

Quantum Science and Technology

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# Quantum Metrology, Imaging, and Communication

 Springer

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# Quantum Metrology, Imaging, and Communication



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# Preface

For many decades, research in quantum mechanics was largely concentrated on two areas: on methods for computing the energy levels and wavefunctions for states of individual particles in potentials, and on computing the statistical properties of many-particle quantum systems. Studying these two regimes has led to a progressively deeper understanding of the fundamental physics of many types of systems, ranging from single atoms to superconductors. It also led to some of the most important technological advances of the twentieth century, including the development of the laser and the transistor.

In recent decades, there has been a major shift of interest, in which the study of quantum few-particle systems (most commonly two or three particles) has become a primary focus. In particular, the study of *entangled* systems has played an increasingly large role, leading to a number of new, previously unknown effects, such as ghost imaging, quantum teleportation, entanglement swapping, and nonlocal interference. More or less simultaneously, the study of information in quantum systems has gained new prominence. Combined with the idea of entanglement, this has led to an explosion of interest in the topics of quantum computation and quantum communication, as well as new methods for making ultra-precise measurements.

In this book, we explore some of the new developments that have arisen from the idea of entanglement, sometimes in conjunction with quantum information theory, as applied to optical systems. In particular, emphasis is placed on how these developments in fundamental science have led to new or improved methods for carrying out practical applications. The goal is to introduce to nonspecialists a number of these applications. We assume that the reader has only a basic undergraduate-level background in quantum mechanics and classical optics, and so we spend the first few chapters covering the necessary background material on quantum optics, entanglement, and related subjects.

The list of possible topics that could be included here is so long that it would be impractical to try to give a comprehensive review. As a result, the choice of applications covered is determined partly by our own areas of expertise, as well as

by the desire to give prominence to some areas which are less well known among nonspecialists. The most notable area that we chose to exclude is quantum computing, because it is a very widely known topic for which numerous excellent reviews already exist at both technical and popular levels. The applications that we cover fall generally in the areas of communication (including cryptography), imaging, and measurement.

The basic quantum-mechanical ingredients that are used repeatedly in these applications are superposition principle, entanglement, and the inability to unambiguously discriminate between non-orthogonal states. In general, these novel aspects of quantum mechanics enable the methods described here to produce advantages over classical methods in a number of different contexts. For example, quantum methods can lead to:

- Improved contrast in imaging.
- Improved resolution and sensitivity in measurements of quantities such as phase, dispersion, and frequency.
- Increased visibility in interference experiments.
- Improved security in cryptography and communication.

In each of these examples, the use of classical correlation can lead to improvements, but there is usually a limit beyond which a system can only go if it is entangled. For example, it can be shown that when interference fringes are measured in coincidence counting (intensity correlation) experiments, classical correlations between the particles arriving at the detectors can never lead to interference visibilities above  $\frac{1}{\sqrt{2}} \approx 71\%$ ; however, entangled systems can have visibilities approaching 100 %.

The goal is to make the methods presented here accessible to both engineers and physicists from a diverse range of backgrounds, so Chap. 1 and the appendices include much of the required background material needed for a mathematically literate reader from other areas to follow the rest of the book. This includes a very brief overview of quantum mechanics in Chap. 1 and a review of optics in Appendix A. The remaining appendices include additional background material in more specialized topics such as turbulence and phase matching in down conversion; these topics are all used at various points in the main text. The review of quantum mechanics in Chap. 1 places emphasis on entanglement, which is central to many of the subsequent chapters. In Chap. 2, we give a survey of some relevant topics in quantum optics. Chapters 3–9 then give detailed discussions of a number of recent applications, ranging from high-precision aberration-canceled and dispersion-canceled measurements, to ghost imaging and quantum cryptography.

The essential material needed to follow the rest of the book is covered in the first two chapters. The remainder of the chapters can be read more or less independently

of each other, according to the interests of the reader. Some of the chapters in this book form a greatly expanded treatment of material first presented as a series of talks in the *Advanced School on Quantum Foundations and Open Quantum Systems* held in João Pessoa, Brazil, July 16–28, 2012.

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# Chapter 1

## Quantum Optics and Entanglement

### 1.1 Introduction

Although it is now about a century old, in many areas quantum mechanics has become a tool for carrying out practical tasks only in recent decades. Quantum mechanical features such as superposition and entanglement, states with fixed photon number, or single photons in modes tailored to suit a particular application have become prominent tools for applications in communication and cryptography, computing, metrology, lithography, and imaging. A brief sample of these applications includes the following:

- Appropriately chosen entangled states can cancel the effects of aberration and dispersion, allowing improved imaging and measuring abilities. Quantum methods allow the classical Abbé limit and the standard quantum limit to be beaten under some circumstances, achieving resolution and sensitivity levels that are impossible for classical systems. This in turn allows the writing of sub-diffraction limited lithographic structures, holding promise for enabling the construction of smaller computer chips.
- The unavoidable disturbances that measurements introduce to the measured system in quantum mechanics, traditionally viewed as a limitation, has been turned into a useful tool that can open new possibilities such as communication and cryptography protocols that are unconditionally secure against eavesdropping.
- Harnessing the superposition principle offers the hope of one day achieving universal quantum computing, with exponential speed-up relative to classical computers in some types of problems.

In this book, some of these applications are examined, focusing primarily on quantum optical methods in the areas of measurement, communication, and imaging. Most of the applications to be discussed have been carried out successfully in the laboratory, but few of them are yet in common use outside a specialized research setting. Much of the material in this book has been previously discussed in much briefer form in [1, 2].

The novel possibilities that arise in quantum mechanical applications generally follow from some combination of three phenomena that do not occur for classical particles. The first of these is the possibility of superposition: if  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are state vectors representing possible states of a system, then any normalized superposition of them,  $\cos\theta|\psi_1\rangle + \sin\theta|\psi_2\rangle$  is also a possible state of the system. If  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are not orthogonal to each other, then this allows the possibility of interference between the two states. Such interference is one of the main tools for making measurements in quantum optics, and is used in nearly all the applications discussed in this book.

The second new aspect is the inability to distinguish between nonorthogonal states. Suppose that  $\langle\psi_1|\psi_2\rangle \neq 0$ . Then if the incoming state is  $|\psi_1\rangle$  there is a non-zero probability of detecting the state  $|\psi_2\rangle$  instead. If the measurement is nondestructive, then the measurement itself converts  $|\psi_1\rangle$  into  $|\psi_2\rangle$ . The measurement alters the state in a way that is detectable statistically over many trials. This is the basis for the enhanced security that arises in quantum key distribution, quantum secret sharing, and related protocols (Chap. 9).

The third new possibility arising in quantum mechanics is that of *quantum correlation* or *entanglement*, which is the superposition of two or more states of a composite (multiparticle) system. Most of the techniques discussed in the coming chapters involve the use of pairs of photons sharing either classical correlation or its stronger quantum analog of entanglement. Instead of detecting individual photons, correlation- or entanglement-based methods usually detect *pairs* of photons by means of coincidence counting; the joint properties of the pairs only appear in these coincidence measurements and it is these joint properties that often give rise to enhanced measurements. These enhancements include improved resolution and sensitivity, as well as suppression of quantum noise: because the noise experienced in the two detections is uncorrelated, it does not affect correlated measurements. As a result, applications ranging from imaging to phase measurement show improvement (Chaps. 3–8). Further, entanglement can introduce phenomena not present in classical systems, such as the ability to detect tampering by an eavesdropper, which is detected through a resulting drop in correlations manifested by a loss of Bell inequality violations; this ability to detect eavesdropping is a key ingredient in some types of quantum cryptography, as is seen in Chap. 9.

The use of entangled photon states to make practical measurements goes back at least as far as the 1960s. For example, positron-emission tomography (PET) scans rely on the emission of entangled photon pairs. In this case, an atom undergoes beta decay to produce a positron; the positron then annihilates with an electron to produce the outgoing photons. This process, though useful for medical imaging, is unsuitable for many high-precision measurement applications. The discovery and development of spontaneous parametric down-conversion techniques from the late 1960s onward led to better methods of entangled photon generation, allowing the tailoring of the photon pair's quantum state to suit the desired application.

Some of the methods covered in later chapters are uniquely quantum effects, while others were first seen in a quantum context but have since been shown to be reproducible in classical systems. These latter, sometimes called *quantum-inspired effects* are those most likely to achieve widespread use. For example, ghost imaging

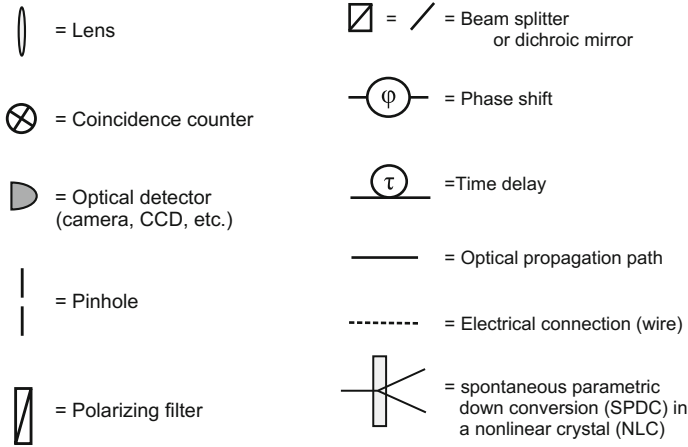


**Fig. 1.1** Symbols that are often used in schematic diagrams throughout this text

(Chap. 6) and sensing of symmetries with correlated orbital angular momentum states (Chap. 8) were first discussed using entangled states, but in fact classical correlations suffice, which allows them to be carried out in a much more practical and robust manner. Together, the various quantum and quantum-inspired effects to be discussed make it possible for classically or quantum-mechanically correlated photon pairs to do things such as evade the standard diffraction limit in microscopy, mitigate (or in some cases completely cancel) the effects of turbulence, aberration, and dispersion, and make measurements of a number of physical quantities with accuracy and precision not possible using traditional methods. Some of these effects, such as ultra-precise measurement of polarization mode dispersion by entangled photon pairs, have already found real-world commercial applications.

In this chapter, we briefly introduce some of the background in quantum mechanics, along with a bit of information theory, that is useful in later chapters.

In coming chapters, there are many schematic diagrams of optical systems. Figure 1.1 displays the symbols that are used most often in these diagrams.

## 1.2 Quantum Mechanics

Quantum theory was developed by Einstein, Bohr, Schrödinger, Heisenberg, Dirac, and others, during the first quarter of the twentieth century in order to explain atomic phenomena. First given a unified formalization by Dirac and later by von Neumann, the relativistic generalization became the basis for quantum field theory, which still serves as our most fundamental theory of known natural processes. Two of the most important early insights from quantum mechanics were the introduction of the photon as the irreducible quantum of electromagnetic energy and the realization that in the

quantum realm, material particles may also exhibit wave-like behavior and so can interfere. Interference effects involving small numbers of photons comprise one of our main tools in the following chapters, along with the ideas of superposition and entanglement. Here, we give a minimalist review of quantum mechanics and emphasize those ideas that are needed in subsequent chapters. More comprehensive discussions can be found in many places, including [3–7], to list just a few.

The state of a quantum system is described by a *state vector* in a complex Hilbert space, denoted  $\mathcal{H}$ . In Dirac notation, these vectors are written as  $|\psi\rangle$ ,  $|\phi\rangle$ , etc., where  $\psi$ ,  $\phi$  are simply names or labels for the states. Depending on the circumstances, the labels, which may be either continuous or discrete, may include the energy, photon number, spin, position, or momentum of the system, among other possibilities. To each state vector  $|\psi\rangle$  (also called a *ket*), there is associated a corresponding dual  $\langle\psi|$  (called a *bra*) which is the Hermitian conjugate (the complex transpose, denoted  $^\dagger$ ) of the corresponding ket. So, for example, if  $|\psi\rangle$  is described by a column vector with components  $\psi_1, \psi_2, \dots$ , then  $\langle\psi|$  is a row vector with components  $\psi_1^*, \psi_2^*, \dots$ :

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \end{pmatrix} \leftrightarrow \langle\psi| = (\psi_1^* \ \psi_2^* \ \dots). \quad (1.1)$$

Inner products are defined between bras and kets:

$$\langle\phi|\psi\rangle = \phi_1^* \psi_1 + \phi_2^* \psi_2 + \phi_3^* \psi_3 + \dots \quad (1.2)$$

The various components of the state vector can be found by taking the inner product with the basis vectors of some orthonormal basis set,  $\{|e_j\rangle\}$ ,

$$\psi_j = \langle e_j|\psi\rangle. \quad (1.3)$$

Outer products such as  $|\phi\rangle\langle\psi|$  act as operators that project out the part of a state in the direction of  $|\psi\rangle$  and redirect it into the direction parallel to  $|\phi\rangle$ . In other words, it represents a transition from state  $|\psi\rangle$  to state  $|\phi\rangle$ . Vectors are usually assumed to be normalized,  $\langle\psi|\psi\rangle \equiv |\psi|^2 = 1$ , so as to provide well-defined probabilities. Transition probabilities are then simply the squares of the inner product or overlap between the initial and final state vectors,  $P_{\psi \rightarrow \phi} = |\langle\phi|\psi\rangle|^2$ .

The vectors above were assumed to be defined on a finite-dimensional Hilbert space, so that their indices were discrete. However, the Hilbert space can also be infinite dimensional, in which case they are labelled by continuous variables. For example, the *wavefunction*  $\psi(\mathbf{x})$  is the projection of the state vector onto position space:  $\psi(\mathbf{x}) = \langle\mathbf{x}|\psi\rangle$ , where  $|\mathbf{x}\rangle$  is basis vector for position space: it is nonzero everywhere except at position  $\mathbf{x}$ . The orthogonality relation of the position eigenstates is of the form  $\langle\mathbf{x}|\mathbf{x}'\rangle = \delta^{(3)}(\mathbf{x} - \mathbf{x}')$ , so the inner product of the wavefunctions is of the form

$$\langle\psi|\phi\rangle = \int \psi^*(\mathbf{x}) \phi(\mathbf{x}) d^3x. \quad (1.4)$$

It can be seen that in this continuous-variable case, the vectors may not be normalizable; for example  $\langle \mathbf{x} | \mathbf{x} \rangle$  diverges. In situations where the lack of normalization may be a problem, the space is often artificially discretized, and then the spacing between lattice points is taken to zero only at the end of the problem.

One particularly useful type of state is the *Fock state*  $|n\rangle$ , which contains exactly  $n$  photons. Creation and annihilation operators,  $\hat{a}^\dagger$  and  $\hat{a}$ , are defined which raise and lower the number of photons in the Fock state:  $\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$ , and  $\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$ . The number operator,  $\hat{N} = \hat{a}^\dagger \hat{a}$ , counts the number of photons in the state:  $\hat{N} |n\rangle = n |n\rangle$ . For such states, the phase is completely indeterminate and yields a different value each time it is measured.

The dynamics in quantum mechanics are governed by the Schrödinger equation, a linear second-order partial differential equation. In accord with the linearity of the Schrödinger equation, quantum systems obey the *superposition principle*: any two possible states of a system, say  $|\psi\rangle$  and  $|\phi\rangle$  can be added to get another allowed state:

$$|\Phi\rangle = \frac{1}{\sqrt{2}} (|\psi\rangle + |\phi\rangle), \quad (1.5)$$

where the  $\frac{1}{\sqrt{2}}$  is included to maintain normalization in the case that  $|\psi\rangle$  and  $|\phi\rangle$  are mutually orthogonal.

Consider the superposition state  $|\Phi\rangle$  above more closely. If  $|\psi\rangle$  and  $|\phi\rangle$  are not orthogonal to each other ( $\langle \phi | \psi \rangle \neq 0$ ), then the two terms in the superposition may interfere:

$$\langle \Phi | \Phi \rangle = \frac{1}{2} (\langle \psi | + \langle \phi |) (|\psi\rangle + |\phi\rangle) = \frac{1}{2} (|\psi|^2 + |\phi|^2 + 2\text{Re}\langle \psi | \phi \rangle) \quad (1.6)$$

$$= 1 + \text{Re}\langle \psi | \phi \rangle. \quad (1.7)$$

The last term is an interference term which leads to many uniquely quantum phenomena. Such interference is put to a number of uses in the coming chapters.

More subtle types of interference effects occur if the superposed states are made up of multiple particles or subsystems. Suppose a pair of particles, A and B, together form a composite system C whose Hilbert space is the product of the two single-particle Hilbert spaces,  $\mathcal{H}_C = \mathcal{H}_A \times \mathcal{H}_B$ . If A is in state  $|\psi\rangle$  and B is in state  $|\phi\rangle$ , then the composite system is in state  $|\Psi\rangle_C = |\psi\rangle_A |\phi\rangle_B$ , where the subscripts are used to indicate which system is in which state. A state that can be written in such a product form in some basis is called a *product state* or a *separable state*.

Often, however, the state of the composite system is known while the states of the individual subsystems are not. For example, the total energy may be known, but it may not be known how it is distributed between the two particles. In such a case, all the possibilities consistent with the available information have to be added or superposed. For instance, suppose that  $|\Phi\rangle_C = \frac{1}{\sqrt{2}} (|\psi\rangle_A |\phi\rangle_B + |\phi\rangle_A |\psi\rangle_B)$ . If the states of the individual subsystems are not measured, then the two possibilities (A in state  $|\psi\rangle$  with B in state  $|\phi\rangle$ , versus A in state  $|\phi\rangle$  with B in state  $|\psi\rangle$ ), can be thought of as *both* existing simultaneously. Such a state cannot be factored into a

single well-defined state for  $A$  and a similarly well-defined state for  $B$ ; this type of nonfactorable state is referred to as *entangled*. Such situations occur very commonly in microscopic physics, for example when two photons arrive at a beam splitter: both photons can exit out through one output port, both can exit at the other output port, or the two photons can exit from opposite ports in two different ways. If it is arranged so that the locations of the outgoing photons cannot be measured with the experimental apparatus, then there is no way to distinguish between the possibilities and they must all be included: the state of the full system is therefore an entangled state formed by the superposition of all four possibilities. If a measurement is made that determines the state of one of the subsystems, then the entangled system collapses to a product state, and the state of the second subsystem is also then known. For example if particle  $A$  is measured and found to be in state  $|\psi\rangle$ , then the measurement causes  $|\Phi\rangle_C = \frac{1}{\sqrt{2}}(|\psi\rangle_A|\phi\rangle_B + |\phi\rangle_A|\psi\rangle_B)$  to collapse to the product state  $|\psi\rangle_A|\phi\rangle_A$ . After the collapse, we know that  $B$  must now be in state  $|\phi\rangle$ . The individual subsystems were not in definite states  $|\psi\rangle_A$  and  $|\phi\rangle_B$  before the measurement; they were indeterminate, in a superposition of two states at the same time. The fact that definite states for each subsystem did not exist before the measurement is one of the more non-intuitive aspects of quantum mechanics, but has been well verified by the fact that the Bell and CHSH inequalities [8–13] are violated.

A common basis to use for bipartite polarization-entangled states is the *Bell basis*, consisting of the four state vectors

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 \pm |V\rangle_1|H\rangle_2), \quad |\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2 \pm |V\rangle_1|V\rangle_2), \quad (1.8)$$

where  $H$  and  $V$  represent horizontal and vertical polarization, while 1 and 2 may be position or particle labels. Clearly, a similar definition may be made replacing polarization by any other two-state variable. Bell states are often taken as the basic unit of entanglement; the entanglement of other systems may then be quantified by determining how many Bell states are needed to build them.

The most common way to produce entangled pairs of photons is via spontaneous parametric down conversion (SPDC) [14], in which nonlinear interactions in a crystal mediate the conversion of an incoming photon (the *pump*) into two lower energy outgoing photons. This process is discussed in detail in Sect. 1.6 and in Appendix D.

Besides state vectors in a Hilbert space, the other basic mathematical object in quantum mechanics is a set of operators. Operators perform actions on states; these actions can include, for example, multiplication by a matrix or differentiation. That an object is an operator is often denoted by a hat (circumflex) over its name. Examples are the linear momentum operator  $\hat{p} = -i\hbar\frac{d}{dx}$  and the spin operator, denoted  $\hat{S}$ . Unlike ordinary numbers, operators do not necessarily commute. The difference of the two possible orderings,  $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ , is called the *commutator*.

Quantities that can be physically measured, such as energy and angular momentum, are eigenvalues of Hermitian operators; an operator  $\hat{A}$  is *Hermitian* (or self-adjoint) if it is equal to its Hermitian conjugate,  $\hat{A}^\dagger = \hat{A}$ . An equivalent definition

is that an operator  $\hat{A}$  is Hermitian if it satisfies  $\langle \psi | \hat{A} \phi \rangle = \langle \hat{A} \psi | \phi \rangle$ , for all states  $|\phi\rangle$  and  $|\psi\rangle$ .

In addition to Hermitian operators, another important class is the set of unitary operators. The *unitary* linear operators,  $\hat{U}$ , are those for which  $\hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = \mathbb{I}$ , where  $\mathbb{I}$  is the identity. These operators describe the effect on the state of various types of transformations, such as movement through space and time or the effect of interaction with another system. The unitary operators preserve the norm of the state vector:  $\langle \hat{U} \psi | \hat{U} \psi \rangle = \langle \psi | \psi \rangle$ . As a simple example of a unitary operator, the time evolution of an energy eigenstate in the Schrödinger picture is given by  $|\psi(0)\rangle \rightarrow |\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$ , where  $\hat{U}(t) = e^{i\hat{H}t/\hbar}$  is the unitary operator obtained by exponentiating the (Hermitian) Hamiltonian operator  $\hat{H}$ .

Suppose that  $a$  and  $b$  are the classical variables associated with two operators. In other words, the possible values that can be measured for  $a$  and  $b$  are the eigenvalues obtained when  $\hat{A}$  and  $\hat{B}$  act on their eigenstates:  $\hat{A}|\psi\rangle = a|\psi\rangle$  and  $\hat{B}|\phi\rangle = b|\phi\rangle$ . If  $\hat{A}$  and  $\hat{B}$  do not commute, then they do not share the same eigenstates: the measurement of one variable (say  $\hat{A}$ ) has the result that the other is rendered indeterminate, since after the measurement the system is in an  $\hat{A}$ -eigenstate but not in a  $\hat{B}$ -eigenstate. The value of  $b$  is no longer well-defined, at least not until a  $\hat{B}$  measurement is made, which then takes the system out of the  $\hat{A}$  eigenstate and into a  $\hat{B}$  eigenstate.

Quantum mechanics is a probabilistic theory: unless the system is in an eigenstate of  $\hat{A}$ , repeated measurements of  $a$  yield possible values indeterministically, fluctuating around some mean value (called the *expectation value*),  $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$ . The typical fluctuation size is given by the standard deviation or *uncertainty* given by the square root of the variance:

$$\Delta a^2 = \left\langle \left( \hat{A} - \langle \hat{A} \rangle \right)^2 \right\rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2. \quad (1.9)$$

When the system is in an  $\hat{A}$  eigenstate,  $a$  has a definite, well-defined value and no uncertainty, while the variable  $b$  corresponding to a noncommuting operator  $\hat{B}$  has a large uncertainty. More precisely, in the case  $[\hat{A}, \hat{B}] \neq 0$ , the two variables obey an *uncertainty relation*, so that there is a minimum value to the product of their uncertainties:  $\Delta a \Delta b$  must exceed a minimum quantity proportional to the commutator  $[\hat{A}, \hat{B}]$ . The most famous example is between position  $x$  and momentum  $p$ . The operators corresponding to these variables,  $\hat{x}$  and  $\hat{p}$ , form a conjugate pair, obeying the so-called canonical commutation relations:  $[\hat{x}, \hat{p}] = i\hbar$ , where  $\hbar$  is Planck's constant. As a result, we find the Heisenberg uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (1.10)$$

These uncertainty relations provide the ultimate fundamental physical limits to all measurements, for example, they are the basis of the Heisenberg limit discussed in Chap. 4. For more general discussions of uncertainty relations in quantum optics, see [15, 16].

If a system can be written in terms of a single Hilbert space state vector  $|\psi\rangle$ , then it is said to be in a *pure state*. More generally, it may be in a statistical mixture of different pure states, each with a different probability; this is called a *mixed state*. In order to treat pure and mixed states on an equal basis, we may describe states by a *density operator*  $\hat{\rho}$  instead of a state vector. For a pure state  $|\psi\rangle$ , the density operator is simply the projection operator onto that state,  $\hat{\rho} = |\psi\rangle\langle\psi|$ . Pure states are defined as those which are *maximally specified* within quantum mechanics. In general, any valid density operator, regardless of the purity of the state it describes, must obey  $\text{tr } \hat{\rho} = 1$ .

A quantitative measure of how close a state described by statistical operator  $\hat{\rho}$  is to being pure may be given by defining the *purity*,  $\mathcal{P}$ , of the state:

$$\mathcal{P}(\hat{\rho}) = \text{tr } \hat{\rho}^2, \quad (1.11)$$

where  $\frac{1}{d} \leq \mathcal{P}(\hat{\rho}) \leq 1$  for a Hilbert space of dimension  $d$ ,  $\mathcal{H}$ . The quantum state is pure if  $\mathcal{P}(\hat{\rho}) = 1$ , that is, if it spans a one-dimensional subspace of  $\mathcal{H}$ . It is mixed if  $\mathcal{P}(\hat{\rho}) < 1$ .

The purity is invariant under unitary transformations of the form  $\hat{\rho} \rightarrow U\hat{\rho}U^\dagger$ . In particular, it is invariant under the *dynamical mapping* that describes time evolution,  $\hat{U}(t, t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$ , where  $\hat{H}$  is the Hamiltonian operator. A quantum state is pure if and only if the statistical operator  $\hat{\rho}$  is *idempotent*, that is,

$$\hat{\rho}^2 = \hat{\rho}, \quad (1.12)$$

providing a convenient test for maximal state purity. As mentioned above, the density operator of a pure state is then simply a projection operator  $|\psi\rangle\langle\psi|$  onto the corresponding one-dimensional subspace of its Hilbert space spanned by the normalized state vector  $|\psi\rangle$ , since a Hermitian operator  $\hat{P}$  acting in a Hilbert space  $\mathcal{H}$  is a *projector* if and only if  $\hat{P}^2 = \hat{P}$ . It follows from this definition that given projector  $\hat{P} = |\psi\rangle\langle\psi|$  then  $\hat{P}^\perp \equiv \mathbb{I} - \hat{P} \equiv |\psi^\perp\rangle\langle\psi^\perp|$  is also a projector. The operators  $\hat{P}$  and  $\hat{P}^\perp$  project onto orthogonal subspaces within  $\mathcal{H}$ ,  $\mathcal{H}_s$ , and  $\mathcal{H}_s^\perp$ , respectively, providing a decomposition of  $\mathcal{H}$  as  $\mathcal{H}_s \oplus \mathcal{H}_s^\perp$ ; two subspaces are said to be *orthogonal* if every vector in one is orthogonal to every vector in the other. In the case of a general state of a single physical *qubit* (a state of a quantum two-level system), one may write

$$\hat{\rho} = p_1|\psi\rangle\langle\psi| + p_2|\psi^\perp\rangle\langle\psi^\perp|, \quad (1.13)$$

where the weights  $p_i$  are the eigenvalues of the statistical operator  $\hat{\rho}$ .

A quantum state is mixed if it is *not* a pure state. Consider a finite set,  $|\psi_i\rangle\langle\psi_i|$ , of projectors corresponding to distinct, orthogonal pure states  $|\psi_i\rangle$ . Any state  $\hat{\rho}$  that can be written

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad (1.14)$$

with  $0 < p_i < 1$  and  $\sum_i p_i = 1$ , is then a normalized mixed state. The superposition principle implies that any (complex) linear combination of qubit basis states, such as  $|0\rangle$  and  $|1\rangle$ , that is,

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \quad (1.15)$$

with  $a_i \in \mathbb{C}$  and  $|a_0|^2 + |a_1|^2 = 1$ , is *also* a pure state of the physical qubit. The coefficients  $a_0$  and  $a_1$  are probability amplitudes, whose squared magnitudes,  $|a_0|^2$  and  $|a_1|^2$ , are the probabilities  $p_0$  and  $p_1$ , respectively, of the physical qubit described by state  $|\psi\rangle$  being found in these basis states  $|0\rangle$  and  $|1\rangle$ , respectively, upon measurement, given that the system was initially prepared in state  $\hat{\rho}$ .

