

The temporal coordinate transformations depend on the position coordinate $r_L(t_{(.)})$ of a light ray relative to the origin $O_{(.)}$ of $R^n_{(.)}$. The basic general generic equations (6.13) through (6.17) are replaced in this setting by the equations (10.61) through (10.64):

$$t_i = \alpha_j^i \left[t_j + \frac{\vartheta^j}{q^j w^j} r_L(t_j) \right],\ \alpha_j^i \in R^+, \tag{10.61}$$

$$t_j = \alpha_i^j \left[t_i - \frac{\vartheta^i}{q^i w^i} r_L(t_i) \right],\ \alpha_i^j \in R^+, \tag{10.62}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u} \right],\ \lambda_j^i \in R^+, \tag{10.63}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - \ v_{ji}^i t_i \mathbf{u} \right],\ \lambda_i^j \in R^+, \tag{10.64}$$

where

$$q^{(.)}, w^{(.)} \in R^+,\ v_{ji}^{(.)}, \vartheta^{(.)} \in R_+,\ q^{(.)}, v_P^{(.)}, w^{(.)} = c_{(.)}^{(.)} \ are\ permitted,$$

$$q^{(.)}, v_P^{(.)}, w^{(.)} = \vartheta^{(.)} \ are\ permitted\ iff\ \vartheta^{(.)} \in R^+. \tag{10.65}$$

Claim 359 ***Time independence and the transformations***

The equations (10.61), (10.62) express completely the independence property of time, (Axiom 47). This is due to the facts that the coordinate $r_L(t_{(.)})$ is the position coordinate of the light signal and not of the arbitrary point P, and the general temporal transfer speed $\vartheta^{(.)} \in R_+$ is the constant temporal transfer speed, which is generally different from the spatial transfer speed $v_{ji}^{(.)}$.

Claim 360 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i and μ_j do not depend (either explicitly or implicitly) on time t. A choice of the initial moment t_0 does not influence the transformations. The transformations (10.60) through (10.65) are time-invariant.

Claim 361 ***Uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ, λ_i^j, μ_i and μ_j do not depend either on the position or on the speed of the arbitrary point P. The position $r_L(t_{(.)})$ represents the position of the light signal in (10.61), (10.62).

The temporal coordinate transformations (10.60) through (10.62) are uniform over space because they do not depend on the characteristics of the arbitrary point P.

Uniformity of the transformations is neither relative or weak or special because the temporal coordinate transformations do not depend on the spatial transfer speed, and the generic speeds can be different from the light speed.

Definition 362 ***The transformations and a time field***

A time-invariant time field is ***a time-invariant uniform time field*** *if and only if the transformations (10.60) through (10.65) hold for every pair of time axes including those from different layers if the time field is multi-layer.*

10.3.10 General uniformity

In order to ensure general uniformity of the temporal coordinate transformations, hence of a *time* field, the transformations should depend on an arbitrarily accepted and then fixed moving reference point P_R, and the temporal transfer speed $\vartheta^{(.)}$ should be in general independent of the spatial transfer speed $v_{ji}^{(.)}$ and of the speed $v_R^{(.)}$ of the reference point P_R. Therefore, the light signal can be, but need not be, accepted as the reference moving point for the temporal coordinate transformations: $P_R \neq L$ is permitted. The general uniformity of the temporal coordinate transformations incorporates their special uniformity and their uniformity.

The scaling coefficients reduce to positive real numbers. Their values are determined by the positive constant values of the speeds $q^{(.)}$, $\vartheta^{(.)}$, v_P^j and $w^{(.)}$ relative to the corresponding frames,

$$\alpha_j^i(q^j, \vartheta^j, \vartheta^j, w^j) \equiv \alpha_j^i = const. \in R^+,\ \alpha_i^j(q^i, \vartheta^i, \vartheta^i, w^i) \equiv \alpha_i^j = const. \in R^+,$$
$$\lambda_j^i(v_P^j, v_{ji}^j) \equiv \lambda_j^i = const. \in R^+,\ \lambda_i^j(v_P^i, v_{ji}^i) \equiv \lambda_i^j = const. \in R^+,$$

and

$$\mu_i(.) \equiv \mu_i = const. \in R^+ \Longrightarrow t_i = \mu_i t. \tag{10.66}$$

The basic general generic equations (6.13) through (6.17) reduce to the equations (10.67) through (10.70),

$$t_i = \alpha_j^i \left[t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j) \right], \tag{10.67}$$

$$t_j = \alpha_i^j \left[t_i - \frac{\vartheta^i}{q^i w^i} r_R(t_i) \right], \tag{10.68}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u}\right], \tag{10.69}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}\right], \tag{10.70}$$

where

$$q^{(.)},\ v_P^{(.)},\ \vartheta^{(.)},\ w^{(.)} \neq \vartheta^{(.)}\ in\ general;\ q^{(.)}, w^{(.)} \in R^+;\ v_{ji}^{(.)}, \vartheta^{(.)} \in R_+,$$

$$q^{(.)}, v_P^{(.)}, w^{(.)} = c_{(.)}^{(.)}\ are\ permitted,\ not\ required,$$

$$q^{(.)}, v_P^{(.)}, w^{(.)} = \vartheta^{(.)}\ are\ permitted\ iff\ \vartheta^{(.)} \in R^+,\ not\ required. \tag{10.71}$$

Claim 363 ***Time independence and the transformations***

The facts that the temporal coordinate transformations (10.67) and (10.68) do not depend on the spatial transfer speed $v_{ji}^{(.)}$ *and that they depend explicitly only on the position of the fixed reference point* P_R *reflect completely the independence property of time, (Axiom 47).*

Claim 364 ***Time invariance of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i *and* μ_j *do not depend (either explicitly or implicitly) on time* t*. A choice of the initial moment* t_0 *does not influence the transformations. The equations (10.66) through (10.70) determine the time-invariant coordinate transformations.*

Claim 365 ***General uniformity of the transformations***

The scaling coefficients α_j^i, α_i^j, λ_j^i, λ_i^j, μ_i *and* μ_j *do not depend either on the position or on the speed of the arbitrary point* P *or on such characteristic of the light signal* L*. The coordinate* $r_R(t_{(.)})$ *represents in (10.67), (10.68) the position coordinate of the fixed reference point* P_R *relative to the origin* $O_{(.)}$ *in* $R_{(.)}^n$*.*

The temporal coordinate transformations (10.67), (10.68) are uniform over space because the point P_R *is invariant relative to a choice of the arbitrary point* P*. Their uniformity is general because the temporal transfer speed* $\vartheta^{(.)}$ *can be independent of the characteristics of the light signal* L*, of the arbitrary point* P *and of the reference point* P_R*, and the generic speeds* $q^{(.)}$ *and* $w^{(.)}$ *can be different from the light speed* $c_{(.)}^{(.)}$*.*

Definition 366 ***The transformations and a time field***

The transformations (10.66) through (10.71) determine ***a time-invariant general uniform time field*** *if and only if every pair of the time axes, including those from different layers if the time field is multi-layer, obeys the transformations.*

Chapter 11

Light Speed of the Arbitrary Point

11.1 General nonuniformity

11.1.1 Transformations of temporal and spatial coordinates

Basic relationships

The scaling coefficients should be positive real numbers. Their values are possibly determined by constant positive speed values $q^{(.)}$ and $w^{(.)}$ of generic velocities $\mathbf{q}^{(.)} = q^{(.)}\mathbf{u}$ and $\mathbf{w}^{(.)} = w^{(.)}\mathbf{u}$ relative to different integral spaces $I_{(.)} = \mathfrak{T}_{(.)} \times R^n_{(.)}$. They are constant. The second equation in (10.15) is valid, i.e.

$$t_i = \mu_i t, \; \mu_i = const. \in R^+. \tag{11.1}$$

The transformations (10.16) through (10.20), which constitute the basis in this framework, and which are *fully generalized time-invariant basic Lorentz transformations*, have the following forms:

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j) \right], \; \alpha_j^i \in R^+, \tag{11.2}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i) \right], \; \alpha_i^j \in R^+, \tag{11.3}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u} \right], \; \lambda_j^i \in R^+, \tag{11.4}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right], \; \lambda_i^j \in R^+, \tag{11.5}$$

where

$$q^{(.)}, w^{(.)} \in R^+, \; q^{(.)} w^{(.)} \neq \left[c_{(.)}^{(.)} \right]^2, \; v_{ji}^{(.)} \in R_+,$$

$$q^{(.)}w^{(.)} = \left[v_P^{(.)}\right]^2 \ \textit{is permitted but not required.} \tag{11.6}$$

The product $q^{(.)}w^{(.)}$ can be equal to $\left[v_P^{(.)}\right]^2$, which is not permitted in the framework of *time*-invariant weakly nonuniform transformations.

Einsteinian approach demands that the position and the speed of the arbitrary point P be the position and the speed of a light ray. This means that both $\mathbf{r}_P\left(t_{(.)}\right) \equiv \mathbf{r}_L(t_{(.)})$ and $\mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$ should be used in the proofs based on Einsteinian approach.

Einstein's condition (7.1), [144] through [154], for the validity of Lorentz transformations in this framework of the generalized *time*-invariant transformations (11.2) through (11.6), takes the following form:

$$\begin{bmatrix} \mathbf{r}_P^T(t_i) & t_i\mathbf{c}_i^{i^T} \end{bmatrix} D \begin{bmatrix} \mathbf{r}_P(t_i) \\ t_i\mathbf{c}_i^i \end{bmatrix} \equiv \begin{bmatrix} \mathbf{r}_P^T(t_j) & t_j\mathbf{c}_j^{j^T} \end{bmatrix} D \begin{bmatrix} \mathbf{r}_P(t_j) \\ t_j\mathbf{c}_j^j \end{bmatrix},$$

$$D = blockdiag\left\{A \quad -B\right\} \in R^{2n\text{x}2n},$$

$$A \in R^{n\text{x}n} \textit{ and } B \in R^{n\text{x}n} \textit{ are positive definite and possibly different.} \tag{11.7}$$

This condition expresses the preservation of the generalized length in the integral spaces $I_i = \mathfrak{T}_i\text{x}R_i^n$ and $I_j = \mathfrak{T}_j\text{x}R_j^n$ under the transformations applied to the coordinates of the arbitrary point P moving with the light speed. When the matrices A and B are different, then they express jointly *time* independence of the space. We will refer to the condition (11.7) throughout this chapter.

The numerical value of the light speed may depend on a choice of a *time* unit, hence, on a choice of a *time* axis. Therefore, the condition (11.7) generalizes and incorporates that by Einstein, (7.1), which is established only for the case when the arbitrary point P moves with the light speed that is invariant relative to *time* axes and to spatial coordinate systems.

The goal of this Subsection is to determine the values of the scaling coefficients α_j^i, α_i^j, λ_j^i and λ_i^j in the general, special and singular case, by considering the movement of the arbitrary point P with the light speed.

Solutions for the general case

Theorem 367 *Let the time scaling coefficient μ_i be defined by (11.1). In order for the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, determined for the light speed of the arbitrary point P, to be positive real numbers and to obey (11.2) through (11.6), and for (11.1) through (11.6) to imply (11.7) it is necessary and sufficient that the following relationships hold for any choice of the time scaling coefficients $\mu_i \in R^+$ and $\mu_j \in R^+$:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j}\frac{1}{1+\frac{v_{ji}^j c_j^j}{q^j w^j}} = \frac{c_j^j}{c_i^i}\frac{1}{1+\frac{v_{ji}^j c_j^j}{q^j w^j}}, \tag{11.8}$$

$$\alpha_i^j = \frac{\mu_j}{\mu_i}\frac{1}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}} = \frac{c_i^i}{c_j^j}\frac{1}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}, \tag{11.9}$$

$$\lambda_j^i = \frac{1}{1+\frac{v_{ji}^j}{c_j^j}}, \tag{11.10}$$

$$\lambda_i^j = \frac{1}{1-\frac{v_{ji}^i}{c_i^i}}, \tag{11.11}$$

$$\min\left\{\frac{q^i w^i}{c_i^i}, c_i^i\right\} > v_{ji}^i \geq 0, \tag{11.12}$$

$$\frac{\mu_j}{\mu_i} = \frac{c_i^i}{c_j^j},\ v_{ji}^{(.)} \in R^+ \Longrightarrow \frac{\mu_j}{\mu_i} = \frac{v_{ji}^i}{v_{ji}^j}. \tag{11.13}$$

The transformations (11.2) through (11.5) become:

$$t_i = \frac{\mu_i}{\mu_j}\frac{t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)}{1+\frac{v_{ji}^j c_j^j}{q^j w^j}}, \tag{11.14}$$

$$t_j = \frac{\mu_j}{\mu_i}\frac{t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i)}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}, \tag{11.15}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + \mathbf{v}_{ji}^j t_j}{1+\frac{v_{ji}^j}{c_j^j}}, \tag{11.16}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - \mathbf{v}_{ji}^i t_i}{1-\frac{v_{ji}^i}{c_i^i}}. \tag{11.17}$$

They are partially both entirely and pairwise compatible.

Theorem is proved in Appendix 22.1.

Note 368 *The preceding theorem permits mutually independent selection of the basic time scaling coefficients μ_i and μ_j.*

Note 369 *The time scaling coefficients α_i^j and α_j^i depend on the relative values $q^{(.)}$ and $w^{(.)}$ of the generic speeds. They are allowed to depend on the value $v_P^{(.)}$ of the speed of the arbitrary point P in this framework due to (11.6).*

Note 370 *Sufficiency of the conditions is proved only for the case when the arbitrary point P moves with the light speed, but not for its arbitrary speed. This caused restrictive compatibility of the transformations (11.14) through (11.17).*

Conclusion 371 *The formulae (11.8) through (11.11), i.e. the transformations (11.2) through (11.17), are essentially different from the existing formulae for the scaling coefficients in Einsteinian relativity theory, i.e. from Lorentz transformations, respectively. The above formulae do not contain either square roots or the squared quotients* $v_{ji}^{(.)}/c_{(.)}^{(.)}$*, which characterize Einsteinian relativity theory.*

The equations (11.13) prove the noninvariance of both the light speed and the spatial transfer speed.

The above formulae are expressed in terms of the values of both the light speed and the spatial transfer speed relative to the integral spaces, hence relative to the time axes. This is a priory rejected in Einsteinian relativity theory.

Consequently, the preceding results restrict the relative value of the spatial transfer speed $v_{ji}^{(.)}$ *by the value of the relative light speed* $c_{(.)}^{(.)}$*, and by* $q^{(\cdot)}w^{(\cdot)}$*,* $v_{ji}^{(.)} < \min\left\{c_{(.)}^{(.)}, q^{(\cdot)}w^{(\cdot)}\left(c_{(.)}^{(.)}\right)^{-1}\right\}$*. This condition does not exist in Einsteinian relativity theory.*

The transformations (11.14) through (11.17) are partially entirely and pairwise compatible, which holds also for pairwise compatibility of the Lorentz transformations (7.20) and (7.21).

Moreover, the above theorem introduces the basic time scaling coefficient μ_i *into the temporal coordinate transformations (11.14) and (11.15), which does not exist in Lorentz transformations (7.20) and (7.21) of the temporal coordinates. The basic time scaling coefficient* μ_i*, (11.1), is completely missed in Einsteinian relativity theory.*

Theorem 372 *Let the spatial frames* R_i^n *and* R_j^n *move with the same speed in the same direction and in the same sense. Let the positive real valued time scaling coefficient* μ_i *be defined by (11.1). In order for the scaling coefficients* α_i^j*,* α_j^i*,* $\alpha_j^i \neq \alpha_i^j$*,* λ_i^j *and* λ_j^i*, determined for the light speed of the arbitrary point P, to be positive real numbers and to obey (11.2) through (11.6), and for (11.1) through (11.6) to imply (11.7) it is necessary and sufficient that the following equations hold for any choice of the time scaling coefficients* $\mu_i \in R^+$ *and* $\mu_j \in R^+$*:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j},\ \alpha_i^j = \frac{\mu_j}{\mu_i},\ \lambda_j^i = \lambda_i^j = 1,\ \frac{\mu_i}{\mu_j} = \frac{c_j^j}{c_i^i}. \tag{11.18}$$

The equations (11.2) through (11.5), and the equations (11.14) through (11.17) reduce to

$$t_i = \frac{\mu_i}{\mu_j}t_j,\ t_j = \frac{\mu_j}{\mu_i}t_i,\ \mathbf{r}_P\left(t_i\right) = \mathbf{r}_P\left(t_j\right). \tag{11.19}$$

They are completely both pairwise and entirely compatible.

Proof. *Necessity and sufficiency.* Necessity and sufficiency of the conditions of the above theorem statement come out from Theorem 367 for $v_{ji}^{(.)} \equiv 0$.

Compatibility. The complete pairwise compatibility follows directly from $t_i = \frac{\mu_i}{\mu_j}t_j$ and $t_j = \frac{\mu_j}{\mu_i}t_i$, $\mathbf{r}_P\left(t_i\right) = \mathbf{r}_P\left(t_j\right)$ and $\mathbf{r}_P\left(t_j\right) = \mathbf{r}_P\left(t_i\right)$. Since these two pairs of the equations are fully decoupled, then their pairwise compatibility implies their complete entire compatibility. Q. E. D ■

The unity value of the space scaling coefficients, $\lambda_j^i = \lambda_i^j = 1$, is natural since the frames R_i^n and R_j^n move in parallel with the same speed in the same sense.

Remark 373 ***Simple case beyond Einsteinian relativity theory***

This corollary shows that the integral spaces $I_i = \mathfrak{T}_i \times R_i^n$ *and* $I_j = \mathfrak{T}_j \times R_j^n$ *can have different time scales,* $\mathfrak{T}_i \neq \mathfrak{T}_j$*, in spite the coordinate systems* R_i^n *and* R_j^n *move with the same speed:* $\mathbf{v}_{O_i}^{O} = \mathbf{v}_{O_j}^{O}$*, which implies* $v_{ji}^i = v_{ji}^j = 0$*. This agrees with the time independence of the space, with the theory of singularly perturbed systems and with the theory of dynamical systems with multiple time scales, [210] through [212], [273], [274], [350], [356], [447], [462].*

This case (Theorem 372) is beyond Einsteinian relativity theory. Lorentz transformations of all coordinates reduce for $v_{ji}^{(.)} \equiv 0$ *to the identity transformations,* $t_i \equiv t_j$, $\mathbf{r}_P(t_i) \equiv \mathbf{r}_P(t_j)$*, which means that they do not allow any nonidentity coordinate transformation. This is due to both* $v_{ji}^i = v_{ji}^j = v_{ji} = v = 0$ *and the fact that Einsteinian relativity theory has not recognized the existence and the significance of the basic time scaling coefficient* μ_i*. For more details see Case 216, which is summarized in Conclusion 218.*

Note 374 *The transformations (11.14) through (11.17) are not completely (either entirely or pairwise) compatible in general. They become completely both entirely and pairwise compatible in the case when the coordinate systems* R_i^n *and* R_j^n *move with the same speed, (11.19). This is natural because in this case the time coordinate transformations (11.14), (11.15) become independent of the spatial coordinates, and the spatial coordinate transformations (11.16), (11.17) become the trivial identity transformation.*

Solution for the special case

By accepting partially Einsteinian conditions we also adopt a priory the same *time* scaling coefficients and the same space scaling coefficients.

Theorem 375 *Let the time scaling coefficient* $\mu_i \in R^+$ *be defined by (11.1). Let* $B = A$ *in D, (11.7). In order for the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i *and* λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$*, determined for the case when the arbitrary point P moves with the speed of light, to be positive real numbers and to obey (11.2) through (11.6), and for (11.1) through (11.6) to imply (11.7) it is necessary and sufficient that the relationships (11.20) through (11.23) hold for any choice of the time scaling coefficient* $\mu_j \in R^+$*:*

$$c_j^j = c_i^i = c_{ij} = c_{ji}, \ \sqrt{q^i w^i} = \sqrt{q^j w^j} = \sqrt{q^{ji} w^{ji}} = c_{ij}, \tag{11.20}$$

$$v_{ji}^i = v_{ji}^j = v_{ji} = -v_{ij}, \tag{11.21}$$

$$\alpha_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}}{\sqrt{q^{ji}w^{ji}}}\right)^2}} = \lambda_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}}, \ \sqrt{q^{ji}w^{ji}} = c_{ij} > v_{ji}, \quad (11.22)$$

$$\mu_i = \mu_j \sqrt{\frac{1 + \frac{v_{ji}}{\sqrt{q^{ji}w^{ji}}}}{1 - \frac{v_{ji}}{\sqrt{q^{ji}w^{ji}}}}} = \mu_j \sqrt{\frac{1 - \frac{v_{ij}}{\sqrt{q^{ji}w^{ji}}}}{1 + \frac{v_{ij}}{\sqrt{q^{ji}w^{ji}}}}}. \quad (11.23)$$

The equations (11.20) through (11.22) transform the equations (11.2) through (11.5) into the equations (11.24) through (11.27):

$$t_i = \frac{t_j + \frac{v_{ji}}{q^{ji}w^{ji}} r_P(t_j)}{\sqrt{1 - \left(\frac{v_{ji}}{\sqrt{q^{ji}w^{ji}}}\right)^2}}, \quad (11.24)$$

$$t_j = \frac{t_i - \frac{v_{ji}}{q^{ji}w^{ji}} r_P(t_i)}{\sqrt{1 - \left(\frac{v_{ji}}{\sqrt{q^{ji}w^{ji}}}\right)^2}}, \quad (11.25)$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + \mathbf{v}_{ji} t_j}{\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}}, \quad (11.26)$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - \mathbf{v}_{ji} t_i}{\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}}. \quad (11.27)$$

The transformations (11.24) through (11.27) are partially both entirely and pairwise compatible.

We prove Theorem 375 in Appendix 22.2.

Remark 376 *Since we extend Einsteinian relativity theory then the sufficiency of the conditions is proved only for the case when the arbitrary point P moves with the light speed. The proof is not valid for an arbitrary speed of the point P. The consequence is the restrictive rather then complete pairwise compatibility of the transformations (11.24) through (11.27).*

Note 377 *This theorem emphasizes that we should use the light speed value* c_{ij} *relative to the time axes* $\mathfrak{T}_i$ *and* $\mathfrak{T}_j$. *The requirement for all the time scaling coefficients to be mutually equal and for all the space scaling coefficients to be also mutually equal causes the restriction on the light speed value to be equal relative to the time axes* $\mathfrak{T}_i$ *and* $\mathfrak{T}_j$.

Note 378 *The basic time scaling coefficients* μ_i *and* μ_j *are mutually dependent, (11.23).*

Solution for the singular case

Einsteinian approach demands a priory, not only the same *time* scaling coefficients and the same space scaling coefficients, but also the invariance of the light speed value and the invariance of the spatial transfer speed relative to integral spaces. This means that we should replace: α_{ji} by α, λ_{ij} by λ, c_{ij} by c, and v_{ij} by v in the preceding theorem:

Corollary 379 *Let the time scaling coefficient $\mu_i \in R^+$ be defined by (11.1). Let $B = A$ in D, (11.7). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji} = \alpha$ and $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji} = \lambda$, determined for the case when the arbitrary point P moves with the speed of light $c_{ij} \equiv c$, to be positive real numbers and to obey (11.2) through (11.6) for $v_{ji} \equiv v$, and for (11.1) through (11.6) to imply (11.7), it is necessary and sufficient that the equations (11.20) for $c_{ij} \equiv c$ and for $q^i w^i = q^j w^j = qw$, (11.21) for $v_{ij} \equiv v$, (11.28) and (11.29) hold for any choice of the time scaling coefficient $\mu_j \in R^+$,*

$$\alpha = \frac{1}{\sqrt{1-\left(\frac{v}{\sqrt{qw}}\right)^2}} = \lambda = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}},\ v < \sqrt{qw} = c, \tag{11.28}$$

$$\mu_i = \mu_j \sqrt{\frac{1+\frac{v}{\sqrt{qw}}}{1-\frac{v}{\sqrt{qw}}}}. \tag{11.29}$$

The equations (11.20) for $c_{ij} \equiv c$ and for $q^{(\cdot)} w^{(\cdot)} \equiv qw$, (11.21) for $v_{ij} \equiv v$, (11.28) and (11.29) transform the equations (11.2) through (11.5) into the equations (11.30) through (11.33):

$$t_i = \frac{t_j + \frac{v}{qw} r_P(t_j)}{\sqrt{1-\left(\frac{v}{\sqrt{qw}}\right)^2}}, \tag{11.30}$$

$$t_j = \frac{t_i - \frac{v}{qw} r_P(t_i)}{\sqrt{1-\left(\frac{v}{\sqrt{qw}}\right)^2}}, \tag{11.31}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + \mathbf{v} t_j}{\sqrt{1-\left(\frac{v}{c}\right)^2}}, \tag{11.32}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - \mathbf{v} t_i}{\sqrt{1-\left(\frac{v}{c}\right)^2}}. \tag{11.33}$$

The transformations (11.30) through (11.33) are partially both entirely and pairwise compatible.

Comment 380 ***Rebuttal to Einstein's claim on the time dependence. Axiom 47 leads to Lorentz transformations***

This shows, in view of $\sqrt{qw} = c$, (11.28), that we can deduce Lorentz transformations from Axiom 47 as soon as we adopt all the restrictions a priory accepted by Lorentz, Einstein and Poincaré. However, differently than they, but consistently with Newton, hence by starting with the independent nature of time, (Axiom 47), we have reproved the original Lorentz transformations. They may not be used as an argument for the wrong claim on time dependence of space.

11.1.2 Transformations of velocity

Velocity transformations will be derived from the *time* and space coordinate transformations (11.14) through (11.17). We remind ourselves that they were obtained by applying Einsteinian approach, i.e. exclusively for the light speed of the arbitrary point P. However, by continuing to follow Einsteinian methodology, we will use them as they were determined for an arbitrary value of the speed of the arbitrary point P.

The general case

Theorem 381 *Let the time scaling coefficient μ_i be defined by (11.1). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i \neq \alpha_i^j$, λ_j^i and λ_i^j, $\lambda_j^i \neq \lambda_i^j$, obey (11.8) through (11.11), and let (11.14) through (11.17) hold. Then, the velocity $\mathbf{v}_P^i \neq \mathbf{0}$ of the arbitrary point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the velocity $\mathbf{v}_P^j \neq \mathbf{0}$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i} \frac{1 + \frac{v_{ji}^j c_j^j}{q^j w^j}}{1 + \frac{v_{ji}^j}{c_j^j}} \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1 + \frac{v_{ji}^j v_P^j}{q^j w^j}},\ \mathbf{v}_P^j = \frac{\mu_i}{\mu_j} \frac{1 - \frac{v_{ji}^i c_i^i}{q^i w^i}}{1 - \frac{v_{ji}^i}{c_i^i}} \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{1 - \frac{v_{ji}^i v_P^i}{q^i w^i}}. \tag{11.34}$$

In order for them to be compatible it is necessary and sufficient that

$$v_P^i = c_i^i,\ v_P^j = c_j^j, \tag{11.35}$$

i.e. the transformations are only partially compatible.

For the proof see Appendix 22.3.

Comment 382 *The condition (11.35) is natural since the coordinate transformations (11.14) through (11.17) are determined exclusively for the light speed of the arbitrary point P.*

Remark 383 ***Inapplicability of Einstein's law of velocity composition***

The velocity transformations (11.34) differ essentially from Einstein's law of the velocity composition. The former permit all velocities to be relative to integral spaces. Hence, they permit noninvariance of both the light velocity and *the transfer velocity. They contain the basic time scaling factors μ_i and μ_j, and the relative values of the generic speeds. Einstein's law is valid only for invariant values of both the light speed and the transfer speed. It is inapplicable in this framework.*

Conclusion 384 ***Noninvariance of the light speed***

If the arbitrary point P moves with the light speed then the equations (11.34) take the following forms:

$$\mathbf{c}_i^i = \frac{\mu_j}{\mu_i}\mathbf{c}_j^j,\ \mathbf{c}_j^j = \frac{\mu_i}{\mu_j}\mathbf{c}_i^i.$$

This confirms (11.13) and the former conclusions that the numerical value of the light speed depends on a choice of the integral space. It verifies the noninvariance of the light speed.

Conclusion 385 *The velocity transformations (11.34) are valid also if there are only time coordinate transformations, or only spatial coordinate transformations (in which case $q^{(\cdot)} \equiv w^{(\cdot)} \equiv \infty$ should be formally set in the formulae). However, Lorentz transformations, and by them induced Einstein's law of the velocity composition, then do not hold (Case 216, Conclusion 218, Case 219 and Conclusion 227). Besides, the former depend on the relative values $q^{(\cdot)}$ and $w^{(\cdot)}$ of the generic speeds, which do not exist in Einsteinian formulae.*

Theorem 386 *If the velocity transformations (11.34) and $\mathbf{v}_P^{(\cdot)} \equiv \mathbf{c}_{(\cdot)}^{(\cdot)}$ are valid then for the light speed $c_{(\cdot)}^{(\cdot)}$ to be invariant it is necessary and sufficient that any of the following equations holds:*

$$\mu_j/\mu_i = 1 \text{ or equivalently } t_j = t_i \text{ or equivalently } v_{ji}^{(\cdot)} \equiv v_{ji}. \tag{11.36}$$

Then,

$$\mathbf{r}_L(t_i) \equiv \mathbf{r}_L(t_j) \equiv \mathbf{r}_L(t). \tag{11.37}$$

Appendix 22.4 contains the proof.

Conclusion 387 ***Light speed invariance implies the identity transformations***

If the integral spaces $I_i = \mathfrak{T}_i x R_i^n$ and $I_j = \mathfrak{T}_j x R_j^n$ are related by the transformations (11.2) through (11.5), then the light speed $c_{(\cdot)}^{(\cdot)}$ is invariant with respect to them if, and only if, all the coordinate transformations become the identity transformation. There is not essentially any coordinate transformation. This illustrates that the light speed is noninvariant in general.

Moreover, we can easily conclude from the equations (11.34) that the spatial transfer speed is noninvariant in general, too.

Special case

Theorem 388 *Let $A = B$ in D (11.7). Let the time scaling coefficient μ_i be defined by (11.1). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, obey (11.22), and let (11.24) through (11.27) hold. Then, the velocity $\mathbf{v}_P^i \neq \mathbf{0}$ of the arbitrary point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the velocity $\mathbf{v}_P^j \neq \mathbf{0}$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}}{1 + \frac{v_{ji} v_P^j}{(c_{ij})^2}}, \; \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}}{1 - \frac{v_{ji} v_P^i}{(c_{ij})^2}}. \tag{11.38}$$

They are partially compatible.

If the arbitrary point P moves with the light speed then

$$\mathbf{c}_i^i = \mathbf{c}_j^j = \mathbf{c}_{ij} = \mathbf{c}_{ji}. \tag{11.39}$$

The proof is in Appendix 22.5.

Note 389 ***Generalization of Einstein's law of the velocity composition***
The velocity transformations (11.38) generalize Einstein's law of the velocity composition. The former permit all velocities to be relative to integral spaces. Hence, they permit noninvariance of the transfer velocity. Einstein's law is valid only for invariant values of both the light speed and the transfer speed.

Remark 390 ***Light speed invariance is the consequence of Lorentz' restrictions***
The proof of the velocity transformations (11.38) shows that the same scaling factor $1/\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}$ for the temporal coordinate transformations and for the spatial coordinate transformations cancels in the quotients of their derivatives that determine the velocities. The scaling factor is determined for the light speed of the arbitrary point P. These facts explain why only the numerical vector value of the light velocity rests unchanged, i.e. invariant, when it is transformed by using the coordinate transformations (11.24) through (11.27).

The fact that the coordinate transformations (11.24) through (11.27) change the temporal coordinate and the spatial coordinate of a light signal in the same ratio implies the invariance of the numerical vector value of the light velocity transformed by these coordinate transformations. The fact that the scaling coefficients were determined exclusively for the light speed of the arbitrary point clarifies why only the value of the light speed is then invariant.

Singular case

The preceding theorem takes the following particular form in the singular case:

Corollary 391 *Let $A = B$ in D, (11.7). Let the numerical value of the light speed be invariant relative to integral spaces, $c_{ij} \equiv c_{ji} \equiv c$. Let also $v_{ji} \equiv v$. Let the time scaling coefficient $\mu_i \in R^+$ be defined by (11.1). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i \equiv \alpha_i^j \equiv \alpha_{ij} \equiv \alpha_{ji} \equiv \alpha$, λ and λ_i^j, $\lambda_j^i \equiv \lambda_i^j \equiv \lambda_{ij} \equiv \lambda_{ji} \equiv \lambda$, obey (11.28), and let (11.30) through (11.33) be valid. Then, the velocity $\mathbf{v}_P^i \neq \mathbf{0}$ of the point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the velocity $\mathbf{v}_P^j \neq \mathbf{0}$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated by the equations (11.40):*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}}{1 + \frac{v v_P^j}{c^2}},\ \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}}{1 - \frac{v v_P^i}{c^2}}. \tag{11.40}$$

The transformations are partially compatible.

Note 392 ***Axiom 47 leads to Einstein's law of the velocity composition***
This theorem shows once more that Einstein's law of the composition of velocities holds just in the singular case. Its proof results from Axiom 47.

Note 393 *Remark 390 is applicable also in the singular case if we replace the equations numbers (11.24) through (11.27) by (11.30) through (11.33), and (11.38) by (11.40).*

11.2 Nonuniformity

11.2.1 Transformations of temporal and spatial coordinates

Basic relationships

All the scaling coefficients are positive real numbers. The *time* scaling coefficient μ_i, (11.41), determines the basic temporal coordinate transformation,

$$t_i = \mu_i t. \tag{11.41}$$

In this framework, both $q^{(.)}$ and $w^{(.)}$ are replaced by the value $v_P^{(.)}$ of the constant velocity $\mathbf{v}_P^{(.)}$ of the arbitrary point P so that the basic general generic transformations (6.13) through (6.17) take the form of the equations (10.11) through (10.14), which are repeated as:

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{\left(v_P^j\right)^2} r_P(t_j) \right],\ v_{ji}^j \in R_+,\ v_P^j \in R^+, \tag{11.42}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{\left(v_P^i\right)^2} r_P(t_i) \right],\ v_{ji}^i \in R_+,\ v_P^i \in R^+, \tag{11.43}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u}\right], \tag{11.44}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - \ v_{ji}^i t_i \mathbf{u}\right]. \tag{11.45}$$

Note 394 *A particular form of the transformations (11.42) through (11.45) obtained for* $v_P^{(.)} \equiv c_{(.)}^{(.)}$ *in (11.42) and in (11.43) are the time - invariant slightly generalized basic Lorentz transformations (7.6) through (7.9).*

The condition for the preservation of the (generalized) length in different integral spaces holds unchanged in the form of the identity (11.7).

Solutions for the general case

Theorem 395 *Let the constant time scaling coefficient* $\mu_i \in R^+$ *be defined by (11.41). In order for the positive real valued scaling coefficients* α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j *and* λ_j^i, $\lambda_j^i \neq \lambda_i^j$, *determined for the light speed of the arbitrary point P,* $v_P^{(.)} \equiv c_{(.)}^{(.)}$, *to obey (11.42) through (11.45), and for (11.41) through (11.45) to imply (11.7) it is necessary and sufficient that the following relationships hold for any choice of the time scaling coefficients* $\mu_i \in R^+$ *and* $\mu_j \in R^+$*:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{v_{ji}^j}{c_j^j}} = \frac{c_j^j}{c_i^i} \frac{1}{1 + \frac{v_{ji}^j}{c_j^j}} = \frac{c_j^j}{c_i^i} \lambda_j^i, \tag{11.46}$$

$$\alpha_i^j = \frac{\mu_j}{\mu_i} \frac{1}{1 - \frac{v_{ji}^i}{c_i^i}} = \frac{c_i^i}{c_j^j} \frac{1}{1 - \frac{v_{ji}^i}{c_i^i}} = \frac{c_i^i}{c_j^j} \lambda_i^j, \tag{11.47}$$

$$\lambda_j^i = \frac{1}{1 + \frac{v_{ji}^j}{c_j^j}} = \frac{c_i^i}{c_j^j} \alpha_j^i, \tag{11.48}$$

$$\lambda_i^j = \frac{1}{1 - \frac{v_{ji}^i}{c_i^i}} = \frac{c_j^j}{c_i^i} \alpha_i^j, \tag{11.49}$$

$$v_{ji}^i < c_i^i, \tag{11.50}$$

$$\frac{\mu_j}{\mu_i} = \frac{c_i^i}{c_j^j},\ v_{ji}^{(.)} \in R^+ \Longrightarrow \frac{\mu_j}{\mu_i} = \frac{v_{ji}^i}{v_{ji}^j}. \tag{11.51}$$

The transformations (11.42) through (11.45) take the following forms:

$$t_i = \frac{\mu_i}{\mu_j} \frac{t_j + \frac{v_{ji}^j}{\left(v_P^j\right)^2} r_P\left(t_j\right)}{1 + \frac{v_{ji}^j}{c_j^j}}, \tag{11.52}$$

$$t_j = \frac{\mu_j}{\mu_i} \frac{t_i - \frac{v_{ji}^i}{\left(v_P^i\right)^2} r_P\left(t_i\right)}{1 - \frac{v_{ji}^i}{c_i^i}}, \tag{11.53}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + \mathbf{v}_{ji}^j t_j}{1 + \frac{v_{ji}^j}{c_j^j}}, \tag{11.54}$$

$$\mathbf{r}_P\left(t_j\right) = \frac{\mathbf{r}_P\left(t_i\right) - \mathbf{v}_{ji}^i t_i}{1 - \frac{v_{ji}^i}{c_i^i}}. \tag{11.55}$$

They are partially both entirely and pairwise compatible.

The proof is in Appendix 22.6.

Note 396 *The transformations (11.42) through (11.45) represent the time-invariant generalized Lorentz transformations in the general case (see Conclusion 400).*

Comment 397 *If we accept* $q^{(\cdot)}w^{(\cdot)} = \left[c_{(\cdot)}^{(\cdot)}\right]^2$ *and set it in (11.8) and (11.9) then they become (11.46) and (11.47), respectively. Besides, the equations (11.10), (11.11) are the equations (11.48), (11.49). However, the transformations (10.16) and (10.17) do not become, respectively, (11.42) and (11.43) for* $q^{(\cdot)}w^{(\cdot)} = \left[c_{(\cdot)}^{(\cdot)}\right]^2$*. Theorem 395 does not follow from Theorem 367.*

Note 398 *The basic time scaling coefficients* μ_i *and* μ_j *can be accepted mutually independently in the general case*

Note 399 *The partial compatibility of the transformations (11.42) - (11.45) is a consequence of the determination of the time and space scaling coefficients exclusively for the light speed of the arbitrary point P.*

Conclusion 400 *The formulae (11.46) through (11.49) for the time and space scaling coefficients are crucially different from the corresponding formulae for the scaling coefficients in Einsteinian relativity theory. The former do not contain either square roots or squared quotients of speed values, while the latter do. The former are expressed in terms of the relative values of both the light speed and the spatial transfer speed, while the latter are not. Consequently, the transformations (11.42) through (11.45) are essentially different from Lorentz transformations (7.20) through (7.23). The former represent the final forms of the solutions in the general case for the time-invariant generalized basic Lorentz transformations (11.42) through (11.45).*

Conclusion 401 ***Noninvariance of the light speed***

Both the light speed and the spatial transfer speed are not invariant in view of (11.51).

Note 402 *Theorem 372, Remark 373, and the Note 374 hold also in this setting when we replace the equation numbers (11.1) through (11.6) by (11.41) through (11.45), and (11.2) through (11.5) by (11.42) through (11.45), respectively.*

Solution for the special case

By referring to Lorentz, Einstein and Poincaré we adopt a priory the same *time* scaling coefficients and the same space scaling coefficients.

Theorem 403 *Let the time scaling coefficient* $\mu_i \in R^+$ *be defined by (11.41). Let* $A = B$ *in D, (11.7). In order for the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i *and* λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, *which are determined for the case when the arbitrary point P moves with the speed of light,* $v_P^{(.)} \equiv c_{(.)}^{(.)}$, *to be positive real numbers and to obey (11.42) through (11.45), and for (11.41) through (11.45) to imply (11.7) it is necessary and sufficient that the relationships (11.56) through (11.59),*

$$c_i^i = c_j^j = c_{ij} = c_{ji}, \tag{11.56}$$

$$v_{ji}^j = v_{ji}^i = v_{ji} = -v_{ij}, \tag{11.57}$$

$$\alpha_{ij} = \lambda_{ij} = \frac{1}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}}, \tag{11.58}$$

$$\mu_j = \mu_i \sqrt{\frac{1-\frac{v_{ji}}{c_{ij}}}{1+\frac{v_{ji}}{c_{ij}}}} = \mu_i \sqrt{\frac{1+\frac{v_{ij}}{c_{ij}}}{1-\frac{v_{ij}}{c_{ij}}}},\ v_{ji} < c_{ij}, \tag{11.59}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. *The equations (11.42) through (11.45) become the equations (11.60) through (11.63):*

$$t_i = \frac{t_j + \frac{v_{ji}}{\left(v_P^j\right)^2} r_P\left(t_j\right)}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}}, \tag{11.60}$$

$$t_j = \frac{t_i - \frac{v_{ji}}{\left(v_P^i\right)^2} r_P\left(t_i\right)}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}}, \tag{11.61}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + \ \mathbf{v}_{ji} t_j}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}}, \tag{11.62}$$

$$\mathbf{r}_P\left(t_j\right) = \frac{\mathbf{r}_P\left(t_i\right) - \ \mathbf{v}_{ji} t_i}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}}. \tag{11.63}$$

The transformations (11.60) through (11.63) are partially both entirely and pairwise compatible.

The proof of Theorem 403 occupies Appendix 22.7.

Remark 404 *The transformations (11.60) through (11.63) are only **partially generalized Lorentz transformations** (see the Definition 315, the Note 316, Note 317 and Note 394). They represent the solutions in the special case to the time-invariant generalized basic Lorentz transformations (11.42) through (11.45).*

The transformations (11.60) through (11.63) generalize Lorentz transformations (7.20) through (7.23), but only partially. The temporal coordinate transformations (11.60), (11.61) contain the values $v_P^{(.)}$ of the speed of the arbitrary point P, while in Lorentz temporal coordinate transformations (7.20), (7.21) the invariant value of the light speed is accepted for the value of the speed of the arbitrary point P, $v_P^{(.)} \equiv c$. The spatial coordinate transformations (11.62) and (11.63) contain the relative value c_{ij} of the light speed instead of its invariant value c that characterizes Lorentz spatial coordinate transformations (7.22), (7.23), too.

*If we replace $v_P^{(.)}$ by $c_{(.)}^{(.)}$ in (11.60) and (11.61) then (11.60) through (11.63) become **the slightly generalized Lorentz transformations** that represent the solution in the special case to the slightly generalized basic Lorentz transformations (7.6) through (7.9) (Definition 176).*

Note 405 *Theorem 403 cannot be deduced from Theorem 375 because (11.60) and (11.61) do not follow from (11.24) and (11.25) for $q^{(.)}w^{(.)} \equiv c_{(.)}^{(.)}$ that should be used since the proof is worked out only for $q^{(.)}w^{(.)} \equiv v_P^{(.)} \equiv c_{(.)}^{(.)}$.*

Note 406 *Sufficiency of the conditions is proved only for the case when the arbitrary point P moves with the light speed. It does not hold for an arbitrary speed of the point P. This caused restrictive pairwise compatibility of the transformations (11.60) through (11.63).*

Note 407 *The Notes 377 and 378 are valid also in this case as soon as we replace the equation (11.23) by (11.59).*

Solution for the singular case

The preceding result provides the solution for the singular case, which is the fundamental of Einsteinian relativity theory. We should replace $c_{ij} = c_{ji}$ by c, and $\mathbf{v}_{ji}$ by $\mathbf{v}$, i.e. v_{ji} by v, everywhere in Theorem 403:

Corollary 408 *Let the numerical value of the light speed be invariant relative to integral spaces, $c_{ij} = c_{ji} = c$. Let also $v_{ji} = v$. Let $A = B$ in D, (11.7). Let the time scaling coefficient $\mu_i \in R^+$ be defined by (11.41). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji} = \alpha$, λ and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji} = \lambda$, determined for the case when the arbitrary point P moves with the speed of light, to be positive real numbers and to obey (11.42) through (11.45), and for (11.41) through (11.45) to imply (11.7) it is necessary and sufficient* *that the relationships (11.64) and (11.65),*

$$\alpha = \lambda = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}},\ v < c, \tag{11.64}$$

$$\mu_j = \mu_i \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}, \tag{11.65}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. *The equations (11.42) through (11.45) become the equations (11.66) through (11.69):*

$$t_i = \frac{t_j + \frac{v}{\left(v_P^j\right)^2} r_P\left(t_j\right)}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{11.66}$$

$$t_j = \frac{t_i - \frac{v}{\left(v_P^i\right)^2} r_P\left(t_i\right)}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{11.67}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + \ \mathbf{v} t_j}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{11.68}$$

$$\mathbf{r}_P\left(t_j\right) = \frac{\mathbf{r}_P\left(t_i\right) - \ \mathbf{v} t_i}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{11.69}$$

The transformations (11.66) through (11.69) are partially both entirely and pairwise compatible.

Remark 409 ***Lorentz transformations do not prove time dependence on space***

The transformations (11.66) through (11.69) become Lorentz transformations (7.20) through (7.23) when we replace the values v_P^i *and* v_P^j *by the light speed invariant value c in (11.66), (11.67) since Lorentz transformations (7.20) through (7.23) are proved exclusively for* $v_P^{(.)} \equiv c$ *(see Remark 404). They are obtained above by starting with the time independence of space (Axiom 47). This shows that there is not any collision between the time independence of space and Lorentz transformations. Lorentz transformations and from them obtained other results do not prove, and cannot prove, the time dependence of space.*

Note 410 *Corollary 408 shows for* $v_P^{(.)} \equiv c$ *that Lorentz transformations correspond to the case when the basic time scaling coefficients* μ_i *and* μ_j *are mutually dependent. Their dependence is linear, (11.65). The coefficient of the proportionality is equal to* $\sqrt{1-\frac{v}{c}}/\sqrt{1+\frac{v}{c}}$.

11.2.2 Transformations of velocity

Velocity transformations will be derived from the temporal and spatial coordinate transformations obtained by applying Lorentz - Einstein - Poincaré approach. This is contradictory because Einsteinian approach assumes the light speed of the arbitrary point. In order to follow their approach, we should ignore this fact.

General case

Theorem 411 *Let the time scaling coefficient* $\mu_i \in R^+$ *be defined by (11.41). Let* $A \neq B$ *in* D *(11.7) hold a priory. Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i \neq \alpha_i^j$, λ_j^i *and* λ_i^j, $\lambda_j^i \neq \lambda_i^j$, *obey (11.46) through (11.49), and let (11.42) through (11.45) hold. Then, the velocity* $\mathbf{v}_P^i$ *of the point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated by the equations (11.70):*

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i}\mathbf{v}_P^j,\ \mathbf{v}_P^j = \frac{\mu_i}{\mu_j}\mathbf{v}_P^i. \tag{11.70}$$

The transformations are completely compatible.

The proof is in Appendix 22.8.

Note 412 *The transformations (11.70) cannot be deduced from the velocity transformations (11.34) for* $q^{(\cdot)}w^{(\cdot)} \equiv v_P^{(\cdot)}$.

Note 413 *The equations (11.70) are beyond both Galilean - Newtonian kinematics and Einsteinian relativity theory.*

The equations (11.70) exist in the corresponding formulae of Galilean - Newtonian kinematics if, and only if, the spatial frames move with the same velocities, ($v_{ji}^{(\cdot)} \equiv 0$), *i.e. if, and only if, there are the transformations only of the temporal (but not of the spatial) coordinates.*

The equations (11.70) contain the basic time scaling coefficients μ_i *and* μ_j, *and relative values of all speeds, while Einsteinian formulae do not. The former are beyond Einsteinian relativity theory.*

Conclusion 414 ***Noninvariance of the light speed***

In the case the arbitrary point P *represents a light signal then the equations (11.70) become*

$$\mathbf{c}_i^i = \frac{\mu_j}{\mu_i}\mathbf{c}_j^j,\ \mathbf{c}_j^j = \frac{\mu_i}{\mu_j}\mathbf{c}_i^i.$$

This verifies (11.51) and confirms the former conclusions that the light speed is not invariant. It is not an exceptional speed.

Remark 415 *The equations (11.70) represent the first equation in (11.51). To verify the second equation (11.51) we accept at first the origin* O_i *for the arbitrary point* P*, in which case the transfer speed equals zero so that the equations (11.70) become* $\mathbf{v}_{O_i}^{O,i} = (\mu_j/\mu_i)\,\mathbf{v}_{O_i}^{O,j}$, $\mathbf{v}_{O_i}^{O,j} = (\mu_i/\mu_j)\,\mathbf{v}_{O_i}^{O,i}$. *If we adopt the origin* O_j *for the arbitrary point* P *then* $\mathbf{v}_{O_j}^{O,i} = (\mu_j/\mu_i)\,\mathbf{v}_{O_j}^{O,j}$, $\mathbf{v}_{O_j}^{O,j} = (\mu_i/\mu_j)\,\mathbf{v}_{O_j}^{O,i}$. *These equations and the definition of the spatial transfer velocity* $\mathbf{v}_{ji}^{(.)}$, $\mathbf{v}_{ji}^{(.)} \equiv \mathbf{v}_{O_j}^{O,(.)} - \mathbf{v}_{O_i}^{O,(.)}$, *furnish* $\mathbf{v}_{ji}^{i} = (\mu_j/\mu_i)\,\mathbf{v}_{ji}^{j}$, *which proves the second equation (11.51). The equations (11.70) confirm the equations (11.51).*

The following result verifies once more the noninvariance of the light velocity.

Theorem 416 *If the velocity transformations (11.70) and* $\mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$ *hold then for the light speed* $c_{(.)}^{(.)}$ *to be invariant it is necessary and sufficient that any of the following equations is true:*

$$\mu_j/\mu_i = 1, \text{ or equivalently } t_j = t_i, \text{ or equivalently } v_{ji}^{(.)} \equiv v_{ji}.$$

Proof. Let the equations (11.70) hold and let $\mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$. In order for the velocity of light $\mathbf{c}_{(.)}^{(.)}$ to be invariant, i.e. $\mathbf{c}_i^i \equiv \mathbf{c}_j^j$, it is necessary and sufficient that both

$$\mathbf{v}_P^i \equiv \mathbf{c}_i^i \equiv \frac{\mu_j}{\mu_i}\mathbf{c}_j^j \equiv \mathbf{c}_j^j \text{ and } \mathbf{v}_P^j \equiv \mathbf{c}_j^j \equiv \mathbf{c}_i^i \equiv \mathbf{c}_i^i.$$

For these identities to hold it is necessary and sufficient that $\mu_j/\mu_i = 1$, i.e. $t_j = t_i$ due to (11.41) and $\mu_j/\mu_i = 1$, or equivalently $v_{ji}^{(.)} \equiv v_{ji}$ due to (11.51) and $\mu_j/\mu_i = 1$. Q. E. D ■

The invariance of the light speed $c_{(.)}^{(.)}$ with respect to the integral spaces $I_i = \mathfrak{T}_i \text{x} R_i^n$ and $I_j = \mathfrak{T}_j \text{x} R_j^n$, which are related by the transformations (11.42) through (11.45), is possible if, and only if, there is not essentially any transformation because the transformations are reduced to the identity transformation, i.e. $I_i = \mathfrak{T}_i \text{x} R_i^n \equiv I_j = \mathfrak{T}_j \text{x} R_j^n$.

The equations (11.70) verify obviously the noninvariance of the spatial transfer speed $v_{ji}^{(.)}$, too.

Special case

Theorem 417 *Let the time scaling coefficient* μ_i *be defined by (11.41). Let* $A = B$ *in* D *(11.7). Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i *and* λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, *obey (11.58), and let (11.60) through (11.63) hold. Then, the velocity* $\mathbf{v}_P^i$ *of the point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated by the equations (11.70) that take the following forms:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}}{1 + \frac{v_{ji}}{v_P^j}} = \mathbf{v}_P^j,\; \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}}{1 - \frac{v_{ji}}{v_P^i}} = \mathbf{v}_P^i. \tag{11.71}$$

The transformations are completely compatible.

Proof. The proof procedure is the same as of the proof of Theorem 388. Q. E. D ■

Comment 418 *The numerical vector value of the light velocity is invariant relative to the integral spaces* $I_i = \mathfrak{T}_i \times R_i^n$ *and* $I_j = \mathfrak{T}_j \times R_j^n$ *in this case.*

Remark 419 ***Light speed invariance is a consequence of Lorentz restrictions. Light speed is not a special speed***

The proof of the velocity transformations (11.71) shows that the same scaling factor $1/\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}$ *for the temporal coordinate transformation and for the spatial coordinate transformation cancels in the quotient of the derivatives of these coordinates, which determines the velocity. After its cancellation, there is not the light speed value in the case of an arbitrary speed* $v_P^{(.)}$ *of* P*. This explains why the (numerical) value of every speed including the light speed rests unchanged, i.e. invariant, under the coordinate transformations (11.60) through (11.63). The light signal is not an exceptional speed. Every velocity is invariant due to (11.71).*

Singular case

In the singular case the preceding theorem takes the following form:

Corollary 420 *Let the numerical value of the light speed be invariant relative to integral spaces,* $c_{ij} = c_{ji} = c$*. Let also* $v_{ji} = v$*. Let* $A = B$ *in D (11.7). Let the time scaling coefficient* $\mu_i \in R^+$ *be defined by (11.41). Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji} = \alpha$, λ_j^i *and* λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji} = \lambda$*, obey (11.64), and let (11.66) through (11.69) be correct. Then, the velocity* $\mathbf{v}_P^i$ *of the point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$*, and the velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated by the equations (11.71) in the following particular form:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}}{1 + \frac{v}{v_P^j}} = \mathbf{v}_P^j, \ \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}}{1 - \frac{v}{v_P^i}} = \mathbf{v}_P^i. \tag{11.72}$$

The transformations are completely compatible.

Note 421 *Every velocity is invariant in view of (11.72).*

11.3 Weak nonuniformity

11.3.1 Transformations of temporal and spatial coordinates

Basic relationships

The *time* and space scaling coefficients are positive real numbers, $\alpha_j^i \in R^+$, $\alpha_i^j \in R^+$, $\lambda_j^i \in R^+$, $\lambda_i^j \in R^+$, and $\mu_i \in R^+$ (Subsection 10.2.2: "Weak nonuniformity"). We start with the simplest equation (10.3) repeated as (11.73),

$$\mu_i \in R^+ \Longrightarrow t_i = \mu_i t. \tag{11.73}$$

The constant positive speed values $q^{(.)}$ and $w^{(.)}$ of the generic velocities $\mathbf{q}^{(.)} = q^{(.)}\mathbf{u}$ and $\mathbf{w}^{(.)} = w^{(.)}\mathbf{u}$ relative to integral spaces $I_{(.)} = \mathfrak{T}_{(.)} \times R^n_{(.)}$ determine the values of the *time* and space scaling coefficients. Either $q^{(.)}$ or $w^{(.)}$, but not both, can be equal to the speeds $c^{(.)}_{(.)}$ and/or $v_P^{(.)}$ of the light velocity $\mathbf{c}^{(.)}_{(.)}$ and/or of a constant velocity $\mathbf{v}_P^{(.)}$ of the arbitrary point P in this setting. The extensions (10.4) through (10.7) of the generalized basic Lorentz transformations (7.6) through (7.9) are repeated as the equations (11.74) through (11.77):

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j) \right],\ \alpha_j^i \in R^+,\ v_{ji}^j \in R_+, \tag{11.74}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i) \right],\ \alpha_i^j \in R^+,\ v_{ji}^i \in R_+, \tag{11.75}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u} \right],\ \lambda_j^i \in R^+, \tag{11.76}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j [\mathbf{r}_P(t_i) - \ v_{ji}^i t_i \mathbf{u}],\ \lambda_i^j \in R^+, \tag{11.77}$$

where, [see (10.8)],

$$q^{(.)},\ w^{(.)} \in R^+,\ q^{(.)} w^{(.)} \notin \left\{ \ \left[c^{(.)}_{(.)} \right]^2, \left[v_P^{(.)} \right]^2 \right\}. \tag{11.78}$$

Solutions for the general case

The general case is of our primary interest. Its solution results from Theorem 367.

Theorem 422 *Let the constant time scaling coefficient μ_i be defined by (11.73). In order for the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, determined for the case when the arbitrary point P moves with the light speed, to be positive real numbers and to obey (11.74) through (11.78), and for (11.73) through (11.78) to imply (11.7) it is necessary and sufficient that the relationships (11.8) through (11.13) hold for any choice of the time scaling coefficients*

$\mu_i \in R^+$ and $\mu_j \in R^+$. The equations (11.74) through (11.77) completed by (11.8) through (11.11) become (11.14) through (11.17). They are partially both entirely and pairwise compatible.

Proof. The proof is the same as the proof of Theorem 367 because (11.74) through (11.78) differ from (11.2) through (11.6) only in the condition (11.78) imposed on $q^{(\cdot)}w^{(\cdot)}$, which does not influence the proof. Q. E. D ∎

Note 423 *Note 368, Note 370, and Conclusion 371 hold also in this setting.*

Note 424 *Theorem 372 modified by changing the equation numbers (11.1) - (11.6) to (11.73) through (11.78), respectively, holds herein. The same applies to Remarks 373 and 374.*

Solution for the special case

Theorem 425 *Let the time scaling coefficient $\mu_i \in R^+$ be defined by (11.73). Let $A = B$ in D, (11.7). Then, it is impossible for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i \equiv \alpha_i^j \equiv \alpha_{ij} \equiv \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i \equiv \lambda_i^j \equiv \lambda_{ij} \equiv \lambda_{ji}$, determined for the case when the arbitrary point P moves with the speed of light, to be positive real numbers and to obey (11.74) through (11.78), and for (11.73) through (11.78) to imply (11.7).*

Proof. The only difference between the transformations (11.74) through (11.78) and the transformations (11.2) through (11.6) is in the restriction (11.78) imposed on $q^{(\cdot)}w^{(\cdot)}$, which is not demanded in (11.6). Since the equations (11.74) through (11.77) are, respectively, the same as (11.2) through (11.5), then the proof of Theorem 375 and the theorem itself are valid. Hence, the equations (11.20) hold. They violate (11.78). This means that there are not positive real numbers μ_i, μ_j, α_i^j, α_j^i, λ_i^j and λ_j^i, which satisfy (11.73) through (11.78), and for which (11.73) through (11.78) imply (11.7). Q. E. D ∎

This is a very peculiar case. There is not a solution for the scaling coefficients in the special case in spite there is the solution for the general case. This is due to the constraint on $q^{(\cdot)}w^{(\cdot)}$ in (11.78) that $q^{(\cdot)}w^{(\cdot)} \neq \left(c_{(\cdot)}^{(\cdot)}\right)^2$.

Solution for the singular case

The preceding result implies that the singular case cannot hold due to the demand for $q^{(\cdot)}w^{(\cdot)} \neq \left(c_{(\cdot)}^{(\cdot)}\right)^2$. In Einsteinian relativity theory $q^{(\cdot)}w^{(\cdot)} \equiv c^2$.

Hence, the following is valid:

Conclusion 426 *There is not a solution of (11.74) through (11.78) for the scaling coefficients under a priory accepted constraints of Einsteinian relativity theory. The time invariant weakly nonuniform coordinate transformations cannot be established under such restrictions.*

11.3.2 Transformations of velocity

General case

We will use the temporal and spatial coordinate transformations (11.74) through (11.77) in order to determine the velocity transformations. Since they are the same as (11.2) through (11.5) then Theorem 381 is directly applicable herein.

Special case and singular case

Theorem 425 shows that the coordinate transformations do not hold in the special and in the singular case in the framework of the Lorentz - Einstein - Poincaré approach. Hence, we cannot establish velocity transformations in these cases.

11.4 Uniformity: general through special

11.4.1 Transformations of temporal and spatial coordinates

Basic relationships

All the scaling coefficients should be positive real numbers. The equation (11.79) determines the basic *time* coordinate transformation due to (10.66),

$$\mu_i \in R^+ \Longrightarrow t_i = \mu_i t. \tag{11.79}$$

In order to define completely the basic generic equations (10.67) through (10.70), we should determine the scaling coefficients in them. They are repeated as

$$t_i = \alpha_j^i \left[t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j) \right],\ \alpha_j^i \in R^+,\ \vartheta^j \in R_+, \tag{11.80}$$

$$t_j = \alpha_i^j \left[t_i - \frac{\vartheta^i}{q^i w^i} r_R(t_i) \right],\ \alpha_i^j \in R^+,\ \vartheta^i \in R_+, \tag{11.81}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u} \right],\ \lambda_j^i \in R^+,\ v_{ji}^j \in R_+, \tag{11.82}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j [\mathbf{r}_P(t_i) - \ v_{ji}^i t_i \mathbf{u}],\ \lambda_i^j \in R^+,\ v_{ji}^i \in R_+, \tag{11.83}$$

where

$$\begin{gathered} q^{(\cdot)},\ v_P^{(\cdot)},\ v_R^{(\cdot)},\ w^{(\cdot)} \neq \vartheta^{(\cdot)}\ \textit{in general};\ q^{(\cdot)}, w^{(\cdot)} \in R^+;\ v_{ji}^{(\cdot)}, \vartheta^{(\cdot)}, v_R^{(\cdot)} \in R_+, \\ q^{(\cdot)}, v_P^{(\cdot)}, w^{(\cdot)} = c_{(\cdot)}^{(\cdot)}\ \textit{are permitted, not required,} \\ v_P^{(\cdot)}, v_R^{(\cdot)} = \vartheta^{(\cdot)}\ \textit{are permitted iff}\ \vartheta^{(\cdot)} \in R^+,\ \textit{not required.} \end{gathered} \tag{11.84}$$

Note 427 ***On time-invariant uniform transformations***

If we replace $r_R(t_{(.)}) \equiv v_R t_{(.)}$ *by* $r_L(t_{(.)}) \equiv c_{(.)}^{(.)} t_{(.)}$ *in the temporal coordinate transformations (11.80), (11.81) and correspondingly in all the following results deduced from them in this section then the section becomes the section on the time-invariant uniform transformations based on Einsteinian approach.*

Note 428 ***On time-invariant specially uniform transformations***

If, additionally to the above replacements (Note 427), we replace $q^{(\cdot)}w^{(\cdot)}$ *by* $[c_{(.)}^{(.)}]^2$ *then the obtained section on the time-invariant uniform transformations becomes the section on the time-invariant specially uniform transformations based on Einsteinian approach.*

Solutions for the general case

Theorem 429 *Let the constant time scaling coefficient* μ_i *be defined by (11.79). In order for the constant scaling coefficients* α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j *and* λ_j^i, $\lambda_j^i \neq \lambda_i^j$, *determined for the case when the arbitrary point P moves with the light speed, to be positive real numbers and to obey (11.80) through (11.84), and for (11.79) through (11.84) to imply (11.7) it is necessary and sufficient that the following relationships hold for any choice of the time scaling coefficients* $\mu_i \in R^+$ *and* $\mu_j \in R^+$*:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}} = \frac{c_j^j}{c_i^i} \frac{1}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}, \tag{11.85}$$

$$\alpha_i^j = \frac{\mu_j}{\mu_i} \frac{1}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}} = \frac{c_i^i}{c_j^j} \frac{1}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}}, \tag{11.86}$$

$$\lambda_j^i = \frac{1}{1 + \frac{v_{ji}^j}{c_j^j}}, \tag{11.87}$$

$$\lambda_i^j = \frac{1}{1 - \frac{v_{ji}^i}{c_i^i}}, \tag{11.88}$$

$$0 \leq v_{ji}^i < c_i^i, \ 0 \leq \vartheta^i v_R^i < q^i w^i, \tag{11.89}$$

$$\frac{\mu_j}{\mu_i} = \frac{c_i^i}{c_j^j}, \ v_{ji}^{(.)} \in R^+ \Longrightarrow \frac{\mu_j}{\mu_i} = \frac{v_{ji}^i}{v_{ji}^j} \ . \tag{11.90}$$

The transformations (11.80) through (11.83) completed by (11.85) through (11.88) take the following forms:

$$t_i = \frac{\mu_i}{\mu_j} \frac{t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j)}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}, \tag{11.91}$$

$$t_j = \frac{\mu_j}{\mu_i} \frac{t_i - \frac{\vartheta^i}{q^i w^i} r_R(t_i)}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}}, \tag{11.92}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + \mathbf{v}_{ji}^j t_j}{1 + \frac{v_{ji}^j}{c_j^j}}, \tag{11.93}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - \mathbf{v}_{ji}^i t_i}{1 - \frac{v_{ji}^i}{c_i^i}}. \tag{11.94}$$

The temporal coordinate transformations (11.91) and (11.92) are completely compatible. The spatial coordinate transformations (11.93) and (11.94) are only partially compatible. The transformations (11.91) through (11.94) are also only partially entirely compatible.

We present the proof in Appendix 22.9.

Comment 430 *The full compatibility of the temporal coordinate transformations (11.91) and (11.92) results from their independence of any characteristic of the arbitrary point P.*

Note 431 *The above Theorem allows mutually independent choice of the basic time scaling coefficients μ_i and μ_j.*

Comment 432 *The time scaling coefficients α_i^j, (11.85), and α_j^i, (11.86), depend on the relative values $q^{(.)}$ and $w^{(.)}$ of the generic speeds, on the relative values of the general temporal transfer speed $\vartheta^{(.)}$ and on the relative values of the speed $v_R^{(.)}$ of the reference point P_R. They need not be expressed in terms of the relative values $c_{(.)}^{(.)}$ of the light speed. This is one of the differences between them and the time scaling coefficients α_i^j, (11.8), and α_j^i, (11.9). Such difference holds also between the temporal coordinate transformations (11.91), (11.92), and (11.14), (11.15).*

Comment 433 *The conditions of the above theorem are proved sufficient only for the light speed of the arbitrary point P, rather than for its arbitrary speed. This caused both the partial compatibility of the spatial coordinate transformations (11.93) and (11.94), and the partial entire compatibility of the transformations (11.91) through (11.94).*

Conclusion 434 *The transformations (11.91) through (11.94) are inherently different from Lorentz transformations (7.20) through (7.23). The above formulae do not contain either square roots or the squared quotients $v_{ji}^{(.)}/c_{(.)}^{(.)}$, which characterize the corresponding formulae of Einsteinian relativity theory.*

The temporal coordinate transformations (11.91) and (11.92) need not be expressed in the forms that contain the light speed values.

The equations (11.90) confirm the noninvariance of both the light speed and the spatial transfer speed.

The above formulae are expressed in terms of the relative, rather than invariant, values of all the speeds, including both the light speed and the spatial transfer speed. Their relative values are with respect to the used integral spaces. Einsteinian relativity theory rejects a priory such noninvariance of both the light speed and the spatial transfer speed.

The preceding result restricts the relative value of the temporal transfer speed $v_R^{(.)}$ *by* $q^{(.)}w^{(.)}/\vartheta^{(.)}$*, and of the spatial transfer speed* $v_{ji}^{(.)}$ *by the value of the relative light speed* $c_{(.)}^{(.)}$*. Einsteinian relativity theory does not recognize these conditions.*

The above theorem introduces the basic time scaling coefficient μ_i *into the temporal coordinate transformations (11.91) and (11.92). Lorentz transformations (7.20) and (7.21) of the temporal coordinates do not contain the coefficient* μ_i*. Consequently, Einsteinian relativity theory has not discovered its existence and importance.*

Theorem 435 *Let the coordinate systems* R_i^n *and* R_j^n *move with the same velocity:* $\mathbf{v}_{O_i}^O = \mathbf{v}_{O_j}^O$*. Let the time scaling coefficient* μ_i *be defined by (11.79). In order for the scaling coefficients* α_i^j*,* α_j^i*,* $\alpha_j^i \neq \alpha_i^j$*,* λ_i^j *and* λ_j^i*, determined for the case when the arbitrary point P moves with the speed of light, to be positive real numbers, to obey (11.80) through (11.84), and for (11.79) through (11.84) to imply (11.7) it is necessary and sufficient that the following equations hold for any choice of the time scaling coefficients* $\mu_i \in R^+$ *and* $\mu_j \in R^+$*:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j}\frac{1}{1+\frac{\vartheta^j v_R^j}{q^j w^j}},\ \alpha_i^j = \frac{\mu_j}{\mu_i}\frac{1}{1-\frac{\vartheta^i v_R^i}{q^i w^i}},\ \lambda_j^i = \lambda_i^j = 1,\ \frac{\mu_j}{\mu_i} = \frac{c_i^i}{c_j^j}. \tag{11.95}$$

The transformations (11.91) through (11.94), hence, (11.80) through (11.83), reduce to

$$t_i = \frac{\mu_i}{\mu_j}\frac{t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j)}{1+\frac{\vartheta^j v_R^j}{q^j w^j}},\ t_j = \frac{\mu_j}{\mu_i}\frac{t_i - \frac{\vartheta^i}{q^i w^i} r_R(t_i)}{1-\frac{\vartheta^i v_R^i}{q^i w^i}},\ \mathbf{r}_P(t_i) = \mathbf{r}_P(t_j). \tag{11.96}$$

They are completely both entirely and pairwise compatible.

For the proof see Appendix 22.10 "Proof of Theorem 435".

Note 436 *Let us make the following changes in Remark 373 and in Note 374: the reference to Theorem 372 is replaced by the reference to Theorem 435, and the equations numbers (11.14) through (11.17) are respectively replaced by the equations numbers (11.91) through (11.94). Then Remark 373 and Note 374 become valid herein.*

Solution for the special case

Theorem 437 *Let the time scaling coefficient $\mu_i \in R^+$ be defined by (11.79). Let $A = B$ in D, (11.7). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, determined for the light speed of the arbitrary point P, to be positive real numbers and to obey (11.80) through (11.84), and for (11.79) through (11.84) to imply (11.7) it is necessary and sufficient that the relationships (11.97) through (11.101),*

$$c_i^i = c_j^j = c_{ij} = c_{ji} \in R^+, \tag{11.97}$$

$$v_{ji}^j = v_{ji}^i = v_{ji} = -v_{ij} \in R_+, \tag{11.98}$$

$$\frac{\vartheta^i v_R^i}{q^i w^i} = \frac{\vartheta^j v_R^j}{q^j w^j} = \left(\frac{\vartheta v_R}{qw}\right)^{ij} = \left(\frac{\vartheta v_R}{qw}\right)^{ji} = \frac{v_{ji}}{c_{ij}}, \tag{11.99}$$

$$\begin{aligned} \alpha_{ij} &= \alpha_{ji} = \frac{1}{\sqrt{1 - \left[\left(\frac{\vartheta v_R}{qw}\right)^{ij}\right]^2}} = \frac{1}{\sqrt{1 - \left[\left(\frac{\vartheta v_R}{qw}\right)^{ji}\right]^2}} = \lambda_{ij} = \\ &= \lambda_{ji} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}},\ v_{ji} < c_{ij},\ (\vartheta v_R)^{(\cdot)} < (qw)^{(\cdot)}, \end{aligned} \tag{11.100}$$

$$\mu_j = \mu_i \sqrt{\frac{1 - \left(\frac{\vartheta v_R}{qw}\right)^{ij}}{1 + \left(\frac{\vartheta v_R}{qw}\right)^{ij}}} = \mu_i \sqrt{\frac{1 - \left(\frac{\vartheta v_R}{qw}\right)^{ji}}{1 + \left(\frac{\vartheta v_R}{qw}\right)^{ji}}} = \mu_i \sqrt{\frac{1 - \frac{v_{ji}}{c_{ij}}}{1 + \frac{v_{ji}}{c_{ij}}}}, \tag{11.101}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. The equations (11.80) through (11.83) become the equations (11.102) through (11.105):

$$t_i = \frac{t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j)}{\sqrt{1 - \left[\left(\frac{\vartheta v_R}{qw}\right)^{ij}\right]^2}}, \tag{11.102}$$

$$t_j = \frac{t_i - \frac{\vartheta^i}{q^i w^i} r_R(t_i)}{\sqrt{1 - \left[\left(\frac{\vartheta v_R}{qw}\right)^{ij}\right]^2}}, \tag{11.103}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + v_{ji} t_j \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}}, \tag{11.104}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - v_{ji} t_i \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}}. \tag{11.105}$$

The transformations (11.102) through (11.105) are partially both entirely and pairwise compatible.

Proof. Since the transformations (11.1) through (11.5) become (11.79) through (11.83) for $v_{ji}^{(.)} \equiv \vartheta^{(.)}$ in (11.2) and (11.3), then the replacement of $v_{ji}^{(.)}$ by $\vartheta^{(.)}$ in (11.2) and (11.3) transforms the proof of Theorem 375 into this proof of Theorem 437. Q. E. D ■

Note 438 *Note 377 and Note 378 apply to this case with the change of the equations numbers (11.23) through (11.27) into (11.101) through (11.105).*

Solution for the singular case

The preceding theorem takes the following form in the singular case.

Theorem 439 *Let the time scaling coefficient $\mu_i \in R^+$ be defined by (11.79). Let $B = A$ in D, (11.7). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji} = \alpha$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji} = \lambda$, determined for the light speed $c_{ij} = c_{ji} = c$ of the arbitrary point P and for $v_{ji}^j = v_{ji}^i = v_{ji} = v$, to be positive real numbers and to obey (11.80) through (11.84), and for (11.79) through (11.84) to imply (11.7) it is necessary and sufficient that the relationships (11.106) through (11.108),*

$$\frac{\vartheta^i v_R^i}{q^i w^i} = \frac{\vartheta^j v_R^j}{q^j w^j} = \left(\frac{\vartheta v_R}{qw}\right)^{ij} = \left(\frac{\vartheta v_R}{qw}\right)^{ji} = \frac{\vartheta v_R}{qw} = \frac{v}{c}, \tag{11.106}$$

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{1-\left(\frac{\vartheta v_R}{qw}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{\vartheta v_R}{qw}\right)^2}} = \\ &= \lambda = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}},\ v < c,\ \vartheta v_R < qw, \end{aligned} \tag{11.107}$$

$$\mu_j = \mu_i \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}, \tag{11.108}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. The equations (11.80) through (11.83) become the equations (11.109) through (11.112):

$$t_i = \frac{t_j + \frac{\vartheta}{qw} r_R(t_j)}{\sqrt{1-\left(\frac{\vartheta v_R}{qw}\right)^2}}, \tag{11.109}$$

$$t_j = \frac{t_i - \frac{\vartheta}{qw} r_R(t_i)}{\sqrt{1-\left(\frac{\vartheta v_R}{qw}\right)^2}}, \tag{11.110}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + \ vt_j\mathbf{u}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \tag{11.111}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - \ vt_i\mathbf{u}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}. \tag{11.112}$$

The transformations (11.109) through (11.112) are partially both entirely and pairwise compatible.

11.4.2 Transformations of velocity

General case

Theorem 440 *Let the time scaling coefficient μ_i be defined by (11.79). Let the scaling coefficients $\alpha_j^i \in R^+$, $\alpha_i^j \in R^+$, $\alpha_j^i \neq \alpha_i^j$, $\lambda_j^i \in R^+$ and $\lambda_i^j \in R^+$, $\lambda_j^i \neq \lambda_i^j$, be constant and obey (11.85) through (11.88), and let (11.91) through (11.94) hold. Then, the velocity $\mathbf{v}_P^i \neq \mathbf{0}$ of the arbitrary point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the velocity $\mathbf{v}_P^j \neq \mathbf{0}$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i}\frac{\mathbf{v}_P^j + \ \mathbf{v}_{ji}^j}{1 + \frac{v_{ji}^j}{c_j^j}}, \ \mathbf{v}_P^j = \frac{\mu_i}{\mu_j}\frac{\mathbf{v}_P^i - \ \mathbf{v}_{ji}^i}{1 - \frac{v_{ji}^i}{c_i^i}}. \tag{11.113}$$

For these transformations to be compatible it is necessary and sufficient that

$$v_P^i = c_i^i, \ v_P^j = c_j^j. \tag{11.114}$$

They are only partially compatible.

The methodology of the proof is the same as of the proof of Theorem 381.

Comment 441 *The conditions (11.114) are a natural consequence of the determination of the coordinate transformations (11.91) through (11.94) exclusively for the light speed of the arbitrary point P.*

Theorem 442 *If the velocity transformations (11.113), and $\mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$ are valid then for the light speed $c_{(.)}^{(.)}$ to be invariant it is necessary and sufficient that any of the following equations is true:*

$$\mu_j/\mu_i = 1, \ t_j = t_i, \ or \ v_{ji}^{(.)} = v_{ji}.$$

This theorem is proved in the same manner as Theorem 386.

Remark 443 ***Noninvariance of the light speed***

This verifies the noninvariance of the light speed $c_{(.)}^{(.)}$ relative to the integral spaces $I_i = \mathfrak{T}_i x R_i^n$ and $I_j = \mathfrak{T}_j x R_j^n$ when they are interrelated by the transformations (11.91) through (11.94). Einsteinian use of the light speed as invariant is incorrect in general. Its satisfaction restricts the validity of Einsteinian relativity theory.

Remark 444 ***Noninvariance of the transfer speed***

The equations (11.113) imply the noninvariance also of the spatial transfer speed. Its invariance restricts Einsteinian relativity theory.

Special case

Theorem 445 *Let the time scaling coefficient μ_i be defined by (11.79). Let $B = A$ in D, (11.7). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, be positive real numbers, be determined for the light speed of the arbitrary point P, and obey (11.80) through (11.84), and let (11.79) through (11.84) imply (11.7). Then, the velocity $\mathbf{v}_P^i \neq \mathbf{0}$ of the arbitrary point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the velocity $\mathbf{v}_P^j \neq \mathbf{0}$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1 + \frac{v_{ji}^j}{c_j^j}},\ c_j^j = c_i^i = c_{ij} = c_{ji},\ \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{1 - \frac{v_{ji}^i}{c_i^i}}. \tag{11.115}$$

For these transformations to be compatible it is necessary and sufficient that

$$v_P^i = v_P^j = v_P^{ij} = v_P^{ji} = c_{ij}. \tag{11.116}$$

They are only partially compatible.

The proof is analogous to the proof of Theorem 388.

Comment 446 *From the proof of the velocity transformations (11.115) we conclude that the same scaling factor $1/\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}$ for the temporal coordinate transformation and for the spatial coordinate transformation cancels in the quotient of the derivatives of these coordinates that determine the velocity. The fact that the scaling factors are calculated exclusively for the light speed of the arbitrary point P explains why only the numerical value of the light speed rests unchanged, i.e. why only it is invariant, under the coordinate transformations (11.102) through (11.105).*

The singular case

Corollary 447 *Let the time scaling coefficient μ_i be defined by (11.79). Let $B = A$ in D, (11.7). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i \equiv \alpha_i^j \equiv \alpha_{ij} \equiv \alpha_{ji} \equiv \alpha$, λ_j^i and λ_i^j, $\lambda_j^i \equiv \lambda_i^j \equiv \lambda_{ij} \equiv \lambda_{ji} \equiv \lambda$, be positive real numbers, be determined for the light speed $c_{ij} \equiv c$ of the arbitrary point P and for $v_{ji} \equiv v$. Let they obey (11.80) through (11.84), and let (11.79) through (11.84) imply (11.7). Then, the velocity $\mathbf{v}_P^i$ of the arbitrary point P with respect to the origin O_i of R_i^n and* *relative to* $\mathfrak{T}_i$*, and the velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \ \mathbf{v}}{1 + \frac{v}{c}},\ \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \ \mathbf{v}}{1 - \frac{v}{c}}. \tag{11.117}$$

For these transformations to be compatible it is necessary and sufficient that

$$v_P^i = v_P^j = v_P = c. \tag{11.118}$$

They are only partially compatible.

Remark 448 ***Invalidity of Einstein's law***

The velocity transformations (11.113), (11.115) and (11.117) do not contain the value $v_P^{(.)}$ *of the speed of the arbitrary point* P *in the denominators on their right-hand sides. They are essentially different from Einstein's law of the velocity composition. The former show the invalidity of the latter in the framework of integral spaces mutually related by the time-invariant general uniform transformations.*

11.5 Weak uniformity results

Note 449 ***On the time-invariant generally weakly uniform transformations***

If we accept $\vartheta^{(.)} \equiv v_R^{(.)}$ *then the transformations (10.67) through (10.70) become (10.50) through (10.53). With such replacements we transform Section 11.4 into the section on the time-invariant generally weakly uniform transformations based on Einsteinian approach.*

Note 450 ***On the time-invariant weakly uniform transformations***

If, additionally to the replacement in Note 449, we accept the light signal L *for the reference temporal point* P_R*, then we replace, additionally,* $r_R(t_{(.)}) \equiv v_R t_{(.)}$ *by* $r_L(t_{(.)}) \equiv c_{(.)}^{(.)} t_{(.)}$*. The result is that the obtained section on the time-invariant general weakly uniform transformations becomes the section on the time-invariant weakly uniform transformations based on Einsteinian approach.*

Note 451 ***On the time-invariant specially weakly uniform transformations***

If, further additionally to the replacements in Notes 449 and 450, we replace $q^{(.)}w^{(.)}$ *by* $[c_{(.)}^{(.)}]^2$ *then the deduced section on the time-invariant weakly uniform transformations becomes the section on the time-invariant specially weakly uniform transformations based on Einsteinian approach.*

11.6 Relative uniformity results

Note 452 ***On the time-invariant generally relatively uniform transformations***

If we accept $\vartheta^{(.)} \equiv v_{ji}^{(.)}$ *then the transformations (10.67) through (10.70) become (10.33) through (10.36). Such replacements transform Section 11.4 into the section on the time-invariant generally relatively uniform transformations based on Einsteinian approach.*

Note 453 ***On the time-invariant relatively uniform transformations***

If, additionally to the replacement in Note 452, we accept the light signal L *for the reference temporal point* P_R*, then we replace, additionally,* $r_R(t_{(.)}) \equiv v_R t_{(.)}$ *by* $r_L(t_{(.)}) \equiv c_{(.)}^{(.)} t_{(.)}$*. The result is that the obtained section on the time-invariant general relatively uniform transformations becomes the section on the time-invariant relatively uniform transformations based on Einsteinian approach.*

Note 454 ***On the time-invariant specially relatively uniform transformations***

If, further additionally to the above replacements in Notes 452 and 453, we replace $q^{(.)}w^{(.)}$ *by* $\left[c_{(.)}^{(.)}\right]^2$ *then the deduced section on the time-invariant relatively uniform transformations becomes the section on the time-invariant specially relatively uniform transformations based on Einsteinian approach.*

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Chapter 12

Any Speed of the Arbitrary Point

12.1 General spatial uniformity

12.1.1 Transformations of temporal and spatial coordinates

Basic relationships

The basic equations for the coordinate transformations are the same as the equations (11.1) through (11.5) for the general *time*-invariant nonuniform transformations,

$$t_i = \mu_i t,\ \mu_i = const. \in R^+. \tag{12.1}$$

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j}\ r_P(t_j) \right],\ \alpha_j^i \in R^+, \tag{12.2}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i}\ r_P(t_i) \right],\ \alpha_i^j \in R^+, \tag{12.3}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u} \right],\ \lambda_j^i \in R^+, \tag{12.4}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right],\ \lambda_i^j \in R^+, \tag{12.5}$$

where

$$q^{(\cdot)}, w^{(\cdot)} \in R^+,\ v_{ji}^{(\cdot)} \in R_+,$$

$$q^{(\cdot)} w^{(\cdot)} \in \left\{ \left[c_{(\cdot)}^{(\cdot)}\right]^2, \left[v_P^{(\cdot)}\right]^2, \left[v_{SU}^{(\cdot)}\right]^2 \right\}\ \textit{is permitted but not required.} \tag{12.6}$$

We will change only the spatial reference point. It will not be the light signal L, but another freely chosen and fixed point denoted by P_{SU}. Since it will ensure uniformity of space scaling coefficients over space, then it is denoted by the subscript SU (for: $Spatial\ \ Uniformity$). This spatial uniformity is general because any point with any constant speed can be chosen and fixed for the reference point P_{SU}. The choice of a light signal L for P_{SU}, hence $v_{SU}^{(.)} \equiv c_{(.)}^{(.)}$, represents a singular case. The selection of the reference point P_{SU} to be different from the light signal L, i.e. $v_{SU}^{(.)} \neq c_{(.)}^{(.)}$, makes what follows to be beyond Einsteinian relativity theory.

The temporal transformations (12.2) and (12.3) are nonuniform due to their dependence on the position $r_P\left(t_{(.)}\right)$ of the arbitrary point P.

All characteristics (position, speed, velocity) of the spatial reference point P_{SU} carry also the subscript SU. Therefore, the condition for the preservation of distance in integral spaces takes the following form:

$$\begin{gathered}\left[\begin{array}{cc}\mathbf{r}_P^T(t_i) & t_i\mathbf{v}_{SU}^{i^T}\end{array}\right] D \left[\begin{array}{c}\mathbf{r}_P(t_i) \\ t_i\mathbf{v}_{SU}^i\end{array}\right] \equiv \left[\begin{array}{cc}\mathbf{r}_P^T(t_j) & t_j\mathbf{v}_{SU}^{j^T}\end{array}\right] D \left[\begin{array}{c}\mathbf{r}_P(t_j) \\ t_j\mathbf{v}_{SU}^j\end{array}\right], \\ D = blockdiag\left\{A \quad -B\right\} \in R^{2n\mathrm{x}2n}, \\ A \in R^n \ and \ B \in R^n \ are \ positive \ definite \ and \ possibly \ different. \end{gathered} \tag{12.7}$$

This replaces (11.7).

In this setting, $q^{(.)}$ and $w^{(.)}$ can simultaneously belong to $\{c_{(.)}^{(.)}, v_P^{(.)}, v_{SU}^{(.)}\}$.

Solutions for the general case

Theorem 455 *Let the time scaling coefficient μ_i be defined by (12.1). In order for the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, determined for the case when the arbitrary point P moves with the nonzero constant speed $v_{SU}^{(.)}$ of the spatial reference point P_{SU}, to be positive real numbers and to obey (12.2) through (12.6), and for (12.1) through (12.6) to imply (12.7) it is necessary and sufficient that the following relationships hold for any choice of the time scaling coefficients $\mu_i \in R^+$ and $\mu_j \in R^+$:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j}\frac{1}{1+\frac{v_{ji}^j v_{SU}^j}{q^j w^j}} = \frac{v_{SU}^j}{v_{SU}^i}\frac{1}{1+\frac{v_{ji}^j v_{SU}^j}{q^j w^j}}, \tag{12.8}$$

$$\alpha_i^j = \frac{\mu_j}{\mu_i}\frac{1}{1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}} = \frac{v_{SU}^i}{v_{SU}^j}\frac{1}{1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}}, \tag{12.9}$$

$$\lambda_j^i = \frac{1}{1+\frac{v_{ji}^j}{v_{SU}^j}}, \tag{12.10}$$

$$\lambda_i^j = \frac{1}{1-\frac{v_{ji}^i}{v_{SU}^i}}, \tag{12.11}$$

$$\min\left\{\frac{q^i w^i}{v_{SU}^i}, v_{SU}^i\right\} > v_{ji}^i \geq 0, \tag{12.12}$$

$$\frac{\mu_j}{\mu_i} = \frac{v_{SU}^i}{v_{SU}^j},\ v_{ji}^{(.)} \in R^+ \Longrightarrow \frac{\mu_j}{\mu_i} = \frac{v_{ji}^i}{v_{ji}^j}. \tag{12.13}$$

The transformations (12.2) through (12.5) become:

$$t_i = \frac{\mu_i}{\mu_j} \frac{t_j + \frac{v_{ji}^j}{q^j w^j}\ r_P\left(t_j\right)}{1 + \frac{v_{ji}^j v_{SU}^j}{q^j w^j}}, \tag{12.14}$$

$$t_j = \frac{\mu_j}{\mu_i} \frac{t_i - \frac{v_{ji}^i}{q^i w^i}\ r_P\left(t_i\right)}{1 - \frac{v_{ji}^i v_{SU}^i}{q^i w^i}}, \tag{12.15}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + \mathbf{v}_{ji}^j t_j}{1 + \frac{v_{ji}^j}{v_{SU}^j}}, \tag{12.16}$$

$$\mathbf{r}_P\left(t_j\right) = \frac{\mathbf{r}_P\left(t_i\right) - \mathbf{v}_{ji}^i t_i}{1 - \frac{v_{ji}^i}{v_{SU}^i}}, \tag{12.17}$$

They are partially both entirely and pairwise compatible.

Appendix 22.11 "Proof of Theorem 455" presents the proof.

Comment 456 *The light speed does not intervene in the above formulae (12.8) through (12.17) in general. However, the transformations (12.14) through (12.17) transform to the general time-invariant nonuniform transformations (11.14) through (11.17) if we accept the light signal L to be the spatial reference point P_{SU}.*

Comment 457 *Theorem 455 shows that the relative values of the speed $v_{SU}^{(.)}$ of the spatial reference point P_{SU} and of the spatial transfer speed $v_{ji}^{(.)}$ intervene in the general case of the time invariant general temporally nonuniform and spatially uniform transformations (12.14) through (12.17). It determines the formulae for the transformations in the form in which there are not square roots at all, or squared ratios $v_{ji}^{(.)}/v_{SU}^{(.)}$. Besides, the formulae contain the basic time scaling coefficients μ_i and μ_j.*

Formulae (12.8) through (12.17) do not contain a value of the light speed $c_{(.)}^{(.)}$ in general. However, if we accept a light signal L for the spatial reference point P_{SU}, i.e. $L = P_{SU}$ and $v_{SU}^{(.)} \equiv c_{(.)}^{(.)}$, then the results (12.8) through (12.17) reduce to (11.8) through (11.17). Then Theorem 455 transforms into Theorem 367, and the general spatial uniformity degenerates into the spatial uniformity. The nonuniformity in the temporal domain rests unchanged.

Note 458 *The pure time scaling coefficients μ_i and μ_j can be mutually independently selected.*

Comment 459 *The transformations (12.14) through (12.17) are only partially (entirely and pairwise) compatible. This agrees with the fact that sufficiency of the values of the scaling coefficients $\alpha_{(.)}^{(.)}$ and $\lambda_{(.)}^{(.)}$ is proved only in the case the arbitrary point P moves with the speed of the spatial reference point P_{SU}.*

Solution for the special case

Theorem 460 *Let the time scaling coefficient $\mu_i \in R^+$ be defined by (12.1). Let $B = A$ in D, (12.7). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i \equiv \alpha_i^j \equiv \alpha_{ij} \equiv \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i \equiv \lambda_i^j \equiv \lambda_{ij} \equiv \lambda_{ji}$, determined for the case when the arbitrary point P moves with the nonzero constant speed of the spatial reference point P_{SU}, to be positive real numbers and to obey (12.2) through (12.6), and for (12.1) through (12.6) to imply (12.7) it is necessary and sufficient that the relationships (12.18) through (12.21) hold for any choice of the time scaling coefficient $\mu_j \in R^+$:*

$$v_{SU}^j \equiv v_{SU}^i \equiv v_{SU}^{ij} \equiv v_{SU}^{ji}, \ \sqrt{q^i w^i} \equiv \sqrt{q^j w^j} \equiv \sqrt{(qw)^{ji}} \equiv v_{SU}^{ij}, \tag{12.18}$$

$$v_{ji}^i \equiv v_{ji}^j \equiv v_{ji} = -v_{ij}, \tag{12.19}$$

$$\alpha_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}}{\sqrt{(qw)^{ji}}}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} = \lambda_{ij}, \ v_{ji} < v_{SU}^{ij}, \tag{12.20}$$

$$\mu_i = \mu_j \sqrt{\frac{1 + \frac{v_{ji}}{\sqrt{(qw)^{ji}}}}{1 - \frac{v_{ji}}{\sqrt{(qw)^{ji}}}}} = \mu_j \sqrt{\frac{1 - \frac{v_{ij}}{v_{SU}^{ij}}}{1 + \frac{v_{ij}}{v_{SU}^{ij}}}}. \tag{12.21}$$

The equations (12.18) through (12.20) transform the equations (12.2) through (12.5) into the equations (12.22) through (12.25):

$$t_i = \frac{t_j + \frac{v_{ji}}{(qw)^{ji}} \ r_P\left(t_j\right)}{\sqrt{1 - \left(\frac{v_{ji}}{\sqrt{(qw)^{ji}}}\right)^2}}, \tag{12.22}$$

$$t_j = \frac{t_i - \frac{v_{ji}}{(qw)^{ji}} \ r_P\left(t_i\right)}{\sqrt{1 - \left(\frac{v_{ji}}{\sqrt{(qw)^{ji}}}\right)^2}}, \tag{12.23}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + \mathbf{v}_{ji} t_j}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}}, \tag{12.24}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - \mathbf{v}_{ji}t_i}{\sqrt{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}}. \quad (12.25)$$

The transformations (12.22) through (12.25) are partially both entirely and pairwise compatible.

The proof is in Appendix 22.12 "Proof of Theorem 460".

