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Time-Dependent Switched Discrete-Time Linear Systems: Control and Filtering

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To my parents and my sisters

—Lixian Zhang

To my parents and my sister

—Yanzheng Zhu

To Fengmei, Lisa and Michael

—Peng Shi

To my family

—Qiugang Lu

Preface

Switched systems, which are used to model many physical or man-made systems displaying switching features, have been extensively studied since the 1990s. Typical applications of switched systems can be seen in engineering practice such as in the vehicle industry, process control, biological systems, and flight control systems. These kinds of systems commonly contain a finite number of subsystems and a switching signal governing the switching among them. The switching signals are therefore crucial in dominating the behaviors of switched systems, which differentiate switched systems from the general time-varying systems, since the solutions of the former are dependent on both system initial conditions and switching signals.

From a perspective of whether a stochastic process is attached to the representation of switching signals, the diverse switching can be categorized as nondeterministic switching versus stochastic switching. Unlike stochastic switching systems where a stochastic process, e.g., a Markov chain can greatly avail the analysis and synthesis of the systems, nondeterministic switched systems are relatively more challenging to cope with. A typical way in this area is to classify the switching signals into different sets with certain time regularities while ignoring the concrete generation mechanism, such as the state-dependent switching or time-dependent switching with the latter being able to represent the former in general. This book refines attention on nondeterministic switched systems that are commonly briefly termed as *switched systems* in the literature, and mainly considers four types of time-dependent switching signals: the arbitrary switching, dwell time (DT) switching, average dwell time (ADT) switching, and persistent dwell time (PDT) switching (the mode-dependent forms of the latter three cases are also considered). The aim is to present the authors' previous findings on the discrete-time switched systems with various types of time-dependent switching signals, as well as to provide some new results by relaxing some assumptions required in existing works which usually introduce conservatism and/or restrict the applications of developed approaches in practice.

Focused on the basic control and filtering synthesis problems for discrete-time switched linear systems under the four typical above-said switching signals, the book is organized into the following seven chapters.

Chapter 1 introduces the backgrounds and motivations of this book. Some preliminaries, the classification of switching signals, and comparisons with other types of hybrid systems are also provided, which aims at giving readers an understanding of the results that will be presented in the book.

In Chap. 2, we focus on the stability and stabilization issues of switched systems. First, the frequently used multiple Lyapunov-like functions (MLFs) approach is discussed, and four typical forms evolved from the general MLFs are introduced by comparing their advantages and disadvantages. Then, considering several classes of time-dependent switching signals, i.e., arbitrary switching, DT switching, ADT switching, PDT switching, and their mode-dependent forms, the corresponding stability and stabilization conditions are obtained based on the general MLFs or the evolved ones.

In Chap. 3, the performance analysis issue is investigated for switched systems with four types of switching signals in the discrete-time context. The results on l_2-gain analysis are first given for the switched systems under arbitrary switching and with l_2 disturbances. The weighted/non-weighted l_2-gain analyses are then studied considering the ADT and PDT switching, respectively. In light of the set-theoretic method, the tube-based robustness analysis is carried out when the modal PDT (MPDT) switching and the l_∞ disturbances are considered simultaneously.

Chapter 4 is concerned with the control synthesis problems for discrete-time switched linear systems. First, for the switched systems under arbitrary switching and with polytopic parameter uncertainties, the design of parameter-independent or parameter-dependent H_∞ controller is addressed. A μ-dependent approach is then introduced to design the H_∞ state-feedback controllers for uncertain switched systems with ADT switching. The parameter μ defines an upper bound for the increasing times of the MLFs at switching instants. Finally, considering the redundant channels existed in the data transmission, the non-weighted H_∞ control problem for discrete-time switched linear systems with MPDT switching is studied via the quasi-time-dependent (QTD) Lyapunov function approach.

In Chap. 5, filtering issues for discrete-time switched linear systems with typical switching signals are investigated in $H_\infty/l_2 - l_\infty$ sense. Considering the switched systems under arbitrary switching signals and with polytopic uncertainties, the robust filter is designed. Then, under ADT switching and considering the systems with time-varying parameters or polytopic uncertainties, the μ-dependent approach is also used to design the weighted (or called exponential) filter. Furthermore, a class of non-weighted QTD H_∞ filter is designed with less conservatism for discrete-time switched linear systems considering the PDT switching property.

Chapter 6 investigates a special problem in switched systems, the so-called asynchronous switching phenomena. Considering that the mode-dependent controllers/filters (less conservative than mode-independent ones) as well as the switching signals that are designed in the case assuming synchronous switching may cause instability or a performance reduction, the asynchronous switched control/filtering problems for discrete-time switched systems with ADT switching are treated with the aid of the techniques developed in Chaps. 4 and 5. In addition, the H_∞ control problem is dealt with for a class of discrete-time switched linear parameter-varying systems under modal ADT switching in the presence of asynchronous switching phenomenon.

Chapter 7 deals with control and filtering issues for discrete-time switched linear systems with time delays. First, instead of considering state feedback control, the output feedback controller design is addressed for the switched systems under arbitrary switching and with polytopic uncertainties. Under cyclic switching, the stability conditions are derived allowing for the constraints on the DT of each time-delay subsystem by virtue of the MLFs approach, and a numerical searching algorithm is explored to compute the feasible values of DT of the subsystems. Finally, considering the PDT switching regularities and mode-dependent time-varying delays, the filtering problem is treated for a class of discrete-time neural networks with the corresponding switching and time-delay dynamics in H_∞ sense.

To summarize, this book presents the most recent theoretical findings on control and filtering issues for time-dependent discrete-time switched linear systems. By integrating novel ideas, fresh insights, and rigorous results in a systematic way, this book is aimed at providing a base for further theoretical research as well as a design guide for engineering applications. This book can serve as a reference for the main research issues and results on switched systems for researchers devoting to various areas of control theory, as well as a material for graduate and undergraduate students interested in switched systems and their applications. Some prerequisites for reading this book include linear system theory, matrix theory, mathematics, set theory, and so on.

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Symbols

T	Matrix transposition
$\mathbb{R}^n$	The n dimensional Euclidean space
$\mathbb{R}^{n\times m}$	$n \times m$ dimensional real matrices
$\mathbb{R}_+$	Set of nonnegative real numbers
$\mathbb{Z}_+$	Set of nonnegative integers
$\mathbb{Z}_{\geq s_1}$	$\{k \in \mathbb{Z}_+ \mid k \geq s_1\}$
$\mathbb{Z}_{[s_1,s_2]}$	$\{k \in \mathbb{Z}_+ \mid s_2 \geq k \geq s_1\}$
$\|\cdot\|$	Euclidean vector norm
$\|w\|_2$	$\sqrt{\sum_{k=0}^{\infty} w^T(k)w(k)}$ for $\{w(k)\} \in l_2[0,\infty)$
$\|e\|_\infty$	$\sqrt{\sup_k\{e^T(k)e(k)\}}$ for $\{e(k)\} \in l_\infty[0,\infty)$
$\|x\|_S$	Distance of a vector x to set S, $\|x\|_S \triangleq \inf_{y\in S}\|x-y\|$
$l_2[0,\infty)$	Space of square summable infinite sequence
$l_\infty[0,\infty)$	Space of all essentially bounded functions
$\lceil a \rceil$	Nearest integer greater than or equal to a
$\star$	An ellipsis for the terms that are introduced by symmetry
$S^n_{\succ 0}$	Set of $n \times n$ symmetric positive definite matrices
$S_1 \ominus S_2$	$\{s \in \mathbb{R}^n \mid s + s_2 \in S_1, \forall s_2 \in S_2\}$
$S_1 \oplus S_2$	$\{s_1 + s_2 \in \mathbb{R}^n \mid s_1 \in S_1, s_2 \in S_2\}$
$co\{S\}$	Convex-hull of S
$\mathcal{B}^n$	$\{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$
$\mathbb{C}^1$	Space of continuously differentiable functions
$A \Leftrightarrow B$	A is equivalent to B
$diag\{\cdots\}$	The block-diagonal matrix
I	Identity matrix
0	Zero matrix
$\square$	The end of proof
$\rightarrow$	Tends to
$\Rightarrow$	Imply or implies (be implied by)
$\lambda_{\max}(P)$	The maximum eigenvalue of P
$\lambda_{\min}(P)$	The minimum eigenvalue of P

$P > 0$	P is real symmetric and positive definite matrix
$P \geq 0$	P is real symmetric and semi-positive definite matrix
$eig(A)$	The set of eigenvalues of matrix A
$\arg\{\max(\cdot)\}$	The index of the maximum element of an ordered set
min	Minimum
max	Maximum
sup	Supremum or least upper bound
inf	Infimum or greatest lower bound
$d(a, S)$	Distance between element a and set S
$d_H(S_1, S_2)$	Hausdorff distance between sets S_1 and S_2
$\triangleq$	Equal by definition or is defined by
$\equiv$	Identically equal
M_i	$M(i)$
$\mathbb{E}\{\mathcal{S}\}$	Mathematical expectation of $\mathcal{S}$
$sym(U)$	$U + U^{\mathrm{T}}$
$\Pr\{\mathcal{A}\}$	Occurrence probability of the event $\mathcal{A}$
class $\mathcal{K}$	The set of continuous and strictly increasing functions that vanish at zero
class $\mathcal{K}_\infty$	The set of unbounded class $\mathcal{K}$ functions
class $\mathcal{KL}$	The set of functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\beta(\cdot, t) \in \mathcal{K}\ \forall t \geq 0$ and $\lim_{t\to\infty} \beta(r, t) = 0\ \forall r \geq 0$

Chapter 1
Introduction

Abstract This chapter introduces the backgrounds and motivations of this book. Some preliminaries, the classification of switching signals and the comparisons with other types of hybrid systems, are also provided which aims at giving readers an understanding on the results that will be presented in the book.

1.1 Switched Systems

Before deeply understanding what the *switched systems* are, it is usually suggested to know about *hybrid systems* which are relatively more complex. Hybrid feature in dynamic systems means a system that consists of interacting continuous and discrete dynamics, with both of which could be described in either continuous-time or discrete-time contexts. Concerned by both control and computer communities, hybrid systems are studied in two different ways with different emphases. Computer scientists commonly care about the discrete dynamics of hybrid systems, and simplify the continuous dynamics for the sake of designing the relevant computer programs. On the contrary, the researchers in the area of control and systems focus on the continuous dynamics of hybrid systems, and abstract the evolutions of the discrete dynamics to a set of supervisory switching laws. In such a way, the systems are referred to as *switched systems*and described by a group of indexed subsystems, represented by differential or difference equations, with a switching rule governing them. For formal definitions of hybrid systems and their application examples and recent progress, readers are referred to [1] and the references therein. Note that hybrid systems also cover other types of systems without switching patterns, such as sampled-data control systems, c.f., [2]. In a later section, the detailed comparisons of switched systems with the general hybrid systems, and other typical dynamic systems displaying switching features but with different terminologies will be elaborated.

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1.2 Backgrounds and Motivations

As has been well summarized in the literature, the motivations in physical world to the studies on the switched systems are in twofold. First, switched systems can be used to model a class of systems with multi-mode feature themselves. This kind of systems includes piecewise linearized nonlinear systems, time-varying systems, the systems with large-scale but segmentable uncertainties, systems with random faults, etc. Typical examples include automotive geared box transmission systems [3], power converter [4, 5], wind turbine regulation [6], networked control systems [7], vertical and/or short take-off and landing (VSTOL) aircrafts [8], on-off continuously stirred tank reactors (CSTR) [9, 10], activated sludge wastewater processes [11], etc., among many other fields.

As pointed out in [12–14], the piecewise-linearization is a simple modeling method which approximates the nonlinear mapping by each linear/affine model for each divided region. In this sense, the piecewise-linearization can be viewed as a class of state-dependent switched system with one local model in each division of the state space. The resulting piecewise affine (PWA) systems switch among the linear/affine models when the systems state reaches the boundary of some region. As there is an additional constant affine term for each subsystem in the PWA system (some domains do not contain the origin after being piecewise linearized), the complexity of the system analysis and synthesis is greatly increased. In [15, 16], the modeling method and the equivalent system model for PWA systems are summarized, and the analysis for these systems using multi-logical dynamic systematization method is given. The formulation of the piecewise continuous Lyapunov function for a class of PWA systems and the optimal control performance analysis [17] for these systems are given, in combination with linear matrix inequalities (LMIs) in [18]. The stability analysis and filtering results are also presented for the discrete-time piecewise affine systems in [19, 20]. Besides, the parameter-varying systems can be modeled via switching behaviors especially when they are subject to large parameter-varying scope. The most representative is the switched linear parameter varying (LPV) system modeling problem [21] in which the parameter variation is not smooth. In particular, the studies related to the analysis and control of the switched LPV system under multi-switching rules are carried out in [22]. Therefore, sometimes these switched LPV systems can be considered as switched systems with multi-mode themselves. One typical example for this case is the utilization of switched LPV systems to deal with the dramatic parameter variation and wide flight range inherent in the model of the F-16 aircraft [22, 23]. Note that in LPV or switched LPV systems, the varying parameters are often measurable. On the other hand, it is well recognized that uncertainties (unmeasurable) always exist in many practical systems, and the robust control theory based on quadratic framework generally solves the analysis and synthesis problems of uncertain systems. As pointed out in [24], for the systems with large-scale uncertainties, the scope of uncertainty can be partitioned into several regions, and each region is associated with one subsystem, resulting in a class of switched systems. Moreover, in the fault detection and isolation (FDI) and fault-tolerant control (FTC), the potential faults in a system often range over a large region [25]. In this case, a single controller (even an adaptive one) is generally hard to find for stabilizing all faulty situations effectively. In order to overcome this difficulty, the supervisory FTC approach is proposed by assuming the plant to have a set of models (including the nominal situation and all possible faulty situation) and designing a family of controller candidates so as to stabilize the system by switching among them.

On the other hand, the switched systems are formed in a broader multi-controller switching control mechanism. Strictly speaking, the resulting systems are better called *switched (hybrid) control systems*. It is well known that, a controlled process can achieve desired behavior by designing static or dynamic feedback controllers. However, under some circumstances, a continuous feedback is absent or inappropriate, which can be roughly divided into the following three categories:

(1) Due to special characteristics of the controlled plant, continuous control cannot be achieved. For example, when there exist components that are sensitive to environment or easily damaged, appropriate schemes should be used to achieve switching control [26, 27]; and some plants have inherent state space boundaries that require the implementation of switched control [28]. Meanwhile, in the field of nonlinear control, there are corresponding necessary conditions for the stabilization of such systems [29]. The nonholonomic systems [30] belong to a class that does not meet these conditions and cannot be stabilized by using continuous control [31–35], and the switching control strategy provides a simple and effective solution for such systems. Note that in the nonlinear systems, controllability does not imply the continuous feedback stabilization.
(2) Due to restrictions on the actuator or sensor, continuous control cannot be achieved. For example, for the systems with input saturation, the Bang-Bang control with switching between upper and lower bound is adopted so as to realize the minimum-time control. In addition, sometimes the quantification of transmission information due to the limitation on data transfer channel bandwidth [36] makes the control or state information switch in the different quantify regions [37, 38], and thus the continuous control cannot be realized.
(3) Due to large-scale uncertainties in some of the plants to be controlled, it is inappropriate to apply traditional adaptive controller to the design of general uncertain dynamical systems. A more direct solution is to partition the uncertainty into several regions, and use switched control strategies to achieve effective control. This scheme can overcome the shortcomings of traditional parameters adjustment based on adaptive control algorithm, and has evident advantages in modular design, simplified analysis and practical applications, cf. [39].

In addition to the application examples mentioned above that are suitable for switched control strategy, even for the systems that can be applied by continuous control, such as general linear time-invariant (LTI) systems, switching control strategy can improve their performances as well. The benefits from switched or hybrid control are discussed in [40]:

(1) Improving the system transient performance. For example, the strategy of using the temporal optimal controller when error is larger, and the linear feedback controllers when error is small, can not only reduce the overshoots but also shorten the response time.
(2) Expanding the domain of attraction (DOA). DOA is a region where the system state can be attracted to the equilibrium point. Switched control scheme can overcome the drawback that one single controller could not shift the system from one equilibrium point to another and finally drive the systems to the stable equilibrium [41, 42], and therefore can expand the DOA.

A more complete survey on the state-of-the-art of switched systems, in particular, with respect to the control and filtering issues will be given in Sect. 1.6

1.3 Mathematical Descriptions

In general, a switched system can be expressed by a family of subsystems and a rule that orchestrates the switching sequence among these subsystems. In mathematics, the switched systems can be described as the following equations:

$$\begin{aligned} \delta x(t) &= f_\sigma(x(t), u(t), w(t)) \\ y(t) &= g_\sigma(x(t), w(t)) \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the input vector and $y(t) \in \mathbb{R}^{n_y}$ is the output vector. $w(t)$ represents the disturbance input vector. σ is the switching signal, which is a piecewise constant function of time and takes its values in the finite set $\mathcal{I} = \{1, \ldots, N\}$, $N > 1$ is the number of subsystems. f_σ and g_σ are vector functions, and the symbol δ denotes the derivative operator in the continuous time case ($\delta x(t) = \dot{x}(t)$) and the difference operator in the discrete-time case ($\delta x(t) = x(t+1)$).

Generally, the switching signal is a piecewise constant function of time, its past values, the state/output, and possibly the external signal. A general representation of the switching signal can be given as [7]

$$\sigma(t) = \psi([t_0, \infty), \sigma([t_0, \infty)), x([t_0, \infty))/y([t_0, \infty)), z([t_0, \infty))) \quad t \geq t_0 \tag{1.2}$$

where t_0 is the initial time, and $z : [t_0, \infty) \to \mathbb{R}^l$ is an external signal generated by devices such as observer.

In particular, if the vector field is linear, and there exists no external disturbance input, the switched system can be modeled as

$$\begin{aligned} \delta x(t) &= A_\sigma x(t) + B_\sigma u(t) \\ y(t) &= C_\sigma x(t) \end{aligned} \tag{1.3}$$

From the previous discussions, it can be easily seen that the studies on switched system is focused on the continuous dynamic part including either the continuous-time or discrete-time dynamics. The discrete events of the hybrid systems are hidden in the switching signals which are assumed belonging to some sets. We will elaborate the classifications on the sets of the switching signals later, and the regularities of time-dependent switching signals.

In this book, our attention is confined to discrete-time linear switched systems, which can be explicitly expressed as

$$\begin{aligned} x(k+1) &= A_\sigma x(k) + B_\sigma u(k) \\ y(k) &= C_\sigma x(k) \end{aligned} \tag{1.4}$$

where the system state and output, control input and the switching signal are described previously. At an arbitrary time k, σ may be dependent on k or $x(k)$, or both, or other logic rules. For a switching sequence $k_0 < k_1 < k_2 < \cdots$, σ is continuous from right everywhere and may be either autonomous or controlled. The switching signal σ is assumed to be unknown a priori, but its instantaneous value is available in real time. When $k \in [k_l, k_{l+1})$, the $\sigma(k_l)$th subsystem is active and therefore the trajectory $x(k)$ of system (1.4) is the trajectory of the $\sigma(k_l)$th subsystem.

For the discrete-time switched systems with polytopic uncertainties, the model can be expressed as

$$x(k+1) = A_{\sigma(k)}(\lambda)x(k) \tag{1.5}$$

where λ is the uncertainty parameter. For $\sigma(k) = i$, $A_i(\lambda)$ can be expressed as

$$A_i(\lambda) = \sum_{m=1}^{s} \lambda_m A_{i,m} \tag{1.6}$$

where $\sum_{m=1}^{s} \lambda_m = 1$, $\lambda_m \geq 0$, $i \in \mathcal{I}$, and $m \in S \triangleq \{1,2,3,\ldots,s\}$. s is the number of the vertices of the polytope.

If the uncertainty is norm-bounded, the corresponding uncertain discrete-time switched system can be written as

$$x(k+1) = A_{\sigma(k)}x(k) + \Delta A_{\sigma(k)}x(k) \tag{1.7}$$

where $\Delta A_{\sigma(k)}$ represents the system parameter uncertainty, and it is assumed in this book that the norm-bounded uncertainty can be decomposed as

$$\Delta A_{\sigma(k)} = D\Gamma_{\sigma(k)}(k)E, \sigma(k) \in \mathcal{I} \tag{1.8}$$

where D and E are known constant matrices with appropriate dimensions. $\Gamma_{\sigma(k)}(k)$ are known matrices with Lebesgue measurable elements and satisfy $\Gamma^T_{\sigma(k)}(k)$ $\Gamma_{\sigma(k)}(k) \leq I$.

In this book, the following discrete time-delay switched systems will also be considered:

$$\begin{aligned} x(k+1) &= A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d(k)) \\ x(k) &= \phi(k), k = -d_M, -d_M+1, \ldots, 0 \end{aligned} \tag{1.9}$$

The time delay is considered to be time-varying and has lower and upper bounds, $0 < d_m \leq d(k) \leq d_M$, which is very common in practice.

Before further proceeding, two basic hypotheses throughout this book are illustrated as follows

Assumption 1.1 The jumps of the states for discrete-time system (1.4), namely, a continuous signal can not be reconstructed everywhere, are also not considered here.

Assumption 1.2 The switching instants are assumed to be exactly the sampling instants of system (1.4).

1.4 Classifications of Switching Signals

Switching signals are crucial in switched systems, as summarized in [43–45], which distinguish switched systems from other types of dynamic systems. The switching signals can be classified from different perspective, and the types of switching signals are fruitful, either developed by the systems themselves or invented by the designers, making switched systems diverse.

1.4.1 Different Perspectives for Classifications

According to different standards, the switching in switched systems can be classified as state-dependent versus time-dependent, autonomous versus controlled, nondetermined versus stochastic, or arbitrary versus constrained, etc.

If the state space is partitioned into several operating regions, and the switching takes place when the states hit the switching surfaces (the boarder of the operating region corresponding to a subsystem), we define such switching as state-dependent switching. If the switching among subsystems is governed by a piecewise constant function of time taking values in the indices set of all the subsystems, we define such switching signal as time-dependent switching signal.

Similar to the definition of autonomous system, the autonomous switching means that the mechanism of the switching is self-triggered (there is no external intervention). The autonomous switching is widespread in the industrial processes that are subject to random component failures or repairs, sudden environmental disturbances. On the other hand, if the designers impose any laws to the switching in order to achieve desired behavior and performance, the switchings are called nonautonomous in this situation.

Note that for the above switching signals, in either the pair "time-dependent" versus "state-dependent" or the pair "autonomous" versus "controlled", are nondeterministic. If both the switching sequence and the switching instants are fixed, the switching is deterministic. If the autonomous switching in the above mentioned nondeterministic switching signals is further attached with descriptions of a certain stochastic process (e.g. Markov process or Markov chain), then the switching can be regarded as stochastic switching. The readers can refer to Sect. 1.5 for the differences and links.

Our attention will be focused on the time-dependent switching in this book. The time-dependent switching concerned here can not only cover the switching signals where the trigger mechanism is time, but also represent the state-dependent switching in a general sense. In fact, as pointed out in [43], the time-dependent switching can be regarded as coarse approximation of state-dependent switching in the case that the locations of switching surfaces are unknown. According to different constraints imposed on the switching signal so as to guarantee the stability (or performance), the time-dependent switched systems can be generally grouped as arbitrary (fast) switched systems and constrained (slow) switched systems. Clearly, one necessary condition for the (asymptotic) stability of arbitrary switched system is that all the individual subsystems are (asymptotically) stable, which will be a default assumption throughout the research on the arbitrary switching in this book. Besides, for the switched systems, a more desirable property is the *uniformly* asymptotic stability, where the *uniformly* is referred to the fact that the stability is *uniform* over all switching signals with a certain regularity instead of the initial conditions in the case of conventional non-switched systems.

In the following, several types of typical time-dependent switching signals are first introduced and then their mode-dependence will be addressed. The arbitrary switching is intuitive and the definition is omitted here. The inclusion relationships among the switching signals are elaborated as well.

1.4.2 Several Typical Time-Dependent Switching Signals

As is well known that under the assumption that each subsystem is stable, the switched system can be stable if the switching signal is sufficiently slow. Then, the "dwell time" switching logic is firstly proposed in [46], and the studies in the literature are devoted to calculating the lower bound of the dwell time (DT) as a basic concern in this area.

Definition 1.1 ([47]) Let $t_0, t_1, t_2, \ldots, t_k, \ldots$, with $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, denote the switching instants. The positive constant τ is called the dwell time if for all $k \geq 0$ such that $t_{k+1} - t_k \geq \tau_D$.

Specifically, given a positive constant τ_D, let $\mathcal{S}_{dwell}[\tau_D]$ denote the set of all switching signals with interval between consecutive switchings being no smaller than τ_D. Then the constant τ_D is called the "dwell time".

However, in some occasions, it is too restrictive if requiring the activation time of each subsystem to be larger than τ_D. Thus it will be attractive if the possibility of "fast" switching is allowed for some subsystems with unacceptable performance while the DT of other subsystems can be sufficiently large as compensation such that the overall system behaviors can be satisfied. The concept of "average dwell time" arises in this context in [48], which can be denoted as $\mathcal{S}_{average}[\tau_D, N_0]$, $\tau_D > 0$, $N_0 > 0$.

Definition 1.2 ([48]) For a switching signal σ and any $K > k > k_0$, let $N_\sigma(K, k)$ be the switching numbers of σ over the interval $[k, K)$. If for any given $N_0 > 0$ and $\tau_D > 0$, the inequality $N_\sigma(K, k) \leq N_0 + (K - k)/\tau_D$ holds, then τ_D and N_0 are called average dwell time and the chatter bound, respectively.

It has been analyzed in [43] that $N_0 > 1$ gives switching signals with ADT and $N_0 = 1$ corresponds exactly to those switching signals with DT. Also, as an extreme case, $\tau_D \to 0$ implies that the constraint on the switching times is almost eliminated and the resulting switching can be arbitrary. Therefore, as a typical set of switching signals with regularities, the DT switching is contained in the ADT switching as a special case.

A more general category of switching signals is called "persistent dwell time" [49], denoted as $\mathcal{S}_{pdwell}[\tau_D, T]$, $\tau_D > 0$, $T \in [0, \infty]$. It can be defined as follows.

Definition 1.3 ([49]) The switching σ belongs to the set $\mathcal{S}_{pdwell}[\tau_D, T]$, if there is an infinite number of disjoint intervals with length no smaller than τ_D on which the switching signal σ is constant, and the consecutive intervals with this property are separated by no more than T. The constant τ_D is called the persistent dwell-time and T is called the period of persistence.

An intuitive understanding about the persistent dwell time (PDT) $\mathcal{S}_{pdwell}[\tau_D, T]$ is that the switched system is running alternatively between the "normal mode" with interval not less than τ_D and the "failure mode" with interval not longer than T. In the normal mode, the switched system is running in one subsystem all the time without any switching. In the failure mode, the switched system is switching among subsystems arbitrarily instead of staying in one subsystem more than τ_D. Also note that the set $\mathcal{S}_{pdwell}[\tau_D, T]$ is equivalent to $\mathcal{S}_{dwell}[\tau_D]$ in the case of T = 0.

Therefore, as well summarized in [49], the relationship among these switching signals for the same constant $\tau_D > 0$ is

$$\begin{aligned}\mathcal{S}_{dwell}[\tau_D] &= \mathcal{S}_{average}[\tau_D, 1] = \mathcal{S}_{pdwell}[\tau_D, 0] \\ &\subset \mathcal{S}_{average}[\tau_D, N_0] \subset \mathcal{S}_{pdwell}[\delta\tau_D, \eta\tau_D]\end{aligned}$$

$\forall\ \tau_D > 0, N_0 \geq 1, \delta \in (0, 1), \eta \triangleq \delta(N_0 - 1)/(1 - \delta)$.

Remark 1.4 The major difference between PDT switching and ADT switching is that the PDT is more general since there is no requirement on the frequency of switching during the period of persistence (as long as the switching belongs to $\mathcal{S}_{nonchatt}$, where $\mathcal{S}_{nonchatt}$ denotes the sets of switching signals excluding chatter phenomenon), but in the framework of the ADT switching, the parameter N_0 strictly limits the upper bound of switching times within an interval of length less than τ_a.

Remark 1.5 It should be noted that the relationship only makes sense when all the types of these switching signals to share a same constant τ_D. Otherwise, it is hard to say which class of switching signals is wider. This argument also holds for the following analysis of mode-dependent switching signals.

The mode-dependent switching signals are a class of more flexible switching signals relative to the mode-independent ones since they no longer require all the subsystems share one common DT, ADT, or PDT value. In [47], the definition of modal dwell time (MDT) switching is proposed as a mode-dependent extension of the typical DT switching.

Definition 1.6 ([47]) Let $k_0, k_1, k_2, \ldots, k_s, \ldots$, with $0 = k_0 < k_1 < k_2 < \cdots < k_s < \cdots$, denote the switching instants. The positive constant τ_j is the modal dwell time associated with the jth mode if for all $s \geq 0$ such that $\sigma(k) = j$ for $k \in [k_s, k_{s+1})$, $k_{s+1} - k_s \geq \tau_j$.

For convenience, the set of MDT switching signals can be denoted as $\mathcal{S}_{M-dwell}[\tau_{Di}]$, $\tau_{Di} > 0, \forall i \in \mathcal{I}$, in which the subscript i in τ_{Di} means that this class of switching signals are mode-dependent. In other words, each subsystem i has its own DT τ_{Di}. In particular, if $\tau_{Di} \equiv \tau_D$, $\forall i \in \mathcal{I}$, the MDT switching signal will reduce to the typical DT switching. Thus, it can be concluded that $\forall \tau_{Di} > 0$, $\forall i \in \mathcal{I}$,

$$\mathcal{S}_{dwell}\left[\max_{i \in \mathcal{I}}(\tau_{Di})\right] \subset \mathcal{S}_{M-dwell}[\tau_{Di}]$$

On the other hand, the property in the ADT switching $\mathcal{S}_{average}[\tau_D, N_0]$ requires the average time interval between any two consecutive switching to be at least τ_D, which is independent of the system modes. Besides, it has been well shown in the literature that the lower bound of ADT can be calculated by two mode-independent parameters, which gives rise to certain conservativeness due to the two common parameters for all the subsystems. The concept of "modal average dwell time (MADT)", denoted as $\mathcal{S}_{M-average}[\tau_{Di}, N_{0i}]$, was firstly introduced in [50] as an relaxation of the typical ADT switching logic.

Definition 1.7 ([50]) For a switching signal σ and any $K > k > k_0$, let $N_{\sigma i}(K, k)$, $i \in \mathcal{I}$ be the switching numbers that the ith subsystem is activated over the interval $[k, K]$ and $T_i[K, k]$ denote the total running time of the ith subsystem over the interval $[k, K]$. The switching signal σ is said to have a modal average dwell time τ_{Di} if there exist positive numbers N_{0i} (N_{0i} is called the mode-dependent chatter bound) and τ_{Di} such that $N_{\sigma i}(K, k) \leq N_{0i} + T_i[K, k]/\tau_{Di}$, $\forall K \geq k \geq 0$.

The definition of MADT means that if there exist positive numbers τ_{Di}, $\forall i \in \mathcal{I}$, such that the switching signal has the MADT property, we only require the ADT among the intervals associated with the ith subsystem to be larger than τ_{Di}, where the intervals here are not adjacent. Also, if $\tau_{Di} \equiv \tau_D$, $N_{0i} \equiv N_0$, $\forall i \in \mathcal{I}$, the inclusion relationship between the ADT switching and MADT switching can be shown as $\forall \tau_{Di} > 0, \forall i \in \mathcal{I}$,

$$\mathcal{S}_{average}\left[\max_{i \in \mathcal{I}}(\tau_{Di}), \max(N_{0i})\right] \subset \mathcal{S}_{M-average}[\tau_{Di}, N_{0i}]$$

Similarly, the definition of modal persistent dwell-time (MPDT) can be given as

Definition 1.8 The switching signal σ belongs to the set $\mathcal{S}_{M-pdwell}[\tau_{Di}, T]$, if there is an infinite number of disjoint intervals with length no smaller than τ_{Di}, $i \in \mathcal{I}$, on which the switching signal σ is constant $\sigma = i \in \mathcal{I}$, and the consecutive intervals with this property are separated by no more than T. The constant τ_{Di} is called the modal persistent dwell-time of the ith subsystem, and T is called the period of persistence.

The MPDT can be interpreted as follows. For a switched system, each subsystem has its own PDT, and if the switched system is running in the normal mode and stays in the ith subsystem, $i \in \mathcal{I}$, the dwell time on this subsystem is not less than τ_{Di}. The running intervals in the normal mode are separated by a mode-independent period of persistence T in which the switched system can switch arbitrarily among all the subsystems. Therefore, it can be concluded that, $\forall \tau_{Di} > 0, \forall i \in \mathcal{I}$,

$$\mathcal{S}_{pdwell}\left[\min_{i \in \mathcal{I}}(\tau_{Di}), T\right] \subset \mathcal{S}_{M-pdwell}[\tau_{Di}, T]$$

In this book, our attention is mainly focused on the arbitrary switching, DT switching, ADT switching, PDT switching and their mode-dependent form (i.e., MDT, MADT and MPDT switching, respectively). The transitional dwell time (TDT) switching, coined in [47], requires the knowledge of the index of next subsystem a priori, will be not considered in the book.

1.5 Comparisons with Other Types of Hybrid Systems

Before further proceeding with this introduction, the comparisons between switched systems and other typical dynamic systems with hybrid characteristics would be addressed to further specify what a switched system is.

(1) *Switched systems versus general hybrid systems.* As mentioned above, switched systems can be viewed as a class of control-oriented hybrid systems. The general hybrid system involves continuous-time dynamics and discrete-event dynamics, with more complex models, definitions and broader spectrums, such as sampled-data control systems [51–54]. To simplify the studies on hybrid systems in the control community, the discrete-event dynamics are commonly abstracted and assumed a priori as a certain class of switching events. This defines the context of switched systems by ignoring the evolutions of discrete-event dynamics. Note that the abstractions, depending on the original hybrid systems, may be various and probably bring more system solutions. The great advantage of using switched systems, however, is that the classical control theories in linear or nonlinear systems could be fully utilized to probe hybrid systems. Moreover, the mode switching in switched systems merely depends on current discrete event excluding the former one. That is, the hybrid systems are much more complicated, considering that the mode transitions of continuous dynamics can be governed by complex discrete-event dynamics.
(2) *Switched systems versus general time-varying systems.* Due to the time-dependency of switching signals, switched systems can be viewed as a subset of time-varying systems [49]. However, in the spectrum of switched systems, the switching excludes the Zeno behavior. Thus, switched systems can not be equivalent to the general time-varying systems even when the switching signals are arbitrary. In addition, the solutions of time-varying systems are only parameterized by the initial states of the system, but for switched systems, the solutions are parameterized both by initial states and the switching signals. Note also that, the methodologies of studying the general time-varying systems are established upon that the variations are known a priori, but the switching signals are considered to be unknown (generally assumed to be instantly known) in the switched systems.
(3) *Switched systems versus Markov jump systems.* If the switching signal in the switched systems is autonomous and further attached with Markovian stochastic descriptions, the resulting system is commonly termed as "Markov jump system (MJS)" [55]. It means that on the switching signals, the stochastic information descriptions are further modeled, and the behavior of a system can be considered in the stochastic sense. Moreover, in the Markov jump systems, the transition probabilities are assumed to be known a priori in order to further conduct the analysis and synthesis [56, 57]. Actually, it is difficult and time-consuming to get the exact transition probabilities [58–60]. In this case, the study on MJSs has to be carried out by resorting to the switched systems theory with arbitrary switching signals. In particular, if the transition probabilities are completely unknown, the MJSs are equivalent to the switched systems with arbitrary switching. In addition, in MJSs, the issues with respect to the DT are commonly implicitly considered, but explicitly addressed in switched systems as a basic concern. Note that in MJSs, the concept of sojourn time is the running time of each mode, which differs from the DT.

(4) *Switched systems versus LPV systems.* The LPV systems have attracted extensive attention during the past decades due to the widespread existence of time-varying parameters in many practical systems as well as the great demand in the design of gain-scheduling controllers [61, 62]. As introduced above, the variations of such time-varying parameters are known a *priori* to the system analysis and synthesis. The purpose of investigations on LPV systems is considered to overcome the conservativeness arising in the research of uncertain dynamic processes, when the uncertainties are measurable online or subject to some known rules in some industrial applications [63, 64]. In the LPV systems theory, the system matrices vary with the parameters. Thus the LPV systems can be viewed as a class of switched systems in some sense. To be specific, the impacts of parameter variations are equivalent to those of switching signals to the switched systems. However, the parameter variations of LPV systems are required to be smooth, while the variations of system matrices in switched systems are comparatively flexible.

(5) *Switched control systems versus adaptive control systems.* In essence, the generalized adaptive control is based on the switching, but the adaptive control in a narrow sense emphasizes the design of controllers based on the regulation of continuous parameters [65–69]. If the parameters are not continuous or not estimable, the controllers are not easy to obtain. Relatively speaking, the switched control theory can design discontinuous controllers based on switching logics. Moreover, different from traditional adaptive control systems, the controllers, state estimators, and design of switching logics in switched systems are mutually independent. Thus the modularization design approach can be adopted. Also, we can utilize some controllers with good performance when realizing the control units without the restrictions of continuous parameter adaptive algorithms.

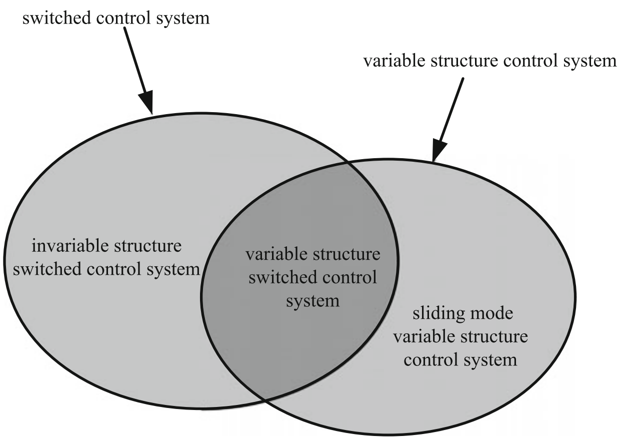

Fig. 1.1 Differences between switched control systems and variable structure control systems

(6) *Switched control systems versus sliding mode control systems.* The generalized variable structure control systems mean that the controllers or structure can change in some situations, but the narrowly defined variable structure control systems refer to that the switching takes place in the control, and the systems will give rise to a class of sliding modes on the switching surface along the fixed trajectory [70–73]. The switched systems induced by multi-controllers do not involve the variable control systems with sliding modes. The differences between switched control systems and variable structure control systems are shown in Fig. 1.1.

1.6 Overview of Control and Filtering of Switched Systems

The stability of switched systems is an issue that needs to be considered before studying problems of both control and filtering, thus we would like to give the literature review on the stability results first.

The stability problem caused by various switching is the basic concern [43, 44, 74, 75]. So far two stability issues have been addressed in the literature, i.e., the stability under arbitrary switching and the stability under constrained switching. The former case is mainly investigated based on constructing a common Lyapunov function for all subsystems [76–78]. In the discrete-time domain, an improved approach by using the switched Lyapunov function (SLF) is proposed in [79, 80]. On the other hand, for the switched systems under constrained switching (either autonomous or to be designed), it is well known that the multiple Lyapunov-like functions (MLFs) approaches are more efficient in offering greater freedom for demonstrating stability of the systems [81]. Some more general techniques in MLFs theory have also been put forward allowing the latent energy function to moderately increase even during the active time of certain subsystems except at the switching instants [51, 75]. As one typical constrained switching signal, the DT switching means that the interval between consecutive switchings is no smaller than a constant. The heuristic behind the DT switching lies in that a switched system is stable if all the individual subsystems are stable and the switching is sufficiently slow, in order to allow the dissipation of the transient effects after each switching. Note that determining the minimum DT for a given switched system is not straightforward and is an important research issue in the DT switched systems. Particularly, it is worth mentioning that a class of quasi-time-dependent (QTD) Lyapunov function approach recently developed in [82] is shown to be less conservative in obtaining shorter dwell-time. The corresponding stabilization problem of switched systems has been investigated in [83, 84]. Besides, the ADT switching means that the number of switches in a finite interval is bounded and the average time between consecutive switching is not less than a constant [48]. The ADT switching can cover the DT switching [43], and its extreme case is actually the arbitrary switching [85]. Therefore, it is of practical and theoretical significance to probe the stability of the switched systems with ADT, and the corresponding results have been also available in [62, 86] for discrete-time version and [23, 87] for related applications. As an mode-dependent form of the typical ADT switching, recently, a novel switching logic named MADT switching has also been proposed in [50]. The corresponding stability conditions for both continuous-time and discrete-time switched systems are given in [50]. As pointed out in [49], another important switching signal, the PDT switching, is more general than the DT switching and the ADT switching. This type of switching signal means that there exists an infinite number of disjoint intervals of length no smaller than a dwell-time on which the system remains unswitched, and the consecutive intervals with this property are separated by no more than a period of persistence. Therefore, the PDT switching has the capability of describing a switched system with hybrid slow and fast switching feature, such as the systems that may encounter abrupt and intermittent faults [88]. Certain conditions ensuring the uniform asymptotic stability of switched linear systems with PDT switching property have been primarily explored in [49], and the results have also been further extended to nonlinear cases in [89–91]. These works provide a solid foundation for investigations on switched systems with PDT switching property. Yet, it is worth noting that in these results, the Lyapunov-like functions during the running time of subsystems are still required to be non-increasing. Recent extensions given in [24, 92] extend the general MLFs defined in [81] by allowing each Lyapunov function associated with each subsystem to be locally increasing. Moreover, a novel class of Lyapunov functions called polynomial and piecewise polynomial Lyapunov functions constructed via using positive polynomials and the sum of squares decomposition have drawn great attention. It should be pointed out that this approach provides a less conservative test for proving stability when switching between subsystems is arbitrary, provided that a finite number of switches occur on every bounded time interval [93, 94].

In addition to the stability issue, the L_2-gain ("l_2" in discrete-time domain) analysis of switched systems has also been frequently reported [24, 85, 95–101]. By the SLF approach, the l_2-gain analysis for a class of discrete-time switched systems under arbitrary switching is given in [102]. It is worth noting that the switched systems comprising of a finite number of linear time-invariant symmetric systems will preserve the stability and L_2 performance of its subsystems under the arbitrary switching. In [103], the L_2-gain properties under arbitrary switching for a class of switched symmetric time-delay systems were studied. Imposing different requirements on the used MLFs, some results on L_2-gain analysis for switched systems with DT or ADT switching have also been obtained in [85, 104]. Likewise, the considered MLFs need mainly to be non-increasing during the running time of the subsystems. In [97], the stability result in [92] was further extended to weighted L_2-gain analysis. Afterwards, the L_2-gain characterization and a design method for stabilizing switching laws are proposed based on the necessary and sufficient stability criteria by extending the general MLFs approach in [98]. More recently, in [105], a non-weighted norm of l_2-gain which is of explicit physical sense as usual with less conservatism is obtained via constructing a both mode-dependent and quasi-time-dependent (QTD) Lyapunov function for PDT switched systems. Moreover, in [106], the L_2-gain for a class of switched systems subject to actuator saturation and time delays is given. The criteria are in the form of delay-dependent and the analysis is under the ADT switching logic.

By considering the individual L_2-gain during the time interval when a subsystem is active, the concept of vector L_2-gain was put forward in [107], and a small-gain theorem for feedback systems with vector L_2-gain assured for each subsystem is established as well.

The control problems for switched systems have captured extensive interests. In this book, instead of the pure state feedback, output feedback control strategies that are generally concerned in the literature, we turn to review some results based on advanced and/or complex control methodologies. Model predictive control (MPC) is a model-based predictive control approach that has its origins in the process industry and that has mainly been developed for linear or nonlinear systems [108]. The MPC has been proven to be powerful when addressing the control problems with constraints. Thus it is of great interest to control the switched systems closed-loop via MPC as the constraints are frequently encountered in practice. For state-dependent switching, MPC of a class of piecewise affine systems—which is also termed *hybrid MPC* in the literature—has been extensively investigated in [109, 110]. As for time-dependent switching, if the switching sequence consisting of both switching instants and switching indices is exactly prescribed, the results of the hybrid MPC literature can be directly used [9, 111–113]. The switched MPC of a class of discrete-time switched linear systems with MDT is investigated in [114]. In addition, adaptive control is a methodology for controlling systems with large modeling uncertainties which render robust control design tools inapplicable and thus require adaptation [39]. One of the most notable limitations in the classical adaptive control is that the continuously parameterized controllers are difficult to construct if unknown parameters enter the process model. In [39], a novel framework is proposed by involving the logic-based switching among candidate controllers with deterministic adaptive control. The controller selection is carried out by means of logic-based switching rather than continuous tuning. In [115], a leakage-type adaptive control is adopted for a class of switched nonlinear systems. Other techniques and results related to the adaptive switching control are available in [116–119] and the references therein. Third, the Takagi–Sugeno (T-S) fuzzy model has recently attracted lots of attention [120, 121], since it is regarded as a powerful solution to bridge the gap between the fruitful linear control and the fuzzy logic control targeting complex nonlinear systems [122–124]. This fuzzy model is described by a family of IF-THEN rules which represent the local linear input-output relations of the system. The overall fuzzy model of the system is achieved by smoothly blending these local linear models together through the membership functions. On the control of switched T-S fuzzy systems, the readers could refer to [125–127] for more details. Before ending the review part on control of switched systems, we would like to mention some results on switched systems and fault-tolerant control that has become the focus in a wide range of industrial and academic communities (many control methods have been put forward and investigated [128]). Especially, in the field of active fault-tolerant control, some controller reconfiguration mechanisms based on switching system theories have been introduced and studied, such as propulsion controlled aircraft [129].

As a dual problem to control design, the problem of state estimation for switched systems has been widely studied in the field of control and signal processing in the past decades. When the states of the systems are not available or measurable, an observer can be designed so as to estimate the states of the original systems. As the classical filtering techniques, the Kalman filtering and extended Kalman filtering have been widely applied in the industrial applications [130–132]. However, when there exists uncertainties in the system parameters, the performance of the standard Kalman filtering could be greatly degraded [133, 134]. In order to investigate the robust filtering for switched systems with uncertainties, the H_∞ filtering has been extensively studied [135–140]. Recently, the moving horizon estimation (MHE) and particle filtering (PF) methods have also been paid increasing attention [141–143]. They have been proved to be effective in dealing with the constrained state estimation and filtering of non-Gaussian, nonlinear systems. For the filtering problem of switched systems, the methodologies aforementioned have been widely applied as well. For example, in [144], an exponential filtering approach was proposed, and the exponential stability is achieved in the filtering processes for some complex systems. The problem of exponential H_∞ filtering for a class of continuous-time switched linear system with interval time-varying delays is investigated in [99]. Other results can be found in [145, 146] and the references therein. For the MHE of hybrid or switched systems, a state-smoothing algorithm for hybrid systems is proposed based on MHE in [147]. With respect to the PF for hybrid or switched systems, in [148], a particle filtering based estimation algorithm for a class of distributed hybrid systems is presented. What is worth mentioning is that there has been some results on using the particle filtering approach to approximate the arrival cost in MHE, c.f. [149, 150], which demonstrates attractive potential. Similar to the FTC, there have been some literature using switching or switching relevant techniques to model and address the problem of fault detection and diagnosis (FDD) [151, 152]. For the FDD of switched systems, in [153], the fault detection problem for a class of switched nonlinear systems with asynchronous switching is addressed. The H_∞ fault detection for continuous-time linear switched systems with its application to turntable systems is investigated in [154].

Regarding the systems perturbed by parameter uncertainties, numerous efficient control techniques have emerged up to now. Uncertainties always exist in many practical systems, and the robust control theory based on quadratic framework is an earlier method to solve the analysis and synthesis problems of uncertain systems. Nowadays it has been well known that the quadratic framework is considered to be conservative. The quadratic stability requires that there exists a common Lyapunov function for all the admissible uncertain parameters [155]. In recent years, many researchers have been devoted to the investigation of parameter-dependent Lyapunov function approach. The target is to establish the Lyapunov functions related to the uncertainty parameters so as to reduce the conservativeness inherent in the common Lyapunov function, and some results have been reported in [156–158]. In the area of switched systems, the studies on the influence caused by uncertainties to the switched systems have been also launched. In general, the class of switched systems with uncertainties on each subsystem can be called uncertain switched systems. The studies on uncertain switched systems generally combine the techniques in uncertain dynamic systems with those methodologies commonly used in switched systems.

Up to now, some issues such as stability analysis, control synthesis, l_2-gain analysis, and observer design have been preliminarily resolved for uncertain switched systems. For example, feedback controller design and l_2-gain analysis for continuous-time switched systems with norm-bounded uncertainty [102, 159], control synthesis for discrete-time uncertain switched systems [160], stabilization of polytopic uncertain switched systems [155], stability and stabilizability of switched systems with uncertainty and time-delay [161], etc. Besides, some results for switching signal with uncertainty can be found in [162] and the references therein.

On the other hand, the existence of time delays is a common phenomenon in engineering control design (see, for example, [163–166]). The studies on the analysis and synthesis of time-delay systems can be classified into delay-dependent and delay-independent approaches. The delay-dependent approach has become prevalent since it facilitates greatly the further analysis and synthesis by making full use of the time delays information. A switched system with time-delay individual subsystems is called a switched time-delay system. Switched systems with time delays have strong engineering backgrounds, such as power systems [85, 167], time-delay systems with controller and/or actuator failures [168], and networked control systems [169]. For the case of constant delay, the L_2-gain properties under arbitrary switching for a class of switched symmetric delay systems are studied in [103]. In [58], sufficient conditions of the asymptotical stability are established for switched linear delay systems. In [161], sufficient conditions for the robust stability and stabilization for a class of uncertain linear switched systems with constant state delays are presented. With the aid of piecewise Lyapunov functions, in [170], the stability of piecewise affine time-delay system is analyzed. For the case of time-varying delay, many results have also been published. For example, the delay-dependent minimum DT for exponential stability of uncertain switched delay systems is given in [171]. In [172], the problem of stability analysis for a class of switched neural networks with time-varying delay is solved. The stabilization of arbitrary switched linear systems with unknown time-varying delays is addressed in [173]. By including both the uncertainties and time-varying delays, the problems of stability and stabilization for uncertain discrete-time switched systems with mode-dependent time-varying delays are studied in [174]. More findings regarding the switched time-delay systems are available in [175–177] and the references therein.

Finally, in switched systems, another noteworthy issue in the control/filtering problems is the so-called asynchronous switching that has received widespread attention. A common assumption in most of existing results is that the detection of the switching signal is instant. However, in many real switched systems, the switching signal is created by some unknown or nondeterministic functions, for example, unknown abrupt phenomena such as component and interconnection failures. One cannot detect the changing of the switching signal instantly, but only after a time period. In fact, the necessities of considering the asynchronous switching for efficient control design have been shown in mechanical or chemical systems [9, 178] with determining the admissible delay of asynchronous switching. Under the assumption that all state variables are available for feedback, the state-feedback stabilization problem is investigated in [179] for switched linear systems with constant time delay in the detection of switching signal. The case of the constant time delay is extended to the time-varying case in [180]. The results in [179, 180] is generalized in [181] to the observer-based state feedback. In [182], the exponential stabilizability of switched linear systems with time-varying delays is investigated in the detection of switching signals via state feedback. In [183], the L_2-gain control synthesis problem is addressed for switched systems in the presence of delay in detection of switching signal. Considering the time delays present in both the feedback state and switching signal, the asynchronous finite-time stabilization for a class of switched systems with ADT switching is explored in [184]. Moreover, it should be pointed out that all of these results are derived for the time-dependent switching with asynchronous behavior, the corresponding results for state-dependent switching still remain open. Besides, in the field of robust controller design for fuzzy systems, it is usually difficult to derive an accurate fuzzy model by imposing that all the premise variables are exactly measurable. Consequently, the problem of asynchronous switching of the premise variables between the fuzzy plant and the observer/controller occurs, especially for the situations where general T-S fuzzy dynamic models are used to represent complex nonlinear systems via offline modeling [120, 185]. Allowing for the asynchronous switching of the premise variables, in [186], the stability of the closed-loop fuzzy system is investigated based on the piecewise fuzzy affine approach and the global static output-feedback controller approach, respectively.

Despite the fruitful results reviewed above, we are aware that it is hardly possible to cover all the past contributions in studies on the control and filtering issues, even in the aspect of time-dependent switching. Also the works are persistently updated although we are trying to include more latest ones. We also have not thoroughly commented on the pros and cons of the various developed methodologies, but hope the literature review could give readers a basic understanding on the key problems and methodologies in control and filtering of switched systems.

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Chapter 2
Stability and Stabilization

Abstract This chapter is devoted to the stability analysis for the four types of time-dependent switched systems. We first summarize the mainly used multiple Lyapunov-like functions (MLFs) approaches, sort out several typical MLFs and review their characteristics. Then, the corresponding methodologies by different MLFs such as switched Lyapunov function, MLFs with μ-times increase at switching instants are developed to study the stability conditions for the underlying system with arbitrary switching, dwell time (DT) switching, average dwell time (ADT) switching, persistent dwell time (PDT) switching and their mode-dependent forms, respectively. The generally analytic results in discrete-time context will be concretely formulated in terms of linear matrix equalities (LMIs). Numerical examples are provided to verify the obtained criteria and to compare the four kinds of time-dependent switched systems. The results obtained in this chapter will lay a foundation for future developments in this book.

2.1 Multiple Lyapunov-Like Functions

In this chapter, consider the unforced discrete-time switched linear systems given by

$$x(k+1) = A_{\sigma(k)}x(k) \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $\sigma(k)$ is the switching signal, which is a piecewise constant function of time and takes its values in the finite set $\mathcal{I} = \{1, \ldots, M\}$, $M > 1$ is the number of subsystems. The switching sequences $k_0, k_1, k_2, \ldots, k_s, \ldots$ are unknown a priori, but are known instantly, in which the switching instant is denoted as $k_s, s \in \mathbb{Z}_+$. When $k \in [k_s, k_{s+1})$, the $\sigma(k_s)$th subsystem (or system mode) is said to be *activated* and the length of the current running time of the subsystem is $k_{s+1} - k_s$. Other descriptions about system (2.1) can be referred to Sect. 1.3 of Chap. 1.

Our aim is to find the stability criteria for the time-dependent switched system (2.1). The following definitions are needed to precisely state what is "stable" in the context of switched systems.

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Definition 2.1 ([1]) The switched system (2.1) is globally uniformly asymptotically stable (GUAS) if there exists a class $\mathcal{KL}$ function $\beta(\|\cdot\|, \cdot)$ such that for all switching signals σ and all initial conditions $x(k_0)$, the solutions of (2.1) satisfy the inequality $\|x(k)\| \leq \beta(\|x(k_0)\|, k - k_0), \forall k \geq k_0$.

Definition 2.2 ([1]) The switched system (2.1) is globally uniformly exponentially stable (GUES) if for constants $c > 0$, $0 < \lambda < 1$, all switching signals σ and all initial conditions $x(k_0)$, the solutions of (2.1) satisfy the inequality $\|x(k)\| \leq c\lambda^{(k-k_0)} \|x(k_0)\|, \forall k \geq k_0$.

Note that for switched systems with above sets of switching signals, the uniformity in above definitions means the uniformity over all switching signals and the set of switching signals with the DT, ADT, PDT properties and their mode-dependent forms, respectively. This differs from the general time-varying system, where the uniformity is only with respect to the initial conditions.

In the stability analysis of switched systems, a common situation is that a global Lyapunov function (GLF) for all subsystems may not exist, or although it does exist, it may be hard to construct and the techniques based on GLF approach are also conservative. Therefore, the so-called MLFs approach is proposed and gradually improved. The key point of MLFs approach is to construct individual Lyapunov (or energy) function for each subsystem and appropriately concatenate these functions at switching instants, aiming to offer more possibilities to demonstrate the stability. Here, the "Lyapunov-like" means that the function associated with each subsystem is nonincreasing within that subsystem, but may increase during the running time of other subsystems. It can be seen that a distinct characteristic of MLFs is that the function values are allowed to jump unlike the continuity in the setting of GLF.

In this section, both the general MLF approach and several special MLFs approaches will be reviewed which are useful to analyze the systems in the discrete-time context. The conservatism and applicability of each MLF will be shown such that the appropriate MLFs can be selected for the time-dependent switched systems with specific switching signals.

Lemma 2.3 *([2]) For the discrete-time nonlinear system $\Sigma_{\sigma(k)} : x(k+1) = f_{\sigma(k)}(x(k))$, $\forall \sigma(k) = i \in \mathcal{I}$, $\mathcal{I} = (1, \ldots N)$. Suppose that the equilibrium point is at the origin. For the Lyapunov functions V_i, $\forall i \in \mathcal{I}$, if at the switching instants k_s and k_v, we have*

$$\sigma(k_s) = \sigma(k_v), \quad \forall s < v, \tag{2.2}$$

where $\sigma(k_s)$ is the switching signal, and the following conditions hold

(a) $\triangle V \triangleq V_i(x(k+1)) - V_i(x(k)) \leq 0, \forall i \in \mathcal{I}$, *when subsystem Σ_i is active;*
(b) $V_{\sigma(k_v)}(x(k_v)) - V_{\sigma(k_s)}(x(k_s)) < 0$;
then the switched system composed by $\Sigma_1, \ldots, \Sigma_N$ is Lyapunov-stable under a certain switching signal.

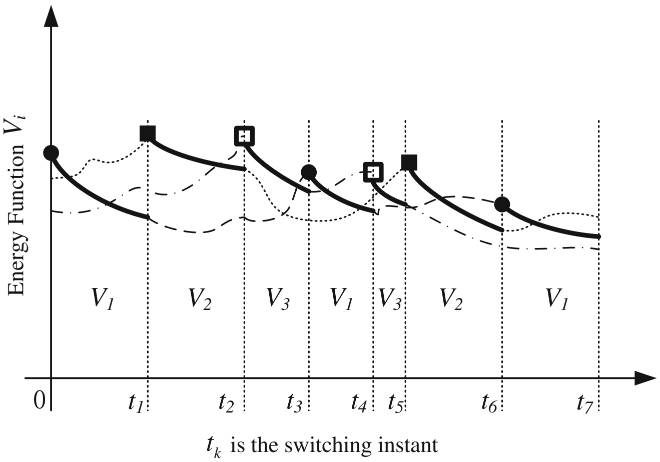

Fig. 2.1 Multiple Lyapunov function ($N = 3$)

The general idea of MLFs approach stated in Lemma 2.3 can be illustrated in Fig. 2.1, taking $N = 3$ for an example. As shown there, it does not only require the Lyapunov stability of each subsystem, but also the monotonically decreasing Lyapunov function values at two consecutive switching instants to a single system. A solid line in the figure denotes that the corresponding subsystem is active.

It can be seen that Lemma 2.3 gives a Lyapunov stability criterion for switched systems from a general perspective. However, owing to the requirement of comparing the Lyapunov function values at two consecutive switching instants to a signal subsystem, Lemma 2.3 is merely able to be applied in the qualitative description of the stability instead of giving the specific and easily-checked conditions. In particular, it is hardly applicable to the cases when the systems are involved with complex behaviors, e.g., uncertainty, nonlinearity and time delays.

Based on the requirements in the two conditions of Lemma 2.3, the evolutions of the general MLFs never ceased during the past decades, targeting the easily-checked stability criteria with less conservatism for switched systems with different regularities. Figure 2.2 shows four typical forms evolved from the general MLFs in Fig. 2.1. We would introduce them one by one by comparing their advantages and disadvantages, respectively.

(1) As shown in Fig. 2.2a, during the running time of each subsystem, the prescribed MLF relaxes condition (a) in Lemma 2.3 into the following condition

$$V_i\left(x(k)\right) \leq h\left(V_i\left(x(k_s)\right)\right), k \in (k_s, k_{s+1}) \tag{2.3}$$

The corresponding MLF is called weak Lyapunov function. From (2.3), allowing the value of the weak Lyapunov function to increase during the running time of each subsystem, the stability criterion is less conservative. However, the rising amplitude h of Lyapunov function value during the running time of each subsystem is hard to be determined, then it is difficult to derive an easily-checked stability criterion, even though the switched system without any complex dynamics, such as uncertainties, time delays.

(2) As shown in Fig. 2.2b, the Lyapunov function is monotonically decreasing during the running time of each subsystem, which is consistent with condition (a) in Lemma 2.3. However, the Lyapunov function value at the switching instant is required to be not higher than that at last switching instant, i.e., condition (b) in Lemma 2.3 can be rewritten as

$$V_j\left(x(k_{s+1})\right) - V_i\left(x(k_s)\right) < 0 \tag{2.4}$$

Therefore, the stability criterion is more conservative, in that this kind of MLF demands a further requirement of general MLF at the switching instant. Such an approach would be very applicable to the stability analysis for switched systems with DT and PDT switching. It could establish a good tradeoff between the less conservative stability criteria (compared with the MLF in Fig. 2.2c) and the easily-checked conditions. However, the investigations based on this approach remain largely open.

(3) As shown in Fig. 2.2c, the Lyapunov function value at instant $k_s + 1$ is non-increasing compared with that at instant k_s, then the corresponding condition (2) in Lemma 2.3 can be rewritten as

$$V_j\left(x(k_s + 1)\right) - V_i\left(x(k_s)\right) \leq 0, \quad i \neq j \tag{2.5}$$

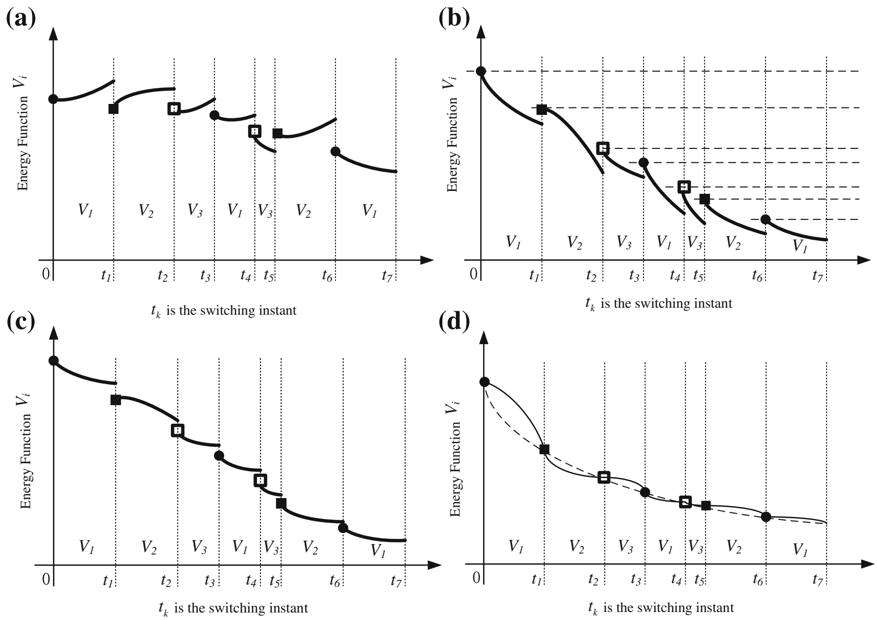

Fig. 2.2 Several MLFs with different forms

Note that k_{s+1} is the next switching instant and $k_s + 1$ is the next sample time to the switching instant.

The MLF with this form is called switched Lyapunov function (SLF) that is coined in [3]. The corresponding stability criterion is much more conservative, since the SLF gives a further requirement between the two function values at the switching instant. However, in the case of discrete time, each subsystem must satisfy $\triangle V \triangleq V_i(x(k+1)) - V_i(x(k)) \leq 0$. Then the requirement at the switching instant in (2.5) and condition a) in Lemma 2.3 can be uniformly rewritten as

$$\triangle V \triangleq V_j(x(k+1)) - V_i(x(k)) \leq 0 \tag{2.6}$$

where $i = j$ denotes that the switched system is during the running time of ith subsystem, and $i \neq j$ means that the switched system is at the switching instant from ith subsystem to jth subsystem. Therefore, the characteristic of unifying the requirement for Lyapunov functions at the switching instant with that during the running time of each subsystem reduces the difficulty in deriving the stability criterion of discrete-time switched linear system. Then the obtained criterion is quite easy to be checked. As a result, based on the SLF method, other analysis and synthesis problems for the switched linear system under arbitrary switching in the discrete-time domain can be solved effectively, e.g., controller design, filter design, model reduction and so forth.

(4) The MLF shown in Fig. 2.2d, requires constructing the Lyapunov function on each subsystem. Meanwhile, the Lyapunov function values are required to be continuous at the switching instant, which is similar to that in the common Lyapunov function. However, the rates of decay may vary among different running time of each subsystem and the required Lyapunov function is still multiple, which is generally defined as piecewise Lyapunov function in some literature. This kind of form facilitates the analysis for state-dependent switched linear systems in the continuous-time domain according to the continuous feature of function value at the switching instant.

From the above analysis, it can be observed that although the analysis approach via weak Lyapunov function is less conservative, it is inconvenient to derive the stability conditions. The general MLF in Fig. 2.1 requires the comparison between the Lyapunov function values at two consecutive switching instants to a single subsystem, and the MLF in Fig. 2.2b requires the comparison between the Lyapunov function values at two consecutive switching instants. However, in the discrete-time domain, for the nominal linear systems, the system state at each sample time can be given by iterating the state space expression. Thus it is possible to obtain the specific stability criterion for the discrete-time switched linear system. However, when the result is extended to the uncertain switched systems or the stabilization issue, the power of system matrices may appear in the iterative process, and it becomes hard to eliminate the power only based on the MLF introduced above (a further quasi-time-dependent Lyapunov function can be resorted to for a solution). In addition, for the general MLF, note that in the situation that the switched system is under arbitrary switching and the number of subsystems $N = 2$, or under a certain switching rule or a periodic switching sequence, the DT can be employed to derive the stability criterion. Finally, for the SLF, although the requirement of the Lyapunov function values at the switching instant is relatively strict, the method unifies the requirement for Lyapunov function at the switching instant with that during the running time of subsystems, since it is unnecessary to use system state at each sample time, which is used to compare the Lyapunov function at the switching instant. In short, the easily-checked stability criterion can be obtained by the SLF approach. In the following sections, the stability analysis of the approaches based on the aforementioned MLFs will be studied and the comparison of their advantages in concern of the conservatism, easily-checked feature, extensibility, etc., will be addressed.

2.2 Arbitrary Switching

This section will utilize the SLF in Fig. 2.2c to conduct the stability analysis for a class of discrete-time switched nominal linear systems. Consider a class of discrete-time switched linear system (2.1), our objective here is to achieve the stability criterion by constructing the SLF with the requirement at the switching instants in Fig. 2.2c as well as the following expression

$$V(x(k)) = x(k)^T P_i x(k) \tag{2.7}$$

Firstly, the Schur complement lemma is recalled, which will be used in the proof of the main results.

Lemma 2.4 ([4]) *The linear matrix inequality*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} < 0$$

where $S_{11} = S_{11}^T$ *and* $S_{22} = S_{22}^T$ *is equivalent to*

$$S_{11} < 0,\ S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0;\ S_{22} < 0,\ S_{11} - S_{12}^T S_{22}^{-1} S_{12} < 0.$$

By Lemma 2.4, the following theorem presents the asymptotic stability conditions for system (2.1).

Theorem 2.5 *Under arbitrary switching, the discrete-time switched linear system (2.1) is asymptotically stable if there exist matrices* $P_i > 0, \forall i \in \mathcal{I}$ *such that*

$$\begin{bmatrix} -P_j & P_j A_i \\ \star & -P_i \end{bmatrix} < 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I} \tag{2.8}$$

Proof Consider positive constants $\alpha > 0$ and $\beta > 0$. Then the SLF in (2.7) satisfies,

$$\alpha \parallel x(k) \parallel \leq V(x(k)) \leq \beta \parallel x(k) \parallel \tag{2.9}$$

Note that, at the sampling instant $k+1$, $k \in [0, \infty)$, the system may switch into another subsystem. Thus, the following equation can be obtained, $\forall(i, j) \in \mathcal{I} \times \mathcal{I}$

$$\Delta V = V(x(k+1)) - V(x(k)) = x^T(k)(A_i^T P_j A_i - P_i)x(k) \tag{2.10}$$

where, $i = j$ denotes the switched system is during ith subsystem and $i \neq j$ means that the switched system is at the switching instant from the ith subsystem to jth subsystem. Thus, if

$$A_i^T P_j A_i - P_i < 0, \quad \forall(i, j) \in \mathcal{I} \times \mathcal{I} \tag{2.11}$$

we have $\Delta V < 0$, which means that system (2.1) is asymptotically stable. By Lemma 2.4, (2.8) is equivalent to (2.11). □

The stability criteria in Theorem 2.5 are obtained via the SLF approach. If some constraints are placed on the matrices, e.g., letting $P_i \equiv P, \forall i \in \mathcal{I}$, the corresponding stability conditions based on the GLF approach can be obtained as follows.

Proposition 2.6 *If there exists matrix $P > 0$ such that*

$$\begin{bmatrix} -P & PA_i \\ \star & -P \end{bmatrix} < 0, \quad \forall i \in \mathcal{I}$$

then there exist GLFs in the form of $V(k, x(k)) = x(k)^T P x(k)$, which indicates that the system (2.1) is asymptotically stable.

Remark 2.7 Theorem 2.5 presents the stability condition for discrete-time switched systems under arbitrary switching, which is in the form of LMIs. By using the SLF approach, the corresponding results can be easily derived for discrete-time switched systems with complex dynamics such as uncertainty, time delays and so forth, which we will show part of extensions in the following chapters.

For discrete-time switched system (2.1), it should be pointed out that the reason why the switched system is not stable under arbitrary switching is due to the fact that some (at least one) of the subsystems satisfy $\|A_i\| \geq 1, i \in \mathcal{I}$. Actually, the following facts hold for system (2.1)

$$\|x(k+1)\| = \|A_i x(k)\| \leq \|A_i\| \, \|x(k)\|$$

If $\|A_i\| < 1, \forall i \in \mathcal{I}$, we have $\|x(k+1)\| < \|x(k)\|$, and it is straightforward that $\|x(k)\| \to 0$ as $k \to \infty$ for arbitrary switching signal. Therefore, our discussion for the stability analysis of switched system under arbitrary switching is based on the assumption that at least one of the subsystems satisfies $\|A_i\| \geq 1, i \in \mathcal{I}$.

Furthermore, for switched system with polytopic uncertainty in (1.5) and (1.6), similarly,

$$\|x(k+1)\| = \|A_i(\lambda)x(k)\| = \left\|\sum_{m=1}^{s} \lambda_m A_{i,m} x(k)\right\|$$
$$\leq \sum_{m=1}^{s} \lambda_m \left\|A_{i,m}\right\| \|x(k)\|$$

If $\left\|A_{i,m}\right\| < 1$, $\forall i \in \mathcal{I}$, $\forall m \in S$, we have $\|x(k+1)\| < \|x(k)\|$, which indicates the asymptotic stability of uncertain switched system. Thus it can be concluded that at least one vertex matrix satisfies $\left\|A_{i,m}\right\| > 1$, $i \in \mathcal{I}$, $m \in S$, for the discrete-time switched system with polytopic uncertainties.

Considering the discrete time-delay switched system (1.9), if $\|A_i\| + \|A_{di}\| \leq 1$, $\forall \sigma(k) = i \in \mathcal{I}$, we have

$$\begin{aligned}
\|x(1)\| &= \left\|A_{\sigma(k)}x(0) + A_{d\sigma(k)}x(0)\right\| \leq \|x(0)\| \\
\|x(2)\| &= \left\|A_{\sigma(k)}x(1) + A_{d\sigma(k)}x(1-d(1))\right\| \leq \|x(0)\| \\
&\vdots \\
\|x(d_M)\| &= \left\|A_{\sigma(k)}x(d_M-1) + A_{d\sigma(k)}x(d_M-1-d(d_M-1))\right\| \leq \|x(0)\| \\
\|x(d_M+1)\| &= \left\|A_{\sigma(k)}x(d_M) + A_{d\sigma(k)}x(d_M-d(d_M))\right\| \leq \|x(0)\|
\end{aligned}$$

Therefore, for a time-delay switched system, if the condition $\|A_i\| + \|A_{di}\| \leq 1$ is satisfied, the stability can be guaranteed straightforwardly.

More details on stability of uncertain switched systems and switched systems with time delays will be elaborated in later chapters. The discussions on the norm of A can be used for a simple judgement or pre-check on the stability conditions.

Example 2.8 Consider the discrete-time switched linear system (2.1) comprising of two subsystems

$$A_1 = \begin{bmatrix} 0.4 & -0.8 \\ 0.5 & 1.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.1 & -0.3 \\ 0.7 & 0.4 \end{bmatrix}$$

First we can check that $\|A_1\| > 1$, $\|A_2\| > 1$, which indicates that the stability cannot be obtained directly. Our objective here is to test the stability of the system under arbitrary switching. According to Theorem 2.5, by using LMI Toolbox, we can get that there exist feasible solutions to (2.8). The obtained Lyapunov matrices are shown as follows.

$$P_1 = \begin{bmatrix} 124.8798 & 95.2966 \\ 95.2966 & 199.9269 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 126.1364 & 70.7540 \\ 70.7540 & 114.1096 \end{bmatrix}$$

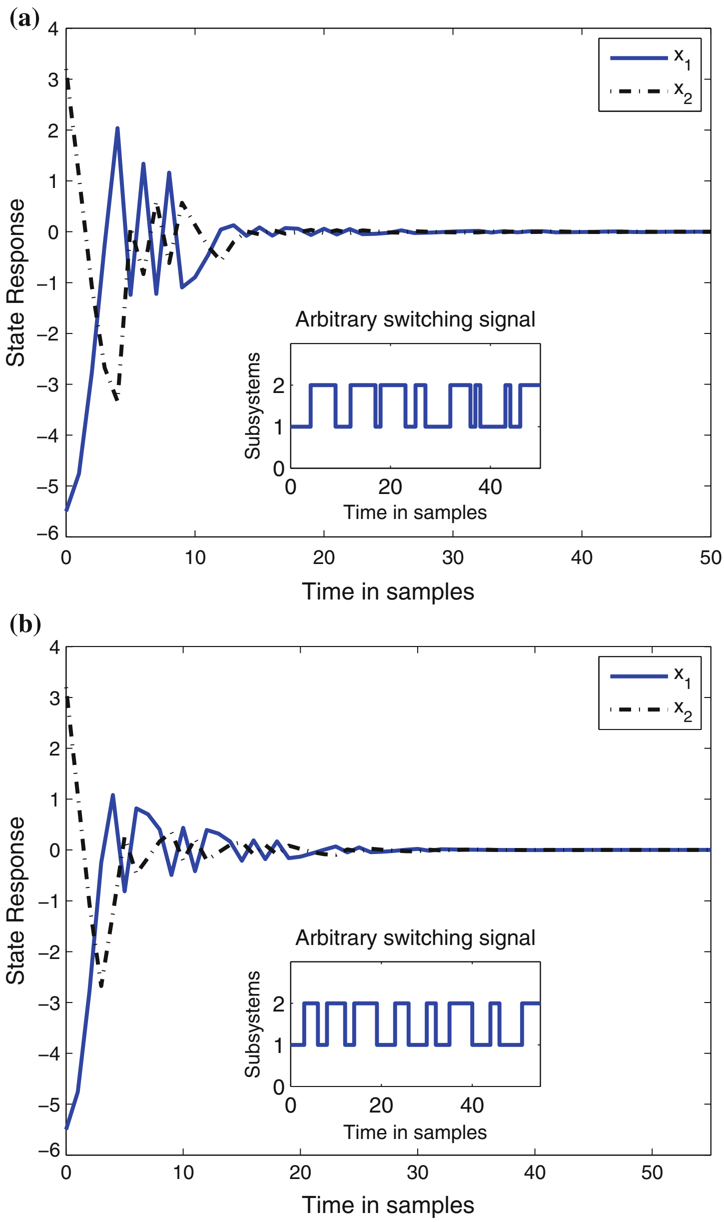

Fig. 2.3 State responses under two arbitrary switching signals

Then it can be concluded that this switched system is stable for any switching signals. In order to test this conclusion, two random switching sequences are generated by the following algorithm

Algorithm 2.1 (*Switching signal generation*) **for** $sample\_T = 1$ **to** $Time\_Length$
  $switching\_value =$ **rand**
  if $swithing\_value \geq Con$;
    $switching\_signal\ (sample\_T) = 2$;
  else
    $switching\_signal(sample\_T) = 1$;
  end
end

where, the **rand** function generates random numbers which are uniformly distributed in the interval (0, 1). For given initial condition $x_0 = [\,-5.5\ 3.2\,]^T$, Fig. 2.3 illustrates the state responses of the switched system under the two arbitrary switching signals, respectively. It can be seen that all the trajectories converge to zero, which proves the validity of Theorem 2.5.

2.3 Dwell Time (DT) Switching

This section will employ the MLFs in Figs. 2.1 and 2.2b, respectively, to investigate the stability problem for system (2.1) with dwell time (DT) constraint on the switching. The corresponding stability criteria of such systems are derived via LMI formulation.

To begin with, based on the general MLF in Fig. 2.1, the same Lyapunov function as shown in (2.7) can be constructed. By Lemma 2.3, one feature of the general MLF is that the values of the Lyapunov function at two consecutive switching instants should be compared as a requirement. However, if $N > 2$, it is almost impossible to determine, under arbitrary switching, which subsystem among $\{A_1, \ldots A_N\}$ the real system will switch into. Thus, although the DT of each subsystem is available, it is still difficult to formulate the state expression of one subsystem at two consecutive switching instants through iteration. Correspondingly, the Lyapunov function in (2.7) can not be obtained, let alone the stability analysis based on Lemma 2.3. On the other hand, if $N = 2$, namely, there exist only two subsystems, in which M_1 and M_2, respectively, representing the DT of each subsystem, and k is one of the switching instant, one can obtain that $x(k + M_1 + M_2) = A_2^{M_2} A_1^{M_1} x(k)$ (or $A_1^{M_2} A_2^{M_1} x(k)$) and consequently, both the state expression at two consecutive switching instants and the corresponding Lyapunov function in (2.7) are available. In addition, if the switching signal of the system (2.1) is under a certain switching rule and the DT is available, it is straightforward to obtain the state values at the corresponding switching instants and the values of the Lyapunov function through iteration. Then, if the switching signals are regulated by the following switching rule:

$$i \xrightarrow{i+1,\ldots,N,1,\ldots,i-1} i, \quad \forall i \in \mathcal{I} \tag{2.12}$$

i.e., one rule of the cyclic switching, we have

$$x(k + M_i + M_{i+1} + \cdots + M_N + M_1 + \cdots + M_{i-1})$$
$$= A_{i-1}^{M_{i-1}} \ldots A_1^{M_1} A_N^{M_N} \ldots A_{i+1}^{M_{i+1}} A_i^{M_i} x(k)$$

where k is assumed to be the instant of the system switching into subsystem A_i. Furthermore, the stability analysis for the corresponding switched system can also be conducted. The two-mode system under arbitrary switching (namely, $N = 2$) will be covered as a special case of the rule in (2.12). In what follows, the switched system under cyclic switching will be coped with and a stability criterion by using the general MLF will be presented.

Theorem 2.9 *Suppose that the switched system (2.1) switches into subsystem A_i at the switching instant k_s, then after switching into subsystem $A_i, A_{i+1}, \ldots, A_N, A_1, \ldots, A_{i-1}$ consecutively under cyclic switching (2.12), and finally, switches into subsystem A_i again at the switching instant k_v. Let $M_i, \forall i \in \mathcal{I}$ denote the DT of each subsystem. Then the system (2.1) is asymptotically stable under cyclic switching (2.12) if there exist matrices $P_i > 0, \forall i \in \mathcal{I}$ such that*

$$A_i^T P_i A_i - P_i \leq 0 \tag{2.13}$$

$$\Upsilon_i^T P_i \Upsilon_i - P_i \leq 0 \tag{2.14}$$

where $\Upsilon_i \triangleq A_{i-1}^{M_{i-1}} \ldots A_1^{M_1} A_N^{M_N} \ldots A_{i+1}^{M_{i+1}} A_i^{M_i}$.

Proof $\forall i \in \mathcal{I}$, system (2.1) can be described by the model of subsystem A_i. Thus if (2.13) holds, one obtains

$$\Delta V_i \triangleq V_i(x(k+1)) - V_i(x(k)) = x^T(k)(A_i^T P_i A_i - P_i)x(k) < 0$$

which means that subsystem A_i is stable and satisfies condition (a) in Lemma 2.3. Meanwhile, in Fig. 2.1, we have $V_i(x(k_s)) = x^T(k_s)P_i x(k_s)$ at the switching instant k_s when the system switches into subsystem A_i. Then at the instant k_v when the system switches into subsystem A_i again, system (2.1) has passed through subsystems $A_i, A_{i+1}, \ldots, A_N, A_1, \ldots, A_{i-1}$ consecutively under cyclic switching (2.12). Thus, at the switching instant k_v,

$$x(k_v) = A_{i-1}^{M_{i-1}} \ldots A_1^{M_1} A_N^{M_N} \ldots A_{i+1}^{M_{i+1}} A_i^{M_i} x(k_s) = \Upsilon_i x(k_s)$$

and $V_i(x(k_v)) = x^T(k_v)P_i x(k_v) = (\Upsilon_i x(k_s))^T P_i \Upsilon_i x(k_s)$. Therefore,

$$V_i(x(k_v)) - V_i(x(k_s)) = x^T(k_s)(\Upsilon_i^T P_i \Upsilon_i - P_i)x(k_s)$$

It is obvious to see that if (2.14) holds, $V_i(x(k_v)) - V_i(x(k_s)) < 0$ and condition (2) in Lemma 2.3 is satisfied. Thus, according to Lemma 2.3, system (2.1) is Lyapunov-stable under cyclic switching (2.12). □

By (2.14) in Theorem 2.9, it is clear that when the switching signals are arbitrary, Υ_i is unavailable, which indicates that the general MLF approach is inapplicable to conduct the stability analysis for the switched system under arbitrary switching. According to (2.4), it can be seen that for the underlying MLF, the approach does not require the comparison between the Lyapunov function values at two consecutive switching instants, and instead, requires the comparison between the Lyapunov function values at two neighboring switching instants. Suppose that the system switches into subsystem A_i at the switching instant k_s, then switches into subsystem A_j $(k_{s+1} - k_s = M_i)$, $\forall(i, j) \in \mathcal{I} \times \mathcal{I}$ at the switching instant k_{s+1}. Thus, with the availability of the DT M_i, $i \in \mathcal{I}$, we obtain that $x(k_s + M_i) = A_i^{M_i} x(k_s)$, which is the state expression between two neighboring switching instants. Furthermore, the expression of the corresponding Lyapunov function can be obtained by (2.7). Through the above analysis, the stability criterion of switched system (2.1) with the MLF in Fig. 2.2b can be derived as follows.

Theorem 2.10 *Suppose that the M_i, $\forall i \in \mathcal{I}$ denote the DT of each subsystem. Then system (2.1) is asymptotically stable if there exist matrices $P_i > 0$, $\forall i \in \mathcal{I}$ such that*

$$A_i^T P_i A_i - P_i \leq 0, \quad \forall i \in \mathcal{I} \tag{2.15}$$

$$(A_i^{M_i})^T P_j A_i^{M_i} - P_i < 0, \quad \forall(i, j) \in \mathcal{I} \times \mathcal{I}, i \neq j \tag{2.16}$$

Proof $\forall i \in \mathcal{I}$, system (2.1) can be described by A_i. Thus if the inequality (2.15) holds, one obtains

$$\Delta V_i \triangleq V_i(x(k+1)) - V_i(x(k)) = x^T(k)(A_i^T P_i A_i - P_i)x(k) \leq 0$$

which means that subsystem A_i is stable and satisfies condition (a) in Lemma 2.3. Meanwhile, in Fig. 2.2b, it is assumed that the system switches into subsystem A_i at the switching instant k_s, then switches into subsystem A_j $(k_{s+1} - k_s = M_i)$ at the switching instant k_{s+1}. Thus, $V_i(x(k_s)) = x^T(k_s)P_i x(k_s)$, $V_j(x(k_{s+1})) = x^T(k_{s+1})P_j x(k_{s+1})$. If (2.16) holds, one obtains

$$V_j(x(k_{s+1})) - V_i(x(k_s)) = x^T(k_s)((A_i^{M_i})^T P_j A_i^{M_i} - P_i)x(k_s) \leq 0 \tag{2.17}$$

Now suppose that system (2.1) switches into subsystem A_i at the switching instant k_{s+n}, and through iteration of (2.17), we have $V_i(x(k_{s+n})) \leq V_i(x(k_s))$, which satisfies condition (b) in Lemma 2.3 obviously. Thus, according to Lemma 2.3, system (2.1) is stable. □

Remark 2.11 Note that, one mode-independent case of Theorem 2.10 can be found in [5].

As shown in Theorems 2.9 and 2.10, with the availability of the DT of the subsystems, whether or not the solutions of the corresponding stability conditions in (2.13)–(2.14) and (2.15)–(2.16) are feasible can be verified by the corresponding functions of LMIs toolbox in MATLAB (also, other toolboxes like sedumi and Yalmip). It is worth mentioning that Theorems 2.9 and 2.10 merely give the sufficient conditions. In other words, for certain given values of the DT, the system has the potential to be stable even without the feasible solutions of the stability conditions.

So far, the stability analysis for the discrete-time switched linear systems with the general MLF and the MLF in Fig. 2.2b has been discussed. From Theorems 2.9 and 2.10, it can be seen that the derivations of the stability criteria are based on the assumption of viewing the DT of each subsystem as the known condition. However, when there exist uncertainties in the system matrices, the corresponding form of $A_i^{M_i}$ will appear, which inevitably brings the computational complexity of the matrices. Correspondingly, it will be difficult to conduct the stability analysis for the uncertain switched system, let alone dealing with other control and filtering issues. Thus, it is difficult to extend the approaches of the stability analysis for the system with such two sorts of MLFs if without additional tricks to the uncertain switched system. In Sect. 2.5, the so-called quasi-time-dependent (QTD) Lyapunov function will be constructed to overcome this difficulty that also exists in the switched systems with PDT switching. The readers can refer to [6] for more discussions how the QTD technique evolves and how the conservatism in non-QTD techniques can be reduced.

Example 2.12 Consider the discrete-time switched linear system (2.1), the system matrices of which are shown as follows

$$A_1 = \begin{bmatrix} 1.00 & 0.01 \\ -0.05 & 0.99 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.84 & 0.30 \\ -1.00 & 0.82 \end{bmatrix} \tag{2.18}$$

First, suppose that the DT of the subsystem $M_1 = 1 \sim 20$, $M_2 = 1 \sim 20$. Based on Theorem 2.9 derived from the general MLF and Theorem 2.10 derived from the MLF in Fig. 2.2b, the corresponding DT pairs guaranteeing the stability of system (2.18) are illustrated in Fig. 2.4a, b, respectively, ("•" represents the feasible DT). Obviously, the stability "region" of Fig. 2.4a is larger than that of Fig. 2.4b, which indicates that the general MLF approach is less conservative than the approach of the MLF in Fig. 2.2b when the number of subsystems $N = 2$. In addition, the stability criterion in Theorem 2.5 shows that the switched system (2.18) is unstable. Similarly, in Fig. 2.4, the case with the DT $M_1 = M_2 = 1$ also shows that the system is unstable. Note that in this case, Theorem 2.10 is equivalent to Theorem 2.9.

As shown from the above verification, although the stability analysis based on the SLF shows that the system is unstable, switched system (2.18) under arbitrary switching, i.e., the switching instant and the switching sequence of the subsystems are arbitrary, possesses the potential to become stable as the running time of the DT increases. This demonstrates that the SLF approach is more strict and conservative. However, as known in the conditions (2.14), (2.16) and Fig. 2.4, the DT of each subsystem has to be treated as the known condition in both Theorems 2.9 and 2.10, which means, the switching instant is not completely arbitrary. Therefore, the stability criteria for the systems based on the general MLF and the MLF in Fig. 2.2b suffer the computational complexity and application limitations (i.e., constrained switching) to a certain degree.

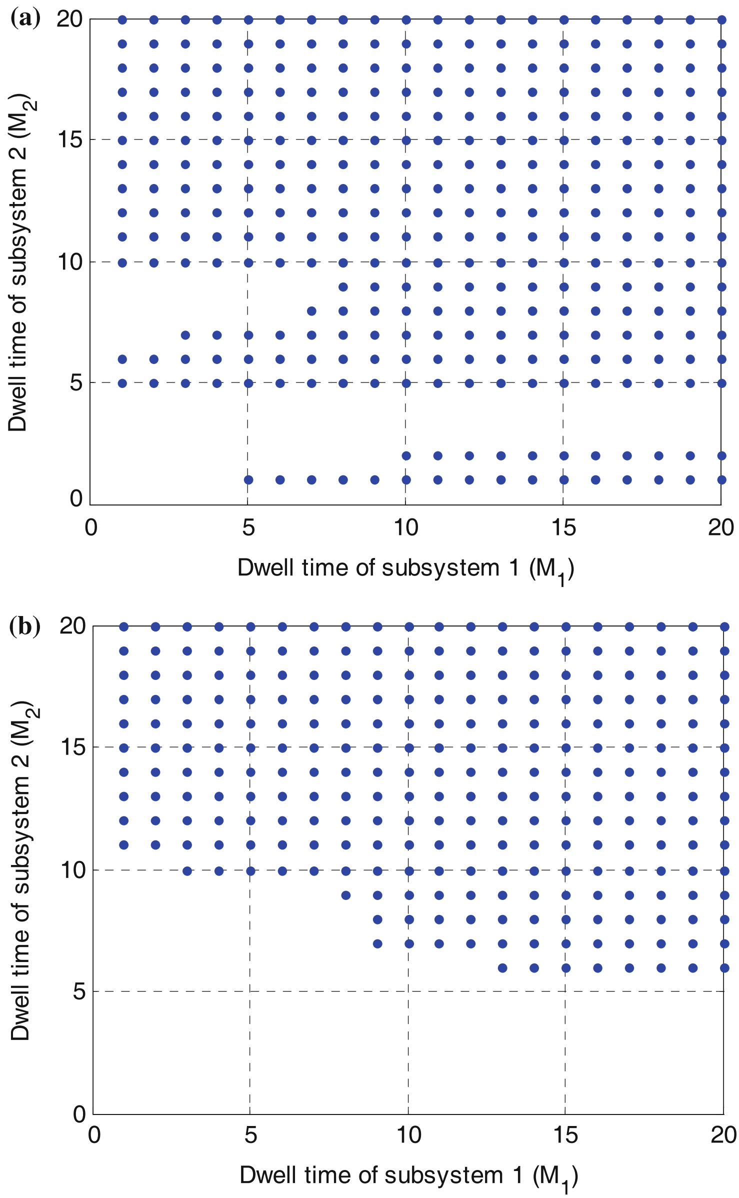

Fig. 2.4 The pairs of MDT such that the switched system is stable

2.4 Average Dwell Time (ADT) Switching

In what follows, the stability conditions for switched systems with average dwell time (ADT) switching will be given. For the convenience of a comparison, we also present the result in continuous-time domain that is first arrived at in [7]. The used Lyapunov function in [7] is sort of MLFs with μ-times increase at switching instants, as can be clearly seen from the derivations in the criteria and thus not listed in Fig. 2.2.

Theorem 2.13 *([7]) Consider the continuous-time switched system* $\dot{x}(t) = f_{\sigma(t)}(x(t))$, $\sigma(t) \in \mathcal{I}$, *and let* $\lambda > 0$, $\mu > 1$ *be given constants. Suppose that there exist* $\mathbb{C}^1$ *functions* $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, *and two class* $\mathcal{K}_\infty$ *functions* κ_1, κ_2 *such that,*

$$\kappa_1(x(t)) \le V_i(x(t)) \le \kappa_2(x(t)) \tag{2.19}$$

$$\dot{V}_i(x(t)) \le -\lambda V_i(x(t)) \tag{2.20}$$

and $\forall(\sigma(t_i) = i, \sigma(t_i^-) = j) \in \mathcal{I} \times \mathcal{I}, i \ne j,$

$$V_i(x(t_i)) \le \mu V_j(x(t_i)) \tag{2.21}$$

then the system is GUAS for any switching signal with ADT

$$\tau_a \ge \tau_a^* = \frac{\ln \mu}{\lambda} \tag{2.22}$$

Proof The proof of this theorem can be referred to [7] and is omitted here. □

Similar to the stability conditions for continuous-time switched systems, the corresponding results for the discrete-time case are given in the following theorem.

Theorem 2.14 *([8]) Consider the discrete-time switched system* $x(k+1) = f_{\sigma(k)}(x(k))$, $\sigma(k) \in \mathcal{I}$ *and let* $0 < \lambda < 1$ *and* $\mu > 0$, $\forall i \in \mathcal{I}$ *be given constants. Suppose that there exist positive definite* $\mathbb{C}^1$ *functions* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$ *and two class* $\mathcal{K}_\infty$ *functions* κ_1, κ_2 *such that,*

$$\kappa_1(\|x(k)\|) \le V_i(x(k)) \le \kappa_2(\|x(k)\|) \tag{2.23}$$

$$\Delta V_i(x(k)) \le -\lambda V_i(x(k)) \tag{2.24}$$

and $\forall(\sigma(k_i) = i, \sigma(k_{i-1}) = j) \in \mathcal{I} \times \mathcal{I}, i \ne j,$

$$V_i(x(k_i)) \le \mu V_j(x(k_i)) \tag{2.25}$$

then the system is GUES for any switching signal with ADT

$$\tau_a > \tau_a^* = -\frac{\ln \mu}{\ln(1-\lambda)} \tag{2.26}$$

Proof For $k \in \left[k_l, k_{l+1}\right)$, it follows from (2.24) that

$$V_{\sigma(k)}\left(x(k)\right) \leqslant (1-\lambda)^{k-k_l} V_{\sigma(k_l)}\left(x(k_l)\right) \tag{2.27}$$

Then, according to (2.25) and (2.27), one can obtain

$$\begin{aligned} V_{\sigma(k)}\left(x(k)\right) &\leqslant (1-\lambda)^{(k-k_l)} \mu V_{\sigma(k_{l-1})}\left(x(k_l)\right) \\ &\leqslant \cdots \leqslant (1-\lambda)^{(k-k_0)} \mu^{N_\sigma(k,k_0)} V_{\sigma(k_0)}\left(x(k_0)\right) \end{aligned}$$

From

$$N_\sigma(k, k_0) \leq N_0 + \frac{k-k_0}{\tau_a}$$

it is straightforward to get

$$V_{\sigma(k)}\left(x(k)\right) \leq \mu^{N_0}\left((1-\lambda)\mu^{1/\tau_\alpha}\right)^{(k-k_0)} V_{\sigma(k_0)}\left(x(k_0)\right)$$

In addition, for the considered Lyapunov function, it is trivial to know that

$$a_\sigma \|x(k)\| \leqslant V_\sigma(x) \leqslant b_\sigma \|x(k)\|, \sigma \in \mathcal{I}$$

for some $a_\sigma > 0$ and $b_\sigma > 0$. Then we have

$$a\|x(k)\| \leqslant V(x) \leqslant b\|x(k)\|$$

where $a \triangleq \inf(a_\sigma)$ and $b \triangleq \sup(b_\sigma)$.

Therefore, if the ADT satisfies (2.26), one can readily obtain

$$(1-\lambda)\mu^{1/\tau_\alpha} \leqslant (1-\lambda)\mu^{-\ln(1-\gamma)/\ln\mu} \leqslant \frac{(1-\lambda)}{(1-\lambda)} = 1$$

Denoting $\beta \triangleq \sqrt{(1-\lambda)\mu^{1/\tau_\alpha}}$, the system state satisfies

$$\|x(k)\|^2 \leqslant \frac{1}{a} V_{\sigma(k)}\left(x(k)\right) \leqslant \frac{b}{a}\mu^{N_0}\beta^{2(k-k_0)}\|x(k_0)\|^2$$

which means

$$\|x(k)\| \leqslant \sqrt{\frac{b}{a}\mu^{N_0}\beta^{(k-k_0)}}\,\|x(k_0)\|$$

thus the considered system is GUES, which completes the proof. □

Remark 2.15 It can be seen from Theorem 2.14 that when we increase the value of μ, the existence likelihood of the multiple Lyapunov function for the system stability will be increased, which means the stability of system can be ensured at the expense of increasing μ. In other words, for a given λ, the system stability will be directly dependent on μ. Note that the stability will also depend on decay rate of Laypunov function λ, however, it is not regarded as a design parameter in the section for simplicity.

In the following, the above results will be extended to the case of discrete-time switched systems with modal (MADT) switching.

Theorem 2.16 *Consider the discrete-time switched nonlinear system*

$$x(k+1) = f_{\sigma(k)}(x(k)), \quad \sigma(k) \in \mathcal{I} \tag{2.28}$$

and let $0 < \lambda_i < 1$ *and* $\mu_i \geq 1$, $i \in \mathcal{I}$ *be given constants. Suppose that there exist* $\mathbb{C}^1$ *functions* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, *and class* $\mathcal{K}_\infty$ *functions* κ_{1i} *and* κ_{2i}, $i \in \mathcal{I}$, *such that* $\forall\sigma(k) = i \in \mathcal{I}$

$$\kappa_{1i}(\|x(k)\|) \leq V_i(x(k)) \leq \kappa_{2i}(\|x(k)\|) \tag{2.29}$$

$$\Delta V_i(x(k)) \leq -\lambda_i V_i(x(k)) \tag{2.30}$$

and $\forall(\sigma(k_p) = i,\ \sigma(k_{p-1}) = j) \in \mathcal{I} \times \mathcal{I},\ i \neq j$,

$$V_i(x(k_i)) \leq \mu_i V_j(x(k_i)) \tag{2.31}$$

then the system is GUAS for any switching signal with MADT

$$\tau_{ai} > \tau_{ai}^* = -\frac{\ln \mu_i}{\ln(1-\lambda_i)} \tag{2.32}$$

Proof For any $K > 0$, let $k_0 = 0$ and denote $k_1, k_2, \ldots, k_p, k_{p+1}, \ldots, k_{N_\sigma(K,0)}$ the switching times on the interval $[0, K]$, where $N_\sigma(K, 0) = \sum_{i=1}^N N_{\sigma i}(K, 0)$.