The pure states of the qubit can be represented by vectors in the two-dimensional complex Hilbert space,  $\mathcal{H} = \mathbb{C}^2$ . Any orthonormal basis for this space can be put in correspondence with two bit values, 0 and 1, in order to act as the single-qubit *computational basis*, written  $\{|0\rangle, |1\rangle\}$ . The vectors of the computational basis can be represented in matrix form as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1.16)$$

Consider the normalized sums

$$|\nearrow\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \text{ and } |\searrow\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \quad (1.17)$$

These are again pure states. The corresponding projectors are  $\hat{P}(|\nearrow\rangle) = |\nearrow\rangle\langle\nearrow|$ ,  $\hat{P}(|\searrow\rangle) = |\searrow\rangle\langle\searrow|$ . However, the normalized sum of a pair of *projectors*, for example,  $\hat{P}(|0\rangle)$  and  $\hat{P}(|1\rangle)$  corresponding to pure states  $|0\rangle$  and  $|1\rangle$ , namely,

$$\hat{\rho}_+ = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}(|\nearrow\rangle\langle\nearrow| + |\searrow\rangle\langle\searrow|), \quad (1.18)$$

is a *mixed* state. Finally, note that the statistical operator corresponding to the normalized sum of  $|\nearrow\rangle$  and  $|\searrow\rangle$  is  $P(|0\rangle) \neq \hat{\rho}_+$ .

Other commonly used bases are the *diagonal basis*,  $\{|\nearrow\rangle, |\searrow\rangle\}$ , sometimes also written  $\{|+\rangle, |-\rangle\}$ , and the *circular basis*  $\{|R\rangle, |L\rangle\}$ :

$$|R\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \quad |L\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle), \quad (1.19)$$

sometimes also written  $\{|\odot\rangle, |\oslash\rangle\}$ , is also useful for quantum cryptography, being conjugate to both the computational and diagonal bases.

The three bases (computational, diagonal, and circular) above are mutually conjugate and are used in protocols for quantum key distribution (Chap. 9); the probabilities of qubits in the states  $|R\rangle$  and  $|L\rangle$  being found in the states  $|0\rangle$ ,  $|1\rangle$ ,  $|\nearrow\rangle$ , and  $|\searrow\rangle$  are all  $\frac{1}{2}$ , and vice-versa. The generic mixed state,  $\hat{\rho}$ , lies in the interior of the Bloch ball

(see below) and can be written as a convex combination of basis-element projectors corresponding to the pure-state bases described above. The pure states are those that reside on the surface of the ball. The effect of a general operation on a qubit can be viewed as a (possibly stochastic) transformation within this ball; for illustrations of this in practical context, see [17].

The density matrix and the *Stokes four-vector*,  $S_\mu$ , are related by

$$\hat{\rho} = \frac{1}{2} \sum_{\mu=0}^3 S_\mu \sigma_\mu, \quad (1.20)$$

where  $\sigma_\mu$  ( $\mu = 1, 2, 3$ ) are the *Pauli operators* which, together with the identity  $\sigma_0 = \mathbb{I}_2$ , are represented in the matrix space  $H(2)$  by the Pauli matrices. The Pauli matrices form a basis for  $H(2)$ , which contains the qubit density matrices. The qubit density matrices themselves are the positive-definite, trace-class elements of the set of  $2 \times 2$  complex Hermitian matrices  $H(2)$  of unit trace, that is, for which the total probability  $S_0$  is one, as prescribed by the Born rule for quantum probabilities and the well-definedness of quantum probabilities as such. Density matrices are similarly defined for systems of countable dimension. The products of the three nontrivial Pauli matrices—those between the  $\sigma_i$  for  $i = 1, 2, 3$ —are given by

$$\sigma_i \sigma_j = \delta_{ij} \sigma_0 + i \epsilon_{ijk} \sigma_k, \quad (1.21)$$

which defines their algebra. Appropriately exponentiating the Pauli matrices provides the rotation operators,  $R_i(\xi) = e^{-i\xi\sigma_i/2}$ , for Stokes vectors about the corresponding directions  $i$  [3]; these rotations realize the group  $SO(3)$ .

The Stokes parameters  $S_\mu$  ( $\mu = 0, 1, 2, 3$ ) also allow one to visualize the qubit state geometrically in the Bloch ball via  $S_1, S_2, S_3$ . The Euclidean length of this three-vector (also known as the *Stokes vector*, or *Bloch vector*) is the radius  $r = (S_1^2 + S_2^2 + S_3^2)^{1/2}$  of the sphere produced by rotations of this vector. With the matrix vector  $\mathbf{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  and the three-vector  $\mathbf{S} = (S_1, S_2, S_3)$ , one has

$$\hat{\rho} = \frac{1}{2} (S_0 \mathbb{I}_2 + S_1 \sigma_1 + S_2 \sigma_2 + S_3 \sigma_3), \quad (1.22)$$

known as the *Bloch-vector representation* of the statistical operator, in accord with Eq. (1.15). In optical situations, where  $\mathbf{S}$  describes a polarization state of a photon, the degree of polarization is given by  $P = r/S_0$ , where  $S_0$  is positive. For the qubit, when the state is normalized so that  $S_0 = 1$ ,  $S_0$  corresponds to total quantum probability. The density matrix of a single qubit is then of the form

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}, \quad (1.23)$$

where  $\rho_{00} + \rho_{11} = 1$ ,  $\rho_{ii} = \rho_{ii}^*$  with  $(i = 0, 1)$ , and  $\rho_{10} = \rho_{01}^*$ , where  $*$  indicates complex conjugation. One can write the Pauli matrices for  $\mu = 1, 2, 3$  in terms of outer products of computational basis vectors, as follows. The Stokes parameters are expressed in terms of the density matrix as

$$S_\mu = \text{tr}(\hat{\rho}\sigma_\mu), \quad (1.24)$$

which are probabilities corresponding to ideal normalized counting rates of measurements in the standard eigenbases.

A key feature of quantum mechanics that serves as the basis for quantum cryptography (Chap. 9) is the fact that it is impossible to distinguish unambiguously between two non-orthogonal state vectors. For example, suppose that a photon is known to be polarized either vertically ( $|\uparrow\rangle = |1\rangle$ ) or horizontally ( $|\rightarrow\rangle = |0\rangle$ ). Suppose further that the photon is transmitted by a polarizer oriented at  $45^\circ$ , so that it is detected in the state  $|\nearrow\rangle$ . Since  $|\nearrow\rangle$  has nonzero overlap with each of the possible initial states,  $\langle\nearrow|\uparrow\rangle = \langle\nearrow|\rightarrow\rangle = \frac{1}{\sqrt{2}}$ , there is no way that the initial state can be determined from this measurement. The problem of distinguishing quantum states is discussed in more detail below (Sect. 1.11).

In the following, the main concern is with bipartite entangled states. However, entanglement can also occur in composite systems with three or more subsystems. For example, two commonly used classes of entangled states formed from tripartite compositions of two-level systems are the *Greenberg–Horne–Zeilinger (GHZ)* states,

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle), \quad (1.25)$$

and the Werner states

$$|W\rangle = \lambda_1|001\rangle + \lambda_2|010\rangle + \lambda_3|001\rangle, \quad (1.26)$$

where  $|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2 = 1$ . The GHZ state, in particular, has a number of uses in areas such as quantum secret sharing (Chap. 9). GHZ states of up to eight photons have been produced experimentally [18]. For more detailed discussions of these states, see [4].

### 1.3 Classical and Quantum Information

Because quantum mechanics intersects with information theory at several points in this book, a very brief introduction to classical and quantum information is given here. Quantum information theoretical aspects are of use especially in Chap. 9. Further details may be found in [4, 19–22].

Classically, information is measured in units of *bits*. One bit is the maximum information that can be derived from measuring the state of a single classical two-state system. The states of the system are conventionally taken to 0 or 1 (the *computational*

*basis*). More generally, for an  $n$ -state system a measurement can produce  $\log_2 n$  bits of information, where  $\log_2$  is the base-two logarithm. Given a probability distribution  $p(x)$  for some random variable  $x$ , the *Shannon information* is defined to be

$$H(x) = - \sum_x p(x) \log_2 p(x). \quad (1.27)$$

Shannon information is a gauge of how much is learned from a measurement of the random variable: if the probability is zero for all but one of the values of  $x$ , then the information vanishes, since the outcome could have been predicted with certainty in advance. On the other hand, the information is maximum for the uniform distribution, in which all outcomes have the same probability, since that is the case in which the initial uncertainty in  $x$  is largest and the most information is gained by measurement.

In quantum physics, a two state system is not necessarily in one state or the other, but instead could be in a superposition of both:

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad (1.28)$$

with  $|a|^2 + |b|^2 = 1$ . One defines the basic unit of quantum information to be the quantum bit or *qubit* [23]. Note that in the physics literature, the word “qubit” is often used to mean both the unit of quantum information and a (two-level) physical system capable of encoding that quantum information. Unlike the classical bit, which is two-valued in the computational basis, a qubit system can be in one of an infinite number of physically different states. It is only when a measurement is made in the computational basis that it is definitely in one of the two states. Further, unless one knows how the basis was chosen that define the  $|0\rangle$  and  $|1\rangle$  states, an unknown state of a qubit system cannot generally be found by a single measurement, but rather state determination requires multiple measurements on an ensemble of identically prepared systems. It is this which provides the possibility of secure quantum key distribution, as is seen in Chap. 9.

The quantum mechanical analog of the Shannon information is the *von Neumann entropy*,

$$S(\hat{\rho}) = -\text{tr}(\hat{\rho} \log_2(\hat{\rho})), \quad (1.29)$$

which vanishes on pure states and takes a maximum value of  $\log_2 N$  for maximally entangled states on an  $N$ -dimensional Hilbert space. The von Neumann entropy figures into many quantitative measures of entanglement, such as those described in Sect. 1.5.

## 1.4 Bits and Qubits in Quantum Optics

For specificity, let us now take the system in question to be a photon. Light is easy to produce and to detect, propagates well over long distances, and has properties that are both well understood and easily controlled. As a result, many experiments in quantum

information and communication are carried out on optical systems. Consequently, the focus henceforth is exclusively on quantum optical systems. We begin by describing one example of how optical qubits can be created.

### 1.4.1 An Example: Spatial Qubits

Consider the beam splitter shown in Fig. 1.2a. A *beam splitter* (BS) is a device for splitting a single optical beam into two outgoing beams: a portion of the beam is transmitted through the BS, while a portion is reflected (see Appendix A.3 for more detail). Throughout, we assume that all beam splitters used are nonpolarizing and 50–50 (light has equal probability to be reflected and transmitted). A beam splitter is a linear, passive four-port device, with two input ports ( $a$  and  $b$ ) and two output ports ( $c$  and  $d$ ). To describe its action, form the operator-valued column vectors

$$\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{c}^\dagger \\ \hat{d}^\dagger \end{pmatrix}, \quad (1.30)$$

where  $\hat{a}^\dagger$ ,  $\hat{b}^\dagger$ ,  $\hat{c}^\dagger$ ,  $\hat{d}^\dagger$  are the creation operators for photon states at the corresponding ports. One may then denote the action of the beam splitter by a matrix  $B$  relating ingoing and outgoing operators,

$$\begin{pmatrix} \hat{c}^\dagger \\ \hat{d}^\dagger \end{pmatrix} = B \begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix}. \quad (1.31)$$

The form of this matrix is easy to determine: the photon is unchanged when it is transmitted and picks up a phase of  $\frac{\pi}{2}$  when reflected, so the BS matrix is

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (1.32)$$

The photons entering or leaving from above the BS (i.e. ports  $a$  and  $d$ ) may now be thought of as representing state  $|0\rangle$ , while those entering or leaving below the beam splitter (i.e.  $b$  and  $c$ ) represent  $|1\rangle$  states. This provides a representation of physical qubits as spatial modes, and allows us to think of the BS matrix  $B$  as taking combinations of input bits to combinations of output bits. In particular, if the *bit-value* 0 is input, the resulting output is the *qubit* state  $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ . Thus, we have a simple way of producing spatial qubit states from classical bit states.

It is important to note that the beam splitter *does not* randomly reflect half of the photons and transmit the other half. Rather, *each individual photon* has an amplitude of being reflected and of simultaneously being transmitted. The state of each photon after the beam splitter is therefore a superposition of these two possibilities; it is in some sense *both* reflected and transmitted at the same time.

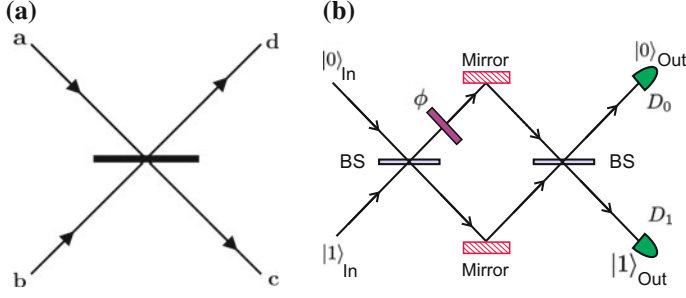


**Fig. 1.2** **a** A 50/50 beamsplitter. A photon entering either input port,  $a$  or  $b$ , has equal probability of being transmitted or reflected out either output port,  $c$  or  $d$ . **b** The Mach-Zehnder interferometer provides a range of qubit states as the input qubit amplitudes  $a_i$  and phases  $\phi_i$  are changed. The detectors provide count rates proportional to the probability of lying in the output computational-basis states described by state-projectors  $|0\rangle\langle 0|$  and  $|1\rangle\langle 1|$ , for input amplitudes  $a_0 = 0$ ,  $a_1 = 1$ , namely,  $p(0) = \sin^2[(\phi_0 - \phi_1)/2]$  and  $p(1) = \cos^2[(\phi_0 - \phi_1)/2]$ . Without loss of generality, the phase shift can be restricted to one arm:  $\phi_0 = \phi$ ,  $\phi_1 = 0$

More general spatial qubit states may be constructed with the Mach-Zehnder interferometer (Fig. 1.2b). This is analogous to a Young double-slit arrangement where only two directions are available to the self-interfering system, so that the two paths inside a beam-splitter act as “slits.” In this interferometer, a photon enters from the left into a beam-splitter, with two exit paths on the right. The interferometer provides a spatial qubit state consisting of occupation of one and/or the other interior beam path. After splitting at the first beam splitter, each of the resulting beams encounters a mirror, a phase shifter, a second beam-splitter, and finally a particle detector, with the two beams being mixed by the second beam splitter before detection. Since only the relative phase between arms matters, the phase shift in one path can be set to zero without loss of generality. One can also use this interferometer to prepare a *phase qubit* by selecting only those systems entering a single initial input port and exiting a single final output port: the  $|0\rangle$  and  $|1\rangle$  states are phase shifted by either 0 or  $\pi$ .

The action of the MZ interferometer may be described by the matrix  $M = B\Phi B$ , where  $B$  is the BS matrix above and the phase shift is described by the matrix

$$\Phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.33)$$

Multiplying out the matrices, it is found that the action on an incoming bit  $|0\rangle$  is:

$$|0\rangle \rightarrow \frac{1}{2} \left[ (e^{i\phi} - 1) |0\rangle + i (e^{i\phi} + 1) |1\rangle \right], \quad (1.34)$$

allowing construction of a family of phase qubits.

### 1.4.2 Types of Optical Qubits

Qubits may be encoded into physical degrees of freedom in many different ways. For communication and metrology applications, optical qubits are often the most convenient, so in this section, we discuss some of the ways that qubits can be encoded into optical degrees of freedom.

- One approach is to use different *spatial paths* or different *phases* to represent  $|0\rangle$  and  $|1\rangle$  states, as was done with the upper and lower branches of the Mach–Zehnder interferometer in the previous section. However, this is only practical when the photons are not being transmitted over long distances through free space. The Mach–Zehnder interferometer has many measurement applications, in addition to being a means of generating qubits, and it is discussed in more detail in Chap. 2.

- *Polarization* is probably the most common degree of freedom used for optical qubits. Once a basis is chosen in space, polarization along one axis is taken to represent  $|0\rangle$ , while polarization along the perpendicular axis represents  $|1\rangle$ . Preparation of such polarization states is particularly easy, requiring only a polarizing filter. Similarly, discriminating between  $|0\rangle$  and  $|1\rangle$  can be carried out by passing the photon through a birefringent crystal, which separates the two polarization states into different spatial modes, so they can be distinguished by arrival at different detectors. Polarization is robust against disruption by turbulence in free-space transmission. However, in fiber systems only polarization states aligned with the birefringent axes of the fiber are preserved, leading to problems for applications such as quantum key distribution, where multiple polarization bases must be used.

For a basis with horizontal and vertical axes, a polarization qubit is represented as

$$|\psi\rangle = a_0|\uparrow\rangle + a_1|\rightarrow\rangle, \quad (1.35)$$

with the arrows representing the polarization axis. In the diagonal basis, this becomes

$$|\psi\rangle = a'_0|\nearrow\rangle + a'_1|\searrow\rangle. \quad (1.36)$$

- *Photon number* can also be used; for example,  $|0\rangle$  can be represented by the vacuum state with no photons, and  $|1\rangle$  by a state with a single photon Fock state. However, detector noise, losses in transmission, and other complications often mean that it is hard to discriminate between the two states without high error rates. Also, it is problematic to superpose states with different energy unless a second photon (or an atom) is entangled to the system to take up the excess energy.

- *Optical orbital angular momentum (OAM)* can not only be used to represent two-level qubits, but also multilevel *qudits* (Sect. 1.8). A photon has nonzero OAM when its wavefront has an azimuthally-varying phase and a phase singularity along the propagation axis. The linear dependence of phase on azimuthal angle tilts the wavefront and gives it a corkscrew shape. The value of OAM is quantized and is in principle unlimited in size, allowing large amounts of information to be extracted from a single photon. However, practical problems often arise in applications. For

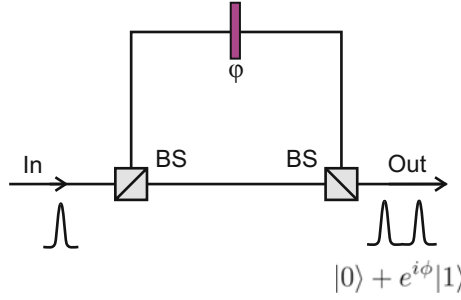


**Fig. 1.3** A superposition of time bin states can be produced using an unbalanced Mach-Zehnder interferometer. The two paths have different lengths: a photon following the short path is in state  $|0\rangle$ , while one following the long path is in state  $e^{i\phi}|1\rangle$ . If it is impossible to tell which path the photon takes, then the output is in state  $|0\rangle + e^{i\phi}|1\rangle$ . Replacing the beam splitter by a controllable optical switch, the time bin state can be switched between  $|0\rangle$  and  $e^{i\phi}|1\rangle$  as desired

example, OAM is easily disrupted by turbulence in free space transmission, and fibers that faithfully propagate a large range of OAM values are difficult to engineer. OAM and its uses are discussed in more detail in Chap. 8.

- *Time bins* form another means of encoding qubits or qudits. The idea is to divide the transmission into a set of discrete time periods of size  $T$ , with each period representing one bit of the message. Within each of these periods there are two smaller time bins of size  $\Delta t$ , separated by a time  $T_{\text{gap}} = T - 2\Delta t$ . A photon emitted during the earlier time bin represents  $|0\rangle$ , one emitted in the later bin represents  $|1\rangle$ .  $T_{\text{gap}}$  must be large enough to clearly distinguish between the pulses in the two bins. A qubit state can then be prepared by sending a photon into an unbalanced Mach-Zehnder interferometer, as in Fig. 1.3. (Unbalanced means the two arms of the interferometer are of different lengths.) The figure assumes the use of 50/50 beam splitters. If instead, the first beam splitter has transmissivity  $t = \cos \theta$  (for some fixed value  $\theta$ ), then a completely general time bin qubit

$$|\psi\rangle = \cos \theta |0\rangle + i \sin \theta e^{i\phi} |1\rangle \quad (1.37)$$

can be prepared.

Since the time bins are not affected by turbulence or by propagation through fibers (assuming that dispersion is minimal), this is a potentially useful form of qubit for long-range quantum communication. It is possible to build “plug and play” time bin generation and detection components for communication systems that also automatically maintain their alignment [24].

- All of the degrees of freedom listed above are discrete. *Continuous variables* may be used as well. For example, two frequency or momentum ranges may be used to represent the two binary values. In other applications, a choice between  $x$  or  $p$  quadratures (see Appendix B) may be used; some examples of this occur in Chap. 9.

## 1.5 Detecting and Quantifying Entanglement

It is valuable to be able to distinguish whether a state is separable or entangled. Further, for some applications it is essential to be able to quantify whether one state is more entangled than another. This section briefly introduces a number of means for either detecting or measuring the degree of entanglement. Here we only discuss a few of the simplest measures; for others and for more detail, see the reviews [25–28].

A central concept in the study of entanglement is the idea of *local operations and classical correlation* (LOCC). If two spatially separated experimenters act on two subsystems of a composite system, they may each apply local operations (rotations, translations, measurements, etc.) that affect only their own subsystem without affecting the other, distant subsystem. These local operations may be coordinated by the exchange of classical communications (telephone calls, radio transmissions, etc.). In general, the quantum correlations found in entangled systems differ from more pedestrian classical correlations by the fact that classical correlations can be generated or increased by LOCC operations. Quantum correlations never increase when LOCC operations are applied.

### 1.5.1 Bell-Type Inequalities

The Bell inequality [8, 9] and similar relations such as the Clauser–Horne–Shimony–Holt (CHSH) inequality [10, 11] are often used to determine whether entanglement is present in bipartite systems. However, violation of Bell-type inequalities are only sufficient, not necessary, conditions for the presence of entanglement; many non-separable mixed states do not exhibit Bell inequality violation. Strictly speaking, violation of Bell inequalities therefore indicates the presence the property of *Bell nonlocality*, which is distinct from entanglement. Bell inequalities are discussed in more detail in Sect. 1.9.

### 1.5.2 Schmidt Decomposition, Schmidt Number

The *Schmidt decomposition* [29] is a useful description of a bipartite system, which makes the presence or absence of entanglement more transparent. Consider two systems  $A$  and  $B$  that together form a composite system. In Schmidt form, such a bipartite state vector is “diagonal,” in the sense that basis vectors of the two Hilbert spaces are matched up in a one-to-one manner,

$$|\Psi\rangle = \sum_{i=1}^{d_{\min}} \sqrt{\lambda_i} |u_i\rangle |v_i\rangle. \quad (1.38)$$

Here,  $d_{min}$  is the dimension of the smaller of the two Hilbert spaces.  $\lambda_i$  is the  $i$ th eigenvalue of the density matrix, and so gives the probability of measuring the  $i$ th term in the expansion,  $p_i = \lambda_i$ .  $|u_i\rangle$  and  $|v_i\rangle$  are basis states in the two Hilbert spaces. Quantum correlations present in entangled systems now become manifest: whenever the first system is measured and found to be in state  $|u_i\rangle$ , the second system is guaranteed to be in state  $|v_i\rangle$ . The number of nonzero terms in the expansion is known as the *Schmidt number*,  $K$ .  $K$  serves as a simple measure of entanglement:  $K = 1$  for a product state, while  $K$  increases as more entangled state terms appear in the sum. Interpreting  $\lambda_k$  as the probability of the  $k$ th state in the sum, the *average* probability per state in the sum is  $\sum_k p(k) \lambda_k = \sum_k \lambda_k^2$ . The average effective number of nonzero components in the decomposition is therefore  $1/\sum \lambda_i^2$ . So, if the Schmidt decomposition is known, then the Schmidt number can be computed from the coefficients [30]:

$$K = 1/\sum \lambda_i^2. \quad (1.39)$$

Each state of a quantum system occupies a minimum space volume  $\geq \hbar$ . Therefore, the Schmidt number has an upper bound given by the number of states that can fit into the phase space volume accessible to the system.  $K$  is therefore finite, even for continuous degrees of freedom, as long as the available phase space volume is finite. The quantity  $\log_2 K$  is sometimes called the *Schmidt measure*.

An explicit recipe can be constructed for putting a state into Schmidt form. Consider some pure state  $|\Psi\rangle = \sum_{ij} C_{ij} |u'_i\rangle |v'_j\rangle$ , so that the density operator is of the form

$$\hat{\rho} = \sum_{ijkl} C_{ij} C_{kl}^* |u'_i\rangle \langle u'_k| \otimes |v'_j\rangle \langle v'_l|. \quad (1.40)$$

First rotate from the  $|u'_i\rangle$  basis to the basis  $|u_i\rangle$  in which  $\hat{\rho}_u = \text{Tr}_v \hat{\rho}$  is diagonal.  $\sum_{ij} C_{ij} C_{kl}^* = \delta_{ik} |g_i|^2$  in this basis, for some constants  $g_i$ , and so  $\hat{\rho}_u = \sum_i |g_i|^2 |u_i\rangle \langle u_i|$ . For each nonzero  $g_i$ , also define a new basis for the second Hilbert space,  $|v_i\rangle = \sum_j \frac{C_{ij}}{|g_i|} |v'_j\rangle$ . One then finds that

$$\hat{\rho} = \sum_{ik} |g_i| |g_k| |u_i\rangle \langle u_k| \otimes |v_i\rangle \langle v_k| = |\Psi\rangle \langle \Psi|, \quad (1.41)$$

with corresponding state vector  $|\Psi\rangle = \sum_i |g_i| |u_i\rangle |v_i\rangle$ , which is of Schmidt form. Therefore,  $\sqrt{\lambda_i} = |g_i|$ .

The Schmidt number for the state emitted by the most common entangled photon source, spontaneous parametric down conversion, are discussed in Sect. 1.6.

### 1.5.3 Negativity and the Positive Partial Transpose Condition

Another entanglement measure that works for non-bipartite states and for some mixed states is the negativity, which is closely related to the positive partial transpose condition (PPT).

The *partial transpose* of a bipartite system is the transpose of the density operator with respect to one of the composite subsystems. For example, given the state  $\rho$  of a composite system with subsystems A and B, the partial transpose  $T_A$  with respect to subsystem A has matrix elements

$$\langle i_A j_B | \rho^{T_A} | k_A l_B \rangle = \langle k_A j_B | \rho | i_A l_B \rangle. \quad (1.42)$$

Equivalently, if the density operator is of the form

$$\hat{\rho} = \sum_{ijkl} p_{ijkl} (|i\rangle \langle j|)_A \otimes (|k\rangle \langle l|)_B, \quad (1.43)$$

then the partial transpose with respect to A is obtained by interchanging the A indices:

$$\hat{\rho}^{T_A} = \sum_{ijkl} p_{ijkl} (|j\rangle \langle i|)_A \otimes (|k\rangle \langle l|)_B. \quad (1.44)$$

Similarly, the partial transpose with respect to subsystem B is

$$\hat{\rho}^{T_B} = \sum_{ijkl} p_{ijkl} (|i\rangle \langle l|)_A \otimes (|k\rangle \langle j|)_B. \quad (1.45)$$

The *Peres–Horodecki positive partial transpose criterion* [31–33], or PPT criterion, then says that the system is entangled if the partial transpose  $\rho^{T_A}$  (or equivalently  $\rho^{T_B}$ ) has a negative eigenvalue.