Comment 461 *This theorem emphasizes that we should use the speed values* v_{ji} *and* v_{SU}^{ij} *relative to the time axes* $\mathfrak{T}_i$ *and* $\mathfrak{T}_j$*. This is a consequence of the requirement for all the time scaling coefficients to be mutually equal and for all space scaling coefficients to be also mutually equal.*

Note 462 *The pure time scaling coefficients* μ_i *and* μ_j *are mutually dependent, (12.21).*

Note 463 *If we accept* $P_{SU} = L$*, thus* $v_{SU}^{(.)} \equiv c_{(.)}^{(.)}$*,* $v_{SU}^{ij} \equiv c_{ij}$*, then Theorem 460 becomes Theorem 375, and the general uniformity reduces to the spatial uniformity, but without changing the nonuniformity of the temporal coordinate transformations.*

Note 464 *The equations (12.20) through (12.25) become undefined, hence, inapplicable, for* $v_{ji}^{(.)} \equiv v_{SU}^{(.)}$*. The same holds for Theorem 460.*

Solution for the singular case

Corollary 465 *Let the time scaling coefficient* $\mu_i \in R^+$ *be defined by (12.1). Let* $B = A$ *in D, (12.7). In order for the scaling coefficients* α_j^i*,* α_i^j*,* $\alpha_j^i \equiv \alpha_i^j \equiv \alpha_{ij} \equiv \alpha_{ji} \equiv \alpha$ *and* $\lambda_j^i \equiv \lambda_i^j \equiv \lambda_{ij} \equiv \lambda_{ji} \equiv \lambda$*, determined for the case when the arbitrary point P moves with the nonzero constant speed* $v_P^{ij} \equiv v_{SU}^{ij}$ *of the spatial reference point* P_{SU}*, to be positive real numbers and to obey (12.2) through (12.6) for* $v_{ji} \equiv v$*, and for (12.1) through (12.6) to imply (12.7), it is necessary and sufficient that the equations (12.18) for* $v_{SU}^{ij} \equiv v_{SU}$ *and* $q^i w^i \equiv q^j w^j \equiv qw$*, (12.19) for* $v_{ij} \equiv v$*, (12.26) and (12.27) hold for any choice of the time scaling coefficient* $\mu_j \in R^+$*,*

$$\alpha = \frac{1}{\sqrt{1-\left(\frac{v}{\sqrt{qw}}\right)^2}} = \lambda = \frac{1}{\sqrt{1-\left(\frac{v}{v_{SU}}\right)^2}},\ v < \sqrt{qw} = v_{SU}, \quad (12.26)$$

$$\mu_i = \mu_j \sqrt{\frac{1+\frac{v}{\sqrt{qw}}}{1-\frac{v}{\sqrt{qw}}}}. \quad (12.27)$$

The equations (12.18) for $v_{SU}^{ij} \equiv v_{SU}$*, (12.19) for* $v_{ij} \equiv v$*, (12.26) and (12.27) transform the equations (12.2) through (12.5) into the equations (12.28)* *through (12.31):*

$$t_i = \frac{t_j + \frac{v}{qw}\ r_P\left(t_j\right)}{\sqrt{1 - \left(\frac{v}{\sqrt{qw}}\right)^2}}, \tag{12.28}$$

$$t_j = \frac{t_i - \frac{v}{qw}\ r_P\left(t_i\right)}{\sqrt{1 - \left(\frac{v}{\sqrt{qw}}\right)^2}}, \tag{12.29}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + \mathbf{v}t_j}{\sqrt{1 - \left(\frac{v}{v_{SU}}\right)^2}}, \tag{12.30}$$

$$\mathbf{r}_P\left(t_j\right) = \frac{\mathbf{r}_P\left(t_i\right) - \mathbf{v}t_i}{\sqrt{1 - \left(\frac{v}{v_{SU}}\right)^2}}. \tag{12.31}$$

The transformations (12.28) through (12.31) are partially both entirely and pairwise compatible.

Remark 466 ***Lorentz transformations are a singular case***

If we accept the light signal L for the spatial reference point P_{SU}, then we should replace $v_{SU} = \sqrt{qw}$ by the light speed value c, i.e. $v_{SU} = \sqrt{qw} = c$, in (12.28) through (12.31). Then they become Lorentz transformations (7.20) through (7.20). In such a case, Corollary 465 transforms into Corollary 379.

Note 467 ***Axiom 47 leads to Lorentz transformations***

This shows once more that we can deduce Lorentz transformations from Axiom 47 as soon as we adopt all the restrictions a priory accepted by Lorentz, Einstein and Poincaré. However, differently than they, but consistently with Newton, hence by starting with the independent nature of time, (Axiom 47), and with any point accepted for the reference spatial point, we have reproved the original Lorentz transformations. This shows that they may not be used as an argument to try to prove the wrong claim that time depends on spatial frames, i.e. on space.

Note 468 *The equations (12.26) through (12.31) become undefined, hence, inapplicable, for $v_{ji}^{(.)} \equiv v_{SU}^{(.)}$. The same holds for Corollary 465.*

12.1.2 Transformations of velocity

The general case

Velocity transformations will be derived from the *time* and space coordinate transformations (12.14) through (12.17).

Theorem 469 *Let the time scaling coefficient* μ_i *be defined by (12.1). Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i \neq \alpha_i^j$, λ_j^i *and* λ_i^j, $\lambda_j^i \neq \lambda_i^j$, *obey (12.8) through (12.11), and let (12.14) through (12.17) be valid. Then, the velocity* $\mathbf{v}_P^i \neq \mathbf{0}$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the velocity* $\mathbf{v}_P^j \neq \mathbf{0}$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i} \frac{1 + \frac{v_{ji}^j v_{SU}^j}{q^j w^j}}{1 + \frac{v_{ji}^j}{v_{SU}^j}} \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1 + \frac{v_{ji}^j v_P^j}{q^j w^j}},\ \mathbf{v}_P^j = \frac{\mu_i}{\mu_j} \frac{1 - \frac{v_{ji}^i v_{SU}^i}{q^i w^i}}{1 - \frac{v_{ji}^i}{v_{SU}^i}} \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{1 - \frac{v_{ji}^i v_P^i}{q^i w^i}}. \tag{12.32}$$

In order for them to be compatible it is necessary and sufficient that

$$v_P^i \equiv v_{SU}^i,\ v_P^j \equiv v_{SU}^j, \tag{12.33}$$

i.e. the transformations are only partially compatible.

Proof. The proof of this theorem is the same as the proof of Theorem 381 after replacing $c_{(.)}^{(.)}$ by $v_{SU}^{(.)}$. Q. E. D ■

Theorem 470 *If the velocity transformations (12.32) and* $\mathbf{v}_P^{(.)} \equiv v_{SU}{}_{(.)}^{(.)}$ *are valid then for the speed* $v_{SU(.)}^{(.)}$ *to be invariant it is necessary and sufficient that any of the following equations holds:*

$$\mu_j/\mu_i = 1 \text{ or } t_j = t_i \text{ or } v_{ji}^{(.)} \equiv v_{ji}.$$

Proof. Let the equations (12.32) be valid and let $\mathbf{v}_P^{(.)} \equiv v_{SU}{}_{(.)}^{(.)}$. In order for the velocity $v_{SU}{}_{(.)}^{(.)}$ to be invariant, i.e. $\mathbf{v}_{SU}^i \equiv \mathbf{v}_{SU}^j$, it is necessary and sufficient that both

$$\mathbf{v}_P^i = \mathbf{v}_{SU}^i = \frac{\mu_j}{\mu_i} \frac{1 + \frac{v_{ji}^j v_{SU}^j}{q^j w^j}}{1 + \frac{v_{ji}^j}{v_{SU}^j}} \frac{1 + \frac{v_{ji}^j}{v_{SU}^j}}{1 + \frac{v_{ji}^j v_{SU}^j}{q^j w^j}} \mathbf{v}_{SU}^j \equiv \mathbf{v}_{SU}^j,$$

and

$$\mathbf{v}_P^j = \mathbf{v}_{SU}^j = \frac{\mu_i}{\mu_j} \frac{1 - \frac{v_{ji}^i v_{SU}^i}{q^i w^i}}{1 - \frac{v_{ji}^i}{v_{SU}^i}} \frac{1 - \frac{v_{ji}^i}{v_{SU}^i}}{1 - \frac{v_{ji}^i v_{SU}^i}{q^i w^i}} \mathbf{v}_{SU}^i \equiv \mathbf{v}_{SU}^i.$$

For these identities to be correct it is necessary and sufficient that $\mu_j/\mu_i = 1$. i.e. $t_j = t_i$, or equivalently $v_{ji}^{(.)} \equiv v_{ji}$, due to (12.1) through (12.3), and (12.13). Q. E. D ■

Remark 471 ***Noninvariance of the speed of the spatial reference point***

If the integral spaces $I_i = \mathfrak{T}xR_i^n$ *and* $I_j = \mathfrak{T}xR_j^n$ *are related by the transformations (12.2) through (12.5), then the speed* $v_{SU}^{(.)}$ *of the spatial reference point*

P_{SU} is invariant with respect to them if, and only if, the temporal coordinate transformations become the identity transformation. There is not essentially any temporal coordinate transformation. This illustrates that the speed $v_{SU}^{(.)}$ is noninvariant in general.

Remark 472 ***Noninvariance of the light speed***

The equations (12.32) show for $\mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$ that the light speed is also noninvariant in general.

Special case

Theorem 473 *Let $A = B$ in D (12.7). Let the time scaling coefficient μ_i be defined by (12.1). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i \equiv \alpha_i^j \equiv \alpha_{ij} \equiv \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i \equiv \lambda_i^j \equiv \lambda_{ij} \equiv \lambda_{ji}$, be positive real numbers and obey (12.2) through (12.6), and let (12.1) through (12.6) imply (12.7). Then, the velocity $\mathbf{v}_P^i$ of the arbitrary point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the velocity $\mathbf{v}_P^j$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}}{1 + \frac{v_{ji} v_P^j}{(v_{SU}^{ij})^2}},\ \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \ \mathbf{v}_{ji}}{1 - \frac{v_{ji} v_P^i}{(v_{SU}^{ij})^2}}. \tag{12.34}$$

They are partially compatible.

The velocity $\mathbf{v}_{SU}^{(.)}$ of the spatial reference point P_{SU} is invariant,

$$\mathbf{v}_{SU}^i \equiv \mathbf{v}_{SU}^j \equiv \mathbf{v}_{SU}^{ij} \equiv \mathbf{v}_{SU}^{ji}, \tag{12.35}$$

relative to the integral spaces I_i and I_j.

Appendix 22.13 "Proof of Theorem 473" contains the proof.

The preceding theorem verifies Corollary 275.

Singular case

The preceding theorem takes the following particular form in the singular case:

Corollary 474 *Let $A = B$ in D, (12.7). Let the numerical value of the speed of the spatial reference point P_{SU} be invariant relative to integral spaces, $v_{SU}^{ij} \equiv v_{SU}^{ji} \equiv \ v_{SU}$. Let also $v_{ji} \equiv v$. Let the time scaling coefficient $\mu_i \in R^+$ be defined by (12.1). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i \equiv \alpha_i^j \equiv \alpha_{ij} \equiv \alpha_{ji} \equiv \alpha$, λ and λ_i^j, $\lambda_j^i \equiv \ \lambda_i^j \equiv \lambda_{ij} \equiv \lambda_{ji} \equiv \lambda$, be positive real numbers and obey (12.2) through (12.6), and let (12.1) through (12.6) imply (12.7). Then, the velocity $\mathbf{v}_P^i \neq \mathbf{0}$ of the point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the* *velocity* $\mathbf{v}_P^j \neq \mathbf{0}$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated by the equations (12.36):*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}}{1 + \frac{v v_P^j}{v_{SU}^2}}, \ \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}}{1 - \frac{v v_P^i}{v_{SU}^2}}. \tag{12.36}$$

The transformations are partially compatible.

The velocity $\mathbf{v}_{SU}^{(.)}$ *of the spatial reference point* P_{SU} *is invariant,*

$$\mathbf{v}_{SU}^i \equiv \mathbf{v}_{SU}^j \equiv v_{SU}^{ij} \equiv \mathbf{v}_{SU}. \tag{12.37}$$

Theorem 475 ***For every nonzero constant speed there are transformations linking integral spaces relative to which the speed is invariant.***

Proof. Let $w^{(\cdot)}$ be an arbitrary nonzero constant speed (with the value $w^{(\cdot)}$ measured relative to the integral space $I_{(.)}$). Let it be accepted for the speed $v_{SU}^{(.)}$ of the spatial reference point P_{SU}, $v_{SU}^{(.)} \equiv w^{(\cdot)}$. Then, the speed $w^{(\cdot)}$ is invariant, which follows from the equations (12.34), also from (12.35) (Theorem 473), and from the equations 12.37 (Corollary 474). These equations follow from the transformations (12.1) through (12.5) in the special and the singular case resolved for the spatial reference point P_{SU} moving with the speed $v_{SU}^{(.)} \equiv w^{(\cdot)}$. Q. E. D ■

Note 476 *For the important consequences of Theorem 473, its Corollary 474, and Theorem 475 on mass and energy see the books [226], [231].*

Remark 477 ***Any spatial reference velocity is invariant***

Theorem 473 and its Corollary 474 point out that any velocity accepted for spatial reference velocity is invariant.

The spatial reference velocity in Lorentz transformations, hence in Einsteinian relativity theory, is the light velocity $\mathbf{c}_{(.)}^{(.)}$*. It is invariant because it is the spatial reference velocity and because the formula guarantees the invariance of the spatial reference velocity whatever is its choice, but not because it expresses a property of light.*

Remark 478 ***Einstein's law of the composition of velocities is singular case***

Einstein's law of the composition of velocities represents just a singular case. To verify this we should accept that the light signal represents the spatial reference point P_{SU}*, i.e. we should replace* v_{SU}^2 *by* c^2 *in (12.34), or in (12.36).*

Remark 479 ***Light speed is noninvariant***

The equations (12.34) and (12.36) confirm the noninvariance of the light speed.

Remark 480 ***Lorentz-Einstein invariance of light speed is a feature of the formulae***

The equations (12.34) and (12.36) show that the invariance of velocity is not a property of the velocity, but the feature of the formulae.

Corollary 474 and Remarks 477 through 480 confirm Corollary 275.

12.2 General complete uniformity

12.2.1 Temporal and spatial coordinate transformations

Basic relationships

The temporal reference point P_R and the spatial reference point P_{SU} will be in general mutually different and different from the light signal L. The reference velocities $\mathbf{v}_R^{(.)}$ and $\mathbf{v}_{SU}^{(.)}$ will be different from the light velocity $\mathbf{c}_{(.)}^{(.)}$ and will be mutually different in general.

The basic coordinate transformations are (11.79), (11.80) through (11.84) (the Section 11.4). We wish to ensure their complete uniformity over the space. The obtained results (11.85) and (11.86) for the *time* scaling coefficients α_j^i and α_i^j, as well as (11.91) and (11.92) for the temporal coordinate transformations, guarantee their general uniformity over the space. In order to get also the general uniform spatial coordinate transformations over the space, we should determine the space scaling coefficients for an accepted and fixed reference point P_{SU}. This means that we should replace the values of the speed $c_{(.)}^{(.)}$ of the light signal L in the formulae (11.87), (11.88), (11.93) and (11.94) obtained for the general *time*-invariant uniform coordinate transformations by the values of the speed $v_{SU}^{(.)}$ of the spatial reference point P_{SU}.

General case

Theorem 481 *Let the time scaling coefficient μ_i be defined by (11.79). In order for the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, to be positive real numbers, to obey (11.80) through (11.84), and for (11.79) through (11.84) to imply (12.7), all determined for the fixed spatial reference point P_{SU} moving with a nonzero constant speed, it is necessary and sufficient that the following relationships hold for any choice of the time scaling coefficients $\mu_i \in R^+$ and $\mu_j \in R^+$:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}, \tag{12.38}$$

$$\alpha_i^j = \frac{\mu_j}{\mu_i} \frac{1}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}}, \tag{12.39}$$

$$\lambda_j^i = \frac{1}{1 + \frac{v_{ji}^j}{v_{SU}^j}}, \tag{12.40}$$

$$\lambda_i^j = \frac{1}{1 - \frac{v_{ji}^i}{v_{SU}^i}}, \tag{12.41}$$

$$0 \leq v_{ji}^{(.)} < v_{SU}^{(.)},\ q^{(.)} w^{(.)} > \vartheta^{(.)} v_R^{(.)}, \tag{12.42}$$

$$v_{ji}^{(.)},\ v_R^{(.)},\ v_{SU}^{(.)} \in R^+ \Longrightarrow\ \frac{\mu_j}{\mu_i} = \frac{v_{ji}^i}{v_{ji}^j} = \frac{v_R^i}{v_R^j} = \frac{v_{SU}^i}{v_{SU}^j} = \frac{c_i^i}{c_j^j}. \tag{12.43}$$

The transformations (11.80) through (11.83) specified by (12.38) through (12.41) become (12.44) through (12.47),

$$t_i = \frac{\mu_i}{\mu_j} \frac{t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j)}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}, \tag{12.44}$$

$$t_j = \frac{\mu_j}{\mu_i} \frac{t_i - \frac{\vartheta^i}{q^i w^i} r_R(t_i)}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}}, \tag{12.45}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}}{1 + \frac{v_{ji}^j}{v_{SU}^j}}, \tag{12.46}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}}{1 - \frac{v_{ji}^i}{v_{SU}^i}}. \tag{12.47}$$

The transformations (12.44) and (12.45) are completely compatible, but (12.46) and (12.47) are partially compatible. The equations (12.44) through (12.47) are partially entirely compatible.

The proof can be found in Appendix 22.14 "Proof of Theorem 481".

Comment 482 *The formulae (12.38) through (12.41) for the scaling coefficients α_j^i, α_i^j, λ_j^i and λ_i^j, which determine them all mutually different, and the coordinate transformations (12.44) through (12.47), do not contain either the values of the light speed $c_{(.)}^{(.)}$ or square roots or squared quotients of speed values. Besides, the formulae (12.38) and (12.39) for the time scaling coefficients α_j^i and α_i^j, as well as the temporal coordinate transformations (12.44) and (12.45), contain the basic time scaling factors μ_i and μ_j. These their features make them essentially different from the formulae for Lorentz scaling factors α and $\lambda = \alpha$, and from Lorentz transformations (7.20) and (7.21).*

Note 483 *The time scaling factors μ_i and μ_j are mutually independent.*

Comment 484 *The inequalities (12.42) show that the value of the light speed does not restrict the value of any speed in this framework.*

The value of the speed $v_{SU}^{(.)}$ of the spatial reference point P_{SU} restricts the value $v_{ji}^{(.)}$ of the spatial transfer speed, (12.42).

The value of the product $q^{(\cdot)}w^{(\cdot)}$ of the generic speeds $q^{(\cdot)}$ and $w^{(\cdot)}$ restricts the product $\vartheta^{(\cdot)}v_R^{(\cdot)}$ of the values of the general temporal transfer speed $\vartheta^{(\cdot)}$ and of the speed $v_R^{(\cdot)}$ of the temporal reference point P_R.

Remark 485 ***The light speed is not invariant***

The equations (12.43) show that the light speed is not any exceptional speed, but that it undergoes the general rule on the relationship among the relative values of any speed. They verify once more the noninvariance of the light speed.

Special case

Theorem 486 *Let the time scaling coefficient μ_i be defined by (11.79). Let $A = B$ be permitted in D, (12.7). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, to be positive real numbers and to obey (11.80) through (11.84), and for (11.79) through (11.84) to imply (12.7), all for the fixed spatial reference point P_{SU} moving with a nonzero constant speed, it is necessary and sufficient that the relationships (12.48) through (12.52),*

$$v_{ji}^i = v_{ji}^j = v_{ji} = -v_{ij},\ v_{[.]}^i = v_{[.]}^j = v_{[.]ij} = v_{[.]ji},\ [.] \in \{R, SU\},$$

$$q^i w^i = q^j w^j = (qw)^{ij} = (qw)^{ji} = v_{SU}^{ji} v_R^{ij},\ c_i^i = c_j^j = c_{ij} = c_{ji},$$

$$v_{ji}^{(\cdot)} = \vartheta^{(\cdot)} = \vartheta^{ji}. \tag{12.48}$$

$$\alpha_{ij} = \frac{1}{\sqrt{1 - \left(\frac{\vartheta^i v_R^i}{q^i w^i}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{\vartheta^j v_R^j}{q^j w^j}\right)^2}}, \tag{12.49}$$

$$\lambda_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}^i}{v_{SU}^i}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}^j}{v_{SU}^j}\right)^2}} = \alpha_{ij}, \tag{12.50}$$

$$v_{ji} < v_{SU}^{ji},\ \vartheta^{ji} v_R^{ji} < (qw)^{ji}, \tag{12.51}$$

$$\mu_j = \mu_i \sqrt{\frac{1 - \frac{\vartheta^i v_R^i}{q^i w^i}}{1 + \frac{\vartheta^i v_R^i}{q^i w^i}}} = \mu_i \sqrt{\frac{1 - \frac{\vartheta^j v_R^j}{q^j w^j}}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}}, \tag{12.52}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. The equations (11.80) through (11.83) become the equations (12.53) through (12.56),

$$t_i = \frac{t_j + \frac{\vartheta^{ji}}{(qw)^{ji}} r_R(t_j)}{\sqrt{1 - \left[\left(\frac{\vartheta v_R}{qw}\right)^{ji}\right]^2}}, \tag{12.53}$$

$$t_j = \frac{t_i - \frac{\vartheta^{ji}}{(qw)^{ji}} r_R(t_i)}{\sqrt{1 - \left[\left(\frac{\vartheta v_R}{qw}\right)^{ji}\right]^2}}, \tag{12.54}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + v_{ji} t_j \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ji}}\right)^2}}, \tag{12.55}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - v_{ji} t_i \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ji}}\right)^2}}. \tag{12.56}$$

They are only partially both entirely and pairwise compatible.

For the proof see Appendix 22.15 "Proof of Theorem 486".

Remark 487 ***The light speed is not an exceptional speed***
The relative values of every speed with respect to the integral spaces I_i and I_j are equal. The light speed is not any exception. It obeys the rule. The same is true for the spatial transfer speed.

Note 488 *The value of the light speed does not restrict the value of any speed in (12.53) through (12.56).*

Note 489 *The basic time scaling coefficients μ_i and μ_j are mutually dependent, (12.52).*

Comment 490 *The formulae (12.49), (12.50), and (12.53) through (12.56) contain the square roots and squared quotients of speed values. From the structural point of view they have the same structure as the corresponding Lorentz formulae for the scaling coefficients α and λ, and for Lorentz coordinate transformations (7.20) through (7.23). However, the former do not contain values of the light speed, while the latter do.*

The values of all the speeds are relative to the integral spaces in (12.53) through (12.56), while the values of the light speed and of the spatial transfer speed are invariant in Lorentz transformations (7.20) through (7.23).

Singular case

Corollary 491 *Let the time scaling coefficient μ_i be defined by (11.79). Let $A = B$ be a priory accepted in D, (12.7). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji} = \alpha$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji} = \lambda$, to be positive real numbers and to obey (11.80) through (11.84), and for (11.79) through (11.84) to imply (12.7), all determined for the fixed spatial reference point P_{SU} moving with a nonzero constant speed, it is necessary and sufficient that the relationships (12.57) through (12.60),*

$$v_{[.]}^i = v_{[.]}^j = v_{[.]}^{ij} = v_{[.]},\ [.] \in \{R, SU\},\ q^i w^i = q^j w^j = qw = v_{SU} v_R,$$

$$c_j^j = c_i^i = c_{ij} = c_{ji} = c, \tag{12.57}$$

$$v_{ji}^i = \vartheta^i = v_{ji}^j = \vartheta^j = v_{ji} = -v_{ij} = v = \vartheta < v_{SU}, \tag{12.58}$$

$$\alpha = \lambda = \frac{1}{\sqrt{1-\left(\frac{\vartheta v_R}{qw}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{v}{v_{SU}}\right)^2}}, \tag{12.59}$$

$$\mu_j = \mu_i \sqrt{\frac{1-\frac{v}{v_{SU}}}{1+\frac{v}{v_{SU}}}}, \tag{12.60}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. *The equations (11.80) through (11.83) become the equations (12.61) through (12.64):*

$$t_i = \frac{t_j + \frac{\vartheta}{qw} r_R(t_j)}{\sqrt{1-\left(\frac{\vartheta v_R}{qw}\right)^2}}, \tag{12.61}$$

$$t_j = \frac{t_i - \frac{\vartheta}{qw} r_R(t_i)}{\sqrt{1-\left(\frac{\vartheta v_R}{qw}\right)^2}}, \tag{12.62}$$

$$\mathbf{r}_R(t_i) = \frac{\mathbf{r}_R(t_j) + v t_j \mathbf{u}}{\sqrt{1-\left(\frac{v}{v_{SU}}\right)^2}}, \tag{12.63}$$

$$\mathbf{r}_R(t_j) = \frac{\mathbf{r}_R(t_i) - v t_i \mathbf{u}}{\sqrt{1-\left(\frac{v}{v_{SU}}\right)^2}}. \tag{12.64}$$

They are partially both entirely and pairwise compatible transformations.

Note 492 *Notes 488 and 489, and Comment 490 are applicable to the singular case, too.*

12.2.2 Velocity transformations

General case

Theorem 493 *Let the time scaling coefficient* μ_i *be defined by (11.79). Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i \neq \alpha_i^j$, λ_j^i *and* λ_i^j, $\lambda_j^i \neq \lambda_i^j$, *be positive real numbers and obey (11.80) through (11.84), be determined for the fixed spatial reference point* P_{SU} *moving with a constant nonzero speed, and let (11.79) through (11.84) imply (12.7). Then, a constant nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the corresponding constant* *nonzero velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i}\frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j}{v_{SU}^j}},\ \mathbf{v}_P^j = \frac{\mu_i}{\mu_j}\frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{1-\frac{v_{ji}^i}{v_{SU}^i}}. \tag{12.65}$$

These transformations are partially compatible.

Proof. *Necessity and sufficiency.* Let all the conditions of the theorem statement be satisfied. The equations (12.38) through (12.41) and the transformations (12.44) through (12.47) hold (Theorem 481). The equations (12.44), (12.46), $\mathbf{v}_P^{(.)} = v_P^{(.)}\mathbf{u}$, and $\mathbf{v}_{ji}^{(.)} = v_{ji}^{(.)}\mathbf{u}$, yield the following:

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P^i}{dt_i} = \frac{d\left\{\frac{\mathbf{r}_P(t_j)+\ v_{ji}^j t_j \mathbf{u}}{1+\frac{v_{ji}^j}{v_{SU}^j}}\right\} : dt_j}{d\left\{\frac{\mu_i}{\mu_j}\frac{t_j+\frac{\vartheta^j}{q^j w^j}r_R(t_j)}{1+\frac{\vartheta^j v_R^j}{q^j w^j}}\right\} : dt_j} = \frac{\mu_j}{\mu_i}\frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j}{v_{SU}^j}}.$$

This result proves the first equation in (12.65). The second equation (12.65) is analogously proved by starting with (12.45) and (12.47).

Compatibility. The compatibility between the equations in (12.65) is evidently valid only for $v_P^{(.)} \equiv v_{SU}^{(.)}$. It is partial because they are compatible only for the value of the speed of the arbitrary point P equal to the value of the speed of the spatial reference point P_{SU}. Q. E. D ■

Remark 494 ***The light speed is not either invariant or limiting***

The equations (12.65) show that the light speed is not invariant in vacuum with respect to all integral spaces that incorporate inertial spatial frames. Besides, they show that there is not any constraint on the velocity of the arbitrary point P*. However, the spatial transfer speed value* $v_{ji}^{(.)}$ *is limited by the value* $v_{SU}^{(.)}$ *of the speed* $v_{SU}^{(.)}$ *of the spatial reference point* P_{SU}*, which agrees with (12.42). The light speed value is not the limiting one. The value of the speed of the arbitrary point* P *can be greater than the light speed value.*

This verifies once more the noninvariance of the light speed and that its limiting role represents a singular case.

Special case

Theorem 495 *Let the time scaling coefficient* μ_i *be defined by (11.79). Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i *and* λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, *be positive real numbers, be determined for the fixed spatial reference point* P_{SU} *moving with a nonzero constant speed, and obey (11.80) through (11.84), and let (11.79) through (11.84) imply (12.7) for* $A = B$*. Then, a constant nonzero* *velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$*, and the corresponding constant nonzero velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}}{1 + \frac{v_{ji}}{v_{SU}^{ji}}},\ \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}}{1 - \frac{v_{ji}}{v_{SU}^{ji}}}. \tag{12.66}$$

These transformations are only partially compatible.

The velocity $\mathbf{v}_{SU}^{(.)}$ *of the spatial reference point* P_{SU} *is invariant,*

$$\mathbf{v}_{SU}^i = \mathbf{v}_{SU}^j = \mathbf{v}_{SU}^{ij} = \mathbf{v}_{SU}^{ji}, \tag{12.67}$$

relative to the integral spaces I_i *and* I_j*.*

Proof. *Necessity and sufficiency.* Under the conditions of the theorem statement, the equations (12.48) through (12.56) hold. They lead to

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P(t_i)}{dt_i} = \frac{d\left[\frac{\mathbf{r}_P(t_j)+v_{ji}^j t_j \mathbf{u}}{\sqrt{1-\left(\frac{v_{ji}^j}{v_{SU}^j}\right)^2}}\right] : dt_j}{d\left[\frac{t_j+\frac{\vartheta^j}{q^j w^j} r_R(t_j)}{\sqrt{1-\left(\frac{\vartheta^j v_R^j}{q^j w^j}\right)^2}}\right] : dt_j} = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1 + \frac{v_{ji}^j}{v_{SU}^j}} = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}}{1 + \frac{v\vartheta^{ji}}{v_{SU}^{ji}}},$$

$$\mathbf{v}_P^j = \frac{d\mathbf{r}_P(t_j)}{dt_j} = \frac{d\left[\frac{\mathbf{r}_P(t_i)-v_{ji}^i t_i \quad \mathbf{u}}{\sqrt{1-\left(\frac{v_{ji}^i}{v_{SU}^i}\right)^2}}\right] : dt_i}{d\left[\frac{t_i-\frac{\vartheta^i}{q^i w^i} \quad r_R(t_i)}{\sqrt{1-\left(\frac{\vartheta^i v_R^i}{q^i w^i}\right)^2}}\right] : dt_i} = \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{1 - \frac{v_{ji}^i}{v_{SU}^i}} = \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}}{1 - \frac{v_{ji}}{v_{SU}^{ji}}}.$$

These results prove (12.66).

Compatibility. The speed transformations are deduced by applying the constraint on $v_R^{(.)}$ in (12.48). Therefore, their compatibility, which is easy to check, is only partial.

Invariance. By setting $\mathbf{v}_P^{(.)} = \mathbf{v}_{SU}^{(.)}$ in (12.66) we find that $\mathbf{v}_{SU}^i = \mathbf{v}_{SU}^j$. Q. E. D ■

Remark 496 ***Invariance of the velocity is the feature of the formula***

This result proves that the freely chosen velocity $\mathbf{v}_{SU}^{(.)}$ *of the spatial reference point* P_{SU} *is the only invariant velocity relative to the integral spaces* I_i *and* I_j*, and not the light velocity in general.*

If we accept the light velocity $\mathbf{c}_{(.)}^{(.)}$ *for* $\mathbf{v}_{SU}^{(.)}$ *then the light velocity becomes invariant. This illustrates once more that Lorentz - Einstein invariance of the light velocity is not the property of light, but it is the feature of the formula.*

Theorem 495 and Remark 496 confirm Corollary 275.

Singular case

Corollary 497 *Let the time scaling coefficient μ_i be defined by (11.79). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji} = \alpha$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji} = \lambda$, be positive real numbers, be determined for the fixed spatial reference point P_{SU} moving with a constant nonzero speed, and obey (11.80) through (11.84), and let (11.79) through (11.84) imply (12.7) for $A = B$. Then, a constant nonzero velocity $\mathbf{v}_P^i$ of the arbitrary point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the corresponding constant nonzero velocity $\mathbf{v}_P^j$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}}{1 + \frac{v}{v_{SU}}},\ \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}}{1 - \frac{v}{v_{SU}}}. \tag{12.68}$$

These transformations are only partially compatible.

The velocity $\mathbf{v}_{SU}^{(.)}$ of the spatial reference point P_{SU} is invariant,

$$\mathbf{v}_{SU}^i \equiv \mathbf{v}_{SU}^j \equiv \mathbf{v}_{SU}. \tag{12.69}$$

Note 498 *In order to verify invariance of the velocity $\mathbf{v}_{SU}^{(.)}$ of the spatial reference point P_{SU}, (12.69), it is sufficient to to accept it for the velocity $\mathbf{v}_P^{(.)}$ of the arbitrary point P:*

$$\mathbf{v}_P^{(.)} = \mathbf{v}_{SU}^{(.)}.$$

This and (12.69) yield

$$\mathbf{v}_P^i = \mathbf{v}_{SU}^i = \frac{\mathbf{v}_{SU}^j + \mathbf{v}}{1 + \frac{v}{v_{SU}}} = \frac{v_{SU}^j + v}{1 + \frac{v}{v_{SU}}}\mathbf{u} = \frac{1 + \frac{v}{v_{SU}}}{1 + \frac{v}{v_{SU}}} v_{SU}^j \mathbf{u} = \mathbf{v}_{SU}^j,$$
$$\mathbf{v}_P^j = \mathbf{v}_{SU}^j = \frac{\mathbf{v}_{SU}^i - \mathbf{v}}{1 - \frac{v}{v_{SU}}} = \frac{v_{SU}^i - v}{1 - \frac{v}{v_{SU}}}\mathbf{u} = \frac{1 - \frac{v}{v_{SU}}}{1 - \frac{v}{v_{SU}}} v_{SU}^i \mathbf{u} = \mathbf{v}_{SU}^i. \tag{12.70}$$

Remark 499 ***Reference spatial velocity is invariant in both the special and the singular case of the transformations. The light velocity is not any exception.***

The results (12.66) and (12.68), i.e. (12.67) and (12.69), show that the velocity $\mathbf{v}_{SU}^{(.)}$ of the spatial reference point P_{SU} is invariant relative to integral spaces if the coordinate transformations among them obey either Theorem 495 or Corollary 497.

Lorentz - Einstein invariance of the light speed is not any exception. It is a consequence of the feature of the formula.

This explanation agrees fully with that of Remark 419.

Corollary 497 and Remark 499 confirm Corollary 275. Moreover, Corollary 497 and Note 498 imply the following result:

Claim 500 ***Every velocity can be invariant***

For every velocity there exist linear coordinate transformations, which satisfy the distance condition (12.7), such that yield the velocity transformations relative to which the given velocity is invariant.

For the light velocity such coordinate transformations are Lorentz transformations (7.20) through (7.23). The light velocity / the light speed is not an exceptional speed from the kinematic point of view. Einstein‘s postulate that the light speed is invariant is completely wrong.

Chapter 13

Conclusion on PCC Relativity Theory

After relaxing the a priory imposed conditions on the *time* scaling coefficients α_i^j and α_j^i to be equal, and on the space scaling coefficients λ_i^j and λ_j^i to be also equal, we established various new forms of the *time*-invariant coordinate transformations in the general case. They are essentially different from Lorentz transformations. They do not contain either square roots or squared ratios of speed values. They are expressed in terms of relative values of speeds, including the relative values of the light speed and of the spatial transfer speed, with respect to the corresponding integral spaces. They enable a number of various consistent nonuniform and uniform coordinate transformations.

The transformations are valid also for the particular case when the inertial frames move in parallel, in the same sense and with the same speeds. Then they take the well known classical forms, which express the influence of a *time* scale and/or of a *time* unit change. Lorentz transformations do not express such influence. They are inapplicable in this case.

Still in the special case we got new formulae different from Einsteinian.

By starting with the features of *time* (Axiom 47) and by accepting a priory the same restrictions of Einsteinian relativity theory, we reproved Lorentz transformations as the singular case. There is not any contradiction between the *time* independence of the space and Lorentz transformations. This warns that Lorentz transformations, and from them deduced other results, do not and cannot prove *time* dependence of space, and may not be used to claim wrongly such *time* dependence.

New velocity transformations resulted. We determined them for an arbitrary speed of the arbitrary point P by applying Einsteinian methodology to ignore the fact that the temporal and the spatial coordinate transformations were established exclusively either for the light speed or for another fixed reference speed used in the proofs as the speed of the arbitrary point P. Their forms depend on the forms of nonuniformity or on the forms of uniformity of the transformations. They contain the basic *time* scaling coefficient μ_i in the general case, while the velocity transformations resulting from Lorentz transformations, i.e. the formulae of Einstein's law of the composition of velocities, do not. Their partial compatibility is the consequence of the a priory accepted Lorentz - Einstein - Poincaré condition imposed on the scaling coefficients to be determined exclusively for the light speed of the arbitrary point P.

The results show that the light speed is not invariant relative to all integral spaces containing inertial frames. Lorentz - Einstein invariance of the light speed is the consequence of the property of the formula. It is not a property of light or of the light speed. The same holds for the spatial transfer speed $v_{ji}^{(.)}$. These facts explain why the a priory accepted invariance of both the light speed and the spatial transfer speed impose so sever restrictions on Einsteinian relativity theory that it represents a singular case.

We considered only time-invariant coordinate transformations in order to satisfy Einsteinian condition that the scaling coefficients are (positive) real numbers.

Becoming aware of the above facts, we will continue studying the coordinate transformations by omitting completely Lorentz - Einstein - Poincaré constraints.

Part IV

Compatible and Consistent Relativity Theory (CC RT)

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Chapter 14

Colinear Motions: Transformations

14.1 Importance. *Time*-invariance

What follows will introduce both new temporal and spatial coordinate transformations that are essentially different from Lorentz transformations and a new approach to coordinate transformations, which is inherently different from Einsteinian approach. They will express completely the *time* independence property. They will be released from all the constraints that are a priory accepted in Einsteinian relativity theory.

The *time*-invariance of the transformations incorporates the constancy of the spatial transfer speed that corresponds to mutually inertial frames.

The temporal transfer speed in the new temporal coordinate transformations will be an arbitrarily accepted and then fixed constant speed $\vartheta^{(.)}$ if its value is measured with the units of $T_{(.)}$ and of $R^n_{(.)}$. In a particular case $\vartheta^{(.)} = v_R^{(.)}$ is allowed.

In a more special case $\vartheta^{(.)} = v_{ji}^{(.)}$ is permitted. This explains why the new temporal coordinate transformations can incorporate those by Lorentz and Einstein as a special case (in fact, as a singular case). To show this we should, by referring to Einsteinian relativity theory, set $v_P^{(.)} \equiv c_{(.)}^{(.)}$, but by retaining simultaneously $r_P^{(.)}(t_{(.)})$, instead of $r_L^{(.)}(t_{(.)})$, in the temporal coordinate transformations. Their difference is not formal, but crucial. By accepting the transfer speed from the spatial coordinates to be also the temporal transfer speed, Einsteinian relativity theory a priory does not permit to the temporal coordinate transformations to express the *time* independence of the space.

By accepting the speed $\vartheta^{(.)}$ for the temporal transfer speed independently of the spatial transfer speed $v_{ji}^{(.)}$ we will establish the coordinate transformations that will reflect completely the *time* independence of the space (Axiom 47).

Lorentz, Einstein and Poincaré aimed at establishing the coordinate transformations valid for any choice of the arbitrary point P. However, they calculated the scaling factors α and λ for the arbitrary point P moving exclusively with the light speed. This has been an essential restriction on the transformations applicability and a drawback of the approach.

We will determine the scaling coefficients $\alpha_{(.)}^{(.)}$ and $\lambda_{(.)}^{(.)}$ in the coordinate transformations for any nonzero value of the speed of the arbitrary point P, hence for its value different from the value of the light speed in general. This means that the transformations should be completely (pairwise and entirely) *compatible*. Besides, they should reflect fully the independence property of *time* (Axiom
In order to achieve this, we will establish and use a novel methodology that will be called *the consistent relativity methodology*. It will provide the basis for the fundamentals of a novel mathematical relativity theory that will use consistently values of various speeds relative to the corresponding integral spaces, and which will establish completely (entirely and pairwise) compatible coordinate transformations. The novel mathematical theory will be called ***Compatible and Consistent (CC) Relativity Theory***. It will be established for different *time* scales and *time* units of different *time* axes, i.e.

$$\mu_i(.) \neq \mu_j(.),\ \forall i,j \in \{-,1,2,...,s\},\ i \neq j. \tag{14.1}$$

This is reasonable because the same *time* scale coefficients mean that there is not a temporal coordinate transformation, which characterizes Galilean - Newtonian transformations.

This chapter concerns only *time*-invariant transformations and the related *time*-invariant *time* fields.

14.2 Nonuniformity: general

14.2.1 Temporal and spatial coordinate transformations

Basic relationships

The basic temporal coordinate transformation (10.15) preserves its form,

$$\mu_i(.) \equiv \mu_i = const. \in R^+ \Longrightarrow t_i = \mu_i t. \tag{14.2}$$

We should determine the scaling coefficients in the basic generic transformations (10.16) through (10.20) repeated as (14.3) through (14.7),

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j) \right], \tag{14.3}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i) \right], \tag{14.4}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u} \right], \tag{14.5}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}\right], \tag{14.6}$$

$$\begin{gathered} q^{(.)}, w^{(.)} \in R^+,\ q^{(.)}w^{(.)} \neq \left[c_{(.)}^{(.)}\right]^2,\ v_{ji}^{(.)} \in R_+, \\ q^{(.)}w^{(.)} = \left[v_P^{(.)}\right]^2 \ \textit{is permitted but not required.} \end{gathered} \tag{14.7}$$

The condition for the preservation of the generalized length in integral spaces will be used in its general form (6.22) valid for an arbitrary nonzero constant speed $v_P^{(.)}$ of the arbitrary point P. In this framework of *time*-invariant transformations it becomes (14.8),

$$\begin{gathered} \begin{bmatrix} \mathbf{r}_P^T(t_i) \\ t_i \mathbf{v}_P^{i^T} \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(t_i) \\ t_i \mathbf{v}_P^i \end{bmatrix} \equiv \begin{bmatrix} \mathbf{r}_P^T(t_j) \\ t_j \mathbf{v}_P^{j^T} \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(t_j) \\ t_j \mathbf{v}_P^j \end{bmatrix}, \\ D = blockdiag\{A \quad -B\} \in R^{2n\mathrm{x}2n}, \\ A \in R^n \ and\ B \in R^n \ are\ positive\ definite. \end{gathered} \tag{14.8}$$

It is to be noted that there is not in this framework Einsteinian demand for the arbitrary point P to move with the light speed, i.e. $\mathbf{v}_P^{(.)} \neq \mathbf{c}_{(.)}^{(.)}$ is allowed in general. However, $\mathbf{v}_P^{(.)} = \mathbf{c}_{(.)}^{(.)}$ is permitted as a particular case.

Since $\mathbf{v}_P^{(.)} \equiv c_{(.)}^{(.)}$ is permitted then the condition 14.8 is valid also for such a special case and incorporates Einstein's original condition (7.1) for the length preservation under the transformations, [144] through [154].

General case

Theorem 501 *Let the time scaling coefficient μ_i be defined by (14.2). In order for the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, to be positive real numbers, to obey (14.3) through (14.7), and for (14.2) through (14.7) to imply (14.8) it is necessary and sufficient that the following relationships hold for any choice of the time scaling coefficients $\mu_i \in R^+$ and $\mu_j \in R^+$:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{v_{ji}^j v_P^j}{q^j w^j}}, \tag{14.9}$$

$$\alpha_i^j = \frac{\mu_j}{\mu_i} \frac{1}{1 - \frac{v_{ji}^i v_P^i}{q^i w^i}}, \tag{14.10}$$

$$\lambda_j^i = \frac{1}{1 + \frac{v_{ji}^j}{v_P^j}}, \tag{14.11}$$

$$\lambda_i^j = \frac{1}{1 - \frac{v_{ji}^i}{v_P^i}}, \tag{14.12}$$

$$0 \leq v_{ji}^{(.)} < \min \left\{ \left| v_P^{(.)} \right|, \frac{q^{(.)} w^{(.)}}{\left| v_P^{(.)} \right|} \right\}, \tag{14.13}$$

$$\frac{\mu_j}{\mu_i} = \frac{v_P^i}{v_P^j} = \frac{c_i^i}{c_j^j}. \tag{14.14}$$

The transformations (14.3) through (14.6), specified by (14.9) through (14.12), take the following forms:

$$t_i = \frac{\mu_i}{\mu_j} \frac{t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)}{1 + \frac{v_{ji}^j v_P^j}{q^j w^j}}, \tag{14.15}$$

$$t_j = \frac{\mu_j}{\mu_i} \frac{t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i)}{1 - \frac{v_{ji}^i v_P^i}{q^i w^i}}, \tag{14.16}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}}{1 + \frac{v_{ji}^j}{v_P^j}}, \tag{14.17}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}}{1 - \frac{v_{ji}^i}{v_P^i}}. \tag{14.18}$$

They are completely both entirely and pairwise compatible.

The proof is in Appendix 23.1.

Comment 502 *The conditions (14.13) show that the theorem is valid for the arbitrary point P moving with an arbitrary constant speed.*

Note 503 *The complete pairwise compatibility of the transformations permits the separate use of the pair of the temporal coordinate transformations (14.15), (14.16) and of the pair of the spatial coordinate transformations (14.17), (14.18). This is not permitted for Lorentz transformations (7.20) through (7.23).*

Note 504 *The choice of the basic time scaling coefficients $\mu_i \in R^+$ and $\mu_j \in R^+$ is free. They can be chosen mutually independently.*

Comment 505 *Theorem 501 presents the formulae for the scaling coefficients and for the coordinate transformations, which need not be expressed in terms of the light speed values $c_{(.)}^{(.)}$. However, they are valid also in the case when the arbitrary point P moves with the light speed. In such a case we should just replace $v_P^{(.)}$ by $c_{(.)}^{(.)}$ everywhere in the formulae.*

Comment 506 *The expressions (14.9) through (14.12) for the scaling coefficients* α^i_j, α^j_i, λ^i_j *and* λ^j_i, *(14.14) and the resulting transformations (14.15) through (14.18) are inherently different from the corresponding formulae in Einsteinian relativity theory. The former contain relative values of the light speed, of the transfer speed and of the speed of the arbitrary point* P*. They do not contain either square roots or squared quotients in the denominators. Besides, they show that the values of the light speed and of the spatial transfer speed relative to the corresponding integral space should be taken into account rather than to accept them a priory invariant.*

Note 507 *The expressions for the time scaling coefficients* α^i_j *and* α^j_i, *(14.9) and (14.10), guarantee such a change of the time units that the time scaling condition (14.2) is satisfied. This means that (14.3) and (14.4) together with (14.9) and (14.10), that is that (14.15) and (14.16), take the following simple forms*

$$t_i = \mu_i \mu_j^{-1} t_j,\ t_j = \mu_i^{-1} \mu_j t_i. \tag{14.19}$$

They are equivalent to (14.2). We should set $r_P(t_{(.)}) \equiv v_P^{(.)} t_{(.)}$ *in (14.15) and (14.16) in order to verify (14.19). Consequently, both (14.2), and the pair (14.15), (14.16), imply the same result for* dt_i/dt_j*:*

$$\frac{dt_i}{dt_j} = \frac{\mu_i}{\mu_j}. \tag{14.20}$$

This analysis shows that both t_i *as a function of* t_j*, (and vice versa), and* dt_i/dt_j *are independent of the speed of the arbitrary point* P*, of the relative speed between the (spatial) frames and of the light speed.*

Galilean - Newtonian spatial coordinate transformations (8.9), (8.10), which do not permit a change of *time* unit, are fully generalized as follows:

$$\mathbf{r}_P(t_i) = \mathbf{r}_P(t_j) + v^j_{ji} t_j \mathbf{u}, \tag{14.21}$$

$$\mathbf{r}_P(t_j) = \mathbf{r}_P(t_i) - v^i_{ji} t_i \mathbf{u}. \tag{14.22}$$

They permit the basic temporal coordinate transformation (14.2), hence a change of the *time* unit, and preserve simultaneously the same, constant, length unit in different frames. Thus the length units do not change with speed. ***The generalized Galilean - Newtonian transformations (14.21), (14.22)*** can be considered as a special case of the spatial coordinate transformations (14.5), (14.6) in which the space scaling coefficients are a priory accepted unity, $\lambda^i_j = \lambda^j_i = 1$. What are their characteristics?

Claim 508 ***Generalized Galilean- Newtonian transformations are completely compatible and form the Poincaré group***

We prove this claim in Appendix 23.2. The proof verifies, in view of $\lambda^i_j = \lambda^j_i = 1$, and by the equation (23.17), [see also (14.23 in the sequel], that there is not a change of the length unit in spite there is a change of the time unit. This is beyond Einsteinian relativity theory. It reflects the *time* independence of the space (Axiom 47).

Note 509 *Generalized Galilean - Newtonian transformations (14.21), (14.22) do not follow from (14.17), (14.18) because* $\lambda_j^i = \lambda_i^j = 1$ *in the latter implies* $v_{ji}^i = v_{ji}^j = 0$ *due to (14.11) and (14.12). The former result from (14.5), (14.6) only if* $\lambda_j^i = \lambda_i^j = 1$ *is accepted in them a priory, but not a posteriori.*

Note 510 *The equations for the space scaling coefficients* λ_j^i *and* λ_i^j*, (14.11) and (14.12), ensure such changes of the length units of the frames, such their adaptation in terms of both the speed of the arbitrary point P and the relative speed between the frames, that the numerical value of the distance of the point P from the origin of every frame is the same,*

$$\mathbf{r}_P(t_i) = \mathbf{r}_P(t_j). \tag{14.23}$$

This is easy to verify by replacing t_j *by* $r_P(t_j)/v_P^j$ *in (14.17), or* t_i *by* $r_P(t_i)/v_P^i$ *in (14.18), and then by applying (23.2). This is a feature that characterizes the generalized Galilean - Newtonian transformations (14.21), (14.22) as proved by the equation (23.17) (Appendix 23.2).*

As soon as neither $q^{(\cdot)}$ *nor* $w^{(\cdot)}$ *nor* $v_P^{(\cdot)}$ *is equal to the light speed value* $c_{(\cdot)}^{(\cdot)}$*, then the space scaling coefficients are independent of the light speed value. This is an essential difference between the above results for these scaling coefficients and those existing in Einsteinian relativity theory.*

Conclusion 511 ***The speed of the arbitrary point P can be bigger than the light speed***

The relationships (14.13) restrict the value of the speed of the arbitrary point P as follows:

$$0 \leq v_{ji}^{(\cdot)} < \left| v_P^{(\cdot)} \right| < \frac{q^{(\cdot)} w^{(\cdot)}}{v_{ji}^{(\cdot)}},$$

and, therefore,

$$0 \leq v_{ji}^{(\cdot)} < \sqrt{q^{(\cdot)} w^{(\cdot)}}.$$

If $v_{ji}^{(\cdot)} = \zeta c_{(\cdot)}^{(\cdot)}$, $\zeta \in]0,1[$, $q^{(\cdot)} = c_{(\cdot)}^{(\cdot)}$, $w^{(\cdot)} = k^2 c_{(\cdot)}^{(\cdot)}$, $k \in]1,\infty[$, *then these inequalities permit:*

$$0 \leq v_{ji}^{(\cdot)} = \zeta c_{(\cdot)}^{(\cdot)} < v_P^{(\cdot)} < \zeta^{-1} k^2 c_{(\cdot)}^{(\cdot)}.$$

Hence, the value of the speed of the arbitrary point P may be bigger than the light speed value, e.g. for $v_P^{(\cdot)} = \zeta^{-1} c_{(\cdot)}^{(\cdot)} > c_{(\cdot)}^{(\cdot)}$*. This opposes inherently Einstein's limit on the speed value of the arbitrary point, which is the light speed value. The arbitrary point P can move faster than the light signal according to this result, i.e. relative to the integral spaces interrelated by the transformations (14.15) through (14.18).*

Note 512 *The expressions (14.9) through (14.12) restrict the value of the spatial transfer speed* $v_{ji}^{(.)}$ *by the value of the speed* $v_P^{(.)}$ *of the arbitrary point* P *rather than by the value of the light speed. The light speed value does not limit the value of any speed.*

Note 513 *The equations (14.14) prove that the light speed value is noninvariant in general.*

Corollary 514 *Let the coordinate systems* R_i^n *and* R_j^n *move with the same velocity:* $\mathbf{v}_{O_i}^O = \mathbf{v}_{O_j}^O$*. Let the constant time scaling coefficient* μ_i *be defined by (14.2). In order for the scaling coefficients* α_i^j*,* α_j^i*,* $\alpha_j^i \neq \alpha_i^j$*,* λ_i^j *and* λ_j^i *to be positive real numbers, to obey (14.3) through (14.7) and for (14.2) through (14.7) to imply (14.8) it is necessary and sufficient that the following equations hold for any choice of the time scaling coefficients* $\mu_i \in R^+$ *and* $\mu_j \in R^+$*:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j},\ \alpha_i^j = \frac{\mu_j}{\mu_i},\ \lambda_j^i = \lambda_i^j = 1,\ \frac{\mu_j}{\mu_i} = \frac{v_P^i}{v_P^j} = \frac{c_i^i}{c_j^j}.$$

The equations (14.3) through (14.6) become

$$t_i = \frac{\mu_i}{\mu_j} t_j,\ t_j = \frac{\mu_j}{\mu_i} t_i,\ \mathbf{r}_P(t_i) = \mathbf{r}_P(t_j).$$

They represent both entirely and pairwise completely compatible transformations.

Note 515 *Einsteinian relativity theory cannot cope with this case, which returns to the framework of the classical temporal coordinate transformations without spatial coordinate transformations.*

Special case

The same *time* scaling coefficients and the same space scaling coefficients characterize the relativity theory established by Lorentz, Einstein and Poincaré, which we accept in the sequel. However, the value of the speed of the arbitrary point P will be a priory allowed to be arbitrary in the proofs, which is a priory rejected in Einsteinian relativity theory.

Theorem 516 *Let the time scaling coefficient* μ_i *be defined by (14.2). Let* $A \neq B$ *be permitted in D, (14.8). In order for the scaling coefficients* α_j^i*,* α_i^j*,* $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$*,* λ_j^i *and* λ_i^j*,* $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$*, to be positive real numbers and to obey (14.3) through (14.7), and for (14.2) through (14.7) to imply (14.8) it is necessary (but not sufficient) that the relationships (14.24) through (14.30),*

$$\sqrt{q^i w^i} = v_P^i, \sqrt{q^j w^j} = v_P^j, \tag{14.24}$$

$$\frac{v_{ji}^i}{v_{ji}^j} \equiv \frac{v_P^i}{v_P^j} \equiv \frac{c_i^i}{c_j^j}, \tag{14.25}$$

$$\alpha_{ij} = \frac{1}{\sqrt{1-\left(\frac{v_{ji}^{i}}{\sqrt{q^{i}w^{i}}}\right)^{2}}} \equiv \frac{1}{\sqrt{1-\left(\frac{v_{ji}^{j}}{\sqrt{q^{j}w^{j}}}\right)^{2}}}, \tag{14.26}$$

$$\lambda_{ij} = \frac{1}{\sqrt{1-\left(\frac{v_{ji}^{i}}{v_{P}^{i}}\right)^{2}}} = \frac{1}{\sqrt{1-\left(\frac{v_{ji}^{j}}{v_{P}^{j}}\right)^{2}}} = \alpha_{ij}, \tag{14.27}$$

$$\frac{\mathbf{u}^{T}A\mathbf{u}}{\mathbf{u}^{T}B\mathbf{u}} = \left(\frac{v_{ji}^{i}}{v_{ji}^{j}}\right)^{2} = \left(\frac{v_{P}^{i}}{v_{P}^{j}}\right)^{2} = \left(\frac{c_{i}^{i}}{c_{j}^{j}}\right)^{2}, \tag{14.28}$$

$$v_{ji}^{(.)} < v_{P}^{(.)} = \sqrt{q^{(.)}w^{(.)}}, \tag{14.29}$$

$$\mu_{j} = \mu_{i}\sqrt{\frac{1-\frac{v_{ji}^{j}}{\sqrt{q^{j}w^{j}}}}{1+\frac{v_{ji}^{j}}{\sqrt{q^{j}w^{j}}}}} = \mu_{i}\sqrt{\frac{1-\frac{v_{ji}^{i}}{\sqrt{q^{i}w^{i}}}}{1+\frac{v_{ji}^{i}}{\sqrt{q^{i}w^{i}}}}}, \tag{14.30}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. The equations (14.3) through (14.6) become the equations (14.31) through (14.34):

$$t_{i} = \frac{t_{j} + \frac{v_{ji}^{j}}{q^{j}w^{j}}r_{P}\left(t_{j}\right)}{\sqrt{1-\left(\frac{v_{ji}^{j}}{\sqrt{q^{j}w^{j}}}\right)^{2}}}, \tag{14.31}$$

$$t_{j} = \frac{t_{i} - \frac{v_{ji}^{i}}{q^{i}w^{i}}r_{P}\left(t_{i}\right)}{\sqrt{1-\left(\frac{v_{ji}^{i}}{\sqrt{q^{i}w^{i}}}\right)^{2}}}, \tag{14.32}$$

$$\mathbf{r}_{P}\left(t_{i}\right) = \frac{\mathbf{r}_{P}\left(t_{j}\right) + v_{ji}^{j}t_{j}\mathbf{u}}{\sqrt{1-\left(\frac{v_{ji}^{j}}{v_{P}^{j}}\right)^{2}}}, \tag{14.33}$$

$$\mathbf{r}_{P}\left(t_{j}\right) = \frac{\mathbf{r}_{P}\left(t_{i}\right) - v_{ji}^{i}t_{i}\mathbf{u}}{\sqrt{1-\left(\frac{v_{ji}^{i}}{v_{P}^{i}}\right)^{2}}}. \tag{14.34}$$

If, additionally, $A = B$ in D, (14.8), then for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, to be positive real numbers and to obey (14.3) through (14.7), and for (14.2) through (14.7) to imply (14.8) it is necessary and sufficient that the equations (14.24) through (14.30) and the relationships (14.35),

$$\begin{gathered} v_{ji}^{i} = v_{ji}^{j} = v_{ji} = -v_{ij},\ v_{P}^{i} = v_{P}^{j} = v_{P}^{ij} = v_{P}^{ji},\ v_{P}^{ji} > v_{ji}^{i} \geq 0, \\ 0 < \sqrt{q^{i}w^{i}} = \sqrt{q^{j}w^{j}} = \sqrt{(qw)^{ij}} = \sqrt{(qw)^{ji}} = v_{P}^{ij},\ c_{i}^{i} = c_{j}^{j} = c_{ij} = c_{ji}. \end{gathered} \tag{14.35}$$

hold. Then, the transformations (14.3) through (14.6), i.e. (14.31) through (14.34) modified by (14.35), become

$$t_i = \frac{t_j + \frac{v_{ji}}{(qw)^{ji}} r_P(t_j)}{\sqrt{1 - \left(\frac{v_{ji}^j}{\sqrt{(qw)^{ji}}}\right)^2}}, \tag{14.36}$$

$$t_j = \frac{t_i - \frac{v_{ji}}{(qw)^{ji}} r_P(t_i)}{\sqrt{1 - \left(\frac{v_{ji}^i}{\sqrt{(qw)^{ji}}}\right)^2}}, \tag{14.37}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + v_{ji} t_j \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}}{v_P^{ji}}\right)^2}}, \tag{14.38}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - v_{ji} t_i \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}}{v_P^{ji}}\right)^2}}. \tag{14.39}$$

They are then partially both entirely and pairwise compatible.

The proof is given in Appendix 23.3.

Note 517 *If we set $\mathbf{r}_P(t_{(.)}) = \mathbf{v}_{Pji} t_{(.)}$ in (14.36) and (14.37) then they become*

$$t_i = \frac{\mu_i}{\mu_j} t_j,\ t_j = \frac{\mu_j}{\mu_i} t_i, \tag{14.40}$$

due to (14.30) and (14.35). The transformation (14.2) implies also the equations (14.40). The latter express such a change of the time scales regardless of the movements of the spatial frames R_i^n and R_j^n.

When we replace $t_{(.)}$ by $(v_P^{(.)})^{-1} r_P(t_{(.)})$ in (14.38) and (14.39) then they reduce to

$$\mathbf{r}_P(t_i) = \frac{\mu_i}{\mu_j} \mathbf{r}_P(t_j),\ \mathbf{r}_P(t_j) = \frac{\mu_j}{\mu_i} \mathbf{r}_P(t_i). \tag{14.41}$$

The numerical values of the distances of the arbitrary point P from the origin O_i of R_i^n and from the origin O_j of R_j^n change in the same ratio as the numerical values of the temporal coordinates despite of a mutual relative movement of the frames R_i^n and R_j^n. The adjustment of the space scaling coefficients λ_{ij}, (14.27), to the velocities $\mathbf{v}_P^i$ and $\mathbf{v}_P^j$ of the arbitrary point P relative to R_i^n and R_j^n, respectively, ensures such a property of (14.38) and (14.39).

The equations (14.40) and (14.41) show that all the velocities, not only the light velocity, are invariant under the coordinate transformations (14.36) *through (14.39),*

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P(t_i)}{dt_i} = \frac{\left\{d\left[\frac{\mu_i}{\mu_j}\mathbf{r}_P(t_j)\right]\right\}/dt_j}{\left[d\left(\frac{\mu_i}{\mu_j}t_j\right)\right]/dt_j} = \mathbf{v}_P^j,$$

$$\mathbf{v}_P^j = \frac{d\mathbf{r}_P(t_j)}{dt_j} = \frac{\left\{d\left[\frac{\mu_j}{\mu_i}\mathbf{r}_P(t_i)\right]\right\}/dt_i}{\left[d\left(\frac{\mu_j}{\mu_i}t_i\right)\right]/dt_i} = \mathbf{v}_P^i. \tag{14.42}$$

Neither Galilean - Newtonian physics nor Einsteinian relativity theory has treated this special case.

Problem solution in the singular case

The conditions of the singular case reduce Theorem 516 to

Corollary 518 *Let the time scaling coefficient μ_i be defined by (14.2). Let $A = B$ be a priory accepted in D, (14.8). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji} = \alpha$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji} = \lambda$, to be positive real numbers and to obey (14.3) through (14.7), and for (14.2) through (14.7) to imply (14.8) it is necessary and sufficient that the relationships (14.43) through (14.46),*

$$v_P^i = v_P^j = v_P^{ij} = v_p,\ 0 < \sqrt{q^i w^i} = \sqrt{q^j w^j} = \sqrt{qw} = v_P,$$

$$c_j^j = c_i^i = c_{ij} = c_{ji} = c, \tag{14.43}$$

$$0 \leq v_{ji}^i = v_{ji}^j = v_{ji} = -v_{ij} = v < v_p, \tag{14.44}$$

$$\alpha = \lambda = \frac{1}{\sqrt{1-\left(\frac{v}{\sqrt{qw}}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{v}{v_P}\right)^2}}, \tag{14.45}$$

$$\mu_j = \mu_i\sqrt{\frac{1-\frac{v}{v_P}}{1+\frac{v}{v_P}}}, \tag{14.46}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. The equations (14.3) through (14.6), i.e. (14.36) through (14.39), become the equations (14.47) through (14.50):

$$t_i = \frac{t_j + \frac{v}{qw}r_P(t_j)}{\sqrt{1-\left(\frac{v}{\sqrt{qw}}\right)^2}}, \tag{14.47}$$

$$t_j = \frac{t_i - \frac{v}{qw}r_P(t_i)}{\sqrt{1-\left(\frac{v}{\sqrt{qw}}\right)^2}}, \tag{14.48}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + vt_j\mathbf{u}}{\sqrt{1-\left(\frac{v}{v_P}\right)^2}}, \tag{14.49}$$

$$\mathbf{r}_P\left(t_j\right) = \frac{\mathbf{r}_P\left(t_i\right) - vt_i\mathbf{u}}{\sqrt{1-\left(\frac{v}{v_P}\right)^2}}. \tag{14.50}$$

They are partially both entirely and pairwise compatible transformations.

Note 519 ***The temporal coordinate transformations are independent of movements of frames***

If we set $r_P(t_{(.)}) \equiv v_P t_{(.)}$ in (14.47) and (14.48) then they become the equations (14.40) due to (14.43) and (14.46). The transformation (14.2) implies also the equations (14.40) that express such a change of the time units that it is independent of the movements of the spatial frames R_i^n and R_j^n. Einsteinian relativity theory has not recognized the time scaling coefficients μ_i and μ_j. However, Galilean - Newtonian physics has well dealt with them in the time-invariant framework, but only if there are not transformations of spatial coordinates.

Remark 520 ***All velocities are invariant, not only the light velocity***

When we replace $t_{(.)}$ by $(v_P^{(.)})^{-1} r_P(t_{(.)})$ in (14.49) and (14.50) then they reduce to the equations (14.41). The numerical values of the distances of the arbitrary point P from the origin O_i of R_i^n and from the origin O_j of R_j^n change in the same ratio as the numerical values of the temporal coordinates in spite of a mutual relative movement of the frames R_i^n and R_j^n. The adjustment of the space scaling coefficient λ, (14.45), to the velocity $\mathbf{v}_P$ of the arbitrary point P relative to R_i^n and R_j^n, respectively, ensures such a property of (14.49) and (14.50), which is expressed by (14.41).

The equations (14.43) show that all velocities, not only the light velocity, are invariant in this singular case.

14.2.2 Velocity transformations

General case

Theorem 521 *Let the time scaling coefficient μ_i be defined by (14.2). Let the scaling coefficients α_j^i, α_i^j, $\alpha_j^i \neq \alpha_i^j$, λ_j^i and λ_i^j, $\lambda_j^i \neq \lambda_i^j$, be positive real numbers and obey (14.3) through (14.7), and let (14.2) through (14.7) imply (14.8). Then, a constant nonzero velocity $\mathbf{v}_P^i$ of the point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the corresponding constant nonzero velocity $\mathbf{v}_P^j$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i}\frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j}{v_P^j}} = \frac{\mu_j}{\mu_i}\mathbf{v}_P^j,\ \mathbf{v}_P^j = \frac{\mu_i}{\mu_j}\frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{1-\frac{v_{ji}^i}{v_P^i}} = \frac{\mu_i}{\mu_j}\mathbf{v}_P^i. \tag{14.51}$$

These transformations are completely compatible.

Proof. *Necessity and sufficiency.* Let all the conditions of the theorem statement be satisfied. The equations (14.9) through (14.12) and the transformations (14.15) through (14.18) hold (Theorem 501). The velocity of the arbitrary point P with respect to $R^n_{(.)}$ and to its origin $O_{(.)}$, and relative to $\mathfrak{T}_{(.)}$ is determined as follows due to (14.15), (14.17), $\mathbf{v}_P^{(.)} = v_P^{(.)}\mathbf{u}$, and $\mathbf{v}_{ji}^{(.)} = v_{ji}^{(.)}\mathbf{u}$:

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P}{dt_i} = \frac{d\left\{\frac{\mathbf{r}_P(t_j)+\ v_{ji}^j t_j \mathbf{u}}{1+\frac{v_{ji}^j}{v_P^j}}\right\} : dt_j}{d\left\{\frac{\mu_i}{\mu_j}\frac{t_j+\frac{v_{ji}^j}{q^j w^j} r_P(t_j)}{1+\frac{v_{ji}^j v_P^j}{q^j w^j}}\right\} : dt_j} =$$

$$= \frac{\mu_j}{\mu_i}\frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j}{v_P^j}} = \frac{\mu_j}{\mu_i}\frac{1+\frac{v_{ji}^j}{v_P^j}}{1+\frac{v_{ji}^j}{v_P^j}}\mathbf{v}_P^j = \frac{\mu_j}{\mu_i}\mathbf{v}_P^j.$$

This result proves the first two equations in (14.51). The other two equations in (14.51) are analogously proved by beginning with (14.16) and (14.18).

Compatibility. Compatibility between the equations for $\mathbf{v}_P^i$ and $\mathbf{v}_P^j$ in (14.51) is evident. It is complete because they are compatible for any nonzero value of the speed of the arbitrary point P. Q. E. D ■

Remark 522 ***Noninvariant light speed and unconstrained speed of the arbitrary point P***

The equations (14.51) hold also for the light speed of the arbitrary point P.

These equations show that the light speed is not invariant. It does not restrict other speeds. The arbitrary point P can move with a speed bigger than the light speed.

Comment 523 *The equations (14.51) agree fully with (14.14) by understanding that they hold also for the light speed of the arbitrary point P. In fact, we can deduce the former directly from the latter.*

Note 524 *The equations (14.51) reflect only the influence of changes of time units. They do not reflect any influence of the relative speed between the frames. This is a consequence of the adaptation of the length units to the speed of the arbitrary point P and to the relative speed between the frames (the Note 507). This follows from the proof of the theorem:*

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P(t_i)}{dt_i} = \frac{d\mathbf{r}_P[t_i(t_j)]/dt_j}{dt_i(t_j)/dt_j},$$

where

$$\frac{d\mathbf{r}_P[t_i(t_j)]}{dt_j} = \frac{d}{dt_j}\left[\frac{\mathbf{r}_P(t_j) +\ v_{ji}^j t_j \mathbf{u}}{1+\frac{v_{ji}^j}{v_P^j}}\right] = \frac{1+\frac{v_{ji}^j}{v_P^j}}{1+\frac{v_{ji}^j}{v_P^j}} v_P^j \mathbf{u} = \mathbf{v}_P^j, \tag{14.52}$$

and

$$\frac{dt_i(t_j)}{dt_j} = \frac{d}{dt_j}\left[\frac{\mu_i}{\mu_j}\frac{t_j + \frac{v^j_{ji}}{q^j w^j} r_P\left(t_j\right)}{1 + \frac{v^j_{ji} v^j_P}{q^j w^j}}\right] = \frac{\mu_i}{\mu_j}\frac{1 + \frac{v^j_{ji} v^j_P}{q^j w^j}}{1 + \frac{v^j_{ji} v^j_P}{q^j w^j}} = \frac{\mu_i}{\mu_j},$$

which is (14.20). By the definition

$$\mathbf{v}^j_P = \frac{d\mathbf{r}_P\left(t_j\right)}{dt_j}.$$

This and (14.52) show that:

$$\frac{d\mathbf{r}_P\left[t_i(t_j)\right]}{dt_j} = \frac{d\mathbf{r}_P\left(t_j\right)}{dt_j}.$$

This result appears natural due to (14.23). It shows that the spatial coordinate transformations in fact do not influence the numerical value of the speed because the length units in different spatial frames are fully adjusted to the speed of the arbitrary point P and to the relative speed between the frames. Consequently, only the temporal coordinate transformations influence numerical speed values, in view of (14.20), which is expressed by the right-hand sides of the equations (14.51).

The generalized Galilean - Newtonian velocity transformations (8.29), (8.30), i.e. (23.14), express the influence of the relative velocity between the frames by the nonzero spatial transfer velocity. This influence does not exist in (14.51). It is compensated by the adaptation of the length units to the speeds. Such adaptation is not involved in (8.29), (8.30), and in (23.14).

Note 525 ***Validity of the clock principle***

The equations (14.51), together with (14.2), show that the scaling factors determined by (14.9) through (14.12) ensure the invariance of the distance $r_P(t_{(.)}) = v^{(.)}_P t_{(.)}$*, which is expressed by, [see (14.23)]:*

$$r_P(t_i) = v^i_P t_i = \frac{\mu_j}{\mu_i} v^j_P \frac{\mu_i}{\mu_j} t_j = v^j_P t_j = r_P(t_j).$$

These equations reflect also the clock principle: ***the product of the speed of a clock hand and of time units corresponding to 360° is constant, i.e. it is equal to the clock circumference*** *(Section 4.8: "Clock principles"). This is the invariance of distance (of length) under the transformations established herein. The bigger speed value, the smaller time unit. A change of a speed value of a clock hand causes only the corresponding change of the time unit. This verifies once more Newton's explanation of the relativity of time.*

The change of the speed value of the clock hand cannot change either time or time value (so as a change of the voltmeter scale and/or of its unit cannot change either voltage or voltage value). It changes only the time unit and, correspondingly, the numerical value of time.

Special case

Theorem 526 *Let the time scaling coefficients* μ_i *be defined by (14.2). Let* $A = B$ *in D, (14.8). Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i *and* λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, *be constant and obey (14.3) through (14.7), and let (14.2) through (14.7) imply (14.8). Then, a constant nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point P with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the corresponding constant nonzero velocity* $\mathbf{v}_P^j$ *of the same point P with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated as follows:*

$$\mathbf{v}_P^i = \mathbf{v}_P^j = \mathbf{v}_P^{ij} = \mathbf{v}_P^{ji}. \tag{14.53}$$

For these transformations to be compatible it is necessary and sufficient that

$$v_P^{ji} = \sqrt{(qw)^{ji}}.$$

They are partially compatible.

We present the proof in Appendix 23.4.

Note 527 *The equations (14.53) agree with those equations in (14.35) which are related to the speed of the arbitrary point P.*

Problem solution in the singular case

It follows directly from the preceding result that the value of the speed of the arbitrary point P is independent of a choice of the *time* axis in the singular case:

$$\mathbf{v}_P^i = \mathbf{v}_P^j = \mathbf{v}_P^{ij} = \mathbf{v}_P^{ji} \equiv \mathbf{v}_P. \tag{14.54}$$

The speed transformations are trivial. They are reduced to the unity transformation under the condition (14.35). Hence, they are partially compatible.

Remark 528 ***Every velocity is invariant in both the special and the singular case of the transformations. The light velocity is not any exception.***

The results (14.53) and (14.54) show that every velocity is invariant relative to the integral spaces among which the coordinate transformations obey either Theorem 516 or Corollary

This agrees with Remark 419.

14.3 Nonuniformity: ordinary

Note 529 *The coordinate transformations (10.10), i.e. (14.2), (10.11) through (10.14) characterize time-invariant nonuniform transformations and time fields. They represent a special form of the transformations (14.3) through (14.6) obtained for* $q^{(\cdot)}w^{(\cdot)} \equiv \left(v_P^{(\cdot)}\right)^2$. *When we set* $q^{(\cdot)}w^{(\cdot)} \equiv \left(v_P^{(\cdot)}\right)^2$ *in the temporal* *coordinate transformations (10.16) and (10.17) then we can easily transform Section 14.2 into this section on time-invariant nonuniform transformations and time fields in the framework of the CC relativity theory.*

14.4 Nonuniformity: weak

14.4.1 Transformations of temporal and spatial coordinates

Basic relationships

The following equations (14.55) through (14.58) are the basic generic transformations (10.4) through (10.7),

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j) \right], \tag{14.55}$$

$$t_j = \alpha_i^j \left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i) \right], \tag{14.56}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + \ v_{ji}^j t_j \mathbf{u} \right], \tag{14.57}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - \ v_{ji}^i t_i \mathbf{u} \right], \tag{14.58}$$

where, due to (10.8),

$$q^{(.)}, w^{(.)} \in R^+,\ q^{(.)} w^{(.)} \notin \left\{ \ \left[c_{(.)}^{(.)} \right]^2, \left[v_P^{(.)} \right]^2 \right\}, v_{ji}^{(.)} \in R_+. \tag{14.59}$$

The *time* scaling coefficient $\mu_i(.)$ is constant in view of the aimed *time*-invariance of the transformations, (10.3) and (14.2).

General case

Note 530 *The difference between (14.3) through (14.7) and (14.55) through (14.59) is only in the condition* $q^{(.)} w^{(.)} \neq \left[v_P^{(.)} \right]^2$, *(14.59), which does not exist in (14.7). This condition does not influence the results in the general case as shown in the proof of Theorem 501. Therefore, the whole Subsubsection "General case" in 14.2.1 holds also herein.*

Special and singular case

Theorem 531 *Let the time scaling coefficient* μ_i *be defined by (14.2). Let the speed of the arbitrary point P be arbitrary. It is impossible for the scaling coefficients* α_i^j, α_j^i, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_i^j *and* λ_j^i, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, *to be positive real numbers, to obey (14.55) through (14.59), and for (14.2), (14.55) through (14.59) to imply (14.8).*

Proof. Since (14.55) through (14.58) are, respectively, the same as (14.3) through (14.6), and the conditions (14.59) cannot influence the obtained results for the scaling factors and for the coordinate transformations, then Theorem 516 holds also herein. Its conditions (14.24) and (14.35) violate the condition $q^{(.)}w^{(.)} \neq \left[v_P^{(.)}\right]^2$ in (14.59). Consequently, there are not such scaling coefficients α_i^j, α_j^i, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_i^j and λ_j^i, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, to be positive real numbers, to obey (14.55) through (14.59), and for which (14.2), (14.55) through (14.59) imply (14.8). Q. E. D ■

14.4.2 Transformations of velocity

General case

Since the whole Subsubsection "General case" in 14.2.1 holds also for the weak nonuniform coordinate transformations (14.55) - (14.59) (Note 530), then the Subsubsection "General case" in 14.2.2 is valid also in this general case.

Special and singular case

The special and singular case of the velocity transformations cannot take place in this framework due to Theorem 531.

14.5 General uniformity

14.5.1 Temporal and spatial coordinate transformations

Basic relationships

The basic generic coordinate transformations (10.66) through (10.71) incorporate various uniform coordinate transformations. They are

$$t_i = \mu_i t,\ \mu_i \in R^+, \tag{14.60}$$

$$t_i = \alpha_j^i \left[t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j) \right], \tag{14.61}$$

$$t_j = \alpha_i^j \left[t_i - \frac{\vartheta^i}{q^i w^i} r_R(t_i) \right], \tag{14.62}$$

$$\mathbf{r}_P(t_i) = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u} \right], \tag{14.63}$$

$$\mathbf{r}_P(t_j) = \lambda_i^j \left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u} \right], \tag{14.64}$$

where

$$q^{(.)},\ v_P^{(.)},\ v_R^{(.)},\ w^{(.)} \neq \vartheta^{(.)}\ in\ general;\ q^{(.)}, w^{(.)} \in R^+;\ v_{ji}^{(.)}, \vartheta^{(.)}, v_R^{(.)} \in R_+,$$
$$q^{(.)}, v_P^{(.)}, w^{(.)} = c_{(.)}^{(.)}\ are\ permitted,\ not\ required,$$
$$v_P^{(.)}, v_R^{(.)} = \vartheta^{(.)}\ are\ permitted\ iff\ \vartheta^{(.)} \in R^+,\ not\ required. \tag{14.65}$$

General case

Theorem 532 *Let the time scaling coefficient μ_i be defined by (14.60). In order for the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, to be positive real numbers, to obey (14.61) through (14.65), and for (14.60) through (14.65) to imply (14.8) it is necessary and sufficient that the following relationships hold for any choice of the time scaling coefficients $\mu_i \in R^+$ and $\mu_j \in R^+$:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}, \tag{14.66}$$

$$\alpha_i^j = \frac{\mu_j}{\mu_i} \frac{1}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}}, \tag{14.67}$$

$$\lambda_j^i = \frac{1}{1 + \frac{v_{ji}^j}{v_P^j}}, \tag{14.68}$$

$$\lambda_i^j = \frac{1}{1 - \frac{v_{ji}^i}{v_P^i}}, \tag{14.69}$$

$$q^{(.)} w^{(.)} > \vartheta^{(.)} v_R^{(.)}, \tag{14.70}$$

$$v_P^{(.)},\ v_R^{(.)} \in R^+ \Longrightarrow \frac{\mu_j}{\mu_i} = \frac{v_P^i}{v_P^j} = \frac{v_R^i}{v_R^j} = \frac{c_i^i}{c_j^j}. \tag{14.71}$$

The transformations (14.61) through (14.64) specified by (14.66) through (14.69) are completely both entirely and pairwise compatible. They take the following forms:

$$t_i = \frac{\mu_i}{\mu_j} \frac{t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j)}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}, \tag{14.72}$$

$$t_j = \frac{\mu_j}{\mu_i} \frac{t_i - \frac{\vartheta^i}{q^i w^i} r_R(t_i)}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}}, \tag{14.73}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}}{1 + \frac{v_{ji}^j}{v_P^j}} = \mathbf{r}_P(t_j), \tag{14.74}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}}{1 - \frac{v_{ji}^i}{v_P^i}} = \mathbf{r}_P(t_i). \tag{14.75}$$

Proof. We replace $v_{ji}^{(.)}$ by $\vartheta^{(.)}$ and $r_P\left(t_{(.)}\right)$ by $r_R\left(t_{(.)}\right)$ in (14.3) and (14.4). They become (14.61) and (14.62), and Theorem 501 becomes Theorem 532 because (14.2), (14.5) and (14.6) are the same as (14.60), (14.63) and (14.64), respectively. Q. E. D ■

Note 533 *The basic time scaling coefficients μ_i and μ_j are mutually independent. We may chose them separately.*

Comment 534 *The inequalities (14.70) show that the value of the light speed does not limit the values of other speeds. This opposes crucially the claim of Einsteinian relativity theory that the light speed value is the upper bound of the values of all other speeds in general.*

Note 535 *The equations (14.71) verify once more that the light speed is not invariant. They hold for arbitrary nonzero speeds of the arbitrary point P, of the reference point P_R, and for the light speed. They illustrate also the importance of the time scaling coefficient μ_i, which is missed in Einsteinian relativity theory.*

Comment 536 *The pair [(14.72), (14.73)] of the temporal coordinate transformations and the pair (14.74), (14.75) of the spatial coordinate transformations may be used separately. Their complete pairwise compatibility allows their separate applications.*

Note 537 *The transformations (14.72) through (14.75) do not contain either the position coordinate $r_L(t_{(.)})$ of the light signal L or the light speed value $c_{(.)}^{(.)}$.*

Note 538 *The transformations (14.72) through (14.75) do not contain either square roots or squared values of speeds, which characterize Lorentz transformations (7.20) through (7.23). Besides, the former contain consistently relative values of all the speeds, while the latter do not. Moreover, the former do not restrict the maximal value of the speed of the arbitrary point P, while the latter do.*

Note 539 *If we set $r_P(t_{(.)}) \equiv v_P^{(.)} t_{(.)}$ in (14.72) and (14.73) then they become*

$$t_i = \mu_i \mu_j^{-1} t_j,\ t_j = \mu_i^{-1} \mu_j t_i. \tag{14.76}$$

The transformation (14.60) implies also the equations (14.76). They both show the same changes of the time units regardless of the movements of the spatial frames R_i^n and R_j^n. Einsteinian relativity theory is not applicable to this case.

The equations (14.74) and (14.75) reduce the spatial coordinate transformations to the identity

$$\mathbf{r}_P(t_i) = \mathbf{r}_P(t_j). \tag{14.77}$$

The numerical values of the distances of the arbitrary point P from the origin O_i of R_i^n and from the origin O_j of R_j^n are equal in spite of a mutual relative movement of the frames R_i^n and R_j^n. The adjustment of space scaling coefficients λ_j^i, (14.68), and λ_i^j, (14.69), to the velocities $\mathbf{v}_P^i$ and $\mathbf{v}_P^j$ of the arbitrary point P relative to R_i^n and R_j^n, respectively, ensures such invariance of the distance numerical value.

Corollary 540 *Let the coordinate systems R_i^n and R_j^n move with the same velocity: $\mathbf{v}_{O_i}^O = \mathbf{v}_{O_j}^O$. Let the constant time scaling coefficient μ_i be defined by (14.60). In order for the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i to be positive real numbers, to obey (14.61) through (14.65), and for (14.60) through (14.65) to imply (14.8) it is necessary and sufficient that the following equations hold for any choice of the time scaling coefficients $\mu_i \in R^+$ and $\mu_j \in R^+$:*

$$\alpha_j^i = \frac{\mu_i}{\mu_j}\frac{1}{1+\frac{\vartheta^j v_R^j}{q^j w^j}},\ \alpha_i^j = \frac{\mu_j}{\mu_i}\frac{1}{1-\frac{\vartheta^i v_R^i}{q^i w^i}},$$
$$\lambda_j^i = \lambda_i^j = 1,\ \frac{\mu_j}{\mu_i} = \frac{v_P^i}{v_P^j} = \frac{v_R^i}{v_R^j} = \frac{c_i^i}{c_j^j}.$$

The equations (14.61) through (14.64) become (14.76) and (14.77). They are both entirely and pairwise completely compatible.

Special case

Theorem 541 *Let the time scaling coefficient μ_i be defined by (14.60). Let $A \neq B$ be permitted in D, (14.8). In order for the scaling coefficients α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, to be positive real numbers and to obey (14.61) through (14.65), and for (14.60) through (14.65) to imply (14.8) it is necessary (but not sufficient) that the relationships (14.78) through (14.84),*

$$q^{(\cdot)} w^{(\cdot)} = v_P^{(\cdot)} v_R^{(\cdot)},\ \vartheta^{(\cdot)} v_P^{(\cdot)} v_R^{(\cdot)} = v_{ji}^{(\cdot)} q^{(\cdot)} w^{(\cdot)}, \tag{14.78}$$

$$0 \leq v_{ji}^{(\cdot)} = \vartheta^{(\cdot)},\ \frac{v_{ji}^i}{v_{ji}^j} \equiv \frac{v_P^i}{v_P^j} \equiv \frac{v_R^i}{v_R^j} \equiv \frac{c_i^i}{c_j^j}, \tag{14.79}$$

$$\alpha_{ij} = \frac{1}{\sqrt{1-\left(\frac{\vartheta^i v_R^i}{q^i w^i}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{\vartheta^j v_R^j}{q^j w^j}\right)^2}}, \tag{14.80}$$

$$\lambda_{ij} = \frac{1}{\sqrt{1-\left(\frac{v_{ji}^i}{v_P^i}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{v_{ji}^j}{v_P^j}\right)^2}} = \alpha_{ij}, \tag{14.81}$$

$$\frac{\mathbf{u}^T A \mathbf{u}}{\mathbf{u}^T B \mathbf{u}} = \left(\frac{v_{ji}^i}{v_{ji}^j}\right)^2 = \left(\frac{v_P^i}{v_P^j}\right)^2 = \left(\frac{v_R^i}{v_R^j}\right)^2 = \left(\frac{\vartheta^i}{\vartheta^j}\right)^2 = \left(\frac{c_i^i}{c_j^j}\right)^2, \tag{14.82}$$

$$v_{ji}^{(\cdot)} < v_P^{(\cdot)},\ \vartheta^{(\cdot)} v_R^{(\cdot)} < q^{(\cdot)} w^{(\cdot)}, \tag{14.83}$$

$$\mu_j = \mu_i \sqrt{\frac{1-\frac{\vartheta^i v_R^i}{q^i w^i}}{1+\frac{\vartheta^i v_R^i}{q^i w^i}}} = \mu_i \sqrt{\frac{1-\frac{\vartheta^j v_R^j}{q^j w^j}}{1+\frac{\vartheta^j v_R^j}{q^j w^j}}}, \tag{14.84}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$*. The equations (14.61) through (14.64) become the equations (14.85) through (14.88):*

$$t_i = \frac{t_j + \frac{\vartheta^j}{q^j w^j} r_R\left(t_j\right)}{\sqrt{1 - \left(\frac{\vartheta^j v_R^j}{q^j w^j}\right)^2}}, \tag{14.85}$$

$$t_j = \frac{t_i - \frac{\vartheta^i}{q^i w^i} r_R\left(t_i\right)}{\sqrt{1 - \left(\frac{\vartheta^i v_R^i}{q^i w^i}\right)^2}}, \tag{14.86}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + v_{ji}^j t_j \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}^j}{v_P^j}\right)^2}}, \tag{14.87}$$

$$\mathbf{r}_P\left(t_j\right) = \frac{\mathbf{r}_P\left(t_i\right) - v_{ji}^i t_i \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}^i}{v_P^i}\right)^2}}. \tag{14.88}$$

If, additionally, $A = B$ *in* D*, (14.8), then for the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i *and* λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$*, to be positive real numbers and to obey (14.61) through (14.65), and for (14.60) through (14.65) to imply (14.8) it is necessary and sufficient that the relationships (14.78) through (14.84) and the equations (14.89),*

$$\begin{gathered} v_{ji}^i = v_{ji}^j = v_{ji} = -v_{ij},\ v_{[.]}^i = v_{[.]}^j = v_{[.]}^{ij} = v_{[.]}^{ji},\ [.] \in \{P, R, SU\}, \\ q^i w^i = q^j w^j = (qw)^{ij} = (qw)^{ji} = v_P^{ji} v_R^{ij},\ c_i^i = c_j^j = c_{ij} = c_{ji}, \\ v_{ji}^{(.)} = \vartheta^{(.)} = \vartheta^{ji},\ \vartheta^{ji} v_P^{ji} v_R^{ij} = v_{ji}\,(qw)^{ji}. \end{gathered} \tag{14.89}$$

hold. The transformations (14.61) through (14.64), i.e. (14.85) through (14.88) modified by (14.89), become

$$t_i = \frac{t_j + \frac{\vartheta^{ji}}{(qw)^{ji}} r_R\left(t_j\right)}{\sqrt{1 - \left(\frac{\vartheta^{ji} v_R^{ji}}{(qw)^{ji}}\right)^2}}, \tag{14.90}$$

$$t_j = \frac{t_i - \frac{\vartheta^{ji}}{(qw)^{ji}} r_R\left(t_i\right)}{\sqrt{1 - \left(\frac{\vartheta^{ji} v_R^{ji}}{(qw)^{ji}}\right)^2}}, \tag{14.91}$$

$$\mathbf{r}_P\left(t_i\right) = \frac{\mathbf{r}_P\left(t_j\right) + v_{ji} t_j \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}}{v_P^{ji}}\right)^2}}, \tag{14.92}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - v_{ji}t_i\mathbf{u}}{\sqrt{1-\left(\frac{v_{ji}}{v_P^{ji}}\right)^2}}. \tag{14.93}$$

They are partially both entirely and pairwise compatible.

Proof. When we replace $v_{ji}^{(.)}$ by $\vartheta^{(.)}$ in (14.3) and (14.4) then we transform easily the proof of Theorem 516 into this proof since (14.60), (14.63) and (14.64) are the same as (14.2), (14.5) and (14.6), respectively. Q. E. D ■

Note 542 *If we set $r_P(t_{(.)}) = v_P^{ji}t_{(.)}$ in (14.90) and (14.91) then they become*

$$t_i = \frac{\mu_i}{\mu_j}t_j,\ t_j = \frac{\mu_j}{\mu_i}t_i \tag{14.94}$$

due to (14.84) and (14.89). The transformation (14.60) implies also the equations (14.94) that express such change of the time scales regardless of the movements of the spatial frames R_i^n and R_j^n.

When we replace $t_{(.)}$ by $(v_P^{(.)})^{-1}r_P(t_{(.)})$ in (14.92) and (14.93) then they reduce to

$$\mathbf{r}_P(t_i) = \frac{\mu_i}{\mu_j}\mathbf{r}_P(t_j),\ \mathbf{r}_P(t_j) = \frac{\mu_j}{\mu_i}\mathbf{r}_P(t_i). \tag{14.95}$$

The numerical values of the distances of the arbitrary point P from the origin O_i of R_i^n and from the origin O_j of R_j^n change in the same ratio as the numerical values of the temporal coordinates do in spite of a mutual relative movement of the frames R_i^n and R_j^n. The adjustment of space scaling coefficients $\lambda_{ji} = \lambda_{ij}$, (14.81), to the velocities $\mathbf{v}_P^i$ and $\mathbf{v}_P^j$ of the arbitrary point P relative to R_i^n and R_j^n, respectively, ensures such property of (14.92) and (14.93).

The equations (14.94) and (14.95) show that all the velocities, not only the light velocity, are invariant under the coordinate transformations (14.90) through (14.93),

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P(t_i)}{dt_i} = \frac{\left\{d\left[\frac{\mu_i}{\mu_j}\mathbf{r}_P(t_j)\right]\right\}/dt_j}{\left[d\left(\frac{\mu_i}{\mu_j}t_j\right)\right]/dt_j} = \mathbf{v}_P^j,$$
$$\mathbf{v}_P^j = \frac{d\mathbf{r}_P(t_j)}{dt_j} = \frac{\left\{d\left[\frac{\mu_j}{\mu_i}\mathbf{r}_P(t_i)\right]\right\}/dt_i}{\left[d\left(\frac{\mu_j}{\mu_i}t_i\right)\right]/dt_i} = \mathbf{v}_P^i. \tag{14.96}$$

Neither Galilean - Newtonian physics nor Einsteinian relativity theory has treated this special case.

Singular case

Corollary 543 *Let the time scaling coefficient μ_i be defined by (14.60). Let $A = B$ be a priory accepted in D, (14.8). In order for the scaling coefficients*

α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji} = \alpha$, λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji} = \lambda$, to be positive real numbers and to obey (14.61) through (14.65), and for (14.60) through (14.65) to imply (14.8) it is necessary and sufficient that the relationships (14.97) through (14.100),

$$v_{[.]}^i = v_{[.]}^j = v_{[.]}^{ij} = v_{[.]},\ [.] \in \{P, R\},\ q^i w^i = q^j w^j = qw = v_P v_R,$$
$$0 \leq v_{ji}^i = \vartheta^i = v_{ji}^j = \vartheta^j = v_{ji} = \vartheta = -v_{ij} = v < v_P, \tag{14.97}$$

$$c_j^j = c_i^i = c_{ij} = c_{ji} = c,\ \vartheta v_P v_R = vqw \tag{14.98}$$

$$\alpha = \lambda = \frac{1}{\sqrt{1 - \left(\frac{\vartheta v_R}{qw}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{v}{v_P}\right)^2}}, \tag{14.99}$$

$$\mu_j = \mu_i \sqrt{\frac{1 - \frac{v}{v_P}}{1 + \frac{v}{v_P}}}, \tag{14.100}$$

hold for any choice of the time scaling coefficient $\mu_i \in R^+$. The equations (14.61) through (14.64) become the equations (14.101) through (14.104):

$$t_i = \frac{t_j + \frac{\vartheta}{qw} r_R(t_j)}{\sqrt{1 - \left(\frac{\vartheta v_R}{qw}\right)^2}}, \tag{14.101}$$

$$t_j = \frac{t_i - \frac{\vartheta}{qw} r_R(t_i)}{\sqrt{1 - \left(\frac{\vartheta v_R}{qw}\right)^2}}, \tag{14.102}$$

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + v t_j \mathbf{u}}{\sqrt{1 - \left(\frac{v}{v_P}\right)^2}}, \tag{14.103}$$

$$\mathbf{r}_P(t_j) = \frac{\mathbf{r}_P(t_i) - v t_i \mathbf{u}}{\sqrt{1 - \left(\frac{v}{v_P}\right)^2}}. \tag{14.104}$$

They are partially both entirely and pairwise compatible transformations.