By (2.30), $\forall i \in \mathcal{I}$,

$$V_i(x(k+1)) - V_i(x(k)) < 0 \tag{2.33}$$

and

$$V_i(x(k+1)) \leq (1-\lambda_i)V_i(x(k)) \tag{2.34}$$

(2.34) together with (2.31) imply

$$\begin{aligned}V_{\sigma(k_{p+1})}(x(k_{p+1})) &\le \mu_{\sigma(k_{p+1})}V_{\sigma(k_{p+1}-1)}(x(k_{p+1}))\\ &\le \mu_{\sigma(k_{p+1})}V_{\sigma(k_{p+1}-1)}(x(k_{p+1}-1))(1-\lambda_{\sigma(k_{p+1}-1)})\\ &= \mu_{\sigma(k_{p+1})}(1-\lambda_{\sigma(k_p)})V_{\sigma(k_p)}(x(k_{p+1}-1))\\ &\le \mu_{\sigma(k_{p+1})}(1-\lambda_{\sigma(k_p)})^{k_{p+1}-k_p}V_{\sigma(k_p)}(x(k_p))\\ &\quad \cdots\\ &\le \prod_{q=0}^{p}\mu_{\sigma(k_{q+1})}\prod_{q=0}^{p}(1-\lambda_{\sigma(k_q)})^{k_{q+1}-k_q}V_{\sigma(k_0)}(x(k_0))\end{aligned}$$

Then, by (2.34), one obtains

$$\begin{aligned}&V_{\sigma(K)}(x(K))\\ &\quad\le (1-\lambda_{\sigma(k_{N_\sigma})})^{K-k_{N_\sigma}}V_{\sigma(k_{N_\sigma})}(x(k_{N_\sigma}))\\ &\quad\le (1-\lambda_{\sigma(k_{N_\sigma})})^{K-k_{N_\sigma}}\prod_{j=0}^{N_\sigma-1}\mu_{\sigma(k_{j+1})}\prod_{j=0}^{N_\sigma-1}(1-\lambda_{\sigma(k_j)})^{k_{j+1}-k_j}V_{\sigma(0)}(x(0))\\ &\quad=\prod_{i=1}^{N}\mu_i^{N_{\sigma i}}\prod_{i=1}^{N}(1-\lambda_i)^{T_i}V_{\sigma(0)}(x(0))\\ &\quad=\prod_{i=1}^{N}\mu_i^{N_{\sigma i}}\exp\left\{\sum_{i=1}^{N}[T_i\ln(1-\lambda_i)]\right\}V_{\sigma(0)}(x(0))\\ &\quad\le \exp\left\{\sum_{i=1}^{N}N_{0i}\ln\mu_i\right\}\exp\left\{\sum_{i=1}^{N}\frac{T_i}{\tau_{ai}}\ln\mu_i+\sum_{i=1}^{N}\ln(1-\lambda_i)T_i\right\}V_{\sigma(0)}(x(0))\end{aligned}$$

Thus, if there exist constants τ_{ai}, $i\in\mathcal{I}$ satisfying (2.32), the following holds

$$\begin{aligned}&V_{\sigma(K)}(x(K))\\ &\quad\le \exp\left\{\sum_{i=1}^{N}N_{0i}\ln\mu_i\right\}\exp\left\{\max_{i\in\mathcal{I}}\left[\frac{\ln\mu_i}{\tau_{ai}}+\ln(1-\lambda_i)\right]K\right\}V_{\sigma(0)}(x(0))\end{aligned}$$

Then, it can be concluded that $V_{\sigma(K)}(x(K))$ converges to zero as $K\to\infty$ if the MADT satisfies (2.32), and the asymptotic stability can be obtained with the aid of (2.29). □

Remark 2.17 It can be seen from Theorems 2.13 and 2.14 that the parameters λ and μ are the same for all subsystems, i.e., mode-independent. However, the parameters in Theorem 2.16 are mode-dependent. It can be concluded that $\tau_{ai}^*\le\tau_a^*$, $\forall i\in\mathcal{I}$, and the mode-dependent features would reduce the conservativeness existed in Theorems 2.13 and 2.14. In fact, note that if $\tau_a=\tau_{ai}$, $\forall i\in\mathcal{I}$, one readily knows from

Definition 1.7 that

$$\sum_{i\in\mathcal{I}} N_{\sigma i}(T,t) \le \sum_{i\in\mathcal{I}} N_{0i} + \sum_{i\in\mathcal{I}} \frac{T_i}{\tau_a}, \quad \forall T \ge t \ge 0$$

Thus, there exist positive numbers $N_0 = \sum_{i\in\mathcal{I}} N_{0i}$ and $\tau_a = \tau_{ai}$ such that

$$N_\sigma(T,t) \le N_0 + \frac{T-t}{\tau_a}, \ \ \forall T \ge t \ge 0$$

That is, a switching signal with bounded MADT τ_{ai}^* also has bounded ADT $\tau_a^* \equiv \tau_{ai}^*$, $\forall i \in \mathcal{I}$ in the special case of $\lambda \equiv \lambda_i$, $\mu \equiv \mu_i$, $\forall i \in \mathcal{I}$. From this, it can be concluded that the MADT switching has the advantage of flexibility for a switched system where the switching is able to or needs be designed.

For the issue of stabilizing controller design, consider the switched linear system given as

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{2.35}$$

Our objective here is to find an admissible controller in the form of

$$u(k) = K_{\sigma(k)}x(k) \tag{2.36}$$

where $K_{\sigma(k)}$ is to be determined. Then, the resulting closed-loop system is given by

$$x(k+1) = \bar{A}_{\sigma(k)}x(k) \tag{2.37}$$

where $\bar{A}_{\sigma(k)} \triangleq A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma(k)}$.

Next, based on the results obtained above, we first give the stability conditions for switched systems (2.35) with MADT switching.

Theorem 2.18 *Consider the switched linear system (2.35) when $u(k) \equiv 0$ and let $0 < \lambda_i < 1$ and $\mu_i \ge 1$, $\forall i \in \mathcal{I}$ be given constants. If there exist matrices $P_i > 0$, $\forall i \in \mathcal{I}$ such that $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$,*

$$A_i^T P_i A_i + \lambda_i P_i - P_i \le 0 \tag{2.38}$$

$$P_i - \mu_i P_j \le 0 \tag{2.39}$$

then the switched linear system (2.35) is GUES with MADT satisfying (2.32).

Proof Construct the Lyapunov function as follows

$$V_i(x(k)) = x^T(k)P_i x(k), \forall \sigma(k) = i \in \mathcal{I} \tag{2.40}$$

where P_i, $\forall i \in \mathcal{I}$, is a positive definite matrix satisfying (2.38) and (2.39). Then, from (2.30), (2.31), (2.35) and (2.40), it is not hard to obtain, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$,

$$\begin{aligned}
&\Delta V_i(x(k)) + \lambda_i V_i(x(k)) \\
&\quad = \lambda_i x^T(k) P_i x(k) + x^T(k) A_i^T P_i A_i x(k) - x^T(k) P_i x(k) \\
&\quad = x^T(k) \left(A_i^T P_i A_i + \lambda_i P_i - P_i \right) x(k)
\end{aligned}$$

and

$$V_i(x(k_i)) - \mu_i V_j(x(k_i)) = x^T(k_i) \left(P_i - \mu_i P_i \right) x(k_i)$$

Thus, if (2.38) and (2.39) hold, system (2.35) is GUES for any switching signal with MADT (2.32). □

Now, we are in a position to give the existence conditions of a stabilizing controller for system (2.35) with the MADT switching.

Theorem 2.19 *Consider the switched linear system (2.35) and let* $0 < \lambda_i < 1$ *and* $\mu_i \geq 1$, $\forall i \in \mathcal{I}$ *be given constants. If there exist matrices* $U_i > 0$ *and* T_i, $\forall i \in \mathcal{I}$ *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$\begin{bmatrix} -U_i & A_i U_i + B_i T_i \\ \star & -(1 - \lambda_i U_i) \end{bmatrix} \leq 0 \tag{2.41}$$

$$U_j \leq \mu_i U_i \tag{2.42}$$

then there exists a stabilizing controller such that system (2.35) is GUAS for any switching signal with MADT satisfying (2.32). Moreover, if (2.41) and (2.42) have a solution, the admissible controller can be given by

$$K_i = T_i U_i^{-1} \tag{2.43}$$

Proof Theorem 2.18 implies that if

$$\begin{aligned}
\bar{A}_i^T P_i \bar{A}_i + \lambda_i P_i - P_i &\leq 0 \\
P_i - \mu_i P_j &\leq 0
\end{aligned}$$

system (2.35) is GUAS for any switching signal with MADT satisfying (2.32). Considering (2.36), setting $U_i \triangleq P_i^{-1}$ and $T_i \triangleq K_i P_i^{-1}$, it can be seen that, if (2.41) holds, (2.38) is satisfied. Moreover, if (2.42) holds, one can obtain that $U_j - \mu_i U_i \leq 0$. By Lemma 2.4, $U_j - \mu_i U_i \leq 0$ can be rewritten as

$$\Lambda \triangleq \begin{bmatrix} -\mu_i U_i & I \\ I & -U_j^{-1} \end{bmatrix} \leq 0$$

Furthermore, note that $\Lambda \leq 0$ is equivalent to $-U_j^{-1} - I(\mu_i U_i)^{-1} I \leq 0$ by Lemma 2.4. Additionally, if the inequalities (2.41) and (2.42) have feasible solutions, the admissible controller gains can be given by (2.43) since $T_i = K_i P_i^{-1}$, which ends the proof. □

Remark 2.20 From the above analysis mentioned, the ADT switching can be viewed as a special case of MADT switching. The stabilizing conditions in the situation of ADT switching can be achieved directly from Theorem 2.19 and therefore are omitted here.

In the following, a numerical example in discrete-time domain will be presented to demonstrate the potential and validity of the results obtained in Sect. 2.4.

Example 2.21 Consider the discrete-time switched linear system (2.35) consisting of three subsystems described by

$$A_1 = \begin{bmatrix} 3.9 & 1.5 \\ 2.5 & 2.3 \end{bmatrix}, B_1 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.4 & 0.3 \\ 1 & -2.7 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, A_3 = \begin{bmatrix} -2.2 & 0.1 \\ -2 & -0.4 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

Our purpose here is to design a mode-dependent stabilizing controller and find the admissible switching signals with MADT such that the resulting closed-loop system is GUAS.

To illustrate the advantages of the proposed MADT switching scheme, the design results of both controllers and switching signals should be presented for the systems with ADT switching for the sake of comparison. By different approaches and setting the relevant parameters appropriately, the computation results for system (2.35) with two different switching schemes are listed in Table 2.1.

It can be seen from Table 2.1 that the minimal MADT are reduced to $\tau_{a1}^* = 1$, $\tau_{a2}^* = 1$, $\tau_{a3}^* = 4$, for given $\mu = \mu_1 = \mu_2 = \mu_3 = 2$, and one special case of MADT switching is $\tau_a^* = \tau_{a1}^* = \tau_{a2}^* = \tau_{a3}^* = 4$ by setting $\lambda = \lambda_1 = \lambda_2 = \lambda_3 = 0.2$, which is the ADT switching, i.e., the designed MADT switching is more general.

To further show the merit of MADT switching, let us now consider the resulting closed-loop system performances. Applying the obtained controller, under the scheme of ADT switching and MADT switching, respectively, the obtained state responses for each closed-loop subsystem are shown in Fig. 2.5. For each closed-loop subsystem $\bar{A}_i$, it is clear to see that the transient behavior for both subsystem 2 and 3 under controllers Γ_1 and Γ_2 are similar, while the response of subsystem 1 under Γ_2 is much better than that under Γ_1.

Table 2.1 Computation results for the system under two different switching schemes

Switching schemes	ADT switching	MADT switching
Controller gains	Γ_1: $K_1 = [36.16\ 18.90]$ $K_2 = [-7.94\ -8.16]$ $K_3 = [21.08\ 1.30]$	Γ_2: $K_1 = [41.67\ 22.69]$ $K_2 = [-8.64\ 6.83]$ $K_3 = [21.18\ 1.06]$
Switching signals	$\tau_a^* = 4$ ($\mu = 2, \lambda = 0.2$)	$\tau_{a1}^* = 1, \tau_{a2}^* = 1, \tau_{a3}^* = 4$ ($\mu_1 = \mu_2 = \mu_3 = 2$, $\lambda_1 = 0.97, \lambda_2 = 0.8, \lambda_3 = 0.2$)

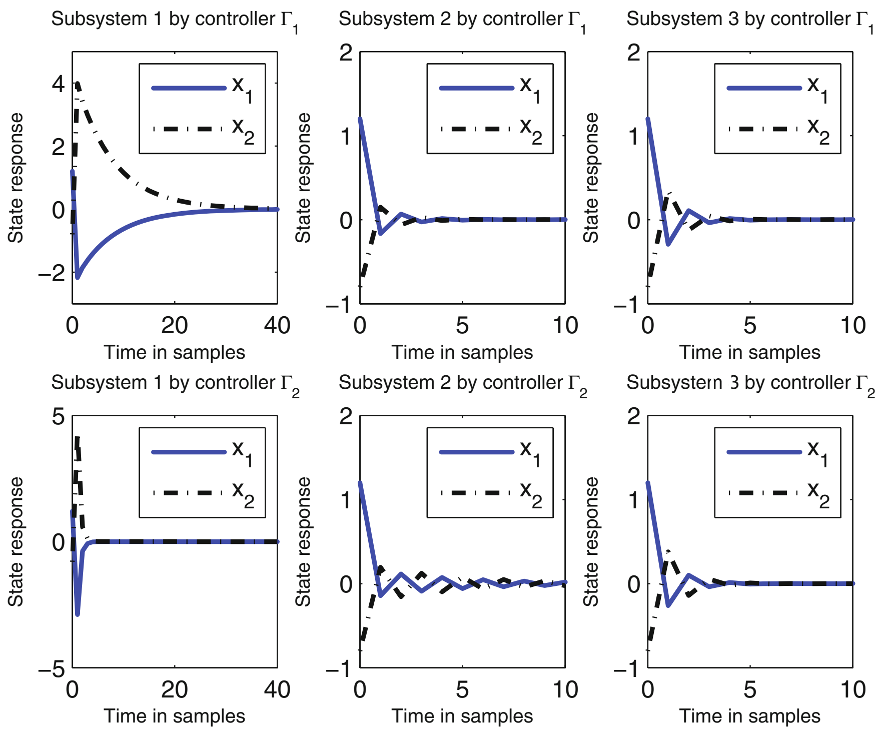

Fig. 2.5 The state response comparisons of the closed-loop subsystems by controllers Γ_1 and Γ_2

Then, generating one possible switching sequence with the ADT property and the MADT property, one can obtain the corresponding state responses of the closed-loop system as shown in Figs. 2.6 and 2.7, respectively, for the same initial state condition. It can be seen from the curves that the state response of closed-loop system is fluctuated under the ADT switching scheme, but can converge to zero in a short time under the MADT switching scheme. To present the reason more clearly, denote the running time of the ith subsystem at the lth working as $t_{i,l}, \forall i \in \mathcal{I}, l \in \mathbb{N}^+$, and use $t_{i,l}^A$ and $t_{i,l}^M$ to represent the running time of the subsystem under ADT and MADT switching schemes, respectively. It can be observed that the state responses both begin with subsystem 2 and in Fig. 2.6, $t_{2,1}^A = 2$ and in Fig. 2.7, $t_{2,1}^M = 2$. Then, due to the constraint of ADT switching ($\tau_a = 4$ in Fig. 2.6), we need $t_{1,1}^A \geq 6$. For the case of MADT switching, the constraint on $t_{1,1}^M$ can be removed. The comparison of the switching signals in Figs. 2.6 and 2.7 shows that even for $t_{1,1}^M < t_{1,1}^A$ (we want $t_{1,1}^A$ to be a little shorter), we can attain $t_{1,1}^M < t_{1,1}^A$. This will better the state response because of the shorter running time needed on the subsequent subsystem 1, which demands longer time to converge to zero as shown in Fig. 2.5.

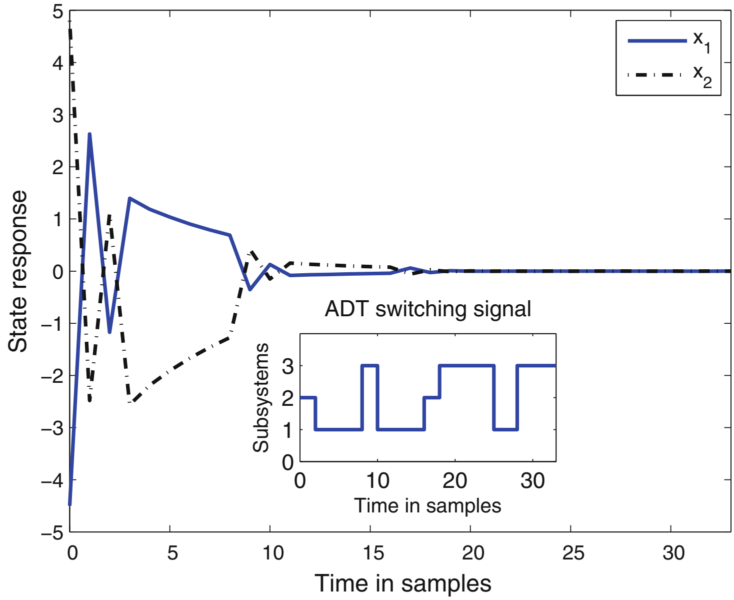

Fig. 2.6 State response of the closed-loop system by controllers Γ_1 under switching signal σ with $\tau_a = 4$

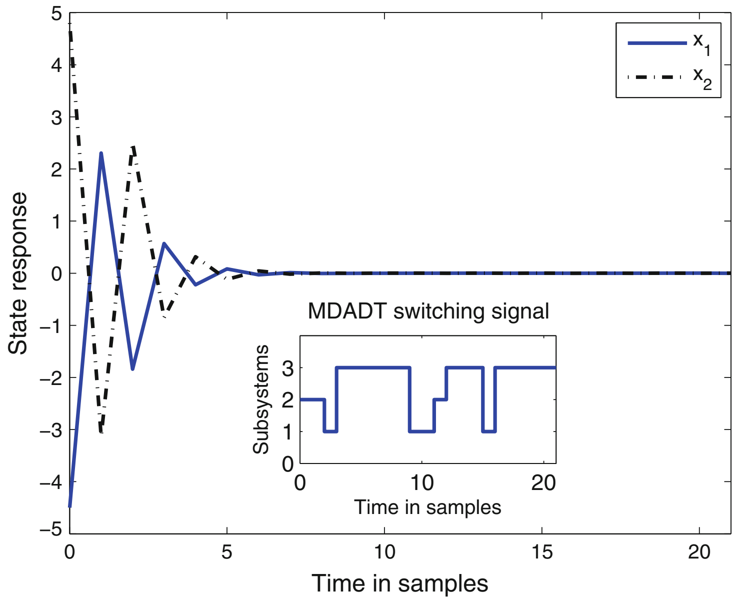

Fig. 2.7 State response of the closed-loop system by controllers Γ_2 under switching signal σ with $\tau_{a1} = 1$, $\tau_{a2} = 1$, $\tau_{a3} = 4$

Thus, from the above discussions, it can be concluded that it will be more flexible in practice to design an MADT switching to perfect or improve the system performances with fewer constraints.

2.5 Persistent Dwell Time (PDT) Switching

Consider the discrete-time switched linear system (2.35), a more general switching signal, "modal persistent dwell-time (MPDT)", is introduced in this section, which not only generalizes the commonly studied DT and ADT switchings, but also further attaches mode-dependency to the PDT switching. The definition of MPDT has been given in Sect. 1.4 of Chap. 1 and is therefore omitted here.

The example below is used to show what an admissible MPDT switchingsequence (we use $\bar{\xi}_{\tau^{[T]}}(k)$ denote inadmissible switching sequences) is. Consider a switched system consisting of three subsystems. The admissible MPDT set, with the period of persistence $\mathbb{T} = 3$, is supposed to be $\tau^{[3]} = \{4, 3, 5\}$. Then $\xi^a_{\tau^{[3]}}(11) = \{1, 1, 1, 1, 2, 1, 3, 2, 2, 2, 2\}$ is an admissible sequence, but both $\bar{\xi}^a_{\tau^{[3]}}(11) = \{1, 1, 1, 3, 2, 1, 3, 3, 3, 3, 3\}$ and $\bar{\xi}^b_{\tau^{[3]}}(11) = \{1, 1, 1, 1, 2, 1, 3, 1, 2, 2, 2\}$ are not since the requirements of $\tau_1 \geq 4$ and $\mathbb{T} \leq 3$ are not satisfied in the former and latter cases, respectively.

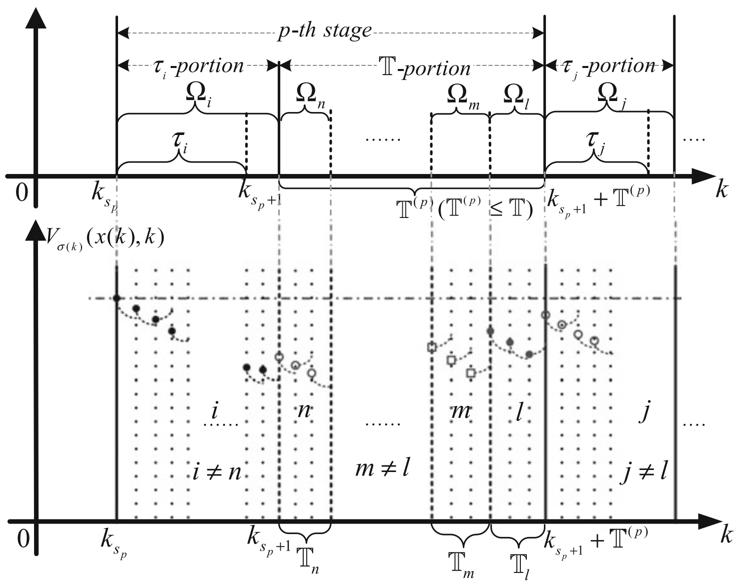

Fig. 2.8 A scenario of MPDT switching (on the *top*), where the period of persistence is $\mathbb{T}$, $i \neq n$, $m \neq l$, $j \neq l$ and $\mathbb{T}^{(p)} \leq \mathbb{T}$. The figure on the *bottom* illustrates the variation of the Lyapunov function used in Theorem 2.22

An illustration on MPDT is given in Fig. 2.8, where the interval consisting of the running time (τ_i-portion) of a certain subsystem and the period of persistence ($\mathbb{T}$-portion) is considered as an *MPDT stage*,[1] and k_{s_p} is denoted as the initial instant of the pth stage, $p \in \mathbb{Z}_{\geq 1}$ with $k_{s_1} \geq k_0$ (here "$\geq$" means a period of persistence may exist before the 1st stage). Let the actual running time of the $\mathbb{T}$-portion at the pth stage be denoted as $\mathbb{T}^{(p)}$, $p \in \mathbb{Z}_{\geq 1}$, it holds that

$$\mathbb{T}^{(p)} \triangleq \sum_{r=1}^{\mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})} \mathbb{T}_{\sigma(k_{s_p+r})} \leq \mathbb{T} \tag{2.44}$$

where $\mathbb{T}_{\sigma(k_{s_p+r})} < \tau_i$ denotes the running time of the subsystem activated at the switching instant $k_{s_p+r} \in \left[k_{s_p+1}, k_{s_{p+1}}\right)$, $r \in \mathbb{Z}_{\geq 1}$, and $\mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})$ stands for the switching times within $\left[k_{s_p+1}, k_{s_{p+1}}\right)$.

In this section, we would like to directly present the stabilization result for the underlying system. The used methodology is similar to the one in Fig. 2.2b, but compares the MLFs at the instants entering into two consecutive stages (can be seen in the derivations of later Theorem 2.18).The specific objectvies are to develop a control policy $\mathcal{F}_{\sigma(k)}(\cdot)$, and find a set of switching signals with admissible MPDT. Here, we are interested in developing a fundamental stabilizing state-feedback policy, but with the *quasi-time-dependent* (QTD) form below, as adopted in [6]

$$\mathcal{F}_{\sigma(k)}(x(k)) \triangleq F_{\sigma(k)}(\vartheta)x(k) \tag{2.45}$$

where ϑ is a scheduled index for the activated subsystem and can be computed online according to the following rules: $\forall \sigma(k) = i \in \mathcal{I}$,

(i) in the τ_i-portion,

$$\vartheta = \begin{cases} k - k_{s_p}, & k \in \left[k_{s_p}, k_{s_p} + \tau_i\right) \\ \tau_i, & k \in \left[k_{s_p} + \tau_i, k_{s_p+1}\right) \end{cases} \tag{2.46}$$

(ii) in the $\mathbb{T}$-portion,

$$\vartheta = k - H_r, k \in \left[k_{s_p+1}, k_{s_{p+1}}\right) \tag{2.47}$$

where $H_r \triangleq \arg\{\max(k_{s_p+r}, r \in \mathbb{Z}_{\geq 1} | k_{s_p+r} \leq k, k_{s_p+r} \in [k_{s_p+1}, k_{s_{p+1}}))\}$ satisfies $\sigma(H_r) = i$.

It has been demonstrated in [6] for switched systems with DT switching that the QTD state-feedback law outperforms the conventional one with less conservatism in achieving minimal DT ensuring the stability of the underlying system. In order to obtain the stabilization criterion by using (2.45) for system (2.35) under MPDT switching, we consider the corresponding QTD Lyapunov function as $V_{\sigma(k)}(x(k), \vartheta)$,

[1]We will slightly abuse the concept as a stage in this book.

where ϑ has been defined in (2.46) and (2.47). Then the stability conditions for the nominal system in nonlinear case can be first arrived at.

Theorem 2.22 *Consider a discrete-time switched nonlinear system* $x(k+1) = f_{\sigma(k)}(x(k))$*, and* $0 < \alpha_i < 1$*,* $\mu_i > 0$ *to be given constants. For a prescribed period of persistence* $\mathbb{T}$*, suppose that there exist functions* $V_{\sigma(k)} : (\mathbb{R}^{n_x}, \mathbb{Z}_{[0,\tau_{\sigma(k)}]}) \to \mathbb{R}$*,* $\sigma(k) \in \mathcal{I}$*, and two class* $\mathcal{K}_\infty$ *functions* κ_1 *and* κ_2 *such that* $\forall \sigma(k) = i \in \mathcal{I}$ *and* $r \in \mathbb{Z}_{[2,\mathcal{Q}(k_{s_p+1},k_{s_{p+1}})+1]}$

(i) $\forall \vartheta \in \mathbb{Z}_{[0,\tau_i]}$,

$$\kappa_1(\|x(k)\|) \le V_i(x(k), \vartheta) \le \kappa_2(\|x(k)\|) \tag{2.48}$$

(ii) $\forall k \in \left[k_{s_p}, k_{s_p} + \tau_i\right)$,

$$V_i(x(k+1), k+1-k_{s_p}) \le \alpha_i V_i(x(k), k-k_{s_p}) \tag{2.49}$$

(iii) $\forall k \in \left[k_{s_p} + \tau_i, k_{s_p+1}\right)$,

$$V_i(x(k+1), \tau_i) \le \alpha_i V_i(x(k), \tau_i) \tag{2.50}$$

(iv) $\forall k \in [k_{s_p+1}, k_{s_{p+1}})$, $r \in \mathbb{Z}_{[1,\mathcal{Q}(k_{s_p+1},k_{s_{p+1}})]}$

$$V_i(x(k+1), k+1-H_r) \le \alpha_i V_i(x(k), k-H_r) \tag{2.51}$$

(v) $\forall \sigma(k_{s_p+1}) = i \ne j = \sigma(k_{s_p+1} - 1)$,

$$V_i(x(k_{s_p+1}), 0) \le \mu_j V_j(x(k_{s_p+1}), \tau_j) \tag{2.52}$$

(vi) $\forall \sigma(k_{s_p+r}) = i \ne j = \sigma(k_{s_p+r} - 1)$,

$$V_i(x(k_{s_p+r}), 0) \le \mu_j V_j(x(k_{s_p+r}), \mathbb{T}_j) \tag{2.53}$$

where $\mathbb{T}_j \in \left[1, \min(\tau_i - 1, \mathbb{T}^{(p)})\right]$*,* $\forall i \in \mathcal{I}$*,* $\mathbb{T}^{(p)} \in \mathbb{Z}_{[1,\mathbb{T}]}$*. Then the switched nonlinear system is GUAS for MPDT switching signals satisfying (2.48)–(2.53) and*

$$\tau_i \ge \frac{(\mathbb{T}+1)\ln \mu_{\max} + \mathbb{T} \ln \alpha_{\max}}{-\ln \alpha_i} \tag{2.54}$$

where $\mu_{\max} \triangleq \max_{i \in \mathcal{I}} \mu_i$*,* $\alpha_{\max} \triangleq \max_{i \in \mathcal{I}} \alpha_i$*.*

Proof First of all, if $\mu_{\max}\alpha_{\max} < 1$, then it is straightforward that a switched system is GUAS with $\tau_i = 1$, i.e., under arbitrarily switching. If (2.54) holds, τ_i is at least 1 in discrete-time domain. Then the proof boils down to the case $\mu_{\max}\alpha_{\max} \ge 1$.

Considering $\sigma(k_{s_p}) = i, \sigma(k_{s_p+1} + \mathbb{T}^{(p)}) = j$ in the pth stage of MPDT switching, and supposing an arbitrary switching occurs within $\mathbb{T}^{(p)}$, it follows from (2.49)–(2.53) that

$$
\begin{aligned}
&V_j(x(k_{s_p+1}+\mathbb{T}^{(p)}),0) \\
&\quad\le \mu_l V_l(x(k_{s_p+1}+\mathbb{T}^{(p)}),\mathbb{T}_l) \\
&\quad\le \mu_l {\alpha_l}^{\mathbb{T}_l} V_l(x(k_{s_p+1}+\mathbb{T}^{(p)}-\mathbb{T}_l),0) \\
&\quad\le \mu_m\mu_l {\alpha_l}^{\mathbb{T}_l} V_m(x(k_{s_p+1}+\mathbb{T}^{(p)}-\mathbb{T}_l),\mathbb{T}_m) \\
&\quad\le \mu_i\mu_n\cdots\mu_m\mu_l {\alpha_l}^{\mathbb{T}_l}{\alpha_m}^{\mathbb{T}_m}\cdots{\alpha_n}^{\mathbb{T}_n}{\alpha_i}^{k_{s_p+1}-k_{s_p}} V_i(x(k_{s_p}),0) \\
&\quad\le \mu_{\max}^{\mathcal{Q}(k_{s_p},k_{s_p+1}+\mathbb{T}^{(p)})}{\alpha_{\max}}^{\mathbb{T}_l+\mathbb{T}_m+\cdots+\mathbb{T}_n}{\alpha_i}^{\tau_i} V_i(x(k_{s_p}),0) \\
&\quad\le \mu_{\max}^{\mathbb{T}^{(p)}+1}\alpha_{\max}^{\mathbb{T}^{(p)}}{\alpha_i}^{\tau_i} V_i(x(k_{s_p}),0)
\end{aligned} \tag{2.55}
$$

where $l, m, \ldots, n$ denote all the possible indices of subsystems being switched within $\mathbb{T}^{(p)}$.

Thus since $\mu_{\max}\alpha_{\max} \ge 1$ and $\mu_{\max}^{\mathbb{T}^{(p)}+1}\alpha_{\max}^{\mathbb{T}^{(p)}} \le \mu_{\max}^{\mathbb{T}+1}\alpha_{\max}^{\mathbb{T}^{(p)}}$ hold. From (2.55), it follows that

$$V_j(x(k_{s_p+1}+\mathbb{T}^{(p)}),0) \le \mu_{\max}^{\mathbb{T}+1}\alpha_{\max}^{\mathbb{T}}{\alpha_i}^{\tau_i}V_i(x(k_{s_p}),0)$$

Then, if (2.54) is satisfied, $\mu_{\max}^{\mathbb{T}+1}\alpha_{\max}^{\mathbb{T}}{\alpha_i}^{\tau_i} \le 1$ holds. Letting $\lambda_i \triangleq \mu_{\max}^{\mathbb{T}+1}\alpha_{\max}^{\mathbb{T}}{\alpha_i}^{\tau_i}$, $\lambda_{\max} \triangleq \max_{i\in\mathcal{I}}\lambda_i$, $\forall i \in \mathcal{I}$, and considering the fact that a period of persistence may exist before the 1st stage, it follows

$$
\begin{aligned}
V_{\sigma(k_{s_p})}(x(k_{s_p}),0) &\le \lambda_{\max}V_{\sigma(k_{s_{p-1}})}(x(k_{s_{p-1}}),0) \le \cdots \\
&\le \lambda_{\max}^{p-1}V_{\sigma(k_{s_1})}(x(k_{s_1}),0) \le \lambda_{\max}^{p-1}\mu_{\max}^{\mathbb{T}}\alpha_{\max}^{\mathbb{T}}V_{\sigma(k_0)}(x(k_0),0).
\end{aligned}
$$

From (2.48),

$$||x(k_{s_p})|| \le \kappa_1^{-1}(\lambda_{\max}^{p-1}\mu_{\max}^{\mathbb{T}}\alpha_{\max}^{\mathbb{T}}\kappa_2(||x(k_0)||))$$

holds. Thus, due to (2.48)–(2.53), $||x(k)|| \le \kappa_3(||x(k_0)||)$ holds, $\forall k \in (k_{s_p}, k_{s_{p+1}}]$, where

$$\kappa_3(\cdot) \triangleq \kappa_1^{-1}(\mu_{\max}^{\mathbb{T}}\alpha_{\max}^{\mathbb{T}}\kappa_2(\kappa_1^{-1}(\lambda_{\max}^{p-1}\mu_{\max}^{\mathbb{T}}\alpha_{\max}^{\mathbb{T}}\kappa_2(\cdot)))).$$

Thus the GUAS can be inferred by the denotation of $\lambda_{\max}$ and Definition 2.1. This completes the proof. □

Remark 2.23 It should be noted that since the running time of each activated subsystem during the period of persistence is unknown a priori, the worst case of using $\mu_{\max}$, $\alpha_{\max}$ in the derivation of (2.55) is taken into account, as well as the consideration of $\mathbb{T}$ times of switching during the period of persistence.

In Theorem 2.22, if $V_i(x(k),\vartheta) \equiv V_i(x(k))$, $\mu_i \equiv \mu$, $\alpha_i \equiv \alpha$, and $\mathbb{T} \equiv 0$, i.e., the time-dependent Lyapunov function and the PDT switching are considered, and the period of persistence vanishes, then the corresponding MPDT switching reduces to the DT case in the end, and the corresponding stability criterion is simplified to the following corollary.

Corollary 2.24 *Consider nominal system (2.35) with* $u(k) \equiv 0$*, and let* $0 < \alpha < 1$*,* $\mu > 1$ *are given constants. If there exist matrices* $P_i \in \mathcal{S}^n_{>0}$*,* $\forall i \in \mathcal{I}$*, such that* $\forall \sigma(k) = i \in \mathcal{I}$*,*

$$V_i(x(k+1)) \leq \alpha V_i(x(k))$$

holds and $\forall i \times j \in \mathcal{I} \times \mathcal{I}$*,* $i \neq j$*,*

$$V_i(x(k_{s_p+1})) \leq \mu V_j(x(k_{s_p+1}))$$

holds, then the switched system is GUAS for any switching signal with DT satisfying $\tau \geq -\ln \mu / \ln(1-\alpha)$*.*

Remark 2.25 A noteworthy fact is that the conditions in Corollary 2.24 are also the ones ensuring that the underlying switched systems are GUAS with ADT switching, as have been derived in [9]. Therefore, Theorem 2.22 obtained in this section is more general than the existing stability results on switched systems with either DT or ADT switching.

Remark 2.26 Note that, in the frame of ADT switching, the requirement on the ADT ensuring that the switched system is GUAS is also $\tau \geq -\ln \mu / \ln(1-\alpha)$ (cf. [10]), which holds for any $N_0 \geq 2$. However, the requirement in the DT case reduced from PDT switching when $\mathbb{T} \equiv 0$ only holds for $N_0 = 1$.

Then, by considering the QTD Lyapunov function as $V_{\sigma(k)}(x(k), \vartheta) \triangleq x^T(k) P_i(\vartheta) x(k)$, the stabilization criterion for nominal system (2.35) can be readily established in the following theorem.

Theorem 2.27 *Consider system (2.35) and let* $0 < \alpha_i < 1$*,* $\mu_i > 0$ *be given constants,* $i \in \mathcal{I}$*. Suppose there exist matrices* $S_i(\vartheta) \in \mathbb{S}^{n_x}_{>0}$ *and* $U_i(\vartheta)$*,* $\vartheta = 0, 1, \ldots, \tau_i$*,* $\forall i \in \mathcal{I}$*, such that* $\forall\ \vartheta = 0, 1, \ldots, \tau_i - 1$*,* $\forall i \in \mathcal{I}$

$$\begin{bmatrix} -S_i(\tau_i) & A_i S_i(\tau_i) + B_i U_i(\tau_i) \\ \star & -\alpha_i S_i(\tau_i) \end{bmatrix} \leq 0 \tag{2.56}$$

$$\begin{bmatrix} -S_i(\vartheta+1) & A_i S_i(\vartheta) + B_i U_i(\vartheta) \\ \star & -\alpha_i S_i(\vartheta) \end{bmatrix} \leq 0 \tag{2.57}$$

and $\forall (i \times j) \in \mathcal{I} \times \mathcal{I}$*,* $i \neq j$*,*

$$S_j(\mathbb{T}_j) - \mu_j S_i(0) \leq 0 \tag{2.58}$$

$$S_j(\tau_j) - \mu_j S_i(0) \leq 0 \tag{2.59}$$

hold, where $\mathbb{T}_j \in \mathbb{Z}_{[1,\min(\tau_j - 1, \mathbb{T}^{(p)})]}$*,* $\mathbb{T}^{(p)} \in \mathbb{Z}_{[1,\mathbb{T}]}$ *with* $\mathbb{T}$ *be the given period of persistence. Then, the resulting closed-loop system is GUAS for MPDT switching signals* *satisfying (2.54). Moreover, the QTD stabilizing controller gain can be obtained by*

$$F_i(\vartheta) = U_i(\vartheta)S_i^{-1}(\vartheta).$$

Proof Based on Theorem 2.22, the proof can be completed by basic matrix manipulations and Lemma 2.4, cf. [11] and is omitted here.

Remark 2.28 In Theorem 2.27, a small τ_i corresponding to fast switching may not guarantee a feasible solution of the admissible controller, then considering α_i and μ_i to be variables, the MPDT can be minimized by solving the following minimization problem.

Problem 2.1

$$\min_{\mu_i,\alpha_i,S_i,U_i} \tau_i,\ s.t.(2.54),\ (2.56)\text{–}(7.84) \tag{2.60}$$

The minimum of τ_i can be trivially found by bisection method. Note that, for a fixed $\mathbb{T}$, the minimal MPDT means to be the one with smallest $||\tau^{[\mathbb{T}]}||_1$. Furthermore, if the minimal MPDT obtained in such a way are many, the smallest variance of $\tau^{[\mathbb{T}]}$ can be further used to refine them.

If setting $U_i(\vartheta) \equiv U_i$ and $S_i(\vartheta) \equiv S_i$ in Theorem 2.27, one can obtain the corresponding control policy with "non-QTD" controller gains $F_i = U_i S_i^{-1}$, $i \in \mathcal{I}$. As a result, for a certain switched system, the minimal MPDT obtained by an optimization problem similar to Problem 2.1, denoted by θ_i, will be generally greater than the minimal τ_i derived from the QTD control policy. Nevertheless, such non-QTD F_i can be directly used as the stabilizing state-feedback gains for system (2.35).

Example 2.29 Consider system (2.35) consisting of two subsystems described by

$$A_1 = \begin{bmatrix} 1.00 & -0.70 \\ 0.50 & -0.70 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.89 & 0.38 \\ 1.65 & 1.14 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

Our purpose here is to design a QTD stabilizing controller for the nominal system, find out the admissible MPDT switching such that the corresponding closed-loop system is GUAS. Firstly, it can be checked that the nominal switched system does not admit a stabilizing controller under arbitrary switching. By Theorem 2.27 and solving Problem 2.1 for given $\alpha_i = 0.15$, however, the minimal admissible MPDT τ_i can be solved as shown in Table 2.2 for given different $\mathbb{T}$, as well as the θ_i corresponding

Table 2.2 Minimal MPDT by QTD and non-QTD stabilizing controller for different $\mathbb{T}$

$\mathbb{T}$	2	3
(τ_i, θ_i)	$\tau_1 = 3, \tau_2 = 4$; $\theta_1 = 4, \theta_2 = 5$	$\tau_1 = 4, \tau_2 = 5$; $\theta_1 = 5, \theta_2 = 6$

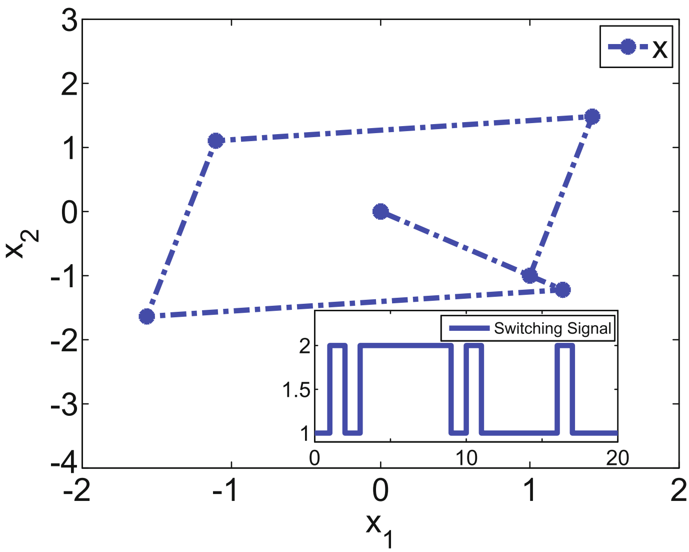

Fig. 2.9 State trajectories of the closed-loop system with MPDT switching ($\mathbb{T} = 3$)

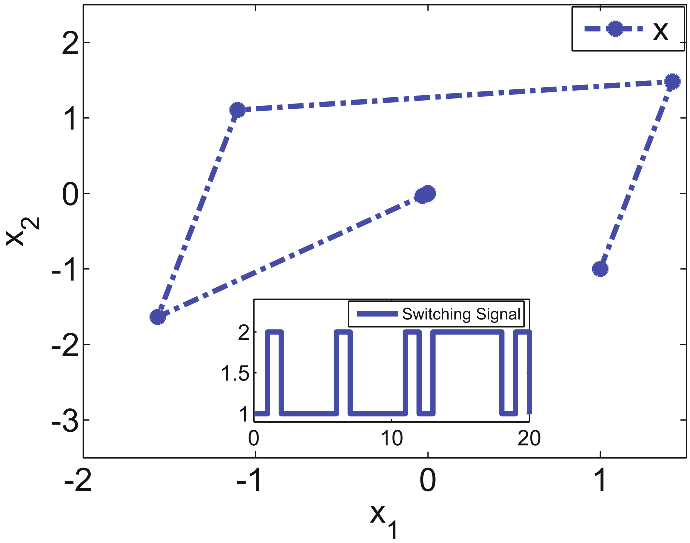

Fig. 2.10 State trajectories of the closed-loop system with MPDT switching ($\mathbb{T} = 2$)

to the non-QTD stabilizing control. It can be seen that the QTD controller has less conservatism in achieving shorter admissible MPDT. The associated controller gains in both cases are omitted here. Given the initial condition $x_0 = [1\ -1]^T$, considering the running time equivalent to the MPDT at each time of switching, and supposing that there exists a period of persistence before the first MPDT stage, the resulting switching signals and the state responses of the corresponding closed-loop system under $\mathbb{T} = 3$ and $\mathbb{T} = 2$ are presented in Figs. 2.9 and 2.10, respectively. It can be seen from Figs. 2.9 and 2.10 that the state trajectory of the resulting closed-loop system converges, verifying the validity of the QTD stabilizing controller.

2.6 Conclusion

In this chapter, we have addressed the stability and stabilization problems of switched systems with several typical time-dependent switching signals. The multiple Lyapunov functions (MLFs) including several evolved forms are introduced to serve as the tools for the stability analysis and stabilizing controller synthesis of switched systems. Specifically, the switched Lyapunov functions are utilized to derive the stability criteria for switched systems under arbitrary switching; the general MLFs and an evolved one (with the comparisons between the Lyapunov function values at two consecutive switching instants) for systems with DT switching; the MLFs with μ-times increase at switching instants for ADT switched systems; and that evolved MLFs but with the comparisons between the MLFs at the instants entering into two consecutive stages for PDT switched systems. Finally, four numerical examples were provided to illustrate the effectiveness of the obtained results.

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Chapter 3
Performance Analysis

Abstract In this chapter, the issue of performance analysis for the time-dependent switched systems with several typical switching signals will be studied. We will first focus our attention on the switched systems with l_2 disturbances and present the results of weighted/non-weighted l_2-gain analyses for the switching signals subject to arbitrary switching, ADT switching, and PDT switching, respectively. Then, considering the switching signals are to have MPDT property, we will give the tube-based robustness analysis for switched systems with l_∞ disturbances with the aid of set-theoretic method. Finally, one example is given to verify the effectiveness of developed results on the section of tube-based robustness analysis for discrete-time switched systems with MPDT switching; the verifications of the results corresponding to other switching signals will be illustrated in later chapters coping with H_∞ control or filtering.

3.1 l_2-Gain Analysis: Arbitrary Switching

Consider a class of switched linear discrete-time systems given by

$$x(k+1) = A_{\sigma(k)}x(k) + E_{\sigma(k)}w(k), \tag{3.1}$$

$$z(k) = C_{\sigma(k)}x(k) + F_{\sigma(k)}w(k) \tag{3.2}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $z(k)$ is the objective signal to be attenuated, $w(k) \in \mathbb{R}^l$ is the disturbance input which belongs to $l_2[0, \infty)$, $\sigma(k)$ is the switching signal, which is a piecewise constant function of time and takes its values in the finite set $\mathcal{I} = \{1, \ldots, N\}$, $N > 1$ is the number of subsystems. At an arbitrary discrete time k, the switching signal $\sigma(k)$, is dependent on k or $x(k)$, or both, or other switching rules. We assume that the sequence of subsystems in the switching signal σ is unknown a priori, but its instantaneous value is available in real time. Meanwhile, for the switching times sequence $k_0 < k_1 < k_2 < \cdots$ of switching signal $\sigma(k)$, the

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interval $[k_l, k_{l+1}]$ is called the running time of the currently engaged subsystem, where $l \in \mathbb{N}$. In addition, when $\sigma(k) = i \in \mathcal{I}$, the matrices (A_i, C_i, E_i, F_i) denote the ith subsystem. At an arbitrary discrete time k, the switching signal $\sigma(k)$, denoted by i for simplicity, is dependent on k or $x(k)$, or both, or other switching rules.

Our objective in this section is to find out the conditions to guarantee

(1) system (3.1) is asymptotically stable;

(2) a prescribed noise attenuation level γ is guaranteed in H_∞ sense, i.e. under zero-initial condition, we have that $\|z\|_2 < \gamma \|w\|_2$ for all nonzero $w \in l_2[0, \infty)$.

By Lemma 2.4, the following sufficient conditions are derived such that system (3.1)–(3.2) satisfies H_∞ performance.

Theorem 3.1 *A switched linear system (3.1)–(3.2) will achieve an H_∞ performance index $\gamma > 0$, $\forall i \in \mathcal{I}$, if there exist matrices $\mathcal{P}_i > 0$, $\forall i \in \mathcal{I}$, satisfying:* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{bmatrix} -\mathcal{P}_j & 0 & \mathcal{P}_j A_i & \mathcal{P}_j E_i \\ \star & -I & C_i & F_i \\ \star & \star & -\mathcal{P}_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \tag{3.3}$$

Proof Construct a Lyapunov function as

$$\mathcal{V}(k, x(k)) = x^T(k)\mathcal{P}_i x(k). \tag{3.4}$$

Hence, along the trajectory of system (3.1)–(3.2), we have

$$\begin{aligned} \Delta\mathcal{V} &= \mathcal{V}(k+1, x(k+1)) - \mathcal{V}(k, x(k)) \\ &= x^T(k)\left[A_i^T \mathcal{P}_j A_i - \mathcal{P}_i\right] x(k) + 2x^T(k)\left[A_i^T \mathcal{P}_j E_i\right] w(k) \\ &\quad + w^T(k)\left[E_i^T \mathcal{P}_j E_i\right] w(k). \end{aligned} \tag{3.5}$$

In (3.5), the case when $i = j$ shows that the switched system is described by the ith mode, while the case when $i \neq j$ represents the switched system is at the switching times from mode i to mode j. For more details, we refer readers to [1].

When assuming the zero disturbance input to system (3.1)–(3.2), we have

$$\begin{aligned} \Delta\mathcal{V} &= \mathcal{V}(k+1, x(k+1)) - \mathcal{V}(k, x(k)) \\ &= x^T(k)\left[A_i^T \mathcal{P}_j A_i - \mathcal{P}_i\right] x(k), \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \end{aligned} \tag{3.6}$$

Thus if

$$A_i^T \mathcal{P}_j A_i - \mathcal{P}_i < 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \tag{3.7}$$

then $\Delta\mathcal{V} < 0$ and the asymptotic stability of system (3.1)–(3.2) is guaranteed. By Lemma 2.4, the condition (3.7) is equivalent to: $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{bmatrix} -\mathcal{P}_j & \mathcal{P}_j A_i \\ \star & -\mathcal{P}_i \end{bmatrix} < 0. \tag{3.8}$$

On the other hand, if the inequality (3.3) holds, that is,

$$\begin{bmatrix} -\mathcal{P}_j & 0 & \mathcal{P}_j A_i & \mathcal{P}_j E_i \\ \star & -I & C_i & F_i \\ \star & \star & -\mathcal{P}_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \tag{3.9}$$

Then from basic matrices manipulations, we have the following inequality

$$\begin{bmatrix} -\mathcal{P}_j & \mathcal{P}_j A_i \\ \star & -\mathcal{P}_i \end{bmatrix} < 0$$

which is the formula (3.8), thus the asymptotic stability of the system (3.1)–(3.2) is ensured.

Now, to establish the H_∞ performance for system (3.1)–(3.2), assume zero-initial condition, and consider the following performance index

$$J \triangleq \sum_{k=0}^{\infty} \left[z^T(k) z(k) - \gamma^2 w^T(k) w(k) \right]$$

under zero initial condition, $\mathcal{V}(k, x(k)) \mid_{k=0} = 0$, and we have

$$\begin{aligned} J &= \sum_{k=0}^{\infty} \left[z^T(k) z(k) - \gamma^2 w^T(k) w(k) + \Delta \mathcal{V} \right] - \mathcal{V}(\infty, x(\infty)) \\ &< \sum_{k=0}^{\infty} \left[z^T(k) z(k) - \gamma^2 w^T(k) w(k) + \Delta \mathcal{V} \right] \\ &= \sum_{k=0}^{\infty} \theta^T(k) \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \star & \Lambda_{22} \end{bmatrix} \theta(k) \end{aligned}$$

where $\theta(k) \triangleq \left[x^T(k) \;\; w^T(k) \right]^T$, and

$$\begin{aligned} &\Lambda_{11} \triangleq A_i^T \mathcal{P}_j A_i - \mathcal{P}_i + C_i^T C_i, \; \Lambda_{12} \triangleq A_i^T \mathcal{P}_j E_i + C_i^T F_i \\ &\Lambda_{22} \triangleq -\gamma^2 I + E_i^T \mathcal{P}_j E_i + F_i^T F_i. \end{aligned}$$

By applying Lemma 2.4 twice, it can be shown that inequality (3.3) is equivalent to

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \star & \Lambda_{22} \end{bmatrix} < 0$$

which guarantees $J < 0$, i.e. $\|z\|_2 < \gamma \|w\|_2$, and the proof is completed. □

The analysis of H_∞ performance of uncertain switched linear systems under arbitrary switching signals will be conducted in what follows. The uncertain description of switched linear discrete-time systems (3.1)–(3.2) is given by

$$x(k+1) = A_{\sigma(k)}(\lambda)x(k) + E_{\sigma(k)}(\lambda)w(k), \tag{3.10}$$

$$z(k) = C_{\sigma(k)}(\lambda)x(k) + F_{\sigma(k)}(\lambda)w(k) \tag{3.11}$$

when $\sigma(k) = i \in \mathcal{I}$, the matrices $(A_i(\lambda),\ C_i(\lambda),\ E_i(\lambda),\ F_i(\lambda))$ denote the ith subsystem and λ is a varying uncertain parameter.

The matrices of each subsystem have appropriate dimensions with partially unknown parameters. It is assumed that $(A_i(\lambda),\ C_i(\lambda),\ E_i(\lambda),\ F_i(\lambda)) \in \Re_i$, where $\Re_i$ is a given convex bounded polyhedral domain described by s vertices in the ith subsystem.

$$\begin{aligned}\Re_i \triangleq \{&[(A_i(\lambda),\ C_i(\lambda),\ E_i(\lambda),\ F_i(\lambda))] \\ &= \sum_{m=1}^{s} \lambda_m [A_{i,m},\ C_{i,m},\ E_{i,m},\ F_{i,m}];\ \sum_{m=1}^{s} \lambda_m = 1,\ \lambda_m \geq 0,\ i \in \mathcal{I}.\Big\}\end{aligned}$$

Without loss of generality, the number of vertices in each subsystem is assumed to be equal here. Also, in this section, it is assumed that all the system state is measurable and all the system mode is observable for the later use.

Remark 3.2 As shown in [2], the uncertainty with polytopic type can describe the parametric uncertainty more precisely, thus less conservative than the norm-bounded uncertainty. It is a generalization of the so-called matching condition.

Based on Lemma 2.4, sufficient conditions are derived as follows such that system (3.10)–(3.11) achieves H_∞ performance.

Theorem 3.3 *Uncertain switched system (3.10)–(3.11) has an H_∞ performance index $\gamma > 0$ over $\Re_i$, $\forall i \in \mathcal{I}$, if there exist matrices $\mathcal{P}_{i,m} > 0$, $\forall i \in \mathcal{I}$, $1 \leq m \leq s$ satisfying:* $\forall (i,\ j) \in \mathcal{I} \times \mathcal{I}$

$$\Phi_m^{i,j} \triangleq \begin{bmatrix} -\mathcal{P}_{j,m} & 0 & \mathcal{P}_{j,m}A_{i,m} & \mathcal{P}_{j,m}E_{i,m} \\ \star & -I & C_{i,m} & F_{i,m} \\ \star & \star & -\mathcal{P}_{i,m} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \tag{3.12}$$

Proof Assume matrix function $\mathcal{P}_i(\lambda)$ to be the following form

$$\mathcal{P}_i(\lambda) = \sum_{m=1}^{s} \lambda_m \mathcal{P}_{i,m}, \quad \forall i \in \mathcal{I} \tag{3.13}$$

where $\mathcal{P}_{i,m} > 0$ satisfies (3.12). Construct a Lyapunov function as

$$\mathcal{V}(k, x(k)) \triangleq x^T(k)\mathcal{P}_i(\lambda)x(k) \tag{3.14}$$

Then, if the inequality (3.12) holds, according to (3.12), we have $\Phi^{i,j}(\lambda) \triangleq \sum_{m=1}^{s} \lambda_m \Phi_m^{i,j} < 0$, i.e.

$$\Phi^{i,j}(\lambda) = \begin{bmatrix} -\mathcal{P}_j(\lambda) & 0 & \mathcal{P}_j(\lambda)A_i(\lambda) & \mathcal{P}_j(\lambda)E_i(\lambda) \\ \star & -I & C_i(\lambda) & F_i(\lambda) \\ \star & \star & -\mathcal{P}_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{3.15}$$

Similar to the proof for Theorem 3.1, it is straightforward to prove that (3.15) ensures that the system is stable and has an H_∞ noise attenuation performance. We omit the proof for conciseness. □

3.2 Weighted l_2-Gain Analysis: ADT Switching

In this section, we are interested in investigating a class of discrete-time switched linear systems with average dwell time (ADT) switching, where the state-space form of the presented discrete-time switched linear systems is given as (3.1)–(3.2). In addition, the definition of ADT switching has been stated in Sect. 1.4 of Chap. 1, and therefore is omitted here.

Note that the issue of l_2-gain analysis has already been intensively addressed for switched systems with diverse switching, see for example, [3–9]. In the frame of ADT switching, most results of adopting mode-dependent Lyapunov function (towards less conservatism) only admit a weighted noise attenuation level—a weaker attenuation property, cf. [3, 4]. To present the main objective of this section more clearly, we introduce the following exponential H_∞ performance definition for the switched linear system (3.1)–(3.2), which will be essential for our later development.

Definition 3.4 Given scalars $\gamma > 0$ and $0 < \alpha < 1$, system (3.1)–(3.2) is said to be robustly exponentially stable with an exponential H_∞ performance γ if it is robustly exponentially stable and under zero initial condition, $\sum_{s=0}^{\infty}(1-\alpha)^s z^T(s)z(s) \le \sum_{s=0}^{\infty}\gamma^2 w^T(s)w(s)$ for all nonzero $w(s) \in l_2[0, \infty)$.

Remark 3.5 For the switched systems under ADT switching, the Lyapunov function values at switching instants are often considered to increase μ times ($\mu > 1$) to reduce the conservatism in system analysis and synthesis, which will imply that the normal noise attenuation performance is hard to compute or check even in linear setting. Here we adopt the exponential H_∞ performance criterion here (see [4, 10] for more details) to evaluate the underlying system while obtaining the expected exponential stability. We will show the techniques how to obtain a non-weighted H_∞ performance in the later section for switched systems with PDT switching. Note that the scalar α in the sequel symbolizes the decreasing rate of the Lyapunov-like function within each subsystem. Then, if $\alpha \to 0$, the evaluated performance index will approach the normal H_∞ performance for the whole time domain.

The objective of this section is to find the admissible ADT switching such that the system (3.1)–(3.2) is exponentially stable and achieves exponential H_∞ performance to some degree. Inspired by the stability results for the general continuous-time switched systems in [11], we first give the exponential stability analysis for the discrete-time system without switching in the following Lemma, which will be used to derive our main results in the sequel.

The H_∞ performance analysis for the underlying systems in this section is based on the following unforced non-switched system

$$x(k+1) = Ax(k) + Ew(k) \tag{3.16}$$

$$z(k) = Cx(k) + Fw(k) \tag{3.17}$$

Constructing a Lyapunov function $V(x_k) = x_k^T P x_k$ for this system, we have the following Lemma.

Lemma 3.6 *For given $\alpha > 0$ and $\gamma > 0$, if there exists a matrix function such that*

$$\begin{bmatrix} -P & 0 & PA & PE \\ \star & -I & C & F \\ \star & \star & -(1-\alpha)P & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{3.18}$$

then, along with the trajectory of system (3.16)–(3.17), we have

$$V(x(k)) \leq (1-\alpha)^{k-k_0} V(x(k_0)) - \sum_{s=k_0}^{k-1} (1-\alpha)^{k-s-1} \Gamma(s) \tag{3.19}$$

where,

$$\Gamma(s) \triangleq z^T(s)z(s) - \gamma^2 w^T(s)w(s)$$

Proof Setting $\Delta V(x_k) \triangleq V(x(k+1)) - V(x(k))$, we have

$$\begin{aligned} &\Delta V(x(k)) + \alpha V(x(k)) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\ &\quad = x^T(k)\left[A^T PA + \alpha P - P + C^T C\right]x(k) + 2x^T(k)\left[A^T PE + C^T F\right]w(k) \\ &\quad\quad + w^T(k)E^T PEw(k) + w^T(k)\left[F^T F - \gamma^2 I\right]w(k) \\ &\quad = \theta^T(k)\Phi\theta(k) \end{aligned}$$

where

$$\theta(k) \triangleq [\, x^T(k)\ w^T(k)\,]^T$$
$$\Phi \triangleq \begin{bmatrix} A^T PA + \alpha P - P + C^T C & A^T PE + C^T F \\ \star & E^T PE + F^T F - \gamma^2 I \end{bmatrix}$$

If (3.18) holds, by Lemma 2.4, we can readily know $\Phi < 0$, then

$$\Delta V(x(k)) + \alpha V(x(k)) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) < 0 \tag{3.20}$$

Therefore, from (3.20), one can get that

$$V(x(k_0+1)) < (1-\alpha)V(x(k_0)) - \left(z^T(k_0)z(k_0) - \gamma^2 w^T(k_0)w(k_0)\right) \tag{3.21}$$

Iterating (3.21) gives (3.19), which completes the proof. □

Then, the exponential H_∞ performance analysis for system (3.1)–(3.2) with ADT switching is presented based on Lemma 3.6 as follows.

Theorem 3.7 *Consider system (3.1)–(3.2) and let $\alpha > 0$, $\gamma > 0$ and $\mu > 1$ be given constants. If there exist matrix functions $P_i > 0$, $\forall i \in \mathcal{I}$ such that*

$$\begin{bmatrix} -P_i & 0 & P_i A_i & P_i E_i \\ \star & -I & C_i & F_i \\ \star & \star & -(1-\alpha)P_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{3.22}$$
$$P_i - \mu P_j \le 0 \tag{3.23}$$

then the system (3.1)–(3.2) is exponentially stable and has a prescribed exponential H_∞ performance index γ for switching signals with ADT satisfying (2.26).

Proof Construct a Lyapunov function as below.

$$V(x_k) \triangleq x^T(k)P_i x(k), \quad i \in \mathcal{I} \tag{3.24}$$

It is easy to obtain

$$a\,\|x(k)\|^2 \le V(x(k)) \le b\,\|x(k)\|^2 \tag{3.25}$$

where positive scalars a and b can be given by

$$a \triangleq \inf_i \left(\inf_m (\theta_{\min}(P_{i,m}))\right), b \triangleq \sup_i \left(\sup_m(\theta_{\max}(P_{i,m}))\right) \tag{3.26}$$

Considering firstly the zero disturbance input to the system, along with the trajectory of system (3.1)–(3.2), one has $\forall(i, j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,

$$\begin{aligned} \Delta V_{\sigma(k)}(x(k)) + \alpha V_{\sigma(k)}(x(k)) &= x^T(k)\left[A_i^T P_i A_i + \alpha P_i - P_i\right]x(k), \\ V_{\sigma(k_l)}(x(k_l)) - \mu V_{\sigma(k_l-1)}(x(k_l)) &= x^T(k_l)\left[P_i - \mu P_j\right]x(k_l) \end{aligned}$$

$\forall k \in [k_l, k_{l+1})$. Thus if

$$A_i^T P_i A_i + \alpha P_i - P_i < 0 \tag{3.27}$$

$$P_i - \mu P_j \leq 0 \tag{3.28}$$

system (3.1)–(3.2) is globally asymptotically stable according to Theorem 2.14. By Lemma 2.4, the condition (3.27) is equivalent to

$$\Psi_i \triangleq \begin{bmatrix} -P_i & P_i A_i \\ \star & -(1-\alpha)P_i \end{bmatrix} < 0. \tag{3.29}$$

Then, (3.22) gives (3.29), Therefore, the asymptotic stability of system (3.1)–(3.2) is ensured. Meanwhile, from the derivation of Theorem 2.14 and (3.25), we have

$$\begin{aligned} ||x(k)||^2 &\leq \frac{1}{a}V_{\sigma(k)}(x(k)) \leq \frac{\mu^{N_0}}{a}\left((1-\alpha)\mu^{1/\tau_a}\right)^{(k-k_0)} V_{\sigma(k_0)}(x(k_0)) \\ &\leq \frac{\mu^{N_0} b}{a}\left((1-\alpha)\mu^{1/\tau_a}\right)^{(k-k_0)} ||x(k_0)||^2 \end{aligned} \tag{3.30}$$

which means system (3.1)–(3.2) is robustly exponentially stable for any switching signals with ADT satisfying (2.26).

Now, to establish the exponential H_∞ performance for the system, consider the following performance index

$$J \triangleq \sum_{s=0}^{\infty}(1-\alpha)^s z^T(s)z(s) - \gamma^2 w^T(s)w(s) \tag{3.31}$$

From Lemma 3.6 and (3.23), one can get that

$$\begin{aligned} V_{\sigma(k)}(x(k)) &\leq (1-\alpha)^{k-k_l}V_{\sigma(k)}(x(k_l)) - \sum_{s=k_l}^{k-1}(1-\alpha)^{k-s-1}\Gamma(s) \\ &\leq (1-\alpha)^{k-k_l}\mu V_{\sigma(k_{l-1})}(x(k_l)) - \sum_{s=k_l}^{k-1}(1-\alpha)^{k-s-1}\Gamma(s) \\ &\leq (1-\alpha)^{k-k_l}\mu\Bigg[(1-\alpha)^{k_l-k_{l-1}}V_{\sigma(k_{l-1})}(x(k_{l-1})) - \sum_{s=k_{l-1}}^{k_l-1} \end{aligned}$$

$$
\begin{aligned}
&\times(1-\alpha)^{k_l-s-1}\Gamma(s)\Big]-\sum_{s=k_l}^{k-1}(1-\alpha)^{k-s-1}\Gamma(s)\\
&\leq(1-\alpha)^{k-k_{l-1}}\mu^2\Big[(1-\alpha)^{k_{l-1}-k_{l-2}}V_{\sigma(k_{l-2})}(x(k_{l-2}))\\
&-\sum_{s=k_{l-2}}^{k_{l-1}-1}(1-\alpha)^{k_{l-1}-s-1}\Gamma(s)\Bigg]-(1-\alpha)^{k-k_l}\mu\\
&\times\sum_{s=k_{l-1}}^{k_l-1}(1-\alpha)^{k_l-s-1}\Gamma(s)-\sum_{s=k_l}^{k-1}(1-\alpha)^{k-s-1}\Gamma(s)\\
&\leq(1-\alpha)^{k-k_0}\mu^{N(k_0,k)}V_{\sigma(k_0)}(x(k_0))-(1-\alpha)^{k-k_1}\mu^{N(k_0,k)}\\
&\times\sum_{s=k_0}^{k_1-1}(1-\alpha)^{k_1-s-1}\Gamma(s)-(1-\alpha)^{k-k_2}\mu^{N(k_1,k)}\\
&\times\sum_{s=k_1}^{k_2-1}(1-\alpha)^{k_2-s-1}\Gamma(s)\ldots-\sum_{s=k_l}^{k-1}(1-\alpha)^{k-s-1}\Gamma(s)\\
&=(1-\alpha)^{k-k_0}\mu^{N(k_0,k)}V_{\sigma(k_0)}(x(k_0))-\sum_{s=k_0}^{k-1}\mu^{N(s,k)}\\
&\times(1-\alpha)^{k-s-1}\Gamma(s)
\end{aligned}
$$

then, under zero initial condition, the above formula gives

$$
\sum_{s=k_0}^{k-1}\mu^{N(s,k)}(1-\alpha)^{k-s-1}\Gamma(s)\leq 0
$$

Multiplying both sides of the above inequality by $\mu^{-N_\sigma(k_0,k)}$, one can get that

$$
\begin{aligned}
&\mu^{-N_\sigma(k_0,k)}\sum_{s=k_0}^{k-1}\mu^{N_\sigma(s,k)}(1-\alpha)^{k-s-1}z^T(s)z(s)\\
&\quad\leq\mu^{-N_\sigma(k_0,k)}\sum_{s=k_0}^{k-1}\mu^{N_\sigma(s,k)}(1-\alpha)^{k-s-1}\gamma^2w^T(s)w(s)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\sum_{s=k_0}^{k-1}\mu^{-N_\sigma(k_0,s)}(1-\alpha)^{k-s-1}z^T(s)z(s)\\
&\quad\leq\sum_{s=k_0}^{k-1}\mu^{-N_\sigma(k_0,s)}(1-\alpha)^{k-s-1}\gamma^2w^T(s)w(s)
\end{aligned}
$$

Then, due to the fact

$$N_\sigma(k_0, s) \le N_0 + \frac{s - k_0}{\tau_a} \le N_0 + \frac{-(s - k_0)\ln(1-\alpha)}{\ln\mu} \tag{3.32}$$

we can know that

$$\begin{aligned}&\sum_{s=k_0}^{k-1} \mu^{\frac{(s-k_0)\ln(1-\alpha)}{\ln\mu} - N_0}(1-\alpha)^{k-s-1} z^T(s)z(s)\\ &\quad\le \sum_{s=k_0}^{k-1} \mu^{-N_\sigma(k_0,s)}(1-\alpha)^{k-s-1} z^T(s)z(s)\\ &\quad\le \sum_{s=k_0}^{k-1} \mu^{-N_\sigma(k_0,s)}(1-\alpha)^{k-s-1}\gamma^2 w^T(s)w(s)\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{s=k_0}^{k-1}(1-\alpha)^{s-k_0}(1-\alpha)^{k-s-1} z^T(s)z(s)\\ &\quad\le \sum_{s=k_0}^{k-1} \mu^{N_0-N_\sigma(k_0,s)}(1-\alpha)^{k-s-1}\gamma^2 w^T(s)w(s)\end{aligned}$$

By Definition 1.2 and $\mu > 1$, we know that $\mu^{N_0 - N_\sigma(k_0,s)} \le 1$, which yields

$$\begin{aligned}&\sum_{s=k_0}^{k-1} \mu^{N_0-N_\sigma(k_0,s)}(1-\alpha)^{k-s-1}\gamma^2 w^T(s)w(s)\\ &\quad\le \sum_{s=k_0}^{k-1}(1-\alpha)^{k-s-1}\gamma^2 w^T(s)w(s)\end{aligned}$$

thus we can obtain

$$\sum_{s=0}^{\infty}(1-\alpha)^s z^T(s)z(s) \le \sum_{s=0}^{\infty}\gamma^2 w^T(s)w(s) \tag{3.33}$$

i.e. the considered system has an exponential H_∞ performance index, which completes the proof. □

Based on Theorem 3.7, the exponential H_∞ performance of uncertain switched system (3.10)–(3.11) with ADT switching is analyzed, and the following theorem is a sufficient condition such that system (3.10)–(3.11) is robustly exponentially stable and has a prescribed exponential H_∞ performance index.

Theorem 3.8 *Consider the uncertain switched linear system (3.10)–(3.11) and let $\alpha > 0, \gamma > 0$ and $\mu > 1$ be given constants. If there exist matrix functions $P_i(\lambda) > 0$, $\forall i \in \mathcal{I}$ such that*

$$\begin{bmatrix} -P_i(\lambda) & 0 & P_i(\lambda)A_i(\lambda) & P_i(\lambda)E_i(\lambda) \\ \star & -I & C_i(\lambda) & F_i(\lambda) \\ \star & \star & -(1-\alpha)P_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{3.34}$$

$$P_i(\lambda) - \mu P_j(\lambda) \leq 0 \tag{3.35}$$

then the system (3.10)–(3.11) is robustly exponentially stable and has a prescribed exponential H_∞ performance index γ for all admissible uncertainties satisfying (3.12) and any switching signals with ADT satisfying (2.26).

Proof Now, by adopting the classical parameter-dependent stability idea in coping with uncertainties for general dynamic systems [12], we further construct a class of parameter-dependent MLFs with the form

$$V(x_k) \triangleq x_k^T P_i(\lambda) x_k, \quad i \in \mathcal{I} \tag{3.36}$$

Assume there exist $P_{i,m} > 0$, $\forall i \in \mathcal{I}$ such that $P_i(\lambda) \triangleq \sum_{m=1}^{s} \lambda_m P_{i,m}$ satisfy (3.34) and (3.35). Then, from the quadratic form of Lyapunov function in (3.36), we can know that

$$a \|x_k\|^2 \leq V(x_k) \leq b \|x_k\|^2 \tag{3.37}$$

where positive scalars a and b can be given by

$$a \triangleq \inf_i \left(\inf_m (\theta_{\min}(P_{i,m})) \right), b \triangleq \sup_i \left(\sup_m (\theta_{\max}(P_{i,m})) \right) \tag{3.38}$$

Considering firstly the zero disturbance input to the system, along with the trajectory of system (3.10)–(3.11), one has $\forall (i, j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,

$$\Delta V_{\sigma(k)}(x_k) + \alpha V_{\sigma(k)}(x_k) = x_k^T \left[A_i^T(\lambda) P_i(\lambda) A_i(\lambda) + \alpha P_i(\lambda) - P_i(\lambda) \right] x_k,$$
$$V_{\sigma(k_l)}(x_{k_l}) - \mu V_{\sigma(k_l - 1)}(x_{k_l}) = x_{k_l}^T \left[P_i(\lambda) - \mu P_j(\lambda) \right] x_{k_l}$$

where $\forall k \in [k_l, k_{l+1})$. Thus if

$$A_i^T(\lambda) P_i(\lambda) A_i(\lambda) + \alpha P_i(\lambda) - P_i(\lambda) < 0 \tag{3.39}$$

$$P_i(\lambda) - \mu P_j(\lambda) \leq 0 \tag{3.40}$$

system (3.10)–(3.11) is globally asymptotically stable according to Theorem 2.14. By Lemma 2.4, the condition (3.39) is equivalent to

$$\Psi_i(\lambda) \triangleq \begin{bmatrix} -P_i(\lambda) & P_i(\lambda)\bar{A}_i(\lambda) \\ \star & -(1-\alpha)\,P_i(\lambda) \end{bmatrix} < 0. \tag{3.41}$$

Then, (3.34) gives (3.41), Therefore, the asymptotic stability of system (3.10)–(3.11) is ensured. Meanwhile, from the derivation of Theorem 2.14 and (3.37), (3.30) is obtained which means system (3.10)–(3.11) is robustly exponentially stable for all admissible uncertainties satisfying (3.12) and any switching signals with ADT satisfying (2.26).

Now, to establish the exponential H_∞ performance for the system, consider the same performance index (3.31) as used in the proof of Theorem 3.7, it is straightforward that (3.33) holds, i.e., the considered uncertain switched system has an exponential H_∞ performance index, which completes the proof. □

3.3 Non-Weighted l_2-Gain Analysis: PDT Switching

In this section, we focus our study of system (3.1)–(3.2) on a class of switching signals with persistent dwell-time (PDT) property. The definition of PDT switching has been stated in Sect. 1.4 of Chap. 1, and therefore is omitted here.

In the PDT switching, the interval consisting of the running time (τ-portion) of a certain subsystem and the period of persistence ($\mathbb{T}$-portion) can be regarded as a *stage* of switching. In the τ-portion, one subsystem is activated and the running time is at least τ. In the $\mathbb{T}$-portion, if letting the actual running time at the pth stage be denoted as $\mathbb{T}^{(p)}$, $p \in \mathbb{Z}_{\geq 1}$, it holds that,

$$\mathbb{T}^{(p)} = \sum_{r=1}^{\mathcal{Q}(k_{s_p+1},k_{s_{p+1}})} \mathbb{T}_{\sigma(k_{s_p+r})} \leq \mathbb{T}$$

where $\mathbb{T}_{\sigma(k_{s_p+r})} < \tau$ denotes the running time of the subsystem activated at the switching instant $k_{s_p+r} \in \left[k_{s_p+1}, k_{s_{p+1}}\right), r \in \mathbb{Z}_{[1,\mathcal{Q}(k_{s_p+1},k_{s_{p+1}})]}$; $\mathcal{Q}(k_t, k_v)$ stands for the switching times within $[k_t, k_v)$, and it hereby holds that $\mathcal{Q}(k_t, k_v) = \mathcal{Q}(k_t, k_g) + \mathcal{Q}(k_g, k_v)$ for $0 \leq k_t \leq k_g \leq k_v$. Note that k_{s_p+1} denotes the next switching instant after k_{s_p} at the pth stage and $k_{s_{p+1}}$ represents the instant switching into the $(p+1)$th stage.

In addition, the following definition is required for proceeding further.

Definition 3.9 ([13]) For $\gamma > 0$, system (3.1)–(3.2) is said to be GUAS with an l_2-gain, if under zero initial condition, system (3.1)–(3.2) is GUAS and

$$\sum_{s=k_0}^{\infty} z^T(s)z(s) \le \sum_{s=k_0}^{\infty} \gamma^2 w^T(s)w(s)$$

holds for all nonzero $w(k) \in l_2[0, \infty)$.

As the set of ADT switching signals is a subset of PDT switching signals [14], it will be more difficult to obtain a non-weighted norm of l_2-gain which is of explicit physical sense with less conservatism, if only mode-dependency of the Lyapunov function to be constructed is invoked. In this section, a both mode-dependent and *quasi-time-dependent* (QTD) Lyapunov function will be explored to overcome the aforesaid difficulty.

The problems of stability and l_2-gain analysis of the switched systems with PDT switching will be addressed in this section. A stability criterion is first established by constructing a time-dependent Lyapunov function.

Theorem 3.10 *Consider a class of discrete-time switched system* $x(k+1) = f_{\sigma(k)}(x(k))$*, and* $0 < \alpha < 1$, $\mu > 0$ *are given constants. For a prescribed period of persistence* $\mathbb{T}$*, suppose that there exist a family of functions* $V_{\sigma(k)} : (\mathbb{R}^{n_x}, \mathbb{Z}_+) \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$*, and two class* $\mathcal{K}_\infty$ *functions* κ_1 *and* κ_2 *such that* $\forall \sigma(k) = i \in \mathcal{I}$*,*

$$\kappa_1(\|x(k)\|) \le V_i(x(k), k) \le \kappa_2(\|x(k)\|) \tag{3.42}$$

$$V_i(x(k+1), k+1) \le \alpha V_i(x(k), k) \tag{3.43}$$

for any $(\sigma(k_s) = i, \sigma(k_s - 1) = j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$

$$V_i(x(k_s), k_s) \le \mu V_j(x(k_s), k_s) \tag{3.44}$$

Then the switched system is GUAS for PDT switching signals satisfying

$$(\mathbb{T} + 1)\ln \mu + \mathbb{T} \ln \alpha + \tau \ln \alpha < 0 \tag{3.45}$$

Proof First of all, if $\mu\alpha < 1$, then it is straightforward that a discrete-time switched system is GUAS with $\tau \ge 1$, i.e., under arbitrarily switching. Note that if (3.45) further holds, it holds that τ is at least 1. Thus the proof boils down to the case $\mu\alpha \ge 1$.

Supposing that $\sigma(k_{s_p}) = i$, $\sigma(k_{s_p+1} + T^{(p)}) = j$ holds and considering an arbitrary switching occurs within $\mathbb{T}^{(p)}$, it follows from (3.43)–(3.44) that

$$\begin{aligned}
&V_j(x(k_{s_p+1} + \mathbb{T}^{(p)}), k_{s_p+1} + \mathbb{T}^{(p)}) \\
&\le \mu V_l(x(k_{s_p+1} + \mathbb{T}^{(p)}), k_{s_p+1} + \mathbb{T}^{(p)}) \\
&\le \mu\alpha^{\mathbb{T}_l} V_l(x(k_{s_p+1} + \mathbb{T}^{(p)} - \mathbb{T}_l), k_{s_p+1} + \mathbb{T}^{(p)} - \mathbb{T}_l) \\
&\le \mu^2\alpha^{\mathbb{T}_l} V_m(x(k_{s_p+1} + \mathbb{T}^{(p)} - \mathbb{T}_l), k_{s_p+1} + \mathbb{T}^{(p)} - \mathbb{T}_l) \\
&\le \cdots \\
&\le \mu^{\mathcal{Q}(k_{s_p}, k_{s_p+1} + \mathbb{T}^{(p)})} \alpha^{\mathbb{T}_l + \mathbb{T}_m + \cdots + \mathbb{T}_n} \alpha^\tau V_i(x_{k_{s_p}}, k_{s_p})
\end{aligned} \tag{3.46}$$

where $l, m, \ldots, n$ denote all the possible indices of subsystems being switched within $\mathbb{T}^{(p)}$, with corresponding running time to be $\mathbb{T}_l, \mathbb{T}_m, \ldots, \mathbb{T}_n$, respectively.

Since $\mu\alpha \geq 1$, $\mu^{\mathbb{T}^{(p)}+1}\alpha^{\mathbb{T}^{(p)}} \leq \mu^{\mathbb{T}+1}\alpha^{\mathbb{T}}$ holds. From (3.46), it follows that

$$V_j(x(k_{s_p+1}+\mathbb{T}^{(p)}), k_{s_p+1}+\mathbb{T}^{(p)}) \leq \mu^{\mathbb{T}+1}\alpha^{\mathbb{T}+\tau}V_i(x(k_{s_p}), k_{s_p}).$$

Then, if (3.45) is satisfied, $\mu^{\mathbb{T}+1}\alpha^{\mathbb{T}+\tau} < 1$ holds. Let $\lambda \triangleq \mu^{\mathbb{T}+1}\alpha^{\mathbb{T}+\tau}$, it holds that

$$V_{\sigma(k_{s_p})}(x(k_{s_p}), k_{s_p}) \leq \lambda^{p-1}V_{\sigma(k_{s_1})}(x(k_{s_1}), k_{s_1}) = \lambda^{p-1}V_{\sigma(k_0)}(x(k_0), k_0)$$

(note that $k_{s_1} = k_0$). From (3.42),

$$||x_{k_{s_p}}|| \leq \kappa_1^{-1}(\lambda^{p-1}\kappa_2(||x_{k_0}||))$$

holds. Thus, due to (3.43) and (3.44), $||x(k)|| \leq \kappa_3(||x(k_0)||)$ holds, $\forall k \in \left[k_{s_p}, k_{s_{p+1}}\right)$, where

$$\kappa_3(\cdot) \triangleq \kappa_1^{-1}(\mu^{\mathbb{T}}\alpha^{\mathbb{T}}\kappa_2(\kappa_1^{-1}(\lambda^{p-1}\kappa_2(\cdot)))).$$

Thus the GUAS can be inferred by Definition 2.1. This completes the proof. □

Remark 3.11 It should be noted that the worst case of $\mathbb{T}$ times of switching during the period of persistence is taken into account in the proof of Theorem 3.10 since the actual switching times within the $\mathbb{T}$-portion are unknown.

It can be seen from Theorem 3.10 that the invoked Lyapunov function is not only mode-dependent, but also time-dependent, which will be further less conservative for controllers/filters design in the later chapter. However, the resulting controller/filter via such a Lyapunov function will be also time-dependent, then an infinite number of computations and/or storage of the filter gains will be necessary and accordingly unpractical. To circumvent the problem, we shall limit the Lyapunov function used in Theorem 3.10 to be a QTD one by

$$V_i(x(k), k) = V_i(x(k), q_k), \quad \forall i \in \mathcal{I} \tag{3.47}$$

which has been defined in Sect. 2.5.

Note that by the definition of H_r, the actual running time of the $\sigma(k)$th subsystem in the $\mathbb{T}$-portion, $\mathbb{T}_{\sigma(k)} \in \left[1, \min(\tau-1, \mathbb{T}^{(p)})\right]$, satisfies that

$$\mathbb{T}_{\sigma(k)} = \begin{cases} k_{s_p+r+1} - H_r, & k \in \left[k_{s_p+1}, k_{s_{p+1}-1}\right) \\ k_{s_{p+1}} - H_r, & k \in \left[k_{s_{p+1}-1}, k_{s_{p+1}}\right) \end{cases}.$$

Such a setting will give rise to a simplified version of Theorem 3.10 as below.

Theorem 3.12 *Consider a class of discrete-time switched system* $x(k+1) = f_{\sigma(k)}(x(k))$*, and* $0 < \alpha < 1$*,* $\mu > 0$ *are given constants. For a prescribed period of persistence* $\mathbb{T}$*, suppose that there exist a family of functions* $V_{\sigma(k)} : (\mathbb{R}^{n_x}, \mathbb{Z}_{[0,\tau]}) \to \mathbb{R}$*,* $\sigma(k) \in \mathcal{I}$*, and two class* $\mathcal{K}_\infty$ *functions* κ_1 *and* κ_2 *such that* $\forall \sigma(k) = i \in \mathcal{I}$

(i) $\forall \varphi \in \mathbb{Z}_{[0,\tau]}$,

$$\kappa_1(\|x(k)\|) \le V_i(x(k), \varphi) \le \kappa_2(\|x(k)\|) \tag{3.48}$$

(ii) $\forall k \in \left[k_{s_p}, k_{s_p} + \tau\right)$,

$$V_i(x(k+1), k+1-k_{s_p}) \le \alpha V_i(x(k), k-k_{s_p}) \tag{3.49}$$

(iii) $\forall k \in \left[k_{s_p} + \tau, k_{s_p+1}\right)$,

$$V_i(x(k+1), \tau) \le \alpha V_i(x(k), \tau) \tag{3.50}$$

(iv) $\forall k \in [k_{s_p+1}, k_{s_{p+1}})$, $r \in \mathbb{Z}_{[1, \mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})]}$

$$V_i(x(k+1), k+1-H_r) \le \alpha V_i(x(k), k-H_r) \tag{3.51}$$

(v) $\forall \sigma(k_{s_p+1}) = i \neq j = \sigma(k_{s_p+1} - 1)$,

$$V_i(x(k_{s_p+1}), 0) \le \mu V_j(x(k_{s_p+1}), \tau) \tag{3.52}$$

(vi) $\forall \sigma(k_{s_p+r}) = i \neq j = \sigma(k_{s_p+r} - 1)$,

$$V_i(x(k_{s_p+r}), 0) \le \mu V_j(x(k_{s_p+r}), \mathbb{T}_j) \tag{3.53}$$

where $\mathbb{T}_j \in \left[1, \min(\tau-1, \mathbb{T}^{(p)})\right]$*,* $\mathbb{T}^{(p)} \in [1, \mathbb{T}]$ *and* $r \in \mathbb{Z}_{[2, \mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})+1]}$*. Then the switched system is GUAS for PDT switching signals satisfying (3.45) and (3.48)–(3.53).*

Proof Let the Lyapunov function be given as (3.47). From (3.49), we have

$$\begin{aligned} &V_i(x(k+1), k+1) - \alpha V_i(x(k), k) \\ &\quad = V_i(x(k+1), k+1-k_{s_p}) - \alpha V_i(x(k), k-k_{s_p}) \\ &\quad \le 0 \end{aligned}$$

when $k \in \left[k_{s_p}, k_{s_p} + \tau\right)$, $\forall i \in \mathcal{I}$. In addition, since $V_i(x(k), k) = V_i(x(k), \tau)$, for $k \in \left[k_{s_p} + \tau, k_{s_p+1}\right)$, $\forall i \in \mathcal{I}$, it yields from (3.50) that

$$\begin{aligned} &V_i(x(k+1), k+1) - \alpha V_i(x(k), k) \\ &\quad = V_i(x(k+1), \tau) - \alpha V_i(x(k), \tau) \\ &\quad \le 0. \end{aligned}$$

By (3.51) and the definition of H_r, it holds that, $k \in [k_{s_p+1}, k_{s_{p+1}})$,

$$\begin{aligned}
& V_i(x(k+1), k+1) - \alpha V_i(x(k), k) \\
& = V_i(x(k+1), k+1-H_r) - \alpha V_i(x(k), k-H_r) \\
& \leq 0.
\end{aligned}$$

Then, (3.49)–(3.51) imply that (3.43) in Theorem 3.10 is satisfied.

On the other hand, suppose that the system switches from jth subsystem to ith subsystem at switching instant k_{s_p+r} when $r \in \mathbb{Z}_{[2, \mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})+1]}$. By (3.53), it can be obtained that

$$V_i(x(k_{s_p+r}), k_{s_p+r}) \leq \mu V_j(x(k_{s_p+r}), \mathbb{T}_j) = \mu V_j(x(k_{s_p+r}), k_{s_p+r}).$$

Likewise, it follows from (3.52) that

$$V_i(x(k_{s_p+1}), k_{s_p+1}) \leq \mu V_j(x(k_{s_p+1}), k_{s_p+1}).$$

Thus, (3.52) together with (3.53) guarantees (3.44) in Theorem 3.10, and the GUAS of the switched system with any switching signals satisfying (3.45) and (3.48)–(3.53) can be therefore ensured. □

Further, by the QTD Lyapunov function in (3.47), the criterion on l_2-gain analysis of switched systems can be also obtained as follows.

Theorem 3.13 *Consider a discrete-time switched system*

$$\begin{aligned}
x(k+1) &= f_{\sigma(k)}(x(k), w(k)) \\
z(k) &= g_{\sigma(k)}(x(k), w(k))
\end{aligned}$$

and $0 < \alpha < 1$, $\mu > 1$ *are given constants. For a prescribed period of persistence* $\mathbb{T}$, *suppose that there exist a family of functions* $V_{\sigma(k)} : (\mathbb{R}^{n_x}, \mathbb{Z}_{[0,\tau]}) \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, *two class* $\mathcal{K}_\infty$ *functions* κ_1 *and* κ_2, *and a scalar* γ *such that* $\forall \sigma(k) = i \in \mathcal{I}$, $\varphi = 0, 1, \ldots, \tau$, *(3.48), (3.52)–(3.53) are satisfied,*

(i) $\forall k \in \left[k_{s_p}, k_{s_p} + \tau\right)$,

$$V_i(x(k+1), k+1-k_{s_p}) \leq \alpha V_i(x(k), k-k_{s_p}) - \Gamma(k) \tag{3.54}$$

(ii) $\forall k \in \left[k_{s_p} + \tau, k_{s_p+1}\right)$,

$$V_i(x(k+1), \tau) \leq \alpha V_i(x(k), \tau) - \Gamma(k) \tag{3.55}$$

(iii) $\forall k \in [k_{s_p+1}, k_{s_{p+1}})$, $r \in \mathbb{Z}_{[1, \mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})]}$

$$V_i(x(k+1), k+1-H_r) \leq \alpha V_i(x(k), k-H_r) - \Gamma(k) \tag{3.56}$$

where $\Gamma(k) \triangleq z^T(k)z(k) - \gamma^2 w^T(k)w(k)$. Then the switched system is GUAS and has an l_2-gain no greater than $\gamma_l = \gamma\beta$, where $\beta = \sqrt{\mu^{\mathbb{T}+1}\frac{h(1-\alpha)}{1-h\alpha}}$ with $h = \mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}$ for PDT switching signals satisfying (3.45), (3.48) and (3.52)–(3.56).

Proof First of all, for $w(k) \equiv 0$, if (3.54)–(3.56) hold, then (3.49)–(3.51) hold, thus the stability of the underlying systems will be ensured by Theorem 3.12.

Now, consider $w(k) \neq 0$, if (3.54) holds, we have

$$\begin{aligned}&V_i(x(k+1), k+1) - \alpha V_i(x(k), k) + \Gamma(k)\\&\quad = V_i(x(k+1), k+1-k_{s_p}) - \alpha V_i(x(k), k-k_{s_p}) + \Gamma(k)\\&\quad \leq 0\end{aligned}$$

when $k \in \left[k_{s_p}, k_{s_p}+\tau\right)$. Also, when $k \in \left[k_{s_p}+\tau, k_{s_{p+1}}\right)$, one has

$$\begin{aligned}&V_i(x(k+1), k+1) - \alpha V_i(x(k), k) + \Gamma(k)\\&\quad = V_i(x(k+1), \tau) - \alpha V_i(x(k), \tau) + \Gamma(k)\\&\quad \leq 0.\end{aligned}$$

From (3.56), $\forall k \in [k_{s_p+1}, k_{s_{p+1}})$,

$$\begin{aligned}&V_i(x(k+1), k+1) - \alpha V_i(x(k), k) + \Gamma(k)\\&\quad = V_i(x(k+1), k+1-H_r) - \alpha V_i(x(k), k-H_r) + \Gamma(k)\\&\quad \leq 0.\end{aligned}$$

Moreover, from Theorem 3.12, we can know that, $\forall r \in \mathbb{Z}_{[1, \mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})+1]}$,

$$V_i(x(k_{s_p+r}), k_{s_p+r}) \leq \mu V_j(x(k_{s_p+r}), k_{s_p+r}).$$

Then, for the pth stage of switching, basic algebraic operations yield that

$$\begin{aligned}&V_{\sigma(k_{s_{p+1}})}(x(k_{s_{p+1}}), k_{s_{p+1}})\\&\quad - \mu^{\mathcal{Q}(k_{s_p}, k_{s_{p+1}})}\alpha^{k_{s_{p+1}}-k_{s_p}} V_{\sigma(k_{s_p})}(x(k_{s_p}), k_{s_p})\\&\quad + \sum_{l=k_{s_p}}^{k_{s_{p+1}}-1} \mu^{\mathcal{Q}(l, k_{s_{p+1}})}\alpha^{k_{s_{p+1}}-l-1}\Gamma(l) \leq 0.\end{aligned}$$

Then consider $n \in \mathbb{Z}_{\geq 2}$, it follows that

$$\begin{aligned}&V_{\sigma(k_{s_n})}(x(k_{s_n}), k_{s_n}) - \mu^{\mathcal{Q}(k_{s_1}, k_{s_n})}\alpha^{k_{s_n}-k_{s_1}} V_{\sigma(k_{s_1})}(x(k_{s_1}), k_{s_1})\\&\quad + \sum_{l=k_{s_1}}^{k_{s_n}-1} \mu^{\mathcal{Q}(l, k_{s_n})}\alpha^{k_{s_n}-l-1}\Gamma(l) \leq 0.\end{aligned}$$

Therefore, under zero initial condition, one has $V_{\sigma(k_{s_1})}(x(k_{s_1}), k_{s_1}) = 0$, then

$$\sum_{l=k_{s_p}}^{k_{s_n}-1} \mu^{\mathcal{Q}(l,k_{s_n})}\alpha^{k_{s_n}-l-1}\Gamma(l) \leq 0$$

which means that

$$\sum_{l=k_0}^{k-1} \mu^{\mathcal{Q}(l,k)}\alpha^{k-l-1}z^T(l)z(l) \leq \gamma^2 \sum_{l=k_0}^{k-1} \mu^{\mathcal{Q}(l,k)}\alpha^{k-l-1}w^T(l)w(l).$$

Due to the fact that

$$0 \leq \mathcal{Q}(l,k) \leq \left\lceil \frac{k-l}{\tau+\mathbb{T}} \right\rceil (\mathbb{T}+1) \leq \left(\frac{k-l}{\tau+\mathbb{T}}+1\right)(\mathbb{T}+1) \tag{3.57}$$

it follows that

$$\sum_{l=k_0}^{k-1} \alpha^{k-l-1}z^T(l)z(l) \leq \gamma^2 \sum_{l=k_0}^{k-1} \mu^{\left(\frac{k-l}{\tau+\mathbb{T}}+1\right)(\mathbb{T}+1)}\alpha^{k-l-1}w^T(l)w(l) \tag{3.58}$$

then we have

$$\begin{aligned}
&\sum_{l=k_0}^{k-1} \alpha^{k-l-1}z^T(l)z(l) \\
&\leq \gamma^2 \sum_{l=k_0}^{k-1} \mu^{\left(\frac{k-l-1}{\tau+\mathbb{T}}\right)(\mathbb{T}+1)+(\mathbb{T}+1)+\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\alpha^{k-l-1}w^T(l)w(l) \\
&\Rightarrow \sum_{l=k_0}^{k-1} \alpha^{k-l-1}z^T(l)z(l) \\
&\leq \gamma^2 \mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\mu^{\mathbb{T}+1} \sum_{l=k_0}^{k-1} \left(\mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\alpha\right)^{k-l-1} w^T(l)w(l).
\end{aligned} \tag{3.59}$$

From (3.45), we have $\mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\alpha < 1$. Thus, it holds that

$$\begin{aligned}
&\sum_{k=k_0+1}^{\infty} \sum_{l=k_0}^{k-1} \alpha^{k-l-1}z^T(l)z(l) \\
&\leq \gamma^2 \mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\mu^{\mathbb{T}+1} \sum_{k=k_0+1}^{\infty} \sum_{l=k_0}^{k-1} (\mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\alpha)^{k-l-1}w^T(l)w(l)
\end{aligned}$$

$$\Rightarrow \sum_{l=k_0}^{\infty}\sum_{k=l+1}^{\infty}\alpha^{k-l-1}z^T(l)z(l)$$
$$\leq \gamma^2\mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\mu^{\mathbb{T}+1}\sum_{l=k_0}^{\infty}\sum_{k=l+1}^{\infty}(\mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\alpha)^{k-l-1}w^T(l)w(l).$$

Then, we have

$$\sum_{l=k_0}^{\infty}z^T(l)z(l) \leq \gamma^2\mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\mu^{\mathbb{T}+1}\frac{1-\alpha}{1-\mu^{\frac{\mathbb{T}+1}{\tau+\mathbb{T}}}\alpha}\sum_{l=k_0}^{\infty}w^T(l)w(l).$$

Therefore, the underlying system is GUAS with an l_2-gain no greater than $\gamma_l = \gamma\beta$. □

Remark 3.14 It can be seen from Theorem 3.13 that, contrast to the existing weighted l_2-gain for the ADT switched systems, the achieved l_2-gain is non-weighted in this section for the more general PDT switching, which benefits from the techniques explored in (3.57)–(3.59).

3.4 Tube-Based Robustness Analysis: Modal PDT Switching

In this section, we will consider l_∞ disturbance involved with the systems. The robustness performance analysis for relatively general modal persistent dwell-time (MPDT) switching signals that could cover DT, ADT, and PDT switching are only addressed.

Consider a class of discrete-time switched linear systems with bounded additive disturbances

$$x(k+1) = A_{\sigma(k)}x(k) + w(k) \tag{3.60}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the system state, $w(k) \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$ is the additive disturbance and $\mathbb{W}$ is a compact polyhedral set containing the origin in its interior.

In this section, an MPDT generalized robust positive invariant (GRPI) set of system (3.60) under MPDT switching will be determined to address the system stability in the sense of set theory. To this end, an MPDT robust positive invariant (RPI) set of system (3.60) needs to be firstly determined, and the following definitions and operators are required.

Let one step reachable set from a set $\mathcal{X}$ along subsystem i be denoted as $\mathcal{P}_1^i(\mathcal{X}, \mathbb{W}) \triangleq \{A_i x + w : x \in \mathcal{X},\ w \in \mathbb{W}\} = A_i\mathcal{X} \oplus \mathbb{W}$, then the H-step reachable set $\mathcal{P}_H^i(\mathcal{X}, \mathbb{W})$ is defined as

$$\mathcal{P}_{y+1}^i(\mathcal{X}, \mathbb{W}) \triangleq \mathcal{P}_1^i\left(\mathcal{P}_y^i(\mathcal{X}, \mathbb{W}), \mathbb{W}\right), \quad y \in \mathbb{Z}_{[0,H-1]}$$

where $\mathcal{P}_0^i(\mathcal{X}, \mathbb{W}) \triangleq \mathcal{X}$. Thus $\mathcal{P}_H^i(\mathcal{X}, \mathbb{W}) = A_i^H \mathcal{X} \oplus A_i^{H-1}\mathbb{W} \oplus \cdots \oplus A_i\mathbb{W} \oplus \mathbb{W}$. In addition, define two operators $\bar{\mathcal{P}}(\cdot, \mathbb{W})$ and $\hat{\mathcal{P}}(\cdot, \mathbb{W})$ as

$$\begin{aligned}\bar{\mathcal{P}}(\cdot, \mathbb{W}) &\triangleq \bigcup\nolimits_{i\in\mathcal{I}} \bigcup\nolimits_{t_0\in\Theta_i} \mathcal{P}_{t_0}^i(\cdot, \mathbb{W})\\ \hat{\mathcal{P}}(\cdot, \mathbb{W}) &\triangleq \Big\{\bigcup\nolimits_{t_Q\in\mathbb{Z}_{[0,\mathbb{T}]}} \bigcup\nolimits_{l\in\mathcal{I}} \mathcal{P}_{t_Q}^l \Big(\bigcup\nolimits_{t_{Q-1}\in\mathbb{Z}_{[0,\mathbb{T}]}} \bigcup\nolimits_{m\in\mathcal{I}}\\ &\qquad \mathcal{P}_{t_{Q-1}}^m \Big(\cdots \bigcup\nolimits_{t_1\in\mathbb{Z}_{[0,\mathbb{T}]}} \bigcup\nolimits_{n\in\mathcal{I}} \mathcal{P}_{t_1}^n(\cdot, \mathbb{W}), \cdots,\\ &\qquad \mathbb{W}), \mathbb{W}) : l \neq m, n \neq i, 0 \leq \sum_{q=1}^{Q} t_q \leq \mathbb{T}\Big\}\end{aligned}$$

Definition 3.15 ([15]) A set $\mathcal{O} \subseteq \mathbb{R}^{n_x}$ is said to be a robust positive invariant (RPI) set for system $x(k+1) = f(x(k), w(k))$, $w(k) \in \mathbb{W}$, if $x(k) \in \mathcal{O}$ implies $x(t) \in \mathcal{O}$ for any $w(t) \in \mathbb{W}$, $t \in \mathbb{Z}_{\geq k+1}$.

Definition 3.16 A set $\mathcal{O}(\boldsymbol{\tau}^{[\mathbb{T}]}) \subseteq \mathbb{R}^n$ is said to be an MPDT RPI set for system (3.60) with MPDT set $\boldsymbol{\tau}^{[\mathbb{T}]} \triangleq \{\tau_1, \tau_2, \ldots, \tau_M\}$, if $x(0) \in \mathcal{O}(\boldsymbol{\tau}^{[\mathbb{T}]})$ implies $x(k) \in \mathcal{O}(\boldsymbol{\tau}^{[\mathbb{T}]})$ for every admissible switching $\xi_{\boldsymbol{\tau}^{[\mathbb{T}]}}(k)$ and for $w(t) \in \mathbb{W}$, $t \in \mathbb{Z}_{[0,k-1]}$.

Then, the MPDT GRPI set for the switched system (3.60) with MPDT switching is defined as follows.