The *trace norm*  $\|\cdot\|_1$  of an operator is the sum of the absolute values of its eigenvalues. Then the *negativity* is defined by

$$\mathcal{N}(\hat{\rho}) = \frac{1}{2} (\|\hat{\rho}^{T_A}\|_1 - 1) = \left| \sum_i \lambda_i \right|, \quad (1.46)$$

where the sum runs only over the negative eigenvalues. By the partial transpose criterion, the system is then entangled if  $\mathcal{N}(\hat{\rho}) > 0$ . Unfortunately, the negativity is not additive, so that often one also defines the logarithmic negativity,  $\log(\mathcal{N}(\hat{\rho}))$ , which is greater than 1 for entangled systems. The negativity and logarithmic negativity are examples of entanglement monotones (see below).

### 1.5.4 Entanglement Monotones and Entanglement of Formation

For a bipartite system, an entanglement monotone is a function  $E(\hat{\rho})$  of the state  $\hat{\rho}$  that satisfies the following conditions:

- (i)  $E(\hat{\rho}) = 0$  if  $\hat{\rho}$  is separable.
- (ii)  $E(\hat{\rho})$  is invariant under local unitary transformations:

$$(U_A \otimes U_B) \hat{\rho} (U_A \otimes U_B)^\dagger = \hat{\rho}. \quad (1.47)$$

- (iii)  $E(\hat{\rho})$  cannot be increased by LOCC operations.

A particularly useful entanglement monotone is the *von Neumann entropy*,  $S(\hat{\rho})$ , which was defined in Sect. 1.3. In classical systems, the Shannon information gives a measure of the spread of the probability distribution over the possible states of the system. Similarly, in quantum systems, the spread of the probability distribution of measurement outcomes for an entangled state is given by  $S(\hat{\rho})$ . Once the system is put into Schmidt form,  $S(\hat{\rho})$  can then be computed from it:

$$S(\hat{\rho}) = -\text{Tr} \hat{\rho} \log_2 \hat{\rho} = -\sum_i \lambda_i \log_2 \lambda_i. \quad (1.48)$$

The von Neumann entropy is a measure of the mixedness of a state:  $S(\hat{\rho}) = 0$  for a pure state  $\hat{\rho} = |\psi\rangle\langle\psi|$  and attains a maximum of  $\log_2 d$  for the maximally mixed state  $\hat{\rho} = \frac{1}{d} \hat{I}$ . Like the classical Shannon entropy, the von Neumann entropy is a measure of the information gained by measurement of the state.  $S(\hat{\rho})$  is the unique entanglement monotone that in addition to the three requirements above, also satisfies the additional desirable conditions that it is additive,

$$S(\hat{\rho}^{\otimes n}) = nS(\hat{\rho}), \quad (1.49)$$

and that asymptotically the entropy of products of copies of the state converge to the entropy of a projection operator: if  $\langle \psi^{\otimes n} | \hat{\rho}_n | \psi^{\otimes n} \rangle \rightarrow 1$  as  $n \rightarrow \infty$  for some joint state  $\hat{\rho}_n$  of  $n$  pairs, then

$$\frac{1}{n} |S(|\psi^{\otimes n}\rangle\langle\psi^{\otimes n}|) - S(\hat{\rho}_n)| \rightarrow 0. \quad (1.50)$$

The latter property is called asymptotic continuity.

Another entanglement monotone, which works for mixed as well as pure states, is the *entanglement of formation*,

$$E_F(\hat{\rho}) = \text{Min}_{p_i, |\psi_i\rangle} \sum_i p_i S(\hat{\rho}_i) \quad (1.51)$$

where the minimum runs over all possible partitions of the system as statistical mixtures of pure states  $\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|$ . For a pure state, this reduces to

$$E_F(\hat{\rho}) = S(\hat{\rho}_A) = S(\hat{\rho}_B). \quad (1.52)$$

For a product state, this vanishes because  $\hat{\rho}_A$  and  $\hat{\rho}_B$  are pure states. If the subsystems are maximally mixed, then the resulting maximally entangled composite state has

$$E_F(\hat{\rho}) = S\left(\frac{1}{N}I\right) = 1. \quad (1.53)$$

$E_F$  is not additive in general. The limit  $\lim_{n \rightarrow \infty} E_F(\rho^{\otimes n})/n$  is equal to the entanglement cost ( $E_C$ ), the maximum rate at which blocks of two-qubit maximally entangled states can be converted into blocks of copies of  $\hat{\rho}$  by LOCC operations. The entanglement of formation is also related to the concurrence (Sect. 1.5.6).

### 1.5.5 Quantum Discord

*Quantum discord* [34–37] is another measure of quantum correlation that differs from entanglement. It measures the effect on the mutual information shared by subsystems of a composite quantum system when measurements are made on one of the subsystems.

Classically, there are two equivalent definitions of the mutual information, given by the two expressions

$$I(X : Y) = H(X) + H(Y) - H(X, Y) \quad (1.54)$$

$$J(X : Y) = H(X) - H(X|Y), \quad (1.55)$$

where  $H(X)$  is the Shannon entropy and  $H(X|Y)$  is the conditional information. Substitution of the definition

$$H(X|Y) = H(X, Y) - H(Y) \quad (1.56)$$

into  $J(X : Y)$  immediately verifies that  $I(X : Y) = J(X : Y)$ . However, in the quantum case this is no longer necessarily true: The fact that measuring one variable affects the value of the other means that the two expressions may give different results. The discord measures the difference between the quantum analogs of  $I(X : Y)$  and  $J(X : Y)$ .

In the quantum case, the difference between the von Neumann entropies of a composite system  $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$  and of its subsystems,

$$I(\hat{\rho}) = S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}) \quad (1.57)$$

is the quantum analog of the mutual information. This vanishes if the two subsystems are statistically independent and is positive if there is any correlation between them. A second function, which reduces to the same result for classical systems, may be defined as

$$J(\hat{\rho}) = S(\hat{\rho}_A) - S\left(\hat{\rho} \left| \left\{ \Pi_B^j \right\} \right.\right), \quad (1.58)$$

where  $\left\{ \Pi_B^j \right\}$  is a set of projective measurements on B. The quantum conditional entropy is defined as

$$S\left(\hat{\rho} \left| \left\{ \Pi_B^j \right\} \right.\right) = \sum_j p_j S(\hat{\rho}_{A|j}), \quad (1.59)$$

where  $\hat{\rho}_{A|j}$  is the reduced density matrix of subsystem A after the measurements on B, and  $p_j$  is the probability of outcome  $j$  when a measurement is made. The quantum discord is then the difference between these two expressions:

$$\mathcal{D}_A(\hat{\rho}) = I(\hat{\rho}) - \text{Max}\left(J(\hat{\rho})\right), \quad (1.60)$$

where the maximum is over all sets of possible measurements  $\left\{ \Pi_B^j \right\}$  on B.  $\mathcal{D}_A(\hat{\rho})$  vanishes on separable states, and on pure states it is equal to the entropy of entanglement. Note also that the definition is asymmetric: in general,  $\mathcal{D}_A(\hat{\rho}) \neq \mathcal{D}_B(\hat{\rho})$ . The discord is often difficult to calculate, due to the maximization required, but it can be computed for some special states; for references to a number of these, see the review [38].

Ghost imaging (Chap. 6), was initially believed to require entanglement, but was later found to be possible with non-entangled, but classically-correlated, light. Recent analysis has indicated that, despite the absence of entanglement, the weaker condition of nonzero quantum discord may still play a role in this semi-classical ghost imaging [39]. More generally, nearly all non-entangled, “classically”—correlated two-mode Gaussian states still have nonzero quantum discord [40]. The role of quantum discord in quantum information processing and in correlation sharing has been discussed in [41–43], and its relation to some other related measures of quantum correlation is discussed in [44].

### 1.5.6 Concurrence

The concurrence  $C(\hat{\rho})$  is again a measure of bipartite entanglement, and is defined by

$$C(\hat{\rho}) = \langle \psi | \tilde{\psi} \rangle \quad (1.61)$$

$\hat{\rho} = |\psi\rangle\langle\psi|$ , where  $|\tilde{\psi}\rangle = \sigma_2 \otimes \sigma_2 |\psi^*\rangle$ .  $|\tilde{\psi}\rangle$  is called the spin-flipped state. For mixed states,

$$C(\hat{\rho}) = \text{Min}(C(|\psi\rangle)) = \text{Max}\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (1.62)$$

where the minimum is taken over all possible ensemble decompositions of  $\hat{\rho}$  and the  $\lambda_i$  are the square roots of the non-negative eigenvalues of  $\rho\tilde{\rho}$ , listed in descending order by size.

It can be shown that the concurrence provides an upper bound for the negativity,  $\mathcal{N}(\hat{\rho}) \leq C(\hat{\rho})$ , and it is related to the entanglement of formation by

$$E_f(\hat{\rho}) = h(C(\hat{\rho})), \quad (1.63)$$

where

$$h(x) = -x \log_2 x - (1-x) \log_2 (1-x) \quad (1.64)$$

is the binary entropy function.

For a bipartite two-level system in a pure state of the form

$$|\psi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle, \quad (1.65)$$

the concurrence takes the simple form

$$C = 2|ad - bc|, \quad (1.66)$$

which is clearly non-negative.

### 1.5.7 Tangle

Most of the quantities introduced so far detect entanglement for bipartite systems but fail for systems with three or more qubits. One quantity that can detect entanglement in three-qubit systems is the *tangle*,  $\tau$ . For a bipartite system, the tangle is simply the square of the concurrence,  $\tau_2(A, B) = C^2(\hat{\rho}_{AB})$ . For a tripartite system let  $\hat{\rho}_i$  be the density operator for the  $i$ th subsystem ( $i = A, B, C$ ), obtained by tracing over the other two. The bipartite concurrence

$$C_{i(jk)} = \sqrt{2 - 2 \text{Tr}(\hat{\rho}_i^2)} \quad (1.67)$$

can measure the entanglement of the  $i$ th subsystem with the composite system formed by the other two, but it does not distinguish between the possibility that subsystem  $i$  is entangled with only *one* of the other two subsystems and is separable from the other. The tangle, on the other hand, measures true three-way entanglement by subtracting off any bipartite entanglement with the other subsystems:

$$\tau_3(i, j, k) = C_{i(jk)}^2 - C_{ij}^2 - C_{ik}^2 \equiv \tau_2(i : j, k) - \tau_2(i, j) - \tau_2(i, k). \quad (1.68)$$

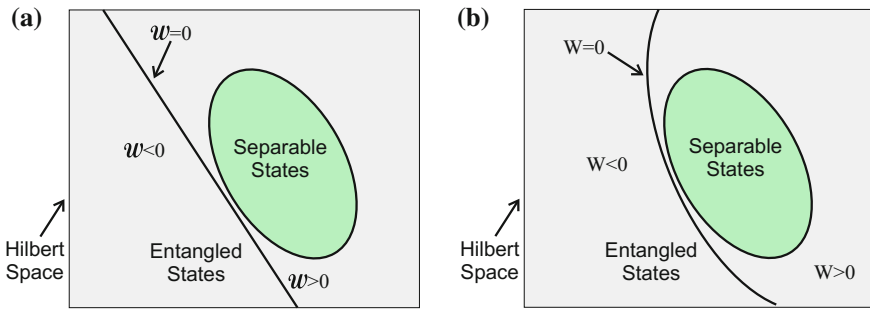


**Fig. 1.4** **a** A linear entanglement witness. The separable states form a convex subspace within the full Hilbert space. The hyperplane on which the entanglement witness  $\mathcal{W}$  vanishes divides the Hilbert space into two disjoint pieces, with the separable subspace on the  $\mathcal{W} \geq 0$  side. In general, entangled states may have positive or negative values of  $\mathcal{W}$ . **b** A nonlinear entanglement witness still divides the Hilbert space into two disjoint portions, but the boundary between them is now a curved surface, which excludes more entangled states from the  $\mathcal{W} > 0$  side

### 1.5.8 Entanglement Witnesses

An entanglement witness [45–48] does not provide a quantitative measure of the amount of entanglement, but simply gives (sometimes) a yes or no answer to the question of whether or not a state is entangled. Specifically, an *entanglement witness*  $\mathcal{W}$  is any scalar function on the system's Hilbert space that obeys  $\mathcal{W} > 0$  for all separable states.  $\mathcal{W} \leq 0$  is therefore a signal of entanglement. Often, the function  $\mathcal{W}$  is defined in terms of some operator  $\hat{W}$ , such that  $\mathcal{W}(\hat{\rho}) = \text{Tr}(\hat{\rho}\hat{W})$ . Although  $\mathcal{W} \leq 0$  is a sufficient condition for entanglement, it is not a necessary condition. In general, when  $\mathcal{W} > 0$ , nothing can be said regarding entanglement or separability. However, in some cases a *strong entanglement witness* may be found, which provides both a necessary *and* sufficient condition for entanglement: a strong witness satisfies  $\mathcal{W} \leq 0$  if and only if the state is entangled. Often a witness is strong only for a certain category of states (Gaussian states, for example), but not for others.

An entanglement witness divides the Hilbert space into two disjoint parts, with all of the entangled states on one side (Fig. 1.4). Unless  $\mathcal{W}$  is a strong witness, there are entangled states with  $\mathcal{W} > 0$  that fall on the same side of the plane as all of the separable states. One witness,  $\mathcal{W}_1$ , is called *finer* than another,  $\mathcal{W}_2$ , if  $\mathcal{W}_1$  excludes more of the entangled states from the positive side than  $\mathcal{W}_2$ . Much early work on entanglement witnesses focused on linear witnesses, which divide the Hilbert space into parts by slicing it with a planar surface, as in Fig. 1.4. In recent years, much work has been done to produce finer entanglement witnesses by using nonplanar surfaces to divide the space; such witnesses are referred to as *nonlinear* entanglement witnesses [49, 50].

## 1.6 Entanglement in Practice: Spontaneous Parametric Down Conversion

### 1.6.1 The Biphoton State

The most reliable and versatile means of producing entangled photon pairs is via *spontaneous parametric down conversion* (SPDC) inside a nonlinear crystal, such as  $\beta$ -barium borate (BBO) or potassium titanyl phosphate (KTP). In this process, a high frequency incoming photon (the *pump*) is converted into a pair of lower frequency outgoing photons (known for historical reasons as the *signal* and *idler* photons). Although the signal and idler beams are individually in spatially and temporally incoherent thermal states, memory of the pump coherence remains hidden in them; when the two photons are jointly measured via coincidence detection, the coherence properties of the pump become apparent in the resulting second-order interference pattern. And, although each beam individually has strong intensity (photon-number) fluctuations, the fluctuations are identical in the two beams. The entangled two-photon outgoing state is often called a *biphoton* state.

The photons within a given pair are entangled in a number of different variables, including position, momentum, frequency, time, polarization, and orbital angular momentum. In fact, the eigenstates of these multiple variables for the two photons are intertwined with each other through entanglement, a phenomenon known as *hyperentanglement* [51–53]; for example, the signal polarization and idler momentum can be jointly entangled. Note further that the output is entangled in both continuous *and* discrete degrees of freedom, allowing SPDC to be used as a source for a wide range of entanglement-based effects. In this section, the entangled two-photon state produced by down conversion is discussed. The actual down conversion process in the crystal, with emphasis on phase matching, is discussed in more detail in Appendix D. For further information on the physics of biphoton states and their applications, see also [14, 54, 55].

When an electric field is applied to a material with a nonlinear response, the polarization may be expanded in powers of the field. Here, we concentrate on the second-order term,  $\hat{P}_i^{(2)} = \chi_{ijk}^{(2)} \hat{E}_j \hat{E}_k$ , where the indices label spatial components and repeated indices are summed over. The corresponding interaction Hamiltonian is

$$\hat{H}_{\text{int}}(t) = \epsilon_0 \int d^3r \hat{\mathbf{P}}^{(2)} \cdot \hat{\mathbf{E}} = \epsilon_0 \chi_{jkl}^{(2)} \int d^3r \hat{E}_{pj} \hat{E}_{sk} \hat{E}_{il}. \quad (1.69)$$

The labels p, s, i have been added to distinguish the pump, signal and idler fields. Each field may be expanded in terms of plane wave components (see Appendix B),

$$\hat{E}_j(\mathbf{r}, t) = \int d^3k \left[ \hat{E}_j^-(\mathbf{k}) e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})} + \hat{E}_j^+(\mathbf{k}) e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \right], \quad (1.70)$$

where, for quantization volume  $V$ , the positive and negative frequency parts are given in terms of creation and annihilation operators by

$$\hat{E}_j^{(-)}(\mathbf{k}) = i\sqrt{\frac{2\pi\hbar\omega}{V}}\hat{a}_j^\dagger(\mathbf{k}), \quad \hat{E}_j^{(+)}(\mathbf{k}) = -i\sqrt{\frac{2\pi\hbar\omega}{V}}\hat{a}_j(\mathbf{k}). \quad (1.71)$$

Substituting Eqs. (1.70)–(1.71) into Eq. (1.69) and keeping only the terms that give a nonzero result when wedged between a one-photon incoming state and two-photon outgoing state, the result is:

$$\begin{aligned} \hat{H}_{\text{int}}(t) = C \int d^3k_s d^3k_i e^{i(\omega_p - \omega_s - \omega_i)t} \int_0^L dz e^{i(k_{pz} - k_{sz} - k_{iz})z} \\ \times \int_A d^2r_\perp e^{i(\mathbf{k}_{s\perp} + \mathbf{k}_{i\perp}) \cdot \mathbf{r}_\perp} \hat{a}^\dagger(\mathbf{k}_s) \hat{a}^\dagger(\mathbf{k}_i) + h.c., \end{aligned} \quad (1.72)$$

where h.c. indicates a term which is the Hermitian conjugate of the displayed term. Here, it has been assumed that the incoming intensity is high enough to treat the pump as a classical field, so that it was replaced by a  $c$ -number constant instead of being treated as an operator. It has also been assumed that the pump is a plane wave aligned along the  $z$ -axis, with no transverse momentum.  $\mathbf{r}_\perp$  is a two-dimensional vector in the plane perpendicular to the  $z$ -axis. In addition, the  $\sqrt{\omega}$  terms coming from Eq. (1.71) are very slowly varying compared to the exponentials, and so were treated as constants. All of the overall constants have been swept into a single constant,  $C$ .  $L$  is the length of the crystal in the  $z$  direction, and  $A$  is the area of the interaction region, i.e. the region of the crystal where the pump is intense enough for significant down conversion to take place. Since the interaction area  $A$  is normally much larger than the wavelength, it is common to approximate by taking  $A \rightarrow \infty$ , making the transverse integral trivial:

$$\int_A d^2r_\perp e^{i(\mathbf{k}_{s\perp} + \mathbf{k}_{i\perp}) \cdot \mathbf{r}_\perp} = (2\pi)^2 \delta^{(2)}(\mathbf{k}_{s\perp} + \mathbf{k}_{i\perp}). \quad (1.73)$$

Defining the longitudinal momentum mismatch,  $\Delta k = k_{pz} - k_{sz} - k_{iz}$ , the  $z$  integration may also be carried out:

$$\Phi(\Delta k, L) \equiv \int_0^L dz e^{i\Delta k z} \equiv 2e^{i\Delta k L} \frac{\sin \frac{\Delta k L}{2}}{\Delta k L} = e^{i\Delta k L} \text{sinc}\left(\frac{\Delta k L}{2}\right), \quad (1.74)$$

where the sinc function is defined by  $\text{sinc}(x) = \frac{\sin x}{x}$ . In the limit of a long crystal,  $L \rightarrow \infty$ , this phase-matching function becomes a delta function for the longitudinal momenta:  $\lim_{L \rightarrow \infty} \Phi(\Delta k, L) = \pi \delta(\Delta k)$ .

The result, finally, is that the relevant part of the interaction Hamiltonian may be written as

$$H_{\text{int}}(t) = C' \int d^3 k_s d^3 k_i \Phi(\Delta k L) e^{i(\omega_p - \omega_s - \omega_i)t} \delta^{(2)}(\mathbf{k}_{s\perp} + \mathbf{k}_{i\perp}) \hat{a}^\dagger(\mathbf{k}_s) \hat{a}^\dagger(\mathbf{k}_i) + h.c. \quad (1.75)$$

This, of course, must be supplemented by the appropriate dispersion relations connecting the frequencies to the wavevectors in the birefringent crystal. The resulting *phase-matching conditions* (equivalent to energy-momentum conservation) that must be satisfied by the outgoing fields are thus dependent on the polarizations of the photons. The down conversion is *Type I* if the signal and idler have the same polarization (opposite to the pump), and *Type II* if the signal and idler have opposite polarizations to each other. Henceforth in this section, we assume Type II parametric down conversion,  $e \rightarrow \{e, o\}$ , with  $o$  being the idler and with the pump and signal both being  $e$ -polarized. ( $o$  and  $e$  denote ordinary and extraordinary polarizations.)

A down conversion time-scale can be defined,

$$\tau_- = ((u_o)^{-1} - (u_e)^{-1}) L, \quad (1.76)$$

where  $L$  is the thickness of the crystal, while  $u_o$ , and  $u_e$  are the group velocities of the ordinary and extraordinary polarizations inside the crystal. This is the maximum time difference between signal and idler photons leaving the crystal. Then, similar to the momentum distribution function of Eq. (1.74), one can define a spectral function describing the frequency distribution of the down conversion pairs:

$$\Phi(\omega) = \text{sinc}\left(\frac{1}{2}\tau_-\omega\right), \quad (1.77)$$

where  $\omega$  is the deviation from the central frequency  $\omega_0$ .

For a weak interaction Hamiltonian  $\hat{H}_{\text{int}}$  which is only nonzero for times in some interval  $-T < t < T$ , perturbation theory tells us that  $\hat{H}_{\text{int}}$  transforms an initial vacuum state (before the interaction)  $|vac\rangle$  into a new state  $|\Psi\rangle$  afterwards, given by

$$|\Psi\rangle = e^{-\frac{i}{\hbar} \int_{-T}^T dt \hat{H}_{\text{int}} t} |vac\rangle = \left(1 - \frac{i}{\hbar} \int_{-T}^T dt H_{\text{int}} + \dots\right) |vac\rangle. \quad (1.78)$$

Taking  $T \rightarrow \infty$  (the time for the pump to cross the crystal is much longer than the characteristic time scale of the nonlinear interactions), the time integration becomes  $\int_{-\infty}^{\infty} dt e^{i(\omega_p - \omega_s - \omega_i)t} = 2\pi\delta(\omega_p - \omega_s - \omega_i)$ . Using the Hamiltonian of Eq. (1.75), we have the biphoton state:

$$|\Psi\rangle = -\frac{iC'}{\hbar} \int d^3 k_e d^3 k_o \delta(\omega_e + \omega_o - \omega_p) \times \delta^{(2)}(\mathbf{k}_{s\perp} + \mathbf{k}_{i\perp}) \Phi(\Delta k L) \hat{a}_H^\dagger(\mathbf{k}_e) \hat{a}_V^\dagger(\mathbf{k}_o) |vac\rangle + \dots, \quad (1.79)$$

where the dots represent the vacuum term and the higher-order ( $>2$  photon) terms that have been dropped. Using the dispersion relations, the  $k$  integrations may be

rewritten as frequency integrations, so that up to constants the two-photon part of the state may also be written:

$$|\Psi\rangle = \int d\omega_e d\omega_o e^{-i\Delta k L/2} E(\omega_e + \omega_o) \Phi\left(\frac{\omega_e - \omega_o}{2}\right) |\omega_e\rangle_H |\omega_o\rangle_V, \quad (1.80)$$

$$\equiv \int d\omega_e d\omega_o e^{-i\Delta k L/2} \mathcal{A}(\omega_e, \omega_o) |\omega_e\rangle_H |\omega_o\rangle_V \quad (1.81)$$

where the situation has been generalized to include a non-monochromatic pump with spectral envelope  $E(\omega_p) = E(\omega_e + \omega_o)$ .

### 1.6.2 Entanglement in SPDC

Due to the nonfactorability of  $\Phi(\omega_e, \omega_o) = \Phi(\frac{\omega_e - \omega_o}{2})$  into a product of the form  $\Phi_e(\omega_e) \cdot \Phi_o(\omega_o)$ , the state of Eq. (1.80) is clearly entangled in terms of the various frequency states. It is also entangled in polarization; in particular, if the frequencies are held fixed (by means of filters, for example), then for type II SPDC, the polarization part of the biphoton state is  $|\Psi\rangle = \frac{1}{\sqrt{2}} (|H\rangle_s |V\rangle_i + |V\rangle_s |H\rangle_i)$ . The latter is a realization of the Bell state  $|\Psi^+\rangle$ , i.e. a maximally entangled bipartite state.

A general frequency-entangled state of oppositely-polarized photons (not necessarily created through down conversion) can be written

$$|\psi\rangle = \int \mathcal{A}(\omega_1, \omega_2) a_H^\dagger(\omega_1) a_V^\dagger(\omega_2) |vac\rangle_H |vac\rangle_V d\omega_1 d\omega_2, \quad (1.82)$$

and the spectral amplitude  $\mathcal{A}(\omega_1, \omega_2)$  can then be decomposed into Schmidt form:

$$\mathcal{A}(\omega_1, \omega_2) = \sum_n \sqrt{\lambda_n} \psi_n(\omega_1) \phi_n(\omega_2). \quad (1.83)$$

The eigenvalues and eigenfunctions  $\lambda_n$ ,  $\psi_n$ , and  $\phi_n$  are solutions to the integral equations

$$\int K_1(\omega, \omega') \psi_n(\omega') d\omega' = \lambda_n \psi_n(\omega) \quad (1.84)$$

$$\int K_2(\omega, \omega') \phi_n(\omega') d\omega' = \lambda_n \phi_n(\omega), \quad (1.85)$$

with integral kernels

$$K_1(\omega, \omega') = \int \mathcal{A}(\omega, \omega_2) \mathcal{A}^*(\omega', \omega_2) d\omega_2 \quad (1.86)$$

$$K_2(\omega, \omega') = \int \mathcal{A}(\omega_1, \omega) \mathcal{A}^*(\omega_1, \omega') d\omega_1. \quad (1.87)$$

The eigenfunctions  $\psi_n$  and  $\phi_n$  define a new set of effective creation operators for horizontally and vertically polarized photons,

$$\hat{b}_n^\dagger = \int \psi_n(\omega_1) \hat{a}_H^\dagger(\omega_1) d\omega_1, \quad \hat{c}_n^\dagger = \int \phi_n(\omega_2) \hat{a}_V^\dagger(\omega_2) d\omega_2. \quad (1.88)$$

In terms of these, the Schmidt decomposition of the biphoton state can be written as

$$|\psi\rangle = \sum_n \sqrt{\lambda_n} \hat{b}_n^\dagger \hat{c}_n^\dagger |vac\rangle_H |vac\rangle_V. \quad (1.89)$$

For SPDC, the amplitude may be split into a pump envelope and a phase-matching function  $\Phi$ :  $\mathcal{A}(\omega_1, \omega_2) = \tilde{E}(\omega_1 + \omega_2) \Phi(\omega_1, \omega_2)$ . The eigenvalues for this case have been calculated [56] and it is found that the terms in the sums of Eqs. (1.83) and (1.89) drop rapidly in size, leaving only a small number of non-negligible eigenvalues. The *effective Schmidt number*  $K$  of the spectrally-entangled system may therefore be relatively small. For the parameter values used in [56], 96 % of the state are accounted for by the first six eigenvalues. The von Neumann entropy computed from these first six eigenvalues is  $S = 1.4$ , compared to the large  $K$  limit of 1.8. By narrowing the bandwidth, correlations between the spectral components increases, so that the von Neumann entropy and the effective Schmidt number both increase. Bandwidth, therefore, determines the effective level of entanglement available for use.