Note 544 *If we set $v_P = v_R = c$, $qw = c^2$ and $\vartheta = v$ in the transformations (14.101) through (14.104) then they become Lorentz transformations (7.20) through (7.23). This shows once more that Lorentz transformations do not, and cannot, prove the wrong claim that time depends on space.*

Note 545 *If we set $r_P(t_{(.)}) \equiv v_P t_{(.)}$ in (14.101) and (14.102) then they become the equations (14.94) due to (14.97) and (14.100).*

When we replace $t_{(.)}$ by $(v_P^{(.)})^{-1} r_P(t_{(.)})$ in (14.103) and (14.104) then they reduce to the equations (14.95).

The equations (14.97) and (14.98) show that all the velocities, not only the light velocity, are invariant under the coordinate transformations (14.101) through (14.104).

14.5.2 Velocity transformations

General case

Theorem 546 *Let the time scaling coefficient* μ_i *be defined by (14.60). Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i \neq \alpha_i^j$, λ_j^i *and* λ_i^j, $\lambda_j^i \neq \lambda_i^j$, *be positive real numbers and obey (14.61) through (14.65), and let (14.60) through (14.65) imply (14.8). Then, a constant nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the corresponding constant nonzero velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i}\frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j}{v_P^j}} = \frac{\mu_j}{\mu_i}\mathbf{v}_P^j,\ \mathbf{v}_P^j = \frac{\mu_i}{\mu_j}\frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{1-\frac{v_{ji}^i}{v_P^i}} = \frac{\mu_i}{\mu_j}\mathbf{v}_P^i. \tag{14.105}$$

These transformations are completely compatible.

Proof. *Necessity and sufficiency.* Let all the conditions of the theorem statement be satisfied. The equations (14.66) through (14.69) and the transformations (14.72) through (14.75) hold (Theorem 532). The equations (14.72), (14.74), $\mathbf{v}_P^{(.)} = v_P^{(.)}\mathbf{u}$, and $\mathbf{v}_{ji}^{(.)} = v_{ji}^{(.)}\mathbf{u}$, yield the following:

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P(t_i)}{dt_i} = \frac{d\left\{\frac{\mathbf{r}_P(t_j)+ v_{ji}^j t_j\mathbf{u}}{1+\frac{v_{ji}^j}{v_P^j}}\right\} : dt_j}{d\left\{\frac{\mu_i}{\mu_j}\frac{t_j+\frac{\vartheta^j}{q^j w^j}r_R(t_j)}{1+\frac{\vartheta^j v_R^j}{q^j w^j}}\right\} : dt_j} =$$

$$= \frac{\mu_j}{\mu_i}\frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j}{v_P^j}} = \frac{\mu_j}{\mu_i}\frac{1+\frac{v_{ji}^j}{v_P^j}}{1+\frac{v_{ji}^j}{v_P^j}}\mathbf{v}_P^j = \frac{\mu_j}{\mu_i}\mathbf{v}_P^j.$$

This result proves the first two equations in (14.105). The other two equations (14.105) are analogously proved by starting with (14.73) and (14.75).

Compatibility. Compatibility between the equations for $\mathbf{v}_P^i$ and $\mathbf{v}_P^j$ in (14.105) is evident. It is complete because they are compatible for any nonzero value of the speed of the arbitrary point P. Q. E. D ■

Remark 547 ***Noninvariance of the light speed***

This theorem is valid for an arbitrary velocity of the arbitrary point P. *It is true also for light signals, i.e. for* $P = L$. *The equations (14.105) verify that the light speed is not invariant in vacuum with respect to all integral spaces that incorporate inertial spatial frames. Besides, they show that there is not any constraint on the velocity of the arbitrary point* P. *The light speed value is not the limiting one. The value of the speed of the arbitrary point* P *can be greater than the light speed value.*

Comment 548 *The complete entire and pairwise compatibility of the coordinate transformations (14.72) through (14.75) ensure the complete compatibility of the velocity transformations (14.105).*

Special case

Theorem 549 *Let the time scaling coefficient* μ_i *be defined by (14.60). Let the scaling coefficients* α_j^i, α_i^j, $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, λ_j^i *and* λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, *be positive real numbers and obey (14.61) through (14.65), and let (14.60) through (14.65) imply (14.8) for* $A = B$. *Then, a constant nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the corresponding constant nonzero velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated as follows:*

$$\mathbf{v}_P^i = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{1 + \frac{v_{ji}^j}{v_P^j}} = \mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{1 - \frac{v_{ji}^i}{v_P^i}} = \mathbf{v}_P^{ij} = \mathbf{v}_P^{ji}. \tag{14.106}$$

These transformations are only partially compatible.

The proof is analogous to the proof of Theorem 526.

Singular case

The value of the speed of the arbitrary point is independent of a choice of the *time* axis in the singular case,

$$\mathbf{v}_P^i = \mathbf{v}_P^j = \mathbf{v}_P^{ij} = \mathbf{v}_P^{ji} \equiv \mathbf{v}_P. \tag{14.107}$$

The velocity transformations are trivial and partially compatible.

Remark 550 ***Every velocity is invariant in both the special and the singular case of the transformations. The light velocity is not any exception.***

The results (14.106) and (14.107) show that every velocity is invariant relative to integral spaces if the coordinate transformations among them obey Theorem 541.

Lorentz - Einstein invariance of the light speed is not any exception.

These explanations agree fully with Theorem 139, the Corollary 275, Remarks 276, and 419, Theorem 475, Remarks 477, 480, 494, 496, and 499.

14.6 Uniformity

Note 551 *The equation (14.60) is the same as the equation (10.60). The transformations (10.61) through (10.64) are a special form of the transformations (14.61) through (14.64) for* $r_R(t_{(.)}) \equiv r_L(t_{(.)}) \equiv c_{(.)}^{(.)} t_{(.)}$ *and* $v_R^{(.)} \equiv c_{(.)}^{(.)}$, *or* $v_R^{ji} \equiv c_{ji}$. *With these simplifications Section 14.5 becomes this section. The general uniformity of the transformations then reduces to their uniformity in the framework of CC relativity theory.*

14.7 Special uniformity

Note 552 *The equation (10.55) is the equation (14.60). The transformations (14.61) through (14.64) reduce to the transformations (10.56) through (10.59) for* $r_R(t_{(.)}) \equiv r_L(t_{(.)}) \equiv c_{(.)}^{(.)} t_{(.)}$ *and* $v_R^{(.)} \equiv c_{(.)}^{(.)}$, *or* $v_{Rji} \equiv c_{ji}$, $q^{(.)}w^{(.)} \equiv [c_{(.)}^{(.)}]^2$ *or* $(qw)^{ji} = c_{ji}^2$. *These simplifications transform Section 14.5 into this section. Then the general uniformity of the transformations takes the particular form of their special uniformity in the setting of CC relativity theory.*

14.8 General weak uniformity

Note 553 *The basic temporal coordinate transformation (10.49) is the same as 14.60. When we replace* $\vartheta^{(.)}$ *by* $v_R^{(.)}$, *in (14.61) and (14.62) then the transformations (14.61) through (14.64) become, respectively, (10.50) through (10.53), and Section 14.5 transforms into the section on the time - invariant general weak uniformity of the transformations in the area of CC relativity theory.*

14.9 Weak uniformity

Note 554 *The transformation (10.43) is equal to the transformation 14.60. When we replace* $\vartheta^{(.)}$ *by* $v_R^{(.)}$, *and* $r_R(t_{(.)})$ *by* $r_L(t_{(.)})$ *in (14.61) and (14.62) then the transformations (14.61) through (14.64) reduce, respectively, to the transformations (10.44) through (10.47), and Section 14.5 transforms into the section on the time - invariant weakly uniform transformations in the setting of CC relativity theory.*

14.10 Special weak uniformity

Note 555 *Since the transformation (10.38) is the same as the transformation 14.60, when we replace* $\vartheta^{(.)}$ *by* $v_R^{(.)}$, $r_R(t_{(.)})$ *by* $r_L(t_{(.)})$, *and* $q^{(.)}w^{(.)}$ *by* $\left[c_{(.)}^{(.)}\right]^2$ *in (14.61) and (14.62), then the equations (14.61) through (14.64) transform, respectively, to the equations (10.39) through (10.42), and Section 14.5 becomes the section on the time - invariant specially weakly uniform transformations in the framework of CC relativity theory.*

14.11 General relative uniformity

Note 556 *The basic temporal coordinate transformations (10.32) and (14.60) represent the same transformation. The basic generic transformations (10.33) through (10.36) result from (14.61) through (14.64) when we replace* $\vartheta^{(.)}$ *by* $v_{ji}^{(.)}$

in the temporal coordinate transformations (14.61) and (14.62). Such replacement transforms Section 14.5 into the section on the time - invariant generally relatively uniform transformations in the area of CC relativity theory.

14.12 Relative uniformity

Note 557 *The transformation (10.26) is the same as the transformation (14.60). When we replace* $\vartheta^{(.)}$ *by* $v_{ji}^{(.)}$*, and* $r_R(t_{(.)})$ *by* $r_L(t_{(.)})$ *in the temporal coordinate transformations (14.61) and (14.62) then they become (10.27) and (10.28). Such replacements transform Section 14.5 into the section on the time - invariant relatively uniform transformations in the setting of CC relativity theory.*

14.13 Special relative uniformity

Note 558 *There is not any difference between the transformation (10.21) and the transformation (14.60). We replace* $\vartheta^{(.)}$ *by* $v_{ji}^{(.)}$*,* $r_R(t_{(.)})$ *by* $r_L(t_{(.)})$*, and* $q^{(.)}w^{(.)}$ *by* $\left[c_{(.)}^{(.)}\right]^2$ *in the temporal coordinate transformations (14.61) and (14.62). The results are the transformations (10.22) and (10.23). Such replacements transform Section 14.5 into the section on the time - invariant specially relatively uniform transformations in the framework of CC relativity theory.*

14.14 Conclusion on colinear motions

Complete (entire and pairwise) compatibility of the temporal coordinate transformations and of the spatial coordinate transformations, as well as of the velocity transformations, requires the *time* scaling coefficients and space scaling coefficients to be pairwise mutually different. This agrees with the *time* independence of space (Axiom 47).

The expressions for the scaling coefficients, hence, the final forms of the transformations, are essentially different from the forms of the corresponding Lorentz - Einstein - Poincaré scaling coefficients and transformations. The former introduce relative values of the transfer speeds and of the speed of light with respect to an accepted integral space, while the latter do not. They express the consistent use of the values of all the speeds in the transformations.

The new results established herein do not contain square roots and squared quotients of the values of speeds in the general case. They permit the temporal transfer speed to be different from the spatial transfer speed in the general case. Hence, they reflect the *time* independence of space also in this way, while Lorentz transformations do not.

The new general results show that a point different from the light signal, hence, its speed different from the light speed, can be used as the reference point and the reference speed. Consequently, the values of the transfer speeds need not be bounded by the light speed value if the arbitrary point P and the reference point P_R are not a light signal. The light signal L, hence its speed, represents just a special choice that has been a priory accepted in the Einsteinian relativity theory.

The new formulae on the velocity transformations are essentially different from both Galilean - Newtonian law and Einstein's law of the composition of velocities. They are completely compatible. They do not restrict the value of the speed of the arbitrary point P. It can be bigger than the light speed value. Moreover, they prove the noninvariance of the value of the light speed in vacuum relative to integral spaces that incorporate inertial spatial frames.

The new results in the special case and in the singular one do not preserve complete both entire and pairwise compatibility of the coordinate and of velocity transformations. They are intentionally presented in this framework of the completely compatible transformations in order to show this their essential drawback that is a consequence of the a priory accepted condition of the equality of both the *time* scaling coefficients and space scaling coefficients. The acceptance of the condition on the equality of the scaling coefficients in the transformations of one coordinate system to another one, and vice versa, corresponds essentially to the condition of the equality between a transformation and its inverse. Such condition is very restrictive and does not have any physical justification in the general framework of coordinate transformations. In view of all this we will not treat the special and the singular case in the sequel.

The new results are established for an arbitrary constant speed of the arbitrary point P. Therefore, the problem of acceleration transformations does not exist since acceleration of the arbitrary point then equals zero vector. In view of the constancy of the speed of the arbitrary point P, it results that completely entirely compatible *time*-invariant coordinate transformations, like Lorentz transformations, are ordinary homogeneous linear transformations with constant gains (for details see Subsection 7.2.2: "Homogenous forms of Lorentz transformations" in Section 7.2).

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Chapter 15

Noncolinear Motions: Transformations

15.1 Generic forms

15.1.1 Vector variables and *time* scales

Every entry of the n-dimensional vector variables $\mathbf{a}(.)$, $\mathbf{g}(.)$, $\mathbf{p}(.)$, $\mathbf{r}(.)$, $\mathbf{v}(.)$, $\boldsymbol{\vartheta}(.)$, $\mathbf{w}(.)$, $\mathbf{x}(.)$, can be, in principle, measured relative to a *time* axis associated with that entry. This characterizes complex processes and systems (automatic control systems of aircrafts, of power stations, of ships, and of robots; biological control systems of beings, and control systems of economic processes). The dimension of the state vector $\mathbf{x}$ of such systems can be (much) greater than three. This holds also for the vector physical variable $\mathbf{g}(.)$ and for the vector variable $\mathbf{p}(.)$ composed of different physical variables. We will use $\mathbf{r}(.)$ in the broad sense so that it can represent, and can be replaced by $\mathbf{g}(.)$ or $\mathbf{p}(.)$ or $\mathbf{x}(.)$. This means that what follows is valid for complex processes and systems whatever is their nature.

15.1.2 Notational preliminaries

For the explication of the notation in the scalar cases, and in the vector cases when the same *time* scale and unit hold for all entries of a vector (variable), see Subsections "*Time* axes and space" in 4.6, "*Time*, relative velocities and their values" in 4.7, and "*Time*, velocity and generic transformations" in 6.2. We should refine the notation in more details due to the existence of different *time* scales and units for different entries of a vector (variable).

Let us remind ourselves that all the entries of the unity vector $\mathbf{u}$ are constant and different from zero,

$$\mathbf{u} = (u_1 \; u_2 \; ... \; u_n)^T, \; u_i = const. \neq 0, \; \forall i = 1, 2, ..., n, \; \|\mathbf{u}\| = 1.$$

The use of different, mutually independent, scales and units for different entries of the vector variables requires us to replace in the coordinate transformations:

° the scalar *time* scaling coefficients α_i^j and α_j^i by *the diagonally positive time scaling diagonal matrices* A_i^j *and* A_j^i,

$$\mathrm{A}_i^j = diag\left\{\alpha_{i_1}^{j_1} \quad \alpha_{i_2}^{j_2} \quad ... \quad \alpha_{i_n}^{j_n}\right\} \in R_+^{nxn},\ \alpha_{i_k}^{j_k} \in R^+,$$
$$\mathrm{A}_j^i = diag\left\{\alpha_{j_1}^{i_1} \quad \alpha_{j_2}^{i_2} \quad ... \quad \alpha_{j_n}^{i_n}\right\} = R_+^{nxn},\ \alpha_{j_k}^{i_k} \in R^+,$$
$$i \leq j,\ i_k, j_k \in \{1, 2, ..., s\},\ i_k \leq j_k,\ k = 1, 2, .., n,$$

° the light speed relative value c_j^i by *the light speed (directional) matrix* $\mathrm{C}_j^i \in R^{+^{nxn}}$,

$$\mathrm{C}_j^i = diag\left\{c_{j_1}^{i_1} \quad c_{j_2}^{i_2} \quad ... \quad c_{j_n}^{i_n}\right\},\ c_{j_k}^{i_k}(.) \in R^+,$$
$$c_{j_k}^{i_k} = \frac{\varsigma_{j_k}^{i_k}}{u_{1k}},\ i_k, j_k \in \{1, 2, ..., s\},\ i_k \leq j_k,\ k = 1, 2, .., n,$$

where $\varsigma_{j_k}^{i_k}$ is the k-th entry of the representation of the vector $\mathbf{c}_j^i$ with respect to the unity basis $\{\mathbf{e}_{u1}, \mathbf{e}_{u2}, ..., \mathbf{e}_{un}\}$,

$$\mathbf{c}_j^i = \left(\varsigma_{j_1}^{i_1} \quad \varsigma_{j_2}^{i_2} \quad ... \quad \varsigma_{j_n}^{i_n}\right)^T = \varsigma_{j_1}^{i_1}\mathbf{e}_{u1} + \varsigma_{j_2}^{i_2}\mathbf{e}_{u2} + ... + \varsigma_{j_n}^{i_n}\mathbf{e}_{un},$$

° the light velocity $\mathbf{c}_j^i = c_j^i\mathbf{u}$ in the direction of the unity vector $\mathbf{u}$ by *the light speed vector* $\mathbf{c}_j^i \in R^{+^n}$ in an arbitrary direction determined by the matrix C_j^i relative to $\mathbf{u}$,

$$\mathbf{c}_j^i = \left(c_{j_1}^{i_1} \quad c_{j_2}^{i_2} \quad ... \quad c_{j_n}^{i_n}\right)^T = \mathrm{C}_j^i\mathbf{u},$$

° the scalar space scaling coefficients λ_i^j and λ_j^i by *the diagonally positive space scaling diagonal matrices* Λ_i^j *and* Λ_j^i,

$$\Lambda_i^j = diag\left\{\lambda_{i_1}^{j_1} \quad \lambda_{i_2}^{j_2} \quad ... \quad \lambda_{i_n}^{j_n}\right\} \in R_+^{nxn},\ \lambda_{i_k}^{j_k} \in R^+,$$
$$\Lambda_j^i = diag\left\{\lambda_{j_1}^{i_1} \quad \lambda_{j_2}^{i_2} \quad ... \quad \lambda_{j_n}^{i_n}\right\} \in R_+^{nxn},\ \lambda_{j_k}^{i_k} \in R^+,$$
$$i \leq j,\ i_k, j_k \in \{1, 2, ..., s\},\ i_k \leq j_k,\ k = 1, 2, .., n,$$

° the basic *time* scaling factors $\mu_{(.)}$ by *the diagonally positive time scaling diagonal matrices* $\mathrm{M}_{(.)}$,

$$\mathrm{M}_{(.)} = diag\left\{\mu_{(.)_1} \quad \mu_{(.)_2} \quad ... \quad \mu_{(.)_n}\right\} \in R_+^{nxn},\ \mu_{(.)_k} \in R^+,$$
$$(.)_k \in \{-, 1, 2, ..., s\},\ k = 1, 2, ..., n,\ \mathrm{M} \equiv I,$$

° *the (position) function* $r_{(..)}(.)$ *by the (position) matrix function* $\mathrm{R}_{(..)}(.)$,

$$\mathrm{R}_{(..)}(.) : \mathfrak{T}^n \mathsf{x} .... \longrightarrow R^{nxn},$$

which determines in general the vector $\mathbf{r}_{(..)}(\mathbf{t}^n_{(.)})$ of a point (..) relative to the unity vector $\mathbf{u}$, in spite they are not colinear in general, $(..) \in \{G, L, P, P_R, P_{SU}\}$,

$$\mathrm{R}_{(..)}(\mathbf{t}^n_{(.)}) = diag\left\{r_{(..)_1}(t_{(.)_1}) \quad r_{(..)_2}(t_{(.)_2}) \ ... \ r_{(..)_n}(t_{(.)_n})\right\},$$
$$r_{(..)_k}(t_{(.)_k}) \equiv \frac{\rho_{(..)k}(t_{(..)_k})}{u_{1k}},\ (.)_k \in \{-, 1, 2, ..., s\},\ k = 1, 2, ..., n,$$

where $\rho_k(t_{(.)_k})$ is the k-th entry of the representation of the vector $\mathbf{r}_{(.)}(\mathbf{t}^n_{(.)})$ with respect to the basis $\{\mathbf{e}_{u1}, \mathbf{e}_{u2}, ..., \mathbf{e}_{un}\}$ given by

$$\mathbf{r}_{(..)}(\mathbf{t}^n_{(.)}) = \rho_{(..)_1}(t_{(.)_1})\mathbf{e}_{u1} + \ \rho_{(..)_2}(t_{(.)_2})\mathbf{e}_{u2} + ... \ \rho_{(..)_n}(t_{(.)_n})\mathbf{e}_{un} =$$
$$= (\rho_{(..)_1}(t_{(.)_1}) \quad \rho_{(..)_2}(t_{(.)_2}) \ ... \ \rho_{(..)_n}(t_{(.)_n}))^T = \mathrm{R}_{(..)}(\mathbf{t}^n_{(.)})\mathbf{u}.$$

If, and only if, additionally, the vectors $\mathbf{r}_{(..)}(\mathbf{t}^n_{(.)})$ and $\mathbf{u}$ are colinear, and if $\mathbf{t}^n_{(.)} = t_{(.)}\mathbf{u}$, then

$$\mathrm{R}_{(..)}(\mathbf{t}^n_{(.)}) \equiv r_{(..)}(t_{(.)})I,\ \mathbf{t}^n_{(.)} = t_{(.)}\mathbf{u},\ r_{(..)}(t_{(.)}) = \rho_{(..)}(t_{(.)}).$$

Besides, we will use in general

° the formally mathematically introduced n-dimensional *temporal (* or*, time) vector* $\mathbf{t}^n_{(.)}$ (in $\mathfrak{T}^n_{(.)} \cup R^n$) , the entries of which are the *time* values measured relative to

$$\mathfrak{T}^n_{(.)} = \mathfrak{T}_{(.)_1} \times \mathfrak{T}_{(.)_2} \times ... \times \mathfrak{T}_{(.)_n},$$

i.e. its k-th entry is measured relative to $\mathfrak{T}_{(.)_k}$, where $(.)_k \in \{-, 1, 2, ..., s\}$, $k = 1, 2, ..., n$, and $(.)_l \neq (.)_m$ for $l \neq m$ in general, but $(.)_l = (.)_m$ is permitted for $l \neq m$,

$$\mathbf{t}^n_{(.)} = diag\{t_{(.)_1} \quad t_{(.)_2} \quad ... \quad t_{(.)_n}\}\mathbf{u} = \mathrm{T}_{(.)}\mathbf{u} = t\mathrm{M}_{(.)}\mathbf{u},\ \mathbf{t}^n = \mathrm{T}\mathbf{u} = tI\mathbf{u} = t\mathbf{u},$$

$\mathbf{t}^n_{(.)0}$ denotes the initial $\mathbf{t}^n_{(.)}$; besides, we use $\mathbf{t}^n_{(.)}$ for $\mathbf{t}^n_{(.)}(...)$ in order to emphasize the independent nature of *time* t,

° *the temporal matrix function*

$$\mathrm{T}_{(.)}(..) : \mathfrak{T}^n_{(.)} \times .... \times R^{(...)} \longrightarrow \mathfrak{T}^{nxn}_{(.)} \cup R^{nxn},$$

which determines in general the *time* vector $\mathbf{t}^n_{(.)}$ relative to the unity vector $\mathbf{u}$,

$$\mathbf{t}^n_{(.)} = \mathrm{T}_{(.)}\mathbf{u},\ \mathrm{T}_{(.)} = diag\left\{t_{(.)_1} \quad t_{(.)_2} \quad ... \quad t_{(.)_n}\right\} =$$
$$= t\mathrm{M}_{(.)} = diag\left\{\mu_{(.)_1}t \quad \mu_{(.)_2}t \quad ... \quad \mu_{(.)_n}t\right\},$$
$$(.)_k \in \{-, 1, 2, ..., s\},\ k = 1, 2, ..., n,\ s \in \{1, 2, ..., n\},\ \mathrm{T} = tI,$$

and

° the *velocity matrix*

$$\mathrm{V}_{(..)}(\mathbf{t}^n_{(.)}) \in R^{nxn},$$

which determines in general the velocity $\mathbf{v}_{(..)}(\mathbf{t}^n_{(.)})$ of a point (..) relative to the unity vector $\mathbf{u}$,

$$\mathbf{v}_{(..)}(\mathbf{t}^n_{(.)}) = \left(v_{(..)1}(t_{(.)_1}) \quad v_{(..)2}(t_{(.)_2}) \ ... \ v_{(..)n}(t_{(.)_n})\right)^T = \mathrm{V}_{(..)}(\mathbf{t}^n_{(.)})\mathbf{u} \Longrightarrow$$
$$\mathrm{V}_{(..)}(\mathbf{t}^n_{(.)}) = diag\left\{v_{(..)1}(t_{(.)_1}) \quad v_{(..)2}(t_{(.)_2}) \ ... \ v_{(..)n}(t_{(.)_n})\right\} \equiv \mathrm{V}^{(.)}_{(..)\mathbf{t}},$$
$$\left|\mathrm{V}_{(..)}(\mathbf{t}^n_{(.)})\right| = diag\left\{\left|v_{(..)1}(t_{(.)_1})\right| \quad \left|v_{(..)2}(t_{(.)_2})\right| \ ... \ \left|v_{(..)n}(t_{(.)_n})\right|\right\},$$
$$v_{(..)k} \equiv \frac{v^e_{(..)k}}{u_{1k}}, \ (.)_k \in \{-,1,2,...,s\}, \ k \in \{1,2,...,n\},$$

where $v^e_{(..)k}$ is the k-th entry of the representation of the vector $\mathbf{v}_{(..)}(\mathbf{t}^n_{(.)})$ with respect to the basis $\{\mathbf{e}_{u1},\ \mathbf{e}_{u2},\ ...,\ \mathbf{e}_{un}\}$, which is determined by

$$\mathbf{v}_{(..)}(\mathbf{t}^n_{(.)}) = v^e_{(..)1}(t_{(.)_1})\mathbf{e}_{u1} + v^e_{(..)2}(t_{(.)_2})\mathbf{e}_{u2} + ... + v^e_{(..)n}(t_{(.)_n})\mathbf{e}_{un} =$$
$$= \left(v^e_{(..)1}(t_{(.)_1}) \quad v^e_{(..)2}(t_{(.)_2}) \ ... \ v^e_{(..)n}(t_{(.)_n})\right)^T.$$

If, and only if the vectors $\mathbf{v}_{(..)}(\mathbf{t}^n_{(.)})$ and $\mathbf{u}$ are colinear and $\mathbf{t}^n_{(.)} = t_{(.)}\mathbf{u}$ then

$$\mathrm{V}_{(..)}(\mathbf{t}^n_{(.)}) \equiv v_{(..)}(t_{(.)})I.$$

In special cases we find the following.

If $\mathbf{v}(\mathbf{t}^n_{(.)}) \equiv \mathbf{q}(\mathbf{t}^n_{(.)})$ then

$$\mathbf{q}(\mathbf{t}^n_{(.)}) = \left(q^{(.)}_1(t_{(.)_1}) \quad q^{(.)}_2(t_{(.)_2}) \ ... \ q^{(.)}_n(t_{(.)_n})\right)^T = \mathrm{Q}^{(.)}(\mathbf{t}^n_{(.)})\mathbf{u}\ ,$$

where

$$\mathrm{Q}^{(.)}(\mathbf{t}^n_{(.)}) = diag\left\{q^{(.)}_1(t_{(.)_1}) \quad q^{(.)}_2(t_{(.)_2}) \ ... \ q^{(.)}_n(t_{(.)_n})\right\} \equiv \mathrm{Q}^{(.)}_{\mathbf{t}},$$
$$q^{(.)}_k(t_{(.)_k}) \equiv \frac{\theta^{(.)}_k(t_{(.)_k})}{u_{1k}}, \ (.)_k \in \{-,1,2,...,s\}, \ k = 1,2,..,n, \ s \in \{1,2,...,n\},$$

and $\theta^{(.)}_k(t_{(.)_k})$ is the k-th entry of the representation of $\mathbf{q}(\mathbf{t}^n_{(.)})$ relative to the unity basis $\{\mathbf{e}_{u1},\ \mathbf{e}_{u2},\ ...,\ \mathbf{e}_{un}\}$,

$$\mathbf{q}_{(.)}(\mathbf{t}^n_{(.)}) = \theta_1(t_{(.)_1})\mathbf{e}_{u1} + \theta_2(t_{(.)_2})\mathbf{e}_{u2} + ... + \theta_n(t_{(.)_n})\mathbf{e}_{un} =$$
$$= \left(\theta_1(t_{(.)_1}) \quad \theta_2(t_{(.)_2}) \ ... \ \theta_n(t_{(.)_n}\right)^T.$$

If $\mathbf{v}(\mathbf{t}^n_{(.)}) \equiv \boldsymbol{\vartheta}(\mathbf{t}^n_{(.)})$ then

$$\boldsymbol{\vartheta}(\mathbf{t}^n_{(.)}) = \left(\vartheta^{(.)}_1(t_{(.)_1}) \quad \vartheta^{(.)}_2(t_{(.)_2}) \ ... \ \vartheta^{(.)}_n(t_{(.)_n})\right)^T = \mathrm{V}^{(.)}_{\vartheta}(\mathbf{t}^n_{(.)})\mathbf{u}\ ,$$
$$\mathrm{V}^{(.)}_{\vartheta}(\mathbf{t}^n_{(.)}) = diag\left\{\vartheta^{(.)}_1(t_{(.)_1}) \quad \vartheta^{(.)}_2(t_{(.)_2}) \ ... \ \vartheta^{(.)}_n(t_{(.)_n}))\right\} \equiv \mathrm{V}^{(.)}_{\vartheta\mathbf{t}}.$$

If $\mathbf{v}(\mathbf{t}^n_{(.)}) \equiv \mathbf{w}^{(.)}(\mathbf{t}^n_{(.)})$ then

$$\mathbf{w}^{(.)}(\mathbf{t}^n_{(.)}) = \left(w^{(.)}_1(t_{(.)_1}) \quad w^{(.)}_2(t_{(.)_2}) \; ... \; w^{(.)}_n(t_{(.)_n})\right)^T \equiv \mathrm{W}^{(.)}(\mathbf{t}^n_{(.)})\mathbf{u} \ ,$$

where

$$\mathrm{W}^{(.)}(\mathbf{t}^n_{(.)}) = diag\left\{w^{(.)}_1(t_{(.)_1}) \quad w^{(.)}_2(t_{(.)_2}) \; ... \; w^{(.)}_n(t_{(.)_n})\right\} \equiv \mathrm{W}^{(.)}_{\mathbf{t}},$$

$$w^{(.)}_k(t) \equiv \frac{\omega^{(.)}_k(t_{(.)_k})}{u_{1k}}, \ k \in \{1, 2, .., n\}\ , \ (.)_k \in \{-, 1, 2, ..., s\}\ , \ k \in \{1, 2, .., n\}\ ,$$

and $\omega^{(.)}_k(t_{(.)_k})$ is the k-th entry of the representation of the vector $\mathbf{w}_{(..)}(\mathbf{t}^n_{(.)})$ with respect to the basis $\{\mathbf{e}_{u1},\ \mathbf{e}_{u2},\ ...,\ \mathbf{e}_{un}\}$,

$$\mathbf{w}^{(.)}(\mathbf{t}^n_{(.)}) = \omega^{(.)}_1(t_{(.)_1})\mathbf{e}_{u1} + \omega^{(.)}_2(t_{(.)_2})\mathbf{e}_{u2} + ... + \omega^{(.)}_n(t_{(.)_n})\mathbf{e}_{un} =$$
$$= \left(\omega^{(.)}_1(t_{(.)_1}) \quad \omega^{(.)}_2(t_{(.)_2}) \; ... \; \omega^{(.)}_n(t_{(.)_n})\right)^T .$$

The *time* independent scaling matrix $\mathrm{M}_{(.)}$ can be determined in general by the initial matrix value $\mathrm{R}_{G\mathbf{0}}$ of the initial position vector $\mathbf{r}_{G\mathbf{0}}$ (relative to $\mathbf{u}$) of the generic point G, $\mathbf{r}_{G\mathbf{0}} = \mathrm{R}_{G\mathbf{0}}\mathbf{u}$,

$$\mathbf{t}^n_i = \mathrm{M}_i(\mathrm{R}_{G\mathbf{0}})\mathbf{t}^n,\ \mathrm{M}_i(\mathrm{R}_{G\mathbf{0}}) \in R^{n\mathrm{x}n}_+,\ \mathbf{t}^n_{i0} = \mathrm{M}_i(\mathrm{R}_{G\mathbf{0}})\mathbf{t}^n_0,\ i \in \{-, 1, 2,\ ...\ , s\}\,. \tag{15.1}$$

We accept, if it is not otherwise stated, $\mathbf{r}_{G\mathbf{0}} = \mathrm{R}_{G\mathbf{0}}\mathbf{u} = \mathbf{0}$ for the sake of simplicity so that $\mathrm{M}_{(.)}(\mathrm{R}_{G\mathbf{0}}) = \mathrm{M}_{(.)}$.

15.1.3 Generic coordinate transformations

The basic generic *time*-invariant vector-matrix coordinate transformations are defined by

$$\mathbf{t}^n_i = \mathrm{A}^i_j\left[\mathbf{t}^n_j + \mathrm{V}^j_\vartheta\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathbf{r}_G(\mathbf{t}^n_j)\right],\ diagmin\mathrm{A}^i_j \in R^+, \tag{15.2}$$

$$\mathbf{t}^n_j = \mathrm{A}^j_i\left[\mathbf{t}^n_i - \mathrm{V}^i_\vartheta\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\mathbf{r}_G(\mathbf{t}^n_i)\right],\ diagmin\mathrm{A}^j_i \in R^+, \tag{15.3}$$

$$\mathbf{r}_P(\mathbf{t}^n_i) = \Lambda^i_j\left[\mathbf{r}_P(\mathbf{t}^n_j) + \mathrm{V}^j_{ji}\mathbf{t}^n_j\right],\ diagmin\Lambda^i_j \in R^+, \tag{15.4}$$

$$\mathbf{r}_P(\mathbf{t}^n_j) = \Lambda^j_i\left[\mathbf{r}_P(\mathbf{t}^n_i) - \mathrm{V}^i_{ji}\mathbf{t}^n_i\right],\ diagmin\Lambda^j_i \in R^+, \tag{15.5}$$

where

$$G \in \{L, P, P_R, P_{SU}\}\ \textit{ is permitted in special cases},$$
$$\mathbf{q}^{(.)} \in R^{+^n},\ \mathbf{w}^{(.)} \in R^{+^n}, \tag{15.6}$$

and *diagmin*(.) is the minimal diagonal entry of a matrix (.).

15.2 General nonuniformity

15.2.1 Transformations of temporal and spatial coordinates

Basic relationships

The basic *time* scaling matrix coefficient M_i is defined by (15.7),

$$\mathbf{t}_i^n = \mathrm{M}_i \mathbf{t}^n,\ \mathbf{t}_{i0}^n = \mathrm{M}_i \mathbf{t}_0^n = \mathrm{M}_i \mathbf{0} = \mathbf{0},\ \mathrm{M}_i \in R_+^{nxn},\ diagmin\ \mathrm{M}_i \in R^+. \tag{15.7}$$

It is elementwise constant. Its diagonal entries are positive real numbers.

The basic general generic equations (15.2) through (15.5) reduce to

$$\mathbf{t}_i^n = \mathrm{A}_j^i \left[\mathbf{t}_j^n + \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \mathbf{r}_P(\mathbf{t}_j^n)\right],$$
$$\mathrm{A}_j^i \in R_+^{nxn},\ diagmin \mathrm{A}_j^i \in R^+,\ \mathbf{q}^j \in R^{+^n}, \mathbf{w}^j \in R^{+^n},\ \mathrm{V}_{ji}^j \in R_+^{nxn}, \tag{15.8}$$

$$\mathbf{t}_j^n = \mathrm{A}_i^j \left[\mathbf{t}_i^n - \mathrm{V}_{ji}^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1} \mathbf{r}_P(\mathbf{t}_i^n)\right],$$
$$\mathrm{A}_i^j \in R_+^{nxn},\ diagmin \mathrm{A}_i^j \in R^+,\ \mathbf{q}^i \in R^{+^n}, \mathbf{w}^i \in R^{+^n},\ \mathrm{V}_{ji}^i \in R_+^{nxn}, \tag{15.9}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \Lambda_j^i \left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j \mathbf{t}_j^n\right],\ \Lambda_j^i \in R_+^{nxn},\ diagmin \Lambda_j^i \in R^+, \tag{15.10}$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \Lambda_i^j \left[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}_{ji}^i \mathbf{t}_i^n\right],\ \Lambda_i^j \in R_+^{nxn},\ diagmin \Lambda_i^j \in R^+, \tag{15.11}$$

where

$$\mathbf{q}^{(.)}, \mathbf{w}^{(.)} = \mathbf{c}_{(.)}^{(.)}\ \textit{are allowed but not simultaneously},$$
$$\textit{i.e.}\ \mathrm{Q}^{(.)} \mathrm{W}^{(.)} \neq \left[\mathrm{C}_{(.)}^{(.)}\right]^2,$$
$$\mathbf{v}_P^{(.)} \neq \mathbf{0}\ \textit{elementwise permits}\ \mathbf{q}^{(.)}, \mathbf{w}^{(.)} = \mathbf{v}_P^{(.)}. \tag{15.12}$$

Either $\mathbf{q}^{(.)}$ or $\mathbf{w}^{(.)}$, but not both, can be identically equal to the light velocity $\mathbf{c}_{(.)}^{(.)}$. They can be equal to the velocity $\mathbf{v}_P^{(.)}$ of the arbitrary point P if, and only if, $\mathbf{v}_P^{(.)}$ is elementwise nonzero.

Note 559 ***Characteristics***

The vector - matrix coordinate transformations (15.8) through (15.12) retain all the characteristics of the transformations (10.16) through (10.20).

Definition 560 ***Nonuniformity of the time field***

The transformations (15.7) through (15.12) determine ***time-invariant generally nonuniform time field*** *if, and only if, the transformations hold for every pair of the time axes including those from different layers in the case the time field is multi-layer.*

The condition for the preservation of the generalized length in integral spaces will be used in its general vector – matrix *time* - invariant form (15.13),

$$\begin{bmatrix} \mathbf{r}_P(\mathbf{t}_i^n) \\ \mathrm{V}_P^i \mathbf{t}_i^n \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(\mathbf{t}_i^n) \\ \mathrm{V}_P^i \mathbf{t}_i^n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_P(\mathbf{t}_j^n) \\ \mathrm{V}_P^j \mathbf{t}_j^n \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P(\mathbf{t}_j^n) \\ \mathrm{V}_P^j \mathbf{t}_j^n \end{bmatrix},$$
$$D = blockdiag\{A \quad -B\} \in R^{2n\mathrm{x}2n},$$
$$A \in R^{n\mathrm{x}n} \ and\ B \in R^{n\mathrm{x}n} \ are\ positive\ definite. \tag{15.13}$$

The position vector $\mathbf{r}_P(\mathbf{t}_{(.)}^n)$ reflects the instantaneous position of the arbitrary point P relative to the origin $O_{(.)}$ of $R_{(.)}^n$ at a vector moment $\mathbf{t}_{(.)}^n$ provided the initial vector instant was $\mathbf{t}_{(.)0}^n = \mathbf{0}$. Its variation is expressed in terms of the instantaneous average speed matrix $\mathrm{V}_P^{(.)}$ of the same point P and the *time* vector $\mathbf{t}_{(.)}^n$.

Note 561 *If we replace* $Q^{(.)}W^{(.)}$ *by* $\left[\mathrm{V}_P^{(.)}\right]^2$ *everywhere then the general nonuniformity of the transformations reduces to their nonuniformity, i.e. this section then represents the section on the time-invariant nonuniform vector - matrix transformations.*

Note 562 *If we restrict* $Q^{(.)}W^{(.)}$ *to obey* $Q^{(.)}W^{(.)} \notin \left\{\left[\mathrm{C}_{(.)}^{(.)}\right]^2, \left[\mathrm{V}_P^{(.)}\right]^2\right\}$ *then this section becomes the section on the time-invariant weakly nonuniform vector - matrix transformations.*

Solutions for the general case

Theorem 563 *Let the scaling diagonal matrix factor* $\mathrm{M}_i \in R_+^{n\mathrm{x}n}$, *diagmin*$\mathrm{M}_i \in R^+$, *be defined by (15.7). In order for the diagonal matrix scaling coefficients* A_j^i, A_i^j, $\mathrm{A}_j^i \neq \mathrm{A}_i^j$, Λ_j^i, Λ_i^j, $\Lambda_j^i \neq \Lambda_i^j$, *to be diagonally elementwise constant and positive real valued, to obey (15.8) through (15.12), and for (15.7) through (15.12) to imply (15.13) it is necessary and sufficient that both the velocity of the arbitrary point P is elementwise constant and different from the zero vector, and that the following relationships*

$$\mathrm{A}_j^i = \mathrm{M}_i \mathrm{M}_j^{-1} \left[I + \mathrm{V}_{ji}^j \mathrm{V}_P^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right]^{-1}, \tag{15.14}$$

$$\mathrm{A}_i^j = \mathrm{M}_i^{-1} \mathrm{M}_j \left[I - \mathrm{V}_{ji}^i \mathrm{V}_P^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right]^{-1}, \tag{15.15}$$

$$\Lambda_j^i = \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}, \tag{15.16}$$

$$\Lambda_i^j = \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right]^{-1}, \tag{15.17}$$

$$O < diagmin \left\{ \begin{array}{c} \left\{ \begin{array}{c} I + \mathrm{V}_{ji}^{j}\mathrm{V}_{P}^{j}\left(\mathrm{Q}^{j}\mathrm{W}^{j}\right)^{-1}, \\ I - \mathrm{V}_{ji}^{i}\mathrm{V}_{P}^{i}\left(\mathrm{Q}^{i}\mathrm{W}^{i}\right)^{-1} \end{array} \right\}, \\ I + \mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1}, I - \mathrm{V}_{ji}^{i}\left(\mathrm{V}_{P}^{i}\right)^{-1} \end{array} \right\}, \tag{15.18}$$

$$\begin{gathered} \mathrm{M}_i\mathrm{M}_j^{-1} = \left(\mathrm{V}_P^i\right)^{-1}\mathrm{V}_P^j = \left(\mathrm{C}_i^i\right)^{-1}\mathrm{C}_j^j = CONST., \\ and\ \mathrm{M}_i^{-1}\mathrm{M}_j = \mathrm{V}_P^i\left(\mathrm{V}_P^j\right)^{-1} = \mathrm{C}_i^i\left(\mathrm{C}_j^j\right)^{-1} = CONST., \end{gathered} \tag{15.19}$$

hold for any choice of the time scaling diagonal matrix coefficients $\mathrm{M}_i \in R_+^{nxn}$, $diagmin\mathrm{M}_i \in R^+$, *and* $\mathrm{M}_j \in R_+^{nxn}$, $diagmin\mathrm{M}_j \in R^+$. *The transformations (15.8) through (15.11) specified by (15.14) through (15.17) are completely both entirely and pairwise compatible. They take the following final forms:*

$$\mathbf{t}_i^n = \mathrm{M}_i\mathrm{M}_j^{-1}\left\{ \begin{array}{c} \left[I + \mathrm{V}_{ji}^{j}\mathrm{V}_{P}^{j}(\mathrm{Q}^{j}\mathrm{W}^{j})^{-1}\right]^{-1}\bullet \\ \bullet\left[\mathbf{t}_j^n + \mathrm{V}_{ji}^{j}(\mathrm{Q}^{j}\mathrm{W}^{j})^{-1}\mathbf{r}_P(\mathbf{t}_j^n)\right] \end{array} \right\}, \tag{15.20}$$

$$\mathbf{t}_j^n = \mathrm{M}_i^{-1}\mathrm{M}_j\left\{ \begin{array}{c} \left[I - \mathrm{V}_{ji}^{i}\mathrm{V}_{P}^{i}\left(\mathrm{Q}^{i}\mathrm{W}^{i}\right)^{-1}\right]^{-1}\bullet \\ \bullet\left[\mathbf{t}_i^n - \mathrm{V}_{ji}^{i}\left(\mathrm{Q}^{i}\mathrm{W}^{i}\right)^{-1}\mathbf{r}_P(\mathbf{t}_i^n)\right] \end{array} \right\}, \tag{15.21}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \left[I + \mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right]^{-1}\left[\mathbf{r}_P(\mathbf{t}_j^n) + \ \mathrm{V}_{ji}^{j}\mathbf{t}_j^n\right], \tag{15.22}$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \left[I - \mathrm{V}_{ji}^{i}\left(\mathrm{V}_{P}^{i}\right)^{-1}\right]^{-1}\left[\mathbf{r}_P(\mathbf{t}_i^n) - \ \mathrm{V}_{ji}^{i}\mathbf{t}_i^n\right]. \tag{15.23}$$

We present the detailed proof in Appendix 23.5.

Remark 564 ***The light velocity is not any exception***

Condition (15.18) discovers a phenomenon that is not known in Einsteinian relativity theory. It is not the light velocity that limits all other velocities. There is not at all the light velocity matrix $\mathrm{C}_{(.)}^{(.)}$ *in this condition. Hence, there is not any influence of the light velocity on the limits of all other velocities. The light velocity does not have any exceptional role in the integral spaces mutually related by the transformations (15.20) through (15.23).*

Comment 565 *The above formulae for the scaling matrix coefficients, (15.14) through (15.17), and for the coordinate transformations, (15.20) through (15.23), need not (but, can) be expressed in terms of the light velocity matrices* $\mathrm{C}_{(.)}^{(.)}$. *However, they are valid also in the case when the arbitrary point P moves with the light velocity. In such a case we should just replace* $\mathrm{V}_P^{(.)}$ *by* $\mathrm{C}_{(.)}^{(.)}$ *everywhere in the formulae.*

Note 566 *If we set* $\mathbf{r}_P(\mathbf{t}^n_{(.)}) \equiv \mathrm{V}^{(.)}_P \mathbf{t}^n_{(.)}$ *in (15.20) and (15.21) then they become*

$$\mathbf{t}^n_i = \mathrm{M}_i \mathrm{M}_j^{-1} \mathbf{t}^n_j,\ \mathbf{t}^n_j = \mathrm{M}_i^{-1} \mathrm{M}_j \mathbf{t}^n_i. \tag{15.24}$$

These equations follow also from (15.7). They show the change of the time units independently of the movements of the spatial frames R^n_i *and* R^n_j*. Einsteinian relativity is inapplicable to this case.*

When we replace $\mathbf{t}^n_{(.)}$ *by* $(\mathrm{V}^{(.)}_P)^{-1}\mathbf{r}_P(\mathbf{t}^n_{(.)})$ *in (15.22) and (15.23) then they reduce to the identity*

$$\mathbf{r}_P(\mathbf{t}^n_i) \equiv \mathbf{r}_P(\mathbf{t}^n_j). \tag{15.25}$$

The numerical values of the distances of the arbitrary point P *from the origin* O_i *of* R^n_i *and from the origin* O_j *of* R^n_j *are equal in spite the frames* R^n_i *and* R^n_j *are in a mutual relative movement. The adjustment of space scaling matrix coefficients* Λ^i_j*, (15.16), and* Λ^j_i*, (15.17), to the velocities* $\mathbf{v}^i_P$ *and* $\mathbf{v}^j_P$ *of the arbitrary point* P *relative to* R^n_i *and* R^n_j*, respectively, enables this invariance of the numerical value of the distance.*

Comment 567 *The transformations (15.20) through (15.23) allow in general different time scales as shown by (15.24) in spite the spatial frames move with the same velocity. This is not possible in the framework of Einsteinian relativity theory.*

Solution for the special case

We adopt now a priory the same *time* scaling diagonal matrix coefficients, $\mathrm{A}^i_j = \mathrm{A}^j_i = \mathrm{A}_{ij} = \mathrm{A}_{ji}$, and the same space scaling diagonal matrix coefficients, $\Lambda^i_j = \Lambda^j_i = \Lambda_{ij} = \Lambda_{ji}$. However, we permit a priory to the light speed matrix $\mathrm{C}^{(.)}_{(.)}$ and to the matrix value $\mathrm{V}^{(.)}_{ji}$ of the spatial transfer velocity, which is also the temporal transfer velocity, to be relative to the corresponding integral space $T_{(.)} \times R^n_{(.)}$ rather than to accept a priory their invariance. The scaling diagonal matrix coefficients will be determined for an arbitrary elementwise nonzero velocity $\mathbf{v}^{(.)}_P$ of the arbitrary point P, $\mathbf{v}^{(.)}_P = \mathrm{V}^{(.)}_P \mathbf{u}$.

In the sequel $diagO \leq diag\mathrm{V}_{ji} < diag\left|\mathrm{V}^{ji}_P\right|$ mean that the diagonal matrix V_{ji} is not diagonally elementwise less than the zero matrix O, and that it is diagonally elementwise less than $\left|\mathrm{V}^{ji}_P\right|$,

Theorem 568 *Let the time scaling diagonal matrix coefficient* M_i *be defined by (15.7). Let* $A = B$ *be positive diagonal matrix in D, (15.13). In order for the scaling diagonal matrix coefficients* A^i_j*,* A^j_i*,* $\mathrm{A}^i_j = \mathrm{A}^j_i = \mathrm{A}_{ij} = \mathrm{A}_{ji} \in R^{n\mathrm{x}n}_+$*,* $diagmin\mathrm{A}_{ij} \in R^+$*,* Λ^i_j*,* Λ^j_i*,* $\Lambda^i_j = \Lambda^j_i = \Lambda_{ij} = \Lambda_{ji} \in R^{n\mathrm{x}n}_+$*,* $diagmin\Lambda_{ij} \in R^+$*, to be diagonally elementwise constant and positive real valued, to obey (15.8) through (15.12), and for (15.7) through (15.12) to imply (15.13) it is necessary* *and sufficient that both the velocity of the arbitrary point P is elementwise constant and different from the zero vector, and that the following relationships, (15.26) through (15.31),*

$$\mathrm{V}_P^i = \mathrm{V}_P^j = \mathrm{V}_P^{ij} = \mathrm{V}_P^{ji}, \ \left|\mathrm{V}_P^{ji}\right| \in R^{+^{nxn}}, \tag{15.26}$$

$$\mathrm{V}_{ji}^j = \mathrm{V}_{ji}^i = \mathrm{V}_{ji} = -\mathrm{V}_{ij} \in R_+^{nxn}, \tag{15.27}$$

$$\begin{gathered}\mathrm{V}_{ji}^i \mathrm{V}_P^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1} = \mathrm{V}_{ji}^j \mathrm{V}_P^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} = \\ = \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW}\right)^{-1}\right)^{ij} = \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW}\right)^{-1}\right)^{ji} = \\ = \mathrm{V}_{ji} \left(\mathrm{V}_P^{ji}\right)^{-1},\end{gathered} \tag{15.28}$$

$$\begin{gathered}\mathrm{A}_{ij} = \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P^{ji} \left(\mathrm{QW}\right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} = \mathrm{A}_{ji} = \\ = \Lambda_{ij} = \Lambda_{ji} = \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2},\end{gathered} \tag{15.29}$$

$$diag O \leq diag \mathrm{V}_{ji} < diag \left|\mathrm{V}_P^{ji}\right|, \ diag \mathrm{V}_{ji} \left|\mathrm{V}_P^{ji}\right| < diag \left(\mathrm{QW}\right)^{ji}, \tag{15.30}$$

$$\mathrm{M}_j = \mathrm{M}_i \left\{ \begin{array}{c} \left[I - \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW}\right)^{-1} \right)^{ji} \right]^{1/2} \bullet \\ \bullet \left[I + \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW}\right)^{-1} \right)^{ji} \right]^{-1/2} \end{array} \right\}, \tag{15.31}$$

hold for any choice of the time scaling diagonal matrix coefficient $\mathrm{M}_i \in R_+^{nxn}$, $diagmin \mathrm{M}_i \in R^+$. *The equations (15.8) through (15.11) become the equations (15.32) through (15.35):*

$$\mathbf{t}_i^n = \left\langle \begin{array}{c} \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW}\right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[\mathbf{t}_j^n + \mathrm{V}_{ji} \left(\left(\mathrm{QW}\right)^{ji} \right)^{-1} \mathbf{r}_P(\mathbf{t}_j^n) \right] \end{array} \right\rangle, \tag{15.32}$$

$$\mathbf{t}_j^n = \left\langle \begin{array}{c} \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW}\right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[\mathbf{t}_i^n - \mathrm{V}_{ji} \left(\left(\mathrm{QW}\right)^{ji} \right)^{-1} \mathbf{r}_P\left(\mathbf{t}_i^n\right) \right] \end{array} \right\rangle, \tag{15.33}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji} \mathbf{t}_j^n \right], \tag{15.34}$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \left[\mathbf{r}_P(\mathbf{t}_i^n) - \ \mathrm{V}_{ji} \mathbf{t}_i^n \right]. \tag{15.35}$$

The transformations (15.32) through (15.35) are partially both entirely and pairwise compatible.

Please see the proof in Appendix 23.6 in order to verify the above Theorem 568.

Remark 569 ***Light speed is not the maximal speed***
This theorem shows, through the conditions in (15.30), that it is not the light speed that restricts other speeds. The diagonal matrix velocity product $(\mathrm{QW})^{ji}$ *is the upper matrix bound for* $(\mathrm{V}_{ji}\mathrm{V}_P)^{ji}$*. This confirms that the light velocity is not any exceptional velocity and that it need not necessarily restrict other velocities, as concluded in the general case in Remark 564.*

In this connection see Note 189.

Comment 570 *The equations (15.26) and (15.27) show that the transfer velocity and the velocity of the arbitrary point P are invariant relative to the integral spaces related by the transformations (15.32) through (15.35).*

Note 571 *Let* $\mathbf{r}_P(\mathbf{t}_{(.)}^n) \equiv \mathrm{V}_P^{ji}\mathbf{t}_{(.)}^n$ *be set in (15.32) and (15.33), which then become the equations (15.24) due to (15.31). They show the change of the time units independently of the movements of the spatial frames* R_i^n *and* R_j^n*. Einsteinian relativity theory is inapplicable to this case. Galilean - Newtonian physics has not treated this case, but it is broadened by this result.*

When we replace $\mathbf{t}_{(.)}^n$ *by* $(\mathrm{V}_P^{ji})^{-1}\mathbf{r}_P(\mathbf{t}_{(.)}^n)$ *in (15.34) and (15.35) then they reduce to (15.36),*

$$\mathbf{r}_P(\mathbf{t}_i^n) = \mathrm{M}_i\mathrm{M}_j^{-1}\mathbf{r}_P(\mathbf{t}_i^n),\ \mathbf{r}_P(\mathbf{t}_i^n) = \mathrm{M}_i^{-1}\mathrm{M}_j\mathbf{r}_P(\mathbf{t}_i^n). \tag{15.36}$$

The numerical values of the distances of the arbitrary point P from the origin O_i *of* R_i^n *and from the origin* O_j *of* R_j^n *change in the same ratio as the temporal coordinates. The adjustment of space scaling matrix coefficients* $\Lambda_j^i = \Lambda_i^j = \Lambda_{ij} = \Lambda_{ji}$*, (15.29), to the velocities* $\mathbf{v}_P^i$ *and* $\mathbf{v}_P^j$ *of the arbitrary point P relative to* R_i^n *and* R_j^n *, respectively, is such to enable that the numerical values of the distances vary in the same manner as the temporal coordinates. This is due to* $\mathrm{A}_{ij} = \Lambda_{ij}$*, (15.29). Consequently, the velocity* $\mathbf{v}_P^{(.)}$ *of the arbitrary point does not change, i.e. it becomes invariant, as expressed by (15.26). This follows directly from (15.24) and (15.25),*

$$\begin{gathered}
\mathbf{v}_P^i = \left[dR_P(\mathbf{t}_i^n)\right]\left[dT_i\right]^{-1}\mathbf{u} = \\
= \left\{\left[dR_P(\mathbf{t}_i^n)\right]\left[dT_j\right]^{-1}\right\}\left\{\left[dT_i\right]\left[dT_j\right]^{-1}\right\}^{-1}\mathbf{u} = \\
= \left\{d\left[\mathrm{M}_i\mathrm{M}_j^{-1}R_P(\mathbf{t}_j^n)\right]\left[dT_j\right]^{-1}\right\}\bullet \\
\bullet\left\{\left[d\mathrm{M}_i\mathrm{M}_j^{-1}T_j\right]\left[dT_j\right]^{-1}\right\}^{-1}\mathbf{u} = \mathbf{v}_P^j = \mathbf{v}_P^{ji}.
\end{gathered} \tag{15.37}$$

This is beyond both Galilean - Newtonian physics and Einsteinian relativity theory.

Solution for the singular case

Corollary 572 *Let the time scaling diagonal matrix coefficient* M_i *be defined by (15.7). Let* $A = B$ *be positive diagonal matrix in D, (15.13). In order for the scaling diagonal matrix coefficients* $\mathrm{A}_j^i,\ \mathrm{A}_i^j,\ \mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}_{ij} = \mathrm{A}_{ji} = \mathrm{A} \in R_+^{nxn}$, *diagmin*$\mathrm{A} \in R^+$, $\Lambda_j^i,\ \Lambda_i^j,\ \Lambda_j^i = \Lambda_i^j = \Lambda_{ij} = \Lambda_{ji} = \Lambda \in R_+^{nxn}$, *diagmin*$\Lambda \in R^+$, *to be diagonally elementwise constant and positive real valued, to obey (15.8) through (15.12), and for (15.7) through (15.12) to imply (15.13) it is necessary and sufficient that both the velocity of the arbitrary point P is elementwise constant and different from the zero vector, and that the following relationships, (15.38) through (15.43),*

$$\mathrm{V}_P^i = \mathrm{V}_P^j = \mathrm{V}_P,\ |\mathrm{V}_P| \in {R^+}^{nxn}, \tag{15.38}$$

$$\mathrm{V}_{ji}^j = \mathrm{V}_{ji}^i = \mathrm{V}_{ji} = -\mathrm{V}_{ij} = \mathrm{V} \in R_+^{nxn}, \tag{15.39}$$

$$\mathrm{Q}^i\mathrm{W}^i = \mathrm{Q}^j\mathrm{W}^j = \mathrm{QW} = \mathrm{V}_P^2, \tag{15.40}$$

$$\mathrm{A} = \left\{I - \left(\mathrm{VV}_P\left(\mathrm{QW}\right)^{-1}\right)^2\right\}^{-1/2} = \\ = \Lambda = \left\{I - \left[\mathrm{V}\left(\mathrm{V}_P\right)^{-1}\right]^2\right\}^{-1/2}, \tag{15.41}$$

$$diagO \leq diag\mathrm{V} < diag\left|\mathrm{V}_P\right|,\ diag\mathrm{V}\left|\mathrm{V}_P\right| < diag\mathrm{QW}, \tag{15.42}$$

$$\mathrm{M}_j = \mathrm{M}_i \left\{ \begin{array}{c} \left[I - \mathrm{VV}_P\left(\mathrm{QW}\right)^{-1}\right]^{1/2} \bullet \\ \bullet \left[I + \mathrm{VV}_P\left(\mathrm{QW}\right)^{-1}\right]^{-1/2} \end{array} \right\}, \tag{15.43}$$

hold for any choice of the time scaling diagonal matrix coefficient $\mathrm{M}_i \in R_+^{nxn}$, *diagmin*$\mathrm{M}_i \in R^+$. *The equations (15.8) through (15.11) become the equations (15.44) through (15.47):*

$$\mathbf{t}_i^n = \left\langle \begin{array}{c} \left\{I - \left[\mathrm{VV}_P\left(\mathrm{QW}\right)^{-1}\right]^2\right\}^{-1/2} \bullet \\ \bullet \left[\mathbf{t}_j^n + \mathrm{V}\left(\mathrm{QW}\right)^{-1}\mathbf{r}_P(\mathbf{t}_j^n)\right] \end{array} \right\rangle, \tag{15.44}$$

$$\mathbf{t}_j^n = \left\langle \begin{array}{c} \left\{I - \left[\mathrm{VV}_P\left(\mathrm{QW}\right)^{-1}\right]^2\right\}^{-1/2} \bullet \\ \bullet \left[\mathbf{t}_i^n - \mathrm{V}\left(\mathrm{QW}\right)^{-1}\mathbf{r}_P\left(\mathbf{t}_i^n\right)\right] \end{array} \right\rangle, \tag{15.45}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \left\{I - \left[\mathrm{V}\left(\mathrm{V}_P\right)^{-1}\right]^2\right\}^{-1/2}\left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}\mathbf{t}_j^n\right], \tag{15.46}$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \left\{ I - \left[\mathrm{V} \left(\mathrm{V}_P \right)^{-1} \right]^2 \right\}^{-1/2} \left[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}\mathbf{t}_i^n \right]. \tag{15.47}$$

The transformations (15.44) through (15.47) are partially both entirely and pairwise compatible.

Note 573 *The sense of Remarks 564 and 569, and of Comment 570 is valid also in the singular case.*

Comment 574 *If* $\mathbf{v}_P^{(.)} = \mathrm{V}_P^{(.)}\mathbf{u} = \mathbf{c}_{(.)}^{(.)} = \mathrm{C}_{(.)}^{(.)}\mathbf{u}$*, then the transformations (15.44) through (15.47) become the* ***vector-matrix forms (15.48) through (15.51) of Lorentz transformations (7.20) through (7.23)****,*

$$\mathbf{t}_i^n = \left[I - \left(\mathrm{VC}^{-1} \right)^2 \right]^{-1/2} \left[\mathbf{t}_j^n + \mathrm{VC}^{-2}\mathbf{r}_P(\mathbf{t}_j^n) \right], \tag{15.48}$$

$$\mathbf{t}_j^n = \left[I - \left(\mathrm{VC}^{-1} \right)^2 \right]^{-1/2} \left[\mathbf{t}_i^n - \mathrm{VC}^{-2}\mathbf{r}_P\left(\mathbf{t}_i^n\right) \right], \tag{15.49}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \left[I - \left(\mathrm{VC}^{-1} \right)^2 \right]^{-1/2} \left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}\mathbf{t}_j^n \right], \tag{15.50}$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \left[I - \left(\mathrm{VC}^{-1} \right)^2 \right]^{-1/2} \left[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}\mathbf{t}_i^n \right]. \tag{15.51}$$

Their pairwise compatibility is only restrictive in this case due to (15.40), which has not been recognized in Einsteinian relativity theory.

If we follow Einsteinian relativity theory so that we ignore the origin of (15.48) through (15.51) by considering them as the unconstrained transformations, then we ignore the constraint (15.40). Consequently, we verify easily that (15.48) through (15.51) considered as unconstrained transformations [i.e. without the constraint (15.40)] would be, as Lorentz transformations themselves, partially pairwise and entirely compatible.

This shows that the fundamentals of CC relativity theory enable us, when we accept the a priory adopted restrictions in Einsteinian relativity theory, to determine the mathematical fundamentals of Einsteinian relativity theory as a deep singular case.

Note 575 *Note 571 with the relevant changes of the equations numbers rests valid in the singular case. It is applicable also to (15.48) through (15.51).*

Note 576 *Theorem 568 and its Corollary 572 are inapplicable if* $Q^{(.)}W^{(.)}$ *is restricted by* $Q^{(.)}W^{(.)} \neq \left[\mathrm{V}_P^{(.)} \right]^2$.

15.2.2 Transformations of velocity

General case

The velocities $\mathbf{q}^{(.)}$ and $\mathbf{w}^{(.)}$ are elementwise positive in the coordinate transformations (15.8) through (15.11), hence in (15.20) through (15.23). Their matrix values are diagonally positive. We allow the same in the velocity transformations.

Theorem 577 *Let the velocities* $\mathbf{q}^{(.)}$ *and* $\mathbf{w}^{(.)}$ *be elementwise positive. Let the time scaling diagonal matrix coefficient* $\mathrm{M}_i \in R_+^{nxn}$, *diagmin*$\mathrm{M}_i \in R^+$, *be defined by (15.7). Let the scaling diagonal matrix coefficients* $\mathrm{A}_j^i \in R_+^{nxn}$, *diagmin*$\mathrm{A}_j^i \in R^+$, $\mathrm{A}_i^j \in R_+^{nxn}$, *diagmin*$\mathrm{A}_i^j \in R^+$, $\mathrm{A}_j^i \neq \mathrm{A}_i^j$, $\Lambda_j^i \in R_+^{nxn}$, *diagmin*$\Lambda_j^i \in R^+$, $\Lambda_i^j \in R_+^{nxn}$, *diagmin*$\Lambda_i^j \in R^+$, $\Lambda_j^i \neq \Lambda_i^j$, *obey (15.8) through (15.12), and let (15.7) through (15.12) imply (15.13). Then, an elementwise nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the corresponding elementwise nonzero velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated as follows:*

$$\mathbf{v}_P^i = \mathrm{M}_i^{-1}\mathrm{M}_j\mathbf{v}_P^j = \mathrm{C}_i^i\left(\mathrm{C}_j^j\right)^{-1}\mathbf{v}_P^j,$$
$$\mathbf{v}_P^j = \mathrm{M}_i\mathrm{M}_j^{-1}\mathbf{v}_P^i = \left(\mathrm{C}_i^i\right)^{-1}\mathrm{C}_j^j\mathbf{v}_P^i. \tag{15.52}$$

These transformations are completely compatible.

The proof is in Appendix 23.7.

Note 578 *The equations for* $\mathbf{v}_P^i$ *and for* $\mathbf{v}_P^j$ *in (15.52) confirm (15.19) since they hold also for the light velocity.*

Note 579 ***Validity of the clock principle***

The equations (15.19), equivalently (15.52), together with (15.7), show that the scaling factors determined by (15.14) through (15.17) ensure the invariance of the numerical vector value $\mathbf{r}_P(\mathbf{t}_{(.)}^n) = \mathrm{V}_P^{(.)}\mathbf{t}_{(.)}^n$, *which is expressed by:*

$$\mathbf{r}_P(\mathbf{t}_i^n) = \mathrm{V}_P^i\mathbf{t}_i^n = \mathrm{M}_i^{-1}\mathrm{M}_j\mathrm{V}_P^j\mathrm{M}_i\mathrm{M}_j^{-1}\mathbf{t}_j^n =$$
$$= \mathrm{V}_P^j\mathbf{t}_j^n = \mathbf{r}_P(\mathbf{t}_j^n).$$

These equations verify the clock principle in this setting (Section 4.8).

Remark 580 ***Light speed is not invariant***

The equations (15.52) show that the light speed is not invariant in general.

Special case

Theorem 581 *Let the velocities* $\mathbf{q}^{(.)}$ *and* $\mathbf{w}^{(.)}$ *be elementwise positive. Let* $A = B$ *be positive diagonal matrix in D, (15.13). Let the time scaling diagonal matrix coefficient* $\mathrm{M}_i \in R_+^{nxn}$ *be defined by (15.7). Let the scaling matrix coefficients* $\mathrm{A}_j^i \in R_+^{nxn}$, $\mathrm{A}_i^j \in R_+^{nxn}$, $\mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}_{ji} = \mathrm{A}_{ij}$, *diagmin*$\mathrm{A}_{ji} \in R^+$, $\Lambda_j^i \in R_+^{nxn}$, $\Lambda_i^j \in R_+^{nxn}$, $\Lambda_j^i = \Lambda_i^j = \Lambda_{ji} = \Lambda_{ij}$, *diagmin*$\Lambda_{ji} \in R^+$, *obey (15.8) through (15.12), and let (15.7) through (15.12) imply (15.13). Then, an elementwise nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$, *and the corresponding elementwise nonzero velocity* $\mathbf{v}_P^j$

of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated by

$$\mathbf{v}_P^i = \mathbf{v}_P^j = \mathbf{v}_P^{ji} = \mathbf{v}_P^{ij}. \tag{15.53}$$

These transformations are partially compatible.

For the proof see Appendix 23.8.

Comment 582 *The equations (15.53) confirm the equations (15.26). They show that the velocity of the arbitrary point P is invariant relative to the integral spaces mutually related by the transformations (15.32) through (15.35). It is a consequence of the equal transformation of the time unit and of the length unit, which results from the equality of the time scaling matrix coefficient A_{ij} and space scaling matrix coefficient Λ_{ij}, (15.29).*

Singular case

Theorem 583 *Let the velocities $\mathbf{q}^{(\cdot)}$ and $\mathbf{w}^{(\cdot)}$ be elementwise positive. Let the time scaling diagonal matrix coefficient $\mathrm{M}_i \in R_+^{n\times n}$, diagmin$\mathrm{M}_i \in R^+$, be defined by (15.7). Let $A = B$ be positive diagonal matrix in D, (15.13). Let*

$$\mathrm{V}_{ji}^j = \mathrm{V}_{ji}^i = \mathrm{V}_{ji} = -\mathrm{V}_{ij} = \mathrm{V} \in R_+^{n\times n},$$

Let the scaling diagonal matrix coefficients $\mathrm{A}_j^i \in R_+^{n\times n}$, diagmin$\mathrm{A}_j^i \in R^+$, $\mathrm{A}_i^j \in R_+^{n\times n}$, diagmin$\mathrm{A}_i^j \in R^+$, $\Lambda_j^i \in R_+^{n\times n}$, diagmin$\Lambda_j^i \in R^+$, $\Lambda_i^j \in R_+^{n\times n}$, diagmin$\Lambda_i^j \in R^+$, $\mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}$, $\Lambda_j^i = \Lambda_i^j = \Lambda$, obey (15.8) through (15.12), and let (15.7) through (15.12) imply (15.13). Then, an elementwise nonzero velocity $\mathbf{v}_P^i$ of the arbitrary point P with respect to the origin O_i of R_i^n and relative to $\mathfrak{T}_i$, and the corresponding elementwise nonzero velocity $\mathbf{v}_P^j$ of the same point P with respect to the origin O_j of R_j^n and relative to $\mathfrak{T}_j$ are interrelated by

$$\mathbf{v}_P^i = \mathbf{v}_P^j = \mathbf{v}_P. \tag{15.54}$$

These transformations are partially compatible.

The proof of this Theorem is essentially the same as for the special case.

Remark 584 *The invariance of the velocity, which characterizes the light velocity only in Einsteinian relativity theory as the unique velocity with such a feature, holds for every velocity in the singular case in this framework.*

Claim 585 *If the vector-matrix Lorentz transformations (15.48) through (15.51) hold, then an elementwise nonzero velocity $\mathbf{v}_P^i$ of the arbitrary point P with respect to the origin O_i of R_i^n and relative to T_i, and the corresponding elementwise nonzero velocity $\mathbf{v}_P^j$ of the same point P with respect to the origin O_j of R_j^n and relative to T_j are interrelated by*

$$\mathbf{v}_P^i = \left[I + \mathrm{V}\mathrm{V}_P^j\,(\mathrm{C})^{-2}\right]^{-1}\left(\mathbf{v}_P^j + \mathbf{v}\right),$$
$$\mathbf{v}_P^j = \left[I - \mathrm{V}\mathrm{V}_P^i\,(\mathrm{C})^{-2}\right]^{-1}\left(\mathbf{v}_P^i - \mathbf{v}\right). \tag{15.55}$$

They are completely compatible.

The proof can be found in Appendix 23.9.

The equations (15.55) are the vector - matrix generalization of Einstein's law of the composition of velocities, (7.43) and (7.45).

At first glance it might seem that the equations (15.55) contradict the equations (15.54), but they do not. This is explained in what follows.

Conclusion 586 *The preceding two proofs permit us to conclude that (matrix) Einstein's law of the composition of velocities has a particular form, different from (15.54), because the (matrix) scaling coefficients in the (matrix) Lorentz transformations are determined just for the light velocity of the arbitrary point P and used afterwards for its arbitrary velocity. However, when they are applied to the velocity of the arbitrary point P for which the (matrix) Lorentz transformations are determined, i.e. for the light velocity, then the equations (15.55) and Einsteinian inconsistent use of the numerical values of the light speed and of the spatial transfer speed yield the following:*

$$\mathbf{v}_P^i = \mathbf{c}^i = \left[I + \mathrm{VC}\,(\mathrm{C})^{-2} \right]^{-1} (\mathbf{c} + \mathbf{v}) =$$

$$= \left[I + \mathrm{VC}^{-1} \right]^{-1} \left[I + \mathrm{VC}^{-1} \right] \mathbf{c} = \mathbf{c} = \mathbf{v}_P,$$

$$\mathbf{v}_P^j = \mathbf{c}^j = \left[I - \mathrm{VC}\,(\mathrm{C})^{-2} \right]^{-1} (\mathbf{c} - \mathbf{v}) =$$

$$= \left[I - \mathrm{VC}^{-1} \right]^{-1} \left[I - \mathrm{VC}^{-1} \right] \mathbf{c} = \mathbf{c} = \mathbf{v}_P = \mathbf{v}_P^i = \mathbf{c}^i.$$

Hence, (15.54) is satisfied. This illustrates once more the consequence of Einsteinian inconsistent use of both the numerical values of the speeds in Lorentz transformations and the transformations themselves. On the left hand sides of the above formulae of the matrix Einstein law of the composition of velocities there are the relative light velocities, but on their right hand sides there are invariant numerical matrix values of the light speed and of the spatial transfer speed. The consistent use of the transformations is only under the conditions under which they are proved, i.e. only for the light velocity of the arbitrary point P. But this then disables their use for an arbitrary speed of the arbitrary point. Such a consistent their implementation is impossible in Einsteinian relativity theory that forbids to anybody and anything, different than light, to move with the light velocity.

Besides, this shows again that the fundamentals of CC relativity theory, which result from Axiom 47, permit us to derive the mathematical fundamentals of Einsteinian relativity theory as a singular case. Consequently, Einsteinian relativity theory has not disproved, and cannot disprove, the time independence of space.

Note 587 *Theorem 581 and Theorem 583 do not hold if $Q^{(\cdot)}W^{(\cdot)}$ is restricted by $Q^{(\cdot)}W^{(\cdot)} \neq \left[\mathrm{V}_P^{(\cdot)} \right]^2$.*

15.3 General uniformity

15.3.1 Transformations of temporal and spatial coordinates

Basic relationships

All the *time* scaling matrix coefficients are diagonally positive real valued diagonal matrices,

$$\begin{aligned} diagmin\mathrm{A}_j^i &\in R^+,\, diagmin\mathrm{A}_i^j \in R^+, \\ diagmin\Lambda_j^i &\in R^+,\, diagmin\Lambda_i^j \in R^+, \end{aligned}$$

and

$$\begin{gathered} \mathbf{t}_i^n = \mathrm{M}_{(.)}\mathbf{t}_{(.)}^n,\ \mathbf{t}_{i0}^n = \mathrm{M}_{(.)}\mathbf{t}_0^n, \\ \mathrm{M}_{(.)} \in R_+^{nxn},\, diagmin\mathrm{M}_{(.)} \in R^+. \end{gathered} \tag{15.56}$$

The basic general generic equations (15.2) through (15.5) rest unchanged in this framework,

$$\mathbf{t}_i^n = \mathrm{A}_j^i\left[\mathbf{t}_j^n + \mathrm{V}_\vartheta^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathbf{r}_R(\mathbf{t}_j^n)\right], \tag{15.57}$$

$$\mathbf{t}_j^n = \mathrm{A}_i^j\left[\mathbf{t}_i^n - \mathrm{V}_\vartheta^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\mathbf{r}_R(\mathbf{t}_i^n)\right], \tag{15.58}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \Lambda_j^i\left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j\mathbf{t}_j^n\right],\ \mathrm{V}_{ji}^j \in R_+^{nxn}, \tag{15.59}$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \Lambda_i^j\left[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}_{ji}^i\mathbf{t}_i^n\right],\ \mathrm{V}_{ji}^i \in R_+^{nxn}, \tag{15.60}$$

where

$$\begin{gathered} \mathbf{q}^{(.)} \in R^{+^n}, \mathbf{v}_R^{(.)} \in R_+^n,\ \boldsymbol{\vartheta}^{(.)} \in R_+^n,\ \mathbf{w}^{(.)} \in R^{+^n}, \\ \mathbf{q}^{(.)}, \mathbf{v}_P^{(.)}, \mathbf{w}^{(.)} \in \{\mathbf{c}_{(.)}^{(.)}\} \cup \left\{\mathbf{v}_R^{(.)} : \mathbf{v}_R^{(.)} \in R^{+^n}\right\} \\ and\ \boldsymbol{\vartheta}^{(.)} = \mathbf{v}_R^{(.)} \\ are\ permitted,\ \mathbf{v}_P^{(.)} \neq \mathbf{0}\ elementwise. \end{gathered} \tag{15.61}$$

The velocity $\mathbf{q}^{(.)}$ and/or $\mathbf{w}^{(.)}$ can be equal to the light velocity $\mathbf{c}_{(.)}^{(.)}$. Either of them, or both, can be equal to the velocity $\mathbf{v}_R^{(.)}$ of the reference point P_R if and only if $\mathbf{v}_R^{(.)}$ is elementwise nonzero vector. The generic transfer speed $\vartheta^{(.)}$ is the temporal transfer speed. It is in general independent of the spatial transfer speed $v_{ji}^{(.)}$, of the positions $\mathbf{r}_P(\mathbf{t}_{(.)}^n)$ of the arbitrary point P, and $\mathbf{r}_R(\mathbf{t}_{(.)}^n)$ of the reference point P_R, and of their speeds $v_P^{(.)}$ and $v_R^{(.)}$.

Note 588 ***Characteristics***

The characteristics of the vector - matrix coordinate transformations (15.57) through (15.61) are the same as those of the transformations (10.67) through (10.71).

Definition 589 ***Transformations and time field***

The equations (15.56) through (15.61) determine ***time-invariant generally uniform time field*** *if and only if every pair of time axes obeys the transformations, including axes from different layers if the time field is multi-layer.*

Note 590 *For* $P_R = L$, *(i.e. for* $\mathbf{r}_R(\mathbf{t}^n_{(.)}) \equiv \mathbf{r}_L(\mathbf{t}^n_{(.)})$ *and* $V_R^{(.)} \equiv C_{(.)}^{(.)}$*), the transformations (15.56) through (15.61) become the time-invariant uniform vector - matrix transformations.*

If, additionally, $Q^{(.)}\mathrm{W}^{(.)} \equiv \left[\mathrm{C}_{(.)}^{(.)}\right]^2$ *then the transformations simplify to the time-invariant specially uniform vector - matrix transformations.*

Solutions

Theorem 591 *Let the time scaling diagonal matrix coefficient* M_i *be defined by (15.56). In order for the scaling diagonal matrix coefficients* A_j^i, A_i^j, $\mathrm{A}_j^i \neq \mathrm{A}_i^j$, Λ_j^i, Λ_i^j, $\Lambda_j^i \neq \Lambda_i^j$, *to be diagonally elementwise constant and positive real valued, to obey (15.57) through (15.61), and for (15.56) through (15.61) to imply (15.13) it is necessary and sufficient that both the velocity of the arbitrary point* P *is elementwise constant and different from the zero vector, and that the following relationships*

$$\mathrm{A}_j^i = \mathrm{M}_i\mathrm{M}_j^{-1}\left[I + \mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}, \tag{15.62}$$

$$\mathrm{A}_i^j = \mathrm{M}_i^{-1}\mathrm{M}_j\left[I - \mathrm{V}_\vartheta^i\mathrm{V}_R^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]^{-1}, \tag{15.63}$$

$$\Lambda_j^i = \left[I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}, \tag{15.64}$$

$$\Lambda_i^j = \left[I - \mathrm{V}_{ji}^i\left(\mathrm{V}_P^i\right)^{-1}\right]^{-1}, \tag{15.65}$$

$$O < diagmin\left\{\begin{array}{c} I - \mathrm{V}_\vartheta^i\mathrm{V}_R^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}, \\ I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}, \\ I - \mathrm{V}_{ji}^i\left(\mathrm{V}_P^i\right)^{-1} \end{array}\right\}, \tag{15.66}$$

$$\begin{gathered}\mathrm{M}_i\mathrm{M}_j^{-1} = \left(\mathrm{V}_P^i\right)^{-1}\mathrm{V}_P^j = \left(\mathrm{C}_i^i\right)^{-1}\mathrm{C}_j^j = CONST., \text{ and} \\ \mathrm{M}_i^{-1}\mathrm{M}_j = \mathrm{V}_P^i\left(\mathrm{V}_P^j\right)^{-1} = \mathrm{C}_i^i\left(\mathrm{C}_j^j\right)^{-1} = CONST.,\end{gathered} \tag{15.67}$$

hold for any choice of the time scaling diagonal matrix coefficients $\mathrm{M}_i \in R_+^{nxn}$, $diagmin\mathrm{M}_i \in R^+$, *and* $\mathrm{M}_j \in R_+^{nxn}$, $diagmin\mathrm{M}_j \in R^+$. *The transformations (15.57) through (15.60) specified by (15.62) through (15.65) are completely* *both entirely and pairwise compatible. They are determined by (15.68) through (15.71),*

$$\mathbf{t}_i^n = \mathrm{M}_i\mathrm{M}_j^{-1}\left\{\begin{array}{c}\left[I+\mathrm{V}_\vartheta^j\mathrm{V}_R^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\right]^{-1}\bullet\\ \bullet\left[\mathbf{t}_j^n+\mathrm{V}_\vartheta^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\mathbf{r}_R(\mathbf{t}_j^n)\right]\end{array}\right\}, \tag{15.68}$$

$$\mathbf{t}_j^n = \mathrm{M}_i^{-1}\mathrm{M}_j\left\{\begin{array}{c}\left[I-\mathrm{V}_\vartheta^i\mathrm{V}_R^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]^{-1}\bullet\\ \bullet\left[\mathbf{t}_i^n-\mathrm{V}_\vartheta^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\mathbf{r}_R(\mathbf{t}_i^n)\right]\end{array}\right\}, \tag{15.69}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}\left[\mathbf{r}_P\left(\mathbf{t}_j^n\right)+\ \mathrm{V}_{ji}^j\mathbf{t}_j^n\right] = \mathbf{r}_P\left(\mathbf{t}_j^n\right), \tag{15.70}$$

$$\mathbf{r}_P\left(\mathbf{t}_j^n\right) = \left[I-\mathrm{V}_{ji}^i\left(\mathrm{V}_P^i\right)^{-1}\right]^{-1}\left[\mathbf{r}_P(\mathbf{t}_i^n)-\ \mathrm{V}_{ji}^i(\mathbf{t}_i^n)\right] = \mathbf{r}_P(\mathbf{t}_i^n). \tag{15.71}$$

Appendix 23.10 exposes the proof in details.

Remark 592 ***The light speed is not any exception***

The light speed does not limit all other speeds, which follows from (15.66).

The light speed is not any exceptional speed in the integral spaces interrelated by the transformations (15.68) through (15.71).

Comment 593 *The equations (15.62) through (15.65), as well as the equations (15.68) through (15.71), can, but need not, be expressed in terms of the light velocity matrices* $\mathrm{C}_{(.)}^{(.)}$. *They are valid also when the velocity of the arbitrary point* P *is the light velocity.*

Corollary 594 *Let the coordinate systems* R_i^n *and* R_j^n *move with the same velocity:* $\mathbf{v}_{O_i}^O = \mathbf{v}_{O_j}^O$. *Let the scaling diagonal matrix coefficients* M_i *be defined by (15.56). In order for the time and the space scaling diagonal matrix coefficients* A_i^j, A_j^i, $\mathrm{A}_j^i \neq \mathrm{A}_i^j$, Λ_i^j *and* Λ_j^i *to be diagonally elementwise positive real valued, to obey (15.57) through (15.61), and for (15.56) through (15.61) to imply (15.13) it is necessary and sufficient that the equations (15.62), (15.63) and the following equations:*

$$\Lambda_j^i = \Lambda_i^j = I,\ \mathrm{M}_i^{-1}\mathrm{M}_j = \mathrm{V}_P^i\left(\mathrm{V}_P^j\right)^{-1} = \mathrm{C}_i^i\left(\mathrm{C}_j^j\right)^{-1},$$

hold for any choice of the time scaling diagonal matrix coefficient $\mathrm{M}_i \in R_+^{n\mathrm{x}n}$, *diagmin*$\mathrm{M}_i \in R^+$. *The temporal coordinate transformations (15.57) and (15.58) become, respectively, (15.68) and (15.69), which can be set into the following simple forms:*

$$\mathbf{t}_i^n = \mathrm{M}_i\mathrm{M}_j^{-1}\mathbf{t}_j^n,\ \mathbf{t}_j^n = \mathrm{M}_j\mathrm{M}_i^{-1}\mathbf{t}_i^n. \tag{15.72}$$

The spatial coordinate transformations (15.59) and (15.60) reduce to the identity transformation,

$$\mathbf{r}_P(\mathbf{t}_i^n) = \mathbf{r}_P(\mathbf{t}_j^n). \tag{15.73}$$

They are completely both entirely and pairwise compatible.

Comment 595 *The preceding Corollary 594 verifies once more the possibility for the existence of different time units and time scales in spite the spatial frames move with the same velocities. This follows from the equations (15.68) and (15.69). Lorentz transformations do not permit such a possibility. Einsteinian relativity theory is not applicable to such a case.*

Solution for the special case

Theorem 596 *Let the time scaling diagonal matrix coefficient* M_i *be defined by (15.56). Let* $A = B$ *be positive diagonal matrix in* D*, (15.13). In order for the scaling diagonal matrix coefficients* A_j^i, A_i^j, $\mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}_{ij} = \mathrm{A}_{ji}$, Λ_i^j, $\Lambda_j^i = \Lambda_i^j = \Lambda_{ij} = \Lambda_{ji}$, *to be diagonally elementwise constant and positive real valued, to obey (15.57) through (15.61), and for (15.56) through (15.61) to imply (15.13) it is necessary and sufficient that both the velocity of the arbitrary point* P *is elementwise constant and different from the zero vector, and that the following relationships, (15.74) through (15.79),*

$$\mathrm{V}_P^i = \mathrm{V}_P^j = \mathrm{V}_P^{ij} = \mathrm{V}_P^{ji} \in R_+^{n\times n},\ diagmin\mathrm{V}_P^{ij} \in R^+, \tag{15.74}$$

$$\mathrm{V}_{ji}^j = \mathrm{V}_{ji}^i = \mathrm{V}_{ji} = -\mathrm{V}_{ij} = \mathrm{V}_\vartheta^j = \mathrm{V}_\vartheta^i = \mathrm{V}_\vartheta^{ij} \in R_+^{n\times n}, \tag{15.75}$$

$$\begin{gathered}\mathrm{V}_R^i\mathrm{V}_\vartheta^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1} = \mathrm{V}_R^j\mathrm{V}_\vartheta^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1} = \\ = \left(\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right)^{ij} = \left(\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right)^{ji} = \\ = \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1},\end{gathered} \tag{15.76}$$

$$\begin{gathered}\mathrm{A}_{ij} = \left\{I - \left[\left(\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} = \mathrm{A}_{ji} = \\ = \Lambda_{ij} = \Lambda_{ji} = \left\{I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\}^{-1/2},\end{gathered} \tag{15.77}$$

$$\begin{gathered}diagO \leq diag\mathrm{V}_{ji} < diag\mathrm{V}_P^{ji}, \\ diag\left(\mathrm{V}_\vartheta\mathrm{V}_R\right)^{ji} < diag\left(\mathrm{QW}\right)^{ji},\end{gathered} \tag{15.78}$$

$$\mathrm{M}_j = \mathrm{M}_i\left\{\begin{array}{c}\left[I - \left(\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^{1/2}\bullet \\ \bullet\left[I + \left(\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^{-1/2}\end{array}\right\}, \tag{15.79}$$

hold for any choice of the time scaling diagonal matrix coefficient $\mathrm{M}_i \in R_+^{n\times n}$, $diagmin\mathrm{M}_i \in R^+$. *The equations (15.57) through (15.60) become the equations*

(15.80) through (15.83):

$$\mathbf{t}_i^n = \left\langle \begin{array}{c} \left\{ I - \left[\left(\mathrm{V}_R \mathrm{V}_\vartheta \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[\mathbf{t}_j^n + \mathrm{V}_\vartheta^j \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \mathbf{r}_R \left(\mathbf{t}_j^n \right) \right] \end{array} \right\rangle, \tag{15.80}$$

$$\mathbf{t}_j^n = \left\langle \begin{array}{c} \left\{ I - \left[\left(\mathrm{V}_R \mathrm{V}_\vartheta \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[\mathbf{t}_i^n - \mathrm{V}_\vartheta^i \left(\mathrm{Q}^i \mathrm{W}^i \right)^{-1} \mathbf{r}_R \left(\mathbf{t}_i^n \right) \right] \end{array} \right\rangle, \tag{15.81}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \left\langle \begin{array}{c} \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[\mathbf{r}_P \left(\mathbf{t}_j^n \right) + \mathrm{V}_{ji} \mathbf{t}_j^n \right] \end{array} \right\rangle, \tag{15.82}$$

$$\mathbf{r}_P \left(\mathbf{t}_j^n \right) = \left\langle \begin{array}{c} \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \\ \bullet \left[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}_{ji} \mathbf{t}_i^n \right] \end{array} \right\rangle. \tag{15.83}$$

The transformations (15.80) through (15.83) are partially both entirely and pairwise compatible.

Appendix 23.11 contains the proof.

Remark 597 ***Light speed is not the upper bound of speeds***
The light speed is not the upper bound of other speeds, (15.78). The light speed is not any exceptional speed.

Note 598 *Let us set* $\mathbf{r}_R(\mathbf{t}_{(.)}^n) \equiv \mathrm{V}_R^{(.)} \mathbf{t}_{(.)}^n$ *in (15.80) and (15.81). They become (15.72) due to (15.79). The change of the time units is independent of the movements of the spatial frames* R_i^n *and* R_j^n*. Einsteinian relativity theory cannot treat this case. This result extends Galilean - Newtonian physics.*

Let us replace $\mathbf{t}_{(.)}^n$ *by* $(\mathrm{V}_P^{ji})^{-1} \mathbf{r}_P(\mathbf{t}_{(.)}^n)$ *in (15.82) and (15.83). Then they reduce to*

$$\mathbf{r}_P(\mathbf{t}_i^n) = \mathrm{M}_i \mathrm{M}_j^{-1} \mathbf{r}_P(\mathbf{t}_j^n). \tag{15.84}$$

The numerical values of the distances of the arbitrary point P *from the origins* O_i *of* R_i^n *and* O_j *of* R_j^n *obey the same rule as the numerical values of the temporal coordinates. The adjustment of the space scaling matrix coefficients* $\Lambda_j^i = \Lambda_i^j = \Lambda_{ij} = \Lambda_{ji}$*, (15.77), to the velocities* $\mathbf{v}_P^i$ *and* $\mathbf{v}_P^j$ *of the arbitrary point* P *with respect to* R_i^n *and* R_j^n *, respectively, enables that the numerical values of the distances vary in the same manner as of the temporal coordinates. This is due to* $\mathrm{A}_{ij} = \Lambda_{ij}$*, (15.77). Therefore, the velocity* $\mathbf{v}_P^{(.)}$ *of the arbitrary point does not change, i.e. it becomes invariant, as expressed by (15.74). This statement results directly from (15.72) and (15.84) as shown in (15.37). Such adjustment is beyond both Galilean - Newtonian physics and Einsteinian relativity theory.*

Note 599 ***Invariance of the arbitrary speed***

The equations (15.74) and (15.75) show that the spatial transfer velocity $\boldsymbol{v}_{ji}^{(.)}$ *and the velocity* $\boldsymbol{v}_{P}^{(.)}$ *of the arbitrary point* P *are invariant relative to the integral spaces that are mutually related by the transformations (15.80) through (15.83). The light speed is not an exceptional speed.*

Solution for the singular case

Corollary 600 *Let the time scaling diagonal matrix coefficients* M_i *be defined by (15.56). Let* $A = B$ *be positive diagonal matrix in D, (15.13). In order for the scaling matrix coefficients* A_j^i, A_i^j, $\mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}_{ij} = \mathrm{A}_{ji} = \mathrm{A} \in R_+^{nxn}$, $diagmin\mathrm{A} \in R^+$, Λ_j^i, Λ_i^j, $\Lambda_j^i = \Lambda_i^j = \Lambda_{ij} = \Lambda_{ji} = \Lambda \in R_+^{nxn}$, $diagmin\Lambda \in R^+$, *to be diagonally elementwise constant and positive real valued, to obey (15.57) through (15.61), and for (15.56) through (15.61) to imply (15.13) it is necessary and sufficient that both the velocity of the arbitrary point* P *is elementwise constant and different from the zero vector, and that the following relationships, (15.85) through (15.90),*

$$\mathrm{V}_P^i = \mathrm{V}_P^j = \mathrm{V}_P \in R_+^{nxn},\ diagmin\mathrm{V}_P \in R^+, \tag{15.85}$$

$$\mathrm{V}_{ji}^j = \mathrm{V}_{ji}^i = \mathrm{V}_{ji} = -\mathrm{V}_{ij} = \mathrm{V} \in R_+^{nxn}, \tag{15.86}$$

$$\begin{gathered}\mathrm{V}_R^i\mathrm{V}_\vartheta^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1} = \mathrm{V}_R^j\mathrm{V}_\vartheta^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1} = \\ = \mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1} = \mathrm{V}\left(\mathrm{V}_P\right)^{-1}\end{gathered} \tag{15.87}$$

$$\begin{aligned}\mathrm{A} &= \left\{I - \left(\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right)^2\right\}^{-1/2} = \\ &= \Lambda = \left\{I - \left[\mathrm{V}\left(\mathrm{V}_P\right)^{-1}\right]^2\right\}^{-1/2},\end{aligned} \tag{15.88}$$

$$\begin{gathered}diagO \leq diag\mathrm{V} < diag\mathrm{V}_P, \\ diag\mathrm{V}_R\mathrm{V}_\vartheta < diag\mathrm{QW},\end{gathered} \tag{15.89}$$

$$\mathrm{M}_j = \left\{\begin{array}{c}\mathrm{M}_i\left[I - \mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right]^{1/2} \bullet \\ \bullet\left[I + \mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right]^{-1/2}\end{array}\right\}, \tag{15.90}$$

hold for any choice of the time scaling diagonal matrix coefficient $\mathrm{M}_i \in R_+^{nxn}$, $diagmin\mathrm{M}_i \in R^+$. *The equations (15.57) through (15.60) become the equations (15.91) through (15.94):*

$$\mathbf{t}_i^n = \left\langle\begin{array}{c}\left\{I - \left[\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right]^2\right\}^{-1/2} \bullet \\ \bullet\left[\mathbf{t}_j^n + \mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\mathbf{r}_R\left(\mathbf{t}_j^n\right)\right]\end{array}\right\rangle, \tag{15.91}$$

$$\mathbf{t}_j^n = \left\langle \begin{array}{c} \left\{I - \left[\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right]^2\right\}^{-1/2} \bullet \\ \bullet \left[\mathbf{t}_i^n - \mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\mathbf{r}_R\left(\mathbf{t}_i^n\right)\right] \end{array} \right\rangle, \tag{15.92}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \left\{I - \left[\mathrm{V}\left(\mathrm{V}_P\right)^{-1}\right]^2\right\}^{-1/2}\left[\mathbf{r}_P\left(\mathbf{t}_j^n\right) + \mathrm{V}\mathbf{t}_j^n\right], \tag{15.93}$$

$$\mathbf{r}_P\left(\mathbf{t}_j^n\right) = \left\{I - \left[\mathrm{V}\left(\mathrm{V}_P\right)^{-1}\right]^2\right\}^{-1/2}\left[\mathbf{r}_P(\mathbf{t}_i^n) - \ \mathrm{V}\mathbf{t}_i^n\right]. \tag{15.94}$$

The transformations (15.91) through (15.94) are partially both entirely and pairwise compatible.