Definition 3.17 A set $\mathcal{G}(\boldsymbol{\tau}^{[\mathbb{T}]}) \subseteq \mathbb{R}^{n_x}$ is said to be an MPDT generalized robust positive invariant (GRPI) set for system (3.60) with MPDT set $\boldsymbol{\tau}^{[\mathbb{T}]} \triangleq \{\tau_1, \tau_2, \ldots, \tau_M\}$, if $x(k) \in \mathcal{O}(\boldsymbol{\tau}^{[\mathbb{T}]}) \subseteq \mathcal{G}(\boldsymbol{\tau}^{[\mathbb{T}]})$ implies $x(t) \in \mathcal{G}(\boldsymbol{\tau}^{[\mathbb{T}]})$ for any $w(t) \in \mathbb{W}$, $t \in \mathbb{Z}_{\geq k+1}$, where $\mathcal{O}(\boldsymbol{\tau}^{[\mathbb{T}]})$ is an MPDT RPI set for system (3.60).

Definition 3.18 An MPDT GRPI set $\mathcal{G}(\boldsymbol{\tau}^{[\mathbb{T}]}) \subseteq \mathbb{R}^n$ is said to be GUAS for system (3.60) with MPDT switching, if for all $k \in \mathbb{Z}_+$, $\|x(k)\|_{\mathcal{G}(\boldsymbol{\tau}^{[\mathbb{T}]})} \leq \kappa(\|x(0)\|_{\mathcal{G}(\boldsymbol{\tau}^{[\mathbb{T}]})})$ and $\|x(k)\|_{\mathcal{G}(\boldsymbol{\tau}^{[\mathbb{T}]})} \to 0$ as $k \to \infty$, where $\kappa \in \mathcal{K}_\infty$.

The following theorem demonstrates the existence of an MPDT RPI set for system (3.60).

Theorem 3.19 *Suppose that system (3.60) with MPDT θ_i for a given $\mathbb{T}$ is GUAS, then an MPDT RPI set $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$ exists for system (3.60).*

Proof Let

$$\begin{aligned}R_{t,i,g_i} \triangleq &\left(\prod_{l=1}^{t} A_{[l]}\right) A_i^{g_i-1}\mathbb{W} \oplus \left(\prod_{l=1}^{t} A_{[l]}\right) A_i^{g_i-2}\mathbb{W}\\ &\oplus \cdots \oplus \left(\prod_{l=1}^{t} A_{[l]}\right)\mathbb{W} \oplus \left(\prod_{l=2}^{t} A_{[l]}\right)\mathbb{W} \oplus \cdots\\ &\oplus A_{[t]}\mathbb{W} \oplus \mathbb{W}\end{aligned}$$

where $\prod_{l=1}^{t} A_{[l]}$ stands for $A_{[t]}A_{[t-1]}\cdots A_{[1]}$ and $A_{[l]}$ varying with l, $l \in \mathbb{Z}_{\geq 1}$ denotes a matrix taken in set $\mathcal{A}_1 \triangleq \{A_1, A_2, \ldots, A_M\}$. Then define $\Theta_i \triangleq \{\theta_i, \theta_i + 1, \ldots, 2\theta_i - 1\}$ and consider

$$\mathcal{O}_{v+1} \triangleq co\left\{\left(\prod_{l=1}^{t} A_{[l]}\right) A_i^{g_i} \mathcal{O}_v \oplus R_{t,i,g_i} : t \in \mathbb{Z}_{[0,\mathbb{T}]}, i \in \mathcal{I},\ g_i \in \Theta_i\right\} \tag{3.61}$$

where $\mathcal{O}_0$ is $\Lambda \triangleq co\{(\prod_{l=1}^{t-1} A_{[l]})\mathbb{W} \oplus (\prod_{l=2}^{t-1} A_{[l]})\mathbb{W} \oplus \cdots \oplus A_{[t-1]}\mathbb{W} \oplus \mathbb{W} : t \in \mathbb{Z}_{[1,\mathbb{T}]}\}$ or $\{0\}$, respectively, for the cases that a period of persistence exists or not before the 1st stage. Let $\mathcal{R} \triangleq co\{R_{t,i,g_i} : t \in \mathbb{Z}_{[0,\mathbb{T}]}, i \in \mathcal{I}, g_i \in \Theta_i\}$, $\mathcal{O}_{v+1}$ satisfies

$$\mathcal{O}_{v+1} \subseteq co\left\{\left(\prod_{l=1}^{t} A_{[l]}\right) A_i^{g_i} \mathcal{O}_v \oplus \mathcal{R} : t \in \mathbb{Z}_{[0,T]}, i \in \mathcal{I}, g_i \in \Theta_i\right\} \tag{3.62}$$

Then, iterating (3.62) from v to 0 yields that, $\forall v \in \mathbb{Z}_{\geq 1}$

$$\begin{aligned}\mathcal{O}_v \subseteq \Psi_v \triangleq co\Bigg\{&\left(\prod_{l=1}^{t_v} A_{[l]}\right) A_h^{g_h} \left(\prod_{l=1}^{t_{v-1}} A_{[l]}\right) A_i^{g_i} \cdots \\ &\left(\prod_{l=1}^{t_2} A_{[l]}\right) A_j^{g_j} \left(\prod_{l=1}^{t_1} A_{[l]}\right) A_k^{g_k} \mathcal{O}_0 \\ &\oplus \left(\prod_{l=1}^{t_v} A_{[l]}\right) A_h^{g_h} \cdots \left(\prod_{l=1}^{t_2} A_{[l]}\right) A_j^{g_j} \mathcal{R} \\ &\oplus \cdots \oplus \left(\prod_{l=1}^{t_v} A_{[l]}\right) A_h^{g_h} \mathcal{R} \oplus \mathcal{R} \\ &:\ t_c \in \mathbb{Z}_{[0,\mathbb{T}]}, c \in \mathbb{Z}_{[1,v]}, g_d \in \Theta_d, d \in \mathcal{I}, \\ &\quad (h \times i \times \cdots \times j \times k) \in \mathcal{I} \times \mathcal{I} \times \cdots \times \mathcal{I} \times \mathcal{I}\Bigg\}\end{aligned}$$

Since system (3.60) under MPDT θ_i for a given $\mathbb{T}$ is GUAS, then the system $\hat{z}(k+1) = \hat{A}\hat{z}(k)$ is asymptotically stable under arbitrary switching, where $\hat{A} \in \Phi(\Theta_i, \mathbb{T}) \triangleq \{(\prod_{l=1}^{t} A_{[l]})A_i^r : t \in \mathbb{Z}_{[0,\mathbb{T}]}, i \in \mathcal{I}, r \in \Theta_i\}$. Here the finite set Θ_i is invoked with a similar usage in [16] such that all the admissible switching sequences during $[k_{s_p}, k_{s_{p+1}})$, $p \in \mathbb{Z}_+$ can be represented equivalently by combinations of matrices in $\Phi(\Theta_i, \mathbb{T})$ where t can be zero. Therefore, there exists a constant $\varepsilon \in (0, 1)$ and $\eta > 0$ satisfying $\mathcal{R} \subseteq \eta\mathcal{B}^n$ such that $\hat{A}\mathcal{R} \subseteq \eta\varepsilon\mathcal{B}^n$. Then

$$\mathcal{O}_v \subseteq \Psi_v \subseteq \eta(\varepsilon^n + \varepsilon^{n-1} + \cdots + \varepsilon + 1)\mathcal{B}^n \tag{3.63}$$

where $n = v$ or $n = v - 1$ corresponds to the case $\mathcal{O}_0 = \Lambda$ $(\subseteq \mathcal{R})$ or $\mathcal{O}_0 = \{0\}$, respectively. Hence, from (3.61) and (3.63), it holds that $\mathcal{O}_v \subseteq \mathcal{O}_{v+1}$ and $\mathcal{O}_v$ is bounded above by $\frac{\eta}{1-\varepsilon}\mathcal{B}^n$ as $v \to \infty$, respectively. Thus the set sequence $\{\mathcal{O}_v : v \in \mathbb{Z}_+\}$ has a limit $\mathcal{O}_\infty$ that is dependent on the MPDT θ_i and $\mathbb{T}$. Therefore, for system (3.60), it follows from the computations of $\mathcal{O}_v$ that for any $x(0) \in \mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]}) \triangleq \mathcal{O}_\infty$, $x(k) \in \mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$ for the admissible MPDT switching with $\xi_{\boldsymbol{\theta}^{[\mathbb{T}]}}(k)$ and for $w(t) \in \mathbb{W}$, $t \in \mathbb{Z}_{[0,k-1]}$. This completes the proof. □

Based on Theorem 3.19 and the definitions of both $\bar{\mathcal{P}}(\cdot, \mathbb{W})$ and $\hat{\mathcal{P}}(\cdot, \mathbb{W})$, an algorithm to compute the MPDT RPI set for system (3.60) can be obtained as shown in what follows.

Algorithm 3.1 (*Computation of* $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$) **Input**: $\mathbb{W}$, $\mathbb{T}$, M, $\boldsymbol{\theta}^{[\mathbb{T}]}$, A_i, $i \in \mathcal{I}$.

(i) Set $v = 0$ and $\mathcal{O}_v = co\{\hat{\mathcal{P}}(\{0\}, \mathbb{W})\}$.
(ii) Set $\mathcal{O}_{v+1} = co\{\hat{\mathcal{P}}(\bar{\mathcal{P}}(\mathcal{O}_v, \mathbb{W}), \mathbb{W})\}$.
(iii) If $\mathcal{O}_{v+1} \equiv \mathcal{O}_v$, set $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]}) = \mathcal{O}_v$, **exit** and **output** $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$; else, set $v = v + 1$ and go to step (ii).

Remark 3.20 Without loss of generality, in Algorithm 3.1, $\mathcal{O}_0 = co\{\hat{\mathcal{P}}(\{0\}, \mathbb{W})\}$ $(= \Lambda)$ is taken into account since $\{0\} \subseteq co\{\hat{\mathcal{P}}(\{0\}, \mathbb{W})\}$, though it brings conservatism to the case that a period of persistence does not exist before the 1st stage. Also, note that the existence of an MPDT RPI set ensures the convergence of Algorithm 3.1.

It can be seen from Definition 3.16 that the MPDT RPI set $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$ has the properties that, $\forall s \in \mathbb{Z}_{\geq \theta_i}$,

$$\mathcal{P}_s^i(\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]}), \mathbb{W}) \subseteq \mathcal{O}(\boldsymbol{\theta}^{[T]})$$

and

$$\hat{\mathcal{P}}(\mathcal{P}_s^i(\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]}), \mathbb{W}), \mathbb{W}) \subseteq \mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]}) \tag{3.64}$$

Then if letting

$$\mathcal{G}^i(\boldsymbol{\theta}^{[\mathbb{T}]}) \triangleq co\{\mathcal{P}_{\tau_i-1}^i(\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]}), \mathbb{W}), \mathcal{P}_{\tau_i-2}^i(\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]}), \mathbb{W}), \cdots, \mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})\} \tag{3.65}$$

it concludes that $x(t) \in \mathcal{G}^i(\boldsymbol{\theta}^{[\mathbb{T}]})$, $t \in \mathbb{Z}_{\geq k+1}$ for any $x(k) \in \mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]}) \subseteq \mathcal{G}^i(\boldsymbol{\theta}^{[\mathbb{T}]})$.

Therefore, if the nominal system with MPDT τ_i for a given T is GUAS, the MPDT GRPI set for system (3.60) can be obtained by $\mathcal{G}(\tau^{[T]}) = \bigcup_{i \in \mathcal{I}} \mathcal{G}^i(\tau^{[T]})$, upon which the stability of system (3.60) can be analyzed in the sense of Definition 3.18. In the following theorem, we shall use an extended set $\mathcal{G}(\tau^{[T]}) \times \{0\}$ to establish the stability criterion of the composite switched system augmented by (3.60) and the corresponding nominal system. To present more clearly, we rewrite the nominal system with $w(k) \equiv 0$ by

$$z(k+1) = A_{\sigma(k)} z(k), \tag{3.66}$$

and the error switched system is described as

$$e(k+1) = A_{\sigma(k)} e(k) + w(k), \tag{3.67}$$

where $e(k) \triangleq x(k) - z(k)$, and $e(0) = 0$ is considered.

Theorem 3.21 *Consider system (3.60) and the corresponding nominal system. Suppose that the nominal system with MPDT τ_i for a given $\mathbb{T}$ is GUAS. Then the set $\hat{\mathcal{G}} \triangleq \mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]}) \times \{0\}$ is GUAS for the composite switched system with the admissible MPDT switching satisfying $\Delta_i \triangleq max\{\theta_i, \tau_i\}$.*

Proof If the nominal system with MPDT τ_i for a given $\mathbb{T}$ is GUAS, then it follows from Definition 2.1 that $\|z(k)\| \leq \kappa(\|z(k_0)\|), \forall k \in \mathbb{Z}_{\geq k_0}$ and $\|z(k)\| \to 0$ as $k \to \infty$, where $\kappa \in \mathcal{K}_\infty$. Since for $k \in [k_s, k_{s+1})$, $x(k) = z(k) + e(k)$ where $e(k) \in \mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})$, it holds that $\|x(k)\|_{\mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})} = d(z(k) + e(k), \mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})) \leq d(z(k) + e(k), e(k)) = \|z(k)\| \leq \kappa(\|z(k_0)\|)$ and $\|x(k)\|_{\mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})} \to 0$ as $k \to \infty$.

Denoting $||(x(k), z(k))|| \triangleq ||x(k)|| + ||z(k)||$, it follows that the extended state $(x(k), z(k))$ of the composite system satisfies

$$\begin{aligned}
\|(x(k), z(k))\|_{\hat{\mathcal{G}}} &= \inf{}_{\hat{x} \in \mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})} ||(x(k), z(k)) - (\hat{x}, 0)|| \\
&= \inf{}_{\hat{x} \in \mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})} ||(x(k) - \hat{x}, z(k))|| \\
&= \inf{}_{\hat{x} \in \mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})} ||(x(k) - \hat{x})|| + ||z(k)|| \\
&= ||x(k)||_{\mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})} + ||z(k)|| \leq 2\kappa(\|z(k_0)\|) \\
&\leq 2\kappa(||x(k_0)||_{\mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})} + \|z(k_0)\|) \\
&= 2\kappa(||(x(k_0), z(k_0))||_{\hat{\mathcal{G}}})
\end{aligned}$$

which implies that $\hat{\mathcal{G}}$ is GUAS for the composite switched system in the sense of Definition 3.18. □

Remark 3.22 It can be concluded from Theorem 3.21 and the definition of $\mathcal{G}^i(\boldsymbol{\theta}^{[\mathbb{T}]})$ in (3.65) that the trajectory of the error switched system will always remain inside $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$ at switching instants k_s, $s \in \mathbb{Z}_+$ and $\mathcal{G}^i(\boldsymbol{\theta}^{[\mathbb{T}]})$ within subsystem $\Xi_i, \forall i \in \mathcal{I}$, respectively. Such a fact implies that system (3.60), as well as the error system, possesses a tube whose cross section displays as $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$ or $\mathcal{G}^i(\boldsymbol{\theta}^{[\mathbb{T}]})$ at each sampling instant. The tube can be therefore viewed as an "uniform tube" as it is uniformly valid for the whole set of switching signals satisfying MPDT property.

Remark 3.23 Two noteworthy observations can be further made. First, for a concrete MPDT switching signal, we can conclude that the error system trajectory will be contained in a tighter tube whose cross section belongs to $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$ and $\mathcal{G}^i(\boldsymbol{\theta}^{[\mathbb{T}]})$ at k_s and within (3.67), respectively. The reason is that $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$ is offline determined, which requires all the possible switching within $\mathbb{T}$ and all the possible values in Θ_i to be considered (see Algorithm 3.1 and the definitions of the two operators $\bar{\mathcal{P}}(\cdot, \mathbb{W})$ and $\hat{\mathcal{P}}(\cdot, \mathbb{W})$) to meet the uniformity of the asymptotic stability. Consequently, it will be somewhat conservative to use $\mathcal{G}(\boldsymbol{\theta}^{[\mathbb{T}]})$ to evaluate the system stability as far as a concrete switching signal is concerned. Second, since any activated subsystem will dwell less than the admissible MPDT within $\mathbb{T}$, the tube may expand during $\mathbb{T}$ and therefore tends to be rather tighter at the very switching instant of entering $\mathbb{T}$ to prevent its subsequent evolution during $\mathbb{T}$ getting out of $\mathcal{O}(\boldsymbol{\theta}^{[\mathbb{T}]})$ (see (3.64)).

In the following, a numerical example is presented to demonstrate the validity of the tube-based robustness analysis results for discrete-time switched systems with MPDT switching.

Example 3.24 Consider the discrete-time switched linear system (3.60) consisting of two subsystems described by

$$A_1 = \begin{bmatrix} 0.5598 & -0.6162 \\ 0.9402 & -0.7838 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.2618 & -0.2517 \\ 0.5002 & 0.5103 \end{bmatrix}$$

where $A_i, i = 1, 2$, are the stable matrices of closed-loop system (2.35), which can be obtained by Theorem 2.27 considering the "non-QTD" Lyapunov function. By Theorem 2.27 and solving Problem 2.1, for given $\|w\|_\infty = 0.1$, $\mathbb{T} = 3$, and $\mu_i = 1.22$, $\alpha_1 = 0.085$, $\alpha_2 = 0.105$, the admissible MPDT can be computed as $\theta_1 = 5$, $\theta_2 = 6$. Given $x_0 = [-51.8]^T$, consider one admissible switching sequence (shown in the subfigure in Fig. 3.1) where the running time of subsystems are equivalent to the MPDT and a period of persistence exists before the first MPDT stage, Figs. 3.1 and 3.2 show the cluster of state trajectories of the practical system, and Fig. 3.3 the error system for 30 realizations of the random disturbance sequences. Also, Fig. 3.4 shows the MPDT RPI set $\mathcal{O}(\boldsymbol{\theta}^{[3]})$ and the two components of the MPDT GRPI set $\mathcal{G}(\boldsymbol{\theta}^{[3]})$, $\mathcal{G}^1(\boldsymbol{\theta}^{[3]})$ and $\mathcal{G}^2(\boldsymbol{\theta}^{[3]})$, which can be obtained by Algorithm 3.1 and by (3.65), respectively. The evolution of the uniform tube (displays as $\mathcal{O}(\boldsymbol{\theta}^{[3]})$, $\mathcal{G}^1(\boldsymbol{\theta}^{[3]})$ or $\mathcal{G}^2(\boldsymbol{\theta}^{[3]})$ at each sampling instant, see Remark 3.23 is also illustrated in Figs. 3.1 and 3.2 for the practical system, and Fig. 3.3 for the error system. Finally, for one realization of the random disturbance sequences till $k = 1000$, Fig. 3.4 also shows the projection of a state trajectory of the error system at switching instants and within subsystems into one 2-dimension coordinate.

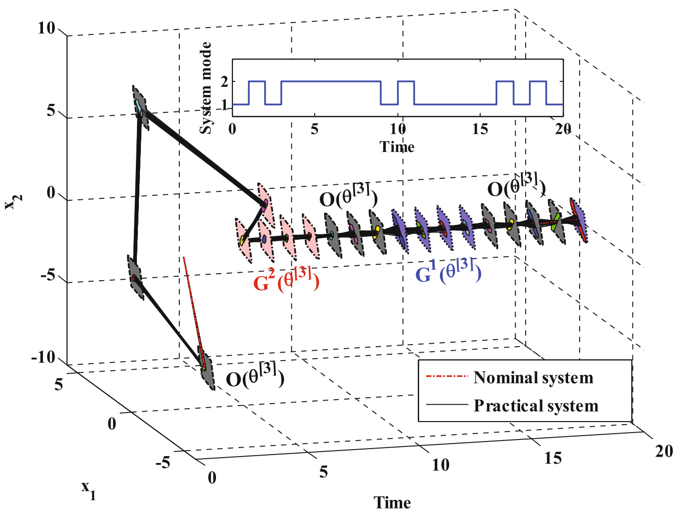

Fig. 3.1 Practical system for 30 realizations of random disturbance sequences in 3-D

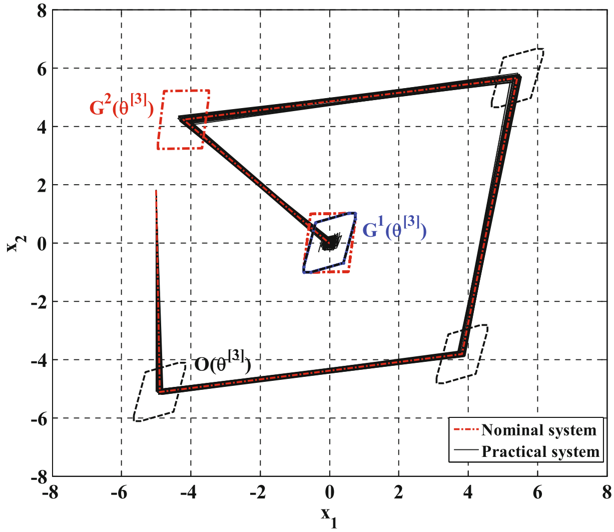

Fig. 3.2 Practical system for 30 realizations of random disturbance sequences in 2-D

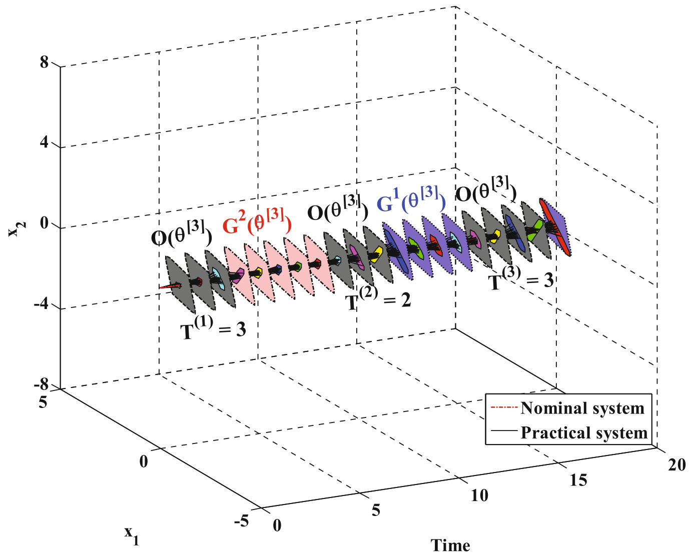

Fig. 3.3 Error system for 30 realizations of random disturbance sequences in 3-D

It can be first seen from Figs. 3.1 and 3.2 that the state trajectory of nominal system converges. Also, Figs. 3.1, 3.2, 3.3 and 3.4 show that the state trajectories either at switching instants or within error subsystems remain inside $\mathcal{G}(\boldsymbol{\theta}^{[3]})$, illustrating that the designed algorithm is effective against the random disturbances. Besides, as also shown in Fig. 3.1 (or Fig. 3.3), all the system trajectories fall into a certain inner set of

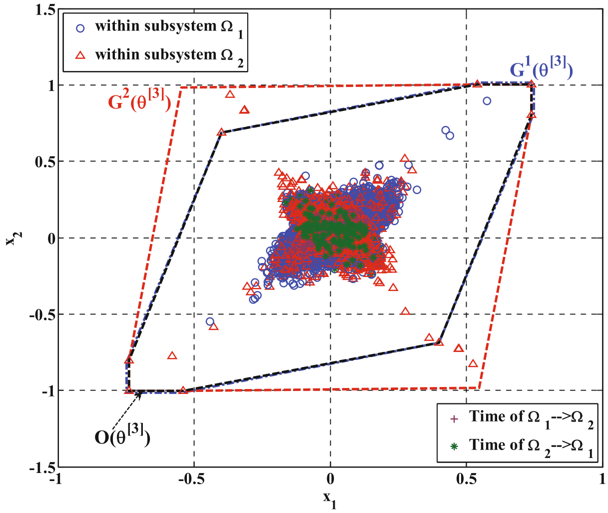

Fig. 3.4 Projection of state trajectory of error system into one coordinate for one realization of random disturbance sequences

$\mathcal{O}(\boldsymbol{\theta}^{[3]})$ at switching instants, and inner set of $\mathcal{G}^i(\boldsymbol{\theta}^{[3]})$ within Ξ_i, respectively. That is, for a concrete realization of the MPDT switching signal, a tighter tube for the error system rather than the uniform tube exists for the practical system, which is consistent to the first observation in Remark 3.23. The second observation in Remark 3.23 is also verified in Figs. 3.1 and 3.3, where the tube expanding during $\mathbb{T}$ is relatively "smaller" at switching instants of entering $\mathbb{T}$ ($k = 9$ and $k = 16$ within the first and second stage, respectively).

3.5 Conclusion

This chapter firstly gives the results on the l_2-gain analysis for switched systems with arbitrary switching and the weighted l_2-gain analysis for switched systems with ADT switching, respectively. Then, a quasi-time-dependent (QTD) Lyapunov function is constructed to address the issues of stability and non-weighted l_2-gain analysis for the PDT switched systems. With the aid of set-theoretic method, an MPDT robust positive invariant (RPI) set is determined for the underlying system allowing for the l_∞ additive disturbance. A concept of generalized robust positive invariant (GRPI) set under MPDT switching is proposed and it is demonstrated that the disturbed system is asymptotically stable in the sense of converging to the MPDT GRPI set.

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Chapter 4
Control

Abstract This chapter is concerned with the control problem for discrete-time switched systems with several typical switching signals. Firstly, the problem of designing H_∞ state-feedback controllers is investigated for switched linear discrete-time systems with arbitrary switching and polytopic uncertainties. Two approaches on designing parameter-independent (robust) and parameter-dependent H_∞ controllers are proposed and the existence conditions of the desired controllers are derived and formulated in terms of a set of linear matrix inequalities (LMIs). Then, considering the average dwell time (ADT) switching, an μ-dependent approach is then introduced for the underlying systems to solve the H_∞ controller, and the obtained conditions are dependent on the admissible increasing level μ of Lyapunov-like function values at switching instants. Finally, in a network-based environment, the quasi-time-dependent (QTD) H_∞ control problem is investigated for a class of discrete-time switched linear systems with modal persistent dwell time (MPDT) switching. One redundant channel is introduced in the data transmission from sensor to controller to reduce the probabilities of packet dropouts occurred in the single channel case. Several examples are used to demonstrate the effectiveness of the developed theoretical results.

4.1 Robust H_∞ Control: Arbitrary Switching

Consider a class of uncertain switched linear discrete-time systems given by

$$x(k+1) = A_{\sigma(k)}(\lambda)x(k) + B_{\sigma(k)}(\lambda)u(k) + E_{\sigma(k)}(\lambda)w(k), \tag{4.1}$$

$$z(k) = C_{\sigma(k)}(\lambda)x(k) + D_{\sigma(k)}(\lambda)u(k) + F_{\sigma(k)}(\lambda)w(k) \tag{4.2}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $z(k)$ is the objective signal to be attenuated, $u(k) \in \mathbb{R}^m$ is the control input vector, $w(k) \in \mathbb{R}^l$ is the disturbance input which belongs to $l_2[0, \infty)$, $\sigma(k)$ is the switching signal, which is a piecewise constant function of time and takes its values in the finite set $\mathcal{I} = \{1, \ldots, N\}$, $N > 1$ is the number of subsystems. We assume that the sequence of subsystems in switching signal σ is unknown a priori, but its instantaneous value is available in real

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time. Meanwhile, for the switching times sequence $k_0 < k_1 < k_2 < \ldots$ of switching signal $\sigma(k)$, the interval $[k_l, k_{l+1}]$ is called the running time of the currently engaged subsystem, where $l \in \mathbb{N}$. When $\sigma(k) = i \in \mathcal{I}$, the matrices $(A_i(\lambda),\ B_i(\lambda),\ C_i(\lambda),\ D_i(\lambda),\ E_i(\lambda),\ F_i(\lambda))$ denote the ith subsystem and λ is a varying uncertain parameter. In addition, at an arbitrary discrete time k, the switching signal $\sigma(k)$ is dependent on k or $x(k)$, or both, or other switching rules.

The matrices of each subsystem have appropriate dimensions with partially unknown parameters. It is assumed that $(A_i(\lambda),\ B_i(\lambda),\ C_i(\lambda),\ D_i(\lambda),\ E_i(\lambda),\ F_i(\lambda)) \in \Re_i$, where $\Re_i$ is a given convex bounded polyhedral domain described by s vertices in the ith subsystem.

$$\begin{aligned}\Re_i \triangleq \Bigg\{ & [(A_i(\lambda),\ B_i(\lambda),\ C_i(\lambda),\ D_i(\lambda),\ E_i(\lambda),\ F_i(\lambda))] \\ & = \sum_{m=1}^{s} \lambda_m [A_{i,m},\ B_{i,m},\ C_{i,m},\ D_{i,m},\ E_{i,m},\ F_{i,m}]; \\ & \sum_{m=1}^{s} \lambda_m = 1,\ \lambda_m \geq 0,\ i \in \mathcal{I}. \Bigg\}\end{aligned} \tag{4.3}$$

Without loss of generality, the number of vertices in each subsystem is assumed to be equal here. Also, in this section, it is assumed that all the system state is measurable and all the system mode is observable for feedback and control design purposes.

Remark 4.1 As shown in [1], the polytopic uncertainty can describe the parametric uncertainty more precisely, thus less conservative than the norm-bounded uncertainty.

Our objective in this section is to design a state-feedback controller such that for all admissible uncertainties in each subsystem

(1) system (4.1)–(4.2) is asymptotically stable;
(2) a prescribed noise attenuation level γ is guaranteed in H_∞ sense, i.e. under zero-initial condition, we have that $\|z\|_2 < \gamma \|w\|_2$ for all nonzero $w \in l_2[0, \infty)$.

In addition, the H_∞ state-feedback controller to be designed has two kinds of form here, one is the robust controller containing constant controller gain despite all variations of parameter λ in each subsystem, and another is the parameter-dependent controller, where the control gain of each subsystem varies with different parameter λ if it is measurable in real time.

4.1.1 Parameter-Independent Control

For general uncertain systems, a commonly used approach in robust control theory is to design a robust controller containing constant controller gain, likewise, for the uncertain switched system (4.1)–(4.3), we consider the controller with the following structure

$$u(k) = \mathcal{K}_i x(k) \tag{4.4}$$

where the controller gain $\mathcal{K}_i$ is constant for fixed ith subsystem.

By applying the above controller, we obtain the corresponding closed-loop system

$$x(k+1) = \bar{A}_i(\lambda)x(k) + E_i(\lambda)w(k) \tag{4.5}$$

$$z(k) = \bar{C}_i(\lambda)x(k) + F_i(\lambda)w(k) \tag{4.6}$$

where $\bar{A}_i(\lambda) \triangleq A_i(\lambda) + B_i(\lambda)\mathcal{K}_i$, $\bar{C}_i(\lambda) \triangleq C_i(\lambda) + D_i(\lambda)\mathcal{K}_i$.

The following theorem presents sufficient conditions for the existence of an admissible robust H_∞ controller with the form (4.4).

Theorem 4.2 *Consider uncertain switched system (4.1)–(4.3). There exists a controller (4.4) that asymptotically stabilizes the resulting closed-loop system (4.5)–(4.6) and achieves an H_∞ performance index $\gamma > 0$ over $\Re_i$, $\forall i \in \mathcal{I}$ if there exist matrices $\mathcal{S}_{i,m} > 0$, matrices $\mathcal{G}_i$ and $\mathcal{U}_i$, $\forall i \in \mathcal{I}$, $1 \le m \le s$ satisfying: $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$*

$$\Phi_m^{i,j} = \begin{bmatrix} -\mathcal{S}_{j,m} & 0 & A_{i,m}\mathcal{G}_i + B_{i,m}\mathcal{U}_i & E_{i,m} \\ \star & -I & C_{i,m}\mathcal{G}_i + D_{i,m}\mathcal{U}_i & F_{i,m} \\ \star & \star & \mathcal{S}_{i,m} - \mathcal{G}_i - \mathcal{G}_i^T & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0, \ 1 \le m \le s. \tag{4.7}$$

If (4.7) has a solution, the controller is given by (4.4) with

$$\mathcal{K}_i = \mathcal{U}_i \mathcal{G}_i^{-1}, \ \forall i \in \mathcal{I}. \tag{4.8}$$

Proof Assume matrix function $\mathcal{S}_i(\lambda)$ to be the following form

$$\mathcal{S}_i(\lambda) = \sum_{m=1}^{s} \lambda_m \mathcal{S}_{i,m}, \ \forall i \in \mathcal{I} \tag{4.9}$$

where $\mathcal{S}_{i,m} > 0$ satisfies (4.7).

Set matrix functions $\mathcal{P}_i(\lambda) \triangleq \mathcal{S}_i^{-1}(\lambda)$ and construct a Lyapunov functional as

$$\begin{aligned} \mathcal{V}(k, x(k)) &\triangleq x^T(k)\mathcal{S}_i^{-1}(\lambda)x(k) \\ &= x^T(k)\mathcal{P}_i(\lambda)x(k). \end{aligned} \tag{4.10}$$

Hence, along the trajectory of system (4.5)–(4.6), we have

$$\begin{aligned} \Delta\mathcal{V} &= \mathcal{V}(k+1, x_{k+1}) - \mathcal{V}(k, x_k) \\ &= x^T(k)\left[\bar{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\bar{A}_i(\lambda) - \mathcal{P}_i(\lambda)\right]x(k) \end{aligned}$$

$$
\begin{aligned}
&+2x^T(k)\left[\bar{A}_i^T(\lambda)\mathcal{P}_j(\lambda)E_i(\lambda)\right]w(k)\\
&+w^T(k)\left[E_i^T(\lambda)\mathcal{P}_j(\lambda)E_i(\lambda)\right]w(k). \qquad (4.11)
\end{aligned}
$$

In formula (4.11), the case when $i = j$ shows that the switched system is described by the ith mode, while the case when $i \neq j$ represents the switched system is at the switching times from mode i to mode j. For more details, we refer readers to [2].

When assuming the zero disturbance input to system (4.5)–(4.6), we have

$$
\begin{aligned}
\Delta\mathcal{V} &= \mathcal{V}(k+1, x_{k+1}) - \mathcal{V}(k, x_k)\\
&= x_k^T\left[\bar{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\bar{A}_i(\lambda) - \mathcal{P}_i(\lambda)\right]x_k, \quad \forall(i,j) \in \mathcal{I}\times\mathcal{I}. \qquad (4.12)
\end{aligned}
$$

Thus if

$$
\bar{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\bar{A}_i(\lambda) - \mathcal{P}_i(\lambda) < 0, \quad \forall(i,j) \in \mathcal{I}\times\mathcal{I}, \qquad (4.13)
$$

then $\Delta\mathcal{V} < 0$ and the asymptotic stability of system (4.5)–(4.6) is guaranteed. By Lemma 2.4, the condition (4.13) is equivalent to: $\forall(i,j) \in (\mathcal{I}\times\mathcal{I})$

$$
\begin{bmatrix} -\mathcal{P}_j(\lambda) & \mathcal{P}_j(\lambda)\bar{A}_i(\lambda) \\ \star & -\mathcal{P}_i(\lambda) \end{bmatrix} < 0. \qquad (4.14)
$$

On the other hand, if the inequality (4.7) holds, according to (4.3), we have $\Phi^{i,j}(\lambda) \triangleq \sum_{m=1}^{s}\lambda_m\Phi_m^{i,j} < 0$, i.e.

$$
\Phi^{i,j}(\lambda) \triangleq \begin{bmatrix} -\mathcal{S}_j(\lambda) & 0 & A_i(\lambda)\mathcal{G}_i + B_i(\lambda)\mathcal{U}_i & E_i(\lambda) \\ \star & -I & C_i(\lambda)\mathcal{G}_i + D_i(\lambda)\mathcal{U}_i & F_i(\lambda) \\ \star & \star & \mathcal{S}_i(\lambda) - \mathcal{G}_i - \mathcal{G}_i^T & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \qquad (4.15)
$$

From (4.15), we can explore the fact that $\mathcal{S}_i(\lambda) - \mathcal{G}_i - \mathcal{G}_i^T < 0$ so that the matrices $\mathcal{G}_i$ are nonsingular. In addition, we have $(\mathcal{S}_i(\lambda) - \mathcal{G}_i)^T\mathcal{S}_i^{-1}(\lambda)(\mathcal{S}_i(\lambda) - \mathcal{G}_i) \geq 0$, which implies $\mathcal{S}_i(\lambda) - \mathcal{G}_i - \mathcal{G}_i^T \geq -\mathcal{G}_i^T\mathcal{S}_i^{-1}(\lambda)\mathcal{G}_i$. Therefore, assuming the controller gain to be of the form (4.8), we conclude

$$
\begin{bmatrix} -\mathcal{S}_j(\lambda) & 0 & \bar{A}_i(\lambda)\mathcal{G}_i & E_i(\lambda) \\ \star & -I & \bar{C}_i(\lambda)\mathcal{G}_i & F_i(\lambda) \\ \star & \star & -\mathcal{G}_i^T\mathcal{S}_i^{-1}(\lambda)\mathcal{G}_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \qquad (4.16)
$$

Performing a congruence transformation to above formula via $diag\{\mathcal{S}_j^{-1}(\lambda), I, \mathcal{G}_i^{-1}, I\}$, and changing the matrix variables $\mathcal{S}_i(\lambda) \triangleq \mathcal{P}_i^{-1}(\lambda)$, we obtain

$$\begin{bmatrix} -\mathcal{P}_j(\lambda) & 0 & \mathcal{P}_j(\lambda)\bar{A}_i(\lambda) & \mathcal{P}_j(\lambda)E_i(\lambda) \\ \star & -I & \bar{C}_i(\lambda) & F_i(\lambda) \\ \star & \star & -\mathcal{P}_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \tag{4.17}$$

If (4.17) holds, then from basic matrices manipulations, we have the following inequality

$$\begin{bmatrix} -\mathcal{P}_j(\lambda) & \mathcal{P}_j(\lambda)\bar{A}_i(\lambda) \\ \star & -\mathcal{P}_i(\lambda) \end{bmatrix} < 0$$

which is the formula (4.14), thus the asymptotic stability of the closed-loop system (4.5)–(4.6) is ensured.

Now, to establish the H_∞ performance for system (4.5)–(4.6), assume zero-initial condition, and consider the following performance index

$$J \triangleq \sum_{k=0}^{\infty} \left[z^T(k)z(k) - \gamma^2 w^T(k)w(k) \right]$$

under zero initial condition, $\mathcal{V}(k, x(k)) \mid_{k=0} = 0$, and we have

$$\begin{aligned} J &= \sum_{k=0}^{\infty} \left[z^T(k)z(k) - \gamma^2 w^T(k)w(k) + \Delta\mathcal{V} \right] - \mathcal{V}(\infty, x(\infty)) \\ &< \sum_{k=0}^{\infty} \left[z^T(k)z(k) - \gamma^2 w^T(k)w(k) + \Delta\mathcal{V} \right] \\ &= \sum_{k=0}^{\infty} \theta^T(k) \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \star & \Lambda_{22} \end{bmatrix} \theta(k) \end{aligned}$$

where $\theta(k) \triangleq \left[x^T(k) \; w^T(k) \right]^T$, and

$$\begin{aligned} \Lambda_{11} &\triangleq \bar{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\bar{A}_i(\lambda) - \mathcal{P}_i(\lambda) + \bar{C}_i^T(\lambda)\bar{C}_i(\lambda), \\ \Lambda_{12} &\triangleq \bar{A}_i^T(\lambda)\mathcal{P}_j(\lambda)E_i(\lambda) + \bar{C}_i^T(\lambda)F_i(\lambda), \\ \Lambda_{22} &\triangleq -\gamma^2 I + E_i^T(\lambda)\mathcal{P}_j(\lambda)E_i(\lambda) + F_i^T(\lambda)F_i(\lambda). \end{aligned}$$

By applying Lemma 2.4 twice, it can be shown that inequality (4.17) is equivalent to $\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \star & \Lambda_{22} \end{bmatrix} < 0$ in above formula, which guarantees $J < 0$, i.e. $\|z\|_2 < \gamma \|w\|_2$, and the proof is completed. □

Remark 4.3 In Theorem 4.2, although a parameter-dependent Lyapunov function is constructed for system (4.5)–(4.6), as is shown in (4.10), the uncertain parameter λ need not to be known in a priori for the robust controller design. However, if we choose the Lyapunov functional based on quadratic framework, we will get more conservative results, for instance, if the matrices $\mathcal{P}_{i,m} \equiv \mathcal{P}_i$ or further $\mathcal{P}_i \equiv \mathcal{P}$ are selected, note that the corresponding controller form for the latter is given by

$$u(k) = \mathcal{K}x(k) \tag{4.18}$$

then we will get different existence conditions for robust controller, which is shown in the following corollaries.

Corollary 4.4 *There exists a controller (4.4) that asymptotically stabilizes the resulting closed-loop system (4.5)–(4.6) and achieves an H_∞ performance index $\gamma > 0$ over $\Re_i$, $\forall i \in \mathcal{I}$ if there exist matrices $\mathcal{S}_i > 0$, matrices $\mathcal{U}_i$, $\forall i \in \mathcal{I}$, $1 \leq m \leq s$ satisfying: $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$*

$$\begin{bmatrix} -\mathcal{S}_j & 0 & A_{i,m}\mathcal{S}_i + B_{i,m}\mathcal{U}_i & E_{i,m} \\ \star & -I & C_{i,m}\mathcal{S}_i + D_{i,m}\mathcal{U}_i & F_{i,m} \\ \star & \star & -\mathcal{S}_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \tag{4.19}$$

If (4.19) has a solution, the controller is given by (4.4) with

$$\mathcal{K}_i = \mathcal{U}_i\mathcal{S}_i^{-1}, \forall i \in \mathcal{I}. \tag{4.20}$$

Corollary 4.5 *There exists a controller (4.18) that asymptotically stabilizes the resulting closed-loop system (4.5)–(4.6) and achieves an H_∞ performance index $\gamma > 0$ over $\Re_i$, $\forall i \in \mathcal{I}$ if there exist matrices $\mathcal{S} > 0$, matrix $\mathcal{U}$ satisfying: $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$*

$$\begin{bmatrix} -\mathcal{S} & 0 & A_{i,m}\mathcal{S} + B_{i,m}\mathcal{U} & E_{i,m} \\ \star & -I & C_{i,m}\mathcal{S} + D_{i,m}\mathcal{U} & F_{i,m} \\ \star & \star & -\mathcal{S} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \tag{4.21}$$

If (4.21) has a solution, the controller is given by (4.18) with

$$\mathcal{K} = \mathcal{U}\mathcal{S}^{-1}, \forall i \in \mathcal{I}. \tag{4.22}$$

The performance index γ described in the above theorem and corollaries can be respectively optimized by the convex optimization procedures given in Table 4.1. Then, the corresponding robust controller gains can be computed by (4.8), (4.20) and (4.22), respectively.

Obviously, the robust controllers given in Theorem 4.2 and Corollaries 4.4 and 4.5 are accessible in practice since the controller gain of each subsystem is constant, however, the results might be conservative if the information on the uncertain parameters can be obtained and utilized in real time. Therefore, in the next subsection, we will hinge on the use of a parameter-dependent idea to design the desired H_∞

Table 4.1 Robust H_∞ performance index optimization procedures

Methods	Convex optimization procedures
Corollary 4.4	$min\ \gamma^2$, s.t. (4.19)
Corollary 4.5	$min\ \gamma^2$, s.t (4.21)
Theorem 4.2	$min\ \gamma^2$, s.t. (4.7)

controller containing a variable gain, namely, the parameter λ under the assumption that the uncertain parameter is available online.

4.1.2 Parameter-Dependent Control

Consider the control input to polytopic uncertain switched systems (4.1)–(4.3) with the following structure

$$u(k) = \mathcal{K}_i(\lambda)x(k). \tag{4.23}$$

By applying the above controller, we obtain the corresponding closed-loop system

$$x(k+1) = \bar{A}_i(\lambda)x(k) + E_i(\lambda)w(k) \tag{4.24}$$
$$z(k) = \bar{C}_i(\lambda)x(k) + F_i(\lambda)w(k) \tag{4.25}$$

where $\bar{A}_i(\lambda) \triangleq A_i(\lambda) + B_i(\lambda)\mathcal{K}_i(\lambda)$, $\bar{C}_i(\lambda) \triangleq C_i(\lambda) + D_i(\lambda)\mathcal{K}_i(\lambda)$.

The following theorem presents sufficient conditions for the existence of an admissible parameter-dependent controller with the form (4.23).

Theorem 4.6 *Consider uncertain switched system (4.1)–(4.3), there exists a controller (4.23) that asymptotically stabilizes the resulting closed-loop system and achieves an H_∞ performance index $\gamma > 0$ over $\Re_i$, $\forall i \in \mathcal{I}$ if there exist matrices $\mathcal{S}_{i,m} > 0$, matrices $\mathcal{U}_{i,m}$ $\forall i \in \mathcal{I}$, $1 \le m \le s$ and matrices*

$$\Omega_{m,n}^{i,j} \triangleq \begin{bmatrix} \mathcal{X}_{m,n}^{i,j} & \mathcal{Y}_{m,n}^{i,j} & \mathcal{Z}_{m,n}^{i,j} & \mathcal{O}_{m,n}^{i,j} \\ \mathcal{R}_{m,n}^{i,j} & \mathcal{H}_{m,n}^{i,j} & \mathcal{T}_{m,n}^{i,j} & \mathcal{J}_{m,n}^{i,j} \\ \mathcal{U}_{m,n}^{i,j} & \mathcal{V}_{m,n}^{i,j} & \mathcal{W}_{m,n}^{i,j} & \mathcal{Q}_{m,n}^{i,j} \\ \mathcal{L}_{m,n}^{i,j} & \mathcal{M}_{m,n}^{i,j} & \mathcal{N}_{m,n}^{i,j} & \mathcal{K}_{m,n}^{i,j} \end{bmatrix},$$

$\forall(i, j) \in \mathcal{I} \times \mathcal{I}$, $1 \le m < n \le s$, satisfying,

$$\begin{bmatrix} (1,1) & -\mathcal{Y}_{m,n}^{i,j} - (\mathcal{R}_{m,n}^{i,j})^T & (1,3) & \tilde{E}_{m,n}^{i} - \mathcal{O}_{m,n}^{i,j} - (\mathcal{L}_{m,n}^{i,j})^T \\ \star & -2I - sym(\mathcal{H}_{m,n}^{i,j}) & (2,3) & \tilde{F}_{m,n}^{i} - \mathcal{J}_{m,n}^{i,j} - (\mathcal{M}_{m,n}^{i,j})^T \\ \star & \star & (3,3) & -\mathcal{Q}_{m,n}^{i,j} - (\mathcal{N}_{m,n}^{i,j})^T \\ \star & \star & \star & -2\gamma^2 I - sym(\mathcal{K}_{m,n}^{i,j})^T \end{bmatrix} \le 0, \tag{4.26}$$

$$\Omega^{i,j} \triangleq \begin{bmatrix} \Omega_1^{i,j} & \Omega_{1,2}^{i,j} & \cdots & \Omega_{1,s}^{i,j} \\ \star & \Omega_2^{i,j} & \cdots & \Omega_{2,s}^{i,j} \\ \star & \star & \ddots & \vdots \\ \star & \star & \star & \Omega_s^{i,j} \end{bmatrix} < 0 \qquad (4.27)$$

where

$$\begin{aligned}
(1,1) &\triangleq -\mathcal{S}_{j,m} - \mathcal{S}_{j,n} - sym(\mathcal{X}_{m,n}^{i,j}),\\
(1,3) &\triangleq \tilde{A}_{m,n}^{i} - \mathcal{Z}_{m,n}^{i,j} - (\mathcal{U}_{m,n}^{i,j})^T,\\
(2,3) &\triangleq \tilde{C}_{m,n}^{i} - \mathcal{T}_{m,n}^{i,j} - (\mathcal{V}_{m,n}^{i,j})^T,\\
(3,3) &\triangleq -\mathcal{S}_{i,m} - \mathcal{S}_{i,n} - sym(\mathcal{W}_{m,n}^{i,j}),\\
\tilde{A}_{m,n}^{i} &\triangleq A_{i,n}\mathcal{S}_{i,m} + A_{i,m}\mathcal{S}_{i,n} + B_{i,n}\mathcal{U}_{i,m} + B_{i,m}\mathcal{U}_{i,n},\\
\tilde{C}_{m,n}^{i} &\triangleq C_{i,n}\mathcal{S}_{i,m} + C_{i,m}\mathcal{S}_{i,n} + D_{i,n}\mathcal{U}_{i,m} + D_{i,m}\mathcal{U}_{i,n},\\
\tilde{E}_{m,n}^{i} &\triangleq E_{i,m} + E_{i,n},\\
\tilde{F}_{m,n}^{i} &\triangleq F_{i,m} + F_{i,n},\\
\Omega_m^{i,j} &\triangleq \begin{bmatrix} -\mathcal{S}_{j,m} & 0 & A_{i,m}\mathcal{S}_{i,m} + B_{i,m}\mathcal{U}_{i,m} & E_{i,m} \\ \star & -I & C_{i,m}\mathcal{S}_{i,m} + D_{i,m}\mathcal{U}_{i,m} & F_{i,m} \\ \star & \star & -\mathcal{S}_{i,m} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix}.
\end{aligned}$$

If (4.26)–(4.27) have solutions, the controller is given by (4.23) with

$$\mathcal{K}_i(\lambda) = \left(\sum_{m=1}^{s} \lambda_m \mathcal{U}_{i,m}\right)\left(\sum_{m=1}^{s} \lambda_m \mathcal{S}_{i,m}\right)^{-1}, \ \forall i \in \mathcal{I},\ 1 \le m \le s. \qquad (4.28)$$

Proof According to (4.17) in Theorem 4.2, system (4.24)–(4.25) is asymptotically stable with an H_∞ noise-attenuation level bound γ if there exist matrix functions $\mathcal{P}_i(\lambda)$ satisfying

$$\Xi^{i,j}(\lambda) \triangleq \begin{bmatrix} -\mathcal{P}_j(\lambda) & 0 & \mathcal{P}_j(\lambda)\bar{A}_i(\lambda) & \mathcal{P}_j(\lambda)E_i(\lambda) \\ \star & -I & \bar{C}_i(\lambda) & F_i(\lambda) \\ \star & \star & -\mathcal{P}_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0.$$

By performing a congruence transformation to above formula via $diag\{\mathcal{P}_j^{-1}(\lambda), I, \mathcal{P}_i^{-1}(\lambda), I\}$, and changing the matrix variables with

$$\mathcal{S}_i(\lambda) \triangleq \mathcal{P}_i^{-1}(\lambda), \mathcal{U}_i(\lambda) \triangleq \mathcal{K}_i(\lambda)\mathcal{P}_i^{-1}(\lambda), \qquad (4.29)$$

we have

$$\varXi^{i,j}(\lambda) \triangleq \begin{bmatrix} -\mathcal{S}_j(\lambda) & 0 & A_i(\lambda)\mathcal{S}_i(\lambda) + B_i(\lambda)\mathcal{U}_i(\lambda) & E_i(\lambda) \\ \star & -I & C_i(\lambda)\mathcal{S}_i(\lambda) + D_i(\lambda)\mathcal{U}_i(\lambda) & F_i(\lambda) \\ \star & \star & -\mathcal{S}_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0.$$

Assume the matrix functions $\mathcal{S}_i(\lambda)$ and $\mathcal{U}_i(\lambda)$ to be the following forms

$$\mathcal{S}_i(\lambda) = \sum_{m=1}^{s} \lambda_m \mathcal{S}_{i,m}, \mathcal{U}_i(\lambda) = \sum_{m=1}^{s} \lambda_m \mathcal{U}_{i,m}, \ \forall i \in \mathcal{I}, \ 1 \le m \le s. \tag{4.30}$$

Then, according to (4.3) and (4.30), we have

$$\begin{aligned} \varXi^{i,j}(\lambda) &\triangleq \sum_{m=1}^{s}\sum_{n=1}^{s} \lambda_m \lambda_n \begin{bmatrix} -\mathcal{S}_{j,m} & 0 & A_{i,m}\mathcal{S}_{i,n} + B_{i,m}\mathcal{U}_{i,n} & E_{i,m} \\ \star & -I & C_{i,m}\mathcal{S}_{i,n} + D_{i,m}\mathcal{U}_{i,n} & F_{i,m} \\ \star & \star & -\mathcal{S}_{i,n} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \\ &= \sum_{m=1}^{s} \lambda_m^2 \begin{bmatrix} -\mathcal{S}_{j,m} & 0 & A_{i,m}\mathcal{S}_{i,m} + B_{i,m}\mathcal{U}_{i,m} & E_{i,m} \\ \star & -I & C_{i,m}\mathcal{S}_{i,m} + D_{i,m}\mathcal{U}_{i,m} & F_{i,m} \\ \star & \star & -\mathcal{S}_{i,m} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \\ &\quad + \sum_{m=1}^{s-1}\sum_{n=m+1}^{s} \lambda_m \lambda_n \left\{ \begin{bmatrix} -\mathcal{S}_{j,m} & 0 & A_{i,m}\mathcal{S}_{i,n} + B_{i,m}\mathcal{U}_{i,n} & E_{i,m} \\ \star & -I & C_{i,m}\mathcal{S}_{i,n} + D_{i,m}\mathcal{U}_{i,n} & F_{i,m} \\ \star & \star & -\mathcal{S}_{i,n} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} -\mathcal{S}_{j,n} & 0 & A_{i,n}\mathcal{S}_{i,m} + B_{i,n}\mathcal{U}_{i,m} & E_{i,n} \\ \star & -I & C_{i,n}\mathcal{S}_{i,m} + D_{i,n}\mathcal{U}_{i,m} & F_{i,n} \\ \star & \star & -\mathcal{S}_{i,m} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \right\}. \end{aligned} \tag{4.31}$$

On the other hand, the inequality (4.26) is equivalent to

$$\begin{aligned} &\begin{bmatrix} -\mathcal{S}_{j,m} & 0 & \bar{A}^i_{m,n} & E_{i,m} \\ \star & -I & \bar{C}^i_{m,n} & F_{i,m} \\ \star & \star & -\mathcal{S}_{i,n} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} -\mathcal{S}_{j,n} & 0 & \bar{A}^i_{n,m} & E_{i,n} \\ \star & -I & \bar{C}^i_{n,m} & F_{i,n} \\ \star & \star & -\mathcal{S}_{i,m} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \\ &\le \begin{bmatrix} \mathcal{X}^{i,j}_{m,n} & \mathcal{Y}^{i,j}_{m,n} & \mathcal{Z}^{i,j}_{m,n} & \mathcal{O}^{i,j}_{m,n} \\ \mathcal{R}^{i,j}_{m,n} & \mathcal{H}^{i,j}_{m,n} & \mathcal{T}^{i,j}_{m,n} & \mathcal{J}^{i,j}_{m,n} \\ \mathcal{U}^{i,j}_{m,n} & \mathcal{V}^{i,j}_{m,n} & \mathcal{W}^{i,j}_{m,n} & \mathcal{Q}^{i,j}_{m,n} \\ \mathcal{L}^{i,j}_{m,n} & \mathcal{M}^{i,j}_{m,n} & \mathcal{N}^{i,j}_{m,n} & \mathcal{K}^{i,j}_{m,n} \end{bmatrix} + \begin{bmatrix} \mathcal{X}^{i,j}_{m,n} & \mathcal{Y}^{i,j}_{m,n} & \mathcal{Z}^{i,j}_{m,n} & \mathcal{O}^{i,j}_{m,n} \\ \mathcal{R}^{i,j}_{m,n} & \mathcal{H}^{i,j}_{m,n} & \mathcal{T}^{i,j}_{m,n} & \mathcal{J}^{i,j}_{m,n} \\ \mathcal{U}^{i,j}_{m,n} & \mathcal{V}^{i,j}_{m,n} & \mathcal{W}^{i,j}_{m,n} & \mathcal{Q}^{i,j}_{m,n} \\ \mathcal{L}^{i,j}_{m,n} & \mathcal{M}^{i,j}_{m,n} & \mathcal{N}^{i,j}_{m,n} & \mathcal{K}^{i,j}_{m,n} \end{bmatrix}^T \end{aligned} \tag{4.32}$$

where $\bar{A}^i_{m,n} \triangleq A_{i,m}\mathcal{S}_{i,n} + B_{i,m}\mathcal{U}_{i,n}$, $\bar{C}^i_{m,n} \triangleq C_{i,m}\mathcal{S}_{i,n} + D_{i,m}\mathcal{U}_{i,n}$, $\bar{A}^i_{n,m} \triangleq A_{i,n}\mathcal{S}_{i,m} + B_{i,n}\mathcal{U}_{i,m}$, $\bar{C}^i_{n,m} \triangleq C_{i,n}\mathcal{S}_{i,m} + D_{i,n}\mathcal{U}_{i,m}$, $1 \leq m < n \leq s$. Then from (4.31) and (4.32) we obtain

$$\begin{aligned}\Xi^{i,j}(\lambda) &\leq \sum_{m=1}^{s} \lambda_m^2 \Omega_m^{i,j} + \sum_{m=1}^{s-1} \sum_{n=m+1}^{s} \lambda_m \lambda_n \left\{\Omega_{m,n}^{i,j} + (\Omega_{m,n}^{i,j})^T\right\} \\ &= \eta^T \Omega^{i,j} \eta\end{aligned}$$

where $\eta \triangleq \left[\, \lambda_1 I \;\; \lambda_2 I \;\; \cdots \;\; \lambda_s I \,\right]^T$. Then, the inequality (4.27) guarantees $\Xi^{i,j}(\lambda) < 0$. Therefore, if the inequalities (4.26) and (4.27) are satisfied, then the close-loop system (4.24)–(4.25) is asymptotically stable with an H_∞ noise-attenuation level bound γ, meanwhile, if a solution exists, then according to (4.29), the gain matrix function of stabilizing controller is given by (4.28). This completes the proof. □

Likewise, the H_∞ performance index can also be optimized by the following convex optimization problem.

Problem 4.1

$$\min \ \gamma^2, \ \text{s.t. } (4.26) \text{ and } (4.27)$$

and the corresponding parameter-dependent H_∞ controller gain (4.28) can be computed.

Remark 4.7 The derivation of Theorem 4.6 employs the parameter-dependent idea. It is observed from (4.23) that in each subsystem, the solution of H_∞ state-feedback controller depends on the parameter λ. Actually, in many practical applications, some parameters can be measured online without difficulty, therefore, in contrast to the robust controllers discussed in Sect. 4.1.1, the controller designed in Theorem 4.6 should be less conservative.

Remark 4.8 It is worth noting that the designed parameter-dependent H_∞ controller is analogous to gain scheduling control strategy, which is different from robust control idea regardless of variations of uncertain parameters. In addition, when the uncertain parameters and switching signals are all measurable in real time, the underlying switched systems can be viewed as a special class of linear parameter varying (LPV) systems. It is obvious that the variations of parameters in the system studied in this subsection are not restricted to be smooth, which differs from that of the general LPV systems. Furthermore, the parameters vary in N switched convex polytopes according to (4.3) (N is the number of subsystems), which is also different from the hybrid (switched) LPV systems proposed and studied in [3, 4].

In the following, an illustrative example emerging in networked control system (NCS) is introduced to show the validity and respective advantages of the above designed robust H_∞ controller and parameter-dependent H_∞ controller.

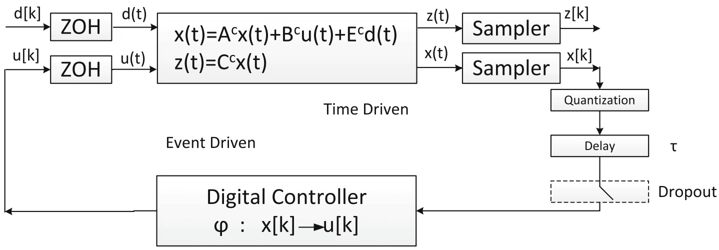


Fig. 4.1 Networked control systems modelled as switched systems

Example 4.9 Considering the three cases varying delays, packets dropout and quantization errors in network transmission, Fig. 4.1 shows a reasonable modeling procedure for widely studied NCS, which is given and explained in [5]. Firstly, the NCS can be formulated as a certain switched system with $D_{\max} + 2$ modes as follows ($D_{\max}$ is the maximal delay bound in network)

$$x[k+1] = A_h x[k] + B_h u[k] + E_h d[k] \tag{4.33}$$

$$z[k] = C_h x[k] \tag{4.34}$$

where,

$$A_h \triangleq \begin{bmatrix} 0 & I \\ hB & A \end{bmatrix}, \quad B_h \triangleq \begin{bmatrix} I \\ (N-h)B \end{bmatrix}, \quad E_h \triangleq \begin{bmatrix} 0 \\ E \end{bmatrix}, \quad C_h \triangleq \begin{bmatrix} 0 & C \end{bmatrix}$$

for $h = 0, 1, 2 \ldots D_{\max}, N$. Then, the matrices (A_h, B_h, E_h, C_h) represent the hth subsystem, where A, B, C and E are the discretization system matrices of the considered linear continuous-time plant. Besides, due to the finite bit-rate constraints in network data communication, quantization effect has to be taken into consideration in NCS. Hence, the system in (4.33)–(4.34) can be further modeled as a polytopic uncertain switched system with the following form:

$$x[k+1] = \bar{A}_h x[k] + \bar{B}_h u[k] + \bar{E}_h d[k] \tag{4.35}$$

$$z[k] = \bar{C}_h x[k] \tag{4.36}$$

Here, we consider the developed uncertain switched system (4.35)–(4.36) and design two kinds of H_∞ controllers for such systems, i.e. robust and parameter-dependent controllers studied respectively in Theorems 4.2 and 4.6. Note that in designing parameter-dependent H_∞ controller, we assume that the "uncertain" parameter λ derived from quantization be measurable in real time.

For simplicity, we consider the uncertain discrete-time switched linear system (4.35)–(4.36) consisting of two uncertain subsystems, where there are two groups of vertex matrices in subsystem 1

$$A_{11} = \rho \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.0341 & -0.2571 & 0.7769 \end{bmatrix}, \quad B_{11} = \rho \begin{bmatrix} 0 \\ -0.1 \\ -0.5 \end{bmatrix},$$

$$E_{11} = \rho \begin{bmatrix} 0.3 \\ 0.1 \\ 0.8 \end{bmatrix}, \quad C_{11} = \rho \begin{bmatrix} 0.2 & 0.1 & 1 \end{bmatrix}, \quad D_{11} = F_{11} = 0,$$

$$A_{12} = \rho \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.0341 & -0.2571 & -0.7769 \end{bmatrix}, \quad B_{12} = \rho \begin{bmatrix} 0 \\ -0.2 \\ -0.3 \end{bmatrix},$$

$$E_{12} = \rho \begin{bmatrix} 0.3 \\ 0.1 \\ -0.8 \end{bmatrix}, \quad C_{12} = \rho \begin{bmatrix} -0.2 & 0.1 & 1 \end{bmatrix}, \quad D_{12} = F_{12} = 0.$$

and the two groups of vertex matrices in the subsystem 2

$$A_{21} = \rho \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0.0341 & -0.2571 & 0.7769 \end{bmatrix}, \quad B_{21} = \rho \begin{bmatrix} 0 \\ -0.1 \\ -0.8 \end{bmatrix},$$

$$E_{21} = \rho \begin{bmatrix} -0.3 \\ 0.1 \\ 0.8 \end{bmatrix}, \quad C_{21} = \rho \begin{bmatrix} 0.2 & 0.1 & -1 \end{bmatrix}, \quad D_{21} = F_{21} = 0,$$

$$A_{22} = \rho \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -0.0341 & -0.2571 & -0.7769 \end{bmatrix}, \quad B_{22} = \rho \begin{bmatrix} 0 \\ -1 \\ -0.5 \end{bmatrix},$$

$$E_{22} = \rho \begin{bmatrix} -0.3 \\ 0.1 \\ -0.8 \end{bmatrix}, \quad C_{22} = \rho \begin{bmatrix} -0.2 & 0.1 & -1 \end{bmatrix}, \quad D_{22} = F_{22} = 0.$$

where ρ is a scalar parameter implying the size of convex polytope each uncertain subsystem can be expanded into.

Our purpose is to design an H_∞ state-feedback controller for the above uncertain switched system for given ρ and to check the H_∞ performance of the system. By solving the corresponding convex optimization problems in the theorems and corollaries given in Table 4.1 and Problem 4.1, respectively, we can obtain different minimum H_∞ noise-attenuation level bounds γ, and the different calculation results by different methods are shown in Table 4.2.

Meanwhile, the admissible controller gains can be constructed by necessary matrices solved in the convex optimization procedures as well. For instance, when $\rho = 1.00$, the robust H_∞ controller gains solved respectively by Theorem 4.2, Corollaries 4.4 and 4.5 are listed in Table 4.3.

As for the parameter-dependent H_∞ controller, we can get the matrices for the calculation of the controller gain as follows

Table 4.2 Minimum γ by different H_∞ controllers

Methods		$\rho = 1.00$	$\rho = 1.25$	$\rho = 1.40$
Robust H_∞ controllers	Corollary 4.5	4.3831	Infeasible	Infeasible
	Corollary 4.4	3.8389	33.1361	Infeasible
	Theorem 4.2	3.4810	29.2426	Infeasible
Parameter-dependent H_∞ controller	Theorem 4.6	1.2145	2.2145	3.0593

Table 4.3 Robust H_∞ controller gains

Methods	Controller gains
Corollary 4.5	$\mathcal{K} = \begin{bmatrix} 0.0044 & -0.4812 & 0.1150 \end{bmatrix}$
Corollary 4.4	$\mathcal{K}_1 = \begin{bmatrix} 0.0969 & -0.5386 & 0.1560 \end{bmatrix}$; $\mathcal{K}_2 = \begin{bmatrix} -0.0572 & -0.4393 & 0.0876 \end{bmatrix}$
Theorem 4.2	$\mathcal{K}_1 = \begin{bmatrix} 0.1180 & -0.4696 & 0.1958 \end{bmatrix}$; $\mathcal{K}_2 = \begin{bmatrix} -0.0606 & -0.3514 & 0.0647 \end{bmatrix}$

$$\mathcal{S}_{11} = \begin{bmatrix} 17.8650 & -0.3173 & -1.4947 \\ -0.3173 & 8.5616 & -0.8476 \\ -1.4947 & -0.8476 & 0.7861 \end{bmatrix}, \quad \mathcal{S}_{12} = \begin{bmatrix} 17.5781 & -0.1827 & 1.1777 \\ -0.1827 & 9.4150 & -0.9129 \\ 1.1777 & -0.9129 & 0.6303 \end{bmatrix},$$

$$\mathcal{S}_{21} = \begin{bmatrix} 18.3237 & -1.0429 & 1.4505 \\ -1.0429 & 8.6960 & 0.6604 \\ 1.4505 & 0.6604 & 0.7244 \end{bmatrix}, \quad \mathcal{S}_{22} = \begin{bmatrix} 19.5579 & 0.5428 & -1.3274 \\ 0.5428 & 6.7499 & -0.1360 \\ -1.3274 & -0.1360 & 0.5156 \end{bmatrix},$$

$$\mathcal{U}_{11} = \begin{bmatrix} -0.9270 & -5.7068 & 1.6573 \end{bmatrix}, \quad \mathcal{U}_{12} = \begin{bmatrix} -2.0878 & -5.6798 & -0.5206 \end{bmatrix},$$

$$\mathcal{U}_{21} = \begin{bmatrix} 2.2280 & -2.2691 & 0.5152 \end{bmatrix}, \quad \mathcal{U}_{22} = \begin{bmatrix} -0.5100 & -2.7960 & -0.4610 \end{bmatrix}.$$

Then, the gain matrix functions for an admissible switched parameter-dependent H_∞ controller are given by

$$\mathcal{K}_i(\lambda) = \left(\sum_{m=1}^{2} \lambda_m \mathcal{U}_{i,m} \right) \left(\sum_{m=1}^{2} \lambda_m \mathcal{S}_{i,m} \right)^{-1}, \quad i = 1, 2.$$

Thus, for given different parameter λ, the corresponding three components of the controller gain functions in two subsystems $\mathcal{K}_1(\lambda)$ and $\mathcal{K}_2(\lambda)$ are depicted respectively in Figs. 4.2 and 4.3.

In addition, for above given $\rho = 1.00$, the different time required are listed in Table 4.4 to compute such robust H_∞ controller gains and parameter-dependent H_∞ controller gain matrix function values for given parameter λ by different theorems and corollaries. The computation time can be easily obtained by the function **cputime** in Matlab.

Remark 4.10 From Tables 4.3 and 4.4 and Figs. 4.2 and 4.3, it can be seen that the realization of parameter-dependent H_∞ controller is complex with longer computa-

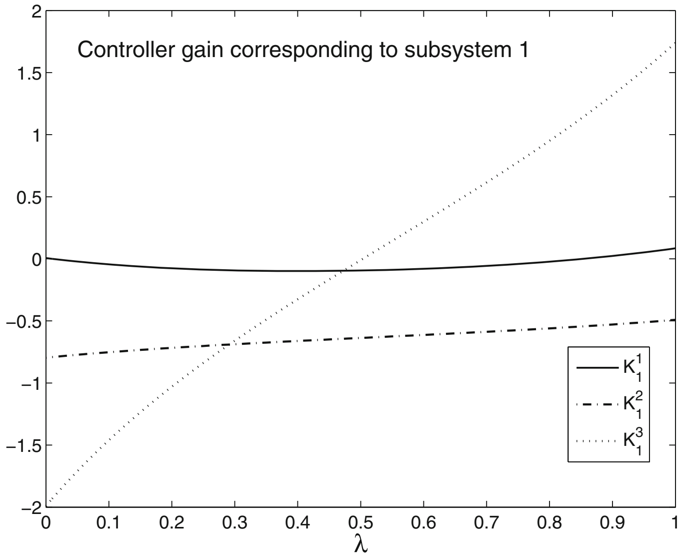


Fig. 4.2 Variations of parameter-dependent H_∞ controller gain in subsystem 1

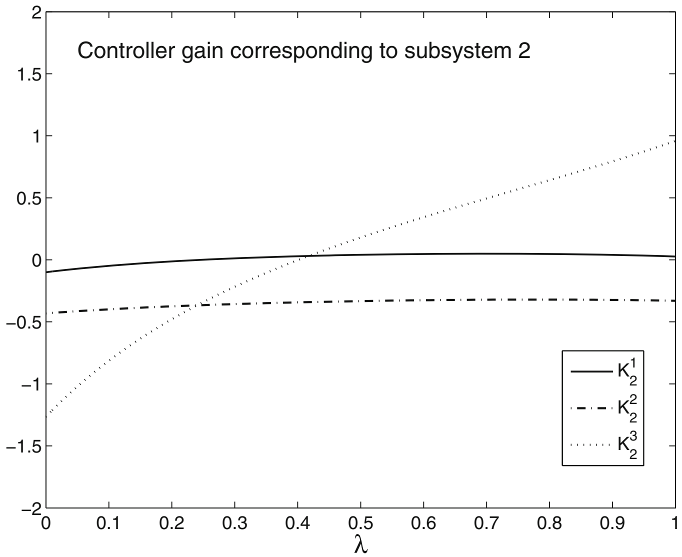


Fig. 4.3 Variations of parameter-dependent H_∞ controller gain in subsystem 2

Table 4.4 Different CPU computation times

Methods	Theorem 4.6	Theorem 4.2	Corollary 4.5	Corollary 4.4
CPU time (s)	5.5781	2.3906	2.0312	1.9687

tion time since the solution of the controller gain is dependent on uncertain parameter λ and calculated by more matrix variables in Theorem 4.6. On the other hand, the robust H_∞ controller gain is constant in each subsystem such that it can be easily obtained with less computation burden. However, from Table 4.2, it is not hard to conclude that the parameter-dependent controller is less conservative than robust controllers in ensuring an optimal H_∞ performance index for the underlying uncertain switched system. Therefore, both robust controller and parameter-dependent controller have their own advantages and disadvantages regarding conservatism and realization complexity. In the practical situation such as above discussed NCS with uncertain switched system model, we can decide to choose which type of controller to use according to the online measurability of the parameter λ.

4.2 Robust H_∞ Control: ADT Switching

In this section, considering the average dwell time (ADT) switching, an H_∞ state-feedback controller is designed for the underlying system (4.1)–(4.3) and the corresponding existence conditions are derived via LMIs formulation. The obtained conditions are dependent on the admissible increasing level μ of Lyapunov-like function values at switching instants. The minimal μ and the desired controller gains are obtained for a given decay degree such that the resulting closed-loop system (4.5)–(4.6) is robustly exponentially stable and ensures a prescribed exponential H_∞ performance index.

The following theorem presents sufficient conditions for the existence of an admissible mode-dependent exponential H_∞ controller with the form (4.4).

Theorem 4.11 *Consider the uncertain switched linear system (4.1)–(4.3) and let $\alpha > 0$, $\gamma > 0$ and $\mu \geq 1$ be given constants. If there exist matrices $S_{i,m} > 0$, $\forall i \in \mathcal{I}$, $1 \leq m \leq s$ and matrices U_i, G_i such that*

$$\begin{bmatrix} -S_{i,m} & 0 & A_{i,m}G_i + B_{i,m}U_i & E_{i,m} \\ \star & -I & C_{i,m}G_i + D_{i,m}U_i & F_{i,m} \\ \star & \star & (1-\alpha)\left(S_{i,m} - G_i - G_i^T\right) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{4.37}$$

$$S_{j,m} - \mu S_{i,m} \leq 0 \tag{4.38}$$

Then, there exists a stabilizing state feedback controller such that the corresponding closed-loop system (4.5)–(4.6) is robustly exponentially stable with a prescribed exponential H_∞ performance index γ for all admissible uncertainties satisfying (4.3) and any switching signals with the ADT satisfying (2.26). Moreover, if (4.37) and (4.38) have a solution, the mode-dependent controller is given by (4.4) with

$$\mathcal{K}_i = U_i G_i^{-1}, \ \forall i \in \mathcal{I}. \tag{4.39}$$

Proof Assume matrix functions $S_i(\lambda)$ to be the following form

$$S_i(\lambda) \triangleq \sum_{m=1}^{s} \lambda_m S_{i,m}, \ \forall i \in \mathcal{I} \tag{4.40}$$

where $S_{i,m} > 0$ satisfies (4.37) and (4.38).

Then, if (4.38) holds, we can get that

$$\sum_{m=1}^{s} \lambda_m \left(S_{j,m} - \mu S_{i,m}\right) \leq 0$$

i.e. $S_j(\lambda) \leq \mu S_i(\lambda)$. Setting matrix functions $P_i(\lambda) \triangleq S_i^{-1}(\lambda), \forall i \in \mathcal{I}$, one can get that (3.35) is satisfied.

In addition, if (4.37) holds, according to (4.3) and (4.40), one can also have

$$\begin{bmatrix} -S_i(\lambda) & 0 & A_i(\lambda)G_i + B_i(\lambda)U_i & E_i(\lambda) \\ \star & -I & C_i(\lambda)G_i + D_i(\lambda)U_i & F_i(\lambda) \\ \star & \star & (1-\alpha)\left(S_i(\lambda) - G_i - G_i^T\right) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \tag{4.41}$$

From (4.41), one can readily explore the fact that $S_i(\lambda) - G_i - G_i^T < 0$ so that the matrices G_i are nonsingular. In addition, we have $(S_i(\lambda) - G_i)^T S_i^{-1}(\lambda)(S_i(\lambda) - G_i) \geq 0$, which implies $S_i(\lambda) - G_i - G_i^T \geq -G_i^T S_i^{-1}(\lambda) G_i$. Therefore, assuming the controller gain to be of the form (4.39), together with $\bar{A}_i(\lambda)$ and $\bar{C}_i(\lambda)$ in (4.5) and (4.6), we conclude

$$\begin{bmatrix} -S_i(\lambda) & 0 & \bar{A}_i(\lambda)G_i & E_i(\lambda) \\ \star & -I & \bar{C}_i(\lambda)G_i & F_i(\lambda) \\ \star & \star & -(1-\alpha)G_i^T S_i^{-1}(\lambda) G_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0.$$

Performing a congruence transformation to above formula via $diag\{S_i^{-1}(\lambda), I, G_i^{-1}, I\}$, and changing the matrix variables $S_i(\lambda) \triangleq P_i^{-1}(\lambda)$, we obtain

$$\begin{bmatrix} -P_i(\lambda) & 0 & P_i(\lambda)\bar{A}_i(\lambda) & P_i(\lambda)E_i(\lambda) \\ \star & -I & \bar{C}_i(\lambda) & F_i(\lambda) \\ \star & \star & -(1-\alpha)P_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0.$$

Therefore, by Theorem 3.8, one can know that the underlying system is robustly exponentially stable with an exponential H_∞ performance for any uncertainties satisfying (4.3) and any switching signals satisfying (2.26) by applying the mode-dependent controller with gains (4.39), which completes the proof. □

Remark 4.12 It can be seen from (2.26) that the ADT in the solved switching signals will be not less than τ_a^*. Then, we actually need to determine the minimal μ for a given system decay degree α to find the minimal ADT for the underlying systems, which to some extent, is analogous to that delay-dependent issues in time-delay system to determine delay bounds. Therefore, it can be viewed as an μ-dependent approach in switched systems to obtain some results on system analysis and synthesis, which will present less conservativeness compared to the ones based on the global Lyapunov function (GLF) or the switched Lyapunov function (SLF) approaches.

In the following, a numerical example is given to demonstrate the effectiveness of the developed theoretical results in this section.

Example 4.13 Consider the following uncertain discrete-time switched linear systems consisting of two uncertain subsystems, where there are two vertices in subsystem 1

$$A_{11} = \begin{bmatrix} -0.9500 & 0.0095 \\ -0.0475 & 0.9405 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0.2850 \\ -0.0475 \end{bmatrix},$$
$$E_{11} = \begin{bmatrix} 0.0285 \\ 0.0190 \end{bmatrix}, \quad C_{11} = \begin{bmatrix} 0.7600 & -0.0665 \end{bmatrix}, \; D_{11} = 0.3800, \; F_{11} = 0.0760,$$
$$A_{12} = \begin{bmatrix} 0.9500 & 0.0095 \\ -0.0475 & 0.8740 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.9500 \\ 0.3800 \end{bmatrix},$$
$$E_{12} = \begin{bmatrix} 0.0950 \\ 0.6650 \end{bmatrix}, \quad C_{12} = \begin{bmatrix} 0.0950 & 0.5700 \end{bmatrix}, \quad D_{12} = 0.2850, \; F_{12} = 0.0095,$$

and the two vertices in the subsystem 2

$$A_{21} = \begin{bmatrix} 0.5985 & 0.2850 \\ -0.9500 & 0.7790 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.5700 \\ -0.9500 \end{bmatrix},$$
$$E_{21} = \begin{bmatrix} 0.0570 \\ 0.0190 \end{bmatrix}, \quad C_{21} = \begin{bmatrix} 0.8550 & -0.9500 \end{bmatrix}, \quad D_{21} = 0.5700, \quad F_{21} = 0.0285,$$
$$A_{22} = \begin{bmatrix} 0.6080 & 0.2850 \\ -0.9500 & 0.7600 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.3800 \\ 0.2850 \end{bmatrix},$$
$$E_{22} = \begin{bmatrix} 0.1900 \\ 0.4750 \end{bmatrix}, \quad C_{22} = \begin{bmatrix} 0.3800 & 0.4750 \end{bmatrix}, \quad D_{22} = 0.7600, \quad F_{22} = 0.0475.$$

Table 4.5 Minimum μ corresponding to different given decay degree α

α	0.060	0.065	0.066	0.067	0.068
μ^*	1.0280	1.3390	1.4070	1.4760	1.5460
τ_a^*	0.4463	4.3435	5.0010	5.6141	6.1865

Our purpose here is to design a mode-dependent state-feedback controller in the form of (4.4) and to find out the admissible switching signals for the above uncertain switched system such that the resulting closed-loop system (4.5)–(4.6) is robustly exponentially stable with a prescribed exponential H_∞ performance for a given decay degree α. In this example, we assume the noise attenuation level $\gamma = 3.3$. By solving LMIs in Theorem 4.11, we can obtain the different minimal μ (μ^*) and the corresponding minimal ADT for given different α as shown in Table 4.5.

Meanwhile, the desired controller gains can be solved correspondingly. For instance, if $\alpha = 0.065$, we can obtain the controller with

$$K_1 = [-1.7466 \;\; 0.8137], \quad K_2 = [-1.9686 \;\; 0.4284]$$

Furthermore, consider input signal $w(k) = 0.01\exp(-0.03k)\sin(0.02\pi k)$, and by using the solved controller and by randomly giving different uncertain parameters λ in (4.3), we can obtain the state response of the closed-loop system in Figs. 4.4, 4.5, 4.6 and 4.7 for two different switching signals and the given initial condition $x = [0.1 \;\; -0.05]^T$. It is clearly observed from the simulation curves that for given energy bounded disturbance $w(k)$, the closed-loop system (4.5)–(4.6) is stable against the variations of uncertain parameters and two different switching signals (both are with $\tau_a = 5 > \tau_a^* = 4.3435$ for $\alpha = 0.065$), which thereby implies that the designed mode-dependent controller is feasible and effective.

4.3 Observer-Based H_∞ Control: Modal PDT Switching

This section studies the observer-based robust control issue for a class of discrete-time switched linear system described as

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) + E_{\sigma(k)}\omega(k) \tag{4.42}$$

$$z(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}\omega(k) \tag{4.43}$$

when $\sigma(k) = i \in \mathcal{I}$, the matrices $(A_i,\ B_i,\ C_i,\ D_i,\ E_i)$ denote the ith subsystem. The switching signals are considered to have modal persistent dwell-time (MPDT) property in this section, and the more details on the concept of MPDT switching can be seen in Sect. 1.4.

The control problem in this section is considered to be realized via communication network with redundant channels. As is well known, the single channel of data transmission may be invalid off and on owing to low reliability, frequent traffic congestion

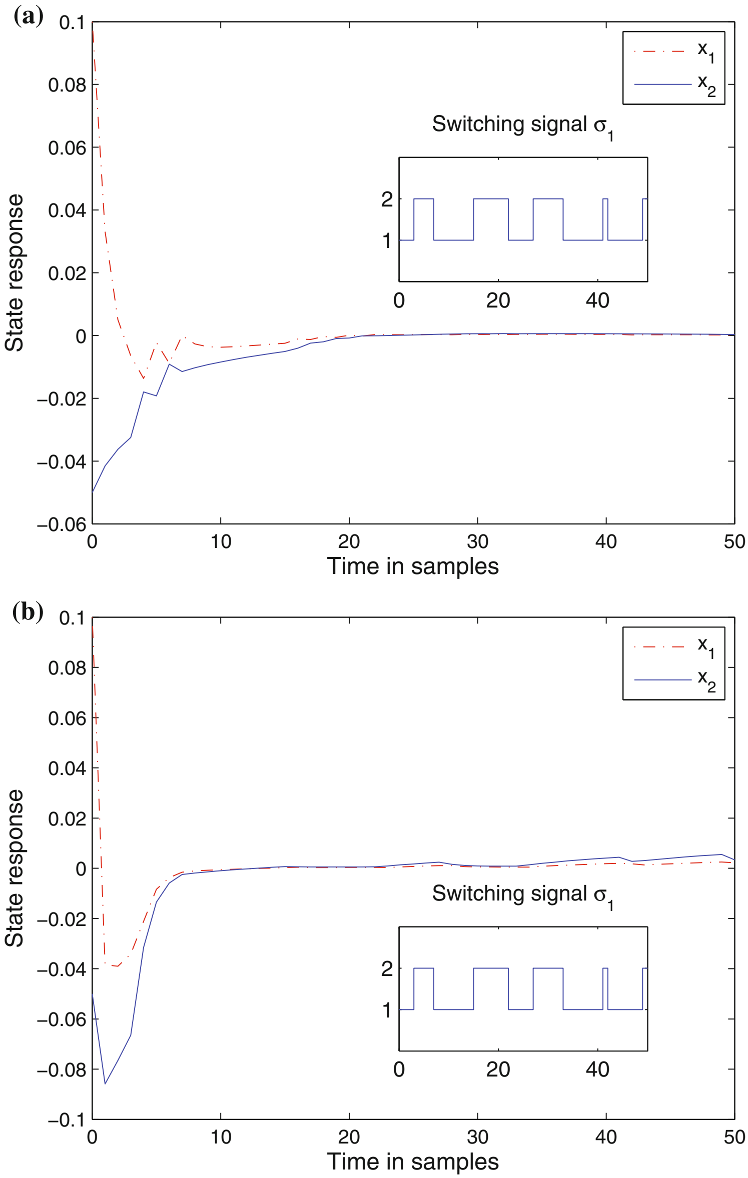


Fig. 4.4 State response of the closed-loop system under switching signal σ_1 with different λ_1 and λ_2. **a** $\lambda_1 = 1$, $\lambda_2 = 0$; **b** $\lambda_1 = 0.5$, $\lambda_2 = 0.5$

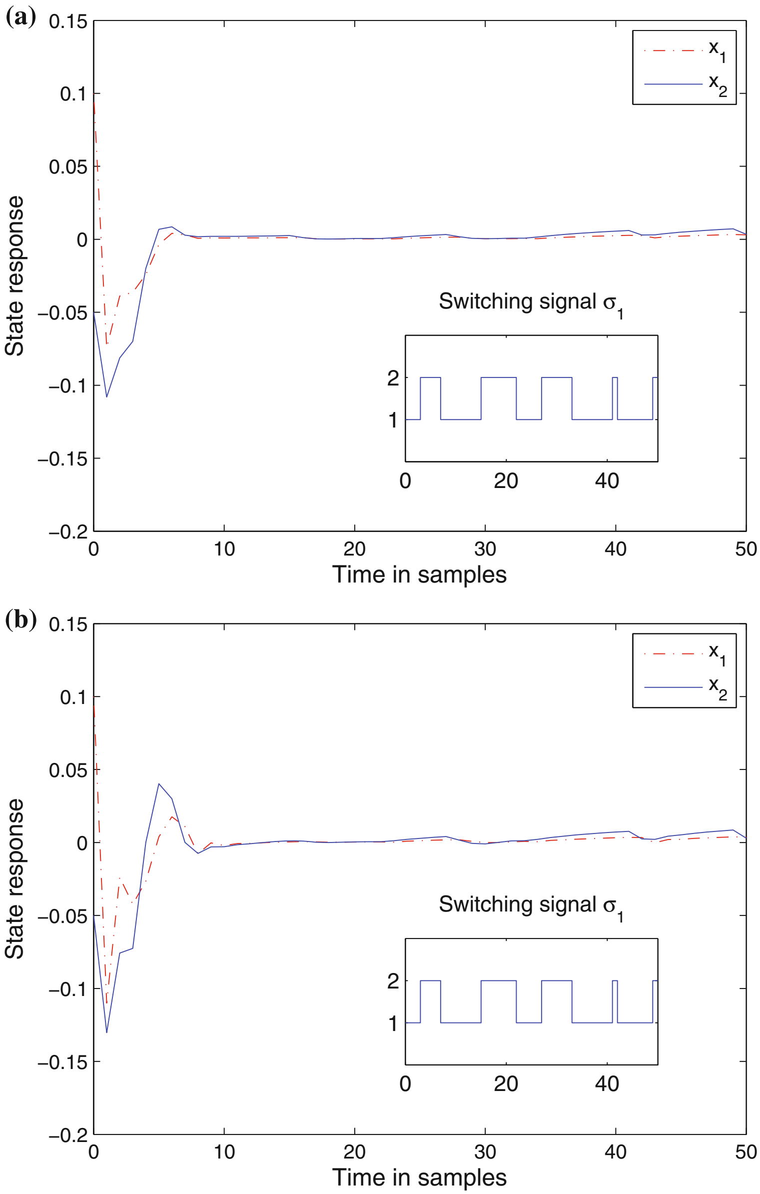


Fig. 4.5 State response of the closed-loop system under switching signal σ_1 with different λ_1 and λ_2. **a** $\lambda_1 = 0.25$, $\lambda_2 = 0.75$; **b** $\lambda_1 = 0$, $\lambda_2 = 1$

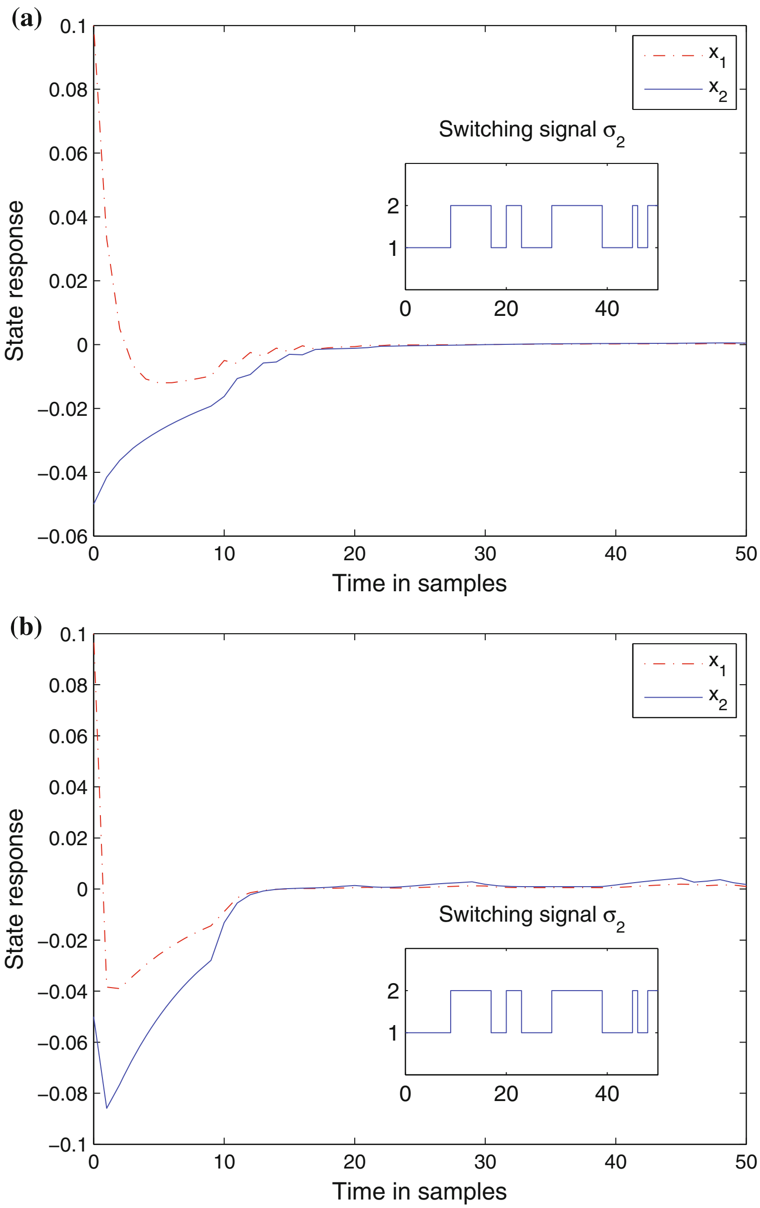
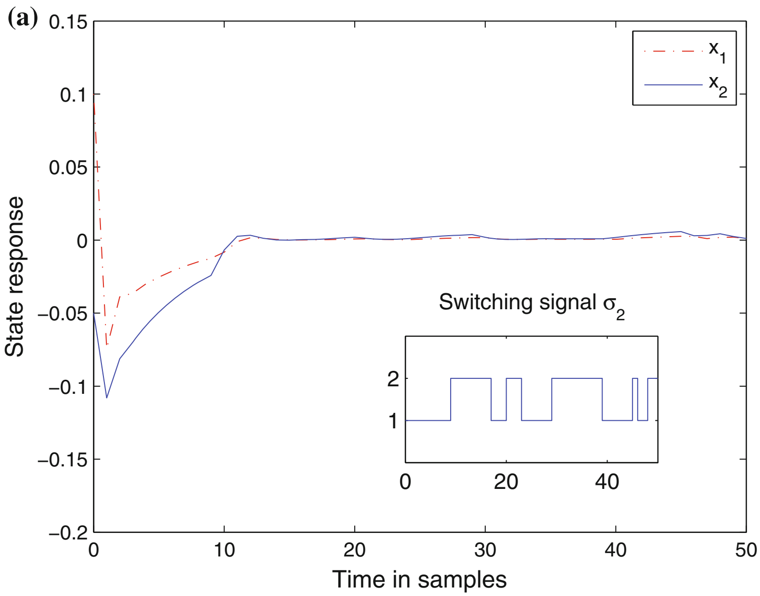


Fig. 4.6 State response of the closed-loop system under switching signal σ_2 with different λ_1 and λ_2. **a** $\lambda_1 = 1$, $\lambda_2 = 0$; **b** $\lambda_1 = 0.5$, $\lambda_2 = 0.5$



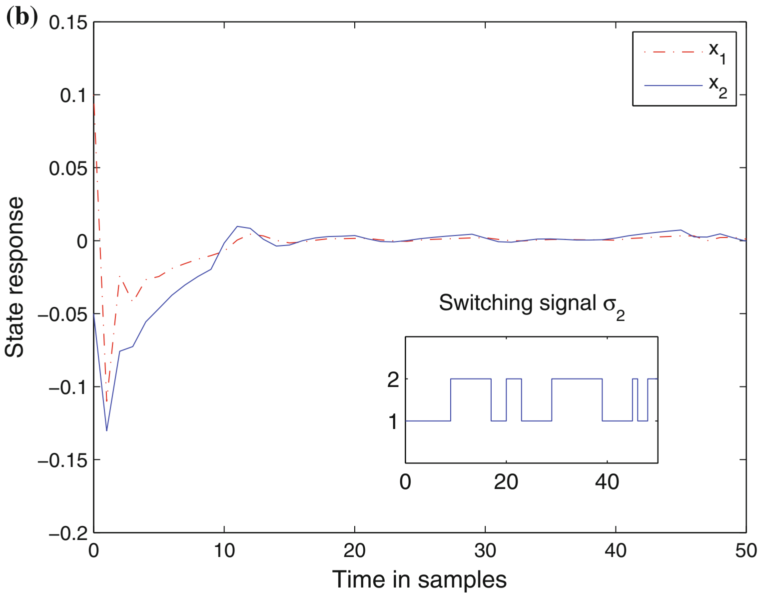


Fig. 4.7 State response of the closed-loop system under switching signal σ_2 with different λ_1 and λ_2. **a** $\lambda_1 = 0.25$, $\lambda_2 = 0.75$; **b** $\lambda_1 = 0$, $\lambda_2 = 1$

and poor anti-interference ability of the communication network, which gives rise to the random occurrence of packet dropouts. In order to reduce the possibility of packet dropouts in single-channel case, one redundant channel will be considered in the communication network in this section.

The actual output $y(k) \in \mathbb{R}^p$ subject to random packet dropouts in the communication network with redundant channels is described by

$$y(k) = \delta_{\sigma(k)}(k)F_{\sigma(k)}x(k) + (1 - \delta_{\sigma(k)}(k))\beta_{\sigma(k)}(k)F_{\sigma(k)}x(k) + G_{\sigma(k)}\omega(k) \quad (4.44)$$

where $F_{\sigma(k)}$ and $G_{\sigma(k)}$ are known real matrices with appropriate dimensions. The stochastic variables $\delta_{\sigma(k)}(k)$ and $\beta_{\sigma(k)}(k)$, are assumed to be subject to Bernoulli binary distribution taking values on 0 and 1 with

$$\begin{aligned} &\Pr\{\delta_{\sigma(k)}(k) = 1\} = \delta_{\sigma(k)}, \quad \Pr\{\delta_{\sigma(k)}(k) = 0\} = 1 - \delta_{\sigma(k)}, \\ &\Pr\{\beta_{\sigma(k)}(k) = 1\} = \beta_{\sigma(k)}, \quad \Pr\{\beta_{\sigma(k)}(k) = 0\} = 1 - \beta_{\sigma(k)} \end{aligned}$$

where $\delta_{\sigma(k)}, \beta_{\sigma(k)} \in [0, 1], \forall \sigma(k) = i \in \mathcal{I}$. Throughout the section, the stochastic variables $\delta_{\sigma(k)}(k), \beta_{\sigma(k)}(k)$ and the switching signals $\sigma(k)$ are assumed to be mutually independent. In addition, we have

$$\begin{aligned} &\mathbb{E}\{\delta_{\sigma(k)}(k)\} = \delta_{\sigma(k)}, \quad \mathbb{E}\{\delta^2_{\sigma(k)}(k)\} = \delta_{\sigma(k)}, \quad \mathbb{E}\{\beta_{\sigma(k)}(k)\} = \beta_{\sigma(k)}, \\ &\mathbb{E}\{\beta^2_{\sigma(k)}(k)\} = \beta_{\sigma(k)}. \end{aligned}$$

Remark 4.14 The stochastic variables $\delta_{\sigma(k)}(k)$ and $\beta_{\sigma(k)}(k)$, which are mutually independent, are introduced to model the random occurring packet dropouts phenomena in (4.44). In particular, at each time instant, if $\delta_{\sigma(k)}(k) = 1$, $\beta_{\sigma(k)}(k) = 1/0$, the measurement output is normal through the primary channel; and if $\delta_{\sigma(k)}(k) = 0$, $\beta_{\sigma(k)}(k) = 1$, the measurement output is normal through the redundant channel; while if $\delta_{\sigma(k)}(k) = 0$, $\beta_{\sigma(k)}(k) = 0$, the packet dropouts occur which means only noises are contained in the measurement of output. Hence, the influences of packet dropouts occurred in (4.44) will be weakened via the proposed redundant channels design approach even inactivation of the primary channel, and its effectiveness will be demonstrated via the numerical examples.

The dynamic observer-based control scheme for system (4.42)–(4.43) is described by

$$\begin{aligned} \hat{x}(k+1) &= A_{\sigma(k)}\hat{x}(k) + B_{\sigma(k)}u(k) + L_{\sigma(k)}(q_k)(y(k) - \delta_{\sigma(k)}F_{\sigma(k)}\hat{x}(k) \\ &\quad - (1 - \delta_{\sigma(k)})\beta_{\sigma(k)}F_{\sigma(k)}\hat{x}(k)), \quad (4.45) \\ u(k) &= K_{\sigma(k)}(q_k)\hat{x}(k) \quad (4.46) \end{aligned}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the state estimate of the system (4.42)–(4.43), $u(k) \in \mathbb{R}^m$ is the control input vector, and $L_i(q_k)$ and $K_i(q_k)$ are the observer and controller gains to be designed, respectively, $\forall \sigma(k) = i$, and q_k is a scheduler for the activated subsystem and can be simply computed online according to the following rules: (i) in the τ_i-portion, $q_k = k - k_{s_p}$, $k \in [k_{s_p}, k_{s_p} + \tau_i)$ and $q_k = \tau_i$, $k \in [k_{s_p} + \tau_i, k_{s_p+1})$; (ii) in the $\mathbb{T}$-portion, $q_k = k - H_r$, $k \in [k_{s_p+1}, k_{s_{p+1}})$, where $H_r \triangleq \arg\{\max(k_{s_p+r}, r \in \mathbb{Z}_{[1,\mathcal{G}(k_{s_p+1},k_{s_{p+1}})]} | k_{s_p+r} \leq k, k_{s_p+r} \in [k_{s_p+1}, k_{s_{p+1}}))\}$. Since the scheduler q_k only requires a finite number of computations (reset after each switching), the controller of the above structure is referred to as quasi-time-dependent (QTD) controller in this section. Note that by the definition of H_r, the actual running time of the $\sigma(k)$th subsystem in the $\mathbb{T}$-portion, $\mathbb{T}_{\sigma(k)} \in [1, \min(\tau_i - 1, \mathbb{T}^{(p)})]$ satisfies that $\mathbb{T}_{\sigma(k)} = k_{s_p+r+1} - H_r$, $k \in [k_{s_p+1}, k_{s_{p+1}-1})$ and $\mathbb{T}_{\sigma(k)} = k_{s_{p+1}} - H_r$, $k \in [k_{s_{p+1}-1}, k_{s_{p+1}})$.