Rather than frequency entanglement, entanglement in other degree of freedom can be quantified in similar manner; for example, the spatial entanglement carried by the momentum vectors has also been investigated [57]. Let  $\mathbf{k}$  and  $\mathbf{q}$  be the transverse spatial momenta of the two photons, perpendicular to the direction of the pump beam (the  $z$ -axis). Approximating the biphoton amplitude in transverse momentum space to be of Gaussian form,

$$\mathcal{A}(\mathbf{k}, \mathbf{q}) = E(\mathbf{k} + \mathbf{q}) \Phi(\mathbf{k} - \mathbf{q}) = C e^{-\frac{|\mathbf{k}+\mathbf{q}|^2}{\sigma^2}} e^{-b^2|\mathbf{k}-\mathbf{q}|^2}, \quad (1.90)$$

allows analytic calculation of the Schmidt number. Here, the two terms represent the pump envelope and the phase matching function in momentum space. The exact Schmidt number can then be determined [57]:  $K = \frac{1}{4} \left( b\sigma + \frac{1}{b\sigma} \right)^2$ . The degree of entanglement in this approximation depends only on  $b\sigma$ , the ratio of widths of the two exponentials.  $K$  becomes large whenever  $b\sigma \gg 1$  or  $b\sigma \ll 1$ , with a minimum at  $b\sigma = 1$ .

A more realistic approximation for the amplitude is given by replacing the second exponential (the phase-matching term) of the Gaussian form by a sinc function:

$$\mathcal{A}(\mathbf{k}, \mathbf{q}) = E_p(\mathbf{k} + \mathbf{q}) \Phi(\mathbf{k} - \mathbf{q}) = C e^{-\frac{|\mathbf{k}+\mathbf{q}|^2}{\sigma^2}} \text{sinc}(b^2|\mathbf{k}-\mathbf{q}|^2), \quad (1.91)$$

where  $b^2 = L/4k_{\text{pump}}$  and  $C$  is a constant. The Schmidt number now has to be calculated numerically, but the result is qualitatively similar to the Gaussian model, with  $K$  becoming large whenever  $b\sigma$  is either much larger or much smaller than 1 [57]. Spatial entanglement can, therefore, be increased by increasing the transverse momentum spread. For some parameter ranges, the effective number of states  $K$  can be in the hundreds, but not all of these states are necessarily accessible; this issue is discussed further in Sect. 1.8.

The analysis of [57] has been generalized [58], showing that a one-dimensional Schmidt number  $K_{1d}$  can be calculated for photon pairs confined to propagate in a single plane, and that the full two-dimensional Schmidt number for the perpendicular plane is simply  $K_{2d} = K_{1d}^2$ . Adding in the effect of a finite-sized detection aperture (diameter  $a$ ) leads to:

$$K_{2d} = \frac{(1/\sigma^2 + b^2 + a^2)^2}{(1/\sigma^2 + b^2 + a^2)^2 - \left(\frac{1}{\sigma^2} - b^2\right)^2}. \quad (1.92)$$

This decreases asymptotically to  $K_{2d} = 1$  as  $a \rightarrow \infty$ , demonstrating the role of spatial filtering by the detector. This should serve as a reminder that in quantum mechanics the degree of entanglement and the information content are not entirely intrinsic to the system being measured, but also depend on the measurements being made.

It can be shown [59] that for transverse spatial modes in the quasi-homogeneous approximation the Schmidt number can be written in a form analogous the étendue [60] of an optical system:

$$K = \frac{1}{\lambda^2} \frac{[\int I_S(\mathbf{x})d\mathbf{x}]^2}{\int I_S^2(\mathbf{x})d\mathbf{x}} \times \frac{[\int I_{FF}(\boldsymbol{\theta})d\boldsymbol{\theta}]^2}{\int I_{FF}^2(\boldsymbol{\theta})d\boldsymbol{\theta}}, \quad (1.93)$$

where  $I_S$  and  $I_{FF}$  are the near-field (source) and far-field intensities. Thus intensity measurements in two planes suffice to determine the Schmidt number experimentally.

It is found that the Schmidt number for biphoton states depends strongly on the properties of both the pump beam and the crystal. For some parameter ranges, it can be extremely large; for example, in the experiment of [61], the number of product states superposed in the outgoing spatially-entangled biphoton state was  $K \sim 1400$ . In contrast, we have seen that for the parameter values considered in [56], the effective number of polarization-entangled terms was very small, on the order of  $K \approx 2$ . This is one of the reasons that down conversion is such an important source for optical experiments: By appropriately tuning the input parameters or by measuring different variables, the experimenter can exert a great deal of control over the output state and can vary many of the properties (such as degree of entanglement) of the biphoton state over a very wide range.

## 1.7 Other Sources of Single Photons and Entangled-Photon Pairs

In addition to photon pairs, it is useful for many applications to be able to produce single photons. The standard means of doing this is to strongly attenuate a coherent state (a laser beam), until the probability of finding two photons in a single pulse is sufficiently low to treat the pulses as single photons.

Although attenuated coherent states and SPDC are the most common means of creating single photon pulses and entangled photon pairs, other methods are also available, each with its own advantages and disadvantages in a given setting. In order for them to be sufficiently well-separated to be treated as distinct entities from their neighbors, these photons or photon pairs must be emitted from their source with sub-Poisson (antibunched) statistics (see Sect. 2.2.2). In this section, we briefly mention some other methods for producing single photons and entangled photon pairs and provide references for those seeking further detail.

Note that sources of entangled photon pairs can always be used as single-photon sources [62]. When a pair is produced, one can be sent to a detector in order to herald the existence of the other photon, which can then be used for the application at hand. For this to work, it is necessary to have some means to ensure that the detector is receiving *only* one photon; the simplest means of doing this is to use photon-counting detectors.

For additional general discussion of methods for entangled photon pair generation, see also [63]. More detailed surveys of single-photon sources may be found in a number of reviews, such as [64–67].

### 1.7.1 Atomic Cascades

The earliest source of entangled photon pairs was from two-photon decay cascades [68–71] in atoms, such as mercury or, more commonly, calcium. Such sources were used for the first tests of the CHSH inequalities [12, 13].

The situation for calcium is illustrated in Fig. 1.5. In the ground state, calcium has two electrons in its outermost, partially filled shell; these electrons are in  $s$  states, with oppositely aligned spins, so that the total angular momentum is  $J = 0$  (a singlet state). Consider then an electron excited to a higher energy singlet state. Intermediate between the ground and excited singlet states, there is a  $J = 1$  triplet state. The excited electron can decay in two steps, via this intermediate state. Two photons are emitted in the process. These photons may be emitted in any direction, but if we look only at photons emitted back-to-back, i.e. look at coincidence counts from detectors separated by  $180^\circ$ , then angular momentum conservation requires the two photons to have the same circular polarization (left-handed or right-handed). The two-photon states can arise in two different ways: two left-handed photons passing through the intermediate  $m = -1$  state, or two right-handed photons with

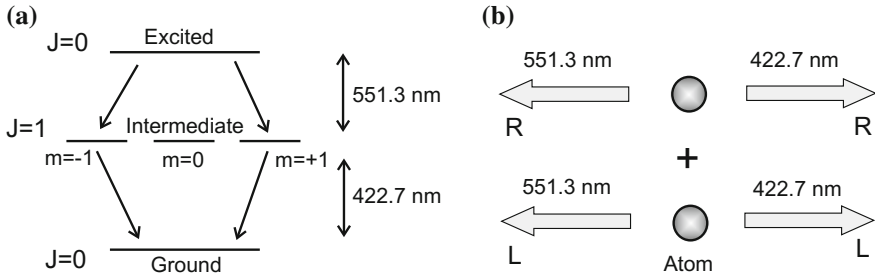


**Fig. 1.5** The production of entangled photon pairs in calcium cascades. The two-photon decay can occur via an intermediate  $m = +1$  or  $m = -1$  state. The amplitudes for the possibilities must be added, leading to polarization-entangled output states

an intermediate  $m = +1$  state. As long as the polarizations of the outgoing photons are not measured, these two possibilities are indistinguishable and they are described by an entangled state. The polarization part of the outgoing state is therefore of the form:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|LL\rangle + |RR\rangle) = \frac{1}{\sqrt{2}} (|HH\rangle - |VV\rangle). \quad (1.94)$$

Although it is possible to produce high-quality entangled states in this manner, the atomic cascade method fell out of favor when SPDC became practical for two main reasons: (i) the difficulty of manipulating single atom sources, and (ii) low event rates and correspondingly long data-collection times. The latter is due to the need to post-select pairs of photons along a single detection line when the emitted photons are going out in all directions.

### 1.7.2 Additional Atomic and Solid-State Sources

There are a number of ways to produce single photons using atomic or solid state sources. Many of these involve optically driving a quantum system via laser excitation and then obtaining the photon via fluorescence. Systems in which this can be done include single atoms or ions [72–74], dye molecules [75–79] and quantum dots [80–86].

Instead of driving the system optically, several methods have been developed in which it is instead driven electrically. One method involves using injections of single electrons and holes into etched double-barrier mesoscopic heterojunctions [87, 88]. However, this has the disadvantage that the system must be maintained at millikelvin temperatures. Another approach uses electrical pulses to stimulate recombination of excitons (electron-hole bound states) in layers of quantum dots [82, 89–92]. By using periodic triggering pulses to produce two photons on consecutive pulses, one photon can be delayed to make it impossible to distinguish which photon was produced

in which pulse; in this way, such quantum dot schemes can also produce entangled photon pairs [93]. Quantum dots can also be used to generate entangled photon states using two-exciton systems in a manner similar to the two-step atomic cascades of Sect. 1.7.1 [94].

There is currently great interest in the use of nitrogen-vacancy (NV) centers in diamonds. These are atom-like defects introduced into diamond structures, which couple strongly to several degrees of freedom, including photons, phonons, and the spins of neighboring atoms. NV centers can serve as optically driven fluorescent single-photon sources [96, 97]. Work has been done showing that in principle NV centers can also serve as sources of entangled photon pairs and  $n$ -photon entangled strings [98]. See [95] for a review of the subject.

### 1.7.3 Fiber and Photonic Crystal Sources

Periodically poled fibers and waveguides made of  $\chi^{(2)}$  nonlinear materials have been used to generate entangled photon pairs [99–103]. Several related schemes have been devised for generating entangled pairs using a Sagnac interferometer (Fig. 1.6) formed from nonlinear fibers. Two orthogonally-polarized time-delayed pump pulses are used to pump the fiber, producing signal and idler photons in the fiber. A birefringent fiber is then used to remove the temporal distinguishability of the photons [104–107].

Photonic crystals [108–110] are periodic lattices formed from nanostructures, which play a similar role for propagating photons that a usual crystal lattice plays for propagating electrons in a solid. In particular, these structures can have photonic bandgaps that control the photon frequencies that can propagate in a material and the frequencies that are absorbed. These crystals can be constructed in the form

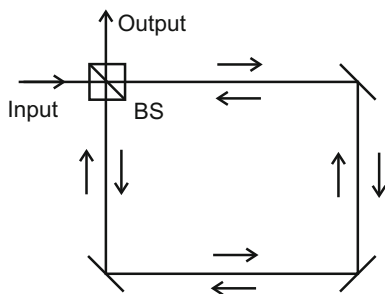


**Fig. 1.6** A Sagnac interferometer. The beam splitter causes each input photon to have amplitudes to go around the loop in either direction. Interference between these two counterpropagating amplitudes is insensitive to instabilities in the distances between the mirrors. By using a Sagnac interferometer with the propagation regions constructed from fibers made of nonlinear material, entangled photon pairs can be generated

of waveguides or fibers. The phase matching conditions that must be satisfied in nonlinear crystals in order to enable spontaneous parametric down-conversion to occur can also be arranged in photonic crystal fibers, which can then be used to generate entangled photon pairs [111–113]. One recent photonic crystal fiber source [114] has reached an intensity level about three times higher than previous fiber sources (about 2500 photons per mode per pulse) with much lower noise, by means of an argon-filled hollow-core fiber; this source produces femtosecond pulse pairs of squeezed states with strongly correlated photon number per pulse.

## 1.8 Qudits

A two-state classical system can be used to encode  $\ln_2 2 = 1$  bit of information. A qubit system can exist in a superposition of the corresponding two states, generally encoding less than that amount of information, with 1 as its limiting capacity. However, encoding more than one bit of information into a quantum system would allow more rapid information exchanges in quantum communication systems without needing to exchange more photons per unit time. For example, when a photon can be transmitted which is randomly in one of a set of  $d$  states ( $d > 2$ ), then  $\ln_2 d$  bits of information could be encoded into each photon. Such  $d$  level states are sometimes called *qudits*; special cases are known as *qutrits* for  $d = 3$  and *ququats* for  $d = 4$ . As is seen in Chap. 9, not only does the expansion from 2-state to  $d$ -state systems increase the information capacity per photon, it also increases the security of quantum mechanics-based encryption systems.

The generalization from physical qubits built from a two-dimensional effective Hilbert space spanned by states  $|0\rangle$  and  $|1\rangle$  to a  $d$ -dimensional *qudit* on a space spanned by states  $|0\rangle, \dots, |d-1\rangle$  is straightforward. The states are of the form

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \cdots + a_{d-1}|d-1\rangle, \quad (1.95)$$

with  $\sum_{i=0}^{d-1} |a_i|^2 = 1$ . There are number of ways to create such qudit states, and a great deal of current work is devoted to generating such systems rapidly and efficiently, with the information encoded robustly in a form that can survive transmission over large distances under real world conditions.

One promising possibility is to use photons with nonzero orbital angular momentum (OAM). Such states are discussed in detail in Chap. 8. Here, we simply note that the orbital angular momentum is distinct from the photon's spin angular momentum (circular polarization), its value about any axis is quantized ( $\langle \hat{L}_z \rangle = l\hbar$ , for integer  $l$ ), and the corresponding quantum number  $l$  is in principle unlimited in size. As a result, the amount of information that can be encoded per photon is also unlimited. Of course, in practice, higher  $l$  states become progressively harder to generate and detect efficiently. Moreover, the states are fragile: when transmitted through the air, they tend to be disturbed by turbulence, with a corresponding degradation of the information transfer [115–118]. Instead of transmission through the air, optical fiber

transmission can be considered, but it is also currently difficult to create multimode fibers that can transmit a wide range of  $l$  values.

A number of other realizations of qudit states have been carried out experimentally, including: polarization entangled four-photon states [119], time-energy entangled qutrits using single photon in a three-arm interferometer [120], and time-bin-entangled photons produced by a train of laser pulses [121]. An additional possibility is to create hybrid qudits from combinations of two variables. For example, a system with two polarization states and three possible OAM states can serve as a six-state qudit, allowing 2.585 bits of information transfer per photon.

## 1.9 “Local Reality” and Bell-Type Inequalities

Entangled states are highly correlated. However, the correlation is of a different nature than classical correlations [122]. In particular, inequalities may be derived that must be obeyed by all classically correlated systems, but that can be violated by some entangled quantum systems. Inequalities of this type were first developed by Bell [9], so all such inequalities are often generically called Bell inequalities. (It is interesting to note that in the 19th George Boole derived, from a mathematical logic perspective, a set of inequalities that must be obeyed by what he referred to as “logically connected events”, and that the Bell and CHSH inequalities below form a subset of these conditions. See [123, 124] for details.)

In the 1930s, Einstein, Podolsky, and Rosen (EPR) [125] argued that quantum mechanics was incomplete as a local “realistic” theory, so that a more comprehensive underlying theory that was classically deterministic would be required. In their point of view, quantum mechanics may appear to be probabilistic simply because underlying local parameters (*hidden variables*) which might determine measurement outcomes may have gone unmeasured; once these variables were taken into account, the apparent randomness of quantum theory would disappear.

To demonstrate this, EPR attempted to show that a pair of complementary variables could both be measured to arbitrary accuracy, violating the uncertainty principle. The argument is based on the consideration of a pair of entangled particles that are allowed to propagate far away from each other before being subjected to measurement at wide separation, far enough apart so that they can not causally affect each other during the duration of measurement. In contemporary terms, the corresponding laboratories would be called those of experimenters Alice and Bob. The two particles are assumed to be in a state that we would now called an entangled state involving two conjugate variables, momentum and position. For the purposes of our exposition here, let us assume that Alice and Bob measure perfectly anti-correlated values for these variables: if Alice measures position  $x$ , then Bob should measure position  $-x$ ; a similar situation holds for momentum  $p$ . Quantum mechanically, there is a nonzero lower limit to the product of the corresponding variances,  $\Delta x \Delta p$ , since  $\hat{x}$  and  $\hat{p}$  are conjugate and therefore have nonzero commutators. EPR’s line of thought points out, however, that if Alice measures  $\hat{x}$  and finds value  $x_0$ , then its Bob must

have value  $-x_0$ , while if the Bob measures  $\hat{p}$  and finds value  $p_0$ , then its Alice must have value  $-p_0$ . Under a local “realist” theory, neither measurement should disturb the values measured at the other, distant location, so that after communicating their results to each other the experimenters should now know the values of both  $\hat{x}$  and  $\hat{p}$ , to as high an accuracy as their measuring devices can achieve. In other words, there is no fundamental nonzero lower limit to the value of  $\Delta x \Delta p$ . Later, Bohm [126] provided a simpler, analogous argument, involving instead a dichotomic spin variable, as a more straightforward illustration of this situation of separated measurements of a subatomic system.

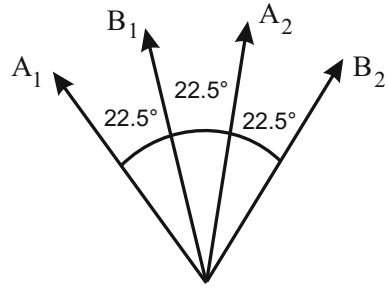
The current consensus is that the problem with such an argument in resolving the situation of distant measurements is the assumption that the variables being measured have well defined, pre-existing values before being measured. In quantum mechanics, until a measurement is made, each observable only has *probability amplitudes* for each of the possible values, rather than having a definite value. The EPR position is usually referred to as the assumption of so-called *local reality*: ‘locality’ means that events can’t affect spacetime points outside their light cone (no superluminal propagation of effects), while ‘reality’ here means that variables have pre-existing values that are simply uncovered, not created or otherwise unaccountably disturbed by measurements, cf. [127], Sect. 1.6. The standard view is, therefore, that quantum mechanics violates this second assumption and that the inability to assign well-defined values to quantum mechanical observables before they are measured is seen as the most important implication of Bell inequality violation.

In the mid 1960s, Bell realized that the assumption of “local realism” and the potential existence of local hidden variables underlying quantum mechanics could be tested experimentally, by checking whether an inequality composed from measured quantities was always obeyed. There are now a number of variants of this original Bell inequality, the best known being the Clauser–Horne–Shimony–Holt (CHSH) inequality, which is better suited to experimental testing. Here we briefly describe these two inequalities; besides the original papers, derivations may be found in many places, including [19, 127–129].

**Original Bell inequality.** We describe the Bell inequality in the context of photon polarization measurements. Consider an entangled photon source sending oppositely-polarized photons to Alice and Bob in a polarization singlet state. Let Alice and Bob separately measure the polarization of each photon they receive along some axis. Both of them randomly and independently choose one of three axes,  $\hat{\alpha}$ ,  $\hat{\beta}$ , or  $\hat{\gamma}$ , along which to make their measurement. Alice then defines a variable  $A$  which takes the value  $+1$  for polarization along the chosen axis and  $-1$  for the orthogonal polarization. Bob similarly defines a variable  $B$  for his measurement. They then communicate the results of their measurements and calculate  $E(\hat{a}, \hat{b})$ , where  $\hat{a}$ ,  $\hat{b}$  are respectively Alice’s and Bob’s axes (chosen from among  $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\}$ ), and  $E(\hat{a}, \hat{b})$  is the expectation value of the product of  $A$  and  $B$  along those axes,

$$E(\hat{a}, \hat{b}) = \langle AB \rangle. \quad (1.96)$$

**Fig. 1.7** Optimal angles between polarization measurement axes for CHSH inequality violation. At these angles, the quantum mechanical violation of the inequality is largest



Then Bell’s theorem says that for any theory with local realism, the expectation values obeys:

$$\left| E(\hat{\alpha}, \hat{\beta}) - E(\hat{\alpha}, \hat{\gamma}) \right| \leq 1 + E(\hat{\beta}, \hat{\gamma}), \quad (1.97)$$

for any choice of  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{\gamma}$ . This remains true no matter how strong the classical anti-correlations between the polarizations.

Quantum mechanics, however can violate this bound. Quantum mechanically,  $E(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b} = -\cos \phi$ , where  $\phi$  is the angle between axes  $\hat{a}$  and  $\hat{b}$ ; so for example, if  $\hat{\beta}$  and  $\hat{\gamma}$  are at  $60^\circ$  and  $120^\circ$  from  $\hat{\alpha}$ , then it is found that

$$\left| E(\hat{\alpha}, \hat{\beta}) - E(\hat{\alpha}, \hat{\gamma}) \right| = 1 \quad \text{and} \quad 1 + E(\hat{\beta}, \hat{\gamma}) = \frac{1}{2}, \quad (1.98)$$

in clear violation of the Bell inequality. This demonstrates that local hidden-variable theories are unable to mimic the predictions of quantum theory.

**CHSH inequality.** The CHSH inequality gives similar results. Alice sends her photon through a measuring apparatus, which tells her the value of some observable  $\hat{A}$ , which has values  $\pm 1$  for the two possible linear polarizations in any basis. The measuring device also has two possible settings, 1 and 2; for example, these could consist of measuring the polarizations in two different bases. Let  $A_1$  and  $A_2$  be the results of measuring  $\hat{A}$  at these two settings. In a similar manner, Bob measures a variable  $B$  of the same form on his photon, with values  $B_1$  and  $B_2$  on the two settings. Define  $E(A, B)$  to be the expectation value of the product  $AB$  over all their measurements. Then the CHSH inequality states that

$$S \equiv |E(A_1, B_1) + E(A_2, B_2) + E(A_2, B_1) - E(A_1, B_2)| \leq 2 \quad (1.99)$$

for any deterministic local hidden-variables theory. The bound is once again violated by two particles in the polarization singlet state. If the settings on the detectors represent the directions of polarization analyzers, then  $E(A, B)$  is the same as Bell’s  $E(\hat{\alpha}, \hat{\beta})$ , and the optimal violation occurs when the axes are separated by  $22.5^\circ$  as shown in Fig. 1.7; in this case,  $S = 2\sqrt{2} = 2.84$ .

It should be noted that, for a given entangled state, the inequality may be violated for some choices of axes and obeyed for other axes. Furthermore, although all separable states obey the inequality, not all entangled states violate it: Entanglement is a necessary but not sufficient condition for Bell violation.

**Experimental tests.** The first experimental demonstration of Bell violation was by Freedman and Clauser [68]; but potential loopholes left room for doubt until more definitive tests a decade later [12, 13] were considered more convincing. The violation of Bell-type inequalities is now sufficiently well-established that it is used as a tool for applications, such as testing for eavesdropping in quantum key distribution schemes (see Chap. 9). Rapid technological advances in recent decades have now made Bell inequality experiments easy enough to do in undergraduate physics labs [130].

Let  $\alpha$  and  $\beta$  be the angles of the axes  $\hat{\alpha}$  and  $\hat{\beta}$  from some fixed axis. Quantum mechanically, the probability of measuring the various possible polarizations when the polarization analyzers are in directions  $\alpha$  and  $\beta$  are given by

$$P_{HH}(\hat{\alpha}, \hat{\beta}) = P_{VV}(\hat{\alpha}, \hat{\beta}) = \frac{1}{2} \cos^2(\alpha - \beta) \quad (1.100)$$

$$P_{HV}(\hat{\alpha}, \hat{\beta}) = P_{VH}(\alpha, \beta) = \frac{1}{2} \sin^2(\alpha - \beta). \quad (1.101)$$

Therefore,

$$E(\hat{\alpha}, \hat{\beta}) = P_{HH} + P_{VV} - P_{HV} - P_{VH} \quad (1.102)$$

$$= \cos^2(\alpha - \beta) - \sin^2(\alpha - \beta) \quad (1.103)$$

$$= \cos(2(\alpha - \beta)). \quad (1.104)$$

Up to an overall constant, the probabilities give the coincidence count rates at the different settings, so that the value of  $E(\hat{\alpha}, \hat{\beta})$  can be extracted from coincidence rates:

$$E(\hat{\alpha}, \hat{\beta}) = \frac{N(\alpha, \beta) + N(\alpha_{\perp}, \beta_{\perp}) - N(\alpha_{\perp}, \beta) - N(\alpha, \beta_{\perp})}{N_{\text{total}}}, \quad (1.105)$$

where  $N(\cdot, \cdot)$  is the number of coincidence counts and  $N_{\text{total}}$  the total number of counts obtained.  $H$  and  $V$  for direction  $\alpha$  correspond respectively to polarization in the direction parallel or perpendicular to  $\alpha$ , and similarly for  $\beta$ . The total count is

$$N_{\text{total}} = N(\alpha, \beta) + N(\alpha_{\perp}, \beta_{\perp}) + N(\alpha_{\perp}, \beta) + N(\alpha, \beta_{\perp}). \quad (1.106)$$

As the angle between the polarizers is varied, the coincidence rates should oscillate. Classically, the visibility of this oscillation can never exceed  $\frac{1}{\sqrt{2}} \approx 70.7\%$ , so that any visibility greater than this indicates violation of the EPR assumptions, plus an assumption of fair sampling.

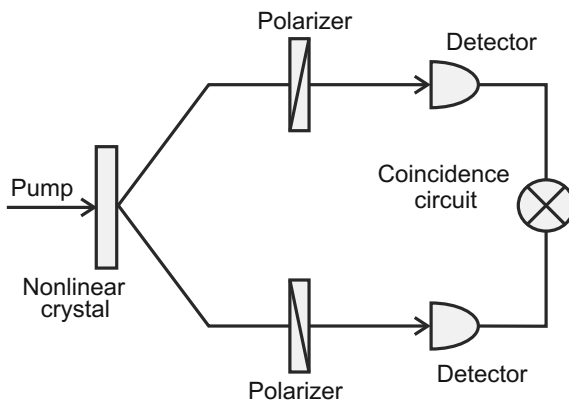


**Fig. 1.8** Schematic of a setup for Bell violation test. A source of polarization-entangled photon pairs (type II SPDC in the version shown here) sends photons through a pair of polarizers to two detectors. The two polarizers are oriented so that an angle  $\theta$  is made between their orientations. As this angle is varied, the coincidence rate between the two detectors oscillates, as shown in the following figure. The degree of Bell violation then is extracted from the visibility of the resulting oscillatory pattern of coincidences

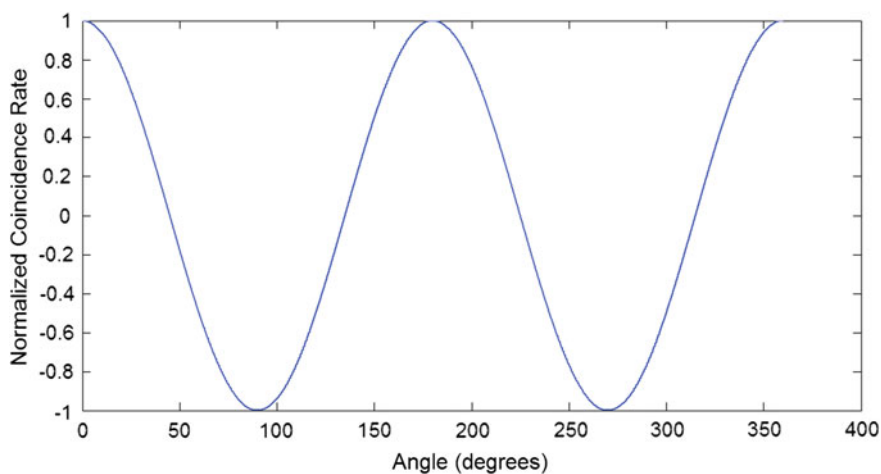


**Fig. 1.9** The coincidence rate  $P_{VV}$  oscillates as the angle between the polarizers varies in the Bell measurement setup of the previous figure

The simplest example of an experimental apparatus for testing for Bell violation is shown schematically in Fig. 1.8. As the angle between the polarizers is varied, the coincidence rate oscillates, as in Fig. 1.9.