Note 601 *The essence of Remark 597 and of Note 599 applies to the singular case.*

15.3.2 Transformations of velocity

General case

Theorem 602 *Let the velocities* $\mathbf{q}^{(.)}$ *and* $\mathbf{w}^{(.)}$ *be elementwise positive. Let the time scaling diagonal matrix coefficients* M_i *be defined by (15.56). Let the scaling diagonal matrix coefficients* A_j^i, A_i^j, $\mathrm{A}_j^i \neq \mathrm{A}_i^j$, Λ_j^i, Λ_i^j, $\Lambda_j^i \neq \Lambda_i^j$, *obey (15.57) through (15.61), and let (15.56) through (15.61) imply (15.13). Then, an elementwise nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$*, and the corresponding elementwise nonzero velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated by (15.95),*

$$\begin{aligned} \mathbf{v}_P^i &= \mathrm{M}_i^{-1}\mathrm{M}_j\mathbf{v}_P^j = \mathrm{C}_i^i(\mathrm{C}_j^j)^{-1}\mathbf{v}_P^j, \\ \mathbf{v}_P^j &= \mathrm{M}_j^{-1}\mathrm{M}_i\mathbf{v}_P^i = \mathrm{C}_j^j\left(\mathrm{C}_i^i\right)^{-1}\mathbf{v}_P^i. \end{aligned} \tag{15.95}$$

The transformations are completely compatible.

We prove this theorem in Appendix 23.12.

Note 603 ***Noninvariance of the light speed***

The equations (15.95) verify once more the noninvariance of both the light speed $c_{(.)}^{(.)}$ *and the spatial transfer speed* $v_{ji}^{(.)}$*.*

Note 604 ***Validity of the clock principle***

The equations (15.67), i.e. the equations (15.95), together with (15.56), reflect the fact that the scaling factors determined by (15.62) through (15.65) guarantee the invariance of the distance $\mathbf{r}_P(\mathbf{t}_{(.)}^n) = \mathrm{V}_P^{(.)}\mathbf{t}_{(.)}^n$*, which is expressed by:*

$$\begin{aligned} \mathbf{r}_P(\mathbf{t}_i^n) &= \mathrm{V}_P^i\mathbf{t}_i^n = \mathrm{M}_i^{-1}\mathrm{M}_j\mathrm{V}_P^j\mathrm{M}_i\mathrm{M}_j^{-1}\mathbf{t}_j^n = \\ &= \mathrm{V}_P^j\mathbf{t}_j^n = \mathbf{r}_P(\mathbf{t}_j^n). \end{aligned} \tag{15.96}$$

These equations verify the clock principle in this setting (Section 4.8).

Special case

Theorem 605 *Let the velocities* $\mathbf{q}^{(\cdot)}$ *and* $\mathbf{w}^{(\cdot)}$ *be elementwise positive. Let* $A = B$ *be positive diagonal matrix in* D*, (15.13). Let the time scaling matrix coefficients* M_i *be defined by (15.56). Let the scaling matrix coefficients* A_j^i, A_i^j, $\mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}_{ij} = \mathrm{A}_{ji}$, Λ_j^i, Λ_i^j, $\Lambda_j^i = \Lambda_i^j = \Lambda_{ij} = \Lambda_{ji}$, *obey (15.57) through (15.61), and let (15.56) through (15.61) imply (15.13). Then, an elementwise nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$*, and the corresponding elementwise nonzero velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated by (15.97),*

$$\mathbf{v}_P^i = \mathbf{v}_P^j = \mathbf{v}_P^{ji} = \mathbf{v}_P^{ij}. \tag{15.97}$$

The transformations are partially compatible.

The proof is in Appendix 23.13.

Comment 606 *The equations (15.97) agree with the equations (15.74). The velocity of the arbitrary point* P *is invariant relative to the integral spaces mutually related by the transformations (15.80) through (15.83). The equality between the time scaling matrix coefficient* A_{ij} *and the space scaling matrix coefficient* Λ_{ij}*, (15.88), implies the equal transformation of the time unit and of the length unit. This results in the invariance of every velocity considered in such integral spaces.*

Singular case

Corollary 607 *Let the velocities* $\mathbf{q}^{(\cdot)}$ *and* $\mathbf{w}^{(\cdot)}$ *be elementwise positive. Let* $A = B$ *be positive diagonal matrix in* D*, (15.13). Let the time scaling diagonal matrix coefficient* M_i *be defined by (15.56). Let* $\mathrm{V}_{ji}^j = \mathrm{V}_{ji}^i = \mathrm{V}_{ji} = -\mathrm{V}_{ij} = \mathrm{V} \in R_+^{n\times n}$*. Let the scaling diagonal matrix coefficients* A_j^i, A_i^j, $\mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}$, Λ_j^i, Λ_i^j, $\Lambda_j^i = \Lambda_i^j = \Lambda$, *obey (15.57) through (15.61), and let (15.56) through (15.61) imply (15.13). Then, an elementwise nonzero velocity* $\mathbf{v}_P^i$ *of the arbitrary point* P *with respect to the origin* O_i *of* R_i^n *and relative to* $\mathfrak{T}_i$*, and the corresponding elementwise non-zero velocity* $\mathbf{v}_P^j$ *of the same point* P *with respect to the origin* O_j *of* R_j^n *and relative to* $\mathfrak{T}_j$ *are interrelated by (15.98),*

$$\mathbf{v}_P^i = \mathbf{v}_P^j = \mathbf{v}_P. \tag{15.98}$$

The transformations are partially compatible.

This Corollary results directly from Theorem 605.

Comment 608 *What holds in Einsteinian relativity theory for the light velocity, i.e. its invariance, it holds for every velocity in the singular case in this framework.*

15.4 General weak uniformity

The scaling matrix coefficients are diagonally elementwise positive real valued diagonal matrices,

$$\mathrm{A}_j^i \in R_+^{nxn},\ diagmin\mathrm{A}_j^i \in R^+,$$
$$\mathrm{A}_i^j \in R_+^{nxn},\ diagmin\mathrm{A}_i^j \in R^+,$$
$$\Lambda_j^i \in R_+^{nxn},\ diagmin\Lambda_j^i \in R^+,$$
$$\Lambda_i^j \in R_+^{nxn},\ diagmin\Lambda_i^j \in R^+,$$

and

$$\begin{aligned}\mathbf{t}_i^n &= \mathrm{M}_{(.)}\mathbf{t}_{(.)}^n,\ \mathbf{t}_{i0}^n = \mathrm{M}_{(.)}\mathbf{t}_0^n,\\ \mathrm{M}_{(.)} &\in R_+^{nxn},\ diagmin\mathrm{M}_{(.)} \in R^+.\end{aligned} \tag{15.99}$$

The velocity of the reference point P_R is the general constant temporal transfer velocity $\boldsymbol{\vartheta}^{(.)}$, $\boldsymbol{\vartheta}^{(.)} \equiv \mathbf{v}_R^{(.)}$. The temporal coordinate transformations depend on the position of the reference point P_R. The basic general generic equations (15.2) through (15.5) become the following in this framework:

$$\mathbf{t}_i^n = \mathrm{A}_j^i \left[\mathbf{t}_j^n + \mathrm{V}_R^j \left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1} \mathbf{r}_R(\mathbf{t}_j^n)\right], \tag{15.100}$$

$$\mathbf{t}_j^n = \mathrm{A}_i^j \left[\mathbf{t}_i^n - \mathrm{V}_R^i \left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1} \mathbf{r}_R(\mathbf{t}_i^n)\right], \tag{15.101}$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \Lambda_j^i \left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j\mathbf{t}_j^n\right], \tag{15.102}$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \Lambda_i^j \left[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}_{ji}^i\mathbf{t}_i^n\right], \tag{15.103}$$

where

$$\begin{gathered}\mathbf{q}^{(.)}, \mathbf{w}^{(.)} \in R^{+^n},\ \mathbf{v}_{ji}^{(.)},\ \mathbf{v}_R^{(.)} \in R_+^n,\\ \mathbf{q}^{(.)}, \mathbf{v}_P^{(.)}, \mathbf{w}^{(.)} \in \{\mathbf{c}_{(.)}^{(.)}\} \cup \{\mathbf{v}_R^{(.)} : \mathbf{v}_R^{(.)} \in R^{+^n}\}\\ are\ \ permitted.\end{gathered} \tag{15.104}$$

Either $\mathbf{q}^{(.)}$ or $\mathbf{w}^{(.)}$, or both, can be equal to the light velocity $\mathbf{c}_{(.)}^{(.)}$. Another possibility for them is to be equal to the velocity $\mathbf{v}_R^{(.)}$ of the reference point P_R if and only if $\mathbf{v}_R^{(.)}$ is elementwise nonzero vector.

Note 609 ***Characteristics***

The temporal vector - matrix coordinate transformations (15.100) through (15.104) preserve all the characteristics of the transformations (10.50) through (10.54).

Definition 610 ***Transformations and time field***

The vector - matrix transformations (15.99) through (15.104) determine ***time-invariant generally weakly uniform time field*** *if and only if every pair of time axes obeys the transformations, including axes from different layers if the time field is multi-layer.*

Note 611 *The choice of the light signal L for the reference point P_R, $P_R = L$, (i.e. $\mathbf{r}_R(\mathbf{t}^n_{(.)}) \equiv \mathbf{r}_L(\mathbf{t}^n_{(.)})$ and $V^{(.)}_R \equiv C^{(.)}_{(.)}$), reduces the transformations (15.99) through (15.104) to the time-invariant weakly uniform vector - matrix transformations.*

If, additionally, $Q^{(.)}\mathrm{W}^{(.)} \equiv \left[\mathrm{C}^{(.)}_{(.)}\right]^2$ then the transformations reduce further to the time-invariant specially weakly uniform vector - matrix transformations.

Note 612 *The only difference between the time-invariant generally weakly uniform matrix transformations (15.99) through (15.104) and the time-invariant general uniform matrix transformations (15.56) through (15.61) is in the temporal transfer velocity that is $\mathbf{v}^{(.)}_R$ in the former and $\boldsymbol{\vartheta}^{(.)}$ in the latter. When we replace $\boldsymbol{\vartheta}^{(.)}$ and $\mathrm{V}^{(.)}_{\vartheta}$ by $\mathbf{v}^{(.)}_R$ and $\mathrm{V}^{(.)}_R$, respectively, in the temporal coordinate transformations (15.57) and (15.58) then they become (15.100) and (15.101), respectively. Such substitutions transform Section 15.3 into this section on the time-invariant generally weakly uniform matrix transformations.*

15.5 General relative uniformity

The *time* scaling diagonal matrix coefficient $\mathrm{M}_{(.)}$ is determined by

$$\mathbf{t}^n_{(.)} = \mathrm{M}_{(.)}\mathbf{t}^n,\ \mathbf{t}^n_{(.)0} = \mathrm{M}_{(.)}\mathbf{t}^n_0 = \mathrm{M}_{(.)}\mathbf{0} = \mathbf{0},$$
$$diagmin\ \mathrm{M}_{(.)} \in R^+. \tag{15.105}$$

The basic general generic equations (15.2) through (15.5) have the following forms in this framework:

$$\mathbf{t}^n_i = \mathrm{A}^i_j \left[\mathbf{t}^n_j + \mathrm{V}^j_{ji}\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathbf{r}_R(\mathbf{t}^n_j)\right],$$
$$\mathrm{A}^i_j \in R^{nxn}_+,\ diagmin\ \mathrm{A}^i_j \in R^+,$$
$$\mathbf{q}^j \in R^{+^n},\ \mathbf{w}^j \in R^{+^n},\ \mathrm{V}^j_{ji} \in R^{nxn}_+, \tag{15.106}$$

$$\mathbf{t}^n_j = \mathrm{A}^j_i \left[\mathbf{t}^n_i - \mathrm{V}^i_{ji}\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\mathbf{r}_R(\mathbf{t}^n_i)\right],$$
$$\mathrm{A}^j_i \in R^{n\times n}_+,\ diagmin\ \mathrm{A}^j_i \in R^+,$$
$$\mathbf{q}^i \in R^{+^n}, \mathbf{w}^i \in R^{+^n},\ \mathrm{V}^i_{ji} \in R^{nxn}_+, \tag{15.107}$$

$$\mathbf{r}_P(\mathbf{t}^n_i) = \Lambda^i_j \left[\mathbf{r}_P(\mathbf{t}^n_j) + \mathrm{V}^j_{ji}\mathbf{t}^n_j\mathbf{u}\right],$$
$$\Lambda^i_j \in R^{nxn}_+,\ diagmin\ \Lambda^i_j \in R^+, \tag{15.108}$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \Lambda_i^j \left[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}_{ji}^i \mathbf{t}_i^n \mathbf{u}\right],$$
$$\Lambda_i^j \in R_+^{nxn},\ diagmin\Lambda_i^j \in R^+, \tag{15.109}$$

where

$$\mathbf{q}^{(\cdot)}, \mathbf{v}_P^{(.)}, \mathbf{w}^{(\cdot)} \in \{\mathbf{c}_{(.)}^{(\cdot)}\} \cup \left\{\mathbf{v}_R^{(.)},\ iff\ \mathbf{v}_R^{(.)} \neq \mathbf{0}\ elemetwise\right\},$$
$$are\ permitted. \tag{15.110}$$

Either $\mathbf{q}^{(\cdot)}$ or $\mathbf{w}^{(\cdot)}$, or both, can be identically equal to the light velocity $\mathbf{c}_{(.)}^{(\cdot)}$. They can be equal to the velocity $\mathbf{v}_R^{(.)}$ of the reference point P_R if and only if $\mathbf{v}_R^{(.)}$ is elementwise nonzero.

Note 613 ***Characteristics***
The vector-matrix coordinate transformations (15.106) through (15.110) possess the same characteristics as the transformations (10.33) through (10.37).

Definition 614 ***Transformations and time field***
The equations (15.105) through (15.110) determine ***time-invariant generally relatively uniform time field*** *if and only if every pair of time axes obeys the transformations, including axes from different layers if the time field is multi-layer.*

Note 615 *When we replace* $\mathrm{V}_{\vartheta}^{(.)}$ *by* $V_{ji}^{(.)}$ *in the temporal coordinate transformations (15.57) and (15.58) then they become the above temporal coordinate transformations (15.106) and (15.107), respectively. The transformations (15.56), (15.59) and (15.60) are, respectively, the transformations (15.105), (15.108) and (15.109). This means that when* $\mathrm{V}_{\vartheta}^{(.)}$ *is replaced by* $V_{ji}^{(.)}$ *in (15.57) and (15.58) then Section 15.3 becomes this section on the time-invariant generally relatively uniform transformations.*

Note 616 *If* $P_R = L$*, i.e.* $\mathbf{r}_R(\mathbf{t}_{(.)}^n) \equiv \mathbf{r}_L(\mathbf{t}_{(.)}^n)$ *and* $V_R^{(.)} \equiv \mathrm{C}_{(.)}^{(.)}$*, then the time-invariant generally relatively uniform transformations (15.105) through (15.110) become the time-invariant relatively uniform vector - matrix transformations.*

If, additionally, $\mathrm{Q}^{(\cdot)}\mathrm{W}^{(\cdot)} \equiv [\mathrm{C}_{(.)}^{(\cdot)}]^2$*, then the transformations (15.105) through (15.110) become the time-invariant specially relatively uniform vector - matrix transformations.*

15.6 Conclusion on noncolinear motions

Definition 46 and Axiom 47 form the basis for the coordinate transformations. They express partially (in the case of nonuniform and relatively uniform transformations) or completely (in the case of weakly uniform and uniform transformations) the *time* independence of the space.

The vector-matrix coordinate transformations permit arbitrary direction of the movement of the arbitrary point P. It can be in any direction rather than to be restricted only to the direction of the accepted unity vector $\mathbf{u}$. The transformations are in all the general cases completely both entirely and pairwise compatible. They enable also different *time* and velocity units to different entries of the corresponding vector variables. This opens possibilities for their effective applications to complex systems and processes with multiple *time* scales. They preserve consistent use of the values of all speeds relative to the corresponding integral spaces. This includes both the light speed and the transfer speeds.

The vector-matrix coordinate transformations are beyond Einsteinian relativity theory. They confirm once more that the light speed and the transfer speeds are noninvariant relative to both *time* axes and inertial frames. They are not exceptional speeds. They obey the general speed laws. The light speed value does not limit the values of other speeds, in general. This opposes crucially Einsteinian relativity theory and simultaneously verifies Galilean - Newtonian physics.

The vector-matrix coordinate transformations, as their scalar counter-parts, change *time* units so that $\mathfrak{T}_i^n$ and $\mathfrak{T}_j^n$ carry different multiple *time* units. The vector *time* values $\mathbf{t}_i^n \in \mathfrak{T}_i^n$ and $\mathbf{t}_j^n \in \mathfrak{T}_j^n$ are mutually linked by the *time* scaling matrix coefficients M_i and M_j [or, coefficient functions $\mathrm{M}_i(.)$ and $\mathrm{M}_j(.)$] in all the cases (in the general, special and singular cases). Galilean - Newtonian physics has well recognized them in the scalar form if and only if the *time* units are only changed (i.e. without a change of the length unit). However, Einsteinian relativity theory has not recognized the existence of the *time* scaling matrix coefficients M_i and M_j [or, coefficient functions $\mathrm{M}_i(.)$ and $\mathrm{M}_j(.)$].

This holds regardless of the mutual movement (i.e. $\mathbf{v}_{ji}^{(.)} \neq \mathbf{0}$) or rest (i.e. $\mathbf{v}_{ji}^{(.)} = \mathbf{0}$) of the frames R_i^n and R_j^n, which expresses the independence property of *time* (Axiom 47). Einsteinian relativity theory is inapplicable to such cases. Galilean - Newtonian physics has successfully dealt with such cases in the scalar form provided that the *time* scaling coefficients are constant, and that there are not transformations of spatial coordinates. It is generalized herein to *time* scaling coefficients regardless of their dimensionality [i.e. whether they are scalar functions $\mu_i(.)$ and $\mu_j(.)$, or matrix functions $\mathrm{M}_i(.)$ and $\mathrm{M}_j(.)$].

The vector-matrix coordinate transformations, as their scalar predecessors, do not change the numerical values of distances in the general cases. The distance of the arbitrary point P from the origin O_i of the frame R_i^n and its distance from the origin O_j of the frame R_j^n have the same numerical values in spite the frames R_i^n and R_j^n are in a mutual relative motion (i.e. $\mathbf{v}_{ji}^{(.)} \neq \mathbf{0}$). The space scaling matrix coefficients Λ_j^i and Λ_i^j [or, the coefficient functions $\Lambda_j^i(.)$ and $\Lambda_i^j(.)$] are so adjusted, to the movement of the arbitrary point P relative to the origins and to the movements of the frames, that the numerical values of its distances from the origins of the frames is invariant in the general cases. In the special and singular cases, the space scaling matrix coefficients Λ_j^i and Λ_i^j are so adjusted, to the movements of the arbitrary point P relative to the origins of the frames, that the relationship between the numerical values of its distances from the origins of the frames is the same as the relationship between the numerical values of the corresponding vector temporal coordinates.

The vector-matrix coordinate transformations imply new laws of the velocity transformations. They are inherently different from Einstein's law of the composition of velocities. They have the forms of the velocity transformations of Galilean - Newtonian physics (which are valid when the temporal coordinates are only transformed, but not the spatial coordinates). They concern herein both the temporal and the spatial coordinate transformations. Such formulae are not known in Galilean - Newtonian physics for the cases of the simultaneous transformations of the temporal and of the spatial coordinates.

All the results verify once more the noninvariance of both the light speed and the spatial transfer speed. Besides, they show that the light speed value is not the upper bound of the values of other speeds. The light velocity is not a special velocity in general (from the kinematic point of view).

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Chapter 16

Conclusion on CC Relativity Theory

16.1 Common features

The CC relativity theory does not accept a priory any restriction on the values of speeds and on the scaling coefficients. It is completely relaxed of such constraints. It discovers the existence of the basic *time* scaling coefficients in the mathematical relativity theories including both Galilean - Newtonian relativity theory and Einsteinian relativity theory. It introduces them from the theory of dynamical systems with multiple *time* scales into the mathematical relativity theories. It overcomes the problems of inconsistencies and of paradoxes caused by Einstein's a priory accepted sever constraints.

The fundamentals of the CC relativity theory are established so that the transformations obey the generalized distance preservation condition. Each of them forms the Poincaré group.

The CC relativity theory uses consistently the values of all the variables relative to integral spaces. It does not accept a priory the invariance of any speed (e.g. of the light speed and/or of the temporal or spatial transfer speed). It establishes a great variety of the coordinate and of the velocity transformations. It presents the proofs of necessity and of sufficiency of their forms, structures and arguments. It provides also the proofs of complete both entire and pairwise compatibility of the transformations. They open various new directions for future developments of the mathematical relativity theories of *time*. Their common physical basis are the properties of *time* expressed in Axiom 47. The obtained results agree fully with this axiom.

The transformations of the coordinates that do not obey Einstein's a priory accepted constraints are permissible for correct applications. Consequently, the formulae valid in terms of such coordinates need not agree with those of Einsteinian relativity theory. It is not the form of a mathematical model that is crucial in general. If the frames are mutually inertial then the transformed mathematical model should be equivalent to the original mathematical model provided the units are constant. The relationships among coordinates are essential for mathematical models related either to noninertial or to inertial frames. They should adequately express the modelled physical phenomena, process or system. The mathematical models should be mutually equivalent. The physical laws, the used coordinates and the relationships among them, determine the forms of the mathematical models.

We may apply any coordinate transformation to every mathematical model. Whether such application will be useful, we cannot always conclude a priory.

16.2 On applications of the CC Relativity Theory

Applications of the fundamentals of the CC-relativity theory escape the scope of this book. It is left open for future research to investigate relationships among every form of the *time* field and coordinate transformations, on the one side, and mathematical models of various natural phenomena, of natural laws, of physical processes and/or of physical systems, on the other side. This can be carried out for the following examples of applications of Lorentz transformations to Galilean - Newtonian - Maxwellean mathematical models of physical processes and systems:

° the equations of dynamics of the material point [116] and [160, p. 274],

° the equations of mechanics and thermodynamics of systems [116] and [160, p. 284],

° the Maxwell-Lorentz equations describing the electromagnetic field [116] and [160, p. 269], [119] and [161, p. 140],

° the Maxwell-Hertz equations for empty space [114, p. 907], [160, p. 156],

° the theory of Doppler's principle and of aberration [114, p. 910], [160, p. 160],

° the transformation of the energy of light rays and the radiation pressure on mirrors [114, p. 913], [160, p. 162],

° the transformation of the Maxwell-Hertz equations when convection currents are taken into consideration [114, p. 916], [160, p. 165],

° the dynamics of the (slowly accelerated) electron [114, p. 917], [160, p. 167],

° the kinetic energy of a rigid body in uniform translation subject to external forces [115, p. 373], [160, p. 240],

° the inertia of an electrically charged rigid body [115, p. 377], [160, p. 243],

° the energy of a system consisting of a number of mass points moving force-free [115, p. 382], [160, p. 248].

Part V

General Conclusion

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Chapter 17

Problem Solutions

The analyses, the conclusions and the obtained results enable us to reply to the questions and to present the solutions to the problems posed in Preface.

Although the solutions to many raised problems are summarized in the preceding concluding parts (Chapter 9: "Conclusion on *Time* and *Time* Fields", Section 7.4: "Conclusion on Einstein's Theory", Chapter 13: "Conclusion on PCC Relativity Theory", Chapter 16: "Conclusion on CC Relativity Theory"), we will present their synthesis in order to emphasize them due to their clear solutions of the problems, and due to their importance to revaluate objectively Galilean - Newtonian physics in general, hence Galilean - Newtonian relativity theory in particular, in order to evaluate adequately Einsteinian Relativity Theory, and new Consistent *Time* Relativity Theory, as well as to explain the differences among them or what is common for them.

Problem 1 **Does *time* exist as a physical variable or it is an abstract (mathematical) variable or even, only a parameter?**

Solution 617 *Time is a physical variable the value of which increases permanently, smoothly, strictly monotonously continuously, equally in all the spatial directions everywhere, independently of everybody and everything.*

Problem 2 **Are there several *times* or there is the unique *time*?**

Solution 618 *Time is the unique temporal physical variable. There are not several times.*

If we speak about the multiplicity related to time, then we can speak only about the multiplicity of the numerical time value with respect to differently accepted (relative) zero moment, (relative) initial instant, time scale and/or time unit. Every their choice is tied with the same variable - time. Their different choices cannot, hence do not, change either time or its value. Their different choices influence only the numerical value of time.

Problem 3 **How is *time* explained, interpreted and exactly defined in science in general and in Einsteinian relativity theory established by Lorentz, Einstein and Poincaré in particular?**

Solution 619 *Time is treated as independent variable in classical physics, in mathematics, in classical mechanics, in fluid mechanics, in thermodynamics, in control theory, in systems theory and in engineering. These scientific or technical disciplines have rarely clearly explicitly and fully explained the characteristics of time, whether it is a physical variable, or only a pure mathematical variable, or even a conventionally introduced parameter.*

However, time is determined as a variable dependent on spatial frames, hence on space, in Einsteinian relativity theory. By following Einstein, it is therein accepted that time is that what the clock hands show. This explanation identifies time (the temporal variable) with its (numerical) value. It does not distinguish between variable and variable (numerical) value. Even more. It does not recognize the crucial differences among variable, variable value and variable numerical value.

Problem 4 **Has Einsteinian relativity theory succeeded to explain *time* so that the explanation agrees with the physical reality and with our experience?**

Solution 620 *Einsteinian relativity theory has created a deep confusion about the nature, the sense and the properties of time, rather than to present their explanations that agree with the physical reality and with our experience, with the common sense.*

Problem 5 **Is *time* really dependent variable (e.g. on space, on the light velocity) or it is an independent variable?**

Solution 621 *Time is an independent variable. Being independent of everybody and of everything, time is independent also of spatial frames, of space and of velocity.*

Problem 6 **What are the properties of *time* and of the speed of its values flow?**

Solution 622 *Time is unique physical variable. There do not exist several different times. It is untouchable. Nobody and nothing can influence time or its values flow.*

Problem 7 **Can velocity be defined before *time* has been explained and defined? If it can, then: how?**

Solution 623 *The sense, the meaning and the nature of velocity (of an oriented and directional change of a variable value per time), hence of speed (of a change of a variable value per time) as the algebraic value of velocity, has demanded us to define and to explain velocity and speed exclusively in terms of a variation of the variable value per time unit, more precisely, an infinitesimal variation (differential) of the variable value per the infinitesimal increase (positive differential) of the time value. Consequently, velocity and speed cannot be completely and well defined and explained before time itself was completely and well defined and explained. This is the well known elementary fact of the basis of physics, which Einstein ignored.*

Problem 8 **Can a speed numerical value (including the numerical value of the light speed) be determined without having specified a *time* scale and a *time* unit? If it can, then: how?**

Solution 624 *Being defined as the ratio of an (infinitesimal) oriented and directional variation of a variable value and of an (infinitesimal) increase of a time value, velocity must be measured in terms of the variable unit per the time unit. Therefore, it is not possible to measure the values of velocity and of speed if a time scale and a time unit have not been well specified. This is also the well known elementary fact of the fundamentals of physics, which Einstein did not respect.*

Problem 9 **What is the influence of a *time* scale change on a speed (numerical) value [including the (numerical) value of the light speed], if any?**

Solution 625 *Any change of a scale and/or of a unit cannot and does not change the value of the corresponding variable, including the light speed. However, a change of a scale and/or of a unit imply/implies a change of the numerical value of the corresponding variable, including the light speed, in general. These facts are well known in physics, and in engineering. Einstein ignored them.*

In this concern, Lorentz transformations guarantee the changes of the time unit and of the length unit in the same ratio relative to the reference units in the case the arbitrary point moves with the light speed. Consequently, they ensure that the numerical value of the light speed and the numerical vector value of the light velocity do not change. This is a consequence of the a priory accepted restrictions under which Lorentz transformations hold. More precisely, it is a consequence of their determination exclusively for the light speed of the arbitrary point, of the a priory accepted spatial transfer speed for the temporal transfer speed, and of the a priory adopted both the equality of the time scaling factors and the equality of space scaling factors, which caused the mutual equality of the time and space scaling factors. Consequently, the scaling factors mutually cancel in the position - time ratio that determines velocity and speed.

Problem 10 **Is there any relationship between Einstein's interpretation of *time* relativity and Newton's explanation of relative *time*?**

Solution 626 *Newton, and conditionally Einstein, explained the relative meaning of time in the same sense, if we interpreted Einstein's meaning of the time relativity only in the sense of the numerical time value. It is the sense of relativeness of time units and of numerical time values, but not of time itself. However, Einstein did not leave a clear possibility for such an interpretation of his sense of time relativity.*

Problem 11 **Is it impossible to interpret *time* in Newton's sense so that it satisfies Einstein's interpretation of *time*, or vice versa?**

Solution 627 *Newton distinguished time from its value, from its numerical value, and from its units. Einstein equalized time with its numerical value. By understanding this, by assuming that Einstein treated only the numerical time value and not time itself, we could conclude that Newton's explication of time incorporates that of Einstein. However, vice versa does not hold.*

Problem 12 **Is it impossible to characterize *time* so that it agrees with: physical reality, our experience and knowledge, Newton's explanation of *time* and Einstein's interpretation of *time*?**

Solution 628 *Definition 46 and Axiom 47 express the physical reality of time and our experience of it. They agree essentially completely with Newton's explanation of time.*

Problem 13 **What is the relationship between properties of *time* and human (biological and/or psychological) feeling of *time*?**

Solution 629 *Being the unique variable, time possesses the property that its value has been permanently changing (strictly monotonously continuously increasing) in such a manner that the value of its speed is invariant constant, and the numerical value of its speed equals one whatever are the accepted time unit, time scale, and the spatial frame.*

Speeds of propagation of various biological processes are different. They influence us to accept a time unit that corresponds to the speed of our biological rhythm. They can influence us also to think wrongly, and occasionally to speak wrongly, of the speed of time as of the speed of our biological rhythm, of our biological processes. This creates a wrong personal impression that everybody possesses her/his own time and speed of time value evolution. What is only personal related to time, it is our personal choice of a time unit that is adequate to our biological rhythm, or our selection of time units adequate to different biological and psychological processes in ourselves.

When we say that "time passed fast/slowly" then it means that the corresponding process was fast/slow by creating our psychological impression that the value of the time speed increase was big/small. As it was shown, the speed of the time value increase is constant and universal invariant. Its numerical value, which is one, is independent of our biological processes.

When we speak about the biological or psychological age of an organism then it has a sense only if it means the age that corresponds the most adequately to the real (to the actual) biological or psychological state of the organism (Definition 94).

Problem 14 **What are the meanings of the biological age and of the psychological age? What are the speeds of the biological aging and of the psychological aging?**

Solution 630 *The age of the organism at some moment t is the time value at that moment relative to the moment of the birth of the organism, which is the* *initial instant* t_0 *for the organism, i.e. the age of the organism* $= t - t_0$. *The speed of the aging of the organism equals 1 (one) time unit per the same time unit independently of the organism, of everybody else and of everything. The duration of the existence of the organism at the moment* t *is the duration during the time interval* $[t_0, t]$ *and equals the age of the organism* $(t - t_0)$ *at the moment* t.

The biological (the psychological) age at a moment t *represents, respectively, the age of an organism that corresponds the most adequately to the real biological (psychological) state of the organism at the moment* t. *It can be sometimes equal to the age of the organism, but it is different from the age of the organism very often.*

The speed of the biological (the psychological) aging of an organism represents the speed of the change of the biological (the psychological) state of the organism, respectively. It is not the speed of the aging. The speed of the (numerical) time value evolution ("the time speed" for short) that equals 1 (one) time unit per the same time unit regardless of the choice of the time unit is the universal (simple) constant and invariant. It is the speed of the aging of the organism.

The speed of the biological (the psychological) aging is not the time speed, hence, it is not the speed of the aging. The former is time-varying in general, while the latter is constant. The former depends on many factors, the latter is completely independent of everybody and everything.

Problem 15 **Is the light speed invariant in vacuum?**

Solution 631 *The light speed is not invariant in vacuum in general. It is invariant only relative to the integral spaces mutually interrelated by Lorentz coordinate transformations. The light speed is not invariant relative to other integral spaces considered herein.*

Problem 16 **Can the speed of an arbitrary material point (of a particle), or of a body, be greater than the light speed in some coordinate systems?**

Solution 632 *Lorentz - Einstein invariance of the light speed value and its appearance as the supremum of all physically realizable speed values are not properties of light or of its speed. The former is the feature of the structure of the formula expressing Einstein's law of the composition of velocities and the consequence of the choice of the light speed for the reference speed therein. The latter is due to both Lorentz scaling coefficients and Einstein's famous formula for mass in terms of the speed.*

There are other integral spaces with inertial frames, e.g. Galilean - Newtonian, in which such a limitation by the light speed does not exist. A number of such other various cases fill out Part III and Part IV. The speed can have a bigger value than the value of the light speed (in vacuum). Lorentz transformations are inapplicable to such cases.

Problem 17 **Do there exist coordinate transformations and from them deduced speed transformations relative to which an arbitrary speed is invariant in vacuum?**

Solution 633 *Yes, they exist. For every velocity/speed there exist linear coordinate transformations that satisfy the general, or Einstein's generalized, distance condition, and from them deduced velocity/speed transformations, relative to which the chosen velocity/speed is invariant. For the light velocity/speed such transformations are Lorentz transformations and Einstein's velocity/speed transformation that is deduced from them.*

Problem 18 **Under what necessary and sufficient conditions Lorentz transformations permit invariance of an arbitrary speed in general and of the spatial transfer speed in particular?**

Solution 634 *For Lorentz transformations to permit invariance of an arbitrary speed it is necessary and sufficient that the speed is equal to the light speed. This is a paradox and an absurd of Lorentz transformations and of Einsteinian relativity theory because Lorentz transformations do not permit for the spatial transfer speed to be equal to the light speed.*

Problem 19 **Under what necessary and sufficient conditions on the spatial transfer speed can Lorentz transformations obey Einstein's distance condition?**

Solution 635 *The invariance of the spatial transfer speed is necessary and sufficient for Lorentz transformations to obey Einstein's distance condition. This is another paradox Einsteinian relativity theory. The spatial transfer speed can be invariant in Lorentz transformations if, and only if it is the light speed, but in that case Lorentz transformations become undefined. Einstein's theory becoming undefined vanishes in infinity!*

Problem 20 D**o Galilei-Newton's transformations of coordinates satisfy Einstein's special condition for the distance preservation in the *time*-space environment?**

Solution 636 *Yes, they do and reject Einstein's claim that they do not.*

Problem 21 **Do Galilei-Newton's transformations of coordinates satisfy Einstein's generalized condition for the distance preservation in the time-space environment?**

Solution 637 *They do. This rejects Einstein's claim that they do not.*

Problem 22 **What are the consequences of the proposed definition and characterization of time for the relativity theory from the point of view of the physical reality, the common human experience and knowledge?**

Solution 638 *Definition 46 and Axiom 47 express the properties of time. They enable us to understand the restrictions and limitations of Einsteinian relativity theory, to understand why it concerns only the singular case that is a physical absurd, and they enable us to develop further the relativity theory in diverse directions beyond both Galilean - Newtonian physics and Einsteinian relativity theory.*

Problem 23 **Can such a characterization of *time* permit new *time* coordinate transformations, new space coordinate transformations and new velocity transformations?**

Solution 639 *The characterization of time presented in Axiom 47 enabled us to establish new coordinate transformations, new velocity transformations and new acceleration transformations, which discover new relationships among temporal and velocity coordinates.*

Problem 24 **Do there exist non-Lorentzian linear coordinate transformations that satisfy general or Einstein's distance condition?**

Solution 640 *Yes, they exist. They are shown in the book.*

Problem 25 **Do the accepted definition and characterization of *time* permit a relaxation of all a priory accepted assumptions and constraints in Einsteinian relativity theory?**

Solution 641 *Yes. The presented time definition and characterization opened a possibility to allow a priory different time and space scaling coefficients, non-invariance of the light speed and of the spatial transfer speed, and a series of new forms of coordinate transformations. Consequently, they led to a family of new formulae on coordinate and velocity transformations. This is achieved by a partial relaxation of Lorentz - Einstein - Poincaré limitations in the framework of the novel mathematical Partially Compatible but Consistent (PCC) Relativity Theory. It enabled the complete relaxation of all a priory accepted constraints in Einsteinian relativity theory. Consequently, the fundamentals are established for the novel Compatible and Consistent (CC) Relativity Theory. It involves a family of various coordinate transformations and related time fields.*

Problem 26 **What is the relationship between *time* and multiple *time* scale dynamical systems?**

Solution 642 *The theory of dynamical systems with multiple time scales exploits different time scaling coefficients $\mu_{(.)}$ in order to express different speeds of propagation of different processes, which induce different time scales and units. It is shown herein that Einsteinian relativity theory involves different time scaling coefficients $\mu_{(.)}$, but it has not recognized them, hence it has not used them. Their application to the relativity theory enables us to avoid all a priory accepted constraints that form the basis and the limits of the validity of Einsteinian relativity theory. Consequently, the obtained new results constitute the fundamentals* *of various directions of the novel CC relativity theory. They explain, from the relativity theory point of view, when a dynamical system can possess multiple time scales.*

Problem 27 **What are the meanings of partial and of complete compatibility of the transformations?**

Solution 643 *The transformations (6.13) through (6.16) are* ***partially compatible*** *if and only if they are compatible, respectively, exclusively either when the arbitrary point P moves with the speed restricted to be equal to a specific speed, or when the generic point G, hence the reference point* P_R *or* P_{SU}*, moves with the speed restricted to be equal to a specific speed, or the product value* $q(t_{(.)}; t_{(.)0}$ *of the generic speeds* $q(t_{(.)}; t_{(.)0})$ *and* $w(t_{(.)}; t_{(.)0})$ *should be equal to a squared specific speed value.*

The transformations (6.13) through (6.16) are ***completely (pairwise, entirely) compatible*** *if and only if they are, respectively, (pairwise, entirely) compatible for any nonzero speed of the arbitrary point P and for any nonnegative speed of the generic point G, hence, of the reference point* P_R *or* P_{SU}*, and for any positive value of the product* $q(t_{(.)}; t_{(.)0})w(t_{(.)}; t_{(.)0})$ *of the generic speeds* $q(t_{(.)}; t_{(.)0})$ *and* $w(t_{(.)}; t_{(.)0})$*; hence, which are not restricted to be equal to specific speeds.*

Problem 28 **What is the meaning of the consistency of the transformations?**

Solution 644 *The transformations are consistent if, and only if, the values of every variable are measured with its same corresponding unit relative to the used coordinate system. Einstein's relativity theory does not satisfy this condition. Consistent Relativity Theory does satisfy the condition.*

Problem 29 **What are the forms of partially compatible but consistent transformations that satisfy the general or Einstein's distance condition?**

Solution 645 *They are essentially different from Lorentz transformations. The book present them in the general case, in which they are crucially different from Lorentz transformations, in the special case and in the singular case in which they either generalize Lorentz transformations or reduce to Lorentz transformations. In all the cases they are based on, and result from, the accepted definition and characterization of time.*

Problem 30 **What are from them deduced velocity transformations like?**

Solution 646 *They are crucially different from Einstein's speed transformations. In the general case they are essentially beyond Einstein's relativity theory. In the special case and in the singular case they either generalize Einstein's formulae or reduce to them. In all the cases they are based on, and result from, the accepted definition and characterization of time.*

Problem 31 **What are the forms of completely compatible and consistent transformations that satisfy the general or Einstein's distance condition?**

Solution 647 *They are substantially different from Lorentz transformations. The book present them in the general case, in which they are crucially different from Lorentz transformations, in the special case and in the singular case in which they either generalize Lorentz transformations or reduce to Lorentz transformations. In all the cases they are based on, and result from, the accepted definition and characterization of time.*

Problem 32 **What are from them deduced velocity and acceleration transformations like?**

Solution 648 *They are inherently different from Einstein's speed and acceleration transformations, respectively. In the general case they are essentially beyond Einstein's relativity theory, and they generalize Galilean - Newton's velocity transformations. In the special case and in the singular case they either generalize Einstein's formulae or reduce to them. In all the cases they are based on, and result from, the accepted definition and characterization of time.*

Problem 33 **Do they lead to a new relativity theory**, **the basis of which are consistent transformations?**

Solution 649 *Yes, they do. It is Consistent Relativity Theory.*

Problem 34 **What is the relationship of the new results relative to the corresponding Galilei-Newton's and Einstein's results?**

Solution 650 *The new results in general case are substantially different from those in Galilei-Newton's physics and Einsteinian physics. They rely fully on the proposed definition and characterization of time. They overcome all inconsistencies, paradoxes, absurds, and mistakes of Einstein's relativity theory.*

When all Lorentz - Einstein's (tacit) assumptions and restrictions are accepted, under which Lorentz and Einstein established their results, then Consistent Relativity Theory reproves them formally, mathematically, but starting from the accepted definition and characterization of time.

The new results incorporate the results of Galilean - Newtonian physics. The former generalize the latter in the general cases.

Problem 35 **What are implications of the properties of *time* on: physical variables, dynamical systems and control?**

Solution 651 *The multiplicity of time scales and units can be linked with the multiplicity of time scales of dynamical systems. The results of the relativity theory, when it is linked with the theory of dynamical physical (including biological) systems with multiple time scales, clarify the conditions under which a dynamical physical system can possess multiple different time scales.*

An intrinsic time property is the permanent continuity of the strictly monotonous time value increase. When it is linked with the physical continuity and uniqueness of physical variables, then we become aware that every physical variable can change its value exclusively continuously in time. This holds without an exception for control variables, hence for control vector, for short, for control. Consequently, control algorithms that determine control as continuous in time, are potentially realizable exactly. Those control algorithms that determine control as discontinuous in time are unrealizable exactly.

Problem 36 **What is the relationship between the properties of *time* and human (biological and/or psychological) feeling of *time*?**

Solution 652 *We know that time value increases irreversibly and continuously, and that nobody and nothing can influence this temporal flow of time values. Consequently, everybody has been aging with the constant speed that is the speed of the time value evolution, one time unit per the same time unit.*

Although we age with the same speed, the speeds of our biological aging and of our psychological aging (i.e. the speeds of the changes of the biological and psychological states of ourselves) are different, are individual, and can be variable during our lives, which are consequences of our biological rhythm, its variations and various influences.

Problem 37

What is *time*?
What is relativity of *time*?
Is the light velocity invariant?
Is the light speed the limiting speed?
Is it possible to establish a consistent relativity theory?
Can the physical properties of *time* be the base of such theory?
Can the theory be relaxed of all Lorentz'-Einstein's assumptions?
Can such theory reflect the common human experience with *time* ?

Solution 653 *Time is a physical variable. It is an elementary primary constituent of the existence of everybody and of everything. It is its own single component. Its nature cannot be explained in terms of another variable, notion, category, or phenomenon. It cannot be explained in terms of energy, matter or space. It is a basic physical variable. Its nature is unique, original, inherently different from the nature of any other physical variable. Its nature is axiomatically called temporal.*

Time is a unique physical variable with the properties explained in Axiom 47. One of its crucial properties is its total independence. Another one is the complete invariance of the speed of its value variation, its constancy and the independence of its numerical value 1 (one) of all time units and time scales, as well as its full independence of all spatial frames, hence, of space, of energy, of matter, of anybody and anything. Time and the time speed are untouchable. Nobody and nothing can influence them, and vice versa, they do not influence *anybody and anything. However, the existence of everybody and everything is in time. Time value penetrates everybody and everything.*

Relativity of time is relativity only of its zero value, of its initial value, of its scale and of its unit accepted to measure its value. Their choices are unbounded. We can establish different transformations of time numerical values. Classes of such new transformations are discovered herein. They can be partially or completely compatible among themselves, or jointly with spatial coordinate transformations. The new transformations open novel directions in the relativity theory area. They enabled us to establish the fundamentals of the mathematical theory of time relativity that is compatible and consistent. They imply new velocity / speed transformations, which are beyond Galilean - Newtonian physics and Einstein's relativity theory.

The light speed is noninvariant in general. It obeys the herein generalized Galilean - Newtonian law of the composition of velocity in many cases that are not restricted by Einstein's a priory accepted constraints. This holds for example as soon as the same time axis holds for all inertial frames under the consideration. It is formally invariant in integral spaces over which the coordinates are mutually related by Lorentz transformations. However, such its invariance, called herein its Lorentz - Einstein invariance, is not its own property, but it reflects both the feature of the formulae of Einstein's law of the composition of velocities and the selection of the light speed value for the reference speed value in the formulae.

Moreover, for every velocity there exist coordinate and velocity transformations under which the given velocity is invariant. For the light velocity such transformations are Lorentz transformations and Einstein's law of the velocities composition.

This book establishes the foundations of the new relativity theory that is Consistent Relativity Theory (CRT). The use of units is consistent relative to the time-space frame for every variable. All transformations are consistent. Various types of transformations compatibility are discovered. Consistent Relativity Theory comprises two parts. One is on partially compatible consistent transformations. The other part is on completely compatible transformations..

The physical properties of time constitute the basis of Consistent Relativity Theory. It reflects the time independence of space. It verifies the time uniqueness.

Einstein's relativity theory is strictly restricted by the (tacit) assumptions accepted as postulates (the light speed invariance, the spatial transfer speed invariance) or as incontestable, a priory accepted, relations (the equality of temporal and spatial scaling coefficients). It is also crucially restricted by the use of the light speed for the speed of the arbitrary point P. Another its restriction is the acceptance, a priory, of the spatial transfer speed for the temporal transfer speed. All its formulae and their proofs hold exclusively for the light speed of the arbitrary point P. The consequences of the restrictions and of physically unacceptable Einstein's a priory posed postulates on the light speed invariance, on time nonuniqueness and on its speed nonuniqueness are various mistakes, inconsistencies, paradoxes and/or absurds. The overall result is the invalidity *of Einstein's theory of time relativity.*

Consistent Relativity Theory permits an arbitrary speed of the arbitrary point P. It allows all scaling coefficients to be mutually different a priory. It does not demand the invariance either of the light speed or of the spatial transfer speed. The spatial transfer speed and the temporal transfer speed are mutually independent, in general, in Consistent Relativity Theory.

Consistent Relativity Theory reflects fully the human experience with time. It recognizes all properties of time as determined in Definition 46 and in Axiom 47, which result from the physical reality and from the common human experience with time. It explains that time does not influence any clock, and that a kind of energy is the source for every clock functioning. It explains also that properties of time such as the constant and invariant speed of the time flow, the time independence and its speed flow independence of everybody and everything verify its uniqueness; i.e., that there do not exist two or more time flow speeds or two or more different times. It expresses the common human experience that time itself is not relative and that time relativeness concerns only relativeness of time value and time numerical value relative to the choice of the zero instant, initial instant, time scale and time unit.

Chapter 18

Summary on *time*

If two people meet twice they must have lived the same time between the two meetings, even if one of them has travelled to a distant part of the universe and back in interim.
Arthur EDDINGTON [110, p. 38]

Summary 654 *Time does exist regardless of the (non)existence of a movement, energy, matter and energy-matter space. It has ever existed and will exist for ever, everywhere, independently of everybody and of everything. It is a physical variable.*

Summary 655 *The set of all time values is strictly ordered, infinite, unbounded, equivalent to the set of real numbers, but has its dynamics while the set of real numbers does not possess such dynamics. It is the eternity.*

Summary 656 *Time does not have a total zero value. Any moment can be freely accepted for its relative zero value.*

Summary 657 *Time does not have either a minimal or a maximal value. It does not have the beginning or the end. Its total initial value has not existed. Its total final value will not occur. Any moment can be accepted for a relative initial moment.*

Summary 658 *Time is the unique physical variable with the flow of its values independently of everybody and of everything, everywhere and always.*

Summary 659 *Time and its values cannot be created or destroyed.*

Summary 660 *The nature of time cannot be explained in terms of other variables. It is self-contained. Time does not contain either energy or matter or space in itself. It cannot be explained in terms of energy, matter, space and other physical or mathematical variables or categories.*

Summary 661 *Every time value can happen exactly once.*

Summary 662 *The numerical value of the time speed equals one, (1), with respect to all integral spaces.*

Summary 663 *The time values and the time numerical values flow permanently, strictly monotonously continuously increasingly.*

Summary 664 *Time value occupies (imbues, impregnates, penetrates, is in and over, passes temporally through) every cell of everybody, and everything, everywhere and always.*

Summary 665 *Any variation of a value of any physical variable can occur only continuously, during a time interval.*

Summary 666 *Time and its values are untouchable. Nobody and nothing can influence time, its value, its values flow and the speed of its values flow.*

Summary 667 *Time and its values have not been ever forcing and will not ever force anybody and anything (including the clock), anywhere, for any action. It is a kind of energy, not time, which forces the clock to work. Any clock can not influence time. The clock operation does not have any influence either on time, or on time values flow or on its speed.*

Summary 668 *The time independence, uniqueness, strictly monotonous flow of time values, the constant and invariant speed of the time values flow form the basis for the clock design and functioning.*

Summary 669 *There are infinitely many different time units and time scales. They determine infinitely many different time sets and axes in this sense.*

Summary 670 *Relativity of time is meaningful only in the sense of relativity of the accepted zero moment, of the adopted initial moment, of the used time unit and/or of the selected time scale. This is the physical sense of time relativity.*

Summary 671 *Time is bridge for, and cross-roads of, all beings, movements, processes and variables.*

Summary 672 *The human does not have a particular organ for sensing time, the value of time and/or its flow. The body feels the flow of time values with every cell.*

Summary 673 *Time appears psychologically the most intimate physical variable for the human.*

Fugit irreparabile tempus

Part VI

Subsidiary Parts

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Chapter 19

Notational Details

19.1 Introductory comment

The meaning of the notation is explained in the text at its first use.

Two level indexes (two level subscripts and two level superscripts) are used in order to denote precisely values of variables with respect to *time* axes and relative to spatial coordinate systems (frames).

The norm of a vector together with the vector direction do not represent, in general, completely the vector concerned because its sense can be opposite to the sense of the colinear unity vector that determines the positive sense. We will use the algebraic value of a vector, which is measured along the vector direction and with respect to the sense of the unity vector. It and the unity vector determine completely the corresponding colinear vector. However, if a vector is not colinear with the accepted and fixed constant unity vector then it is completely determined by both its matrix value and the unity vector.

19.2 Indexes

In principle, a single superscript at the letter denoting a variable designates the used *time* axis, hence the used *time* unit. A single subscript at the letter denoting the variable determines the corresponding spatial coordinate system with respect to which the variable value is (to be) measured.

19.2.1 In general

(.), (..) indexes in general, if it is not otherwise stated, where

$$(.) \in \{-, i, j\} \ \ or \ (.) \in \{-, 1, 2, ...s\} \, .$$

"$(.) = -$" means that (.) may be omitted. Throughout the book where it is not explicitly written otherwise then it means that $(.) \in \{-, i, j\}$ holds. If $(.) \in \{-, 1, 2, ...s\}$ then (.) stands for the *time* scale (.).

$i, j \in \{-, 1, 2, ..., s\}, \ i \leq j,$ is valid throughout the book if it is not written otherwise. It is omitted for the sake of the simplicity.

19.2.2 Subscripts

d the subscript d denotes "discrete",

e the subscript e denotes "extension", "extended", or "energy",

i the subscript i denotes "the i-th", or "with respect to the origin O_i of R_i^n",

j the subscript j denotes "the j-th", or "with respect to the origin O_j of R_j^n",

m the subscript m denotes "movable" or "moving", or "mass",

P the subscript P denotes "arbitrary point P",

r the subscript r denotes "at rest",

R the subscript R denotes "reference P_R",

SU the subscript SU denotes "spatial uniformity",

t the subscript t denotes "dependent on *time* t", "function of *time* t",

$\mathbf{t}$ the subscript $\mathbf{t}$ denotes "dependent on the *time* vector $\mathbf{t}^n$", "function of the *time* vector $\mathbf{t}^n$",

u the subscript u denotes "unity",

0 the subscript 0 (zero) associated with a variable (.) denotes its "initial value"; however, if $(.) \in \{\mathfrak{T}, T\}$ then the subscript 0 (zero) associated with (.) denotes the *time* set $\mathfrak{T}_0$ or the *time* axis T_0, or it denotes "at rest",

zero the subscript *zero* denotes "the zero value", or it denotes "at rest",

$-$ denotes the blank place, and if it is used as a subscript then the blank space has the meaning of the subscript 0 (zero),

(..) denotes a point (..), $(..) \in \{G, L, P, P_R, P_{SU}\}$ if not stated otherwise.

19.2.3 Superscripts

i the superscript i denotes "with respect to the *time* axis T_i", i.e. "*time* value measured with the *time* unit 1_{t_i}",

j the superscript j denotes "with respect to the *time* axis T_j", i.e. "*time* value measured with the *time* unit 1_{t_j}",

k the superscript k denotes "k-dimensional", $k \in \{1, 2, ..., n, ...\}$,

$O_{(.)}$ means "with respect to the origin $O_{(.)}$ of $R_{(.)}^n$",

0 the superscript 0 (which can be omitted) denotes that the *time* value is measured with the *time* unit 1_t of the *time* axis T,

$-$ denotes the blank space with the meaning of the superscript 0 (zero).

19.3 Letters

Lower case block or italic letters are used for scalars. Lower case bold block letters denote vectors. Upper-case letters denote matrices, or points or sets or spaces.

The notation "$;t_{(.)0}$" will be omitted as an argument of a variable if, and only if, a choice of the initial moment $t_{(.)0}$ does not have any influence on the value of the variable.

19.3.1 *Caligraphic letters*

Capital calligraphic letters denote sets.
$\mathcal{A} \subseteq R^n$ *a nonempty subset of* R^n,
$\mathcal{B} \subseteq R^n$ *a nonempty subset of* R^n.

19.3.2 Fraktur letters

Capital fraktur letters are used for spaces or for sets.
$\mathfrak{A}^3$ *the angle* (i.e. *the orientation*) *space*, the mathematical model of which is R^3,
$\mathfrak{c} \in R^+$ *the numerical value of the light speed measured in* Kms^{-1} *in vacuum*, $\mathfrak{c} = 2.99792458\text{x}10^8$,
$\mathfrak{C}$ *the set of all complex numbers*, *the family of all continuous functions*,
$\mathfrak{E}^3$ *an energy-matter space*, the mathematical model of which is R^3,
$\mathfrak{G}^n$ *n-dimensional* **g-** *physical variable space* is the set of all numerical vector values of a physical variable $\mathbf{g}(.)$, its mathematical representation G^n is a subset of R^n, $G^n \subseteq R^n$,
$\mathfrak{I}$ *the* 1+*n dimensional integral space*,

$$\mathfrak{I} = \mathfrak{T}\text{x}\mathfrak{R}^n = \{(t, \mathbf{x}) : t \in \mathfrak{T},\, \mathbf{x} \in R^n\},$$

its mathematical representation is I,
$\mathfrak{I}_{(.)}$ *the* 1+*n dimensional (.)-integral space*,

$$\mathfrak{I}_{(.)} = \mathfrak{T}_{(.)}\text{x}\mathfrak{R}^n_{(.)} = \left\{\left(t_{(.)}, \mathbf{x}\right) : t_{(.)} \in \mathfrak{T}_{(.)},\, \mathbf{x} \in \mathfrak{R}^n_{(.)}\right\},$$

$\mathfrak{M}^{12}$ *the motion space*, the mathematical description of which is R^{12},
$\mathfrak{N}(P)$ *a neighborhood of a point* P is a set that contains a set $\mathfrak{N}_\alpha(P)$ of all points P_i such that $\left\|\mathbf{r}^O_{P_i} - \mathbf{r}^O_P\right\| < \alpha$ for any accepted norm $\|.\|$ and for some $\alpha > 0$,

$$\mathfrak{N}_\alpha(\mathbf{x}) = \{\mathbf{x} : \|\mathbf{x}\| < \alpha\},\ \mathfrak{N}_\alpha(P) = \left\{P_i\text{: } \left\|\mathbf{r}^O_{P_i} - \mathbf{r}^O_P\right\| < \alpha\right\},$$
$$\exists\, \alpha > 0 \Longrightarrow \mathfrak{N}_\alpha(P) \subseteq \mathfrak{N}(P),$$

where
$\mathfrak{N}_\alpha(P)$ is *the* $\alpha-$*neighborhood of* P,
$\mathfrak{P}^P$ a *p-dimensional* **p**-*physical variable space*; its mathematical representation P^p is a subset of R^p, $P^p \subseteq R^p$,
$\mathfrak{R}^3$ *the position space,*

$$\mathfrak{R}^3 = \left\{ \begin{array}{c} \mathbf{x} : \text{dim}\mathbf{x} = 3,\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ x_i \in \mathfrak{R}, \\ phdimx_i = L,\ \|\mathbf{x}\| < \infty \end{array} \right\},$$

equivalently,

$$\mathfrak{R}^3 = \underset{P}{\cup}\left[\underset{\alpha>0}{\cup}\mathfrak{N}_\alpha(P),\ \dim\mathbf{r}_P^O = 3\right].$$

which is equalized formally (by ignoring the physical dimension of the position) with its mathematical model - the real vector space R^3, i.e. formally $\mathfrak{R}^3 = R^3$,

$\mathfrak{R}^n$ an *n-dimensional real space*; its mathematical representation is R^n,

$\mathfrak{T}$ *the accepted reference time set*, the arbitrary element of which is an arbitrary moment t and the *time* unit of which is second, $1_t = s$, $t\left\langle s\right\rangle$,

$$\mathfrak{T} = \left\{t : t[T]\left\langle s\right\rangle,\ numt \in R,\ dt > 0\right\},\ \ inf\ \mathfrak{T} = -\infty,\ \ sup\ \mathfrak{T} = \infty,$$

$\mathfrak{T}_{(.)}$ *the accepted time set*, the arbitrary element of which is an arbitrary moment $t_{(.)}$, and the *time* unit of which is $1_{t_{(.)}}$, $t_{(.)}\left\langle 1_{t_{(.)}}\right\rangle$,

$$\mathfrak{T}_{(.)} = \left\{t_{(.)} : t_{(.)}[T]\left\langle 1_{t_{(.)}}\right\rangle,\ numt_{(.)} \in R,\ dt_{(.)} > 0\right\},$$
$$inf\ \mathfrak{T}_{(.)} = -\infty,\ \ sup\ \mathfrak{T}_{(.)} = \infty,$$

$\mathfrak{T}_{(.)0}$ *the subset of* $\mathfrak{T}_{(.)}$, which has the minimal element $min\mathfrak{T}_{(.)0}$ that is the initial instant t_0,

$$\mathfrak{T}_{(.)0} = \left\{t_{(.)} : t_{(.)} \in \mathfrak{T}_{(.)},\ t_{(.)} \geq t_{(.)0}\right\}, \mathfrak{T}_{(.)0} \subset \mathfrak{T}_{(.)},$$
$$min\mathfrak{T}_{(.)0} = t_{(.)0} \in \mathfrak{T}_{(.)} \cup R,\ \ sup\ \mathfrak{T}_{(.)0} = \infty,$$

$\mathfrak{T}^1$ *the accepted reference vector time* set, the arbitrary element of which is an arbitrary one dimensional vector moment $\mathbf{t}$ and the *time* unit of which is second, $1_t = s$, $t\left\langle s\right\rangle$,

$$\mathfrak{T}^1 = \left\{\mathbf{t} :\ \mathbf{t} = (t)\,, t \in \mathfrak{T}\right\},\ \ inf\ \mathfrak{T}^1 = (-\infty),\ \ sup\ \mathfrak{T}^1 = (\infty),$$

$\mathfrak{T}^k$ *the Cartesian product of* $\mathfrak{T}^1$ *by itself k - times,*

$$\mathfrak{T}^k = \mathfrak{T}^1 \mathsf{x} \mathfrak{T}^1 \mathsf{x}\ ...\ \mathsf{x}\mathfrak{T}^1\ (k - times),$$

$\mathfrak{T}_M^s$ *the s-dimensional time space,*

$$\mathfrak{T}_M^s = \left\{\mathbf{t}^s : \mathbf{t}^s = M\mathbf{t}_1^s = tM\mathbf{1}^s\right\}, \mathfrak{T}_I^s = \mathfrak{T}^s\ for\ short,$$

$\mathfrak{T}_{(.)}^n$ *the product time set,* $\mathfrak{T}_{(.)_1} \mathsf{x} \mathfrak{T}_{(.)_2} \mathsf{x}\ ...\ \mathsf{x}\mathfrak{T}_{(.)_n}$,

$\mathfrak{T}_{(.)}^n$ *the space of the nxn diagonal matrices, the i-th diagonal entry of which is* $t_{(.)i}$,

$\mathfrak{V}^3$ *the translational velocity space* (**v**-*physical space*) is the set of all vector numerical values of the velocity **v**; its mathematical description is R^3,

$\mathfrak{V}^6$ *the full (translational and angular) velocity space*; its mathematical description is R^6.

19.3.3 Greek letters

$\alpha_i^j(.)\ :\ \mathfrak{T}\text{x}...\text{x}R^{(..)}\ \longrightarrow\ R^+$ a positive real valued *time scaling coefficient function* in general, which determines the corresponding temporal coordinate transformation from $\mathfrak{T}_i$ into $\mathfrak{T}_j$; $\alpha_i^j(.) \equiv \alpha_i^j$ if, and only if, $\alpha_i^j(.)$ is constant,

α_{ij} denotes both α_i^j and α_j^i if, and only if, $\alpha_i^j = \alpha_j^i$ and then $\alpha_i^j = \alpha_j^i = \alpha_{ij} \equiv \alpha_{ji}$,

δ_{ij} *the Kronecker delta*, $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$,

$\eta \in R^+$ *the angle scaling coefficient*, which transforms the angle unit 1rad into $N_{\varphi 1_{rad}}$ units 1_φ,

$$1rad \left\langle rad \right\rangle = \left(N_{\varphi 1_{rad}} 1_\varphi \right) \left\langle 1_\varphi \right\rangle = \eta \left\langle 1_\varphi rad^{-1} \right\rangle 1rad \left\langle rad \right\rangle = \left(\eta 1rad \right) \left\langle 1_\varphi \right\rangle ,$$
$$N_{\varphi 1_{rad}} \left\langle - \right\rangle = numN_{\varphi 1_{rad}} \left\langle - \right\rangle = num\eta \left\langle - \right\rangle ,$$

i.e. in general, it transforms φ radians into φ_φ units 1_φ,

$$\varphi rad \left\langle rad \right\rangle = \left(\varphi_\varphi \right) \left\langle 1_\varphi \right\rangle = \eta \left\langle 1_\varphi rad^{-1} \right\rangle \varphi \left\langle rad \right\rangle = \left(\eta \varphi \right) \left\langle 1_\varphi \right\rangle ,$$

for short $\varphi_\varphi = \eta\varphi$,

θ *temperature* or *the temperature value*,

$\vartheta^{(.)} \in R^+$ an arbitrarily accepted and then fixed constant *temporal transfer speed*, the value of which is measured with the length unit $1_{L_{(.)}}$ and with the *time* unit $1_{t_{(.)}}$, $\vartheta^{(.)} \left\langle 1_{L_{(.)}} 1_{t_{(.)}}^{-1} \right\rangle$, and in special cases we permit $\vartheta^{(.)} \in \{ v_{ji}^{(.)}, v_R^{(.)} \}$,

$\lambda_i^j(.)\ :\ \mathfrak{T}\text{x}...\text{x}R^{(..)}\ \longrightarrow\ R^+$ a positive real valued *space scaling coefficient function* in general, which determines the corresponding spatial coordinate transformation from R_i^n into R_j^j; $\lambda_i^j(.) \equiv \lambda_i^j$ if, and only if, $\lambda_i^j(.)$ is constant,

$\lambda_{ij}(.)$ denotes both $\lambda_i^j(.)$ and $\lambda_j^i(.)$ if, and only if, $\lambda_i^j(.) \equiv \lambda_j^i(.)$ and then $\lambda_i^j(.) \equiv \lambda_j^i(.) \equiv \lambda_{ij}(.) \equiv \lambda_{ji}(.)$,

$\Lambda_i^j(.) : \mathfrak{T}^n\text{x}...\text{x}R^{(..)} \longrightarrow R_+^{nxn}$ a diagonally elementwise positive real valued *space scaling diagonal matrix function* in general,

$$\Lambda_i^j(.) = diag \left\{ \lambda_{i_1}^{j_1}(.) \quad \lambda_{i_2}^{j_2}(.) \quad ... \quad \lambda_{i_n}^{j_n}(.) \right\},\ \lambda_{i_k}^{j_k}(.) : \mathfrak{T}\text{x}...\text{x}R^{(..)} \rightarrow R^+,$$
$$i_k, j_k \in \{1, 2, ..., s\},\ k = 1, 2, .., n,$$

if, and only if, all $\lambda_{i_k}^{j_k}(.)$ are constant, i.e. if $\lambda_{i_k}^{j_k}(.) \equiv \lambda_{i_k}^{j_k}$ then $\Lambda_i^j(.) \equiv \Lambda_i^j$,

$\Lambda_{ij}(.)$ denotes both $\Lambda_i^j(.)$ and $\Lambda_j^i(.)$ if, and only if, $\Lambda_i^j(.) \equiv \Lambda_j^i(.)$ and then $\Lambda_i^j(.) \equiv \Lambda_j^i(.) \equiv \Lambda_{ij}(.) \equiv \Lambda_{ji}(.)$,

$\mu_{(.)}(..) \in \mathfrak{T}\text{x}...\text{x}R^{(...)} \longrightarrow R^+$ the positive real valued (*basic*) *time scaling coefficient function* that transforms the *time* unit $1_t = s = 1 \left\langle s \right\rangle$ of the reference *time* axis T into $N_{(.)1_t}$ *time* units $1_{(.)}$ of the *time* axis $T_{(.)}$, i.e. it transforms t units 1_t into $t_{(.)}$ units $1_{(.)}$,

$$1_t = \left(N_{(.)1_t} 1_{(.)} \right) \left\langle 1_{(.)} \right\rangle,\ t \left\langle 1_t \right\rangle = t_{(.)} \left\langle 1_{(.)} \right\rangle = \mu_{(.)}(..) \left\langle 1_{(.)} 1_t^{-1} \right\rangle t \left\langle 1_t \right\rangle =$$
$$= \left(\mu_{(.)}(..)t \right) \left\langle 1_{(.)} \right\rangle ,$$

or for short $t_{(.)} \equiv \mu_{(.)}(..)t$, if, and only if, $\mu_{(.)}(..)$ is constant then $\mu_{(.)}(..) = \mu_{(.)}$,

$\mu_{ij} \in R^+$ *the time scaling coefficient* that transforms the *time* unit 1_j of the *time* axis T_j into N_{i1_j} *time* units 1_i of the *time* axis T_i, i.e. it transforms t_j units 1_j into t_i units 1_i,

$$1_j \langle 1_j \rangle = (N_{i1_j} 1_i) \langle 1_i \rangle, \; t_j \langle 1_j \rangle = t_i \langle 1_i \rangle = \mu_{ij} \left\langle 1_i 1_j^{-1} \right\rangle t_j \langle 1_j \rangle = \left(\mu_{ij} t_j \right) \langle 1_i \rangle,$$

or for short, $t_i = \mu_{ij} t_j$,

$\mu_{ji} \in R^+$ *the time scaling coefficient* that transforms the *time* unit 1_i of the *time* axis T_i into N_{j1_i} *time* units 1_j of the *time* axis T_j, i.e. it transforms t_i units 1_i into t_j units 1_j,

$$1_i \langle 1_i \rangle = (N_{j1_i} 1_j) \langle 1_j \rangle, \; t_i \langle 1_i \rangle = t_j \langle 1_j \rangle = \mu_{ji} \left\langle 1_j 1_i^{-1} \right\rangle t_i \langle 1_i \rangle = \left(\mu_{ji} t_i \right) \langle 1_j \rangle,$$

or for short $t_j = \mu_{ji} t_i$, where the *time* scaling coefficients μ_{ij} and μ_{ji} obey

$$\mu_{ji} = \mu_{ij}^{-1} \in R^+,$$

$\mu_{\tau t} \in R^+$ *the time scaling coefficient* that transforms the unit $1_t = s = 1 \langle s \rangle$ of the reference *time* axis T into $N_{\tau 1_t} = N_{\tau s}$ units $1_\tau = s_\tau = 1 \langle s_\tau \rangle$ of the *time* axis T_τ, $\tau_{1_\tau} = \tau_s$, i.e. it transforms t units $1_t = s$ into τ units 1_τ,

$$1_t \langle s \rangle = (N_{\tau 1_t} 1_\tau) \langle 1_\tau \rangle, \; t \langle 1_t \rangle = \tau \langle 1_\tau \rangle = \mu_{\tau t} \left\langle 1_\tau 1_t^{-1} \right\rangle t \langle 1_t \rangle = (\mu_{\tau t} t) \langle 1_\tau \rangle,$$

or for short $\tau = \mu_{\tau t} t$,

$\mu_{t\tau} \in R^+$ *the time scaling coefficient* that transforms the *time* unit $1_\tau = s_\tau = 1 \langle s_\tau \rangle$ of the *time* axis T_τ into N_{t1_τ} units $1_t = s = 1 \langle s \rangle$ of the reference *time* axis T, i.e. it transforms τ units 1_τ into t units $1_t = s$,

$$1_\tau \langle 1_\tau \rangle = (N_{t1_\tau} 1_t) \langle s \rangle, \; \tau \langle 1_\tau \rangle = t \langle 1_t \rangle = t \langle s \rangle = \mu_{t\tau} \left\langle 1_t 1_\tau^{-1} \right\rangle \tau \langle 1_\tau \rangle = (\mu_{t\tau} \tau) \langle 1_t \rangle,$$

or for short $t = \mu_{t\tau} \tau$, so that the *time* scaling coefficients $\mu_{t\tau}$ and $\mu_{\tau t}$ undergo the following relationship:

$$\mu_{t\tau} = \mu_{\tau t}^{-1} \in R^+,$$

$\mu_{(.)_k}(..) \in \mathfrak{T} \text{x} ... \text{x} R^{(...)} \longrightarrow R^+$ the positive real valued (*basic*) *time scaling coefficient function* $\mu_{(.)_k}(..)$ associated with the k-th element of the *time* vector $\mathbf{t}^n$, $(.)_k \in \{-, 1, 2, ..., s\}$, $k = 1, 2, ..., n$,

$\rho_{(..)k}$ *the k-th coordinate* of the vector $\mathbf{r}_{(..)}$ relative to the unity basis $\{\mathbf{e}_{u1}, \mathbf{e}_{u2}, ..., \mathbf{e}_{un}\}$, if, and only if, $\rho_{(..)i} \equiv \rho_{(..)j}$ then $\rho_{(..)i} \equiv \rho_{(..)j} \equiv \rho_{(..)k} \equiv \rho_{(..)}$, $(..) \in \{G, L, P, P_R, P_{SU}\}$,

τ a subsidiary notation for *time t,*

$\boldsymbol{\tau}_{(.)}^n$ a subsidiary notation for the n-dimensional *time* vector $\mathbf{t}_{(.)}^n$,

$\boldsymbol{\tau}_u \in R^1$ *the time unity vector* in R^1, $\boldsymbol{\tau}_u^e$ is its extension *in* R^{n+1},

$$\boldsymbol{\tau}_u = (1), \; \boldsymbol{\tau}_u^e = (0 \; 0 \; ...0 \; 1)^T \in R^{n+1},$$

$\boldsymbol{\tau}_{ue} \in R^{1+n}$ is *the extension of* $\boldsymbol{\tau}_u$ *in* R^{1+n},

$$\boldsymbol{\tau}_{ue} = (1 \; 0 \; ... \; 0)^T \in R^{1+n},$$

$\upsilon^{(.)} \in R^{+}$ an arbitrarily accepted and then fixed constant *speed of a clock measurement*, the value of which is measured with the length unit $1_{L_{(.)}}$ and with the *time* unit $1_{t_{(.)}}$, $\upsilon^{(.)} \left\langle 1_{L_{(.)}} 1_{t_{(.)}}^{-1} \right\rangle$,

ϕ *the empty set*,

$\varphi \in R$ *angle of a clock hand*,

$\varphi_u \in R$ *a unity angle of the clock hand*, which corresponds to the relevant *time* unit,

$\boldsymbol{\chi}(.; t_0, \mathbf{x}_0) : \mathfrak{T}_0 \text{x} \mathfrak{T} \text{x} R^n \rightarrow R^n$ *a motion of a dynamical system*, which passes through $\mathbf{x}_0$ at t_0,

$\boldsymbol{\chi}(t; t_0, \mathbf{x}_0) \in R^n$ *the instantaneous vector value of the motion* $\boldsymbol{\chi}(.; t_0, \mathbf{x}_0)$ *at a moment* t, $\boldsymbol{\chi}(t_0; t_0, \mathbf{x}_0) \equiv \mathbf{x}_0$,

$\omega \in R^{+}$ *a constant angular speed* (*of a clock hand*).

19.3.4 Roman letters

$A \in R^{n\text{x}n}$ *a positive definite matrix*, $\mathbf{x}^T A \mathbf{x} > \mathbf{0}$, $\forall \left(\mathbf{x} \neq \mathbf{0} \right) \in R^n$,

$\mathrm{A}_i^j(.)$ a diagonally elementwise positive real valued *time scaling diagonal matrix coefficient function* in general,

$$\mathrm{A}_i^j(.) : \mathfrak{T}^n \text{x} ... \text{x} R^{(..)} \longrightarrow R_{+}^{n\text{x}n},\ diagmin \mathrm{A}_i^j(.) : \mathfrak{T} \text{x} ... \text{x} R^{(..)} \longrightarrow R^{+},$$

$$\mathrm{A}_i^j(.) = diag \left\{ \alpha_{i_1}^{j_1}(.) \quad \alpha_{i_2}^{j_2}(.) \quad ... \quad \alpha_{i_n}^{j_n}(.) \right\},\ \alpha_{i_k}^{j_k}(.) : \mathfrak{T}^n \text{x} ... \text{x} R^{(..)} \rightarrow R^{+},$$

$$i_k, j_k \in \{1, 2, ..., s\},\ k = 1, 2, .., n,$$

if, and only if, all $\alpha_{i_k}^{j_k}(.)$ are constant, i.e. if $\alpha_{i_k}^{j_k}(.) \equiv \alpha_{i_k}^{j_k}$ then $\mathrm{A}_i^j(.) \equiv \mathrm{A}_i^j$,

$\mathrm{A}_{ij}(.)$ denotes both $\mathrm{A}_i^j(.)$ and $\mathrm{A}_j^i(.)$ if, and only if, $\mathrm{A}_i^j(.) \equiv \mathrm{A}_j^i(.)$ and then $\mathrm{A}_i^j(.) \equiv \mathrm{A}_j^i(.) \equiv \mathrm{A}_{ij}(.) \equiv \mathrm{A}_{ji}(.)$,

B *a positive definite matrix*, $B \in R^{n\text{x}n}$, $B \neq A$ in general, $B = A$ is permitted in a special case, $B = A = I$ is the singular case,

$\mathbf{c}(t, \mathbf{x})$ *the light velocity in* $\mathbf{x}$ *at a moment* t in general, $\mathbf{c}(t, \mathbf{x}) = d\mathbf{r}_L(t)/dt$, $\mathbf{x} = \mathbf{r}_L(t)$; if, and only if, $\mathbf{c}(t, \mathbf{x})$ and $\mathbf{u}$ are colinear all the *time* then $\mathbf{c}(t, \mathbf{x}) \equiv c(t, \mathbf{x})\mathbf{u}$, otherwise $\mathbf{c}(t, \mathbf{x}) \equiv C(t, \mathbf{x})\mathbf{u}$,

$c \in R^{+}$ *the constant light speed in vacuum*, or *the constant value of the light speed in vacuum*, $c = 2.99792458\text{x}10^8 \left\langle Kms^{-1} \right\rangle$,

$\mathbf{c}$ *the constant light velocity in vacuum*,

$$\mathbf{c} = c\mathbf{u}\ if, and only if,\ \mathbf{c}\ and\ \mathbf{u}\ are\ colinear,\ otherwise\ \mathbf{c} = C\mathbf{u},$$

$\mathbf{c}^{O_j, i} \in R^n$ *the relative light velocity in vacuum with respect to* R_j^n *and its origin* O_j, when the length value is measured with the length unit 1_{L_j} of R_j^n and the *time* value is measured with the *time* unit 1_{t_i} of t_i, where $i = j$ is permitted,

$$\mathbf{c}^{O_j, i} = \mathrm{C}^{O_j, i}\mathbf{u}\ in\ general,\ \mathbf{c}^{O_j, i} \equiv \mathbf{c}_j^i,\ \mathrm{C}^{O_j, i} \equiv \mathrm{C}_j^i,$$

$$\mathbf{c}^{O_j, i} = c^{O_j, i}\mathbf{u}\ if, and only if,\ \mathbf{c}^{O_j, i}\ and\ \mathbf{u}\ are\ colinear,$$

for short $\mathbf{c}^{O_j,i} \equiv \mathbf{c}_j^i$, and $c^{O_j,i}\left\langle 1_{L_j}1_{t_i}^{-1}\right\rangle$ is *the corresponding relative light speed*, for short $c^{O_j,i} \equiv c_j^i$,

$$\mathbf{c}_j^i \equiv \mathbf{c}^{O_j,i} = c_j^i\mathbf{u}\ if, and only if,\ \mathbf{c}_j^i\ and\ \mathbf{u}\ are\ colinear,\ otherwise\ \mathbf{c}_j^i = \mathrm{C}_j^i\mathbf{u},$$

the coordinates of $\mathbf{c}_j^i$ relative to the unity basis $\{\mathbf{e}_{u1},\ \mathbf{e}_{u2},\ ...,\ \mathbf{e}_{un}\}$ are $\varsigma_{j_k}^{i_k}$, $k = 1, 2, ..., n$,

$$\mathbf{c}_j^i = \begin{pmatrix}\varsigma_{j_1}^{i_1} & \varsigma_{j_2}^{i_2} & ... & \varsigma_{j_n}^{i_n}\end{pmatrix}^T = \varsigma_{j_1}^{i_1}\mathbf{e}_{u1} + \varsigma_{j_2}^{i_2}\mathbf{e}_{u2} + ... + \varsigma_{j_n}^{i_n}\mathbf{e}_{un},$$

and its coordinates relative to the unity vector $\mathbf{u}$ are $c_{j_k}^{i_k}$, $k = 1, 2, ..., n$,

$$\mathbf{c}_j^i = \begin{pmatrix}c_{j_1}^{i_1} & c_{j_2}^{i_2} & ... & c_{j_n}^{i_n}\end{pmatrix}^T,\ c_{j_k}^{i_k} = \frac{\varsigma_{j_k}^{i_k}}{u_k} \in R^+,$$
$$i \leq j,\ i_k, j_k \in \{1, 2, ..., s\},\ k = 1, 2, .., n,$$

$\mathrm{C}_j^i \in R_+^{nxn}$ *the light speed matrix relative to* the unity vector $\mathbf{u}$,

$$\mathrm{C}_j^i = diag\left\{c_{j_1}^{i_1} \quad c_{j_2}^{i_2} \quad ... \quad c_{j_n}^{i_n}\right\} \Longrightarrow \mathbf{c}_j^i = \mathrm{C}_j^i\mathbf{u},$$
$$i \leq j,\ i_k, j_k \in \{1, 2, ..., s\},\ k = 1, 2, .., n,$$

c_i and c_j denote, respectively, *the light speeds relative to* R_i^n *and* R_j^n if, and only if, we use the same *time* axis for both R_i^n and R_j^n, $T_i = T_j$, i.e.

$$t_i = t_j,\ 1_{t_i} = 1_{t_j} \Longrightarrow c_i^i = c_i^j = c_i\left\langle 1_{L_i}1_{t_i}^{-1}\right\rangle\ and\ c_j^i = c_j^j = c_j\left\langle 1_{L_j}1_{t_i}^{-1}\right\rangle,$$

$c^{(.)} \in R^+$ denotes *the light speed with respect to the origin* $O_{(.)}$ *of* $R_{(.)}^n$ if it is measured with the length unit $1_{L_{(.)}}$ of $R_{(.)}^n$ and the *time* value is measured with the *time* unit $1_{t_{(.)}}$ of $T_{(.)}$,

$$c_{(.)}^{(.)} = c^{(.)}\ if, and only if,\ v_{R_{(.)}^n}^{R^n} = v_{O_{(.)}}^{O} = 0,\ c^{(.)}\left\langle 1_{L_{(.)}}1_{t_{(.)}}^{-1}\right\rangle,$$

$c^{ji} = c^{ij} \in R^+$ denotes both c^i and c^j if, and only if, $c^i = c^j$,

$$c^i = c^j = c^{ij} = c^{ji}\ if, and only if,\ c^i = c^j,$$

c^t, c^τ *the light speed value measured with the time unit*, respectively, $1_t = s$, $1_\tau = s_\tau$,

$c_{ij} = c_{ji} \in R^+$ denotes both c_i^i and c_j^j if, and only if, $c_i^i = c_j^j$,

$$c_i^i = c_{ij}\ and\ c_j^j = c_{ji} \Longleftrightarrow c_i^i = c_j^j = c_{ij} = c_{ji},$$

C *Celsius (degree)*,

C *the light speed matrix relative to* $\mathbf{u}$,

$D \in R^{2nx2n}$ *a block diagonal matrix*,

$$D = \begin{pmatrix} A & O \\ O & -B \end{pmatrix} = blockdiag\{A \quad -B\} \in R^{2nx2n},$$

$\mathbf{e}_{ui} \in R^n$ *the i-th unity vector of an orthonormal vector basis* $\{\mathbf{e}_{u1}, \mathbf{e}_{u2}, ..., \mathbf{e}_{un}\}$ of R^n, $\mathbf{e}_{ui} = (\delta_{1i} \;\; \delta_{2i} \; ... \; \delta_{ni})^T$, $\mathbf{e}_{ui}^T \mathbf{e}_{uj} = \delta_{ij}$,

E *event*, or *the matrix of Einstein's equations* (26a) in [150, pp. 32, 33.], or *energy*,

$\mathbf{f}(.) : \mathfrak{T} \text{x} R^n \rightarrow R^n$ *a vector function defining the internal dynamics of a dynamical system* represented in the Cauchy (normal, state) form,

$\mathbf{f}_1(.) : \mathfrak{T} \text{x} R^{n_x} \text{x} R^s \text{x} R^{s\text{x}s} \rightarrow R^{n_x}$ *a vector function defining the internal dynamics of the (slow) subsystem of a dynamical system with multiple time scales*,

$\mathbf{f}_2(.) : \mathfrak{T} \text{x} R^{n_x} \text{x} R^s \text{x} R^{s\text{x}s} \rightarrow R^s$ *a vector function defining the internal dynamics of the (fast) subsystem of the dynamical system with multiple time scales*,

F *Fahrenheit* (*degree*),

$\mathbf{g}(.) : R^k \rightarrow R^n$ *a vector physical variable* all entries of which are the same scalar physical variable $g(.) : R^k \rightarrow R$ with the values possibly measured in different units along n different directions; the vector value of $\mathbf{g}(.)$ depends on a vector variable $\mathbf{z}$ in general,

$\mathbf{g}(\mathbf{z}) \in R^n$ *the vector value of* $\mathbf{g}(.)$ *at* $\mathbf{z}$; its representation relative to the unity basis $\{\mathbf{e}_{u1}, \mathbf{e}_{u2}, ..., \mathbf{e}_{un}\}$ is

$$\mathbf{g}(\mathbf{z}) = \gamma_1(\mathbf{z})\mathbf{e}_{u1} + \gamma_2(\mathbf{z})\mathbf{e}_{u2} + ... \gamma_n(\mathbf{z})\mathbf{e}_{un} = (\gamma_1(\mathbf{z}) \;\; \gamma_2(\mathbf{z}) \;\; ... \;\; \gamma_n(\mathbf{z}))^T = \Gamma(\mathbf{z})\mathbf{1};$$

$\Gamma(\mathbf{z})$ is the matrix value of $\mathbf{g}(\mathbf{z})$ relative to the same basis, $\Gamma(\mathbf{z}) = diag\{\gamma_1(\mathbf{z}) \;\; \gamma_2(\mathbf{z}) \;\; ... \;\; \gamma_n(\mathbf{z})\}$,

$g(\mathbf{z})$ *the algebraic value*, for short: *the value*, of $\mathbf{g}(\mathbf{z})$ at $\mathbf{z}$ relative to the unity vector $\mathbf{u}$ collinear with $\mathbf{g}(\mathbf{z})$,

$$\mathbf{g}(\mathbf{z}) = g(\mathbf{z})\mathbf{u}, \; g(\mathbf{z}) = \|\mathbf{g}(\mathbf{z})\| \, sign\left[\mathbf{g}^T(\mathbf{z})\mathbf{u}\right], \; \gamma_i(\mathbf{z}) \equiv g(\mathbf{z})u_i,$$

$G(.) : R^k \rightarrow R^n$ *the matrix function induced by* $\mathbf{g}(.)$ *relative to the unity vector* $\mathbf{u}$; its matrix value at $\mathbf{z}$ is the matrix value $G(\mathbf{z})$ of $g(\mathbf{z})$, $\mathbf{g}(\mathbf{z}) = G(\mathbf{z})\mathbf{u}$,

$$G(\mathbf{z}) = diag\{g_1(\mathbf{z}) \quad g_2(\mathbf{z}) \quad ... \quad g_n(\mathbf{z})\}, \;\; g_k(\mathbf{z}) = \frac{\gamma_k(\mathbf{z})}{u_k}, \; k = 1, 2, .., n,$$

G a freely accepted and then fixed *generic point*, which can be a light signal L, or an arbitrary point P, or the temporal reference point P_R, or the spatial reference point P_{SU},

$$G \in \{L, \; P, \, P_R, \, P_{SU}\} \; \textit{is permitted in special cases},$$

H(t;T) *a hyperplane in the integral space I at the moment t*, which is parallel with the space (hence, symbolically parallel with the frame R^n) and orthogonal to the *time* axis T at its temporal point *t,*

$$H(t;T) = \{(\sigma, \mathbf{x}) : \; \sigma \in T, \; \sigma = numt, \; \mathbf{x} \in R^n\} \subset I,$$

if, and only if, the *time* axis is known and fixed then "$;T$" is omitted so that then

$$H(t;T) = H(t) = \{(\sigma, \mathbf{x}) : \; \sigma \in T, \; \sigma = numt, \; \mathbf{x} \in R^n\},$$

i *an arbitrary natural number*, or *the imaginary unit* $\sqrt{-1}$,

I, I *the identity matrix of the n-th order*, I= $I = diag\{1 \ \ 1 \ ... \ 1\} \in R^{nxn}$, or

I *the mathematical representation of the 1+n dimensional integral space* $\mathfrak{I}$,

$$I = T{\times}R^n = \{(\sigma, \mathbf{x}) : \sigma \in T, \, \mathbf{x} \in R^n\} \, ,$$

$I_{(.)}$ *the mathematical representation of the* 1+n *dimensional (.)-integral space*,

$$I_{(.)} = T_{(.)}{\times}R^n_{(.)} = \left\{\left(\sigma_{(.)}, \mathbf{x}\right) : \sigma_{(.)} \in T_{(.)}, \, \mathbf{x} \in R^n_{(.)}\right\} ,$$

j *an arbitrary natural number*,

k *an arbitrary natural number*,

K *Kelvin* (*degree*),

l *arc*, or *length*,

l_u *the unity arc*, or *the unity length*,

L *light*, or *length*, or *length dimension*,

m *mass*, or *an arbitrary natural number*,

m_v *moving mass*, $m_{v_{(.)}}$-*mass moving in* $R^n_{(.)}$, $m_{0_{(.)}}$-*mass at rest in* $R^n_{(.)}$,

M *an sxs diagonal matrix*, the diagonal entries of which are the (basic) *time* scaling coefficients $\mu_{(.)}$,

$$M = diag\{\mu_1 \ \mu_2 \ ... \ \mu_s\} \, , \, \mu_i \in R^+, \, i = 1, 2, \ ... \ , s \ ,$$

n *an arbitrary natural number*,

$\mathrm{N}_{u_{(.)}}$ *the number of the unity angles* $\varphi_{u_{(.)}}$ (*of the unity arcs* $l_{u_{(.)}}$) *contained in* 2π (in $2\pi\mathrm{R}_{(.)}$), respectively, i.e. *the number of the time units* 1_t *contained in the full scale of the corresponding clock scale*, the radius of which equals R, $N_{u_{(.)}} = 2\pi\varphi^{-1}_{u_{(.)}} = 2\pi\mathrm{R}_{(.)}l^{-1}_{u_{(.)}}$,

$\mathrm{N}_{1_{(.)}}$ *the number of the unity angles* $\varphi_{u_{(.)}}$ (*of the unity arcs* $l_{u_{(.)}}$) *contained in an angle* $\varphi_{(.)}$ (*in an arc* $l_{(.)}$), respectively, i.e. *the number of the time units* 1_t *contained in* $\varphi_{(.)}$ (*in* $l_{(.)}$), $N_{1_{(.)}} = \varphi_{(.)}\varphi^{-1}_{u_{(.)}} = l_{(.)}l^{-1}_{u_{(.)}}$,

O *is the origin of* R^n; or *the zero matrix of the appropriate order*,

$O_{(.)}$ *is the origin of the corresponding frame* $R^n_{(.)}$,

P *person*; or *an arbitrary point in* R^n, where the adjective *arbitrary* designates that all the results are valid for any point P in R^n (hence, also for the reference point P_R, or P_{SU}, and for every light signal L); the arbitrary point P can represent a particle (a material point); or *power*,

P_R *an arbitrarily accepted and then fixed temporal reference point* (*in* R^n) *of the temporal coordinate transformations*, where the adjective *fixed* denotes that all results are valid relative to that point P_R,

P_{SU} *an arbitrarily accepted and then fixed spatial reference point* (*in* R^n) *of the spatially uniform spatial coordinate transformations*,

p *a natural number*,

$\mathbf{p}(.)$ *a vector variable*, the entries of which are l different physical variables p_k in general, $\mathbf{p}(.) = (p_1(.) \ \ p_2(.) \ \ ... \ \ p_p(.))^T$ relative to $\mathbf{u}$, $1 < l \leq p$,

$\mathbf{p}(.) = P(.)\mathbf{u}$, $P(.) = diag\{p_1(.)\ \ p_2(.)\ \ ...\ p_p(.)\}$; or *momentum* of a quantity of matter (of a body) of mass m moving with a velocity $\mathbf{v}$, $\mathbf{p} = m\mathbf{v}$,

q *a natural number,*

$q^{(.)} \in R^+$ *a freely accepted and then fixed constant generic, or reference, speed,*

$\mathbf{q}^{(.)} \in R^+$ *a freely accepted and then fixed constant generic, or reference, velocity,* $\mathbf{q}^{(.)} = q^{(.)}\mathbf{u}$ if, and only if, $\mathbf{q}^{(.)}$ and $\mathbf{u}$ are colinear, otherwise $\mathbf{q}^{(.)} = Q^{(.)}\mathbf{u}$,

$r_{(..)k}$ *the k-th coordinate of the vector* $\mathbf{r}_{(..)}$ *relative to the unity vector* $\mathbf{u}$ and *the k-th diagonal entry of the diagonal matrix* $\mathrm{R}_{(..)}$, $(..) \in \{G, P, PR, PSU\}$,

$\mathbf{r}_{(..)} \in R^n$ *the position vector* (for short: *position*) *of a point* (..) *relative to the origin* O of R^n when its length is measured with the unit 1_L of R^n; its representation with respect to the basis $\{\mathbf{e}_{u1}, \mathbf{e}_{u2}, ..., \mathbf{e}_{un}\}$ is given by

$$\mathbf{r}_{(..)} = \rho_{(..)1}\mathbf{e}_{u1} + \ \rho_{(..)2}\mathbf{e}_{u2} + ...\ \rho_{(..)n}\mathbf{e}_{un} = (\rho_{(..)1}\ \rho_{(..)2}\ ...\ \rho_{(..)n})^T,$$
$$\mathbf{r}_{(..)} = \rho_{(..)}\mathbf{1}\ if, and only if,\ \rho_{(..)i} \equiv \rho_{(..)k}\ \equiv \rho_{(..)},\ (..) \in \{G, P, P_R, P_{SU}\},$$

and its representation relative to the unity vector $\mathbf{u}$, $\mathbf{r}_{(..)} = \mathrm{R}_{(.)}\mathbf{u}$, has the following form in general:

$$\mathbf{r}_{(..)} = (r_{(..)1}\ r_{(..)2}\ ...\ r_{(..)n})^T\ in\ general,\ \mathrm{R}_{(..)} = diag\{r_{(..)1}\ r_{(..)2}\ ...\ r_{(..)n}\},$$
$$r_{(..)k} \equiv \frac{\rho_{(..)_k}}{u_k},\ (.)_k \in \{-, 1, 2, ..., s\},\ k = 1, 2, ..., n,\ (..) \in \{G, P, P_R, P_{SU}\},$$

if, and only if, $\mathbf{r}_{(..)}$ and $\mathbf{u}$ are colinear then $r_{(..)1} = \ r_{(..)2} = \ ... \ = r_{(..)n} = r_{(..)}$, i.e. $\mathrm{R}_{(..)} = r_{(..)}I$, which imply

$$\mathbf{r}_{(..)} = r_{(..)}\mathbf{u},\ r_{(..)} = \left\|\mathbf{r}_{(..)}\right\| sign\left(\mathbf{r}^T_{(..)}\mathbf{u}\right),\ (..) \in \{G, P, P_R, P_{SU}\},$$