Let the estimation error to be $e(k) \triangleq x(k) - \hat{x}(k)$, the closed-loop system can be obtained by substituting (4.44)–(4.46) into (4.42)–(4.43)

$$x(k+1) = \bar{A}_i(q_k)x(k) - B_i K_i(q_k)e(k) + E_i\omega(k) \tag{4.47}$$

$$\begin{aligned} e(k+1) = &\tilde{A}_i(q_k)e(k) - \bar{\delta}_i(k)L_i(q_k)F_i x(k) \\ &- \tilde{\delta}_i(k)L_i(q_k)F_i x(k) + \bar{E}_i(q_k)\omega(k) \end{aligned} \tag{4.48}$$

where

$$\begin{aligned} &\bar{A}_i(q_k) \triangleq A_i + B_i K_i(q_k),\ \tilde{A}_i(q_k) \triangleq A_i - \delta_i L_i(q_k)F_i - (1-\delta_i)\beta_i L_i(q_k)F_i, \\ &\bar{E}_i(q_k) \triangleq E_i - L_i(q_k)G_i,\ \tilde{\delta}_i(k) \triangleq (1-\delta_i(k))\beta_i(k) - (1-\delta_i)\beta_i, \\ &\bar{\delta}_i(k) \triangleq \delta_i(k) - \delta_i. \end{aligned}$$

To more precisely describe the main objective in this section, we also introduce the following definitions for system (4.47)–(4.48).

Definition 4.15 ([6]) The closed-loop system (4.47)–(4.48) is said to be exponentially mean-square stable, if with $\omega(k) \equiv 0$, there exist constants $\phi > 0$ and $\varrho \in (0, 1)$, such that

$$\mathbb{E}\{\|\eta(k)\|^2\} \leq \phi\varrho^k \mathbb{E}\{\|\eta(0)\|^2\},$$

for all $\eta(0) \in \mathbb{R}^n$, $k \in \mathbb{Z}_+$, where $\eta(k) \triangleq \begin{bmatrix} x^T(k) & e^T(k) \end{bmatrix}^T$.

Definition 4.16 Given a scalar $\gamma > 0$, system (4.47)–(4.48) is said to be exponentially mean-square stable with a prescribed H_∞ error performance γ if it is exponentially mean-square stable and under zero initial condition, the controlled output $z(k)$ satisfies

$$\sum_{k=0}^{\infty} \mathbb{E}\{\ \|z(k)\|^2\ \} \leq \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\ \|\omega(k)\|^2\ \}$$

for all nonzero $\omega(k) \in l_2[0, \infty)$.

The problem to be addressed in this section will be to design the QTD controller and observer (4.45)–(4.46), such that, in the presence of redundant channels subject to random packet dropouts, the resulting closed-loop system (4.47)–(4.48) is exponentially mean-square stable under admissible MPDT switching sequence and has a guaranteed non-weighted H_∞ performance γ.

Throughout this section, without loss of generality, we will make the following assumption for technical convenience.

Assumption 4.1 The matrix B_i in (4.42) is of full column rank, i.e., $rank(B_i) = m$.

By Assumption 4.1, we have the singular value decomposition (SVD) of B_i as follows,

$$U_i B_i V_i = \begin{bmatrix} U_{1i}^T & U_{2i}^T \end{bmatrix}^T B_i V_i = \begin{bmatrix} \Sigma_i^T & 0 \end{bmatrix}^T \tag{4.49}$$

where $U_i \in \mathbb{R}^{n\times n}$ and $V_i \in \mathbb{R}^{m\times m}$ are unitary matrices, $\Sigma_i = diag\{\varrho_{1i}, \varrho_{2i}, \ldots, \varrho_{mi}\}$ is a diagonal matrix with nonnegative real numbers on the diagonal, ϱ_{ji}, $j = 1, 2, \ldots, m$ are nonzero singular values of B_i.

Lemma 4.17 ([6]) *For the matrix $B_i \in \mathbb{R}^{n\times m}$ that is of full-column rank, if matrix P_i is of the following structure*

$$P_i = U_i^T \begin{bmatrix} P_{11i} & 0 \\ 0 & P_{22i} \end{bmatrix} U_i = U_{1i}^T P_{11i} U_{1i} + U_{2i}^T P_{22i} U_{2i},$$

where $P_{11i} \in \mathbb{R}^{m\times m} > 0$, and $P_{22i} \in \mathbb{R}^{(n-m)\times(n-m)} > 0$, and U_{1i}, U_{2i} are defined in (4.49), then there exists a nonsingular matrix $\bar{P}_i \in \mathbb{R}^{m\times m}$, such that $B_i \bar{P}_i = P_i B_i$.

The following theorem presents a sufficient criterion ensuring that the closed-loop system (4.47)–(4.48) is exponentially mean-square stable and has a guaranteed non-weighted H_∞ performance index γ under admissible MPDT switching.

Theorem 4.18 *Consider the discrete-time switched linear system (4.42)–(4.43) and let $\alpha_i > 0$, $\mu_i > 1$, $\gamma > 0$ be given constants, $i \in \mathcal{I}$. Suppose that both the QTD controller gain matrix $K_i(\xi_i)$ and the QTD observer gain matrix $L_i(\xi_i)$ are given. For a prescribed period of persistence $\mathbb{T}$, the closed-loop system (4.47)–(4.48) is exponentially mean-square stable and has a non-weighted H_∞ performance index*

$$\gamma_{non} = \gamma \sqrt{\left(\mu_{\max}^{\frac{\mathbb{T}+1}{\tau_{\min}+\mathbb{T}}}(1-\alpha_{\min})\mu_{\max}^{(\mathbb{T}+1)}\right) \Big/ \left(1 - \mu_{\max}^{\frac{\mathbb{T}+1}{\tau_{\min}+\mathbb{T}}}\alpha_{\min}\right)}$$

for MPDT switching signals satisfying

$$\tau_i \geq \frac{(\mathbb{T}+1)\ln \mu_{\max} + \mathbb{T}\ln(1-\alpha_{\min})}{-\ln(1-\alpha_i)} \tag{4.50}$$

where $\mu_{\max} \triangleq \max_{i\in\mathcal{I}} \mu_i$, $\alpha_{\min} \triangleq \min_{i\in\mathcal{I}} \alpha_i$, if there exist matrices $P_i(\xi_i) \in S_{>0}^n$ and $Q_i(\xi_i) \in U_{>0}^n$, $\xi_i = 0, 1, \ldots, \tau_i$, $\forall i \in \mathcal{L}$, such that

$$\tilde{\Pi}(\varphi_i) \triangleq \begin{bmatrix} \tilde{\Pi}_{11}(\varphi_i) & \star \\ \tilde{\Pi}_{21}(\varphi_i) & \tilde{\Pi}_{22}(\varphi_i+1) \end{bmatrix} < 0 \tag{4.51}$$

$$\breve{\Pi}(\tau_i) \triangleq \begin{bmatrix} \breve{\Pi}_{11}(\tau_i) & \star \\ \breve{\Pi}_{21}(\tau_i) & \breve{\Pi}_{22}(\tau_i) \end{bmatrix} < 0 \tag{4.52}$$

where

$$\begin{aligned}
\tilde{\Pi}_{11}(\varphi_i) &\triangleq diag\{-\alpha_i P_i(\varphi_i), -\alpha_i Q_i(\varphi_i), -\gamma^2 I\},\\
\tilde{\Pi}_{22}(\varphi_i+1) &\triangleq diag\{-Q_i(\varphi_i+1), -Q_i(\varphi_i+1), -P_i(\varphi_i+1), -Q_i(\varphi_i+1), -I\},\\
\breve{\Pi}_{11}(\tau_i) &\triangleq diag\{-\alpha_i P_i(\tau_i), -\alpha_i Q_i(\tau_i), -\gamma^2 I\},\\
\breve{\Pi}_{22}(\tau_i) &\triangleq diag\{-Q_i(\tau_i), -Q_i(\tau_i), -Q_i(\tau_i), -P_i(\tau_i), -Q_i(\tau_i), -I\},\\
\tilde{\Pi}_{21}(\varphi_i) &\triangleq \left[\, \tilde{\Pi}_{211}(\varphi_i)\ \tilde{\Pi}_{212}(\varphi_i)\ \tilde{\Pi}_{213}(\varphi_i) \,\right],\\
\breve{\Pi}_{21}(\tau_i) &\triangleq \left[\, \breve{\Pi}_{211}(\tau_i)\ \breve{\Pi}_{212}(\tau_i)\ \breve{\Pi}_{213}(\tau_i) \,\right],\\
\tilde{\Pi}_{211}(\varphi_i) &\triangleq \left[\, \bar{\delta}_i Q_i(\varphi_i+1)L_i(\varphi_i)F_i;\ \tilde{\delta}_i Q_i(\varphi_i+1)L_i(\varphi_i)F_i; \right.\\
&\quad \left. P_i(\varphi_i+1)\bar{A}_i(\varphi_i);\ 0;\ C_i \,\right],\\
\tilde{\Pi}_{212}(\varphi_i) &\triangleq \left[\, 0;\ 0;\ -P_i(\varphi_i+1)B_i K_i(\varphi_i);\ Q_i(\varphi_i+1)\tilde{A}_i(\varphi_i);\ 0 \,\right],\\
\tilde{\Pi}_{213}(\varphi_i) &\triangleq \left[\, 0;\ 0;\ P_i(\varphi_i+1)E_i;\ Q_i(\varphi_i+1)\bar{E}_i(\varphi_i);\ D_i \,\right],\\
\breve{\Pi}_{211}(\tau_i) &\triangleq \left[\, \bar{\delta}_i Q_i(\tau_i)L_i(\tau_i)F_i;\ \tilde{\delta}_i Q_i(\tau_i)L_i(\tau_i)F_i;\ P_i(\tau_i)\bar{A}_i(\tau_i);\ 0\ C_i \,\right],\\
\breve{\Pi}_{212}(\tau_i) &\triangleq \left[\, 0;\ 0;\ -P_i(\tau_i)B_i K_i(\tau_i);\ Q_i(\tau_i)\tilde{A}_i(\tau_i);\ 0 \,\right],\\
\breve{\Pi}_{213}(\tau_i) &\triangleq \left[\, 0;\ 0;\ P_i(\tau_i)E_i;\ Q_i(\tau_i)\bar{E}_i(\tau_i);\ D_i \,\right],
\end{aligned}$$

and for any $(\sigma(k_s) = i, \sigma(k_s - 1) = j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$P_i(0) - \mu_j P_j(\mathbb{T}_j) < 0 \tag{4.53}$$

$$P_i(0) - \mu_j P_j(\tau_j) < 0 \tag{4.54}$$

hold, where $\mathbb{T}_j \in \mathbb{Z}_{[1,\min(\tau_j - 1, \mathbb{T}^{(p)})]}$, $\mathbb{T}^{(p)} \in \mathbb{Z}_{[1,\mathbb{T}]}$.

Proof In terms of the QTD scheduler q_k, we construct the following QTD Lyapunov function

$$V_i(x(k), k) = V_{1,i}(x(k), k) + V_{2,i}(e(k), k) \tag{4.55}$$

where

$$V_{1,i}(x(k), k) \triangleq \begin{cases} x^T(k)P_i(k - k_{s_p})x(k), k \in [k_{s_p}, k_{s_p} + \tau_i) \\ x^T(k)P_i(\tau_i)x(k), k \in [k_{s_p} + \tau_i, k_{s_p+1}) \\ x^T(k)P_i(k - H_r)x(k), k \in [k_{s_p+1}, k_{s_{p+1}}) \end{cases} \tag{4.56}$$

$$V_{2,i}(e(k),k) \triangleq \begin{cases} e^T(k)Q_i(k-k_{s_p})e(k), k \in [k_{s_p}, k_{s_p}+\tau_i) \\ e^T(k)Q_i(\tau_i)e(k), k \in [k_{s_p}+\tau_i, k_{s_p+1}) \\ e^T(k)Q_i(k-H_r)e(k), k \in [k_{s_p+1}, k_{s_{p+1}}) \end{cases} \tag{4.57}$$

Then it follows that, for $k \in [k_{s_p}, k_{s_p}+\tau_i) \cup [k_{s_p+1}, k_{s_{p+1}})$,

$$\begin{aligned}
&\Delta V_{1,i}(x(k),k) \\
&\triangleq V_{1,i}(x(k+1),k+1) - \alpha_i V_{1,i}(x(k),k) \\
&= x^T(k+1)P_i(k+1-k_s)x(k+1) - \alpha_i x^T(k)P_i(k-k_s)x(k) \\
&= \left(\left[\bar{A}_i(q_k)\ -B_iK_i(q_k)\right]\eta(k)\right)^T P_i(k+1-k_s)\left(\left[\bar{A}_i(q_k)\ -B_iK_i(q_k)\right]\eta(k)\right) \\
&\quad -x^T(k)\alpha_i P_i(k-k_s)x(k), \\
&\quad \Delta V_{2,i}(e(k),k) \\
&\triangleq V_{2,i}(e(k+1),k+1) - \alpha_i V_{2,i}(e(k),k) \\
&= e^T(k+1)Q_i(k+1-k_s)e(k+1) - \alpha_i e^T(k)Q_i(k-k_s)e(k) \\
&= \left(\left[0\ \tilde{A}_i(q_k)\right]\eta(k)\right)^T Q_i(k+1-k_s)\left(\left[0\ \tilde{A}_i(q_k)\right]\eta(k)\right) \\
&\quad + \left(\left[L_i(q_k)F_i\ 0\right]\eta(k)\right)^T \bar{\delta}_i Q_i(k+1-k_s)\left(\left[L_i(q_k)F_i\ 0\right]\eta(k)\right) \\
&\quad + \left(\left[L_i(q_k)F_i\ 0\right]\eta(k)\right)^T \tilde{\delta}_i Q_i(k+1-k_s)\left(\left[L_i(q_k)F_i\ 0\right]\eta(k)\right) \\
&\quad -e^T(k)\alpha_i Q_i(k-k_s)e(k),
\end{aligned}$$

where $k_s = k_{s_p}$ in $[k_{s_p}, k_{s_p}+\tau_i)$, and $k_s = H_r$ in $[k_{s_p+1}, k_{s_{p+1}})$, respectively. Then, let $\varphi_i = 0, 1, \ldots, \tau_i - 1$, we have

$$\Delta V_i(x(k),k) = \eta^T(k)\Phi(\varphi_i)\eta(k)$$

where $\eta(k) \triangleq \left[x^T(k)\ e^T(k)\right]^T$, and

$$\begin{aligned}
\Phi(\varphi_i) = &\begin{bmatrix} \bar{A}_i(\varphi_i) & -B_iK_i(\varphi_i) \\ 0 & \tilde{A}_i(\varphi_i) \end{bmatrix}^T \begin{bmatrix} P_i(\varphi_i+1) & 0 \\ 0 & Q_i(\varphi_i+1) \end{bmatrix} \begin{bmatrix} \bar{A}_i(\varphi_i) & -B_iK_i(\varphi_i) \\ 0 & \tilde{A}_i(\varphi_i) \end{bmatrix} \\
&+ \begin{bmatrix} \bar{\delta}_i L_i(\varphi_i)F_i & 0 \\ \tilde{\delta}_i L_i(\varphi_i)F_i & 0 \end{bmatrix}^T \begin{bmatrix} Q_i(\varphi_i+1) & 0 \\ 0 & Q_i(\varphi_i+1) \end{bmatrix} \begin{bmatrix} \bar{\delta}_i L_i(\varphi_i)F_i & 0 \\ \tilde{\delta}_i L_i(\varphi_i)F_i & 0 \end{bmatrix} \\
&- \begin{bmatrix} \alpha_i P_i(\varphi_i) & 0 \\ 0 & \alpha_i Q_i(\varphi_i) \end{bmatrix}.
\end{aligned}$$

By Lemma 2.4, we have

$$\Phi(\varphi_i) \triangleq \begin{bmatrix} \Phi_{11}(\varphi_i) & \star \\ \Phi_{21}(\varphi_i) & \Phi_{22}(\varphi_i+1) \end{bmatrix}$$

where

$$\begin{aligned}\Phi_{22}(\varphi_i+1) &\triangleq diag\{-P_i^{-1}(\varphi_i+1), -Q_i^{-1}(\varphi_i+1), -Q_i^{-1}(\varphi_i+1),\\ &\quad -Q_i^{-1}(\varphi_i+1)\},\\ \Phi_{11}(\varphi_i) &\triangleq diag\{-\alpha_i P_i(\varphi_i), -\alpha_i Q_i(\varphi_i)\}, \Phi_{21}(\varphi_i) \triangleq \left[\, \Phi_{211}(\varphi_i)\ \Phi_{212}(\varphi_i) \,\right],\\ \Phi_{211}(\varphi_i) &\triangleq \left[\, \bar{A}_i(\varphi_i);\ 0;\ \bar{\delta}_i L_i(\varphi_i) F_i;\ \tilde{\delta}_i L_i(\varphi_i) F_i \,\right],\\ \Phi_{212}(\varphi_i) &\triangleq \left[\, -B_i K_i(\varphi_i);\ \tilde{A}_i(\varphi_i);\ 0;\ 0 \,\right]\end{aligned}$$

Now consider system (4.47)–(4.48) with $\omega(k) \equiv 0$. If $\Phi(\varphi_i) < 0$ holds, we can obtain that $\Delta V_i(x(k), k) < 0, \forall k \in [k_{s_p}, k_{s_p} + \tau_i) \cup [k_{s_p+1}, k_{s_{p+1}})$. Likewise, for $k \in [k_{s_p} + \tau_i, k_{s_p+1})$, (4.52) ensures $\Delta V_i(x(k), k) \equiv \Delta V_i(x(k), \tau_i) < 0$. By the proof of Theorem 1 in [6], it is not hard to find $\phi > 0$ and $\varrho \in (0, 1)$, such that

$$\mathbb{E}\{\|\eta(k)\|^2\} \le \phi \varrho^k \mathbb{E}\{\|\eta(0)\|^2\}, \text{ for all } \eta(0) \in \mathbb{R}^n, k \in \mathbb{Z}_+,$$

which guarantees that the system (4.47)–(4.48) is exponentially mean-square stable with $\omega(k) \equiv 0$.

To establish the H_∞ performance for system (4.47)–(4.48), assume the zero initial condition $V_i(x(k), k)|_{k=0} = 0$, we consider the following performance index

$$J(k) \triangleq z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k) \tag{4.58}$$

for any non-zero $\omega \in l_2[0, \infty)$ and $k > 0$. Because of the zero initial condition, it can be verified that

$$V_i(x_{k+1}, k+1) - (1-\alpha_i)V_i(x_k, k) + z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) = \xi^T(k)\tilde{\Pi}(\varphi_i)\xi(k)$$

where

$$\begin{aligned}\xi(k) &\triangleq \begin{bmatrix} \eta(k) \\ \omega(k) \end{bmatrix}, \quad \tilde{\Pi}(\varphi_i) \triangleq \begin{bmatrix} \Pi_{11}(\varphi_i) & \star \\ \Pi_{21}(\varphi_i) & \Pi_{22}(\varphi_i) \end{bmatrix},\\ \Pi_{11}(\varphi_i) &\triangleq \Phi(\varphi_i) + \left[\, C_i\ 0 \,\right]^T \left[\, C_i\ 0 \,\right],\\ \Pi_{211}(\varphi_i+1) &\triangleq diag\{P_i(\varphi_i+1), Q_i(\varphi_i+1)\},\end{aligned}$$

$$\Pi_{21}(\varphi_i) \triangleq \left[\, E_i^T\ \bar{E}_i^T(\varphi_i) \,\right] \Pi_{211}(\varphi_i+1) \begin{bmatrix} \bar{A}_i(\varphi_i) & -B_i K_i(\varphi_i) \\ 0 & \tilde{A}_i(\varphi_i) \end{bmatrix} + D_i^T \left[\, C_i\ 0 \,\right],$$

$$\Pi_{22}(\varphi_i) \triangleq \left[\, E_i^T\ \bar{E}_i^T(\varphi_i) \,\right] \Pi_{211}(\varphi_i+1) \begin{bmatrix} E_i \\ \bar{E}_i(\varphi_i) \end{bmatrix} + D_i^T D_i - \gamma^2 I.$$

If $\tilde{\Pi} < 0$, $J_k < 0$. Then for the pth stage of switching, it holds that

$$V_{\sigma(k_{s_{p+1}})}(x_{k_{s_{p+1}}}, k_{s_{p+1}}) - \mu_{\max}^{\mathcal{G}(k_{s_p}, k_{s_{p+1}})}(1-\alpha_{\min})^{k_{s_{p+1}}-k_{s_p}} V_{\sigma(k_{s_p})}(x_{k_{s_p}}, k_{s_p}) + \sum_{l=k_{s_p}}^{k_{s_{p+1}}-1} \mu_{\max}^{\mathcal{G}(l, k_{s_{p+1}})}(1-\alpha_{\min})^{k_{s_{p+1}}-l-1} J(k) \leq 0 \tag{4.59}$$

Iterating from $n \geq 2$, $n \in \mathbb{Z}$, it follows that

$$V_{\sigma(k_{s_n})}(x_{k_{s_n}}, k_{s_n}) - \mu_{\max}^{\mathcal{G}(k_{s_1}, k_{s_n})}(1-\alpha_{\min})^{k_{s_n}-k_{s_1}} V_{\sigma(k_{s_1})}(x_{k_{s_1}}, k_{s_1}) + \sum_{l=k_{s_1}}^{k_{s_n}-1} \mu^{\mathcal{G}(l, k_{s_n})}(1-\alpha_{\min})^{k_{s_n}-l-1} J(k) \leq 0 \tag{4.60}$$

Therefore, under zero initial condition, one has $V_{\sigma(k_{s_1})}(x_{k_{s_1}}, k_{s_1}) = 0$, then

$$\sum_{l=k_{s_1}}^{k_{s_n}-1} \mu^{\mathcal{G}(l, k_{s_n})}(1-\alpha_{\min})^{k_{s_n}-l-1} J(k) \leq 0 \tag{4.61}$$

which means that

$$\sum_{l=k_0}^{k-1} \mu^{\mathcal{G}(l,k)}(1-\alpha_{\min})^{k-l-1} z^T(l)z(l) \leq \gamma^2 \sum_{l=k_0}^{k-1} \mu^{\mathcal{G}(l,k)}(1-\alpha_{\min})^{k-l-1} \omega^T(l)\omega(l)$$

Similar to the derivations of Theorem 3.13, we have

$$\sum_{l=k_0}^{\infty} z^T(k)z(k) \leq \gamma_{non}^2 \sum_{l=k_0}^{\infty} \omega^T(l)\omega(l),$$

where $\gamma_{non} = \gamma\sqrt{\left(\mu_{\max}^{\frac{\mathbb{T}+1}{\tau_{\min}+\mathbb{T}}}(1-\alpha_{\min})\mu_{\max}^{(\mathbb{T}+1)}\right) \Big/ \left(1-\mu_{\max}^{\frac{\mathbb{T}+1}{\tau_{\min}+\mathbb{T}}}\alpha_{\min}\right)}$, which implies that $\|z(k)\|_2 \leq \gamma_{non}\|\omega(k)\|_2$ for any $k \in \mathbb{Z}^+$ and $\omega(k) \in l_2[0, \infty)$. □

Then, based on Theorem 4.18, the observer-based controller can be designed and its existence conditions are given in the following theorem.

Theorem 4.19 *Consider the discrete-time switched linear system (4.42)–(4.43) and let $\alpha_i > 0$, $\mu_i > 1$, $\gamma > 0$ be given constants, $i \in \mathcal{I}$. The observer-based QTD controller (4.45)–(4.46) can be designed if there exist positive definite matrices $P_{11,i}(\xi_i)$, $Q_i(\xi_i)$, $M_i(\xi_i)$, $N_i(\xi_i)$, such that*

$$\Upsilon(\tau_i) \triangleq \begin{bmatrix} \Upsilon_{11}(\tau_i) & \star \\ \Upsilon_{21}(\tau_i) & \Upsilon_{22}(\tau_i) \end{bmatrix} < 0 \tag{4.62}$$

$$\tilde{\Upsilon}(\varphi_i) \triangleq \begin{bmatrix} \tilde{\Upsilon}_{11}(\varphi_i) & \star \\ \tilde{\Upsilon}_{21}(\varphi_i) & \tilde{\Upsilon}_{22}(\varphi_i+1) \end{bmatrix} < 0 \tag{4.63}$$

hold, where

$$\begin{aligned}
\tilde{\Upsilon}_{11}(\varphi_i) &\triangleq diag\{-\alpha_i P_i(\varphi_i), -\alpha_i Q_i(\varphi_i), -\gamma^2 I\}, \\
\Upsilon_{22}(\tau_i) &\triangleq diag\{-Q_i(\tau_i), -Q_i(\tau_i), -P_i(\tau_i), -Q_i(\tau_i), -I\}, \\
\Upsilon_{7,2}(\tau_i) &\triangleq Q_i(\tau_i)A_i - \delta_i N_i(\tau_i)F_i - (1-\delta_i)\beta_i N_i(\tau_i)F_i, \\
\tilde{\Upsilon}_{7,2}(\varphi_i+1) &\triangleq Q_i(\varphi_i+1)A_i - \delta_i N_i(\varphi_i+1)F_i - (1-\delta_i)\beta_i N_i(\varphi_i+1)F_i, \\
\Upsilon_{11}(\tau_i) &\triangleq diag\{-\alpha_i P_i(\tau_i), -\alpha_i Q_i(\tau_i), -\gamma^2 I\}, \\
\tilde{\Upsilon}_{7,3}(\varphi_i+1) &\triangleq Q_i(\varphi_i+1)E_i - N_i(\varphi_i+1)G_i, \\
\tilde{\Upsilon}_{21}(\varphi_i) &\triangleq \left[\, \tilde{\Upsilon}_{211}(\varphi_i)\ \tilde{\Upsilon}_{212}(\varphi_i)\ \tilde{\Upsilon}_{213}(\varphi_i) \,\right], \\
\Upsilon_{21}(\tau_i) &\triangleq \left[\, \Upsilon_{211}(\tau_i)\ \Upsilon_{212}(\tau_i)\ \Upsilon_{213}(\tau_i) \,\right], \\
\tilde{\Upsilon}_{211}(\varphi_i) &\triangleq \left[\, \tilde{\Upsilon}_{4,1}(\varphi_i+1);\ \tilde{\Upsilon}_{5,1}(\varphi_i+1);\ \tilde{\Upsilon}_{6,1}(\varphi_i+1);\ 0;\ C_i \,\right], \\
\tilde{\Upsilon}_{212}(\varphi_i) &\triangleq \left[\, 0;\ 0;\ \tilde{\Upsilon}_{6,2}(\varphi_i+1);\ \tilde{\Upsilon}_{7,2}(\varphi_i+1);\ 0 \,\right], \\
\tilde{\Upsilon}_{213}(\varphi_i) &\triangleq \left[\, 0;\ 0;\ \tilde{\Upsilon}_{6,3}(\varphi_i+1);\ \tilde{\Upsilon}_{7,3}(\varphi_i+1);\ D_i \,\right], \\
\Upsilon_{211}(\tau_i) &\triangleq \left[\, \bar{\delta}_i N_i(\tau_i)F_{1,i};\ \tilde{\delta}_i N_i(\tau_i)F_{2,\rho,i};\ \Upsilon_{6,1}(\tau_i);\ 0;\ C_i \,\right], \\
\Upsilon_{212}(\tau_i) &\triangleq \left[\, 0;\ 0;\ -B_i M_i(\tau_i);\ \Upsilon_{7,2}(\tau_i);\ 0 \,\right], \\
\Upsilon_{213}(\tau_i) &\triangleq \left[\, 0;\ 0;\ P_i(\tau_i)E_i;\ \Upsilon_{7,3}(\tau_i);\ D_i \,\right], \\
\tilde{\Upsilon}_{4,1}(\varphi_i+1) &\triangleq \bar{\delta}_i N_i(\varphi_i+1)F_i, \tilde{\Upsilon}_{5,1}(\varphi_i+1) \triangleq \tilde{\delta}_i N_i(\varphi_i+1)F_i, \\
\Upsilon_{7,3}(\tau_i) &\triangleq Q_i(\tau_i)E_i - N_i(\tau_i)G_i, \tilde{\Upsilon}_{6,2}(\varphi_i+1) \triangleq -B_i M_i(\varphi_i+1), \\
\Upsilon_{6,1}(\tau_i) &\triangleq P_i(\tau_i)A_i + B_i M_i(\tau_i), \tilde{\Upsilon}_{6,3}(\varphi_i+1) \triangleq P_i(\varphi_i+1)E_i, \\
\tilde{\Upsilon}_{6,1}(\varphi_i+1) &\triangleq P_i(\varphi_i+1)A_i + B_i M_i(\varphi_i+1), \\
\tilde{\Upsilon}_{22}(\varphi_i+1) &\triangleq diag\{-Q_i(\varphi_i+1), -Q_i(\varphi_i+1), -P_i(\varphi_i+1), -Q_i(\varphi_i+1), -I\},
\end{aligned}$$

where $\varphi_i = 0, 1, \ldots, \tau_i - 1$, $\forall i \in \mathcal{I}$, *and for any* ($\sigma(k_s) = i, \sigma(k_s - 1) = j$)$\in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$P_i(0) - \mu_j P_j(\mathbb{T}_j) < 0 \tag{4.64}$$

$$P_i(0) - \mu_j P_j(\tau_j) < 0 \tag{4.65}$$

hold, where $\mathbb{T}_j \in \mathbb{Z}_{[1,\min(\tau_j-1,\mathbb{T}^{(p)})]}$, $\mathbb{T}^{(p)} \in \mathbb{Z}_{[1,\mathbb{T}]}$. *Then, the resulting closed-loop system (4.47)–(4.48) is exponentially mean-square stable and has an* H_∞ *performance index* $\gamma_l = \gamma\epsilon$ *for MPDT switching signals satisfying (4.50), where* $\epsilon \triangleq$

$\sqrt{\left(\mu_{\max}^{\frac{\mathbb{T}+1}{\tau_{\min}+\mathbb{T}}}(1-\alpha_{\min})\mu_{\max}^{(\mathbb{T}+1)}\right) \Big/ \left(1-\mu_{\max}^{\frac{\mathbb{T}+1}{\tau_{\min}+\mathbb{T}}}\alpha_{\min}\right)}$. *Moreover, if a feasible solution exists, both the QTD controller gains and the QTD observer gains can be obtained by*

$$L_i(\varphi_i) = Q_i^{-1}(\varphi_i+1)N_i(\varphi_i+1),\ K_i(\varphi_i) = \bar{P}_i^{-1}(\varphi_i+1)M_i(\varphi_i+1), \tag{4.66}$$

where $\bar{P}_i(\varphi_i+1) = (V_i^T(\varphi_i+1))^{-1}\Sigma_i^{-1}(\varphi_i+1)P_{11,i}(\varphi_i+1)\Sigma_i^{-1}(\varphi_i+1)V_i^T(\varphi_i+1)$.

Proof Since there exist $P_{11i}(\varphi_i+1) > 0$ and $P_{22i}(\varphi_i+1) > 0$, such that

$$\begin{aligned} P_i(\varphi_i+1) = {} & U_{1i}^T(\varphi_i+1)P_{11i}(\varphi_i+1)U_{1i}(\varphi_i+1) \\ & + U_{2i}^T(\varphi_i+1)P_{22i}(\varphi_i+1)U_{2i}(\varphi_i+1), \end{aligned}$$

where $U_{1i}(\varphi_i+1)$ and $U_{2i}(\varphi_i+1)$ are defined in (4.49), it follows from Lemma 4.17 that there exists a nonsingular matrix $\bar{P}_i(\varphi_i+1)$ such that

$$P_i(\varphi_i+1)B_i = B_i\bar{P}_i(\varphi_i+1). \tag{4.67}$$

Then, let us calculate such a matrix $\bar{P}_i(\varphi_i+1)$ from the relation (4.67) as follows

$$\begin{aligned} & P_i(\varphi_i+1)U_i^T(\varphi_i+1)\begin{bmatrix} \Sigma_i(\varphi_i+1) \\ 0 \end{bmatrix} V_i^T(\varphi_i+1) \\ & = U_i^T(\varphi_i+1)\begin{bmatrix} \Sigma_i(\varphi_i+1) \\ 0 \end{bmatrix} V_i^T(\varphi_i+1)\bar{P}_i(\varphi_i+1) \\ & \Rightarrow U_i^T(\varphi_i+1)\begin{bmatrix} P_{11i}(\varphi_i+1) & 0 \\ 0 & P_{22i}(\varphi_i+1) \end{bmatrix}\begin{bmatrix} \Sigma_i(\varphi_i+1) \\ 0 \end{bmatrix} V_i^T(\varphi_i+1) \\ & = U_i^T(\varphi_i+1)\begin{bmatrix} \Sigma_i(\varphi_i+1) \\ 0 \end{bmatrix} V_i^T(\varphi_i+1)\bar{P}_i(\varphi_i+1), \end{aligned}$$

which implies that

$$\begin{aligned} \bar{P}_i(\varphi_i+1) = {} & (V_i^T(\varphi_i+1))^{-1}\Sigma_i^{-1}(\varphi_i+1)P_{11,i}(\varphi_i+1) \\ & \times \Sigma_i^{-1}(\varphi_i+1)V_i^T(\varphi_i+1). \end{aligned} \tag{4.68}$$

Therefore, we can conclude from (4.66) and (4.68) that

$$\begin{aligned} & P_i(\varphi_i+1)B_i = B_i\bar{P}_i(\varphi_i+1),\ N_i(\varphi_i+1) = Q_i(\varphi_i+1)L_i(\varphi_i), \\ & M_i(\varphi_i+1) = \bar{P}_i(\varphi_i+1)K_i(\varphi_i), \end{aligned}$$

then, it could be seen that (4.63) is equivalent to (4.51). Similarly, it could be directly obtained that (4.62) is equivalent to (4.52). □

Remark 4.20 In Theorem 4.19, the parameters α_i and μ_i are considered to be variables since a small $\tau_i, \forall i \in \mathcal{I}$ corresponding to fast switching may not guarantee the existence of a feasible solution with admissible MPDT. Hence, the MPDT can be minimized by solving the following minimization problem

Problem 4.2

$$\begin{aligned}&\min_{\alpha_i,\mu_i} \ \ \tau_i\\ &\text{s.t. } \alpha_i > 0, \mu_i > 1, \ (4.50), (4.62) - (4.65)\end{aligned}$$

In addition, the minimum of τ_i can be trivially found by bisection method.

It is worth pointing out that the constructed Lyapunov function (4.55) containing the QTD scheduler yields the controller design with less conservatism. For the sake of a comparison, the time-independent controller is designed based on the corresponding "q_k-independent" Lyapunov function in the following corollary.

Corollary 4.21 *Consider the discrete-time switched linear system (4.42)–(4.43) and let $\alpha_i > 0$, $\mu_i > 1$, $\gamma > 0$ be given constants, $i \in \mathcal{I}$. For a prescribed period of persistence $\mathbb{T}$, if there exist matrices $P_i \in S^n_{>0}$, $Q_i \in U^n_{>0}$, M_i and N_i, $\forall i \in \mathcal{I}$, such that*

$$\Upsilon \triangleq \begin{bmatrix} \Upsilon_{11i} & \star \\ \Upsilon_{21i} & \Upsilon_{22i} \end{bmatrix} < 0$$

and for any ($\sigma(k_s) = i, \sigma(k_s - 1) = j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,

$$P_i - \mu_j P_j < 0 \tag{4.69}$$

hold, where

$$\begin{aligned}
&\Upsilon_{11i} \triangleq diag\{-\alpha_i P_i, -\alpha_i Q_i, -\gamma^2 I\}, \quad \bar{\Upsilon}_{4,1} \triangleq \bar{\delta}_i N_i F_i, \quad \bar{\Upsilon}_{5,1} \triangleq \tilde{\delta}_i N_i F_i,\\
&\bar{\Upsilon}_{6,2} \triangleq -B_i M_i, \quad \bar{\Upsilon}_{6,3} \triangleq P_i E_i, \quad \Upsilon_{22i} \triangleq diag\{-Q_i, -Q_i, -P_i, -Q_i, -I\},\\
&\bar{\Upsilon}_{7,2} \triangleq Q_i A_i - \delta_i N_i F_i - (1-\delta_i)\beta_i N_i F_i, \quad \bar{\Upsilon}_{7,3} \triangleq Q_i E_i - N_i G_i,\\
&\bar{\Upsilon}_{6,1} \triangleq P_i A_i + B_i M_i, \quad \Upsilon_{21i} \triangleq \left[\, \Upsilon_{211i} \ \Upsilon_{212i} \ \Upsilon_{213i} \,\right],\\
&\Upsilon_{211i} \triangleq \left[\, \bar{\Upsilon}_{4,1};\ \bar{\Upsilon}_{5,1};\ \bar{\Upsilon}_{6,1};\ 0;\ C_i \,\right], \quad \Upsilon_{212i} \triangleq \left[0;\ 0;\ \bar{\Upsilon}_{6,2};\ \bar{\Upsilon}_{7,2};\ 0\right],\\
&\Upsilon_{213i} \triangleq \left[0;\ 0;\ \bar{\Upsilon}_{6,3};\ \bar{\Upsilon}_{7,3};\ D_i \,\right].
\end{aligned}$$

Then, the resulting closed-loop system (4.47)–(4.48) is exponentially mean-square stable and has a guaranteed non-weighted H_∞ performance index $\gamma_c = \gamma\epsilon$, where

$$\epsilon \triangleq \sqrt{\left(\mu_{\max}^{\frac{\mathbb{T}+1}{\eta_{\min}+\mathbb{T}}}(1-\alpha_{\min})\mu_{\max}^{(\mathbb{T}+1)}\right) \Big/ \left(1-\mu_{\max}^{\frac{\mathbb{T}+1}{\eta_{\min}+\mathbb{T}}}\alpha_{\min}\right)},$$

for MPDT switching signals satisfying (4.50), where $\eta_{\min} \triangleq \min_{i \in \mathcal{L}} \eta_i$, η_i is the minimal MPDT. Moreover, if a feasible solution exists, the mode-dependent controller *gains can be obtained by* $L_i = Q_i^{-1} N_i$, $K_i = \bar{P}_i^{-1} M_i$ *with* $\bar{P}_i = (V_i^T)^{-1} \Sigma_i^{-1} P_{11,i} \Sigma_i^{-1} V_i^T$.

Proof The proof can be obtained in a similar vein to the one for Theorem 4.19 and omitted here. □

Remark 4.22 Note that the designed QTD controllers and observers in Theorem 4.19 contain the ones in Corollary 4.21 as a special case, which implies that the methodology for the QTD controller and observer design offers more freedom and will be consequently less conservative than the case used in Corollary 4.21. That is to say, a class of QTD controllers and observers exists but the time-dependent ones may not; in addition, even though the two different controllers and observers exist simultaneously, the H_∞ performance achieved by the former one can be better, which will be verified as below.

Here we will present two examples to demonstrate the validity of the QTD controller and observer design approach in the presence of MPDT switching and random packet dropouts in the redundant channel case. The first numerical example is used to show the effectiveness of the designed QTD controllers and observers, and the second one is a PWM-driven DC–DC boost converter that is used to illustrate the applicability of the theoretical results.

Example 4.23 Consider the following discrete-time switched linear system (4.42)–(4.43) consisting of two subsystems given by

$$
\begin{aligned}
&A_1 = \begin{bmatrix} 1.6 & -0.8 \\ 0.8 & -0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 1.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \\
&B_2 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ 1.4 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \\
&F_1 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, G_1 = 0.1, G_2 = 0.1, \\
&C_1 = \begin{bmatrix} 0.02 & 0.45 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1 & 0.04 \end{bmatrix}, \quad D_1 = 0.1, D_2 = 0.1.
\end{aligned}
$$

Our purpose here is to design a QTD observer-based controller and to find out the admissible switching signals with minimal MPDT such that the resulting closed-loop system is exponentially mean-square stable with a guaranteed non-weighted H_∞ disturbance attenuation performance. Consider that the conditional probabilities representing the occurrences of packet dropouts in two different channels are taken as $\delta_1 = 0.6$, $\delta_2 = 0.8$, $\beta_1 = 0.4$, $\beta_2 = 0.5$, respectively, and designate $\alpha_1 = 0.105$, $\alpha_2 = 0.085$, $\mu_1 = \mu_2 = 1.165$. The minimization procedure given in Problem 4.2 can be realized in cases of $\mathbb{T} = 1, 2, 3$ by solving LMIs in Theorem 4.19 and Corollary 4.21, respectively, and the corresponding MPDT and optimal performance indices are listed in Table 4.6. From Table 4.6 , the results by Theorem 4.19 are less conservatism than the ones by Corollary 4.21. Meanwhile, the desired controller and observer gains can be solved accordingly, for instance, the controller gain matrices are given as follows for the case of $\mathbb{T} = 1$ by Theorem 4.19. Moreover, from

Table 4.6 The minimal MPDT and performance index obtained by different criteria

$\mathbb{T}$	Theorem 4.19		Corollary 4.21	
	(τ_1^*, τ_2^*)	γ_t^*	(η_1^*, η_2^*)	γ_c^*
3	(4, 4)	3.3247	(5, 6)	5.8367
2	(3, 4)	3.0877	(4, 5)	5.3282
1	(2, 3)	2.8768	(3, 4)	4.8639

Table 4.7 The optimal H_∞ error performance index γ_l^* for different period of persistence $\mathbb{T}$

Cases	Redundant channel	Single channel
$\mathbb{T} = 3$	3.3247	7.2582
$\mathbb{T} = 2$	3.0877	6.7408
$\mathbb{T} = 1$	2.8768	6.2804

Table 4.7, it can be observed that the values of γ_l through the redundant channels approach are better than the corresponding ones in single channel case, which shows that the inclusion of one redundant channel alleviates the effect of packet dropouts even inactivation of the primary channel.

$$\begin{aligned}
&K_1(0) = \begin{bmatrix} 2.7563 & 0.7782 \end{bmatrix} \quad K_1(1) = \begin{bmatrix} 2.1170 & 0.8891 \end{bmatrix}\\
&K_1(2) = \begin{bmatrix} 2.1345 & 0.8902 \end{bmatrix} \quad K_2(0) = \begin{bmatrix} 0.7741 & -1.7743 \end{bmatrix}\\
&K_2(1) = \begin{bmatrix} -0.1173 & -0.7406 \end{bmatrix} \quad K_2(2) = \begin{bmatrix} -0.1635 & -0.8142 \end{bmatrix}\\
&K_2(3) = \begin{bmatrix} -0.1548 & -0.8031 \end{bmatrix}\\
&L_1(0) = \begin{bmatrix} 0.5462 \\ 1.2720 \end{bmatrix} \quad L_1(1) = \begin{bmatrix} 3.6319 \\ -0.0095 \end{bmatrix} \quad L_1(2) = \begin{bmatrix} 3.6213 \\ -0.0112 \end{bmatrix}\\
&L_2(0) = \begin{bmatrix} -0.0928 \\ 1.1433 \end{bmatrix} \quad L_2(1) = \begin{bmatrix} 0.6944 \\ 4.1948 \end{bmatrix} \quad L_2(2) = \begin{bmatrix} 0.7977 \\ 5.0922 \end{bmatrix}\\
&L_2(3) = \begin{bmatrix} 0.8011 \\ 5.1046 \end{bmatrix}.
\end{aligned}$$

Further, given the initial condition $x_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$ and the disturbance signal $\omega(k) = \exp(-0.5k)$, consider the running time equivalent to the MPDT at each time of switching, and suppose that there exists a period of persistence before the first MPDT stage, the resulting switching signals and the state response of the corresponding closed-loop system under $\mathbb{T} = 3$ and $\mathbb{T} = 2$ are presented in Figs. 4.8 and 4.9, respectively, for one realization of the disturbance sequence.

Example 4.24 In this example, a PWM (Pulse-Width-Modulation)-driven DC–DC boost converter as shown in Fig. 4.10 can be alternatively modeled as switched systems and will be used to verify the effectiveness of proposed control approaches. In Fig. 4.10, the switch $S(k)$ is controlled by a PWM device and can switch at most once in each period T_s. Here, R is the load resistance, C the capacitance, D the diode, L the inductance, and V_{in} the source voltage, and u_c the output voltage, i.e., $y = u_c$. There are two stable variables, the capacitor voltage u_c and the inductor current i_L, if

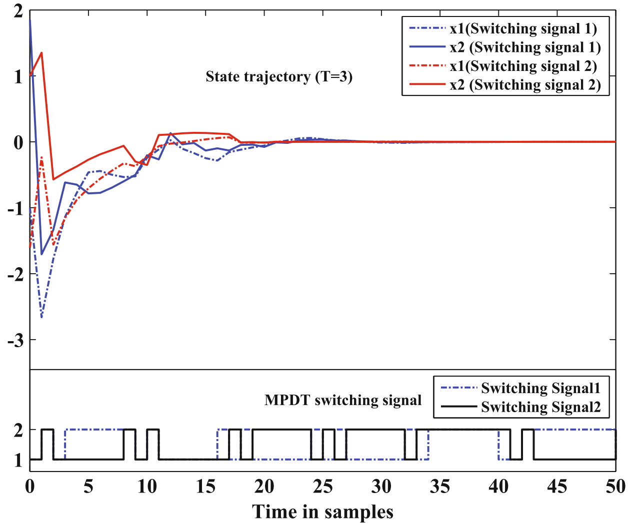


Fig. 4.8 State trajectories of the closed-loop system with two different MPDT switching signals ($\mathbb{T} = 3$)

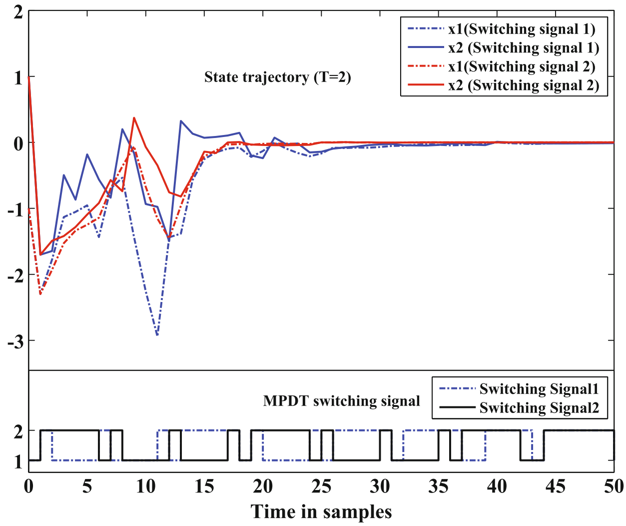


Fig. 4.9 State trajectories of the closed-loop system with two different MPDT switching signals ($\mathbb{T} = 2$)

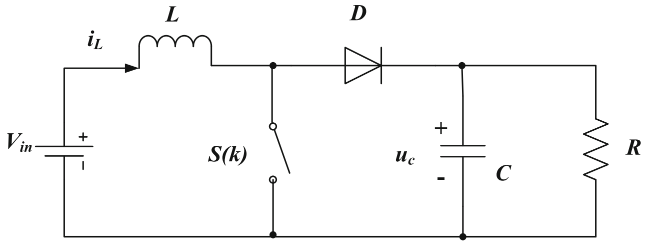


Fig. 4.10 DC–DC boost converter

we choose the state vector as $x = \begin{bmatrix} i_L & u_c \end{bmatrix}^T$, the system dynamics can be expressed in continuous-time switched form as follows,

$$\dot{x} = \begin{cases} A_1 x + B_1 u & \text{ON} \\ A_2 x + B_2 u & \text{OFF} \end{cases} \quad \text{and} \quad y = \begin{cases} C_1 x & \text{ON} \\ C_2 x & \text{OFF} \end{cases}$$

where

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = C_1.$$

Given $V_{in} = 5\,\text{V}$, $L = 500\,\mu\text{H}$, $C = 4.4\,\mu\text{F}$, $R = 28\,\Omega$. By setting the sampling time $T_s = 20\,\mu\text{s}$ and considering that the disturbance input exists in the underlying system, one can obtain that

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.8502 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.9151 & -0.0358 \\ 4.0697 & 0.7698 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.0400 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.0388 \\ 0.0849 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = C_1, \quad G_1 = G_2 = 0.1.$$

Suppose other system parameters to be

$$E_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}, \quad F_1 = F_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}^T, \quad D_1 = D_2 = 0.$$

In this example, the conditional probabilities representing the occurrences of packet dropouts in two different channels are taken as $\delta_1 = 0.6$, $\delta_2 = 0.8$, $\beta_1 = 0.7$, $\beta_2 = 0.6$, designating $\alpha_1 = 0.105$, $\alpha_2 = 0.085$, $\mu_1 = \mu_2 = 1.165$, and $\mathbb{T} = 1$, by Theorem 4.19 and solving Problem 4.2, the optimum (μ_i^*, α_i^*) of (μ_i, α_i), $i = 1, 2$ can

Table 4.8 The minimal MPDT and performance index obtained by different criteria

$\mathbb{T}$	Theorem 4.19		Corollary 4.21	
	(τ_1^*, τ_2^*)	γ_t^*	(η_1^*, η_2^*)	γ_c^*
3	(4, 4)	5.0689	(4, 5)	5.6571
2	(3, 4)	4.7035	(4, 4)	5.1504
1	(2, 3)	4.3820	(3, 3)	4.7214

Table 4.9 The optimal H_∞ error performance index γ_l^* for different period of persistence $\mathbb{T}$

Cases	Redundant channel	Single channel
$\mathbb{T} = 3$	5.0689	6.1502
$\mathbb{T} = 2$	4.7035	5.7083
$\mathbb{T} = 1$	4.3820	5.3183

be obtained and the feasible controller and observer gains associated with (μ_i^*,α_i^*), $i = 1, 2$ can be solved as

$$
\begin{aligned}
&K_1(0) = \begin{bmatrix} -12.8317 & -0.3729 \end{bmatrix}, \quad K_1(1) = \begin{bmatrix} -5.7334 & -0.5972 \end{bmatrix},\\
&K_1(2) = \begin{bmatrix} -5.7326 & -0.6053 \end{bmatrix}, \quad K_2(0) = \begin{bmatrix} -5.2793 & -0.9156 \end{bmatrix},\\
&K_2(1) = \begin{bmatrix} -0.5801 & -0.2095 \end{bmatrix}, \quad K_2(2) = \begin{bmatrix} -0.2966 & -0.1686 \end{bmatrix},\\
&K_2(3) = \begin{bmatrix} -0.3025 & -0.1712 \end{bmatrix};\\
&L_1(0) = \begin{bmatrix} 0.2189 \\ 0.2308 \end{bmatrix}, \quad L_1(1) = \begin{bmatrix} 0.9016 \\ 0.6432 \end{bmatrix}, \quad L_1(2) = \begin{bmatrix} 0.9286 \\ 0.6158 \end{bmatrix},\\
&L_2(0) = \begin{bmatrix} 0.9868 \\ 7.0469 \end{bmatrix}, \quad L_2(1) = \begin{bmatrix} 2.0852 \\ 16.2775 \end{bmatrix}, \quad L_2(2) = \begin{bmatrix} 1.9062 \\ 15.6429 \end{bmatrix},\\
&L_2(3) = \begin{bmatrix} 1.8056 \\ 15.4325 \end{bmatrix}.
\end{aligned}
$$

Then, for different $\mathbb{T}$, the minimal MPDT can be computed and three cases are listed in Table 4.8, and the similar observations can be shown in Table 4.9 as discussed about the Table 4.7 in Example 4.23. Consider zero initial condition and the disturbance input $\omega(k) = \exp(-0.5k)$, the reference state $x_f = \begin{bmatrix} 0.2 & 10 \end{bmatrix}^T$. Then by generating one admissible switching sequence, where the running time of subsystems are equivalent to the MPDT and a period of persistence exists before the first MPDT phase, the state trajectories of the closed-loop systems of DC–DC boost converter and the corresponding switching signal under $\mathbb{T} = 3$, and $\mathbb{T} = 2$ are shown in Figs. 4.11 and 4.12, respectively.

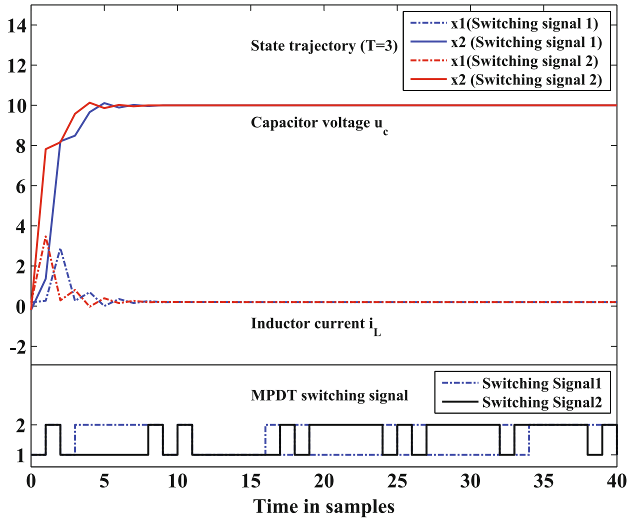


Fig. 4.11 State trajectories of DC–DC boost converter with two different MPDT switching signals. ($\mathbb{T} = 3$)

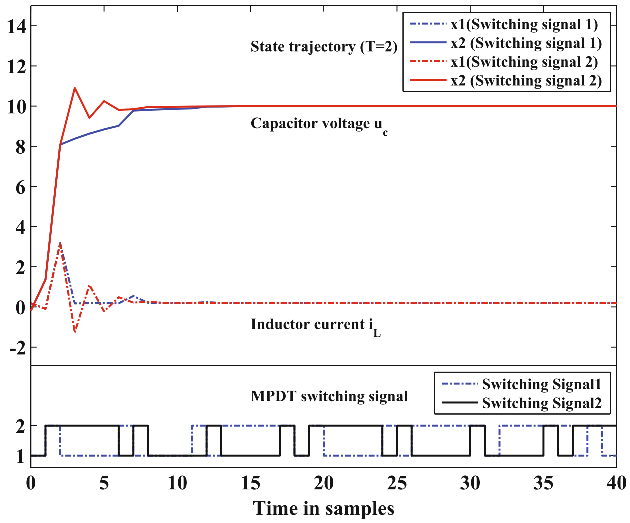


Fig. 4.12 State trajectories of DC–DC boost converter with two different MPDT switching signals. ($\mathbb{T} = 2$)

4.4 Conclusion

In this chapter, we have dealt with the H_∞ control problem for discrete-time switched systems with different types of switching signals. Considering the arbitrary switching, both robust H_∞ controller and parameter-dependent H_∞ controller have been designed respectively for switched linear discrete-time systems with polytopic uncertainties; the controller gains can be obtained by solving convex optimization problems subject to a set of LMIs. Moreover, the optimal H_∞ noise-attenuation level bounds of corresponding closed-loop system are computed as well. Then, an μ-dependent approach has been introduced to the control synthesis for the underlying systems with ADT switching, and the minimal μ and the corresponding controller gains can be obtained from such conditions for a given system decay degree. Finally, the observer-based H_∞ control has been investigated for the MPDT switched linear systems, and the redundant channels design approach has been adopted in the measurement output to reduce the influence of random occurring packet dropouts. A QTD observer-based controller has been designed such that the resulting closed-loop system is exponentially mean-square stable with a guaranteed non-weighted H_∞ disturbance attenuation performance.

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Chapter 5
Filtering

Abstract This chapter first studies the problem of robust H_∞ filtering for switched linear discrete-time systems with arbitrary switching and polytopic uncertainties. Based on the mode-dependent idea and parameter-dependent stability result, a robust switched linear filter is designed such that the corresponding filtering error system achieves robust asymptotic stability and guarantees a prescribed H_∞ performance index for all admissible uncertainties. The existence condition of such filters is derived and formulated in terms of a set of linear matrix inequalities (LMIs) by the introduction of slack variables to eliminate the cross coupling of system matrices and Lyapunov matrices among different subsystems. Then, an μ-dependent approach proposed in Chap. 2 is used to investigate the exponential H_∞ filtering problem for discrete-time uncertain switched systems with average dwell time (ADT) switching, and a mode-dependent full-order filter is designed to guarantee that the resulting filtering error system is robustly exponentially stable and has an exponential H_∞ performance. Moreover, a class of discrete-time switched linear parameter varying (LPV) systems under ADT switching is considered to investigate the H_∞ filtering problem, and a mode-dependent full-order parameterised filter is then designed and the corresponding existence conditions of such filters are derived via LMIs formulation. Finally, the non-weighted H_∞ filtering problem is studied for a class of switched linear systems with persistent dwell-time (PDT) switching in discrete-time domain. A proper Lyapunov function suitable to the PDT switching is constructed, which is not only mode-dependent but also quasi-time-dependent (QTD). Then, a QTD filter is designed such that the resulting filtering error system is globally uniformly asymptotically stable and has a guaranteed H_∞ noise attenuation performance. Several examples are illustrated to show the validity of the obtained theoretical results.

5.1 Robust H_∞ Filtering: Arbitrary Switching

Consider a class of uncertain switched linear discrete-time systems given by

$$x(k+1) = A_{\sigma(k)}(\lambda)x(k) + B_{\sigma(k)}(\lambda)\omega(k) \tag{5.1}$$

$$y(k) = C_{\sigma(k)}(\lambda)x(k) + D_{\sigma(k)}(\lambda)\omega(k) \tag{5.2}$$

$$z(k) = H_{\sigma(k)}(\lambda)x(k) + L_{\sigma(k)}(\lambda)\omega(k) \tag{5.3}$$

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where $x(k) \in \mathbb{R}^n$ is the state vector, $\omega(k) \in \mathbb{R}^l$ is the disturbance input which belongs to $l_2[0, +\infty)$, $y(k) \in \mathbb{R}^m$ is the measurement, $z(k)$ is the objective signal to be attenuated, $\sigma(k)$ is the switching signal, which is a piecewise constant function of time, and takes its values in the finite set $\mathcal{I} = \{1, \ldots, N\}$, $N > 1$ is the number of subsystems. When $\sigma(k) = i \in \mathcal{I}$, the matrices $(A_i(\lambda), B_i(\lambda), C_i(\lambda), D_i(\lambda), H_i(\lambda), L_i(\lambda))$ denote the ith subsystem. As in [1], we assume that the switching signal i is unknown a priori, but its instantaneous value is available in real time.

The matrices of each subsystem have appropriate dimensions with partially unknown parameters. It is assumed that $(A_i(\lambda), B_i(\lambda), C_i(\lambda), D_i(\lambda), H_i(\lambda), L_i(\lambda)) \in \Re_i$, where $\Re_i$ is a given convex bounded polyhedral domain described by s vertices in the ith subsystem.

$$\begin{aligned}\Re_i \triangleq \{&[(A_i(\lambda), B_i(\lambda), C_i(\lambda), D_i(\lambda), H_i(\lambda), L_i(\lambda))] \\ &= \sum_{m=1}^{s} \lambda_m [A_{i,m}, B_{i,m}, C_{i,m}, D_{i,m}, H_{i,m}, L_{i,m}]; \sum_{m=1}^{s} \lambda_m = 1, \\ &\lambda_m \geq 0, i \in \mathcal{I}\}\end{aligned} \tag{5.4}$$

Without loss of generality, the number of vertices in each subsystem is assumed to be equal here. Our objective in this section is to design a filter being the form:

$$x_f(k+1) = A_{fi} x_f(k) + B_{fi} y(k) \tag{5.5}$$

$$z_f(k) = C_{fi} x_f(k) + D_{fi} y(k) \tag{5.6}$$

In the filter with the above structure, the switching signal i is also assumed unknown a priori but available in real-time with the switching signal in system (5.1)–(5.3).

Augmenting the model of (5.1)–(5.3) to include the states of the filter, denoting $\tilde{x}(k) \triangleq \left[\, x^T(k)\ x_f^T(k) \,\right]^T$, $e(k) \triangleq z(k) - z_f(k)$, we obtain the filtering error system:

$$\tilde{x}(k+1) = \tilde{A}_i(\lambda)\tilde{x}(k) + \tilde{B}_i(\lambda) w(k) \tag{5.7}$$

$$e(k) = \tilde{C}_i(\lambda)\tilde{x}(k) + \tilde{D}_i(\lambda) w(k) \tag{5.8}$$

where,

$$\tilde{A}_i(\lambda) \triangleq \begin{bmatrix} A_i(\lambda) & 0 \\ B_{fi} C_i(\lambda) & A_{fi} \end{bmatrix}, \tilde{B}_i(\lambda) \triangleq \begin{bmatrix} B_i(\lambda) \\ B_{fi} D_i(\lambda) \end{bmatrix},$$
$$\tilde{C}_i(\lambda) \triangleq \left[\, H_i(\lambda) - D_{fi} C_i(\lambda)\ \ -C_{fi} \,\right], \tilde{D}_i(\lambda) \triangleq L_i(\lambda) - D_{fi} D_i(\lambda).$$

Then, the robust H_∞ filtering problem addressed in this section can be formulated as follows: given uncertain switched system (5.1)–(5.3) and a prescribed level of noise attenuation $\gamma > 0$, determine a robust switched linear filter (5.5)–(5.6) such that the filtering error system is robustly asymptotically stable and

$$\|e\|_2 < \gamma \|w\|_2 \tag{5.9}$$

under zero-initial conditions for any nonzero $w \in l_2[0, +\infty)$ and all admissible uncertainties.

In this section, a sufficient condition for the existence of robust H_∞ filter for uncertain switched system (5.1)–(5.3) will be formulated in terms of a set of LMIs. The following lemma is first presented which will be used in the sequel.

Lemma 5.1 *Consider the uncertain switched system (5.1)–(5.3) and let $\gamma > 0$ be a given scalar. If there exist matrix functions $\mathcal{P}_i(\lambda) > 0$ and matrices $\mathcal{R}_i(\lambda)$ satisfying*

$$\Phi^{i,j}(\lambda) \triangleq \begin{bmatrix} \mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda) & 0 & \mathcal{R}_i(\lambda)A_i(\lambda) & \mathcal{R}_i(\lambda)B_i(\lambda) \\ \star & -I & C_i(\lambda) & D_i(\lambda) \\ \star & \star & -\mathcal{P}_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{5.10}$$

then, system (5.1)–(5.3) is robustly asymptotically stable with an H_∞ noise-attenuation level bound γ.

Proof Construct a Lyapunov functional as

$$\mathcal{V}(k, x(k)) \triangleq x^T(k)\mathcal{P}_i(\lambda)x(k).$$

Then, along the trajectory of system (5.1)–(5.3), we have

$$\begin{aligned} \Delta\mathcal{V} &= \mathcal{V}(k+1, x(k+1)) - \mathcal{V}(k, x(k)) \\ &= x^T(k)\left[A_i^T(\lambda)\mathcal{P}_j(\lambda)A_i(\lambda) - \mathcal{P}_i(\lambda)\right]x(k) \\ &\quad + 2x^T(k)\left[A_i^T(\lambda)\mathcal{P}_j(\lambda)B_i(\lambda)\right]w(k) \\ &\quad + w^T(k)\left[B_i^T(\lambda)\mathcal{P}_j(\lambda)B_i(\lambda)\right]w(k) \end{aligned} \tag{5.11}$$

When assuming the zero disturbance input to system (5.1)–(5.3), we have

$$\begin{aligned} \Delta\mathcal{V} &= \mathcal{V}(k+1, x(k+1)) - \mathcal{V}(k, x(k)) \\ &= x^T(k)\left[A_i^T(\lambda)\mathcal{P}_j(\lambda)A_i(\lambda) - \mathcal{P}_i(\lambda)\right]x(k), \quad \forall (i,j) \in \mathcal{I} \times \mathcal{I}. \end{aligned}$$

Thus if

$$A_i^T(\lambda)\mathcal{P}_j(\lambda)A_i(\lambda) - \mathcal{P}_i(\lambda) < 0, \quad \forall (i,j) \in \mathcal{I} \times \mathcal{I} \tag{5.12}$$

then $\Delta\mathcal{V} < 0$ and the robust asymptotic stability of system (5.1)–(5.3) is guaranteed. By Lemma 2.4, condition (5.12) is equivalent to: $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\Xi^{i,j}(\lambda) \triangleq \begin{bmatrix} -\mathcal{P}_j(\lambda) & \mathcal{P}_j(\lambda)A_i(\lambda) \\ \star & -\mathcal{P}_i(\lambda) \end{bmatrix} < 0. \tag{5.13}$$

On the other hand, if the inequality (5.10) holds, we can explore the fact that $\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda) < 0$ so that the matrices $\mathcal{R}_i(\lambda)$ are nonsingular for each i. In addition, we have $(\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda))\mathcal{P}_j^{-1}(\lambda)(\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda))^T \geq 0$, which implies $\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda) \geq -\mathcal{R}_i(\lambda)\mathcal{P}_j^{-1}(\lambda)\mathcal{R}_i^T(\lambda)$. Hence, we conclude

$$\Phi^{i,j}(\lambda) \triangleq \begin{bmatrix} -\mathcal{R}_i(\lambda)\mathcal{P}_j^{-1}(\lambda)\mathcal{R}_i^T(\lambda) & 0 & \mathcal{R}_i(\lambda)A_i(\lambda) & \mathcal{R}_i(\lambda)B_i(\lambda) \\ \star & -I & C_i(\lambda) & D_i(\lambda) \\ \star & \star & -\mathcal{P}_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0.$$

Then, by performing a congruence transformation to above inequality via $diag\{\mathcal{R}_i^{-T}(\lambda)\mathcal{P}_j(\lambda), I, I, I\}$, we obtain

$$\Phi^{i,j}(\lambda) \triangleq \begin{bmatrix} -\mathcal{P}_j(\lambda) & 0 & \mathcal{P}_j(\lambda)A_i(\lambda) & \mathcal{P}_j(\lambda)B_i(\lambda) \\ \star & -I & C_i(\lambda) & D_i(\lambda) \\ \star & \star & -\mathcal{P}_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0. \tag{5.14}$$

LMI (5.14) implies (5.13), thus the robust asymptotic stability of system (5.1)–(5.3) is ensured.

Now, to establish the H_∞ performance for the uncertain switched system (5.1)–(5.3), consider the same performance index J as given in the proof of Theorem 3.1, under zero initial condition, i.e., $\mathcal{V}(k, x(k)) \mid_{k=0} = 0$, we have

$$J < \sum_{k=0}^{\infty} \theta^T(k) \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \star & \Lambda_{22} \end{bmatrix} \theta(k)$$

where $\theta(k) \triangleq \begin{bmatrix} x^T(k) & w^T(k) \end{bmatrix}^T$, and

$$\begin{aligned} \Lambda_{11} &\triangleq A_i^T(\lambda)\mathcal{P}_j(\lambda)A_i(\lambda) - \mathcal{P}_i(\lambda) + C_i^T(\lambda)C_i(\lambda), \\ \Lambda_{12} &\triangleq A_i^T(\lambda)\mathcal{P}_j(\lambda)B_i(\lambda) + C_i^T(\lambda)D_i(\lambda), \\ \Lambda_{22} &\triangleq -\gamma^2 I + B_i^T(\lambda)\mathcal{P}_j(\lambda)B_i(\lambda) + D_i^T(\lambda)D_i(\lambda). \end{aligned}$$

By Lemma 2.4, inequality (5.14) guarantees $J < 0$ which means that $\|y\|_2 < \gamma \|w\|_2$, this completes the proof. □

Remark 5.2 As shown in (5.13)–(5.14), the coupling of product terms is cross among different subsystems so that the filter design is hard for whole switched systems. The introduction of matrix variables $\mathcal{R}_i(\lambda)$ overcomes this difficulty, which transfers the interaction among subsystems from terms $\mathcal{P}_j(\lambda)A_i(\lambda)$ or $\mathcal{P}_j(\lambda)B_i(\lambda)$ to another form $\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda)$, and the resulting terms $\mathcal{R}_i(\lambda)A_i(\lambda)$ and $\mathcal{R}_i(\lambda)B_i(\lambda)$ are much easier to be dealt with.

Now, based upon the above criterion for filtering analysis, we give the existence condition of robust H_∞ filter in the following theorem.

Theorem 5.3 *Consider the uncertain switched linear system (5.1)–(5.3) and let $\gamma > 0$ be a given scalar. Then, there exists a robust switched linear filter (5.5)–(5.6) such that, for all admissible uncertainties, the filtering error system (5.7)–(5.8) is robustly asymptotically stable and (5.9) holds for any nonzero $w \in l_2[0, +\infty)$, if for $i \in \mathcal{I}$, $1 \le m \le s$ there exist matrices $\mathcal{X}_{i,m}$, $\mathcal{Y}_i$, $\mathcal{Z}_{i,m}$, $\bar{A}_{fi}$, $\bar{B}_{fi}$, $\bar{C}_{fi}$, $\bar{D}_{fi}$, $\mathcal{P}_{2i,m}$, positive definite matrix $\mathcal{P}_{1i,m}$, $\mathcal{P}_{3i,m}$, and scalars ε_i such that*

$$\Xi^{ij}_{m,n} + \Xi^{ij}_{n,m} < 0, \;\; (1 \le m \le n \le s), \quad \forall (i,j) \in \mathcal{I} \times \mathcal{I}, \tag{5.15}$$

where,

$$\Xi^{ij}_{m,n} \triangleq \begin{bmatrix} \mathcal{P}_{1j,n} - \mathcal{X}_{i,n} - \mathcal{X}_{i,n}^T & \mathcal{P}_{2j,n} - \varepsilon_i \mathcal{Y}_i - \mathcal{Z}_{i,n}^T & 0 & \mathcal{X}_{i,n} A_{i,m} + \varepsilon_i \bar{B}_{fi} C_{i,m} & \varepsilon_i \bar{A}_{fi} & \mathcal{X}_{i,n} B_{i,m} + \varepsilon_i \bar{B}_{fi} D_{i,m} \\ \star & \mathcal{P}_{3j,n} - \mathcal{Y}_i - \mathcal{Y}_i^T & 0 & \mathcal{Z}_{i,n} A_{i,m} + \bar{B}_{fi} C_{i,m} & \bar{A}_{fi} & \mathcal{Z}_{i,n} B_{i,m} + \bar{B}_{fi} D_{i,m} \\ \star & \star & -I & H_{i,m} - \bar{D}_{fi} C_{i,m} & -\bar{C}_{fi} & L_{i,m} - \bar{D}_{fi} D_{i,m} \\ \star & \star & \star & -\mathcal{P}_{1i,m} & -\mathcal{P}_{2i,m} & 0 \\ \star & \star & \star & \star & -\mathcal{P}_{3i,m} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0$$

In this case, a suitable robust filter in the form (5.5) and (5.6) has the parameters as follows

$$A_{fi} = \mathcal{Y}_i^{-1} \bar{A}_{fi}, \;\; B_{fi} = \mathcal{Y}_i^{-1} \bar{B}_{fi}, \;\; C_{fi} = \bar{C}_{fi}, \;\; D_{fi} = \bar{D}_{fi}, \;\; i \in \mathcal{I}. \tag{5.16}$$

Proof As shown in (5.7) and (5.8), the filtering error system is a switched linear system with same structure of polytopic uncertainties as well as uncertain switched system (5.1)–(5.3), thus, by Lemma 5.1, system (5.7)–(5.8) is robustly asymptotically stable with a prescribed H_∞ noise-attenuation level bound γ if the following inequality holds

$$\Xi^{i,j}(\lambda) = \begin{bmatrix} \mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda) & 0 & \mathcal{R}_i(\lambda)\tilde{A}_i(\lambda) & \mathcal{R}_i(\lambda)\tilde{B}_i(\lambda) \\ \star & -I & \tilde{C}_i(\lambda) & \tilde{D}_i(\lambda) \\ \star & \star & -\mathcal{P}_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0, \tag{5.17}$$

$\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, where, $\tilde{A}_i(\lambda)$, $\tilde{B}_i(\lambda)$, $\tilde{C}_i(\lambda)$, $\tilde{D}_i(\lambda)$ are described in (5.7) and (5.8).

Then, by defining matrix functions

$$\mathcal{P}_i(\lambda) \triangleq \begin{bmatrix} \mathcal{P}_{1i}(\lambda) & \mathcal{P}_{2i}(\lambda) \\ \star & \mathcal{P}_{3i}(\lambda) \end{bmatrix}, \quad \mathcal{R}_i(\lambda) \triangleq \begin{bmatrix} \mathcal{X}_i(\lambda) & \varepsilon_i \mathcal{Y}_i \\ \mathcal{Z}_i(\lambda) & \mathcal{Y}_i \end{bmatrix}$$

and matrix variables

$$\bar{A}_{fi} \triangleq \mathcal{Y}_i A_{fi}, \; \bar{C}_{fi} \triangleq C_{fi}, \; \bar{B}_{fi} \triangleq \mathcal{Y}_i B_{fi}, \; \bar{D}_{fi} \triangleq D_{fi},$$

respectively, and by some matrix manipulations, it can be readily established that (5.17) is equivalent to: $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\Xi^{ij}(\lambda) \triangleq \begin{bmatrix} \mathcal{P}_{1j}(\lambda) - \mathcal{X}_i(\lambda) - \mathcal{X}_i^T(\lambda) & \mathcal{P}_{2j}(\lambda) - \varepsilon_i \mathcal{Y}_i - \mathcal{Z}_i^T(\lambda) & 0 & \mathcal{X}_i(\lambda)A_i(\lambda) + \varepsilon_i \bar{B}_{fi} C_i(\lambda) & \varepsilon_i \bar{A}_{fi} & \mathcal{X}_i(\lambda)B_i(\lambda) + \varepsilon_i \bar{B}_{fi} D_i(\lambda) \\ \star & \mathcal{P}_{3j}(\lambda) - \mathcal{Y}_i - \mathcal{Y}_i^T & 0 & \mathcal{Z}_i(\lambda)A_i(\lambda) + \bar{B}_{fi} C_i(\lambda) & \bar{A}_{fi} & \mathcal{Z}_i(\lambda)B_i(\lambda) + \bar{B}_{fi} D_i(\lambda) \\ \star & \star & -I & H_i(\lambda) - \bar{D}_{fi} C_i(\lambda) & -\bar{C}_{fi} & L_i(\lambda) - \bar{D}_{fi} D_i(\lambda) \\ \star & \star & \star & -\mathcal{P}_{1i}(\lambda) & -\mathcal{P}_{2i}(\lambda) & 0 \\ \star & \star & \star & \star & -\mathcal{P}_{3i}(\lambda) & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0$$

Hence, further assuming matrix functions $\mathcal{P}_i(\lambda)$ and $\mathcal{R}_i(\lambda)$ to be the following forms

$$\mathcal{P}_i(\lambda) \triangleq \sum_{m=1}^{s} \lambda_m \mathcal{P}_{i,m} = \sum_{m=1}^{s} \lambda_m \begin{bmatrix} \mathcal{P}_{1i,m} & \mathcal{P}_{2i,m} \\ \star & \mathcal{P}_{3i,m} \end{bmatrix}, \tag{5.18}$$

$$\mathcal{R}_i(\lambda) \triangleq \sum_{m=1}^{s} \lambda_m \mathcal{R}_{i,m} = \sum_{m=1}^{s} \lambda_m \begin{bmatrix} \mathcal{X}_{i,m} & \varepsilon_i \mathcal{Y}_i \\ \mathcal{Z}_{i,m} & \mathcal{Y}_i \end{bmatrix}, \tag{5.19}$$

and taking (5.4) and (5.18)–(5.19) into account, we have

$$\begin{aligned} \Xi^{ij}(\lambda) &= \sum_{m=1}^{s} \sum_{n=1}^{s} \lambda_m \lambda_n \Xi_{m,n}^{ij} \\ &= \sum_{m=1}^{s} \lambda_m^2 \Xi_{m,m}^{ij} + \sum_{m=1}^{s-1} \sum_{n=m+1}^{s} \lambda_m \lambda_n (\Xi_{m,n}^{ij} + \Xi_{n,m}^{ij}). \end{aligned}$$

Thus, if condition (5.15) holds, then $\Xi^{ij}(\lambda) < 0$, which implies (5.17) holds, i.e. the filtering error system is robustly asymptotically stable with an H_∞ noise-attenuation level bound γ. Meanwhile, if a solution exists, the parameters of admissible filter is given by (5.16). □

Remark 5.4 From the expression $\Xi^{ij}_{m,n}$, it can be easily seen that a parameter-dependent quadratic Lyapunov function is actually constructed in Theorem 5.3 for the developed filtering error system, and all vertex systems in each subsystem are considered by means of matrix variables $\mathcal{P}_{i,m}$. However, if we choose the Lyapunov function based on quadratic framework, and correspondingly, the matrices $\mathcal{P}_{i,m}$ and $\mathcal{R}_{i,m}$ in Theorem 5.3 are selected as $\mathcal{P}_i$ and $\mathcal{R}_i$, we will get the following simpler (but will be conservative) result.

Corollary 5.5 *Consider the uncertain switched linear system (5.1)–(5.3) and let $\gamma > 0$ be a given scalar. Then, there exists a robust switched linear filter (5.5)–(5.6) such that, for all admissible uncertainties, the filtering error system (5.7)–(5.8) is robustly asymptotically stable and (5.9) holds for any nonzero $w \in l_2[0, +\infty)$, if for each $i \in \mathcal{I}$ there exist matrices $\mathcal{X}_i$, $\mathcal{Y}_i$, $\mathcal{Z}_i$, $\bar{A}_{fi}$, $\bar{B}_{fi}$, $\bar{C}_{fi}$, $\bar{D}_{fi}$, $\mathcal{P}_{2i}$, positive definite matrix $\mathcal{P}_{1i}$, $\mathcal{P}_{3i}$, and scalars ε_i such that*

$$\Xi^{ij}_m < 0, \ \ (1 \le m \le s)\,, \ \forall (i,j) \in \mathcal{I} \times \mathcal{I}$$

where,

$$\Xi^{ij}_m \triangleq \begin{bmatrix} \mathcal{P}_{1j} - \mathcal{X}_i - \mathcal{X}_i^T & \mathcal{P}_{2j} - \varepsilon_i \mathcal{Y}_i - \mathcal{Z}_i^T & 0 & \mathcal{X}_i A_{i,m} + \varepsilon_i \bar{B}_{fi} C_{i,m} & \varepsilon_i \bar{A}_{fi} & \mathcal{X}_i B_{i,m} + \varepsilon_i \bar{B}_{fi} D_{i,m} \\ \star & \mathcal{P}_{3j} - \mathcal{Y}_i - \mathcal{Y}_i^T & 0 & \mathcal{Z}_i A_{i,m} + \bar{B}_{fi} C_{i,m} & \bar{A}_{fi} & \mathcal{Z}_i B_{i,m} + \bar{B}_{fi} D_{i,m} \\ \star & \star & -I & H_{i,m} - \bar{D}_{fi} C_{i,m} & -\bar{C}_{fi} & L_{i,m} - \bar{D}_{fi} D_{i,m} \\ \star & \star & \star & -\mathcal{P}_{1i} & -\mathcal{P}_{2i} & 0 \\ \star & \star & \star & \star & -\mathcal{P}_{3i} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \qquad (5.20)$$

Conditions (5.15) in Theorem 5.3 and (5.20) in Corollary 5.5 are all formulated in terms of a set of LMIs. These LMIs can be solved by means of numerically efficient convex programming algorithms [2]. Moreover, the performance index γ described in these conditions can be respectively optimized by the corresponding convex optimization procedures.

In the following, a numerical example is presented to demonstrate the validity and the less conservativeness of proposed filter design approach in Theorem 5.3.

Example 5.6 Consider the uncertain discrete-time switched linear system (5.1)–(5.3) consisting of two uncertain subsystems, where there are two groups of vertex matrices in subsystem 1

$$\begin{aligned}
&A_{11}=\rho\begin{bmatrix}0.82 & 0.10\\ -0.06 & 0.77\end{bmatrix},\ B_{11}=\rho\begin{bmatrix}0\\ 0.1\end{bmatrix},\\
&C_{11}=\rho\begin{bmatrix}1 & 0\end{bmatrix},\ D_{11}=\rho,\ H_{11}=\rho\begin{bmatrix}1 & 0\end{bmatrix},\ L_{11}=0\\
&A_{12}=\rho\begin{bmatrix}0.82 & 0.10\\ -0.06 & -0.75\end{bmatrix},\ B_{12}=\rho\begin{bmatrix}0\\ -0.1\end{bmatrix},\\
&C_{12}=\rho\begin{bmatrix}1 & 0.2\end{bmatrix},\ D_{12}=0.8\rho,\ H_{12}=\rho\begin{bmatrix}1 & 0\end{bmatrix},\ L_{12}=0
\end{aligned}$$

two groups of vertex matrices in subsystem 2

$$\begin{aligned}
&A_{21}=\rho\begin{bmatrix}0.82 & 0.06\\ -0.10 & 0.77\end{bmatrix},\ B_{21}=\rho\begin{bmatrix}0.1\\ 0\end{bmatrix},\\
&C_{21}=\rho\begin{bmatrix}0 & -1\end{bmatrix},\ D_{21}=-\rho,\ H_{21}=\rho\begin{bmatrix}1 & 0\end{bmatrix},\ L_{21}=0\\
&A_{22}=\rho\begin{bmatrix}0.82 & 0.06\\ -0.10 & -0.75\end{bmatrix},\ B_{22}=\rho\begin{bmatrix}-0.1\\ 0\end{bmatrix},\\
&C_{22}=\rho\begin{bmatrix}0.2 & -1\end{bmatrix},\ D_{22}=-0.8\rho,\ H_{22}=\rho\begin{bmatrix}1 & 0\end{bmatrix},\ L_{22}=0
\end{aligned}$$

where ρ is a scalar parameter.

Our purpose is to design a robust H_∞ filter for the above uncertain switched system for a given ρ and check the H_∞ performance of the resulting filtering error system. The disturbance $w(k)=0.001e^{-0.003k}\sin(0.002\pi k)$ and the switching signal is generated randomly by Algorithm 2.1. Then, assuming $Time\_Length=200$ and $Con=0.6$ in Algorithm 2.1 in the example, the switching signal can be realized by Matlab and a possible case is shown in Fig. 5.1.

By choosing $\varepsilon_1=\varepsilon_2=1$ and solving the corresponding convex optimization problems in Theorem 5.3 and Corollary 5.5, we can obtain different minimum H_∞ noise-attenuation level bounds γ as listed in Table 5.1 which lists the different calculation results by different methods. From obtained γ, it can be clearly seen that the condition in Theorem 5.3 using parameter-dependent stability idea for the existence of a robust switched linear filter is less conservative.

In addition, for given $\rho=1.20$, the admissible filter parameters can be obtained by Theorem 5.3 as

$$\begin{aligned}
&A_{f1}=\begin{bmatrix}0.0391 & -0.0400\\ 0.0703 & 0.4807\end{bmatrix},\ B_{f1}=\begin{bmatrix}-0.7140\\ 0.0432\end{bmatrix},\ A_{f2}=\begin{bmatrix}0.2043 & 0.4062\\ 0.2493 & 0.5090\end{bmatrix},\\
&B_{f2}=\begin{bmatrix}-1.2090\\ 0.6217\end{bmatrix},\ C_{f1}=\begin{bmatrix}0.0186\\ 0.2056\end{bmatrix}^T,\ C_{f2}=\begin{bmatrix}-1.1651\\ -0.1088\end{bmatrix}^T,\\
&D_{f1}=0.9059,\ D_{f2}=0.1191.
\end{aligned}$$

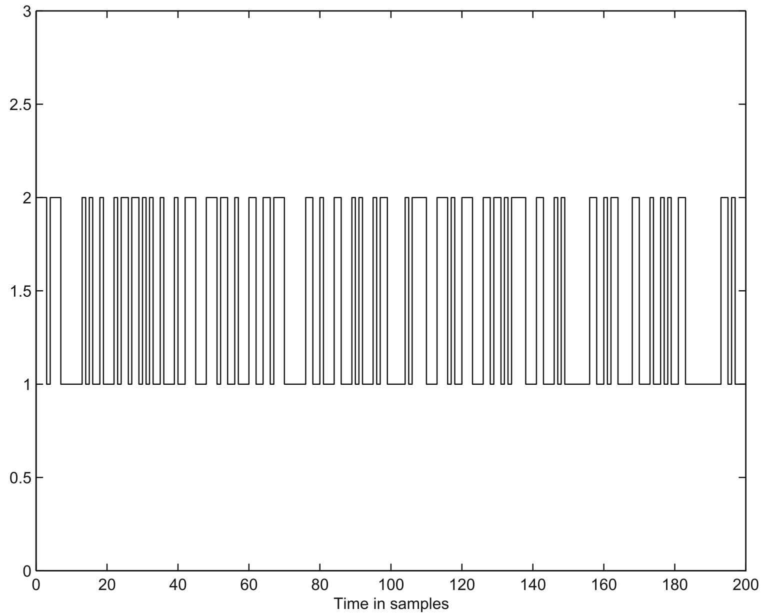


Fig. 5.1 Switching signal

Table 5.1 Different minimum γ of filtering error system

Methods	$\rho = 1.00$	$\rho = 1.10$	$\rho = 1.20$
Corollary 5.5	0.5440	1.1819	6.7103
Theorem 5.3	0.5375	1.1414	4.4493

Then, given H_∞ noise-attenuation level bound $\gamma = 4.4493$, Fig. 5.2 shows the error response of the resulting filtering error system by applying above obtained filter gains for given uncertain parameters $\lambda_1 = 0.4$, $\lambda_2 = 0.6$ in (5.4), and Fig. 5.3 shows the corresponding result if the uncertain parameters $\lambda_1 = 0.2$, $\lambda_2 = 0.8$. It can be observed from simulation curves in Figs. 5.2 and 5.3 that the designed switched linear filter is feasible and effective against the variations of uncertain parameter under the given arbitrary switching signal.

5.2 Robust H_∞ Filtering: ADT Switching

5.2.1 Uncertain Switched Linear Systems

In this subsection, with the aid of Theorem 3.8, our objective is to design a full-order mode-dependent exponential H_∞ filter of form (5.5)–(5.6), and find admissible

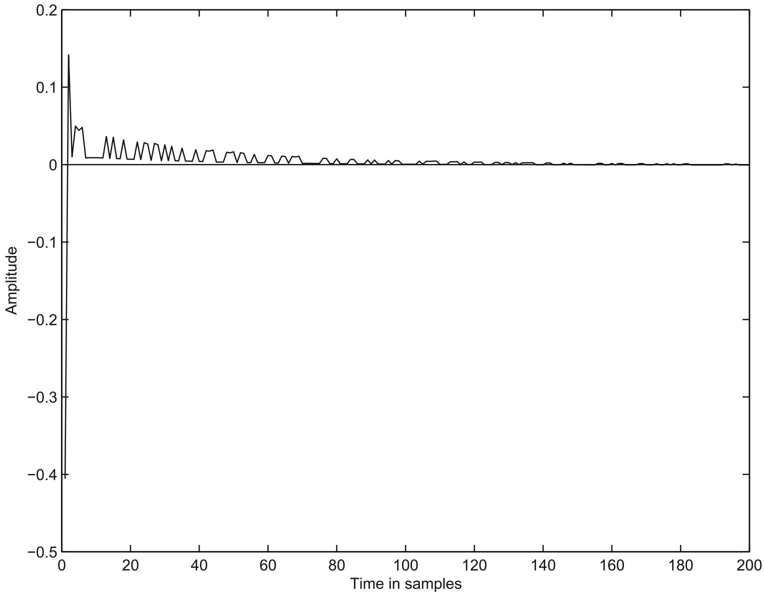


Fig. 5.2 Filtering error response corresponding to uncertain parameters. $\lambda_1 = 0.4,\ \lambda_2 = 0.6$

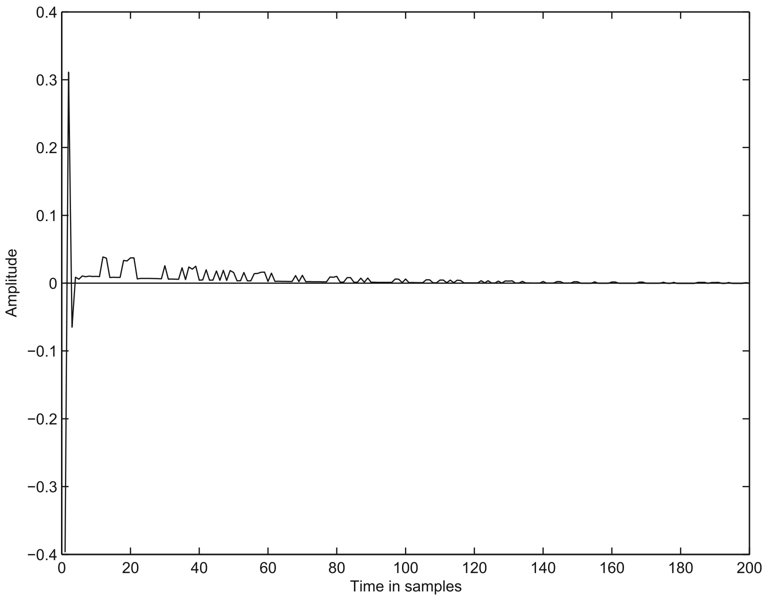


Fig. 5.3 Filtering error response corresponding to uncertain parameters. $\lambda_1 = 0.2,\ \lambda_2 = 0.8$

switching signals with the minimal average dwell time (ADT) such that the polytopic uncertain switched filtering error system (5.7)–(5.8) is robustly exponentially stable and achieves a prescribed exponential H_∞ performance index.

Theorem 5.7 *Consider the uncertain switched linear system (5.1)–(5.3) and let* $\alpha > 0$*,* $\gamma > 0$ *and* $\mu > 1$ *be given constants. If there exist matrices* $\bar{P}_{1i,m} > 0$, $\bar{P}_{3i,m} > 0$ *and matrices* $\bar{P}_{2i,m}$, $R_{i,m}$, $S_{i,m}$, T_i, $\hat{A}_{fi}$, $\hat{B}_{fi}$, $\hat{C}_{fi}$, $\hat{D}_{fi}$, $\forall i \in \mathcal{I}$, $1 \le m \le s$ *such that*

$$\Xi^i_{m,n} + \Xi^i_{n,m} < 0, \ \ (1 \le m \le n \le s) \tag{5.21}$$

$$\begin{bmatrix} \Upsilon^i_{11,m} & \bar{P}_{2i,m} - \mu S_{i,m} - \mu T_i & R^T_{i,m} & T_j \\ \star & \bar{P}_{3i,m} - \mu T_i^T - \mu T_i & S^T_{i,m} & T_j \\ \star & \star & -\mu^{-1}\bar{P}_{1j,m} & -\mu^{-1}\bar{P}_{2j,m} \\ \star & \star & \star & -\mu^{-1}\bar{P}_{3j,m} \end{bmatrix} \le 0, i \ne j \tag{5.22}$$

where,

$$\Xi^i_{m,n} \triangleq \begin{bmatrix} \Lambda^i_{11,m} & \Lambda^i_{12,m} & 0 & R^T_{i,m}A_{i,n} + \hat{B}_{fi}C_{i,m} & \hat{A}_{fi} & R^T_{i,m}B_{i,n} + \hat{B}_{fi}D_{i,m} \\ \star & \Lambda^i_{22,m} & 0 & S^T_{i,m}A_{i,n} + \hat{B}_{fi}C_{i,m} & \hat{A}_{fi} & S^T_{i,m}B_{i,n} + \hat{B}_{fi}D_{i,m} \\ \star & \star & I & H_{i,m} - \hat{D}_{fi}C_{i,m} & -\hat{C}_{fi} & L_{i,m} - \hat{D}_{fi}D_{i,m} \\ \star & \star & \star & -(1-\alpha)\bar{P}_{1i,m} & \Lambda^i_{45,m} & 0 \\ \star & \star & \star & \star & \Lambda^i_{55,m} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix}$$

$$\Upsilon^i_{11,m} \triangleq \bar{P}_{1i,m} - \mu R^T_{i,m} - \mu R_{i,m}, \ \Lambda^i_{11,m} \triangleq \bar{P}_{1i,m} - R^T_{i,m} - R_{i,m},$$
$$\Lambda^i_{12,m} \triangleq \bar{P}_{2i,m} - S_{i,m} - T_i, \ \ \Lambda^i_{22,m} \triangleq \bar{P}_{3i,m} - T_i^T - T_i,$$
$$\Lambda^i_{45,m} \triangleq -(1-\alpha)\bar{P}_{2i,m}, \ \Lambda^i_{55,m} \triangleq -(1-\alpha)\bar{P}_{3i,m}.$$

Then, there exists a mode-dependent full-order filter such that the corresponding filtering error system (5.7)–(5.8) is robustly exponentially stable with an exponential H_∞ *performance index* γ *for all admissible uncertainties satisfying (5.4) and any switching signal with ADT satisfying (2.26). Moreover, if the LMIs (5.21)–(5.22) have a feasible solution, then the admissible filter in the form (5.5)–(5.6) can be given by*

$$\begin{bmatrix} A_{fi} & B_{fi} \\ C_{fi} & D_{fi} \end{bmatrix} \triangleq \begin{bmatrix} T_i^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}_{fi} & \hat{B}_{fi} \\ \hat{C}_{fi} & \hat{D}_{fi} \end{bmatrix} \tag{5.23}$$

Proof By Theorem 3.8, system (5.7)–(5.8) is robustly asymptotically stable with a prescribed exponential H_∞ noise-attenuation level bound γ if the following inequalities hold

$$\begin{bmatrix} -P_i(\lambda) & 0 & P_i(\lambda)\tilde{A}_i(\lambda) & P_i(\lambda)\tilde{B}_i(\lambda) \\ \star & -I & \tilde{C}_i(\lambda) & \tilde{D}_i(\lambda) \\ \star & \star & -(1-\alpha)P_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{5.24}$$

$$P_i(\lambda) - \mu P_j(\lambda) \leq 0 \tag{5.25}$$

where $\tilde{A}_i(\lambda)$, $\tilde{B}_i(\lambda)$, $\tilde{C}_i(\lambda)$, $\tilde{D}_i(\lambda)$ are described in (5.7)–(5.8).

Then, for a matrix function $G_i(\lambda)$, $\forall i \in \mathcal{I}$, we have the fact that

$$(P_i(\lambda) - G_i(\lambda))^T P_i^{-1}(\lambda)(P_i(\lambda) - G_i(\lambda) \geq 0$$
$$(P_j(\lambda) - G_i(\lambda))^T P_j^{-1}(\lambda)(P_j(\lambda) - G_i(\lambda) \geq 0$$

namely,

$$P_i(\lambda) - G_i(\lambda) - G_i^T(\lambda) \geq -G_i^T(\lambda)P_i^{-1}(\lambda)G_i(\lambda)$$
$$P_j(\lambda) - G_i(\lambda) - G_i^T(\lambda) \geq -G_i^T(\lambda)P_j^{-1}(\lambda)G_i(\lambda)$$

Therefore, if the following inequalities hold

$$\begin{bmatrix} P_i(\lambda) - G_i(\lambda) - G_i^T(\lambda) & 0 & G_i^T(\lambda)\tilde{A}_i(\lambda) & G_i^T(\lambda)\tilde{B}_i(\lambda) \\ \star & -I & \tilde{C}_i(\lambda) & \tilde{D}_i(\lambda) \\ \star & \star & -(1-\alpha)P_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{5.26}$$

$$P_i(\lambda) - \mu\left[G_i(\lambda) + G_i^T(\lambda) - G_i^T(\lambda)P_j^{-1}(\lambda)G_i(\lambda)\right] \leq 0 \tag{5.27}$$

then (5.27) implies (5.25). Also, from (5.26), we can obtain

$$\begin{bmatrix} -G_i^T(\lambda)P_i^{-1}(\lambda)G_i(\lambda) & 0 & G_i^T(\lambda)\tilde{A}_i(\lambda) & G_i^T(\lambda)\tilde{B}_i(\lambda) \\ \star & -I & \tilde{C}_i(\lambda) & \tilde{D}_i(\lambda) \\ \star & \star & -(1-\alpha)P_i(\lambda) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0$$

By performing a congruence transformation to above formula via $diag\{G_i^{-1}(\lambda) \times P_i(\lambda), I, I, I\}$ yields (5.24). In addition, by Lemma 2.4, (5.27) is equivalent to

$$\begin{bmatrix} P_j(\lambda) - \mu G_i(\lambda) - \mu G_i^T(\lambda) & G_i^T(\lambda) \\ \star & -\mu^{-1}P_j(\lambda) \end{bmatrix} \leq 0 \tag{5.28}$$

Now, let us show that conditions (5.21) and (5.22) ensure respectively that (5.26) and (5.28) are satisfied. Firstly, if (5.22) holds, we have

$$\begin{bmatrix} \tilde{P}_{1i}(\lambda) & \bar{P}_{2i}(\lambda)-\mu S_i(\lambda)-\mu T_i & R_i^T(\lambda) & T_j \\ \star & \bar{P}_{3i}(\lambda)-\mu T_i^T-\mu T_i & S_i^T(\lambda) & T_j \\ \star & \star & -\mu^{-1}\bar{P}_{1j}(\lambda) & -\mu^{-1}\bar{P}_{2j}(\lambda) \\ \star & \star & \star & -\mu^{-1}\bar{P}_{3j}(\lambda) \end{bmatrix} \leq 0 \tag{5.29}$$

where $\tilde{P}_{1i}(\lambda) \triangleq \bar{P}_{1i}(\lambda) - \mu R_i^T(\lambda) - \mu R_i(\lambda)$. Also, if (5.21) hold, we have

$$\begin{aligned} \Xi^i(\lambda) &= \sum_{m=1}^{s}\sum_{n=1}^{s}\lambda_m\lambda_n\Xi^i_{m,n} \\ &= \sum_{m=1}^{s}\lambda_m^2\Xi^i_{m,m} + \sum_{m=1}^{s-1}\sum_{n=m+1}^{s}\lambda_m\lambda_n(\Xi^i_{m,n}+\Xi^i_{n,m}) < 0 \end{aligned}$$

i.e.

$$\begin{bmatrix} \Lambda^i_{11}(\lambda) & \Lambda^i_{12}(\lambda) & 0 & \Lambda^i_{14}(\lambda) & \hat{A}_{fi} & R_i^T(\lambda)B_i(\lambda)+\hat{B}_{fi}D_i(\lambda) \\ \star & \Lambda^i_{22}(\lambda) & 0 & \Lambda^i_{24}(\lambda) & \hat{A}_{fi} & S_i^T(\lambda)B_i(\lambda)+\hat{B}_{fi}D_i(\lambda) \\ \star & \star & I & \Lambda^i_{34}(\lambda) & -\bar{C}_{fi} & L_i(\lambda)-\hat{D}_{fi}D_i(\lambda) \\ \star & \star & \star & \Lambda^i_{44}(\lambda) & -(1-\alpha)\bar{P}_{2i}(\lambda) & 0 \\ \star & \star & \star & \star & -(1-\alpha)\bar{P}_{3i}(\lambda) & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{5.30}$$

where,

$$\begin{aligned} &\Lambda^i_{11}(\lambda) \triangleq \bar{P}_i(\lambda) - R_i^T(\lambda) - R_i(\lambda),\ \Lambda^i_{12}(\lambda) \triangleq \bar{P}_{2i}(\lambda) - S_i(\lambda) - T_i, \\ &\Lambda^i_{14}(\lambda) \triangleq R_i^T(\lambda)A_i(\lambda) + \hat{B}_{fi}C_i(\lambda),\ \Lambda^i_{22}(\lambda) \triangleq \bar{P}_{3i}(\lambda) - T_i^T - T_i, \\ &\Lambda^i_{24}(\lambda) \triangleq S_i^T(\lambda)A_i(\lambda) + \hat{B}_{fi}C_i(\lambda),\ \Lambda^i_{34}(\lambda) \triangleq H_i(\lambda) - \hat{D}_{fi}C_i(\lambda), \\ &\Lambda^i_{44}(\lambda) \triangleq -(1-\alpha)\bar{P}_{1i}(\lambda) \end{aligned}$$

Note that from (5.30), we also know that

$$\bar{P}_{3i}(\lambda) - T_i^T - T_i < 0$$

thus we can infer that $T_i^T + T_i > 0$, which implies T_i is nonsingular. Then one can always find nonsingular matrices G_{3i} and G_4 satisfying $T_i = G_4^T G_{3i}^{-1} G_4$, $\forall i \in \mathcal{I}$. Now, introduce the following matrix variables related to G_{3i} and G_4

$$V_i \triangleq \begin{bmatrix} I & 0 \\ 0 & G_{3i}^{-1}G_4 \end{bmatrix},\quad G_i(\lambda) = \begin{bmatrix} R_i(\lambda) & S_i(\lambda)G_4^{-1}G_{3i} \\ G_4 & G_{3i} \end{bmatrix}$$

Then, by further performing a congruence transformation to (5.29) and (5.30) via $diag\{V_i^{-1}, V_j^{-1}\}$ and $diag\{V_i^{-1}, I, V_i^{-1}, I\}$, respectively, and setting matrix functions

$$P_i(\lambda) \triangleq V_i^{-T}\bar{P}_i(\lambda)V_i^{-1} = V_i^{-T}\begin{bmatrix} \bar{P}_{1i}(\lambda) & \bar{P}_{2i}(\lambda) \\ \star & \bar{P}_{3i}(\lambda) \end{bmatrix} V_i^{-1}$$

$$\begin{bmatrix} A_{fi} & B_{fi} \\ C_{fi} & D_{fi} \end{bmatrix} \triangleq \begin{bmatrix} G_4^{-T} & 0 \\ \star & I \end{bmatrix}\begin{bmatrix} \hat{A}_{fi} & \hat{B}_{fi} \\ \hat{C}_{fi} & \hat{D}_{fi} \end{bmatrix}\begin{bmatrix} G_4^{-1}G_{3i} & 0 \\ \star & I \end{bmatrix} \tag{5.31}$$

we can obtain (5.45) and (5.47).