## 1.10 Classical Versus Quantum Correlations

Entanglement implies perfect correlation (or anti-correlation) between measurements made of some property on two different, possibly widely-separated particles. But perfect correlation of widely separated random events can be arranged in a completely classical manner, as well. For example, imagine an experimenter in San Francisco sealing two balls, one red and one blue, into two boxes. The boxes are then shuffled, and one of them is randomly chosen to be flown to Tokyo. The remaining box is then flown to London. If someone opens the Tokyo box to see which color is inside, the outcome is completely random and unpredictable in advance. The same is true for the London box. However, once one box is opened, the color in the other box is known with absolute certainty (assuming nobody has tampered with the boxes). The measurements made when the two boxes are opened are ideally perfectly random, yet perfectly correlated, even though everything here is perfectly classical. So the question is: what is special about entanglement that goes beyond ordinary, classical correlation?

The answer resides in the fact that, although the properties being measured do not necessarily have definite values before measurement, as seen in the previous section, in the singlet state the correlation between the two outcomes is found regardless of the basis in which the measurement is performed; it is true for *all possible* joint measurement outcomes. For classical correlations, the correlations are only apparent in the basis in which the state was prepared.

To see this more generally, consider the following example. Suppose type I SPDC is used to prepare a polarization-entangled two-photon state in which the two polarizations are perfectly correlated:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|H\rangle_A |H\rangle_B + |V\rangle_A |V\rangle_B), \quad (1.107)$$

where  $H$  and  $V$  are the polarizations with respect the horizontal and vertical axes. If a measurement is made of the polarizations of particles  $A$  and  $B$  with respect to this basis, it is always found that the results are the same: the polarizations are perfectly correlated in this basis. Now imagine making the measurements in a different basis instead: define a unit vector  $\hat{n}$  at an angle of  $\theta$  from the  $x$  axis, and let  $|H_\theta\rangle$  and  $|V_\theta\rangle$ , respectively, represent the states polarized parallel or perpendicular to  $\hat{n}$ . The transformation between bases is simply a straightforward application of a rotation matrix:

$$|H_\theta\rangle = \cos \theta |H\rangle + \sin \theta |V\rangle \quad (1.108)$$

$$|V_\theta\rangle = -\sin \theta |H\rangle + \cos \theta |V\rangle. \quad (1.109)$$

Inverting these relations and substituting them into Eq.(1.107), it is found that the original state can also be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|H_\theta\rangle_A |H_\theta\rangle_B + |V_\theta\rangle_A |V_\theta\rangle_B). \quad (1.110)$$

It is clear that the correlation persists in the rotated basis as well.

Contrast this behavior to the classical (separable) case. Two polarizations are prepared in such a way that they are of the same polarization with respect to the vertical-horizontal basis, but the measurement polarization for each pair is chosen randomly. If measured in the original basis, the results are exactly the same as before. But in a rotated basis, each initially horizontal photon  $|H\rangle$  has a probability  $\cos^2 \theta$  of being found in the  $|H_\theta\rangle$  state and probability  $\sin^2 \theta$  of being found in state  $|V_\theta\rangle$ . (For an initial  $|V\rangle$  photon, the probabilities are reversed.) But the outcomes for two photons are independent of each other. More explicitly, if the initial state was  $|H\rangle_A |H\rangle_B$  then the following four outcomes are all possible for the outgoing state after measurement:

| State                                   | Amplitude                  | Probability                   |
|-----------------------------------------|----------------------------|-------------------------------|
| $ H_\theta\rangle_A  H_\theta\rangle_B$ | $\cos^2 \theta$            | $\cos^4 \theta$               |
| $ H_\theta\rangle_A  V_\theta\rangle_B$ | $-\sin \theta \cos \theta$ | $\sin^2 \theta \cos^2 \theta$ |
| $ V_\theta\rangle_A  H_\theta\rangle_B$ | $-\sin \theta \cos \theta$ | $\sin^2 \theta \cos^2 \theta$ |
| $ V_\theta\rangle_A  V_\theta\rangle_B$ | $\sin^2 \theta$            | $\sin^4 \theta$               |

The correlation between the two polarizations is reduced in this basis, and completely vanishes at  $\theta = 45^\circ$ , where all four outcomes are equally likely.

Thus, in the classical case the amount of correlation between the measurements depends on the basis chosen for measurement; in the quantum case, the correlation is basis-independent. This fact is used in Chap. 5, where the quantum approach to polarization mode dispersion does not require alignment of measurement axes, due to this effect. The classical case corresponds to a statistical distribution of two different states, or in other words, in a mixed state. In the quantum case, the input for each photon pair constitutes a single entangled state, formed from a coherent superposition of two amplitudes. Coherent here means that there is a stable phase difference between the amplitudes. If the phase becomes unstable, due to random phase fluctuations added by interaction with the surroundings for example, then the superposition degrades and is replaced by a mixed state.

For the most part in this book (and more generally, in most of the literature on quantum optics), “quantum correlation” and “entanglement” are used interchangeably. However, it should be kept in mind that there are other types of quantum correlation that exist even in the absence of entanglement. The most prominent example is quantum discord which, as was mentioned in Sect. 1.5.5, is often nonzero even for correlated, but separable states. See [36, 40] for more details.

A further variation on the concept of quantum correlation is the concept of *steerability*, which was first mentioned by Schrödinger [131] and was further elaborated on in [132, 133]. Entanglement is symmetric (if  $A$  is entangled with  $B$ , then  $B$  is entangled with  $A$ ), whereas steerable states can be asymmetric; if Alice and Bob share a state it is steerable if one party (say Alice) can make measurements that can nonlocally collapse the other party’s subsystem into a desired eigenstate (see

[132, 133] for a more precise definition); this collapsibility need only go one way, not necessarily both. Steerable states are intermediate between entangled and Bell-nonlocal states, in the sense that if  $B$ ,  $S$ , and  $E$  represent, respectively, the sets of Bell-nonlocal, steerable, and entangled states, then  $B \subset S \subset E$ .

In addition to the distinction between classical and quantum correlation, there is also the issue of phase-sensitive versus phase-insensitive correlations. *Phase-sensitive correlations* are based on correlations between products of the field with itself,  $\sim \langle E(x)E(y) \rangle$ , while *phase-insensitive correlations* involve correlations between a field and its complex conjugate,  $\sim \langle E^*(x)E(y) \rangle$ . It can be shown [134, 135] that thermal ghost imaging is due entirely to phase-insensitive correlations, while quantum ghost imaging is primarily due to phase-sensitive correlations, which is responsible for the different properties that appear in these two cases; ghost imaging is discussed in Chap. 6.

Classical correlations can be destroyed by noise, turbulence, and other influences, providing limits to the distances over which they remain useful. The same is true of quantum correlations. As a pair of entangled particles propagates, interactions with the environment induce a loss of quantum coherence. As a result, the degree of entanglement decreases over time. Bell violations also tend to decay for the same reasons. As a result, there is a limit to the distances over which entanglement and Bell nonlocality can be used for applications. Generally, both entanglement and Bell nonlocality decay asymptotically, with exponentially decaying values of concurrence, Bell inequality violation and other similar measures. However, it has been found that under some conditions these measures can drop to zero within a finite distance, phenomena known as entanglement sudden death (ESD) and Bell nonlocality sudden death (BNSD) [136–142]. Such effects need to be accounted for in considering the usefulness of quantum correlations in applications, especially in quantum communications over long distances or quantum computations requiring many operations. In the long term, mitigating loss of entanglement and nonlocality need to be seriously addressed for such applications [143].

One further comment should be made regarding the idea of nonlocal correlations such as those addressed by Bell inequalities. Although they involve correlations between widely-separated particles, they cannot be used to communicate or influence events in a superluminal manner. In order for these “nonlocal” correlations to be utilized in any way, it is always necessary to bring information about the two particles back together, and this is done by ordinary, classical means. In Bell inequality experiments, for example, the nonlocal correlations only become apparent when the signals at the two detectors are brought together in the coincidence-counting circuit.

## 1.11 State Discrimination in Quantum Mechanics

The security of the quantum cryptography protocols in Chap. 9 relies on the fact that two quantum states are impossible to discriminate under some circumstances. In classical physics it is in principle always possible to make a set of measurements

that unambiguously decide whether a system is in one physical state or in a different one. If the states only differ by a small amount in some of their physical variables (temperature, pressure, energy, etc.) then it may be difficult in practice to tell them apart, but given sufficiently sensitive measuring devices it is always possible.

In quantum mechanics the situation is very different. Consider two states,  $|\psi_1\rangle$  and  $|\psi_2\rangle$ . These can be unambiguously distinguished only if they are orthogonal:  $\langle\psi_1|\psi_2\rangle = 0$ . More generally, however, there may be nonzero overlap between the states:  $\langle\psi_1|\psi_2\rangle \neq 0$ ; in this case, there is no set of measurements that allow an experimenter to distinguish unambiguously between the states without loss or error. In particular, if one state is given, the error probability (the chance of concluding it is the other state) on a given measurement is given by  $P_E(\psi_1, \psi_2) = |\langle\psi_1|\psi_2\rangle|^2$ . For a mixed state,  $\hat{\rho}$ , the analogous probability of mistaking  $\hat{\rho}$  for the pure state  $|\psi\rangle$  is

$$P_E(\psi, \hat{\rho}) = \langle\psi|\hat{\rho}|\psi\rangle = \text{Tr } \hat{\rho}|\psi\rangle\langle\psi|. \quad (1.111)$$

Suppose that we are for some reason required to make a definite decision as to whether a given state is  $|\psi_1\rangle$  or  $|\psi_2\rangle$ ; we are not allowed to declare that we are undecided. Then we must try to construct a set of measurement operators that give the maximum average probability of correctly identifying the states over a series of trials. Suppose  $|\psi_1\rangle$  and  $|\psi_2\rangle$  occur with probabilities  $p_1$  and  $p_2$ , respectively. The two states can always be written as

$$|\psi_1\rangle = \cos\theta |x\rangle + \sin\theta |y\rangle \quad (1.112)$$

$$|\psi_2\rangle = \cos\theta |x\rangle - \sin\theta |y\rangle \quad (1.113)$$

for some orthogonal pair of vectors  $|x\rangle$  and  $|y\rangle$ , and some angle  $0 \leq \theta \leq \frac{\pi}{4}$ . It can be shown that the optimal error probability (the *Helstrom bound* [144]) is

$$P_E^{opt} = \frac{1}{2} \left( 1 - \sqrt{1 - 4 p_1 p_2 |\langle\psi_1|\psi_2\rangle|^2} \right) \quad (1.114)$$

This bound is achieved by using an experimental set-up arranged to detect the two states

$$|\omega_{\pm}\rangle = \frac{1}{\sqrt{2}} \left( \sqrt{1 \pm \xi} |x\rangle \pm \sqrt{1 \mp \xi} |y\rangle \right), \quad (1.115)$$

where  $\xi = \Delta \cos(2\theta) (1 + \cos^2 2\theta (\Delta^2 - 1))^{-1/2}$  and  $\Delta = p_1 - p_2$ . If  $|\omega_+\rangle$  is detected, the conclusion is that the state was  $|\psi_1\rangle$ , while detection of  $|\omega_-\rangle$  leads to the conclusion that the state was  $|\psi_2\rangle$ .

For the special case that  $p_1 = p_2 = \frac{1}{2}$ , the optimal detection states simplify to

$$|\omega_{\pm}\rangle = \frac{1}{\sqrt{2}} (|x\rangle \pm |y\rangle), \quad (1.116)$$

with optimal error probability

$$P_E^{opt} = \frac{1}{2} \left( 1 - (1 - |\langle \psi_1 | \psi_2 \rangle|^2)^{1/2} \right). \quad (1.117)$$

The discussion above assumes that there are only two possible conclusions: the state is  $|\psi_1\rangle$  or it is  $|\psi_2\rangle$ , in which case there is always some nonzero error probability as long as the states are nonorthogonal. On the other hand, it is still possible to guarantee a zero error rate, as long as the loss of some of the states is tolerated. This is done by assuming that there are *three* possible outcomes for the measurement: (i) “State =  $|\psi_1\rangle$ ”, (ii) “State is  $|\psi_2\rangle$ ”, and (iii) “State is undetermined”. In this case (known as *unambiguous state discrimination*), the starting point is to define vectors perpendicular to the states,

$$|\psi_1^\perp\rangle = \sin\theta|x\rangle - \cos\theta|y\rangle \quad \text{and} \quad |\psi_2^\perp\rangle = \sin\theta|x\rangle + \cos\theta|y\rangle. \quad (1.118)$$

Then the measurements correspond to the three detection operators

$$\Pi_1 = \frac{P_1}{|\langle \psi_2^\perp | \psi_1 \rangle|^2} |\psi_2^\perp\rangle \langle \psi_2^\perp|, \quad (1.119)$$

$$\Pi_2 = \frac{P_2}{|\langle \psi_1^\perp | \psi_2 \rangle|^2} |\psi_1^\perp\rangle \langle \psi_1^\perp|, \quad (1.120)$$

$$\Pi_? = 1 - \Pi_1 - \Pi_2. \quad (1.121)$$

The denominators  $P_{1,2}$  are the expectation values

$$P_1 = \langle \psi_1 | \Pi_1 | \psi_1 \rangle \quad P_2 = \langle \psi_2 | \Pi_2 | \psi_2 \rangle \quad (1.122)$$

in other words,  $P_i$  is the probability that the state was identified correctly, given that it was prepared as  $|\psi_i\rangle$ , for  $i = 1, 2$ . For  $p_1 = p_2$ , the optimized probability  $P_?$  for the state to be undetermined is given by

$$P_?^{opt} = |\langle \psi_1 | \psi_2 \rangle| \quad (1.123)$$

(the Ivanovic–Dieks–Peres bound) [145–147], and occurs when  $P_1 = P_2$ . The case of  $p_1 \neq p_2$  has also been studied [148, 149], and the optimal value of  $P_?$  is found to be given by the Jaeger–Shimony bound:

$$P_?^{opt} = 2\sqrt{p_1 p_2} |\langle \psi_1 | \psi_2 \rangle|. \quad (1.124)$$

More detailed discussions of quantum state discrimination (and more generally of quantum state estimation and quantum hypothesis tests) may be found in [144, 150–152].

## References

1. D.S. Simon, G. Jaeger, A.V. Sergienko, *Int. J. Quantum Inf.* **12**, 1430004 (2014)
2. D.S. Simon, *J. Sens.* Article ID 6051286 (2016)
3. J.J. Sakurai, *Modern Quantum Mechanics*, Revised edn. (Addison-Wesley, Reading, 1994)
4. G. Jaeger, *Quantum Information: An Overview* (Springer, Berlin, 2007)
5. R. Shankar, *Principles of Quantum Mechanics*, 2nd edn. (Plenum Press, New York, 2011)
6. A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, 1995)
7. D.J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edn. (Pearson Education Ltd., Essex, 2004)
8. J.S. Bell, *Rev. Mod. Phys.* **38**, 447 (1966) (Reprinted in [128])
9. J.S. Bell, *Physics* **1**, 195 (1964) (Reprinted in [128])
10. J.F. Clauser, M.A. Horne, *Phys. Rev. D* **10**, 526 (1974)
11. J.F. Clauser, A. Shimony, *Rep. Prog. Phys.* **41**, 1881 (1978)
12. A. Aspect, P. Grangier, G. Roger, *Phys. Rev. Lett.* **47**, 460 (1981)
13. A. Aspect, P. Grangier, G. Roger, *Phys. Rev. Lett.* **49**, 91 (1982)
14. Y. Shih, *Rep. Prog. Phys.* **66**, 1009 (2003)
15. E. Arthurs, J.L. Kelly Jr., *Bell Syst. Technol. J.* **44**, 725 (1965)
16. E. Arthurs, M.S. Goodman, *Phys. Rev. Lett.* **60**, 2447 (1988)
17. N.A. Peters, J.B. Altepeter, E.R. Jeffrey, D. Branning, P.G. Kwiat, *Quantum Inf. Comput.* **3**, 503 (2003)
18. Y.F. Huang, B.H. Liu, L. Peng, Y.H. Li, L. Li, C.F. Li, G.C. Guo, *Nat. Commun.* **2**, 546 (2012)
19. M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information*, 10 Anniversary edn. (Oxford University Press, Oxford, 2011)
20. S.M. Barnett, *Quantum Information* (Oxford University Press, Oxford, 2009)
21. M.M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, 2013)
22. J.A. Bergou, M. Hillery, *Introduction to the Theory of Quantum Information Processing* (Springer, New York, 2013)
23. B.W. Schumacher, *Phys. Rev. A* **51**, 2738 (1995)
24. A. Muller, T. Herzog, B. Huttner, W. Tittel, H. Zbinden, N. Gisin, *Appl. Phys. Lett.* **70**, 793 (1997)
25. R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009)
26. M. Li, S.M. Fei, X. Li-Jost, *Adv. Math. Phys.* **2010**, Article ID 301072 (2010)
27. O. Gühne, G. Tóth, *Phys. Rep.* **474**, 1 (2009)
28. M.B. Plenio, S.S. Virmani, in *Quantum Information and Coherence, Scottish Graduate Series*, ed. by E. Andersson, P. Öhberg (Springer International Publishing, Switzerland, 2014), p. 173
29. A. Ekert, P.L. Knight, *Am. J. Phys.* **63**, 415 (1995)
30. B. Grobe, K. Rzaewski, J.H. Eberly, *J. Phys. B At. Mol. Opt. Phys.* **27**, L503 (1994)
31. A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996)
32. P. Horodecki, *Phys. Lett. A* **232**, 333 (1997)
33. M. Horodecki, P. Horodecki, R. Horodecki, *Phys. Rev. Lett.* **80**, 5239 (1998)
34. H. Ollivier, W.H. Zurek, *Phys. Rev. Lett.* **88**, 017901 (2001)
35. L. Henderson, V. Vedral, *J. Phys. A* **34**, 6899 (2001)
36. G. Adesso, A. Datta, *Phys. Rev. Lett.* **105**, 030501 (2010)
37. P. Gorda, M.G.A. Paris, *Phys. Rev. Lett.* **105**, 020503 (2010)
38. J.S. Zhang, A.X. Chen, *Quantum Phys. Lett.* **1**, 69 (2012)
39. S. Ragy, G. Adesso, *Sci. Rep.* **2**, 651 (2012)
40. A. Ferraro, L. Aolita, D. Cavalcanti, F.M. Cucchietti, A. Acín, *Phys. Rev. A* **81**, 052318 (2010)
41. A. Datta, A. Shaji, C.M. Caves, *Phys. Rev. Lett.* **100**, 050502 (2008)
42. B.P. Lanyon, M. Barbieri, M.P. Almeida, A.G. White, *Phys. Rev. Lett.* **101**, 200501 (2008)
43. M. Piani, P. Horodecki, R. Horodecki, *Phys. Rev. Lett.* **100**, 090502 (2008)
44. A. Brodutch, D.R. Terno, *Phys. Rev. A* **81**, 062103 (2010)

45. D. Bruß, J.I. Cirac, P. Horodecki, F. Hulpke, B. Kraus, M. Lewenstein, A. Sanpera, J. Mod. Opt. **49**, 1399 (2002)
46. M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A **223**, 1 (1996)
47. B.M. Terhal, Phys. Lett. A **271**, 319 (2000)
48. M. Lewenstein, B. Kraus, J.I. Cirac, P. Horodecki, Phys. Rev. A **62**, 052310 (2000)
49. O. Gühne, N. Lütkenhaus, Phys. Rev. Lett. **96**, 170502 (2006)
50. O. Gühne, N. Lütkenhaus, J. Phys. Conf. Ser. **67**, 012004 (2007)
51. M. Atatüre, G. Di Giuseppe, M.D. Shaw, A.V. Sergienko, B.E.A. Saleh, M.C. Teich, Phys. Rev. A **66**, 023822 (2002)
52. A.V. Sergienko, G.S. Jaeger, G. Giuseppe, B.E.A. Saleh, M.C. Teich, Quantum metrology and quantum information processing with hyper-entangled quantum states, in *Quantum Communication and Information Technologies (NATO Science Series)*, ed. by A.S. Shumovsky, V.I. Rupasov (Springer, Dordrecht, 2003)
53. C. Bonato, D. Simon, P. Villaresi, A.V. Sergienko, Phys. Rev. A **79**, 062304 (2009)
54. Y. Shih, *An Introduction to Quantum Optics: Photon and Biphoton Physics* (Taylor and Francis, Boca Raton, 2011)
55. Z.Y.J. Ou, *Multi-Photon Quantum Interference* (Springer, New York, 2007)
56. C.K. Law, I.A. Walmsley, J.H. Eberly, Phys. Rev. Lett. **84**, 5304 (2000)
57. I.A. Law, J.H. Eberly, Phys. Rev. Lett. **92**, 127903 (2004)
58. M.P. van Exter, A.R. Aiello, S.S. Oemrawsingh, G. Nienhuis, J.P. Woerdman, Phys. Rev. A **74**, 012309 (2006)
59. H. Di Lorenzo Pires, C.H. Monken, M.P. van Exter, Phys. Rev. A **80**, 022307 (2009)
60. J. Mertz, *Introduction to Optical Microscopy* (Robert and Co., Greenwood, 2010)
61. P.B. Dixon, G.A. Howland, J. Schneeloch, J.C. Howell, Phys. Rev. Lett. **108**, 143603 (2012)
62. C.K. Hong, L. Mandel, Phys. Rev. Lett. **56**, 58 (1986)
63. K. Edamatsu, Jpn. J. Appl. Phys. **46**, 7175 (2007)
64. B. Lounis, M. Orrit, Rep. Prog. Phys. **68**, 1129 (2005)
65. M. Oxborrow, A.G. Sinclair, Contemp. Phys. **46**, 173 (2005)
66. M.D. Eisaman, J. Fan, A. Migdall, S.V. Polyakov, Rev. Sci. Instr. **82**, 071101 (2011)
67. C.J. Chunnillall, I.P. Degiovanni, S. Kück, I. Müller, A.G. Sinclair, Opt. Eng. **53**, 081910 (2014)
68. S.J. Freedman, J.F. Clauser, Phys. Rev. Lett. **28**, 938 (1972)
69. C.A. Kocher, E.D. Commins, Phys. Rev. Lett. **18**, 575 (1967)
70. J.F. Clauser, Phys. Rev. Lett. **36**, 1223 (1976)
71. E.S. Fry, R.C. Thompson, Phys. Rev. Lett. **37**, 465 (1976)
72. B. Darquie, M.P.A. Jones, J. Dingjan, J. Beugnon, S. Bergamini, Y. Sortais, G. Messin, A. Browaeys, P. Grangier, Science **309**, 454 (2005)
73. S. Chen, Y.-A. Chen, T. Strassel, Z.S. Yuan, B. Zhao, J. Schmiedmayer, J.W. Pan, Phys. Rev. Lett. **97**, 173004 (2006)
74. A. Kuhn, M. Hennrich, G. Rempe, Phys. Rev. Lett. **89**, 067901 (2002)
75. Th. Basche, W.E. Moerner, M. Orrit, H. Talon, Phys. Rev. Lett. **69**, 1516 (1992)
76. F. De Martini, G. Di Giuseppe, M. Marrocco, Phys. Rev. Lett. **76**, 900 (1996)
77. C. Brunel, B. Lounis, P. Tamarat, M. Orrit, Phys. Rev. Lett. **83**, 2722 (1999)
78. B. Lounis, W.E. Moerner, Nature **407**, 491 (2000)
79. L. Fluery, J.-M. Segura, G. Zumofen, B. Hecht, U.P. Wild, Phys. Rev. Lett. **84**, 1148 (2000)
80. P. Michler, A. Imamoglu, M.D. Mason, P.J. Carson, G.F. Strouse, S.K. Buratto, Nature **406**, 968 (2000)
81. P. Michler, A. Kiraz, C. Becher, W.V. Schoenfeld, P.M. Petroff, L. Zhang, E. Hu, A. Imamoglu, Science **290**, 2282 (2000)
82. C. Santori, M. Pelton, G. Solomon, Y. Dale, Y. Yamamoto, Phys. Rev. Lett. **86**, 1502 (2001)
83. C. Santori, D. Fattal, J. Vučković, G.S. Solomon, Y. Yamamoto, Nature **419**, 594 (2002)
84. E. Moreau, I. Robert, J.M. Gérard, I. Abram, L. Manin, V. Thierry-Mieg, Appl. Phys. Lett. **79**, 2865 (2001)

85. V. Zwiller, H. Blom, P. Jonsson, N. Panev, S. Jeppesen, T. Tsegaye, E. Goobar, M. Pistol, L. Samuelson, G. Björk, Appl. Phys. Lett. **78**, 2476 (2001)
86. R.M. Thompson, R.M. Stevenson, A.J. Shields, I. Farrer, C.J. Lobo, D.A. Ritchie, M.L. Leadbeater, M. Pepper, Phys. Rev. B **64**, 201302R (2001)
87. A. Imamoglu, Y. Yamamoto, Phys. Rev. Lett. **72**, 210 (1994)
88. J. Kim, O. Benson, H. Kan, Y. Yamamoto, Nature **397**, 500 (1999)
89. R.M. Stevenson, R.J. Young, P. Atkinson, K. Cooper, D.A. Ritchie, A.J. Shields, Nature **439**, 179 (2006)
90. J. Claudon, J. Bleuse, N.S. Malik, M. Bazin, P. Jaffrennou, N. Gregersen, C. Sauvan, P. Lalanne, J.M. Gérard, Nat. Photonics **4**, 174 (2010)
91. P. Michler, A. Imamoglu, M.D. Mason, P.J. Carson, G.F. Strouse, S.K. Buratto, Nature **406**, 968 (2000)
92. Z. Yuan, B.E. Kardynal, R.M. Stevenson, A.J. Shields, C.J. Lobo, K. Cooper, N.S. Beattie, D.A. Ritchie, M. Pepper, Science **295**, 102 (2002)
93. D. Fattal, K. Inoue, J. Vuckovic, C. Santori, G.S. Solomon, Y. Yamamoto, Phys. Rev. Lett. **92**, 037903 (2004)
94. O. Benson, C. Santori, M. Pelton, Y. Yamamoto, Phys. Rev. Lett. **84**, 2513 (2000)
95. L. Childress, R. Walsworth, M. Lukin, Phys. Today **67**, 38 (2014)
96. C. Kurtsiefer, S. Mayer, P. Zarda, H. Weinfurter, Phys. Rev. Lett. **85**, 290 (2000)
97. R. Brouri, A. Beveratos, J.-P. Poizat, P. Grangier, Opt. Lett. **25**, 1294 (2000)
98. D.D.B. Rao, S. Yang, J. Wrachtrup, Phys. Rev. B **92**, 081301(R) (2015)
99. G. Bonfrate, V. Pruneiri, P. Kazanski, P.R. Tapster, J.G. Rarity, Appl. Phys. Lett. **75**, 2356 (1999)
100. S. Tanzilli, H. de Riedmatten, W. Tittel, H. Zbinden, P. Baldi, M. de Micheli, D.B. Ostrowski, N. Gisin, Electron. Lett. **37**, 26 (2001)
101. M. Fiorentino, P.L. Voss, J.E. Sharping, P. Kumar, E.E.E. Photon, Technol. Lett. **14**, 983 (2002)
102. X. Li, J. Chen, P. Voss, J.E. Sharping, P. Kumar, Opt. Express **12**, 373 (2004)
103. J.E. Sharping, J. Chen, X. Li, P. Kumar, Opt. Express **12**, 3086 (2004)
104. T. Kim, M. Fiorentino, F.N.C. Wong, Phys. Rev. A **73**, 012316 (2006)
105. X. Li, P.L. Voss, J.E. Sharping, P. Kumar, Phys. Rev. Lett. **94**, 053601 (2005)
106. J. Fan, M.D. Eisaman, A. Migdall, Phys. Rev. A **76**, 043836 (2007)
107. A. Fedrizzi, T. Herbst, A. Poppe, T. Jennewein, A. Zeilinger, Opt. Exp. **15**, 15377 (2007)
108. E. Yablonovitch, Phys. Rev. Lett. **58**, 2059 (1987)
109. S. John, Phys. Rev. Lett. **58**, 2486 (1987)
110. J.D. Joannopoulos, S.G. Johnson, *Photonic Crystals: Molding the Flow of Light*, 2nd edn. (Princeton University Press, Princeton, 2008)
111. M.J.A. de Dood, W.T.M. Irvine, D. Bouwmeester, Phys. Rev. Lett. **93**, 040504 (2004)
112. J.G. Rarity, J. Fulconis, J. Duligall, W.J. Wadsworth, P.S.J. Russell, Opt. Exp. **13**, 534 (2005)
113. H. Takesue, N. Matsuda, E. Kuramochi, M. Notomi, Sci. Rep. **4**, 3913 (2014)
114. M.A. Finger, T.S. Iskhakov, N.Y. Joly, M.V. Chekhova, P.S.J. Russell, Phys. Rev. Lett. **115**, 143602 (2015)
115. C. Paterson, Phys. Rev. Lett. **94**, 153901 (2005)
116. G. Gbur, R.K. Tyson, JOSA A **25**, 225 (2008)
117. G.A. Tyler, R.W. Boyd, Opt. Lett. **34**, 142 (2009)
118. B.J. Pors, C.H. Monken, E.R. Eliel, J.P. Woerdman, Opt. Exp. **7**, 6671 (2011)
119. J.C. Howell, A. Lamas-Linares, D. Bouwmeester, Phys. Rev. Lett. **88**, 030401 (2002)
120. R. Thew, A. Acin, H. Zbinden, N. Gisin, Quantum Inf. Comput. **4**, 93 (2004)
121. D. Stucki, H. Zbinden, N. Gisin, J. Mod. Opt. **52**, 2637 (2005)
122. J.S. Bell, J. Phys. C2, suppl. au no. **3**, tome 42, 41 (1981) (Reprinted in [128])
123. I. Pitowsky, Br. J. Philos. Sci. **45**, 95 (1994)
124. G. Jaeger, Mathematics **4**, 40 (2016)
125. A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **47**, 777 (1935)
126. D. Bohm, *Quantum Theory* (Prentice Hall, New York, 1951)