$\mathbf{r}^{(.)}_{(..)} \in R^n$ *the position vector* (for short: *position*) *of a point* (..) *relative to the origin* $O_{(.)}$ *of* $R^n_{(.)}$ when its length is measured with the unit $1_{L_{(.)}}$ of $R^n_{(.)}$,

$\mathbf{r}_{(..)e} \in I$ *the extension of the vector* $\mathbf{r}_{(.)}$, i.e. it is the extended representation of the vector $\mathbf{r}_{(..)}$ if it is considered as a vector in the integral space I,

$$\mathbf{r}_{(..)e} = (0\ \mathbf{r}^T_{(..)}) = (0\ \ \rho_{(..)1}\ \ \rho_{(..)2}\ \ ...\ \ \rho_{(..)n})^T,\ (..) \in \{G, P, P_R, P_{SU}\},$$

$\mathbf{r}^{O_{(...)},(.)}_{(..)}(t_{(.)}; t_{(.)0})$ *the position vector of a point* (..) *relative to* $O_{(...)}$ *at a moment* $t_{(.)}$ provided the initial moment was $t_{(.)0} \in \mathfrak{T}_{(.)}$, for the sake of the simplicity $\mathbf{r}^{O_{(...)},(.)}_{(..)}(t_{(.)}; t_{(.)0}) \equiv \mathbf{r}^{O_{(...)}}_{(..)}(t_{(.)}; t_{(.)0})$, if, and only if, (...) = (.) then we omit "$O_{(...)}, (.)$", i.e. we omit "$O_{(...)}$", respectively, from the superscript as follows,

$\mathbf{r}_{(..)}(t_{(.)}; t_{(.)0})$ the vector position of a point (..) relative to $O_{(.)}$ at a moment t provided the initial moment was $t_{(.)0} \in \mathfrak{T}_{(.)}$, for short $\mathbf{r}^{(.)}_{(..)t}$, i.e.

$$\mathbf{r}^{(.)}_{(..)t} \equiv \mathbf{r}_{(..)}(t_{(.)}; t_{(.)0}), \mathbf{r}_{(..)}(t_{(.)0}; t_{(.)0}) \equiv \mathbf{r}^{(.)}_{(..)0},\ (..) \in \{G, P, PR, PSU\},$$

it can be the position vector $\mathbf{r}_{Lt}^{(.)} \equiv \mathbf{r}_L(t_{(.)};t_{(.)0})$ of a light signal L for $(..) = L$, or the position vector $\mathbf{r}_{Pt}^{(.)} \equiv \mathbf{r}_P(t_{(.)0};t_{(.)0})$ of an arbitrary point P if $(..) = P$, the position vector $\mathbf{r}_{Rt}^{(.)} \equiv \mathbf{r}_R(t_{(.)};t_{(.)0})$ of the temporal reference point P_R in the case $(..) = P_R$, or the position vector $\mathbf{r}_{SUt}^{(.)} \equiv \mathbf{r}_{SU}(t_{(.)};t_{(.)0})$ of the spatial reference point P_{SU} when $(..) = P_{SU}$,

$$\mathbf{r}_{(..)}^{(.)}(...) \in \left\{\mathbf{r}_L^{(.)}(...),\, \mathbf{r}_P^{(.)}(...),\, \mathbf{r}_R^{(.)}(...),\, \mathbf{r}_{SU}^{(.)}(...)\right\},$$

$r_{(..)}(t_{(.)};t_{(.)0})$ *the* (*algebraic*, or, *scalar*) *value of the vector* $\mathbf{r}_{(..)t}^{(.)}$ *of a point* (..) *relative to* $O_{(.)}$ *at a moment* $t_{(.)}$ provided the initial moment was $t_{(.)0} \in \mathfrak{T}_{(.)}$ if, and only if, $\mathbf{r}_{(..)t}^{(.)}$ is colinear with the unity vector $\mathbf{u}$, which determines the position of a point (..) relative to the origin $O_{(.)}$ along the direction of $\mathbf{u}$ at the moment $t_{(.)}$, $r_{(..)t}^{(.)} \equiv r_{(..)}(t_{(.)};t_{(.)0})$, $r_{(..)0}^{(.)} = r_{(..)}(t_{(.)0};t_{(.)0}) \equiv 0$ if it is not otherwise stated, $(..) \in \{G,L,P,PR,PSU\}$,

$$r_{(..)}^{(.)}(...) \in \left\{r_G^{(.)}(...),\, r_L^{(.)}(...),\, r_P^{(.)}(...),\, r_R^{(.)}(...),\, r_{SU}^{(.)}(...)\right\},$$

R *the set of all real numbers*,
R^+ *the set of all positive real numbers*,
R_+ *the set of all nonnegative real numbers*,
R^n *the n- dimensional real vector space*,
$R^{(n)}$ *the axis* that symbolically, graphically represents $\mathfrak{R}^n$ and R^n,
$R_0^{(n)}$ *the initial position of the* $R^{(n)}$*-axis*,
R_i *the i-th scale radius and the radius of the i-th clock hand*,
R^{1+n} *the* $1+n$ *dimensional real vector space*, $R^{1+n} = R^1 \times R^n$,
R^{n+1} *the* $n+1$ *dimensional real vector space*, $R^{n+1} = R^n \times R^1$,
s *the basic time unit*: *second*, or *a natural number*,
$sign(.) : R \to \{-1, 0, 1\}$ *the signum function*,

$$sign(x) = |x|^{-1}\, x \text{ if } x \neq 0, \text{ and } sign(0) = 0,$$

$S \subseteq R^n$ *a nonempty subset of* R^n,
t *time* (*temporal variable*), or *an arbitrary time value* (*an arbitrary moment, an arbitrary instant*); and formally mathematically, t denotes for short also the numerical time value *numt* if it does not create a confusion,

$$t[\mathrm{T}]\,\langle s\rangle\,,\ numt \in R,\ dt > 0\ ,\ \text{or equivalently: } t \in \mathfrak{T},$$

t_0 a conventionally accepted *initial value of time* (*initial instant*, *initial moment*), $t_0 \in \mathfrak{T}$,
$t_{\inf}$ *the first instant*, which has not happened, $t_{\inf} = -\infty$,
$t_{\sup}$ *the last instant*, which will not occur, $t_{\sup} = \infty$,
$t_{ZeroTotal}$ *the total zero value of time*, which has not existed and will not happen,

t_{zero} *a conventionally accepted relative zero value of time*,

$\mathbf{t} \in T^1 \cup R^1$ *the one dimensional time vector* represented in T^1 and in R^1 in the sense that $t \in \mathfrak{T}$, $numt \in T$ and $num\mathbf{t} \in T^1 \cup R^1$,

$$\mathbf{t} = (t) = t\mathbf{t}_u \in T^1 \cup R^1 \text{ meaning } t \in \mathfrak{T},\ num\mathbf{t} \in T^1 \cup R^1,$$

$\mathbf{t}_e \in T \times R^n$ *the extension of the time vector* $\mathbf{t}$ in $T \times R^n$, respectively,

$$\mathbf{t}_e = (\sigma \ \ 0 \ \ ... \ 0)^T = \sigma \mathbf{t}_{ue} \in T \times R^n,\ \sigma = numt,\ t \in \mathfrak{T},$$

$\mathbf{t}_{\mathbf{1}}^s$ *the formally mathematically introduced s-dimensional time vector (in* $\mathfrak{T}^s$), the entries of which are the same *time* value t,

$$\mathbf{t}_{\mathbf{1}}^s = (t\ t\ \ ...\ t)^T = t\mathbf{1}^s \in \mathfrak{T}^s,$$

$\mathbf{t}^n$ *the formally mathematically introduced n-dimensional time vector (in* $T^n \cup R^n$), the entries of which are the same *time* value t,

$$\mathbf{t}^n = diag\{t\ \ t\ \ ...\ t\}\mathbf{u} = \mathrm{T}\mathbf{u} = tI\mathbf{u} = t\mathbf{u} \in T^n \cup R^n,$$
$$\mathbf{t}_0^n = diag\{t_0\ \ t_0\ \ ...\ t_0\}\mathbf{u} = t_0\mathbf{u} \in T^n \cup R^n,$$

$\mathbf{t}_{(.)}^n \in \mathfrak{T}_{(.)}^n \cup R^n$ *the formally mathematically introduced n-dimensional time vector (* in $\mathfrak{T}_{(.)}^n$ and in R^n), the entries of which are the *time* values measured relative to $\mathfrak{T}_{(.)}^n$, i.e. the k-th entry is measured relative to $\mathfrak{T}_{(.)_k}$, where $(.)_k \in \{-, 1, 2, ..., s\}$, $k = 1, 2, ...\ n$, and $(.)_l \neq (.)_m$ for $l \neq m$ in general, but $(.)_l = (.)_m$ is permitted for $l \neq m$ in special cases,

$$\mathbf{t}_{(.)}^n = diag\{t_{(.)_1}\ \ \ t_{(.)_2}\ \ \ ...\ \ \ t_{(.)_n}\}\mathbf{u} = \mathrm{T}_{(.)}\mathbf{u} = t\mathrm{M}_{(.)}\mathbf{u} = \mathrm{M}_{(.)}\mathbf{t}^n,$$

$\mathbf{t}_{(.)0}^n$ denotes *the initial* $\mathbf{t}_{(.)}^n$, $\mathbf{t}_{(.)0}^n = \mathrm{T}_{(.)0}\mathbf{u} = t_0\mathrm{M}_{(.)}\mathbf{u} = \mathrm{M}_{(.)}\mathbf{t}_0^n$,

$\mathbf{t}_M^s$ *time (temporal) vector variable*, $\mathbf{t}_M^s = M\mathbf{t}_1^s$,

$\mathbf{t}_u \in R^1$ *the time unity vector* represented in R^1,

$$\mathbf{t}_u = (1)\,,$$

$\mathbf{t}_u^e \in R^{n+1}$ is *the extension*, i.e. *the representation*, *of the time unity vector* $\mathbf{t}_u$ *in the n+1 dimensional real vector space* R^{n+1}, which is

$$\mathbf{t}_u^e = (0\ 0\ ...0\ 1)^T \in R^{n+1},$$

$\mathbf{t}_{ue} \in R^{1+n}$ is *the extension*, i.e. *the representation*, *of the time unity vector* $\mathbf{t}_u$ *in the* $1{+}n$ *real vector space* R^{1+n},

$$\mathbf{t}_{ue} = (1\ 0\ ...\ \ 0)^T \in R^{1+n},$$

T *the temporal dimension*, "the *time* dimension", which is the physical dimension of *time*,

T *the accepted reference time axis*, which is the geometrical representation of the *time* set $\mathfrak{T}$ in R^1, the arbitrary element of which is the numerical value $\sigma = numt$ of an arbitrary moment $t \in \mathfrak{T}$, $t\left<s\right>$, and the *time* unit of which is second, $1_t = s$,

$$T = \left\{ \sigma : \begin{array}{cc} \sigma \in R, & \forall \sigma \in R \Longrightarrow \exists! t \in \mathfrak{T},\, numt = \sigma,\, numdt = d\sigma; \\ d\sigma \in R^+, & \forall t \in \mathfrak{T} \Longrightarrow \exists! \sigma \in R,\, \sigma = numt,\, d\sigma = numdt \end{array} \right\},$$

$$inf\ T = -\infty,\ \ sup\ T = \infty,$$

T^1 *the accepted reference one dimensional vector time axis*, which is the geometrical representation of the *time* set $\mathfrak{T}$ in R^1, the arbitrary element of which is an arbitrary vector moment $\mathbf{t}$, and the *time* unit of which is second, $1_t = s$,

$$T^1 = \left\{ (\sigma) : (\sigma) \in R^1,\ \sigma \in T \right\},\ \ inf\ T^1 = (-\infty),\ \ sup\ T^1 = (\infty),$$

T_e *the extension of the time axes* T *and* T^1, which is their representation in the integral space I,

$$T_e = \{\mathbf{t}_e :\ \mathbf{t}_e = \sigma \mathbf{t}_{ue} \in T \times R^n,\, \sigma \in T,\, \sigma = numt,\, t \in \mathfrak{T}\},$$

$T^1_{(.)}$ *the vector time axis* that is the geometrical representation of the *time* set $\mathfrak{T}_{(.)}$ in R^1, and the *time* unit of which is $1_{t_{(.)}}$, $\left(t_{(.)}\left<1_{t_{(.)}}\right>\right)$,

$$T^1_{(.)} = \left\{ \left(\sigma_{(.)}\right) : \left(\sigma_{(.)}\right) \in R^1,\ \sigma_{(.)} \in T_{(.)} \right\},\ \ inf\ T^1_{(.)} = (-\infty),\ \ sup\ T^1_{(.)} = (\infty),$$

$T_{(.)e}$ *the extension of the time axes* $T_{(.)}$ *and* $T^1_{(.)}$, which is their representation in the integral space $I_{(.)}$,

$$T_{(.)e} = \{\mathbf{t}_{(.)e} :\ \mathbf{t}_{(.)e} = \sigma_{(.)} \mathbf{t}_{ue} \in R^{1+n},\, \sigma_{(.)} \in T_{(.)}\},$$

$T_{(.)0}$ *the subset of* $T_{(.)}$, which has the minimal element $minT_{(.)0}$ that is the numerical value $\sigma_0 = numt_{(.)0}$ of the initial instant t_0,

$$\begin{array}{c} T_{(.)0} = \left\{ \sigma_{(.)} : \sigma_{(.)} \in T_{(.)},\ \sigma_{(.)} \geq \sigma_0 = numt_{(.)0} \right\}, T_{(.)0} \subset T_{(.)}, \\ minT_{(.)0} = \sigma_0 = numt_{(.)0} \in T_{(.)} \cup R,\ \ supT_{(.)0} = \infty, \end{array}$$

$T_m(t; S_i)$ *the m-th time axis* $T_m(t; S_i)$ (in R^1) is valid over the set S_i at the moment $t \in \mathfrak{T}$, that is that

$$\forall \mathbf{x} \in S_i,\, T_m \textit{is valid time axis at } \mathbf{x} \textit{ at the moment } t \in \mathfrak{T},\, m \in \{-, 1, 2, ...\},$$

$T_m(t; \mathbf{x})$ *the time axis* $T_m(t; \mathbf{x})$ (in R^1) is valid at the point $\mathbf{x}$ at the moment t, that is that

$$T_m(t; \mathbf{x}) = T_m(t; S_i) \mid_{S_i = \{\mathbf{x}\}},\ m \in \{-, 1, 2, ...\},$$

$T_{me}(t; S_i)$ *the extended time axis* $T_{me}(t; S_i)$ (in R^{1+n}) is valid over the set S_i (i.e. at every $\mathbf{x} \in S_i$) at the moment $t \in \mathfrak{T}$,

$T_{me}(t;\mathbf{x})$ *the extended time axis* $T_{me}(t;\mathbf{x})$ (in R^{1+n}) is valid at the point $\mathbf{x}$ at the moment $t \in \mathfrak{T}$,

$\mathbf{T}_{(.)}(t)$ *the temporal hyperplane* $\mathbf{T}_{(.)}(t)$ determined by an arbitrarily chosen and then fixed n-vector $\mathbf{a}$ and by a real number κ, $\mathbf{a} \in R^n$, $\kappa \in R$,

$$\mathbf{T}_{(.)}(t) = \mathbf{T}_{(.)}(\mathbf{a},\kappa,t) = \left\{T_{(.)}(t;\mathbf{x}): \ \mathbf{a}^T\mathbf{x} = \kappa,\, t \in \mathfrak{T},\, \mathbf{x} \in R^n\right\},$$
$$(.) \in \{-,1,2,...\},$$

$\mathsf{T}_{(.)}$ *the temporal environment* $\mathsf{T}_{(.)}$,

$$\mathsf{T}_{(.)} = \left\{T_k : T_k = T_{(.)},\ k \in \{1,2,...\}\right\},\, (.) \in \{-,1,\, 2,\, ...\},$$

$\mathsf{T}_{(.)} = \mathsf{T}_{(.)}(t;S)$ *the* $\mathsf{T}_{(.)}$*-environment of the set* S *at the moment* t,

$$\mathsf{T}_{(.)}(t;S) = \left\{T_k(t;S): \ T_k(t;S) = T_{(.)}(t;S),\ k \in \{1,2,...\}\right\},\, (.) \in \{-,1,2,...\},$$

$\mathsf{T}_{(.)}(t;\mathbf{x})$ *the* $\mathsf{T}_{(.)}$*-environment of the point* $\mathbf{x}$ *at the moment* t,

$$\mathsf{T}_{(.)} = \left\{T_i(t;\mathbf{x}): \ T_i(t;\mathbf{x}) = T_{(.)}(t;\mathbf{x}),\right\},\, (.) \in \{-,1,2,...\},$$

$\mathbb{T}_{(.)}(t_{(.)};A)$ *the time field over a set* A, $A \subseteq R^n$, [*over the* R^n*-space* if, and only if, $A = R^n$] *at a moment* $t_{(.)}$,

$$\mathbb{T}_{(.)}(t_{(.)};A) = \left\{ \begin{array}{c} \mathsf{T}(t_{(.)},\mathbf{x}): \ \exists\,(S \neq \phi) \subseteq A,\, \exists k \in \{-,1,2,...\} \Longrightarrow \\ \exists \mathsf{T}_k(t_{(.)};S;A) \neq \phi,\ and\ \mathsf{T}(t_{(.)},\mathbf{x}) = \mathsf{T}_k(t_{(.)};S;A),\, \forall \mathbf{x} \in S \end{array} \right\},$$
$$\mathbb{T}_{(.)}(t_{(.)};R^n) = \left\{ \begin{array}{c} \mathsf{T}(t_{(.)},\mathbf{x}): \ \exists\,(S \neq \phi) \subseteq R^n,\, \exists k \in \{-,1,2,...\} \Longrightarrow \\ \exists \mathsf{T}_k(t_{(.)};S) \neq \phi,\ and\ \mathsf{T}(t_{(.)},\mathbf{x}) = \mathsf{T}_k(t_{(.)};S),\, \forall \mathbf{x} \in S \end{array} \right\} \equiv$$
$$\equiv \mathbb{T}_{(.)}(t_{(.)}),$$

$u_i \in R^+$ *the i-th element of the unity vector* $\mathbf{u}$,

$\mathbf{u} \in R^n$ *the physically dimensionless fixed and constant unity vector* that can determine the direction of all other vectors and of all translations in the space, its representation relative to the unity basis $\{\mathbf{e}_{u1},\, \mathbf{e}_{u2},\, ...,\, \mathbf{e}_{un}\}$ reads

$$\mathbf{u} = u_1\mathbf{e}_{01} + \ u_2\mathbf{e}_{02} + ...\ u_n\mathbf{e}_{0n},\, \mathbf{u} = (u_1\ u_2\ ...\ u_n)^T,\, u_i \neq 0,\, \|\mathbf{u}\| = 1,$$

$\mathbf{u}^e \in R^n \times R^1$ *the physically dimensionless extension of the unity vector* $\mathbf{u}$ *in* R^{n+1}, i.e. its representation in the *n+1* dimensional vector space $R^n \times R^1$,

$$\mathbf{u}^e = (\mathbf{u}\ \ 0)^T = (u_1\ u_2\ ...\ u_n\ \ 0)^T,$$

$\mathbf{u}_e \in R^1 \times R^n$ *the physically dimensionless extension of the unity vector* $\mathbf{u}$ *in* R^{1+n}, i.e. its representation in the $1+n$ dimensional vector space $R^1 \times R^n$,

$$\mathbf{u}_e = (0\ \ \mathbf{u}) = (0\ \ u_1\ u_2\ ...\ u_n)^T,$$

v *speed*, the value of which is measured with the length unit 1_L and with the *time* unit $1_t = 1s$, $v^{(.)}\left\langle 1_L 1_t^{-1}\right\rangle$; or *vehicle*,

$v^{(.)} \in R^{+}$ *speed*, the value of which is measured with the length unit $1_{L_{(.)}}$ and with the *time* unit $1_{t_{(.)}}$, $v^{(.)}\left\langle 1_{L_{(.)}} 1_{t_{(.)}}^{-1}\right\rangle$, if, and only if, $\mathbf{u}$ and $\mathbf{v}^{(.)}$ are colinear then

$$v^{(.)}{=}\|\mathbf{v}^{(.)}\| sign\left(\mathbf{v}^{(.)T}\mathbf{u}\right), \left|v^{(.)}\right| = \|\mathbf{v}^{(.)}\|,$$

$\mathbf{v}^{(.)} \in \mathfrak{V}^{3}$ *translational velocity*, $\mathbf{v}^{(.)} = (v_1^{(.)}\ v_2^{(.)}\ v_3^{(.)})^T$,

$$\mathbf{v}^{(.)} = v^{(.)}\mathbf{u}\ if, andonlyif,\ \mathbf{u}\ and\ \mathbf{v}^{(.)}\ are\ colinear,\ otherwise\ \mathbf{v}^{(.)} = \mathrm{V}^{(.)}\mathbf{u},$$

$\mathbf{v}_{(..)}^{O_i,j}$ *the constant velocity of a point* $(..)$ *with respect to the origin* O_i *of* R_i^n if the length value is measured with the length unit 1_{L_i} of R_i^n and if the *time* value is measured with the *time* unit 1_{t_j} of the *time* axis T_j,

$$\mathbf{v}_{(..)}^{O_i,j} = \mathrm{V}_{(..)}^{O_i,j}\mathbf{u}\ in\ general,\ if, andonlyif,\ \mathbf{v}_{(..)}^{O_i,j}\ and\ \mathbf{u}\ are\ colinear\ then$$
$$\mathbf{v}_{(..)}^{O_i,j} = v_{(..)}^{O_i,j}\mathbf{u} = \|\mathbf{v}_{(..)}^{O_i,j}\| sign\left(\mathbf{u}^T\mathbf{v}_G^{O_i,j}\right)\mathbf{u},\ (..) \in \{G, P, P_R, P_{SU}\},$$

if $i = j$ then $\mathbf{v}_{(..)}^{O_i,i} \equiv \mathbf{v}_{(..)}^{i}$,

$\mathbf{v}_{(..)}^{ij}$ *the constant velocity* $\mathbf{v}_{(..)}^{O_i,j}$ if, and only if, $\mathbf{v}_{(..)}^{O_i,i} = \mathbf{v}_{(..)}^{O_j,j}$,

$$\mathbf{v}_{(..)}^{O_i,i} = \mathbf{v}_{(..)}^{O_j,j} \Longrightarrow \mathbf{v}_{(..)}^{O_i,i} = \mathbf{v}_{(..)}^{O_j,j} = \mathbf{v}_{(..)}^{ij} = \mathbf{v}_{(..)}^{ji},\ (..) \in \{G, P, P_R, P_{SU}\},$$

v_{ji}^{i} *the spatial transfer speed* that is constant relative speed of both O_j and R_j^n with respect to both O_i and R_i^n; its value is measured with the length unit 1_{L_i} of R_i^n and with the *time* unit 1_{t_i} of T_i,

$$v_{ji}^{i} = v_{O_j}^{i} - v_{O_i}^{i} = (-v_{ij}^{i}) \in R_{+},\ v_{ji}^{i}\left\langle 1_{L_i}1_{t_i}^{-1}\right\rangle,$$

$\mathbf{v}_{ji}^{i}$ *the spatial transfer velocity* that is constant relative velocity of both O_j and R_j^n with respect to both O_i and R_i^n measured with the length unit 1_{L_i} of R_i^n and with the *time* unit 1_{t_i} of T_i, $\mathbf{v}_{ji}^{i}\left\langle 1_{L_i}1_{t_i}^{-1}\right\rangle$,

$$\mathbf{v}_{ji}^{i} \equiv \mathbf{v}_{O_j}^{i} - \mathbf{v}_{O_i}^{i} \equiv -\mathbf{v}_{ij}^{i} \equiv \mathbf{v}_{ji}^{i} = \mathrm{V}_{ji}^{i}\mathbf{u}\ in\ general,$$
$$\mathbf{v}_{ji}^{i} = v_{ji}^{i}\mathbf{u}\ if, andonlyif,\ \mathbf{v}_{ji}^{i}\ and\ \mathbf{u}\ are\ colinear,$$

$\mathbf{v}_{ji}$ denotes the relative velocities $\mathbf{v}_{ji}^{i}$ and $\mathbf{v}_{ji}^{j}$ if, and only if, $\mathbf{v}_{ji}^{i} = \mathbf{v}_{ji}^{j}$,

$$\mathbf{v}_{ji}^{i} = \mathbf{v}_{ji}^{j} \Longrightarrow \mathbf{v}_{ji}^{i} = \mathbf{v}_{ji}^{j} = \mathbf{v}_{ji} = -\mathbf{v}_{ij},$$

$\mathbf{v}_{ji}^{0}$ the zero superscript denotes that the length value is measured with the length unit 1_L of R^n and the *time* value is measured with the unit 1_t of the *time* axis *T*,

$$\mathbf{v}_{ji}^{0} \equiv \left(\mathbf{v}_{O_j}^{O} - \mathbf{v}_{O_i}^{O}\right) \equiv -\mathbf{v}_{ij}^{0},\ \mathbf{v}_{ji}^{0} \in R_{+}^{n},$$

$\mathbf{v}_{(..)}^{(.)}$ *the constant velocity of a point* (..) *with respect to the origin* $O_{(.)}$ *of* $R_{(.)}^{n}$ measured with the length unit $1_{L_{(.)}}$ of $R_{(.)}^{n}$ and with the *time* unit $1_{t_{(.)}}$ of $T_{(.)}$, $\mathbf{v}_{(..)}^{(.)}\left\langle 1_{L(.)}1_{t_{(.)}}^{-1}\right\rangle$,

$$\mathbf{v}_{(..)}^{(.)} \equiv \mathrm{V}_{(..)}^{(.)}\mathbf{u}\ in\ general,\ \mathrm{V}_{(..)}^{(.)}\left\langle 1_{L(.)}1_{t_{(.)}}^{-1}\right\rangle,$$

$$\mathbf{v}_{(..)}^{(.)} \equiv v_{(..)}^{(.)}\mathbf{u}\ if, and only if,\ \mathbf{v}_{(..)}^{(.)}\ and\ \mathbf{u}\ are\ colinear,\ \mathbf{v}_{(..)}^{(.)}\left\langle 1_{L(.)}1_{t_{(.)}}^{-1}\right\rangle,$$

$\mathrm{V}_{(..)}^{(.)} = O$, or $v_{(..)}^{(.)} = 0$, means that the point (..) is at rest relative to R^n, we adopt in general

$$O \le diagV_{(..)}^{i} \le diagV_{(..)}^{j},\ 0 \le v_{(..)}^{i} \le\ v_{(..)}^{j},$$

$\mathrm{V}_{(..)}^{(.)} \in R^{nxn}$ *the elementwise nonzero constant diagonal matrix* that determines in general the constant velocity $\mathbf{v}_{(..)}^{(.)}$ of a point (..) relative to $\mathbf{u}$, it is *the speed matrix*; if, and only if, $\mathbf{v}_{(..)}^{(.)}$ and $\mathbf{u}$ are colinear then $\mathrm{V}_{(..)}^{(.)} = v_{(..)}^{(.)}I$, $(..) \in \{G, P, P_R, P_{SU}\}$,

$w^{(.)} \in R^{+}$ *a freely accepted and then fixed constant generic, or reference, speed,*

$\mathbf{w}^{(.)} \in (R^{+})^{n}$ *a freely accepted and then fixed constant generic, or reference, velocity*, its representation relative to the unity basis $\{\mathbf{e}_{u1},\ \mathbf{e}_{u2},\ ...,\ \mathbf{e}_{un}\}$ is determined by

$$\mathbf{w}^{(.)} = \omega_1^{(.)}\mathbf{e}_{u1} + \omega_2^{(.)}\mathbf{e}_{u2} + ... + \omega_n^{(.)}\mathbf{e}_{un} = \left(\omega_1^{(.)}\quad \omega_2^{(.)}\ ...\ \omega_n^{(.)}\right)^T,$$

$\mathbf{w}^{(.)} = \mathrm{W}^{(.)}\mathbf{u}$ in general relative to $\mathbf{u}$, in particular $\mathbf{w}^{(.)} = w^{(.)}\mathbf{u}$ if, and only if, $\mathbf{w}^{(.)}$ and $\mathbf{u}$ are colinear,

$\mathrm{W}^{(.)}$ *the diagonal velocity matrix of the velocity* $\mathbf{w}^{(.)}$,

$$\mathrm{W}^{(.)} = diag\left\{w_1^{(.)}\quad w_2^{(.)}\ ...\ w_n^{(.)}\right\},\ w_k^{(.)} \equiv \frac{\omega_k^{(.)}}{u_k},$$
$$(.)_k \in \{-, 1, 2, ..., s\},\ k = 1, 2, ..., n,$$

$x \in R$ *a real valued scalar variable,*

$\mathbf{x} \in R^n$ *the real state vector of a system,* $\mathbf{x} = (x_1\ \ x_2\ \ ...\ x_n)^T$,

$\mathbf{x}_e \in R^n$ *an extended vector variable in* R^{1+n}, $\mathbf{x}_e = (0\quad x_1\ \ x_2\ \ ...\ x_n)^T$,

$\mathbf{x}_r \in R^n$ *the reference state vector of a plant,*

$X(t;E) \subseteq R^n$ *the set* X_E is occupied by the characteristics (existence, features, attributes,...) of the E-event at the moment t, $X(t;E) \equiv X_E(t) \equiv X_E$,

$\mathbf{z} \in R^k$ $\mathbf{z} = (z_1\ z_2\ ...\ z_k)^T$.

19.4 Names

Consistent Time Relativity Theory *(**CTRT**) is the new relativity theory, which is exposed in this book, which is general and reduces to Galilean - Newtonian and to Einsteinian relativity theories under the corresponding assumptions and restrictions.* For short, ***Consistent Relativity Theory*** (***CRT***).

Einsteinian relativity theory means the relativity theory established by Lorentz [297] - [301], Einstein [114] - [157] and Poincaré [383], [386], which has been further developed on the fundamentals founded by them.

Lorentz-Einstein invariance of the light velocity denotes the light speed invariance that is claimed to be proved by Einstein's law of the velocity composition.

PCP is the abbreviation of ***Physical Continuity Principle.***

PCUP is the abbreviation of ***Physical Continuity and Uniqueness Principle.***

PUP is the abbreviation of ***Physical Uniqueness Principle.***

19.5 Symbols

(.) *an arbitrary variable*, or *an index*, or *a physical unit* " . ",

$(.) \in \{-\}$ means that (.) should be omitted,

$(.) = -$ means that (.) should be omitted,

(..) *an arbitrary variable*, or *a point (..)*, if not stated otherwise,

$|(.)| : R \rightarrow R_+$ *the absolute value* (*module*) *of a (complex valued) scalar variable* (.),

$\|.\| : R^n \rightarrow R_+$ *an accepted norm on* R^n, which can be the *Euclidean norm on* R^n in a special case:

$$||\mathbf{r}|| = \sqrt{\mathbf{r}^T\mathbf{r}} = \sqrt{\sum_{i=1}^{i=n} \rho_i^2}, \ \mathbf{r} \in R^n,$$

$\|\mathbf{v}\| \in R_+$ *the accepted norm of the vector* $\mathbf{v}$ is its *length* (its *intensity*),

$\langle 1_{..} \rangle$ shows the units 1... of a physical variable,

$[\ \alpha, \beta\ ] \subset R$ *a compact interval*, $[\alpha, \beta] = \{x: \ x \in R, \ \alpha \leq x \leq \beta\}$,

$[\ \alpha, \beta\ [\ \subseteq R$ *a left closed, right open interval*, $[\alpha, \beta[= \{x : x \in R, \alpha \leq x < \beta\}$,

$]\ \alpha, \beta\ ] \subseteq R$ *a left open, right closed interval*, $[\alpha, \beta[= \{x : x \in R, \alpha < x \leq \beta\}$,

$]\ \alpha, \beta\ [\ \subseteq R$ *an open interval*, $]\alpha, \beta[= \{x: \ x \in R, \ \alpha < x < \beta\}$,

$(\ \alpha, \beta\ ) \subseteq R$ *a general interval*, $(\ \alpha, \beta\ ) \in \{[\alpha, \beta], \ [\alpha, \beta[, \ ]\alpha, \beta], \ ]\alpha, \beta[\}$,

[A..] shows *the physical dimension* A... *of a physical variable*,

$\mathbf{1} \in R^n$ *the elementwise unity vector*, $\mathbf{1} = (1 \ \ 1 \ ... \ 1)^T$, $\mathbf{1}^s = (1\ 1 ... 1)^T \in R^s$, $\mathbf{1}^n = \mathbf{1}$,

$\mathbf{1}_e \in R^{1+n}$ *the extended elementwise unity vector*, $\mathbf{1}_e = (1 \ \ 1 \ ... \ 1)^T = (1 \ \ \mathbf{1}^T)^T$,

$1_{\theta C}$ *the temperature unit of the Celsius scale*,
$1_{\theta F}$ *the temperature unit of the Fahrenheit scale*,
$1_{\theta K}$ *the temperature unit of the Kelvin scale*,
$\forall$ for every,
*diagmin*V denotes *the minimal diagonal element of the diagonal matrix* V, V $=diag\{\mathrm{v}_1 \ \ \mathrm{v}_2 \ ... \ \mathrm{v}_n\}$ implies

$$diagmin\mathrm{V} = min\left(\mathrm{v}_1 \ \mathrm{v}_2 ... \mathrm{v}_n\right),$$

$diag\mathrm{V}_{ji} < diag\mathrm{V}_P$ means that the diagonal matrix V_{ji} is diagonally elementwise less than the diagonal matrix V_P, i.e. every diagonal entry of V_{ji} is less than the corresponding diagonal entry of V_P, for $\mathrm{V}_{ji} = diag\{\mathrm{v}_{ji1} \ \ \mathrm{v}_{ji2} \ ... \ \mathrm{v}_{jin}\}$ and $\mathrm{V}_P = diag\{\mathrm{v}_{P1} \ \ \mathrm{v}_{P2} \ ... \ \mathrm{v}_{Pn}\}$, $diag\mathrm{V}_{ji} < diag\mathrm{V}_P$ if, and only if, $\mathrm{v}_{jik} < \mathrm{v}_{Pk}$, $\forall k = 1, 2, ..., n$,
dim **z** *the mathematical dimension of a vector* **z**, $\mathbf{z} \in R^n \Longrightarrow dim\ \mathbf{z} = n$,
$\exists$ there exist(s),
$\exists!$ there exists exactly one,
$\in$ belong(s) to, are (is) members (a member) of, respectively,
$\subset$ a proper subset of (it can not be equal to),
$\subseteq$ a subset of (it can be equal to),
$\sqrt{-1}$ *the imaginary unit* denoted by i, $i = \sqrt{-1}$,
inf *infimum*,
max *maximum*,
min *minimum*, or the time unit *minute*,
numx *the numerical value of* x, $numx = \frac{|x|}{1_x}$ if $0 < |x| < \infty$ and $0 < 1_x < \infty$, e.g. if $x = 50V$ then $numx = 50$,
phdim $x(.)$ *the physical dimension of a variable* $x(.)$,

$$x(.) = t \Longrightarrow phdim\ x(.) = phdim\ t = \mathrm{T},\ but\ dim\ t = 1,$$

rad *radian*,
sup *supremum*,
$|x|$ *the absolute value of* $x \in R$, $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x \leq 0$,
$|\mathrm{V}|$ *the elementwise absolute value matrix of the matrix* $\mathrm{V} = diag\{\mathrm{v}_1 \ \ \mathrm{v}_2 \ ... \ \mathrm{v}_n\}$, $|\mathrm{V}| = diag\{|\mathrm{v}_1| \ \ |\mathrm{v}_2| ... |\mathrm{v}_n|\}$.

19.6 Units

$1_{(.)}$ *the unit of a physical variable* (.); *the time unit of the time axis* $T_{(.)}$,
1_i the short notation for the *time* unit 1_{t_i} of the *time* axis T_i, $1_i = 1_{t_i}$,
$1_{L_{(.)}}$ *the length* (*distance, position*) *unit of the length scale* (.) *of* $R^n_{(.)}$,
1_{tbasic} is *the basic time unit*, which is second, s, $1_{tbasic} = s$,
1_t *the time unit of the reference time axis* T, $1_t = 1_{tbasic} = s$,
$1_{t_{(.)}}$ *the time unit of the time axis* $T_{(.)}$, for short $1_{t_{(.)}} \equiv 1_{(.)}$.

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Chapter 20

Appendices: Proofs for Part 1

20.1 *Time* Uniqueness

20.1.1 Einstein‘s postulate

Einstein (Section) accepted tacitly the following hypothesis and liftrd it to the rank of (undoubtfull) postulate:

Hypothesis 674 ***Time is not unique***
There exist several different times.

20.1.2 Mathematical expression of Hypothesis 674

In view of Hypothesis 674, let us assume, conversely to Axiom 47 (Subsection 4.2: "Characterization of *time*"), that Lorentz transformations and/or the transformations established herein prove the existence of two different, linearly independent, *times* (*time* variables) denoted by t and by τ. Then their values can increase with different variable speeds. Let the speed of the increase of τ relative to t be a function $f(.) : R \rightarrow R^{+}$ of t, which expresses a possibility of its *time*-varying speed:

$$\frac{d\tau}{dt} = f(t) \neq constant. \tag{20.1}$$

This is a mathematical expression of Hypothesis 674.

In order to disprove Hypothesis 674, i.e. the equation (20.1), we should prove the following equation:

$$\frac{d\tau}{dt} = f(t) \equiv \ \eta = constant. \ and \ phdim\eta = -, \ i.e. \ \eta \ [-]. \tag{20.2}$$

The first equation (20.2) will be called *the basic equation*.

The proof will be presented in four different ways.

20.1.3 Direct proof of (20.2) via the light speed

Proof. We do not change the space coordinate system in this case. The light speed is defined as follows:

$$\mathbf{c}^t = \frac{d\mathbf{r}_L(t;t_0)}{dt} = c^t\mathbf{u},\ c^t = \mathit{constant\ in\ vacuum,\ relative\ to\ time}\ t, \quad (20.3)$$

$$\mathbf{c}^\tau = \frac{d\mathbf{r}_L(\tau;\tau_0)}{d\tau} = c^\tau\mathbf{u},\ c^\tau = \mathit{constant\ in\ vacuum,\ relative\ to\ time}\ \tau. \quad (20.4)$$

The equation (20.3) can be modified in view of (20.1) and (20.4) as follows:

$$\mathbf{c}^t = c^t\mathbf{u} = \frac{d\mathbf{r}_L(t;t_0)}{dt} = \frac{d\mathbf{r}_L[t(\tau);t(\tau_0)]}{d\tau}\frac{d\tau}{dt} = \mathbf{c}^\tau f(t) = f(t)c^\tau\mathbf{u}. \quad (20.5)$$

(20.1) and (20.5) imply (20.2) with $\eta = (c^\tau)^{-1}c^t$ $[-]$, which contradicts $f(t) \neq$ constant. Since $\eta = (c^\tau)^{-1}c^t = const.$ is correct, and $phdim\eta = -$, *i.e.* η $[-]$, then $f(t) \equiv \eta = const.$ and (20.2) is correct. ■

20.1.4 Proof of (20.2) by using (7.22)

Proof. We apply Lorentz transformation (7.22) to the light signal L accepted for the arbitrary point P, i.e. for $\mathbf{r}_P(t_{(.)};t_{(.)0}) \equiv \mathbf{r}_L(t_{(.)};t_{(.)0})$, $t_i = t$ and $t_j = \tau$,

$$\begin{aligned}\mathbf{r}_L(t;t_0) &= \mathbf{r}_L(t_i;t_{i0}) = \frac{\mathbf{r}_L(t_j;t_{j0}) + v(t_j - t_{j0})\mathbf{u}}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{\mathbf{r}_L(\tau;\tau_0) + v(\tau - \tau_0)\mathbf{u}}{\sqrt{1-\frac{v^2}{c^2}}},\end{aligned} \quad (20.6)$$

$$\mu_{t\tau} = \mu_{ij} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}[-],\ \mu_{\tau t} = \mu_{ji} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}[-], \quad (20.7)$$

so that, for $c^t = c^\tau = c^{t\tau} = c$ according to Einsteinian relativity theory:

$$\begin{aligned}\mathbf{c}^t &= c^t\mathbf{u} = c\mathbf{u} = \frac{d\mathbf{r}_L(t;t_0)}{dt} = \frac{d\mathbf{r}_L[t(\tau);t(\tau_0)]}{d\tau}\frac{d\tau}{dt} \\ &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\left[\mathbf{r}_L(\tau;\tau_0) + v(\tau-\tau_0)\mathbf{u}\right]}{d\tau}f(t) = \\ &= \frac{c^\tau + v}{\sqrt{1-\frac{v^2}{c^2}}}f(t)\mathbf{u} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}cf(t)\mathbf{u} = \mu_{t\tau}cf(t)\mathbf{u} \Longrightarrow \\ & f(t) = \mu_{t\tau}^{-1} = \mu_{ij}^{-1} = const. = \eta\ [-].\end{aligned}$$

This and (20.1) show that (20.2) is valid for $\eta = \mu_{ij}^{-1} = const.$ $[-]$. ■

20.1.5 Proof of (20.2) by using (11.54)

Proof. Let us now change the space coordinate system by using the time-invariant generalized Lorentz transformation (11.54), which is determined for the light signal L as the arbitrary point P, i.e. for $v_P^{(.)} \equiv c_{(.)}^{(.)}$,

$$\mathbf{r}_P(t_i) = \frac{\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}}{1 + \frac{v_{ji}^j}{c_j^j}}. \tag{20.8}$$

We get the following from (20.1) and (20.8), in which t_i and t_j are replaced by t and τ, the subscripts / superscripts i and j are replaced by t and τ, $\mathbf{r}_P$ and r_P by $\mathbf{r}_L$ and r_L, respectively:

$$\mathbf{c}^t = c^t \mathbf{u} = \frac{d\mathbf{r}_L(t)}{dt} = \frac{d\mathbf{r}_L[t(\tau)]}{d\tau}\frac{d\tau}{dt} = \frac{d\left\{\frac{[\mathbf{r}_L(\tau) + v_{\tau t}^{\tau}\tau \mathbf{u}]}{1 + \frac{v_{\tau t}^{\tau}}{c^{\tau}}}\right\}}{d\tau} f(t) \Longrightarrow$$

$$\mathbf{c}^t = c^t \mathbf{u} = \frac{c^{\tau} + v_{\tau t}^{\tau}}{1 + \frac{v_{\tau t}^{\tau}}{c^{\tau}}} f(t)\mathbf{u} = c^{\tau} f(t)\mathbf{u}. \tag{20.9}$$

This result and equality in (20.1) yield the following in view of $(c^{\tau})^{-1}c^t = constant = \eta$:

$$\frac{d\tau}{dt} = f(t) \equiv (c^{\tau})^{-1} c^t = constant = \eta \ [-], \tag{20.10}$$

which once more contradicts $f(t) \neq constant$ and proves (20.2). ■

20.1.6 Termination of the proof via (20.2)

Proof. In all four typical cases we proved (20.2). We should proceed with it:

$$d\tau/dt = \eta \ [-].$$

Integrating this equation we find:

$$\tau - \tau_0 = \eta(t - t_0).$$

Evidently, t and τ differ only for the constant scaling dimensionless coefficient η. They are linearly dependent. They represent the same *time*, (temporal variable), the values of which are measured with respect to two different *time* scales and/or by two different *time* units, i.e. with respect to the *time* axes T and T_τ. Q. E. D ■

20.1.7 Proof via *time* speed

Proof. The speed of the *time* value flow, i.e., the *time* speed, invariant, independent of everybody and everything. Its numerical is invariant 1 (one). This follows from Theorem 128. Therefore, *time* is unique and untouchable. There are not two or more different times. Nobody and nothing can influence *time* or its speed. Only relative zero value of *time*, and relative initial moment, and *time* scale and *time* unit can be relative, touchable and multiple (Note 130) ■

20.2 Proof of Theorem 82

Proof. Einsteinian proof of the relativity theory based clock principle in the general form

Let Clock conditions 79 hold.

Einstein used the following Lorentz relationships for the temporal and spatial coordinates of a moving point P (the subscript "P") relative to the coordinate system at rest (the subscript r) and relative to the inertial coordinate system moving (the subscript m) with the speed v, [144, p. 28], [153, p. 36], [154, pp. 32, 33],

$$t_{Pm} = \frac{t_{Pr} - \frac{v}{c^2} x_{Pr}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{20.11}$$

$$x_{Pm} = \frac{x_{Pr} - v t_{Pr}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{20.12}$$

Let the moving clock (the subscript "C" for "clock C") be taken for the arbitrary point P, P=C=O_m. Hence,

$$x_{pm} = x_{Cm} = 0, \; and \; v = v_m.$$

These equations and (20.12) express the following fact in view of $t_{Cr} = t_{Pr}$ and $x_{cr} = x_{Pr}$:

$$x_{Cr} = v_m t_{Cr}.$$

The equation (20.11) becomes now

$$t_{Cm} = \sqrt{\frac{1 - \frac{v_m}{c}}{1 + \frac{v_m}{c}}} t_{Cr},$$

due to $t_{Cm} = t_{Pm}$, or equivalently,

$$t_{Cm} = \mu_{mr} t_{Cr}, \; \mu_{mr} = \sqrt{\frac{1 - \frac{v_m}{c}}{1 + \frac{v_m}{c}}} = const.,$$

$$0 < v_m < c \Longrightarrow \mu_{mr} \in]0, \; 1[.$$

The constant coefficient μ_{mr} is the *time* scaling coefficient that satisfies (4.68). It relates the numerical *time* value t_{Cr} measured relative to the *time* axis T_r of the clock at rest to the numerical *time* value t_{Cm} measured relative to the *time* axis T_m of the moving clock. Therefore,

$$\begin{aligned} (t_m - t_{m0}) \langle 1_{t_m} \rangle = \left(N_{1_{t_m}} 1_{t_m} \right) \langle 1_{t_m} \rangle = \left(\mu_{mr} N_{1_{t_r}} 1 t_r \right) \langle 1_{t_m} \rangle = \\ = \left(N_{1_{tr}} 1_{t_r} \right) \langle 1_{t_r} \rangle = (t_r - t_{r0}) \langle 1_{t_r} \rangle = \\ = [\mu_{mr}(t_r - t_{r0})] \langle 1_{t_m} \rangle , \; \mu_{mr} \left\langle 1_{t_m} 1_{t_r}^{-1} \right\rangle \; [-], \end{aligned} \tag{20.13}$$

for short

$$t_m - t_{m0} = \mu_{mr}(t_r - t_{r0}). \tag{20.14}$$

These equations verify the equations (4.67). Notice that the angle units of the clock at rest and of the moving clock are 1_{φ_r} and 1_{φ_m}, respectively. Let $\eta_{mr}\left\langle 1_{\varphi_m}1_{\varphi_r}^{-1}\right\rangle$ be the angle scaling coefficient,

$$\left(N_{1_{\varphi_m}}1_{\varphi_m}\right)\left\langle 1_{\varphi_m}\right\rangle = \left(\eta_{mr}N_{1_{\varphi_r}}1_{\varphi_r}\right)\left\langle 1_{\varphi_m}\right\rangle \Longrightarrow$$
$$\varphi_m - \varphi_{m0} = \eta_{mr}\left(\varphi_r - \varphi_{r0}\right). \tag{20.15}$$

Therefore, the equations (4.57), (20.13) through (20.15) yield

$$\omega_m\left\langle 1_{\varphi_m}1_{t_m}^{-1}\right\rangle = \frac{d\varphi_m\left\langle 1_{\varphi_m}\right\rangle}{dt_m\left\langle 1_{t_m}\right\rangle} = \frac{\eta_{mr}\left\langle 1_{\varphi_m}1_{\varphi_r}^{-1}\right\rangle d\varphi_r\left\langle 1_{\varphi_r}\right\rangle}{\mu_{mr}\left\langle 1_{t_m}1_{t_r}^{-1}\right\rangle dt_r\left\langle 1_{t_r}\right\rangle} =$$
$$= \eta_{mr}\left\langle 1_{\varphi_m}1_{\varphi_r}^{-1}\right\rangle \mu_{mr}^{-1}\left\langle 1_{t_m}^{-1}1_{t_r}\right\rangle \omega_r\left\langle 1_{\varphi_r}1_{t_r}^{-1}\right\rangle,$$

so that

$$\omega_m\left\langle 1_{\varphi_m}1_{t_m}^{-1}\right\rangle = \left(\eta_{mr}\mu_{mr}^{-1}\omega_r\right)\left\langle 1_{\varphi_m}1_{t_m}^{-1}\right\rangle \tag{20.16}$$

The equation (4.58) is valid for each of the clocks,

$$\frac{\varphi_{(.)} - \varphi_{(.)0}}{\omega_{(.)}} = t_{(.)} - t_{(.)_0},\ \forall\omega_{(.)} \in R^+,\ (.) = m,\, r. \tag{20.17}$$

This, (20.14) through (20.16) show that

$$\frac{\varphi_m - \varphi_{m_0}}{\omega_m} = \frac{\eta_{mr}\left(\varphi_r - \varphi_{r_0}\right)}{\eta_{mr}\mu_{mr}^{-1}\omega_r} = \mu_{mr}\frac{\varphi_r - \varphi_{r_0}}{\omega_r} = \mu_{mr}(t_r - t_{r_0}),$$

or equivalently, by using $N_m\left\langle -\right\rangle = \left(\varphi_m - \varphi_{m_0}\right)\left\langle 1_{\varphi_m}\right\rangle / \varphi_{u_m}\left\langle 1_{\varphi_m}\right\rangle$,

$$\mu_{mr}^{-1}\frac{\varphi_m - \varphi_{m_0}}{\omega_m} = \mu_{mr}^{-1}\frac{N_m\varphi_{u_m}}{\omega_m} = t_r - t_{r_0} = t - t_0,\ \forall\omega_m \in R^+. \tag{20.18}$$

This proves the equations (4.69) since the *time* axes T and T_r are identical. We multiply the equations (20.18) by μ_{mr} and then we apply the equation (4.67). The result is

$$\frac{\varphi_m - \varphi_{m_0}}{\omega_m} = \frac{N_m\varphi_{u_m}}{\omega_m} = t_m - t_{m_0}. \tag{20.19}$$

These are the equations (4.66) for $i_2 = m$ and $i_1 = m_0$. The equations (4.69), (20.18) and (20.19) lead to the following relationships:

$$v_{t_m} = \frac{d\left(t_m - t_{m_0}\right)}{dt_m} = \frac{d\mu_{mr}\left(t_r - t_{r_0}\right)}{d\mu_{mr}t_r} =$$
$$= \frac{d\left(t_r - t_{r_0}\right)}{dt_r} = v_{t_r} = \frac{d\left(t - t_0\right)}{dt} = v_t,$$
$$\forall(v_m,\, \omega_m,\, \omega_r) \in R^+ \times R^+ \times R^+,$$

which prove (4.70). Q. E. D ■

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Chapter 21

Appendices: Proofs for Part 2

21.1 Proof of Theorem 292

Proof. Let the same *time* and length units hold for all integral spaces so that the *time* axes are equal and the same for all the frames,

$$T_i = T_j = T, \text{ } i.e. \text{ } t_i = t_j = t. \tag{21.1}$$

Let $A = B$ in the block diagonal matrix D, (6.22). Let $\mathbf{v}_G^{(.)} \equiv \mathbf{v}_P^{O_{(.)}}$ and (8.8) hold in (6.22). Let the velocity $\mathbf{v}_P^{O_{(.)}}$ of the arbitrary point P be arbitrary. We use

$$\mathbf{r}_P^{O_i}(t_i) \equiv \mathbf{r}_P^{O_i}(t) \equiv r_P^{O_i}(t)\mathbf{u},\ \mathbf{r}_P^{O_j}(t;t_0) \equiv t\mathbf{v}_P^{O_j},\ \mathbf{v}_P^{O_{(.)}} \equiv v_P^{O_{(.)}}\mathbf{u},$$

in (6.22). The equations (8.2) through (8.6), and (21.1) transform the left-hand side of (6.22) as follows:

$$\left[\begin{array}{c} \mathbf{r}_P^{O_i^T}(t) \\ t\mathbf{v}_P^{O_i^T} \end{array}\right]^T D \left[\begin{array}{c} \mathbf{r}_P^{O_i}(t) \\ t\mathbf{v}_P^{O_i} \end{array}\right] \equiv \left[\begin{array}{c} r_P^{O_i}(t)\mathbf{u}^T \\ tv_P^{O_i}\mathbf{u}^T \end{array}\right]^T \begin{array}{cc} A & O \\ O & -A \end{array} \left[\left[\begin{array}{c} r_P^{O_i}(t)\mathbf{u} \\ tv_P^{O_i}\mathbf{u} \end{array}\right]\right] \equiv$$

$$\equiv \left[\begin{array}{c} \left[r_P^{O_j}(t)\right]^2 + 2r_P^{O_j}(t)v_{ji}^0 t + \left(v_{ji}^0 t\right)^2 - \\ -t_i^2\left\{\left(v_P^{O_j}\right)^2 + 2v_P^{O_j}v_{ji}^0 + \left(v_{ji}^0\right)^2\right\} \end{array}\right]\mathbf{u}^T A\mathbf{u} \equiv$$

$$\equiv \left[\begin{array}{c} \left[r_P^{O_j}(t)\right]^2 + 2v_P^{O_j}tv_{ji}^0 t + + \left(v_{ji}^0 t\right)^2 - \\ -\left(v_P^{O_j}t\right)^2 - 2v_{ji}^0 v_P^{O_j}t^2 - \left(v_{ji}^0 t\right)^2 \end{array}\right]\mathbf{u}^T A\mathbf{u} \equiv \left[\begin{array}{c} \mathbf{r}_P^{O_j^T}(t) \\ t\mathbf{v}_P^{O_j^T} \end{array}\right]^T D \left[\begin{array}{c} \mathbf{r}_P^{O_j}(t) \\ t\mathbf{v}_P^{O_j} \end{array}\right].$$

This holds also for the light velocity $\mathbf{c}_{(.)}$ if and only if $P = L$, for the velocity $\mathbf{v}_R^{O_{(.)}}$ of the reference point P_R if and only if $P = P_R$, and for the velocity $\mathbf{v}_{SU}^{O_{(.)}}$ of the reference point P_{SU} if and only if $P = P_{SU}$, (8.8). Q. E. D ■

21.2 Proof of Theorem 309

Proof. Let the equations (8.17) be satisfied. Let $A = B$ in D, (6.22), and let $G = P$. Let the velocity of the arbitrary point P be arbitrary. We transform the left-hand side of (6.22) by applying (8.17), (8.18), (8.19) through (8.23):

$$\begin{bmatrix} \mathbf{r}_P^{O_i}(t_i;t_{i0}) \\ (t_i - t_{i0})\widetilde{\mathbf{v}}_P^{O_i,i}(t_i;t_{i0}) \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P^{O_i}(t_i;t_{i0}) \\ (t_i - t_{i0})\widetilde{\mathbf{v}}_P^{O_i,i}(t_i;t_{i0}) \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} \mathbf{r}_P^{O_i}(t_j;t_{j0}) \\ \frac{\mu_i}{\mu_j}(t_j - t_{j0})\frac{\mu_j}{\mu_i}\widetilde{\mathbf{v}}_P^{O_i,j}(t_j;t_{j0}) \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P^{O_i}(t_j;t_{j0}) \\ \frac{\mu_i}{\mu_j}(t_j - t_{j0})\frac{\mu_j}{\mu_i}\widetilde{\mathbf{v}}_P^{O_i,j}(t_j;t_{j0}) \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} r_P^2(t_j;t_{j0}) + 2v_{ji}^j r_P(t_j;t_{j0})(t_j - t_{j0}) + \\ + \left[v_{ji}^j(t_j - t_{j0})\right]^2 - \left[\widetilde{v}_P^{O_j,j}(t_j;t_{j0})(t_j - t_{j0})\right]^2 - \\ -2v_{ji}^j\widetilde{v}_P^{O_j,j}(t_j;t_{j0})(t_j - t_{j0})^2 - \left[v_{ji}^j(t_j - t_{j0})\right]^2 \end{bmatrix} \mathbf{u}^T A\mathbf{u} \equiv$$

$$\equiv \begin{bmatrix} \mathbf{r}_P^{O_j}(t_j;t_{j0}) \\ (t_j - t_{j0})\widetilde{\mathbf{v}}_P^{O_j,j}(t_j;t_{j0}) \end{bmatrix}^T D \begin{bmatrix} \mathbf{r}_P^{O_j}(t_j;t_{j0}) \\ (t_j - t_{j0})\widetilde{\mathbf{v}}_P^{O_j,j}(t_j;t_{j0}) \end{bmatrix}.$$

Q. E. D ■

Chapter 22

Appendices: Proofs for Part 3

22.1 Proof of Theorem 368

Proof. *Necessity*. Let the scaling coefficients $\alpha_{(.)}^{(..)}$ and $\lambda_{(.)}^{(..)}$ be determined for the case when the arbitrary point P moves with the velocity of light:

$$\mathbf{v}_P^{(.)} = \mathbf{c}_{(.)}^{(.)} = c_{(.)}^{(.)}\mathbf{u},\ i.e.\ v_P^{(.)} = c_{(.)}^{(.)}. \tag{22.1}$$

Let the scaling coefficient μ_i satisfy (11.1). Let the point P start moving from $O = O_{i0}$ at the initial instant $t_{(.)0} = 0$. In view of (22.1),

$$\mathbf{r}_P\left(t_{(.)}\right) \equiv \mathbf{r}_L(t_{(.)}) = c_{(.)}^{(.)} t_{(.)}\mathbf{u}. \tag{22.2}$$

The position vectors can be expressed also in terms of their (algebraic, i.e. scalar) values,

$$\mathbf{r}_{(..)}\left(t_{(.)}\right) = r_{(..)}\left(t_{(.)}\right)\mathbf{u},\ (..) \in \{G, L, P, P_R, P_{SU}\}. \tag{22.3}$$

Let the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, obey (11.2) through (11.6) so that they together with (11.1) imply (11.7). The equations (11.1), (22.2) and (22.3) give the next scalar forms to the equations (11.4) and (11.5):

$$r_L(t_i) = \lambda_j^i[r_L(t_j) + v_{ji}^j t_j] = c_i^i t_i = c_i^i \mu_i t, \tag{22.4}$$

$$r_L(t_j) = \lambda_i^j[r_L(t_i) - v_{ji}^i t_i] = c_j^j t_j = c_j^j \mu_j t. \tag{22.5}$$

After replacing $r_L(t_j)$ by $c_j^j t_j$, (22.2), and by using both $t_i = \mu_i t$ and $t_j = \mu_j t$, (11.1), we set the right-hand side of the first equation (22.4) into the form (22.6) as follows:

$$r_L(t_i) \equiv \lambda_j^i[c_j^j t_j + v_{ji}^j t_j] \equiv \lambda_j^i \frac{c_j^j}{c_i^i}\left(1 + \frac{v_{ji}^j}{c_j^j}\right) c_i^i t_j \equiv$$
$$\equiv \lambda_j^i \frac{c_j^j}{c_i^i}\left(1 + \frac{v_{ji}^j}{c_j^j}\right) c_i^i \frac{\mu_j}{\mu_i} t_i \equiv \lambda_j^i \frac{c_j^j}{c_i^i}\left(1 + \frac{v_{ji}^j}{c_j^j}\right) \frac{\mu_j}{\mu_i} r_L(t_i). \tag{22.6}$$

The solution of (22.6) for λ_j^i is:

$$\lambda_j^i = \frac{c_i^i \mu_i}{c_j^j \mu_j} \frac{1}{1 + \frac{v_{ji}^j}{c_j^j}}. \tag{22.7}$$

By applying the same procedure to the equation (22.5) we get:

$$\lambda_i^j = \frac{c_j^j \mu_j}{c_i^i \mu_i} \frac{1}{1 - \frac{v_{ji}^i}{c_i^i}}. \tag{22.8}$$

By combining (11.1), (11.2) and (22.2) we find:

$$t_i \equiv \alpha_j^i \left(1 + \frac{v_{ji}^j c_j^j}{q^j w^j}\right) \frac{\mu_j}{\mu_i} t_i.$$

This identity implies:

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{v_{ji}^j c_j^j}{q^j w^j}}. \tag{22.9}$$

This equation proves the first equation in (11.8). The first equation (11.9) is proved along the same lines. The equations (11.2), (11.4), (22.2), (22.3), and (11.7) imply the following:

$$[\mathbf{r}_P^T(t_i) \quad t_i \mathbf{c}_i^{i^T}] D [\mathbf{r}_P^T(t_i) \quad t_i \mathbf{c}_i^{i^T}]^T \equiv [\mathbf{r}_L^T(t_i) \quad t_i \mathbf{c}_i^{i^T}] D [\mathbf{r}_L^T(t_i) \quad t_i \mathbf{c}_i^{i^T}]^T \equiv$$
$$\equiv [\lambda_j^i \left(1 + \frac{v_{ji}^j}{c_j^j}\right) \mathbf{r}_L^T(t_j) \quad \alpha_j^i \left(1 + \frac{v_{ji}^j c_j^j}{q^j w^j}\right) \frac{c_i^i}{c_j^j} t_j \mathbf{c}_j^{j^T}] D \bullet$$
$$\bullet [\lambda_j^i \left(1 + \frac{v_{ji}^j}{c_j^j}\right) \mathbf{r}_L^T(t_j) \quad \alpha_j^i \left(1 + \frac{v_{ji}^j c_j^j}{q^j w^j}\right) \frac{c_i^i}{c_j^j} t_j \mathbf{c}_j^{j^T}]^T \equiv$$
$$\equiv [\mathbf{r}_L^T(t_j) \quad t_j \mathbf{c}_j^{j^T}] D [\mathbf{r}_L^T(t_j) \quad t_j \mathbf{c}_j^{j^T}]^T \equiv [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{c}_j^{j^T}] D [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{c}_j^{j^T}]^T. \tag{22.10}$$

These identities, $\alpha_j^i \in R^+$ and $\lambda_j^i \in R^+$ yield:

$$\alpha_j^i = \frac{c_j^j}{c_i^i}\left(1 + \frac{v_{ji}^j c_j^j}{q^j w^j}\right)^{-1}, \; \lambda_j^i = \left(1 + \frac{v_{ji}^j}{c_j^j}\right)^{-1}. \tag{22.11}$$

The first equation (22.11) verifies the second equation (11.8). The proof of the second equation (11.9) is analogous. The second equation (22.11) proves the equation (11.10). The equations (11.10) and (22.7) imply:

$$\frac{c_i^i}{c_j^j}\frac{\mu_i}{\mu_j} = 1. \tag{22.12}$$

The equations (22.8) and (22.12) prove the equation (11.11). The condition that all the scaling coefficients, as well as q^i, q^j, w^i and w^j, are positive real valued, that $v_{ji}^j \in R_+$ by the definition, together with (11.8) through (11.11), imply $v_{ji}^i < \min\left\{c_i^i, \frac{q^i w^i}{c_i^i}\right\}$ and $v_{ji}^j > \max\left\{-c_j^j, -\frac{q^j w^j}{c_j^j}\right\}$, which, together with $v_{ji}^j \geq 0$ and $\max\left\{-c_j^j, -\frac{q^j w^j}{c_j^j}\right\} < 0$, prove (11.12). The first equation in (11.13) results directly from the equation (22.12). The definitions of the speed and of the spatial transfer speed $v_{ji}^{(.)}$, together with (11.1), (11.3), (11.5), (11.9), (11.11), and (22.2) permit the following proof of the second equation in (11.13):

$$v_{ji}^j = \frac{d\left[r_{O_j}^O(t_j) - r_{O_i}^O(t_j)\right]}{dt_j} = \frac{d\left[\frac{r_{O_j}^O(t_i)-0t_i}{1-\frac{0}{c_i^i}} - \frac{r_{O_i}^O(t_i)-0t_i}{1-\frac{0}{c_i^i}}\right] : dt_i}{d\left[\frac{\mu_j}{\mu_i}\frac{t_i - \frac{v_{ji}^i}{q^i w^i} r_L(t_i)}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}\right] : dt_i} =$$

$$= \frac{\mu_i}{\mu_j}\frac{v_{O_j}^{O,i} - v_{O_i}^{O,i}}{\frac{1-\frac{v_{ji}^i}{q^i w^i}c_i^i}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}} = \frac{\mu_i}{\mu_j} v_{ji}^i. \tag{22.13}$$

The equations (11.8) through (11.11) transform the equations (11.2) through (11.5) into the equations (11.2) through (11.17).

Sufficiency. Let (11.8) through (11.17) be valid. Let μ_i obey (11.1). We transform the equation (22.2) as follows by using (11.1), (11.10), (11.13), and (22.2):

$$\mathbf{r}_P(t_i) = c_i^i t_i \mathbf{u} = \frac{c_j^j \mu_j}{\mu_i} t_i \mathbf{u} = c_j^j (1 + \frac{v_{ji}^j}{c_j^j})^{-1} (1 + \frac{v_{ji}^j}{c_j^j}) t_j \mathbf{u} =$$
$$= \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}\right]. \tag{22.14}$$

This proves (11.4). We prove the equation (11.5) along the same lines by beginning with (22.2). We proceed by rearranging (11.1):

$$t_i = \mu_i t = \frac{\mu_i}{\mu_j}\left(1 + \frac{v_{ji}^j c_j^j}{q^j w^j}\right)^{-1}\left(1 + \frac{v_{ji}^j c_j^j}{q^j w^j}\right) t_j =$$
$$= \frac{\mu_i}{\mu_j}\frac{1}{1 + \frac{v_{ji}^j c_j^j}{q^j w^j}}\left(t_j + \frac{v_{ji}^j c_j^j}{q^j w^j} t_j\right). \tag{22.15}$$

Now, (11.8), (22.2) and (22.15) imply:

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j) \right]. \tag{22.16}$$

This is the equation (11.2). By repeating this procedure applied to (11.1) for i replaced by j we prove the equation (11.3). We continue to transform the left-hand side of the identity (11.7) by using (11.8), (11.10), (11.14), (11.16), (22.2), (22.3) and $\mathbf{c}_{(.)}^{(.)} = c_{(.)}^{(.)}\mathbf{u}$,

$$\begin{gathered}
[\mathbf{r}_P^T(t_i) \quad t_i \mathbf{c}_i^{i^T}] D [\mathbf{r}_P^T(t_i) \quad t_i \mathbf{c}_i^{i^T}]^T \equiv \\
\equiv \left\{ \lambda_j^i \left[\mathbf{r}_P^T(t_j) + v_{ji}^j t_j \mathbf{u} \right] \quad t_i \mathbf{c}_i^{i^T} \right\} D \left\{ \lambda_j^i \left[\mathbf{r}_P^T(t_j) + v_{ji}^j t_j \mathbf{u} \right] \quad t_i \mathbf{c}_i^{i^T} \right\}^T \equiv \\
\equiv [\lambda_j^i \left(1 + \frac{v_{ji}^j}{c_j^j} \right) \mathbf{r}_P^T(t_j) \quad t_i \mathbf{c}_i^{i^T}] D [\lambda_j^i \left(1 + \frac{v_{ji}^j}{c_j^j} \right) \mathbf{r}_P^T(t_j) \quad t_i \mathbf{c}_i^{i^T}]^T \equiv \\
\equiv [\mathbf{r}_P^T(t_j) \quad \alpha_j^i \frac{c_i^i}{c_j^j} \left(1 + \frac{v_{ji}^j c_j^j}{q^j w^j} \right) t_j \mathbf{c}_j^{j^T}] D [\mathbf{r}_P^T(t_j) \quad \alpha_j^i \frac{c_i^i}{c_j^j} \left(1 + \frac{v_{ji}^j c_j^j}{q^j w^j} \right) t_j \mathbf{c}_j^{j^T}]^T \equiv \\
\equiv [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{c}_j^{j^T}] D [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{c}_j^{j^T}]^T.
\end{gathered}$$

This proves (11.7).

Compatibility. In order to test the transformations for their complete compatibility we replace $r_{(.)}(t_{(.)})$ by $v_{P(.)}^{(.)} t_{(.)}$ in (11.14) and (11.15),

$$t_i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{v_{ji}^j c_j^j}{q^j w^j}} \left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j} \right) t_j, \; t_j = \frac{\mu_j}{\mu_i} \frac{1}{1 - \frac{v_{ji}^i c_i^i}{q^i w^i}} \left(1 - \frac{v_{ji}^i v_P^i}{q^i w^i} \right) t_i. \tag{22.17}$$

By eliminating, for example, t_j from (22.17), we get:

$$t_i \equiv \frac{\mu_i}{\mu_j} \frac{\mu_j}{\mu_i} \frac{1}{1 + \frac{v_{ji}^j c_j^j}{q^j w^j}} \left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j} \right) \frac{1}{1 - \frac{v_{ji}^i c_i^i}{q^i w^i}} \left(1 - \frac{v_{ji}^i v_P^i}{q^i w^i} \right) t_i.$$

For this identity to hold it is necessary and sufficient that

$$v_P^i = c_i^i \text{ and } v_P^j = c_j^j. \tag{22.18}$$

This proves partial compatibility of (11.14) and (11.15) because they are compatible if and only if the speed of the arbitrary point P equals the light speed. We replace now $t_{(.)}$ by $\left(v_P^{(.)} \right)^{-1} r_{(.)}(t_{(.)})$ in (11.16) and (11.17), which imply:

$$\mathbf{r}_P(t_i) \equiv \frac{1 + \frac{v_{ji}^j}{v_P^j}}{1 + \frac{v_{ji}^j}{c_j^j}} \frac{1 - \frac{v_{ji}^i}{v_P^i}}{1 - \frac{v_{ji}^i}{c_i^i}} \mathbf{r}_P(t_i).$$

For this identity to hold it is necessary and sufficient that (22.18) is valid. Hence, (11.4) and (11.5) are also only partially compatible. Let us now test their entire compatibility. We eliminate at first t_j and $r_j(t_j)$ from (11.14) by using (11.13), (11.3) and (11.5) in order to express $r_i(t_i)$ in terms of v_P^i and t_i,

$$t_i \equiv \frac{\frac{1-\frac{v_{ji}^i v_P^i}{q^i w^i}}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}} + \frac{\mu_i}{\mu_j}\frac{v_{ji}^j v_P^i}{q^j w^j}\frac{1-\frac{v_{ji}^i}{v_P^i}}{1-\frac{v_{ji}^i}{c_i^i}}}{1+\frac{v_{ji}^j c_j^j}{q^j w^j}} t_i.$$

For this identity to hold it is necessary and sufficient that $v_P^{(.)} = c_{(.)}^{(.)}$. The transformations (11.14), (11.15) and (11.17) are partially compatible. We prove analogously partial compatibility of the transformations ((11.14)) through (11.16). The transformations ((11.14)) through (11.17) are partially compatible in the temporal domain. Let us now eliminate t_j and $\mathbf{r}_P(t_j)$, $r_{(.)}(t_{(.)}) = v_P^{(.)} t_{(.)}$, from (11.16) by using (11.13), (11.15) and (11.17),

$$\mathbf{r}_P(t_i) \equiv \frac{\frac{1-\frac{v_{ji}^i}{v_P^i}}{1-\frac{v_{ji}^i}{c_i^i}} + \frac{v_{ji}^j}{v_P^i}\frac{\mu_j}{\mu_i}\frac{1-\frac{v_{ji}^i v_P^i}{q^i w^i}}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}}{1+\frac{v_{ji}^j}{c_j^j}} \mathbf{r}_P(t_i).$$

For this identity to hold it is necessary and sufficient that the arbitrary point P moves with the light speed, (22.18). The equations (11.15) through (11.17) are partially compatible. By applying the same procedure to ((11.14)), (11.16) and (11.17) we prove their partial compatibility. This and the preceding result show that ((11.14)) through (11.17) are partially entirely compatible. Altogether, the transformations ((11.14)) through (11.17) are partially both entirely and pairwise compatible. Q. E. D ■

22.2 Proof of Theorem 376

Proof. *Necessity.* Let us accept that $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$ and $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$ are positive real numbers. Let the arbitrary point P move with the light speed, i.e. let (22.1) be valid. Let the *time* scaling coefficient μ_i be positive real number and be defined by (11.1). Let $B = A$ in D, (11.7). Let the scaling coefficients λ_{ij} and α_{ij} obey (11.2) through (11.5), and let (11.1) through (11.6) imply (11.7). The equations (11.2) through (11.5), (22.1) through (22.3), and

(11.7) together with $D = blockdiag\{A \quad -A\}$ enable the following:

$$
\begin{aligned}
&[\mathbf{r}_P^T(t_i) \qquad t_i \mathbf{c}_i^{i^T}] D [\mathbf{r}_P^T(t_i) \qquad t_i \mathbf{c}_i^{i^T}]^T \equiv \\
&\equiv \begin{bmatrix} \lambda_{ij}\left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}\right] \\ \alpha_{ij}\left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)\right] \mathbf{c}_i^i \end{bmatrix}^T D \begin{bmatrix} \lambda_{ij}\left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}\right] \\ \alpha_{ij}\left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)\right] \mathbf{c}_i^i \end{bmatrix} \equiv \\
&\equiv (\lambda_{ij})^2 \left\{ \begin{array}{c} \left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}} \frac{v_{ji}^j c_i^i}{q^j w^j}\right)^2\right] r_P(t_j) + \\ +2 v_{ji}^j \left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^2 \frac{(c_i^i)^2}{q^j w^j}\right] r_P(t_j) t_j \\ + \left[\left(v_{ji}^j\right)^2 - \left(\frac{\alpha_{ij}}{\lambda_{ij}} c_i^i\right)^2\right] t_j^2 \end{array} \right\} \mathbf{u}^T A \mathbf{u} \equiv \\
&\equiv \lambda_{ij} \begin{bmatrix} \sqrt{\begin{array}{c} \left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}} \frac{v_{ji}^j c_i^i}{q^j w^j}\right)^2\right] r_P(t_j) + \\ +2 v_{ji}^j \left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^2 \frac{(c_i^i)^2}{q^j w^j}\right] r_P(t_j) t_j \end{array}} \mathbf{u} \\ \sqrt{\left[1 - \left(\frac{v_{ji}^j}{c_i^i} \frac{\lambda_{ij}}{\alpha_{ij}}\right)^2\right] \left(\frac{\alpha_{ij}}{\lambda_{ij}} \frac{c_i^i}{c_j^j}\right)^2} \left(c_j^j\right) t_j \mathbf{u} \end{bmatrix}^T D \bullet \\
&\bullet \lambda_{ij} \begin{bmatrix} \sqrt{\begin{array}{c} \left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}} \frac{v_{ji}^j c_i^i}{q^j w^j}\right)^2\right] r_P(t_j) + \\ +2 v_{ji}^j \left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^2 \frac{(c_i^i)^2}{q^j w^j}\right] r_P(t_j) t_j \end{array}} \mathbf{u} \\ \sqrt{\left[1 - \left(\frac{v_{ji}^j}{c_i^i} \frac{\lambda_{ij}}{\alpha_{ij}}\right)^2\right] \left(\frac{\alpha_{ij}}{\lambda_{ij}} \frac{c_i^i}{c_j^j}\right)^2} \left(c_j^j\right) t_j \mathbf{u} \end{bmatrix} \equiv \\
&\equiv [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{c}_j^{j^T}] D [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{c}_j^{j^T}]^T.
\end{aligned} \tag{22.19}
$$

The last identity implies:

$$1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^2 \frac{\left(c_i^i\right)^2}{q^j w^j} = 0, \ \ hence \ \frac{\alpha_{ij}}{\lambda_{ij}} = \frac{\sqrt{q^j w^j}}{c_i^i}, \tag{22.20}$$

$$\lambda_{ij} \sqrt{1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}} \frac{v_{ji}^j c_i^i}{q^j w^j}\right)^2} = 1, \tag{22.21}$$

$$\lambda_{ij} \sqrt{\left[1 - \left(\frac{v_{ji}^j}{c_i^i} \frac{\lambda_{ij}}{\alpha_{ij}}\right)^2\right] \left(\frac{\alpha_{ij}}{\lambda_{ij}} \frac{c_i^i}{c_j^j}\right)^2} = 1. \tag{22.22}$$

The equations 22.20 through 22.22 yield:

$$\lambda_{ij}\sqrt{1-\left(\frac{\sqrt{q^j w^j}}{c_i^i}\frac{v_{ji}^j c_i^i}{q^j w^j}\right)^2}=1 \Longleftrightarrow \lambda_{ij}\sqrt{1-\left(\frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2}=1, \qquad (22.23)$$

and

$$\lambda_{ij}\sqrt{\left[1-\left(\frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2\right]\left(\frac{\sqrt{q^j w^j}}{c_j^j}\right)^2}=1. \qquad (22.24)$$

The last two results, (22.23) and (22.24), demand:

$$1-\left(\frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2 \equiv \left[1-\left(\frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2\right]\left(\frac{\sqrt{q^j w^j}}{c_j^j}\right)^2.$$

For this identity to hold it is necessary and sufficient that:

$$\sqrt{q^j w^j}=c_j^j. \qquad (22.25)$$

This, (22.20), (22.23) and (22.24) imply:

$$\frac{\alpha_{ij}}{\lambda_{ij}}=\frac{c_j^j}{c_i^i}, \qquad (22.26)$$

$$\lambda_{ij}=\frac{1}{\sqrt{1-\left(\frac{v_{ji}^j}{c_j^j}\right)^2}}, \qquad (22.27)$$

$$\alpha_{ij}=\frac{c_j^j}{c_i^i}\lambda_{ij}=\frac{c_j^j}{c_i^i}\frac{1}{\sqrt{1-\left(\frac{v_{ji}^j}{c_j^j}\right)^2}}. \qquad (22.28)$$

Let us now transform the right-hand side of (11.7) by repeating the preceding procedure and by using (22.1):

$$[\mathbf{r}_P^T(t_j) \qquad t_j \mathbf{c}_j^{j^T}]D[\mathbf{r}_P^T(t_j) \qquad t_j \mathbf{c}_j^{j^T}]^T \equiv$$

$$\equiv \lambda_{ij}^2\begin{bmatrix}\left[r_P(t_i)-v_{ji}^i(t_i)\right]\mathbf{u}\\ \frac{\alpha_{ij}}{\lambda_{ij}}\left[c_j^j t_i-\frac{v_{ji}^i c_j^j}{q^i w^i}r_P(t_i)\right]\mathbf{u}\end{bmatrix}^T D\begin{bmatrix}\left[r_P(t_i)-v_{ji}^i(t_i)\right]\mathbf{u}\\ \frac{\alpha_{ij}}{\lambda_{ij}}\left[c_j^j t_i-\frac{v_{ji}^i c_j^j}{q^i w^i}r_P(t_i)\right]\mathbf{u}\end{bmatrix}\equiv$$

$$\equiv \lambda_{ij}\left[\sqrt{\begin{array}{c}\left[1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{ji}^{i}c_{j}^{j}}{q^{i}w^{i}}\right)^{2}\right]r_{P}\left(t_{i}\right)+\\+2v_{ji}^{i}\left[1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^{2}\frac{\left(c_{j}^{j}\right)^{2}}{q^{i}w^{i}}\right]r_{P}\left(t_{i}\right)\left(t_{i}\right)\end{array}}\mathbf{u}\atop\sqrt{\left[1-\left(\frac{v_{ji}^{i}}{c_{j}^{j}}\frac{\lambda_{ij}}{\alpha_{ij}}\right)^{2}\right]\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{c_{j}^{j}}{c_{i}^{i}}\right)^{2}\left(c_{i}^{i}\right)\left(t_{i}\right)^{2}\mathbf{u}}}\right]^{T}D\bullet$$

$$\bullet\lambda_{ij}\left[\sqrt{\begin{array}{c}\left[1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{ji}^{i}c_{j}^{j}}{q^{i}w^{i}}\right)^{2}\right]r_{P}\left(t_{i}\right)+\\+2v_{ji}^{i}\left[1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^{2}\frac{\left(c_{j}^{j}\right)^{2}}{q^{i}w^{i}}\right]r_{P}\left(t_{i}\right)\left(t_{i}\right)\end{array}}\mathbf{u}\atop\sqrt{\left[1-\left(\frac{v_{ji}^{i}}{c_{j}^{j}}\frac{\lambda_{ij}}{\alpha_{ij}}\right)^{2}\right]\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{c_{j}^{j}}{c_{i}^{i}}\right)^{2}\left(c_{i}^{i}\right)\left(t_{i}\right)^{2}\mathbf{u}}}\right]\equiv$$

$$\equiv[\mathbf{r}_{P}^{T}\left(t_{i}\right)\qquad(t_{i})\mathbf{c}_{i}^{i^{T}}]D[\mathbf{r}_{P}^{T}\left(t_{i}\right)\qquad(t_{i})\mathbf{c}_{i}^{i^{T}}]^{T}. \tag{22.29}$$

The last identity implies:

$$1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^{2}\frac{\left(c_{j}^{j}\right)^{2}}{q^{i}w^{i}}=0,\text{ hence }\frac{\alpha_{ij}}{\lambda_{ij}}=\frac{\sqrt{q^{i}w^{i}}}{c_{j}^{j}}, \tag{22.30}$$

$$\lambda_{ij}\sqrt{1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{ji}^{i}c_{j}^{j}}{q^{i}w^{i}}\right)^{2}}=1, \tag{22.31}$$

$$\lambda_{ij}\sqrt{\left[1-\left(\frac{v_{ji}^{i}}{c_{j}^{j}}\frac{\lambda_{ij}}{\alpha_{ij}}\right)^{2}\right]\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{c_{j}^{j}}{c_{i}^{i}}\right)^{2}}=1. \tag{22.32}$$

The equations (22.30) through (22.32) yield:

$$\lambda_{ij}\sqrt{1-\left(\frac{\sqrt{q^{i}w^{i}}}{c_{j}^{j}}\frac{v_{ji}^{i}c_{j}^{j}}{q^{i}w^{i}}\right)^{2}}=1\Longleftrightarrow\lambda_{ij}\sqrt{1-\left(\frac{v_{ji}^{i}}{\sqrt{q^{i}w^{i}}}\right)^{2}}=1, \tag{22.33}$$

and

$$\lambda_{ij}\sqrt{\left[1-\left(\frac{v_{ji}^{i}}{\sqrt{q^{i}w^{i}}}\right)^{2}\right]\left(\frac{\sqrt{q^{i}w^{i}}}{c_{i}^{i}}\right)^{2}}=1. \tag{22.34}$$

The last two results, (22.33) and (22.34), demand:

$$1-\left(\frac{v_{ji}^{i}}{\sqrt{q^{i}w^{i}}}\right)^{2}\equiv\left[1-\left(\frac{v_{ji}^{i}}{\sqrt{q^{i}w^{i}}}\right)^{2}\right]\left(\frac{\sqrt{q^{i}w^{i}}}{c_{i}^{i}}\right)^{2}.$$

For this identity to hold it is necessary and sufficient that:

$$\sqrt{q^i w^i} = c_i^i. \tag{22.35}$$

This, (22.30), (22.33) and (22.34) imply:

$$\frac{\alpha_{ij}}{\lambda_{ij}} = \frac{c_i^i}{c_j^j}, \tag{22.36}$$

$$\lambda_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}^i}{c_i^i}\right)^2}}, \tag{22.37}$$

$$\alpha_{ij} = \frac{c_j^j}{c_i^i}\lambda_{ij} = \frac{c_j^j}{c_i^i}\frac{1}{\sqrt{1 - \left(\frac{v_{ji}^j}{c_j^j}\right)^2}}. \tag{22.38}$$

The equations (22.25), (22.35), (22.26), (22.36), (22.27), (22.37), (22.28) and (22.38) prove (11.20) through (11.22). We continue with (11.1) and $r_P(t_{(.)}) \equiv c_{ij}t_{(.)}$,

$$r_P(t_i) \equiv c_{ij}\frac{\mu_i}{\mu_j}t_j \equiv \lambda_{ij}\left[r_P(t_j) + v_{ji}^j t_j\right] \equiv \lambda_{ij}c_{ij}\left(1 + c_{ij}^{-1}v_{ji}^j\right)t_j. \tag{22.39}$$

The equations (22.39) and (11.20) through (11.22) show that:

$$\frac{\mu_i}{\mu_j} = \frac{\left(1 + \frac{v_{ji}}{c_{ij}}\right)}{\sqrt{\left(1 - \frac{v_{ji}}{c_{ij}}\right)\left(1 + \frac{v_{ji}}{c_{ij}}\right)}} = \sqrt{\frac{1 + \frac{v_{ji}}{c_{ij}}}{1 - \frac{v_{ji}}{c_{ij}}}} = \sqrt{\frac{1 + \frac{v_{ji}}{\sqrt{q^i w^i}}}{1 - \frac{v_{ji}}{\sqrt{q^i w^i}}}} = \sqrt{\frac{1 + \frac{v_{ji}}{\sqrt{q^j w^j}}}{1 - \frac{v_{ji}}{\sqrt{q^j w^j}}}}.$$

This proves (11.23).

Sufficiency. Let all the conditions of the theorem statement hold. We start by transforming (11.1) with the help of (11.20) and (11.23):

$$t_i = \frac{\mu_i}{\mu_j}t_j = \frac{\left(1 + \frac{v_{ji}}{c_{ij}}\right)}{\sqrt{\left(1 - \frac{v_{ji}}{c_{ij}}\right)\left(1 + \frac{v_{ji}}{c_{ij}}\right)}}t_j. \tag{22.40}$$

Now we apply (11.22), $t_{(.)} = c_{ij}^{-1}r_P(t_{(.)})$, again (11.20), (11.21) and $\alpha_{ij} = \alpha_{ji} = \alpha_i^j = \alpha_j^i$ to (22.40):

$$t_i = \alpha_j^i\left[t_j + \frac{v_{ji}^j}{q^j w^j}r_P(t_j)\right].$$

This is (11.2). We prove the equation (11.3) in the same way by starting with (11.1) in which i and j exchange places. We proceed by using $\mathbf{r}_P\left(t_{(.)}\right) =$

$c_{ij}t_{(.)}\mathbf{u}$, (11.1), (11.20) through (11.23), and $\lambda_{ij} = \lambda_{ji} = \lambda_i^j = \lambda_j^i$:

$$\mathbf{r}_P(t_i) = c_{ij}\frac{\mu_i}{\mu_j}t_j\mathbf{u} = c_{ij}\sqrt{\frac{1+\frac{v_{ji}}{c_{ij}}}{1-\frac{v_{ji}}{c_{ij}}}}t_j\mathbf{u} = c_{ij}\frac{1+\frac{v_{ji}}{c_{ij}}}{\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}}t_j\mathbf{u} =$$
$$= \lambda_j^i\left(c_{ij}+v_{ji}^j\right)t_j\mathbf{u} = \lambda_j^i\left[\mathbf{r}_P(t_j)+v_{ji}^j t_j\right].$$

This is (11.4). The equation (11.5) is proved in the same way by beginning with $\mathbf{r}_P(t_j) = c_{ij}t_j\mathbf{u}$.

Compatibility. Let us check the transformations (11.24) through (11.27) for their complete pairwise compatibility. We eliminate at first, for example, t_j from (11.24) and (11.25) by using (11.20) and by applying $r_P(t_{(.)}) = v_P^{(.)}t_{(.)}$,

$$t_i \equiv \frac{\left(1-\frac{v_{ji}v_P^i}{c_{ij}^2}\right)\left(1+\frac{v_{ji}v_P^j}{c_{ij}^2}\right)}{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}t_i.$$

For this identity to hold it is necessary and sufficient that $v_P^i = v_P^j = c_{ij}$. The equations (11.24) and (11.25) are only partially compatible. We eliminate now, for example, $\mathbf{r}_P(t_j)$ from (11.26) and (11.27) and apply $t_{(.)} = \left[v_P^{(.)}\right]^{-1} r_P(t_{(.)})$:

$$\mathbf{r}_P(t_i) \equiv \frac{\left(1-\frac{v_{ji}}{v_P^i}\right)\left(1+\frac{v_{ji}}{v_P^j}\right)}{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}\mathbf{r}_P(t_i).$$

For this identity to hold it is necessary and sufficient that $v_P^i = v_P^j = c_{ij}$. The equations (11.26) and (11.27) are only partially compatible. Hence, the equations (11.24) through (11.27) are only partially pairwise compatible. Let us check them for their entire compatibility. At first we use (11.20), (11.24), (11.25) and (11.27) from which we eliminate both t_j and $r_P(t_j)$ in view of $\mathbf{r}_P(t_j) = r_P(t_j)\,\mathbf{u}$:

$$t_i \equiv \frac{\frac{t_i-\frac{v_{ji}}{c_{ij}^2}r_P(t_i)}{\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}}+\frac{v_{ji}}{c_{ij}^2}\frac{r_P(t_i)-v_{ji}(t_i)}{\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}}}{\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}} \equiv \frac{1-\frac{v_{ji}^2}{c_{ij}^2}}{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}t_i \equiv t_i.$$

This proves compatibility of (11.24), (11.25) and (11.27), which is partial due to the constraint on $q^{ji}w^{ji}$ in (11.20). In the same way we prove partial compatibility of (11.24) through (11.26). The transformations (11.24) through (11.27) are partially entirely compatible in the temporal domain. We exploit now (11.20), (11.24), (11.26) and (11.27) from which we eliminate both t_i and $\mathbf{r}_P(t_i)$ and apply $t_{(.)}\mathbf{u} = \left[v_P^{(.)}\right]^{-1} \mathbf{r}_P\left(t_{(.)}\right)$:

$$\mathbf{r}_P(t_j) \equiv \frac{\left(1+\frac{v_{ji}}{v_P^j}\right)\mathbf{r}_P(t_j) - \frac{v_{ji}}{v_P^j}\left(1+\frac{v_{ji}v_P^j}{c_{ij}^2}\right) r_P(t_j)\,\mathbf{u}}{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2} \equiv \mathbf{r}_P(t_j).$$

This proves partial compatibility of (11.24),(11.26) and (11.27) due to the restriction on $q^{ji}w^{ji}$ in (11.20). We prove analogously partial compatibility of (11.24) through (11.26), which proves partial entire compatibility of the transformations (11.24) through (11.27). Q. E. D ∎

22.3 Proof of Theorem 382

Proof. *Necessity and sufficiency.* The velocity of the arbitrary point P with respect to $R^n_{(.)}$ and its origin $O_{(.)}$ and relative to $\mathfrak{T}_{(.)}$ is defined by (22.43):

$$\mathbf{v}_P^{(.)}(t_{(.)}) = \frac{d\mathbf{r}_P^{(.)}}{dt_{(.)}}. \tag{22.43}$$

This equation, (11.14) and (11.16) hold. They yield:

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P^{O_i}}{dt_i} = \frac{d\mathbf{r}_P}{dt_i} = \frac{d\left[\frac{\mathbf{r}_P(t_j)+\ v_{ji}^j t_j \mathbf{u}}{1+\frac{v_{ji}^j}{c_j^j}}\right] : dt_j}{d\left[\frac{\mu_i}{\mu_j}\frac{t_j+\frac{v_{ji}^j}{q^j w^j} r_P(t_j)}{1+\frac{v_{ji}^j c_j^j}{q^j w^j}}\right] : dt_j} \Longrightarrow$$

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i}\frac{\frac{\mathbf{v}_P^j+\ v_{ji}^j \mathbf{u}}{1+\frac{v_{ji}^j}{c_j^j}}}{\frac{1+\frac{v_{ji}^j v_P^j}{q^j w^j}}{1+\frac{v_{ji}^j c_j^j}{q^j w^j}}} = \frac{\mu_j}{\mu_i}\frac{1+\frac{v_{ji}^j c_j^j}{q^j w^j}}{1+\frac{v_{ji}^j}{c_j^j}}\frac{\mathbf{v}_P^j+\ \mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j v_P^j}{q^j w^j}},$$

which proves the first equation (11.34). From (11.15) and (11.17), which hold, we get

$$\mathbf{v}_P^j = \frac{d\left[\frac{\mathbf{r}_P(t_i)-\ v_{ji}^i t_i \mathbf{u}}{1-\frac{v_{ji}^i}{c_i^i}}\right] : dt_i}{d\left[\frac{\mu_j}{\mu_i}\frac{t_i-\frac{v_{ji}^i}{q^i w^i} r_P(t_i)}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}\right] : dt_i} = \frac{\mu_i}{\mu_j}\frac{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}{1-\frac{v_{ji}^i}{c_i^i}}\frac{\mathbf{v}_P^i-\ \mathbf{v}_{ji}^i}{1-\frac{v_{ji}^i v_P^i}{q^i w^i}}.$$

This proves the second equation (11.34).