Meanwhile, from (5.31) we know that an admissible filter for the underlying system can be given by

$$A_{fi} = G_4^{-T}\hat{A}_{fi}G_4^{-1}G_{3i},\;\; B_{fi} = G_4^{-T}\hat{B}_{fi},\;\; C_{fi} = \hat{C}_{fi}G_4^{-1}G_{3i},\;\; D_{fi} = \hat{D}_{fi} \tag{5.32}$$

Now, denote the filter transfer function from $y(k)$ to $z(k)$ by

$$T(\mathbf{z}) = C_{fi}(\mathbf{z}I - A_{fi})^{-1}B_{fi} + D_{fi}$$

By substituting the matrices $(A_{fi}, B_{fi}, C_{fi}, D_{fi})$ in (5.32) and considering $T_i = G_4^T G_{3i}^{-1} G_4$, we have

$$\begin{aligned} T(\mathbf{z}) &= \hat{C}_{fi}G_4^{-1}G_{3i}(\mathbf{z}I - G_4^{-T}\hat{A}_{fi}G_4^{-1}G_{3i})^{-1}G_4^{-T}\hat{B}_{fi} + \hat{D}_{fi} \\ &= \hat{C}_{fi}(\mathbf{z}I - T_i^{-1}\hat{A}_{fi})^{-1}T_i^{-1}\hat{B}_{fi} + \hat{D}_{fi} \end{aligned}$$

which implies an admissible filter is given by (5.23), and the proof is completed. □

Remark 5.8 Note that the matrices of the desired H_∞ filter can be obtained from the LMIs in (5.21)–(5.23), which will be applied to the corresponding system (5.1)–(5.3) under the switching signal from (2.26), (5.21)–(5.22), i.e. both the underlying subsystem and the obtained filter will be switched by the obtained ADT switching signals.

Remark 5.9 In addition, conditions (5.21)–(5.22) are formulated in terms of a set of LMIs, which are not only over the matrix variables but also the scalar γ^2. Therefore, the scalar can be optimized by a μ-dependent convex optimization problem for a fixed system decay degree as follows.

Problem 5.1

$$\min \delta, \text{ s.t.}(5.21)\text{–}(5.22), \forall i \in \mathcal{I}, 1 \le m \le s$$

with $\delta = \gamma^2$ over $R_{i,m}$, $S_{i,m}$, T_i, $\hat{A}_{fi}$, $\hat{B}_{fi}$, $\hat{C}_{fi}$, $\hat{D}_{fi}$, $\bar{P}_{1i,m}$, $\bar{P}_{2i,m}$, $\bar{P}_{3i,m}$. The minimum exponential noise attenuation level bound is then obtained by $\gamma^* = \sqrt{\delta^*}$, where δ^* is the optimal value of δ, and the corresponding filter matrices are given by (5.23).

In the following, we borrow the numerical example from Sect. 5.1 to demonstrate the potential and reduced conservatism of our developed theoretical results in Sect. 5.2.1.

Example 5.10 In this example, our first purpose is to design a mode-dependent full-order filter in the form of (5.5)–(5.6) and find out the admissible switching signals for the above uncertain switched system such that the resulting filtering error system is robustly exponentially stable with a μ-dependent exponential H_∞ performance index, for a given decay degree α. By giving $\rho = 1.2$ and solving Problem 5.1, we can obtain different optimal γ^* and minimal ADT τ_a^* for different μ as shown in Table 5.2.

It can be easily seen from Table 5.2 that the obtained exponential H_∞ performance γ^* is dependent on μ for a given system decay degree α (it is straightforward from (2.26) that the minimal ADT also depends on μ). Moreover, it can be observed that the larger μ corresponds to the smaller γ^*, but it will be at the expense of longer ADT in the system.

Now, to further demonstrate the less conservatism of our results, we list the corresponding μ-independent H_∞ performance indexes, which is obtained by the switched Lyapunov function approach in Theorem 5.3, and the μ-dependent results in Table 5.3 for given $\alpha = 0.0005$ and different ρ. It is clearly demonstrated that even if $\tau_a^* = 1$, the least time interval between two consecutive subsystems in arbitrary switched systems correspondingly (within discrete-time context), better noise attenuation performance can be achieved by Theorem 5.7, showing that our μ-dependent approach is less conservative.

Table 5.2 μ-dependent optimal γ^* for given different α

μ	1.001	1.002	1.003	1.004	1.005
(a) $\alpha = 0.0001$					
τ_a^*	10	20	30	40	50
γ^*	4.0080	3.9864	3.9727	3.9649	3.9591
(b) $\alpha = 0.0005$					
τ_a^*	2	4	6	8	10
γ^*	4.0333	4.0129	3.9992	3.9916	3.9853
(c) $\alpha = 0.0015$					
τ_a^*	1	2	2	3	4
γ^*	4.1014	4.0807	4.0671	4.0599	4.0538
(d) $\alpha = 0.005$					
τ_a^*	1	1	1	1	1
γ^*	4.3663	4.3451	4.3316	4.3240	4.3182

Table 5.3 Minimum γ^* by different approaches

ρ	μ,τ_a^*	1.0	1.1	1.2
Theorem 5.3 (μ-independent)	$\tau_a^* \geq 1$	0.5375	1.1414	4.4493
Theorem 5.7 (μ-dependent)	μ =1.0005,τ_a^*=1	0.5134	1.0282	4.0636
	μ =1.001,τ_a^*=2	0.5133	1.0268	4.0333

By giving $\rho = 1.2$ and $\alpha = 0.0015$, the corresponding μ-dependent full-order filter can be also solved by Problem 5.1, e.g., for $\mu = 1.004$, the desired filter for the underlying system with the matrices is obtained as follows

$$A_{F1} = \begin{bmatrix} 0.0275 & 0.0452 \\ 0.5123 & 0.4102 \end{bmatrix}, \ B_{F1} = \begin{bmatrix} -0.7074 \\ 0.3935 \end{bmatrix},$$
$$C_{F1} = \begin{bmatrix} 0.2544 & 0.2263 \end{bmatrix}, \ D_{F1} = 1.0033,$$
$$A_{F2} = \begin{bmatrix} 0.7809 & 0.7847 \\ -0.0393 & -0.1712 \end{bmatrix}, \ B_{F2} = \begin{bmatrix} -0.4268 \\ 0.2870 \end{bmatrix},$$
$$C_{F2} = \begin{bmatrix} -0.8610 & -0.6773 \end{bmatrix}, \ D_{F2} = 0.5822.$$

Furthermore, applying the solved filter and giving different uncertain parameters λ in (5.4) randomly, we can obtain the error response of the resulting filtering error system in Figs. 5.4, 5.5 and 5.6 for given three different switching signals (all are with $\tau_a^* = 3$), initial condition $x = [-0.8 \ \ 0.5 \ \ 0 \ \ 0]^T$ and input signal $w(k) = 0.01 \exp(-0.03k) \sin(0.02\pi k)$. It is clearly observed from the simulation curves that for given energy bounded disturbance, the filtering error system is stable against the variations of uncertain parameters under the different switching signals, which thereby implies that the designed filter is feasible and effective.

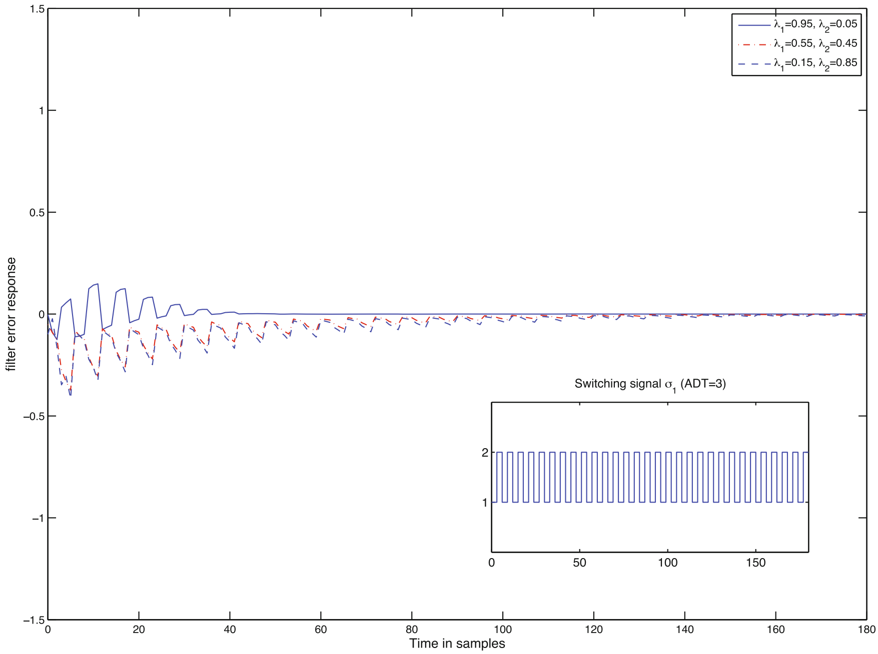


Fig. 5.4 Filtering error response under switching signal σ_1

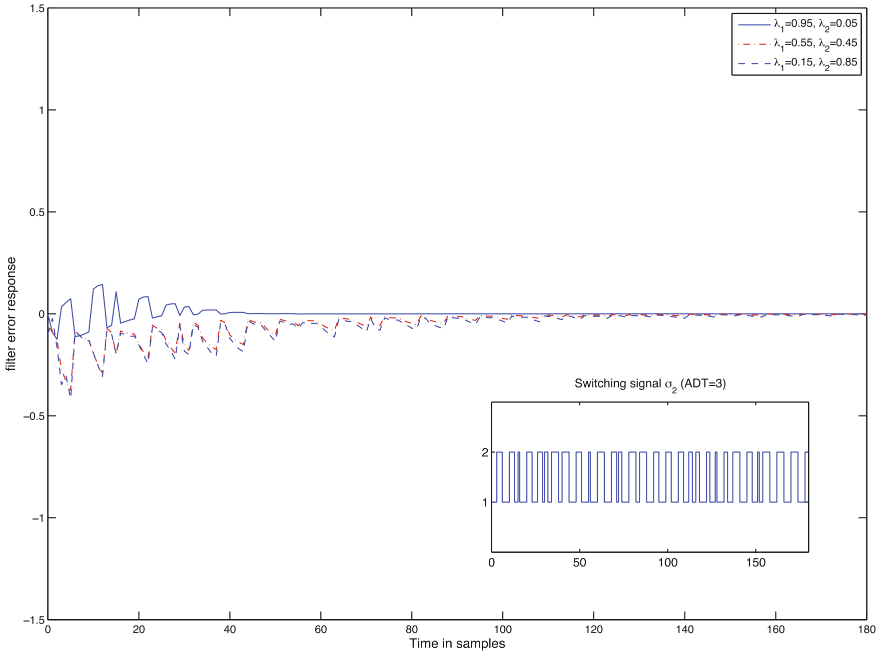


Fig. 5.5 Filtering error response under switching signal σ_2

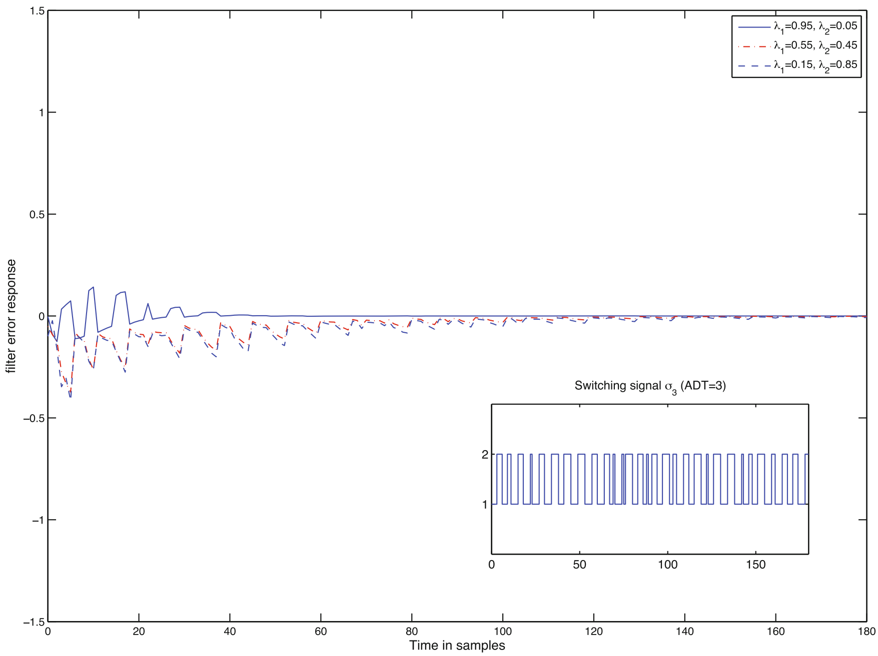


Fig. 5.6 Filtering error response under switching signal σ_3

5.2.2 *Switched Linear Parameter Varying (LPV) Systems*

Consider a class of discrete-time switched linear parameter varying (LPV) systems given by

$$x(k+1) = A_{\sigma(k)}(\rho(k))x(k) + B_{\sigma(k)}(\rho(k))w(k) \tag{5.33}$$

$$y(k) = C_{\sigma(k)}(\rho(k))x(k) + D_{\sigma(k)}(\rho(k))w(k) \tag{5.34}$$

$$z(k) = H_{\sigma(k)}(\rho(k))x(k) + L_{\sigma(k)}(\rho(k))w(k) \tag{5.35}$$

when $\sigma(k) = i \in \mathcal{I}$, the matrices $A_i(\rho(k))$, $B_i(\rho(k))$, $C_i(\rho(k))$, $D_i(\rho(k))$, $H_i(\rho(k))$, $L_i(\rho(k))$, denoting the ith subsystem, are known functions of measurable $\rho(k)$, where $\rho(k) = [\rho_1(k), \ldots, \rho_s(k)]^T$, $|\rho_v(k)| \le \bar{\rho}_v, \forall 1 \le v \le s$ is a vector of time-varying parameters which belong to a compact set $\Re \in \mathbb{R}^s$.

In this subsection, we are interested in designing a full-order mode-dependent filter for system (5.33)–(5.35) under ADT switching. Since the time-varying parameters are real-time measurable, our desired filter can be constructed by

$$x_F(k+1) = A_{Fi}(\rho_k)x_F(k) + B_{Fi}(\rho_k)y(k) \tag{5.36}$$

$$z_F(k) = C_{Fi}(\rho_k)x_F(k) + D_{Fi}(\rho_k)y(k) \tag{5.37}$$

where $x_F(k)$ is the filter state and $A_{Fi}(\rho_k)$, $B_{Fi}(\rho_k)$, $C_{Fi}(\rho_k)$ and $D_{Fi}(\rho_k)$, $i \in \mathcal{I}$ (we write $\rho(k)$ as ρ_k for notation simplicity) are the filter matrices to be determined with the same parameter dependence of system (5.33)–(5.35). Also, the filter with the above form is assumed to be switched synchronously by the switching signal σ in system (5.33)–(5.35).

Augmenting the model (5.33)–(5.35) to include the states of the filter, denoting $\tilde{x}(k) \triangleq [\, x^T(k)\ x_F^T(k)\,]^T$, $e(k) \triangleq z(k) - z_F(k)$, we can obtain the following filtering error system

$$\tilde{x}(k+1) = \tilde{A}_i(\rho_k)\tilde{x}(k) + \tilde{B}_i(\rho_k)w(k) \tag{5.38}$$

$$e(k) = \tilde{C}_i(\rho_k)\tilde{x}(k) + \tilde{D}_i(\rho_k)w(k) \tag{5.39}$$

where,

$$\tilde{A}_i(\rho_k) \triangleq \begin{bmatrix} A_i(\rho_k) & 0 \\ B_{Fi}(\rho_k)C_i(\rho_k) & A_{Fi}(\rho_k) \end{bmatrix}, \ \tilde{B}_i(\rho_i) \triangleq \begin{bmatrix} B_i(\rho_k) \\ B_{Fi}(\rho_k)D_i(\rho_k) \end{bmatrix},$$
$$\tilde{C}_i(\rho_k) \triangleq \begin{bmatrix} H_i(\rho_k) - D_{Fi}(\rho_k)C_i(\rho_k) & -C_{Fi}(\rho_k) \end{bmatrix},$$
$$\tilde{D}_i(\rho_k) \triangleq L_i(\rho_k) - D_{Fi}(\rho_k)D_i(\rho_k).$$

Thus, our objective here is to calculate matrices $(A_{Fi}(\rho_k), B_{Fi}(\rho_k), C_{Fi}(\rho_k), D_{Fi}(\rho_k))$ of the parameterized filter, and find out admissible switching signals such that the filtering error system (5.38)–(5.39) is exponentially stable and has a guaranteed exponential H_∞ performance index.

Remark 5.11 Note that if we restrict $[A_{Fi}(\rho_k), B_{Fi}(\rho_k), C_{Fi}(\rho_k), D_{Fi}(\rho_k)] \triangleq [A_{Fi}, B_{Fi}, C_{Fi}, D_{Fi}]$ or select $[A_{Fi}, B_{Fi}, C_{Fi}, D_{Fi}] \triangleq [A_F, B_F, C_F, D_F]$ in (5.36)–(5.37), one will readily obtain the different non-parameterized filters with different conservatism and computational complexity.

Lemma 5.12 *Consider switched linear system (5.38)–(5.39) and let* $0 < \alpha < 1$, $\gamma > 0$ *and* $\mu > 1$ *be given constants. If there exist matrix functions* $P_i(\rho_k) > 0$, $\forall i \in \mathcal{I}$ *such that*

$$\begin{bmatrix} -P_i(\rho_{k+1}) & 0 & P_i(\rho_{k+1})\tilde{A}_i(\rho_k) & P_i(\rho_{k+1})\tilde{B}_i(\rho_k) \\ \star & -I & \tilde{C}_i(\rho_k) & \tilde{D}_i(\rho_k) \\ \star & \star & -(1-\alpha)P_i(\rho_k) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{5.40}$$

$$P_i(\rho_k) - \mu P_j(\rho_k) \le 0 \tag{5.41}$$

then over the entire parameter set, the filtering error system (5.38)–(5.39) is exponentially stable and has an exponential H_∞ *performance index* γ *for any switching signals with ADT satisfying (2.26).*

Remark 5.13 In Lemma 5.12, the desired exponential H_∞ performance index for the underlying system in the subsection is achieved by $\gamma = \max\{\gamma_i\}_{i\in\mathcal{I}}$, where γ_i corresponds to the performance index for each subsystem. The proof of Lemma 5.12 can be completed by referring to Theorems 3.8 and 3 in [3]. More specifically, the time-varying parameters considered here can be dealt with as done in Sect. 3.2 for the considered polytopic uncertainties. Then, the difference of the corresponding parameterized Lyapunov function will present the variations of ρ from ρ_k to ρ_{k+1}, as shown in Theorem 3 in [3] with ρ_k and ρ_{k+1} mixed.

The following theorem presents a sufficient existence condition of an admissible filter for the underlying systems.

Theorem 5.14 *Consider switched linear system (5.33)–(5.35) and let* $0 < \alpha < 1$, $\gamma > 0$ *and* $\mu > 1$ *be given constants. If there exist matrices* $\bar{P}_{1i}(\rho_k) > 0$, $\bar{P}_{3i}(\rho_k) > 0$, *and matrices* $\bar{P}_{2i}(\rho_k)$, $R_i(\rho_k)$, $S_i(\rho_k)$, T_i, $\hat{A}_{Fi}(\rho_k)$, $\hat{B}_{Fi}(\rho_k)$, $\hat{C}_{Fi}(\rho_k)$, $\hat{D}_{Fi}(\rho_k)$, $\forall i \in \mathcal{I}$, *such that the following parameterized LMIs hold*

$$\begin{bmatrix} \tilde{P}_{1i}(\rho_k) & \bar{P}_{2i}(\rho_k) - \mu S_i(\rho_k) - \mu T_i & R_i^T(\rho_k) & T_j \\ \star & \bar{P}_{3i}(\rho_k) - \mu T_i^T - \mu T_i & S_i^T(\rho_k) & T_j \\ \star & \star & -\mu^{-1}\bar{P}_{1j}(\rho_k) & -\mu^{-1}\bar{P}_{2j}(\rho_k) \\ \star & \star & \star & -\mu^{-1}\bar{P}_{3j}(\rho_k) \end{bmatrix} \le 0,\ i \ne j \tag{5.42}$$

$$\begin{bmatrix} \Lambda_{11}^i & \Lambda_{12}^i & 0 & \Lambda_{14}^i & \hat{A}_{Fi}(\rho_k) & \Lambda_{16}^i \\ \star & \Lambda_{22}^i & 0 & \Lambda_{24}^i & \hat{A}_{Fi}(\rho_k) & \Lambda_{26}^i \\ \star & \star & I & \Lambda_{34}^i & -\hat{C}_{Fi}(\rho_k) & \Lambda_{36}^i \\ \star & \star & \star & -(1-\alpha)\bar{P}_{1i}(\rho_k) & -(1-\alpha)\bar{P}_{2i}(\rho_k) & 0 \\ \star & \star & \star & \star & -(1-\alpha)\bar{P}_{3i}(\rho_k) & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{5.43}$$

where,

$$\begin{aligned}
&\Lambda_{11}^i \triangleq \bar{P}_{1i}(\rho_{k+1}) - R_i^T(\rho_k) - R_i(\rho_k),\ \Lambda_{12}^i \triangleq \bar{P}_{2i}(\rho_{k+1}) - S_i(\rho_k) - T_i,\\
&\Lambda_{22}^i \triangleq \bar{P}_{3i}(\rho_{k+1}) - T_i^T - T_i,\ \Lambda_{14}^i \triangleq R_i^T(\rho_k)A_i(\rho_k) + \hat{B}_{Fi}(\rho_k)C_i(\rho_k),\\
&\Lambda_{24}^i \triangleq S_i^T(\rho_k)A_i(\rho_k) + \hat{B}_{Fi}(\rho_k)C_i(\rho_k),\ \Lambda_{34}^i \triangleq H_i(\rho_k) - \hat{D}_{Fi}(\rho_k)C_i(\rho_k),\\
&\Lambda_{16}^i \triangleq R_i^T(\rho_k)B_i(\rho_k) + \hat{B}_{Fi}(\rho_k)D_i(\rho_k),\ \Lambda_{26}^i \triangleq S_i^T(\rho_k)B_i(\rho_k) + \hat{B}_{Fi}(\rho_k)D_i(\rho_k),\\
&\Lambda_{36}^i \triangleq L_i(\rho_k) - \hat{D}_{Fi}(\rho_k)D_i(\rho_k),\ \tilde{P}_{1i}(\rho_k) \triangleq \bar{P}_{1i}(\rho_k) - \mu R_i^T(\rho_k) - \mu R_i(\rho_k).
\end{aligned}$$

Then, there exists a mode-dependent full-order parameterized filter such that the corresponding filter error system (5.38)–(5.39) is exponentially stable with a guaranteed exponential H_∞ performance index γ for any switching signal with the ADT satisfying (2.26). Moreover, if the LMIs (5.42)–(5.43) have a feasible solution, then the admissible filter in the form (5.36)–(5.37) can be given by

$$\begin{bmatrix} A_{Fi}(\rho_k) & B_{Fi}(\rho_k) \\ C_{Fi}(\rho_k) & D_{Fi}(\rho_k) \end{bmatrix} \triangleq \begin{bmatrix} T_i^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}_{Fi}(\rho_k) & \hat{B}_{Fi}(\rho_k) \\ \hat{C}_{Fi}(\rho_k) & \hat{D}_{Fi}(\rho_k) \end{bmatrix} \tag{5.44}$$

Proof By Lemma 5.12, system (5.38)–(5.39) is exponentially stable with a prescribed exponential H_∞ noise-attenuation level bound γ if (5.40) and (5.41) hold. Then, consider an arbitrary matrix function $G_i(\rho_k)$, $\forall i \in \mathcal{I}$ with compatible dimension, we have the fact

$$(P_i(\rho_{k+1}) - G_i(\rho_k))^T P_i^{-1}(\rho_{k+1})(P_i(\rho_{k+1}) - G_i(\rho_k)) \geq 0$$

$$(P_j(\rho_k) - G_i(\rho_k))^T P_j^{-1}(\rho_k)(P_j(\rho_k) - G_i(\rho_k)) \geq 0$$

thus one has

$$P_i(\rho_{k+1}) - G_i(\rho_k) - G_i^T(\rho_k) \geq -G_i^T(\rho_k)P_i^{-1}(\rho_{k+1})G_i(\rho_k)$$

$$P_j(\rho_k) - G_i(\rho_k) - G_i^T(\rho_k) \geq -G_i^T(\rho_k)P_j^{-1}(\rho_k)G_i(\rho_k)$$

Therefore, if the following inequalities hold

$$\begin{bmatrix} P_i(\rho_{k+1}) - G_i(\rho_k) - G_i^T(\rho_k) & 0 & G_i^T(\rho_k)\tilde{A}_i(\rho_k) & G_i^T(\rho_k)\tilde{B}_i(\rho_k) \\ \star & -I & \tilde{C}_i(\rho_k) & \tilde{D}_i(\rho_k) \\ \star & \star & -(1-\alpha)P_i(\rho_k) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{5.45}$$

$$P_i(\rho_k) - \mu\left[G_i(\rho_k) + G_i^T(\rho_k) - G_i^T(\rho_k)P_j^{-1}(\rho_k)G_i(\rho_k)\right] \leq 0 \tag{5.46}$$

then (5.46) implies (5.41). Also, from (5.45), we can obtain

$$\begin{bmatrix} -G_i^T(\rho_k)P_i^{-1}(\rho_{k+1})G_i(\rho_k) & 0 & G_i^T(\rho_k)\tilde{A}_i(\rho_k) & G_i^T(\rho_k)\tilde{B}_i(\rho_k) \\ \star & -I & \tilde{C}_i(\rho_k) & \tilde{D}_i(\rho_k) \\ \star & \star & -(1-\alpha)P_i(\rho_k) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0$$

By performing a congruence transformation to above formula via $diag\{G_i^{-1}(\rho_k) \times P_i(\rho_{k+1}), I, I, I\}$ yields (5.40) (note that $G_i(\rho_k)$ will be invertible if it satisfies (5.45)). In addition, by Lemma 2.4, (5.46) is equivalent to

$$\begin{bmatrix} P_i(\rho_k) - \mu G_i(\rho_k) - \mu G_i^T(\rho_k) & G_i^T(\rho_k) \\ \star & -\mu^{-1}P_j(\rho_k) \end{bmatrix} \leq 0 \tag{5.47}$$

Now, let us show that conditions (5.42) and (5.43) ensure respectively that (5.45) and (5.47) are satisfied. Firstly, if (5.43) holds, we know that $\bar{P}_{3i}(\rho_{k+1}) - T_i^T - T_i < 0$, thus we can infer that $T_i^T + T_i > 0$, which implies T_i is nonsingular. Then one can always find nonsingular matrices G_{3i} and G_4 satisfying $T_i = G_4^T G_{3i}^{-1} G_4, \forall i \in \mathcal{I}$. Now, introduce the following matrix variables related to G_{3i} and G_4

$$V_i \triangleq \begin{bmatrix} I & 0 \\ 0 & G_{3i}^{-1}G_4 \end{bmatrix}, G_i(\rho_k) \triangleq \begin{bmatrix} R_i(\rho_k) & S_i(\rho_k)G_4^{-1}G_{3i} \\ G_4 & G_{3i} \end{bmatrix}$$

By further performing a congruence transformation to (5.42) and (5.43) via $diag\{V_i^{-1}, I, V_i^{-1}, I\}$ and $diag\{V_i^{-1}, V_j^{-1}\}$, respectively, and setting matrix functions

$$P_i(\rho_k) \triangleq V_i^{-T}\bar{P}_i(\rho_k)V_i^{-1} = V_i^{-T}\begin{bmatrix} \bar{P}_{1i}(\rho_k) & \bar{P}_{2i}(\rho_k) \\ \star & \bar{P}_{3i}(\rho_k) \end{bmatrix}V_i^{-1}$$

$$\begin{bmatrix} A_{Fi}(\rho_k) & B_{Fi}(\rho_k) \\ C_{Fi}(\rho_k) & D_{Fi}(\rho_k) \end{bmatrix} \triangleq \begin{bmatrix} G_4^{-T} & 0 \\ \star & I \end{bmatrix}\begin{bmatrix} \hat{A}_{Fi}(\rho_k) & \hat{B}_{Fi}(\rho_k) \\ \hat{C}_{Fi}(\rho_k) & \hat{D}_{Fi}(\rho_k) \end{bmatrix}\begin{bmatrix} G_4^{-1}G_{3i} & 0 \\ \star & I \end{bmatrix} \tag{5.48}$$

we can obtain (5.45) and (5.47).

Meanwhile, from (5.48) we know that an admissible filter for the underlying system is given by

$$A_{Fi}(\rho_k) = G_4^{-T}\hat{A}_{Fi}(\rho_k)G_4^{-1}G_{3i}, B_{Fi}(\rho_k) = G_4^{-T}\hat{B}_{Fi}(\rho_k), \tag{5.49}$$

$$C_{Fi}(\rho_k) = \hat{C}_{Fi}(\rho_k)G_4^{-1}G_{3i}, \ D_{Fi}(\rho_k) = \hat{D}_{Fi}(\rho_k). \tag{5.50}$$

Then, denoting the filter transfer function from $y(k)$ to $z_F(k)$ by

$$T(\mathbf{z}) = C_{Fi}(\rho_k)(\mathbf{z}I - A_{Fi}(\rho_k))^{-1}B_{Fi}(\rho_k) + D_{Fi}(\rho_k),$$

substituting the matrices $(A_{Fi}, B_{Fi}, C_{Fi}, D_{Fi})$ in (5.49)–(5.50) and considering $T_i = G_4^T G_{3i}^{-1} G_4$, we have

$$\begin{aligned} T(\mathbf{z}) &= \hat{C}_{Fi}(\rho_k) G_4^{-1} G_{3i}(\mathbf{z}I - G_4^{-T} \hat{A}_{Fi}(\rho_k) G_4^{-1} G_{3i})^{-1} G_4^{-T} \hat{B}_{Fi}(\rho_k) + \hat{D}_{Fi}(\rho_k) \\ &= \hat{C}_{Fi}(\rho_k)(\mathbf{z}I - T_i^{-1} \hat{A}_{Fi}(\rho_k))^{-1} T_i^{-1} \hat{B}_{Fi}(\rho_k) + \hat{D}_{Fi}(\rho_k), \end{aligned}$$

which implies an admissible filter can be given by (5.44), and the proof is completed. □

Remark 5.15 From (5.42)–(5.44), it can be obviously seen that the filter gains will be dependent on μ, which resembles, on some level, the delay-dependent issues in time-delay system to determine delay-dependent filter. Therefore, a μ-dependent approach for the underlying system is introduced here, and the results obtained via this approach will be less conservative than those based on global Lyapunov function and switched Lyapunov function [1] approaches, which one may refer to "μ-independent". In addition, conditions (5.42)–(5.43) are formulated in terms of a set of parameterized LMIs, which are not only over the matrix variables but also the scalar γ^2. Therefore, the scalar can be optimized by a μ-dependent convex optimization problem for a fixed system decay degree α as follows.

Problem 5.2

$$\min\ \delta,\ \text{s.t.}(5.42)\text{–}(5.43),\ \forall i \in \mathcal{I},$$

with $\delta = \gamma^2$ over $\bar{P}_{1i}(\rho_k)$, $\bar{P}_{3i}(\rho_k)$, $\bar{P}_{2i}(\rho_k)$, $R_i(\rho_k)$, $S_i(\rho_k)$, T_i, $\hat{A}_{Fi}(\rho_k)$, $\hat{B}_{Fi}(\rho_k)$, $\hat{C}_{Fi}(\rho_k)$, $\hat{D}_{Fi}(\rho_k)$. The minimum noise attenuation level bound is then obtained by $\gamma^* = \sqrt{\delta^*}$, where δ^* is the optimal value of δ, and the corresponding filter matrices are given by (5.44).

As shown in the most of LPV literature [4], by choosing appropriate basis functions $\{f_l(\rho_k)\}_{l=1}^{n_f}$, the matrix function variables $\mathcal{Y}_i(\rho) = \{\bar{P}_{1i}(\rho_k), \bar{P}_{2i}(\rho_k), \bar{P}_{3i}(\rho_k), R_i(\rho_k), S_i(\rho_k), \hat{A}_{Fi}(\rho_k), \hat{B}_{Fi}(\rho_k), \hat{C}_{Fi}(\rho_k), \hat{D}_{Fi}(\rho_k)\}$ in Problem 5.2 can be decomposed as the following affine fashion

$$\mathcal{Y}_i(\rho) = \sum_{l=1}^{n_f} f_l(\rho_k) \mathcal{Y}_i^l \tag{5.51}$$

where $f_l(\rho_k)$ and n_f can be chosen by designers according to the dependence structure in system (5.33)–(5.35), and $\mathcal{Y}_i^l = \{\bar{P}_{1i}^l, \bar{P}_{2i}^l, \bar{P}_{3i}^l, R_i^l, S_i^l, \hat{A}_{Fi}^l, \hat{B}_{Fi}^l, \hat{C}_{Fi}^l, \hat{D}_{Fi}^l\}$ will be the corresponding decision variables in Problem 5.2. Also, one can utilize the gridding technique in non-switched LPV systems such that the above infinite-dimension convex problem is solvable (more details can be referred to [4]).

In the following, an illustrate example is presented to demonstrate the feasibility and efficiency of the designed filter in this subsection.

Example 5.16 Consider the following discrete-time switched linear systems consisting of two subsystems with time-varying parameters

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.95 & 1.10+0.1\rho_k \\ -0.06 & 0.75 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} 0.95 & -1.10 \\ 0.06 & -0.75+0.1\rho_k \end{bmatrix},\ B_2 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & -1 \end{bmatrix},\ D_1 = 1,\ , \\
H_1 &= H_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D_2 = -1, L_1 = L_2 = 0
\end{aligned}
$$

where $\rho_k = \cos(0.2\pi k)$ is the time-varying measurable parameters.

Our purpose is to design a full-order mode-dependent filter in the form of (5.36)–(5.37) and find out the admissible ADT switching signals for the above switched system such that the resulting filter error system is exponentially stable and has a guaranteed exponential H_∞ performance for a given decay degree α .

According to the structure of the parameter dependence in above system, we choose the basic functions in (5.51) as $f_1(\rho_k) = 1$ and $f_2(\rho_k) = \cos(0.2\pi k)$. Further, grid the parameter space of ρ_k with 10 uniform grids, which means to uniformly partition the value set of ρ_k, $[-1, 1]$ with 10 parts. Then, giving $\alpha = 0.1$ and different μ, and solving Problem 5.2, we can obtain the different μ-dependent optimal γ^* and the corresponding minimal ADT τ_a^*, as shown in Table 5.4. It is clear that the obtained exponential H_∞ performance γ^* is actually dependent on μ for a given system decay degree α. Also, the larger μ corresponds to the smaller γ^*, but the longer ADT will be demanded in the system.

In addition, the desired mode-dependent full-order filter can be also obtained by solving Problem 5.2. We omit the filter gains for brevity. Furthermore, consider the input signal $w(k) = 0.1\exp(-0.03k)\sin(0.02\pi k)$, and by applying the solved filter, we can obtain the error response of the resulting filtering error system in Figs. 5.7 and 5.8 for given four different switching signals (all are with $\tau_a = 4 > 3.19 = \tau_a^*$ for $\mu = 1.4$) and initial condition $x = [-0.8\ 0.5\ 0\ 0]^T$. It is clearly observed from the simulation curves that for given energy bounded disturbance $w(k)$, the filtering error system is stable against time-varying parameters under the different switching signals, which thereby implies that the designed filter is valid.

Table 5.4 μ-dependent optimal γ^* for given $\alpha = 0.1$

μ	1.20	1.25	1.30	1.40
τ_a^*	1.73	2.18	2.49	3.19
γ^*	2.47	2.35	2.23	2.01

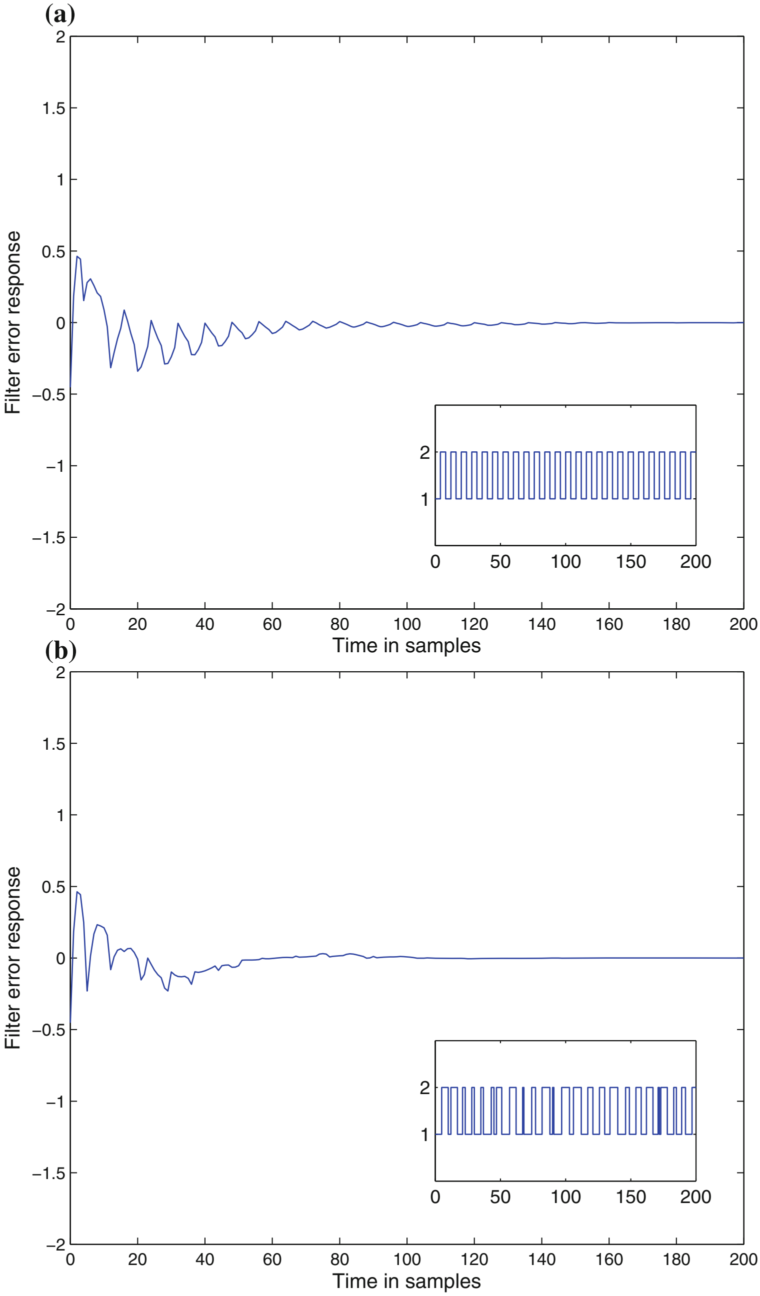
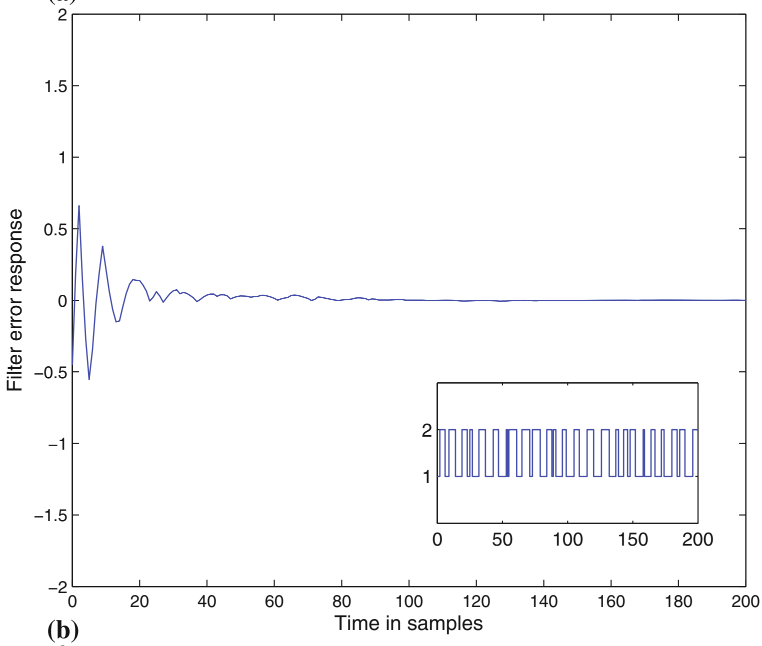


Fig. 5.7 Filtering error response under $\tau_a = 4$ with different switching signals $\sigma_i, i = 1, 2$. **a** Switching signal $\sigma_1(\text{ADT} = 4)$. **b** Switching signal $\sigma_2(\text{ADT} = 4)$

(a)



(b)

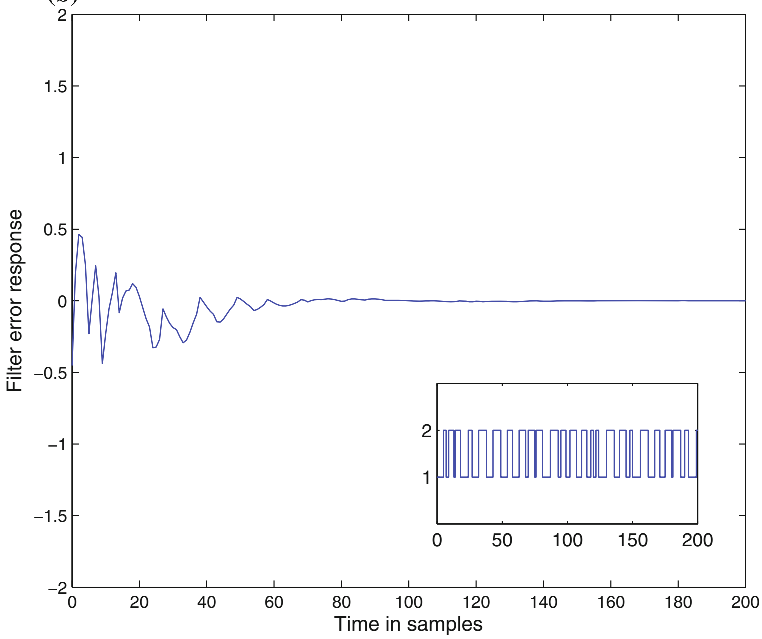


Fig. 5.8 Filtering error response under $\tau_a = 4$ with different switching signals $\sigma_i, i = 3, 4$. **a** Switching signal $\sigma_3(\text{ADT} = 4)$. **b** Switching signal $\sigma_4(\text{ADT} = 4)$

5.3 Quasi-Time-Dependent (QTD) H_∞ Filtering: PDT Switching

Consider a class of discrete-time switched linear systems

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}w(k) \tag{5.52}$$
$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}w(k) \tag{5.53}$$
$$z(k) = H_{\sigma(k)}x(k) + L_{\sigma(k)}w(k) \tag{5.54}$$

when $\sigma(k) = i \in \mathcal{I}$, the matrices $(A_i,\ B_i,\ C_i,\ D_i,\ H_i,\ L_i)$ denote the ith subsystem. The persistent dwell-time (PDT) switching is introduced in this section. The definition of PDT has been given in Sect. 1.4 and is therefore omitted here.

Based on the discussions in Sect. 3.3, a both mode-dependent and quasi-time-dependent (QTD) Lyapunov function will be also explored in this section, upon which a full-order switched filter with the following structure for system (5.52)–(5.54) will be considered

$$x_F(k+1) = A_{F_{\sigma(k)}}(q_k)x_F(k) + B_{F_{\sigma(k)}}(q_k)y(k) \tag{5.55}$$
$$z_F(k) = C_{F_{\sigma(k)}}(q_k)x_F(k) + D_{F_{\sigma(k)}}(q_k)y(k) \tag{5.56}$$

where $A_{F_{\sigma(k)}}(q_k)$, $B_{F_{\sigma(k)}}(q_k)$, $C_{F_{\sigma(k)}}(q_k)$ and $D_{F_{\sigma(k)}}(q_k)$, $\forall\sigma(k) \in \mathcal{I}$, are filter gains to be determined, and q_k is a scheduler for the activated subsystem and can be simply computed online according to the rules given as in Sect. 2.5.

Augmenting the model of (5.52)–(5.54) to include the states of the filter (5.55)–(5.56), we obtain the following filtering error system

$$\tilde{x}(k+1) = \bar{A}_i(q_k)\tilde{x}(k) + \bar{E}_i(q_k)w(k) \tag{5.57}$$
$$e(k) = \bar{C}_i(q_k)\tilde{x}(k) + \bar{F}_i(q_k)w(k) \tag{5.58}$$

where $\tilde{x}(k) \triangleq \begin{bmatrix} x^T(k) \ x_F^T(k) \end{bmatrix}^T$, $e(k) \triangleq z(k) - z_F(k)$ and

$$\begin{aligned}
\bar{A}_i(q_k) &\triangleq \begin{bmatrix} A_i & 0 \\ B_{Fi}(q_k)C_i & A_{Fi}(q_k) \end{bmatrix}, \bar{E}_i(q_k) \triangleq \begin{bmatrix} B_i \\ B_{Fi}(q_k)D_i \end{bmatrix}, \\
\bar{C}_i(q_k) &\triangleq \begin{bmatrix} H_i - D_{Fi}(q_k)C_i & -C_{Fi}(q_k) \end{bmatrix}, \\
\bar{F}_i(q_k) &\triangleq L_i - D_{Fi}(q_k)D_i.
\end{aligned} \tag{5.59}$$

Then, our objective in this section is to design such a QTD full-order filter and find a set of admissible PDT switching signals such that the resulting filtering error system (5.57)–(5.58) is globally uniformly asymptotically stable (GUAS) and has a guaranteed non-weighted H_∞ noise attenuation performance, i.e., the l_2-gain holds $\|e\|_2^2 \leq \gamma^2 \|w\|_2^2$.

In this section, the desired filter design will be carried out for system (5.52)–(5.54), for which we will first present the linear case of Lemma 5.17 as below, by considering $V_i(\tilde{x}(k), q_k) \triangleq \tilde{x}^T(k)P_i(q_k)\tilde{x}(k)$, $P_i(q_k) \in \mathcal{S}^{2n_x}_{>0}$, $\forall i \in \mathcal{I}$ in (3.47).

Lemma 5.17 *Consider switched linear system (5.57)–(5.58) and let* $0 < \alpha < 1$, $\mu > 1$ *be given constants. For a prescribed period of persistence* $\mathbb{T}$*, suppose that there exist a set of matrices* $P_i(\varphi) \in \mathcal{S}^{2n_x}_{>0}$, $\varphi = 0, 1, \ldots, \tau$, $\forall i \in \mathcal{I}$ *and a scalar* $\gamma > 0$ *such that* $\varphi = 0, 1, \ldots, \tau - 1$,

$$\Theta(\tau, \tau) < 0 \tag{5.60}$$

$$\Theta(\varphi + 1, \varphi) < 0 \tag{5.61}$$

and $\forall (i \times j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$

$$P_i(0) - \mu P_j(\mathbb{T}_j) \leq 0 \tag{5.62}$$

$$P_i(0) - \mu P_j(\tau) \leq 0 \tag{5.63}$$

where $\mathbb{T}_j$, $\forall i \in \mathcal{I}$ *is denoted in Theorem 3.12, the period of persistence* $\mathbb{T}$ *is given and* $\Theta(\theta_1, \theta_2) \triangleq \bar{\mathcal{A}}_i^T(\theta_2)\Theta_1\bar{\mathcal{A}}_i(\theta_2) - \Theta_2$ *with* $\Theta_1 \triangleq diag\{P_i(\theta_1), I\}$, $\Theta_2 \triangleq diag\{\alpha P_i(\theta_2), \gamma^2 I\}$ *and*

$$\bar{\mathcal{A}}_i(\theta) \triangleq \begin{bmatrix} \bar{A}_i(\theta) & \bar{E}_i(\theta) \\ \bar{C}_i(\theta) & \bar{F}_i(\theta) \end{bmatrix}.$$

Then switched system (5.57)–(5.58) is GUAS and has an H_∞ *performance index no greater than* $\gamma_t = \gamma\beta$*, where* β *is defined in Theorem 3.13, for PDT switching signals satisfying (3.45) and (5.60)–(5.63).*

Proof Consider $\varphi = 0, 1, \ldots, \tau - 1$ and $k - k_{s_p} = \varphi$, $\forall k \in [k_{s_p}, k_{s_p} + \tau)$, $\sigma(k) = i \in \mathcal{I}$. Letting $\zeta(k) \triangleq \begin{bmatrix} \tilde{x}^T(k) & w^T(k) \end{bmatrix}^T$, it follows that

$$\begin{aligned}
\digamma(k, \varphi) &\triangleq \zeta^T(k)\Theta(\varphi + 1, \varphi)\zeta(k) \\
&= \zeta^T(k)\left[\bar{\mathcal{A}}_i^T(\varphi)\Theta_1\bar{\mathcal{A}}_i(\varphi) - \Theta_2\right]\zeta(k) \\
&= \zeta^T(k)\left\{\begin{bmatrix} \bar{A}_i^T(\varphi) & \bar{C}_i^T(\varphi) \\ \bar{E}_i^T(\varphi) & \bar{F}_i^T(\varphi) \end{bmatrix}\begin{bmatrix} P_i(\varphi+1) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \bar{A}_i(\varphi) & \bar{E}_i(\varphi) \\ \bar{C}_i(\varphi) & \bar{F}_i(\varphi) \end{bmatrix}\right. \\
&\quad \left. - \begin{bmatrix} \alpha P_i(\varphi) & 0 \\ 0 & \gamma^2 I \end{bmatrix}\right\}\zeta(k) \\
&= \tilde{x}^T(k)[\bar{A}_i^T(k - k_{s_p})P_i(k + 1 - k_{s_p})\bar{A}_i(k - k_{s_p}) + \bar{C}_i^T(k - k_{s_p})\bar{C}_i(k - k_{s_p}) \\
&\quad - \alpha P_i(k - k_{s_p})]\tilde{x}(k) + 2\tilde{x}^T(k)[\bar{A}_i^T(k - k_{s_p})P_i(k + 1 - k_{s_p})\bar{E}_i(k - k_{s_p}) \\
&\quad + \bar{C}_i^T(k - k_{s_p})\bar{F}_i(k - k_{s_p})]w(k) + w^T(k)[\bar{E}_i^T(k - k_{s_p})P_i(k + 1 - k_{s_p}) \\
&\quad \times \bar{E}_i(k - k_{s_p}) + \bar{F}_i^T(k - k_{s_p})\bar{F}_i(k - k_{s_p}) - \gamma^2 I]w(k)
\end{aligned}$$

If (5.61) is satisfied, then $\digamma(k, \varphi) < 0$, $\varphi = 0, 1, \ldots, \tau - 1$, which yields (3.54) according to (5.57), (5.58) and the construction of $V_i(\tilde{x}(k), q_k)$ while replacing the system state and output, $x(k)$ and $y(k)$ in Theorem 3.13 with $\tilde{x}(k)$ and $e(k)$ of system (5.57)–(5.58), respectively. By the similar manipulation, it can be shown that (3.55) is guaranteed by (5.60), $\forall k \in \left[k_{s_p} + \tau, k_{s_p+1}\right)$. In addition, when $k \in \left[k_{s_p+1}, k_{s_{p+1}}\right)$, $\forall \sigma(k) = i \in \mathcal{I}$, since $q_k = k - H_r < \tau$ holds, (5.61) ensures (3.56) for $\varphi = 0, 1, \ldots, \tau - 2$. Then together with (5.62) and (5.63), which ensures (3.52)–(3.53), respectively, the proof is completed by Theorem 3.13. □

Based on Lemma 5.17, we are in a position to give the existence conditions of the QTD H_∞ filter for the underlying system (5.52)–(5.54) in the following theorem.

Theorem 5.18 *Consider switched linear system (5.52)–(5.54) and let* $0 < \alpha < 1$, $\mu > 1$ *be given constants. For a prescribed period of persistence* $\mathbb{T}$*, suppose that there exist a set of matrices* $X_i(\varphi)$, $Y_i(\varphi)$, $Z_i(\varphi)$, $\bar{A}_{Fi}(\varphi)$, $\bar{B}_{Fi}(\varphi)$, $\bar{C}_{Fi}(\varphi)$, $\bar{D}_{Fi}(\varphi)$, $P_{2i}(\varphi)$, $P_{1i}(\varphi) \in \mathcal{S}^{n_x}_{>0}$, $P_{3i}(\varphi) \in \mathcal{S}^{n_x}_{>0}$, $\varphi = 0, 1, \ldots, \tau$, $\forall i \in \mathcal{I}$ *and a scalar* $\gamma > 0$ *such that* $\forall \varphi = 0, 1, \ldots, \tau - 1$,

$$\Psi_i(\tau, \tau) < 0 \tag{5.64}$$

$$\Psi_i(\varphi + 1, \varphi) < 0 \tag{5.65}$$

and (5.62)–(5.63) hold for any $(i \times j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, *where*

$$\Psi_i(\theta_1, \theta_2) \triangleq \begin{bmatrix} diag\{\Lambda_i, -I\} & diag\{\mathcal{X}_i(\theta_2), I\}\hat{\mathcal{A}}_i(\theta_2) \\ \star & -diag\{\alpha P_i(\theta_2), \gamma^2 I\} \end{bmatrix}$$

with $\Lambda_i \triangleq P_i(\theta_1) - diag\{I, Y_i^T(\theta_2)\}\mathcal{X}_i^T(\theta_2) - \mathcal{X}_i(\theta_2)\, diag\{I, Y_i(\theta_2)\}$ *and*

$$\begin{aligned}
P_i(\theta) &\triangleq \begin{bmatrix} P_{1i}(\theta) & P_{2i}(\theta) \\ \star & P_{3i}(\theta) \end{bmatrix}, \mathcal{X}_i(\theta) \triangleq \begin{bmatrix} X_i(\theta) & I \\ Z_i(\theta) & I \end{bmatrix}, \\
\hat{\mathcal{A}}_i(\theta) &\triangleq \begin{bmatrix} \hat{A}_i(\theta) & \hat{E}_i(\theta) \\ \hat{C}_i(\theta) & \hat{F}_i(\theta) \end{bmatrix}, \hat{F}_i(\theta) \triangleq L_i - \bar{D}_{Fi}(\theta)D_i, \\
\hat{A}_i(\theta) &\triangleq \begin{bmatrix} A_i & 0 \\ \bar{B}_{Fi}(\theta)C_i & \bar{A}_{Fi}(\theta) \end{bmatrix}, \hat{E}_i(\theta) \triangleq \begin{bmatrix} B_i \\ \bar{B}_{Fi}(\theta)D_i \end{bmatrix}, \\
\hat{C}_i(\theta) &\triangleq \begin{bmatrix} H_i - \bar{D}_{Fi}(\theta)C_i & -\bar{C}_{Fi}(\theta) \end{bmatrix}.
\end{aligned}$$

Then the switched system (5.57)–(5.58) is GUAS and has an H_∞ *performance index* $\gamma_t = \gamma\beta$, *where* β *is defined in Theorem 3.13, for PDT switching signals satisfying (3.45) and (5.62)–(5.65). Moreover, if a feasible solution exists, then the admissible filter gains are given by,* $\varphi = 0, 1, \ldots, \tau$,

$$\begin{aligned}
A_{Fi}(\varphi) &= Y_i^{-1}(\varphi)\bar{A}_{Fi}(\varphi), C_{Fi}(\varphi) = \bar{C}_{Fi}(\varphi), \\
B_{Fi}(\varphi) &= Y_i^{-1}(\varphi)\bar{B}_{Fi}(\varphi), D_{Fi}(\varphi) = \bar{D}_{Fi}(\varphi).
\end{aligned} \tag{5.66}$$

Proof First of all, for matrix $P_i(\varphi+1)$, $\varphi = 0, 1, \ldots, \tau-1$, $\forall i \in \mathcal{I}$, from the fact that $(P_i(\varphi+1) - \mathcal{X}_i(\varphi)diag\{I, Y_i(\varphi)\})P_i^{-1}(\varphi+1)(P_i(\varphi+1) - \mathcal{X}_i(\varphi)diag\{I, Y_i^T(\varphi)\}) \geq 0$, we have $P_i(\varphi+1) - diag\{I, Y_i^T(\varphi)\}\mathcal{X}_i^T(\varphi) - \mathcal{X}_i(\varphi)diag\{I, Y_i(\varphi)\} \geq -\mathcal{X}_i(\varphi)\ diag\{I, Y_i(\varphi)\}P_i^{-1}(\varphi+1)diag\{I, Y_i^T(\varphi)\}\mathcal{X}_i^T(\varphi)$. Then if (5.64) and (5.65) hold, the following inequalities hold

$$\Upsilon(\tau, \tau) < 0 \tag{5.67}$$

$$\Upsilon(\varphi+1, \varphi) < 0 \tag{5.68}$$

where

$$\Upsilon(\theta_1, \theta_2) \triangleq \begin{bmatrix} diag\{\Xi_i, -I\} & diag\{\mathcal{X}_i(\theta_2), I\}\hat{\mathcal{A}}_i(\theta_2) \\ \star & -diag\{\alpha P_i(\theta_2), \gamma^2 I\} \end{bmatrix}$$

with $\Xi_i \triangleq -\mathcal{X}_i(\theta_2)diag\{I, Y_i(\theta_2)\}P_i^{-1}(\theta_1)diag\{I, Y_i^T(\theta_2)\}\mathcal{X}_i^T(\theta_2)$.

Next, by performing congruence transformations to (5.67) via $diag\{\mathcal{X}_i^{-T}(\tau) \times diag\{I, Y_i^{-T}(\tau)\}P_i(\tau), I\}$ and (5.68) via $diag\{\mathcal{X}_i^{-T}(\varphi)diag\{I, Y_i^{-T}(\varphi)\}P_i(\varphi+1), I\}$, respectively, setting matrix variables, $\forall \varphi = 0, 1, \ldots, \tau$,

$$\begin{aligned} \bar{A}_{Fi}(\varphi) &= Y_i(\varphi)A_{Fi}(\varphi), \bar{C}_{Fi}(\varphi) = C_{Fi}(\varphi) \\ \bar{B}_{Fi}(\varphi) &= Y_i(\varphi)B_{Fi}(\varphi), \bar{D}_{Fi}(\varphi) = D_{Fi}(\varphi) \end{aligned} \tag{5.69}$$

and considering (5.3), one can obtain (5.60) and (5.61). Then by Lemma 5.17, the filtering error system (5.57)–(5.58) is GUAS for PDT switching signals satisfying (3.45) and (3.52)–(3.56) and has an H_∞ performance index γ_t. In addition, from (5.69), the QTD H_∞ filter gains are given by (5.66). □

Remark 5.19 In (5.62), $\mathbb{T}_j$ is not known a priori, therefore it is required to check (5.62) for all the possible cases of $\mathbb{T}_j$ among $\left[1, \min(\tau-1, \mathbb{T}^{(p)})\right]$ with $\mathbb{T}^{(p)} \in [1, \mathbb{T}]$, but note that, equivalently, all the resulting conditions will be covered in the case of $\mathbb{T}^{(p)} \equiv \mathbb{T}$.

For a comparison with the above QTD filter, we also present as below the time-independent filter that can be obtained based on the corresponding "q_k-independent" Lyapunov function $V_i(x(k))$, $\forall i \in \mathcal{I}$, reduced from (3.47). The proof can be obtained in a similar vein to the one for Theorem 5.18 using the techniques (3.57)–(3.59) and omitted here.

Corollary 5.20 *Consider switched linear system (5.52)–(5.54) and let* $0 < \alpha < 1$, $\mu > 1$ *be given constants. For a prescribed period of persistence* $\mathbb{T}$, *suppose that there exist a set of matrices* X_i , Y_i, Z_i, $\bar{A}_{Fi}$, $\bar{B}_{Fi}$, $\bar{C}_{Fi}$, $\bar{D}_{Fi}$, P_{2i}, $P_{1i} \in \mathcal{S}_{>0}^{n_x}$, $P_{3i} \in \mathcal{S}_{>0}^{n_x}$, $\forall i \in \mathcal{I}$ *and a scalar* $\gamma > 0$ *such that* $\Pi_i < 0$ *and* $\forall (i \times j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $P_i \leq \mu P_j$, *where*

$$\Pi_i \triangleq \begin{bmatrix} diag\{\Omega_i, -I\} & diag\{\mathcal{X}_i, I\}\hat{\mathcal{A}}_i \\ \star & -diag\{\alpha P_i, \gamma^2 I\} \end{bmatrix}$$

with $\Omega_i \triangleq P_i - diag\{I, Y_i^T\}\mathcal{X}_i^T - \mathcal{X}_i diag\{I, Y_i\}$ *and*

$$P_i \triangleq \begin{bmatrix} P_{1i} & P_{2i} \\ \star & P_{3i} \end{bmatrix}, \mathcal{X}_i \triangleq \begin{bmatrix} X_i & I \\ Z_i & I \end{bmatrix}, \hat{\mathcal{A}}_i \triangleq \begin{bmatrix} \hat{A}_i & \hat{E}_i \\ \hat{C}_i & \hat{F}_i \end{bmatrix}$$
$$\hat{F}_i \triangleq L_i - \bar{D}_{Fi}D_i, \hat{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ \bar{B}_{Fi}C_i & \bar{A}_{Fi} \end{bmatrix}, \hat{E}_i \triangleq \begin{bmatrix} B_i \\ \bar{B}_{Fi}D_i \end{bmatrix}$$
$$\hat{C}_i \triangleq \begin{bmatrix} H_i - \bar{D}_{Fi}C_i & -\bar{C}_{Fi} \end{bmatrix}.$$

Then switched system (5.52)–(5.54) is GUAS with an H_∞ performance index $\gamma_c = \gamma\beta$, for PDT switching signals satisfying (3.45), where β is defined in Theorem 3.13. Moreover, if a feasible solution exists, then the admissible filter gains are given by $A_{Fi} = Y_i^{-1}\bar{A}_{Fi}$, $B_{Fi} = Y_i^{-1}\bar{B}_{Fi}$, $C_{Fi} = \bar{C}_{Fi}$, $D_{Fi} = \bar{D}_{Fi}$.

Remark 5.21 Clearly, if setting $(A_{Fi}(\varphi), B_{Fi}(\varphi), C_{Fi}(\varphi), D_{Fi}(\varphi)) \equiv (A_{Fi}, B_{Fi}, C_{Fi}, D_{Fi})$, then the QTD filter designed by Theorem 5.18 reduces to the one by Corollary 5.20, which implies that the QTD filtering methodology offers more freedom and will be accordingly less conservative than the time-independent one used in Corollary 5.20. In other words, for a same PDT switched system, a QTD filter exists but a time-independent one may not; in addition, even though the two distinct filters both exist, the H_∞ performance achieved by the former can be better. The two observations will be verified in next section.

Example 5.22 Consider a mass-spring system as shown in Fig. 5.9, where x_1 and x_2 are the positions of the masses M_1 and M_2, respectively; the spring stiffness K^c is constant and $K^d_{\sigma(k)}$ is assumed to be changeable with $\sigma(k) \in \{1, 2\}$; c is the viscous friction coefficient between the masses and the horizontal surface, and w the disturbance input to the system. We suppose that $K^d_{\sigma(k)}$ is capable of automatically being replaced according to PDT switching sequences, to reflect a scenario in certain mechatronic systems that may encounter both low-frequency and high-frequency

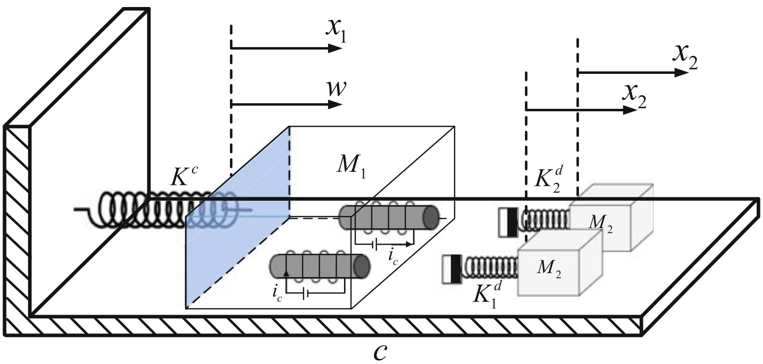


Fig. 5.9 A mass-spring system

motions in one control task, e.g., [5]. The purpose here is to design a QTD H_∞ filter to estimate x_2 by measuring x_1 against the switching of $K^d_{\sigma(k)}$, and show the advantage of QTD methodology by comparison.

Suppose that the measurement noise on x_1 is also w, the state-space realization of the system is given by (5.52)–(5.54) with

$$A_{\sigma(k)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K^c+K^d_{\sigma(k)}}{M_1} & \frac{K^d_{\sigma(k)}}{M_1} & -\frac{c}{M_1} & 0 \\ \frac{K^d_{\sigma(k)}}{M_2} & -\frac{K^d_{\sigma(k)}}{M_2} & 0 & -\frac{c}{M_2} \end{bmatrix}, B_{\sigma(k)} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix},$$

$$C_{\sigma(k)} = \begin{bmatrix} 1 \; 0 \; 0 \; 0 \end{bmatrix}, H_{\sigma(k)} = \begin{bmatrix} 0 \; 1 \; 0 \; 0 \end{bmatrix}, D_{\sigma(k)} = 0.1, L_{\sigma(k)} = 0.$$

Assign parameters $M_1 = 1\,kg$, $M_2 = 0.5\,kg$, $c = 0.5\,kg/s$, $K^c = K^d_1 = 1N/m$, $K^d_2 = 6N/m$ and consider the sampling period $T_s = 1\,s$.

For given $\alpha = 0.87$, by Theorem 5.18 and Corollary 5.20, respectively, the admissible PDTs corresponding to feasible solutions of the filters can be obtained while varying $\mathbb{T}$ and μ, as shown in Fig. 5.10. It can be clearly seen that the region of the admissible PDTs determined by Theorem 5.18 (QTD filtering) completely cover those by Corollary 5.20 (time-independent filtering). In addition, fix $\mu = 1.25$ and $\mathbb{T} = 1\,$s, the H_∞ performance indices are minimized based on Theorem 5.18 and Corollary 5.20, respectively, with $\gamma_t^* = 1.0154$ and $\gamma_c^* = 3.0492$.

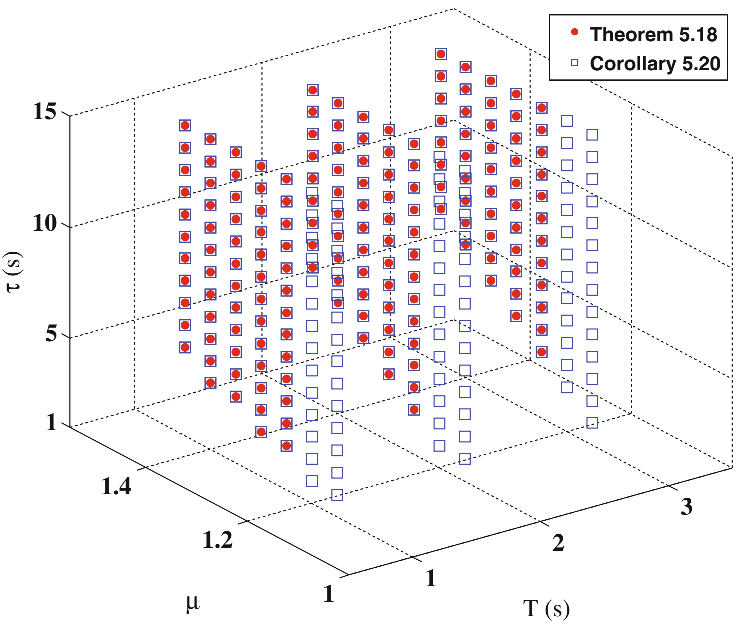


Fig. 5.10 Admissible PDTs for different $\mathbb{T}$ and μ

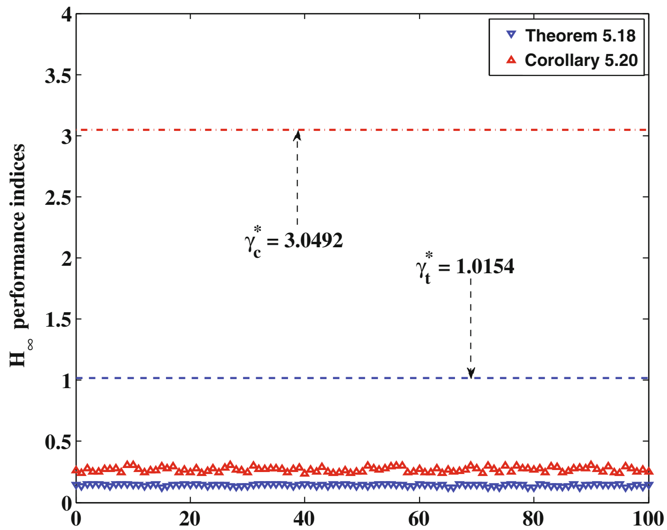


Fig. 5.11 Actual H_∞ performance indices by QTD filter (by Theorem 5.18) and time-independent filter (by Corollary 5.20) for given $\mathbb{T} = 1$ s

Consider zero initial condition and the noise input $w(k) = 0.9\exp(-0.1k)$, Fig. 5.11 gives the computation results of actual H_∞ performance indices for 100 realizations of random PDT switching sequences, by applying the filters obtained from Theorem 5.18 and Corollary 5.20, respectively. All the actual H_∞ performance indices in both cases are below the respective optimal ones, illustrating the effectiveness of the two filters. By comparison, however, the less conservatism of the QTD filter is clearly shown by the three facts that γ_t^* is less than γ_c^*, all the actual H_∞ performance indices achieved by QTD filter are lower than those by time-independent filter, and the gap between the optimum and the actual indices is smaller in the former case.

Figure 5.12 further presents three filtering error responses while randomly generating three realizations of PDT switching sequence for $\mathbb{T} = 2$ s and $\mathbb{T} = 3$ s, respectively, by applying the corresponding QTD filters. The convergence of the curves demonstrates that the designed filter is valid despite the faster switching during the period of persistence. Moreover, for given different μ and $\mathbb{T}$, Fig. 5.13 shows the variations of β and γ_t^* as τ changes, where two remarkable monotonicities can be observed. Specifically, for a fixed admissible PDT τ, a smaller γ_t^* corresponds to smaller μ and $\mathbb{T}$, as shown in Fig. 5.13a, b, respectively; in addition, γ_t^* is decreasing as τ increases in either case. Such phenomena verify the truth that, for a fixed α, the achieved H_∞ performance becomes better when the three relevant factors μ, $\mathbb{T}$ and τ tend to be more positive, i.e., the jump of Lyapunov function values at switching instants is smaller, the length of $\mathbb{T}$-portion containing faster switching is shorter, and the length of τ-portion of no switching is longer.

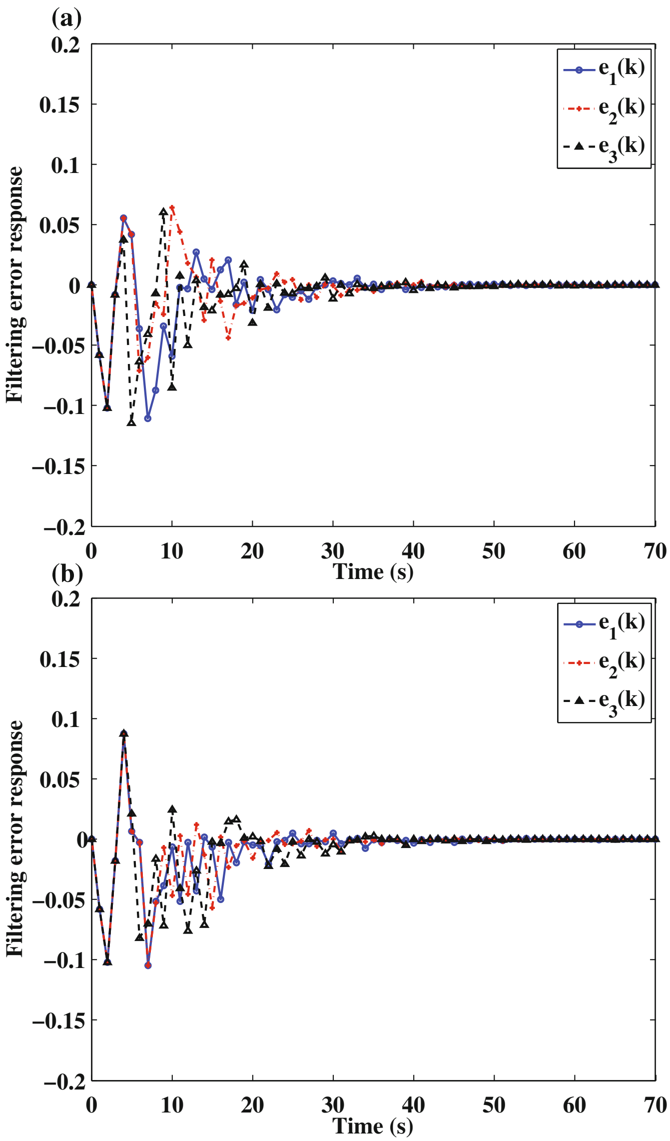


Fig. 5.12 Three realizations $e_1(k)$, $e_2(k)$ and $e_3(k)$ of filtering error response by QTD filters. **a** Filtering error response in the case of $\mathbb{T} = 2\,\text{s}$. **b** Filtering error response in the case of $\mathbb{T} = 3\,\text{s}$

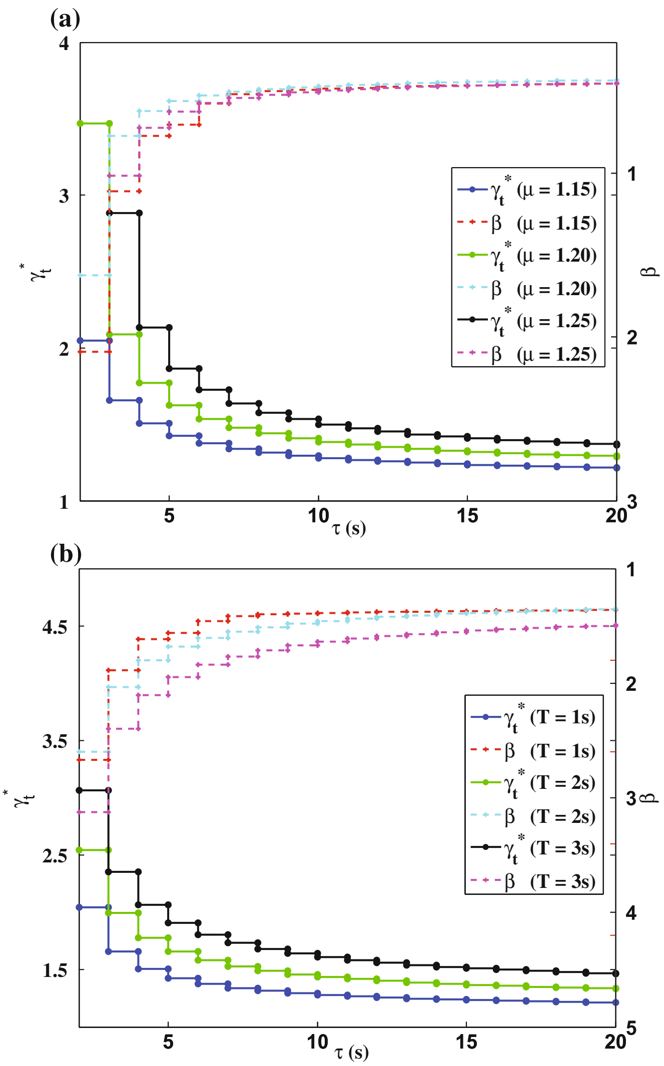


Fig. 5.13 H_∞ performance indices with corresponding β for given different $\mathbb{T}$ or μ. **a** The cases of $\mu = 1.15, 1.20, 1.25$ for $\mathbb{T} = 1$ s. **b** The cases of $\mathbb{T} = 1$ s, 2 s, 3 s for $\mu = 1.15$

5.4 Conclusion

In this chapter, the problem of robust H_∞ filtering for switched linear discrete-time systems with polytopic uncertainties has been first studied under arbitrary switching. A robust switched linear filter has been designed based on mode-dependent and parameter-dependent stability approaches such that the corresponding filtering error system achieves robust asymptotic stability and guarantees a prescribed exponential H_∞ performance index for all admissible uncertainties. Then, considering the ADT switching, an μ-dependent approach has been used, in which the analysis and synthesis of the underlying system are dependent on the increase degree μ of the piecewise Lyapunov function at the switching instants. Moreover, the filtering problem has been investigated for a class of discrete-time switched LPV systems under ADT switching. A mode-dependent full-order parameterized filter is designed and the corresponding existence conditions of such filters are derived via LMI formulation. Finally, the H_∞ filtering problem for a class of discrete-time switched linear systems with PDT switching has been dealt with. A QTD Lyapunov function is constructed to address the proposed problems for the PDT switched systems, upon which a QTD filter is designed such that the filtering error system is globally uniformly asymptotically stable with a guaranteed non-weighted H_∞ noise attenuation performance.

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Chapter 6
Asynchronous Switched Systems: ADT Switching

Abstract This chapter first investigates the stability and l_2-gain analysis problems for a class of discrete-time switched systems with average dwell time (ADT) switching by allowing the Lyapunov-like functions to increase during the running time of subsystems. The obtained results then facilitate the studies on the issues of asynchronous control, where "asynchronous" means the switching of the controllers has a lag to the switching of system modes. The basic asynchronous stabilization and asynchronous H_∞ control problem are both studied and the case for the system with time-varying parameter is further addressed under the modal average dwell time (MADT). Finally, the asynchronous H_∞ filter design problem is dealt with for the underlying switched linear systems with ADT switching. The phenomenon of "asynchronous" switching will unavoidably deteriorate the control performance such as the H_∞ noise attenuation index. However, it can be verified that the designed controller/filter considering the synchronous switching will be not necessarily valid in the presence of asynchronous switching. Several examples are provided to show the potential of the developed results.

6.1 New Stability Analysis

Consider a class of discrete-time switched systems given by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{6.1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $\sigma(k)$ is a piecewise constant function of time, called a switching signal, which takes its values in a finite set $\mathcal{I} = \{1, \ldots, N\}$, $N > 1$ is the number of subsystems. At an arbitrary time k, $\sigma(k)$ is dependent on k or $x(k)$, or both, or other logic rules. Also, for a switching time sequence $0 < k_1 < k_2 < \cdots$, $\sigma(k)$ is continuous from right everywhere and may be either autonomous or controlled. When $k \in [k_l, k_{l+1})$, we say the $\sigma(k_l)$th subsystem is active and therefore the trajectory $x(k)$ of system (6.1) is the trajectory of the $\sigma(k_l)$th subsystem. The two matrices pair (A_i, B_i), $\forall\sigma(k) = i \in \mathcal{I}$, represents the ith subsystem or ith mode of (6.1). In addition, we assume that the state of the

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system (6.1) does not jump at the switching instants, i.e., a continuous signal $x(k)$ can not be reconstructed everywhere. In this chapter, we focus our study of system (6.1) on two classes of switching signals with average dwell time (ADT) switching and modal average dwell time (MADT) switching, respectively, and their definitions have been given in Sect. 1.4, and therefore are omitted here.

It has been well recognized that the Multiple Lyapunov-like functions (MLFs) approach is an efficient stability analysis tool for switched systems [1–4], especially for slowly switched systems with DT or ADT [5]. In MLFs theory, each Lyapunov-like function constructed for each active subsystem is generally considered to be decreasing. An interesting extension also gives the so-called weak Lyapunov function, where the Lyapunov-like function can rise to a limited extent [1].

In this section, by further considering a class of Lyapunov-like functions allowed to increase with bounded increase rate, the improved results of Theorem 2.14 can be obtained as below. For concise notation, let k_l and $k_{l+1}, \forall l \in \mathbb{N}$ denote the starting time and ending time of some active subsystem, while $\mathcal{T}_{\uparrow}(k_l, k_{l+1})$ and $\mathcal{T}_{\downarrow}(k_l, k_{l+1})$ represent the unions of the dispersed intervals during which Lyapunov function is increasing and decreasing within the interval $[k_l, k_{l+1})$. The division gives that $[k_l, k_{l+1}) = \mathcal{T}_{\uparrow}(k_l, k_{l+1}) \cup \mathcal{T}_{\downarrow}(k_l, k_{l+1})$ and Fig. 6.1 illustrates the considered Lyapunov-like function. Also, we use $\mathcal{T}_{\uparrow}(k_{l+1} - k_l)$ and $\mathcal{T}_{\downarrow}(k_{l+1} - k_l)$ to denote the length of $\mathcal{T}_{\uparrow}(k_l, k_{l+1})$ and $\mathcal{T}_{\downarrow}(k_l, k_{l+1})$, respectively.

Theorem 6.1 *Consider switched system $x_{k+1} = f_\sigma(x_k, u_k)$ with $u_k \equiv 0$ and let $0 < \alpha < 1$, $\beta \geq 0$ and $\mu \geq 1$ be given constants. Suppose that there exist $\mathbb{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2 such that $\forall \sigma(k) = i \in \mathcal{I}$,*

$$\kappa_1(\|x_k\|) \leq V_i(x_k) \leq \kappa_2(\|x_k\|) \tag{6.2}$$

$$\Delta V_i(x_k) \leq \begin{cases} -\alpha V_i(k), & \forall k \in \mathcal{T}_{\downarrow}(k_l, k_{l+1}) \\ \beta V_i(k), & \forall k \in \mathcal{T}_{\uparrow}(k_l, k_{l+1}) \end{cases} \tag{6.3}$$

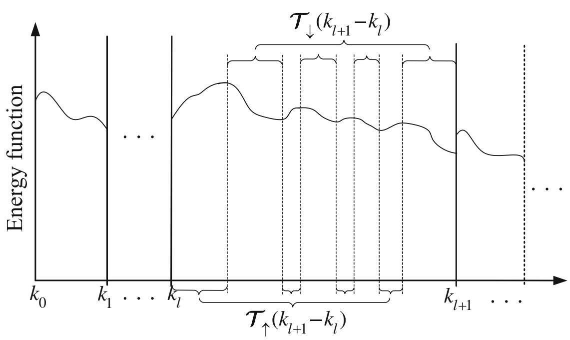

Fig. 6.1 Extended Lyapunov-like function. The sets $\mathcal{T}_{\uparrow}(k_l, k_{l+1})$ and $\mathcal{T}_{\downarrow}(k_l, k_{l+1})$ denote the unions of the dispersed intervals during which Lyapunov function is increasing and decreasing within the interval $[k_l, k_{l+1})$, respectively

and $\forall(i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$V_i(x_{k_l}) \leq \mu V_j(x_{k_l}) \tag{6.4}$$

then the system is globally uniformly asymptotically stable (GUAS) for any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{\mathcal{T}_{\max}\left[\ln(1+\beta) - \ln(1-\alpha)\right] + \ln\mu}{-\ln(1-\alpha)} \tag{6.5}$$

where $\mathcal{T}_{\max} \triangleq \max_l \mathcal{T}_\uparrow(k_{l+1} - k_l), \forall l \in \mathbb{N}$.

Proof For $k \in [k_l, k_{l+1})$, denoting $\theta \triangleq \frac{1+\beta}{1-\alpha}$ and $\bar{\alpha} \triangleq 1 - \alpha$, it holds from (6.3) that

$$\begin{aligned} V_{\sigma(k)}(x_k) &\leq \bar{\alpha}^{\mathcal{T}_\downarrow(k-k_l)}(1+\beta)^{\mathcal{T}_\uparrow(k-k_l)} V_{\sigma(k_l)}(x_{k_l}) \\ &\leq \bar{\alpha}^{\left[\mathcal{T}_\downarrow(k-k_l)+\mathcal{T}_\uparrow(k-k_l)\right]}\theta^{\mathcal{T}_\uparrow(k-k_l)} V_{\sigma(k_l)}(x_{k_l}) \\ &= \bar{\alpha}^{(k-k_l)}\theta^{\mathcal{T}_\uparrow(k-k_l)} V_{\sigma(k_l)}(x_{k_l}) \end{aligned} \tag{6.6}$$

Then, according to Definition 2.1, together with (6.4) and (6.6), one obtains

$$\begin{aligned} & V_{\sigma(k)}(x_k) \\ \leq\ & \bar{\alpha}^{(k-k_l)}\theta^{\mathcal{T}_\uparrow(k-k_l)}\mu V_{\sigma(k_l-1)}(x_{k_l}) \\ \leq\ & \bar{\alpha}^{(k-k_l)}\theta^{\mathcal{T}_{\max}}\mu V_{\sigma(k_l-1)}(x_{k_l}) \\ \leq\ & \ldots \leq \bar{\alpha}^{(k-k_0)}\left(\theta^{\mathcal{T}_{\max}}\right)^{N_\sigma(k_0,k)}\mu^{N_\sigma(k_0,k)} V_{\sigma(k_0)}(x_{k_0}) \\ \leq\ & \mu^{N_0}\theta^{N_0\mathcal{T}_{\max}}\left(\bar{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}\mu^{1/\tau_a}\right)^{(k-k_0)} V_{\sigma(k_0)}(x_{k_0}) \end{aligned}$$

Now if the ADT satisfies (6.5), one has

$$\begin{aligned} & \bar{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}\mu^{1/\tau_a} \\ <\ & \bar{\alpha}\theta^{\frac{-\mathcal{T}_{\max}\ln\bar{\alpha}}{\mathcal{T}_{\max}\ln\theta+\ln\mu}}\mu^{-\frac{\ln\bar{\alpha}}{\mathcal{T}_{\max}\ln\theta+\ln\mu}} = \bar{\alpha}\left(\theta^{\mathcal{T}_{\max}}\mu\right)^{-\frac{\ln\bar{\alpha}}{\mathcal{T}_{\max}\ln\theta+\ln\mu}} \\ =\ & \bar{\alpha}\left(e^{\mathcal{T}_{\max}\ln\theta+\ln\mu}\right)^{-\frac{\ln\bar{\alpha}}{\mathcal{T}_{\max}\ln\theta+\ln\mu}} = \bar{\alpha}/\bar{\alpha} = 1 \end{aligned}$$

Therefore, we conclude that $V_{\sigma(k)}(x_k)$ converges to zero as $k \to \infty$, then the asymptotic stability can be deduced with the aid of (6.2). □

Remark 6.2 Note that the hypothesis (6.3) relaxes the counterpart of Theorem 2.14, namely, the considered energy function in Theorem 6.1 can be increased both at switching instants and during the running time of subsystems. However, the possible increment will be compensated by the more specific decrement (by limiting the lower bound of ADT), therefore, the system energy is decreasing from a whole perspective and the system stability is guaranteed accordingly.

Remark 6.3 In Theorem 6.1, if $\mathcal{T}_{\max} = 0$, one can readily get Theorem 2.14. Therefore, Theorem 6.1 presents a more general version of stability results for the switched systems with ADT in discrete-time case. In addition, if one regards the increasing and decreasing intervals in one mode as two different modes (one stable and one unstable), a similar study for linear cases in continuous-time context can be found in [6].

In the following, we extend the results of Theorem 6.1 to the MADT case.

Theorem 6.4 *Consider the discrete-time switched system* $x_{k+1} = f_\sigma(x_k, u_k)$ *with* $u_k \equiv 0$ *and let* $0 < \alpha_i < 1$, $\beta_i > 0$ *and* $\mu_i > 1$ *be given constants. Suppose that there exist* $\mathbb{C}^1$ *functions* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, *and two class* $\mathcal{K}_\infty$ *functions* κ_{1i} *and* κ_{2i}, $\forall i \in \mathcal{I}$, *such that* $\forall \sigma(k) = i \in \mathcal{I}$,

$$\kappa_{1i}(\|x_k\|) \le V_i(x_k) \le \kappa_{2i}(\|x_k\|) \tag{6.7}$$

$$\Delta V_i(x_k) \le \begin{cases} -\alpha_i V_i(k), \forall k \in [k_l + \mathcal{T}_l, k_{l+1}) \\ \beta_i V_i(k), \forall k \in [k_l, k_l + \mathcal{T}_l) \end{cases} \tag{6.8}$$

and for $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$V_i(x_{k_l}) \le \mu_i V_j(x_{k_l}) \tag{6.9}$$

then the system is GUAS for switching signal with MADT

$$\tau_{ai} > \tau_{ai}^* = -\{\mathscr{T}_i[\ln(1 + \beta_i) - \ln(1 - \alpha_i)] + \ln \mu_i\}/\ln(1 - \alpha_i) \tag{6.10}$$

where $\mathscr{T}_i$ *denotes the increasing interval of the* ith *subsystem.*

Proof $\forall k \in [k_l, k_{l+1})$, $\forall l \in \mathbb{N}$, denoting $\tilde{\alpha}_i \triangleq (1 - \alpha_i)$, $\theta_i \triangleq (1 + \beta_i)/\tilde{\alpha}_i$, we can get

$$\begin{aligned}
& V_{\sigma(k)}(x_k) \\
& \le \tilde{\alpha}_{\sigma(k_l)}^{k-k_l-\mathcal{T}_l}(1 + \beta_{\sigma(k_l)})^{\mathcal{T}_l} V_{\sigma(k_l)}(x_{k_l}) \\
& = \tilde{\alpha}_{\sigma(k_l)}^{k-k_l} \theta_{\sigma(k_l)}^{\mathcal{T}_l} V_{\sigma(k_l)}(x_{k_l}) \\
& \le \tilde{\alpha}_{\sigma(k_l)}^{k-k_l} \theta_{\sigma(k_l)}^{\mathcal{T}_l} \mu_{\sigma(k_l)} V_{\sigma(k_{l-1})}(x_{k_l}) \\
& \le \tilde{\alpha}_{\sigma(k_l)}^{(k-k_l)} \theta_{\sigma(k_l)}^{\mathcal{T}_l} \mu_{\sigma(k_l)} \tilde{\alpha}_{\sigma(k_{l-1})}^{(k_l-k_{l-1})} \theta_{\sigma(k_{l-1})}^{\mathcal{T}_{l-1}} V_{\sigma(k_{l-1})}(x_{k_{l-1}}) \\
& = \tilde{\alpha}_{\sigma(k_l)}^{(k-k_l)} \tilde{\alpha}_{\sigma(k_{l-1})}^{(k_l-k_{l-1})} \theta_{\sigma(k_l)}^{\mathcal{T}_l} \theta_{\sigma(k_{l-1})}^{\mathcal{T}_{l-1}} \mu_{\sigma(k_l)} V_{\sigma(k_{l-1})}(x_{k_{l-1}}) \\
& \le \cdots \\
& \le \tilde{\alpha}_{\sigma(k_l)}^{(k-k_l)} \tilde{\alpha}_{\sigma(k_{l-1})}^{(k_l-k_{l-1})} \cdots \tilde{\alpha}_{\sigma(k_0)}^{(k_1-k_0)} \theta_{\sigma(k_l)}^{\mathcal{T}_l} \cdots \theta_{\sigma(k_0)}^{\mathcal{T}_0} \mu_{\sigma(k_l)} \mu_{\sigma(k_{l-1})} \\
& \quad \cdots \mu_{\sigma(k_1)} V_{\sigma(k_0)}(x_{k_0}) \\
& = \exp\big\{(k - k_l) \ln \tilde{\alpha}_{\sigma(k_l)} + (k_l - k_{l-1}) \ln \tilde{\alpha}_{\sigma(k_{l-1})} \\
& \quad + \cdots + (k_1 - k_0) \ln \tilde{\alpha}_{\sigma(k_0)}\big\}
\end{aligned}$$

$$\times \exp\left\{\mathcal{T}_l \ln(\theta_{\sigma(k_l)}) + \mathcal{T}_{l-1} \ln(\theta_{\sigma(k_{l-1})}) + \cdots + \mathcal{T}_0 \ln(\theta_{\sigma(k_0)})\right\}$$
$$\times \mu_{\sigma(k_l)}\mu_{\sigma(k_{l-1})}\cdots\mu_{\sigma(k_1)} V_{\sigma(k_0)}(x_{k_0})$$
$$= \exp\left\{\sum_{i=1}^{M} N_{\sigma i}(\ln \tilde{\alpha}_i \tau_{ai} + \ln \theta_i \mathscr{T}_i + \ln \mu_i)\right\} V_{\sigma(k_0)}(x_{k_0})$$
$$= \exp\left\{\sum_{i=1}^{M} N_{\sigma i}(\ln \tilde{\alpha}_i \tau_{ai} + \ln \theta_i \mathscr{T}_i + \ln \mu_i)\right\} V_{\sigma(k_0)}(x_{k_0})$$

where we use $N_{\sigma i}$ to denote $N_{\sigma(k)i}(k_0, k)$ for simplicity. If supposing

$$\ln(1-\alpha_i)\tau_{ai} + \ln \theta_i \mathscr{T}_i + \ln \mu_i < 0$$

we get a sufficient condition which can guarantee the GUAS of the switched system. The inequality above is equivalent to

$$\tau_{ai} > -(\mathscr{T}_i \ln(\theta_i) + \ln \mu_i)/\ln(1-\alpha_i)$$

Therefore, we conclude that $V_{\sigma(k)}(x_k)$ converges to zero as $k \to \infty$ if the above condition is satisfied, then the asymptotic stability can be deduced with the aid of (6.7). □

Remark 6.5 Theorem 6.4 relaxes the requirements in Theorem 6.1 to be mode-dependent. Since each subsystem has its own properties (stability or performance), it is inevitable to be more conservative if we require all subsystems have the same ADT while neglecting the properties of each subsystem. More specifically, e.g., for the ith subsystem, if its decaying rate α_i is relatively large while the increasing rate β_i is small, a shorter MADT can achieve the required performance, which reduces the conservatism of the analysis and synthesis process.

6.2 New Performance Analysis

To facilitate the performance analysis in this section, we first recall the following definition on the weighted l_2-gain analysis for switched systems with synchronous switching.

Definition 6.6 For $\gamma_s > 0$, the system

$$x(k+1) = f_\sigma(x(k), u(k)) \tag{6.11}$$
$$y(k) = h_\sigma(x(k)) \tag{6.12}$$

is said to be GUAS with weighted l_2-gain no greater than γ_s, if under zero initial condition, the system is GUAS and the inequality $\sum_{k=0}^{\infty}(1-\alpha)^k y_k^T y_k \le \sum_{k=0}^{\infty} \gamma_s^2 u_k^T u_k$, $0 < \alpha < 1$ holds for all nonzero $u(k) \in l_2[0, \infty)$.

Now, further invoking the extended Lyapunov-like function illustrated in Fig. 6.1, the following theorem can be obtained on the weighted l_2-gain analysis for system (6.11)–(6.12).