127. G. Jaeger, *Entanglement, Information, and the Interpretation of Quantum Mechanics* (Springer, Berlin, 2009)
128. J.S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987)
129. N.D. Mermin, *Boojums All the Way Through: Communicating Science in a Prosaic Age* (Cambridge University Press, Cambridge, 1990)
130. D. Dehlinger, M.W. Mitchell, *Am. J. Phys.* **70**, 903 (2002)
131. E. Schrödinger, *Proc. Camb. Philos. Soc.* **31**, 553 (1935)
132. H.M. Wiseman, S.J. Jones, A.C. Doherty, *Phys. Rev. Lett.* **98**, 140402 (2007)
133. S.J. Jones, H.M. Wiseman, A.C. Doherty, *Phys. Rev. A* **76**, 052116 (2007)
134. B.I. Erkmen, J.H. Shapiro, *Phys. Rev. A* **77**, 043809 (2008)
135. B.I. Erkmen, J.H. Shapiro, *Adv. Opt. Photonics* **2**, 405 (2010)
136. T. Yu, J.H. Eberly, *Phys. Rev. Lett.* **93**, 140404 (2004)
137. T. Yu, J.H. Eberly, *Phys. Rev. Lett.* **97**, 140403 (2006)
138. T. Yu, J.H. Eberly, *Quantum Inf. Comput.* **7**, 459 (2007)
139. M.P. Almeida, F. de Melo, M. Hor-Myell, A. Salles, S.P. Walborn, P.H. Souto Ribero, L. Davidovich, *Science* **316**, 579 (2007)
140. K. Ann, G. Jaeger, *Phys. Rev. A* **76**, 044101 (2007)
141. K. Ann, G. Jaeger, *Phys. Lett. A* **372**, 579 (2008)
142. K. Ann, G. Jaeger, *Found. Phys.* **39**, 790 (2009)
143. G.S. Jaeger, A.V. Sergienko, *Nat. Comput.* **13**, 459 (2014)
144. C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976)
145. I.D. Ivanovic, *Phys. Lett. A* **123**, 257 (1987)
146. D. Dieks, *Phys. Lett. A* **126**, 303 (1988)
147. A. Peres, *Phys. Lett. A* **128**, 19 (1988)
148. G. Jaeger, A. Shimony, *Phys. Lett. A* **197**, 83 (1995)
149. M. Ban, *Phys. Lett. A* **213**, 235 (1996)
150. J.A. Bergou, U. Herzog, M. Hillery, Discrimination of quantum states, in *Quantum State Estimation, Lect. Notes Phys.*, vol. 649, ed. by M. Paris, J. Rehacek (Springer, Berlin, 2004), p. 417
151. A. Chefles, *Contemp. Phys.* **41**, 401 (2000)
152. S.M. Barnett, S. Croke, *Adv. Opt. Photonics* **1**, 238 (2009)

## Chapter 2

# Two-Photon Interference

### 2.1 Classical Interferometry

Interferometry using various portions of the electromagnetic spectrum has a long history, extending back to the beginning of the nineteenth century in the optical case, and expanding far into other parts of the electromagnetic spectrum by the mid-twentieth century. During this time, interferometry has been used both as a probe for investigating the properties of light itself and as a practical tool for making high-precision measurements of other objects and phenomena. The applications range from measurement of lattice structures in solids and determination of stellar diameters to searches for gravity waves, a luminiferous ether, and extra space-time dimensions.

The archetypal interference experiment is the Young double slit experiment (Fig. 2.1), in which a wavefront is split into two components. In general, each component travels a slightly different distance as they make their way to a given point of the viewing screen. The two recombined wavefronts thus have different phase values due to their different propagation distances, with the result that bright and dark regions are formed on the screen from the corresponding constructive and destructive interference.

When the double slit experiment is examined at the level of individual photons, it is tempting to think of one photon passing through the top slit and the other passing through the bottom slit, followed by interference of one photon with the other at the final screen. However, this cannot be the case. To see this, suppose that a photographic plate is placed at the screen location to record photon arrival events. Experimentally, even if the intensity is reduced to the point where there is negligible probability of more than one photon reaching the slits at a given time, interference still arises (see [1], Sect. 1.1). Rather, each photon has two paths that can be followed to reach the same point on the screen, one path through each slit. If there is no way to determine which path was taken, then the superposition principle tells us that the amplitudes for both paths must be added; what interferes then are these two amplitudes for

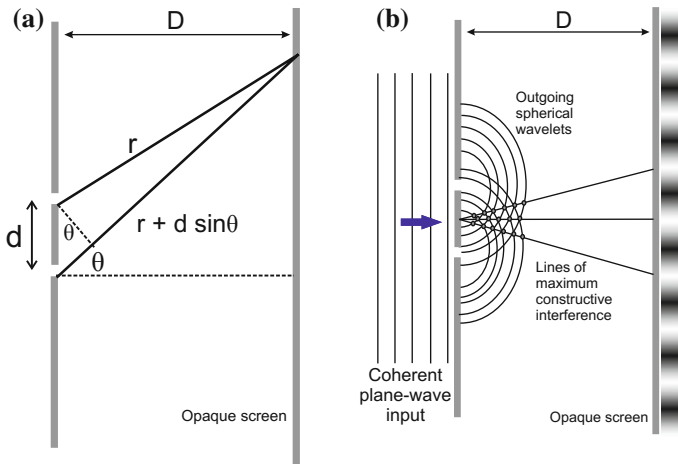


**Fig. 2.1** Young double-slit interference. **a** The beams from the two slits travel slightly different distances to reach a point on the screen, resulting in a relative phase shift between the beams, causing *bright* and *dark bands*, **b** the regions of constructive and destructive interference, constructive interference occurring at angles  $\theta$  such that the phase difference  $\frac{d \sin \theta}{\lambda}$  is an integer multiple of  $2\pi$

different behaviors of the *same* photon. Turning up the intensity, all interference arising in classical optics can be viewed as interference of light with itself in this manner. In the case both of single photon interference and of high-intensity classical interference, this situation is captured by the definition of first-order correlation  $g^{(1)}$  given in the next section. For such interference to arise, it is necessary for the different intermediate histories of the photons to be indistinguishable with respect to the experimental apparatus: if there is any way to determine which history is followed, then the interference pattern becomes washed out (see [1], Sect. 3.6, [2]).

A more versatile way to produce interference for many applications is through the use of interferometers, such as the Mach–Zehnder and Hong–Ou–Mandel interferometers in the case of spatial interference. When combined with entangled two-photon sources like spontaneous parametric down conversion discussed, these open up a number of new possibilities, including true multi-photon interference of a kind that does not occur in classical interferometry. In these processes, different potential histories of the same multi-photon set interfere. In order to quantify the non-classical nature of these interference effects, we must define a new type of correlation function, the second-order correlation function  $g^{(2)}$ .

The first and second-order correlation functions are introduced in the next section, with a more detailed discussion of interferometers in the following sections.

## 2.2 Correlation Functions

### 2.2.1 First-Order Correlations

We wish to be able to compare the field amplitude at one space-time point (position  $\mathbf{x}_1$ , time  $t_1$ ) to the field at a different point ( $\mathbf{x}_2$ ,  $t_2$ ). Young two-slit interference corresponds to a special case where  $\mathbf{x}_1 = \mathbf{x}_2$  and  $t_1 = t_2$ ; in other words, the two fields were directed to the same point before measuring the combined intensity.

To compare fields at different points, we define the *first-order correlation function* or *amplitude correlation function* [3],

$$G^{(1)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) = \langle \hat{E}^{(-)}(\mathbf{r}_1, t_1) E^{(+)}(\mathbf{r}_2, t_2) \rangle, \quad (2.1)$$

where  $\langle \dots \rangle$  denotes the quantum expectation value in the relevant state and  $\hat{E}^{(\pm)}$  are the positive and negative frequency parts of the electric field operator (see “Appendix B”). This function obeys

$$G^{(1)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) = (G^{(1)}(\mathbf{r}_2, \mathbf{r}_1; t_2, t_1))^* \quad (2.2)$$

$$G^{(1)}(\mathbf{r}, \mathbf{r}; t, t) \geq 0, \quad (2.3)$$

$$|G^{(1)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)|^2 \leq G^{(1)}(\mathbf{r}_1, \mathbf{r}_1; t_1, t_1) G^{(1)}(\mathbf{r}_2, \mathbf{r}_2; t_2, t_2). \quad (2.4)$$

The normalized correlation function

$$\begin{aligned} g^{(1)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) &= \frac{G^{(1)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)}{[G^{(1)}(\mathbf{r}_1, \mathbf{r}_1; t_1, t_1) G^{(1)}(\mathbf{r}_2, \mathbf{r}_2; t_2, t_2)]^{1/2}} \\ &= \frac{\langle \hat{E}^{(-)}(\mathbf{r}_1, t_1) E^{(+)}(\mathbf{r}_2, t_2) \rangle}{\sqrt{\langle \hat{E}^{(-)}(\mathbf{r}_1, t_1) E^{(+)}(\mathbf{r}_1, t_1) \rangle \langle \hat{E}^{(-)}(\mathbf{r}_2, t_2) E^{(+)}(\mathbf{r}_2, t_2) \rangle}}, \end{aligned} \quad (2.5)$$

therefore, satisfies  $0 \leq g^{(1)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) \leq 1$ . Optical correlation functions are also sometimes called *coherence functions* and the absolute value  $|g^{(1)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)|$  is called the *degree of coherence*.

Setting the two positions equal to each other gives the temporal correlation function,

$$g^{(1)}(\tau) = \frac{G^{(1)}(\mathbf{r}, \mathbf{r}; t, t + \tau)}{G^{(1)}(\mathbf{r}, \mathbf{r}; t, t)} = \frac{G^{(1)}(\mathbf{r}, \mathbf{r}; 0, \tau)}{G^{(1)}(\mathbf{r}, \mathbf{r}; 0, 0)}. \quad (2.7)$$

Here, we have assumed that the fields are stationary, i.e. that their correlations depend only on coordinate differences  $\Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$  and  $\tau = t_2 - t_1$ , not on the individual values  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ ,  $t_1$ , and  $t_2$  themselves. It is clear that  $g^{(1)}(0) = 1$ . Except for the case of perfect coherence (see below), the temporal coherence function decays at long times:  $g^{(1)}(\tau) \rightarrow 0$  as  $\tau \rightarrow \infty$ .

A thermal light source consists of many atoms, usually in the form of a gas, radiating independently. Collisions between the atoms cause random phase jumps in the emitted radiation, so that the light emitted remains coherent only over short time periods determined by the mean time between collisions. In addition, the light may scatter multiple times before leaving the source, adding further randomness to the phases. For a quasi-monochromatic field of the form (suppressing the spatial dependence for simplicity)  $\hat{E}^{(-)}(t) = E_0 e^{-i\omega t} e^{i\phi(t)} \hat{a}$ , the temporal correlation function has the form

$$g^{(1)}(\tau) = e^{-i\omega\tau} \langle e^{i(\phi(t+\tau) - \phi(t))} \rangle. \quad (2.8)$$

The first exponential factor provides a rapid oscillation, while the expectation value provides a much more slowly varying envelope, so that  $g^{(1)}(\tau)$  generally has an overall form as in Fig. 2.2. The time at which the correlation begins rapidly decreasing in amplitude is called the *coherence time*,  $\tau_c$ . The coherence time is generally inversely proportional to the bandwidth  $\Delta\omega$  of the light.

Light can be classified by its coherence properties. If the light is perfectly monochromatic ( $\Delta\omega = 0$ ), then it is called *perfectly coherent* or simply *coherent*, with coherence time  $\tau_c = \infty$ . A laser is often considered to be a good approximation to a coherent light source, with a narrow bandwidth and a long coherence time. On the other hand, if  $\tau_c = 0$  (and therefore  $\Delta\omega = \infty$ ), the light is *incoherent*. For

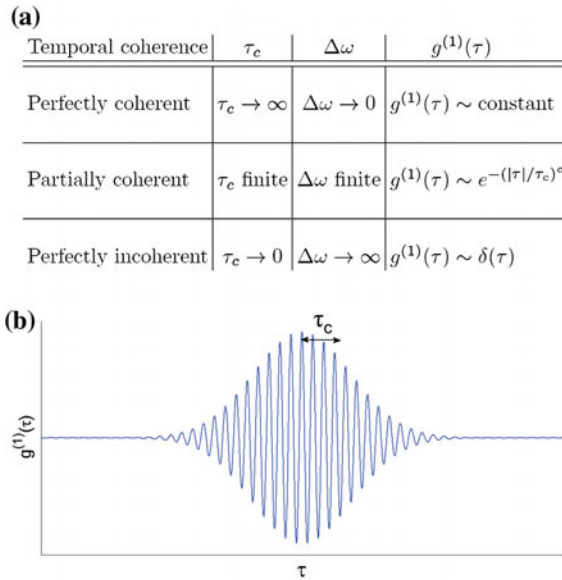


**Fig. 2.2** **a** The possible types of first-order coherence. In the second row,  $\alpha$  is some constant of order 1, determined by the source the frequency spread. **b** The form of  $g^{(1)}(\tau)$  for partially coherent light in the case of Gaussian linewidth. The shape of the envelope is given by  $\langle e^{i(\phi(t+\tau) - \phi(t))} \rangle$ . The peak in the envelope is at  $\tau = 0$

perfectly coherent light,  $g^{(1)}(\tau) = 1$  is constant, while for perfectly incoherent light the correlation function is a Dirac delta function,  $g^{(1)}(\tau) = \delta(\tau)$ . These are shown in Fig. 2.2a

Light that falls between these two extremes and has a finite coherence time is called *partially coherent*. Light produced by a thermal source is partially coherent. If the light source is a hot gas, collisions between molecules and random Doppler shifts broaden the lines to give a spread of frequencies and therefore a finite width to  $g^{(1)}(\tau)$ . Such thermally generated partially coherent light is often referred to as *chaotic light*. Two models are generally used to describe the thermal line broadening, in which the averaged term in Eq. (2.8) is either Gaussian or Lorentzian.

In the Gaussian case, where the frequency spread is primarily dominated by inhomogeneous processes like Doppler broadening, the temporal correlation function is

$$g^{(1)}(\tau) = e^{-i\omega\tau} e^{-\frac{\pi}{2} \left( \frac{\tau}{\tau_c} \right)^2}, \quad (2.9)$$

where  $\tau_c = \frac{\sqrt{8\pi \ln 2}}{\Delta\omega}$ . In the Lorentzian case, due to homogeneous processes like collisional broadening or the natural broadening of an excited state due to its finite lifetime, we instead have

$$g^{(1)}(\tau) = e^{-i\omega\tau} e^{-|\tau|/\tau_c}, \quad (2.10)$$

with coherence time  $\tau_c = \frac{1}{\Delta\omega}$ .

Given an amplitude interference pattern, the interference visibility is defined by

$$\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}, \quad (2.11)$$

where  $I_{\max}$  and  $I_{\min}$  are the maximum and minimum intensities (see Fig. 2.3). One of the reasons that the temporal correlation function is important is that it is directly related to interference visibility. Imagine two fields  $E_1(t)$  and  $E_2(t + \tau)$  interfering, where  $\tau$  is the additional time one wave propagated relative to the other in order to reach the point at which interference is considered. Then the total amplitude is  $E(t) = E_1(t) + E_2(t + \tau)$ , and the intensity averaged over a short time interval is

$$\begin{aligned} \langle I \rangle &= \langle \hat{E}_1^{(-)}(t) \hat{E}_1^{(+)}(t) \rangle + \langle \hat{E}_2^{(-)}(t + \tau) \hat{E}_2^{(+)}(t + \tau) \rangle + 2 \operatorname{Re} \langle \hat{E}_1^{(-)}(t) \hat{E}_2^{(+)}(t + \tau) \rangle \\ &= \langle I_1 \rangle + \langle I_2 \rangle + 2 \operatorname{Re} \langle \hat{E}_1^{(-)}(t) \hat{E}_2^{(+)}(t + \tau) \rangle \end{aligned} \quad (2.12)$$

$$= \sqrt{\langle I_1 \rangle \langle I_2 \rangle} \left[ \sqrt{\frac{\langle I_1 \rangle}{\langle I_2 \rangle}} + \sqrt{\frac{\langle I_2 \rangle}{\langle I_1 \rangle}} + \operatorname{Re} \left( \frac{\langle \hat{E}_1^{(-)}(t) \hat{E}_2^{(+)}(t + \tau) \rangle}{\sqrt{\langle I_1 \rangle \langle I_2 \rangle}} \right) \right] \quad (2.13)$$

$$= \sqrt{\langle I_1 \rangle \langle I_2 \rangle} \left[ \sqrt{\frac{\langle I_1 \rangle}{\langle I_2 \rangle}} + \sqrt{\frac{\langle I_2 \rangle}{\langle I_1 \rangle}} + \operatorname{Re} (g^{(1)}(\tau)) \right]. \quad (2.14)$$

The first two terms in the brackets are constants, and the third term contains the oscillations. Since  $-|g^{(1)}(\tau)| \leq \operatorname{Re} (g^{(1)}(\tau)) \leq +|g^{(1)}(\tau)|$ , it is found that

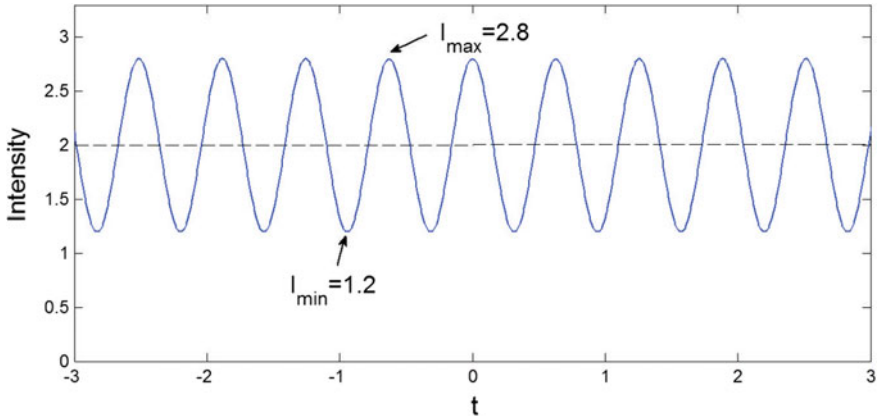


**Fig. 2.3** An interference pattern with mean intensity  $I = 2$  and visibility  $\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2.8 - 1.2}{2.8 + 1.2} = 0.4$ , in units where the intensity is dimensionless

$$\mathcal{V} = |g^{(1)}(\tau)|. \quad (2.15)$$

Spatial correlations may also be considered. Setting the two times equal, but looking at two different spatial points, the spatial correlation function is

$$g_s^{(1)}(\Delta r) = \frac{G^{(1)}(\mathbf{r}, \mathbf{r} + \Delta \mathbf{r}; t, t)}{G^{(1)}(\mathbf{r}, \mathbf{r}; t, t)}, \quad (2.16)$$

where stationary fields have again been assumed. The correlations should decay as the transverse distances (perpendicular to the propagation direction) become large, so that one may define a *transverse coherence length*  $d_c$  as the transverse distance at which decay sets in.  $d_c$  is sometimes defined as the square root of the *coherence area*,  $d_c = \sqrt{A_c}$ .  $A_c$  is defined to be the size of the area around one point in the wave, perpendicular to the propagation direction, in which other points have a stable enough phase relation with the first point to create interference. Then, similarly to the temporal case, a system may be described as spatially coherent or spatially incoherent if its size is, respectively, much less than or much greater than  $d_c$ . A second coherence length can also be defined, the *longitudinal coherence length*,  $l_c$ , which is the spatial distance covered by the light during one coherence time unit:  $l_c = c\tau_c$ .

Because the coherence time tends to be very short for true thermal light (black-body radiation), which often has  $\tau_c < 10^{-9}$  s, in experiments *pseudothermal* light may be used instead. This is produced by scattering coherent light from a rotating ground glass surface. Light scattering from different points in the surface travel different distances, so that spatially and temporally random patterns of constructive and destructive interference occur if the light is shown onto a screen. These patterns are called *speckles*, and they vary randomly over time as the glass disk rotates.

For light from a  $TM_{00}$  laser mode striking the disk at distance  $r$  from the axis, the coherence time of these speckles is given by the formula

$$\tau_c = \frac{w_0}{2\sqrt{\pi}r\Omega}, \quad (2.17)$$

where  $w_0$  is the beam diameter at the disk and  $\Omega$  is the rotational frequency of the disk. In this way, longer coherence times, typically  $1\text{ }\mu\text{s} \leq \tau_c \leq 1\text{ ms}$ , can be obtained. If the light strikes a screen at distance  $z$  from the disk, a typical speckle size is of order  $d = \frac{\lambda z}{2w_0}$ .

### 2.2.2 Second-Order Correlations

In the next section, and in later chapters, it is often useful to consider the correlation between two intensities, rather than between two fields. The corresponding correlation function is called the (normalized) *second-order correlation function* [3],  $g^{(2)}(\tau)$ . Similarly to the first-order case, we may look at both spatial and temporal correlations, but here we restrict ourselves to the temporal case. Suppressing the spatial variables again, the correlation is defined by

$$g^{(2)}(\tau) = \frac{\langle \hat{I}(t)\hat{I}(t+\tau) \rangle}{\langle \hat{I}(t) \rangle \langle \hat{I}(t+\tau) \rangle} = \frac{\langle \hat{I}(t)\hat{I}(t+\tau) \rangle}{\langle \hat{I} \rangle^2}, \quad (2.18)$$

where in the second equality the average intensity has been assumed to be time-independent. Using the fact that the intensity is the absolute square of the field, one may also write

$$g^{(2)}(\tau) = \frac{\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t)\hat{E}^{(+)}(t+\tau) \rangle}{\langle \hat{E}^{(-)}(t)\hat{E}^{(+)}(t) \rangle^2}, \quad (2.19)$$

or, since the intensity is proportional to the number of photons,  $g^{(2)}$  may alternatively be written in the form

$$g^{(2)}(\tau) = \frac{\langle \hat{N}(t)\hat{N}(t+\tau) \rangle}{\left( \langle \hat{N}(t) \rangle \right)^2}, \quad (2.20)$$

where  $\hat{N} = \hat{a}^\dagger \hat{a}$  is the photon number operator.

For perfectly coherent monochromatic light,  $g^{(2)}(\tau) = g^{(2)}(0) = 1$  is constant. For classical light,  $g^{(2)}(\tau)$  is largest at  $\tau = 0$  and decays monotonically with increasing  $\tau$ , just as for the first-order correlation function. However, for anti-bunched, non-classical states of light (see below),  $g^{(2)}(\tau)$  exhibits unusual behavior and actually grows for nonzero  $\tau$ ,  $g^{(2)}(\tau) > g^{(2)}(0)$ .

For partially coherent thermal light, where light scatters multiple times before exiting the source, the first and second-order correlations functions are connected by Siegert's relation

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2. \quad (2.21)$$

Since  $g^{(1)}(\tau)$  decays from 1 at short times to 0 at long times (much greater than  $\tau_c$ ), it follows that  $g^{(2)}(\tau)$  decays from  $g^{(2)}(0) = 2$  to  $g^{(2)}(\infty) = 1$ . For the Lorentzian and Gaussian cases, the Siegert relation implies that:

$$g^{(2)}(\tau) = 1 + e^{-2|\tau|/\tau_c} \quad (\text{Lorentzian}) \quad (2.22)$$

$$g^{(2)}(\tau) = 1 + e^{-\pi(\tau/\tau_c)^2} \quad (\text{Gaussian}), \quad (2.23)$$

for thermal light.

The intensity can always be written as a mean value plus some randomly fluctuating term,  $I(t) = \langle I \rangle + \Delta I(t)$ . Regardless of the source of the fluctuations,  $\Delta I$ , they must have vanishing mean,  $\langle \Delta I \rangle = 0$ , so if it is assumed that the average intensity is time independent then the second-order coherence function can be written as

$$\begin{aligned} g^{(2)}(\tau) &= \frac{\langle (\langle I \rangle + \Delta I(t)) (\langle I \rangle + \Delta I(t + \tau)) \rangle}{\langle I \rangle^2} \\ &= 1 + \frac{\langle \Delta I(t) \Delta I(t + \tau) \rangle}{\langle I \rangle^2}. \end{aligned} \quad (2.24)$$

Therefore,  $g^{(2)}(\tau)$  can be seen as a measure of the correlation between the intensity *fluctuations* at different times. Equivalently, since the intensity is proportional to the number of photons present, it measures correlations between photon number fluctuations.

Looking at the case of zero time delay,  $\tau = 0$ , three possibilities can be distinguished. (i) If  $g^{(2)}(0) = 1$ , then the distribution of times between photon emissions are given by a Poisson distribution. This is the case for light in a coherent state. (ii) If  $g^{(2)}(0) > 1$ , then the spacings between photons are reduced from the Poisson case. In other words, the photons tend to cluster together more than in the Poisson case. This case is called *super-Poisson*, and the light is often referred to as *chaotic* or *photon-bunched*. Thermal light sources produce photon-bunching. (iii) If  $g^{(2)}(0) < 1$ , then the spacings between photons are increased. This is called *sub-Poisson* or *anti-bunched* light.

Both the Poisson and super-Poisson cases are possible classically, however the sub-Poisson case is not. So the appearance of sub-Poisson or anti-bunched light is a clear signal of non-classical behavior.