Compatibility. We allow an arbitrary speed of the arbitrary point P in order to test the complete compatibility of the equations in (11.34), and we eliminate, for example, both $\mathbf{v}_P^j$ and v_P^j, and we apply (11.13),

$$\mathbf{v}_P^i \equiv \frac{1+\frac{v_{ji}^j c_j^j}{q^j w^j}}{1+\frac{v_{ji}^j}{c_j^j}} \frac{\frac{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}{1-\frac{v_{ji}^i}{c_i^i}} \frac{1-\frac{v_{ji}^i}{v_P^i}}{1-\frac{v_{ji}^i v_P^i}{q^i w^i}} + \frac{\mu_j}{\mu_i}\frac{v_{ji}^j}{v_P^i}}{1+\frac{v_{ji}^j}{q^j w^j}\frac{\mu_i}{\mu_j}\frac{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}{1-\frac{v_{ji}^i}{c_i^i}} \frac{1-\frac{v_{ji}^i}{v_P^i}}{1-\frac{v_{ji}^i v_P^i}{q^i w^i}} v_P^i} \mathbf{v}_P^i,$$

For this identity to hold it is necessary and sufficient that

$$v_P^i = c_i^i,\ v_P^j = c_j^j.$$

This proves (11.35). Hence, the speed transformations are only partially compatible. Q. E. D ■

22.4 Proof of Theorem 387

Proof. We can prove the first equation from (11.36) in two ways as follows.

a) Let the equations (11.34) be valid and let $\mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$. In order for the velocity of light $\mathbf{c}_{(.)}^{(.)}$ to be invariant, i.e. $\mathbf{c}_i^i \equiv \mathbf{c}_j^j \equiv \mathbf{c}$, it is necessary and sufficient that both

$$\mathbf{v}_P^i = \mathbf{c}_i^i = \frac{\mu_j}{\mu_i} \frac{1+\frac{v_{ji}^j c_j^j}{q^j w^j}}{1+\frac{v_{ji}^j}{c_j^j}} \frac{1+\frac{v_{ji}^j}{c_j^j}}{1+\frac{v_{ji}^j c_j^j}{q^j w^j}} \mathbf{c}_j^j \equiv \mathbf{c}_j^j,$$

and

$$\mathbf{v}_P^j = \mathbf{c}_j^j = \frac{\mu_i}{\mu_j} \frac{1-\frac{v_{ji}^i c_i^i}{q^i w^i}}{1-\frac{v_{ji}^i}{c_i^i}} \frac{1-\frac{v_{ji}^i}{c_i^i}}{1-\frac{v_{ji}^i c_i^i}{q^i w^i}} \mathbf{c}_i^i \equiv \mathbf{c}_i^i.$$

For these identities to hold it is necessary and sufficient that $\mu_j/\mu_i = 1$.

b) The equations (11.13) and $\mathbf{c}_{(.)}^{(.)} \equiv \mathbf{c}$ imply both $\mu_j/\mu_i = 1$ and $v_{ji}^i = v_{ji}^j \equiv v_{ji}$.

The equations in (11.14), (11.15), $\mathbf{c}_{(.)}^{(.)} \equiv \mathbf{c}$ and $v_{ji}^i = v_{ji}^j \equiv v_{ji}$ yield

$$t_i = \frac{1+\frac{v_{ji}c}{q^j w^j}}{1+\frac{v_{ji}c}{q^j w^j}} t_j = t_j,\ t_j = \frac{1-\frac{v_{ji}c}{q^i w^i}}{1-\frac{v_{ji}c}{q^i w^i}} t_i = t_i.$$

And vice versa. If $t_j = t_i$ then (11.1) implies $\mu_j/\mu_i = 1$ that yields both $\mathbf{c}_{(.)}^{(.)} \equiv \mathbf{c}$ and $v_{ji}^i = v_{ji}^j \equiv v_{ji}$ due to (11.13). If $v_{ji}^i = v_{ji}^j \equiv v_{ji}$ then $\mu_j/\mu_i = 1$ and

$\mathbf{c}_{(.)}^{(.)} \equiv \mathbf{c}$ due to (11.13), and $t_j = t_i$ in view of (11.1). This completes the proof of (11.36). Furthermore, $\mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$, (11.16) and (11.17) result in (11.37). Q. E. D ■

22.5 Proof of Theorem 389

Proof. *Necessity and sufficiency.* Let all the conditions of the theorem statement hold. Then, the equations (11.20) through (11.27) are valid. We start with the equation (22.43) for $(.) = i$, in which we replace both $d\mathbf{r}_P^{(.)} = d\mathbf{r}_P^i$ by the differential of the right hand side of the equation (11.26) and $dt_{(.)} = dt_i$ by the differential of the right hand side of the equation (11.24),

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P^{O_i}}{dt_i} = \frac{d\mathbf{r}_P}{dt_i} = \frac{d\left[\frac{\mathbf{r}_P(t_j)+\mathbf{v}_{ji}t_j}{\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}}\right] : dt_j}{d\left[\frac{t_j+\frac{v_{ji}}{(c_{ij})^2}r_P(t_j)}{\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}}\right] : dt_j} = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}}{1+\frac{v_{ji}}{(c_{ij})^2}v_P^j}.$$

This proves the first transformation (11.38). The second transformation (11.38) is analogously proved by starting with (22.43) for $(.) = j$, (11.27) and (11.25).

Compatibility. In order to test their complete compatibility, we eliminate, for example, $\mathbf{v}_P^i$ and v_P^i from (11.38),

$$\mathbf{v}_P^j \equiv \frac{\frac{\mathbf{v}_P^j+\mathbf{v}_{ji}}{1+\frac{v_{ji}v_P^j}{(c_{ij})^2}} - \mathbf{v}_{ji}}{1-\frac{v_{ji}}{(c_{ij})^2}\frac{v_P^j+v_{ji}}{1+\frac{v_{ji}v_P^j}{(c_{ij})^2}}} \equiv \frac{\frac{1+\frac{v_{ji}}{v_P^j}}{1+\frac{v_{ji}v_P^j}{(c_{ij})^2}} - \frac{v_{ji}}{v_P^j}}{1-\frac{v_{ji}v_P^j}{(c_{ij})^2}\frac{1+\frac{v_{ji}}{v_P^j}}{1+\frac{v_{ji}v_P^j}{(c_{ij})^2}}}\mathbf{v}_P^j \Longleftrightarrow v_P^{(.)} = c_{(.)}^{(.)}.$$

The transformations (11.38) are partially compatible. If the arbitrary point P moves with the light speed then the equations (11.38) furnish

$$\mathbf{v}_P^i = \mathbf{c}_i^i = \frac{\mathbf{c}_j^j + \mathbf{v}_{ji}}{1+\frac{v_{ji}}{(c_j^j)^2}c_j^j} = \frac{1+\frac{v_{ji}}{c_j^j}}{1+\frac{v_{ji}}{c_j^j}}\mathbf{c}_j^j = \mathbf{c}_j^j \Longleftrightarrow \mathbf{c}_i^i = \mathbf{c}_j^j = \mathbf{c}_{ij} = \mathbf{c}_{ji}.$$

Q. E. D ■

22.6 Proof of Theorem 396

Proof. *Necessity and sufficiency.* Necessity and sufficiency of the relationships (11.46) through (11.51) follow from necessity and sufficiency of the relationships (11.13) as soon as we replace in them $q^{(.)}w^{(.)}$ by $\left(c_{(.)}^{(.)}\right)^2$ in view of $q^{(\cdot)}w^{(\cdot)} = \left(v_p^{(\cdot)}\right)^2$ due to (11.42) and (11.43), and due to the validity of the proof of necessity and sufficiency only for $v_p^{(\cdot)} = c_{(\cdot)}^{(\cdot)}$ that is imposed by Einsteinian approach. Necessity and sufficiency of (11.46) through (11.51), together with (11.42) through (11.45) prove necessity and sufficiency of (11.52) through (11.55).

Compatibility. In order to test the transformations for their complete compatibility let the arbitrary point P be permitted now to move with an arbitrary constant nonzero velocity $\mathbf{v}_p^{(\cdot)} = v_p^{(\cdot)}\mathbf{u}$. We replace $r_p\left(t_{(\cdot)}\right)$ by $v_p^{(\cdot)}t_{(\cdot)}$ in (11.52) and (11.53), which yield:

$$t_i = \frac{\mu_i}{\mu_j}\frac{1}{1+\frac{v_{ji}^j}{c_j^j}}\left(1+\frac{v_{ji}^j v_p^j}{\left(v_p^j\right)^2}\right)t_j,\ t_j = \frac{\mu_j}{\mu_i}\frac{1}{1-\frac{v_{ji}^i}{c_i^i}}\left(1-\frac{v_{ji}^i v_p^i}{\left(v_p^i\right)^2}\right)t_i.$$

By eliminating, for example, t_j from these equations we find:

$$t_i \equiv \frac{1}{1+\frac{v_{ji}^j}{c_j^j}}\left(1+\frac{v_{ji}^j v_p^j}{\left(v_p^j\right)^2}\right)\frac{1}{1-\frac{v_{ji}^i}{c_i^i}}\left(1-\frac{v_{ji}^i v_p^i}{\left(v_p^i\right)^2}\right)t_i.$$

For this identity to hold it is necessary and sufficient that

$$v_p^i = c_i^i \text{ and } v_p^j = c_j^j, \tag{22.44}$$

i.e., that the arbitrary point P moves with the speed of flight. This proves the compatibility of (11.52) and (11.53) if, and only if the point P moves with the speed of light. They are only partially compatible. We replace now $t_{(\cdot)}$ by $\left(v_p^{(\cdot)}\right)^{-1} r_p\left(t_{(\cdot)}\right)$ in (11.54), which imply

$$\mathbf{r}_p\left(t_i\right) = \frac{1+\frac{v_{ji}^j}{v_p^j}}{1+\frac{v_{ji}^j}{c_j^j}}\frac{1-\frac{v_{ji}^i}{v_p^i}}{1-\frac{v_{ji}^i}{c_i^i}}\mathbf{r}_p\left(t_i\right).$$

For this identity to hold it is necessary and sufficient that the equations (22.44) also hold. Hence, (11.54) and (11.55) are compatible if, and only if the point P moves with the speed of light. Altogether, the transformations (11.52) through (11.55) are only partially compatible. Let us now verify their entire compatibility. We eliminate first t_j and $r_p\left(t_j\right)$ from (11.52) by utilizing (11.51), (11.53), (11.55) and $r_p\left(t_i\right) =$

$$v_p^i t_i : t_i = \frac{\mu_i}{\mu_j}\frac{\frac{\mu_j}{\mu_i}\frac{t_i-\frac{v_{ji}^i}{\left(v_p^i\right)^2}r_P(t_i)}{1-\frac{v_{ji}^i}{c_i^i}}+\frac{v_{ji}^j}{\left(v_P^j\right)^2}\frac{r_P(t_i)-v_{ji}^i t_i}{1-\frac{v_{ji}^i}{c_i^i}}}{1+\frac{v_{ji}^j}{c_j^j}} \Longleftrightarrow$$

$$t_i = \frac{\frac{t_i - \frac{v_{ji}^i}{v_p^i}}{1 - \frac{v_{ji}^i}{c_i^i}} + \frac{v_{ji}^j}{\left(v_P^j\right)^2} v_p^j \frac{1 - \frac{v_{ji}^i}{v_p^i}}{1 - \frac{v_{ji}^i}{c_i^i}}}{1 + \frac{v_{ji}^j}{c_j^j}} t_i \Longleftrightarrow v_p^i = c_i^i.$$

The equations (11.52), (11.53), (11.55) are partially compatible. In the same manner we prove the partial compatibility of (11.52) through (11.54). Let us now eliminate, for example, all the coordinates with the subscript j from (11.54) by using (11.51), (11.53) and (11.55),

$$r_P\left(t_i\right) = \frac{\frac{1 - \frac{v_{ji}^i}{v_p^i} t_i}{1 - \frac{v_{ji}^i}{c_i^i}} + \frac{v_{ji}^j}{v_p^i} \frac{\mu_j}{\mu_i} \frac{1 - \frac{v_{ji}^i}{v_p^i}}{1 - \frac{v_{ji}^i}{c_i^i}}}{1 + \frac{v_{ji}^j}{c_j^j}} r_P\left(t_i\right) \Longleftrightarrow$$

$$r_P\left(t_i\right) = \frac{\frac{1 - \frac{v_{ji}^i}{v_p^i}}{1 - \frac{v_{ji}^i}{c_i^i}} \left(+ \frac{v_{ji}^i}{v_p^j}\right) \frac{1 - \frac{v_{ji}^i}{v_p^i}}{1 - \frac{v_{ji}^i}{c_i^i}}}{1 + \frac{v_{ji}^j}{c_j^j}} r_P\left(t_i\right) \Longleftrightarrow$$

$$1 = \frac{\left(1 - \frac{v_{ji}^i}{v_p^i}\right)\left(1 + \frac{v_{ji}^j}{v_p^j}\right)}{\left(1 - \frac{v_{ji}^i}{c_i^i}\right)\left(1 + \frac{v_{ji}^j}{c_j^j}\right)} 1 \Longleftrightarrow v_p^{(.)} = c_{(.)}^{(.)}, \quad (.) = i, j.$$

Hence, (11.53) through (11.55) are partially compatible. The proof of the partial compatibility of (11.52), (11.54) and (11.55) is analogous. Altogether, the transformations (11.52) through (11.55) are partially entirely compatible. Q. E. D ■

22.7 Proof of Theorem 404

Proof. *Necessity and sufficiency.* We should set $q^{(.)}w^{(.)} \equiv \left(v_P^{(.)}\right)^2 \equiv \left(c_{(.)}^{(.)}\right)^2$ in the proof of (11.20) through (11.27) due to (11.2) through (11.6), (11.42) through (11.45), and due to Einsteinian framework. Then we get the proof of necessity and sufficiency of (11.60) through (11.63).

Compatibility. We replace $r_P(t_{(.)})$ by $v_P^{(.)} t_{(.)}$ in (11.60) and (11.61):

$$t_i = \frac{\left(1 + \frac{v_{ji}}{v_P^j}\right) t_j}{\sqrt{1 - \left(\frac{v_{ji}}{c_{ij}}\right)^2}}, \; t_j = \frac{\left(1 - \frac{v_{ji}}{v_P^i}\right) t_i}{\sqrt{1 - \frac{v_{ji}^2}{c_{ij}^2}}}.$$

By eliminating, for example, t_j from these equations we get:

$$t_i \equiv \frac{\left(1+\frac{v_{ji}}{v_P^j}\right)\left(1-\frac{v_{ji}}{v_P^i}\right)}{\left[\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}\right]^2} t_i.$$

For this identity to hold it is necessary and sufficient that:

$$v_P^j = v_P^i = c_{ij}. \tag{22.45}$$

This proves the partial compatibility of (11.60) and (11.61). We replace now $t_{(.)}$ by $\left(v_P^{(.)}\right)^{-1} r_P(t_{(.)})$ in (11.62) and (11.63):

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{1+\frac{v_{ji}}{v_p^j}}{\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}} \mathbf{r}_P\left(t_j\right), \; \mathbf{r}_P\left(t_j\right) \equiv \frac{1-\frac{v_{ji}}{v_p^i}}{\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}} \mathbf{r}_P\left(t_i\right).$$

These identities imply:

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{\left(1+\frac{v_{ji}}{v_p^j}\right)\left(1-\frac{v_{ji}}{v_p^i}\right)}{\left[\sqrt{1-\left(\frac{v_{ji}}{c_{ij}}\right)^2}\right]^2} \mathbf{r}_P\left(t_i\right).$$

For this identity to hold it is necessary and sufficient that the equations (22.45) hold. Hence, the transformations (11.60) through (11.63) are only partially pairwise compatible. In order to test complete entire compatibility of (11.60) through (11.63) we replace at first t_j from (11.61) into (11.60) and $r_P\left(t_j\right)$ from (11.63), into (11.60):

$$t_i \equiv \frac{\frac{t_i-\frac{v_{ji}}{\left(v_P^i\right)^2} r_P(t_i)}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}} + \frac{v_{ji}}{\left(v_P^j\right)^2}\frac{r_P(t_i)-\ v_{ji}t_i}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}}}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}} \equiv$$

$$\equiv \frac{t_i - \frac{v_{ji}}{\left(v_P^i\right)^2} r_P\left(t_i\right) + \frac{v_{ji}}{\left(v_P^j\right)^2}\left[r_P\left(t_i\right) -\ v_{ji}t_i\right]}{1-\frac{v_{ji}^2}{c_{ij}^2}} \equiv t_i \Longleftrightarrow v_P^i = v_P^j = c_{ij}.$$

The transformations (11.60), (11.61) and (11.63) are partially compatible. The proof of partial compatibility of (11.60) through (11.62) is analogous. The transformations (11.60) through (11.63) are partially compatible in the temporal domain. We replace now t_j from (11.61) into (11.62) and $\mathbf{r}_P\left(t_j\right)$ from (11.63) also into (11.62):

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{\frac{\mathbf{r}_P(t_i)-\ v_{ji}t_i\mathbf{u}}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}} + v_{ji}\frac{t_i-\frac{v_{ji}}{\left(v_P^i\right)^2}r_P(t_i)}{\sqrt{1-\frac{v_{ji}^2}{c_{ij}^2}}}\mathbf{u}}{\sqrt{1-\frac{v_{ji}^2}{{c_{ij}}^2}}} \equiv$$

$$\equiv \frac{\mathbf{r}_P\left(t_i\right) - v_{ji}t_i\mathbf{u} + v_{ji}\left[t_i\mathbf{u} - \frac{v_{ji}}{\left(v_P^i\right)^2}\mathbf{r}_P\left(t_i\right)\right]}{1-\frac{v_{ji}^2}{{c_{ij}}^2}} \equiv \mathbf{r}_P\left(t_i\right) \Longleftrightarrow v_P^i = v_P^j = c_{ij}.$$

The transformations (11.61) through (11.63) are partially compatible. The same proof procedure shows that (11.60), (11.62) and (11.63) are also partially compatible. Altogether, the preceding results show that the transformations (11.60) through (11.63) are partially entirely compatible. Q. E. D ■

22.8 Proof of Theorem 412

Proof. *Necessity and sufficiency.* The equations (11.52) and (11.54) imply:

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P\left(t_i\right)}{dt_i} = \frac{d\left[\frac{\mathbf{r}_P(t_j)+v_{ji}^j t_j\mathbf{u}}{1+\frac{v_{ji}^j}{c_j^j}}\right]:dt_j}{d\left[\frac{\mu_i}{\mu_j}\frac{t_j+\frac{v_{ji}^j}{\left(v_P^j\right)^2}r_P(t_j)}{1+\frac{v_{ji}^j}{c_j^j}}\right]:dt_j} = \frac{\mu_j}{\mu_i}\frac{\mathbf{v}_P^j+\mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j}{v_P^j}} = \frac{\mu_j}{\mu_i}\mathbf{v}_P^j.$$

This is the first equation (11.70). By starting with (11.53) and (11.55) we prove the second equation (11.70) in the same way.

Compatibility. In order to test complete compatibility of the equations (11.70) we eliminate from them, for example, $\mathbf{v}_P^i$:

$$\mathbf{v}_P^j \equiv \frac{\mu_i}{\mu_j}\mathbf{v}_P^i \equiv \frac{\mu_i}{\mu_j}\frac{\mu_j}{\mu_i}\mathbf{v}_P^j \equiv \mathbf{v}_P^j.$$

This shows that the equations (11.70) are completely compatible. Q. E. D ■

22.9 Proof of Theorem 420

Proof. *Necessity.* Let the scaling coefficients $\alpha_{(.)}^{(..)}$ and $\lambda_{(.)}^{(..)}$ be determined for the case when the arbitrary point P moves with the speed of light, (22.1). Let the scaling coefficient μ_i obey (11.79). Let the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, obey (11.80) through (11.84) so that they together with

(11.79) imply (11.7). The equations (11.79), (11.82), (11.83), (22.2) and (22.3) imply [see the proof of (22.7) and (22.8)]

$$\lambda_j^i = \frac{c_i^i \mu_i}{c_j^j \mu_j} \frac{1}{1 + \frac{v_{ji}^j}{c_j^j}}, \tag{22.46}$$

$$\lambda_i^j = \frac{c_j^j \mu_j}{c_i^i \mu_i} \frac{1}{1 - \frac{v_{ji}^i}{c_i^i}}. \tag{22.47}$$

By combining (11.79), (11.80) and $r_R(t_j) = v_R^j t_j$ we find [see the proof of (22.9)]

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}. \tag{22.48}$$

This proves the first equation in (11.85). The first equation (11.86) is proved along the same lines. The equations (11.80), (11.82), (22.2), (22.3), and (11.7) imply the following:

$$\begin{aligned} &[\mathbf{r}_P^T(t_i) \quad t_i \mathbf{c}_i^{i^T}] D [\mathbf{r}_P^T(t_i) \quad t_i \mathbf{c}_i^{i^T}]^T \equiv [\mathbf{r}_L^T(t_i) \quad t_i \mathbf{c}_i^{i^T}] D [\mathbf{r}_L^T(t_i) \quad t_i \mathbf{c}_i^{i^T}]^T \equiv \\ &\equiv [\lambda_j^i \left(1 + \frac{v_{ji}^j}{c_j^j}\right) \mathbf{r}_L^T(t_j) \quad \alpha_j^i \left(1 + \frac{\vartheta^j v_R^j}{q^j w^j}\right) \frac{c_i^i}{c_j^j} t_j \mathbf{c}_j^{j^T}] D \bullet \\ &\bullet [\lambda_j^i \left(1 + \frac{v_{ji}^j}{c_j^j}\right) \mathbf{r}_L(t_j) \quad \alpha_j^i \left(1 + \frac{\vartheta^j v_R^j}{q^j w^j}\right) \frac{c_i^i}{c_j^j} t_j \mathbf{c}_j^{j^T}]^T \equiv \\ &\equiv [\mathbf{r}_L^T(t_j) \quad t_j \mathbf{c}_j^{j^T}] D [\mathbf{r}_L^T(t_j) \quad t_j \mathbf{c}_j^{j^T}]^T \equiv [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{c}_j^{j^T}] D [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{c}_j^{j^T}]^T. \end{aligned} \tag{22.49}$$

These identities, $\alpha_j^i \in R^+$ and $\lambda_j^i \in R^+$ yield:

$$\lambda_j^i = \frac{1}{1 + \frac{v_{ji}^j}{c_j^j}}, \tag{22.50}$$

$$\alpha_j^i = \frac{c_j^j}{c_i^i} \frac{1}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}}. \tag{22.51}$$

This equation verifies the second equation (11.85). The proof of the second equation (11.86) is analogous. The equation (22.50) proves the equation (11.87). The equations (22.46) and (22.50) imply:

$$\frac{c_i^i}{c_j^j} \frac{\mu_i}{\mu_j} = 1. \tag{22.52}$$

The equations (22.47) and (22.52) prove the equation (11.88). The condition that all the scaling coefficients, as well as q^i, q^j, w^i and w^j, are positive real numbers, that $v_R^{(.)} \in R_+$, $\vartheta^{(.)} \in R_+$ and $v_{ji}^{(.)} \in R_+$ by their definitions, together with (11.85) through (11.88), imply $0 \leq v_{ji}^i < c_i^i$ and $0 \leq \vartheta^i v_R^i < q^i w^i$, which prove (11.89). The first equation in (11.90) results directly from the equation (22.52). The definitions of the speed in general and of the spatial transfer speed $v_{ji}^{(.)}$ in particular, together with (11.81), (11.83), (11.86) and (11.88) permit the following proof of the second equation in (11.90) [for details see the proof of (22.13)]:

$$v_{ji}^j = \frac{d\left[r_{O_j}^O(t_j) - r_{O_i}^O(t_j)\right] : dt_i}{dt_j : dt_i} = \frac{\mu_i}{\mu_j} \frac{v_{O_j}^{O,i} - v_{O_i}^{O,i}}{\frac{1-\frac{\vartheta^i}{q^i w^i} v_R^i}{1-\frac{\vartheta^i v_R^i}{q^i w^i}}} = \frac{\mu_i}{\mu_j} v_{ji}^i.$$

The equations (11.85) through (11.88) transform the equations (11.80) through (11.83) into the equations (11.91) through (11.94).

Sufficiency. Let (11.85) through (11.94) hold. Let μ_i obey (11.79). The equation (22.2) can be transformed as follows by using (11.79), (11.87), (11.90) [for details see the proof of (22.14)]:

$$\mathbf{r}_P(t_i) = c_j^j(1 + \frac{v_{ji}^j}{c_j^j})^{-1}(1 + \frac{v_{ji}^j}{c_j^j})t_j\mathbf{u} = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}\right]. \qquad (22.53)$$

This proves (11.82). The equation (11.83) is proved along the same lines by starting with (22.2). We rearrange (11.79) by using (11.85), (22.2), (22.3), and $r_R(t_{(.)}) \equiv v_R^{(.)} t_{(.)}$ [for details see the proof of (22.16)]:

$$t_i = \mu_i t = \frac{\mu_i}{\mu_j}\left(1 + \frac{\vartheta^j v_R^j}{q^j w^j}\right)^{-1}\left(1 + \frac{\vartheta^j v_R^j}{q^j w^j}\right) t_j = \alpha_j^i \left[t_j + \frac{\vartheta^j}{q^j w^j} r_R(t_j)\right]. \quad (22.54)$$

The equations (22.54) prove the equation (11.80). By repeating this procedure we prove the equation (11.81). We transform the left-hand side of the identity (11.7) by using (11.85), (11.87), (11.91), (11.93), (22.2) and (22.3),

$$[\mathbf{r}_P^T(t_i) \;\; t_i \mathbf{c}_i^{i^T}] D [\mathbf{r}_P^T(t_i) \;\; t_i \mathbf{c}_i^{i^T}]^T \equiv$$

$$\equiv \left\{\lambda_j^i \left[\mathbf{r}_L^T(t_j) + v_{ji}^j t_j \mathbf{u}\right] \;\; t_i \mathbf{c}_i^{i^T}\right\} D \left\{\lambda_j^i \left[\mathbf{r}_L^T(t_j) + v_{ji}^j t_j \mathbf{u}\right] \;\; t_i \mathbf{c}_i^{i^T}\right\}^T \equiv$$

$$\equiv [\lambda_j^i \left(1 + \frac{v_{ji}^j}{c_j^j}\right) \mathbf{r}_P(t_j) \;\; t_i \mathbf{c}_i^{i^T}] D [\lambda_j^i \left(1 + \frac{v_{ji}^j}{c_j^j}\right) \mathbf{r}_P(t_j) \;\; t_i \mathbf{c}_i^{i^T}]^T \equiv$$

$$\equiv [\mathbf{r}_P^T(t_j) \;\; \alpha_j^i \left(1 + \frac{\vartheta^j v_R^j}{q^j w^j}\right) t_j \mathbf{c}_i^{i^T}] D [\mathbf{r}_P^T(t_j) \;\; \alpha_j^i \left(1 + \frac{\vartheta^j v_R^j}{q^j w^j}\right) t_j \mathbf{c}_i^{i^T}]^T.$$

This, (11.85) and $\mathbf{c}_{(.)}^{(.)} = c_{(.)}^{(.)} \mathbf{u}$ yield:

$$[\mathbf{r}_P^T(t_i) \;\; t_i \mathbf{c}_i^{i^T}] D [\mathbf{r}_P^T(t_i) \;\; t_i \mathbf{c}_i^{i^T}]^T \equiv [\mathbf{r}_P^T(t_j) \;\; t_j \mathbf{c}_j^{j^T}] D [\mathbf{r}_P^T(t_j) \;\; t_j \mathbf{c}_j^{j^T}]^T.$$

This proves (11.7).

Compatibility. We replace $r_R(t_{(.)})$ by $v_R^{(.)} t_{(.)}$ in (11.91) and (11.92), which yield:

$$t_i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}} \left(1 + \frac{\vartheta^j v_R^j}{q^j w^j}\right) t_j = \frac{\mu_i}{\mu_j} t_j, \tag{22.55}$$

$$t_j = \frac{\mu_j}{\mu_i} \frac{1}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}} \left(1 - \frac{\vartheta^i v_R^i}{q^i w^i}\right) t_i = \frac{\mu_j}{\mu_i} t_i. \tag{22.56}$$

By eliminating, for example, t_j from (22.55) and (22.56), we get:

$$t_i \equiv \frac{\mu_i}{\mu_j} \frac{\mu_j}{\mu_i} t_i \equiv t_i.$$

This proves complete compatibility of (11.91) and (11.92). We replace now $t_{(.)}$ by $\left(v_{P(.)}^{(.)}\right)^{-1} r_{(.)}(t_{(.)})$ in (11.93) and (11.94), which imply:

$$\mathbf{r}_P(t_i) \equiv \frac{1 + \frac{v_{ji}^j}{v_P^j}}{1 + \frac{v_{ji}^j}{c_j^j}} \frac{1 - \frac{v_{ji}^i}{v_P^i}}{1 - \frac{v_{ji}^i}{c_i^i}} \mathbf{r}_P(t_i) \Longleftrightarrow v_P^{(.)} \equiv c_{(.)}^{(.)}.$$

For this identity to hold it is necessary and sufficient that the equations (22.44) are valid. Hence, (11.93) and (11.94) are only partially compatible. In order to test the entire compatibility we use them together with (11.91), (11.92), and (22.2). We eliminate at first t_j and $r_P(t_j)$ from (11.91) by using (11.92), (11.94), (22.2) and (11.79),

$$t_i \equiv \frac{\frac{1 - \frac{\vartheta^i}{q^i w^i} v_R^i}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}} + \frac{\mu_i}{\mu_j} \frac{\vartheta^j}{q^j w^j} v_R^i \frac{1 - \frac{v_{ji}^i}{v_R^i}}{1 - \frac{v_{ji}^i}{c_i^i}}}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}} t_i \Longleftrightarrow v_R^{(.)} \equiv c_{(.)}^{(.)}.$$

The equations (11.91), (11.92) and (11.94) are not compatible in general. The same holds for (11.91) through (11.93), which is easy to verify by repeating the preceding procedure applied to them. The transformations (11.91) through (11.94) are partially entirely compatible in the temporal domain. Let us now eliminate t_j and $\mathbf{r}_P(t_j)$, $r_P(t_{(.)}) = v_P^{(.)} t_{(.)}$, from (11.93) by using (11.92), (11.94), (11.79) and (11.90),

$$\mathbf{r}_P(t_i) \equiv \frac{\frac{1 - \frac{v_{ji}^i}{v_P^i}}{1 - \frac{v_{ji}^i}{c_i^i}} + v_{ji}^j \frac{\mu_j}{\mu_i} \frac{1}{v_P^i} \frac{1 - \frac{\vartheta^i}{q^i w^i} v_R^i}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}}}{1 + \frac{v_{ji}^i}{c_i^i}} \mathbf{r}_P(t_i) \Longleftrightarrow v_P^i = c_i^i.$$

The transformations (11.92) through (11.94) are partially compatible. The same holds for (11.91), (11.93) and (11.94), which is analogously proved. The transformations (11.91) through (11.94) are partially entirely compatible in the spatial domain, too. Finally, we may conclude that they are partially entirely compatible. Q. E. D ■

22.10 Proof of Theorem 436

Proof. *Necessity and sufficiency.* The necessity and sufficiency of (11.95) and (11.96) result from Theorem 429 for $v_{ji}^{(.)} \equiv 0$.

Compatibility. Theorem 429 guarantees the complete compatibility of the temporal coordinate transformations in (11.96) because they are the same as (11.91) and (11.92). The equation $\mathbf{r}_P(t_i) = \mathbf{r}_P(t_j)$ proves the complete compatibility of the spatial coordinate transformations. The above coordinate transformations are completely pairwise compatible. We eliminate at first all the variables related to the integral space I_i, and afterwards all those related to I_j, from the above temporal coordinate transformations by help of the above trivial (i.e. identity) spatial coordinate transformation,

$$t_j \equiv \frac{\frac{1+\frac{\vartheta^j v_R^j}{q^j w^j}}{1+\frac{\vartheta^j v_R^j}{q^j w^j}} - \frac{\mu_j}{\mu_i}\frac{\vartheta^i}{q^i w^i} v_R^i \frac{\mu_i}{\mu_j}\frac{1+\frac{\vartheta^j}{q^j w^j} v_R^j}{1+\frac{\vartheta^j v_R^j}{q^j w^j}}}{1 - \frac{\vartheta^i v_R^i}{q^i w^i}} t_j \equiv t_j,$$

$$t_i \equiv \frac{\mu_i}{\mu_j}\frac{\frac{\mu_j}{\mu_i}\frac{1-\frac{\vartheta^i}{q^i w^i} v_R^i}{1-\frac{\vartheta^i v_R^i}{q^i w^i}} + \frac{\vartheta^j}{q^j w^j} v_R^j \frac{\mu_j}{\mu_i}\frac{1-\frac{\vartheta^i}{q^i w^i} v_R^i}{1-\frac{\vartheta^i v_R^i}{q^i w^i}}}{1 + \frac{\vartheta^j v_R^j}{q^j w^j}} t_i \equiv t_i.$$

The coordinate transformations are completely entirely compatible in the temporal domain. The equations $\mathbf{r}_P(t_i) = \mathbf{r}_P(t_j)$ and $\mathbf{r}_P(t_j) = \mathbf{r}_P(t_i)$ are independent of the temporal coordinate transformations. Altogether, the coordinate transformations are completely entirely compatible. Q. E. D ■

22.11 Proof of Theorem 456

Proof. *Necessity.* Let the scaling coefficients $\alpha_{(.)}^{(..)}$ and $\lambda_{(.)}^{(..)}$ be determined for the case when the arbitrary point P moves with the speed of the spatial reference point P_{SU} so that

$$v_P^{(.)} \equiv v_{SU}^{(.)}. \tag{22.57}$$

Let the scaling coefficient μ_i satisfy (12.1). In view of (22.57),

$$\mathbf{r}_P(t_i) \equiv \mathbf{r}_{SU}(t_i) = v_{SU}^i t_i \mathbf{u}. \tag{22.58}$$

The position vectors can be expressed also in terms of their (algebraic, i.e. scalar) values, (22.3). Let the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and

λ_j^i, $\lambda_j^i \neq \lambda_i^j$, determined for the speed $v_{SU}^{(.)}$ of the spatial reference point P_{SU}, obey (12.2) through (12.6) so that they together with (12.1) imply (12.7). The equations (12.1), (22.3) and (22.58) give the next scalar forms to the equations (12.4) and (12.5):

$$r_{SU}(t_i) = \lambda_j^i \left(v_{SU} t_j + v_{ji}^j t_j \right) = v_{SU}^i t_i = v_{SU}^i \frac{\mu_i}{\mu_j} t_j, \tag{22.59}$$

$$r_{SU}(t_j) = \lambda_i^j \left(v_{SU} t_i - v_{ji}^i t_i \right) = v_{SU}^j t_j = v_{SU}^j \frac{\mu_j}{\mu_i} t_i. \tag{22.60}$$

We find the following results for λ_j^i and λ_i^j:

$$\lambda_j^i = \frac{v_{SU}^i \mu_i}{v_{SU}^j \mu_j} \frac{1}{1 + \frac{v_{ji}^j}{v_{SU}^j}}, \tag{22.61}$$

$$\lambda_i^j = \frac{v_{SU}^j \mu_j}{v_{SU}^i \mu_i} \frac{1}{1 - \frac{v_{ji}^i}{v_{SU}^i}}. \tag{22.62}$$

By combining (12.1), (12.2) and (22.58) we obtain:

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{v_{ji}^j v_{SU}^j}{q^j w^j}}. \tag{22.63}$$

This equation proves the first equation in (12.8). The first equation (12.9) is analogously proved. The equations (22.3), (12.2), (12.4), (12.7) and (22.58) imply the following:

$$\begin{gathered}
[\ \mathbf{r}_P^T(t_i)\ \ t_i \mathbf{v}_{SU}^{i^T}] D [\ \mathbf{r}_P^T(t_i)\ \ t_i \mathbf{v}_{SU}^{i^T}]^T \equiv [\mathbf{r}_{SU}^T(t_i)\ \ t_i \mathbf{v}_{SU}^{i^T}] D [\mathbf{r}_{SU}^T(t_i)\ \ t_i \mathbf{v}_{SU}^{i^T}]^T \equiv \\
\equiv [\lambda_j^i \left(1 + \frac{v_{ji}^j}{v_{SU}^j}\right) \mathbf{r}_{SU}^T(t_j)\ \ \alpha_j^i \left(1 + \frac{v_{ji}^j v_{SU}^j}{q^j w^j}\right) \frac{v_{SU}^i}{v_{SU}^j} t_j \mathbf{v}_{SU}^{j^T}] D \bullet \\
\bullet [\lambda_j^i \left(1 + \frac{v_{ji}^j}{v_{SU}^j}\right) \mathbf{r}_{SU}^T(t_j)\ \ \alpha_j^i \left(1 + \frac{v_{ji}^j v_{SU}^j}{q^j w^j}\right) \frac{v_{SU}^i}{v_{SU}^j} t_j \mathbf{v}_{SU}^{j^T}]^T \equiv \\
\equiv [\mathbf{r}_{SU}^T(t_j)\ \ t_j \mathbf{v}_{SU}^{j^T}] D [\mathbf{r}_{SU}^T(t_j)\ \ t_j \mathbf{v}_{SU}^{j^T}]^T \equiv [\ \mathbf{r}_P^T(t_j)\ \ t_j \mathbf{v}_{SU}^{j^T}] D [\ \mathbf{r}_P^T(t_j)\ \ t_j \mathbf{v}_{SU}^{j^T}]^T.
\end{gathered} \tag{22.64}$$

These identities, $\alpha_j^i \in R^+$ and $\lambda_j^i \in R^+$ yield:

$$\lambda_j^i = \left(1 + \frac{v_{ji}^j}{v_{SU}^j}\right)^{-1}, \tag{22.65}$$

$$\alpha_j^i = \frac{v_{SU}^j}{v_{SU}^i} \left(1 + \frac{v_{ji}^j v_{SU}^j}{q^j w^j}\right)^{-1}. \tag{22.66}$$

The equation (22.66) verifies the second equation (12.8). The proof of the second equation (12.9) is analogous. The equation (22.65) proves the equation (12.10). The equations (22.61) and (22.65) imply:

$$\frac{v_{SU}^{i}}{v_{SU}^{j}}\frac{\mu_i}{\mu_j} = 1. \tag{22.67}$$

The equations (22.62) and (22.67) prove the equation (12.11). The condition that all the scaling coefficients, as well as q^i, q^j, w^i and w^j, are positive real valued, that $v_{ji}^{j} \in R_+$ by the definition, together with (12.8) through (12.11) imply $v_{ji}^{i} < \min\left\{v_{SU}^{i}, \frac{q^i w^i}{v_{SU}^{i}}\right\}$ and $v_{ji}^{j} > \max\left\{-v_{SU}^{j}, -\frac{q^j w^j}{v_{SU}^{j}}\right\}$, which, together with $v_{ji}^{j} \geq 0$ and $\max\left\{-v_{SU}^{j}, -\frac{q^j w^j}{v_{SU}^{j}}\right\} < 0$, prove (12.12). The first equation in (12.13) results directly from the equation (22.67). The second equation in (12.13) comes out from the first equation (12.13) due to an arbitrary choice of v_{SU}. The equations (12.8) through (12.11) transform the equations (12.2) through (12.5) into the equations (12.14) through (12.17).

Sufficiency. Let (12.8) through (12.17) be valid. Let μ_i obey (12.1). The equation (22.58) can be transformed as follows by using (12.1), (12.10), (12.13) and (22.58) itself with i replaced by j,

$$\mathbf{r}_P(t_i) = v_{SU}^{i} t_i\ \mathbf{u} = \frac{v_{SU}^{j}\mu_j}{\mu_i} t_i\ \mathbf{u} = v_{SU}^{j} t_j\ \mathbf{u} =$$
$$= v_{SU}^{j}\left(1+\frac{v_{ji}^{j}}{v_{SU}^{j}}\right)^{-1}\left(1+\frac{v_{ji}^{j}}{v_{SU}^{j}}\right) t_j\ \mathbf{u} = \lambda_j^i \left[\ \mathbf{r}_P(t_j) + v_{ji}^{j} t_j\ \mathbf{u}\right]. \tag{22.68}$$

This proves (12.4). The equation (12.5) is proved analogously by starting with (22.58) for i replaced by j everywhere therein. We rearrange (12.1):

$$t_i = \mu_i t = \frac{\mu_i}{\mu_j}\left(1+\frac{v_{ji}^{j} v_{SU}^{j}}{q^j w^j}\right)^{-1}\left(1+\frac{v_{ji}^{j} v_{SU}^{j}}{q^j w^j}\right) t_j =$$
$$= \frac{\mu_i}{\mu_j}\frac{1}{1+\frac{v_{ji}^{j} v_{SU}^{j}}{q^j w^j}}\left(t_j + \frac{v_{ji}^{j} v_{SU}^{j}}{q^j w^j} t_j\right). \tag{22.69}$$

Now, (12.8), (22.58), and (22.69) imply:

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^{j}}{q^j w^j}\ r_P(t_j)\right]. \tag{22.70}$$

The equation (22.70) is the equation (12.2). By repeating this procedure applied to (12.1) for i replaced by j we prove the equation (12.3). We continue to transform the left-hand side of the identity (12.7) by using (12.8), (12.10), (12.14), (12.16), (22.3), (22.58), and $\mathbf{v}_{SU}^{(.)} = v_{SU}^{(.)}\mathbf{u}$,

$$
\begin{aligned}
&[\ \mathbf{r}_P^T(t_i) \qquad t_i\mathbf{v}_{SU}^{i^T}]D[\ \mathbf{r}_P^T(t_i) \qquad t_i\mathbf{v}_{SU}^{i^T}]^T \equiv \\
&\equiv [\lambda_j^i \left(1+\frac{v_{ji}^j}{v_{SU}^j}\right) \mathbf{r}_P^T(t_j) \qquad t_i\mathbf{v}_{SU}^{i^T}]D[\lambda_j^i \left(1+\frac{v_{ji}^j}{v_{SU}^j}\right) \mathbf{r}_P^T(t_j) \qquad t_i\mathbf{v}_{SU}^{i^T}]^T \equiv \\
&\equiv [\mathbf{r}_P^T(t_j)\ \alpha_j^i \frac{v_{SU}^i}{v_{SU}^j}(1+\frac{v_{ji}^j v_{SU}^j}{q^j w^j})t_j\mathbf{v}_{SU}^{j^T}]D[\mathbf{r}_P^T(t_j)\ \alpha_j^i \frac{v_{SU}^i}{v_{SU}^j}(1+\frac{v_{ji}^j v_{SU}^j}{q^j w^j})t_j\mathbf{v}_{SU}^{j^T}]^T \equiv \\
&\equiv [\ \mathbf{r}_P^T(t_j) \qquad t_j\mathbf{v}_{SU}^{j^T}]D[\ \mathbf{r}_P^T(t_j) \qquad t_j\mathbf{v}_{SU}^{j^T}]^T,
\end{aligned}
$$

which proves (12.7).

Compatibility. In order to test the transformations for their complete compatibility we replace $r_P(t_{(.)};t_{(.)0})$ by $v_{P(.)}^{(.)}(t_{(.)}-t_{(.)0})$ in (12.14) and (12.15),

$$t_i = \frac{\mu_i}{\mu_j}\frac{1}{1+\frac{v_{ji}^j v_{SU}^j}{q^j w^j}}\left(1+\frac{v_{ji}^j v_P^j}{q^j w^j}\right)t_j, \tag{22.71}$$

$$t_j = \frac{\mu_j}{\mu_i}\frac{1}{1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}}\left(1-\frac{v_{ji}^i v_P^i}{q^i w^i}\right)t_i. \tag{22.72}$$

By eliminating, for example, t_j from (22.71) and (22.72), we get:

$$t_i \equiv \frac{\mu_i}{\mu_j}\frac{\mu_j}{\mu_i}\frac{1}{1+\frac{v_{ji}^j v_{SU}^j}{q^j w^j}}\left(1+\frac{v_{ji}^j v_P^j}{q^j w^j}\right)\frac{1}{1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}}\left(1-\frac{v_{ji}^i v_P^i}{q^i w^i}\right)t_i.$$

For this identity to hold it is necessary and sufficient that

$$v_P^i = v_{SU}^i \ \textit{and}\ v_P^j = v_{SU}^j. \tag{22.73}$$

This proves the partial compatibility of (12.14) and (12.15). We replace now $t_{(.)}$ by $\left(v_P^{(.)}\right)^{-1} r_P(t_{(.)})$ in (12.16) and (12.17),

$$\mathbf{r}_P(t_i) \equiv \frac{1+\frac{v_{ji}^j}{v_P^j}}{1+\frac{v_{ji}^j}{v_{SU}^j}}\ \mathbf{r}_P(t_j),\ \ \mathbf{r}_P(t_j) \equiv \frac{1-\frac{v_{ji}^i}{v_P^i}}{1-\frac{v_{ji}^i}{v_{SU}^i}}\ \mathbf{r}_P(t_i),$$

which yield

$$\mathbf{r}_P(t_i) \equiv \frac{1+\frac{v_{ji}^j}{v_P^j}}{1+\frac{v_{ji}^j}{v_{SU}^j}}\frac{1-\frac{v_{ji}^i}{v_P^i}}{1-\frac{v_{ji}^i}{v_{SU}^i}}\ \mathbf{r}_P(t_i).$$

For this identity to be valid it is necessary and sufficient that (22.73) is valid. Hence, (12.16) and (12.17) are also only partially compatible. Let us now test the entire compatibility of (12.14) through (12.17). We use them together with (22.58). We eliminate at first t_j and $r_P(t_j)$ from (12.14) by using (22.3), (12.13), (12.15) and (12.17),

$$t_i \equiv \frac{\left\{\begin{array}{c} \left[t_i - \frac{v_{ji}^i}{q^i w^i}\ r_P(t_i)\right]\left(1-\frac{v_{ji}^i}{v_{SU}^i}\right)+ \\ +\frac{v_{SU}^j}{v_{SU}^i}\frac{v_{ji}^j}{q^j w^j}\left(1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}\right)\left[\ r_P(t_i) - \ v_{ji}^i t_i\right] \end{array}\right\}}{\left(1+\frac{v_{ji}^j v_{SU}^j}{q^j w^j}\right)\left(1-\frac{v_{ji}^i}{v_{SU}^i}\right)\left(1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}\right)} \equiv$$

$$\equiv \frac{\left\{1-\frac{v_{ji}^i}{v_{SU}^i}+\frac{\left(v_{ji}^i\right)^2}{q^i w^i}\left[-\ \left(1-\frac{v_P^i}{v_{SU}^i}\right)-\frac{v_{SU}^i v_P^i}{q^i w^i}\left(1-\frac{v_{ji}^i}{v_P^i}\right)\right]\right\}}{\left(1+\frac{v_{ji}^i v_{SU}^i}{q^i w^i}\right)\left(1-\frac{v_{ji}^i}{v_{SU}^i}\right)\left(1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}\right)} t_i.$$

For these identities to hold it is necessary and sufficient that $v_P^{(.)} = v_{SU}^{(.)}$. The transformations (12.14), (12.15) and (12.17) are partially compatible. We prove analogously partial compatibility of the transformations (12.14) through (12.16). The transformations (12.14) through (12.17) are partially compatible in the temporal domain. Let us now eliminate t_j and $\mathbf{r}_P(t_j)$, $\mathbf{r}_P(t_{(.)}) = v_P^{(.)} t_{(.)}\mathbf{u}$, from (12.16) by using (12.13), (12.15) and (12.17),

$$\mathbf{r}_P(t_i) \equiv \frac{\left(1-\frac{v_{ji}^i}{v_P^i}\right)\left(1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}\right)+\ \frac{v_{ji}^i}{v_P^i}\left(1-\frac{v_{ji}^i v_P^i}{q^i w^i}\right)\left(1-\frac{v_{ji}^i}{v_{SU}^i}\right)}{\left(1-\frac{v_{ji}^i}{v_{SU}^i}\right)\left(1+\frac{v_{ji}^j}{v_{SU}^j}\right)\left(1-\frac{v_{ji}^i v_{SU}^i}{q^i w^i}\right)}\ \mathbf{r}_P(t_i).$$

For this identity to hold it is necessary and sufficient that the equations (22.73) are valid. The same holds for (12.14), (12.16) and (12.17). This and the preceding result show that (12.14) through (12.17) are only partially entirely compatible. Altogether, the transformations (12.14) through (12.17) are only partially both entirely and pairwise compatible. Q. E. D ■

22.12 Proof of Theorem 461

Proof. *Necessity.* Let $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$ and $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$ be positive real numbers. Let the arbitrary point P move with the speed of the spatial reference point P_{SU}. Hence, (22.57) is valid. Let the *time* scaling coefficients μ_i be positive real numbers and be defined by (12.1). Let $B = A$ in D, (12.7). Let the scaling coefficients λ_{ij} and α_{ij} obey (12.2) through (12.6), and let (12.1) through (12.6) imply (12.7). The equations (22.3), (12.2) through (12.5), (12.7), (22.57) and (22.58), together with $D = blockdiag\{A \ \ -A\}$, enable the following:

$$[\ \mathbf{r}_P^T(t_i) \qquad t_i\mathbf{v}_{SU}^{i^T}]D[\ \mathbf{r}_P^T(t_i) \qquad t_i\mathbf{v}_{SU}^{i^T}]^T \equiv$$

$$\equiv \lambda_{ij}\left[\frac{\sqrt{\begin{array}{c}\left[1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{ji}^{j}v_{SU}^{i}}{q^{j}w^{j}}\right)^{2}\right]r_{P}\left(t_{j}\right)+\\+2v_{ji}^{j}\left[1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^{2}\frac{\left(v_{SU}^{i}\right)^{2}}{q^{j}w^{j}}\right]r_{P}\left(t_{j}\right)t_{j}\end{array}}\,\mathbf{u}}{\sqrt{\left[1-\left(\frac{v_{ji}^{j}}{v_{SU}^{i}}\frac{\lambda_{ij}}{\alpha_{ij}}\right)^{2}\right]\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{SU}^{i}}{v_{SU}^{j}}\right)^{2}\left(v_{SU}^{j}\right)t_{j}\mathbf{u}}}\right]^{T} D\bullet$$

$$\bullet\lambda_{ij}\left[\frac{\sqrt{\begin{array}{c}\left[1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{ji}^{j}v_{SU}^{i}}{q^{j}w^{j}}\right)^{2}\right]r_{P}\left(t_{j}\right)+\\+2v_{ji}^{j}\left[1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^{2}\frac{\left(v_{SU}^{i}\right)^{2}}{q^{j}w^{j}}\right]r_{P}\left(t_{j}\right)t_{j}\end{array}}\,\mathbf{u}}{\sqrt{\left[1-\left(\frac{v_{ji}^{j}}{v_{SU}^{i}}\frac{\lambda_{ij}}{\alpha_{ij}}\right)^{2}\right]\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{SU}^{i}}{v_{SU}^{j}}\right)^{2}\left(v_{SU}^{j}\right)t_{j}\mathbf{u}}}\right]\equiv$$

$$\equiv [\ r_{P}\left(t_{j}\right)\mathbf{u}^{T}\quad t_{j}v_{SU}^{j}\ \mathbf{u}^{T}]D[\ r_{P}\left(t_{j}\right)\mathbf{u}^{T}\quad t_{j}v_{SU}^{j}\ \mathbf{u}^{T}]^{T}. \tag{22.74}$$

The last identity implies:

$$1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\right)^{2}\frac{\left(v_{SU}^{i}\right)^{2}}{q^{j}w^{j}}=0,\ \textit{hence}\ \frac{\alpha_{ij}}{\lambda_{ij}}=\frac{\sqrt{q^{j}w^{j}}}{v_{SU}^{i}}, \tag{22.75}$$

$$\lambda_{ij}\sqrt{1-\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{ji}^{j}v_{SU}^{i}}{q^{j}w^{j}}\right)^{2}}=1, \tag{22.76}$$

and

$$\lambda_{ij}\sqrt{\left[1-\left(\frac{v_{ji}^{j}}{v_{SU}^{i}}\frac{\lambda_{ij}}{\alpha_{ij}}\right)^{2}\right]\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{SU}^{i}}{v_{SU}^{j}}\right)^{2}}=1. \tag{22.77}$$

The equations under (22.75) through (22.77) yield:

$$\lambda_{ij}=\frac{1}{\sqrt{1-\left(\frac{v_{ji}^{j}}{\sqrt{q^{j}w^{j}}}\right)^{2}}}, \tag{22.78}$$

and

$$\lambda_{ij}=\frac{1}{\sqrt{\left[1-\left(\frac{v_{ji}^{j}}{\sqrt{q^{j}w^{j}}}\right)^{2}\right]\left(\frac{\sqrt{q^{j}w^{j}}}{v_{SU}^{j}}\right)^{2}}}. \tag{22.79}$$

The last two results, (22.78) and (22.79), demand:

$$1-\left(\frac{v_{ji}^{j}}{\sqrt{q^{j}w^{j}}}\right)^{2}\equiv\left[1-\left(\frac{v_{ji}^{j}}{\sqrt{q^{j}w^{j}}}\right)^{2}\right]\left(\frac{\sqrt{q^{j}w^{j}}}{v_{SU}^{j}}\right)^{2}.$$

For this identity to hold it is necessary and sufficient that:

$$\sqrt{q^j w^j} = v^j_{SU}. \tag{22.80}$$

This, (22.75), (22.78) and (22.79) imply:

$$\frac{\alpha_{ij}}{\lambda_{ij}} = \frac{v^j_{SU}}{v^i_{SU}}, \tag{22.81}$$

$$\lambda_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v^j_{ji}}{v^j_{SU}}\right)^2}}, \tag{22.82}$$

$$\alpha_{ij} = \frac{v^j_{SU}}{v^i_{SU}}\lambda_{ij} = \frac{v^j_{SU}}{v^i_{SU}} \frac{1}{\sqrt{1 - \left(\frac{v^j_{ji}}{v^j_{SU}}\right)^2}}. \tag{22.83}$$

Let us now transform the right-hand side of (12.7). We use (22.57) and we repeat the preceding procedure. The results are the following:

$$\sqrt{q^i w^i} = v^i_{SU}, \tag{22.84}$$

$$\frac{\alpha_{ij}}{\lambda_{ij}} = \frac{v^i_{SU}}{v^j_{SU}}, \tag{22.85}$$

$$\lambda_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v^i_{ji}}{v^i_{SU}}\right)^2}}, \tag{22.86}$$

$$\alpha_{ij} = \frac{v^j_{SU}}{v^i_{SU}}\lambda_{ij} = \frac{v^j_{SU}}{v^i_{SU}} \frac{1}{\sqrt{1 - \left(\frac{v^j_{ji}}{v^j_{SU}}\right)^2}}. \tag{22.87}$$

The equations (22.81) and (22.85), (22.82) and (22.86), (22.83) and (22.87) prove (12.18) through (12.20). We continue with (12.1) and $r_P(t_{(.)}) \equiv v^{ij}_{SU} t_{(.)}$,

$$r_P\left(t_i\right) \equiv v^{ij}_{SU} \frac{\mu_i}{\mu_j} t_j \equiv \lambda_{ij} \left[\, r_P\left(t_j\right) + v^j_{ji} t_j \right] \equiv \lambda_{ij} v^{ij}_{SU} \left(1 + v^{ij}_{SU} v^j_{ji}\right) t_j.$$

This, (12.18) and (12.20) show that:

$$\frac{\mu_i}{\mu_j} = \frac{\left(1 + \frac{v_{ji}}{v^{ij}_{SU}}\right)}{\sqrt{\left(1 - \frac{v_{ji}}{v^{ij}_{SU}}\right)\left(1 + \frac{v_{ji}}{v^{ij}_{SU}}\right)}} = \sqrt{\frac{1 + \frac{v_{ji}}{v^{ij}_{SU}}}{1 - \frac{v_{ji}}{v^{ij}_{SU}}}} =$$

$$= \sqrt{\frac{1 + \frac{v_{ji}}{\sqrt{q^i w^i}}}{1 - \frac{v_{ji}}{\sqrt{q^i w^i}}}} = \sqrt{\frac{1 + \frac{v_{ji}}{v^{ij}_{SU}}}{1 - \frac{v_{ji}}{v^{ij}_{SU}}}}.$$

This proves (12.21). The equations (12.18) through (12.20) transform the equations (12.2) through (12.5) into (12.22) through (12.25).

Sufficiency. Let all the conditions of the theorem statement hold. We start by transforming (12.1) with the help of (12.18) and (12.21):

$$t_i = \frac{\mu_i}{\mu_j} t_j = \frac{\left(1 + \frac{v_{ji}}{v_{SU}^{ij}}\right)}{\sqrt{\left(1 - \frac{v_{ji}}{v_{SU}^{ij}}\right)\left(1 + \frac{v_{ji}}{v_{SU}^{ij}}\right)}} t_j .$$

Now we apply (12.20), $t_{(.)} = (v_{SU}^{ij})^{-1}\ r_P(t_{(.)})$, again (12.18), (12.19) and $\alpha_{ij} = \alpha_{ji} = \alpha_i^j = \alpha_j^i$:

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j}\ r_P\left(t_j\right) \right].$$

This is (12.2). The equation (12.3) is proved in the same way by starting with (12.1) in which i and j exchange places. We proceed by using $\mathbf{r}_P(t_{(.)}) = v_{SU}^{ij} t_{(.)} \mathbf{u}$, (12.1), (12.18), (12.19), (12.21), and $\lambda_{ij} = \lambda_{ji} = \lambda_i^j = \lambda_j^i$:

$$\mathbf{r}_P\left(t_i\right) = v_{SU}^{ij} \frac{\mu_i}{\mu_j} t_j \mathbf{u} = v_{SU}^{ij} \sqrt{\frac{1 + \frac{v_{ji}}{v_{SU}^{ij}}}{1 - \frac{v_{ji}}{v_{SU}^{ij}}}} t_j \mathbf{u} = v_{SU}^{ij} \frac{1 + \frac{v_{ji}}{v_{SU}^{ij}}}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} t_j \mathbf{u} =$$

$$= \lambda_j^i \left(v_{SU}^{ij} + v_{ji}^j\right) t_j \mathbf{u} = \lambda_j^i \left[\ \mathbf{r}_P\left(t_j\right) + v_{ji}^j t_j\right] \mathbf{u}.$$

This is (12.4). The equation (12.5) is proved in the same way by starting with $\mathbf{r}_P\left(t_j\right) = v_{SU}^{ij} t_j \mathbf{u}$.

Compatibility. Let us check the transformations (12.22) through (12.25) for their complete pairwise compatibility. We eliminate at first, for example, t_j from (12.22) and (12.23) after replacing in them $(qw)^{ji}$ by v_{SU}^{ij} in view of (12.18),

$$t_i \equiv \frac{\frac{t_i - \frac{v_{ji}}{v_{SU}^{ij}}\ r_P(t_i)}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} + \frac{v_{ji}}{(v_{SU}^{ij})^2}\ r_P\left(t_j\right)}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} \equiv \frac{\frac{t_i - \frac{v_{ji}}{v_{SU}^{ij}}\ r_P(t_i)}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} + \frac{v_{ji} v_P^j}{(v_{SU}^{ij})^2} t_j}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}},$$

and we apply $r_P(t_{(.)}) = v_P^{(.)} t_{(.)}$,

$$t_i \equiv \frac{\left(1 - \frac{v_{ji} v_P^i}{v_{SU}^{ij}}\right)\left(1 + \frac{v_{ji} v_P^j}{v_{SU}^{ij}}\right)}{1 - \left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2} t_i\ .$$

For this identity to hold it is necessary and sufficient that $v_P^i = v_P^j = v_{SU}^{ij}$. The equations (12.22) and (12.23) are only partially compatible. We eliminate, for example, $\mathbf{r}_P\left(t_j\right)$ from (12.24) and (12.25), and we apply $t_{(.)}\mathbf{u} =$

$\left[v_P^{(.)}\right]^{-1} \mathbf{r}_P(t_{(.)};t_{(.)0})$,

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{\frac{\left(1-\frac{v_{ji}}{v_P^i}\right)\ \mathbf{r}_P(t_i)}{\sqrt{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} + \frac{v_{ji}}{v_P^j}\mathbf{r}_P\left(t_j\right)}{\sqrt{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} \equiv \frac{\left(1-\frac{v_{ji}}{v_P^i}\right)\left(1+\frac{v_{ji}}{v_P^j}\right)}{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}\mathbf{r}_P\left(t_i\right).$$

For these identities to hold it is necessary and sufficient that $v_P^i = v_P^j = v_{SU}^{ij}$. The equations (12.24) and (12.25) are only partially compatible. Hence, the equations (12.22) through (12.25) are only partially pairwise compatible. Let us check them for their entire compatibility. At first we use (12.18), (12.22), (12.23) and (12.25) from which we eliminate both t_j and $r_P\left(t_j\right)$ in view of $\mathbf{r}_P\left(t_j\right) = r_P\left(t_j\right)\mathbf{u}$:

$$t_i \equiv \frac{\frac{t_i - \frac{v_{ji}}{(v_{SU}^{ij})^2}\ r_P(t_i)}{\sqrt{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} + \frac{v_{ji}}{(v_{SU}^{ij})^2}\frac{r_P(t_i)-v_{ji}(t_i)}{\sqrt{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}}}{\sqrt{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}} \equiv \frac{1-\frac{v_{ji}^2}{(v_{SU}^{ij})^2}}{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}t_i \equiv t_i.$$

This proves the compatibility of (12.22), (12.23) and (12.25), which is partial due to (12.18). We prove in the same manner the partial compatibility of (12.22) through (12.24). The coordinate transformations (12.22) through (12.25) are partially compatible in the temporal domain. We exploit now $t_{(.)} = \left[v_P^{(.)}\right]^{-1} r_P(t_{(.)})$, (12.18), together with (12.22), (12.24) and (12.25), from which we eliminate both t_i and $\mathbf{r}_P\left(t_i\right)$:

$$\mathbf{r}_P\left(t_j\right) \equiv \frac{\left(1+\frac{v_{ji}}{v_P^j}\right)\ \mathbf{r}_P\left(t_j\right) - \frac{v_{ji}}{v_P^j}\left(1+\frac{v_{ji}v_P^j}{(v_{SU}^{ij})^2}\right)\ r_P\left(t_j\right)\ \mathbf{u}}{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2} \equiv \mathbf{r}_P\left(t_j\right).$$

This proves partial compatibility of (12.22), (12.24) and (12.25), in view of (12.18). The same is true for (12.23) through (12.25). These results prove the partial entire compatibility of the transformations (12.22) through (12.25). Q. E. D ∎

22.13 Proof of Theorem 474

Proof. *Necessity and sufficiency.* Let all the conditions of the theorem statement hold. Then, Theorem 460 is valid, i.e. the equations (12.18) through (12.25) hold. We start with the equation (22.43) for $(.) = i$, in which we replace both $d\mathbf{r}_P^{(.)} = d\mathbf{r}_P^i$ by the right hand side of the equation (12.24), and $dt_{(.)} = dt_i$

by the right hand side of the equation (12.22) with $(qw)^{ji} = v_{SU}^{ij}$,

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P^{O_i}}{dt_i} = \frac{d\mathbf{r}_P}{dt_i} = \frac{d\left[\frac{\mathbf{r}_P(t_j)+\mathbf{v}_{ji}t_j}{\sqrt{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}}\right] : dt_j}{d\left[\frac{t_j+\frac{v_{ji}}{(v_{SU}^{ij})^2} \ r_P(t_j)}{\sqrt{1-\left(\frac{v_{ji}}{v_{SU}^{ij}}\right)^2}}\right] : dt_j} = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}}{1+\frac{v_{ji}}{(v_{SU}^{ij})^2}v_P^j}.$$

This proves the first transformation (12.34). The second transformation (12.34) is analogously proved by starting with (22.43) for $(.) = j$, (12.23) and (12.25).

Compatibility. In order to test the complete compatibility of the transformations (12.34), we eliminate, for example, $\mathbf{v}_P^i$ and v_P^i from them,

$$\mathbf{v}_P^j = \frac{\mathbf{v}_P^i - \ \mathbf{v}_{ji}}{1-\frac{v_{ji}^i v_P^i}{(v_{SU}^{ij})^2}} = \frac{\frac{\mathbf{v}_P^j+\mathbf{v}_{ji}}{1+\frac{v_{ji}}{(v_{SU}^{ij})^2}v_P^j} - \frac{\left[1+\frac{v_{ji}}{(v_{SU}^{ij})^2}v_P^j\right]\mathbf{v}_{ji}}{1+\frac{v_{ji}}{(v_{SU}^{ij})^2}v_P^j}}{1-\frac{v_{ji}\left(v_P^j+v_{ji}\right)}{(v_{SU}^{ij})^2\left[1+\frac{v_{ji}}{(v_{SU}^{ij})^2}v_P^j\right]}} =$$

$$= (v_{SU}^{ij})^2 \frac{\mathbf{v}_P^j - \frac{(v_{ji})^2}{(v_{SU}^{ij})^2}\mathbf{v}_P^j}{(v_{SU}^{ij})^2 + v_{ji}v_P^j - v_{ji}^i v_P^j - (v_{ji})^2} = \frac{1-\frac{(v_{ji})^2}{(v_{SU}^{ij})^2}}{1-\frac{(v_{ji})^2}{(v_{SU}^{ij})^2}}\mathbf{v}_P^j = \mathbf{v}_P^j.$$

The transformations are compatible but only partially because they hold under the restriction on $(qw)^{ji}$ in (12.18). The invariance of $\mathbf{v}_{SU}^{(.)}$ results from (12.34) for $\mathbf{v}_P^{(.)} \equiv \mathbf{v}_{SU}^{(.)} \equiv \mathbf{v}_{SU}^{ij}$ Q. E. D ■

22.14 Proof of Theorem 482

Proof. *Necessity and sufficiency.* Since the only difference herein relative to the proof of the general *time*-invariant uniform transformations (11.80) through (11.84) is the replacement of the light signal L (and its characteristics: position and speed) as the reference point for the spatial coordinate transformations by the reference spatial point P_{SU} (and its characteristics: position and speed), (respectively), then we should do such replacements in (11.87), (11.88), (11.93) and (11.94). The results are (12.38) through (12.47) since there is not any change in the *time* scaling coefficients α_j^i and α_i^j, (11.85) and (11.86), and in the temporal coordinate transformations (11.91) and (11.92).

Compatibility. Since (12.44) and (12.45) are the same as (11.91) and (11.92), then they are completely compatible, (Theorem 429). The proof of the partial compatibility of (12.46) and (12.47) is the same as the proof of the partial compatibility of (11.93) and (11.94) when we replace in them $c_{(.)}^{(.)}$ by $v_{SU}^{(.)}$.

We eliminate all the variables related to the integral space I_j from (12.44) by the help of (12.43), (12.45) and (12.47),

$$t_i = \frac{\frac{1-\frac{\vartheta^i}{q^i w^i} v_R^i}{1-\frac{\vartheta^i v_R^i}{q^i w^i}} + \frac{\vartheta^j}{q^j w^j} v_R^j \frac{1-\frac{v_{ji}^i}{v_R^i}}{1-\frac{v_{ji}^i}{v_{SU}^i}}}{1+\frac{\vartheta^j v_R^j}{q^j w^j}} t_i \Longleftrightarrow v_R^i = v_{SU}^i.$$

The transformations (12.44), (12.45) and (12.47) are partially compatible. The same holds for (12.44) through (12.46). Altogether, the coordinate transformations (12.44) through (12.47) are partially entirely compatible in the temporal domain. We continue with (12.45) through (12.47), $t_i = \left(v_P^i\right)^{-1} r_P\left(t_i\right)$ and $r_R\left(t_i\right) = v_R^i t_i$,

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{\frac{1-\frac{v_{ji}^i}{v_P^i}}{1-\frac{v_{ji}^i}{v_{SU}^i}} \mathbf{r}_P\left(t_i\right) + v_{ji}^j \frac{\mu_j}{\mu_i} \frac{1-\frac{\vartheta^i v_R^i}{q^i w^i}}{1-\frac{\vartheta^i v_R^i}{q^i w^i}} t_i \mathbf{u}}{1+\frac{v_{ji}^j}{v_{SU}^j}}.$$

This, $t_i = \mu_i t_j/\mu_j$ due to (11.79), $t_i = \left(v_P^i\right)^{-1} r_P\left(t_i\right)$, and $r_P\left(t_{(.)}\right)\mathbf{u} = v_P^{(.)} t_{(.)}\mathbf{u}$ $= \mathbf{r}_P\left(t_{(.)}\right)$ lead to

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{\left(1-\frac{v_{ji}^i}{v_P^i}\right) + \frac{v_{ji}^j}{v_P^j}\left(1-\frac{v_{ji}^i}{v_{SU}^i}\right)}{\left(1+\frac{v_{ji}^j}{v_P^j}\right)\left(1-\frac{v_{ji}^i}{v_{SU}^i}\right)} \mathbf{r}_P\left(t_i\right).$$

For this identity to hold it is necessary and sufficient that $v_P^i = v_{SU}^i$. The transformations (12.45) through (12.47) are partially compatible. In the same way we prove partial compatibility of (12.44), (12.46) and (12.47). Therefore, the transformations (12.44) through (12.47) are partially entirely compatible in the spatial domain. Q. E. D ■

22.15 Proof of Theorem 487

Proof. Let the *time* scaling coefficient $\mu_i \in R^+$ be defined by (11.79). Let the scaling coefficients α_j^i and α_i^j be equal: $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, as well as λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$.

Necessity and sufficiency. If in the proofs of necessity and of sufficiency of the conditions of Theorem 437 we accept the spatial reference point P_{SU} instead of the light signal L for the arbitrary point P, then the formulae of Theorem 437 take the form of the formulae (12.48) through (12.56).

Compatibility. Since (12.53) and (12.54) are the same as (11.102) and (11.103), then they are partially compatible, Theorem 437. Let us now eliminate, for example, $\mathbf{r}_P\left(t_j\right)$ from (12.55) and (12.56) by applying (22.3) for $v_P^{(.)} \equiv v_{Pji}$ and

$v_{SU}^{(.)} \equiv v_{SU}^{ji}$, (12.48):

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{1 + \frac{v_{ji}}{v_{Pji}}}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ji}}\right)^2}} \frac{1 - \frac{v_{ji}}{v_{Pji}}}{\sqrt{1 - \left(\frac{v_{ji}}{v_{SU}^{ji}}\right)^2}} \mathbf{r}_P\left(t_i\right).$$

This proves the compatibility of (12.55) and (12.56) only for $v_{Pji} = v_{SU}^{ji}$. They are only partially compatible. Let us now eliminate, for example, all the coordinates with the subscript j from (12.53) by the help of $r_{[.]}(t_{(.)}) = v_{[.]}^{(.)} t_{(.)}$, $[.] \in \{R,\, SU\}$, $r_P(t_{(.)}) = v_P^{(.)} t_{(.)}$, (12.48), (12.54) and (12.56),

$$t_i \equiv \frac{1 - \frac{v_{ji}^i}{v_{SU}^i} + \frac{v_{ji}^i}{v_{SU}^i}\frac{v_R^i}{v_R^j}\left(1 - \frac{v_{ji}^i}{v_R^i}\right)}{1 - \left(\frac{v_{ji}^i}{v_{SU}^i}\right)^2} t_i \equiv t_i \Longleftrightarrow$$
$$v_R^i = v_R^j = v_R^{ij} = v_R^{ji} = v_{SU}^i = v_{SU}^j = v_{SU}^{ij} = v_{SU}^{ji}.$$

This, and the used restrictions on v_R^{ij} in (12.48) show that the transformations (12.53), (12.54) and (12.56) are only partially compatible. The same procedure applied to (12.53) through (12.55) verifies their partial compatibility. The transformations (12.53) through (12.56) are partially entirely compatible in the temporal domain. Let us now eliminate, for example, all the coordinates with the subscript j from (12.55) by the help of (12.54) and (12.56):

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{1 - \frac{v_{ji}^i}{v_P^i} + \frac{v_{ji}^j}{v_P^i}\left(1 - \frac{v_{ji}^i}{v_{SU}^i}\right)}{1 - \left(\frac{v_{ji}^i}{v_{SU}^i}\right)^2} \mathbf{r}_P\left(t_i\right) \Longleftrightarrow$$
$$\Longleftrightarrow v_{ji}^i = v_{ji}^j = v_{ij} = v_{ji},\; v_P^i = v_{SU}^i.$$

The transformations (12.54) through (12.56) are partially compatible. The same holds for (12.53), (12.55) and (12.56). The transformations (12.53) through (12.56) are partially entirely compatible in the spatial domain. Altogether, they are partially entirely compatible. Q. E. D ■

Chapter 23

Appendices: Proofs for Part 4

23.1 Proof of Theorem 502

Proof. Let the scaling coefficient μ_i obey (14.2). The following is true:

$$\mathbf{r}_{(..)}(t_{(.)}) = v_{(..)}^{(.)} t_{(.)} \mathbf{u},\ (..) \in \{G, L, P, P_R, P_{SU}\}. \tag{23.1}$$

The position vector can be expressed also in terms of its length $r_{(..)}$,

$$\mathbf{r}_{(..)}(t_{(.)}) = r_{(..)}(t_{(.)}) \mathbf{u},\ (..) \in \{G, L, P, P_R, P_{SU}\}. \tag{23.2}$$

Necessity. Let the scaling coefficients α_i^j, α_j^i, $\alpha_j^i \neq \alpha_i^j$, λ_i^j and λ_j^i, $\lambda_j^i \neq \lambda_i^j$, be positive real numbers and obey (14.3) through (14.7) so that they, together with (14.2), imply (14.8). The equations (14.2), (23.1) and (23.2) give the following scalar form to the equation (14.5):

$$r_P\left(t_i\right) = \lambda_j^i v_P^j \left(1 + \frac{v_{ji}^j}{v_P^j}\right) \mu_j t = v_P^i t_i = v_P^i \mu_i t. \tag{23.3}$$

The solution of (23.3) for λ_j^i is:

$$\lambda_j^i = \frac{v_P^i \mu_i}{v_P^j \mu_j} \frac{1}{1 + \frac{v_{ji}^j}{v_P^j}}. \tag{23.4}$$

By combining (14.2), (14.3) and $r_P\left(t_j\right) = v_P^j t_j$ we find:

$$t_i = \alpha_j^i \left(1 + \frac{v_{ji}^j}{q^j w^j} v_P^j\right) \frac{\mu_j}{\mu_i} t_i.$$

This equation implies

$$\alpha_j^i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{v_{ji}^j v_P^j}{q^j w^j}},$$

which proves the equation (14.9). The equation (14.10) is proved along the same lines. The equations (14.3), (14.5), (14.8) and (23.2) imply the following:

$$\begin{gathered}
[\mathbf{r}_P^T(t_i) \quad t_i \mathbf{v}_P^{i^T}] D [\mathbf{r}_P^T(t_i) \quad t_i \mathbf{v}_P^{i^T}]^T \equiv \\
\equiv \left[\lambda_j^i \left(1 + \frac{v_{ji}^j}{v_P^j}\right) \mathbf{r}_P^T(t_j) \quad \alpha_j^i \left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j}\right) \frac{v_P^i}{v_P^j} t_j \mathbf{v}_P^{j^T} \right] D \bullet \\
\bullet \left[\lambda_j^i \left(1 + \frac{v_{ji}^j}{v_P^j}\right) \mathbf{r}_P^T(t_j) \quad \alpha_j^i \left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j}\right) \frac{v_P^i}{v_P^j} t_j \mathbf{v}_P^{j^T} \right]^T \equiv \\
\equiv [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{v}_P^{j^T}] D [\mathbf{r}_P^T(t_j) \quad t_j \mathbf{v}_P^{j^T}]^T.
\end{gathered}$$

The last identity, $\alpha_j^i \in R^+$ and $\lambda_j^i \in R^+$ yield:

$$\lambda_j^i = \frac{1}{1 + \frac{v_{ji}^j}{v_P^j}}, \tag{23.5}$$

$$\alpha_j^i = \frac{v_P^j}{v_P^i} \frac{1}{1 + \frac{v_{ji}^j v_P^j}{q^j w^j}}. \tag{23.6}$$

The equation (23.5) is the equation (14.11). The proof of the equation (14.12) is analogous. The equations (23.4) and (23.5) imply:

$$\frac{v_P^i}{v_P^j} \frac{\mu_i}{\mu_j} = 1. \tag{23.7}$$

This and (23.6) verify (14.9). Since the point P is arbitrary, then it can also represent a light signal L, in which case (23.7) becomes:

$$\frac{c_i^i}{c_j^j} \frac{\mu_i}{\mu_j} = 1.$$

This and (23.7) prove the equations (14.14). The condition that all the scaling coefficients are positive real numbers, that the speed values $q^{(\cdot)}$ and $w^{(\cdot)}$ are positive real, and the arbitrariness of $\left[v_P^{(\cdot)} \neq 0\right] \in R$ together with (14.9) through (14.12), prove (14.13). The equations (14.9) through (14.12) transform (14.3) through (14.6) into (14.15) through (14.18).

Sufficiency. Let (14.9) through (14.18) be valid. Let μ_i obey (14.2). The equation (23.1) can be transformed as follows by using the relationship between the distance and the speed of the arbitrary point P, (14.2), (14.11), (14.14) and

(23.1) itself:

$$\mathbf{r}_P(t_i) = v_P^j \left(1 + \frac{v_{ji}^j}{v_P^j}\right) \left(1 + \frac{v_{ji}^j}{v_P^j}\right)^{-1} t_j \mathbf{u} = \lambda_j^i \left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}\right].$$

This proves (14.5). The equation (14.6) is proved along the same lines by starting with (23.1). We rearrange (14.2):

$$t_i = \mu_i t = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{v_{ji}^j v_P^j}{q^j w^j}} \left(t_j + \frac{v_{ji}^j v_P^j}{q^j w^j} t_j\right). \tag{23.8}$$

Now, (14.9), (23.1), (23.2) and (23.8) imply:

$$t_i = \alpha_j^i \left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P\left(t_j\right)\right]. \tag{23.9}$$

The equation (23.9) is the equation (14.3). By repeating this procedure applied to (14.2) for i replaced by j, we prove the equation (14.4). We continue with Gaussian transformation, (14.8), by using (14.3), (14.5), (14.11), (23.1), (23.2), and $\mathbf{v}_P^{(.)} = v_P^{(.)} \mathbf{u}$:

$$\begin{gathered}
[\mathbf{r}_P^T(t_i) \quad t_i \mathbf{v}_P^{i^T}] D [\mathbf{r}_P^T(t_i) \quad t_i \mathbf{v}_P^{i^T}]^T \equiv \\
\equiv \left[\lambda_j^i \left(1 + \frac{v_{ji}^j}{v_P^j}\right) \mathbf{r}_P^T(t_j) \quad \alpha_j^i \left(t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)\right) \mathbf{v}_P^{i^T}\right] D \bullet \\
\bullet \left[\lambda_j^i \left(1 + \frac{v_{ji}^j}{v_P^j}\right) \mathbf{r}_P^T(t_j) \quad \alpha_j^i \left(t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)\right) \mathbf{v}_P^{i^T}\right]^T \equiv \\
\equiv \left[\mathbf{r}_P^T(t_j) \quad \alpha_j^i \frac{v_P^i}{v_P^j} \left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j}\right) t_j \mathbf{v}_P^{j^T}\right] D \left[\mathbf{r}_P^T(t_j) \quad \alpha_j^i \frac{v_P^i}{v_P^j} \left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j}\right) t_j \mathbf{v}_P^{j^T}\right]^T,
\end{gathered}$$

which, together with (14.9), yield:

$$[\mathbf{r}_P^T(t_i) \quad t_i \mathbf{v}_P^{i^T}] D [\mathbf{r}_P^T(t_i) \quad t_i \mathbf{v}_P^{i^T}]^T \equiv \left[\mathbf{r}_P^T(t_j) \quad t_j \mathbf{v}_P^{j^T}\right] D \left[\mathbf{r}_P^T(t_j) \quad t_j \mathbf{v}_P^{j^T}\right]^T.$$

This proves (14.8).

Compatibility. We replace $r_P(t_{(.)})$ by $v_P^{(.)} t_{(.)}$ in (14.15) and (14.16), and we eliminate, for example, t_j, which yield:

$$t_i = \frac{\mu_i}{\mu_j} \frac{1}{1 + \frac{v_{ji}^j v_P^j}{q^j w^j}} \left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j}\right) \frac{\mu_j}{\mu_i} \frac{1}{1 - \frac{v_{ji}^i v_P^i}{q^i w^i}} \left(1 - \frac{v_{ji}^i v_P^i}{q^i w^i}\right) t_i = t_i. \tag{23.10}$$

This result proves complete compatibility of (14.15) and (14.16). We replace now $t_{(.)}$ by $\left(v_P^{(.)}\right)^{-1} r_P(t_{(.)})$ in (14.17) and (14.18), and we eliminate, for example,

$\mathbf{r}_P\left(t_j\right)$,

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{1+\frac{v_{ji}^j}{v_P^j}}{1+\frac{v_{ji}^j}{v_P^j}} \frac{1-\frac{v_{ji}^i}{v_P^{o_i,i}}}{1-\frac{v_{ji}^i}{v_P^{o_i,i}}} \mathbf{r}_P\left(t_i\right) \equiv \mathbf{r}_P\left(t_i\right). \tag{23.11}$$

Hence, (14.17) and (14.18) are also completely compatible. Altogether, the transformations (14.15) through (14.18) are completely pairwise compatible. Let us now verify their complete entire compatibility. We eliminate t_j and $r_P\left(t_j\right)$ from (14.15) by utilizing (14.14), (14.16), (14.18), (23.2), and we apply $r_P(t_{(.)}) = v_P^{(.)} t_{(.)}$,

$$t_i \equiv \frac{1}{1+\frac{v_{ji}^j v_P^j}{q^j w^j}}\left(1+\frac{\mu_i}{\mu_j} v_P^i \frac{v_{ji}^j}{q^j w^j}\right) t_i \equiv \frac{1}{1+\frac{v_{ji}^j v_P^j}{q^j w^j}}\left(1+v_P^j \frac{v_{ji}^j}{q^j w^j}\right) t_i \equiv t_i.$$

This proves complete compatibility of (14.15), (14.16) and (14.18). We prove analogously complete compatibility of (14.15) through (14.17). The transformations (14.15) through (14.18) are completely entirely compatible in the temporal domain. In order to verify their entire compatibility in the spatial domain we start with (14.17) and (14.18), and we use afterwards $t_i = \left(v_P^i\right)^{-1} r_P\left(t_i\right)$, (14.14), $r_P\left(t_i\right)\mathbf{u} = \mathbf{r}_P\left(t_i\right)$ and (14.16),

$$\mathbf{r}_P\left(t_i\right) \equiv \frac{\frac{1-\frac{v_{ji}^i}{v_P^i}}{1-\frac{v_{ji}^i}{v_P^i}} + v_{ji}^j \frac{\mu_j}{\mu_i} \frac{1}{v_P^i} \frac{1-\frac{v_{ji}^i}{q^i w^i} v_P^i}{1-\frac{v_{ji}^i v_P^i}{q^i w^i}}}{1+\frac{v_{ji}^j}{v_P^j}} \mathbf{r}_P\left(t_i\right) \equiv \frac{1+\frac{v_{ji}^j}{v_P^j}}{1+\frac{v_{ji}^j}{v_P^j}} \mathbf{r}_P\left(t_i\right) \equiv \mathbf{r}_P\left(t_i\right). \tag{23.13}$$

The transformations (14.16) through (14.18) are completely compatible in view of (23.13). The complete compatibility of (14.15), (14.17) and (14.18) follows in the same way. Altogether, the transformations (14.15) through (14.18) are completely both entirely and pairwise compatible. Q. E. D ■

23.2 Proof of Claim 509

Proof. *Compatibility proof.* Generalized Galilean - Newtonian transformations (14.21), (14.22) imply the following transformations of the velocity of the arbitrary point P:

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P\left(t_i\right)}{dt_i} = \frac{d\mathbf{r}_P\left[t_i(t_j)\right]/dt_j}{dt_i/dt_j} = \frac{d\left[\mathbf{r}_P\left(t_j\right) + v_{ji}^j t_j \mathbf{u}\right]/dt_j}{dt_i/dt_j} = \frac{\mathbf{v}_P^j + \mathbf{v}_{ji}^j}{\mu_i/\mu_j},$$

$$\mathbf{v}_P^j = \frac{d\mathbf{r}_P\left(t_j\right)}{dt_j} = \frac{d\mathbf{r}_P\left[t_j(t_i)\right]/dt_i}{dt_j/dt_i} = \frac{d\left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}\right]/dt_i}{dt_j/dt_i} = \frac{\mathbf{v}_P^i - \mathbf{v}_{ji}^i}{\mu_j/\mu_i},$$

i.e.

$$\mathbf{v}_P^i = \frac{\mu_j}{\mu_i}\left(\mathbf{v}_P^j + \mathbf{v}_{ji}^j\right),\ \mathbf{v}_P^j = \frac{\mu_i}{\mu_j}\left(\mathbf{v}_P^i - \mathbf{v}_{ji}^i\right). \tag{23.14}$$

These equations represent generalized Galilean - Newtonian velocity transformations (8.29) and (8.30). Let us accept $P = O_j$ so that $\mathbf{v}_P^j = \mathbf{v}_{O_j}^j = \mathbf{0}$ and $\mathbf{v}_P^i = \mathbf{v}_{ji}^i$. The equations (23.14) then become, respectively,

$$\mathbf{v}_{ji}^i = \frac{\mu_j}{\mu_i}\ \mathbf{v}_{ji}^j,\ \mathbf{v}_P^j = \mathbf{0}. \tag{23.15}$$

Generalized Galilean - Newtonian transformations (14.21), (14.22) together with (14.2) and $v_{ji}^{(.)}\mathbf{u} \equiv \mathbf{v}_{ji}^{(.)}$ yield

$$\begin{aligned}\mathbf{r}_P(t_i) &\equiv \mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}\ |_{t_j=\mu_j t,\ t_i=\mu_i t,\ \mathbf{r}_P(t_j)=\mathbf{r}_P(t_i)-v_{ji}^i t_i \mathbf{u}} \equiv \\ &\equiv \mathbf{r}_P(t_i) + t_i\left(\frac{\mu_j}{\mu_i}\mathbf{v}_{ji}^j - \mathbf{v}_{ji}^i\right).\end{aligned} \tag{23.16}$$

The first equation (23.15) transforms (23.16) into

$$\mathbf{r}_P(t_i) \equiv \mathbf{r}_P(t_i). \tag{23.17}$$

This identity proves the complete compatibility of generalized Galilean - Newtonian transformations (14.21), (14.22).

Poincaré group proof. We apply (14.2), (14.21), (23.1), (23.2), and (23.14) to the left-hand side of (14.8) for $B = A$ in D,

$$\left[\begin{array}{c}\mathbf{r}_P\,(t_i)\\ t_i\mathbf{v}_P^i\end{array}\right]^T D\left[\begin{array}{c}\mathbf{r}_P\,(t_i)\\ t_i\mathbf{v}_P^i\end{array}\right] \equiv$$

$$\equiv\left[\begin{array}{c}\mathbf{r}_P^T(t_j) + v_{ji}^j t_j \mathbf{u}^T\\ \frac{\mu_i}{\mu_j}t_j\frac{\mu_j}{\mu_i}\left(\mathbf{v}_P^j + \ \mathbf{v}_{ji}^j\right)^T\end{array}\right]^T D\left[\begin{array}{c}\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}\\ \frac{\mu_i}{\mu_j}t_j\frac{\mu_j}{\mu_i}\left(\mathbf{v}_P^j + \ \mathbf{v}_{ji}^j\right)\end{array}\right] \equiv$$

$$\equiv\left[\begin{array}{c}\left[r_P(t_j) + v_{ji}^j t_j\right]\mathbf{u}^T\\ \left(v_P^j t_j + \ v_{ji}^j t_j\right)\mathbf{u}^T\end{array}\right]^T\left(\begin{array}{cc}A & O\\ O & -A\end{array}\right)\left[\begin{array}{c}\left[r_P(t_j) + v_{ji}^j t_j\right]\mathbf{u}\\ \left(v_P^j t_j + \ v_{ji}^j t_j\right)\mathbf{u}\end{array}\right] \equiv$$

$$\equiv\left[\left[r_P(t_j) + v_{ji}^j t_j\right]^2 - \left(v_P^j t_j + \ v_{ji}^j t_j\right)^2\right]\mathbf{u}^T A\mathbf{u} \equiv$$

$$\equiv\left[\begin{array}{c}\left[r_P^2(t_j) + 2r_P(t_j)v_{ji}^j t_j + \left(v_{ji}^j t_j\right)^2\right] -\\ -\left[\left(v_P^j t_j\right)^2 + 2r_P(t_j)v_{ji}^j t_j + \ \left(v_{ji}^j t_j\right)^2\right]\end{array}\right]\mathbf{u}^T A\mathbf{u} \equiv$$

$$\equiv\left[r_P^2(t_j) - \left(v_P^j t_j\right)^2\right]\mathbf{u}^T A\mathbf{u} \equiv \left[\begin{array}{c}r_P(t_j)\mathbf{u}\\ v_P^j t_j\mathbf{u}\end{array}\right]^T\left(\begin{array}{cc}A & O\\ O & -A\end{array}\right)\left[\begin{array}{c}r_P(t_j)\mathbf{u}\\ v_P^j t_j\mathbf{u}\end{array}\right] \equiv$$

$$\equiv\left[\begin{array}{c}r_P(t_j)\mathbf{u}\\ v_P^j t_j\mathbf{u}\end{array}\right]^T D\left[\begin{array}{c}r_P(t_j)\mathbf{u}\\ v_P^j t_j\mathbf{u}\end{array}\right].$$

This proves the identity (14.8). Hence, (14.2), (14.21), (14.22) form the Poincaré group. Q. E. D ■

23.3 Proof of Theorem 517

Proof. Let the *time* scaling coefficients $\mu_i \in R^+$ be defined by (14.2). Let the scaling coefficients α_j^i and α_i^j be equal: $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, as well as λ_j^i and λ_i^j, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$. Let $B \neq A$ be permitted in D, (14.8).