Theorem 6.7 *Consider the switched system (6.11)–(6.12), let* $0 < \alpha < 1$, $\beta \geq 0$, $\mu \geq 1$ *and* $\gamma > 0$ *be given constants. Suppose that there exist positive definite* $\mathbb{C}^1$ *functions* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) = i \in \mathcal{I}$, *with zero initial condition* $V_{\sigma(k_0)}(x_{k_0}) \equiv 0$ *such that* $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $V_i(x_{k_l}) \leq \mu V_j(x_{k_l})$ *and* $\forall i \in \mathcal{I}$, *denoting* $\Gamma(k) \triangleq y_k^T y_k - \gamma^2 u_k^T u_k$ *and*

$$\Delta V_i(x_k) \leq \begin{cases} -\alpha V_i(k) - \Gamma(k), \forall k \in [k_l + \mathcal{T}_M, k_{l+1}) \\ \beta V_i(k) - \Gamma(k), \forall k \in [k_l, k_l + \mathcal{T}_M) \end{cases} \tag{6.13}$$

then the switched system is GUAS for any switching signal with ADT satisfying (6.5) and has weighted l_2*-gain* $\sum_{k=k_0}^{\infty} (1-\alpha)^k y_k^T y_k \leq \gamma_a^2 \sum_{k=k_0}^{\infty} u_k^T u_k$, *where* $\theta \triangleq (1+\beta)/\tilde{\alpha}$, $\tilde{\alpha} \triangleq (1-\alpha)$, $\gamma_a = \sqrt{(\theta^{\mathcal{T}_M}\mu)^{N_0}\theta^{\mathcal{T}_M - 1}}\gamma$ *and* $\mathcal{T}_M$ *has the same definition as (6.5).*

Proof From [7], we can get

$$\begin{aligned}
& V_{\sigma(k_l)}(x_k) \\
& \leq \tilde{\alpha}^{k-k_l}\theta^{\mathcal{T}_M} V_{\sigma(k_l)}(x_{k_l}) - \sum_{s=k_l+\mathcal{T}_M}^{k-1} \tilde{\alpha}^{k-1-s}\Gamma(s) - \sum_{s=k_l}^{k_l+\mathcal{T}_M-1} \tilde{\alpha}^{k-1-s} \\
& \quad \times \theta^{\mathcal{T}_M+k_l-s-1}\Gamma(s) \\
& \leq \tilde{\alpha}^{k-k_l}\theta^{\mathcal{T}_M}\mu V_{\sigma(k_{l-1})}(x_{k_l}) - \sum_{s=k_l+\mathcal{T}_M}^{k-1} \tilde{\alpha}^{k-1-s}\Gamma(s) - \sum_{s=k_l}^{k_l+\mathcal{T}_M-1} \tilde{\alpha}^{k-1-s} \\
& \quad \times \theta^{\mathcal{T}_M+k_l-s-1}\Gamma(s) \\
& \leq \tilde{\alpha}^{k-k_l}\theta^{\mathcal{T}_M}\mu \left[\tilde{\alpha}^{k_l-k_{l-1}}\theta^{\mathcal{T}_M} V_{\sigma(k_{l-1})}(x_{k_{l-1}}) - \sum_{s=k_{l-1}+\mathcal{T}_M}^{k_l+\mathcal{T}_M-1} \tilde{\alpha}^{k_l-1-s}\Gamma(s) \right. \\
& \quad \left. - \sum_{s=k_{l-1}}^{k_{l-1}+\mathcal{T}_M-1} \tilde{\alpha}^{k_l-s-1}\theta^{\mathcal{T}_M+k_{l-1}-s-1}\Gamma(s) \right] - \sum_{s=k_l+\mathcal{T}_M}^{k-1} \tilde{\alpha}^{k-1-s}\Gamma(s) \\
& \quad - \sum_{s=k_l}^{k_l+\mathcal{T}_M-1} \tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_M+k_l-s-1}\Gamma(s) \\
& = \tilde{\alpha}^{k-k_{l-1}}(\theta^{\mathcal{T}_M})^2\mu V_{\sigma(k_{l-1})}(x_{k_{l-1}}) - \sum_{s=k_{l-1}+\mathcal{T}_M}^{k_l-1} \tilde{\alpha}^{k-1-s}\theta^{\mathcal{T}_M}\mu\Gamma(s)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{s=k_{l-1}}^{k_{l-1}+\mathcal{T}_M-1} \tilde{\alpha}^{k-1-s}\theta^{\mathcal{T}_M}\mu\theta^{\mathcal{T}_M+k_{l-1}-s-1}\Gamma(s) - \sum_{s=k_l+\mathcal{T}_M}^{k-1} \tilde{\alpha}^{k-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_l}^{k_l+\mathcal{T}_M-1} \tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_M+k_l-s-1}\Gamma(s) \\
&\leq \cdots \\
&\leq \tilde{\alpha}^{k-k_0}(\theta^{\mathcal{T}_M})^{N_\sigma(k_0,k)}\mu^{N_\sigma(k_0,k)-1}V_{\sigma(k_0)}(x_{k_0}) - \sum_{s=k_0+\mathcal{T}_M}^{k_1-1} \tilde{\alpha}^{k-1-s} \\
&\quad \times\ (\theta^{\mathcal{T}_M}\mu)^{N_\sigma(s,k)}\Gamma(s) - \sum_{s=k_0}^{k_0+\mathcal{T}_M-1} \tilde{\alpha}^{k-1-s}(\theta^{\mathcal{T}_M}\mu)^{N_\sigma(s,k)}\theta^{\mathcal{T}_M+k_0-s-1}\Gamma(s) \\
&\quad - \cdots - \sum_{s=k_l+\mathcal{T}_M}^{k-1} \tilde{\alpha}^{k-s-1}\Gamma(s) - \sum_{s=k_l}^{k_l+\mathcal{T}_M-1} \tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_M+k_l-s-1}\Gamma(s)
\end{aligned}
$$

Since $V_{\sigma(k_0)}(x_{k_0}) = 0$, $V_{\sigma(k_l)}(x_k) \geq 0$, and $\Gamma(k) = y_k^T y_k - \gamma^2 u_k^T u_k$, we have

$$
\begin{aligned}
&\sum_{s=k_0+\mathcal{T}_M}^{k_1-1} \tilde{\alpha}^{k-s-1}(\theta^{\mathcal{T}_M}\mu)^{N_\sigma(s,k)}y_s^T y_s + \sum_{s=k_0}^{k_0+\mathcal{T}_M-1} \tilde{\alpha}^{k-s-1}(\theta^{\mathcal{T}_M}\mu)^{N_\sigma(s,k)} \\
&\quad \times\ \theta^{\mathcal{T}_M+k_0-s-1}y_s^T y_s + \cdots + \sum_{s=k_l+\mathcal{T}_M}^{k-1} \tilde{\alpha}^{k-s-1}y_s^T y_s + \sum_{s=k_l}^{k_l+\mathcal{T}_M-1} \tilde{\alpha}^{k-s-1} \\
&\quad \times\ \theta^{\mathcal{T}_M+k_l-s-1}y_s^T y_s \\
&\leq \sum_{s=k_0+\mathcal{T}_M}^{k_1-1} \tilde{\alpha}^{k-s-1}(\theta^{\mathcal{T}_M}\mu)^{N_\sigma(s,k)}\gamma^2 u_s^T u_s + \sum_{s=k_0}^{k_0+\mathcal{T}_M-1} \tilde{\alpha}^{k-s-1} \\
&\quad \times\ (\theta^{\mathcal{T}_M}\mu)^{N_\sigma(s,k)}\theta^{\mathcal{T}_M+k_0-s-1}\gamma^2 u_s^T u_s + \cdots + \sum_{s=k_l+\mathcal{T}_M}^{k-1} \tilde{\alpha}^{k-s-1}\gamma^2 u_s^T u_s \\
&\quad + \sum_{s=k_l}^{k_l+\mathcal{T}_M-1} \tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_M+k_l-s-1}\gamma^2 u_s^T u_s
\end{aligned}
$$

Due to $1 \leq \theta^{\mathcal{T}_M+k_i-s-1} \leq \theta^{\mathcal{T}_M-1}$, $s \in (k_i,\ k_i + \mathcal{T}_M - 1)$, $i = 1, 2, \ldots, l$, we have

$$
\begin{aligned}
&\sum_{s=k_0}^{k-1} \tilde{\alpha}^{k-s-1}(\theta^{\mathcal{T}_M}\mu)^{N_\sigma(s,k)}y_s^T y_s \\
&\leq \sum_{s=k_0}^{k-1} \tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_M-1}(\theta^{\mathcal{T}_M}\mu)^{N_\sigma(s,k)}\gamma^2 u_s^T u_s
\end{aligned}
$$

which is equivalent to

$$\sum_{s=k_0}^{k-1} \tilde{\alpha}^{k-s-1} (\theta^{\mathcal{T}_M} \mu)^{-N_\sigma(k_0,s)} y_s^T y_s \\ \leq \sum_{s=k_0}^{k-1} \tilde{\alpha}^{k-s-1} \theta^{\mathcal{T}_M - 1} (\theta^{\mathcal{T}_M} \mu)^{-N_\sigma(k_0,s)} \gamma^2 u_s^T u_s$$

Due to $N_\sigma(k_0, s) \leq N_0 + (s - k_0)/\tau_a$, $\theta^{\mathcal{T}_M} \mu \geq 1$, we can get

$$\left(\theta^{\mathcal{T}_M} \mu\right)^{-N_\sigma(k_0,s)} \geq \left(\theta^{\mathcal{T}_M} \mu\right)^{-N_0} \left(\theta^{\mathcal{T}_M} \mu\right)^{-(s-k_0)/\tau_a}$$

From (6.5), we have $-\tau_a (\ln \tilde{\alpha}) > \ln \left(\theta^{\mathcal{T}_M} \mu\right)$, i.e., $\ln \tilde{\alpha} < \ln \left(\theta^{\mathcal{T}_M} \mu\right)^{-1/\tau_a}$. Therefore, the following holds,

$$\left(\theta^{\mathcal{T}_M} \mu\right)^{-N_\sigma(k_0,s)} \geq \left(\theta^{\mathcal{T}_M} \mu\right)^{-N_0} \tilde{\alpha}^{s-k_0}$$

With $0 \leq \left(\theta^{\mathcal{T}_M} \mu\right)^{-N_\sigma(k_0,s)} \leq 1$, we have

$$\sum_{s=k_0}^{k-1} \tilde{\alpha}^{k-s-1} (1-\alpha)^s y_s^T y_s \leq (\theta^{\mathcal{T}_M} \mu)^{N_0} \sum_{s=k_0}^{k-1} \tilde{\alpha}^{k-s-1} \theta^{\mathcal{T}_M - 1} \gamma^2 u_s^T u_s$$

then we can get

$$\sum_{k=k_0}^{\infty} \sum_{s=k_0}^{k-1} \tilde{\alpha}^{k-s-1} (1-\alpha)^s y_s^T y_s \\ \leq (\theta^{\mathcal{T}_M} \mu)^{N_0} \sum_{k=k_0}^{\infty} \sum_{s=k_0}^{k-1} \tilde{\alpha}^{k-s-1} \theta^{\mathcal{T}_M - 1} \gamma^2 u_s^T u_s$$

and

$$\sum_{k=k_0}^{\infty} \sum_{k=s+1}^{k-1} \tilde{\alpha}^{k-s-1} (1-\alpha)^s y_s^T y_s \\ \leq (\theta^{\mathcal{T}_M} \mu)^{N_0} \sum_{k=k_0}^{\infty} \sum_{k=s+1}^{\infty} \tilde{\alpha}^{k-s-1} \theta^{\mathcal{T}_M - 1} \gamma^2 u_s^T u_s$$

Due to $\sum_{k=s+1}^{\infty} \tilde{\alpha}^{k-s-1} = \frac{1}{\alpha}$, we can easily obtain $\sum_{k=k_0}^{\infty} (1-\alpha)^k y_k^T y_k \leq (\theta^{\mathcal{T}_M} \mu)^{N_0}$ $\theta^{\mathcal{T}_M - 1} \gamma^2 \sum_{k=k_0}^{\infty} u_k^T u_k$, which ends the proof. □

To extend the results of Theorem 6.7 to the MADT case, the weighted l_2-gain analysis of the discrete-time switched system (6.11)–(6.12) under the MADT switching signal is given as follows.

Theorem 6.8 *Consider switched system (6.11)–(6.12) and let* $0 < \alpha_i < 1$, $\beta_i \geq 0$, $\mu_i \geq 1$ *and* $\gamma > 0$, $\forall i \in \mathcal{I}$ *be given constants. Suppose that there exists positive definite* $\mathbb{C}^1$ *function* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, *with* $V_{\sigma(k_0)}(x_{k_0}) \equiv 0$ *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $V_i(x_{k_l}) \leq \mu_i V_j(x_{k_l})$, *denoting* $\Gamma(k) \triangleq y_k^T y_k - \gamma^2 u_k^T u_k$, *if the following inequality is satisfied*

$$\Delta V_i(x_k) \leq \begin{cases} -\alpha_i V_i(k) - \Gamma(k), \forall k \in [k_l + \mathcal{T}_l, k_{l+1}) \\ \beta_i V_i(k) - \Gamma(k), \forall k \in [k_l, k_l + \mathcal{T}_l) \end{cases} \tag{6.14}$$

then the switched system is GUAS for any switching signal satisfying (6.10) and has weighted l_2*-gain* $\sum_{k=k_0}^{\infty} (1 - \alpha_{\max})^k y_k^T y_k \leq \gamma_a^2 \sum_{k=k_0}^{\infty} u_k^T u_k$, *where* $\gamma_a = \sqrt{\prod_{i \in \mathcal{I}} (\theta_i^{\mathcal{T}_i} \mu_i)^{N_{0i}} \frac{\alpha_{\max}}{\alpha_{\min}} \theta_{\max}^{\mathcal{T}_{\max}-1}}\gamma$, $\theta_{\max} = \max\{\theta_i\} = \max\{(1 + \beta_i)/(1 - \alpha_i)\}$, *and* $\mathcal{T}_{\max} = \max\{\mathscr{T}_i\}$, $\forall i \in \mathcal{I}$.

Proof It yields from [7] that

$$\begin{aligned}
V_{\sigma(k_l)}(x_k) &\leq \tilde{\alpha}_{\sigma(k_l)}^{k-k_l} \theta_{\sigma(k_l)}^{\mathcal{T}_l} V_{\sigma(k_l)}(x_{k_l}) - \sum_{s=k_l+\mathcal{T}_l}^{k-1} \tilde{\alpha}_{\sigma(k_l)}^{k-1-s} \Gamma(s) \\
&\quad - \sum_{s=k_l}^{k_l+\mathcal{T}_l-1} \tilde{\alpha}_{\sigma(k_l)}^{k-1-s} \theta_{\sigma(k_l)}^{\mathcal{T}_l+k_l-s-1} \Gamma(s) \\
&\leq \tilde{\alpha}_{\sigma(k_l)}^{k-k_l} \theta_{\sigma(k_l)}^{\mathcal{T}_l} \mu_{\sigma(k_l)} V_{\sigma(k_{l-1})}(x_{k_l}) - \sum_{s=k_l+\mathcal{T}_l}^{k-1} \tilde{\alpha}_{\sigma(k_l)}^{k-1-s} \Gamma(s) \\
&\quad - \sum_{s=k_l}^{k_l+\mathcal{T}_l-1} \tilde{\alpha}_{\sigma(k_l)}^{k-1-s} \theta_{\sigma(k_l)}^{\mathcal{T}_l+k_l-s-1} \Gamma(s) \\
&\leq \tilde{\alpha}_{\sigma(k_l)}^{k-k_l} \theta_{\sigma(k_l)}^{\mathcal{T}_l} \mu_{\sigma(k_l)} \Bigg[\tilde{\alpha}_{\sigma(k_{l-1})}^{k_l-k_{l-1}} \theta_{\sigma(k_{l-1})}^{\mathcal{T}_{l-1}} V_{\sigma(k_{l-1})}(x_{k_{l-1}}) - \sum_{s=k_{l-1}+\mathcal{T}_{l-1}}^{k_l-1} \\
&\quad \times \tilde{\alpha}_{\sigma(k_{l-1})}^{k_l-1-s} \Gamma(s) - \sum_{s=k_{l-1}}^{k_{l-1}+\mathcal{T}_{l-1}-1} \tilde{\alpha}_{\sigma(k_{l-1})}^{k_l-k_{l-1}} \theta_{\sigma(k_{l-1})}^{\mathcal{T}_{l-1}+k_{l-1}-s-1} \Gamma(s) \Bigg] \\
&\quad - \sum_{s=k_l+\mathcal{T}_l}^{k-1} \tilde{\alpha}_{\sigma(k_l)}^{k-1-s} \Gamma(s) - \sum_{s=k_l}^{k_l+\mathcal{T}_l-1} \tilde{\alpha}_{\sigma(k_l)}^{k-s-1} \theta_{\sigma(k_l)}^{\mathcal{T}_l+k_l-s-1} \Gamma(s) \\
&\leq \cdots
\end{aligned}$$

$$
\begin{aligned}
&\leq \tilde{\alpha}_{\sigma(k_l)}^{k-k_l}\tilde{\alpha}_{\sigma(k_{l-1})}^{k_l-k_{l-1}}\cdots\tilde{\alpha}_{\sigma(k_0)}^{k_1-k_0}\theta_{\sigma(k_l)}^{\mathcal{T}_l}\theta_{\sigma(k_{l-1})}^{\mathcal{T}_{l-1}}\cdots\theta_{\sigma(k_0)}^{\mathcal{T}_0}\mu_{\sigma(k_l)}\mu_{\sigma(k_{l-1})}\\
&\quad\cdots\mu_{\sigma(k_1)}V_{\sigma(k_0)}(x_{k_0})-\tilde{\alpha}_{\sigma(k_l)}^{k-k_l}\cdots\tilde{\alpha}_{\sigma(k_1)}^{k_2-k_1}\theta_{\sigma(k_l)}^{\mathcal{T}_l}\theta_{\sigma(k_{l-1})}^{\mathcal{T}_{l-1}}\\
&\quad\cdots\theta_{\sigma(k_1)}^{\mathcal{T}_1}\mu_{\sigma(k_l)}\mu_{\sigma(k_{l-1})}\cdots\mu_{\sigma(k_1)}\sum_{s=k_0+\mathcal{T}_0}^{k_1-1}\tilde{\alpha}_{\sigma(k_0)}^{k_1-1-s}\Gamma(s)\\
&\quad-\tilde{\alpha}_{\sigma(k_l)}^{k-k_l}\cdots\tilde{\alpha}_{\sigma(k_1)}^{k_2-k_1}\theta_{\sigma(k_l)}^{\mathcal{T}_l}\theta_{\sigma(k_{l-1})}^{\mathcal{T}_{l-1}}\cdots\theta_{\sigma(k_1)}^{\mathcal{T}_1}\mu_{\sigma(k_l)}\mu_{\sigma(k_1)}\\
&\quad\cdots\mu_{\sigma(k_1)}\sum_{s=k_0}^{k_0+\mathcal{T}_0-1}\tilde{\alpha}_{\sigma(k_0)}^{k_1-1-s}\theta_{\sigma(k_0)}^{k_0+\tau_0-s-1}\Gamma(s)-\cdots\\
&\quad-\sum_{s=k_l+\mathcal{T}_l}^{k-1}\tilde{\alpha}_{\sigma(k_l)}^{k-1-s}\Gamma(s)-\sum_{s=k_l}^{k_l+\mathcal{T}_l-1-s}\tilde{\alpha}_{\sigma(k_l)}^{k-s-1}\theta_{\sigma(k_l)}^{k_l+\mathcal{T}_l-1-s}\Gamma(s)
\end{aligned}
$$

Since $V_{\sigma(k_0)}(x_{k_0})=0$, denoting $\alpha_{\max}=\max\{\alpha_i\}$, $\tilde{\alpha}_{\max}=\{1-\alpha_{\max}\}$, $\alpha_{\min}=\min\{\alpha_i\}$, and $\tilde{\alpha}_{\min}=\{1-\alpha_{\min}\}$, $\forall i\in\mathcal{I}$, similar to the proof of Theorem 6.7, we have

$$
\begin{aligned}
&\sum_{s=k_0+\mathcal{T}_0}^{k_1-1}\tilde{\alpha}_{\max}^{k-1-s}\prod_{i\in\mathcal{I}}(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{\sigma i}(s,k)}y_s^Ty_s+\sum_{s=k_0}^{k_0+\mathcal{T}_0-1}\tilde{\alpha}_{\max}^{k-1-s}\\
&\times\prod_{i\in\mathcal{I}}(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{\sigma i}(s,k)}\theta_{\sigma(k_0)}^{k_0+\mathcal{T}_0-1-s}y_s^Ty_s+\sum_{s=k_1+\mathcal{T}_1}^{k_2-1}\tilde{\alpha}_{\max}^{k-1-s}\prod_{i\in\mathcal{I}}(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{\sigma i}(s,k)}\\
&\times y_s^Ty_s+\sum_{s=k_1}^{k_1+\mathcal{T}_1-1}\tilde{\alpha}_{\max}^{k-1-s}\prod_{i\in\mathcal{I}}(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{\sigma i}(s,k)}\theta_{\sigma(k_1)}^{k_1+\mathcal{T}_1-1-s}y_s^Ty_s+\cdots\\
&+\sum_{s=k_l+\mathcal{T}_l}^{k-1}\tilde{\alpha}_{\max}^{k-1-s}y_s^Ty_s+\sum_{s=k_l}^{k_l+\mathcal{T}_l-1}\tilde{\alpha}_{\max}^{k-1-s}\theta_{\sigma(k_l)}^{k_l+\mathcal{T}_l-1-s}y_s^Ty_s\\
&\leq\sum_{s=k_0+\mathcal{T}_0}^{k_1-1}\tilde{\alpha}_{\min}^{k-1-s}\prod_{i\in\mathcal{I}}(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{\sigma i}(s,k)}u_s^Tu_s+\sum_{s=k_0}^{k_0+\mathcal{T}_0-1}\tilde{\alpha}_{\min}^{k-1-s}\\
&\times\prod_{i\in\mathcal{I}}(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{\sigma i}(s,k)}\theta_{\sigma(k_0)}^{k_0+\mathcal{T}_0-1-s}\gamma^2u_s^Tu_s+\sum_{s=k_1+\mathcal{T}_1}^{k_2-1}\tilde{\alpha}_{\min}^{k-1-s}\prod_{i\in\mathcal{I}}\\
&\times(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{\sigma i}(s,k)}\gamma^2u_s^Tu_s+\sum_{s=k_1}^{k_1+\mathcal{T}_1-1}\tilde{\alpha}_{\min}^{k-1-s}\prod_{i\in\mathcal{I}}(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{\sigma i}(s,k)}\\
&\times\theta_{\sigma(k_1)}^{k_1+\mathcal{T}_1-1-s}\gamma^2u_s^Tu_s+\cdots+\sum_{s=k_l+\mathcal{T}_l}^{k-1}\tilde{\alpha}_{\min}^{k-1-s}\gamma^2u_s^Tu_s\\
&+\sum_{s=k_l}^{k_l+\mathcal{T}_l-1}\tilde{\alpha}_{\min}^{k-1-s}\theta_{\sigma(k_l)}^{k_l+\mathcal{T}_l-1-s}\gamma^2u_s^Tu_s
\end{aligned}
$$

Since $1 < \theta_i^{k_i+\mathcal{T}_i-s-1} < \theta_i^{\mathcal{T}_i-1}$, then the inequalities above can be formulated as

$$\sum_{s=k_0}^{k-1} \tilde{\alpha}_{\max}^{k-1-s} \mathop{\Pi}_{i\in\mathcal{I}} (\theta_i^{\mathscr{T}_i}\mu_i)^{-N_{\sigma i}(k_0,s)} y_s^T y_s$$
$$\leq \sum_{s=k_0}^{k-1} \tilde{\alpha}_{\min}^{k-1-s} \mathop{\Pi}_{i\in\mathcal{I}} (\theta_i^{\mathscr{T}_i}\mu_i)^{-N_{\sigma i}(k_0,s)} \theta_{\sigma(s)}^{\mathcal{T}_{\sigma(s)}-1} u_s^T u_s$$

From Definition 1.7 and (6.10) we get

$$-N_{\sigma i}(k_0, s) \geq -N_{0i} - H_i(k_0, s)/\tau_{ai}$$

hence we can get

$$\sum_{s=k_0}^{k-1} \tilde{\alpha}_{\max}^{k-1-s} \mathop{\Pi}_{i\in\mathcal{I}} (\theta_i^{\mathscr{T}_i}\mu_i)^{-N_{0i}+\frac{H_i(k_0,s)\ln\tilde{\alpha}_i}{\ln(\theta_i^{\mathscr{T}_i}\mu_i)}} y_s^T y_s$$
$$\leq \sum_{s=k_0}^{k-1} \tilde{\alpha}_{\min}^{k-1-s} \theta_{\max}^{\mathcal{T}_{\max}-1}\gamma^2 u_s^T u_s,$$
$$\sum_{s=k_0}^{k-1} \tilde{\alpha}_{\max}^{k-1-s} \mathop{\Pi}_{i\in\mathcal{I}} \tilde{\alpha}_i^{H_i(k_0,s)} y_s^T y_s \leq \Theta_i\gamma^2 \sum_{s=k_0}^{k-1} \tilde{\alpha}_{\min}^{k-1-s} \theta_{\max}^{\mathcal{T}_{\max}-1} u_s^T u_s,$$
$$\sum_{k=k_0}^{\infty}\sum_{s=k_0}^{k-1} \tilde{\alpha}_{\max}^{k-1-s} \tilde{\alpha}_{\max}^{H(k_0,s)} y_s^T y_s \leq \Theta_i\gamma^2 \sum_{k=k_0}^{\infty}\sum_{s=k_0}^{k-1} \tilde{\alpha}_{\min}^{k-1-s} \theta_{\max}^{\mathcal{T}_{\max}-1} u_s^T u_s,$$
$$\sum_{s=k_0}^{\infty}\sum_{k=s+1}^{\infty} \tilde{\alpha}_{\max}^{k-1-s} \tilde{\alpha}_{\max}^{s} y_s^T y_s \leq \Theta_i\gamma^2 \sum_{s=k_0}^{\infty}\sum_{k=s+1}^{\infty} \tilde{\alpha}_{\min}^{k-1-s} \theta_{\max}^{\mathcal{T}_{\max}-1} u_s^T u_s,$$

where $\Theta_i = \mathop{\Pi}_{i\in\mathcal{I}} (\theta_i^{\mathscr{T}_i}\mu_i)^{N_{0i}}$. Due to $\sum_{k=s+1}^{\infty} \tilde{\alpha}_{\max}^{k-1-s} = \frac{1}{\alpha_{\max}}$, $\sum_{k=s+1}^{\infty} \tilde{\alpha}_{\min}^{k-1-s} = \frac{1}{\alpha_{\min}}$, we have

$$\sum_{s=k_0}^{\infty} (1-\alpha_{\max})^s y_s^T y_s$$
$$\leq \mathop{\Pi}_{i\in\mathcal{I}} (\theta_i^{\mathscr{T}_i}\mu_i)^{N_{0i}} \frac{\alpha_{\max}}{\alpha_{\min}} \theta_{\max}^{\mathcal{T}_{\max}-1}\gamma^2 \sum_{s=k_0}^{\infty} u_s^T u_s$$

which completes the proof. □

Remark 6.9 Theorem 6.8 addresses the l_2-gain of switched system (6.11)–(6.12) with MADT switching signal, and Theorem 6.7 for the underlying system with ADT switching. Then, according to the discussions in Sect. 1.4.2 (cf. Definition 1.7 and the relationship between ADT switching and MADT switching), we can conclude that

$$\gamma^*_{ADT}\ (\tau_D, N_0) \leq \gamma^*_{MADT}\ (\tau_{Di}, N_{0i})$$

where $\gamma^*_{ADT}\ (\tau_D, N_0)$ and $\gamma^*_{MADT}\ (\tau_{Di}, N_{0i})$ are the optimal weighted l_2-gains of the switched system under the two switching logics with $\tau_D = \max\{\tau_{Di}\}$ and $N_0 = \max\{N_{0i}\}$ (see the third inclusion relationship in Sect. 1.4.2), respectively.

6.3 Asynchronous Stabilization

In this section, we will consider the issue of asynchronous stabilization for the switched linear system (6.1).

The control input $u(k)$ in (6.1) is used to achieve system stability under certain switching signals, and usually, the mode-dependent control pattern is considered and formed as (if state feedback) $u(k) = K_{\sigma(k)}x(k)$, where K_i ($\forall\sigma(k) = i \in \mathcal{I}$) is the controller gain to be determined. In literature, however, a common assumption is that the switches of $K_{\sigma(k)}$ coincide *real time* with those of system modes, which is hard to satisfy in practice. Then, if the time lag of switched controllers to system modes (asynchronous switching) is T, the control input will become $u(k) = K_{\sigma(k-T)}x(k)$, hence the resulting closed-loop system is given by

$$x(k+1) = \left(A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma(k-T)}\right)x(k) \tag{6.15}$$

Obviously, the mode-unmatched (probably wrong) controllers in the loop, together with the switching signals designed/found in the case of synchronous switching, may cause a worse performance for the underlying system.

Therefore, in this section, we are interested in finding a mode-dependent state-feedback controller and a set of admissible switching signals with ADT such that the resulting closed-loop system (6.15) is globally uniformly exponentially stable (GUES) in the presence of asynchronous switching. It is worth noting that for a practical system, it also takes time to measure the system state besides identifying the system modes. Thus the corresponding state-feedback stabilization problem needs take both switching delays and state delays into account. This general case and other more complex cases considering output-feedback control, H_∞ control, etc., can be further studied based on the basic state-feedback stabilization methods to be developed in this section.

In the presence of asynchronous switching, the mode-unmatched controller will be applied in control loop for a certain time, then the energy function to evaluate the system may be increased. This, together with the inspiration from [1], motivates us to consider a class of Lyapunov-like function allowed to increase but the increase rate is bounded.

Note that for Theorem 6.1, a natural question is how $\mathcal{T}_{\max}$ is known in advance. Generally, this is hard since within $[k_l, k_{l+1}), \forall l \in \mathbb{N}^+$, $\mathcal{T}_\uparrow(k_l, k_{l+1})$ includes all

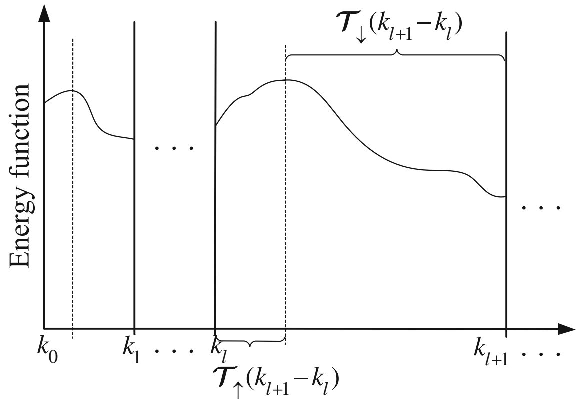

Fig. 6.2 A typical case of the extended Lyapunov-like function in Fig. 6.1. Here, $\mathcal{T}_\uparrow(k_l, k_{l+1})$ is the only interval close to the switching times

the randomly dispersed intervals during which the Lyapunov function is increasing. However, for the asynchronously switched control problem, the corresponding $\mathcal{T}_\uparrow(k_l, k_{l+1})$ will be only the interval close to the switching instants of subsystems as shown in Fig. 6.2, depending on the running time of unmatched controller. In practice, the interval rests with the identification and scheduling process among all the candidates of stabilizing controllers, which may be different in different environments. Here we assume the maximal delay of asynchronous switching, $\mathcal{T}_{\max}$, is known a priori without loss of generality.

Now, we are in a position to give the existence conditions of an asynchronous mode-dependent stabilizing controller for system (6.1).

Theorem 6.10 *Consider the switched linear system (6.1) and let* $0 < \gamma < 1$, $\eta > -1$, $\mu > 1$ *be given constants. If there exist matrices* $S_i > 0$ *and* U_i, $\forall i \in \mathcal{I}$, *such that*, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$

$$\begin{bmatrix} -S_i & A_iS_i + B_iU_i \\ \star & -(1-\gamma)S_i \end{bmatrix} \leq 0 \tag{6.16}$$

$$\begin{bmatrix} -S_i & A_iS_j + B_iU_j \\ \star & (1+\eta)(S_i - S_j - S_j^T) \end{bmatrix} \leq 0 \tag{6.17}$$

$$S_j \leq \mu S_i \tag{6.18}$$

then there exists a mode-dependent stabilizing controller with the asynchronous delay $\mathcal{T}_{\max}$ *such that system (6.15) is GUES for any switching signal with ADT satisfying (6.5). Moreover, if (6.16)–(6.18) has a solution, the admissible controller can be given by* $K_i = U_iS_i^{-1}$.

Proof For mode-dependent controller input $u(k) = K_ix(k)$ in asynchronous switching case, namely, when the subsystem i has been switched, the controller K_j will be still active instead of K_i for $\mathcal{T}_{\max}$, then we have

$$x(k+1) = \begin{cases} \hat{A}_{i,j}x(k+1), \forall k \in [k_l, k_l + \mathcal{T}_{\max}) \\ \bar{A}_i x(k+1), \forall k \in [k_l + \mathcal{T}_{\max}, k_{l+1}) \end{cases} \tag{6.19}$$

where $\bar{A}_i \triangleq A_i + B_i K_i, \hat{A}_{i,j} \triangleq A_i + B_i K_j$.

Then, consider the extended Lyapunov-like function $V_i(x(k)) = x^T(k)P_i x(k)$, together with (6.3) and (6.19), we have $\forall (i, j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,

$$\begin{aligned} \Delta V_i(x_k) + \gamma V_i(x_k) &= x_k^T \left[\bar{A}_i^T P_i \bar{A}_i + \gamma P_i - P_i\right] x_k \\ \Delta V_i(x_k) - \eta V_i(x_k) &= x_k^T \left[\hat{A}_{i,j}^T P_i \hat{A}_{i,j} - \eta P_i - P_i\right] x_k \\ V_i(k_l) - \mu V_j(k_l) &= x_k^T \left[P_i - \mu P_j\right] x_k \end{aligned}$$

Thus if

$$\bar{A}_i^T P_i \bar{A}_i + \gamma P_i - P_i \leq 0 \tag{6.20}$$

$$\hat{A}_{i,j}^T P_i \hat{A}_{i,j} - \eta P_i - P_i \leq 0 \tag{6.21}$$

$$P_i - \mu P_j \leq 0 \tag{6.22}$$

system (6.1) is GUAS for any switching signal with ADT (6.5) according to Theorem 6.1. Replacing $\bar{A}_i, \hat{A}_{i,j}$ in (6.19) and by Lemma 2.4, we have

$$\begin{bmatrix} -P_i & P_i A_i + P_i B_i K_i \\ \star & -(1-\gamma)P_i \end{bmatrix} \leq 0 \tag{6.23}$$

$$\begin{bmatrix} -P_i & P_i A_i + P_i B_i K_j \\ \star & -(1+\eta)P_i \end{bmatrix} \leq 0 \tag{6.24}$$

Setting $S_i \triangleq P_i^{-1}, U_i \triangleq K_i S_i$ and performing a congruence transformation to (6.23) via $diag\{S_i, S_i\}$, we can obtain (6.16). In addition, from the fact $(S_i - S_j)^T S_i (S_i - S_j) \geq 0$, we have $S_i - S_j - S_j^T \geq -S_j^T S_i^{-1} S_j$. Then, if (6.17) holds, one has

$$\begin{bmatrix} -S_i & A_i S_j + B_i U_j \\ \star & -(1+\eta) S_j^T S_i^{-1} S_j \end{bmatrix} \leq 0$$

Performing a congruence transformation to the above inequality via $diag\{S_i^{-1}, S_j^{-1}\}$, we can obtain (6.24). Therefore, (6.16)–(6.18) ensure that (6.20)–(6.22) are satisfied. In addition, by denoting $\zeta = \sqrt{(1-\gamma)\theta^{\mathcal{T}_{\max}/\tau_a}\mu^{1/\tau_a}}$, the system state satisfies $||x_k|| \leq K\zeta^{(k-k_0)}||x_{k_0}||$ for a certain $K > 0$, i.e. the underlying system is GUES. □

Likewise, the following corollary gives the case of synchronous switching in the discrete-time context.

Corollary 6.11 *Consider the switched linear system (6.1) and let* $0 < \gamma < 1$ *and* $\mu > 1$ *be given constants. If there exist matrices* $S_i > 0$ *and* U_i*,* $\forall i \in \mathcal{I}$*, such that,* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$*,* $i \neq j$

$$\begin{bmatrix} -S_i & A_i S_i + B_i U_i \\ \star & -(1-\gamma) S_i \end{bmatrix} \leq 0 \tag{6.25}$$

$$S_j \leq \mu S_i \tag{6.26}$$

then there exists a mode-dependent stabilizing controller such that system (6.1) is GUES for any switching signal with ADT satisfying (2.26). Moreover, if (6.25)–(6.26) has a solution, the admissible controller can be given by $K_i = U_i S_i^{-1}$.

Remark 6.12 The conditions derived in the above Theorems and Corollaries are LMIs for given α, β (or α only) and μ. Then, by providing other two parameters (or one, respectively) a priori, the optimum for the other one can be approximately obtained by the bisection method when a feasible solution of the corresponding LMIs is guaranteed. This is actually due to the latent monotonicity of all of them, e.g., a bigger β corresponding to more possibilities of feasible solutions in Theorem 6.1.

6.4 H_∞ Control

6.4.1 Switched Linear Systems

In this subsection, we will investigate the problem of designing the mode-dependent H_∞ controllers for the underlying systems in the presence of asynchronous switching.

Consider a class of discrete-time switched linear systems given by

$$x(k+1) = A_{\sigma(k)} x(k) + B_{\sigma(k)} u(k) + E_{\sigma(k)} \omega(k) \tag{6.27}$$

$$z(k) = C_{\sigma(k)} x(k) + D_{\sigma(k)} u(k) + F_{\sigma(k)} \omega(k) \tag{6.28}$$

where the system description has been given in the previous sections. $\sigma(k)$ is the switching signal discussed in Sect. 6.1 and we also consider it to be with ADT property. For the system in the presence of asynchronous switching, we are interested in designing an H_∞ state-feedback controller $u(k) = K_{\sigma(k)} x(k)$, where K_i ($\forall \sigma(k) = i \in \mathcal{I}$) is the controller gain to be determined.

If there exists the asynchronous switching, i.e., the switches of $K_{\sigma(k)}$ do not coincide in *real time* with those of system modes, then the control input will become

$u(k) = K_{\sigma(k-\mathcal{T}_{\max})}x(k), \forall k \in [k_l,\ k_l + \mathcal{T}_{\max})$. Hence the resulting closed-loop system is given by $\forall \sigma(k - \mathcal{T}_{\max}) = j, \sigma(k) = i, i \neq j$,

$$\begin{cases} \{\ x(k+1) &= \hat{A}_i x(k) + \hat{E}_i w(k) \\ \quad z(k) &= \hat{C}_i x(k) + \hat{F}_i w(k), \quad \forall k \in [k_l,\ k_l + \mathcal{T}_{\max}) \\ x(k+1) &= \bar{A}_i x(k) + \bar{E}_i w(k) \\ \quad z(k) &= \bar{C}_i x(k) + \bar{F}_i w(k), \quad \forall k \in [k_l + \mathcal{T}_{\max}, k_{l+1}) \end{cases} \tag{6.29}$$

where

$$\hat{A}_i \triangleq A_i + B_i K_j, \bar{A}_i \triangleq A_i + B_i K_i, \hat{C}_i \triangleq A_i + B_i K_j,$$
$$\bar{C}_i \triangleq A_i + B_i K_i, \hat{E}_i \triangleq \bar{E}_i = E_i, \hat{F}_i \triangleq \bar{F}_i = F_i.$$

Then, the controllers as well as the switching signals, designed in the case assuming synchronous switching, may cause a worse performance.

Therefore, our objective in this subsection is to design a mode-dependent state-feedback controller and find a set of admissible switching signals with ADT such that the resulting closed-loop system (6.29) is GUAS and has a guaranteed exponential H_∞ disturbance attenuation performance, i.e., $\|z\|_2^2 \leq \gamma^2 \|w\|_2^2$ for a $\gamma > 0$ in the presence of asynchronous switching. Note that as shown in (6.29), the mismatched controller only appears once during the closed loop interval for the active subsystems.

Using both Theorems 6.1 and 6.7, the above problem can be solved by the following theorem, which is used to give stability and exponential H_∞ performance analyses for system (6.29).

Theorem 6.13 *Consider switched linear system (6.29) and let* $0 < \alpha < 1, \beta \geq 0, \gamma > 0$, *and* $\mu \geq 1$ *be given constants. If there exist matrices* $P_i > 0, \forall i \in \mathcal{I}$, *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $P_i \leq \mu P_j$, $\Theta_i \leq 0$ *and* $\Theta_{ij} \leq 0$, *where*

$$\Theta_i \triangleq \begin{bmatrix} -P_i & 0 & P_i\bar{A}_i & P_i\bar{E}_i \\ \star & -I & \bar{C}_i & \bar{F}_i \\ \star & \star & -(1-\alpha)P_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix},$$
$$\Theta_{ij} \triangleq \begin{bmatrix} -P_i & 0 & P_i\hat{A}_i & P_i\hat{E}_i \\ \star & -I & \hat{C}_i & \hat{F}_i \\ \star & \star & -(1+\beta)P_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix}$$

then under the asynchronous delay $\mathcal{T}_{\max}$, *the corresponding system is GUAS for any switching signal satisfying (6.5) and has a guaranteed exponential* H_∞ *performance index* $\gamma_a = \sqrt{(\theta^{\mathcal{T}_{\max}}\mu)^{N_0}\theta^{\mathcal{T}_{\max}-1}}\gamma$.

Proof Consider the extended Lyapunov-like function shown in Fig. 6.2 as the following quadratic form

$$V_i(x_k) = x_k^T P_i x_k, \forall \sigma(k) = i \in \mathcal{I} \tag{6.30}$$

where P_i is a positive definite matrix. Firstly, it is straightforward to know that (6.30) satisfies the hypothesis (6.2).

Now assuming zero disturbance input to the system, we know from (6.3), (6.4) and (6.29) that $\forall (i, j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,

$$\begin{aligned} \Delta V_i(x_k) - \beta V_i(x_k) &= x_k^T \hat{\Lambda}_i x_k, \forall k \in [k_l,\ k_l + \mathcal{T}_{\max}) \\ \Delta V_i(x_k) + \alpha V_i(x_k) &= x_k^T \bar{\Lambda}_i x_k, \forall k \in [k_l + \mathcal{T}_{\max}, k_{l+1}) \\ V_i(x_{k_l}) - \mu V_j(x_{k_l}) &= x_{k_l}^T \left[P_i - \mu P_j\right] x_{k_l}, \end{aligned}$$

where $\hat{\Lambda}_i \triangleq \hat{A}_i^T P_i \hat{A}_i - \beta P_i - P_i$, $\bar{\Lambda}_i \triangleq \bar{A}_i^T P_i \bar{A}_i + \alpha P_i - P_i$. From $\Theta_i \leq 0$ and $\Theta_{ij} \leq 0$, we readily know that

$$\begin{bmatrix} -P_i & P_i \bar{A}_i \\ \star & -(1-\alpha) P_i \end{bmatrix} \leq 0, \begin{bmatrix} -P_i & P_i \hat{A}_i \\ \star & -(1+\beta) P_i \end{bmatrix} \leq 0$$

which, by Lemma 2.4, imply $\hat{\Lambda}_i \leq 0$ and $\bar{\Lambda}_i \leq 0$. Therefore, if we further have $P_i - \mu P_j \leq 0$, system (6.29) is GUAS for any switching signal satisfying (6.5). Now consider the disturbance input, one has that $\forall k \in [k_l, k_l + \mathcal{T}_{\max})$

$$\begin{aligned} &\Delta V_i(x_k) - \beta V_i(x_k) + z_k^T z_k - \gamma^2 u_k^T u_k \\ &= x_k^T \left[\hat{\Lambda}_i + \hat{C}_i^T \hat{C}_i\right] x_k + 2 x_k^T \left[\hat{A}_i^T P_i \hat{E}_i + \hat{C}_i^T \hat{F}_i\right] w_k \\ &\quad + w_k^T \left[-\gamma^2 I + \hat{E}_i^T P_i \hat{E}_i + \hat{F}_i^T \hat{F}_i\right] w_k \\ &= \zeta^T(k) \Omega_{\uparrow i} \zeta(k) \end{aligned}$$

and $\forall k \in [k_l + \mathcal{T}_{\max}, k_{l+1})$

$$\begin{aligned} &\Delta V_i(x_k) + \alpha V_i(x_k) + z_k^T z_k - \gamma^2 u_k^T u_k \\ &= x_k^T \left[\bar{\Lambda}_i + \bar{C}_i^T \bar{C}_i\right] x_k + 2 x_k^T \left[\bar{A}_i^T P_i \bar{E}_i + \bar{C}_i^T \bar{F}_i\right] w_k \\ &\quad + w_k^T \left[-\gamma^2 I + \bar{E}_i^T P_i \bar{E}_i + \bar{F}_i^T \bar{F}_i\right] w_k \\ &= \zeta^T(k) \Omega_{\downarrow i} \zeta(k) \end{aligned}$$

where $\zeta(k) \triangleq [\,\tilde{x}^T(k)\;\; w^T(k)\,]^T$ and

$$\Omega_{\uparrow i} \triangleq \begin{bmatrix} \hat{\Lambda}_i + \hat{C}_i^T \hat{C}_i & \hat{E}_i^T P_i \hat{E}_i + \hat{C}_i^T \hat{F}_i \\ \star & -\gamma^2 I + \hat{E}_i^T P_i \hat{E}_i + \hat{F}_i^T \hat{F}_i \end{bmatrix}$$
$$\Omega_{\downarrow i} \triangleq \begin{bmatrix} \bar{\Lambda}_i + \bar{C}_i^T \bar{C}_i & \bar{E}_i^T P_i \bar{E}_i + \bar{C}_i^T \bar{F}_i \\ \star & -\gamma^2 I + \bar{E}_i^T P_i \bar{E}_i + \bar{F}_i^T \bar{F}_i \end{bmatrix}$$

By Lemma 2.4, $\Theta_i \le 0$ and $\Theta_{ij} \le 0$ are equivalent to $\Omega_{\downarrow i} \le 0$ and $\Omega_{\uparrow i} \le 0$, respectively. Therefore, one has

$$\Delta V_i(x_k) \le \begin{cases} -\alpha V_i(x_k) + z_k^T z_k - \gamma^2 u_k^T u_k, \forall k \in [k_l + \mathcal{T}_{\max}, k_{l+1}) \\ \beta V_i(k) + z_k^T z_k - \gamma^2 u_k^T u_k, \forall k \in [k_l,\; k_l + \mathcal{T}_{\max}) \end{cases}$$

According to Theorem 6.7, the system has a guaranteed l_2-gain no greater than γ_a, which means the system (6.29) satisfies $\|z\|_2^2 \le \gamma_a^2 \, \|w\|_2^2$. This completes the proof. □

Based on the above results, the following theorem presents a sufficient condition of the existence of the mode-dependent state-feedback H_∞ controllers for system (6.27)–(6.28) in the presence of asynchronous switching.

Theorem 6.14 *Consider switched system (6.27)–(6.28) and let* $0 < \alpha < 1$, $\beta \ge 0$, $\gamma > 0$, *and* $\mu \ge 1$ *be given constants. If there exist matrices* $S_i > 0$ *and* U_i, $\forall i \in \mathcal{I}$, *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$, $S_i \le \mu S_j$, $\Psi_i \le 0$ *and* $\Psi_{ij} \le 0$, *where*

$$\Psi_i \triangleq \begin{bmatrix} -S_i & 0 & A_i S_i + B_i U_i & E_i \\ \star & -I & C_i S_i + D_i U_i & F_i \\ \star & \star & -(1-\alpha) S_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix},$$
$$\Psi_{ij} \triangleq \begin{bmatrix} -S_i & 0 & A_i S_j + B_i U_j & E_i \\ \star & -I & C_i S_j + D_i U_j & F_i \\ \star & \star & -(1+\beta)\left(S_i - S_j - S_j^T\right) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix},$$

then there exists a mode-dependent state-feedback controller with the asynchronous delay $\mathcal{T}_{\max}$ *such that system (6.29) is GUAS for any switching signal with ADT satisfying (6.5) and has an exponential* H_∞ *performance index* $\gamma_a = \sqrt{(\theta^{\mathcal{T}_{\max}} \mu)^{N_0} \theta^{\mathcal{T}_{\max}-1}}\gamma$. *Moreover, if a feasible solution exists, the admissible controller gain is given by*

$$K_i = U_i S_i^{-1} \tag{6.31}$$

Proof Replace $\bar{A}_i$, $\hat{A}_i$ of Θ_i and Θ_{ij} in Theorem 6.13 by the ones in (6.29). Setting $S_i \triangleq P_i^{-1}$, $U_i \triangleq K_i S_i$ and performing a congruence transformation [8] to $\Psi_i \le 0$

via $diag\{S_i^{-1}, I, S_i^{-1}, I\}$, we can obtain $\Theta_i \le 0$. In addition, from the fact $(S_i - S_j)^T S_i (S_i - S_j) \ge 0$, we have $S_i - S_j - S_j^T \ge -S_j^T S_i^{-1} S_j$. Then, if $\Psi_{ij} \le 0$, one has

$$\begin{bmatrix} -S_i & 0 & A_i S_j + B_i U_j & E_i \\ \star & -I & C_i S_j + D_i U_j & F_i \\ \star & \star & -(1+\beta) S_j^T S_i^{-1} S_j & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0$$

Performing a congruence transformation to the above inequality via $diag\{S_i^{-1}, I, S_j^{-1}, I\}$, we can obtain $\Theta_{ij} \le 0$. Therefore, according to Theorem 6.13, we know if $S_i \le \mu S_j$, $\Psi_i \le 0$ and $\Psi_{ij} \le 0$, system (6.29) is GUAS for any switching signal with ADT satisfying (6.5) and has an exponential H_∞ performance index γ_a. Meanwhile, the mode-dependent controller gain is given by $K_i = U_i S_i^{-1}$.This completes the proof. □

Remark 6.15 Note that if setting $\mathcal{T}_{\max} \equiv 0$, then Theorem 6.14 reduces to Definition 6.6, i.e., the synchronous switching case, which we list here again as a corollary for further comparison.

Corollary 6.16 *Consider switched system (6.27)–(6.28) and let* $0 < \alpha < 1$, $\gamma > 0$, *and* $\mu \ge 1$ *be given constants. If there exist matrices* $S_i > 0$ *and* U_i, $\forall i \in \mathcal{I}$, *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$, $S_i \le \mu S_j$, $\Psi_i \le 0$, *where* Ψ_i *is shown in Theorem 6.14, then there exists a mode-dependent state-feedback controller such that system (6.27)–(6.28) is GUAS for any switching signal with ADT satisfying (6.5) and has an exponential* H_∞ *performance index* γ. *Moreover, if a feasible solution exists, the admissible controller gain is given by (6.31).*

Remark 6.17 Solving the convex problems contained in the above Theorem 6.14 and Corollary 6.16, the scalars γ and γ_a can be optimized in terms of the feasibility of the corresponding conditions. In addition, it is obvious that $\gamma_a \ge \gamma$, which means that the H_∞ performance achieved in the presence of asynchronous switching is worse than the one in the case of synchronous switching. However, the controller designed without considering asynchronous switching, even under the admissible switching (6.5), may fail to obtain the prescribed (or optimized) γ or even γ_a, which will be shown via the following example.

Example 6.18 Consider discrete-time switched linear system (6.27)–(6.28) consisting of three subsystems described by

$$A_1 = \begin{bmatrix} 0.88 & -0.05 \\ 0.40 & -0.72 \end{bmatrix}, A_2 = \begin{bmatrix} 0.51 & 0.24 \\ 0.80 & 0.32 \end{bmatrix}, A_3 = \begin{bmatrix} -0.80 & 0.16 \\ 0.80 & 0.64 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.30 \\ -5.0 \end{bmatrix}, B_2 = \begin{bmatrix} -1.4 \\ 0.30 \end{bmatrix}, B_3 = \begin{bmatrix} -1.5 \\ 0.10 \end{bmatrix},$$
$$E_1 = \begin{bmatrix} 0.70 \\ 1.30 \end{bmatrix}, E_2 = \begin{bmatrix} 0.20 \\ 1.40 \end{bmatrix}, E_3 = \begin{bmatrix} -1.10 \\ 0.90 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.20 & 0.10 \end{bmatrix}, C_2 = \begin{bmatrix} 0.30 & 0.40 \end{bmatrix}, C_3 = \begin{bmatrix} -0.10 & 0.20 \end{bmatrix},$$
$$D_1 = 0.40, D_2 = -0.50, D_3 = 0.60, F_1 = 0.20, F_2 = 0.30, F_3 = -1.10.$$

The maximal delay of asynchronous switching $\mathcal{T}_{\max} = 2$.

Our purpose here is to design a mode-dependent state-feedback controller and find out the admissible switching signals such that the resulting closed-loop system is stable with an optimized exponential H_∞ disturbance attenuation performance.

First, we shall demonstrate that if one studies the control problem of the above system assuming synchronous switching, i.e., based on Corollary 6.16, the corresponding design results will be invalid in the presence of asynchronous switching. Given $\mu = 1.05$ and $\alpha = 0.20$ and solving the convex optimization problem in Corollary 6.16 (minimizing γ in the criteria), one can obtain $\tau_a^* = 0.2186$, $\gamma^* = 2.6309$ and the controller gains as

$$K_1 = [0.9505\ 0.1529], K_2 = [0.3657\ 0.1847], K_3 = [-0.8420\ 0.0741] \quad (6.32)$$

Applying controller (6.32) and generating a possible switching sequence satisfying $\tau_a = 1 > 0.2186$, one can get the steady state response of the resulting closed-loop system as shown in Fig. 6.3a for $w(k) = 0.5\exp(-0.5k)$. Now if there exists asynchronous switching in practice with $\mathcal{T}_{\max} = 2$, the state response of the resulting systems for switching sequences with $\tau_a = 1, 2, 3$ are plotted, respectively, in Figs. 6.3b and 6.4. One can observe that although the states become converging as the selected ADT is increasing, all the practical H_∞ performance indices are greater than the optimized one. It is actually hard by trial-and-error to find admissible switching signals since the designed controller may be also wrong.

Thus, we consider the asynchronous switching in the design phase and turn to Theorem 6.14. By further giving $\beta = 0.05$ and solving the corresponding convex optimization problem in Theorem 6.14, we obtain $\tau_a^* = 2.6559$, $\gamma_a^* = \sqrt{(\theta^{\mathcal{T}_{\max}}\mu)^{N_0}}\gamma_d^*$ with $\gamma_d^* = 5.6886$ and $N_0 = 1.2$, and the controller gains as

$$K_1 = [0.2698\ 0.1360], K_2 = [0.2897\ 0.1785], K_3 = [-0.1711\ 0.1343] \quad (6.33)$$

Using the controller (6.33) and giving switching sequences with $\tau_a = 3$ and $\tau_a = 4$ (both are greater than 2.6559), respectively, the state responses of the resulting system are given in Fig. 6.5. In addition, generating randomly 200 switching sequences with $\tau_a = 3$, Fig. 6.6 gives the comparison on the H_∞ performance indices that the resulting closed-loop systems can achieve when applying (6.32) and (6.33), respectively. It can be seen from Figs. 6.5 and 6.6 that the designed controller (6.33) under the admissible switching signals is effective despite asynchronous switching. Also, in Fig. 6.6, it is obvious that the controller (6.32) even can not guarantee $\gamma_a^* = \sqrt{(\theta^{\mathcal{T}_{\max}}\mu)^{N_0}}\gamma_d^*$ with $\gamma_d^* = 5.6886$ and $N_0 = 1.2$, although the given switching sequences with $\tau_a = 3$ are admissible. Therefore, combining with Figs. 6.3 and 6.4, we conclude that only increasing ADT is not sufficient to ensure the system stability and/or performance, which also shows the necessity of Theorem 6.14 and its potential in practice.

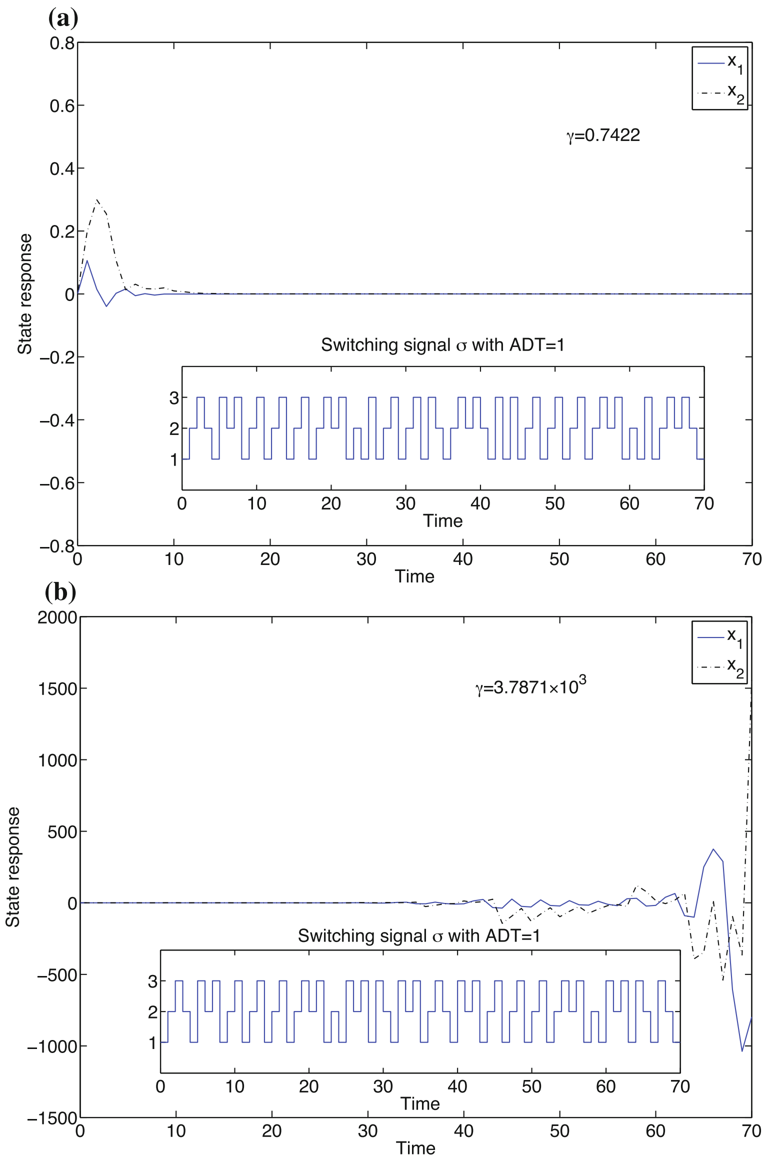

Fig. 6.3 State responses of the closed-loop systems by controller (6.32) with different $\mathcal{T}_{\max}$ and ADT. **a** $\mathcal{T}_{\max} = 0, ADT = 1$; **b** $\mathcal{T}_{\max} = 2, ADT = 1$

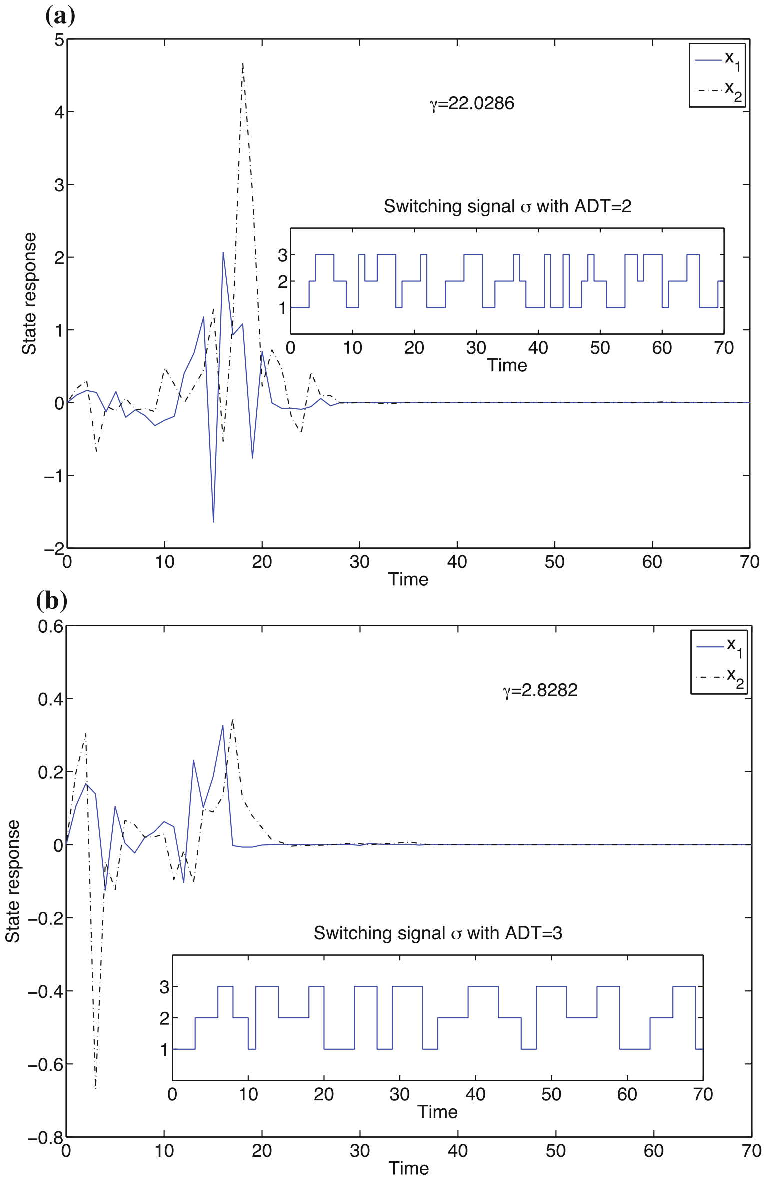

Fig. 6.4 State responses of the closed-loop systems by controller (6.32) with different $\mathcal{T}_{\max}$ and ADT. **a** $\mathcal{T}_{\max} = 2, ADT = 2$; **b** $\mathcal{T}_{\max} = 2, ADT = 3$

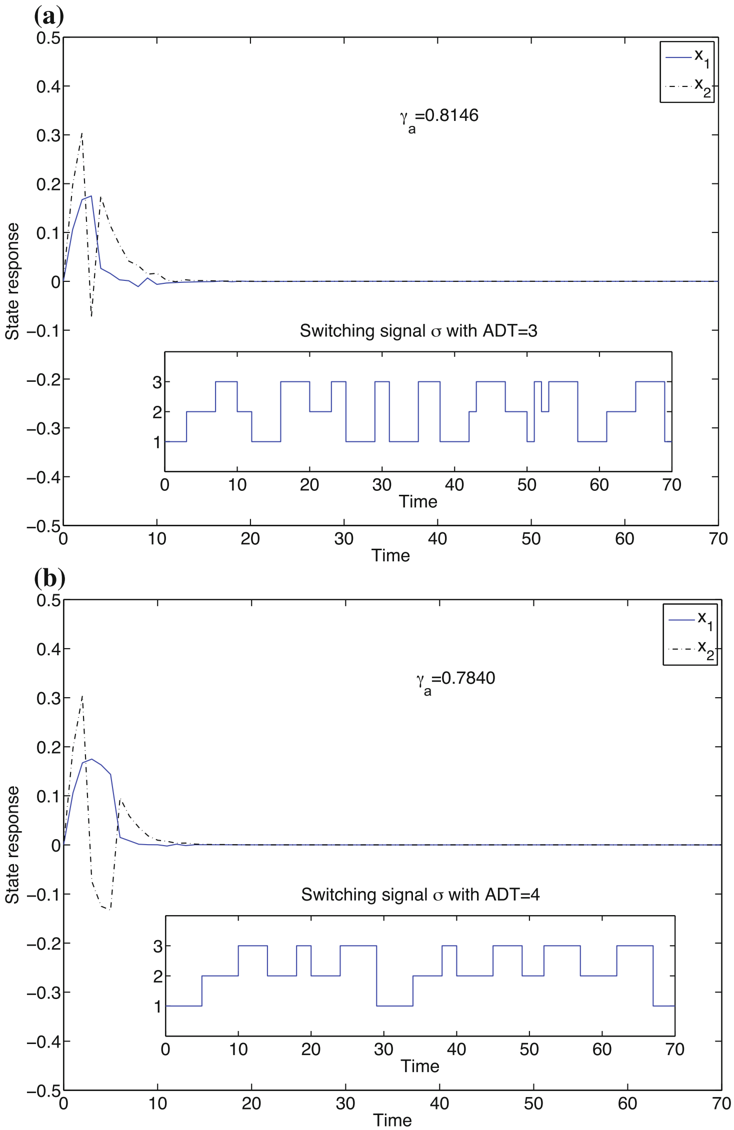

Fig. 6.5 State responses of the closed-loop systems by controller (6.33). **a** $\mathcal{T}_{\max} = 2, ADT = 3$; **b** $\mathcal{T}_{\max} = 2, ADT = 4$

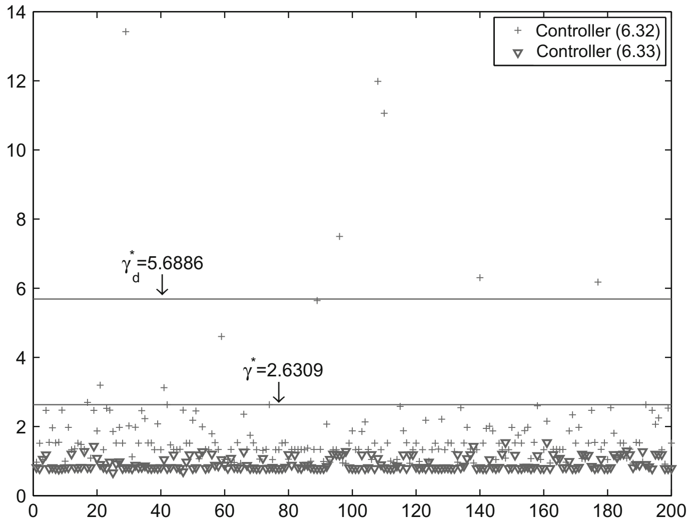

Fig. 6.6 H_∞ performance indices of the closed-loop systems by controller (6.32) and controller (6.33)

6.4.2 Switched LPV Systems

Consider the following discrete-time switched linear parameter varying (LPV) system

$$x_{k+1} = A_{\sigma(k)}(\rho_k)x_k + B_{\sigma(k)}(\rho_k)u_k + E_{\sigma(k)}(\rho_k)\omega_k \tag{6.34}$$

$$y_k = C_{\sigma(k)}(\rho_k)x_k + D_{\sigma(k)}(\rho_k)u_k + F_{\sigma(k)}(\rho_k)\omega_k \tag{6.35}$$

when $\sigma(k) = i \in \mathcal{I}$, the matrices $(A_i(\rho_k),\ B_i(\rho_k),\ C_i(\rho_k),\ D_i(\rho_k),\ E_i(\rho_k),\ F_i(\rho_k))$ denote the ith subsystem, which are known functions of measurable ρ_k, where $\rho_k = [\rho_{1k},\ \ldots,\ \rho_{sk}]$, $|\rho_{zk}| \leq \bar{\rho}_z$, $\forall 1 \leq z \leq s$ is a vector of time-varying parameters which belongs to a compact set.

Note that in the studies in the previous section and other literature, such as [9–13], the delays in the switching of controllers are assumed to be constant or time-varying. In this subsection, we assume that the delay of controllers to be mode-dependent. Specifically, let $\mathscr{T}_i$, $i \in \mathcal{I}$, be the value of the delay for the ith subsystem.

Based on the stability, l_2-gain analysis results in Theorems 6.4 and 6.8, the conditions and the corresponding H_∞ controller ensuring the GUAS and l_2-gain can be obtained for discrete-time switched LPV system (6.34)–(6.35). As studied in Sect. 6.4.1, this subsection is to find mode-dependent state-feedback controllers $K_i(\rho_k), \forall i \in \mathcal{I}$, such that the closed-loop system can achieve anticipated performance in spite of the asynchronous switching. When considering asynchronous switching, the controller input can be expressed as $u_k = K_{\sigma(k-\mathcal{T}_l)}(\rho_k)x_k, \forall k \in [k_l, k_{l+1}), \forall l \in \mathbb{N}$.

Hence, assuming $\sigma(k_l) = i \in \mathcal{I}$, we have $\forall k \in [k_l, k_l + \mathcal{T}_l)$, $K_{\sigma(k-\mathcal{T}_l)}(\rho_k) = K_j(\rho_k)$, $j \neq i$, and $\forall k \in [k_l + \mathcal{T}_l, k_{l+1})$, $K_{\sigma(k-\mathcal{T}_l)}(\rho_k) = K_i(\rho_k)$. The resulting closed-loop system can be written as

$$\begin{cases} x_{k+1} = \hat{A}_i(\rho_k)x_k + E_i(\rho_k)\omega_k \\ y_k = \hat{C}_i(\rho_k)x_k + F_i(\rho_k)\omega_k \end{cases} \forall k \in [k_l, k_l + \mathcal{T}_l) \tag{6.36}$$

$$\begin{cases} x_{k+1} = \bar{A}_i(\rho_k)x_k + E_i(\rho_k)\omega_k \\ y_k = \bar{C}_i(\rho_k)x_k + F_i(\rho_k)\omega_k \end{cases} \forall k \in [k_l + \mathcal{T}_l, k_{l+1}) \tag{6.37}$$

where $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$,

$$\begin{aligned} &\hat{A}_i(\rho_k) \triangleq A_i(\rho_k) + B_i(\rho_k)K_j(\rho_k), \bar{A}_i(\rho_k) \triangleq A_i(\rho_k) + B_i(\rho_k)K_i(\rho_k), \\ &\hat{C}_i(\rho_k) \triangleq C_i(\rho_k) + D_i(\rho_k)K_j(\rho_k), \bar{C}_i(\rho_k) \triangleq C_i(\rho_k) + D_i(\rho_k)K_i(\rho_k), \\ &\hat{E}_i(\rho_k) \triangleq \bar{E}_i(\rho_k) = E_i(\rho_k), \hat{F}_i(\rho_k) \triangleq \bar{F}_i(\rho_k) = F_i(\rho_k). \end{aligned}$$

The controllers designed under the assumption of synchronous switching may cause instability or other worse performance in the presence of asynchronous behavior when controllers switch. To solve this problem, we deduce the following theorems that can ensure the performance.

Theorem 6.19 *Consider discrete-time switched LPV systems (6.36) and (6.37), let* $0 < \alpha_i < 1$, $\beta_i \geq 0$, $\gamma > 0$ *and* $\mu_i \geq 1$, $\forall i \in \mathcal{I}$, *be given constants. If there exist matrices* $P_i(\rho_k) > 0$, $\forall i \in \mathcal{I}$, *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $P_i(\rho_k) \leq \mu_i P_j(\rho_k)$, *and the following parameterized LMIs hold*

$$\begin{bmatrix} -P_i(\rho_{k+1}) & 0 & P_i(\rho_{k+1})\hat{A}_i(\rho_k) & P_i(\rho_{k+1})\hat{E}_i(\rho_k) \\ \star & -I & \hat{C}_i(\rho_k) & \hat{F}_i(\rho_k) \\ \star & \star & -(1+\beta_i)P_i(\rho_k) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \leq 0 \tag{6.38}$$

$$\begin{bmatrix} -P_i(\rho_{k+1}) & 0 & P_i(\rho_{k+1})\bar{A}_i(\rho_k) & P_i(\rho_{k+1})\bar{E}_i(\rho_k) \\ \star & -I & \bar{C}_i(\rho_k) & \bar{F}_i(\rho_k) \\ \star & \star & -(1-\alpha_i)P_i(\rho_k) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \leq 0 \tag{6.39}$$

Then under asynchronous delay $\mathcal{T}_i$, *the corresponding system is GUAS for any MADT switching signal satisfying (6.10) and has an exponential* H_∞ *performance index* $\gamma_s = \sqrt{\prod_{i\in\mathcal{I}} (\theta_i^{\mathcal{T}_i}\mu_i)^{N_{0i}} \frac{\alpha_{\max}}{\alpha_{\min}} \theta_{\max}^{\mathcal{T}_{\max}-1}}\gamma$.

Proof Choose the Lyapunov function of the form

$$V_i(x_k, \rho_k) = x_k^T P_i(\rho_k)x_k, \forall \sigma(k) = i \in \mathcal{I}$$

For the zero disturbance input to the system, we have

$$\Delta V_i(x_k, \rho_k) + \alpha_i V_i(x_k, \rho_k) = x_k^T \bar{\Omega}_i x_k, \forall k \in [k_l + \mathcal{T}_l, k_{l+1})$$
$$\Delta V_i(x_k, \rho_k) - \beta_i V_i(x_k, \rho_k) = x_k^T \hat{\Omega}_i x_k, \forall k \in [k_l, k_l + \mathcal{T}_l)$$

where

$$\bar{\Omega}_i \triangleq \bar{A}_i^T(\rho_k) P_i(\rho_{k+1}) \bar{A}_i(\rho_k) - P_i(\rho_k) + \alpha_i P_i(\rho_k),$$
$$\hat{\Omega}_i \triangleq \hat{A}_i^T(\rho_k) P_i(\rho_{k+1}) \hat{A}_i(\rho_k) - P_i(\rho_k) - \beta_i P_i(\rho_k).$$

In addition, we have $V_i(x_{k_l}, \rho_{k_l}) - \mu_i V_j(x_{k_l}, \rho_{k_l}) = x_{k_l}^T [P_i(\rho_{k_l}) - \mu_i P_j(\rho_{k_l})] x_{k_l}$. By Lemma 2.4, (6.38) and (6.39) imply

$$\Delta V_i(x_k, \rho_k) \leq \begin{cases} -\alpha_i V_i(x_k, \rho_k), & \forall k \in [k_l + \mathcal{T}_l, k_{l+1}) \\ \beta_i V_i(x_k, \rho_k), & \forall k \in [k_l, k_l + \mathcal{T}_l) \end{cases}$$

By $P_i(\rho_k) \leq \mu_i P_j(\rho_k)$, we have $\Delta V_i(x_{k_l}, \rho_k) \leq \mu_i \Delta V_j(x_{k_l}, \rho_k)$. From Theorem 6.4, discrete-time switched LPV systems (6.36) and (6.37) are GUAS under any MADT switching signal satisfying (6.10). Now consider the disturbance input, we have $\forall k \in [k_l + \mathcal{T}_l, k_{l+1})$, $\Delta V_i(x_k, \rho_k) + \alpha_i V_i(x_k, \rho_k) + y_k^T y_k - \gamma^2 \omega_k^T \omega_k = \varsigma_k^T \bar{\Omega}_{\downarrow i} \varsigma_k$, and $\forall k \in [k_l, k_l + \mathcal{T}_l)$, $\Delta V_i(x_k, \rho_k) - \beta_i V_i(x_k, \rho_k) + y_k^T y_k - \gamma^2 \omega_k^T \omega_k = \varsigma_k^T \bar{\Omega}_{\uparrow i} \varsigma_k$, where $\varsigma_k = [\, x_k^T \;\, \omega_k^T \,]$, and

$$\bar{\Omega}_{\downarrow i} = \begin{bmatrix} \bar{\Omega}_i + \bar{C}_i^T(\rho_k)\bar{C}_i(\rho_k) & \bar{A}_i^T(\rho_k)P_i(\rho_{k+1})\bar{E}_i(\rho_k) + \bar{C}_i(\rho_k)\bar{F}_i(\rho_k) \\ \star & -\gamma^2 I + \bar{E}_i^T(\rho_k)P_i(\rho_{k+1})\bar{E}_i(\rho_k) + \bar{F}_i^T(\rho_k)\bar{F}_i(\rho_k) \end{bmatrix}$$
$$\bar{\Omega}_{\uparrow i} = \begin{bmatrix} \hat{\Omega}_i + \hat{C}_i^T(\rho_k)\hat{C}_i(\rho_k) & \hat{A}_i^T(\rho_k)P_i(\rho_{k+1})\bar{E}_i(\rho_k) + \bar{C}_i(\rho_k)\bar{F}_i(\rho_k) \\ \star & -\gamma^2 I + \bar{E}_i^T(\rho_k)P_i(\rho_{k+1})\bar{E}_i(\rho_k) + \bar{F}_i^T(\rho_k)\bar{F}_i(\rho_k) \end{bmatrix}$$

From (6.38) and (6.39), by Lemma 2.4, we have $\bar{\Omega}_{\downarrow i} \leq 0$, $\bar{\Omega}_{\uparrow i} \leq 0$. Therefore, we can get

$$\Delta V_i(x_k, \rho_k) \leq \begin{cases} -\alpha_i V_i(x_k, \rho_k) + y_k^T y_k - \gamma^2 w_k^T w_k, & \forall k \in [k_l + \mathcal{T}_l, k_{l+1}) \\ \beta_i V_i(x_k, \rho_k) + y_k^T y_k - \gamma^2 w_k^T w_k, & \forall k \in [k_l, k_l + \mathcal{T}_l) \end{cases}$$

together with $P_i(\rho_k) \leq \mu_i P_j(\rho_k)$ and Theorem 6.8, we can complete this proof. □

Then, the existence conditions of an asynchronous H_∞ controller for the underlying switched LPV systems can be obtained as follows.

Theorem 6.20 *Consider discrete-time switched LPV system (6.34)–(6.35) with the controllers in the form* $u_k = K_{\sigma(k-\mathcal{T}_l)} x_k$, $\forall k \in [k_l, k_{l+1})$, $\forall l \in \mathbb{N}$. *Let* $0 < \alpha_i < 1$, $\beta_i \geq 0$, $\gamma > 0$ *and* $\mu_i \geq 1$, $\forall i \in \mathcal{I}$, *be given constants. If there exist matrices* $S_i(\rho_k) > 0$, $U_i(\rho_k)$, $\forall i \in \mathcal{I}$, *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $S_i(\rho_k) \leq \mu_i S_j(\rho_k)$, *the following parameterized LMIs hold*

$$\begin{bmatrix} -S_i(\rho_{k+1}) & 0 & A_i(\rho_k)S_j(\rho_k)+B_i(\rho_k)U_j(\rho_k) & E_i(\rho_k) \\ \star & -I & C_i(\rho_k)S_j(\rho_k)+D_i(\rho_k)U_j(\rho_k) & F_i(\rho_k) \\ \star & \star & (1+\beta_i)\left[S_i(\rho_k)-S_j(\rho_k)-S_j^T(\rho_k)\right] & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{6.40}$$

$$\begin{bmatrix} -S_i(\rho_{k+1}) & 0 & A_i(\rho_k)S_i(\rho_k)+B_i(\rho_k)U_i(\rho_k) & E_i(\rho_k) \\ \star & -I & C_i(\rho_k)S_i(\rho_k)+D_i(\rho_k)U_i(\rho_k) & F_i(\rho_k) \\ \star & \star & -(1-\alpha_i)S_i(\rho_k) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{6.41}$$

then there exists a mode-dependent state-feedback controller

$$K_i(\rho_k) = U_i(\rho_k)S_i^{-1}(\rho_k) \tag{6.42}$$

such that the discrete-time switched LPV system is GUAS with weighted H_∞ performance index $\gamma_a = \sqrt{\underset{i\in\mathcal{I}}{\Pi}(\theta_i^{\mathcal{T}_i}\mu_i)^{N_{0i}}\frac{\alpha_{\max}}{\alpha_{\min}}\theta_{\max}^{\mathcal{T}_{\max}-1}}\gamma$ under any switching signal (6.10).

Proof Replace $\hat{A}_i(\rho_k)$ and $\bar{A}_i(\rho_k)$ in (6.38) and (6.39) by the ones in (6.36) and (6.37). Defining $S_i(\rho_k) \triangleq P_i^{-1}(\rho_k)$ and $U_i(\rho_k) \triangleq K_i(\rho_k)S_i(\rho_k)$, and performing the congruence transformation via $diag\{S_i^{-1}(\rho_{k+1}), I, S_i^{-1}(\rho_k), I\}$ to (6.40) and (6.41), we can easily obtain (6.39). Furthermore, from $(S_i(\rho_k)-S_j(\rho_k))^T S_i^{-1}(\rho_k)(S_i(\rho_k) - S_j(\rho_k)) \geq 0$, we have $S_i(\rho_k)-S_j(\rho_k)-S_j^T(\rho_k) \geq -S_j^T(\rho_k)S_i^{-1}(\rho_k)S_j(\rho_k)$. Therefore, from (6.40), we have

$$\begin{bmatrix} -S_i(\rho_{k+1}) & 0 & A_i(\rho_k)S_j(\rho_k)+B_i(\rho_k)U_j(\rho_k) & E_i(\rho_k) \\ \star & -I & C_i(\rho_k)S_j(\rho_k)+D_i(\rho_k)U_j(\rho_k) & F_i(\rho_k) \\ \star & \star & -(1+\beta_i)S_j^T(\rho_k)S_i^{-1}(\rho_k)S_j(\rho_k) & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{6.43}$$

Performing the congruence transformation to (6.43) via $diag\{S_i^{-1}(\rho_{k+1}), I, S_i^{-1}(\rho_k), I\}$, we can obtain (6.38). Furthermore, $\forall(i,j) \in \mathcal{I}\times\mathcal{I}$, $i \neq j$, $S_i(\rho_k) \leq \mu_i S_j(\rho_k)$ implies $P_i(\rho_k) \leq \mu_i P_j(\rho_k)$. From Theorem 6.19, the proof can be completed. □

Remark 6.21 As shown in most of LPV literature (see for instance [14]), by choosing appropriate basis functions $\{f_l(\rho_k)\}_l^{n_f}$, the matrix function variables $\mathcal{Y}_i(\rho) = \{P_i(\rho_k), S_i(\rho_k), U_i(\rho_k)\}$ in Theorems 6.19 and 6.20 can be decomposed as the following affine fashion

$$\mathcal{Y}_i(\rho) = \sum_{i=1}^{n_f} f_l(\rho_k)\mathcal{Y}_i^l \tag{6.44}$$

where $f_l(\rho_k)$ and n_f can be chosen by designer in accordance with the dependence structure in system (6.36)–(6.37). Consequently, $\mathcal{Y}_i^l(\rho) = \{P_i(\rho_k), S_i(\rho_k), U_i(\rho_k)\}$ becomes the decision variables of Theorems 6.19 and 6.20. In this sense, the gridding technique can be utilized to eliminate the dependence on the parameter vector ρ_k in the parameterized LMIs [14].

In the following, we provide an example to illustrate the effectiveness of the controller design method given in this subsection.

Example 6.22 Consider discrete-time switched LPV system (6.34)–(6.35) with the following state-space matrices

$$
\begin{aligned}
&A_1(\rho_k) = \begin{bmatrix} 0.9 & -1.44 + 0.25\rho_k \\ 1.08 & 0.72 \end{bmatrix},\ A_2(\rho_k) = \begin{bmatrix} 1.08 & 0.36 + 0.2\rho_k \\ -0.72 & -1.17 \end{bmatrix}, \\
&B_1(\rho_k) = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}, B_2(\rho_k) = \begin{bmatrix} 0.5 \\ -0.1 \end{bmatrix},\ E_1(\rho_k) = E_1(\rho_k) = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}, \\
&C_1(\rho_k) = C_2(\rho_k) = [\,0.3\ -0.1\,],\ D_1(\rho_k) = D_2(\rho_k) = 0.6, \\
&F_1(\rho_k) = F_2(\rho_k) = 0,
\end{aligned}
$$

where $\rho_k = \cos(0.2\pi k)$ is the time-varying parameter. Our purpose is to design a mode-dependent stabilizing controller and find admissible switching signals with MADT such that the closed-loop asynchronous switched LPV system (6.36)–(6.37) is stable with a guaranteed exponential H_∞ performance for given α_i, β_i, and μ_i, $\forall i \in \mathcal{I}$.

According to the structure of the parameter dependence in the system above, we choose the basic functions in (6.44) as $f_1(\rho_k) = 1$ and $f_2(\rho_k) = \cos(0.2\pi k)$. Gridding the parameter space of ρ_k with 10 uniform grids, for the parameters given in Table 6.1, we can obtain the different MADT τ_{ai}^*, ADT τ_a^* and the optimized H_∞ performance index γ^* via Theorem 6.20. The obtained results under MADT and ADT switching logics are illustrated in Table 6.1.

It can be seen in Table 6.1 that the minimal MADT are reduced to $\tau_{a1}^* = 5.7700$, $\tau_{a2}^* = 2.8904$, and one special case of MADT switching is $\tau_{a1}^* = \tau_{a2}^* = 5.7700$ by setting $\alpha = \alpha_1 = \alpha_2 = 0.12$ and $\beta = \beta_1 = \beta_2 = 0.15$ (note that the ADT switching logic is independent of the special system modes, thus any switching signal satisfying the ADT of a system will satisfy the MADT of all subsystems, i.e., $\tau_a^* \geq \tau_{ai}^*$, $\forall i \in \mathcal{I}$). Therefore, the ADT switching can be viewed as a special case of MADT switching.

Table 6.1 Parameters and computation results for the system under two different switching logics

Switching schemes	MADT switching	ADT switching
Parameters	$\mu_1 = \mu_2 = 1.6$	$\mu = 1.6$
	$\alpha_1 = 0.12$, $\alpha_2 = 0.24$	$\alpha = 0.12$
	$\beta_1 = 0.15$, $\beta_2 = 0.05$	$\beta = 0.15$
	$\mathcal{T}_1 = \mathcal{T}_2 = 1$	$\mathcal{T} = 1$
Optimal H_∞ performance	$\gamma_M^* = 18.2779$	$\gamma_A^* = 7.7377$
Switching signals	$\tau_{a1}^* = 5.7700$, $\tau_{a2}^* = 2.8904$	$\tau_{a1}^* = 5.7700$

It should be noted that in Table 6.1, the optimal H_∞ performance index of MADT switching γ_M^* is larger than that of ADT case γ_A^*, i.e., $\gamma_M^* > \gamma_A^*$, which is consistent with Remark 6.9.

The obtained gain controllers for the MADT switching scheme are as follows

$$
\begin{aligned}
S_1(\rho_k) &= \begin{bmatrix} 0.2142 & -0.0716 \\ -0.0716 & 0.0452 \end{bmatrix} + \rho_k \begin{bmatrix} -0.1911 & 0.0623 \\ 0.0623 & -0.0357 \end{bmatrix} \\
S_2(\rho_k) &= \begin{bmatrix} 0.2019 & -0.0973 \\ -0.0973 & 0.0697 \end{bmatrix} + \rho_k \begin{bmatrix} -0.1823 & 0.0858 \\ 0.0858 & -0.0557 \end{bmatrix} \\
U_1(\rho_k) &= [\,-0.4317\ 0.1693\,] + \rho_k[\,0.3828\ -0.1444\,] \\
U_2(\rho_k) &= [\,-0.4586\ 0.2487\,] + \rho_k[\,0.4125\ -0.2140\,]
\end{aligned}
$$

and for the ADT switching scheme

$$
\begin{aligned}
S_1(\rho_k) &= \begin{bmatrix} 0.2999 & -0.0767 \\ -0.0767 & 0.0603 \end{bmatrix} + \rho_k \begin{bmatrix} -0.2439 & 0.0588 \\ 0.0588 & -0.0437 \end{bmatrix} \\
S_2(\rho_k) &= \begin{bmatrix} 0.2763 & -0.1179 \\ -0.1179 & 0.0942 \end{bmatrix} + \rho_k \begin{bmatrix} -0.2361 & 0.0987 \\ 0.0987 & -0.0707 \end{bmatrix} \\
U_1(\rho_k) &= [\,-0.5605\ 0.1831\,] + \rho_k[\,0.4477\ -0.1371\,] \\
U_2(\rho_k) &= [\,-0.5922\ 0.2977\,] + \rho_k[\,0.5014\ -0.2397\,]
\end{aligned}
$$

In order to show the effectiveness and advantages of the MADT switching, we construct a switching signal, which satisfies $\tau_{a1} = 5.9 > \tau_{a1}^*$, and $\tau_{a2} = 3.9 > \tau_{a2}^*$. It is apparent that this switching signal does not satisfy the requirement of ADT switching since $\tau_a = 4.5714 < \tau_a^*$. The initial conditions are assumed to be $x_0 = [\,0\ 0\,]^T$, which is aimed to guarantee the condition $V_{\sigma(k_0)}(x_{k_0}) \equiv 0$ in Theorem 6.8. The disturbance input is assumed to be $w(k) = 0.5\cos(0.2\pi k)$ $\exp(-0.5k)$. With the controllers obtained via LMI toolbox in Matlab, we can get the state response of the closed-loop switched LPV system, which is demonstrated in Fig. 6.7. In Fig. 6.7, all the states of the system converge to zero. Figure 6.8 shows the response of the ratio $\sqrt{\sum_{i=1}^{k}(1-\alpha_{\max})^i y(i)^T y(i)} \Big/ \sqrt{\sum_{i=1}^{k} w(i)^T w(i)}$ under disturbance input $w(k)$. From Fig. 6.8, it can be observed that the response performances are satisfactory, and it can also be seen from Fig. 6.8 that the ratio $\sqrt{\sum_{i=1}^{k}(1-\alpha_{\max})^i y(i)^T y(i)} \Big/ \sqrt{\sum_{i=1}^{k} w(i)^T w(i)}$ is less than 1.5686 , which is below the minimum disturbance-attenuation level $\gamma_M^* = 18.2779$, thus showing the effectiveness of the controller design. In comparison with the MADT switching scheme, the state response of the closed-loop system for the ADT case under the same switching signal is demonstrated in Fig. 6.9. It can be seen that the state response does not converge to zero. Therefore, from Figs. 6.7 and 6.9, we can conclude the MADT switching logic is less rigid than the ADT switching in terms of the requirement of the switching signal, which shows the advantage of the MADT switching.

∞

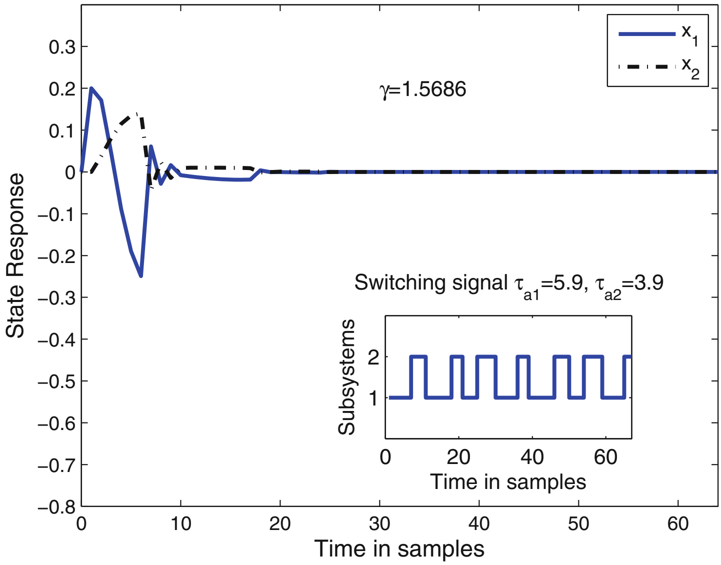

Fig. 6.7 State response of closed-loop system with MADT

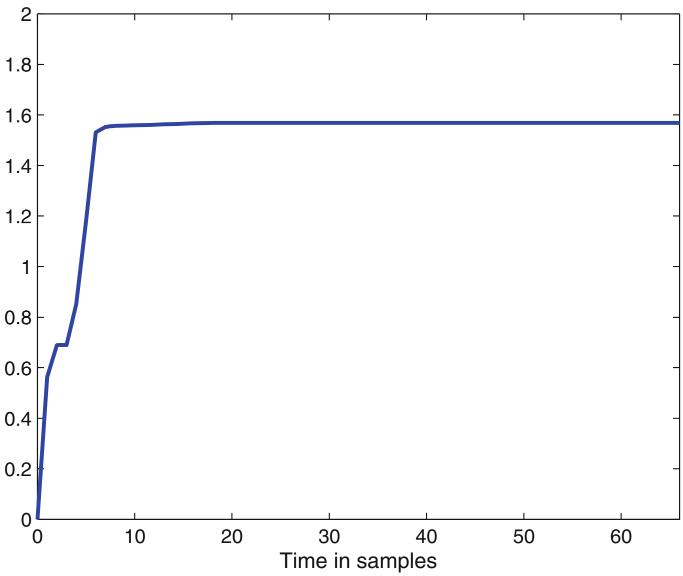

Fig. 6.8 Response of ratio $\sqrt{\sum_{i=1}^{k}(1-\alpha_{\max})^i y(i)^T y(i)} \Big/ \sqrt{\sum_{i=1}^{k} w(i)^T w(i)}$

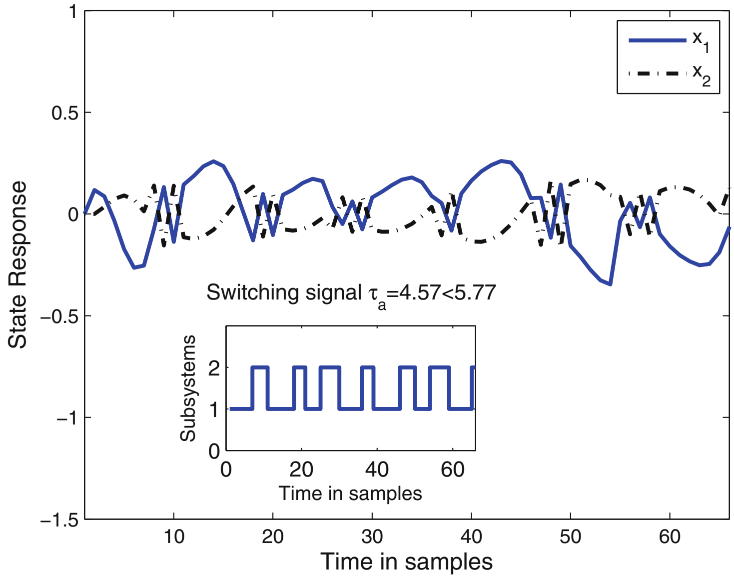

Fig. 6.9 State response of closed-loop system with ADT

6.5 H_∞ Filtering

Consider a class of discrete-time switched linear systems given by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}\omega(k) \quad (6.45)$$

$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}\omega(k) \quad (6.46)$$

$$z(k) = H_{\sigma(k)}x(k) + L_{\sigma(k)}\omega(k) \quad (6.47)$$

where the system description has been given in the previous sections, and when $\sigma(k) = i \in \mathcal{I}$, the matrices $(A_i, B_i, C_i, D_i, H_i, L_i)$ denote the ith subsystem. In this section, we focus our study of system (6.45)–(6.47) on a class of switching signals with ADT property.

Here, we are interested in designing the following mode-dependent full-order filter for system (6.45)–(6.47), $\forall \sigma = i \in \mathcal{I}$

$$x_F(k+1) = A_{Fi}x_F(k) + B_{Fi}y(k) \quad (6.48)$$

$$z_F(k) = C_{Fi}x_F(k) + D_{Fi}y(k) \quad (6.49)$$

where A_{Fi}, B_{Fi}, C_{Fi} and D_{Fi} are the filter gains to be determined. Also, we aim to consider the more practical asynchronous filtering problem, that is, the switches of the filter gains do not coincide in *real time* with those of system modes. Thus, the resulting filtering error system becomes

$$\begin{cases} \tilde{x}(k+1) & = \hat{A}_i\tilde{x}(k) + \hat{E}_i w(k) \\ e(k) & = \hat{C}_i\tilde{x}(k) + \hat{F}_i w(k), \quad \forall k \in [k_l,\ k_l + \mathcal{T}_{\max}) \\ \tilde{x}(k+1) & = \bar{A}_i\tilde{x}(k) + \bar{E}_i w(k) \\ e(k) & = \bar{C}_i\tilde{x}(k) + \bar{F}_i w(k), \quad \forall k \in [k_l + \mathcal{T}_{\max}, k_{l+1}) \end{cases} \tag{6.50}$$

where $\tilde{x}(k) \triangleq \left[x^T(k)\ x_F^T(k)\right]^T$, $e(k) \triangleq z(k) - z_F(k)$, and

$$\hat{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ B_{Fj}C_i & A_{Fj} \end{bmatrix},\ \hat{E}_i \triangleq \begin{bmatrix} B_i \\ B_{Fj}D_i \end{bmatrix},$$
$$\hat{C}_i \triangleq \left[H_i - D_{Fj}C_i\ -C_{Fj}\right],\ \hat{F}_i \triangleq L_i - D_{Fj}D_i,$$
$$\bar{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ B_{Fi}C_i & A_{Fi} \end{bmatrix},\ \bar{E}_i \triangleq \begin{bmatrix} B_i \\ B_{Fi}D_i \end{bmatrix},$$
$$\bar{C}_i \triangleq \left[H_i - D_{Fi}C_i\ -C_{Fi}\right],\ \bar{F}_i \triangleq L_i - D_{Fi}D_i.$$

Then, our objective in this section is to design a mode-dependent full-order filter and find a set of admissible switching signals with ADT such that the resulting filtering error system (6.50) is GUAS and has a guaranteed exponential H_∞ disturbance attenuation performance, i.e., $\|e\|_2^2 \leq \gamma^2 \|w\|_2^2$ for a given $\gamma > 0$ in the presence of asynchronous switching. A sufficient condition of the existence of the mode-dependent full-order H_∞ filters for the underlying system in the presence of asynchronous switching is given in the following theorem.

Theorem 6.23 *Consider system (6.45)–(6.47) and let* $0 < \alpha < 1$, $\beta \geq 0$, $\gamma > 0$, *and* $\mu \geq 1$ *be given constants. If there exist matrices* $P_{1i} > 0$, $P_{3i} > 0$ *and matrices* P_{2i}, X_i, Y_i, Z_i, A_{fi}, B_{fi}, C_{fi}, D_{fi}, $\forall i \in \mathcal{I}$ *such that* $\Phi_i \leq 0$, $\Phi_{ij} \leq 0$ *and*

$$\begin{bmatrix} P_{1i} & P_{2i} \\ \star & P_{3i} \end{bmatrix} - \mu \begin{bmatrix} P_{1j} & P_{2j} \\ \star & P_{3j} \end{bmatrix} \leq 0 \tag{6.51}$$

where

$$\Phi_i \triangleq \begin{bmatrix} \Phi_i^{11} & \Phi_i^{12} & 0 & \Phi_i^{14} & X_iB_i + B_{fi}D_i \\ \star & \Phi_i^{22} & 0 & \Phi_i^{24} & Z_iB_i + B_{fi}D_i \\ \star & \star & -I & \Phi_i^{34} & L_i - D_{fi}D_i \\ \star & \star & \star & \Phi_i^{44} & 0 \\ \star & \star & \star & \star & -\gamma^2 I \end{bmatrix}$$

$$\Phi_{ij} \triangleq \begin{bmatrix} \Phi_{ij}^{11} & \Phi_{ij}^{12} & 0 & \Phi_{ij}^{14} & X_j B_i + B_{fj} D_i \\ \star & \Phi_{ij}^{22} & 0 & \Phi_{ij}^{24} & Z_j B_i + B_{fj} D_i \\ \star & \star & -I & \Phi_{ij}^{34} & L_i - D_{fj} D_i \\ \star & \star & \star & \Phi_{ij}^{44} & 0 \\ \star & \star & \star & \star & -\gamma^2 I \end{bmatrix}$$

with

$$\begin{aligned} &\Phi_i^{11} \triangleq P_{1i} - X_i - X_i^T, \Phi_{ij}^{11} \triangleq P_{1i} - X_j - X_j^T, \Phi_i^{12} \triangleq P_{2i} - Y_i - Z_i^T, \\ &\Phi_{ij}^{12} \triangleq P_{2i} - Y_j - Z_j^T, \Phi_i^{14} \triangleq X_i A_i + B_{fi} C_i A_{fi}, \Phi_{ij}^{14} \triangleq X_j A_i + B_{fj} C_i A_{fj}, \\ &\Phi_i^{24} \triangleq Z_i A_i + B_{fi} C_i A_{fi}, \Phi_{ij}^{24} \triangleq Z_j A_i + B_{fj} C_i A_{fj}, \\ &\Phi_i^{34} \triangleq H_i - D_{fi} C_i - C_{fi}, \Phi_{ij}^{34} \triangleq H_i - D_{fj} C_i - C_{fj}, \end{aligned}$$

and

$$\Phi_i^{44} \triangleq \begin{bmatrix} -\bar{\alpha} P_{1i} & -\bar{\alpha} P_{2i} \\ \star & -\bar{\alpha} P_{3i} \end{bmatrix}, \Phi_{ij}^{44} \triangleq \begin{bmatrix} -\tilde{\beta} P_{1i} & -\tilde{\beta} P_{2i} \\ \star & -\tilde{\beta} P_{3i} \end{bmatrix},$$

$\bar{\alpha} \triangleq 1 - \alpha$, $\tilde{\beta} \triangleq 1 + \beta$, *then there exists a mode-dependent filter with the asynchronous delay $\mathcal{T}_{\max}$ such that the corresponding filtering error system (6.50) is GUAS for any switching signal with ADT satisfying (6.5) and has an exponential H_∞ performance index $\gamma_a = \sqrt{(\theta^{\mathcal{T}_{\max}} \mu)^{N_0} \theta^{\mathcal{T}_{\max} - 1}} \gamma$. Moreover, if the feasible solutions exist, the admissible filter gains are given by*

$$A_{Fi} = Y_i^{-1} A_{fi}, B_{Fi} = Y_i^{-1} B_{fi}, C_{Fi} = C_{fi}, D_{Fi} = D_{fi}. \tag{6.52}$$

Proof First of all, for a matrix R_i, $\forall i \in \mathcal{I}$, from the fact $(P_i - R_i)^T P_i (P_i - R_i) \geq 0$ we have $P_i - R_i - R_i^T \geq -R_i^T P_i^{-1} R_i$, then we know the following inequalities

$$\begin{bmatrix} P_i - R_i - R_i^T & 0 & R_i \bar{A}_i & R_i \bar{E}_i \\ \star & -I & \bar{C}_i & \bar{F}_i \\ \star & \star & -(1-\alpha) P_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \leq 0 \tag{6.53}$$

$$\begin{bmatrix} P_i - R_j - R_j^T & 0 & R_j \hat{A}_i & R_j \hat{E}_i \\ \star & -I & \hat{C}_i & \hat{F}_i \\ \star & \star & -(1+\beta) P_i & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \leq 0 \tag{6.54}$$

guarantee $\Theta_i \leq 0$ and $\Theta_{ij} \leq 0$ in Theorem 6.13, respectively. Then, replace $\bar{A}_i$, $\bar{C}_i$, $\bar{E}_i$, $\bar{F}_i$ and $\hat{A}_i$, $\hat{C}_i$, $\hat{E}_i$, $\hat{F}_i$ in (6.53) and (6.54) by the ones in (6.50) and assume the matrices P_i, R_i to have the following forms

$$P_i \triangleq \begin{bmatrix} P_{1i} & P_{2i} \\ \star & P_{3i} \end{bmatrix}, \ R_i \triangleq \begin{bmatrix} X_i & Y_i \\ Z_i & Y_i \end{bmatrix}$$

Defining matrix variables

$$A_{fi} = Y_i A_{Fi}, B_{fi} = Y_i B_{Fi}, C_{fi} = C_{Fi}, D_{fi} = D_{Fi} \tag{6.55}$$

one can readily obtain Φ_i and Φ_{ij}. Therefore, if $\Phi_i \leq 0$, $\Phi_{ij} \leq 0$ and (6.51) holds, we have $\Theta_i \leq 0$, $\Theta_{ij} \leq 0$ and $P_i \leq \mu P_j$, respectively. According to Theorem 6.13, the filtering error system (6.50) is GUAS for any switching signal with ADT satisfying (6.5) and has an exponential H_∞ performance index γ. In addition, from (6.55), the mode-dependent filter gains are given by (6.52). This completes the proof. □

In the absence of asynchronous switching, i.e., $\mathcal{T}_{\max} = 0$ in Theorem 6.23, we can get the following corollary (cf. Remark 6.15).