First-order correlation functions are measured by comparing the output of a single detector at different locations or times, while second-order correlation functions are measured by detecting two photons in two separate detectors. In a similar manner, higher  $n$ th-order correlation functions can be defined and can be measured using  $n$ -detector arrangements. These have been used, for example, to study higher-order ghost imaging [4–9].

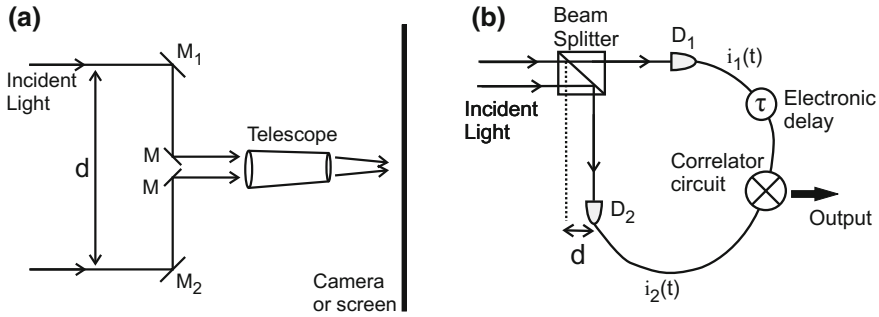


**Fig. 2.4** Two stellar interferometers. **a** The Michelson stellar interferometer, in which coherence between field amplitudes at points separated by distance  $d$  is measured. Mirrors  $M_1$  and  $M_2$  reflect light from different points in the wavefront into a telescope, where they are focused to produce an interference pattern, **b** the Hanbury Brown and Twiss interferometer. In this apparatus, *intensities* rather than merely simple amplitudes are correlated at points separated by  $d$ . Two photodetectors send photo currents  $i_1$  and  $i_2$  to an electronic circuit that correlates the signals. The output signal is the mean product of currents,  $\langle i_1(t)i_2(t + \tau) \rangle$ , which is proportional to the mean product of intensities,  $\langle I_1(t)I_2(t + \tau) \rangle$ . In the case of low photon numbers, the electronic correlator becomes a photon coincidence counter

## 2.3 Hanbury Brown and Twiss: Source Size from Correlation

Classical interference theory is built around first-order (field or amplitude) correlations. The importance of second-order (intensity or photon number) correlations first became clear through the work of Hanbury Brown and Twiss (HBT) on stellar diameter measurements in the 1950s.

Prior to the work of HBT, the principal means of measuring stellar diameters was through measurement of spatial coherence via the field correlations measured by  $g^{(1)}$ . In the Michelson stellar interferometer (Fig. 2.4a) light arriving from a star strikes two mirrors separated by distance  $d$ , and then the two reflected beams are combined and sent through a telescope, finally forming an interference pattern in a camera. Light arriving at different angles have maxima and minima shifted in position by a distance proportional to the angle. So, if the angular size of the star is too large, all of the shifted fringes from different angular directions end up washing out the interference pattern. The pattern is lost when the angular spread  $\delta\theta_S$  of the starlight is smaller than the angular resolution  $\Delta\theta_I$  of the interferometer. The spread  $\delta\theta_S$  is determined by the size of the star and its distance  $L$  from earth,  $\delta\theta_S = \frac{D}{L}$ , for  $L \gg D$ . The angular resolution, on the other hand, is determined by the wavelength used and by the size of the interferometer,  $\Delta\theta_I = \frac{1.22\lambda}{d}$ , where the factor of 1.22 comes from the width of the Bessel function determining the resolution of the circular telescope aperture, as in Eq. A.32 of “Appendix A.” The requirement that  $\delta\theta_S < \Delta\theta_I$  then places a limit on how large  $d$  can be before the interference is lost, since we must have

$$\frac{D}{L} < \frac{1.22\lambda}{d}. \quad (2.25)$$

By varying  $d$  and finding the distance  $d = l_c$  at which the pattern is lost, we then have an estimate for the stellar diameter,  $D = 1.22\lambda \frac{L}{l_c}$ .  $l_c$  is then the transverse spatial coherence length of the light.

Such field interference methods provide a great improvement in angular resolution over other methods, since the distance  $d$  can be made much larger than the diameter of the telescope. However, it has several drawbacks: (i) the ability to collect light is lower, so that dim objects can not be effectively measured, (ii) the measurement is susceptible to atmospheric turbulence, and (iii) a high level of mechanical stability must be maintained in the distances between the mirrors and the telescope, which becomes increasingly difficult as  $d$  becomes larger.

The HBT approach can be seen as replacing the two mirrors of the Michelson interferometer by two independent photodetectors,  $D_1$  and  $D_2$ , separated by distance  $d$  (Fig. 2.4b). The detectors produce two photocurrents proportional to the incident light intensities at those points. After a time delay  $\tau$  is imposed on one current, the two signals are then fed into an electronic circuit that multiplies the currents and integrates over an interval of time in order to produce an average. The net result is proportional to the expectation value  $\langle I_1(t)I_2(t + \tau) \rangle = \langle I(r_1, t)I(r_1 + d, t + \tau) \rangle$ , where  $I_1$  is the intensity in detector  $D_1$  (located at  $r_1$ ), and  $I_2$  is the intensity at  $D_2$  (located at  $r_2 = r_1 + d$ ). From this, the second-order correlation function  $g^{(2)}(d, \tau)$  may be found. By setting  $\tau = 0$  and varying the distance  $d$  between the detectors, the coherence length  $l_c$  can be determined. Alternatively, the distance can be fixed and the time delay may be varied in order to find the coherence time.

## 2.4 From One-Photon to Two-Photon Interference

Yet another important interferometer, which appears in multiple guises in this book, is the Mach–Zehnder interferometer, shown in basic form in Fig. 2.5. It was discussed in the previous chapter, where it was considered as a means of producing spatial-mode qubits. Here, it is treated in more detail. There are two input ports, 1 and 2 on the left, and two output ports, 3 and 4, on the right. Suppose light enters through one input, say port 1, and optical detectors are placed at the outputs. Assume also that the input light is perfectly coherent. Then, the first beam splitter sends the light either along the upper path or the lower path, with equal probability. A phase shift  $\phi$  can be introduced to the portion of the light following the upper path, for example, by changing the path length or by introducing a small piece of glass of refractive index  $n \neq 1$  into the light's path. The two beams recombine at the second beam splitter, now possibly out of phase with each other, leading to constructive or destructive interference as  $\phi$  is varied. Let the input and output states of the light be described by the two-component vectors

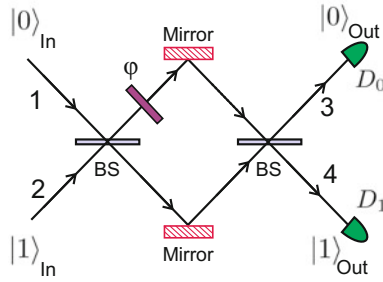


**Fig. 2.5** The Mach–Zehnder interferometer. Light input at one port (1 or 2) has two possible paths through the system before leaving at another port (3 or 4). Depending on the phase shift introduced into one path, the output can be steered in one direction or the other, analogous to the *bright* and *dark spots* the Young experiment

$$|\psi_{in}\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad |\psi_{out}\rangle = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \quad (2.26)$$

where  $\psi_n$  is the amplitude at port  $n$ . The “amplitude” here can be either the classical electric field in the case of bright light, or the quantum mechanical amplitude in the case of single-photon input. The action of each beam splitter can be represented as a matrix  $M_{BS}$  (“Appendix A”) and the phase shift by matrix  $M_\phi$ , where

$$M_{BS} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad M_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.27)$$

so that

$$|\psi_{out}\rangle = M_{BS} M_\phi M_{BS} |\psi_{in}\rangle = \begin{pmatrix} e^{i\phi} - 1 & i(e^{i\phi} + 1) \\ i(e^{i\phi} + 1) & 1 - e^{i\phi} \end{pmatrix} |\psi_{in}\rangle. \quad (2.28)$$

Here, we have ignored the propagation phases  $e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}}$  and mirror reflection phases that are common to all terms. So if the light is all incident on port 1 ( $\psi_2 = 0$ ), we find that the output at port 3 is  $\psi_3 = e^{i\phi} - 1$ . The resulting intensity (in the bright light case) or detection probability (in the single photon case) is, therefore, proportional to  $|\psi_{out}|^2 = |\psi_3|^2 = |e^{i\phi} - 1|^2 = 2(1 - \cos \phi)$ .

However, instead of looking at the output of each detector separately, one may follow the lead of Hanbury Brown and Twiss, by looking at joint detections in the pair of detectors. In other words, coincidence counts are measured, providing measurement of the second-order correlation function. The interference is now between two-photon amplitudes, instead of single-photon amplitudes. The coincidence counting is done by using an electronic gate which only registers a count if there is a signal coming in from both detectors within some very short time window.

It can be shown that there is a complementarity between the single-photon and double-photon visibilities,  $\mathcal{V}_1$  and  $\mathcal{V}_{12}$  in any system [10, 11]. In particular,  $\mathcal{V}_1^2 + \mathcal{V}_{12}^2 \leq$

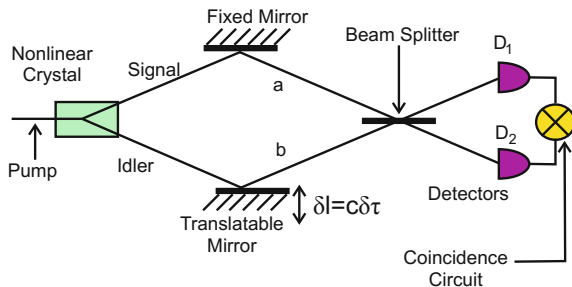
1, or equivalently,  $\mathcal{V}_1 \mathcal{V}_{12} \leq \frac{1}{2}$ . Thus to achieve strong two-photon interference, it is necessary to make the observation of single-photon effects unobservable. (See the Franson interferometer, Sect. 2.6, for an example of this.)

## 2.5 The Hong–Ou–Mandel Dip

The *Hong–Ou–Mandel (HOM) dip* [12] is an interference effect that occurs in the coincidence rate of two-photon interferometers. Besides opening a new arena for the exploration of quantum mechanical two-photon interference effects, the HOM dip has shown itself useful in measuring extremely short (sub-femtosecond) time intervals. It is also used to reconstruct the temporal profile of single photon wave packets [12], to measure the delay times caused by passage of light through materials [13], and to make measurements of tunneling times of single photons through barriers [14]. Of particular importance here is the fact that if a dispersive material is placed in one of the possible photon paths of the interferometer used to observe the dip, the effect of group velocity dispersion (GVD) cancels from the coincidence rate. In other words, there is no dispersive broadening of wave packets. This effect is discussed in more detail in Chap. 3.

Figure 2.6 shows the *HOM interferometer*. The signal and idler from parametric down conversion take separate paths to a 50/50 beam splitter, with a variable delay inserted into one of the paths, before the beam splitter. The signal takes path a to the beam splitter and the idler takes path b. After the beam splitter, the outgoing photons reach two detectors, where the coincidence count is recorded. As the delay time is scanned, the coincidence rate traces out a roughly triangular dip, as shown in Fig. 2.7. Ideally, the dip reaches a minimum of zero (100 % visibility) when the delay vanishes. It is essential that the indistinguishability of the two photons be maintained for this effect to appear. Any factors that increase the ability to distinguish which path was taken by which photon after the beam splitter causes a corresponding decrease in the visibility of the dip. When the translatable mirror is moved a distance  $\delta l$ , a delay of  $\delta\tau = \delta l/c$  is introduced between the two paths, along with a phase difference of  $\delta\phi = \delta l/\lambda$ .

**Fig. 2.6** The Hong–Ou–Mandel (HOM) interferometer. The translatable mirror introduces a phase shift between the signal and idler, which are then mixed at the beam splitter before coincidence detection



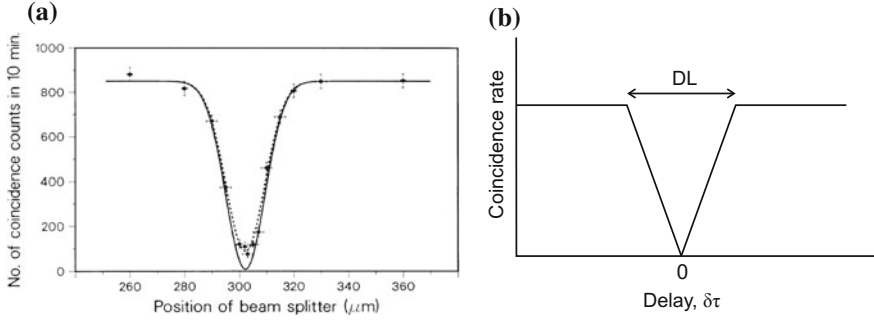


**Fig. 2.7** The HOM dip. The coincidence rate exhibits a *triangular* dip, vanishing at zero time delay. **a** Experimental data showing the dip arising from interference of signal and idler [12], **b** idealized dip arising from interference of two square amplitudes

The explanation for the HOM dip is very simple. Assume for a moment that there is no delay. In order for a coincidence count to be registered, exactly one of the photons must reach each of the two detectors. This can happen in one of two ways: either both photons are reflected at the beam splitter (RR) or both are transmitted (TT). Because each reflection at the beam splitter introduces a phase shift of  $\pi/2$ , the amplitudes for these two possibilities have opposite sign. In the case of a symmetric 50/50 beam splitter (reflectance = transmittance = 50%), the RR and TT amplitudes are also equal in magnitude, so that they exactly cancel, giving a vanishing coincidence rate. Now, if a delay is introduced, this allows the photons to be distinguished by their arrival times. The interference is reduced and the cancelation becomes only partial. As the delay increases from zero, the coincidence rate rises from zero back to the classically expected value.

Quantitatively, the coincidence rate is given by

$$R_c(\delta\tau) = R_0 \left\{ 1 - \frac{1}{4} \frac{\int g^*(\tau)g(2\delta\tau - \tau)d\tau}{\int |g(\tau)|^2 d\tau} \right\}, \quad (2.29)$$

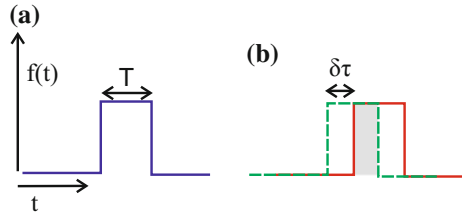
where  $R_0$  is a constant,  $g(\tau) = \frac{G(\tau)}{G(0)}$ , and  $G(\tau)$  is the Fourier transform of the spectral function  $\Phi(\omega)$ . Assuming a Gaussian pump and using the function  $\Phi(\omega) = \text{sinc}(\omega T)$  for SPDC (see Chap. 1), the coincidence rate is

$$R_c(\delta\tau) = R_0 \left( 1 - e^{-\Delta\omega^2 \delta\tau^2} \right), \quad (2.30)$$

for some constant  $R_0$  [12]. Here,  $T \sim \frac{1}{\Delta\omega}$  is the width of the photon wavepacket. The experimental data for the coincidence rate matches this expression well, as seen in Fig. 2.7a.

To give an idea of why the coincidence rate has this form, imagine a simplified situation in which the spectrum is flat, and where the signal and idler amplitudes

**Fig. 2.8** Two rectangular amplitudes of unit height and of width  $T$  (a), have an overlap given as by Eq. (2.32) when one is displaced by time  $\delta\tau$  relative to the other (b)



both have unit-height rectangular shapes (Fig. 2.8a) of the form

$$f(t) = \begin{cases} 1, & \text{for } |t| \leq \frac{T}{2} \\ 0 & \text{for } |t| > \frac{T}{2}, \end{cases} \quad (2.31)$$

where  $T$  is the time for the photons to cross the nonlinear crystal. This is a reasonable first approximation, since the amplitude for pair creation drops suddenly to zero at the edge of the crystal. Suppose that one photon is delayed relative to the other by delay time  $\delta\tau$ . Then, the overlap of these amplitudes is given by the convolution

$$I(t) = \int_{-\infty}^{\infty} f(t)f(t - \delta\tau)dt = \begin{cases} 1 - \frac{|\delta\tau|}{T}, & \text{for } |t| \leq T \\ 0 & \text{for } |t| > T. \end{cases} \quad (2.32)$$

Examining Fig. 2.8b, the reader can verify that this expression is simply the area of overlap between the two shifted square functions. This overlap is triangular, having maximum value at  $\delta\tau = 0$  and decaying linearly to zero with increasing  $|\delta\tau|$ , representing constructive interference between the amplitudes. In the case of destructive interference, this triangular overlap is subtracted from the constant noninterfering value of 1, so that

$$R_c(\delta\tau) = R_0 \left\{ 1 - \frac{1}{2} \Lambda \left( \frac{2\delta\tau}{T} \right) \right\}, \quad (2.33)$$

where

$$\Lambda(x) = \begin{cases} 1 - |x|, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| > 1 \end{cases} \quad (2.34)$$

is the triangle function. The idealized form of Eq. (2.33) provides a good approximation to Eq. (2.30). This ideal coincidence rate is plotted in Fig. 2.7b. The experimental data of Fig. 2.7a roughly matches this and gives a generally triangular shape, but more rounded because the photon creation amplitudes in reality are sinc functions, rather than squares.

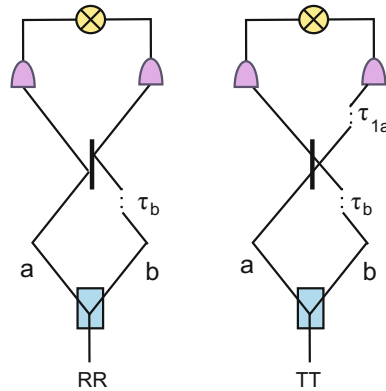


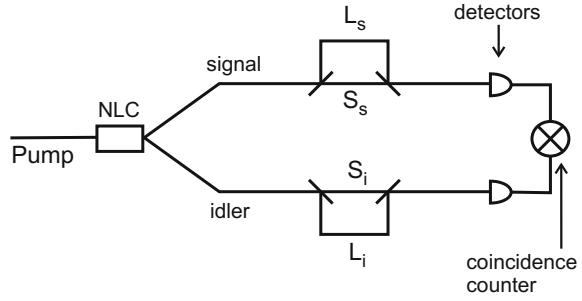
**Fig. 2.9** Signal and idler need not overlap temporally on the beam splitter to interfere and produce an HOM dip. Here,  $\tau_a$  is a time delay that effects only the signal photon in branch  $a$ . If  $\tau_a = 2\tau_b$ , then both the RR and TT cases lead to the same time delay  $\Delta t \equiv t_2 - t_1 = \tau_b$  between detection of photons. The two outcomes are therefore indistinguishable and can interfere

## 2.6 The Franson Interferometer

As was the case in the discussion of single-photon interference, it must be borne in mind that “two-photon” interference is not the result of one photon interfering with a second photon; more generally, it is the interference of one potential two-photon state with another potential state of the same two-photon system. The two photons involved need not arrive at a beam splitter at the same time or ever meet at all for interference to occur. Only indistinguishability, not temporal overlap, is required. This was demonstrated by the experiment of [15], where a fixed delay was introduced in one path before the beam splitter, so that the two photons arrived at the beam splitter well separated in time (Fig. 2.9). In order to restore distinguishability, postponed compensation was used; i.e. a selective time delay was added after the beam splitter that only affects the polarization traveling in branch  $a$ . The RR (reflection/reflection) and TT (transmission/transmission) cases may then be pictured as in Fig. 2.9. In the RR case the idler arrives at detector 2, delayed by a time  $\tau_b$ , so that the time difference between the two detections is  $\Delta t \equiv t_2 - t_1 = \tau_b$ . However, in the TT case the signal arrives at detector 2, delayed by  $\tau_{1a}$ , while the idler arrives at detector 1 delayed by  $\tau_b$ . Thus, in this case, the detection time difference is  $\Delta t \equiv t_2 - t_1 = \tau_a - \tau_b$ . So, if delays are chosen such that  $\tau_a = 2\tau_b$ , both cases have the same time difference,  $\Delta t = \tau_b$ . Since the time difference is the same for the RR and TT possibilities, they cannot be distinguished and the interference dip appears, even though the two photons never overlap on the beam splitter. This apparent nonlocality of the interference again demonstrates that the HOM dip is a purely quantum effect.

A similar effect can be seen in the Franson interferometer [16] shown in Fig. 2.10. The nonlinear crystal at the left is pumped by a beam at frequency  $\omega_p$ . Each photon from a down conversion pair enters a separate path. Each of those photons then has

**Fig. 2.10** The Franson interferometer. Each photon can travel via a long path or a short path. After post-selection to remove long-short interference, the amplitude for both to follow the long path interferes with the amplitude for both following the short path



a choice of two possible subpaths, one long and one short. At the other end, the two photons are then detected in coincidence. The difference of path lengths between the long and short paths is much longer than the photon coherence time, so that there no single-photon interference occurs between the long and short paths.

Since the time at which the pair is created is unknown, the cases where both photons take the long path (LL) and where both take the short path (SS) are indistinguishable, and so these two-photon states should interfere with each other. The visibility of this interference can be greatly enhanced by post-selection: take the coincidence window short enough to reject the two cases  $L_1 S_2$  and  $S_1 L_2$  in which one photon takes the long path and one takes the short path. In this case, the two-photon visibility is ideally 100 %, despite the fact that the two photons are never in the same location and so have no opportunity to interfere in the sense of classical waves.

Let  $\Delta l_s$  and  $\Delta l_i$  be the path difference between the long and short paths for the signal and idler respectively. Then, the phase difference between the  $LL$  path and the  $SS$  path is

$$\phi = \frac{\omega_s \Delta l_s}{c} + \frac{\omega_i \Delta l_i}{c} \approx \frac{1}{2} \omega_p (\Delta l_i + \Delta l_s), \quad (2.35)$$

so long as  $|\omega_s - \omega_i|$  is small compared to the central frequency  $\omega_0 = \frac{\omega_p}{2}$ . The coincidence probability is

$$P_c = |\langle \psi | \psi \rangle|^2 = \frac{1}{4} |1 + e^{i\phi}|^2 \quad (2.36)$$

$$= \frac{1}{2} (1 + \cos \phi) = \frac{1}{2} \left[ 1 + \cos \left( \frac{\omega_p}{2} (\Delta l_s + \Delta l_i) \right) \right]. \quad (2.37)$$

This interference was demonstrated experimentally in [17–20]. The 100 % interference visibility as the path lengths are varied is again an indication of an intrinsically quantum mechanical effect.

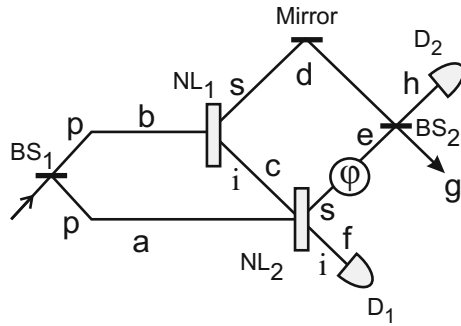


**Fig. 2.11** Schematic of setup for two-crystal interference experiment. The pump can undergo down conversion in either of two crystals. If the two signal modes and the two idler modes are sufficiently well overlapped, it is impossible to tell in which crystal the down conversion occurred. In that case, the amplitudes for both possibilities overlap and first-order interference can occur between the two potential signal photons at detector  $D_2$ . In the figure, p, s, and i stand, respectively, for pump, signal, and idler, and  $a-h$  label points in the possible photon paths

## 2.7 Double-Crystal Experiments and Induced Coherence

The existence of interference between two amplitudes requires that it is impossible to tell which amplitude was actualized; in particular, an interferometer only exhibits interference when it is impossible in principle for one to tell which path the photons took through the arrangement. As the degree of path distinguishability increases, the visibility of the interference pattern goes down. In addition, there is often a form of complementarity between single-photon (field or amplitude) interference and two-photon (intensity) interference [10, 11]: in most experiments, as the visibility of one type of interference goes up, the other goes down.

A highly counter-intuitive experiment that can exhibit first and second order interference simultaneously uses the Zhou–Wang–Mandel two-crystal interferometer shown schematically in Fig. 2.11 [21–23]. A pump beam is passed through a 50/50 beam splitter, so that each photon has equal amplitude to enter either of two nonlinear crystals,  $NL_1$  or  $NL_2$ . Imagine that a pump photon undergoes down conversion in one of the crystals. If the two crystals are very well aligned, so that the paths of the idlers produced by the crystals completely overlap, then it is impossible to tell whether an idler photon detected at detector  $D_1$  came from the second crystal or the first one. Similarly, it is impossible to detect which crystal produced a signal photon detected at  $D_2$ . Because it is impossible to distinguish whether the down conversion occurred in  $NL_1$  or  $NL_2$ , the amplitudes for both possibilities must be added. By varying the phase shift introduced between the two potential signal paths, interference is then seen at  $D_2$ .

However, if the idler path between  $NL_1$  and  $NL_2$  is blocked at point  $c$ , then any idler arriving at  $D_1$  must have come from  $NL_2$ . It is found that the interference at  $D_2$  then disappears, since the presence or absence of an idler detection at  $D_2$

gives information about which crystal the pair was born in. The presence of this path information makes the two amplitudes distinguishable. The interesting (or even “mind-boggling” [24]) thing is that only the *idler* photons are blocked by the filter at c, yet the interference pattern in the *signal* at  $D_2$  disappears, despite the fact that the signal never passed through c. If a beam splitter is placed at c, it is found that the visibility of the interference is decreased in proportion to the reflectance of the beam splitter; the more reflective the beam splitter, the more path information can be obtained by monitoring the presence of idler photons at  $D_1$  (or alternatively, by monitoring those reflected at c). Note that these photons do not actually have to be detected in order for the interference visibility to be affected. The simple *possibility* of their detection is all that is needed to damage the interference.

Second-order interference in the two-detector coincidence counting is to be expected in this setup, between potential biphoton states created in the two different crystals. But the single-photon, first-order interference occurring at a single detector may seem surprising. Because there is no fixed phase relation between the signal and idler photons, there can be no first-order interference effects between them. However, there can be single-photon interference between the two amplitudes for signal photons arising in the two different crystals due to the same pump. The phase of the full biphoton state is determined by the phase of the pump beam [25], so that different potential histories of the signal photons created in the different crystals should be capable of interfering with each other, even though the signal by itself is of low coherence. This hidden information about the pump phase that is retained by the outgoing state has been referred to as a *phase memory effect*. The single-photon and biphoton interferences show up, respectively, in the first and second-order correlation functions  $g^{(1)}(\tau)$  and  $g^{(2)}(\tau)$ .

If the pump intensity is low enough so that the probability of the idler from one crystal inducing down conversion in the other is negligible, the single-photon interference becomes noticeable, since the experimental arrangement cannot distinguish between down conversion photons produced at one crystal and those produced at the other. In this case, if point c is unblocked,  $D_1$  fires every time  $D_2$  fires (assuming perfect detectors and no photon losses). While indistinguishability is maintained, the potential idler photons originating in the two crystals are mutually coherent, as are the two signals; this is referred to as *induced coherence* between the crystals. There should then be first-order interference between the two possible signal photons. The coherence is induced by the act of matching the modes of the two idlers to maintain indistinguishability.

It has been shown [26–28] that interference can be obtained by varying the phase between the two signals, or by varying the relative phase of the two pump beams after  $BS_1$ . In both cases, interference arises in *both* the singles ( $g^{(1)}(\tau)$ ) and coincidence ( $g^{(2)}(\tau)$ ) counting rates. When varying signal phase, the interference has a fringe period corresponding to the frequency of the signal photons. In contrast, if the pump phase is varied, the coincidence rate and signal count rate both have fringes with the period expected from the pump frequency.