Necessity. Let the arbitrary point P move with an arbitrary nonzero constant velocity $\mathbf{v}_P^{(.)} = v_P^{(.)}\mathbf{u}$. Let the scaling coefficients λ_{ij} and α_{ij} obey (14.3) through (14.7), and let (14.2) through (14.7) imply (14.8). At first we replace $\mathbf{r}_P(t_j)$ by the right-hand side of (14.6) into (14.5) and apply (23.1) and (23.2) so that:

$$\mathbf{r}_P(t_i) \equiv \lambda_{ij}\left\{\lambda_{ij}[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}] + \frac{v_{ji}^j}{v_P^j}\mathbf{r}_P(t_j)\right\} \equiv$$
$$\equiv \lambda_{ij}^2\left(1 - \frac{v_{ji}^i}{v_P^i}\right)\left(1 + \frac{v_{ji}^j}{v_P^j}\right)\mathbf{r}_P(t_i),$$

which, together with $\lambda_{ij} \in R^+$, results in:

$$\lambda_{ij} = \frac{1}{\sqrt{\left(1 - \frac{v_{ji}^i}{v_P^i}\right)\left(1 + \frac{v_{ji}^j}{v_P^j}\right)}}. \tag{23.18}$$

Now, we replace t_j by the right-hand side of (14.4) into (14.3), and apply $r_P\left(t_{(.)}\right) = v_P^{(.)} t_{(.)}$, (23.1) and (23.2):

$$t_i = \alpha_{ij}\left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)\right] = \alpha_{ij}^2\left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i)\right] + \alpha_{ij}\frac{v_{ji}^j v_P^j}{q^j w^j} t_j =$$
$$= \alpha_{ij}^2\left(1 - \frac{v_{ji}^i v_P^i}{q^i w^i}\right)\left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j}\right) t_i.$$

This and positivity of α_{ij} imply:

$$\alpha_{ij} = \frac{1}{\sqrt{\left(1 - \frac{v_{ji}^i v_P^i}{q^i w^i}\right)\left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j}\right)}}. \tag{23.19}$$

The equations (14.3) through (14.6), (14.8), (23.1), (23.2) and $D = blockdiag\{A - B\}$ enable the following (since $A \neq B$ is permitted):

$$[\mathbf{r}_P^T(t_i) \quad t_i \mathbf{v}_P^{i^T}] D [\mathbf{r}_P^T(t_i) \quad t_i \mathbf{v}_P^{i^T}]^T \equiv$$

$$\equiv \lambda_{ij}^2 \begin{bmatrix} \left[r_P(t_j) + v_{ji}^j t_j\right]\mathbf{u} \\ \frac{\alpha_{ij}}{\lambda_{ij}}\left[v_P^i t_j + \frac{v_{ji}^j v_P^i}{q^j w^j} r_P(t_j)\right]\mathbf{u} \end{bmatrix}^T \begin{pmatrix} A & O \\ O & -B \end{pmatrix} \bullet$$

$$\bullet \begin{bmatrix} \left[r_P(t_j) + v_{ji}^j t_j\right]\mathbf{u} \\ \frac{\alpha_{ij}}{\lambda_{ij}}\left[v_P^i t_j + \frac{v_{ji}^j v_P^i}{q^j w^j} r_P(t_j)\right]\mathbf{u} \end{bmatrix} \equiv$$

$$\equiv \lambda_{ij} \begin{bmatrix} \sqrt{\begin{matrix} \left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{ji}^j v_P^i}{q^j w^j}\right)^2 \frac{\mathbf{u}^T B\mathbf{u}}{\mathbf{u}^T A\mathbf{u}}\right] r_P(t_j) + \\ +2v_{ji}^j\left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_P^i}{\sqrt{q^j w^j}}\right)^2 \frac{\mathbf{u}^T B\mathbf{u}}{\mathbf{u}^T A\mathbf{u}}\right] r_P(t_j)\, t_j \end{matrix}}\;\mathbf{u} \\ \sqrt{\left[\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_P^i}{v_P^j}\right)^2 \frac{\mathbf{u}^T B\mathbf{u}}{\mathbf{u}^T A\mathbf{u}} - \left(\frac{v_{ji}^j}{v_P^j}\right)^2\right]} t_j v_P^j \mathbf{u} \end{bmatrix} \begin{pmatrix} A & O \\ O & -B \end{pmatrix} \bullet$$

$$\bullet \lambda_{ij} \begin{bmatrix} \sqrt{\begin{matrix} \left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_{ji}^j v_P^i}{q^j w^j}\right)^2 \frac{\mathbf{u}^T B\mathbf{u}}{\mathbf{u}^T A\mathbf{u}}\right] r_P(t_j) + \\ +2v_{ji}^j\left[1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_P^i}{\sqrt{q^j w^j}}\right)^2 \frac{\mathbf{u}^T B\mathbf{u}}{\mathbf{u}^T A\mathbf{u}}\right] r_P(t_j)\, t_j \end{matrix}}\;\mathbf{u} \\ \sqrt{\left[\left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_P^i}{v_P^j}\right)^2 \frac{\mathbf{u}^T B\mathbf{u}}{\mathbf{u}^T A\mathbf{u}} - \left(\frac{v_{ji}^j}{v_P^j}\right)^2\right]} t_j v_P^j \mathbf{u} \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} r_P(t_j)\mathbf{u} \\ t_j v_P^j \mathbf{u} \end{bmatrix}^T D \begin{bmatrix} r_P(t_j)\mathbf{u} \\ t_j v_P^j \mathbf{u} \end{bmatrix}. \tag{23.20}$$

The last identity implies:

a)

$$1 - \left(\frac{\alpha_{ij}}{\lambda_{ij}}\frac{v_P^i}{\sqrt{q^j w^j}}\right)^2 \frac{\mathbf{u}^T B\mathbf{u}}{\mathbf{u}^T A\mathbf{u}} = 0 \iff \frac{\alpha_{ij}}{\lambda_{ij}} = \frac{\sqrt{q^j w^j}}{v_P^i}\sqrt{\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T B\mathbf{u}}} \iff \tag{23.21}$$

$$\iff \alpha_{ij} = \lambda_{ij}\frac{\sqrt{q^j w^j}}{v_P^i}\sqrt{\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T B\mathbf{u}}}, \tag{23.22}$$

b)

$$\lambda_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2}}, \tag{23.23}$$

c)

$$\lambda_{ij} = \frac{1}{\sqrt{1 - \left(\frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2 \frac{\sqrt{q^j w^j}}{v_P^j}}}.$$

This should be identically equal to (23.23),

$$\lambda_{ij} \;=\frac{1}{\sqrt{1-\left(\frac{v^j_{ji}}{\sqrt{q^j w^j}}\right)^2}} \equiv \frac{1}{\sqrt{1-\left(\frac{v^j_{ji}}{\sqrt{q^j w^j}}\right)^2 \frac{\sqrt{q^j w^j}}{v^j_P}}}.$$

Hence,

$$\sqrt{q^j w^j} = v^j_P, \tag{23.24}$$

which is the second equation in (14.24). The preceding equations and (23.18) result in

$$\lambda_{ij} = \frac{1}{\sqrt{\left(1-\frac{v^i_{ji}}{v^i_P}\right)\left(1+\frac{v^j_{ji}}{v^j_P}\right)}} \equiv \frac{1}{\sqrt{1-\left(\frac{v^j_{ji}}{v^j_P}\right)^2}}, \tag{23.25}$$

so that

$$\frac{v^i_{ji}}{v^i_P} = \frac{v^j_{ji}}{v^j_P}. \tag{23.26}$$

These equations and (23.18) prove the first two equations in (14.27). The equations (23.22) through (23.24) imply:

$$\alpha_{ij} = \frac{v^j_P}{v^i_P}\frac{\sqrt{\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T B\mathbf{u}}}}{\sqrt{\left[1-\left(\frac{v^j_{ji}}{v^j_P}\right)^2\right]}},$$

or, together with (23.19), (23.24) and (23.26), result in:

$$\alpha_{ij} = \frac{v^j_P}{v^i_P}\frac{\sqrt{\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T B\mathbf{u}}}}{\sqrt{\left(1-\frac{v^i_{ji}}{v^i_P}\right)\left(1+\frac{v^j_{ji}}{v^j_P}\right)}} \equiv \frac{1}{\sqrt{\left(1-\frac{v^i_{ji}v^i_P}{q^i w^i}\right)\left(1+\frac{v^j_{ji}}{v^j_P}\right)}} \Longleftrightarrow$$

$$\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T B\mathbf{u}} = \left(\frac{v^i_P}{v^j_P}\right)^2, \tag{23.27}$$

$$\frac{v^j_P}{v^i_P}\frac{\sqrt{\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T B\mathbf{u}}}}{\sqrt{1-\frac{v^i_{ji}}{v^i_P}}} \equiv \frac{1}{\sqrt{1-\frac{v^i_{ji}v^i_P}{q^i w^i}}} \Longleftrightarrow v^i_P = \sqrt{q^i w^i}. \tag{23.28}$$

The last equation is the first equation in (14.24). Hence,

$$\alpha_{ij} = \frac{1}{\sqrt{1-\left(\frac{v^i_{ji}}{\sqrt{q^i w^i}}\right)^2}} \equiv \frac{1}{\sqrt{1-\left(\frac{v^j_{ji}}{\sqrt{q^j w^j}}\right)^2}}.$$

These are the equations (14.26). They, (23.24), (23.28) and the first two equations in (14.27) prove the third equation in (14.27). Since the speed of the arbitrary point P is also arbitrary (different from zero), then it can be equal to the light speed so that (23.26) becomes:

$$\frac{v_{ji}^i}{v_{ji}^j} \equiv \frac{v_P^i}{v_P^j} \equiv \frac{c_i^i}{c_j^j}, \tag{23.29}$$

which are the equations (14.25). We deduce the following from (23.27) and (23.29):

$$\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T B\mathbf{u}} = \left(\frac{v_{ji}^i}{v_{ji}^j}\right)^2 = \left(\frac{v_P^i}{v_P^j}\right)^2 = \left(\frac{c_i^i}{c_j^j}\right)^2, \tag{23.30}$$

which, together with (14.24), prove (14.28), and (14.35) for $A = B$ in D. Positivity of α_{ij} and λ_{ij}, together with (14.26) and (14.27), proves (14.29). We continue by combining (14.2) and (14.3) with (23.1):

$$t_i = \mu_i t = \alpha_{ij}\left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P\left(t_j\right)\right] = \alpha_{ij}\mu_j\left(1 + \frac{v_{ji}^j v_P^j}{q^j w^j}\right)t,$$

which, together with (14.24) through (14.26), shows that:

$$\mu_j = \mu_i \frac{\sqrt{\left(1 - \frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)\left(1 + \frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)}}{\sqrt{\left(1 + \frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2}} =$$

$$= \mu_i \sqrt{\frac{1 - \frac{v_{ji}^j}{\sqrt{q^j w^j}}}{1 + \frac{v_{ji}^j}{\sqrt{q^j w^j}}}} = \mu_i \sqrt{\frac{1 - \frac{v_{ji}^j}{v_P^j}}{1 + \frac{v_{ji}^j}{v_P^j}}} = \mu_i \sqrt{\frac{1 - \frac{v_{ji}^i}{v_P^i}}{1 + \frac{v_{ji}^i}{v_P^i}}} = \mu_i \sqrt{\frac{1 - \frac{v_{ji}^i}{\sqrt{q^i w^i}}}{1 + \frac{v_{ji}^i}{\sqrt{q^i w^i}}}}.$$

This proves (14.30). The equations (14.3) through (14.6) together with (14.24) through (14.27) imply directly (14.31) through (14.34). The equations (14.35) transform (14.31) through (14.34) into (14.36) through (14.39). The real value of the space scaling coefficient in (14.39) implies $v_{Pji} > v_{ji}^i$ in (14.35).

Sufficiency. Let $A = B$ in D, (14.8). Let the *time* scaling coefficients μ_i be defined by (14.2). Let the equations (14.24) through (14.30), and the relationships (14.35) through (14.39) hold for any choice of the *time* scaling coefficient $\mu_i \in R^+$. The equations (14.26) and (14.27) can be transformed as follows:

$$\alpha_{ij}^2\left(1 - \frac{v_{ji}^i}{\sqrt{q^i w^i}}\right)\left(1 + \frac{v_{ji}^i}{\sqrt{q^i w^i}}\right) = \alpha_{ij}^2\left(1 - \frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)\left(1 + \frac{v_{ji}^j}{\sqrt{q^j w^j}}\right) = 1,$$

$$\lambda_{ij}^2\left(1 - \frac{v_{ji}^i}{v_P^i}\right)\left(1 + \frac{v_{ji}^i}{v_P^i}\right) = \lambda_{ij}^2\left(1 - \frac{v_{ji}^j}{v_P^j}\right)\left(1 + \frac{v_{ji}^j}{v_P^j}\right) = 1.$$

They, (23.1), (23.2), and (14.35) permit the following transformations:

$$t_i = \alpha_{ij}\left[\alpha_{ij}\left(t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i)\right) + \lambda_{ij}\frac{v_{ji}^j}{q^j w^j}\frac{v_P^j}{v_P^i}\left(r_P(t_i) - v_{ji}^i t_i\right)\right], \quad (23.31)$$

$$t_j = \alpha_{ij}\left[\alpha_{ij}\left(t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)\right) - \lambda_{ij}\frac{v_{ji}^i}{q^i w^i}\frac{v_P^i}{v_P^j}\left(r_P(t_j) + v_{ji}^j t_j\right)\right], \quad (23.32)$$

$$\mathbf{r}_P(t_i) = \lambda_{ij}\left\{\lambda_{ij}\left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}\right] + v_{ji}^j \frac{v_P^i}{v_P^j}\alpha_{ij}\left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i)\right]\mathbf{u}\right\}, \quad (23.33)$$

$$\mathbf{r}_P(t_j) = \lambda_{ij}\left\{\lambda_{ij}\left[\mathbf{r}_P(t_j) + v_{ji}^j t_j \mathbf{u}\right] - v_{ji}^i \frac{v_P^j}{v_P^i}\alpha_{ij}\left[t_j + \frac{v_{ji}^j}{q^j w^j} r_P(t_j)\right]\mathbf{u}\right\}, \quad (23.34)$$

These equations suggest

$$t_j = \alpha_{ij}\left[t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i)\right] = \frac{t_i - \frac{v_{ji}^i}{q^i w^i} r_P(t_i)}{\sqrt{1 - \left(\frac{v_{ji}^i}{\sqrt{q^i w^i}}\right)^2}}. \quad (23.35)$$

and

$$\mathbf{r}_P(t_j) = \lambda_{ij}\left[\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}\right] = \frac{\mathbf{r}_P(t_i) - v_{ji}^i t_i \mathbf{u}}{\sqrt{1 - \left(\frac{v_{ji}^i}{v_P^i}\right)^2}}, \quad (23.36)$$

The equations (23.35) and (23.36) transform (23.31) through (23.34) into (14.3) through (14.6), due to $\alpha_j^i = \alpha_i^j = \alpha_{ij} = \alpha_{ji}$, $\lambda_j^i = \lambda_i^j = \lambda_{ij} = \lambda_{ji}$, $r_P\left(t_{(.)}\right) \equiv v_P^{(.)} t_{(.)}$ and $t_{(.)} \equiv \left(v_P^{(.)}\right)^{-1} r_P\left(t_{(.)}\right)$. Hence, the equations (23.35) and (23.36) are well specified. Since the equations (14.3) through (14.6), hold, then the proof of the identity (14.8) is inherently the same as in the sufficiency proof of Theorem 501.

Compatibility. Let $A = B$ in D, (14.8). Let us eliminate, for example, t_i from (14.36) and (14.37) by using (14.35) and (23.1) for an arbitrary speed $v_P^{(.)} = v_{Pji}$ of the arbitrary point P:

$$t_j \equiv \frac{1 - \frac{v_{ji}}{(qw)_{ji}} v_{Pji}}{\sqrt{1 - \left(\frac{v_{ji}}{\sqrt{(qw)_{ji}}}\right)^2}} \frac{1 + \frac{v_{ji}}{(qw)_{ji}} v_{Pji}}{\sqrt{1 - \left(\frac{v_{ji}}{\sqrt{(qw)_{ji}}}\right)^2}} t_j \Longleftrightarrow v_P^i = v_P^j = v_{Pji} = \sqrt{(qw)_{ji}}.$$

This proves partial compatibility of (14.36) and (14.37). Let us now eliminate, for example, $\mathbf{r}_P(t_j)$ from (14.38) and (14.39) by applying (23.2) and $t_{(.)} = r_P / v_{Pji}$:

$$\mathbf{r}_P(t_i) \equiv \frac{1 + \frac{v_{ji}}{v_{Pji}}}{\sqrt{1 - \left(\frac{v_{ji}}{v_{Pji}}\right)^2}} \frac{1 - \frac{v_{ji}}{v_{Pji}}}{\sqrt{1 - \left(\frac{v_{ji}}{v_{Pji}}\right)^2}} \mathbf{r}_P(t_i) \equiv \mathbf{r}_P(t_i).$$

This proves the compatibility of (14.38) and (14.39), which is partial due to (14.35). Altogether, we have verified the partial pairwise compatibility of (14.36) through (14.39). Let us eliminate all the coordinates with the subscript j from (14.36) by the help of (14.37), (14.39), and $v_{Pji} = \sqrt{(qw)_{ji}}$, (14.35):

$$t_i \equiv \frac{\frac{1-\frac{v_{ji}}{(qw)_{ji}}v_{Pji}}{\sqrt{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}} + \frac{v_{ji}}{(qw)_{ji}}v_{Pji}\frac{1-\frac{v_{ji}}{v_{Pji}}}{\sqrt{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}}}{\sqrt{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}}t_i \equiv \frac{1-\frac{v_{ji}^2}{v_{Pji}^2}}{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}t_i \equiv t_i.$$

This result shows that the transformations (14.36), (14.37) and (14.39) are partially compatible due to the restriction $v_{Pji} = \sqrt{(qw)_{ji}}$, (14.35). By repeating the procedure, we prove the same for (14.36) through (14.38). The transformations (14.36) through (14.39) are partially entirely compatible in the temporal domain. Let us eliminate all the coordinates with the subscript j from (14.38) by the help of (14.37), (14.39), and $v_{Pji} = \sqrt{(qw)_{ji}}$:

$$\begin{aligned}\mathbf{r}_P(t_i) &\equiv \frac{\frac{1-\frac{v_{ji}}{v_{Pji}}}{\sqrt{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}} + \frac{v_{ji}}{v_{Pji}}\frac{1-\frac{v_{ji}}{v_{Pji}}}{\sqrt{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}}}{\sqrt{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}}\mathbf{r}_P(t_i) \equiv \\ &\equiv \frac{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}{1-\left(\frac{v_{ji}}{v_{Pji}}\right)^2}\mathbf{r}_P(t_i) \equiv \mathbf{r}_P(t_i).\end{aligned} \tag{23.38}$$

The coordinate transformations (14.37), (14.38) and (14.39) are partially compatible due to (23.38) and the restriction $v_{Pji} = \sqrt{(qw)_{ji}}$. In the same manner we prove the partial compatibility of (14.36), (14.38) and (14.39). The transformations (14.36) through (14.39) are partially compatible in the spatial domain. They are partially entirely compatible. Q. E. D ■

23.4 Proof of Theorem 527

Proof. *Necessity and sufficiency.* Under the conditions of the theorem statement, the equations (14.36) through (14.39) hold. They lead to

$$\mathbf{v}_P^i = \frac{d\mathbf{r}_P(t_i)}{dt_i} = \frac{d\left[\frac{\mathbf{r}_P(t_j)+v_{ji}^j t_j \mathbf{u}}{\sqrt{1-\left(\frac{v_{ji}^j}{v_P^j}\right)^2}}\right] : dt_j}{d\left[\frac{t_j+\frac{v_{ji}^j}{q^j w^j} r_P(t_j)}{\sqrt{1-\left(\frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2}}\right] : dt_j} =$$

$$= \frac{\frac{\mathbf{v}_P^j+\mathbf{v}_{ji}}{\sqrt{1-\left(\frac{v_{ji}^j}{v_P^j}\right)^2}}}{\frac{1+\frac{v_{ji}^j}{q^j w^j}v_P^j}{\sqrt{1-\left(\frac{v_{ji}^j}{\sqrt{q^j w^j}}\right)^2}}} = \frac{\frac{\mathbf{v}_P^j+\mathbf{v}_{ji}^j}{\sqrt{1-\left(\frac{v_{ji}^j}{v_P^j}\right)^2}}}{\frac{1+\frac{v_{ji}^j}{v_P^j}}{\sqrt{1-\left(\frac{v_{ji}^j}{v_P^j}\right)^2}}} = \frac{\mathbf{v}_P^j+\mathbf{v}_{ji}^j}{1+\frac{v_{ji}^j}{v_P^j}} = \frac{1+\frac{v_{ji}^j}{v_P^j}}{1+\frac{v_{ji}^j}{v_P^j}}\mathbf{v}_P^j = \mathbf{v}_P^j = \mathbf{v}_{Pij}.$$

This result proves (14.53).

Compatibility. The speed transformations are reduced to the unity transformation under the condition (14.35). They are partially compatible due to the restriction $v_{Pij} = \sqrt{(qw)_{ji}}$, (14.35). Q. E. D ■

23.5 Proof of Theorem 564

Proof. Let the scaling diagonal matrix coefficient $\mathrm{M}_i \in R_+^{nxn}$, $diagmin\mathrm{M}_i \in R^+$, obey (15.7). The following is true:

$$\mathbf{r}_{(..)}(\mathbf{t}_{(.)}^n) = \mathrm{V}_{(..)}^{(.)}\mathrm{T}_{(.)}\mathbf{u},\ (..) \in \{G, L, P, P_R, P_{SU}\}. \tag{23.39}$$

The position vector can be expressed also in terms of its matrix algebraic value,

$$\mathbf{r}_{(..)}(\mathbf{t}_{(.)}^n) = \mathrm{R}_{(..)}\left(\mathbf{t}_{(.)}^n\right)\mathbf{u},\ (..) \in \{G, L, P, P_R, P_{SU}\}. \tag{23.40}$$

Necessity. Let the scaling diagonal matrix coefficients

$$\mathrm{A}_j^i \in R_+^{n\mathrm{x}n},\ diagmin\mathrm{A}_j^i \in R^+,\ \mathrm{A}_i^j \in R_+^{n\mathrm{x}n},\ diagmin\mathrm{A}_i^j \in R^+,\ \mathrm{A}_j^i \neq \mathrm{A}_i^j,$$
$$\Lambda_j^i \in R_+^{n\mathrm{x}n},\ diagmin\Lambda_j^i \in R^+,\ \Lambda_i^j \in R_+^{n\mathrm{x}n},\ diagmin\Lambda_i^j \in R^+,\ \Lambda_j^i \neq \Lambda_i^j,$$

be *time*-independent, diagonally elementwise positive real valued and let they obey (15.8) through (15.12) in the general case so that they, together with (15.7), imply (15.13). The equations (15.7), (23.39) and (23.40) give the next matrix forms to the equations (15.10) and (15.11):

$$\mathbf{r}_P(\mathbf{t}_i^n) = \Lambda_j^i[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j \mathbf{t}_j^n] = \Lambda_j^i \left(\mathrm{V}_P^j + \mathrm{V}_{ji}^j\right) \mathbf{t}_j^n =$$
$$= \Lambda_j^i \mathrm{V}_P^j \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right] \mathrm{M}_j \mathrm{T}\mathbf{u} = \mathrm{V}_P^i \mathbf{t}_i^n = \mathrm{V}_P^i \mathrm{T}_i \mathbf{u} = \mathrm{V}_P^i \mathrm{M}_i \mathrm{T}\mathbf{u}, \quad (23.41)$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \Lambda_i^j[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}_{ji}^i \mathbf{t}_i^n] = \Lambda_i^j \left(\mathrm{V}_P^i - \mathrm{V}_{ji}^i\right) \mathbf{t}_i^n =$$
$$= \Lambda_i^j \mathrm{V}_P^i \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right] \mathrm{M}_i \mathrm{T}\mathbf{u} = \mathrm{V}_P^j \mathbf{t}_j^n = \mathrm{V}_P^j \mathrm{T}_j \mathbf{u} = \mathrm{V}_P^j \mathrm{M}_j \mathrm{T}\mathbf{u}. \quad (23.42)$$

The solution of (23.41) for Λ_j^i is

$$\Lambda_j^i = \mathrm{V}_P^i \mathrm{M}_i \left(\mathrm{V}_P^j \mathrm{M}_j^{-1}\right)^{-1} \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}. \quad (23.43)$$

By solving the equation (23.42) for Λ_i^j we get

$$\Lambda_i^j = \mathrm{V}_P^j \mathrm{M}_j \left(\mathrm{V}_P^i \mathrm{M}_i\right)^{-1} \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right]^{-1}. \quad (23.44)$$

These results demand elementwise nonzero vector value of the velocity of the arbitrary point P since the scaling matrix coefficients are well defined. By combining (15.7), (15.8), $\mathbf{r}_P(\mathbf{t}_j^n) = \mathrm{V}_P^j \mathbf{t}_j^n = \mathrm{V}_P^j \mathrm{T}_j \mathbf{u}$ and (23.40) we find

$$\mathbf{t}_i^n = \mathrm{T}_i \mathbf{u} = \mathrm{A}_j^i \left[I + \mathrm{V}_{ji}^j \mathrm{V}_P^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right] (\mathrm{M}_i)^{-1} \mathrm{M}_j \mathrm{T}_i \mathbf{u}.$$

This equation implies

$$\mathrm{A}_j^i = \mathrm{M}_i \mathrm{M}_j^{-1} \left[I + \mathrm{V}_{ji}^j \mathrm{V}_P^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right]^{-1}.$$

This proves the equation (15.14). The equation (15.15) is proved along the same lines. *Time*-independence of A_j^i implies *time*-independence of $\mathrm{V}_P^j(..)$, hence

$$\mathrm{V}_P^{(.)}(..) = \mathrm{V}_P^{(.)} = CONST. \quad (23.45)$$

The equations (15.8), (15.10), (15.13) and (23.40) imply the following:

$$\begin{bmatrix}\mathbf{r}_P^T(\mathbf{t}_i^n) & (\mathbf{t}_i^n)^T\mathrm{V}_P^i\end{bmatrix} D \begin{bmatrix}\mathbf{r}_P^T(\mathbf{t}_i^n) & (\mathbf{t}_i^n)^T\mathrm{V}_P^i\end{bmatrix}^T \equiv$$

$$\equiv \begin{bmatrix} \Lambda_j^i\left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathbf{r}_P(\mathbf{t}_j^n) \\ \mathrm{A}_j^i\left[I+\mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathrm{V}_P^i\mathbf{t}_j^n \end{bmatrix}^T D\bullet$$

$$\bullet \begin{bmatrix} \Lambda_j^i\left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathbf{r}_P(\mathbf{t}_j^n) \\ \mathrm{A}_j^i\left[I+\mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathrm{V}_P^i\mathbf{t}_j^n \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} \Lambda_j^i\left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathbf{r}_P(\mathbf{t}_j^n) \\ \mathrm{A}_j^i\left[I+\mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathrm{V}_P^i\left(\mathrm{V}_P^j\right)^{-1}\mathrm{V}_P^j\mathbf{t}_j^n \end{bmatrix}^T D\bullet$$

$$\bullet \begin{bmatrix} \Lambda_j^i\left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathbf{r}_P(\mathbf{t}_j^n) \\ \mathrm{A}_j^i\left[I+\mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathrm{V}_P^i\left(\mathrm{V}_P^j\right)^{-1}\mathrm{V}_P^j\mathbf{t}_j^n \end{bmatrix} \equiv$$

$$\equiv [\mathbf{r}_P^T(\mathbf{t}_j^n) \quad \mathbf{t}_j^{nT}\mathrm{V}_P^j]D[\mathbf{r}_P^T(\mathbf{t}_j^n) \quad \mathbf{t}_j^{nT}\mathrm{V}_P^j]^T. \tag{23.46}$$

These identities yield:

$$\Lambda_j^i = \left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}, \tag{23.47}$$

$$\mathrm{A}_j^i = \left(\mathrm{V}_P^i\right)^{-1}\mathrm{V}_P^j\left[I+\mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}. \tag{23.48}$$

The equation (23.47) is the equation (15.16). The proof of the equation (15.17) is analogous. The equations (23.43) and (23.47) imply:

$$\mathrm{V}_P^i\mathrm{M}_i\left(\mathrm{V}_P^j\mathrm{M}_j\right)^{-1} = I,\ \mathrm{V}_P^j\mathrm{M}_j\left(\mathrm{V}_P^i\mathrm{M}_i\right)^{-1} = I. \tag{23.49}$$

Since the point P is arbitrary, then it can represent also a light signal in which case the equations (23.49) become:

$$\begin{gathered}\mathrm{C}_i^i\mathrm{M}_i\left(\mathrm{C}_j^j\mathrm{M}_j\right)^{-1} = I, \quad \mathrm{C}_j^j\mathrm{M}_j\left(\mathrm{C}_i^i\mathrm{M}_i\right)^{-1} = I \Longrightarrow \\ \mathrm{M}_i\mathrm{M}_j^{-1} \equiv \left(\mathrm{C}_i^i\right)^{-1}\mathrm{C}_j^j = CONST., \\ \mathrm{M}_i^{-1}\mathrm{M}_j \equiv \mathrm{C}_i^i(\mathrm{C}_j^j)^{-1} = CONST.\end{gathered} \tag{23.50}$$

This and (23.49) complete the proof of (15.19). The condition that all the matrix scaling coefficients are diagonally elementwise constant and positive real valued, together with (15.14) through (15.17), proves (15.18). The equations (15.14) through (15.17) transform (15.8) through (15.11) into (15.20) through (15.23).

Sufficiency. Let (15.14) through (15.23) be valid. Let M_i obey (15.7). The equation (23.39) can be transformed as follows by using (15.7), (15.16), (15.19) and (23.39) itself,

$$\mathbf{r}_P\left(\mathbf{t}_i^n\right) = \mathrm{V}_P^i\mathbf{t}_i^n = \left(\mathrm{M}_i\right)^{-1}\mathrm{M}_j\mathrm{V}_P^j\mathbf{t}_i^n = \mathrm{V}_P^j\mathbf{t}_j^n =$$
$$= \left[I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}\left[I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathrm{V}_P^j\mathbf{t}_j^n =$$
$$= \Lambda_j^i\left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j\mathbf{t}_j^n\right]. \tag{23.51}$$

This proves (15.10). We use the same method to prove the equation (15.11) by starting with (23.39). We rearrange (15.7):

$$\mathbf{t}_i^n = \mathrm{M}_i\mathbf{t}^n =$$
$$= \mathrm{M}_i\mathrm{M}_j^{-1}\left[I + \mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\left[I + \mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathbf{t}_j^n =$$
$$= \mathrm{M}_i\mathrm{M}_j^{-1}\left[I + \mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\left[\mathbf{t}_j^n + \mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathbf{t}_j^n\right]. \tag{23.52}$$

Now, (15.14), (23.39) and (23.52) imply:

$$\mathbf{t}_i^n = \mathrm{A}_j^i[\mathbf{t}_j^n + \mathrm{V}_{ji}^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathbf{r}_P(\mathbf{t}_j^n)]. \tag{23.53}$$

This is the equation (15.8). By repeating this procedure applied to (15.7) for i replaced by j, we prove the equation (15.9). We continue with Gaussean transformation, (15.13), by using (15.7), (15.10), (15.16), (15.19), (15.20), (23.39), (23.40) and $\mathbf{v}_P^{(.)} = \mathrm{V}_P^{(.)}\mathbf{u}$,

$$[\mathbf{r}_P^T(\mathbf{t}_i^n) \quad (\mathbf{t}_i^n)^T\mathrm{V}_P^i]D[\mathbf{r}_P^T(\mathbf{t}_i^n) \quad (\mathbf{t}_i^n)^T\mathrm{V}_P^i]^T \equiv$$
$$\equiv \begin{bmatrix} \Lambda_j^i\left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j\mathbf{t}_j^n\right] \\ \mathrm{V}_P^i\mathbf{t}_i^n \end{bmatrix}^T D \begin{bmatrix} \Lambda_j^i\left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j\mathbf{t}_j^n\right] \\ \mathrm{V}_P^i\mathbf{t}_i^n \end{bmatrix} \equiv$$
$$\equiv \begin{bmatrix} \Lambda_j^i\left[I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathbf{r}_P(\mathbf{t}_j^n) \\ \mathrm{V}_P^i\mathrm{M}_i\mathrm{M}_j^{-1}\left\{\begin{array}{c}\left[I + \mathrm{V}_{ji}^j\mathrm{V}_P^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\right]^{-1}\bullet \\ \bullet\left[I + \mathrm{V}_{ji}^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\mathrm{V}_P^j\right]\end{array}\right\}\mathbf{t}_j^n \end{bmatrix}^T D\bullet$$
$$\bullet \begin{bmatrix} \Lambda_j^i\left[I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathbf{r}_P(\mathbf{t}_j^n) \\ \mathrm{V}_P^i\mathrm{M}_i\mathrm{M}_j^{-1}\left\{\begin{array}{c}\left[I + \mathrm{V}_{ji}^j\mathrm{V}_P^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\right]^{-1}\bullet \\ \bullet\left[I + \mathrm{V}_{ji}^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\mathrm{V}_P^j\right]\end{array}\right\}\mathbf{t}_j^n \end{bmatrix} \equiv$$
$$\equiv \left[\mathbf{r}_P^T(\mathbf{t}_j^n) \quad \left(\mathbf{t}_j^n\right)^T\mathrm{V}_P^j\right] D \left[\mathbf{r}_P^T(\mathbf{t}_j^n) \quad \left(\mathbf{t}_j^n\right)^T\mathrm{V}_P^j\right]^T.$$

This proves (15.13).

Compatibility. We will replace $\mathbf{r}_P(\mathbf{t}^n_{(.)})$ by $\mathrm{V}^{(.)}_P\mathbf{t}^n_{(.)}$ in (15.20) and (15.21),

$$\mathbf{t}^n_i \equiv \mathrm{M}_i\mathrm{M}_j^{-1}\left\{\begin{array}{c}\left[I+\mathrm{V}^j_{ji}\mathrm{V}^j_P(\mathrm{Q}^j\mathrm{W}^j)^{-1}\right]^{-1}\bullet \\ \bullet\left[I+\mathrm{V}^j_{ji}\mathrm{V}^j_P(\mathrm{Q}^j\mathrm{W}^j)^{-1}\right]\end{array}\right\}\mathbf{t}^n_j \equiv \mathrm{M}_i\mathrm{M}_j^{-1}\mathbf{t}^n_j,$$

$$\mathbf{t}^n_j \equiv \mathrm{M}_i^{-1}\mathrm{M}_j\left\{\begin{array}{c}\left[I-\mathrm{V}^i_{ji}\mathrm{V}^i_P\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]^{-1}\bullet \\ \bullet\left[I-\mathrm{V}^i_{ji}\mathrm{V}^i_P\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]\end{array}\right\}\mathbf{t}^n_i \equiv \mathrm{M}_i^{-1}\mathrm{M}_j\mathbf{t}^n_i.$$

By eliminating, for example, $\mathbf{t}^n_j$ from these identities we get

$$\mathbf{t}^n_i \equiv \mathrm{M}_i\mathrm{M}_j^{-1}\mathrm{M}_i^{-1}\mathrm{M}_j\mathbf{t}^n_i \equiv \mathbf{t}^n_i.$$

This proves compatibility of (15.20) and (15.21). Since this holds for an arbitrary speed of the arbitrary point P then compatibility between (15.20) and (15.21) is complete. We replace now $\mathbf{t}^n_{(.)}$ by $\left(\mathrm{V}^{(.)}_P\right)^{-1}\mathbf{r}_P(\mathbf{t}^n_{(.)})$ in (15.22) and (15.23), and we eliminate $\mathbf{r}_P(\mathbf{t}^n_j)$ from them, which imply

$$\mathbf{r}_P(\mathbf{t}^n_i) = \left[I+\mathrm{V}^j_{ji}\left(\mathrm{V}^j_P\right)^{-1}\right]^{-1}\bullet$$

$$\bullet\left\{\begin{array}{c}\left[I-\mathrm{V}^i_{ji}\left(\mathrm{V}^i_P\right)^{-1}\right]^{-1}\left[I-\mathrm{V}^i_{ji}\left(\mathrm{V}^i_P\right)^{-1}\right]+ \\ +\mathrm{V}^j_{ji}\left(\mathrm{V}^j_P\right)^{-1}\left[I-\mathrm{V}^i_{ji}\left(\mathrm{V}^i_P\right)^{-1}\right]^{-1}\left[I-\mathrm{V}^i_{ji}\left(\mathrm{V}^i_P\right)^{-1}\right]\end{array}\right\}\mathbf{r}_P(\mathbf{t}^n_i) \equiv$$

$$\equiv\left[I+\mathrm{V}^j_{ji}\left(\mathrm{V}^j_P\right)^{-1}\right]\left[I+\mathrm{V}^j_{ji}\left(\mathrm{V}^j_P\right)^{-1}\right]^{-1}\mathbf{r}_P(\mathbf{t}^n_i) \equiv \mathbf{r}_P(\mathbf{t}^n_i).$$

Hence, (15.22) and (15.23) are also completely compatible. Altogether, the transformations (15.20) through (15.23) are completely pairwise compatible. Let us now verify their complete entire compatibility. We eliminate, for example, $\mathbf{t}^n_j$ and $\mathbf{r}_P(\mathbf{t}^n_j)$ from (15.20) by using (15.19), (15.21), (15.23) for $\mathbf{r}_P(\mathbf{t}^n_{(.)}) = \mathrm{V}^{(.)}_P\mathbf{t}^n_{(.)}$,

$$\mathbf{t}^n_i \equiv \left[I+\mathrm{V}^j_{ji}\mathrm{V}^j_P\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\bullet$$

$$\bullet\left[\begin{array}{c}\left[I-\mathrm{V}^i_{ji}\mathrm{V}^i_P\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]^{-1}\left[I-\mathrm{V}^i_{ji}\mathrm{V}^i_P\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]+ \\ +\mathrm{M}_i\mathrm{M}_j^{-1}\mathrm{V}^j_{ji}\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\left[I-\mathrm{V}^i_{ji}\left(\mathrm{V}^i_P\right)^{-1}\right]^{-1}\bullet \\ \bullet\mathrm{V}^i_P\left[I-\mathrm{V}^i_{ji}\left(\mathrm{V}^i_P\right)^{-1}\right]\end{array}\right]\mathbf{t}^n_i \equiv$$

$$\equiv\left[I+\mathrm{V}^j_{ji}\mathrm{V}^j_P\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\left[I+\mathrm{V}^j_{ji}\mathrm{V}^j_P\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathbf{t}^n_i \equiv \mathbf{t}^n_i.$$

This shows that (15.20), (15.21) and (15.23) are completely compatible. We prove the same for (15.20) through (15.22) by repeating the preceding procedure.

The transformations (15.20) through (15.23) are completely entirely compatible in the temporal domain. We use now (15.19), (15.21) through (15.23), $\mathbf{t}_i^n = \left(\mathrm{V}_P^i\right)^{-1} \mathbf{r}_P(\mathbf{t}_i^n)$,

$$\mathbf{r}_P(\mathbf{t}_i^n) \equiv \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^{-1} \bullet$$

$$\bullet \left\{ \begin{array}{c} \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right]^{-1} \left[I - \ \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right] + \\ +\mathrm{V}_{ji}^j \mathrm{M}_i^{-1} \mathrm{M}_j \left[I - \mathrm{V}_{ji}^i \mathrm{V}_P^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right]^{-1} \bullet \\ \bullet \left(\mathrm{V}_P^i\right)^{-1} \left[I - \mathrm{V}_{ji}^i \mathrm{V}_P^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right] \end{array} \right\} \mathbf{r}_P(\mathbf{t}_i^n) \equiv$$

$$\equiv \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^{-1} \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right] \mathbf{r}_P(\mathbf{t}_i^n) \equiv \mathbf{r}_P(\mathbf{t}_i^n).$$

The equations (15.21) through (15.23) are fully compatible. We prove complete compatibility of (15.20), (15.22) and (15.23) in the same way. Altogether, the transformations (15.20) through (15.23) are completely both entirely and pairwise compatible. Q. E. D ■

23.6 Proof of Theorem 569

Proof. *Necessity.* Let the arbitrary point P move with an arbitrary velocity, $\mathbf{v}_P^{(.)} = \mathrm{V}_P^{(.)}\mathbf{u}$. Let the basic *time* scaling diagonal matrix coefficient M_i be defined by (15.7). Let the matrix scaling coefficients A_j^i and A_i^j be equal: $\mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}_{ij} = \mathrm{A}_{ji}$, as well as Λ_j^i and Λ_i^j, $\Lambda_j^i = \Lambda_i^j = \Lambda_{ij} = \Lambda_{ji}$. Let $B = A$ in D, (15.13). Let the diagonal matrix coefficients A_{ij} and Λ_{ij} obey (15.8) through (15.12), and let (15.7) through (15.12) imply (15.13). At first we replace $\mathbf{r}_P(\mathbf{t}_j^n)$ by the right-hand side of (15.11) into (15.10), and afterwards we apply both (23.39) and (23.40). The results are the following:

$$\mathbf{r}_P(\mathbf{t}_i^n) \equiv \Lambda_{ij}^2 \left\{ \begin{array}{c} \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right] \bullet \\ \bullet \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right] \end{array} \right\} \mathbf{r}_P(\mathbf{t}_i^n),$$

which, together with diagonal positivity of the diagonal matrix Λ_{ij}, implies:

$$\Lambda_{ij} = \left\{ \begin{array}{c} \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right] \bullet \\ \bullet \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right] \end{array} \right\}^{-1/2}. \tag{23.54}$$

Now we replace $\mathbf{t}_j^n$ by the right-hand side of (15.9) into (15.8), and apply (23.39):

$$\mathbf{t}_i^n \equiv \mathrm{A}_{ij}^2 \left\{ \begin{array}{c} \left[I - \mathrm{V}_P^i \mathrm{V}_{ji}^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right] \bullet \\ \bullet \left[I + \mathrm{V}_P^j \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right] \end{array} \right\} \mathbf{t}_i^n.$$

This and the diagonal positivity of the diagonal matrix A_{ij} imply:

$$\mathrm{A}_{ij} = \left\{ \begin{array}{c} \left[I - \mathrm{V}_P^i \mathrm{V}_{ji}^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right] \bullet \\ \bullet \left[I + \mathrm{V}_P^j \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right] \end{array} \right\}^{-1/2}. \tag{23.55}$$

The equations (15.8) through (15.11), (15.13) together with $D = blockdiag\{A \; -A\}$ due to $A = B$, (23.39) and (23.40) enable the following:

$$\begin{gathered} [\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \mathbf{v}_P^{i^T} \mathrm{T}_i] D [\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \mathbf{v}_P^{i^T} \mathrm{T}_i]^T \equiv \\ \equiv \left[\begin{array}{c} \Lambda_{ij} \left[\mathrm{R}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j \mathrm{T}_j\right] \mathbf{u} \\ \mathrm{A}_{ij} \left[\mathrm{V}_P^i \mathrm{T}_j + \mathrm{V}_P^i \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \mathrm{R}_P(\mathbf{t}_j^n)\right] \mathbf{u} \end{array} \right]^T D \bullet \\ \bullet \left[\begin{array}{c} \Lambda_{ij} \left[\mathrm{R}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j \mathrm{T}_j\right] \mathbf{u} \\ \mathrm{A}_{ij} \left[\mathrm{V}_P^i \mathrm{T}_j + \mathrm{V}_P^i \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \mathrm{R}_P(\mathbf{t}_j^n)\right] \mathbf{u} \end{array} \right]. \end{gathered} \tag{23.56}$$

Let

$$\mathrm{F}^{(..)} \equiv \Lambda_{ij} \left[\mathrm{R}_P\left(\mathbf{t}_{(..)}^n\right) + \mathrm{V}_{ji}^{(..)} \mathrm{T}_{(..)}\right],$$

and, also in view of (23.56),

$$\mathrm{G}^{(.)(..)} = \mathrm{A}_{ij} \left[\mathrm{V}_P^{(.)} \mathrm{T}_{(..)} + \mathrm{V}_P^{(.)} \mathrm{V}_{ji}^{(..)} \left(\mathrm{Q}^{(..)} \mathrm{W}^{(..)}\right)^{-1} \mathrm{R}_P\left(\mathbf{t}_{(..)}^n\right)\right].$$

Hence,

$$\begin{gathered} [\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i\right)^T \mathrm{T}_i] D [\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i\right)^T \mathrm{T}_i]^T \equiv \\ \equiv \left[\mathbf{u}^T \mathrm{F}^j \quad \mathbf{u}^T \mathrm{G}^{ij}\right] \left[\begin{array}{cc} A & O \\ O & -A \end{array} \right] \left[\begin{array}{c} \mathrm{F}^j \mathbf{u} \\ \mathrm{G}^{ij} \mathbf{u} \end{array} \right] \equiv \\ \equiv \mathbf{u}^T A \left[\left(\mathrm{F}^j\right)^2 - \left(\mathrm{G}^{ij}\right)^2\right] \mathbf{u} \equiv \mathbf{u}^T \left[\left(\mathrm{F}^j\right)^2 - \left(\mathrm{G}^{ij}\right)^2\right] A \mathbf{u}, \end{gathered} \tag{23.57}$$

where

$$\left(\mathrm{F}^j\right)^2 - \left(\mathrm{G}^{ij}\right)^2 = \Lambda_{ij}^2 \left[\mathrm{R}_P\left(\mathbf{t}_j^n\right) + \mathrm{V}_{ji}^j \mathrm{T}_j\right]^2 -$$

$$-\mathrm{A}_{ij}^2 \left[\mathrm{V}_P^i \mathrm{T}_j + \mathrm{V}_P^i \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \mathrm{R}_P\left(\mathbf{t}_j^n\right)\right]^2 \equiv$$

$$\equiv \Lambda_{ij}^2 \left\langle I - \left\{ \begin{array}{c} \left[\mathrm{A}_{ij}\left(\Lambda_{ij}\right)^{-1}\right]^2 \bullet \\ \bullet \left[\mathrm{V}_P^i \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right]^2 \end{array} \right\} \right\rangle \mathrm{R}_P^2\left(\mathbf{t}_j^n\right) +$$

$$+2\Lambda_{ij}^2 \mathrm{V}_{ji}^j \left\langle I - \left\{ \begin{array}{c} \left[\mathrm{A}_{ij}\left(\Lambda_{ij}\right)^{-1}\right]^2 \bullet \\ \bullet (\mathrm{V}_P^i)^2 \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \end{array} \right\} \right\rangle \mathrm{R}_P\left(\mathbf{t}_j^n\right) \mathrm{T}_j -$$

$$-\Lambda_{ij}^2 \left\langle -\left[\mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^2 + \left\{ \begin{array}{c} \left[\mathrm{A}_{ij}\left(\Lambda_{ij}\right)^{-1}\right]^2 \bullet \\ \bullet \left[\mathrm{V}_P^i \left(\mathrm{V}_P^j\right)^{-1}\right]^2 \end{array} \right\} \right\rangle \bullet$$

$$\bullet \left(\mathrm{V}_P^j \mathrm{T}_j\right)^2. \tag{23.58}$$

Since

$$[\mathbf{r}_{(.)}^T(\mathbf{t}_{(.)}^n) \qquad \left(\mathbf{v}_P^{(.)}\right)^T \mathrm{T}_{(.)}] D [\mathbf{r}_{(.)}^T(\mathbf{t}_{(.)}^n) \qquad \left(\mathbf{v}_P^{(.)}\right)^T \mathrm{T}_{(.)}]^T,$$

can be set into the following form:

$$[\mathbf{u}^T \mathrm{R}_P(\mathbf{t}_{(.)}^n) \quad \mathbf{u}^T \mathrm{V}_P^{(.)} \mathrm{T}_{(.)}] \left[\begin{array}{cc} A & O \\ O & -A \end{array}\right] [\mathbf{u}^T \mathrm{R}_P(\mathbf{t}_{(.)}^n) \quad \mathbf{u}^T \mathrm{V}_P^{(.)} \mathrm{T}_{(.)}]^T \equiv$$

$$\equiv \mathbf{u}^T A \left\langle \left[\mathrm{R}_P(\mathbf{t}_{(.)}^n)\right]^2 - \left[\mathrm{V}_P^{(.)} \mathrm{T}_{(.)}\right]^2 \right\rangle \mathbf{u} \equiv$$

$$\equiv \mathbf{u}^T \left\langle \left[\mathrm{R}_P(\mathbf{t}_{(.)}^n)\right]^2 - \left[\mathrm{V}_P^{(.)} \mathrm{T}_{(.)}\right]^2 \right\rangle A\mathbf{u}, \tag{23.59}$$

then (23.58) and (23.59) yield

$$[\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i\right)^T \mathrm{T}_i] D [\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i\right)^T \mathrm{T}_i]^T \equiv$$

$$\equiv \mathbf{u}^T A \left\langle [\mathrm{R}_P(\mathbf{t}_i^n)]^2 - \left[\mathrm{V}_P^i \mathrm{T}_i\right]^2 \right\rangle \mathbf{u} \equiv$$

$$= \mathbf{u}^T A \left\langle \left[\mathrm{R}_P\left(\mathbf{t}_j^n\right)\right]^2 - \left[\mathrm{V}_P^j \mathrm{T}_j\right]^2 \right\rangle \mathbf{u} \equiv$$

$$\equiv [\mathbf{r}_P^T(\mathbf{t}_j^n) \qquad \left(\mathbf{v}_P^j\right)^T \mathrm{T}_j] D [\mathbf{r}_P^T(\mathbf{t}_j^n) \qquad \left(\mathbf{v}_P^j\right)^T \mathrm{T}_j]^T.$$

The last identity, (23.57) and (23.58) imply

$$\Lambda_{ij}^2 \left\langle I - \left[\mathrm{A}_{ij}\left(\Lambda_{ij}\right)^{-1}\right]^2 \left[\mathrm{V}_P^i \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right]^2 \right\rangle = I,$$

$$\left\langle I - \left[\mathrm{A}_{ij}\left(\Lambda_{ij}\right)^{-1}\right]^2 (\mathrm{V}_P^i)^2 \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \right\rangle = O,$$

and

$$\Lambda_{ij}^2 \left\langle -\left[\mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^2 + \left\{ \begin{array}{c} \left[\mathrm{A}_{ij} \left(\Lambda_{ij}\right)^{-1}\right]^2 \bullet \\ \bullet \left[\mathrm{V}_P^i \left(\mathrm{V}_P^j\right)^{-1}\right]^2 \end{array} \right\} \right\rangle = I,$$

or equivalently, by noting that the scaling diagonal matrix coefficients A_{ij} and Λ_{ij} are positive diagonal,

$$\Lambda_{ij} = \left\langle I - \left[\mathrm{A}_{ij} \left(\Lambda_{ij}\right)^{-1}\right]^2 \left[\mathrm{V}_P^i \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right]^2 \right\rangle^{-1/2},$$

$$\left[\mathrm{A}_{ij} \left(\Lambda_{ij}\right)^{-1}\right]^2 = \mathrm{Q}^j \mathrm{W}^j \left(\mathrm{V}_P^i\right)^{-2}, \tag{23.60}$$

and

$$\Lambda_{ij} = \left\{ \begin{array}{c} -\left[\mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^2 + \\ + \left\{ \begin{array}{c} \mathrm{Q}^j \mathrm{W}^j \left(\mathrm{V}_P^i\right)^{-2} \bullet \\ \bullet \left[\mathrm{V}_P^i \left(\mathrm{V}_P^j\right)^{-1}\right]^2 \end{array} \right\} \end{array} \right\}^{-1/2},$$

which should be mutually linked.

Hence,

$$\Lambda_{ij} = \left[I - \left\{ \begin{array}{c} \mathrm{Q}^j \mathrm{W}^j \left(\mathrm{V}_P^i\right)^{-2} \bullet \\ \bullet \left[\mathrm{V}_P^i \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right]^2 \end{array} \right\} \right]^{-1/2} \equiv$$

$$= \left[I - \left(\mathrm{V}_{ji}^j\right)^2 \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \right]^{-1/2} \equiv$$

$$\equiv \left[-\left[\mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^2 + \left\{ \begin{array}{c} \mathrm{Q}^j \mathrm{W}^j \left(\mathrm{V}_P^i\right)^{-2} \bullet \\ \bullet \left[\mathrm{V}_P^i \left(\mathrm{V}_P^j\right)^{-1}\right]^2 \end{array} \right\} \right]^{-1/2} \equiv$$

$$\equiv \mathrm{V}_P^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1/2} \left\langle I - \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \left(\mathrm{V}_{ji}^j\right)^2 \right\rangle^{-1/2} \Longleftrightarrow$$

$$\Longleftrightarrow \mathrm{Q}^j \mathrm{W}^j = \left(\mathrm{V}_P^j\right)^2 \Longrightarrow \Lambda_{ij} = \left\{ I - \left[\mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^2 \right\}^{-1/2}.$$

This and (23.54) yield

$$I - \left[\mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^2 = \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right] \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right] \Longleftrightarrow$$

$$\mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1} = \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1} = \left[\mathrm{V}_{ji} \left(\mathrm{V}_P\right)^{-1}\right]^{ij} = \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ij}\right)^{-1}\right]. \tag{23.61}$$

Further, $\mathrm{Q}^j\mathrm{W}^j = \left(\mathrm{V}_P^j\right)^2$ and the second equation in (23.60) yield

$$\begin{aligned}\mathrm{A}_{ij} = \Lambda_{ij}\left[\mathrm{Q}^j\mathrm{W}^j\left(\mathrm{V}_P^i\mathrm{V}_P^i\right)^{-1}\right]^{1/2} = \Lambda_{ij}\left[\left(\mathrm{V}_P^j\right)^2\left(\mathrm{V}_P^i\right)^{-2}\right]^{1/2} = \\ = \mathrm{V}_P^j\left(\mathrm{V}_P^i\right)^{-1}\left\{I - \left[\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^2\right\}^{-1/2}. \qquad (23.62)\end{aligned}$$

This, $\mathrm{Q}^j\mathrm{W}^j = \left(\mathrm{V}_P^j\right)^2$ and (23.55) imply

$$\begin{aligned}\left[I - \mathrm{V}_P^i\mathrm{V}_{ji}^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]\left[I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right] = \\ = \left[\mathrm{V}_P^i\left(\mathrm{V}_P^j\right)^{-1}\right]^2\left\{I - \left[\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^2\right\},\end{aligned}$$

which, together with (23.61), gives

$$\mathrm{V}_P^i \equiv \mathrm{V}_P^j \equiv \mathrm{V}_P^{ij} \equiv \mathrm{V}_P^{ji}, \qquad (23.63)$$

$$\mathrm{Q}^i\mathrm{W}^i \equiv \left(\mathrm{V}_P^i\right)^2 \equiv (\mathrm{V}_P^{ij})^2, \qquad (23.64)$$

$$\mathrm{V}_{ji}^i \equiv \mathrm{V}_{ji}^j \equiv \mathrm{V}_{ji} \equiv -\mathrm{V}_{ij}, \qquad (23.65)$$

and

$$\begin{aligned}\mathrm{V}_P^i\mathrm{V}_{ji}^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1} \equiv \mathrm{V}_P^j\mathrm{V}_{ji}^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1} \equiv \\ \equiv \left[\mathrm{V}_P\mathrm{V}_{ji}\left(\mathrm{QW}\right)^{-1}\right]^{ji} \equiv \left[\mathrm{V}_P\mathrm{V}_{ji}\left(\mathrm{QW}\right)^{-1}\right]^{ij}. \qquad (23.66)\end{aligned}$$

The equations (23.63) through (23.66) prove (15.26) through (15.28). Finally, (23.55), (23.62), (23.63), and (23.66) imply

$$\begin{aligned}\mathrm{A}_{ij} &= \left\{I - \left\langle\left[\mathrm{V}_P\mathrm{V}_{ji}\left(\mathrm{QW}\right)^{-1}\right]^{ji}\right\rangle^2\right\}^{-1/2} = \\ &= \Lambda_{ij} = \left\{I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\}^{-1/2}. \qquad (23.67)\end{aligned}$$

These equations prove (15.29). They and diagonal positivity of scaling diagonal matrix coefficients imply (15.30). The equations (15.7) through (15.9), (23.39), (23.40), and (15.26) through (15.30) enable the following:

$$\mathbf{t}_i^n = \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \bullet$$

$$\bullet \left[I + \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \mathrm{M}_j \mathbf{t}^n = \mathrm{M}_i \mathbf{t}^n,$$

$$\mathbf{t}_j^n = \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \bullet$$

$$\bullet \left[I - \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \mathrm{M}_i \mathbf{t}^n = \mathrm{M}_j \mathbf{t}^n.$$

Their solutions for M_j are the same,

$$\mathrm{M}_j = \left\{ I + \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \right\}^{-1/2} \bullet$$

$$\bullet \left[I - \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^{1/2} \mathrm{M}_i.$$

This equation proves (15.31). The results (15.26) through (15.29) transform (15.8) through (15.11) into (15.32) through (15.35).

Sufficiency. Let all the conditions of the theorem statement be satisfied. The equations (15.7), (23.39), (23.40) and (15.26) through (15.35) lead to

$$\mathbf{t}_i^n = \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \left[I + \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \mathbf{t}_j^n =$$

$$= \mathrm{A}_j^i \left[\mathbf{t}_j^n + \mathrm{V}_{ji}^j \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \mathbf{r}_P(\mathbf{t}_j^n) \right],$$

$$\mathbf{t}_j^n = \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \left[I - \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \mathbf{t}_i^n =$$

$$= \mathrm{A}_i^j \left[\mathbf{t}_i^n - \mathrm{V}_{ji}^i \left(\mathrm{Q}^i \mathrm{W}^i \right)^{-1} \mathbf{r}_P \left(\mathbf{t}_i^n \right) \right],$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \mathrm{V}_P^{ij} \mathbf{t}_i^n = \mathrm{M}_i \mathrm{M}_j^{-1} \mathrm{V}_P^{ji} \mathbf{t}_j^n =$$

$$= \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \left[I + \left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \mathbf{r}_P(\mathbf{t}_j^n) =$$

$$= \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \bullet$$

$$\bullet \left[I + \mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right] \mathbf{r}_P(\mathbf{t}_j^n) = \Lambda_j^i \left[\mathbf{r}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}^j \mathbf{t}_j^n \right],$$

$$\mathbf{r}_P(\mathbf{t}_j^n) = \mathrm{V}_P^{ji}\mathbf{t}_j^n = \mathrm{M}_i^{-1}\mathrm{M}_j\mathrm{V}_P^{ij}\mathbf{t}_i^n =$$

$$= \left[I - \left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right] \bullet$$

$$\bullet \left\{I - \left[\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \mathbf{r}_P(\mathbf{t}_i^n) =$$

$$= \left\{I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\}^{-1/2} \bullet$$

$$\bullet \left[I - \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right] \mathbf{r}_P(\mathbf{t}_i^n) = \Lambda_i^j \left[\mathbf{r}_P(\mathbf{t}_i^n) - \ \mathrm{V}_{ji}^i\mathbf{t}_i^n\right].$$

These equations prove (15.8) through (15.11). In order to verify the validity of (15.13) we apply to its left-hand side (15.7), (23.39), (23.40), (15.26) through (15.28), (15.32), (15.34), $B = A$ in D, the diagonal structure of all the matrices and $\mathbf{v}_P^{(.)} = \mathrm{V}_P^{(.)}\mathbf{u}$,

$$[\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i\right)^T \mathrm{T}_i] D [\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i\right)^T \mathrm{T}_i]^T \equiv$$

$$\equiv \begin{bmatrix} \left\{I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\}^{-1/2} \left[\mathrm{R}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}\mathrm{T}_j\right]\mathbf{u} \\ \mathrm{V}_P^{ji}\left\{I - \left[\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \bullet \\ \bullet \left[\mathrm{T}_j + \mathrm{V}_{ji}\left(\left(\mathrm{QW}\right)^{ji}\right)^{-1}\mathrm{R}_P(\mathbf{t}_j^n)\right]\mathbf{u} \end{bmatrix}^T D \bullet$$

$$\bullet \begin{bmatrix} \left\{I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\}^{-1/2} \left[\mathrm{R}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}\mathrm{T}_j\right]\mathbf{u} \\ \mathrm{V}_P^{ji}\left\{I - \left[\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \bullet \\ \bullet \left[\mathrm{T}_j + \mathrm{V}_{ji}\left(\left(\mathrm{QW}\right)^{ji}\right)^{-1}\mathrm{R}_P(\mathbf{t}_j^n)\right]\mathbf{u} \end{bmatrix} \equiv$$

$$\equiv \mathbf{u}^T \left\{I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\}^{-1} \bullet$$

$$\bullet \begin{bmatrix} \left[\mathrm{R}_P(\mathbf{t}_j^n) + \mathrm{V}_{ji}\mathrm{T}_j\right]^2 - \\ -(\mathrm{V}_P^{ji})^2\left[\mathrm{T}_j + \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-2}\mathrm{R}_P(\mathbf{t}_j^n)\right]^2 \end{bmatrix} A\mathbf{u} \equiv$$

$$\equiv \mathbf{u}^T \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1} \bullet$$

$$\bullet \left[\begin{array}{c} \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\} \mathrm{R}_P^2(\mathbf{t}_j^n) - \\ - \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\} (\mathrm{V}_P^{ji})^2 \mathrm{T}_j^2 \end{array} \right] A\mathbf{u} \equiv$$

$$\equiv [\mathbf{r}_P^T(\mathbf{t}_j^n) \qquad \left(\mathbf{v}_P^j \right)^T \mathrm{T}_j] D [\mathbf{r}_P^T(\mathbf{t}_j^n) \qquad \left(\mathbf{v}_P^j \right)^T \mathrm{T}_j]^T.$$

This proves the validity of (15.13).

Compatibility. In order to verify the compatibility of the temporal coordinate transformations, we eliminate, for example, $\mathbf{t}_j^n$ from (15.32) and (15.33) by applying (23.39), (23.40), and (15.26) through (15.30),

$$\mathbf{t}_i^n \equiv \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \left\{ I + \left[\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right]^{ji} \right\} \bullet$$

$$\bullet \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \left[I - \left[\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right]^{ji} \right] \mathbf{t}_i^n \equiv \mathbf{t}_i^n.$$

The temporal coordinate transformations (15.32) and (15.33) are partially compatible due to the constraint on $\mathrm{V}_P^{(.)}$ and $\mathrm{Q}^{(.)}\mathrm{W}^{(.)}$ in (15.28). The elimination, for example, of $\mathbf{r}_P(\mathbf{t}_i^n)$ from (15.34) by applying (15.35), (23.39), (23.40) and (15.26) through (15.30), results in

$$\mathbf{r}_P(\mathbf{t}_j^n) \equiv \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \left[I - \mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right] \mathbf{r}_P(\mathbf{t}_i^n) \equiv$$

$$\equiv \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \left[I - \mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right] \bullet$$

$$\bullet \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \left[I + \mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right] \mathbf{r}_P(\mathbf{t}_j^n) \equiv \mathbf{r}_P(\mathbf{t}_j^n).$$

The spatial coordinate transformations (15.34) and (15.35) are partially compatible due to the constraint (15.28) on $\mathrm{V}_P^{(.)}$ and $\mathrm{Q}^{(.)}\mathrm{W}^{(.)}$, too. The transformations (15.32) through (15.35) are partially pairwise compatible. Let us now eliminate, for example, $\mathbf{t}_i^n$ and $\mathbf{r}_P(\mathbf{t}_i^n)$ from (15.33) by using (23.39), (23.40), (15.28) through (15.32) and (15.34),

$$\mathbf{t}_j^n \equiv \left\langle I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\rangle^{-1/2} \bullet$$

$$\bullet\{\left\langle I-\left[\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\rangle^{-1/2}\left[\mathbf{t}_j^n+\mathrm{V}_{ji}^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathbf{r}_P(\mathbf{t}_j^n;\mathbf{t}_{j0}^n)\right]-$$

$$-\mathrm{V}_{ji}^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\left\langle I-\left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1/2}\left[\mathbf{r}_P(\mathbf{t}_j^n)+\mathrm{V}_{ji}\mathbf{t}_j^n\right]\}\equiv$$

$$\equiv\left\langle I-\left[\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\rangle^{-1/2}\bullet$$

$$\bullet\{\left\langle I-\left[\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\rangle^{-1/2}\bullet\left[I+\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]\mathbf{t}_j^n-$$

$$-\left[\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right]^{ji}\bullet\left\langle I-\left[\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\rangle^{-1/2}\bullet$$

$$\bullet\left[I+\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]\mathbf{t}_j^n\}\equiv$$

$$=\left\langle I-\left[\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\rangle^{-1}\bullet$$

$$\bullet\left\{\begin{array}{c}\left[I+\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]\bullet\\ \bullet\left[I-\left(\mathrm{V}_{ji}\mathrm{V}_P\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]\mathbf{t}_j^n\end{array}\right\}\equiv\mathbf{t}_j^n. \tag{23.70}$$

The transformations (15.32) through (15.34) are partially compatible due to (23.70) and the constraint on $\mathrm{V}_P^{(.)}$ and $\mathrm{Q}^{(.)}\mathrm{W}^{(.)}$ in (15.28). The same holds for the transformations (15.32), (15.33) and (15.35). Altogether, the whole set of the coordinate transformations (15.32) through (15.35) is partially entirely compatible in the temporal domain. We eliminate, for example, $\mathbf{t}_j^n$ and $\mathbf{r}_P(\mathbf{t}_j^n;\mathbf{t}_{j0}^n)$ from (15.34) by linking it with (23.39), (23.40), (15.33), (15.35), and by applying (15.26) through (15.28),

$$\mathbf{r}_P(\mathbf{t}_i^n)\equiv\left\langle I-\left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1/2}\left[\mathbf{r}_P(\mathbf{t}_j^n)+\mathrm{V}_{ji}\mathbf{t}_j^n\mathbf{u}\right]\equiv$$

$$\equiv\left\langle I-\left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1/2}\bullet$$

$$\bullet\{\left\langle I-\left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1/2}\left[I-\ \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]\mathbf{r}_P(\mathbf{t}_i^n)+$$

$$+\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\left\langle I-\left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1/2}\bullet$$

$$\bullet\left[I-\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]\mathbf{r}_P\left(\mathbf{t}_i^n\right)\}\equiv$$

$$\equiv\left\langle I-\left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1}\bullet$$

$$\bullet\left[I-\ \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]\left[I+\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]\mathbf{r}_P\left(\mathbf{t}_i^n\right)\equiv\mathbf{r}_P\left(\mathbf{t}_i^n\right). \tag{23.71}$$

The transformations (15.33) through (15.35) are partially compatible due to (23.71) and (15.28). In the same manner we prove the partial compatibility of (15.32), (15.34) and (15.35), which then verifies the partial entire compatibility of (15.32) through (15.35) in the spatial domain. The transformations (15.32) through (15.35) are partially both entirely and pairwise compatible. Q. E. D ■

23.7 Proof of Theorem 578

Proof. *Necessity and sufficiency.* Let all the conditions of the statement of the theorem be satisfied. Theorem 568 holds. The definition of the velocity, (23.72),

$$\mathbf{v}_P^{(.)}=\mathrm{V}_P^{(.)}\mathbf{u}=\left[d\mathrm{R}_P(\mathbf{t}_{(.)}^n)\right]\left(d\mathrm{T}_{(.)}\right)^{-1}\mathbf{u}, \tag{23.72}$$

the equations (15.32) and (15.34), the constancy of $\mathrm{M}_i\left(\mathrm{M}_j^{-1}\right)^{-1}$, (15.7), (23.39), (23.40) and $\mathbf{v}_{ji}^{(.)}=\mathrm{V}_{ji}^{(.)}\mathbf{u}$ yield

$$\mathbf{v}_P^i=\left[d\mathrm{R}_P(\mathbf{t}_i^n)\right]\left(d\mathrm{T}_i\right)^{-1}\mathbf{u}=$$

$$=\left\{d\left\langle\left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}\left[\mathrm{R}_P\left(\mathbf{t}_j^n\right)+\ \mathrm{V}_{ji}^j\mathrm{T}_j\right]\right\rangle\right\}\left(d\mathrm{T}_j\right)^{-1}\bullet$$

$$\bullet\left[\left\{d\left\langle\begin{array}{c}\mathrm{M}_i\mathrm{M}_j^{-1}\left[I+\mathrm{V}_{ji}^j\mathrm{V}_P^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\bullet\\ \bullet\left[\mathrm{T}_j+\mathrm{V}_{ji}^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathrm{R}_P\left(\mathbf{t}_j^n\right)\right]\end{array}\right\rangle\right\}\left(d\mathrm{T}_j\right)^{-1}\right]^{-1}\mathbf{u}=$$

$$= \left\{ d \left(\left\langle \begin{array}{c} \left[I + \mathrm{V}^j_{ji} \left(\mathrm{V}^j_P \right)^{-1} \right]^{-1} \bullet \\ \bullet \left[I + \ \mathrm{V}^j_{ji} \left(\mathrm{V}^j_P \right)^{-1} \right] \end{array} \right\rangle \mathrm{R}_P(\mathbf{t}^n_j) \right) \right\} (d\mathrm{T}_j)^{-1} \bullet$$

$$\bullet \left[\left\{ d \left[\begin{array}{c} \mathrm{M}_i \mathrm{M}_j^{-1} \left[I + \mathrm{V}^j_{ji} \mathrm{V}^j_P \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \right]^{-1} \bullet \\ \bullet \left[I + \mathrm{V}^j_{ji} \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \mathrm{V}^j_P \right] \mathrm{T}_j \end{array} \right] \right\} (d\mathrm{T}_j)^{-1} \right]^{-1} \mathbf{u} =$$

$$= \mathrm{M}_i^{-1} \mathrm{M}_j \mathbf{v}^j_P.$$

This result, which proves all the equations in (15.52) since it holds also for $\mathbf{v}_P^{(.)} \equiv \mathbf{c}_{(.)}^{(.)}$, can be deduced directly from (15.19).

Compatibility. The equations for $\mathbf{v}^i_P$ and for $\mathbf{v}^j_P$ in (15.52) are evidently completely compatible. Q. E. D ■

23.8 Proof of Theorem 582

Proof. *Necessity and sufficiency.* Let all the conditions of the statement of the theorem be satisfied. Theorem 568 is valid. The definition of the velocity, (23.72), (23.39), (15.28), (15.32) and (15.34) yield

$$\mathbf{v}^i_P = \mathrm{V}^i_P \mathbf{u} = \left[d\mathrm{R}_P(\mathbf{t}^n_i) \right] (d\mathrm{T}_i)^{-1} \mathbf{u} =$$

$$= \left(d \left[\begin{array}{c} \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}^{ji}_P \right)^{-1} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[\mathrm{R}_P \left(\mathbf{t}^n_j \right) + \mathrm{V}_{ji} \mathrm{T}_j \right] \end{array} \right] (d\mathrm{T}_j)^{-1} \right) \bullet$$

$$\bullet \left(d \left[\begin{array}{c} \left\{ I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_P \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[\mathrm{T}_j + \mathrm{V}^j_{ji} \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \mathrm{R}_P \left(\mathbf{t}^n_j \right) \right] \end{array} \right] (d\mathrm{T}_j)^{-1} \right)^{-1} \mathbf{u} =$$

$$= \left(d \left\langle \begin{array}{c} \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}^{ji}_P \right)^{-1} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[I + \mathrm{V}_{ji} \left(\mathrm{V}^{ji}_P \right)^{-1} \right] \mathrm{R}_P \left(\mathbf{t}^n_j \right) \end{array} \right\rangle (d\mathrm{T}_j)^{-1} \right) \bullet$$

$$\bullet \left(d \left\langle \begin{array}{c} \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}^{ji}_P \right)^{-1} \right]^2 \right\}^{-1/2} \bullet \\ \bullet \left[I + \mathrm{V}_{ji} \left(\mathrm{V}^{ji}_P \right)^{-1} \right] \mathrm{T}_j \end{array} \right\rangle (d\mathrm{T}_j)^{-1} \right)^{-1} \mathbf{u} =$$

$$= \left\langle d\mathrm{R}_P \left(\mathbf{t}^n_j \right) \right\rangle (d\mathrm{T}_j)^{-1} \mathbf{u} = \mathbf{v}^j_P.$$

This proves (15.53).

Compatibility. Since the transformations are the identity transformation, then they are trivially compatible. The compatibility is partial due to the constraint (15.28). Q. E. D ■

23.9 Proof of Claim 586

Proof. Let the vector-matrix Lorentz transformations (15.48) through (15.51) hold.

Necessity and sufficiency. The definition of the velocity, (23.72), (23.39), (15.48) and (15.50) yield

$$\mathbf{v}_P^i = \left[d\mathrm{R}_P(\mathbf{t}_i^n)\right](d\mathrm{T}_i)^{-1}\mathbf{u} =$$
$$= d\left\{\left[I-\left(\mathrm{VC}^{-1}\right)^2\right]^{-1/2}\left[\mathrm{R}_P(\mathbf{t}_j^n)+\mathrm{VT}_j^n\right]\right\}(d\mathrm{T}_j)^{-1}\bullet$$
$$\bullet\left[d\left\{\left[I-\left(\mathrm{VC}^{-1}\right)^2\right]^{-1/2}\left[\mathrm{T}_j^n+\mathrm{V}\left(\mathrm{C}\right)_P^{-2}\mathrm{R}(\mathbf{t}_j^n)\right]\right\}(d\mathrm{T}_j)^{-1}\right]^{-1}\mathbf{u} =$$
$$= \left(\mathrm{V}_P^j+\mathrm{V}\right)\left[I+\mathrm{V}\left(\mathrm{C}\right)^{-2}\mathrm{V}_P^j\right]^{-1}\mathbf{u}.$$

This result proves the first equation (15.55) due to $\mathbf{v}_{(..)}^{(.)} = \mathrm{V}_{(..)}^{(.)}\mathbf{u}$. The second equation (15.55) is analogously proved by starting with (23.72), (15.49), (15.51) and by using (23.39).

Compatibility. The equations (15.55), $\mathbf{v} = V\mathbf{u}$ and $\mathbf{v}_P^i = V_P^i\mathbf{u}$ yield

$$\mathbf{v}_P^i = \left[I+\mathrm{V}\left[I-\mathrm{VV}_P^i\left(\mathrm{C}\right)^{-2}\right]^{-1}\left(\mathrm{V}_P^i-\mathrm{V}\right)\left(\mathrm{C}\right)^{-2}\right]^{-1}\bullet$$
$$\bullet\left\{\left[I-\mathrm{VV}_P^i\left(\mathrm{C}\right)^{-2}\right]^{-1}\left(\mathbf{v}_P^i-\mathbf{v}\right)+\mathbf{v}\right\} =$$
$$= \left[I-\mathrm{VV}_P^i\left(\mathrm{C}\right)^{-2}\right]\left\{\left[I-\mathrm{VV}_P^i\left(\mathrm{C}\right)^{-2}\right]+\mathrm{V}\left(\mathrm{V}_P^i-\mathrm{V}\right)\left(\mathrm{C}\right)^{-2}\right\}^{-1}\bullet$$
$$\bullet\left[I-\mathrm{VV}_P^i\left(\mathrm{C}\right)^{-2}\right]^{-1}\left\{\left(\mathbf{v}_P^i-\mathbf{v}\right)+\left[I-\mathrm{VV}_P^i\left(\mathrm{C}\right)^{-2}\right]\mathbf{v}\right\} =$$
$$= \left(I-\mathrm{V}^2\mathrm{C}^{-2}\right)^{-1}\left\{I-\mathrm{V}^2\left(\mathrm{C}\right)^{-2}\right\}\mathbf{v}_P^i = \mathbf{v}_P^i.$$

The equations (15.55) are completely compatible. Q. E. D ■

23.10 Proof of Theorem 592

Proof. Let the matrix scaling coefficient $\mathrm{M}_i \in R_+^{nxn}$, $diagmin\mathrm{M}_i \in R^+$, obey (15.56).

Necessity. Let the *time*-independent matrix scaling coefficients A_j^i, A_i^j, $\mathrm{A}_j^i \neq \mathrm{A}_i^j$, Λ_j^i, Λ_i^j, $\Lambda_j^i \neq \Lambda_i^j$, be diagonally elementwise constant and positive real valued and let they obey (15.57) through (15.61) so that they, together with (15.56), imply (15.13). Since the equations (15.59) and (15.60) are the same as the equations (15.10) and (15.11), respectively, then the proof of the equations (15.16) and (15.17) is simultaneously the proof of the equations (15.64) and (15.65). These results demand elementwise constant nonzero vector value of the velocity of the arbitrary point P since the scaling matrix coefficients are well defined and elementwise constant and due to $\mathrm{V}_{ji}^{(.)} \in R_{+}^{n \times n}$,

$$\mathrm{V}_{P}^{(.)}(..) = \mathrm{V}_{P}^{(.)} = CONST. \tag{23.73}$$

By combining (15.56), (15.57), $\mathbf{r}_R(\mathbf{t}_j^n) = \mathrm{V}_R^j \mathbf{t}_j^n = \mathrm{V}_R^j \mathrm{T}_j \mathbf{u}$ and (23.40) for $(..) = P_R$ we find

$$\mathbf{t}_i^n = \mathrm{T}_i \mathbf{u} = \mathrm{A}_j^i \left[I + \mathrm{V}_{\vartheta}^j \mathrm{V}_R^j \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \right] (\mathrm{M}_i)^{-1} \mathrm{M}_j \mathrm{T}_i \mathbf{u}.$$

This equation implies

$$\mathrm{A}_j^i = \mathrm{M}_i \mathrm{M}_j^{-1} \left[I + \mathrm{V}_{\vartheta}^j \mathrm{V}_R^j \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \right]^{-1}.$$

This proves the equation (15.62). The equation (15.63) is proved in the same manner. The equations (15.13), (23.40), (15.57) and (15.59) imply the following:

$$\begin{gathered}
\left[\mathbf{r}_P^T(\mathbf{t}_i^n) \quad (\mathbf{t}_i^n)^T \mathrm{V}_P^i \right] D \left[\mathbf{r}_P^T(\mathbf{t}_i^n) \quad (\mathbf{t}_i^n)^T \mathrm{V}_P^i \right]^T \equiv \\
\equiv \left\{ \begin{array}{c}
\left[\begin{array}{c}
\Lambda_j^i \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j \right)^{-1} \right] \mathbf{r}_P\left(\mathbf{t}_j^n\right) \\
\mathrm{A}_j^i \left[I + \mathrm{V}_{\vartheta}^j \mathrm{V}_R^j \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \right] \bullet \\
\bullet \mathrm{V}_P^i \left(\mathrm{V}_P^j \right)^{-1} \mathrm{V}_P^j \mathbf{t}_j^n
\end{array} \right]^T D \bullet \\
\bullet \left[\begin{array}{c}
\Lambda_j^i \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j \right)^{-1} \right] \mathbf{r}_P\left(\mathbf{t}_j^n\right) \\
\mathrm{A}_j^i \left[I + \mathrm{V}_{\vartheta}^j \mathrm{V}_R^j \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \right] \bullet \\
\bullet \mathrm{V}_P^i \left(\mathrm{V}_P^j \right)^{-1} \mathrm{V}_P^j \mathbf{t}_j^n
\end{array} \right]
\end{array} \right\} \equiv \\
\equiv [\mathbf{r}_P^T\left(\mathbf{t}_j^n\right) \quad \mathbf{t}_j^{nT} \mathrm{V}_P^j] D [\mathbf{r}_P^T\left(\mathbf{t}_j^n\right) \quad \mathbf{t}_j^{nT} \mathrm{V}_P^j]^T.
\end{gathered} \tag{23.74}$$

These identities furnish:

$$\Lambda_j^i = \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j \right)^{-1} \right]^{-1}, \tag{23.75}$$

$$\mathrm{A}_j^i = \left(\mathrm{V}_P^i \right)^{-1} \mathrm{V}_P^j \left[I + \mathrm{V}_{\vartheta}^j \mathrm{V}_R^j \left(\mathrm{Q}^j \mathrm{W}^j \right)^{-1} \right]^{-1}. \tag{23.76}$$

The equation (23.75) confirms the equation (15.64). The proof of the equation (15.65) is analogous. The equations (15.62), (15.63) and (23.76) imply:

$$\mathrm{V}_P^i \mathrm{M}_i \left(\mathrm{V}_P^j \mathrm{M}_j^{-1} \right)^{-1} = I \ and \ \left(\mathrm{V}_P^i \mathrm{M}_i \right)^{-1} \mathrm{V}_P^j \mathrm{M}_j^{-1} = I. \tag{23.77}$$

Since the point P is arbitrary, then it can represent also a light signal in which case (23.77) becomes:

$$\mathrm{C}_i^i\mathrm{M}_i\left(\mathrm{C}_j^j\mathrm{M}_j^{-1}\right)^{-1} = I \ and \ \left(\mathrm{C}_i^i\mathrm{M}_i\right)^{-1}\mathrm{C}_j^j\mathrm{M}_j^{-1} = I \Longrightarrow$$
$$\mathrm{M}_i\left(\mathrm{M}_j^{-1}\right)^{-1} \equiv \mathrm{C}_j^j\left(\mathrm{C}_i^i\right)^{-1} = CONST.$$

This and (23.77) complete the proof of (15.67). The condition that all the scaling matrix coefficients are diagonally elementwise positive real valued, together with (15.62) through (15.65), proves (15.66). The equations (15.62) through (15.65) transform (15.57) through (15.60) into (15.68) through (15.71).

Sufficiency. Let (15.62) through (15.71), be valid. Let M_i obey (15.56). The proof of the equations (15.10) and (15.11) is also the proof of the transformations (15.59) and (15.60). We proceed with (15.56) as follows:

$$\mathbf{t}_i^n = \mathrm{M}_i\mathbf{t}^n{=}\mathrm{M}_i\left\{\begin{array}{c}\left[I+\mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\bullet\\ \bullet\left[I+\mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\end{array}\right\}\mathrm{M}_j^{-1}\mathbf{t}_j^n=$$
$$={\mathrm{M}_i\mathrm{M}_j^{-1}}\left\{\begin{array}{c}\left[I+\mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\bullet\\ \bullet\left[\mathbf{t}_j^n+\mathrm{V}_\vartheta^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathrm{V}_R^j\mathbf{t}_j^n\right]\end{array}\right\}. \tag{23.78}$$

Now, (15.62), (23.39) applied for $(..) = P_R$ and (23.78) imply:

$$\mathbf{t}_i^n{=}\mathrm{A}_j^i[\mathbf{t}_j^n{+}\mathrm{V}_{ji}^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathbf{r}_P\left(\mathbf{t}_j^n\right)]. \tag{23.79}$$

The equation (23.79) is the equation (15.57). By repeating this procedure applied to (15.56) for i replaced by j, we prove the equation (15.58). We continue with Gaussean transformation, (15.13), by using (23.39), (23.40), (15.56), (15.67), (15.68), (15.70) and $\mathbf{v}_P^{(.)} = \mathrm{V}_P^{(.)}\mathbf{u}$:

$$[\mathbf{r}_P^T(\mathbf{t}_i^n) \quad (\mathbf{t}_i^n)^T\mathrm{V}_P^i]D[\mathbf{r}_P^T(\mathbf{t}_i^n) \quad (\mathbf{t}_i^n)^T\mathrm{V}_P^i]^T \equiv$$

$$\equiv\left[\begin{array}{c}\left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}\left[I+\ \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathbf{r}_P\left(\mathbf{t}_j^n\right)\\ \mathrm{V}_P^i\mathrm{M}_i\mathrm{M}_j^{-1}\left\{\begin{array}{c}\left[I+\mathrm{V}_\vartheta^j\mathrm{V}_R^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\right]^{-1}\bullet\\ \bullet\left[I+\mathrm{V}_\vartheta^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\mathrm{V}_R^j\right]\end{array}\right\}\mathbf{t}_j^n\end{array}\right]^T D\bullet$$
$$\bullet\left[\begin{array}{c}\left[I+\mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^{-1}\left[I+\ \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathbf{r}_P\left(\mathbf{t}_j^n\right)\\ \mathrm{V}_P^i\mathrm{M}_i\mathrm{M}_j^{-1}\left\{\begin{array}{c}\left[I+\mathrm{V}_\vartheta^j\mathrm{V}_R^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\right]^{-1}\bullet\\ \bullet\left[I+\mathrm{V}_\vartheta^j(\mathrm{Q}^j\mathrm{W}^j)^{-1}\mathrm{V}_R^j\right]\end{array}\right\}\mathbf{t}_j^n\end{array}\right]\equiv$$

$$\equiv \left[\begin{array}{c} \mathbf{r}_P\left(\mathbf{t}_j^n\right) \\ \mathrm{M}_j\left(\mathrm{M}_i\right)^{-1}\mathrm{V}_P^j\mathbf{t}_i^n \end{array} \right]^T D \left[\begin{array}{c} \mathbf{r}_P\left(\mathbf{t}_j^n\right) \\ \mathrm{M}_j\left(\mathrm{M}_i\right)^{-1}\mathrm{V}_P^j\mathbf{t}_i^n \end{array} \right] \equiv$$

$$\equiv \left[\mathbf{r}_P^T\left(\mathbf{t}_j^n\right) \quad \left(\mathbf{t}_j^n\right)^T \mathrm{V}_P^j \right] D \left[\mathbf{r}_P^T\left(\mathbf{t}_j^n\right) \quad \left(\mathbf{t}_j^n\right)^T \mathrm{V}_P^j \right]^T.$$

This proves (15.13).

Compatibility. We replace $\mathbf{r}_P(\mathbf{t}_{(.)}^n)$ by $\mathrm{V}_P^{(.)}\mathbf{t}_{(.)}^n$ in (15.68) and (15.69), and we apply (15.67),

$$\begin{aligned} \mathbf{t}_i^n &\equiv \mathrm{M}_i\mathrm{M}_j^{-1}\left[I + \mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1} \bullet \\ &\bullet \left[I + \mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathbf{t}_j^n \equiv \mathrm{M}_i\mathrm{M}_j^{-1}\mathbf{t}_j^n \equiv \mathbf{t}_i^n, \end{aligned} \tag{23.80}$$

$$\begin{aligned} \mathbf{t}_j^n &\equiv \mathrm{M}_i^{-1}\mathrm{M}_j\left[I - \mathrm{V}_\vartheta^i\mathrm{V}_R^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]^{-1} \bullet \\ &\bullet \left[I - \mathrm{V}_\vartheta^i\mathrm{V}_R^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]\mathbf{t}_i^n \equiv \mathrm{M}_i^{-1}\mathrm{M}_j\mathbf{t}_i^n \equiv \mathbf{t}_j^n. \end{aligned} \tag{23.81}$$

The identities (23.80) and (23.81) prove the complete compatibility of (15.68) and (15.69). The proof of the complete compatibility of (15.70) and (15.71) is the same as the proof of the complete compatibility of (15.22) and (15.23). Altogether, the transformations (15.68) through (15.71) are completely pairwise compatible. Let us now verify their entire compatibility. We eliminate, for example, $\mathbf{t}_j^n$ and $\mathbf{r}_P(\mathbf{t}_j^n)$ from (15.68) by using (15.67), (15.69) and (15.71) for $P = P_R$, hence, for $\mathbf{r}_P(\mathbf{t}_{(.)}^n) \equiv \mathbf{r}_R(\mathbf{t}_{(.)}^n) \equiv \mathrm{V}_R^{(.)}\mathbf{t}_{(.)}^n$:

$$\begin{aligned} \mathbf{t}_i^n &\equiv \mathrm{M}_i\mathrm{M}_j^{-1}\left[I + \mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1} \bullet \\ \bullet &\left[\begin{array}{c} \mathrm{M}_i^{-1}\mathrm{M}_j \left\{ \begin{array}{c} \left[I - \mathrm{V}_\vartheta^i\mathrm{V}_R^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\right]^{-1} \bullet \\ \bullet\left[\mathbf{t}_i^n - \mathrm{V}_\vartheta^i\left(\mathrm{Q}^i\mathrm{W}^i\right)^{-1}\mathbf{r}_R(\mathbf{t}_i^n)\right] \end{array} \right\} + \\ +\mathrm{V}_\vartheta^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\left[I - \mathrm{V}_{ji}^i\left(\mathrm{V}_R^i\right)^{-1}\right]^{-1}\left[\mathbf{r}_R(\mathbf{t}_i^n) - \mathrm{V}_{ji}^i\mathbf{t}_i^n\right] \end{array} \right] \equiv \\ &\equiv \left[I + \mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\left[I + \mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathbf{t}_i^n \equiv \mathbf{t}_i^n. \end{aligned}$$

This shows that (15.68), (15.69) and (15.71) are completely compatible. We prove the same for (15.69) through (15.71) by repeating the preceding procedure. The transformations (15.68) through (15.71) are completely entirely compatible in the temporal domain. In order to verify their entire compatibility in the spatial domain we combine (15.67), (15.69) through (15.71) and we use $\mathbf{t}_i^n =$

$\left(\mathrm{V}_P^i\right)^{-1} \mathbf{r}_P(\mathbf{t}_i^n)$,

$$\mathbf{r}_P(\mathbf{t}_i^n) \equiv \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^{-1} \bullet$$

$$\bullet \left[\begin{array}{c} \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right]^{-1} \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right] \mathbf{r}_P(\mathbf{t}_i^n) + \\ + \mathrm{V}_{ji}^j \mathrm{M}_i^{-1} \mathrm{M}_j \left\{ \begin{array}{c} \left[I - \mathrm{V}_\vartheta^i \mathrm{V}_R^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right]^{-1} \bullet \\ \bullet \left[I - \mathrm{V}_\vartheta^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1} \mathrm{V}_R^i\right] \end{array} \right\} \mathbf{t}_i^n \end{array} \right] \equiv$$

$$\equiv \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right]^{-1} \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right] \mathbf{r}_P(\mathbf{t}_i^n) \equiv \mathbf{r}_P(\mathbf{t}_i^n).$$

The equations (15.69) through (15.71) are fully compatible. We prove complete compatibility of (15.68), (15.70) and (15.71) in the same way. Altogether, the transformations (15.68) through (15.71) are completely both entirely and pairwise compatible. Q. E. D ■

23.11 Proof of Theorem 597

Proof. *Necessity.* Let the arbitrary point P move with an arbitrary velocity $\mathbf{v}_P^{(.)} = \mathrm{V}_P^{(.)}\mathbf{u}$. Let the basic *time* scaling diagonal matrix coefficient M_i be defined by (15.7). Let the scaling diagonal matrix coefficients A_j^i and A_i^j be equal: $\mathrm{A}_j^i = \mathrm{A}_i^j = \mathrm{A}_{ij} = \mathrm{A}_{ji}$, as well as Λ_j^i and Λ_i^j, $\Lambda_j^i = \Lambda_i^j = \Lambda_{ij} = \Lambda_{ji}$. Let $B = A$ in D, (15.13). Let the diagonal matrix coefficients A_{ij} and Λ_{ij} obey (15.57) through (15.61), and let (15.110) through (15.61) imply (15.13). At first we replace $\mathbf{r}_P(\mathbf{t}_j^n)$ by the right-hand side of (15.60) into (15.59). Afterwards we apply (23.39), (23.40) and the diagonal positivity of the diagonal matrix Λ_{ij}. The result is

$$\Lambda_{ij} = \left\{ \left[I - \mathrm{V}_{ji}^i \left(\mathrm{V}_P^i\right)^{-1}\right] \left[I + \mathrm{V}_{ji}^j \left(\mathrm{V}_P^j\right)^{-1}\right] \right\}^{-1/2}. \tag{23.82}$$

Now we replace $\mathbf{t}_j^n$ by the right-hand side of (15.58) into (15.57), and apply (23.39):

$$\mathbf{t}_i^n \equiv \mathrm{A}_{ij} \left[\mathbf{t}_j^n + \mathrm{V}_\vartheta^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \mathbf{r}_R\left(\mathbf{t}_j^n\right)\right] \equiv$$

$$\equiv \mathrm{A}_{ij} \left\{ \begin{array}{c} \mathrm{A}_{ij} \left[I - \mathrm{V}_R^i \mathrm{V}_\vartheta^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right] \mathbf{t}_i^n + \\ + \mathrm{A}_{ij} \mathrm{V}_R^j \mathrm{V}_\vartheta^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \left[I - \mathrm{V}_R^i \mathrm{V}_\vartheta^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right] \mathbf{t}_i^n \end{array} \right\} \equiv$$

$$\equiv \mathrm{A}_{ij}^2 \left[I - \mathrm{V}_R^i \mathrm{V}_\vartheta^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right] \left[I + \mathrm{V}_R^j \mathrm{V}_\vartheta^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right] \mathbf{t}_i^n.$$

The left-hand side of the first identity, the right-hand side of the last identity, and the diagonal positivity of the diagonal matrix A_{ij} imply:

$$\mathrm{A}_{ij} = \left\{ \left[I - \mathrm{V}_R^i \mathrm{V}_\vartheta^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1}\right] \left[I + \mathrm{V}_R^j \mathrm{V}_\vartheta^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1}\right] \right\}^{-1/2}. \tag{23.83}$$

The equations (15.13) together with $D = blockdiag\ \{A \quad -A\}$ due to $A = B$, (23.39), (23.40), (15.57) through (15.60), and $\mathrm{R}_R\left(\mathbf{t}_j^n\right) = \mathrm{V}_R^j\left(\mathrm{V}_P^j\right)^{-1}\mathrm{R}_P\left(\mathbf{t}_j^n\right)$ enable the following:

$$[\mathbf{r}_P(\mathbf{t}_i^n) \qquad \mathbf{v}_P^{i^T}\mathrm{T}_i]D[\mathbf{r}_P(\mathbf{t}_i^n) \qquad \mathbf{v}_P^{i^T}\mathrm{T}_i]^T \equiv$$

$$\equiv \begin{bmatrix} \Lambda_{ij}\left[\mathrm{R}_P\left(\mathbf{t}_j^n\right) + \mathrm{V}_{ji}^j\mathrm{T}_j\right]\mathbf{u} \\ \mathrm{A}_{ij}\left[\mathrm{V}_P^i\mathrm{T}_j + \mathrm{V}_P^i\mathrm{V}_R^j\mathrm{V}_\vartheta^j\left(\mathrm{V}_P^j\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathrm{R}_P\left(\mathbf{t}_j^n\right)\right]\mathbf{u} \end{bmatrix}^T D \bullet$$

$$\bullet \begin{bmatrix} \Lambda_{ij}\left[\mathrm{R}_P\left(\mathbf{t}_j^n\right) + \mathrm{V}_{ji}^j\mathrm{T}_j\right]\mathbf{u} \\ \mathrm{A}_{ij}\left[\mathrm{V}_P^i\mathrm{T}_j + \mathrm{V}_P^i\mathrm{V}_R^j\mathrm{V}_\vartheta^j\left(\mathrm{V}_P^j\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathrm{R}_P\left(\mathbf{t}_j^n\right)\right]\mathbf{u} \end{bmatrix}.$$

The forms of the terms on the right-hand side of the preceding identity suggest the following notation for the sake of simplicity,

$$\mathrm{F}^{(..)} \equiv \Lambda_{ij}\left[\mathrm{R}_P(\mathbf{t}_{(..)}^n) + \mathrm{V}_{ji}^{(..)}\mathrm{T}_{(..)}\right],$$

and,

$$\mathrm{G}^{(.)(..)} = \mathrm{A}_{ij}\left[\mathrm{V}_P^{(.)}\mathrm{T}_{(..)} + \mathrm{V}_P^{(.)}\mathrm{V}_R^{(..)}\mathrm{V}_\vartheta^{(..)}\left(\mathrm{V}_P^{(..)}\mathrm{Q}^{(..)}\mathrm{W}^{(..)}\right)^{-1}\mathrm{R}_P\left(\mathbf{t}_{(..)}^n\right)\right].$$

Hence,

$$[\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i\right)^T\mathrm{T}_i]D[\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i\right)^T\mathrm{T}_i]^T \equiv$$
$$\equiv \left[\mathbf{u}^T\mathrm{F}^j \qquad \mathbf{u}^T\mathrm{G}^{ij}\right]\begin{bmatrix} A & O \\ O & -A \end{bmatrix}\begin{bmatrix} \mathrm{F}^j\mathbf{u} \\ \mathrm{G}^{ij}\mathbf{u} \end{bmatrix} \equiv$$
$$\equiv \mathbf{u}^T A\left[\left(\mathrm{F}^j\right)^2 - \left(\mathrm{G}^{ij}\right)^2\right]\mathbf{u} \equiv \mathbf{u}^T\left[\left(\mathrm{F}^j\right)^2 - \left(\mathrm{G}^{ij}\right)^2\right]A\mathbf{u}, \tag{23.84}$$

where

$$\left(\mathrm{F}^j\right)^2 - \left(\mathrm{G}^{ij}\right)^2 = \Lambda_{ij}^2\left[\mathrm{R}_P\left(\mathbf{t}_j^n\right) + \mathrm{V}_{ji}^j\mathrm{T}_j\right]^2 -$$
$$-\mathrm{A}_{ij}^2\begin{bmatrix} \mathrm{V}_P^i\mathrm{T}_j + \\ +\mathrm{V}_P^i\mathrm{V}_R^j\mathrm{V}_\vartheta^j\left(\mathrm{V}_P^j\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathrm{R}_P\left(\mathbf{t}_j^n\right) \end{bmatrix}^2 \equiv$$

$$= \Lambda_{ij}^2 \left\langle \begin{array}{c} I - \left[A_{ij} \left(\Lambda_{ij} \right)^{-1} \right]^2 \bullet \\ \bullet \left[V_P^i V_R^j V_\vartheta^j \left(V_P^j Q^j W^j \right)^{-1} \right]^2 \end{array} \right\rangle R_P^2 \left(\mathbf{t}_j^n \right) +$$

$$+2\Lambda_{ij}^2 V_{ji}^j \left\langle \begin{array}{c} I - \left[A_{ij} \left(\Lambda_{ij} \right)^{-1} \right]^2 \\ \bullet V_P^i V_P^i V_R^j V_\vartheta^j \left(V_{ji}^j V_P^j Q^j W^j \right)^{-1} \end{array} \right\rangle R_P \left(\mathbf{t}_j^n \right) T_j -$$

$$-\Lambda_{ij}^2 \left\langle \begin{array}{c} - \left[V_{ji}^j \left(V_P^j \right)^{-1} \right]^2 + \\ + \left[A_{ij} \left(\Lambda_{ij} \right)^{-1} \right]^2 \left[V_P^i \left(V_P^j \right)^{-1} \right]^2 \end{array} \right\rangle \left(V_P^j T_j \right)^2 . \qquad (23.85)$$

Since

$$[\mathbf{r}_P^T(\mathbf{t}_{(.)}^n) \qquad \left(\mathbf{v}_P^{(.)} \right)^T T_{(.)}] D [\mathbf{r}_P^T(\mathbf{t}_{(.)}^n) \qquad \left(\mathbf{v}_P^{(.)} \right)^T T_{(.)}]^T,$$

can be set into the following forms:

$$[\mathbf{u}^T R_P(\mathbf{t}_{(.)}^n) \;\; \mathbf{u}^T V_P^{(.)} T_{(.)}] \left[\begin{array}{cc} A & O \\ O & -A \end{array} \right] [\mathbf{u}^T R_P(\mathbf{t}_{(.)}^n) \; \mathbf{u}^T V_P^{(.)} T_{(.)}]^T \equiv$$
$$\equiv \mathbf{u}^T A \left\langle \left[R_P(\mathbf{t}_{(.)}^n) \right]^2 - \left[V_P^{(.)} T_{(.)} \right]^2 \right\rangle \mathbf{u} =$$
$$= \mathbf{u}^T \left\langle \left[R_P(\mathbf{t}_{(.)}^n) \right]^2 - \left[V_P^{(.)} T_{(.)} \right]^2 \right\rangle A \mathbf{u},$$

then they imply

$$[\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i \right)^T T_i] D [\mathbf{r}_P^T(\mathbf{t}_i^n) \qquad \left(\mathbf{v}_P^i \right)^T T_i]^T \equiv$$
$$\equiv \mathbf{u}^T A \left\langle [R_P(\mathbf{t}_i^n)]^2 - \left[V_P^i T_i \right]^2 \right\rangle \mathbf{u} \equiv$$
$$= \mathbf{u}^T A \left\langle \left[R_P(\mathbf{t}_j^n) \right]^2 - \left[V_P^j T_j \right]^2 \right\rangle \mathbf{u} \equiv$$
$$\equiv [\mathbf{r}_P^T \left(\mathbf{t}_j^n \right) \qquad \left(\mathbf{v}_P^j \right)^T T_j] D [\mathbf{r}_P^T \left(\mathbf{t}_j^n \right) \qquad \left(\mathbf{v}_P^j \right)^T T_j]^T.$$

This, (23.84), (23.85), and the diagonal positivity of the scaling diagonal matrix coefficients A_{ij} and Λ_{ij} yield

$$\Lambda_{ij} = \left\langle \begin{array}{c} I - \left[A_{ij} \left(\Lambda_{ij} \right)^{-1} \right]^2 \bullet \\ \bullet \left[V_P^i V_R^j V_\vartheta^j \left(V_P^j Q^j W^j \right)^{-1} \right]^2 \end{array} \right\rangle^{-1/2},$$
$$\left[A_{ij} \left(\Lambda_{ij} \right)^{-1} \right]^2 = V_{ji}^j V_P^j Q^j W^j \left[\left(V_P^i \right)^2 V_R^j V_\vartheta^j \right]^{-1}, \qquad (23.86)$$

and

$$\Lambda_{ij} = \left\langle \begin{array}{c} -\left[\mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right]^{2} + \\ +\mathrm{V}_{P}^{j}\mathrm{Q}^{j}\mathrm{W}^{j}\left[\left(\mathrm{V}_{P}^{i}\right)^{2}\mathrm{V}_{R}^{j}\right]^{-1}\left[\mathrm{V}_{P}^{i}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right]^{2} \end{array} \right\rangle^{-1/2},$$

which should be mutually linked. Hence,

$$\Lambda_{ij} = \left\langle I - \mathrm{V}_{ji}^{j}\mathrm{V}_{R}^{j}\mathrm{V}_{\vartheta}^{j}\left(\mathrm{V}_{P}^{j}\mathrm{Q}^{j}\mathrm{W}^{j}\right)^{-1}\right\rangle^{-1/2} \equiv$$

$$\equiv \left\langle \begin{array}{c} -\left[\mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right]^{2} + \\ +\mathrm{V}_{P}^{j}\mathrm{Q}^{j}\mathrm{W}^{j}\left[\left(\mathrm{V}_{P}^{i}\right)^{2}\mathrm{V}_{R}^{j}\right]^{-1}\left[\mathrm{V}_{P}^{i}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right]^{2} \end{array} \right\rangle^{-1/2} \equiv$$

$$\equiv \left\langle -\left[\mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right]^{2} + \mathrm{Q}^{j}\mathrm{W}^{j}\left(\mathrm{V}_{R}^{j}\mathrm{V}_{P}^{j}\right)^{-1}\right\rangle^{-1/2} \equiv$$

$$\equiv \left[\mathrm{V}_{P}^{j}\mathrm{V}_{R}^{j}\left(\mathrm{Q}^{j}\mathrm{W}^{j}\right)^{-1}\right]^{1/2} \bullet$$

$$\bullet \left\langle I - \mathrm{V}_{R}^{j}\left(\mathrm{V}_{P}^{j}\mathrm{Q}^{j}\mathrm{W}^{j}\right)^{-1}\left(\mathrm{V}_{ji}^{j}\right)^{2}\right\rangle^{-1/2} \Longleftrightarrow$$

$$\Longleftrightarrow \mathrm{Q}^{j}\mathrm{W}^{j} = \mathrm{V}_{P}^{j}\mathrm{V}_{R}^{j} \; and$$

$$\mathrm{V}_{ji}^{j} = \mathrm{V}_{\vartheta}^{j} \Longrightarrow \Lambda_{ij} = \left\{I - \left[\mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right]^{2}\right\}^{-1/2}.$$

This and (23.82) lead to

$$I - \left[\mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right]^{2} \equiv$$

$$= \left[I - \mathrm{V}_{ji}^{i}\left(\mathrm{V}_{P}^{i}\right)^{-1}\right]\left[I + \mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1}\right] \Longleftrightarrow$$

$$\mathrm{V}_{ji}^{j}\left(\mathrm{V}_{P}^{j}\right)^{-1} = \mathrm{V}_{ji}^{i}\left(\mathrm{V}_{P}^{i}\right)^{-1} =$$

$$= \left[\mathrm{V}_{ji}\left(\mathrm{V}_{P}\right)^{-1}\right]^{ji} = \left[\mathrm{V}_{ji}\left(\mathrm{V}_{P}\right)^{-1}\right]^{ij}. \qquad (23.87)$$

Further, $Q^jW^j = V_P^jV_R^j$, $V_{ji}^j = V_\vartheta^j$ and (23.86) yield

$$\begin{aligned}A_{ij} = \Lambda_{ij}\left[V_{ji}^jV_P^jQ^jW^j\left[\left(V_P^i\right)^2V_R^jV_\vartheta^j\right]^{-1}\right]^{1/2} = \\ = \Lambda_{ij}\left[\left(V_P^j\right)^2\left(V_P^i\right)^{-2}\right]^{1/2} = \\ = V_P^j\left(V_P^i\right)^{-1}\left\{I - \left[V_{ji}^j\left(V_P^j\right)^{-1}\right]^2\right\}^{-1/2}.\end{aligned} \quad (23.88)$$

This, $Q^jW^j = V_P^jV_R^j$, (23.83) and (23.87) give

$$V_P^i \equiv V_P^j \equiv V_P^{ij} \equiv V_P^{ji}, \quad (23.89)$$

$$Q^iW^i \equiv V_P^iV_R^i, \quad (23.90)$$

$$V_{ji}^i \equiv V_{ji}^j \equiv V_{ji} \equiv -V_{ij} = V_\vartheta^j = V_\vartheta^i = V_\vartheta^{ij}, \quad (23.91)$$

and

$$\begin{aligned}V_R^iV_\vartheta^i\left(Q^iW^i\right)^{-1} \equiv V_R^jV_\vartheta^j\left(Q^jW^j\right)^{-1} \equiv \\ \equiv \left[V_RV_{ji}\left(QW\right)^{-1}\right]^{ji} \equiv \left[V_RV_{ji}\left(QW\right)^{-1}\right]^{ij}.\end{aligned} \quad (23.92)$$

The equations (23.89) through (23.91) prove (15.74) through (15.76). Finally, the equations (23.88), (23.89) through (23.92) imply

$$\begin{aligned}A_{ij} = \left\{I - \left\langle\left[V_RV_\vartheta\left(QW\right)^{-1}\right]^{ji}\right\rangle^2\right\}^{-1/2} = \\ = \Lambda_{ij} = \left\{I - \left[V_{ji}\left(V_P^{ji}\right)^{-1}\right]^2\right\}^{-1/2}.\end{aligned} \quad (23.93)$$

These equations prove (15.77). They and the diagonal positivity of the scaling diagonal matrix coefficients imply (15.78). The equations (15.110) through (15.58), (15.77), (23.39) and (23.40) enable the following:

$$\begin{aligned}\mathbf{t}_i^n = M_i\mathbf{t}^n = \left\{I - \left[\left(V_\vartheta V_R\left(QW\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \bullet \\ \bullet\left[I + \left(V_\vartheta V_R\left(QW\right)^{-1}\right)^{ji}\right]M_j\mathbf{t}^n,\end{aligned}$$

$$\begin{aligned}\mathbf{t}_j^n = M_j\mathbf{t}^n = \left\{I - \left[\left(V_\vartheta V_R\left(QW\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \bullet \\ \bullet\left[I - \left(V_\vartheta V_R\left(QW\right)^{-1}\right)^{ji}\right]M_i\mathbf{t}^n.\end{aligned}$$

Their solutions for M_j are the same, for which we apply (15.76),

$$\mathrm{M}_j = \left\{I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \bullet$$
$$\bullet \left[I - \left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right] \mathrm{M}_i.$$

This equation proves (15.79). The equations (15.74) through (15.77) transform (15.57) through (15.60) into (15.80) through (15.83). This complete the necessity part of the proof of Theorem 596.