Corollary 6.24 *Consider switched system (6.45)–(6.47) and let* $0 < \alpha < 1$, $\gamma > 0$, *and* $\mu \geq 1$ *be given constants. If there exist matrices* $P_{1i} > 0$, $P_{3i} > 0$ *and* P_{2i}, X_i, Y_i, Z_i, A_{fi}, B_{fi}, C_{fi}, D_{fi}, $\forall i \in \mathcal{I}$ *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $\Phi_i \leq 0$ *and (6.51) holds, where* Φ_i *is shown in Theorem 6.23, then there exists a mode-dependent filter such that the resulting filtering error system is GUAS for any switching signal with ADT satisfying (6.5) and has an exponential* H_∞ *performance index* γ. *Moreover, if a feasible solution exists, the admissible filter gains are given by (6.52).*

In what follows, we will present two examples to demonstrate the validity of the filter design approach in the presence of asynchronous switching. The first numerical example is used to show the necessity of considering asynchronous switching, and the second example is derived from a PWM-driven boost converter, a typical circuit system to illustrate the applicability of the theoretical results.

Example 6.25 Consider a discrete-time switched linear system (6.45)–(6.47) consisting of three subsystems described by

$$\begin{aligned}
A_1 &= \begin{bmatrix} -0.60 & -0.05 \\ 0.38 & 0.68 \end{bmatrix}, A_2 = \begin{bmatrix} 0.63 & 0.23 \\ 0.75 & -0.68 \end{bmatrix}, A_3 = \begin{bmatrix} -0.75 & -0.15 \\ 0.75 & 0.90 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} -0.30 \\ 0.20 \end{bmatrix}, B_2 = \begin{bmatrix} -1.40 \\ -0.30 \end{bmatrix}, B_3 = \begin{bmatrix} 0.10 \\ -0.10 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 0.10 & -0.10 \end{bmatrix}, C_2 = \begin{bmatrix} 0.30 & -0.40 \end{bmatrix}, C_3 = \begin{bmatrix} -0.10 & 0.20 \end{bmatrix}, \\
H_1 &= \begin{bmatrix} 0.70 & 0.30 \end{bmatrix}, H_2 = \begin{bmatrix} 0.20 & 0.40 \end{bmatrix}, H_3 = \begin{bmatrix} -0.10 & 0.20 \end{bmatrix}, \\
L_1 &= 0.20, L_2 = 0.30, L_3 = -0.10, D_1 = 0.40, D_2 = -0.50, D_3 = 0.20.
\end{aligned}$$

The maximal delay of asynchronous switching $\mathcal{T}_{\max} = 2$.

The objective is to design a mode-dependent full-order filter and find out the admissible switching signals such that the resulting filtering error system is stable with an optimized exponential H_∞ disturbance attenuation performance.

We shall first demonstrate that if one studies the filtering problem of the above system assuming synchronous switching, i.e., by Corollary 6.24, the corresponding design results will be invalid in the presence of asynchronous switching. Giving $\mu = 1.05$ and $\alpha = 0.20$ and solving the convex optimization problem in Corollary 6.24, one can get $\tau_a^* = 0.463$, $\gamma^* = 0.427$ and the corresponding filter gains as

$$A_{F1} = \begin{bmatrix} -0.48 & -0.12 \\ 0.39 & 0.65 \end{bmatrix}, A_{F2} = \begin{bmatrix} 1.27 & 1.00 \\ 0.32 & -0.18 \end{bmatrix}, A_{F3} = \begin{bmatrix} -0.25 & -0.24 \\ -0.25 & 0.99 \end{bmatrix} \tag{6.56}$$

Due to the space limit, we omit $B_{Fi}, C_{Fi}, D_{Fi}, i = 1, 2, 3$ here. The filtering error response in Fig. 6.10 (a) shows that the above filter is effective with $\gamma = 0.1536 < 0.4273$ under a switching sequence with $\tau_a = 1 > 0.463$ for given $w(k) = 0.5\exp(-0.05k)$. However, the filtering error responses in the presence of asynchronous switching, plotted in Figs. 6.10 (b) and 6.11 for the switching sequences with $\tau_a = 1, 2, 3$, respectively, show that the filtering error system is stable though, the optimized exponential H_∞ performance can not be guaranteed. In other words, the designed filter can not estimate the state of the original system in a required exponential H_∞ performance index. Now, turn to Theorem 6.23 and consider the asynchronous switching. By further giving $\beta = 0$ and solving the convex optimization problem in Theorem 6.23, we can get $\tau_a^* = 2.463$, $\gamma_a^* = \sqrt{(\theta^{\mathcal{T}_{\max}}\mu)^{N_0}}\gamma_d^*$ with $N_0 = 1.2$ and $\gamma_d^* = 1.872$, and filter gains as ($B_{Fi}, C_{Fi}, D_{Fi}, i = 1, 2, 3$ are omitted)

$$A_{F1} = \begin{bmatrix} -0.15 & -0.18 \\ 0.43 & 0.52 \end{bmatrix}, A_{F2} = \begin{bmatrix} 0.36 & 0.98 \\ 0.26 & -0.13 \end{bmatrix}, A_{F3} = \begin{bmatrix} -0.08 & -0.03 \\ 0.60 & 0.58 \end{bmatrix} \tag{6.57}$$

Then, for the switching sequences with $\tau_a = 3, 4$ (both are greater than 2.463), the filtering error responses using filter (6.57) are given in Fig. 6.12. Also, Fig. 6.13 gives the validation on the exponential H_∞ performance indices that the resulting filter error systems can achieve when applying (6.56) and (6.57), respectively, under randomly 200 switching sequences with $\tau_a = 3$. It can be observed from Figs. 6.10–6.13 that the H_∞ filter (6.56) designed by Corollary 6.24 is invalid (even can not ensure $\gamma_a^* = \sqrt{(\theta^{\mathcal{T}_{\max}}\mu)^{N_0}}\gamma_d^*$ with $N_0 = 1.2$ and $\gamma_d^* = 1.872$), on the contrary, the filter obtained from Theorem 6.23 is effective in spite of asynchronous switching.

Example 6.26 Consider a PWM (Pulse-Width-Modulation)-driven boost converter, shown in Fig. 6.14. The switch $s(t)$ is controlled by a PWM device and can switch at most once in each period T; L is the inductance, C the capacitance, R the load resistance, and $e_s(t)$ the source voltage. As a typical circuit system, the converter is used to transform the source voltage into a higher voltage. The control problems for such power converters have been widely studied in the literature, such as the optimal control [15], the passivity-based control [16], and the sliding mode control [17], etc. In recent years, the class of power converters is alternatively modeled as switched system and the corresponding stabilization problem has also been investigated [18],

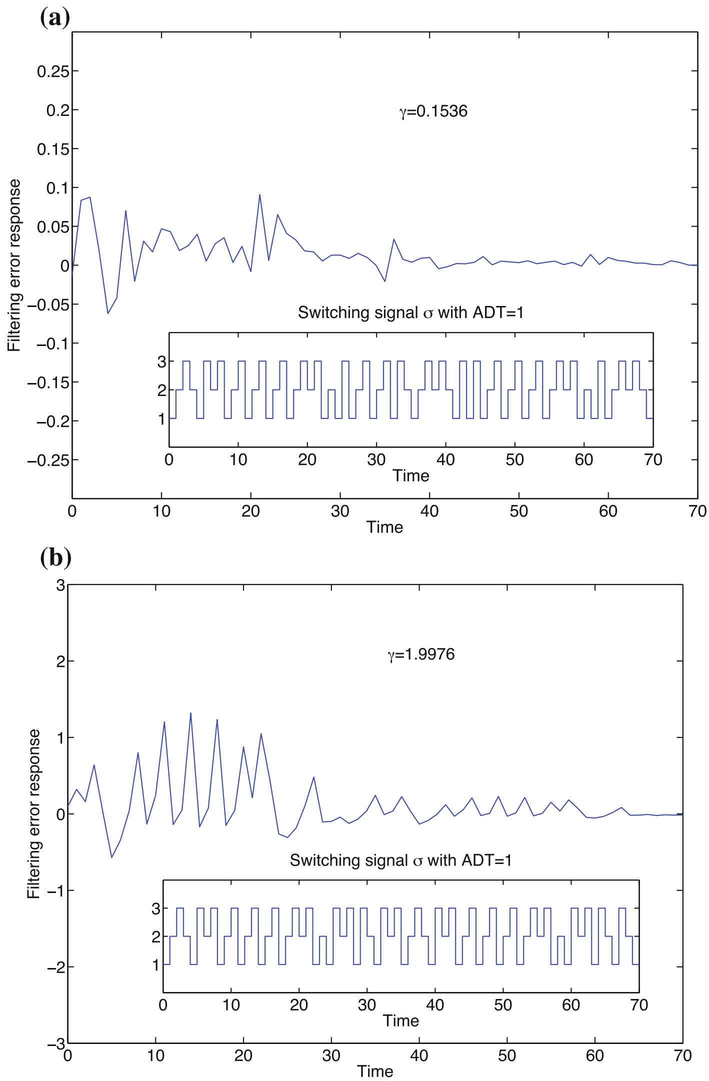

Fig. 6.10 Filtering error response by filter (6.56). **a** $\mathcal{T}_{\max} = 0, ADT = 1$; **b** $\mathcal{T}_{\max} = 2, ADT = 1$

[19]. As done in [18, 19], by introducing variables $\tau = t/T, L_1 = L/T$ and $C_1 = C/T$, the differential equations for the boost converter are as follows

$$\dot{e}_c(\tau) = -\frac{1}{RC_1}e_C(\tau) + (1 - s(\tau))\frac{1}{C_1}i_L(\tau) \tag{6.58}$$

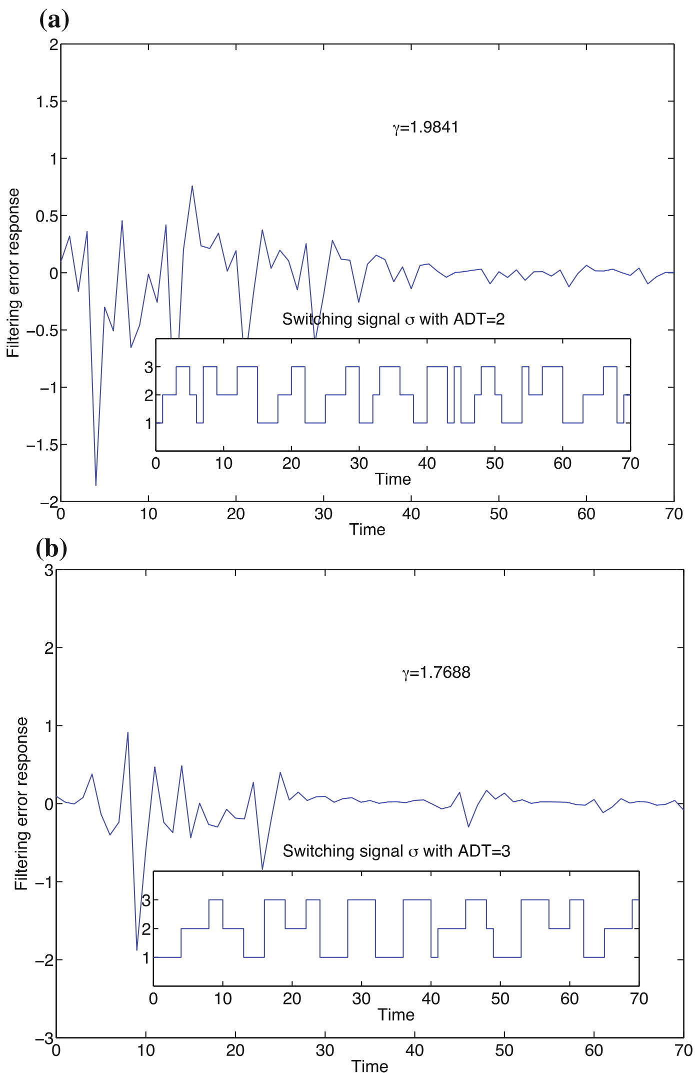

Fig. 6.11 Filtering error response by filter (6.56). **a** $\mathcal{T}_{\max} = 2, ADT = 2$; **b** $\mathcal{T}_{\max} = 2, ADT = 3$

$$\dot{i}_L(\tau) = -(1 - s(\tau))\frac{1}{L_1}e_C(\tau) + s(\tau)\frac{1}{L_1}e_S(\tau) \tag{6.59}$$

Then, (6.58)–(6.59) can be further expressed by

$$\dot{x} = A^c_\sigma x, \sigma \in \{1, 2\}, \tag{6.60}$$

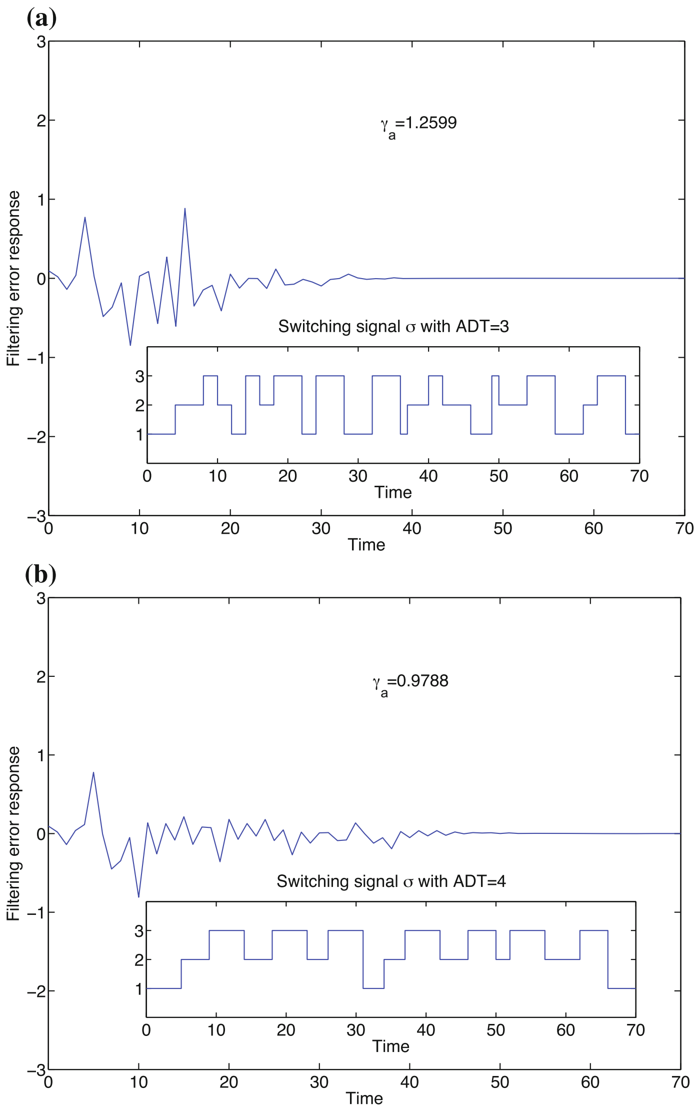

Fig. 6.12 Filtering error response by filter (6.57). **a** $\mathcal{T}_{\max} = 2, ADT = 3$; **b** $\mathcal{T}_{\max} = 2, ADT = 4$

where $x = [e_C, i_L, 1]^T$ and

$$A_1^c = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{C_1} & 0 \\ -\frac{1}{L_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2^c = \begin{bmatrix} -\frac{1}{RC_1} & 0 & 0 \\ 0 & 0 & \frac{1}{L_1} \\ 0 & 0 & 0 \end{bmatrix}$$

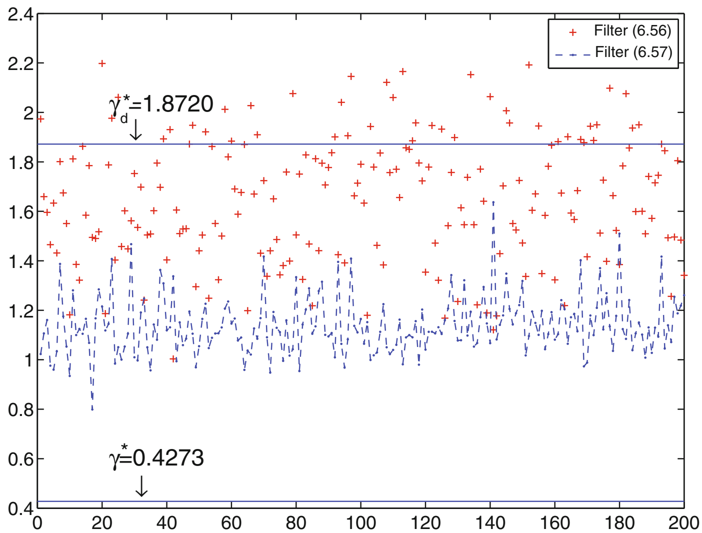

Fig. 6.13 H_∞ performance indices of filtering error system by filter (6.56) and filter (6.57)

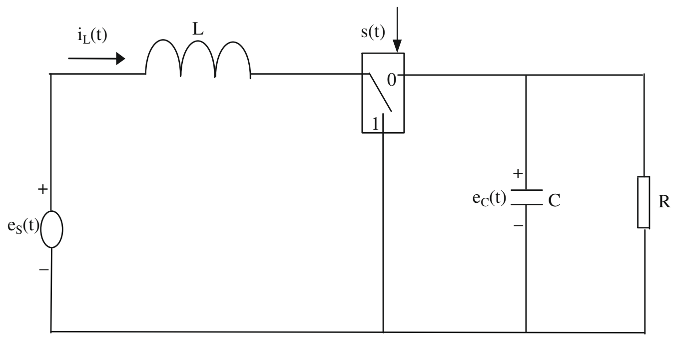

Fig. 6.14 The Boost converter

Note that each mode in (6.60) is non-Hurwitz and the stabilization problem for it is solved in [19] by designing stabilizing switching laws (the result for the buck-boost converter therein is applicable to the boost converter). As a prerequisite of employing the filtering techniques, however, all the modes of the filtered system (6.45)–(6.47) should be stable. Here, differing from [19], we assume that each mode is firstly stabilized by some control law and get a closed-loop continuous-time switched system $\dot{x} = \bar{A}^c_\sigma x$, $\sigma \in \{1, 2\}$, where the two subsystems are both Hurwitz. According to the same normalization technique used in [19], the matrices in (6.60) can be given by

$$A_1^c = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2^c = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the objective in the example is to testify the asynchronous H_∞ filter design techniques and show the potential of the obtained theoretical results in circuit systems, we assume the control matrices for (6.60) to be $B_1^c = B_2^c = [\,-0.1\ 0.4\ 0.5\,]^T$ and a set of admissible controller gains can be solved as $K_1 = [\,-6.61\ -1.07\ -9.32\,]$, $K_2 = [\,-5.37\ -12.42\ -10.07\,]$. Then, the closed-loop system can be obtained with matrices

$$\bar{A}_1^c = \begin{bmatrix} -0.34 & 1.11 & 0.93 \\ -3.65 & -0.43 & -3.73 \\ -3.30 & -0.54 & -4.66 \end{bmatrix}, \bar{A}_2^c = \begin{bmatrix} -0.46 & 1.24 & 1.00 \\ -2.15 & -4.97 & -3.03 \\ -2.68 & -6.21 & -5.03 \end{bmatrix}$$

By setting a certain sampling time $T_s = T/10$ and considering that there exists the disturbance input in the underlying system, one can obtain

$$A_1 = \begin{bmatrix} 0.94 & 0.10 & 0.06 \\ -0.30 & 0.95 & -0.30 \\ -0.25 & -0.06 & 0.63 \end{bmatrix}, A_2 = \begin{bmatrix} 0.93 & 0.08 & 0.07 \\ -0.14 & 0.66 & -0.20 \\ -0.16 & -0.40 & 0.66 \end{bmatrix},$$

in (6.45)–(6.47) and suppose other system matrices to be

$$\begin{aligned} &B_1 = [\,-0.30\ 0.20\ 0.10\,]^T, B_2 = [\,-1.40\ -0.30\ 0.20\,]^T, C_1 = [\,0.10\ -0.10\ 0.10\,], \\ &C_2 = [\,0.30\ -0.40\ 0.10\,], H_1 = [\,0.70\ 0\ 0.30\,], H_2 = [\,0.20\ 0\ 0.40\,], \\ &D_1 = 0.4, D_2 = -0.5, L_1 = L_2 = 0. \end{aligned}$$

Also, we assume the maximal delay of asynchronous switching $\mathcal{T}_{\max} = 2$. Then, by giving $\mu = 1.02$, $\alpha = 0.02$, $\beta = 0.01$ and solving the convex optimization problem in Theorem 6.23, we can get $\tau_a^* = 3.9652$, $\gamma_a^* = \sqrt{(\theta^{\mathcal{T}_{\max}}\mu)^{N_0}}\gamma_d^*$ with $N_0 = 1.2$ and $\gamma_d^* = 2.2359$ and filter gains as

$$A_{F1} = \begin{bmatrix} 0.84 & 0.02 & 0.23 \\ -0.07 & 0.67 & -0.08 \\ -0.13 & -0.01 & 0.48 \end{bmatrix}, A_{F2} = \begin{bmatrix} 0.83 & 0.24 & 0.13 \\ -0.37 & 0.96 & -0.38 \\ -0.28 & -0.14 & 0.49 \end{bmatrix}$$

We also omit $B_{Fi}, C_{Fi}, D_{Fi}, i = 1, 2$ due to space limit. The effectiveness of the desired filter with the above gains can be verified by observing the responses of the filtering error systems in the same rein of Example 6.25. We only demonstrate the applicability of the developed filter design techniques and omit the curves here.

6.6 Conclusion

By allowing the MLFs to increase during the running time of subsystems with a limited increase rate, the more general results have been obtained on stability and l_2-gain analysis for the discrete-time switched systems under ADT switching. Aiming at a class of practical problem that the switching of the controllers/filters may have a lag to the switching of system modes, the problem of the so-called asynchronous switching is then considered. Via LMIs formulation, the existence conditions of the asynchronous H_∞ controller/filter have been derived for the underlying systems in linear cases. The developed approaches are further extended to the asynchronous control of a class of discrete-time switched LPV systems with MADT switching. Several numerical examples verify the necessity of considering the asynchronous switching and the applicability of the obtained theoretical results.

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Chapter 7
Time-Delay Switched Systems

Abstract This chapter first investigates the stability problem of a class of discrete-time linear switched systems with cyclic switching and state delays, and a numerical searching algorithm is explored to compute the feasible values of dwell time of the subsystems. Then, the problem of H_∞ output feedback control for discrete-time switched linear systems with time delays is investigated. The time delay is assumed to be time-varying and has minimum and maximum bounds, which covers the constant delay and mode-dependent constant delay as two special cases. By constructing a switched quadratic Lyapunov function for the underlying system, both static and dynamic H_∞ output feedback controllers are designed respectively such that the corresponding closed-loop switched system under arbitrary switching signals is asymptotically stable and guarantees a prescribed H_∞ noise attenuation level bound. Moreover, under the arbitrary switching, the problem of robust $l_2 - l_\infty$ filtering is studied for discrete-time switched linear systems with polytopic uncertainties and time-varying delays. The robust switched linear filters are designed based on the mode-dependent idea and parameter-dependent stability approach, and the existence conditions of such filters, dependent on the upper and lower bound of time-varying delays, are formulated in terms of a set of linear matrix inequalities. Finally, the state estimation problem is studied for a class of discrete-time switching neural networks (NNs) with persistent dwell time (PDT) switching regularities and mode-dependent time-varying delays in H_∞ sense. The random packet dropouts, which are governed by a Bernoulli distributed white sequence, are considered to exist together for the estimator design of underlying switching NNs. The desired mode-dependent estimators are designed such that the resulting estimation error system is exponentially mean-square stable and achieves a prescribed H_∞ level of disturbance attenuation. The effectiveness and the superiority of the developed results are demonstrated through numerical examples.

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7.1 Stability Analysis: DT Switching

Consider the discrete-time switched linear systems with state delays

$$x(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d) \quad (7.1)$$

$$x(k) = \phi(k), k \in [-d, 0] \quad (7.2)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $\phi(k)$ is the given initial condition sequence, $d > 0$ is the constant time delay, $\sigma(k)$ is the switching signal, which is a piecewise constant function of time and takes its values in the finite set $\mathcal{I} = \{1, \ldots, N\}$, $N > 1$ is the number of subsystems. The switching sequences $k_0, k_1, k_2, \ldots, k_s, \ldots$ are unknown a priori, but are known instantly, in which the switching instant is denoted as k_s, $s \in \mathbb{Z}_+$. When $k \in [k_s, k_{s+1})$, the $\sigma(k_s)$th subsystem (or system mode) is said to be *activated* and the length of the current running time of the subsystem is $k_{s+1} - k_s$. As in [1], we assume that the switching signal $\sigma(k)$ is available in real time. At an arbitrary discrete time k, the switching signal $\sigma(k)$ is dependent on k or $x(k)$, or both, or other switching rules. In this section, the switching signal $\sigma(k)$ is considered to be regulated by the following switching rule

$$i \overset{i+1,\ldots,N,1,\ldots,i-1}{\longrightarrow} i, \forall i \in \mathcal{I}$$

i.e., the cyclic switching, which is denoted as $\sigma_{cs}(k)$. The subsystem model ($A_{\sigma(k)}$, $A_{d\sigma(k)}$) can be chosen in the following set

$$\{(A_1, A_{d1}), \ldots, (A_N, A_{dN})\}$$

where A_i, $A_{di} \in \mathbb{R}^{n\times n}$ are real constant matrices, which represent the different system models of N subsystems. The time interval between two consecutive switchings is denoted as M_i, which represents the running time of ith subsystem. The above-side system (7.1)–(7.2) is therefore called the cyclic switched linear time-delay system. Here we assume that the equilibria of all the subsystems are located at the origin.

In this section, the purpose is to derive the stability condition, and to calculate the corresponding running time M_i ensuring the system (7.1)–(7.2) is stable in the Lyapunov sense under the switching law $\sigma_{cs}(k)$.

As discussed in Chap. 2, the multiple Lyapunov-like functions (MLFs) method [2, 3] is the basic approach to study the stability of switched system, and it describes the conditions which need to be satisfied from the view point of energy during the activated subsystems and at the switching instants. The idea of this approach has been given in Lemma 2.3, which not only requires that each subsystem is stable in the Lyapunov sense, but also needs the value of Lyapunov function to be decreasing in the two consecutive switchings for the same subsystem as shown in Fig. 7.1.

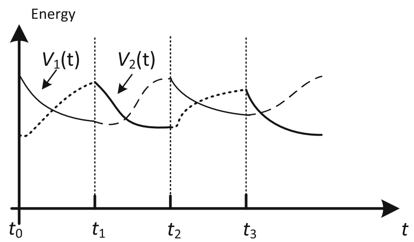

Fig. 7.1 Two Lyapunov functions, where the *thick/thin solid line* stands for the active Lyapunov function, and the *dash/dot line* stands for the non-active Lyapunov function

Consider one subsystem in (7.1)–(7.2), which can be described as a discrete-time linear system with state delays

$$x(k+1) = Ax(k) + A_d x(k-d) \tag{7.3}$$

$$x(k) = \phi(k), k \in [-d, 0] \tag{7.4}$$

For system (7.3)–(7.4), denoting $\delta(k) \triangleq x(k) - x(k-1)$, the following Lyapunov function is constructed

$$V(k) = V_1(k) + V_2(k) + V_3(k) \tag{7.5}$$

where

$$V_1(k) \triangleq x^T(k)Px(k), V_2(k) \triangleq \sum_{j=k-d}^{k-1} x^T(j)Qx(j),$$

$$V_3(k) \triangleq \sum_{\theta=-d}^{-1} \sum_{j=k+\theta+1}^{k} \delta^T(j)R\delta(k).$$

Then, the sufficient condition that ensures the system (7.3)–(7.4) is asymptotically stable is given in the following lemma.

Lemma 7.1 ([4]) *If there exist positive definite matrices P, Q, R, matrices E, X, such that*

$$\begin{bmatrix} -P & PA & PA_d & 0 \\ \star & \Lambda & -X & \bar{d}(A-I)^T R \\ \star & \star & -Q & \bar{d}A_d^T R \\ \star & \star & \star & \bar{d}R \end{bmatrix} < 0$$

$$\begin{bmatrix} E & X \\ \star & R \end{bmatrix} \geq 0$$

where $\Lambda \triangleq \bar{d}E + X + X^T - P + Q$, then $\triangle V(k) = V(k+1) - V(k) < 0$, and the system (7.3)–(7.4) is asymptotically stable if the constant time delay satisfies $0 \leq d \leq \bar{d}$.

Remark 7.2 Note that, when the constant time delay contained in (7.3)–(7.4) is extended to the time-varying delays, or a new Lyapunov function is constructed which is different from (7.5), the different sufficient conditions can be obtained in ensuring the asymptotic stability of (7.3)–(7.4).

To derive the main results more conveniently, the following transformation is made for system (7.1)–(7.2):

$$\tilde{x}(k+1) = \tilde{A}_i \tilde{x}(k)$$

where $\tilde{x}(k) \triangleq \left[\, x(k) \, \ldots \, x^T(k-d) \,\right]^T$ and

$$\tilde{A}_i \triangleq \begin{bmatrix} A_i & 0 & \cdots & A_{di} \\ I & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}_{(d+1)n \times (d+1)n}$$

Then, the following theorem gives the sufficient condition that ensures the system (7.1)–(7.2) is asymptotically stable.

Theorem 7.3 *As shown in Fig. 7.2, suppose that the system (7.1)–(7.2) switches into the subsystem* (A_i, A_{di}) *at the switching instant* k_i *in the cyclic switching law* $\sigma_{cs}(k)$*, the system switches into the subsystem* (A_i, A_{di}) *again at the switching instant* k_j *after passing through the subsystems* $(A_{i+1}, A_{d(i+1)}), \ldots, (A_N, A_{dN}), (A_1, A_{d1}), \ldots, (A_{i-1}, A_{d(i-1)})$*. Let the dwell time corresponding to each subsystem to be denoted as* $M_1, \ldots, M_N$*, respectively. If there exist positive definite matrices* P_i*,* Q_i*,* R_i*, matrices* E_i*,* X_i*,* $i \in \mathcal{I}$*, such that*

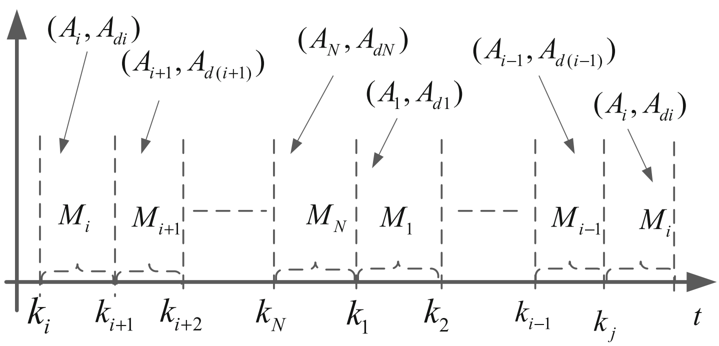

Fig. 7.2 The switching of subsystems

$$\Omega_i \triangleq \begin{bmatrix} -P_i & P_iA_i & P_iA_{di} & 0 \\ \star & \Lambda_i & -X_i & \bar{d}(A_i - I)^T R_i \\ \star & \star & -Q_i & \bar{d}A_{di}^T R_i \\ \star & \star & \star & \bar{d}R_i \end{bmatrix} < 0$$

$$\Delta_i \triangleq \begin{bmatrix} E_i & X_i \\ X_i^T & R_i \end{bmatrix} \geq 0$$

$$\Xi_i \triangleq \Upsilon_i^T \Theta_i \Upsilon_i - \Theta_i < 0, \forall i \in \mathcal{I}$$

where

$$\Lambda_i \triangleq \bar{d}E_i + X_i + X_i^T - P_i + Q_i,$$

$$\Upsilon_i \triangleq \tilde{A}_{i-1}^{M_{i-1}} \cdots \tilde{A}_1^{M_1} \tilde{A}_N^{M_N} \cdots \tilde{A}_{i+1}^{M_{i+1}} \tilde{A}_i^{M_i},$$

$$\Theta_i \triangleq \begin{bmatrix} P_i & 0 & \cdots & 0 \\ \star & Q_i & \ddots & \vdots \\ \star & \star & \ddots & 0 \\ \star & \star & \star & Q_i \end{bmatrix} + \begin{bmatrix} \bar{d}R_i & -\bar{d}R_i & 0 & \cdots & 0 \\ \star & (2\bar{d}-1)R_i & -(\bar{d}-1)R_i & \ddots & \vdots \\ \star & \star & (2\bar{d}-3)R_i & \ddots & 0 \\ \star & \star & \star & \ddots & -R_i \\ \star & \star & \star & \star & R_i \end{bmatrix}$$

then the system (7.1)–(7.2) is asymptotically stable at the switching law $\sigma_{cs}(k)$ if the constant time delay satisfies $0 \leq d \leq \bar{d}$.

Proof For any $i \in \mathcal{I}$, the system (7.1)–(7.2) can be described by the subsystem model (A_i, A_{di}). If both $\Omega_i < 0$ and $\Delta_i \geq 0$ hold, the subsystem (A_i, A_{di}) is asymptotically stable by Lemma 7.1. Thus it yields from the Lyapunov candidate as listed in (7.5), that $\triangle V_i < 0$, which satisfies the condition (a) in Lemma 2.3. Moreover, as shown in Fig. 7.2, once the system (7.1)–(7.2) switches into the subsystem (A_i, A_{di}) at the switching instant k_i, we have

$$V_i(k_i) = \tilde{x}^T(k_i)\Theta_i\tilde{x}(k_i). \tag{7.6}$$

When the system (7.1)–(7.2) switches into the subsystem (A_i, A_{di}) again at the switching instant k_j, the system passes through the subsystems $(A_{i+1}, A_{d(i+1)})$, $\ldots, (A_N, A_{dN}), (A_1, A_{d1}), \ldots, (A_{i-1}, A_{d(i-1)})$ in the role of the cyclic switching law $\sigma_{cs}(k)$, hence at the switching instant k_j, it holds

$$\tilde{x}(k_j) = \tilde{A}_{i-1}^{M_{i-1}} \cdots \tilde{A}_1^{M_1} \tilde{A}_N^{M_N} \cdots \tilde{A}_{i+1}^{M_{i+1}} \tilde{A}_i^{M_i} \tilde{x}(k_i) = \Upsilon_i \tilde{x}(k_i),$$

then, it has

$$V_i(k_j) = \tilde{x}^T(k_j)\Theta_i\tilde{x}(k_j) = \tilde{x}^T(k_i)\Upsilon_i^T\Theta_i\Upsilon_i\tilde{x}(k_i). \tag{7.7}$$

From (7.6) and (7.7), it holds that

$$V_i(k_j) - V_i(k_i) = \tilde{x}^T(k_i)(\Upsilon_i^T \Theta_i \Upsilon_i - \Theta_i)\tilde{x}(k_i) = \tilde{x}^T(k_i)\Xi_i\tilde{x}(k_i),$$

if $\Xi_i < 0$, we can obtain that,

$$V_i(k_j) - V_i(k_i) < 0,$$

which satisfies the condition (b) of Lemma 2.3. Therefore, at the cyclic switching law $\sigma_{cs}(k)$, the system (7.1)–(7.2) comprising of subsystems $(A_1, A_{d1}), \ldots, (A_N, A_{dN})$ is asymptotically stable in the Lyapunov sense. □

Remark 7.4 Due to the complexity of matrix Θ_i, and the uncertainty of dimension of matrix Θ_i before $\bar{d}$ is calculated, it is difficult to solve the required matrix variables via the typical LMIs toolbox. Therefore, we could only first calculate $\bar{d}$ by using the conditions $\Omega_i < 0$ and $\Delta_i \geq 0$, then check the condition $\Xi_i < 0$ is satisfied or not.

Remark 7.5 In system (7.1)–(7.2), the upper bound of time delays maybe different for each subsystem, i.e., there exists $\bar{d}_i$ $(i \in \mathcal{I})$, such that $\tilde{A}_i \neq \tilde{A}_j$, $\Theta_i \neq \Theta_j$, $\forall i, j \in \mathcal{I}$ $(i \neq j)$. To deal with this problem, the state vector $\tilde{x}(k)$ can be raised its dimension as $\tilde{x}(k) = [x^T(k), \ldots, x^T(k - \bar{d}_{\max})]^T$, where $\bar{d}_{\max} = \max(\bar{d}_i)$, that is, the allowed maximum time delay is $\bar{d}_{\max}$ for system (7.1)–(7.2). To simplify the derivation, the maximum time delay allowed in this section equals to the solved $\bar{d}$ satisfying the stability conditions $\Omega_i < 0$ and $\Delta_i \geq 0$, i.e., $\bar{d} = \bar{d}_{\min}$ $(\bar{d}_{\min} = \min(\bar{d}_i))$.

Remark 7.6 The analytic form of computing the admissible dwell time is hard to obtain in accordance with the condition $\Xi_i < 0$ in Theorem 7.3 which is relevant with the dwell time of each subsystem. However, a numerical approach as given in the following algorithm can be used to verify the condition $\Xi_i < 0$ and to compute the dwell time when the value of N is finite.

Algorithm 7.1 (*Computation of the dwell time*)

Step 1: Obtain the maximum time delay $\bar{d}$ ensuring the asymptotic stability of system (7.1)–(7.2) via solving $\Omega_i < 0$ and $\Delta_i \geq 0$, and calculating $(P_i, Q_i, R_i, E_i, X_i)$, then Θ_i;

Step 2: Given the constant $\mathcal{M} \in \mathbb{Z}^+$, consider $1 \leq M_i \leq \mathcal{M}$, obtain the values of $M_1, \ldots, M_N$ while verifying the condition

$$\min(eig(\Xi_i)) < 0, \forall i \in \mathcal{I} \tag{7.8}$$

where $\min(eig(U))$ represents the minimum eigenvalue of matrix U.

Remark 7.7 Because Theorem 7.3 only gives the sufficient condition ensuring the asymptotic stability of system (7.1)–(7.2), and the constraint (7.8) is strict, the system (7.1)–(7.2) may be asymptotically stable in some cases that (7.8) does not hold.

Example 7.8 Consider the following discrete-time cyclic switched linear system with state delays

$$x(k+1) = A_i x(k) + A_{di} x(k-d), i = 1, 2 \tag{7.9}$$

where

$$A_1 = \begin{bmatrix} 0.990 & 0.062 \\ -0.080 & 1.016 \end{bmatrix}, A_{d1} = \begin{bmatrix} -0.050 & 0 \\ -0.056 & -0.048 \end{bmatrix},$$
$$A_2 = A_1^T, A_{d2} = A_{d1}^T.$$

We could first calculate $\bar{d} = 6$ via Algorithm 7.1, and then obtain the values of dwell time of each subsystem which ensure that the system (7.9) is asymptotically stable, as shown in Fig. 7.3. In Fig. 7.3, the shadow area consisting of "·" stands for the feasible values of dwell time, and the x-axis represents the dwell time M_1 of subsystem 1, and the y-axis represents the dwell time M_2 of subsystem 2, respectively. The values of dwell time contained in the blank area enables the system (7.9) is asymptotically stable or not.

Given the initial state $x_0 = \begin{bmatrix} -0.5 & -0.1 \end{bmatrix}^T$, and choosing the values of dwell time located in the blank area of Fig. 7.3, which cannot necessarily ensure the system stability, for instance, $M_1 = 10$, $M_2 = 10$, the Lyapunov function is increasing continuously at the switching time instant although the two Lyapunov functions are decreasing when each corresponding subsystem is running individually; the state trajectories are not convergent as shown in Fig. 7.4.

Next, designating $M_1 = 20$, $M_2 = 20$, as given in Fig. 7.5, we can see that the two Lyapunov functions behave as required in Lemma 2.3 (see Fig. 7.3), and the system is asymptotically stable from the state trajectories.

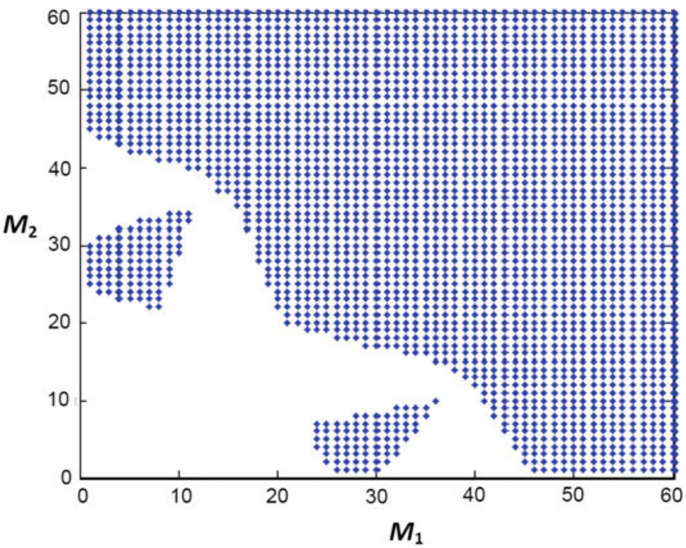

Fig. 7.3 The values of dwell time

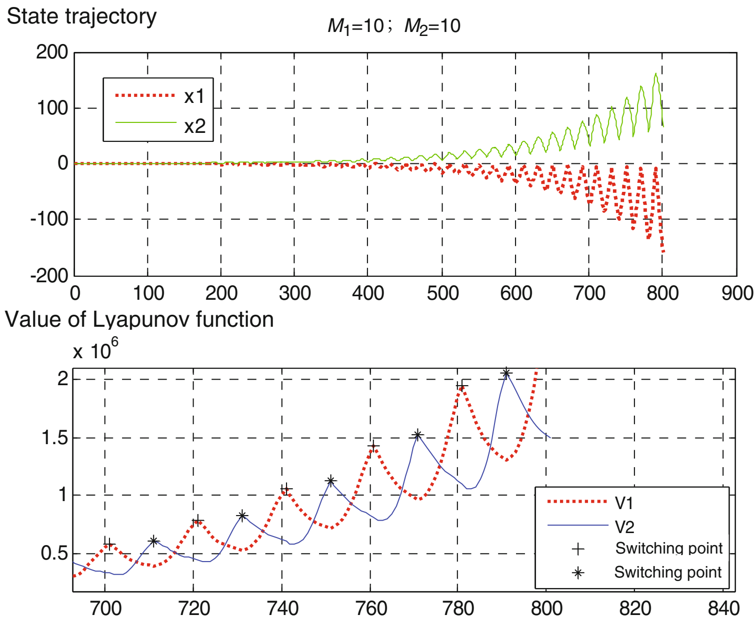

Fig. 7.4 State trajectory and Lyapunov function curve as $M_1 = 10$, $M_2 = 10$

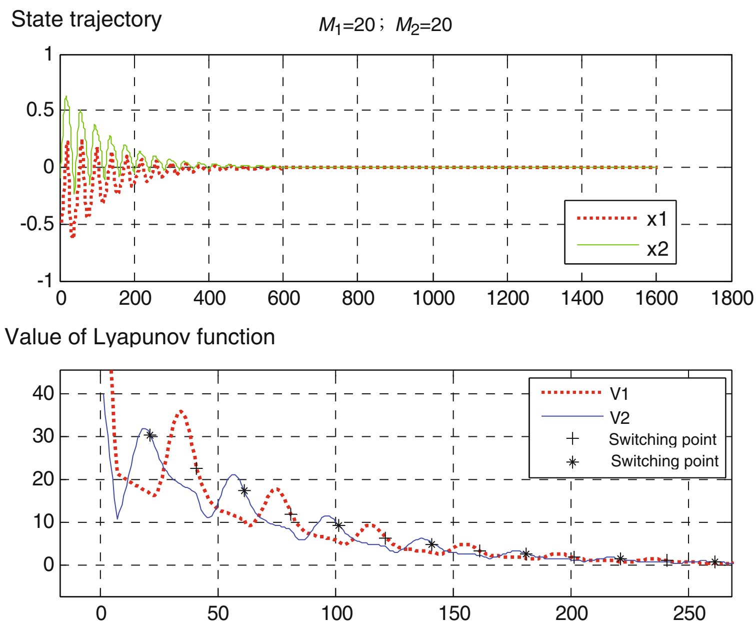

Fig. 7.5 State trajectory and Lyapunov function curve as $M_1 = 20$, $M_2 = 20$

7.2 H_∞ Control: Arbitrary Switching

Consider a class of switched linear discrete-time systems given by

$$x(k+1) = A_\sigma x(k) + A_{1\sigma} x(k-d(k)) + B_\sigma u(k) + B_{1\sigma}\omega(k) \tag{7.10}$$

$$z(k) = C_\sigma x(k) + C_{1\sigma} x(k-d(k)) + D_\sigma u(k) + D_{1\sigma}\omega(k) \tag{7.11}$$

$$y(k) = G_\sigma x(k) + G_{1\sigma} x(k-d(k)) + E_\sigma w(k) \tag{7.12}$$

$$x(k) = \phi(k), \qquad k = -d_M, -d_M+1, \ldots, 0 \tag{7.13}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $\omega(k) \in \mathbb{R}^l$ is the disturbance input which belongs to $l_2[0,\infty)$, $y(k) \in \mathbb{R}^m$ is the measurement output, $z(k)$ is the objective signal to be attenuated, $\phi(k)$ is a given initial condition sequence. Denote $\sigma \triangleq \sigma(k)$, for $\sigma(k) = i \in \mathcal{I}$, the matrices $(A_i, A_{1i}, B_i, B_{1i}, C_i, C_{1i}, D_i, D_{1i}, G_i, G_{1i}, E_i)$ denote the ith subsystem.

In system (7.10)–(7.13), the time delay $d(k)$ is assumed to be time-varying and satisfies $d_m \le d(k) \le d_M$, where d_m and d_M are constant positive scalars representing the lower and upper delay bounds respectively for any subsystem.

Remark 7.9 Note that if the lower and upper delay bounds in system (7.10)–(7.13) become identical, that is $d_m = d_M = d$, then the time delay becomes constant delay. Also, if $d(k)$ only changes when system mode is switched, then the time delay becomes mode-dependent constant delay, thus the time-varying delay considered here covers the previous two cases.

Remark 7.10 It should be also mentioned that in continuous-time context, time delay can be further assumed to be mode-dependent time-varying, as considered in [5]. However, the meaning of mode-dependent in [5] actually is that the delay derivative is different when system mode changes, that is, if the delay derivative of each mode is identical, then the delay is mode-independent and merely time-varying. On the contrary, due to the limitation of classic Lyapunov-Krasovskii technique, the time-delay difference has been rarely considered in discrete-time context and the type of delay is only assumed to be time-varying as a consequence.

In this section, the state variables are assumed to be not available for feedback, thus we are interested in designing an output-feedback controller for switched system (7.10)–(7.13) such that the resulting closed-loop system is asymptotically stable with a prescribed H_∞ performance index. The output-feedback controller to be designed has two classes here, one is the static output-feedback controller with the following form

$$u(k) = D_{si} y(k) \tag{7.14}$$

and another is the dynamic output-feedback controller, which is assumed to have the following structure

$$x_d(k+1) = A_{di}x_d(k) + B_{di}y(k) \tag{7.15}$$
$$u(k) = C_{di}x_d(k) + D_{di}y(k) \tag{7.16}$$
$$x_d(k) = 0, \qquad k \le 0 \tag{7.17}$$

The closed-loop system (7.10)–(7.13) with (7.14) is given by

$$x(k+1) = \hat{A}_i x(k) + \hat{A}_{1i}x(k-d(k)) + \hat{B}_i w(k) \tag{7.18}$$
$$z(k) = \hat{C}_i x(k) + \hat{C}_{1i}x(k-d(k)) + \hat{D}_i w(k) \tag{7.19}$$
$$x(k) = \phi(k), \qquad k = -d_M, -d_M+1, \ldots, 0 \tag{7.20}$$

where

$$\hat{A}_i \triangleq A_i + B_i D_{si} G_i,\ \hat{A}_{1i} \triangleq A_{1i} + B_i D_{si} G_{1i},\ \hat{B}_i \triangleq B_{1i} + B_i D_{si} E_i,$$
$$\hat{C}_i \triangleq C_i + D_i D_{si} G_i,\ \hat{C}_{1i} \triangleq C_{1i} + D_i D_{si} G_{1i},\ \hat{D}_i \triangleq D_{1i} + D_i D_{si} E_i.$$

Likewise, by defining $\xi(k) \triangleq \begin{bmatrix} x^T(k) & x_d^T(k) \end{bmatrix}^T$, for the dynamic output-feedback case, the corresponding closed-loop system resulted from (7.10)–(7.13) and (7.15)–(7.17) is given by

$$\xi(k+1) = \tilde{A}_i \xi(k) + \tilde{A}_{1i}\xi(k-d(k)) + \tilde{B}_i w(k) \tag{7.21}$$
$$z(k) = \tilde{C}_i \xi(k) + \tilde{C}_{1i}\xi(k-d(k)) + \tilde{D}_i w(k) \tag{7.22}$$
$$\xi(k) = \begin{bmatrix} \phi^T(k) & 0 \end{bmatrix}^T, \qquad k = -d_M, -d_M+1, \ldots, 0 \tag{7.23}$$

where

$$\tilde{A}_i \triangleq \begin{bmatrix} A_i + B_i D_{di} C_i & B_i C_{di} \\ B_{di} C_i & A_{di} \end{bmatrix},\ \tilde{A}_{1i} \triangleq \begin{bmatrix} A_{1i} + B_i D_{di} G_{1i} & 0 \\ B_{di} G_{1i} & 0 \end{bmatrix},$$
$$\tilde{B}_i \triangleq \begin{bmatrix} B_{1i} + B_i D_{di} E_i \\ B_{di} E_i \end{bmatrix}, \tilde{C}_i \triangleq \begin{bmatrix} C_i + D_i D_{di} G_i & D_i C_{di} \end{bmatrix},$$
$$\tilde{C}_{1i} \triangleq \begin{bmatrix} C_{1i} + D_i D_{di} G_{1i} & 0 \end{bmatrix}, \tilde{D}_i \triangleq [D_{1i} + D_i D_{di} E_i].$$

Our purpose is to determine the matrix gain D_{si} in (7.14) and the matrix variables A_{di}, B_{di}, C_{di} and D_{di} in (7.15)–(7.17) such that the obtained static and dynamic output-feedback controllers asymptotically stabilize the corresponding closed-loop systems (7.18)–(7.20) and (7.21)–(7.23) with a prescribed H_∞ performance index, respectively. It is noted that the switching signal in the designed controllers is assumed to be homogeneous with the one in system (7.10)–(7.13).

Before ending this section, we recall the following lemmas which will be used in the proof of our main results.

Lemma 7.11 *([4]) Assume that* $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$ *and* $\mathcal{N} \in \mathbb{R}^{n_a \times n_b}$. *Then, for any matrices* $X \in \mathbb{R}^{n_a \times n_a}$, $Y \in \mathbb{R}^{n_a \times n_b}$ *and* $R \in \mathbb{R}^{n_b \times n_b}$ *satisfying* $\begin{bmatrix} X & Y \\ Y^T & R \end{bmatrix} \ge 0$, *the fol-*

lowing inequality holds

$$-2a^T\mathcal{N}b \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y - \mathcal{N} \\ Y^T - \mathcal{N}^T & R \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

7.2.1 Performance Analysis

In this subsection, according to (7.18)–(7.20) and (7.21)–(7.23), we establish a general closed-loop system model for the performance analysis, which is denoted as

$$x(k+1) = \bar{A}_i x(k) + \bar{A}_{1i} x(k - d(k)) + \bar{B}_i w(k) \tag{7.24}$$
$$z(k) = \bar{C}_i x(k) + \bar{C}_{1i} x(k - d(k)) + \bar{D}_i w(k) \tag{7.25}$$
$$x(k) = \phi(k), \qquad k = -d_M, -d_M + 1, \ldots, 0 \tag{7.26}$$

where the state vector $x(k) \in \mathbb{R}^r$ $(r \geq n)$ and $\phi(k)$ is a given initial condition sequence. Note that $r = n$ corresponds to the case of using static output-feedback controllers, and $n < r < 2n$ corresponds to the case of using dynamic output-feedback controllers (including the reduced or full-order output-feedback controller). Also, the construction of system matrices $(\bar{A}_i, \bar{A}_{1i}, \bar{B}_i, \bar{C}_i, \bar{C}_{1i}, \bar{D}_i)$ is different when applying the different types of controllers. For the closed-loop switched system (7.24)–(7.26), we have the following lemma.

Lemma 7.12 *Consider system (7.24)–(7.26) and let $\gamma > 0$ be a given scalar. If there exist appropriate matrices $\mathcal{P}_i > 0$, $\mathcal{Q} > 0$, $\mathcal{Z} > 0$ and matrices $\mathcal{X}_{ij} > 0$, $\mathcal{Y}_{ij}$, $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$ satisfying*

$$\begin{bmatrix} -\mathcal{P}_j & 0 & 0 & \mathcal{P}_j \bar{A}_i & \mathcal{P}_j \bar{A}_{1i} & \mathcal{P}_j \bar{B}_i \\ \star & -d_M^{-1}\mathcal{Z} & 0 & \mathcal{Z}\left(\bar{A}_i - I\right) & \mathcal{Z}\bar{A}_{1i} & \mathcal{Z}\bar{B}_i \\ \star & \star & -I & \bar{C}_i & \bar{C}_{1i} & \bar{D}_i \\ \star & \star & \star & \Upsilon_{ij} & -\mathcal{Y}_{ij} & 0 \\ \star & \star & \star & \star & -\mathcal{Q} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{7.27}$$

$$\begin{bmatrix} \mathcal{X}_{ij} & \mathcal{Y}_{ij} \\ \mathcal{Y}_{ij}^T & \mathcal{Z} \end{bmatrix} \geq 0 \tag{7.28}$$

where $\Upsilon_{ij} \triangleq -\mathcal{P}_i + d_M \mathcal{X}_{ij} + \mathcal{Y}_{ij} + \mathcal{Y}_{ij}^T + (d_M - d_m + 1)\mathcal{Q}$. Then, system (7.24)–(7.26) is robustly asymptotically stable with an H_∞ noise-attenuation level bound γ.

Proof Letting

$$\eta(m) \triangleq x(m+1) - x(m)$$

we have

$$x(k-d(k)) = x(k) - \sum_{m=k-d(k)}^{k-1} \eta(m) \tag{7.29}$$

Then, system (7.24)–(7.26) can be transformed into

$$x(k+1) = \left(\bar{A}_i + \bar{A}_{1i}\right) x(k) - \bar{A}_{1i} \sum_{m=k-d(k)}^{k-1} \eta(m) + \bar{B}_i w(k) \tag{7.30}$$

Construct a Lyapunov functional candidate as

$$V(k) = V_1 + V_2 + V_3 + V_4$$

where

$$\begin{aligned}
V_1 &\triangleq x^T(k)\mathcal{P}_i x(k), \ \ V_2 \triangleq \sum_{l=k-d(k)}^{k-1} x^T(l)\mathcal{Q}x(l), \\
V_3 &\triangleq \sum_{n=-d_M+2}^{-d_m+1} \sum_{l=k+n-1}^{k-1} x^T(l)\mathcal{Q}x(l), \ \ V_4 \triangleq \sum_{n=-d_M}^{-1} \sum_{m=k+n}^{k-1} \eta^T(m)\mathcal{Z}\eta(m)
\end{aligned}$$

and $\mathcal{P}_i$, $\mathcal{Q}$ and $\mathcal{Z}$ satisfy (7.27) and (7.28). Define $\varDelta V \triangleq V(k+1) - V(k)$, then along the solution of (7.30), we have $(\forall (i, j) \in \mathcal{I} \times \mathcal{I})$

$$\begin{aligned}
\varDelta V_1 &= x^T(k+1)\mathcal{P}_j x(k+1) - x^T(k)\mathcal{P}_i x(k) \\
&= x^T(k)\left[(\bar{A}_i + \bar{A}_{1i})^T \mathcal{P}_j (\bar{A}_i + \bar{A}_{1i}) - \mathcal{P}_i\right] x(k) + 2x^T(k)(\bar{A}_i + \bar{A}_{1i})^T \\
&\quad \times \mathcal{P}_j \bar{B}_i w(k) + \left[\bar{A}_{1i} \sum_{m=k-d(k)}^{k-1} \eta(m)\right]^T \mathcal{P}_j [\bar{A}_{1i} \sum_{m=k-d(k)}^{k-1} \eta(m)] \\
&\quad - 2\left[\bar{A}_{1i} \sum_{m=k-d(k)}^{k-1} \eta(m)\right]^T \mathcal{P}_j \bar{B}_i w(k) + w^T(k)\bar{B}_i^T \mathcal{P}_j \bar{B}_i w(k) \\
&\quad + \sum_{m=k-d(k)}^{k-1} \left[-2x^T(k)(\bar{A}_i + \bar{A}_{1i})^T \mathcal{P}_j \bar{A}_{1i}\eta(m)\right]
\end{aligned} \tag{7.31}$$

Now, identify $a \triangleq x(k)$, $b \triangleq \eta(m)$ and $\mathcal{N} \triangleq \left(\bar{A}_i + \bar{A}_{1i}\right)^T \mathcal{P}_j \bar{A}_{1i}$ in Lemma 7.12, we can obtain (7.28) and the following inequality $(\forall (i, j) \in \mathcal{I} \times \mathcal{I})$

$$
\begin{aligned}
&\sum_{m=k-d(k)}^{k-1} \left[-2x^T(k)(\bar{A}_i+\bar{A}_{1i})^T\mathcal{P}_j\bar{A}_{1i}\eta(m)\right] \\
&\quad \le d_M^T x(k)\mathcal{X}_{ij}x(k) + 2x^T(k)\left[\mathcal{Y}_{ij} - (\bar{A}_i+\bar{A}_{1i})^T\mathcal{P}_j\bar{A}_{1i}\right] \\
&\qquad \times \sum_{m=k-d(k)}^{k-1}\eta(m) + \sum_{m=k-d_M}^{k-1}\eta^T(m)\mathcal{Z}\eta(m) \qquad (7.32)
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
\Delta V_2 &= \sum_{l=k-d(k+1)+1}^{k} x^T(l)\mathcal{Q}x(l) - \sum_{l=k-d(k)}^{k-1} x^T(l)\mathcal{Q}x(l) \\
&= x^T(k)\mathcal{Q}x(k) - x^T(k-d(k))\mathcal{Q}x(k-d(k)) \\
&\quad + \sum_{l=k-d(k+1)+1}^{k-1} x^T(l)\mathcal{Q}x(l) - \sum_{l=k-d(k)+1}^{k-1} x^T(l)\mathcal{Q}x(l) \qquad (7.33)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{l=k-d(k+1)+1}^{k-1} x^T(l)\mathcal{Q}x(l) &= \sum_{l=k-d_m+1}^{k-1} x^T(l)\mathcal{Q}x(l) + \sum_{l=k-d(k+1)+1}^{k-d_m} x^T(l)\mathcal{Q}x(l) \\
&\le \sum_{l=k-d(k)+1}^{k-1} x^T(l)\mathcal{Q}x(l) + \sum_{l=k-d_M+1}^{k-d_m} x^T(l)\mathcal{Q}x(l) \qquad (7.34)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\Delta V_2 &\le x^T(k)\mathcal{Q}x(k) - x^T(k-d(k))\mathcal{Q}x(k-d(k)) \\
&\quad + \sum_{l=k-d_M+1}^{k-d_m} x^T(l)\mathcal{Q}x(l) \qquad (7.35)
\end{aligned}
$$

Also, note that,

$$
\begin{aligned}
\Delta V_3 &= \sum_{n=-d_M+2}^{-d_m+1}\left[x^T(n)\mathcal{Q}x(n) - x^T(k+n-1)\mathcal{Q}x(k+n-1)\right] \\
&= (d_M-d_m)x^T(k)\mathcal{Q}x(k) - \sum_{l=k-d_M+1}^{k-d_m} x^T(l)\mathcal{Q}x(l) \qquad (7.36)
\end{aligned}
$$

Finally, due to

$$\begin{aligned}\eta(m) &= x(m+1) - x(m) \\ &= \left(\bar{A}_i - I\right) x(m) + \bar{A}_{1i} x(m - d(m)) + \bar{B}_i w(m)\end{aligned}$$

one has

$$\begin{aligned}\varDelta V_4 &= \sum_{n=-d_M}^{-1} \left[\eta^T(k)\mathcal{Z}\eta(k) - \eta^T(k+n)\mathcal{Z}\eta(k+n)\right] \\ &= d_M \eta^T(k)\mathcal{Z}\eta(k) - \sum_{m=k-d_M}^{k-1} \eta^T(m)\mathcal{Z}\eta(m) \\ &= d_M \Big\{x^T(k)\left(\bar{A}_i - I\right)^T \mathcal{Z}\left(\bar{A}_i - I\right)x(k) + 2x^T(k)\left(\bar{A}_i - I\right)^T \mathcal{Z} \\ &\quad \times \bar{A}_{1i}x(k-d(k) + 2x^T(k)\left(\bar{A}_i - I\right)^T \mathcal{Z}\bar{B}_i w(k) + x^T(k-d(k)) \\ &\quad \times \bar{A}_{1i}^T \mathcal{Z}\bar{A}_{1i}x(k-d(k)) + 2x^T(k-d(k))\bar{A}_{1i}^T\mathcal{Z}\bar{B}_i w(k) + w^T(k)\Big\} \\ &\quad \times \bar{B}_i^T\mathcal{Z}\bar{B}_i w(k) - \sum_{m=k-d_M}^{k-1} \eta^T(m)\mathcal{R}\eta(m) \end{aligned} \tag{7.37}$$

Then, when assuming the zero disturbance input to system (7.24)–(7.26) and from (7.31), (7.32), (7.35), (7.36) and (7.37), we have

$$\varDelta V = \varDelta V_1 + \varDelta V_2 + \varDelta V_3 + \varDelta V_4 \le \lambda^T(k)\varXi\lambda(k) \tag{7.38}$$

where

$$\begin{aligned}\varXi &\triangleq \begin{bmatrix}\varXi_{1,1} & \varXi_{1,2} \\ \star & \varXi_{2,2}\end{bmatrix}, \lambda(k) \triangleq \begin{bmatrix} x(k) \\ x(k-d(k))\end{bmatrix}, \\ \varXi_{1,1} &\triangleq \bar{A}_i{}^T\mathcal{P}_j\bar{A}_i + \varUpsilon_{ij} + d_M\left(\bar{A}_i - I\right)^T \mathcal{Z}\left(\bar{A}_i - I\right), \\ \varXi_{1,2} &\triangleq \bar{A}_i{}^T\mathcal{P}_j\bar{A}_{1i} + \left(\bar{A}_i - I\right)^T\mathcal{Z}\bar{A}_{1i} - \mathcal{Y}_{ij}, \\ \varXi_{2,2} &\triangleq \bar{A}_{1i}^T\mathcal{P}_j\bar{A}_{1i} + \bar{A}_{1i}^T\mathcal{Z}\bar{A}_{1i} - \mathcal{Q},\end{aligned}$$

then by Lemma 2.4 and simple matrices principle, the inequality (7.27) implies $\varXi < 0$, i.e. $\varDelta V < 0$, thus the asymptotic stability of system (7.24)–(7.26) is guaranteed.

Now, to establish the H_∞ performance for switched system (7.24)–(7.26), consider the following performance index

$$J \triangleq \sum_{k=0}^{\infty}\left[z^T(k)z(k) - \gamma^2 w^T(k)w(k)\right]$$

zero initial condition, $V(k)\mid_{k=0}=0$, we have

$$
\begin{aligned}
J &= \sum_{k=0}^{\infty}\left[z^T(k)z(k)-\gamma^2 w^T(k)w(k)+\Delta V\right]-V(\infty)\\
&< \sum_{k=0}^{\infty}\left[z^T(k)z(k)-\gamma^2 w^T(k)w(k)+\Delta V\right]\\
&= \sum_{k=0}^{\infty}\theta^T(k)\begin{bmatrix}\Lambda_{11} & \Lambda_{12} & \Lambda_{13}\\ \star & \Lambda_{22} & \Lambda_{23}\\ \star & \star & \Lambda_{33}\end{bmatrix}\theta(k)
\end{aligned}
$$

where

$$
\begin{aligned}
\theta(k) &\triangleq \left[x^T(k)\ x^T(k-d(k))\ w^T(k)\right]^T,\\
\Lambda_{11} &\triangleq \bar{A}_i^T\mathcal{P}_j\bar{A}_i+\Upsilon_{ij}+d_M\left(\bar{A}_i-I\right)^T\mathcal{Z}\left(\bar{A}_i-I\right)+\bar{C}_i^T\bar{C}_i,\\
\Lambda_{12} &\triangleq \bar{A}_i^T\mathcal{P}_j\bar{A}_{1i}+\left(\bar{A}_i-I\right)^T\mathcal{Z}\bar{A}_{1i}-\mathcal{Y}_{ij}+\bar{C}_i^T\bar{C}_{1i},\\
\Lambda_{13} &\triangleq \bar{A}_i^T\mathcal{P}_j\bar{B}_i+\left(\bar{A}_i-I\right)^T\mathcal{Z}\bar{B}_i+\bar{C}_i^T\bar{D}_i,\\
\Lambda_{22} &\triangleq \bar{A}_{1i}^T\mathcal{P}_j\bar{A}_{1i}+\bar{A}_{1i}^T\mathcal{Z}\bar{A}_{1i}-\mathcal{Q}+\bar{C}_{1i}^T\bar{C}_{1i},\\
\Lambda_{23} &\triangleq \bar{A}_{1i}^T\mathcal{P}_j\bar{B}_i+\bar{A}_{1i}^T\mathcal{Z}\bar{B}_i+\bar{C}_{1i}^T\bar{D}_i,\\
\Lambda_{33} &\triangleq -\gamma^2 I+\bar{B}_i^T\mathcal{Z}\bar{B}_i+\bar{B}_i^T\mathcal{P}_j\bar{B}_i+\bar{D}_i^T\bar{D}_i.
\end{aligned}
$$

Also, by Lemma 2.4, inequality (7.27) guarantees $J<0$, which means that $\|z\|_2 < \gamma\|w\|_2$, this completes the proof. □

Remark 7.13 It is well known that the reasonable construction of Lyapunov functional is very crucial to derive non (or less)-conservative stability conditions in system theory. In the proof of Lemma 7.12, we attract the idea of switched quadratic Lyapunov function proposed in [1] to construct a quadratic Lyapunov functional candidate for switched system (7.24)–(7.26) by the positive definite matrices $\mathcal{P}_i$, $\mathcal{Q}$ and $\mathcal{Z}$. Evidently, the matrices $\mathcal{Q}$ and $\mathcal{Z}$ still are the common variables among all subsystem. However, if we further choose common variables $\mathcal{Q}$ and $\mathcal{Z}$ as piecewise variables $\mathcal{Q}_i$ and $\mathcal{Z}_i$, then the condition will be hard to obtain due to the tight coupling between $\mathcal{Q}$ and $\mathcal{Z}$ and time delay terms.

Remark 7.14 Within the LMIs framework, Lemma 7.12 presents the criterion for switched system (7.24)–(7.26), which can be viewed as an unified model developed by output feedback stabilizing control for system (7.10)–(7.13), hence, we can easily extend Lemma 7.12 to design either static or dynamic output-feedback controllers such that the developed closed-loop system of the form (7.18)–(7.20) or (7.21)–(7.23) is asymptotically stable with a prescribed H_∞ performance index. In addition, it is obvious that when the number of subsystems $s=1$, the stability criterion presented in Lemma 7.12 will cover the output-feedback stabilization problems for common linear discrete-time system with state delays under no switchings as a special case.

7.2.2 Output-Feedback Controller Design

In this subsection, we will present sufficient conditions for the existence of static output-feedback controller of the form (7.14) and dynamic output-feedback controller of the form (7.15)–(7.17) for the underlying switched system (7.10)–(7.13) respectively based on the performance analysis results in Sect. 7.2.1.

Static Output Feedback Control

Theorem 7.15 *Consider system (7.10)–(7.13). A stabilizing static output-feedback controller of the form (7.14) exists if there exist* $n \times n$ *matrices* $\mathcal{J}_i > 0$, $\mathcal{P}_i > 0$, $\mathcal{X}_{ij} > 0$, $\mathcal{Y}_{ij}$, $\mathcal{Q} > 0$, $\mathcal{Z} > 0$, $\mathcal{R} > 0$ $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$ *and* $l \times m$ *matrices* D_{si} *satisfying (7.28) and*

$$\begin{bmatrix} -\mathcal{J}_j & 0 & 0 & (1,4) & A_{1i} + B_i D_{si} G_{1i} & B_{1i} + B_i D_{si} E_i \\ \star & -d_M^{-1}\mathcal{R} & 0 & (2,4) & A_{1i} + B_i D_{si} G_{1i} & B_{1i} + B_i D_{si} E_i \\ \star & \star & -I & (3,4) & C_{1i} + D_i D_{si} G_{1i} & D_{1i} + D_i D_{si} E_i \\ \star & \star & \star & \Upsilon_{ij} & -\mathcal{Y}_{ij} & 0 \\ \star & \star & \star & \star & -\mathcal{Q} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \quad (7.39)$$

$$\mathcal{P}_i \mathcal{J}_i = I, \quad \mathcal{R}\mathcal{Z} = I \quad (7.40)$$

where $(1,4) \triangleq A_i + B_i D_{si} G_i$, $(2,4) \triangleq A_i + B_i D_{si} G_i - I$, $(3,4) \triangleq C_i + D_i D_{si} G_i$, *and* Υ_{ij} *is defined in (7.27). Moreover, if (7.28), (7.39) and (7.40) have solutions, the controller is given by (7.14) with the controller gain* D_{si}.

Proof Consider the corresponding closed-loop system with the control (7.14), and replace $\bar{A}_i$, $\bar{A}_{1i}$, $\bar{B}_i$, $\bar{C}_i$, $\bar{C}_{1i}$ and $\bar{D}_i$ in (7.27) with $\hat{A}S_i$, $\hat{A}_{1i}$, $\hat{B}_i$, $\hat{C}_i$, $\hat{C}_{1i}$ and $\hat{D}_i$ in (7.18)–(7.20), respectively. Now performing a congruence transformation to (7.27) via $diag\{\mathcal{P}_j^{-1}, \mathcal{R}^{-1}, I, I, I, I\}$, we have

$$\begin{bmatrix} -\mathcal{P}_j^{-1} & 0 & 0 & (1,4) & A_{1i} + B_i D_{si} G_{1i} & B_{1i} + B_i D_{si} E_i \\ \star & -d_M^{-1}\mathcal{Z}^{-1} & 0 & (2,4) & A_{1i} + B_i D_{si} G_{1i} & B_{1i} + B_i D_{si} E_i \\ \star & \star & -I & (3,4) & C_{1i} + D_i D_{si} G_{1i} & D_{1i} + D_i D_{si} E_i \\ \star & \star & \star & \gamma_{ij} & -\mathcal{Y}_{ij} & 0 \\ \star & \star & \star & \star & -\mathcal{Q} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \quad (7.41)$$

where $(1,4) \triangleq A_i + B_i D_{si} G_i, (2,4) \triangleq A_i + B_i D_{si} G_i - I, (3,4) \triangleq C_i + D_i D_{si} G_i$. Then, the proof is done by defining $\mathcal{J}_i \triangleq \mathcal{P}_i^{-1}, \mathcal{R} \triangleq \mathcal{Z}^{-1}$. □

It should be noted that although the resultant conditions in Theorem 7.15 are not strict LMIs conditions due to (7.40), we can cope with this nonconvex feasibility problem using the cone complementary linearization (CCL) algorithm developed in [6], which has been proved to be efficient [4, 7]. Now, we first transform the nonconvex feasibility problem in Theorem 7.15 into the following nonlinear minimization problem subject to LMIs constraints.

Problem 7.1 min $Tr\left(\sum_{i=1}^{s}\mathcal{P}_i\mathcal{J}_i+\mathcal{R}\mathcal{Z}\right)$ subject to (7.28), (7.39) and (7.42)

$$\begin{bmatrix}\mathcal{P}_i & I\\ I & \mathcal{J}_i\end{bmatrix}\geq 0,\begin{bmatrix}\mathcal{Z} & I\\ I & \mathcal{R}\end{bmatrix}\geq 0 \tag{7.42}$$

Thus, as discussed in [6], if the solution of the above minimization problem is $(s+1)n$, that is, min $Tr\left(\sum_{i=1}^{s}\mathcal{P}_i\mathcal{J}_i+\mathcal{R}\mathcal{Z}\right)=(s+1)n$, then the conditions in Theorem 7.15 are solvable. Although it is still not possible to always find the global optimal solution, the proposed nonlinear minimization problem is easier to solve than the original nonconvex feasibility problem. In fact, we can easily modify Algorithm 1 in [6] to solve the above nonlinear problem and get the following algorithm.

Algorithm 7.2 (*Solve a stabilizing output-feedback controller*)

Step 1: Find a feasible set $\left(\mathcal{P}_i, \mathcal{J}_i, \mathcal{X}_{ij}, \mathcal{Y}_{ij}, D_{si}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \ \forall(i,j)\in\mathcal{I}\times\mathcal{I}\right)^0$ satisfying (7.28), (7.39) and (7.42). Set $k=0$.

Step 2: Solve the following LMIs problem

$$\min\ Tr\left(\sum_{i=1}^{s}\left(\mathcal{P}_i\mathcal{J}_i^k+\mathcal{P}_i^k\mathcal{J}_i\right)+\left(\mathcal{R}^k\mathcal{Z}+\mathcal{R}^k\mathcal{Z}\right)\right),$$

subject to (7.28), (7.39) and (7.42).

Step 3: Substitute the obtained matrix variables $(\mathcal{P}_i, \mathcal{J}_i, \mathcal{X}_{ij}, \mathcal{Y}_{ij}, D_{si}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}$, $\forall(i,j)\in\mathcal{I}\times\mathcal{I})$ into (7.41). **If** condition (7.41) is satisfied with

$$\left|Tr\sum_{i=1}^{s}(\mathcal{P}_i\mathcal{J}_i+\mathcal{R}\mathcal{Z})-(S+1)n\right|<\delta$$

for some sufficiently small scalar $\delta>0$, then output the feasible solutions $(\mathcal{P}_i, \mathcal{J}_i$, $\mathcal{X}_{ij}, \mathcal{Y}_{ij}, D_{si}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \forall(i,j)\in\mathcal{I}\times\mathcal{I})$, **exit**, **else** Step 4.

Step 4: **If** $k>N$, where N is the maximum number of iterations allowed, **exit**, **else** Step 5.

Step 5: Set $k=k+1$, $\left(\mathcal{P}_i, \mathcal{J}_i, \mathcal{X}_{ij}, \mathcal{Y}_{ij}, D_{si}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \ \forall(i,j)\in\mathcal{I}\times\mathcal{I}\right)^k=(\mathcal{P}_i$, $\mathcal{J}_i, \mathcal{X}_{ij}, \mathcal{Y}_{ij}, D_{si}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \forall(i,j)\in\mathcal{I}\times\mathcal{I})$, and **go to** Step 2.

Remark 7.16 A noteworthy fact is Algorithm 7.2 aims to find the feasible solution of desired controller for given d_m and d_M, then based on this, one can also find the suboptimal H_∞ performance index when an outside loop procedure of minimizing γ is used. For fixed γ and N, the algorithm will be serviceable in finding the feasible solution if there exists. Also, by increasing the values of two variables (positive), one may get the feasible solution for the non-linear minimization problem as given in Algorithm 7.2.

Dynamic Output Feedback Control

Theorem 7.17 *Consider system (7.10)–(7.13), let* $\gamma > 0$. *A stabilizing dynamic output-feedback controller of the form (7.15)–(7.17) exists if there exist* $r \times r$ *matrices* $\mathcal{J}_i > 0$, $\mathcal{P}_i > 0$, $\mathcal{X}_{ij} > 0$, $\mathcal{Y}_{ij}$, $\mathcal{Q} > 0$, $\mathcal{R} > 0$, $\mathcal{Z} > 0$, $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$ *and* $(l + r - n) \times (m + r - n)$ *matrices* $\mathcal{H}_i$ *satisfying (7.28), (7.40) and*

$$\begin{bmatrix} -\mathcal{J}_j & 0 & 0 & \breve{A}_i + \breve{B}_i\mathcal{H}_i\breve{G}_i & \breve{A}_{1i} + \breve{B}_i\mathcal{H}_i\breve{G}_{1i} & \breve{B}_{1i} + \breve{B}_i\mathcal{H}_i\breve{E}_i \\ \star & -d_M^{-1}\mathcal{R} & 0 & \left(\breve{A}_i + \breve{B}_i\mathcal{H}_i\breve{G}_i - I\right) & \breve{A}_{1i} + \breve{B}_i\mathcal{H}_i\breve{G}_{1i} & \breve{B}_{1i} + \breve{B}_i\mathcal{H}_i\breve{E}_i \\ \star & \star & -I & \breve{C}_i + \breve{D}_i\mathcal{H}_i\breve{G}_i & \breve{C}_{1i} + \breve{D}_i\mathcal{H}_i\breve{G}_{1i} & \breve{D}_{1i} + \breve{D}_i\mathcal{H}_i\breve{E}_i \\ \star & \star & \star & \Upsilon_{ij} & -\mathcal{Y}_{ij} & 0 \\ \star & \star & \star & \star & -\mathcal{Q} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{7.43}$$

where Υ_{ij} *is defined in (7.27) and*

$$\mathcal{H}_i \triangleq \begin{bmatrix} D_{di} & C_{di} \\ B_{di} & A_{di} \end{bmatrix},\ \breve{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix},\ \breve{B}_i \triangleq \begin{bmatrix} B_i & 0 \\ 0 & I \end{bmatrix},\ \breve{G}_i \triangleq \begin{bmatrix} G_i & 0 \\ 0 & I \end{bmatrix},$$
$$\breve{A}_{1i} \triangleq \begin{bmatrix} A_{1i} & 0 \\ 0 & 0 \end{bmatrix},\ \breve{G}_{1i} \triangleq \begin{bmatrix} G_{1i} & 0 \\ 0 & 0 \end{bmatrix},\ \breve{B}_{1i} \triangleq \begin{bmatrix} B_{1i} \\ 0 \end{bmatrix},\ \breve{E}_i \triangleq \begin{bmatrix} E_i \\ 0 \end{bmatrix},$$
$$\breve{C}_i \triangleq \begin{bmatrix} C_i & 0 \end{bmatrix},\ \breve{C}_{1i} \triangleq \begin{bmatrix} C_{1i} & 0 \end{bmatrix},\ \breve{D}_i \triangleq \begin{bmatrix} D_i & 0 \end{bmatrix},\ \breve{D}_{1i} \triangleq D_{1i}.$$

Moreover, if (7.28), (7.40) and (7.43) have solutions, the controller is given by (7.15)–(7.17) with the matrix variables A_{di}, B_{di}, C_{di}, D_{di} *in* $\mathcal{H}_i$.

Proof Consider the corresponding closed-loop system (7.10)–(7.13) with the control (7.15)–(7.17), and replace $\bar{A}_i$, $\bar{A}_{1i}$, $\bar{B}_i$, $\bar{C}_i$, $\bar{C}_{1i}$ and $\bar{D}_i$ in (7.27) with $\tilde{A}_i$, $\tilde{A}_{1i}$, $\tilde{B}_i$, $\tilde{C}_i$, $\tilde{C}_{1i}$ and $\tilde{D}_i$ in (7.21)–(7.23), respectively. Now rewrite the matrices in (7.21)–(7.23) in the following forms

$$\tilde{A}_i = \breve{A}_i + \breve{B}_i\mathcal{H}_i\breve{G}_i,\ \tilde{A}_{1i} = \breve{A}_{1i} + \breve{B}_i\mathcal{H}_i\breve{G}_{1i},\ \tilde{B}_i = \breve{B}_{1i} + \breve{B}_i\mathcal{H}_i\breve{E}_i$$
$$\tilde{C}_i = \breve{C}_i + \breve{D}_i\mathcal{H}_i\breve{G}_i,\ \tilde{C}_{1i} = \breve{C}_{1i} + \breve{D}_i\mathcal{H}_i\breve{G}_{1i},\ \tilde{D}_i = \breve{D}_{1i} + \breve{D}_i\mathcal{H}_i\breve{E}_i$$

Then, performing a congruence transformation to (7.27) via $diag\{\mathcal{P}_j^{-1}, \mathcal{Z}^{-1}, I, I, I, I\}$, we have

$$\begin{bmatrix} -\mathcal{P}_j^{-1} & 0 & 0 & \breve{A}_i + \breve{B}_i\mathcal{H}_i\breve{G}_i & \breve{A}_{1i} + \breve{B}_i\mathcal{H}_i\breve{G}_{1i} & \breve{B}_{1i} + \breve{B}_i\mathcal{H}_i\breve{E}_i \\ \star & -d_M^{-1}\mathcal{Z}^{-1} & 0 & (2,4) & \breve{A}_{1i} + \breve{B}_i\mathcal{H}_i\breve{G}_{1i} & \breve{B}_{1i} + \breve{B}_i\mathcal{H}_i\breve{E}_i \\ \star & \star & -I & \breve{C}_i + \breve{D}_i\mathcal{H}_i\breve{G}_i & \breve{C}_{1i} + \breve{D}_i\mathcal{H}_i\breve{G}_{1i} & \breve{D}_{1i} + \breve{D}_i\mathcal{H}_i\breve{E}_i \\ \star & \star & \star & \gamma_{ij} & -\mathcal{Y}_{ij} & 0 \\ \star & \star & \star & \star & -\mathcal{Q} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{7.44}$$

where $(2,4) \triangleq (\breve{A}_i + \breve{B}_i \mathcal{H}_i \breve{G}_i - I)$. Then, the desired result can be worked out by defining $\mathcal{J}_i \triangleq \mathcal{P}_i^{-1}, \mathcal{R} \triangleq \mathcal{Z}^{-1}$. □

Analogous to the static output-feedback case, the obtained conditions in Theorem 7.17 are also not strict LMIs conditions, and the nonconvex feasibility problem in Theorem 7.17 can also be transformed into the following nonlinear minimization problem.

Problem 7.2 $\min Tr\left(\sum_{i=1}^{s} \mathcal{P}_i \mathcal{J}_i + \mathcal{R}\mathcal{Z}\right)$ subject to (7.28), (7.42) and (7.43)

Likewise, if $\min Tr\left(\sum_{i=1}^{s} \mathcal{P}_i \mathcal{J}_i + \mathcal{R}\mathcal{Z}\right) = (s+1)r$, then the conditions in Theorem 7.17 are solvable. Moreover, the Algorithm 7.2 can be easily modified here to design a dynamic output-feedback controller and the detail is omitted here.

In the following, we will present a numerical example to demonstrate the validity of above designed output-feedback controllers, and their respective advantages.

Example 7.18 Consider the switched system (7.10)–(7.13) consisting of two subsystems. For subsystem 1, the dynamics of the system are described as

$$A_1 = \begin{bmatrix} 0.70 & 0 \\ 0.08 & 0.95 \end{bmatrix}, \ A_{11} = \begin{bmatrix} 0.15 & 0 \\ -0.10 & -0.10 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.60 \\ -0.50 \end{bmatrix},$$
$$B_{11} = \begin{bmatrix} 0.10 \\ -0.02 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, G_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, E_1 = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.20 & 0.10 \end{bmatrix}, \ C_{11} = \begin{bmatrix} -0.50 & 0.30 \end{bmatrix}, D_1 = 0.8, D_{11} = 0.01.$$

For subsystem 2, the dynamics of the system are described as

$$A_2 = \begin{bmatrix} 0.70 & 0 \\ -0.08 & 0.90 \end{bmatrix}, \ A_{12} = \begin{bmatrix} 0.14 & 0 \\ -0.04 & -0.05 \end{bmatrix}, \ B_2 = \begin{bmatrix} -0.70 \\ 0.40 \end{bmatrix},$$
$$B_{12} = \begin{bmatrix} 0.08 \\ -0.01 \end{bmatrix}, \ G_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, G_{12} = \begin{bmatrix} 0.6 & 0 \\ 1 & 0.80 \end{bmatrix}, E_2 = \begin{bmatrix} 0.07 \\ -0.01 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.40 & -0.10 \end{bmatrix}, \ C_{12} = \begin{bmatrix} -0.20 & -0.30 \end{bmatrix}, D_2 = 0.4, D_{12} = 0.04.$$

The disturbance $w(k) = 0.05e^{-0.05k}\sin(0.05\pi k)$ and the switching signal is generated randomly by Algorithm 2.1 as given in Example 2.8.

Then, assuming *Time_Length* $= 200$ and *Con* $= 0.6$ in Algorithm 2.1 of Example 2.8 in the example, the switching signal can be realized by Matlab and a possible case is shown in Fig. 7.6. Note that the switching instants are arbitrary in Fig. 7.6 by the **rand** function in Algorithm 2.1 of Example 2.8, and the dwell time in each mode, which is coined in [8] and detailed in [9], might be one sampling instant or longer.

Our aim are to design static and dynamic stabilizing output-feedback controllers for the above uncertain switched system for given time-varying delays $2 \le d \le 5$ and check the H_∞ performance of the resulted closed-loop system.

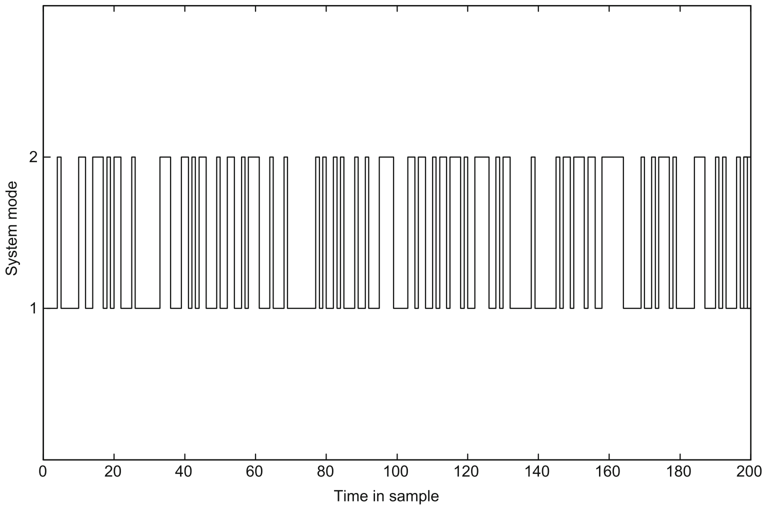

Fig. 7.6 Switching signal

Based on the existence conditions of admissible controller in Theorem 7.15 and Algorithm 7.2, we can obtain static H_∞ output-feedback controller by solving the corresponding CCL problem. The controller gain is calculated as follows

$$D_{s1} = \begin{bmatrix} -0.0571 & -0.0638 \end{bmatrix}, \ D_{s2} = \begin{bmatrix} -0.2826 & 0.1952 \end{bmatrix}.$$

Likewise, by the conditions in Theorem 7.17 and modified Algorithm 7.2, the admissible dynamic H_∞ output-feedback controller also can be obtained. Here, noted that the expected controller is considered to be a full-order form, i.e. in Theorem 7.17, $r = 4$ is selected. The dynamic controller is with the following matrix variables

$$\begin{aligned}
A_{d1} &= \begin{bmatrix} 0.1524 & 0.0441 \\ 0.1148 & 0.2929 \end{bmatrix}, B_{d1} = \begin{bmatrix} 0.0736 & 0.0114 \\ 0.1928 & 0.0287 \end{bmatrix}, \\
C_{d1} &= \begin{bmatrix} 0.0967 & 0.1183 \end{bmatrix}, \ D_{d1} = \begin{bmatrix} -0.0920 & -0.0599 \end{bmatrix}, \\
A_{d2} &= \begin{bmatrix} 0.1136 & 0.0002 \\ 0.0155 & 0.1804 \end{bmatrix}, B_{d2} = \begin{bmatrix} 0.1047 & 0.0085 \\ 0.2906 & 0.0181 \end{bmatrix}, \\
C_{d2} &= \begin{bmatrix} 0.0697 & 0.0583 \end{bmatrix}, \ D_{d2} = \begin{bmatrix} -0.2142 & 0.1562 \end{bmatrix}.
\end{aligned}$$

Furthermore, by applying the static output-feedback controller, we can obtain the control trajectory and the state response of corresponding closed-loop system in Figs. 7.7 and 7.8 respectively for given initial condition $x = [-0.05 \ 0.03]^T$ and time-varying delays $2 \leq d(k) \leq 5$. Similarly, using the dynamic output-feedback controller, we can get the corresponding control trajectory and state response in Figs. 7.9 and 7.10, respectively. It is clearly observed from Figs. 7.8 and 7.10 that

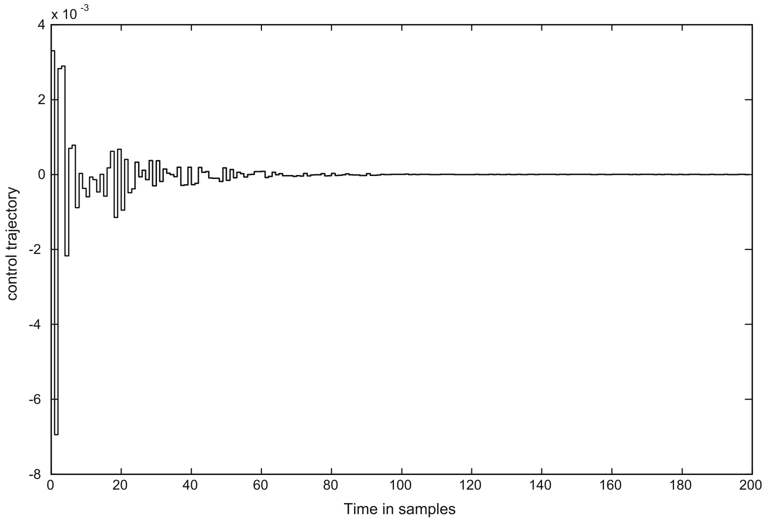

Fig. 7.7 Control trajectory of static H_∞ output-feedback controller

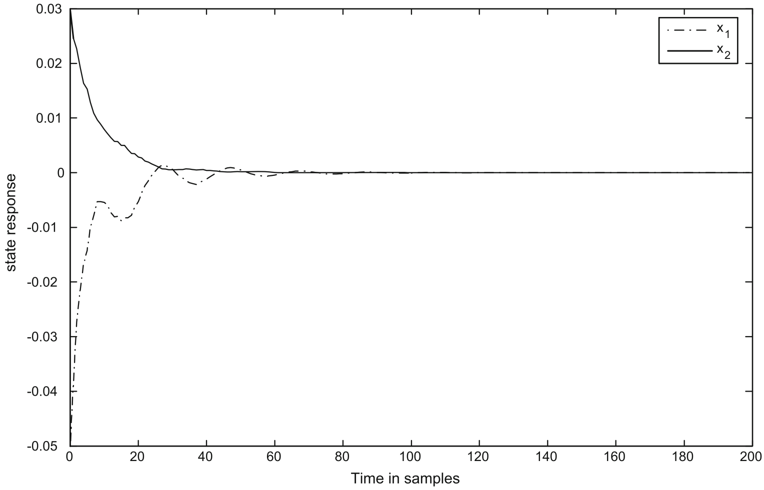

Fig. 7.8 State response by static H_∞ output-feedback controller

under the arbitrary switching signals, the obtained stabilizing output-feedback controller stabilizes the switched system against the given variations of time delays.

In addition, when an outside loop procedure is used in Algorithm 7.2 as mentioned in Remark 7.16, suboptimal H_∞ performance index can be solved as well corresponding to static and dynamic output-feedback controller respectively. Table 7.1 lists the different calculation results by applying different classes of controller.

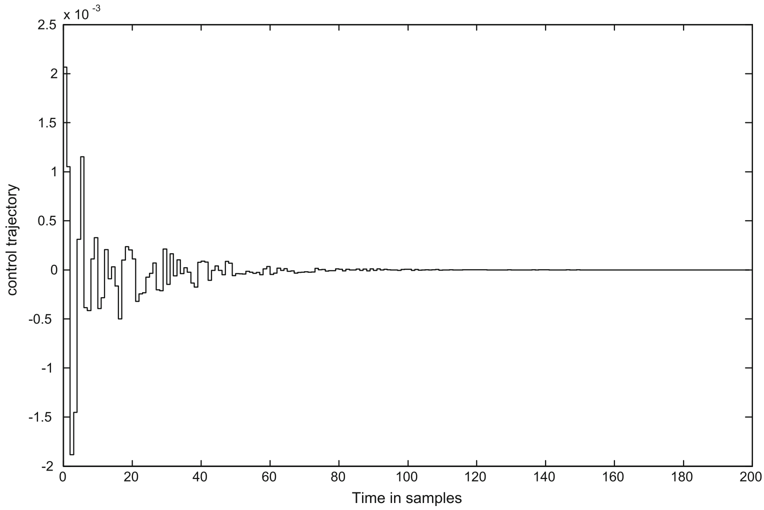

Fig. 7.9 Control trajectory of dynamic H_∞ output-feedback controller

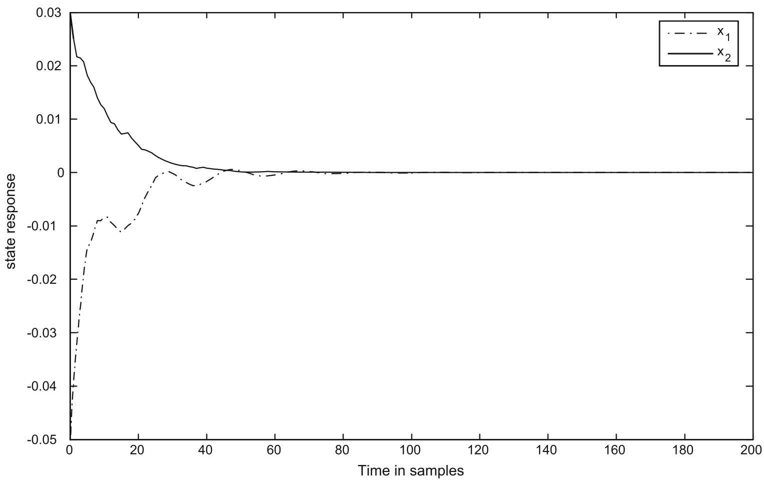

Fig. 7.10 State response by dynamic H_∞ output-feedback controller

Table 7.1 Different suboptimal γ of the resulting closed-loop system

Static output-feedback controller	2.1909
Dynamic output-feedback controller	1.9748

Obviously, it can be seen that the dynamic output-feedback controller is complex in realization since more matrix variables needed to be computed and the static output-feedback controller is easily to be realized with only a constant controller gain in each subsystem. However, from Table 7.1, it concludes that the dynamic output-feedback controller is less conservative in ensuring a suboptimal H_∞ performance index for the underlying switched system. Therefore, both static and dynamic output-feedback controllers have their own advantages and disadvantages for the switched time-delay systems studied in this section regarding conservatism and complexity, which is analogous to the general dynamic systems.

7.3 Filtering

7.3.1 $l_2 - l_\infty$ Sense: Arbitrary Switching

Consider a class of uncertain switched linear discrete-time systems given by

$$x(k+1) = A_{\sigma(k)}(\lambda)x(k) + A_{d\sigma(k)}(\lambda)x(k-d(k)) + B_{\sigma(k)}(\lambda)w(k) \tag{7.45}$$
$$y(k) = C_{\sigma(k)}(\lambda)x(k) + C_{d\sigma(k)}(\lambda)x(k-d(k)) + D_{\sigma(k)}(\lambda)w(k) \tag{7.46}$$
$$z(k) = H_{\sigma(k)}(\lambda)x(k) \tag{7.47}$$

where the system descriptions have been stated in Sect. 7.2, and therefore are omitted here. The matrices of each subsystem have appropriate dimensions with partially unknown parameters. When $\sigma(k) = i \in \mathcal{I}$, it is assumed that $(A_i(\lambda), A_{di}(\lambda), B_i(\lambda), C_i(\lambda), C_{di}(\lambda), D_i(\lambda), H_i(\lambda)) \in \Re_i$, where $\Re_i$ is a given convex bounded polyhedral domain described by s vertices in the ith subsystem.

$$\begin{aligned}\Re_i \triangleq \Bigg\{&[(A_i(\lambda), A_{di}(\lambda), B_i(\lambda), C_i(\lambda), C_{di}(\lambda), D_i(\lambda), H_i(\lambda))]\\ &= \sum_{m=1}^{s} \lambda_m [A_{i,m},\ A_{di,m},\ B_{i,m},\ C_{i,m},\ C_{di,m},\ D_{i,m},\ H_{i,m}];\\ &\sum_{m=1}^{s} \lambda_m = 1,\ \lambda_m \geq 0,\ i \in \mathcal{I}\Bigg\}\end{aligned} \tag{7.48}$$

In this subsection, the filter we shall design is assumed to have the following form

$$x_f(k+1) = A_{fi}x_f(k) + B_{fi}y(k) \tag{7.49}$$
$$z_f(k) = C_{fi}x_f(k) \tag{7.50}$$

Augmenting the model of (7.45)–(7.47) to include the states of the filter, and denoting $\xi(k) \triangleq \left[\, x^T(k)\ x_f^T(k) \,\right]^T$, $e(k) \triangleq z(k) - z_f(k)$, we obtain the filtering error system

$$\xi(k+1) = \tilde{A}_i(\lambda)\xi(k) + \tilde{A}_{di}(\lambda)K\xi(k-d(k)) + \tilde{B}_i(\lambda)w(k) \tag{7.51}$$
$$e(k) = \tilde{C}_i(\lambda)\xi(k) \tag{7.52}$$

where

$$\tilde{A}_i(\lambda) \triangleq \begin{bmatrix} A_i(\lambda) & 0 \\ B_{fi}C_i(\lambda) & A_{fi} \end{bmatrix}, \ \tilde{A}_{di}(\lambda) \triangleq \begin{bmatrix} A_{di}(\lambda) \\ B_{fi}C_{di}(\lambda) \end{bmatrix}, \tilde{B}_i(\lambda) \triangleq \begin{bmatrix} B_i(\lambda) \\ B_{fi}D_i(\lambda) \end{bmatrix},$$
$$\tilde{C}_i(\lambda) \triangleq \begin{bmatrix} H_i(\lambda) & -C_{fi} \end{bmatrix}, \ K \triangleq \begin{bmatrix} I & 0 \end{bmatrix}.$$

Then, the robust $l_2 - l_\infty$ filtering problem addressed in this subsection can be formulated as follows: given uncertain switched system (7.45)–(7.47) and a prescribed level of noise attenuation $\gamma > 0$, determine a robust switched linear filter (7.49)–(7.50) such that the filtering error system (7.51)–(7.52) is robustly asymptotically stable and

$$\|e\|_\infty < \gamma \|w\|_2 \tag{7.53}$$

under zero-initial conditions for any nonzero $w \in l_2[0, \infty)$ and all admissible uncertainties satisfying (7.48). Note that the filtering error system is a switched linear system with same structure of polytopic uncertainties as the uncertain switched system (7.45)–(7.47).

In the following, a sufficient condition for the existence of robust $l_2 - l_\infty$ filter for uncertain switched system (7.45)–(7.47) will be formulated in terms of a set of LMIs. Before presenting our main results, we first conduct the delay-dependent $l_2 - l_\infty$ performance analysis for the filtering error system (7.51)–(7.52).