We continue the proof of Theorem 596 with its sufficiency part and with its compatibility part.

Sufficiency. Let all the conditions of the theorem statement hold. The equations (15.7), the relationships (15.74) through (15.83), (23.39), and (23.40) lead to

$$\mathbf{t}_i^n = \mathrm{M}_i \mathrm{M}_j^{-1} \mathbf{t}_j^n =$$
$$= \left\{I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \left[I + \left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right] \mathbf{t}_j^n =$$
$$= \mathrm{A}_j^i \left[\mathbf{t}_j^n + \mathrm{V}_\vartheta^j \left(\mathrm{Q}^j \mathrm{W}^j\right)^{-1} \mathbf{r}_R \left(\mathbf{t}_j^n\right)\right],$$

$$\mathbf{t}_j^n = \mathrm{M}_i^{-1} \mathrm{M}_j \mathbf{t}_i^n =$$
$$\left\{I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \left[I - \left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right] \mathbf{t}_i^n =$$
$$= \mathrm{A}_i^j \left[\mathbf{t}_i^n - \mathrm{V}_\vartheta^i \left(\mathrm{Q}^i \mathrm{W}^i\right)^{-1} \mathbf{r}_R \left(\mathbf{t}_i^n\right)\right],$$

$$\mathbf{r}_P(\mathbf{t}_i^n) = \mathrm{V}_P^{ij} \mathbf{t}_i^n =$$
$$= \left\{I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \left[I + \left(\mathrm{V}_{ji} \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right] \mathbf{r}_P \left(\mathbf{t}_j^n\right) =$$
$$= \left\{I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\}^{-1/2} \bullet \left[I + \mathrm{V}_{ji} \left(\mathrm{V}_P^{ji}\right)^{-1}\right] \mathbf{r}_P \left(\mathbf{t}_j^n\right) =$$
$$= \Lambda_j^i \left[\mathbf{r}_P \left(\mathbf{t}_j^n\right) + \mathrm{V}_{ji}^j \mathbf{t}_j^n\right],$$

$$\mathbf{r}_P \left(\mathbf{t}_j^n\right) = \mathrm{V}_P^{ji} \mathbf{t}_j^n = \left[I - \left(\mathrm{V}_{ji} \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right] \bullet$$
$$\bullet \left\{I - \left[\left(\mathrm{V}_{ji} \mathrm{V}_R \left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\right\}^{-1/2} \mathbf{r}_P(\mathbf{t}_i^n) =$$

$$= \left\{ I - \left[\mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right]^2 \right\}^{-1/2} \left[I - \mathrm{V}_{ji} \left(\mathrm{V}_P^{ji} \right)^{-1} \right] \mathbf{r}_P(\mathbf{t}_i^n) =$$

$$= \Lambda_i^j \left[\mathbf{r}_P(\mathbf{t}_i^n) - \mathrm{V}_{ji}^i \mathbf{t}_i^n \right].$$

These equations prove (15.57) through (15.60). The proof of the validity of the distance condition (15.13) is the same as in the sufficiency proof in 23.6.

Compatibility. In order to verify the compatibility of the temporal coordinate transformations, we eliminate, for example, $\mathbf{t}_j^n$ from (15.80) and (15.81) by applying (23.39), (23.40),

$$\mathbf{t}_i^n \equiv \left\{ I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \left\{ I + \left[\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right]^{ji} \right\} \bullet$$

$$\bullet \left\{ I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\}^{-1/2} \left[I - \left[\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right]^{ji} \right] \mathbf{t}_i^n \equiv \mathbf{t}_i^n.$$

The temporal coordinate transformations (15.80) and (15.81) are partially compatible because they are valid only under the constraints on $\mathrm{V}_P^{(.)}$, $\mathrm{V}_R^{(.)}$ and $Q^{(.)}W^{(.)}$ in (15.76). The proof of the partial compatibility of (15.34) and (15.35) is also the proof of the partial compatibility of (15.82) and (15.83). The transformations (15.80) through (15.83) are partially pairwise compatible. Let us now eliminate, for example, $\mathbf{t}_i^n$ and $\mathbf{r}_P(\mathbf{t}_i^n)$ from (15.81) by using (23.39), (23.40), (15.80) and (15.82),

$$\mathbf{t}_j^n \equiv \left\langle I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\rangle^{-1/2} \bullet$$

$$\bullet \{ \left\langle I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\rangle^{-1/2} \bullet \left[I + \left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \mathbf{t}_j^n -$$

$$- \left[\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right]^{ji} \left\langle I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\rangle^{-1/2} \bullet$$

$$\bullet \left[I + \left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \mathbf{t}_j^n \} \equiv$$

$$\equiv \left\langle I - \left[\left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right]^2 \right\rangle^{-1} \bullet$$

$$\bullet \left\{ \begin{array}{c} \left[I + \left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \bullet \\ \bullet \left[I - \left(\mathrm{V}_\vartheta \mathrm{V}_R \left(\mathrm{QW} \right)^{-1} \right)^{ji} \right] \mathbf{t}_j^n \end{array} \right\} \equiv \mathbf{t}_j^n.$$

The transformations (15.80) through (15.82) are only partially compatible due to the constraint (15.76). The same holds for the transformations (15.80), (15.81) and (15.83). Altogether, the coordinate transformations (15.80) through (15.83) are partially entirely compatible in the temporal domain. We eliminate, for example, $\mathbf{t}_j^n$ and $\mathbf{r}_P\left(\mathbf{t}_j^n;\mathbf{t}_{j0}^n\right)$ from (15.82) by linking it with (23.39), (23.40), (15.81) and (15.83),

$$\mathbf{r}_P(\mathbf{t}_i^n) \equiv \left\langle I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1/2} \bullet$$

$$\bullet\{\left\langle I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1/2}\left[I - \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]\mathbf{r}_P(\mathbf{t}_i^n) +$$

$$+\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\left\langle I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1/2}\left[I - \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]\mathbf{r}_P\left(\mathbf{t}_i^n\right)\} \equiv$$

$$\equiv \left\langle I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\rangle^{-1} \bullet$$

$$\bullet\left[I - \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]\left[I + \mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]\mathbf{r}_P\left(\mathbf{t}_i^n\right) = \mathbf{r}_P\left(\mathbf{t}_i^n\right).$$

The transformations (15.81) through (15.83) are only partially compatible due to the constraint (15.76). In the same manner we prove the partial compatibility of (15.80), (15.82) and (15.83), which then verifies the partial entire compatibility of (15.80) through (15.83) in the spatial domain. Altogether, the transformations (15.80) through (15.83) are partially entirely compatible. Q. E. D ■

23.12 Proof of Theorem 603

Proof. *Necessity and sufficiency.* Let all the conditions of the statement of the theorem be satisfied. Theorem 591 is applicable. The definition of the velocity, (23.72), (23.39), (23.40), (15.7), (15.68), (15.70), constancy of $\mathrm{M}_i\left(\mathrm{M}_j^{-1}\right)^{-1}$, (15.7), and $\mathbf{v}_{ji}^{(.)} = \mathrm{V}_{ji}^{(.)}\mathbf{u}$ yield

$$\mathbf{v}_P^i = \left[d\mathrm{R}_P(\mathbf{t}_i^n)\right]\left(d\mathrm{T}_i\right)^{-1}\mathbf{u} =$$

$$= \left\{ d\left[\begin{array}{c} \left[I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]^{-1} \bullet \\ \bullet\left[I + \mathrm{V}_{ji}^j\left(\mathrm{V}_P^j\right)^{-1}\right]\mathrm{V}_P^j\mathrm{T}_j \end{array}\right]\right\}\left(d\mathrm{T}_j\right)^{-1} \bullet$$

$$\bullet\left[\begin{array}{c} \left\{ d\left[\begin{array}{c} \mathrm{M}_i\mathrm{M}_j^{-1}\bullet \\ \bullet\left[I + \mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]^{-1}\bullet \\ \bullet\left[I + \mathrm{V}_\vartheta^j\mathrm{V}_R^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\right]\mathrm{T}_j \end{array}\right]\right\}\bullet \\ \bullet\left(d\mathrm{T}_j\right)^{-1} \end{array}\right]^{-1}\mathbf{u} =$$

$$= (\mathrm{M}_i)^{-1}\,\mathrm{M}_j\left[d(\mathrm{V}_P^j\mathrm{T}_j)\right](d\mathrm{T}_j)^{-1}\,\mathbf{u} = (\mathrm{M}_i)^{-1}\,\mathrm{M}_j\mathbf{v}_P^j.$$

This result proves (15.95) since $\mathbf{v}_P^{(.)} = \mathbf{c}_{(.)}^{(.)}$ is permitted.

Compatibility. The complete compatibility of the equations for $\mathbf{v}_P^i$ and $\mathbf{v}_P^j$ in (15.95) is evident. Q. E. D ■

23.13 Proof of Theorem 606

Proof. *Necessity and sufficiency.* Let the conditions of the statement of the theorem be satisfied. We apply Theorem 596. The definition of the velocity, i.e. (23.72), (23.39), and (15.74) through (15.76), (15.80) and (15.82), yield

$$\mathbf{v}_P^i = \mathrm{V}_P^i\mathbf{u} = \left[d\left[\begin{array}{c}\left\{I - \left[\mathrm{V}_{ji}\left(\mathrm{V}_P^{ji}\right)^{-1}\right]^2\right\}^{-1/2} \bullet \\ \bullet\left[\mathrm{R}_P\left(\mathbf{t}_j^n\right) + \mathrm{V}_{ji}\mathrm{T}_j\right]\end{array}\right](d\mathrm{T}_j)^{-1}\right] \bullet$$

$$\bullet\left[d\left[\begin{array}{c}\left[\begin{array}{c} I- \\ -\left[\left(\mathrm{V}_R\mathrm{V}_\vartheta\left(\mathrm{QW}\right)^{-1}\right)^{ji}\right]^2\end{array}\right]^{-1/2} \bullet \\ \bullet\left[\mathrm{T}_j + \mathrm{V}_\vartheta^j\left(\mathrm{Q}^j\mathrm{W}^j\right)^{-1}\mathrm{R}_R\left(\mathbf{t}_j^n\right)\right]\end{array}\right](d\mathrm{T}_j)^{-1}\right]^{-1}\mathbf{u} =$$

$$= \left\langle d\mathrm{R}_P\left(\mathbf{t}_j^n\right)\right\rangle(d\mathrm{T}_j)^{-1}\,\mathbf{u} = \mathbf{v}_P^j.$$

This proves (15.97).

Compatibility. The transformations are trivially compatible. The compatibility is partial due to the restriction (15.76). Q. E. D ■

Chapter 24

Used literature

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Bibliography

[1] E. Achelis, *Du Temps et du Calendrier*, (the original title in English: Of Time and the Calendar), Editions Corrêa, Buchet / Chastel, 1955.

[2] R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, Mc Graw Hill Kogakusha, Ltd., Tokyo, 1975.

[3] S. Alexander, *Space · Time · and Deity*, I, Macmillan and Co., Limited, London, 1927.

[4] S. Alexander, *Space · Time ·* and *Deity*, II, Macmillan and Co., Limited, London, 1927.

[5] M. Alonso and E. J. Finn, *Physique générale, 1, Mécanique*, Editions du Renouveau Pédagogique, Inc., Montréal, 1970.

[6] J. D. Alonso and M. L. Paramo, Ed.'s, *Proceedings of the Relativity Meeting' 93 RELATIVITY IN GENERAL*, held in Salas, Asturias, (Spain), September 7 - 10, 1993, Frontières, Gif-sur-Yvette, France, 1994.

[7] A. Amann, "Individual Complex Quantum Objects and Dynamics of Decompositions", published in *[15]*, pp. 267 - 277.

[8] H. Andrillat, *L'univers sous le regard du temps*, Masson, Paris, 1993.

[9] P. E. Ariotti, "The Concept of Time in Western Antiquity", published in *[189]*, pp. 69 - 80.

[10] Aristotle, *Physics III and IV*, published in [249], pp. 1 - 54.

[11] Aristote, *Physique*, *I - IV*, Les belles lettres, Paris, 1996.

[12] Aristotle, "On Time, Motion and Change", published in *[81]*, pp. 147 - 157.

[13] A. Aspect et P. Grangier, "Des intuitions d'Einstein aux bits quantiques", published in *[46]*, pp. 120 - 125.

[14] H. Atmanspacher and G. J. Dalenoort, Editors, *Inside Versus Outside*, Springer Verlag, Berlin, 1994.

[15] H. Atmanspacher and E. Runhau, *Time, Temporality, Now*, Springer-Verlag, Berlin, 1997.

[16] J. Attali, *Histoires du temps*, Librairie Arthème Fayard, Paris, 1982.

[17] J.-P. Auffray, *Einstein et Poincaré*, Le Pommier, Paris, 1999.

[18] J.-P. Auffray, *Newton*, Le Pommier, Paris, 2000.

[19] R. Balian, "Le temps macroscopique", published in *[72]*, pp. 155 - 211.

[20] C. Bailey, "The Relational Theory of Time in Ancient Atomism", published in *[81]*, pp. 143 - 145.

[21] V. S. Barashenkov, E. Kapuscik and M. V. Lablin, "Nature of Relativistic Effects and Delayed Clock Synchronization"published in *[444]*, pp. 195 - 200.

[22] M. Barone, "Some Almost Unknown Aspects of Special Relativity Theory", published in *[444]*, pp. 93 - 102.

[23] J. F. Barrett, "Review of Problems of Dynamics in the Hyperbolic Theory of Special Relativity", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 17 - 30.

[24] J. F. Barrett, "Relative Motion and Hyperbolic Geometry in Special Relativity, *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, p. 8.

[25] I. Barrow, "Absolute Time", published in [81], pp. 203 - 208.

[26] G. Baumann, *Mathematica for Theoretical Physics. Classical Mechanics and Nonlinear Dynamics*, Springer, HAM, USA, 2005.

[27] G. Baumann, *Mathematica for Theoretical Physics. Electrodynamics, Quantum Mechanics, General Relativity, and Fractals*, Springer, HAM, USA, 2005.

[28] O. C. de Beauregard, "Mesure du temps et relativité", *Cahiers de Physique*, No. 95, 1958, pp. 257 - 260.

[29] O. C. de Beauregard, *Le second principe de la science du temps*, Editions du Seuil, Paris, 1963.

[30] O. C. de Beauregard, *La notion de temps*, HERMANN, Paris, 1963.

[31] R. Bellman, "Vector Lyapunov Functions", *J. SIAM Contro*l, Ser. A., 1, No. 1, 1962, pp. 32 - 34.

[32] P. G. Bergmann, *Introduction to the Theory of Relativity*, Dover Publications, Inc., New York, 1975.

[33] H. Bergson, *Durée et simultanéité*, Librairie Félix Alcan, Paris, 1931.

[34] H. Bergson, "On Zeno's Paradoxes", published in *[81]*, pp. 245 - 250.

[35] H. Bergson, "Discussion with Becquerel of the Paradox of the Twins", published in *[81]*, pp. 433 - 439.

[36] H. Bergson, *Essai sur les données immédiates de la conscience*, Quadrige / PUF, Paris, 2001.

[37] G. Berkeley, "Criticism of Newton", published in *[81]*, pp. 267 - 271.

[38] F. Bernardeau, "Les mirages de la gravitation", published in *[46]*, pp. 84 - 89.

[39] E. Bitsakis, "Space and Time: Who was Right, Einstein or Kant?", published in *[444]*, pp. 115 - 125.

[40] J. Van Bladel, *Relativity and Engineering*, Springer Verlag, Berlin, 1984.

[41] D. Bohm, "Comment on the Paradox of the Twins", published in *[81]*, pp. 451 - 453.

[42] D. Bohm, "Inadequacy of Laplacean Determinism and Irreversibility of Time", published in *[81]*, pp. 547 - 559.

[43] D. Bohm, *The Special Theory of Relativity*, Routledge, London, 1996.

[44] M. Borneas, "On the Concepts of Space and Time", published in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 35 - 38.

[45] R. J. Boscovich, "On the Relativity of Temporal Intervals", published in *[81]*, pp. 225 - 226.

[46] P. Boulanger, Directeur de la rédaction, *Pour la SCIENCE*, No. 326, December, 2004.

[47] P. Boulanger, "Soyez original ! Comme Einstein ?", published in *[46]*, p. 1.

[48] R. Bousso et J. Polchinski, "Le paysage de la théorie des cordes", published in [46], pp. 133 - 139.

[49] P. A. Bowman, "On Conventionality and Simultaneity - Another Reply", published in *[107]*, pp. 433 - 447.

[50] C. M. Bowra, *Problems in Greek Poetry*, The Clarendon Press, Oxford, 1953.

[51] G. Brelet, *Le TEMPS MUSICAL*, Presse universitaire de France, Paris, 1949.

[52] G. S. Brown, *Laws of Form*, The Julian Press, Inc. Publishers, New York, 1972.

[53] G. Bruno, "Hesitations between Absolute and Relational Theory of Time", published in *[81]*, pp. 189 - 193.

[54] G. Builder, "Ether and Relativity", *Australian Journal of Physics*, 11, 1958, pp. 279 - 297.

[55] Bureau des Longitudes, *Encyclopédie scientifique de l'univers*, Gauthier-Villars, Paris, 1981.

[56] P. Busch, "The Time-Energy Uncertainty Relation", published in *[354]*, pp. 69 - 98.

[57] A. Calaprice, Ed., *The Quotable Einstein*, Princeton University Press, Princeton, New Jersey, 1996.

[58] J. E. Carroll, "Relativity with Three Dimensions of Time: Space - Time Votices", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 55 - 68.

[59] Y. Castin, "La saga des condesats quantiques", published in *[46]*, pp. 114 - 119.

[60] A. C. V. Ceapa, *Roots of the Falsification of 20th Century Physics*, Published by A. C. V. Ceapa (PO Box 1-1035, R-70700, Bucharest, Rumania, ISBN 973-9318-06-1), Bucharset, 1998.

[61] A. C. V. Ceapa, "On the Criticism of Einstein's Special Relativity Theory", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 80 - 81.

[62] A. C. V. Ceapa, "Revelation & Progress in Science: Einstein & Relativity", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 82 - 84.

[63] A. C. V. Ceapa, "The Meaning of Mass", *Journal of Theoretics*, 6-3, July 2004, pp. 1 - 2.

[64] A. C. V. Ceapa, "Lorentz Transformation as a Complementary Time-Dependent Coordinate Transformation", *Galilean Electrodynamics*, 16, No. 1, January/Fbruary 2005, pp. 3 - 11.

[65] A. C. V. Ceapa, "Consequences of Founding SRT on the Whole of Einstein's 1905 Paper on Relativity", *Private communication in which it is written The Proceedings of the Natural Philosophy Alliance*, Storrs, 2005, pp. 1 - 3.

[66] C. J. S. Clarke, "Time in General Relativity", published in *[107]*, pp. 94 - 108.

[67] P. Claudel, *Art Poétique*, Gallimard, Paris, 1984.

[68] M. Clavelin, *La philosophie naturelle de Galilée*, Librairie Armand Colin, Paris, 1968.

[69] G. Cohen-Tannoudji, "Le temps des processus élémentaires - I", published in *[72]*, pp. 93 - 130.

[70] G. Cohen-Tannoudji and M. Spiro, *La matière-espace-temps*, Fayard, France, 1986.

[71] C. Collobert, *L'être de Parmènide ou le refus du temps*, Editions Kimé, Paris, 1993.

[72] Colloquium : "Le Temps et sa Flèche", J. T. T. Vân, directeur de l'édition, E. Klein et M. Spiro, organisateurs du colloque, *Proceedings of the Colloquium*, Editions Frontieres, Gif sur Yvette Cedex, 1994. The second edition was published 1995.

[73] A. Comte-Sponville, "L'être-temps. Quelques réflexions sur le temps de la conscience", published in *[72]*, pp. 239 - 270.

[74] M. Conche, *Temps et destin*, Presse universitaire de France, Paris, 1992.

[75] F. M. Cornford, "The elimination of time by Parmenides", published in *[81]*, pp. 137 - 142.

[76] J.-M. Courty et É. Kierlik, "Connaître sa position, un problème de relativité", published in *[46]*, pp. 44 - 48.

[77] W. L. Craig, "Relativity and the "Elimination" of Absolute Time", published in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 47 - 66.

[78] A. C. Crombie, *AUGUSTINE TO GALILEO. The History of Science A. D. 400 - 1650*, Falcon Educational Books, Falcon Press, London, 1952. In French: HISTOIRE DES SCIENCES de Saint Augustin à Galilée, I et II, Presses universitaire de France, Paris, 1959.

[79] M. Crozon, "Introduction", published in *[72]*, pp. 15 - 20.

[80] O. Cullman, *Temps et histoire dans le christianisme primitif*, Imprimerie Delachaux & Niestlé S. A., Neuchatel, 1947.

[81] M. Čapek (editor), *The Concepts of Space and Time*, D. Reidel Publishing Company, Dordrecht, 1976.

[82] M. Čapek, "The Inclusion of Becoming in the Physical World", published in *[81]*, pp. 501 - 524.

[83] J. R. D'Alembert, *Traité de dynamique*, (The first edition: 1743), Johnson Reprint Corporation, New York, 1968.

[84] J. R. D'Alembert, *Essai sur les éléments de philosophie ou sur les principes des connaissances humaines*, (the first edition: 1758), Librairie Arthème Fayard, Paris, 1986.

[85] G. J. Dalenoort, "Cognitive Aspects of the Representation of Time", published in *[15]*, pp. 179 - 188.

[86] A. Das, *The special Theory of Relativity*, Springer-Verlag, New York, 1993.

[87] D. W. Dauer, "Nietzsche and the Concept of Time", published in *[189]*, pp. 81 - 97.

[88] P. C. W. Davies, *The Physics of Time Asymmetry*, The University of California Press, Berkeley, 1974.

[89] H. Dingle, "The 'Clock Paradox' of Relativity", *Nature*, April 27, 1957, pp. 865 - 866.

[90] B. P. Demidovich, *Lectures on Mathematical Theory of Stability*, (in Russian), Nauka, Moscow, 1967.

[91] L. de Broglie, *Matière et lumière*, Albin Michel, Paris, 1937.

[92] L. de Broglie, *Continu et Discontinu en Physique moderne*, Albin Michel, Paris, 1941.

[93] L. de Broglie, *Physique et microphysique*, Albin Michel, Paris, 1947.

[94] L. de Broglie, *Du Temps et du Calendrier*, Corrêa, Buchet/Chastel, 1955.

[95] S. de Grazia, *Of Time Work and Leisure*, The Twentieth Century Fund, New York, 1962.

[96] R. Descartes, *Œuvres philosophiques*, II, Éditions Garnier Frères, Paris, 1967.

[97] R. Descartes, "Discours de la méthode & Essais", in *Œuvres de Descartes*, VI, Librairie philosophique J. Vrin, Paris, 1996.

[98] R. D'Inverno, *Introducing Einstein's Relativity*, Clarendon Press, Oxford, 1995.

[99] P. A. M. Dirac, "The Quantum Theory of the Electron", published in *[269]*, pp. 237 - 256.

[100] P. A. M. Dirac, *General Theory of Relativity*, Princeton University Press, Princeton, New Jersey, 1996.

[101] W. G. Dixon, *Special relativity. The foundation of macroscopic physic*s, Cambridge University Press, Cambridge, 1978.

[102] S. Drake, *Galileo Studies*, The University of Michigan Press, Ann Arbor, 1970.

[103] M. C. Duffy, "Geometrized Space - Time & the Relativistic World Ether", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 162 - 202.

[104] P. Duhem, "The Problem of the Absolute Clock", published in *[81]*, pp. 185 - 186.

[105] P. Duhem, *La théorie physique*, Librairie philosophique J. Vrin, Paris, 1989.

[106] J. Earman, "Till the End of Time", published in *[107]*, pp. 109 - 133.

[107] J. Earman, C. Glymour, and J. Stachel, Editors, *Foundations of Space-Time Theories*, Minnesota Studies in the Philosophy of Science, University of Minnesota Press, Minneapolis, 1977.

[108] A. S. Eddington, *Space Time and Gravitation*, the University Press, Cambridge, 1920.

[109] A. Eddington, *The Philosophy of Physical Science*, Cambridge University Press, Cambridge, 1939.

[110] A. S. Eddington, *The Nature of the Physical World*, The Macmillan Company, New York, 1940.

[111] A. S. Eddington, *The Mathematical Theory of Relativity*, Chelsea Publishing Co., New York, 1975.

[112] A. S. Eddington, "The Arrow of Time, Entropy and the Expansion of the Universe", published in *[81]*, pp. 463 - 470.

[113] J. Ehlers, "Concepts of Time in Classical Physics", published in *[15]*, pp. 191 - 200.

[114] A. Einstein, "Zur Elektrodynamik bewegter Körper", *Annalen der Physik*, 17, June 30, 1905, pp. 891 - 921. English Translation in *[151, pp. 35 - 65]*, *[160, pp. 140 - 171]*, *[269, pp. 187 - 218]*, and *[305, pp. 35 - 65]*. French translation in *[156]*.

[115] A. Einstein, Über die vom Relativitätsprinzip geforderte Trägheit der Energie", *Annalen deer Physik*, 23, 1907, pp. 371 - 384. English translation reads: "On the Inertia of Energy Required by the Relativity Principle", published in *[160, pp. 238 - 250]*.

[116] A. Einstein, "On the Relativity Principle and the Conclusions Drawn from it", (in German), *Jahrbuch der Radioaktivität und Elektronik*, 4, 1907, pp. 411 - 462. English translation in *[160, pp. 252 - 311]*. "Corrections to the Paper: 'On the Relativity Principle and the Conclusions Drawn from it' ", in German, *Jahrbuch der Radioaktivität und Elektronik*, 5, 1907, pp. 98 - 99. English translation in *[160, pp. 316 - 317]*.

[117] A. Einstein, "On the Inertia of Energy required by the Relativity Principle", *Annalen der Physik*, 23, 1907, pp. 371 - 384. Also in [160, pp. 238 - 250].

[118] A. Einstein, "On the Development of our views concerning the Nature and Constitution of Radiation", *Deutsche Physikalische Geselschaft*, Verhandlugen 7, 1909, pp. 482 - 500; *Physikalische Zeitschrift*, 10, 1909, pp. 817 - 826. Also in *[160, pp. 379 - 394]*.

[119] A. Einstein, "The Principle of Relativity and Its Consequences in Modern Physics", *Archives des sciences physiques et naturelles*, 29, 1910, pp. 5-28, 125-144. Also in *[161, pp. 117 - 142]*.

[120] A. Einstein, "The Theory of Relativity", *Naturforschende Gesellschaft in Zürich*. Vierteljahrsschrift 56, 1911, pp. 1 - 14. Also in *[161, pp. 340 - 350]*.

[121] A. Einstein, "On the Ehrenfest Paradox. Comment on V. Varičak's Paper", *Physikalische Zeitschrift,* 12, 1911, pp. 509 - 510. Also in *[161, p. 378]*.

[122] A. Einstein, *Outline of a Generalized Theory of Relativity and of a Theory of Gravitation*, Teubner, Leipzig, 1913. Also in *[161, pp. 302 - 323]*.

[123] A. Einstein, "Comments on "Outline of a Generalized Theory of Relativity and of a Theory of Gravitation" ", *Zeitschrift für Mathematik und Physik*, 62, 1914, pp. 260 - 261. Also in *[161, pp. 579 -581]*.

[124] A. Einstein, "On the Principle of Relativity", original in German in *Vossische Zeitung, Morgen-Ausgabe*, No. 209, April 1914, pp. 33 - 34. Reprinted in *[164, pp. 3 - 5]*.

[125] A. Einstein, "The Formal Foundation of the General Theory of Relativity", original in *German in Königlich Preussiche Akademie der Wissenschaften. Sitzungsberichte*, 1914, pp. 1030 - 1085. Reprinted in *[164, pp. 72 - 128]*.

[126] A. Einstein, "On the Foundations of the Generalized Theory of Relativity and the Theory of Gravitation", *Physikalische Zeitschrift*, 15, 1914, pp. 176 - 180. Also in *[161, pp. 571 - 576]*.

[127] A. Einstein, "On the Relativity Problem", *Scientia*, 15, 1914, pp. 337 - 348. Also in *[161, pp. 608 - 620]*.

[128] A. Einstein, "On the General Theory of Relativity", original in German in *Königlich Preussiche Akademie der Wissenschaften. Sitzungsberichte*, 1915, pp. 778 - 786. Reprinted in *[164, pp. 214 - 223, Addendum: 225 - 228]*.

[129] A. Einstein, "Theory of Relativity", *Die Kultur der Gegenwart. Ihre Entwicklung und ihre Ziele*, Editor: Paul Hinneberg, Part 3, section 3, 1, *Physik*. Editor E. Warburg, Teubner, Leipzig, 1915, pp. 703 - 713. Also in *[161, pp. 535 - 550]*.

[130] A. Einstein, "The Foundation of the General Theory of Relativity", original in German in *Annalen der Physik*, 49, 1916, pp. 769 - 822. Reprinted in *[164, pp. 283 - 337]*.

[131] A. Einstein, *On the Special and the General Theory of Relativity (A Popular Account)*, Vieweg, Braunschweig, 1917. Reprinted in [164, pp. 420 - 534].

[132] A. Einstein, "Cosmological Considerations in the General Theory of Relativity", original in German in *Königlich Preussiche Akademie der Wissenschaften. Sitzungsberichte*, 1917, pp. 145 - 152. Reprinted in *[164, pp. 540 - 551]*.

[133] A. Einstein, "On the Foundations of the General Theory of Relativity", *Annalen der Physi*k, 1918, pp. 241 - 244. Reprinted in *[166, pp. 33- 35]*.

[134] A. Einstein, "Motives for Research", reprinted in *[166, pp. 41- 45]*. Original was published in German in: *"Zu Max Plancks sechzigstem Geburtstag". Ansprachen*, gehalten am 26. April 1918 in *der Deutschen Physikalischen Gesellschaft von E. Warburg*, M. v. Laue, A. Sommerfeld und A. Einstein, C. F. Müllersche Hofbuchhandlung, 1918, pp. 29-32.

[135] A. Einstein, "The Law of Energy Conservation in the General Theory of Relativity" and "Note Added in Proof", reprinted in *[166, pp. 47 - 61]*. Original was published in: *Königlich Preussiche Akademie der Wissenshaften. Sitzungsberichte*, Berlin, 1918, pp. 448 - 459.

[136] A. Einstein, "Review of Hermann Weyl, Space-Time-Matter: Lectures on General relativity", reprinted in *[166, pp. 62 - 63]*. Original was published in: *Die Naturwissenschaften*, 6, 1918, p. 373.

[137] A. Einstein, "Dialogue about Objections to the Theory of Relativity", reprinted in *[166, pp. 66 - 75]*. Original was published in: *Die Naturwissenschaften*, 6, 1918, pp. 697 - 702.

[138] A. Einstein, "A Test of the General Theory of Relativity", reprinted in *[166, Document No. 23]*. Original was published in: *Die Naturwissenschaften*, 7, 1919, p. 776.

[139] A. Einstein,"What is the Theory of Relativity?", reprinted in *[166, pp. 100 - 105]*. Original was published in: The London Times, November 28, 1919. Also in: *Einstein, Ideas and Opinions*, Crown, New York, 1952.

[140] A. Einstein, "Induction and Deduction in Physics", reprinted in *[166, pp. 108, 109]*. Original was published in: *Berliner Tageblatt*, 4. Beiblatt, 25 December, 1919, p. 1.

[141] A. Einstein, "Fundamental Ideas and Methods of the Theory of Relativity, Presented in Their Development", reprinted in *[166, pp. 113 - 150]*.

[142] A. Einstein, "Ether and the Theory of Relativity", reprinted in *[166, pp. 160 - 182]* from: Einstein, Sidelights in Relativity, Dover, New York, 1983. The original was published by Julius Springer, Berlin, 1920.

[143] A. Einstein, "My Response. On the Anti-Relativity Company", reprinted in [166, pp. 197 - 199] from: *Berliner Tageblatt*, Morgen - Ausgabe, 27 August, 1920, pp. 1,2.

[144] A. Einstein, *La théorie de la relativité. Restreinte et généralisée*, Gauthier-Villars et Cie, Paris, 1921.

[145] A. Einstein, *L'éther et la théorie de la relativité*, Gauthier-Villars et Cie, Paris, 1921.

[146] A. Einstein, "On a Natural Addition to the Foundation of the General Theory of Relativity", reprinted in *[166, pp. 224 - 228]*. Original was published in: *Königlich Preussiche Akademie der Wissenshaften. Sitzungsberichte*, Berlin, 1921, pp. 261 - 264.

[147] A. Einstein, "King's College Lecture", *[166, pp. 238 , 239]*.

[148] A. Einstein, "Four Lectures on the Theory of Relativity, Held at Princeton University in May 1921", reprinted in *[166, pp. 261 - 368]*. The original was published by *Vieweg*, Braunschweig, 1921.

[149] A. Einstein, "Physik und realität", *Journal of the Franklin Institute*, 221, No. 3, March, 1936, pp. 313 - 347.

[150] A. Einstein, *The meaning of relativity*, Methuen & Co. Ltd., London, 1950.

[151] "On the Electrodynamics of Moving Bodies" in the book *The Principle of Relativity by H. A. Lorentz*, A. Einstein, H. Minkowski and H. Weyl, Dover Publications, Inc., New York, 1952, pp. 35 - 65. Original in *[114]*. English translation also in *[160, pp. 140 - 171]*.

[152] A. Einstein, *L'ÉTHER et la THÉORIE DE LA RELATIVITÉ*, Gauthier-Villars, Paris, 1953.

[153] A. Einstein, *La théorie de la relativité restreinte et générale*, Gauthier-Villars et Cie, Paris, 1954.

[154] A. Einstein, *RELATIVITY The Special & the General Theory*, University Paperbacks, Methuen & Co. Ltd., London, 1960.

[155] A. Einstein, *RELATIVITY The Special and the General Theory*, Three Rivers Press, New York, 1961.

[156] A. Einstein, S*ur L'ÉLECTRODYNAMIQUE des corps en mouvement*, Paris, Gauthier-Villars, 1965.

[157] A. Einstein, "The Inadequacy of Classical Models of Aether", published in *[81]*, pp. 329 - 337.

[158] A. Einstein, *Autoportrait*, InterÉdition, Paris, 1980.

[159] Albert Einstein, *The Collected Papers. The Early Years: Writings, 1879 - 1902*, 1, Princeton University Press, Princeton, New Jersey, 1987.

[160] Albert Einstein, *The Collected Papers. The Swiss Years: Writings, 1900 - 1909*, 2, Princeton University Press, Princeton, New Jersey, 1989.

[161] Albert Einstein, *The Collected Papers. The Swiss Years: Writings, 1909 - 1911*, 3, Princeton University Press, Princeton, New Jersey, 1993.

[162] Albert Einstein, *The Collected Papers. The Swiss Years: Writings, 1912 - 1914*, 4, Princeton University Press, Princeton, New Jersey, 1995.

[163] Albert Einstein, *The Collected Papers. The Swiss Years: Correspondence*, 1902 - 1914, 5, Princeton University Press, Princeton, New Jersey, 1995.

[164] Albert Einstein, *The Collected Papers. The Berlin Years: Writings, 1914 - 1917*, 6, Princeton University Press, Princeton, New Jersey, 1996.

[165] Albert Einstein, *The Collected Papers. The Berlin Years: Correspondence, 1914 - 1918*, 8, Princeton University Press, Princeton, New Jersey, 1998.

[166] Albert Einstein, *The Collected Papers. The Berlin Years: Writings, 1918 - 1921*, 7, Princeton University Press, Princeton, New Jersey, 2000.

[167] A. Einstein, "The Principal Ideas of the Theory of Relativity", in *[166, pp. 3 - 6]*.

[168] A. Einstein, "King's College Lecture", in *[166, pp. 238, 239]*.

[169] A. Einstein and L. Infeld, *L'évolution des idées en physique*, Flammarion, Paris, 1938.

[170] J. Eisenstaedt, "Le long sommeil de la relativité générale", published in *[46]*, pp. 74 - 78.

[171] G. Esposito-Farèse, "La relativité à l'epreuve des pulsairs binaires", published in *[46]*, pp. 68 - 72.

[172] M. Euler, "Sensations of Temporality: Models and Metaphors from Acoustic Perception", in *[15]*, pp. 159 - 178.

[173] N. Falletta, *Le livre des paradoxes*, Pierre Belfond, Paris, 1985.

[174] A. A. Faus, "Variation of the Speed of Light in an Expanding Cosmological Model", in *[6]*, pp. 355 - 363.

[175] H. Feigl and G. Maxwell, Editors, *Current Issues in the Philosophy of Science*, *Proceedings of the Section L of the American Association for the Advancement of Science*, 1959, Holt, Rinehart and Winston, New York, 1961.

[176] A. Fettweis, "Losslessness in Nonlinear Kirchhoff Circuits and in relativity Theory", *Problems of Nonlinear Analysis in Engineering Systems*, 9, No. 3(19), 2003, pp. 141 - 163. Also: *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 203 - 225.

[177] A. Fettweis, "From wave-digital principles to relativity theory", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 226 - 231.

[178] L. S. Feuer, *einstein et le conflit des générations*, Editions complexe, Bruxelles, 1978, translated (incompletely: except its first part "Les sources philosophiques de Niels BOHR", the chapter 2 was omitted in the French edition) from English: *Einstein and the Generations of Science*, Basic Books, Inc., 1974.

[179] R. P. Feynman, *The Strange Theory of Light and Matter*, Princeton University Press, Princeton, New Jersey, 1985 - the seventh edition 1988. French translation of the first edition: Lumière et matière, InterÉditions, Paris, 1987.

[180] R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman lectures on physics*, Addison-Wesley Publishing Company, London, 1969.

[181] M. H. Fizeau, "Sur les hypothèses relatives à l'éther lumineaux", *Ann. de Chim. et de Phys.*, 3^{e} série, LVII, 1859, pp. 385 - 404.

[182] V. Fock, *The Theory of Space, Time and Gravitation*, Pergamon Press, London, 1959.

[183] J. Foster and J. D. Nightingale, *A Short Course in General Relativity*, Springer-Verlag, New York, 1995.

[184] P. Fraisse, *Psychologie du temps*, Presses universitaires de France, Paris, 1957. English translation: [185].

[185] P. Fraisse, *Psychology of Time*, Eyre & Spottiswoode, London, 1964. Original in French: *[184]*.

[186] G. Franck, "Physical Time and Intrinsic Temporality", published in *[14]*, pp. 63 - 83.

[187] P. Frank, "Is the Future Already Here?", published in *[81]*, pp. 387 - 395.

[188] M.-L. von Franz, *Le temps - le fleuve et la roue*, Chêne, Paris, 1978.

[189] J. T. Fraser and N. Lawrence, Editors, The Study of Time II, *Proceedings of the Second Conference of the International Society for the Study of Time*, Springer Verlag, Berlin, 1975.

[190] M. Friedman, "Simultaneity in Newtoniain Mechanics and Special Relativity", published in *[107]*, pp. 403 - 432.

[191] G. Galilei, *Dialogue Concerning the Two Chief World Systems - Ptolemaic & Copernican*, (the first edition: 1632), (translated by Stillman Drake), University of California Press, Berkeley, 1967. French translation: *Dialogue sur les deux grands systèmes du monde*, Éditions du Seuil, Paris, 1992.

[192] G. Galilei, *Discorsi e dimostrazioni matematiche intorno a DUE NUOVE SCEINZE*, (the first edition: 1638), Editore Boringhieri, Torino, 1958. English translation: *Dialogues concerning Two New Sciences*, Northwestern University, Evanston, 1946.

[193] J.-P. Garnier-Malet, "Modelling and Computing of Anticipatory Embedded System : Application to the Solar System (Speed of Light)", *International Journal of Computing Anticipatory Systems*, 2, 1998, pp. 132 - 156.

[194] J.-P. Garnier-Malet, "Geometrical Model of Anticipatory Embedded Systems", *International Journal of Computing Anticipatory Systems*, 3, 1999, pp. 143 - 160.

[195] J.-P. Garnier-Malet, "The Three Time Flows of Any Quantum of Cosmic Particle", *International Journal of Computing Anticipatory Systems*, 10, 2001, pp. 311 - 321.

[196] J.-P. Garnier-Malet and P. Bobola, "The Relativistic Correction According to the Doubling Theory", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 123 - 140.

[197] J.-P. Garnier-Malet and P. Bobola, "The Explanation of the Big-Bang According to the Doubling Theory", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 232 - 249.

[198] P. Gassendi, "Reality of Absolute Time", published in *[81]*, pp. 195 - 201.

[199] L. Geymonat, *Galilée*, Editions du Seuil, Paris, 1992.

[200] K. Gödel, "Static Interpretation of Space-Time with Einstein's Comment on it", published in *[81]*, pp. 455 - 461.

[201] T. Gold, "The Arrow of Time", *American Journal of Physics*, New York, 30, 1962, pp. 403 - 410.

[202] V. Goldschmidt, *Temps physique et temps tragique chez Aristote*, Librairie philosohique J. Vrin, Paris, 1982.

[203] C. Goldstein et J. Ritter, "*Einstein et la théorie unitaire: 40 ans de perdus* ?", published in [46], pp. 126 - 131.

[204] L. Gouguenheim, *Méthodes de l'astrophysique. Comment connaître et comprendre l'univers*, Hachette, Paris, 1981.

[205] É. Gourgoulhon, "Des ondes gravitationnelles pour scruter le cosmos", published in *[46]*, pp. 102 - 106.

[206] F. Goy, "Non-Invariant Light-Speed and Clock Synchronization in Accelerated Systems", in *Recent Advances in Relativity Theory*, II, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 98 - 106.

[207] H. B. Green, "Temporal Stages in the Development of the Self", published in *[189]*, pp. 1 -19.

[208] Lj. T. Grujić, "Time and modeling", *Proceedins of the 1996 IEEE International Conference on Systems, Man and Cybernetics*, 3, Beijing, China, published by the Institute of Electrical and Electronics Engineers, Beijing - New York, 1996, pp. 2438 - 2443.

[209] Lj. T. Grujić, "Multiple time scale systems, time and modeling", *Proceedings of the IFAC Conference: Control of Industrial Systems*, 1, Pergamon, Elsevier, London, 1997, pp. 189 - 194.

[210] Lj. T. Grujić, "Sets and singularly perturbed systems", *Systems Science*, Wroclaw, Poland, 5, No. 4, 1979, pp. 327 - 338.

[211] Lj. T. Grujić, "Singular perturbations and large-scale systems", *International Journal of Control*, London, 29, No. 1, 1979, pp. 159 - 169.

[212] Lj. T. Grujić, "Singular perturbations, large-scale systems and asymptotic stability of invariant sets", *International Journal of Systems Science*, London, 10, No. 12, 1979, pp. 1323 - 1341.

[213] Lj. T. Grujić, A. A. Martynyuk and M. Ribbens-Pavella, *Large Scale Systems Stability under Structural and Singular Perturbations*, (Russian edition: Naukova Dumka, Kiev, 1984), Springer Verlag, Berlin, 1987.

[214] A. Grünbaum, P*hilosophical Problems of Space and Time*, Routledge & Kegan Paul, Ltd., London, 1963.

[215] A. Grünbaum, "The Exclusion of Becoming from the Physical World", published in *[81]*, pp. 471 - 500.

[216] Ly. T. Gruyitch, "Physical Continuity and Uniqueness Principle. Exponential Natural Tracking Control", *Neural, Parallel & Scientific Computations*, 6, 1998, pp. 143 - 170.

[217] Ly. T. Gruyitch, "Time, relativity and physical principle: generalizations and applications", Proc. V International Conference: Physical Interpretations of Relativity Theory, (Ed. M. C. Duffy), British Society for the Philosophy of Science, (London), 1998, pp. 134 - 155.

[218] Ly. T. Gruyitch, "Time, relativity and physical principle: generalizations and applications", *Neliniyni Koluvannya* (Kiev, Ukraine) 2, N° 4, 1999, 465 - 489.

[219] Ly. T. Gruyitch, "Gaussian generalisations of the relativity theory fundaments with applications", *Proceedings of the VII International Conference: Physical Interpretations of Relativity Theory*, Ed. M. C. Duffy, British Society for the Philosophy of Science, London, 15 - 18 September, 2000, pp. 125 -136.

[220] Ly. T. Gruyitch, "Systems Approach to the Relativity Theory Fundaments", *Nonlinear Analysis*, (London), 2001, 47, 37 - 48.

[221] Ly. T. Gruyitch, "Time and Uniform Relativity Theory Fundaments", *Problems of Nonlinear Analysis in Engineering Systems*, 7, N° 2(14), Kazan, Russia, 2001, pp. 1 - 29.

[222] Ly. T. Gruyitch, "Compatibility and Uniformity in the Relativity Theory: Problems and Solutions", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 176 - 190.

[223] Ly. T. Gruyitch, "Time, Systems, and Control: Qualitative Properties and Methods", in *Stability and Control of Dynamical Systems with Applications, A Tribute to Anthony N. Michel*, Ed's D. Liu and P. J. Antsaklis, Birkhäuser, Boston, 2003, pp. 23 - 46.

[224] Ly. T. Gruyitch, "ON TIME: Einsteinian and non-Einsteinian CC relativity theory", *Problems of Nonlinear Analysis in Engineering Systems*, 9, No. 3(19), 2003, pp. 18 - 54.

[225] Ly. T. Gruyitch, "On the Fundamentals of the CC Relativity Theory", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 257 - 272.

[226] Ly. T. Gruyitch, *EINSTEIN'S RELATIVITY THEORY Correct, Paradoxical, and Wrong*, Trafford, Victoria, 2006.

[227] Ly. T. Gruyitch, *TIME. Fields, Relativity, and Systems*, Llumina Press,Coral Springs, 2006.

[228] Ly. T. Gruyitch, *TIME AND TIME FIELDS. Modeling, Relativity, and Systems Control*, Trafford, Victoria, 2006.

[229] Ly. T. Gruyitch, *Advances in the Linear Dynamic Systems Theory*, Plantation, FL, USA: Llumina Press, 2013.

[230] Ly. T. Gruyitch, *Tracking Control of Linear Systems*, Boca Raton, FL, USA: CRC Press/Taylor & Francis Group, 2013.

[231] Ly. T. Gruyitch, *Galilean - Newtonean Rebuttal to Einstein' sRelativity Theory,* Cambridge: Cambridge International Science Publishing Ltd. Cambridge, UK, 2015.

[232] Ly. T. Gruyitch and A. Kökösy, "Conceptual development of vector Lyapunov functions and control synthesis for mechanical systems", *CD Proceedings of 17th IMACS World Congress*, Paris, paper # T2-I-0572 July 11-15, 2005, pp. 1-8.

[233] Ly. T. Gruyitch and W. P. Mounfield, "Stablewise absolute output natural tracking control with finite reachability time: MIMO Lurie systems", *CD Proceedings of 17th IMACS World Congress*, Paris, paper # T5-I-02-0078 July 11-15, 2005, pp. 1-8. Also in: *Mathematics and Computers in Simulation*, 76, 2008, pp. 330-344.

[234] B. Guy, "The Link Between the Time and Space Concepts and the Meaning of the Theory of Relativity", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 191 - 195.

[235] J.-M. Guyau, La Genèse de l'Idée de Temps, Félix Alcan, Paris, 1890; published also in *[346]*, in French: pp. 37 - 90, in English: pp. 93 - 148.

[236] S. Guye et H. Michel, *Mesures du Temps et de L'Espace*, Office du Livre, Fribourg, 1970.

[237] S. Hajra, "Derivation of Lorentz Transformation Equations and the Exact Equation of Planetary Motion from Maxwell and Newton", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 439 - 444.

[238] S. Hajra, "Alternatives to Einstein in the Light of the Origin and the Development of the Theory of Relativity", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 445 - 462.

[239] G. S. Hall and J. R. Pulham, Editors, General Relativity, *Proceedings of the Forty Sixth Scottish Universities Summer School in Physics*, Aberdeen, 1995.

[240] S. W. Hawking, *A Brief History of Time*, Bantam Press, London, 1988. French translation: *Une brève histoire du temps. Du Big Bang aux trous noirs*, Flammarion, Paris, 1989.

[241] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge, 1991.

[242] S. Hawking, *Black Holes and Baby Universes and Other Essays*, Bantan Books, New York, 1993. French translation: *Trous noirs et bébés univers et autres essais*, Editions Odile Jacob, Paris, 2000.

[243] S. W. Hawking and R. Penrose, *The Nature of Space and Time*, Princeton University Press, Princeton, New Jersey, 1996. In French: La nature de l'espace et du temps, Gallimard, Paris, 1997.

[244] B. J. Hiley and M. Fernandes, "Process and Time", published in *[15]*, pp. 365 - 383.

[245] E. L. Hill, "Quantum physics and the relativity theory", in *Current Issues in the Philosophy of Science, Proceedings of the Section L of the American Association for the Advancement of Science*, (Ed's. H. Feigl and G. Maxwell), Holt, Rinehart and Winston, New York, 1961, pp. 429 - 441.

[246] D. Hoffmann, "L'engagement politique d'Einstein", published in *[46]*, pp. 34 - 38.

[247] L. P. Horwitz, "Time and State-Evolution in Mechanics", in *Recent Advances in Relativity Theory*, II, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 107 - 120.

[248] E. Husserl, *Leçons pour une phénoménologie de LA CONSCIENCE INTIME DU TEMPS*, Press universitaires de France, Paris, 1994.

[249] E. Hussey, *Aristotle's Physics*, Clarendon Press - Oxford University Press, Oxford - New York, 1983.

[250] A. F. Ioffe, "Pamyati Alberta Einsteina" (the original is in Russian, its English translation reads: "In Memory of Albert Einstein"), *Uspehi Fizicheskih Nauk* (published in Russian, its English translation reads: Successes of Physics Sciences), LVII, No. 2, October 1955, pp. 187 - 192.

[251] M. Jacob, "Le temps des processus élémentaires -II", published in *[72]*, pp. 131 - 153.

[252] P. Janet, *L'Évolution de la mémoire et de la notion du temps*, A. Chahine, Paris, 1928.

[253] L. Jánossy, *Theory of Relativity Based on Physical Reality*, Akadémiai Kiadó, Budapest, 1971.

[254] R. C. Jennison, "Proper Time, Proper Length and Some Comments on the Concepts of Time and Distance", published in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 73 - 83.

[255] E. R. Jones and R. L. Childers, *Contemporary College Physics*, Addison-Wesley Publishing Company, Reading, 1993.

[256] A. K. Jonscher and A. de Polignac, "The time dependence of luminescence in solids", *J. Physics C: Solid State Physics*, 17, 1984, pp. 6493 - 6519.

[257] R. E. Kalman, P. L. Falb and Arbib M. A., *Topics in Mathematical System Theory*, Mc Graw Hill, New York, 1969.

[258] G. Kampis, "Biological Evolution as a Process Viewed Internally", published in *[14]*, pp. 85 - 110.

[259] E. Kant, *Critique of pure reason*, Cambridge University Press, Cambridge, 1998. French translation from German: *Critique de la Raison Pure*, Félix Alcan, Paris, 1905.

[260] E. Kant, *Œuvres philosophiques*, II, Gallimard, Paris, 1985.

[261] R. Kastenbaum, "Time, Death and Ritual in Old Age", published in *[189]*, pp. 20 - 38.

[262] A. Katchalsky-Katzir, "Some Thoughts on the Physics and Biology of Time", *Advances in Biological and Medical Physics*, J. H. Lawrence, J. W. Gofman and T. L. Hayes, Eds., 16, *Academic Press, New York*, 1977, pp. 289 - 322.

[263] M. J. Kelly, *Did EINSTEIN Get it WRONG?*, Trafford, Victoria, BC, Canada, 2004.

[264] S. Kern, *The Culture of Time and Space 1880 - 1918*, Weidenfeld and Nicholson, London, 1983.

[265] G. H. Keswani, "A Critique of Relativity", in *Recent Advances in Relativity Theory,* II, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 130 - 155.

[266] G. H. Keswani, "Reflections on Relativity: (1) Intimations of Relativity: Relativity Before Einstein (2) What is Space? (3) Physical Time (4) Is the Space Non-Euclidean in General Relativity?", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 205 - 228.

[267] A. L. Kholmetskii, "The author's collection of relativistic paradoxes in classical electrodynamics", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 314 - 331.

[268] C. Kiefer, "Does Time Exist at the Most Fundamental Level?", published in *[15]*, pp. 227 - 240.

[269] C. W. Kilmister, *Special Theory of Relativity*, Pergamon Press, Oxford, 1970.

[270] E. Klein, *Le Temps*, Dominos, Flammarion, Paris, 1995.

[271] E. Klein et M. Spiro, "L'Esprit du Temps", published in *[72]*, pp. 11 - 13.

[272] J. Klose, "Whitehead's Theory of Perception", published in *[15]*, pp. 23 - 42.

[273] P. V. Kokotović, "Subsystems, Time Scales and Multimodeling", *Automatica*, London, 17, 1981, No. 6, pp. 789 - 795.

[274] P. V. Kokotović, H. K. Halil and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, Academic Press, New York 1986, SIAM, Philadelphia, 1999.

[275] P. V. Kokotović, R. E. O'Malley Jr., and P. Sannuti, "Singular perturbations and order reduction in control theory - An overview", *Automatica*, No. 12, 1976, pp. 123 - 132.

[276] E. I. Kolosova, V. A. Krakau, N. I. Kukulevskaya, L. L. Myasnikov, E. N. Myasnikova and N. I. Stepanova, *Course in physics. First part: Relativity theory. Oscillations and waves. Acoustics. Waves and particle*, (In Russian), LKI, Leningrad, 1971.

[277] A. Kostelecký, "La relativité est-elle inviolable ?", published in *[46]*, pp. 55 - 61.

[278] B. Kouznétsov, *Essais sur la relativité*, Mir, Moscou, 1971.

[279] A. F. Kracklauer, "Palacios, Relativity Without Asymmetric Aging", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 332 - 345.

[280] F. M. Kronz, "Theory and Experience of Time: Philosophical Aspects", published in *[15]*, pp. 7 - 22.

[281] A. Kull, "Self-Reference and Time According to Spencer Brown", published in *[15]*, pp. 71 - 79.

[282] M. Lachièze-Rey, "A la recherche du temps cosmique", published in *[72]*, pp. 81 - 92.

[283] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Butterworth-Heinemann, Oxford, 2003.

[284] P. Langevin, "Le temps, l'espace et la causalité", *Bulletin de la Société Française de Philisophie*, Séanse du 19 Octobre 1911, pp. 2 - 28.

[285] P. Langevin, "L'évolution de l'espace et du temps", *Scienti*a, X, pp. 31 - 54, 1911.

[286] J. Laski, "Comparison of the classical and relativistic Doppler formulae to those of Einstein-Minkowski and Poincare-Lorentz", *Proceedings of Physical Interpretations of Relativity Theory I*X, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 346 - 354.

[287] J.-P. Lasota, "La chasse aux trous noirs", published in *[46]*, pp. 90 - 95.

[288] R. Lehoucq, "Voyager dans le temps ?", published in *[46]*, pp. 140 - 142.

[289] G. W. Leibniz, *Principes de la nature et de la grâce fondés en raison. Principes de la philosophie ou Monadologie*, Presses Universitaires de France, Paris, 2002.

[290] G. W. Leibniz and S. Clarke, "Discussion on the Nature of Space and Time", published in *[81]*, pp. 273 - 288.

[291] J.-M. Lévy-Leblond, "Quel temps fait-on?", published in *[72]*, pp. 271 - 281.

[292] R. B. Lindsay and H. Margenau, "Time: Continuous or Discrete", published in *[81]*, pp. 533 - 534.

[293] F. Le Lionnais, Le Temps, Robert Delpire, Paris, 1959.

[294] J. Locke, "On Succession and Duration", published in *[81]*, pp. 211 - 224.

[295] M. Lorente, "Quantum Processes and the Foundation of Relational Theories of Space and Time", in *[6]*, pp. 297 - 302.

[296] H. A. Lorentz, *De relatieve beweging van de aarde en den aether, Verslagen der Afdeeling Natuurk*, Koninklyke Nederlandse Academie van Wetenschappen, A°, 1892, pp. 74 - 79.

[297] H. A. Lorentz, *Versuch einer theorie der electrischen und optischen Erscheinungen in bewegten Körpern*, E. K. Brill, Leiden, 1895.

[298] H. A. Lorentz, "Electromagnetische verschijnselen in een stelsel dat zich met willekeurige snelheid, kleiner dan die van het licht, beweegt", *Verslagen der Afdeeling Natuurk, Koninklyke Nederlandse Academie van Wetenschappen*, Dl. XII, A°, 1904, pp. 986 - 1009. English translation: "Electromagnetic Phenomena In a System Moving With Any Velocity Less Than That of Light", Reprinted from the English version in *Proceedings of the Academy of Sciences of Amsterdam*, 6, 1904, in [305], pp. 9 - 33; also in [269], pp. 119 - 143.

[299] H. A. Lorentz, *Das Relativitätsprinzip*, B. G. Teubner, Leipzig, 1914.

[300] H. A. Lorentz, *The theory of Electrons*, B. G. Teubner, Leipzig, 1916.

[301] H. A. Lorentz, "Deux mémoires de Henri Poincaré sur la physique mathématique", *Acta Mathematica*, Journal edited by G. Mittag - Leffler, Berlin - Paris, 38, 1921, pp. 293 - 308.

[302] H. A. Lorentz, *Lectures on THEORETICAL PHYSICS*, 1, MacMillan and Co, Ltd. 1927.

[303] H. A. Lorentz, "Michelson's Interference Experiment", published in *[305]*, pp. 3 - 7.

[304] H. A. Lorentz, A. Einstein and H. Minkowski, *Das Relativitätsprinzip*, B. G. Teubner, Leipzig, 1923.

[305] H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, *The Principle of Relativity*, Dover Publications, Inc., New York, 1952.

[306] J. R. Lucas and P. E. Hodgson, *Spacetime and Electromagnetism*, Clarendon Press, Oxford, 1990.

[307] Lucrèce, *La nature des choses*, Arléa, Paris, 1992.

[308] J.-P. Luminet, "Matière, Espace, Temps", published in *[72]*, pp. 59 - 80.

[309] J.-P. Luminet, *Les Trous noirs*, Éditions Belfond, France, 1987.

[310] A. M. Lyapunov, *The General Problem of Stability of Motion*, (in Russian), Kharkov Mathematical Society, Kharkov, 1892; in *Academician A. M. Lyapunov: "Collected Papers"*, U.S.S.R. Academy of Science, Moscow, II, pp. 5 - 263, 1956. French translation: "Problème général de la stabilité du mouvement", *Ann. Fac. Toulouse*, 9, pp. 203 - 474; also in: *Annals of Mathematics Study*, No. 17, Princeton University Press, 1949. English translation: *Intern. J. Control*, 55, pp. 531 - 773, 1992; also the book, *Taylor and Francis*, London, 1992.

[311] H. Lyre, "Time and Information", pubelished in *[15]*, pp. 81 - 89.

[312] G. Mahler, "Temporal Bell Inequalities: A Journey to the Limits of "Consistent Histories" ", published in *[14]*, pp. 195 - 205.

[313] G. Mahler, "Nonlocality in Quantum Dynamics", published in *[15]*, pp. 347 - 363.

[314] K. Mainzer, *The little book of TIME*, Copernicus Books - Srpinger Verlag, New York, 2002.

[315] J.-P. G. Malet and P. Bobola, "The Relativistic Correction According to the Doubling Theory", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 123 - 140.

[316] J.-P. G. Malet and P. Bobola, "The Explanation of the Big-Bang According to the Doubling Theory", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 232 - 249.

[317] S. Malin, "Delayed-Choice Experiments and the Concept of Time", published in *[15]*, pp. 43 - 52.

[318] A. March, *La physique moderne et ses théories*, Gallimard, Paris, Random House, In., New York, 1962.

[319] L. Marder, *Time and the Space-Traveller*, George Allen & Unwin Ltd, London, 1971.

[320] P. Marmet, *Einstein's Theory of Relativity versus Classical Mechanics*, Newton Physics Books, 2401 Ogilvie Road, Gloucester, ON, Canada, K1J 7N4, 1997.

[321] P. Marmet, "Relationship between $E = mc^2$ and $m = \gamma m_o$", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: faq/gamma-mass-13.html*, updated May 2002 (the text cited was copied for this book June 7, 2003).

[322] P. Marmet, "Natural Length Contraction Mechanism Due to Kinetic Energy", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: kinetic/length.html*, June 10, 2003.

[323] P. Marmet, "Natural Physical Length Contraction Due to Gravity", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: Gravity/gravity.html*, June 10, 2003.

[324] P. Marmet, "The GPS and the Constant Velocity of Light", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: Illusion/index.html*, June 10, 2003.

[325] P. Marmet, "Mathematical Error in the Lorentz Transformation", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: Lorentz/lorentz.html*, June 10, 2003.

[326] P. Marmet, "Fundamental Nature of Relativistic Mass and Magnetic Fields", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: magnetic/mass.html*, September 23, 2003. Published in *Problems of Nonlinear Analysis in Engineering Systems*, 9, No. 3(19), 2003, pp. 164 - 186.

[327] P. Marmet, "Big Bang Cosmology Meets an Astronomical Death", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: BIGBANG/Bigbang.html#Author*, February 24, 2004.

[328] P. Marmet, "The Collapse of the Lorentz Transformation", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: lorentz/lorentz.html*, June 8, 2004.

[329] P. Marmet, "The Overlooked Phenomena in the Michelson-Morley Experiment", the Internet site by Paul Marmet: *http://www.newtonphysics.on.ca/, then go to: michelson/michelson.html*, June 8, 2004.

[330] P. Marmet, "Realistic Lorentz Transformation", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 278 - 289.

[331] P. Marmet, "The overlooked phenomena in the Michelson-Morley experiment", *Problems of Nonlinear Analysis in Engineering Systems*, 11, No. 3(24), 2005, pp. 37 - 51.

[332] A. Martin, "Reception of Light Signals in Galilean Space-Time", published in *[444]*, pp. 47 - 55.

[333] W. Mays, "Temporality and Time in Hegel and Marx", published in *[189]*, pp. 98 - 114.

[334] J. C. Maxwell, *Matter and Motion*, Dover Publications, Inc., New York, 1991.

[335] J. C. Maxwell, "Absolute Time and the Order of Nature", published in *[81]*, pp. 231 - 232.

[336] A. Le Méhauté, R. R. Nigmatullin and L. Nivanen, *Flèches du temps et géométrie fractale*, Hermes, Paris, 1998.

[337] G. H. Mehlberg, "Physical laws and time's arrow", published in *[175]*, pp. 105 - 138, 1961.

[338] A. Mercier, "From Metaphisics to Physics: The Reconstruction of Space-Time as Time-Space", in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 123 - 132.

[339] J. Merleau-Ponty and B. Morando, *Les trois étapes de la cosmologi*e, Editions Robert Laffont, Paris, 1971.

[340] H. Meyerhoff, *Time in Literature*, University of California Press, Berkeley, 1955.

[341] E. Meyerson, "The Elimination of Time in Classical Science", published in *[81]*, pp. 255 - 264.

[342] E. Meyerson, "On Various Interpretations of the Relativistic Time", published in *[81]*, pp. 353 - 362.

[343] A. N. Michel and L. Hou, "Modeling and Qualitative Theory for General Hybrid Dynamical and Control Systems", a) *Preprints of the IFAC Conference: Control of Industrial Systems*, 1, IFAC, Belfort, April 1997, pp. 173 - 183, and b) *Proceedings of the IFAC Conference: Control of Industrial Systems*, 1, Pergamon, Elsevier, London, 1997, pp. 77 - 87.

[344] A. A. Michelson, "The Relative Motion of the Earth and the Luminiferous Ether", published in *[269]*, pp. 91 - 104.

[345] A. A. Michelson and E. W. Morley, "On the Relative Motion of the Earth and the Luminiferous Ether", *The American Journal of Science*, Third Series, XXXIV, No. 203, November 1887, pp. 333 - 345.

[346] J. A. Michon, with V. Pouthas and J. L. Jackson, Editors, *Guyau and the Idea of Time*, North-Holland Publishing Company, Amsterdam, 1998.

[347] W. I. Milham, *Time & Timekeepers*, MacMillan andCo., Limited, London, 1923.

[348] H. Minkowski, "Space and Time", Address delivered at the 80th Assembly of German Natural Scientitst and Physicians, at Cologne, 21 September, 1908, which was published in *[305]*, pp. 73 - 91, *[446]*, pp. 297 - 312.

[349] H. Minkowski, "The Union of Space and Time", published in *[81]*, pp. 339 - 351.

[350] E. F. Mischenko, Yu. S. Kolesov, A. Yu. Kolesov and N. Kh. Rozov, *Asymptotic Methods in Singularly Perturbed Systems*, Consultants Bureau, New York, 1994.

[351] R. A. Mould, *Basic Relativity*, Springer, New York, 2002.

[352] A. P. D. Mourelatos, Editor, *The Pre-Socrates*, Anchor Books, Garden City, 1974.

[353] J. G. Muga, "Characteristic Times in One-Dimensional Scattering", published in *[354]*, pp. 29 - 67.

[354] J. G. Muga, R. Sala Mayato and I. L. Egusquiza (Editors), *Time in Quantum Mechanics*, Springer-Verlag, Berlin, 2002.

[355] J. G. Muga, R. Sala Mayato and I. L. Egusquiza (Editors), "Introduction", published in *[354]*, pp. 1 - 27.

[356] D. S. Naidu, *Singular Perturbation Methodology in Control Systems*, Peter Peregrinus Ltd., London, 1988.

[357] C. Neumann, "On the Definition of the Equality of Successive Intervals of Time", published in *[81]*, pp. 233 - 234.

[358] A. V. Nesteruk, "Creation of the Universe & Quantum Cosmology", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 329 - 354.

[359] R. Nevanlinna, *Raum, Zeit und Relativität*, Birkhäuserverlag, Basel, 1964.

[360] Sir Isaac Newton, *Mathematical Principles of Natural Philosophy - BOOK I. The Motion of Bodies*, William Benton, Publisher, ENCYCLOPAEDIA BRITANNICA, Inc., Chicago, (The first publication: 1687) 1952.

[361] H. Nordenson, *Relativity - Time and Reality*, George Allen and Unwin Ltd, London, 1969.

[362] L. Nottale, *Fractal Space - Time and Microphysics*, World Scientific, Singapore, 1993.

[363] L. Nottale, "Scale Relativity: First steps Toward a Field Theory", published in *[6]*, pp. 121 - 132.

[364] L. Nottale, *La relativité dans tous ses états*, Hachette Littératures, Paris, 1998.

[365] L. du Noüy, *Le temps et la vie*, Gallimard, Paris, 1936.

[366] H. Nowotny, "Time Structuring and Time Measurement: On the Interrelation Between Timekeepers and Social Time", published in *[189]*, pp. 325 - 342.

[367] H. C. Ohanian, *Physics*, W. W. Norton & Company, New York, 1989.

[368] R. E. Ornstein, *On the experience of time*, Penguin Books, Middlesex, 1969.

[369] V. Ougarov, *Théorie de la relativité restreinte*, Nauka, Editions Mir, Moscou, 1969.

[370] G. Oyibo, "Generalized Mathematical Proof of Einstein's Theory Using A New Group Theory", *Problems of Nonlinear Analysis in Engineering Systems*, Kazan, Russia, 1, No. 2, 1995, pp. 22 - 29.

[371] A. Pais, "Subtle is the Lord". *The Science and the Life of Albert Einstein*, Oxford University Press, Oxford, 1982.

[372] D. Park, "Laws of Physics and Ideas of Time", published in *[189]*, pp. 258 - 266.

[373] Parmenide, Poème, published in *[71]*, pp. 10 - 25.

[374] M. Paty, "Sur l'histoire du problème du temps. Le temps physique et les phénomènes", published in *[72]*, pp. 21 - 58.

[375] M. Paty, *Einstein philosophe*, Press universitaires de France, Paris, 1993.

[376] M. Paty, "1905, l'année admirable", published in *[46]*, pp. 26 - 33.

[377] R. Penrose, *L'esrpit, l'ordinateur et les lois de la physique*, InterEditions, Paris, 1992.

[378] Y. Pierseaux, "Einstein's Special Relativity (Without Quantum Clocks) and Poincare's Special Relativity (With Classical Clocks) in "Einstein's General Relativity", *Proceedings of Physical Interpretations of Relativity Theory VIII*, (Imperial College, London, 6 - 9 September 2002), Ed. M. C. Duffy, PD Publications, Liverpool, 2003, pp. 397 - 432.

[379] Platon, *Timée*. Critias, Flammarion, Paris, 1992.

[380] Plotinus, "Criticism of the Relational Theories of Time", published in *[81]*, pp. 173 - 178.

[381] H. Poincaré, *La Science et l'Hypothèse*, Ernest Flammarion, (26, rue Racine), Paris, 1902, 1943.

[382] H. Poincaré, "The Principles of Mathematical Physics", An address delivered before t*he International Congress of Arts and Science*, St. Louis, September 1904, published in The MONIST, XV, No. 1, 1905, pp. 1 - 24.

[383] H. Poincaré, "Sur la dynamique de l'éléctron", *Comte rendus de l'Académie des sciences*, Paris, July 23, 1905, pp. 1504 - 1508; also in Rendiconti del circolo matem. di Palermo, 21, Palermo, 1906, pp. 130 - 176, and in *[269]*, pp. 145 - 185.

[384] H. Poincaré, "Les rapports de la matière et de l'éther", *Journal de physique théorique et appliquée*, 2, 1912, pp. 347 - 360.

[385] H. Poincaré, *La Valeur de la Science*, Ernest Flammarion, Paris, 1913.

[386] H. Poincaré, "The Measure of Time", published in [81], pp. 317 - 327.

[387] M. F. Podlaha, "Matter Waves & Universal Fields", in *Recent Advances in Relativity Theory*, II, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 213 - 217.

[388] E. Pöppel, "The Brain's Way to Create "Nowness", published in *[15]*, pp. 107 - 120.

[389] G. Poulet, *Etudes sur le temps humain*, Plon, Paris, 1966.

[390] G. Poulet, *Etudes sur le temps humain. II La distance intérieure*, Plon, Paris, 1961.

[391] G. Poulet, *Etudes sur le temps humain. III Le point de départ*, Plon, Paris, 1961.

[392] G. Poulet, Etudes sur le temps humain. *IV Mesure de l'instant*, Plon, Paris, 1968.

[393] I. Prigogine, *Etude Thermodynamique des Phénomènes Irréversibles*, Dunod, Paris, Editions Desoer, 1947.

[394] I. Prigogine, *Non-Equilibrium Statistical Mechanics*, Interscience Publishers, John Wiley & Sons, New York, 1962.

[395] I. Prigogine, *From Being to Becoming. Time and complexity in the physical sciences,* W. H. Freeman and Co., San Francisco, 1980.

[396] I. Prigogine, *Physique temps et devenir*, Masson, 1982.

[397] I. Prigogine and I. Stengers, *La nouvelle alliance*, Gallimard, Paris, 1979.

[398] I. Prigogine and I. Stengers, *Entre le temps et l'éternité*, Fayard, Paris, 1988.

[399] H. Primas, "The Representation of Facts in Physical Theories", published in *[15]*, pp. 243 - 264.

[400] A. N. Prior, *Papers on Time and Tense*, The Clarendon Press, Oxford, 1968.

[401] S. J. Prokhovnik, *Logique de la relativité restreinte*, Gauthier-Villars, Paris, 1969.

[402] S. Psillos, "Conceptions & Misconceptions of 'Ether' ", in *Recent Advances in Relativity Theory*, II, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 236 - 248.

[403] R. J. Quinones, "Four Phases of Time and Literary Modernism", published in [189], pp. 122 - 135.

[404] H. Reichenbach, *The Philosophy of Space and Time*, Dover Publications, Inc., New York, 1958.

[405] H. Reichenbach, "Comment on the Clock Paradox", published in *[81]*, pp. 447 - 449.

[406] H. Reichenbach, *The Direction of Time*, University of California Press, Berkeley, 1991.

[407] S. Reynaud, "Les fluctuations d'Einstein", published in *[46]*, pp. 108 - 112.

[408] I. W. Richardson, "The Metrical Structure of Aging (Dissipative) Systems", *J. theor. Biol.*, 85, pp. 745 - 756.

[409] I. W. Richardson and R. Rosen, "Aging and the Matrics of Time", *J. theor. Biol.*, 79, pp. 415 - 423.

[410] C. P. Richter, "Astronomical Refrences in Biological Rhytms", published in [189], pp. 39 - 53.

[411] P. Ricœur, *Temps et récit*, I, Éditions du Seuil, Paris, 1983.

[412] P. Ricœur, *Temps et récit*, II, Éditions du Seuil, Paris, 1984.

[413] P. Ricœur, *Temps et récit*, III, Éditions du Seuil, Paris, 1985.

[414] W. Rindler, *Introduction to Special Relativit*y, Clarendon Press, Oxford, 1991.

[415] W. Rindler, *RELATIVITY Special, General, and Cosmological*, Oxford University Press, 2001.

[416] A. Robb, *Geometry of Time and Space*, Cambridge University Press, Cambridge, 1936.

[417] A. A. Robb, "The Conical Order of Time-Space", published in *[81]*, pp. 369 - 386.

[418] L. Robert, "Le temps en biologie - le viellissement", published in *[72]*, pp. 213 - 237.

[419] L. Robert, *Mécanismes cellulaires et moléculaires du vieillissement*, Masson, Paris, 1983.

[420] L. Robert, *Les horloges bilogiques*, Nouvelle Bibliothèque Scientifique - Flammarion, Paris, 1989.

[421] L. Robert, *LE VIEILLISSEMENT. De l'homme à la cellule*, CNRS Editions, Belin, Paris, 1994.

[422] G. Rochberg, "The Structure of Time in Music: Traditional and Contemporary Raminifications and Consequences", published in *[189]*, pp. 136 - 149.

[423] R. Rosen, "Dynamical Similarity and the Theory of Biological Transformations", *Bulletin of Mathematical Biology*, 40, 1978, pp. 549 - 579.

[424] F. Rothen, *Physique générale*, Presses polytechniques et universitaires romandes, Lausanne, 1999.

[425] R. Rucker, *Infinity and Mind*, Princeton University Press, Princeton, New Jersey, 1995.

[426] E. Ruhnau, "The Now - A Hidden Window to Dynamics", published in *[14]*, pp. 291 - 308.

[427] E. Ruhnau, "The Deconstruction of Time and the Emergence of Temporality", published in *[15]*, pp. 53 - 69.

[428] B. Russell, *ABC de la relativié*, Union Général d'Editions, Saint-Amand, 1965.

[429] B. Russell, "On Zeno's Paradoxes", published in *[81]*, pp. 235 - 243.

[430] B. Russell, "On Change, Time and Motion", published in *[81]*, pp. 251 - 254.

[431] V. L. Rvachev, "Non-Archimedean arithmetic and other constructive mathematical tools, based on the ideas of the special relativity theory", *Dokladi Akademii Nauk SSSR*, 316, No. 4, 1991, pp. 884 - 889.

[432] M. Sachs, "On the Meaning of Space and Time", in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 170 - 176.

[433] Saint Augustine, *The City of God*, Image Books, New York, 1958.

[434] Saint Augustin, *Les confessions*, Garnier Frères, Paris, 1964.

[435] S. Sambursky, "The Stoic Views of Time", published in *[81]*, pp. 159 - 166.

[436] S. Sambursky, "The Stoic Doctrine of Eternal Recurrence", published in *[81]*, pp. 167 - 171.

[437] R. M. Santilli, "Direct Universality of Isospecial Relativity for Generalized Spacetimes", in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 177 - 194.

[438] M. Schlick, *Space and Time in Contemporary Physics*, Dover Publications, Inc, New York, 1963.

[439] A. Schopenhauer, "On the Necessary Attributes of Time and Space", published in *[81]*, pp. 227 - 230.

[440] E. Schrödinger, *Space-Time Structure*, Cambridge University Press, Cambridge, 1954.

[441] F. Selleri, "Space-time Transformations in Ether Theories", *Z. Naturforsch*, 46A, 1990, pp. 419 - 425.

[442] F. Selleri, "Inertial Systems and the Transformations of Space and Time", *Physics Essays*, 8, No. 3, 1995, pp. 342 - 349.

[443] F. Selleri, "On the Anisotropy of Light Propagation Observed by Miller and by Kennedy & Thorndike", in *Recent Advances in Relativity Theory*, II, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 279 - 283 .

[444] F. Selleri, Editor, *Open Questions in Relativistic Physics*, Apeiron, Montreal, 1998.

[445] L. Silberstein, *The Size of the Universe*, Oxford University Press, 1930.

[446] J. J. C. Smart, *Problems of Space and Time*, The Macmillan Co., New York, 1973.

[447] D. R. Smith, *Singular-perturbation theory*, Cambridge University Press, Cambridge, 1985.

[448] Society of Physicists of Serbia and Montenegro, *World Year of Physics*, (in Serb), Society of Physicists of Serbia and Montenegro, Galery of Science and Technics of SANU and Swiss Agency for Development and Cooperation, Exhibition Issue No. 22, Belgrade, November 2005, pp. 1 - 17.

[449] Spinoza, *Œuvres, III Éthique*, GF Flammarion, Paris, 1965.

[450] R. K. Sprague, "Parmenides: A Suggested Rearrangement of Fragments in the 'Way of Truth' ", *Classical Philology*, 50, 1995, pp. 124 - 126.

[451] St. Augustine, "Views on Time", published in *[81]*, pp. 179 - 183.

[452] St. Augustine, "Questions about Time", published in *[446]*, pp. 58 - 72.

[453] *St. Augustine's Confessions*, I, Harvard University Press, Cambridge, Massachussetts, 1989.

[454] *St. Augustine's Confessions*, II, Harvard University Press, Cambridge, Massachussetts, 1988.

[455] A. P. Stone, "Time as Chronos and Kairos", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 465 - 466.

[456] A. P. Stone, "Time as Chronos and Kairos. Physical and metapysical time", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 467 - 476.

[457] T. Suntola, "Dynamic space converts relativity into absolute time and distance", *Proceedings of Physical Interpretations of Relativity Theory IX*, (Imperial College, London, 3 - 6 September 2004), Ed. M. C. Duffy, PD Publications, Liverpool, 2004, pp. 477 - 492.

[458] S. H. Talbert, "On Some Intrinsic Attributes of Motion and Propagation", *Proceedings of the VIII International Conference Physical Interptretations of Relativity Theory*, London, September 6 - 9, 2002, pp. 569 - 584.

[459] N. A. Tambakkis, "On the Question of Physical Geometry", published in *[444]*, pp. 141 - 147.

[460] E. F. Taylor and J. A. Wheeler, *Spacetime physics*, W. H. Freeman and Company, San Francisco 1966.

[461] B. Telesio, "Independence of Time from Motion", published in *[81]*, pp. 187 - 188.

[462] A. N. Tikhonov, "Systems of differential equations containing small parameters in the derivatives", *Mathematicheski Sbornik*, (in Russian), Moscow, 31, 1952, pp. 576 -586.

[463] S. C. Tiwari, "Reality, Intuition, and Mind. Reflections on the Philosophy of H. Poincaré", in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 225 - 231.

[464] W. Thirring, *Classical Mathematical Physics*, Springer, New York, 1992.

[465] C. A. Tobias, "Human and Scientific Concepts of Time", *Advances in Biological and Medical Physics*, J. H. Lawrence, J. W. Gofman and T. L. Hayes, Eds., 16, *Academic Press*, New York, 1977, pp. 267 - 287.

[466] B. J. Tonkinson, "Clocks Don't Go Slow. Rods Don't Contract", in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 232 - 244.

[467] M.-A. Tonnelat, *Histoire du principe de relativité*, Flammarion, Paris, 1971.

[468] A. A. Tyapkin, "On the History of the Special Relativity Concept", published in *[444]*, pp. 161 - 174.

[469] J.-P. Uzan, "Que cache la constante cosmologique ?", published in *[46]*, pp. 96 - 100.

[470] A. V. Vasiliev, *Space·Time·Motion*, Chatto & Windus, London, 1924.

[471] C. Vilain, *L'Espace et le mouvement*, U.F.R. de Physique, Paris VII, G.A.AR. Observatoire de Meudon, 1988.

[472] W. Voigt, "Ueber das Doppler'sche Princip", *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg* - Augusts - Universität zu Göttingen - *Königlichen Gesellschaft der Wissenschaften*, No. 2, 1887, pp. 41 - 52.

[473] W. Voisé, "On Historical Time in the Works of Leibniz", published in *[189]*, pp. 114 - 121.

[474] C. F. von Weizsäcker, "Time - Empirical Mathematics - Quantum Theory", published in *[15]*, pp. 91 - 104.

[475] R. M. Wald, Space, *Time and Gravity*, The University of Chicago Press, Chicago, 1992.

[476] E. H. Walker, "Did Einstein Espouse His Spouse's Ideas", *Physics Today*, February 1989, pp. 9 - 11.

[477] M. Wegener, "Ideas of Cosmology. A Philosopher's Synthesis", in *Recent Advances in Relativity Theory*, I, M. C. Duffy and M. Wegener, Editors, Hadronic Press, Palm Harbor, 2000, pp. 255 - 274.

[478] J. P. Wesley, "Evidence for Newtonian Absolute Space and Time", published in *[444]*, pp. 257 - 263.

[479] M. Wetzel, *Le temps*, Editions Quintette, Paris, 1990.

[480] H. Weyl, *Space - Time - Matter*, Dover Publications, Inc., New York, 1952.

[481] A. N. Whitehead, *The Principle of Relativity*, Cambridge University Press, Cambridge, 1922.

[482] A. N. Whitehead, *Science and the Modern World*, The Free Press, New York, 1925.

[483] A. N. Whitehead, *Process and Reality*, the first edition 1922 by Macmillan Publishing Company, Free Press, New York, 1978.

[484] A. N. Whitehead, "Comment on the Paradox of the Twins", published in *[81]*, pp. 441 - 446.

[485] A. N. Whitehead, "The Inapplicability of the Concept of Instant on the Quantum Level", published in *[81]*, pp. 535 - 537.

[486] G. J. Whitrow, "'Becoming' and the Nature of Time", published in *[81]*, pp. 525 - 532.

[487] G. J. Whitrow, *The Natural Philosophy of Time*, Clarendon Press, Oxford, 1980.

[488] J. Whittaker, *God·Time·Being*, Andelsbogtrykkeriet i Odense, Oslaoe, 1971.

[489] E. Wigner, "On Unitary Representations of the Inhomogeneous Lorentz Group", published in *[269]*, pp. 267 - 295.

[490] N. Wiener, "Spatio-Temporal Continuity, Quantum Theory and Music", published in *[81]*, pp. 539 - 546.

[491] J. Wild, "The New Empiricism and Human Time", *Review of Metaphysics*, 7, 1954, pp. 537 - 557.

[492] C. M. Will, "The Confrontation between General Relativity and Experiment. A 1995 Update", published in *[239]*, pp. 239 - 281.

[493] N. M. J. Woodhouse, *Special relativity*, Springer-Verlag, London, 2003.

[494] P. Yam, "Du GPS au DVD", published in *[46]*, pp. 40 - 43.

[495] W. Yourgrau, "Some problems concerning fundamental constants in physics" published in *[175]*, 1961, pp. 319 - 342.

[496] P. Zeeman, "Fresnel's Coefficient for Light of Differrent Colours (First Part)", published in *[269]*, pp. 229 - 236.

[497] H.-D. Zeh, *The Physical Basis of The Direction of Time*, Springer Verlag, Berlin, 1989.

[498] R. Zouckermann, *Galilée, penseur libre*, Éditions de l'Union rationaliste, Paris, 1968.

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ABOUT THE AUTHOR

Lyubomir T. Gruyitch, DSc, was a Professor at the Ecole Nationale d'Ingénieurs, which integrated with the Institut Polytechnique de Sévenans at the University of Technology Belfort–Montbeliard, in France. He was also the AECI Professor of Control in the Department of Electrical Engineering at the University of Natal, Durban, South Africa, and a Professor of Automatic Control in the Faculty of Mechanical Engineering at the University of Belgrade, Serbia, as well as a visiting professor at Ecole Centrale, Lille, France; Louisiana State University, Baton Rouge, Louisiana; and the University of Notre Dame, Notre Dame. He has continued his research, lecturing, and consulting activity.

Dr. Gruyitch is the author of several published books and many scientific papers on dynamical systems, control systems, and time and its relativity. He has participated at many scientific conferences throughout the world and has been honored with several awards and honors, including Doctor *Honoris Causa* by the French Republic, the highest award presented by the Faculty of Mechanical Engineering, University of Belgrade, for teaching and scientific contributions to the faculty, 1964–1992, and an award from the Yugoslav Air Force Academy for teaching achievements in the undergraduate course Foundations of Automatic Control.

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