Lemma 7.19 *Consider the uncertain switched system (7.51)–(7.52) and let $\gamma > 0$ be a given scalar. If there exist matrix functions $\mathcal{P}_i(\lambda) > 0$, $\mathcal{Q}(\lambda) > 0$, $\mathcal{Z}(\lambda) > 0$, $\mathcal{X}_{ij}(\lambda) > 0$ and $\mathcal{Y}_{ij}(\lambda)$ satisfying*

$$\begin{bmatrix} -\mathcal{P}_j(\lambda) & 0 & \mathcal{P}_j(\lambda)\tilde{A}_i(\lambda) & \mathcal{P}_j(\lambda)\,\tilde{A}_{di}(\lambda) & \mathcal{P}_j(\lambda)\tilde{B}_i(\lambda) \\ \star & -d_M^{-1}\mathcal{Z}(\lambda) & \mathcal{Z}(\lambda)K\left(\tilde{A}_i(\lambda) - I\right) & \mathcal{Z}(\lambda)K\tilde{A}s_{di}(\lambda) & \mathcal{Z}(\lambda)K\tilde{B}_i(\lambda) \\ \star & \star & \Upsilon_{ij}(\lambda) & -\mathcal{Y}_{ij}(\lambda) & 0 \\ \star & \star & \star & -\mathcal{Q}(\lambda) & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0 \tag{7.54}$$

$$\begin{bmatrix} \mathcal{P}_i(\lambda) & \tilde{C}_i^T(\lambda) \\ \star & -\gamma^2 I \end{bmatrix} > 0, \tag{7.55}$$

$$\begin{bmatrix} \mathcal{X}_{ij}(\lambda) & \mathcal{Y}_{ij}(\lambda) \\ \star & \mathcal{Z}(\lambda) \end{bmatrix} \geq 0, \tag{7.56}$$

where, $\Upsilon_{ij}(\lambda) \triangleq -\mathcal{P}_i(\lambda) + d_M\mathcal{X}_{ij}(\lambda) + \mathcal{Y}_{ij}(\lambda)K + K^T\mathcal{Y}_{ij}^T(\lambda) + (d_M - d_m + 1)$ $K^T\mathcal{Q}(\lambda)K$, and $\tilde{A}_i(\lambda)$, $\tilde{A}_{di}(\lambda)$, $\tilde{B}_i(\lambda)$, $\tilde{C}_i(\lambda)$, K are as in (7.51) and (7.52), then, system (7.51)–(7.52) is robustly asymptotically stable with an $l_2 - l_\infty$ noise-attenuation level bound γ.

Proof Set

$$\eta(m) \triangleq \xi(m+1) - \xi(m)$$

and we have

$$\xi(k-d(k)) = \xi(k) - \sum_{m=k-d(k)}^{k-1} \eta(m) \tag{7.57}$$

Then, system (7.51)–(7.52) can be transformed into

$$\begin{aligned} \xi(k+1) = {} & (\tilde{A}_i(\lambda) + \tilde{A}_{di}(\lambda)K)\xi(k) + \tilde{B}_i(\lambda)w(k) \\ & - \tilde{A}_{di}(\lambda)K \sum_{m=k-d(k)}^{k-1} \eta(m) \end{aligned} \tag{7.58}$$

Construct a Lyapunov functional as

$$V(k) = V_1 + V_2 + V_3 + V_4$$

where

$$\begin{aligned} V_1 &\triangleq \xi^T(k)\mathcal{P}_i(\lambda)\xi(k),\ V_2 \triangleq \sum_{l=k-d(k)}^{k-1} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l), \\ V_3 &\triangleq \sum_{n=-d_M+2}^{-d_m+1} \sum_{l=k+n-1}^{k-1} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l), \\ V_4 &\triangleq \sum_{n=-d_M}^{-1} \sum_{m=k+n}^{k-1} \eta^T(m)K^T\mathcal{Z}(\lambda)K\eta(m) \end{aligned}$$

Then, along the solution of (7.58), we have

$$\begin{aligned} \varDelta V_1 &= \xi^T(k+1)\mathcal{P}_j(\lambda)\xi(k+1) - \xi^T(k)\mathcal{P}_i(\lambda)\xi(k) \\ &= \xi^T(k)[(\tilde{A}_i(\lambda) + \tilde{A}_{di}(\lambda)K)^T\mathcal{P}_j(\lambda)(\tilde{A}_i(\lambda) + \tilde{A}_{di}(\lambda)K) \\ &\quad - \mathcal{P}_i(\lambda)]\xi(k) + \sum_{m=k-d(k)}^{k-1} [-2\xi^T(k)(\tilde{A}_i(\lambda) + \tilde{A}_{di}(\lambda)K)^T\mathcal{P}_j(\lambda) \\ &\quad \times \tilde{A}_{di}(\lambda)K\eta(m)] + (\tilde{A}_{di}(\lambda)K \sum_{m=k-d(k)}^{k-1} \eta(m))^T\mathcal{P}_j(\lambda)(\tilde{A}_{di}(\lambda) \\ &\quad \times K \sum_{m=k-d(k)}^{k-1} \eta(m)) + 2\xi^T(k)(\tilde{A}_i(\lambda) + \tilde{A}_{di}(\lambda)K)^T\mathcal{P}_j(\lambda) \end{aligned}$$

$$\times \; \tilde{B}_i(\lambda)w(k) - (2\tilde{A}_{di}(\lambda)K \sum_{m=k-d(k)}^{k-1} \eta(m))^T \mathcal{P}_j(\lambda)\tilde{B}_i(\lambda)w(k)$$
$$+ \; w^T(k)\tilde{B}_i^T(\lambda)\mathcal{P}_j(\lambda)\tilde{B}_i(\lambda)w(k) \tag{7.59}$$

Now, identify $a \triangleq \xi(k)$, $b \triangleq K\eta(m)$ and $\mathcal{N} \triangleq (\tilde{A}_i(\lambda) + \tilde{A}_{di}(\lambda)K)^T\mathcal{P}_j\tilde{A}_{di}(\lambda)$ in Lemma 7.11, $\forall(i, j) \in \mathcal{I} \times \mathcal{I}$, we can obtain (7.56) and the following inequality

$$\sum_{m=k-d(k)}^{k-1} (-2\xi^T(k)(\tilde{A}_i(\lambda) + \tilde{A}_{di}(\lambda)K)^T\mathcal{P}_j(\lambda)\tilde{A}_{di}(\lambda)K\eta(m))$$
$$\leq d_M^T\xi(k)\mathcal{X}_{ij}(\lambda)\xi(k) + 2\xi^T(k)(\mathcal{Y}_{ij}(\lambda) - (\tilde{A}_i(\lambda) + \tilde{A}_{di}(\lambda)K)^T\mathcal{P}_j(\lambda)$$
$$\times\tilde{A}_{di}(\lambda)K) \sum_{m=k-d(k)}^{k-1} \eta(m) + \sum_{m=k-d_M}^{k-1} \eta^T(m)K^T\mathcal{Z}(\lambda)K\eta(m) \tag{7.60}$$

In addition, we have

$$\begin{aligned}
\varDelta V_2 &= \sum_{l=k-d(k+1)+1}^{k} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l) - \sum_{l=k-d(k)}^{k-1} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l) \\
&= \xi^T(k)K^T\mathcal{Q}(\lambda)K\xi(k) - \xi^T(k-d(k))K^T\mathcal{Q}(\lambda)K\xi(k-d(k)) \\
&\quad + \sum_{l=k-d(k+1)+1}^{k-1} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l) - \sum_{l=k-d(k)+1}^{k-1} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l)
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{l=k-d(k+1)+1}^{k-1} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l) \\
&= \sum_{l=k-d_m+1}^{k-1} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l) + \sum_{l=k-d(k+1)+1}^{k-d_m} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l) \\
&\leq \sum_{l=k-d(k)+1}^{k-1} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l) + \sum_{l=k-d_M+1}^{k-d_m} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta V_2 \le\ & \xi^T(k)K^T\mathcal{Q}(\lambda)K\xi(k) - \xi^T(k-d(k))K^T\mathcal{Q}(\lambda)K\xi(k-d(k)) \\ & + \sum_{l=k-d_M+1}^{k-d_m} \xi^T(l)K^T\mathcal{Q}(\lambda)K\xi(l)\end{aligned} \tag{7.61}$$

Also, note that,

$$\begin{aligned}\Delta V_3 =\ & \sum_{n=-d_M+2}^{-d_m+1} [\xi^T(n)K^T\mathcal{Q}(\lambda)K\xi(n) - \xi^T(k+n-1)K^T\mathcal{Q}(\lambda) \\ & \times\ K\xi(k+n-1)] \\ =\ & (d_M-d_m)\xi^T(k)K^T\mathcal{Q}(\lambda)K\xi(k) - \sum_{l=k-d_M+1}^{k-d_m} \xi^T(l)K^T\mathcal{Q}(\lambda) \\ & \times\ K\xi(l)\end{aligned} \tag{7.62}$$

Finally, bearing in mind

$$\eta(m) \triangleq \xi(m+1) - \xi(m)$$

then

$$\eta(k) = \left(\tilde{A}_i(\lambda) - I\right)\xi(k) + \tilde{A}_{di}(\lambda)K\xi(k-d(k)) + \tilde{B}_i(\lambda)w(k)$$

thus, one has

$$\begin{aligned}\Delta V_4 =\ & \sum_{n=-d_M}^{-1} \left[\eta^T(k)K^T\mathcal{Z}(\lambda)K\eta(k) - \eta^T(k+n)K^T\mathcal{Z}(\lambda)K\eta(k+n)\right] \\ =\ & d_M\eta^T(k)K^T\mathcal{Z}(\lambda)K\eta(k) - \sum_{m=k-d_M}^{k-1} \eta^T(m)K^T\mathcal{Z}(\lambda)K\eta(m) \\ =\ & d_M\left\{\xi^T(k)\left(\tilde{A}_i(\lambda)-I\right)^T K^T\mathcal{Z}(\lambda)K\left(\tilde{A}_i(\lambda)-I\right)\xi(k)\right. \\ & + 2\xi^T(k)\left(\tilde{A}_i(\lambda)-I\right)^T K^T\mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda)K\xi(k-d(k) \\ & + 2\xi^T(k)\left(\tilde{A}_i(\lambda)-I\right)^T K^T\mathcal{Z}(\lambda)K\tilde{B}_i(\lambda)w(k) \\ & + \xi^T(k-d(k))K^T\tilde{A}_{di}^T(\lambda)K^T\mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda)K\xi(k-d(k)) \\ & + 2\xi^T(k-d(k))K^T\tilde{A}_{di}^T(\lambda)K^T\mathcal{Z}(\lambda)K\tilde{B}_i(\lambda)w(k) + w^T(k)\tilde{B}_i^T(\lambda)K^T \\ & \left.\times\,\mathcal{Z}(\lambda)K\tilde{B}_i(\lambda)w(k)\right\} - \sum_{m=k-d_M}^{k-1} \eta^T(m)K^T\mathcal{Z}(\lambda)K\eta(m)\end{aligned} \tag{7.63}$$

Then, denoting $\chi(k) \triangleq \left[\xi^T(k)\ (K\xi(k-d(k)))^T\right]^T$, when assuming the zero disturbance input to system (7.51)–(7.52) and from (7.59)–(7.63), we have

$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4 \le \chi^T(k)\Xi(\lambda)\chi(k)$$

where

$$\begin{aligned}
\Xi(\lambda) &\triangleq \begin{bmatrix} \Xi_{1,1}(\lambda) & \Xi_{1,2}(\lambda) \\ \star & \Xi_{2,2}(\lambda) \end{bmatrix}, \\
\Xi_{1,1}(\lambda) &\triangleq \tilde{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\tilde{A}_i(\lambda) + d_M\left(\tilde{A}_i(\lambda) - I\right)^T K^T\mathcal{Z}(\lambda)K\left(\tilde{A}_i(\lambda) - I\right) + \Upsilon_{ij}, \\
\Xi_{1,2}(\lambda) &\triangleq \tilde{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\tilde{A}_{di}(\lambda) - \mathcal{Y}_{ij}(\lambda) + \left(\tilde{A}_i(\lambda) - I\right)^T K^T\mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda), \\
\Xi_{2,2}(\lambda) &\triangleq \tilde{A}_{di}^T(\lambda)\mathcal{P}_j(\lambda)\tilde{A}_{di}(\lambda) - \mathcal{Q}(\lambda) + \tilde{A}_{di}^T(\lambda)K^T\mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda).
\end{aligned}$$

Now by Lemma 2.4 and simple matrix manipulations, the inequality (7.27) implies $\Xi(\lambda) < 0$, i.e. $\Delta V < 0$, thus the asymptotic stability of system (7.51)–(7.52) is guaranteed.

Now, to establish the $l_2 - l_\infty$ performance for the switched system (7.51)–(7.52), assume the zero initial condition $V(\xi(k))\mid_{k=0} = 0$ and consider the following performance index

$$J \triangleq V(\xi(k)) - \sum_{v=0}^{k-1} w^T(v)w(v)$$

for any nonzero $w \in l_2[0,\infty)$ and $k > 0$, there holds

$$\begin{aligned}
J &= V(\xi(k)) - V(\xi(k))\mid_{k=0} - \sum_{v=0}^{k-1} w^T(v)w(v) \\
&= \sum_{v=0}^{k-1} \theta^T(v)\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \star & \Lambda_{22} & \Lambda_{23} \\ \star & \star & \Lambda_{33} \end{bmatrix}\theta(v)
\end{aligned}$$

where $\theta(v) \triangleq \left[\xi^T(v)\ \xi^T(v-d(v))K^T\ \ w^T(v)\right]^T$, and

$$\begin{aligned}
\Lambda_{11} &\triangleq \tilde{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\tilde{A}_i(\lambda) + d_M\left(\tilde{A}_i(\lambda) - I\right)^T K^T\mathcal{Z}(\lambda)K\left(\tilde{A}_i(\lambda) - I\right) + \Upsilon_{ij}, \\
\Lambda_{12} &\triangleq \tilde{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\tilde{A}_{di}(\lambda) + \left(\tilde{A}_i(\lambda) - I\right)^T K^T\mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda) - \mathcal{Y}_{ij}(\lambda), \\
\Lambda_{13} &\triangleq \tilde{A}_i^T(\lambda)\mathcal{P}_j(\lambda)\tilde{B}_i(\lambda) + \left(\tilde{A}_i(\lambda) - I\right)^T K^T\mathcal{Z}(\lambda)K\tilde{B}_i(\lambda), \\
\Lambda_{22} &\triangleq \tilde{A}_{di}^T(\lambda)\mathcal{P}_j(\lambda)\tilde{A}_{di}(\lambda) + \tilde{A}_{di}^T(\lambda)K^T\mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda) - \mathcal{Q}(\lambda),
\end{aligned}$$

$$\Lambda_{23} \triangleq \tilde{A}_{di}^T(\lambda)\mathcal{P}_j(\lambda)\tilde{B}_i(\lambda) + \tilde{A}_{di}^T(\lambda)K^T\mathcal{Z}(\lambda)K\tilde{B}_i(\lambda),$$
$$\Lambda_{33} \triangleq \tilde{B}_i^T(\lambda)\mathcal{P}_j(\lambda)\tilde{B}_i(\lambda) + \tilde{B}_i^T(\lambda)K^T\mathcal{Z}(\lambda)K\tilde{B}_i(\lambda) - I.$$

Also, by Lemma 2.4, (7.54) guarantees $J < 0$, which implies that

$$\xi^T(k)\mathcal{P}_i(\lambda)\xi(k) \le V(\xi(k)) \le \sum_{v=0}^{k-1} w^T(v)w(v) \tag{7.64}$$

On the other hand, (7.56) is equivalent to $\tilde{C}_i^T(\lambda)\tilde{C}_i(\lambda) < \gamma^2\mathcal{P}_i(\lambda)$. Then we can conclude that for all $k > 0$,

$$\begin{aligned} e^T(k)e(k) &= \xi^T(k)\tilde{C}_i^T(\lambda)\tilde{C}_i(\lambda)\xi(k) < \gamma^2\xi^T(k)\mathcal{P}_i(\lambda)\xi(k) \\ &\le \gamma^2\sum_{v=0}^{\infty} w^T(v)w(v) \end{aligned}$$

Taking the supremum over $k > 0$ yields $\|e\|_\infty < \gamma\|w\|_2$, which completes the proof. □

Remark 7.20 It should be noted that the filter design based on Lemma 2.4 involves the cross coupling of system matrices and Lyapunov matrices among different subsystems. However, applying the same techniques developed in [10], we can introduce additional matrix variables to overcome this difficulty and present another sufficient condition of robust $l_2 - l_\infty$ performance for system (7.51)–(7.52) in the following lemma.

Lemma 7.21 *Consider the uncertain switched system (7.51)–(7.52) and let $\gamma > 0$ be a given scalar. If there exist matrix functions $\mathcal{P}_i(\lambda) > 0$, $\mathcal{Q}(\lambda) > 0$, $\mathcal{Z}(\lambda) > 0$ and $\mathcal{R}_i(\lambda)$ satisfying (7.55), (7.56) and*

$$\begin{bmatrix} \Upsilon_{i,j}^{1,1}(\lambda) & 0 & \mathcal{R}_i(\lambda)\tilde{A}_i(\lambda) & \mathcal{R}_i(\lambda)\tilde{A}_{di}(\lambda) & \mathcal{R}_i(\lambda)\tilde{B}_i(\lambda) \\ \star & -d_M^{-1}\mathcal{Z}(\lambda) & \Upsilon_i^{2,3}(\lambda) & \mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda) & \mathcal{Z}(\lambda)K\tilde{B}_i(\lambda) \\ \star & \star & \Upsilon_{ij}(\lambda) & -\mathcal{Y}_{ij}(\lambda) & 0 \\ \star & \star & \star & -\mathcal{Q}(\lambda) & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0 \tag{7.65}$$

where $\Upsilon_{i,j}^{1,1}(\lambda) \triangleq \mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda)$, $\Upsilon_i^{2,3}(\lambda) \triangleq \mathcal{Z}(\lambda)K(\tilde{A}_i(\lambda) - I)$, and $\Upsilon_{ij}(\lambda)$ is shown in Lemma 7.19. Then, system (7.51)–(7.52) is robustly asymptotically stable with an $l_2 - l_\infty$ noise-attenuation level bound γ.

Proof we can prove the lemma by showing the equivalence between (7.54) and (7.65). If (7.54) holds, (7.65) is easily established by choosing $\mathcal{P}_j(\lambda) = \mathcal{R}_i(\lambda) = \mathcal{R}_i^T(\lambda)$. On the other hand, if the inequality (7.65) holds, we can explore the fact that $\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda) < 0$ so that the matrices $\mathcal{R}_i(\lambda)$ are nonsingular for each i.

In addition, we have $(\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda))\mathcal{P}_j^{-1}(\lambda)(\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda))^T \geq 0$, which implies $\mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda) \geq -\mathcal{R}_i(\lambda)\mathcal{P}_j^{-1}(\lambda)\mathcal{R}_i^T(\lambda)$. Hence, we conclude

$$\begin{bmatrix} \tilde{\mathcal{R}}_{i,j}(\lambda) & 0 & \mathcal{R}_i(\lambda)\tilde{A}_i(\lambda) & \mathcal{R}_i(\lambda)\ \tilde{A}_{di}(\lambda) & \mathcal{R}_i(\lambda)\tilde{B}_i(\lambda) \\ \star & -d_M^{-1}\mathcal{Z}(\lambda) & \Upsilon_i^{2,3}(\lambda) & \mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda) & \mathcal{Z}(\lambda)K\tilde{B}_i(\lambda) \\ \star & \star & \Upsilon_{ij} & -\mathcal{Y}_{ij}(\lambda) & 0 \\ \star & \star & \star & -\mathcal{Q}(\lambda) & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0 \tag{7.66}$$

where $\tilde{\mathcal{R}}_{i,j}(\lambda) \triangleq -\mathcal{R}_i(\lambda)\mathcal{P}_j^{-1}(\lambda)\mathcal{R}_i^T(\lambda)$, $\Upsilon_i^{2,3}(\lambda) \triangleq \mathcal{Z}(\lambda)K(\tilde{A}_i(\lambda) - I)$. By performing a congruence transformation to (7.66) via $diag\{\mathcal{R}_i^{-T}(\lambda)\mathcal{P}_j(\lambda), I, I, I\}$, we obtain (7.54), thus the proof is completed. □

Now, based upon the above criterion for filtering performance analysis, we give the existence condition of robust $l_2 - l_\infty$ filter for the underlying system (7.45)–(7.47) in the following theorem.

Theorem 7.22 *Consider the uncertain switched linear system (7.45)–(7.47) and let $\gamma > 0$ be a given scalar. Then, there exists a robust switched linear filter (7.49)–(7.50) such that, for all admissible uncertainties satisfying (7.48), the filtering error system (7.51)–(7.52) is robustly asymptotically stable and (7.53) holds for any nonzero $w \in l_2[0, \infty)$, if for $i \in \mathcal{I}$, $1 \leq m \leq s$, there exist matrices $R_{1i,m}$, $R_{3i,m}$, R_{2i}, $\mathcal{Y}_{1ij,m}$, $\mathcal{Y}_{2ij,m}$, $\mathcal{X}_{2ij,m}$, $\bar{A}_{fi}$, $\bar{B}_{fi}$, $\bar{C}_{fi}$, $\mathcal{P}_{2i,m}$, positive definite matrix $\mathcal{P}_{1i,m}$, $\mathcal{P}_{3i,m}$, $\mathcal{Z}_m$, $\mathcal{X}_{1ij,m}$, $\mathcal{X}_{3ij,m}$, $\mathcal{Q}_m$ and scalars ε_i such that*

$$\Xi_{m,n}^{ij} + \Xi_{n,m}^{ij} < 0, (1 \leq m \leq n \leq s), \forall (i, j) \in \mathcal{I} \times \mathcal{I}, \tag{7.67}$$

$$\begin{bmatrix} \mathcal{P}_{1i,m} & \mathcal{P}_{2i,m} & H_{i,m}^T \\ \star & \mathcal{P}_{3i,m} & -\bar{C}_{fi}^T \\ \star & \star & -\gamma^2 I \end{bmatrix} > 0, \quad \begin{bmatrix} \mathcal{X}_{1ij,m} & \mathcal{X}_{2ij,m} & \mathcal{Y}_{1ij,m} \\ \star & \mathcal{X}_{3ij,m} & \mathcal{Y}_{2ij,m} \\ \star & \star & \mathcal{Z}_m \end{bmatrix} \geq 0, \tag{7.68}$$

where,

$$\Xi_{m,n}^{ij} \triangleq \begin{bmatrix} \Pi_{i,j,n}^{1,1} & \Pi_{i,j,n}^{1,2} & 0 & \Pi_{i,m,n}^{1,4} & \varepsilon_i \bar{A}_{fi} & \Pi_{i,m,n}^{1,6} & \Pi_{i,m,n}^{1,7} \\ \star & \Pi_{i,j,n}^{2,2} & 0 & \Pi_{i,m,n}^{2,4} & \bar{A}_{fi} & \Pi_{i,m,n}^{2,6} & \Pi_{i,m,n}^{2,7} \\ \star & \star & -d_M^{-1}\mathcal{Z}_n & \Pi_{i,m,n}^{3,4} & 0 & \mathcal{Z}_n A_{di,m} & \mathcal{Z}_n B_{i,m} \\ \star & \star & \star & \Pi_{i,j,m}^{4,4} & \Pi_{i,j,m,n}^{4,5} & \mathcal{Y}_{1ij,m} & 0 \\ \star & \star & \star & \star & \Pi_{i,j,m,n}^{5,5} & \mathcal{Y}_{2ij,m} & 0 \\ \star & \star & \star & \star & \star & -\mathcal{Q}_m & 0 \\ \star & \star & \star & \star & \star & \star & -I \end{bmatrix} \tag{7.69}$$

with

$$\begin{aligned}
\Pi_{i,j,n}^{1,1} &\triangleq \mathcal{P}_{1j,n} - R_{1i,n} - R_{1i,n}^T, \Pi_{i,j,n}^{1,2} \triangleq \mathcal{P}_{2j,n} - \varepsilon_i R_{2i} - R_{3i,n}^T, \\
\Pi_{i,m,n}^{1,4} &\triangleq R_{1i,n} A_{i,m} + \varepsilon_i \bar{B}_{fi} C_{i,m}, \Pi_{i,j,n}^{2,2} \triangleq \mathcal{P}_{3j,n} - R_{2i} - R_{2i}^T, \\
\Pi_{i,m,n}^{2,4} &\triangleq R_{3i,n} A_{i,m} + \bar{B}_{fi} C_{i,m}, \Pi_{i,m,n}^{3,4} \triangleq \mathcal{Z}_n \left(A_{i,m} - I\right), \\
\Pi_{i,j,m}^{4,4} &\triangleq -\mathcal{P}_{1i,m} + (d_M - d_m + 1)\mathcal{Q}_m + d_M \mathcal{X}_{1ij,m} + \mathcal{Y}_{1ij,m} + \mathcal{Y}_{1ij,m}^T, \\
\Pi_{i,j,m,n}^{4,5} &\triangleq -\mathcal{P}_{2i,m} + d_M \mathcal{X}_{2ij,m} + \mathcal{Y}_{2ij,m}^T, \Pi_{i,j,m,n}^{5,5} \triangleq -\mathcal{P}_{3i,m} + d_M \mathcal{X}_{3ij,m}, \\
\Pi_{i,m,n}^{1,6} &\triangleq R_{1i,n} A_{di,m} + \varepsilon_i \bar{B}_{fi} C_{di,m}, \Pi_{i,m,n}^{1,7} \triangleq R_{1i,n} B_{i,m} + \varepsilon_i \bar{B}_{fi} D_{i,m}, \\
\Pi_{i,m,n}^{2,6} &\triangleq R_{3i,n} A_{di,m} + \bar{B}_{fi} C_{di,m}, \Pi_{i,m,n}^{2,7} \triangleq R_{3i,n} B_{i,m} + \bar{B}_{fi} D_{i,m}
\end{aligned}$$

Moreover, a suitable robust filter in the form (7.49)–(7.50) has the parameters as follows

$$A_{fi} = R_{2i}^{-1} \bar{A}_{fi}, \ B_{fi} = R_{2i}^{-1} \bar{B}_{fi}, \ C_{fi} = \bar{C}_{fi}, \ i \in \mathcal{I}. \tag{7.70}$$

Proof By Lemma 7.19, system (7.51)–(7.52) is robustly asymptotically stable with a prescribed $l_2 - l_\infty$ noise-attenuation level bound γ if the following inequalities hold $(\forall (i, j) \in \mathcal{I} \times \mathcal{I})$

$$\Xi^{i,j}(\lambda) = \begin{bmatrix} \Upsilon_{i,j}^{1,1}(\lambda) & 0 & \Upsilon_i^{1,3}(\lambda) & \mathcal{R}_i(\lambda)\ \tilde{A}_{di}(\lambda) & \mathcal{R}_i(\lambda)\tilde{B}_i(\lambda) \\ \star & -d_M^{-1}\mathcal{Z}(\lambda) & \Upsilon_{i,j}^{2,3}(\lambda) & \mathcal{Z}(\lambda)K\tilde{A}_{di}(\lambda) & \mathcal{Z}(\lambda)K\tilde{B}_i(\lambda) \\ \star & \star & \Upsilon_{ij} & -\mathcal{Y}_{ij}(\lambda) & 0 \\ \star & \star & \star & -\mathcal{Q}(\lambda) & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0 \tag{7.71}$$

$$\begin{bmatrix} \mathcal{P}_i(\lambda) & \tilde{C}_i^T(\lambda) \\ \star & -\gamma^2 I \end{bmatrix} > 0, \ \begin{bmatrix} \mathcal{X}_{ij}(\lambda) & \mathcal{Y}_{ij}(\lambda) \\ \star & \mathcal{Z}(\lambda) \end{bmatrix} \geq 0 \tag{7.72}$$

where $\Upsilon_{i,j}^{1,1}(\lambda) \triangleq \mathcal{P}_j(\lambda) - \mathcal{R}_i(\lambda) - \mathcal{R}_i^T(\lambda)$, $\Upsilon_i^{1,3}(\lambda) \triangleq \mathcal{R}_i(\lambda)\tilde{A}_i(\lambda)$, $\Upsilon_{i,j}^{2,3}(\lambda) \triangleq \mathcal{Z}(\lambda)K\left(\tilde{A}_i(\lambda) - I\right)$, $\tilde{A}_i(\lambda)$, $\tilde{A}_{di}(\lambda)$, $\tilde{B}_i(\lambda)$, $\tilde{C}_i(\lambda)$ and K have been described in (7.51) and (7.52).

Then, by defining matrix functions

$$\mathcal{P}_i(\lambda) \triangleq \begin{bmatrix} \mathcal{P}_{1i}(\lambda) & \mathcal{P}_{2i}(\lambda) \\ \star & \mathcal{P}_{3i}(\lambda) \end{bmatrix}, \ \mathcal{R}_i(\lambda) \triangleq \begin{bmatrix} R_{1i}(\lambda) & \varepsilon_i R_{2i} \\ R_{3i}(\lambda) & R_{2i} \end{bmatrix}$$

$$\mathcal{X}_{ij}(\lambda) \triangleq \begin{bmatrix} \mathcal{X}_{1ij}(\lambda) & \mathcal{X}_{2ij}(\lambda) \\ \star & \mathcal{X}_{3ij}(\lambda) \end{bmatrix}, \ \mathcal{Y}_{ij}(\lambda) \triangleq \begin{bmatrix} \mathcal{Y}_{1ij}(\lambda) \\ \mathcal{Y}_{2ij}(\lambda) \end{bmatrix}$$

and matrix variables

$$\bar{A}_{fi} = R_2 A_{fi}, \ C_{fi} = \bar{C}_{fi}, \ \bar{B}_{fi} = R_2 B_{fi},$$

respectively, and by some matrix manipulations, it can be readily established that (7.71) is equivalent to: $\forall(i,j)\in\mathcal{I}\times\mathcal{I}$

$$\varXi^{ij}(\lambda)\triangleq\begin{bmatrix}\Gamma^{1,1}_{i,j}(\lambda) & \Gamma^{1,2}_{i,j}(\lambda) & 0 & \Gamma^{1,4}_{i,n}(\lambda) & \varepsilon_i\bar{A}_{fi} & \Gamma^{1,6}_{i,n}(\lambda) & \Gamma^{1,7}_{i,n}(\lambda)\\ \star & \Gamma^{2,2}_{i,j}(\lambda) & 0 & \Gamma^{2,4}_{i,n}(\lambda) & \bar{A}_{fi} & \Gamma^{2,6}_{i,n}(\lambda) & \Gamma^{2,7}_{i,n}(\lambda)\\ \star & \star & -d_M^{-1}\mathcal{Z}(\lambda) & \Gamma^{3,4}_{i}(\lambda) & 0 & \Gamma^{3,6}_{i}(\lambda) & \Gamma^{3,7}_{i}(\lambda)\\ \star & \star & \star & \varUpsilon_{1ij}(\lambda) & \varUpsilon_{2ij}(\lambda) & \mathcal{Y}_{1ij}(\lambda) & 0\\ \star & \star & \star & \star & \varUpsilon_{3ij}(\lambda) & \mathcal{Y}_{2ij}(\lambda) & 0\\ \star & \star & \star & \star & \star & -\mathcal{Q}(\lambda) & 0\\ \star & \star & \star & \star & \star & \star & -I\end{bmatrix}<0$$

where,

$$\begin{aligned}
&\Gamma^{1,1}_{i,j}(\lambda)\triangleq\mathcal{P}_{1j}(\lambda)-R_{1i}(\lambda)-R^T_{1i}(\lambda),\ \Gamma^{1,2}_{i,j}(\lambda)\triangleq\mathcal{P}_{2j}(\lambda)-\varepsilon_iR_{2i}-R^T_{3i}(\lambda),\\
&\Gamma^{2,2}_{i,j}(\lambda)\triangleq\mathcal{P}_{3j}(\lambda)-R_{2i}-R^T_{2i},\ \Gamma^{1,4}_{i,n}(\lambda)\triangleq R_{1i,n}A_i(\lambda)+\varepsilon_i\bar{B}_{fi}C_i(\lambda),\\
&\Gamma^{1,6}_{i,n}(\lambda)\triangleq R_{1i,n}A_{di}(\lambda)+\varepsilon_i\bar{B}_{fi}C_{di}(\lambda),\ \Gamma^{1,7}_{i,n}(\lambda)\triangleq R_{1i,n}B_i(\lambda)+\varepsilon_i\bar{B}_{fi}D_i(\lambda),\\
&\Gamma^{2,4}_{i,n}(\lambda)\triangleq R_{3i,n}A_i(\lambda)+\bar{B}_{fi}C_i(\lambda),\ \Gamma^{2,6}_{i,n}(\lambda)\triangleq R_{3i,n}A_{di}(\lambda)+\bar{B}_{fi}C_{di}(\lambda),\\
&\Gamma^{2,7}_{i,n}(\lambda)\triangleq R_{3i,n}B_i(\lambda)+\bar{B}_{fi}D_i(\lambda),\ \Gamma^{3,4}_{i}(\lambda)\triangleq\mathcal{Z}(\lambda)\left(A_i(\lambda)-I\right),\\
&\Gamma^{3,6}_{i}(\lambda)\triangleq\mathcal{Z}(\lambda)A_{di}(\lambda),\ \Gamma^{3,7}_{i}(\lambda)\triangleq\mathcal{Z}(\lambda)B_i(\lambda),\\
&\varUpsilon_{1ij}(\lambda)\triangleq-\mathcal{P}_{1i}(\lambda)+(d_M-d_m+1)\mathcal{Q}(\lambda)+d_M\mathcal{X}_{1ij}(\lambda)+\mathcal{Y}_{1ij}(\lambda)+\mathcal{Y}^T_{1ij}(\lambda),\\
&\varUpsilon_{2ij}(\lambda)\triangleq-\mathcal{P}_{2i}(\lambda)+d_M\mathcal{X}_{2ij}(\lambda)+\mathcal{Y}^T_{2ij}(\lambda),\\
&\varUpsilon_{3ij}(\lambda)\triangleq-\mathcal{P}_{3i}(\lambda)+d_M\mathcal{X}_{3ij}(\lambda).
\end{aligned}$$

Hence, further assuming matrix functions $\mathcal{P}_i(\lambda)$, $\mathcal{Q}(\lambda)$, $\mathcal{Z}(\lambda)$ and $\mathcal{R}_i(\lambda)$ to be the following forms

$$\mathcal{P}_i(\lambda)\triangleq\sum_{m=1}^{s}\lambda_m\mathcal{P}_{i,m}=\sum_{m=1}^{s}\lambda_m\begin{bmatrix}\mathcal{P}_{1i,m} & \mathcal{P}_{2i,m}\\ \star & \mathcal{P}_{3i,m}\end{bmatrix},\tag{7.73}$$

$$\mathcal{Q}(\lambda)\triangleq\sum_{m=1}^{s}\lambda_m\mathcal{Q}_m,\ \ \mathcal{Z}(\lambda)\triangleq\sum_{m=1}^{s}\lambda_m\mathcal{Z}_m\tag{7.74}$$

$$\mathcal{R}_i(\lambda)\triangleq\sum_{m=1}^{s}\lambda_m\mathcal{R}_{i,m}=\sum_{m=1}^{s}\lambda_m\begin{bmatrix}R_{1i,m} & \varepsilon_iR_{2i}\\ R_{3i,m} & R_{2i}\end{bmatrix}\tag{7.75}$$

$$\mathcal{X}_{ij}(\lambda)\triangleq\sum_{m=1}^{s}\lambda_m\mathcal{X}_{ij,m}=\sum_{m=1}^{s}\lambda_m\begin{bmatrix}\mathcal{X}_{1ij,m} & \mathcal{X}_{2ij,m}\\ \star & \mathcal{X}_{3ij,m}\end{bmatrix},\tag{7.76}$$

$$\mathcal{Y}_{ij}(\lambda)\triangleq\sum_{m=1}^{s}\lambda_m\begin{bmatrix}\mathcal{Y}_{1ij,m}\\ \mathcal{Y}_{2ij,m}\end{bmatrix},\tag{7.77}$$

and taking (7.48) and (7.73)–(7.77) into account, we have

$$\begin{aligned}\Xi^{ij}(\lambda) &= \sum_{m=1}^{s}\sum_{n=1}^{s}\lambda_m\lambda_n\Xi_{m,n}^{ij} \\ &= \sum_{m=1}^{s}\lambda_m^2\Xi_{m,m}^{ij} + \sum_{m=1}^{s-1}\sum_{n=m+1}^{s}\lambda_m\lambda_n\left(\Xi_{m,n}^{ij} + \Xi_{n,m}^{ij}\right).\end{aligned}$$

Thus, if conditions (7.67) and (7.68) holds, then $\Xi^{ij}(\lambda) < 0$, which implies (7.71) and (7.72) hold, i.e. the filtering error system is robustly asymptotically stable with an $l_2 - l_\infty$ noise-attenuation level bound γ. Therefore, if a solution to (7.67) and (7.68) exists, the parameters of admissible filters are given by (7.70). This completes the proof. □

Remark 7.23 Conditions (7.67) and (7.68) in Theorem 7.22 are formulated in terms of a set of LMIs, which can be solved by means of numerically efficient convex programming algorithms [11]. Moreover, the performance index γ described in the conditions can be respectively optimized by the corresponding convex optimization procedures. In addition, from (7.69), it can be easily seen that a switched parameter-dependent quadratic Lyapunov function is actually constructed in Theorem 7.22 for the developed filtering error system, and all vertex systems in each subsystem are considered by means of matrix variables $\mathcal{P}_{i,m}$. Therefore, similar to the results in [10], the filter designed in this subsection is less conservative than the one based on switched quadratic framework, in which the matrices $\mathcal{P}_{i,m}$, $\mathcal{Q}_m$, $\mathcal{Z}_m$ and $\mathcal{R}_{i,m}$ in Theorem 7.22 are selected as $\mathcal{P}_i$ and $\mathcal{R}_i$ correspondingly, and the one based on quadratic framework, in which the matrices $\mathcal{P}_i$ and $\mathcal{R}_i$ are further selected as $\mathcal{P}$ and $\mathcal{R}$.

In the following, we present a numerical example to demonstrate the feasibility and validity of the filter developed in Theorem 7.22.

Example 7.24 Consider the uncertain discrete-time switched linear system (7.45)–(7.47) consisting of two uncertain subsystems, where there are two groups of vertex matrices in subsystem 1

$$\begin{aligned}&A_{11} = \rho\begin{bmatrix} 0.72 & 0.10 \\ -0.06 & 0.77 \end{bmatrix},\ B_{11} = \rho\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \\ &C_{11} = \rho\begin{bmatrix} 1 & 0 \end{bmatrix},\ D_{11} = \rho,\ H_{11} = \rho\begin{bmatrix} 0.8 & 0 \end{bmatrix},\ L_{11} = 0 \\ &A_{12} = \rho\begin{bmatrix} 0.72 & 0.10 \\ -0.06 & -0.75 \end{bmatrix},\ B_{12} = \rho\begin{bmatrix} 0 \\ -0.1 \end{bmatrix}, \\ &C_{12} = \rho\begin{bmatrix} 1 & 0.2 \end{bmatrix},\ D_{12} = 0.8\rho,\ H_{12} = \rho\begin{bmatrix} 1 & 0 \end{bmatrix},\ L_{12} = 0\end{aligned}$$

and two groups of vertex matrices in subsystem 2

$$A_{21} = \rho \begin{bmatrix} 0.72 & 0.06 \\ -0.10 & 0.77 \end{bmatrix}, \; B_{21} = \rho \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},$$
$$C_{21} = \rho \begin{bmatrix} 0 & -1 \end{bmatrix}, \; D_{21} = -\rho, \; H_{21} = \rho \begin{bmatrix} 0.8 & 0 \end{bmatrix}, \; L_{21} = 0$$

$$A_{22} = \rho \begin{bmatrix} 0.72 & 0.06 \\ -0.10 & -0.75 \end{bmatrix}, \; B_{22} = \rho \begin{bmatrix} -0.1 \\ 0 \end{bmatrix},$$
$$C_{22} = \rho \begin{bmatrix} 0.2 & -1 \end{bmatrix}, \; D_{22} = -0.8\rho, \; H_{22} = \rho \begin{bmatrix} 1 & 0 \end{bmatrix}, \; L_{22} = 0$$

where ρ is a scalar parameter implying the size of convex polytope each uncertain subsystem can be expanded into.

Our purpose is to design a robust $l_2 - l_\infty$ filter for the above uncertain switched system for different given ρ and time-varying delays, and check the $l_2 - l_\infty$ performance of the resulting filtering error system. The energy bounded disturbance $w(k) = 0.02e^{-0.03k} \sin(0.02\pi k)$ and the switching signal is generated randomly by Algorithm 2.1 of Example 2.8. Then, assuming $Time\_Length = 200$ and $Con = 0.6$ in the example, the switching signal can be realized by Matlab and a possible case is shown in Fig. 7.11.

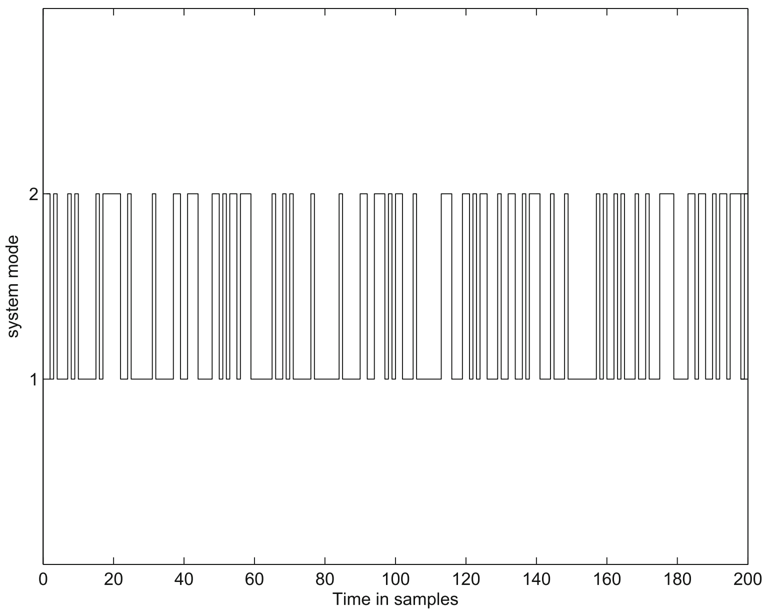

Fig. 7.11 Switching signal

Table 7.2 Different minimum γ of filtering error system

ρ	0.95	1.00	1.05
γ	0.6812	0.8679	1.1885

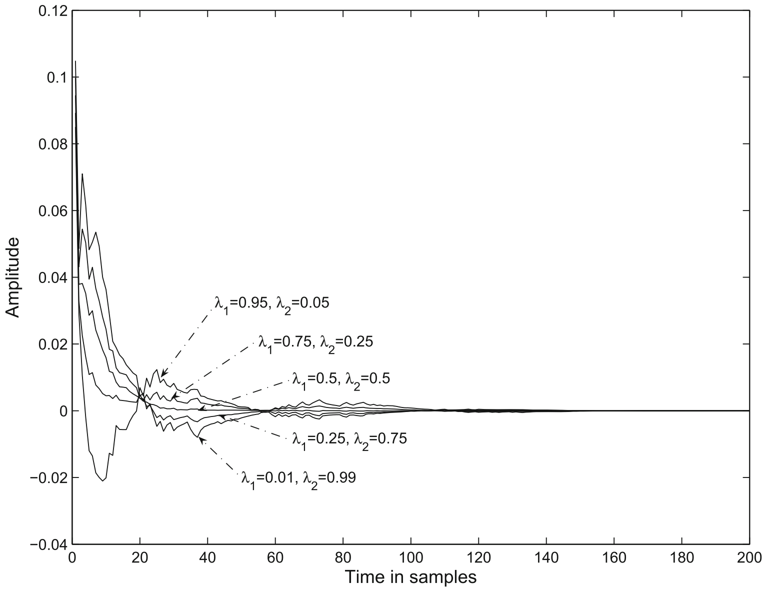

Fig. 7.12 Filtering error response to different uncertain parameters

By choosing $\varepsilon_1 = \varepsilon_2 = 1$ and solving the corresponding convex optimization problem in Theorem 7.22, we can obtain different minimum $l_2 - l_\infty$ noise-attenuation level bounds γ for given $2 \le d \le 5$ and different ρ, which are listed in Table 7.2.

Furthermore, for $\rho = 1.05$, the admissible filter parameters can be obtained according to Theorem 7.22 as

$$A_{f1} = \begin{bmatrix} -0.2275 & 0.0192 \\ -0.5107 & 0.7699 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} -0.9453 \\ -0.2277 \end{bmatrix},$$
$$A_{f2} = \begin{bmatrix} 0.7905 & -0.0433 \\ -0.0903 & 0.6082 \end{bmatrix}, \quad B_{f2} = \begin{bmatrix} -0.3489 \\ 0.0007 \end{bmatrix},$$
$$C_{f1} = \begin{bmatrix} -0.1500 & 0.0922 \end{bmatrix}, \quad C_{f2} = \begin{bmatrix} -0.1590 & 0.0926 \end{bmatrix}.$$

Then, given $\gamma = 1.1885$ and initial condition $\xi(0) = [0.1 \ -0.2 \ 0 \ 0]^T$ in (7.51) and (7.52), Fig. 7.12 shows the error response of the resulting filtering error system by applying above filter for given different uncertain parameters in (7.48). Note that the resulting system is certain when assuming the uncertain parameters are determined. In addition, note that the smaller the λ_1 or λ_2 is, the closer each corresponding subsystem is to some vertex of the convex polytope decided by different given values of ρ. It can be easily observed from Fig. 7.12 that the error response curves will shake with a larger amplitude when λ_1 or λ_2 is smaller for the same disturbance $w(k)$. The reason is that in such cases the two subsystems will both be closer to the vertex systems in their respective convex polytope, and the difference between the modes of two subsystems in the estimated system will be enlarged as a result . However, despite the variations of uncertain parameters within the scope $0 \leq \lambda_{1(2)} \leq 1$, as shown in Fig. 7.12, the resulting filtering error system is asymptotically stable for given $l_2 - l_\infty$ performance index and energy bounded disturbance $w(k)$. Therefore, we can conclude that the designed switched linear filter is feasible and effective for the underlying system under the given switching signal and time-varying delays.

7.3.2 H_∞ Sense: PDT Switching

Consider the following discrete-time switching neural networks with mode-dependent time-varying delays

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}f(x(k)) \\ \qquad\qquad +C_{\sigma(k)}g(x(k-\varphi_{\sigma(k)}(k))) + D_{1,\sigma(k)}\omega(k) \\ z(k) = F_{\sigma(k)}x(k) \\ x(s) = \psi(s),\ s = -\varphi_{\max}, -\varphi_{\max}+1, \ldots 0 \end{cases} \tag{7.78}$$

where $x(k) \triangleq [\,x_1(k)\ x_2(k)\ \cdots\ x_n(k)\,]^T \in \mathbb{R}^n$ is the state vector with n neurons, $z(k) \in \mathbb{R}^p$ is the linear combination of the neuron states to be estimated, $\omega(k) \in \mathbb{R}^q$ is the external disturbance belonging to $l_2[0,\infty)$, and $\psi(s)$ is a given initial condition sequence. $f(x(k)) \triangleq [f_1(x_1(k)), \ldots, f_n(x_n(k))]^T \in \mathbb{R}^n$ and $g(x(k)) \triangleq [g_1(x_1(k)), \ldots, g_n(x_n(k))]^T \in \mathbb{R}^n$ are the nonlinear neuron activation functions. $\varphi_{\sigma(k)}(k)$ denotes the mode-dependent discrete time delays satisfying $0 \leq \varphi_{\min} \leq \varphi_{\sigma(k),m} \leq \varphi_{\sigma(k)}(k) \leq \varphi_{\sigma(k),M} \leq \varphi_{\max}$, where $\varphi_{\max} \triangleq \max\{\varphi_{\sigma(k),M}\}$, $\varphi_{\min} \triangleq \min\{\varphi_{\sigma(k),m}\}$, and $\varphi_{\sigma(k),m}$, $\varphi_{\sigma(k),M}$ are constant positive scalars representing the lower and upper bounds of time delays in mode $\sigma(k)$, respectively. $A_{\sigma(k)} \triangleq diag\{a_{1,\sigma(k)}, a_{2,\sigma(k)}, \ldots, a_{n,\sigma(k)}\}$ describes the rate in which each neuron resets its potential to the resting state in isolation when disconnected from the networks and external inputs; $B_{\sigma(k)}$ and $C_{\sigma(k)}$ are the connection weight matrix and the delayed connection weight matrix, respectively; $D_{1,\sigma(k)}$ and $F_{\sigma(k)}$ are constant matrices with appropriate dimensions. In addition, when $\sigma(k) = i \in \mathcal{L}$, the matrices $(A_i, B_i, C_i, D_{1,i}, F_i)$ denoting the ith subsystem, are known real constant matrices with appropriate dimensions. Specially, we focus our study of system (7.78) on a class of switching regularities with persistent dwell-time (PDT) switching. The definition on the PDT switching has been given in Sect. 1.4, and therefore is omitted here.

The actual output model considered in this subsection is given as follows, which satisfies random occurring packet dropouts,

$$y(k) = \delta_{\sigma(k)}(k)y_{mo}(k) + D_{2,\sigma(k)}\omega(k) \tag{7.79}$$

where $y_{mo}(k) = E_{\sigma(k)}x(k)$ is the measurement output, E_i and $D_{2,i}$ are constant matrices with appropriate dimensions for any $\sigma(k) = i \in \mathcal{L}$.

The stochastic variables $\delta_i(k)$, are assumed to be subject to Bernoulli binary distribution taking values on 0 and 1 with

$$\Pr\{\delta_i(k) = 1\} = \delta_i, \ \Pr\{\delta_i(k) = 0\} = 1 - \delta_i, \tag{7.80}$$

where $\delta_i \in [0, 1]$ is the expectation of packet arrivals. Obviously, it gets from (7.80) that

$$\mathbb{E}\{\delta_i(k) - \delta_i\} = 0, \ \mathbb{E}\{(\delta_i(k) - \delta_i)^2\} = \delta_i(1 - \delta_i).$$

For system (7.78), we are interested in obtaining the estimate of the signal $z(k)$ from the actual measured output $y(k)$. The full-order filter to be considered is given as follows

$$\begin{cases} x_f(k+1) = A^f_{\sigma(k)}x_f(k) + B^f_{\sigma(k)}y(k) \\ z_f(k) = C^f_{\sigma(k)}x_f(k) \\ x_f(s) = \psi(s), \ s = -\varphi_{\max}, -\varphi_{\max} + 1, \ldots 0 \end{cases} \tag{7.81}$$

where $x_f(k) \in \mathbb{R}^n$, $z_f(k) \in \mathbb{R}^p$, are the estimator state and output vectors, respectively, and $\psi(s)$ is the initial estimator state. The matrices A^f_i, B^f_i and C^f_i are the estimator gains to be determined for any $\sigma(k) = i \in \mathcal{L}$.

Then, it follows from (7.78), (7.79) and (7.81) that the estimation error system is given by

$$\begin{cases} \tilde{x}(k+1) = \tilde{A}_i\tilde{x}(k) + \tilde{B}_i f(\tilde{I}_2\tilde{x}(k)) + \tilde{C}_{1,i}g(\tilde{I}_2\tilde{x}(k - \varphi_i(k))) \\ \qquad\qquad +\{\delta_i(k) - \delta_i\}\tilde{I}_1 B^f_i E_i \tilde{I}_2\tilde{x}(k) + \tilde{D}_i\omega(k) \\ \tilde{z}(k) = \tilde{F}_i\tilde{x}(k) \end{cases} \tag{7.82}$$

where

$$\tilde{x}(k) \triangleq \begin{bmatrix} x^T(k) \ x^T_f(k) \end{bmatrix}^T, \tilde{z}(k) \triangleq z(k) - z_f(k), \tilde{I}_1 \triangleq \begin{bmatrix} 0 \ I \end{bmatrix}, \tilde{I}_2 \triangleq \begin{bmatrix} I \ 0 \end{bmatrix},$$

$$\tilde{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ \delta_i B^f_i E_i & A^f_i \end{bmatrix}, \tilde{B}_i \triangleq \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \tilde{C}_i \triangleq \begin{bmatrix} C_i \\ 0 \end{bmatrix}, \tilde{D}_i \triangleq \begin{bmatrix} D_{1,i} \\ B^f_i D_{2,i} \end{bmatrix},$$

$$\tilde{F}_i \triangleq \begin{bmatrix} F_i & -C^f_i \end{bmatrix}.$$

Due to the existences of the nondeterministic variable $\sigma(k)$ and the stochastic variable $\delta_{\sigma(k)}(k)$ in the estimation error system (7.82), the following definitions are needed for the forthcoming issue of the stability and H_∞ performance analysis.

Definition 7.25 ([12]) The estimation error system (7.82) is said to be exponentially mean-square stable with $\omega(k) \equiv 0$ under switching regularity $\sigma(k)$, if there exist constants $\phi > 0$, and $\varrho \in (0, 1)$, such that

$$\mathbb{E}\{\|\tilde{x}(k)\|^2\} \leq \phi\varrho^{k-k_0} \|\tilde{x}(k_0)\|_L^2, \forall k \in \mathbb{Z}_{\geq k_0}.$$

where $\|\tilde{x}(k_0)\|_L^2 \triangleq \sup_{k_0-\varphi_{\max}\leq s\leq k_0} \mathbb{E}\{\|\tilde{x}(s)\|^2\}$ and ϱ is called the decay rate.

Thus, consider system (7.78) subject to PDT switching regularities and actual output model (7.79) with randomly occurring packet dropouts, the purpose of this subsection is to design a mode-dependent estimator in the form of (7.81) such that the resulting estimation error system (7.82) is exponentially mean-square stable and has a non-weighted H_∞ noise attenuation performance.

In what follows, we will thoroughly investigate the stability and H_∞ estimation performance analysis for the estimation error system (7.82). Before proceeding further, let us restate the following lemma which will be useful in the derivation of our main results.

Lemma 7.26 *([13]) Suppose that $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\} > 0$, $\Gamma = diag\{\iota_1, \iota_2, \ldots \iota_n\} > 0$. For $s \neq 0$, $s \in \mathbb{R}$, $i = 1, 2, \ldots, n$, the neuron activation functions satisfy*

$$l_i^- \leq \frac{f_i(s)}{s} \leq l_i^+, \; v_i^- \leq \frac{g_i(s)}{s} \leq v_i^+,$$

where l_i^-, l_i^+, v_i^- and v_i^+ are constant scalars, then

$$\begin{bmatrix} \tilde{I}_2\tilde{x}(k) \\ f(\tilde{I}_2\tilde{x}(k)) \end{bmatrix}^T \begin{bmatrix} \Lambda L_1 & -\Lambda L_2 \\ \star & \Lambda \end{bmatrix} \begin{bmatrix} \tilde{I}_2\tilde{x}(k) \\ f(\tilde{I}_2\tilde{x}(k)) \end{bmatrix} \leq 0, \tag{7.83}$$

$$\begin{bmatrix} \tilde{I}_2\tilde{x}(k) \\ g(\tilde{I}_2\tilde{x}(k)) \end{bmatrix}^T \begin{bmatrix} \Gamma\Upsilon_1 & -\Gamma\Upsilon_2 \\ \star & \Gamma \end{bmatrix} \begin{bmatrix} \tilde{I}_2\tilde{x}(k) \\ g(\tilde{I}_2\tilde{x}(k)) \end{bmatrix} \leq 0, \tag{7.84}$$

where

$$\begin{aligned} L_1 &\triangleq diag\{l_1^+l_1^-, l_2^+l_2^-, \ldots, l_n^+l_n^-\}, \\ L_2 &\triangleq diag\left\{\frac{l_1^+ + l_1^-}{2}, \frac{l_2^+ + l_2^-}{2} \cdots, \frac{l_n^+ + l_n^-}{2}\right\}, \\ \Upsilon_1 &\triangleq diag\{v_1^+v_1^-, v_2^+v_2^-, \ldots, v_n^+v_n^-\}, \\ \Upsilon_2 &\triangleq diag\left\{\frac{v_1^+ + v_1^-}{2}, \frac{v_2^+ + v_2^-}{2}, \ldots, \frac{v_n^+ + v_n^-}{2}\right\}. \end{aligned}$$

Remark 7.27 As discussed in [13], the constants l_i^-, l_i^+, v_i^- and v_i^+ are allowed to be positive, negative, or zero. Hence, the resulting neuron activation functions could be non-monotonic and more general than the usual sigmoid-type functions and

Lipschitz-type conditions. Such a description facilitates obtaining less conservative results since it restricts the lower and upper bounds of the neuron activation functions.

With the aid of Lemma 7.26, we now present the analysis result for the estimation error system (7.82) to be exponentially mean-square stable with an H_∞ performance attenuation level.

Theorem 7.28 *Consider the discrete-time switching neural networks (7.78) with PDT switching regularities and mode-dependent time-varying delays, and let* $\varphi_{\max} \geq \varphi_{\min} \geq 0$, $0 \leq \delta_i \leq 1$, $i \in \mathcal{L}$. *For given scalars* $\mu > 0$ *and* $0 < \alpha < 1$, *the estimation error system (7.82) is exponentially mean-square stable for any switching regularity* $\sigma(k)$ *with PDT property for the given period of persistence* $\mathbb{T}$ *and has a non-weighted* H_∞ *performance attenuation level* $\gamma_{no} > 0$, *where* $\gamma_{no} = \gamma\epsilon$ *with* $\epsilon \triangleq \sqrt{(\mu^{(\frac{\mathbb{T}+1}{\tau+\mathbb{T}})}\mu^{(\mathbb{T}+1)}(1-\alpha))/(1-\mu^{(\frac{\mathbb{T}+1}{\tau+\mathbb{T}})}\alpha)}$, *if there exist symmetric positive definite matrices* P_i, Q_i, *positive diagonal matrices* Λ, Γ, *such that the following LMIs hold for all* $i, j \in \mathcal{L}$, $i \neq j$,

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 & 0 & \Omega_{16} & \Omega_{18} & \tilde{F}_i^T \\ \star & -\Lambda & 0 & 0 & 0 & \Omega_{28} & 0 \\ \star & \star & \Omega_{33} & 0 & 0 & \Omega_{38} & 0 \\ \star & \star & \star & -\gamma^2 I & 0 & \Omega_{58} & 0 \\ \star & \star & \star & \star & \Omega_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & -P_i & 0 \\ \star & \star & \star & \star & \star & \star & -I \end{bmatrix} \leq 0 \tag{7.85}$$

$$P_i \leq \mu P_j,\ Q_i \leq \mu Q_j, \tag{7.86}$$

$$(\mathbb{T}+1)\ln\mu + \mathbb{T}\ln\alpha + \tau\ln\alpha < 0 \tag{7.87}$$

where

$$\Omega_{11} \triangleq -\alpha P_i - \tilde{I}_2^T \Lambda L_1 \tilde{I}_2 - \tilde{I}_2^T \Gamma \Upsilon_1 \tilde{I}_2 + \delta_i(1-\delta_i)(\tilde{I}_1 B_i^f E_i \tilde{I}_2)^T P_i (\tilde{I}_1 B_i^f E_i \tilde{I}_2),$$
$$\Omega_{12} \triangleq \tilde{I}_2^T \Lambda L_2,\ \Omega_{16} \triangleq \tilde{I}_2^T \Gamma \Upsilon_2,\ \Omega_{18} \triangleq \tilde{A}_i^T P_i^T,\ \Omega_{28} \triangleq \tilde{B}_i^T P_i^T,\ \Omega_{33} \triangleq -\alpha^{\varphi_{\max}} Q_i,$$
$$\Omega_{38} \triangleq \tilde{C}_i^T P_i^T,\ \Omega_{58} \triangleq \tilde{D}_i^T P_i^T,\ \Omega_{66} \triangleq (\varphi_{\max} - \varphi_{\min} + 1) Q_i - \Gamma.$$

Moreover, an estimate of the state decay is given by

$$\mathbb{E}\{\|\tilde{x}_k\|^2\} \leq \varrho^{k-k_0}\phi\mathbb{E}\{\|\tilde{x}_{k_0}\|_L^2\},$$

where $\varrho \triangleq \mu^{\mathbb{T}+1}\alpha^{\mathbb{T}+\tau}$ *and* $\phi \triangleq \frac{\kappa_2}{\kappa_1}$ *with* $\kappa_1 \triangleq \min_{\forall i \in \mathcal{L}} \lambda_{\min}(P_i)$, $\kappa_2 \triangleq \max_{\forall i \in \mathcal{L}} \lambda_{\max}(P_i) + (2\varphi_{\max} - \varphi_{\min} + 1)\max_{\forall i \in \mathcal{L}} \lambda_{\max}(Q_i)$.

Proof Consider the following Lyapunov-Krasovskii candidate

$$V_i(\tilde{x}_k, k) = \sum_{z=1}^{3} V_{z,i}(\tilde{x}_k, k)$$

where

$$V_{1,i}(\tilde{x}_k, k) \triangleq \tilde{x}^T(k)P_i\tilde{x}(k),$$
$$V_{2,i}(\tilde{x}_k, k) \triangleq \sum_{s=k-\varphi_i(k)}^{k-1} g^T(\tilde{I}_2\tilde{x}(s))\alpha^{k-s-1}Q_i g(\tilde{I}_2\tilde{x}(s)),$$
$$V_{3,i}(\tilde{x}_k, k) \triangleq \sum_{s=k-\varphi_{\max}+1}^{k-\varphi_{\min}} \sum_{m=s}^{k-1} g^T(\tilde{I}_2\tilde{x}(m))\alpha^{k-m-1}Q_i g(\tilde{I}_2\tilde{x}(m)).$$

Now, denoting $\triangle V_i(\tilde{x}_k, k) = \sum_{z=1}^{3} \triangle V_{z,i}(\tilde{x}_k, k)$, where $\triangle V_{z,i}(\tilde{x}_k, k) \triangleq V_{z,i}(\tilde{x}_{k+1}, k+1) - \alpha V_{z,i}(\tilde{x}_k, k)$, $z = 1, \ldots 3$, calculating the difference of $V_{z,i}(\tilde{x}_k, k)$ along the trajectory of the estimation error system (7.82), $z = 1, \ldots 3$, and taking the mathematical expectation, we have,

$$\begin{aligned}
&\mathbb{E}\{V_{1,i}(\tilde{x}_{k+1}, k+1) - \alpha V_{1,i}(\tilde{x}_k, k)\}\\
&= \mathbb{E}\{\tilde{x}^T(k+1)P_i\tilde{x}(k+1) - \alpha\tilde{x}^T(k)P_i\tilde{x}(k)\}\\
&= \tilde{x}^T(k)\tilde{A}_i^T P_i\tilde{A}_i\tilde{x}(k) + sym(\tilde{x}^T(k)\tilde{A}_i^T P_i\tilde{C}_i g(\tilde{I}_2\tilde{x}(k-\varphi_i(k))))\\
&\quad + sym(\tilde{x}^T(k)\tilde{A}_i^T P_i\tilde{D}_i\omega(k)) + \tilde{x}^T(k)\delta_i(1-\delta_i)(\tilde{I}_1 B_i^f E_i\tilde{I}_2)^T P_i(\tilde{I}_1 B_i^f E_i\tilde{I}_2)\tilde{x}(k)\\
&\quad + f^T(\tilde{I}_2\tilde{x}(k))\tilde{B}_i^T P_i\tilde{B}_i f(\tilde{I}_2\tilde{x}(k)) + sym(f^T(\tilde{I}_2\tilde{x}(k))\tilde{B}_i^T P_i\tilde{C}_i g(\tilde{I}_2\tilde{x}(k-\varphi_i(k))))\\
&\quad + sym(f^T(\tilde{I}_2\tilde{x}(k))\tilde{B}_i^T P_i\tilde{D}_i\omega(k)) + g^T(\tilde{I}_2\tilde{x}(k-\varphi_i(k)))\tilde{C}_i^T P_i\tilde{C}_i\\
&\quad \times g(\tilde{I}_2\tilde{x}(k-\varphi_i(k))) + sym(g^T(\tilde{I}_2\tilde{x}(k-\varphi_i(k)))\tilde{C}_i^T P_i\tilde{D}_i\omega(k))\\
&\quad + sym(\tilde{x}^T(k)\tilde{A}_i^T P_i\tilde{B}_i f(\tilde{I}_2\tilde{x}(k))) + \omega^T(k)\tilde{D}_i^T P_i\tilde{D}_i\omega(k) - \alpha\tilde{x}^T(k)P_i\tilde{x}(k),
\end{aligned} \tag{7.88}$$

$$\begin{aligned}
&\mathbb{E}\left\{V_{2,i}(\tilde{x}_{k+1}, k+1) - \alpha V_{2,i}(\tilde{x}_k, k)\right\}\\
&= \mathbb{E}\left\{\sum_{s=k+1-\varphi_i(k+1)}^{k} g^T(\tilde{I}_2\tilde{x}(s))\alpha^{k-s}Q_i g(\tilde{I}_2\tilde{x}(s)) - \sum_{s=k-\varphi_i(k)}^{k-1} g^T(\tilde{I}_2\tilde{x}(s))\right.\\
&\quad \left. \times \alpha^{k-s}Q_i g(\tilde{I}_2\tilde{x}(s))\right\}\\
&= \mathbb{E}\left\{\tilde{g}^T(\tilde{x}(k))Q_i\tilde{g}(\tilde{x}(k))) - g^T(\tilde{I}_2\tilde{x}(k-\varphi_i(k)))\alpha^{\varphi_i(k)}Q_i g(\tilde{I}_2\tilde{x}(k-\varphi_i(k)))\right.\\
&\quad + \sum_{s=k-\varphi_i(k+1)+1}^{k-1} g^T(\tilde{I}_2\tilde{x}(s))\alpha^{k-s}Q_i g(\tilde{I}_2\tilde{x}(s)) - \sum_{s=k-\varphi_i(k)+1}^{k-1} g^T(\tilde{I}_2\tilde{x}(s))\\
&\quad \left. \times \alpha^{k-s}Q_i g(\tilde{I}_2\tilde{x}(s))\right\}\\
&\leq \mathbb{E}\left\{\tilde{g}^T(\tilde{x}(k))Q_i\tilde{g}(\tilde{x}(k)) - g^T(\tilde{I}_2\tilde{x}(k-\varphi_i(k)))\alpha^{\varphi_{\max}}Q_i g(\tilde{I}_2\tilde{x}(k-\varphi_i(k)))\right.\\
&\quad \left. + \sum_{s=k-\varphi_{\max}+1}^{k-\varphi_{\min}} \tilde{g}^T(\tilde{x}(s))\alpha^{k-s}Q_i\tilde{g}(\tilde{x}(s))\right\},
\end{aligned} \tag{7.89}$$

$$
\begin{aligned}
&\mathbb{E}\left\{V_{3,i}(\tilde{x}_{k+1}, k+1) - \alpha V_{3,i}(\tilde{x}_k, k)\right\} \\
&= \mathbb{E}\left\{\sum_{s=k-\varphi_{\max}+2}^{k+1-\varphi_{\min}} \sum_{m=s}^{k} g^T(\tilde{I}_2\tilde{x}(m))\alpha^{k-m} Q_i g(\tilde{I}_2\tilde{x}(m))\right. \\
&\quad \left. -\alpha \sum_{s=k-\varphi_{\max}+1}^{k-\varphi_{\min}} \sum_{m=s}^{k-1} g^T(\tilde{I}_2\tilde{x}(m))\bar{Q}_{i,m} g(\tilde{I}_2\tilde{x}(m))\right\} \\
&= \mathbb{E}\left\{\sum_{s=k-\varphi_{\max}+1}^{k-\varphi_{\min}} \sum_{m=s+1}^{k} g^T(\tilde{I}_2\tilde{x}(m))\alpha\bar{Q}_{i,m} g(\tilde{I}_2\tilde{x}(m))\right. \\
&\quad \left. - \sum_{s=k-\varphi_{\max}+1}^{k-\varphi_{\min}} \sum_{m=s}^{k-1} g^T(\tilde{I}_2\tilde{x}(m))\alpha\bar{Q}_{i,m} g(\tilde{I}_2\tilde{x}(m))\right\} \\
&= \mathbb{E}\left\{(\varphi_{\max} - \varphi_{\min}) g^T(\tilde{I}_2\tilde{x}(k)) Q_i g(\tilde{I}_2\tilde{x}(k))\right. \\
&\quad \left. - \sum_{s=k-\varphi_{\max}+1}^{k-\varphi_{\min}} g^T(\tilde{I}_2\tilde{x}(s))\alpha Q_{i,s} g(\tilde{I}_2\tilde{x}(s))\right\}.
\end{aligned}
\tag{7.90}
$$

Combining (7.88)–(7.90) together with (7.83)–(7.84), by Lemma 7.26, we get

$$
\mathbb{E}\{\triangle V_i(\tilde{x}_k, k)\} \leq \zeta^T(k)\Phi_1\zeta(k)
$$

where

$$
\zeta^T(k) \triangleq [\tilde{x}^T(k)\;\; f^T(\tilde{I}_2\tilde{x}(k))\;\; g^T(\tilde{I}_2\tilde{x}(k-\varphi_i(k)))\;\; \omega^T(k)\;\; g^T(\tilde{I}_2\tilde{x}(k))],
$$

and

$$
\Phi_1 \triangleq \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{15} & \Phi_{16} \\ \star & \Phi_{22} & \Phi_{23} & \Phi_{25} & 0 \\ \star & \star & \Phi_{33} & \Phi_{35} & 0 \\ \star & \star & \star & \Phi_{55} & 0 \\ \star & \star & \star & \star & \Phi_{66} \end{bmatrix}
$$

with

$$
\begin{aligned}
\Phi_{11} &\triangleq \tilde{A}_i^T P_i \tilde{A}_i - \alpha P_i - \tilde{I}_2^T \Lambda L_1 \tilde{I}_2 - \tilde{I}_2^T \Gamma \Upsilon_1 \tilde{I}_2 + \delta_i(1-\delta_i)(\tilde{I}_1 B_i^f E_i \tilde{I}_2)^T \\
&\quad \times P_i(\tilde{I}_1 B_i^f E_i \tilde{I}_2), \\
\Phi_{12} &\triangleq \tilde{A}_i^T P_i \tilde{B}_i + \tilde{I}_2^T \Lambda L_2,\ \Phi_{13} \triangleq \tilde{A}_i^T P_i \tilde{C}_i,\ \Phi_{15} \triangleq \tilde{A}_i^T P_i \tilde{D}_i, \\
\Phi_{16} &\triangleq \tilde{I}_2^T \Gamma \Upsilon_2,\ \Phi_{23} \triangleq \tilde{B}_i^T P_i \tilde{C}_i,\ \Phi_{25} \triangleq \tilde{B}_i^T P_i \tilde{D}_i,\ \Phi_{35} \triangleq \tilde{C}_i^T P_i \tilde{D}_i, \\
\Phi_{22} &\triangleq \tilde{B}_i^T P_i \tilde{B}_i - \Lambda,\ \Phi_{33} \triangleq \tilde{C}_i^T P_i \tilde{C}_i - \alpha^{\varphi_{\max}} Q_i,\ \Phi_{55} \triangleq \tilde{D}_i^T P_i \tilde{D}_i, \\
\Phi_{66} &\triangleq (\varphi_{\max} - \varphi_{\min} + 1) Q_i - \Gamma.
\end{aligned}
$$

In view of (7.85), by Lemma 2.4, we have $\varPhi_1 \leq 0$. Thus, it is straightforward to obtain

$$\triangle V_i(\tilde{x}_k, k) \leq 0,$$

which is equivalent to

$$V_i(\tilde{x}_{k+1}, k+1) \leq \alpha V_i(\tilde{x}_k, k). \tag{7.91}$$

Consider $\sigma(k_{s_p}) = i$, $\sigma(k_{s_p+1} + \mathbb{T}^{(p)}) = j$ in the pth stage of PDT switching, and suppose an arbitrary switching occurs within $\mathbb{T}^{(p)}$, with the aid of the proof of Theorem 3.10, it follows from (7.86) that

$$\begin{aligned} &V_{\sigma(k_{s_p})}(\tilde{x}_{k_{s_p}}, k_{s_p}) \leq \lambda V_{\sigma(k_{s_{p-1}})}(\tilde{x}_{k_{s_{p-1}}}, k_{s_{p-1}}) \leq \cdots \leq \lambda^{p-1} V_{\sigma(k_{s_1})}(\tilde{x}_{k_{s_1}}, k_{s_1}) \\ &\leq \lambda^{p-1} V_{\sigma(k_0)}(\tilde{x}_{k_0}, k_0). \end{aligned} \tag{7.92}$$

On the other hand, there exist two scalars κ_1 and κ_2 such that

$$V_{\sigma(k_0)}(\tilde{x}_{k_0}, k_0) \leq \kappa_2 \left\| \tilde{x}_{k_0} \right\|_L^2, \tag{7.93}$$

$$V_{\sigma(k_{s_p})}(\tilde{x}_{k_{s_p}}, k_{s_p}) \geq \kappa_1 \left\| \tilde{x}_{k_{s_p}} \right\|^2, \tag{7.94}$$

where $\left\| \tilde{x}_{k_0} \right\|_L^2 \triangleq \sup_{k_0 - \varphi_{\max} \leq l \leq k_0} \mathbb{E}\{\|\tilde{x}(l)\|^2\}$, and

$$\kappa_1 \triangleq \min_{\forall i \in \mathcal{L}} \lambda_{\min}(P_i), \; \kappa_2 \triangleq \max_{\forall i \in \mathcal{L}} \lambda_{\max}(P_i) + (2\varphi_{\max} - \varphi_{\min} + 1) \max_{\forall i \in \mathcal{L}} \lambda_{\max}(Q_i).$$

Then, denoting $k_{s_p} \triangleq k$, $p \triangleq k_{s_p} - k_0 + 1$, $\varrho \triangleq \lambda$ and $\phi \triangleq \frac{\kappa_2}{\kappa_1}$, taking the mathematical expectations and using (7.92), (7.93) and (7.94), we obtain that

$$\mathbb{E}\{\|\tilde{x}_k\|^2\} \leq \varrho^{k-k_0} \phi \mathbb{E}\{\left\| \tilde{x}_{k_0} \right\|_L^2\}. \tag{7.95}$$

Therefore, by Definition 7.25, the estimation error system (7.82) is exponentially mean-square stable with $\omega(k) \equiv 0$.

Now for $\omega(k) \neq 0$, letting $\varGamma(k) \triangleq \tilde{z}^T(k)\tilde{z}(k) - \gamma^2 \omega^T(k)\omega(k)$, we have

$$\triangle V_i(\tilde{x}_k, k) + \varGamma(k) = \zeta^T(k) \varPhi_2 \zeta(k)$$

where $\varPhi_2 \triangleq \varPhi_1 + diag\{\tilde{F}_i^T \tilde{F}_i, 0, 0, -\gamma^2 I, 0\}$. By Lemma 2.4, (7.85) guarantees $\varPhi_2 \leq 0$, which is equivalent to $\triangle V_i(\tilde{x}_k, k) + \varGamma(k) \leq 0$.

Similar to the derivations of Theorem 3.13, it yields that

$$\sum_{l=k_0}^{\infty} \tilde{z}^T(l)\tilde{z}(l) \leq \gamma_{no}^2 \sum_{l=k_0}^{\infty} \omega^T(l)\omega(l)$$

for $\omega(l) \in l_2[0, \infty)$, where $\gamma_{no} = \gamma\epsilon$ with

$$\epsilon \triangleq \sqrt{(\mu^{(\frac{\mathbb{T}+1}{\tau+\mathbb{T}})}\mu^{(\mathbb{T}+1)}(1-\alpha))/(1-\mu^{(\frac{\mathbb{T}+1}{\tau+\mathbb{T}})}\alpha)}.$$

By Definition 4.16, it can be concluded that the estimation error system (7.82) is exponentially mean-square stable with a non-weighted H_∞ performance index γ_{no}. This completes the proof. □

Next, we will continue to present a solution to the H_∞ estimator design problem for the switching neural networks (7.78) with PDT switching regularities and mode-dependent time-varying delays. The following theorem provides sufficient conditions on the existence of the desired H_∞ estimators and explicit expression of the estimator gains.

Theorem 7.29 *Consider the discrete-time switching neural networks (7.78) with PDT switching regularities and mode-dependent time-varying delays, and let $\gamma > 0$, $\varphi_{\max} \geq \varphi_{\min} \geq 0$, $0 \leq \delta_i \leq 1$, $\mu > 0$ and $0 < \alpha < 1$ be given constants, $i \in \mathcal{L}$. For a prescribed period of persistence $\mathbb{T}$, suppose that there exist positive definite matrices $P_{1,i}$, $P_{2,i}$, $P_{3,i}$, Q_i, matrices $R_{1,i}$, $R_{2,i}$, $R_{3,i}$, $\tilde{A}_i^f$, $\tilde{B}_i^f$, $\tilde{C}_i^f$, positive diagonal matrices $\varLambda$, Γ, such that for any $i, j \in \mathcal{L}$, $i \neq j$,*

$$\begin{bmatrix} \varPi_1 & \varPi_2 \\ \star & \varPi_3 \end{bmatrix} \leq 0 \tag{7.96}$$

$$P_{s,i} \leq \mu P_{s,j},\ Q_i \leq \mu Q_j,\ s = 1, 2, 3, \tag{7.97}$$

where

$$\varPi_1 \triangleq \begin{bmatrix} \varPi_{1,1} & \varPi_{1,2} & \varLambda L_2 & 0 & 0 \\ \star & \varPi_{2,2} & 0 & 0 & 0 \\ \star & \star & -\varLambda & 0 & 0 \\ \star & \star & \star & \varPi_{4,4} & 0 \\ \star & \star & \star & \star & -\gamma^2 I \end{bmatrix},$$

$$\varPi_2 \triangleq \begin{bmatrix} \Gamma\varUpsilon_2 & \varPi_{1,7} & \varPi_{1,8} & F_i^T & \varPi_{1,10} & \varPi_{1,11} \\ 0 & (\tilde{A}_i^f)^T & (\tilde{A}_i^f)^T & -(\tilde{C}_i^f)^T & 0 & 0 \\ 0 & B_i^T R_{1,i}^T & B_i^T R_{2,i}^T & 0 & 0 & 0 \\ 0 & C_i^T R_{1,i}^T & C_i^T R_{2,i}^T & 0 & 0 & 0 \\ 0 & \varPi_{5,7} & \varPi_{5,8} & 0 & 0 & 0 \end{bmatrix},$$

$$\varPi_3 \triangleq diag\{\varPi_{6,6}, \hat{\varPi}_{7,7}, -I, \hat{\varPi}_{7,7}\},$$

with

$$\varPi_{1,1} \triangleq -\alpha P_{1,i} - \varLambda L_1 - \Gamma\varUpsilon_1,\ \varPi_{1,2} \triangleq -\alpha P_{2,i},\ \varPi_{2,2} \triangleq -\alpha P_{3,i},$$
$$\varPi_{4,4} \triangleq -\alpha^{\varphi_{\max}} Q_i,\ \varPi_{6,6} \triangleq (\varphi_{\max} - \varphi_{\min} + 1)Q_i - \varGamma,$$
$$\varPi_{5,7} \triangleq D_{1,i}^T R_{1,i}^T + D_{2,i}^T (\tilde{B}_i^f)^T,\ \varPi_{5,8} \triangleq D_{1,i}^T R_{2,i}^T + D_{2,i}^T (\tilde{B}_i^f)^T,$$

$$\Pi_{1,7} \triangleq A_i^T R_{1,i}^T + \delta_i E_i^T (\tilde{B}_i^f)^T, \Pi_{1,8} \triangleq A_i^T R_{2,i}^T + \delta_i E_i^T (\tilde{B}_i^f)^T,$$
$$\Pi_{1,10} \triangleq \sqrt{\delta_i(1-\delta_i)} E_i^T (\tilde{B}_i^f)^T, \Pi_{1,11} \triangleq \sqrt{\delta_i(1-\delta_i)} E_i^T (\tilde{B}_i^f)^T,$$
$$\Pi_{7,7} \triangleq P_{1,i} - R_{1,i} - R_{1,i}^T, \Pi_{7,8} \triangleq P_{2,i} - R_{2,i}^T - R_{3,i},$$
$$\Pi_{8,8} \triangleq P_{3,i} - R_{3,i}^T - R_{3,i}, \hat{\Pi}_{7,7} \triangleq \left[\Pi_{7,7}, \Pi_{7,8}; \Pi_{7,8}^T, \Pi_{8,8}\right],$$

then there exists a mode-dependent estimator (7.81) such that the corresponding estimation error system (7.82) is exponentially mean-square stable for any switching regularities with PDT satisfying (7.87) and has an H_∞ performance index γ_{no}, where γ_{no} has been defined in Theorem 7.28. Moreover, if a feasible solution to (7.96)–(7.97) exists, then the admissible estimator gains are given by

$$A_i^f = R_{3,i}^{-1} \tilde{A}_i^f, B_i^f = R_{3,i}^{-1} \tilde{B}_i^f, C_i^f = \tilde{C}_i^f. \tag{7.98}$$

Proof First of all, for a matrix $R_i, \forall i \in \mathcal{L}$, from the fact $(P_i - R_i)^T P_i (P_i - R_i) \geq 0$, we have $P_i - R_i - R_i^T \geq -R_i^T P_i R_i$, then we know the following inequality

$$\begin{bmatrix} \Theta_1 & \Theta_2 \\ \star & \Theta_3 \end{bmatrix} \leq 0, \tag{7.99}$$

where

$$\Theta_1 \triangleq \begin{bmatrix} \Theta_{1,1} & \tilde{I}_2^T \Lambda L_2 & 0 & 0 \\ \star & -\Lambda & 0 & 0 \\ \star & \star & \Theta_{3,3} & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix}, \Theta_2 \triangleq \begin{bmatrix} \tilde{I}_2^T \Gamma \Upsilon_2 & \tilde{A}_i^T R_i^T & \tilde{F}_i^T & \Theta_{1,9} \\ 0 & \tilde{B}_i^T R_i^T & 0 & 0 \\ 0 & \tilde{C}_i^T R_i^T & 0 & 0 \\ 0 & \tilde{D}_i^T R_i^T & 0 & 0 \end{bmatrix},$$
$$\Theta_3 \triangleq diag\{\Theta_{6,6}, \Theta_{8,8}, -I, \Theta_{8,8}\},$$

with

$$\Theta_{1,1} \triangleq -\alpha P_i - \tilde{I}_2^T \Lambda L_1 \tilde{I}_2 - \tilde{I}_2^T \Gamma \Upsilon_1 \tilde{I}_2, \Theta_{3,3} \triangleq -\alpha^{\varphi_{\max}} Q_i, \Theta_{8,8} \triangleq P_i - R_i^T - R_i,$$
$$\Theta_{6,6} \triangleq (\varphi_{\max} - \varphi_{\min} + 1) Q_i - \Gamma, \Theta_{1,9} \triangleq \sqrt{\delta_i(1-\delta_i)} (\tilde{I}_1 B_i^f E_i \tilde{I}_2)^T R_i^T.$$

By the similar manipulation in the proof of Theorem 3 of [14], it follows that (7.99) guarantees (7.85) holds. Then replacing $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{C}_{1,i}$, $\tilde{C}_{2,i}$, $\tilde{D}_i$ and $\tilde{F}_i$ in (7.99) by the ones in (7.82), considering the matrices P_i, R_i to have the following forms

$$P_i \triangleq \begin{bmatrix} P_{1,i} & P_{2,i} \\ \star & P_{3,i} \end{bmatrix}, R_i \triangleq \begin{bmatrix} R_{1,i} & R_{3,i} \\ R_{2,i} & R_{3,i} \end{bmatrix},$$

and defining the matrix variables

$$\tilde{A}_i^f \triangleq R_{3,i} A_i^f, \tilde{B}_i^f \triangleq R_{3,i} B_i^f, \tilde{C}_i^f \triangleq C_i^f, \tag{7.100}$$

one can readily obtain (7.96)–(7.97). Therefore, if (7.96)–(7.97) hold, we have (7.85)–(7.86), respectively. By Theorem 7.28, the corresponding estimation error system (7.82) is exponentially mean-square stable for any switching regularities with PDT satisfying (7.87) and has a non-weighted H_∞ performance index γ_{no}. In addition, from (7.100), the mode-dependent estimator gains are given by (7.98). This completes the proof. □

Remark 7.30 In Theorem 7.29, the H_∞ estimator design problem is solved for a class of discrete-time switching neural networks with PDT switching regularities and mode-dependent time-varying delays. In the presence of random packet dropouts, an estimator is designed such that the resulting estimation error system is exponentially stable in the sense of mean square and also achieves a prescribed non-weighted H_∞ disturbance attenuation level. It is shown that the estimator design problem under consideration is solvable if the conditions (7.96)–(7.97) are convex. Therefore, a convex optimization problem can be formulated as follows.

Problem 7.3

$$\min \gamma, \text{ s.t. } (7.96) - (7.97).$$

Moreover, if the above optimization problem admits a solution γ^*, then the optimal non-weighted H_∞ performance index can be calculated as $\gamma_{no}^* = \gamma^* \epsilon$, where ϵ has been given in Theorem 7.28.

In the following, based on the work in [15] which presents and experimentally investigates a class of synthetic oscillatory networks of transcriptional regulators as mathematical model of the repressilator in *Escherichia coli*, a biological network model is utilized to illustrate applications of the theoretical findings obtained in this subsection.

Example 7.31 We consider a synthetic oscillatory network of transcriptional regulators with three repressor protein concentrations and their corresponding mRNA concentrations, where,

mode 1:

$$A_1 = \begin{bmatrix} 0.2\mathcal{A} & 0 \\ 0 & 0.1\mathcal{A} \end{bmatrix}, C_1 = \begin{bmatrix} 0 & -0.5\mathcal{C} \\ 0.09I & 0 \end{bmatrix},$$
$$D_{1,1} = \begin{bmatrix} -0.7 & 0 & 0.5 & 0 & 0.6 & 0 \\ 0 & 0.2 & 0.3 & 0 & 0 & 0.1 \end{bmatrix}^T, E_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, D_{2,1} = -0.05,$$

mode 2:

$$A_2 = \begin{bmatrix} 0.1\mathcal{A} & 0 \\ 0 & 0.09\mathcal{A} \end{bmatrix}, C_2 = \begin{bmatrix} 0 & -0.8\mathcal{C} \\ 0.08I & 0 \end{bmatrix},$$

Table 7.3 The optimal performance indices for different upper bounds $\varphi_{\max}$ of time delays and period of persistence $\mathbb{T}$ as fixing $\delta_1 = 0.6$, $\delta_2 = 0.8$ and $\varphi_{\min} = 1$

$(\varphi_{\min}, \varphi_{\max})$	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
$\mathbb{T} = 3$	0.8714	1.2983	2.3641	11.9037	Infeasible
$\mathbb{T} = 2$	0.7794	1.1613	2.1145	10.6470	Infeasible
$\mathbb{T} = 1$	0.6971	1.0387	1.8913	9.5229	Infeasible

$$D_{1,2} = \begin{bmatrix} 0.3 & 0 & -0.5 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0 & 0.5 & 0.2 & 0 \end{bmatrix}^T, E_2 = \begin{bmatrix} 0.3 & 0 & -0.5 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0 & 0.5 & 0.2 & 0 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, D_{2,2} = 0.05,$$

with

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Choosing the regulation function as $g(x(k)) = x^2(k)/(1 + x^2(k))$, we can determine that the maximal value of the derivation of $g(x(k))$ is less than 0.65, it is straightforward to verify that $\Upsilon_1 = diag\{1, 1, 1, 0, 0, 0\}$, $\Upsilon_2 = diag\{1, 1, 1, 0.65, 0.65, 0.65\}$. In addition, without loss of generalization, we assume that $B_1 = B_2 = 0$, and $L_1 = L_2 = 0$, and further discussions about this model can be seen in [12].

The purposes here are to design the mode-dependent estimators for the presented synthetic oscillatory networks with mode-dependent time-varying delays, find out the switching regularities with admissible PDT, such that the estimation error system is exponentially mean-square stable and ensures a prescribed non-weighted H_∞ performance. Firstly, given $\alpha = 0.85$, $\mu = 1.25$, the H_∞ performance index can be minimized by Problem 7.3, and the values of γ_{no}^* can be calculated as listed in Tables 7.3 and 7.4 respectively as designating different pairs (δ_1, δ_2) and different upper bounds $\varphi_{\max}$ of time delays. It can be shown from Table 7.3 that, the increasing of upper bound $\varphi_{\max}$ of time delays has a negative impact on the system performance.

Table 7.4 The optimal performance indices for different probabilities pair (δ_1, δ_2) and period of persistence $\mathbb{T}$ as fixing $\varphi_{\min} = 1$ and $\varphi_{\max} = 4$

(δ_1, δ_2)	(0, 0)	(0.2, 0.2)	(0.4, 0.4)	(0.6, 0.6)	(0.8, 0.8)	(1, 1)
$\mathbb{T} = 3$	13.2986	12.9107	12.5044	12.3379	11.8335	11.4206
$\mathbb{T} = 2$	11.8946	11.5477	11.1843	11.0353	10.5842	10.2149
$\mathbb{T} = 1$	10.6389	10.3286	10.0035	9.8703	9.4668	9.1365

Also, from Table 7.4, it can be seen that, the larger the probabilities δ_1 and δ_2 representing packet arrivals are, the better performance indices γ_{no}^* can be obtained. In addition, one consistent phenomenon has been captured in both Tables 7.3 and 7.4, i.e., the achieved H_∞ performance index becomes better as decreasing the length of period of persistence $\mathbb{T}$.

Further, assume that the mode-dependent time-varying delays satisfy $2 \le \varphi_1(k) \le 4$ and $1 \le \varphi_2(k) \le 3$, respectively. In addition, the conditional probabilities of Bernoulli distributions are taken as $\delta_1 = 0.6$, $\delta_2 = 0.8$, respectively. Next, consider the external disturbance input $\omega(k) = 0.5\exp(-0.1k)$, and designate $\tau = 4$, it yields that the actual H_∞ performance index $\gamma = 0.5057$, which lowers the optimal one γ_{no}^* illustrating the effectiveness of the designed estimators, where $\gamma_{no}^* = 11.9037$. Also, we have $\varrho = 0.7827 < 1$ and $\phi = 1345.6 > 0$, which satisfy the conditions of Definition 7.25. The evolutions of the system modes satisfying PDT switching are generated randomly as shown in Fig. 7.13. Then, using the obtained estimator gains, the trajectories of state, its state estimation and its state estimation error of mRNAs and protein are presented in Figs. 7.14, 7.15, 7.16 and 7.17, respectively. Also, the output, its output estimation and its output estimation error are shown in Fig. 7.18. Finally, the convergence of Figs. 7.14, 7.15, 7.16, 7.17 and 7.18 demonstrates that the designed estimator is valid ensuring a satisfactory tracking in the presence of the faster switching during the period of persistence, the random packet dropouts and the time-varying delays for the proposed discrete-time switching synthetic oscillatory networks.

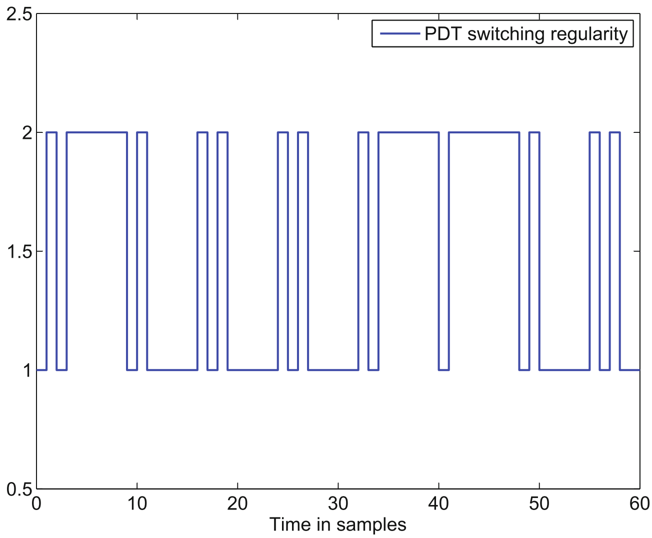

Fig. 7.13 The PDT switching regularity

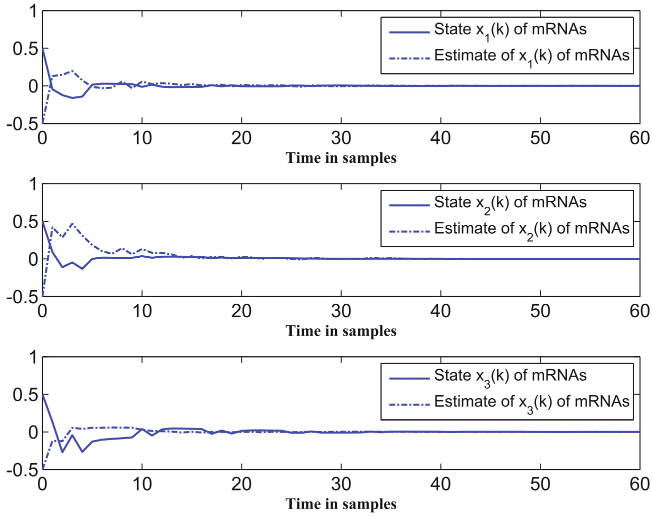

Fig. 7.14 The trajectories of state $x_i(k)$ and its estimate $x_{f,i}(k)$ of mRNAs, respectively, $i = 1, 2, 3$

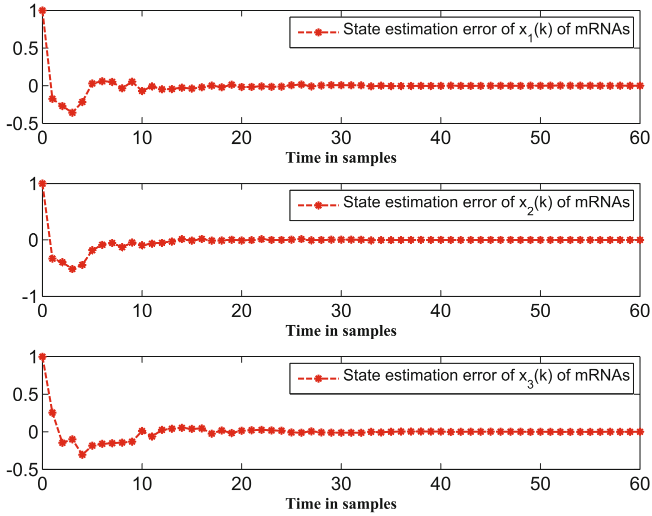

Fig. 7.15 The trajectories of its estimation error $x_i(k) - x_{f,i}(k)$ of mRNAs, respectively, $i = 1, 2, 3$

7.4 Conclusion

In this chapter, the stability problem has been studied for a class of discrete-time linear switched systems with state delays under cyclic switching, and a numerical searching algorithm has been proposed for the determination of dwell time of the subsystems.

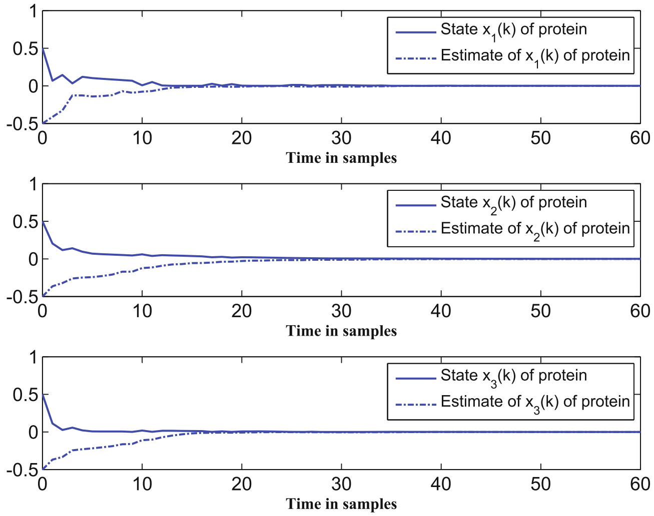

Fig. 7.16 The trajectories of state $x_i(k)$ and its estimate $x_{f,i}(k)$ of protein, respectively, $i = 1, 2, 3$

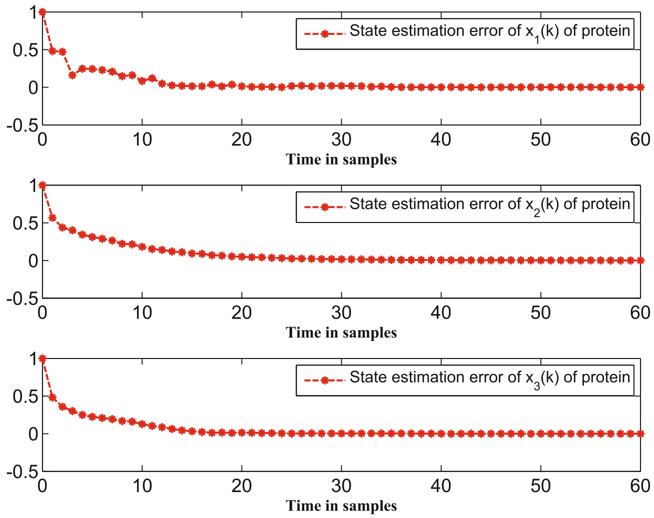

Fig. 7.17 The trajectories of its estimation error $x_i(k) - x_{f,i}(k)$ of protein, respectively, $i = 1, 2, 3$

Then, under arbitrary switching, the H_∞ output feedback control problem has been investigated for a class of switched linear discrete-time systems with time delays. The time delay is assumed to be time-varying and bounded. Both static and dynamic H_∞ output feedback controllers are designed respectively. A CCL algorithm is exploited to obtain the desired controllers and a suboptimal H_∞ performance index such that the

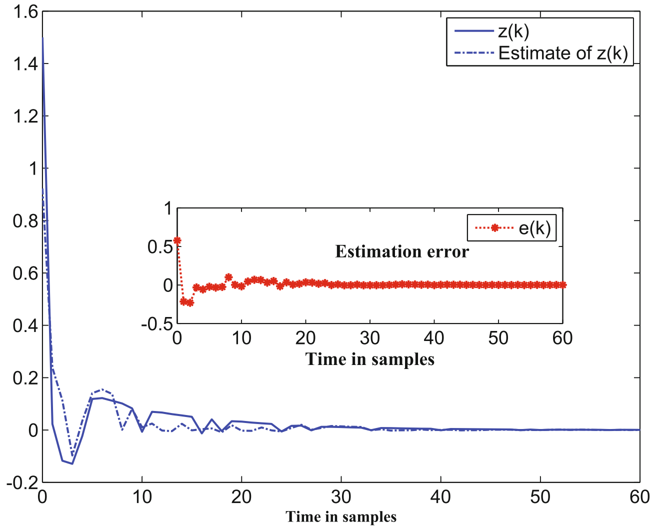

Fig. 7.18 The trajectories of $z(k)$, its estimate $z_f(k)$, and its estimation error $e(k)$

underlying switched systems can be stabilized for given time-varying delays bounds. Furthermore, considering the system with polytopic uncertainties, the problem of robust $l_2 - l_\infty$ filtering has been studied for the underlying systems with arbitrary switching. Based on the mode-dependent and parameter-dependent stability analysis approaches, a robust switched linear filter is designed such that the corresponding filtering error system is robustly asymptotically stable, and a prescribed $l_2 - l_\infty$ performance index is guaranteed for all admissible uncertainties. Finally, under PDT switching regularities, the state estimation problem has been investigated for a class of discrete-time switching neural networks with mode-dependent time-varying delays in H_∞ sense. Sufficient conditions on the existence of the desired mode-dependent estimators, are established such that the estimation error system is exponentially mean-square stable and achieves a guaranteed non-weighted H_∞ noise attenuation performance level. The impacts of the random packet dropouts and the length of period of persistence on the system performance have been found by comparison, respectively.

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