For a more quantitative discussion of the interferometer, note that after  $BS_1$  the state is

$$\frac{1}{\sqrt{2}} (i|1\rangle_a|0\rangle_b + |0\rangle_a|1\rangle_b). \quad (2.38)$$

After crystal  $NL_1$ , the pump photon at  $b$  undergoes down-conversion so that the state becomes

$$\frac{1}{\sqrt{2}} (i|1\rangle_a|0\rangle_c|0\rangle_d + |0\rangle_a|1\rangle_c|1\rangle_d). \quad (2.39)$$

Following the second crystal,  $NL_2$ , and the phase shift, the state becomes:

$$\frac{1}{\sqrt{2}} (ie^{i\phi}|0\rangle_c|0\rangle_d|1\rangle_e|1\rangle_f + |1\rangle_c|1\rangle_d|0\rangle_e|0\rangle_f). \quad (2.40)$$

But the idler at  $c$  continues through  $NL_2$  to appear at  $f$ , so that if the two idlers are indistinguishable, this becomes

$$\frac{1}{\sqrt{2}} (ie^{i\phi}|0\rangle_d|1\rangle_e + |1\rangle_d|0\rangle_e) |1\rangle_f. \quad (2.41)$$

After the final beam splitter, the result is

$$\frac{1}{2} [ie^{i\phi} (|0\rangle_g|1\rangle_h + i|1\rangle_g|0\rangle_h) + (|1\rangle_g|0\rangle_h + i|0\rangle_g|1\rangle_h)] |1\rangle_f \quad (2.42)$$

$$= \frac{1}{2} [i(e^{i\phi} + 1)|0\rangle_g|1\rangle_h - (e^{i\phi} - 1)|1\rangle_g|0\rangle_h] |1\rangle_f. \quad (2.43)$$

The probability of detection at  $D_1$  (point  $h$ ) is thus

$$\frac{1}{4} |e^{i\phi} + 1|^2 = \frac{1}{2} (1 + \cos \phi), \quad (2.44)$$

exhibiting interference with (ideally) 100% visibility. If a filter of transmissivity  $t$  is placed at point  $c$ , this becomes

$$\frac{1}{2} (1 + t \cos \phi). \quad (2.45)$$

It is clear that the interference visibility is proportional to  $t$  and that the pattern disappears completely if the idler path at  $c$  is blocked ( $t \rightarrow 0$ ). This is because if  $c$  is blocked then the path can be determined by checking for idler photons at  $D_1$ ; the existence of this which-path information destroys the interference pattern.

Rather than carrying out an interferometric experiment using the apparatus of Fig. 2.11, it is natural to consider if an object can be imaged with it. Instead of placing a filter of fixed transmissivity  $t$  at point  $c$ , consider an object with a spatially dependent transmissivity,  $T(x, y)$ , where  $x$  and  $y$  are coordinates perpendicular to the propagation axis. Then adding some lenses and beam splitters, the new apparatus



$$m = \frac{\lambda_s f'}{\lambda_i f}, \quad (2.49)$$

where  $\lambda_s$  and  $\lambda_i$  are the signal and idler wavelengths. The strange feature of this arrangement is that only the idler photon ever interacts with the object, but the image arises from measurement of the signal photons; the idlers are discarded without being detected.

The theory of the two-crystal imaging arrangement has been further worked out in [31, 32]. Like many of the initially quantum-based effects discussed in this book, it has been shown [33] that most of its features can be mimicked by a bright-source classical arrangement, with higher signal-to-noise ratio.

## References

1. G. Jaeger, *Entanglement, Information, and the Interpretation of Quantum Mechanics* (Springer, Berlin, 2009)
2. D.J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edn. (Pearson Education Ltd., Essex, 2004)
3. M. Fox, *Quantum Optics: An Introduction* (Oxford University Press, Oxford, 2006)
4. Y.F. Bai, S.S. Han, Phys. Rev. A **76**, 043828 (2007)
5. D.Z. Cao, J. Xiong, S.H. Zhang, L.F. Lin, L. Gao, K.G. Wang, Appl. Phys. Lett. **92**, 201102 (2008)
6. Q. Liu, X.H. Chen, K.H. Luo, W. Wu, L.A. Wu, Phys. Rev. A **79**, 053844 (2009)
7. K.W.C. Chan, M.N. O'Sullivan, R.W. Boyd, Opt. Lett. **34**, 3343 (2009)
8. Y. Zhou, J. Simon, J.B. Liu, Y.H. Shih, Phys. Rev. A **81**, 043831 (2010)
9. B. Cao, C. Zhang, Opt. Lett. **35**, 2091 (2010)
10. G. Jaeger, M.A. Horne, A. Shimony, Phys. Rev. A **48**, 1023 (1993)
11. G. Jaeger, A. Shimony, L. Vaidman, Phys. Rev. A **51**, 54 (1995)
12. C.K. Hong, Z.Y. Ou, L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987)
13. A.M. Steinberg, P.G. Kwiat, R.Y. Chiao, Phys. Rev. Lett. **68**, 2421 (1992)
14. A.M. Steinberg, P.G. Kwiat, R.Y. Chiao, Phys. Rev. Lett. **71**, 708 (1993)
15. T.B. Pittman, D.V. Strekalov, M.H. Migdall, M.H. Rubin, A.V. Sergienko, Y.H. Shih, Phys. Rev. Lett. **77**, 1917 (1996)
16. J.D. Franson, Phys. Rev. Lett. **62**, 7205 (1989)
17. Z.Y. Ou, X.Y. Zou, L.J. Wang, L. Mandel, Phys. Rev. Lett. **65**, 321 (1990)
18. J. Brendel, E. Mohler, W. Martienssen, Euro. Phys. Lett. **20**, 575 (1992)
19. Y.H. Shih, A.V. Sergienko, M.H. Rubin, Phys. Rev. A **47**, 1288 (1993)
20. P.G. Kwiat, A.M. Steinberg, R.Y. Chiao, Phys. Rev. Lett. **47**, R2472 (1993)
21. X.Y. Zhou, L.J. Wang, L. Mandel, Phys. Rev. Lett. **67**, 318 (1991)
22. L.J. Wang, X.Y. Zhou, L. Mandel, Phys. Rev. A **44**, 4614 (1991)
23. X.Y. Zou, T.P. Grayson, G.A. Barbosa, L. Mandel, Phys. Rev. A **47**, 2293 (1993)
24. D.M. Greenberger, M.A. Horne, A. Zeilinger, Phys. Today **46**, 22 (1993)
25. P. Grangier, M.J. Potasek, B. Yurke, Phys. Rev. A **38**, 3132 (1988)
26. A. Heuer, S. Raabe, R. Menzel, Phys. Rev. A **90**, 045803 (2014)
27. A. Heuer, R. Menzel, P.W. Milonni, Phys. Rev. Lett. **114**, 053601 (2015)
28. A. Heuer, R. Menzel, P.W. Milonni, Phys. Rev. A **92**, 033834 (2015)
29. G.B. Lemos, V. Borish, G.D. Cole, S. Ramelow, R. Lapkiewicz, A. Zeilinger, Nature **512**, 409 (2014)
30. D.S. Simon, M.H. Horne, unpublished notes (2014)

31. M. Lahiri, R. Lapkiewicz, G.B. Lemos, A. Zeilinger, *Phys. Rev. A* **92**, 013832 (2015)
32. M. Ghalaii, M. Afsary, S. Alipour, A.T. Rezakhani, [arXiv:1509.02031](https://arxiv.org/abs/1509.02031) [quant-ph] (2015)
33. J. Shapiro, D. Venkatraman, F.N.C. Wong, *Sci. Rep.* **5**, 10329 (2015)

## Chapter 3

# Aberration and Dispersion Cancellation

### 3.1 Introduction: Cancellation of Optical Phase Distortions

In the following sections, we look at the use of correlated-photon methods for the cancellation of distortive effects in optical systems. These effects usually appear in the form of unwanted phase-shifts that arise as the light passes through the system. These phase shifts come in several varieties. (i) If they are functions of frequency, then the resulting distortion occurs in the form of *dispersion*; dispersion cancellation is discussed in Sect. 3.2. (ii) If the phases are time-independent functions of position, then the distortion is *aberration* or *spatial dispersion*. Aberrations commonly occur, as a result of the properties of the optical system itself, such as imperfections in a lens. Aberration is discussed in detail in Sect. 3.4. (iii) Finally, the unwanted phases may also occur as random functions of both position and time, as a result of *turbulence*. Discussion of turbulence is postponed until Sect. 6.4.

Each of the above cases involves the addition of extra phases (either random or deterministic) of the form  $e^{i\phi(x,t)}$ . The goal is to find a way to cancel these phases using pairs of entangled photons or classically correlated beams detected in coincidence. In general, the methods follow a common pattern: (i) Two photons are arranged so that they are distorted by equal and opposite phase shifts,  $\phi_1 = -\phi_2$ . (ii) A coincidence measurement then causes the phase factors to be multiplied, leading to cancellation:  $e^{i\phi_1} \cdot e^{i\phi_2} = e^{i(\phi_1+\phi_2)} = 1$ . A number of variations of this basic idea exist, such as canceling just those parts of the phases which are symmetric or antisymmetric under some transformation; but the essential idea is always the same. This chapter looks at several applications of this approach.

### 3.2 Dispersion Cancellation

Recall that propagating light waves have two different speeds readily associated with them. The *phase velocity* is the ratio of frequency to wavenumber,

$$v_p = \frac{\omega}{k}, \quad (3.1)$$

and gives the speed at which individual wavefronts propagate, while the *group velocity*,

$$v_g = \frac{d\omega}{dk}, \quad (3.2)$$

provides the speed of a narrow wavepacket centered at frequency  $\omega$ . In vacuum these two velocities are equal, but inside a material interactions of the light with optical resonances cause them to differ. This is a result of the fact that the refractive index inside that material is no longer constant, but instead depends (for fixed polarization and propagation direction) on the frequency of the propagating light. As a result, light pulses with broad spectra tend to spread out in time as they propagate: portions of the wave packet with different frequencies travel at different speeds and gain different phase shifts. This phenomenon is known as *dispersion*.

Imagine that light with a spread of frequencies, distributed about a mean frequency  $\omega_0$ , encounters a dispersive element of thickness  $L$ . In a nondispersive material such as vacuum, the photon wavenumber  $k$  and frequency  $\omega$  are connected by a linear dispersion relation,

$$k(\omega) = \frac{n\omega}{c}, \quad (3.3)$$

where the refractive index  $n$  is a frequency-independent constant. However, inside a dispersive material the refractive index becomes a function of frequency,  $n(\omega)$ , so that the dispersion relation becomes nonlinear.

The dispersion relation may be expanded as a Taylor series about the mean frequency of the wavepacket,

$$k(\omega) = k_0 + \alpha(\omega - \omega_0) + \beta(\omega - \omega_0)^2 + \dots, \quad (3.4)$$

where  $k_0 = \frac{n(\omega_0)}{c}\omega_0$  is the wavenumber at the mean frequency. The phase shift  $\phi(\omega)$  acquired by an optical wavepacket as it passes through the dispersive medium may be written as

$$\phi(\omega) = k(\omega)L \quad (3.5)$$

$$= \phi_0 + c_1(\omega - \omega_0) + c_2(\omega - \omega_0)^2 + c_3(\omega - \omega_0)^3 + \dots \quad (3.6)$$

Here,  $\phi_0 = k_0L$  is the phase shift acquired by photons at the mean frequency, while the *group delay* term  $c_1 = \alpha L$  simply represents a time delay for the pulse as a whole; it causes an overall translation of the interference pattern.  $c_2 = \beta L$ , the *group delay dispersion*, causes broadening of wave packets and of interference peaks. The term  $c_3$  and higher terms introduce skewing and other types of distortions into interference patterns.

It is useful to have an expression for the amount of pulse broadening introduced by a material with a given amount of dispersion. Note that inside the material, the group velocity may be found as a function of frequency,

$$v_g(\omega) = \frac{d\omega}{dk} = [\alpha + 2\beta(\omega - \omega_0) + \dots]^{-1} \quad (3.7)$$

$$\approx \frac{1}{\alpha} - \frac{2\beta(\omega - \omega_0)}{\alpha^2}, \quad (3.8)$$

where a binomial approximation has been applied in the last line. The velocity difference between frequencies separated by the full bandwidth  $\Delta\omega$  of the pulse is given by

$$\Delta v_g = |v_g(\omega + \Delta\omega) - v_g(\omega)| = \frac{2\beta}{\alpha^2} \Delta\omega, \quad (3.9)$$

so the temporal broadening of the pulse is given by

$$\Delta t = \left| \frac{L}{v_g} - \frac{L}{v_g + \Delta v_g} \right| = \frac{L}{v_g} \left| 1 - \frac{1}{1 + \frac{\Delta v_g}{v_g}} \right| \quad (3.10)$$

$$\approx \frac{L \Delta v_g}{v_g^2} \quad (3.11)$$

$$= 2\beta L \Delta\omega. \quad (3.12)$$

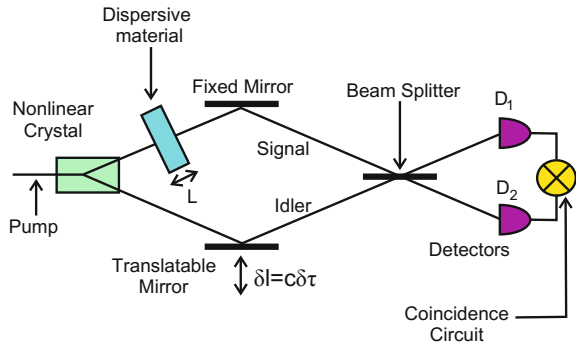
So, as advertised above, the amount of broadening is determined by the second-order dispersion coefficient  $\beta$ .

For more detailed discussion of dispersion, group and phase velocities, and related topics, see [1]. Experimental demonstrations of dispersion cancellation have been carried out up to fifth order [2].

### 3.2.1 Dispersion Cancellation

Quantum dispersion cancellation, demonstrated in quantum optics in the early 1990's [3, 4], relies on the frequency-anticorrelation of entangled photon pairs produced via SPDC. The even-order cancellation effect [4] is seen when dispersive material is added inside the Hong–Ou–Mandel (HOM) interferometer of the previous chapter. There is a triangularly shaped modulation (the ‘‘HOM dip’’) in the measured coincidence rate as the time delay in one arm of the interferometer is varied. The resulting interferogram depends only on the odd-order dispersion coefficients of the material. In particular, the HOM dip is not broadened by the dispersion, demonstrating insensitivity to the second-order group coefficient,  $c_2$  or  $\beta$  [5]. A separate, nonlocal type of dispersion effect occurs when entangled photons are sent through opposite-sign dispersive materials [3].

**Fig. 3.1** HOM interferometer with dispersive material in one arm. The coincidence rate exhibits even-order dispersion cancellation



Dispersion cancellation has found a number applications, including high resolution timing measurements [6], improved clock synchronization [7] and enhanced polarization mode dispersion measurements [8–10] (Chap. 5). This effect is also present in quantum optical coherence tomography (QOCT) [5, 11, 12] (Chap. 4).

It has been shown [13] that a novel type of quantum interferometer can be constructed that separates the effects of even-order and odd-order dispersion terms into different portions of an interferogram, so that they may be measured or manipulated separately. In other words, one part of the interferogram exhibits even-order cancellation, while other parts exhibit odd-order cancellation. This is discussed in more detail in Sect. 3.3.

Although the original demonstrations of even-order dispersion cancellation made use of entangled photons, it has been shown [14–19] that the effect may be mimicked using classically correlated light, so entanglement does not seem to be essential. Below, we follow the original treatments, making use of entangled photon pairs.

In the following subsections, we look briefly at the two forms of dispersion cancellation, those discovered by Franson [3], and by Steinberg, Kwiat and Chiao [4].

### 3.2.2 Steinberg–Kwiat–Chiao Dispersion Cancellation

Consider the apparatus shown in Fig. 3.1. A noncollinear parametric down conversion source feeds entangled photon pairs into an HOM interferometer, with the output ports connected in coincidence. One branch of the interferometer contains a dispersive element of length  $L$  before the beamsplitter; dispersion is assumed to be negligible in the other branch. The dispersion relation is taken to be of the form given in Eq. 3.4. In the lower branch, the mirror is translatable;  $\delta l$  is the distance the mirror has been moved away from the position that gives equal path lengths to the two branches.

The light emitted from the crystal is given by

$$|\Psi\rangle = \int d\omega \Phi(\omega) |\omega_0 + \omega, \omega_0 - \omega\rangle, \quad (3.13)$$

where the first and second arguments of the ket on the right correspond to signal and idler modes respectively. The spectral function of the pump is given in Eq. 1.77, but all that is relevant here is the fact that it is symmetric,  $\Phi(-\omega) = \Phi(\omega)$ .

If the signal and idler annihilation operators are denoted by  $\hat{a}_s(\omega)$  and  $\hat{a}_i(\omega)$ , then after the beam splitter and propagation phase shifts, the annihilation operators at detectors 1 and 2 are given by

$$\hat{a}_1(\omega_1) = \frac{1}{\sqrt{2}} (i\hat{a}_s(\omega_1)e^{ik(\omega_1)L} + \hat{a}_i(\omega_1)e^{i\omega_1\delta l/c}) \quad (3.14)$$

$$\hat{a}_2(\omega_2) = \frac{1}{\sqrt{2}} (i\hat{a}_i(\omega_2)e^{i\omega_2\delta l/c} + \hat{a}_s(\omega_2)e^{ik(\omega_2)L}). \quad (3.15)$$

The positive- and negative-frequency components of the field at the detector  $j$  ( $j = 1, 2$ ) may also be written in terms of the creation and annihilation operators,

$$E_j^+(t_j) = \int d\omega_j^+ \hat{a}_j(\omega_j^+) e^{-i\omega_j^+ y_j} \quad (3.16)$$

$$E_j^-(t_j) = \int d\omega_j^- \hat{a}_j^\dagger(\omega_j^-) e^{i\omega_j^- y_j} = \left(E_j^+(t_j)\right)^\dagger. \quad (3.17)$$

The coincidence rate can be computed from

$$R_c = \int_0^T dt_1 dt_2 \langle \Psi | E_1^-(t_1) E_2^-(t_2) E_1^+(t_1) E_2^+(t_2) | \Psi \rangle, \quad (3.18)$$

where  $T$  is the integration time of the detector. Assuming that the detectors are slow compared to the characteristic down conversion time, one can take  $T \rightarrow \infty$  so that the time integrals become proportional to delta functions. Substituting Eqs. 3.14–3.17 into  $R_c$  and making use of the commutation relations

$$[\hat{a}_j(\omega), \hat{a}_k^\dagger(\omega')] = \delta_{jk} \delta(\omega - \omega') \quad (3.19)$$

$$[\hat{a}_j(\omega), \hat{a}_k(\omega')] = [\hat{a}_j^\dagger(\omega), \hat{a}_k^\dagger(\omega')] = 0, \quad (3.20)$$

gives the result

$$R_c = \int d\omega_1 d\omega_2 d\omega d\omega' \Phi(\omega) \Phi^*(\omega') |\psi|^2, \quad (3.21)$$

where the coincidence amplitude is given by

$$\psi = \langle 0 | \hat{a}_1(\omega_1) \hat{a}_2(\omega_2) \hat{a}_1^\dagger(\omega_0 - \omega) \hat{a}_s^\dagger(\omega_0 + \omega) | 0 \rangle \quad (3.22)$$

$$= \frac{1}{2} \left\{ -\delta(\omega_1 - \omega_0 - \omega) \delta(\omega_2 - \omega_0 + \omega) e^{i[k(\omega_1)L + \omega_1 \delta l / c]} \right. \\ \left. + \delta(\omega_1 - \omega_0 + \omega) \delta(\omega_2 - \omega_0 - \omega) e^{i[k(\omega_2)L + \omega_2 \delta l / c]} \right\}. \quad (3.23)$$

Carrying out three of the frequency integrations in  $R_c$  by means of the delta functions and making use of the fact that  $\Phi(-\omega) = \Phi(\omega)$ , the coincidence rate reduces to

$$R_c = \frac{1}{2} \int d\omega |\Phi(\omega)|^2 \left\{ 1 - \cos \left[ [k(\omega_0 - \omega) - k(\omega_0 + \omega)] L + \frac{2\delta l}{c} \omega \right] \right\} \quad (3.24)$$

$$= \frac{1}{2} \int d\omega |\Phi(\omega)|^2 \left\{ 1 - \cos \left[ 2 \left( \alpha L - \frac{\delta l}{c} \right) \omega \right] \right\}, \quad (3.25)$$

where only terms up to second order in  $\omega$  were kept in the expansion of  $k(\omega)$ . Note that  $\beta$  no longer appears in the coincidence rate, and so there is no broadening of wave packets. More generally, if higher order terms are kept, then it is found that all even-order dispersion coefficients cancel, leaving the coincidence rate dependent only on the odd-order dispersion.

### 3.2.3 Franson Dispersion Cancellation

In the Steinberg–Kwiat–Chiao method of dispersion cancellation, the two photons in each pair are mixed at the beam splitter. A different type of dispersion cancellation occurs in the apparatus shown in Fig. 3.2 [3, 20]. There is now dispersion in both branches and no mixing between the two beams, i.e. no beam splitter before the detectors. Once again the coincidence rate is recorded.

The input light is assumed for simplicity to have a Gaussian spectrum

$$\Phi(\Omega_0 + \omega) = \Phi(\Omega_0) e^{-\omega^2 / 2\eta^2} \quad (3.26)$$

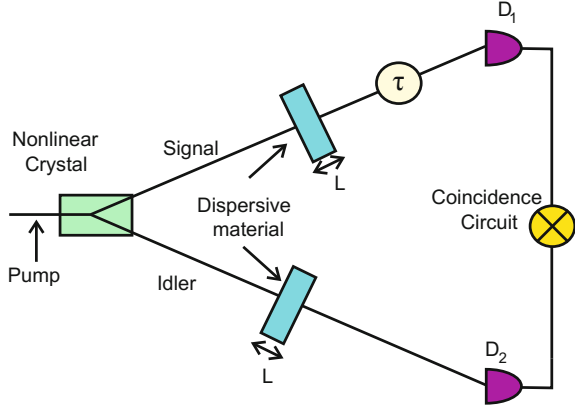
of width  $\eta$ . Due to dispersion, the magnitudes of the entangled beam wave vectors in the two arms are given by

$$k_1 = k_0 + \alpha_1 \omega + \beta_1 \omega^2 \quad (3.27)$$

$$k_2 = k_0 + \alpha_2 \omega + \beta_2 \omega^2, \quad (3.28)$$

where  $\alpha_1, \beta_1$  are the dispersion coefficients in one arm and  $\alpha_2, \beta_2$  are the coefficients in the other arm. The width of the distribution in detection times for a single beam

**Fig. 3.2** Franson-type dispersion cancellation. A parametric down conversion source illuminates a pair of dispersive materials. The coincidence rate exhibits dispersion cancellation if the dispersion coefficients in the two arms are opposite in sign and equal in magnitude



in the absence of dispersion is  $\sigma_0 = \frac{1}{\eta}$ . Normally, the second order coefficient  $\beta_i$  in each arm would broaden the incoming light pulse and increase the width of the temporal distribution. The idea is to look at the distribution of coincidence counts and to compare it to the original width  $\sigma_0$ , to see whether the dispersive broadening of the pulse can be avoided.

By methods similar to those used in earlier sections, it can be shown [3] that the variance in the distribution of coincidence times in Fig. 3.2 is

$$\sigma_Q^2 = \frac{1/\eta^4 + (\beta_1 + \beta_2)^2 L^2}{1/\eta^2} = \frac{\sigma_0^4 + (\beta_1 + \beta_2)^2 L^2}{\sigma_0^2}. \quad (3.29)$$

So it is easily seen that if the second order dispersion coefficients are equal and opposite in the two branches,  $\beta_2 = -\beta_1$ , then the dispersive broadening in the two beams cancels, leaving a coincidence distribution equal in width to that which would occur in the absence of any dispersion at all:

$$\sigma_Q = \sigma_0. \quad (3.30)$$

Thus, there is no dispersive broadening. This is to be compared to the classical case, where the same apparatus is illuminated by a pair of uncorrelated but statistically identical pulses of light:

$$\sigma_C^2 = \frac{1/2\eta^4 + (\beta_1^2 + \beta_2^2)L^2}{1/\eta^2}. \quad (3.31)$$

In the classical case, we see that the  $\beta_i$  are squared *before* adding, rather than after, so that there is no possibility of a similar cancellation.

### 3.3 Separation of Even and Odd Orders

In addition to canceling even and odd orders of dispersion, other sorts of manipulation are possible using similar techniques. For example, it is possible [13] to isolate the effects of even and odd dispersive orders into separate portions of an interferogram.

The basic setup (Fig. 3.3) consists of a Hong-Ou-Mandel (HOM) interferometer (see Sect. 2.5) with one output port feeding into a Mach-Zehnder (MZ) interferometer. These two interferometers have adjustable time delays  $\tau_1$  and  $\tau_2$  respectively. The entire setup is fed with entangled two-photon states generated by SPDC. A sample of some material is placed into one branch of the MZ interferometer, introducing a dispersive phase shift  $\phi(\omega)$  that may be expanded in the form of Eq. 3.6. The HOM delay  $\tau_1$  is scanned to construct an interferogram of coincidence rate versus  $\tau_1$ . During each scan, the MZ delay  $\tau_2$  is held fixed.

Up to overall constants, the coincidence rate can be written in form

$$R_c(\tau_1, \tau_2) = \int |\langle 0 | \hat{E}_1^{(+)}(t_1) \hat{E}_2^{(+)}(t_2) | \Psi \rangle|^2 dt_1 dt_2, \quad (3.32)$$

where  $|\Psi\rangle$  is the entangled down conversion output state. Propagating the field through the interferometer, this can be put in the form

$$R_c(\tau_1, \tau_2) = B + R_0(\tau_1) - R_{\text{even}}(\tau_1, \tau_2) - R_{\text{odd}}(\tau_1, \tau_2), \quad (3.33)$$

where  $B$  is a delay-independent constant. The second term is

$$R_0(\tau_1) = 4 \int d\omega \Phi(\omega) \Phi^*(-\omega) e^{-2i\omega\tau_1}, \quad (3.34)$$

where  $\Phi(\omega)$  is the spectral profile of the down conversion source (Eq. 1.77). This term is simply the power spectrum of the source. Note that  $R_0$  is  $\phi$ -independent, and so is insensitive to dispersive effects.

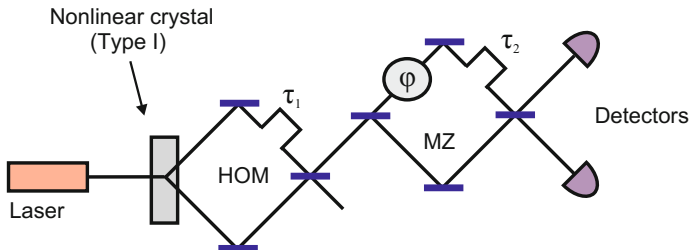


**Fig. 3.3** HOM and MZ interferometers coupled together in order to separate even and odd orders of dispersion

The next term,

$$R_{\text{even}}(\tau_1, \tau_2) = \int d\omega \Phi(\omega) \Phi^*(-\omega) e^{-2i\omega\tau_1} \times [e^{-2i\omega_0\tau_2} e^{-i\phi(\omega_0-\omega)} e^{-i\phi(\omega_0+\omega)} + e^{2i\omega_0\tau_2} e^{i\phi(\omega_0-\omega)} e^{i\phi(\omega_0+\omega)}], \quad (3.35)$$

is a function of the phase shift only through the combination  $\phi(\omega_0-\omega) + \phi(\omega_0+\omega)$  and so only depends on the even-order dispersion coefficients  $\phi_0, c_2, c_4$  appearing in the expansion of Eq. 3.6. This term therefore exhibits odd-order dispersion cancellation.

In a similar manner, the final term

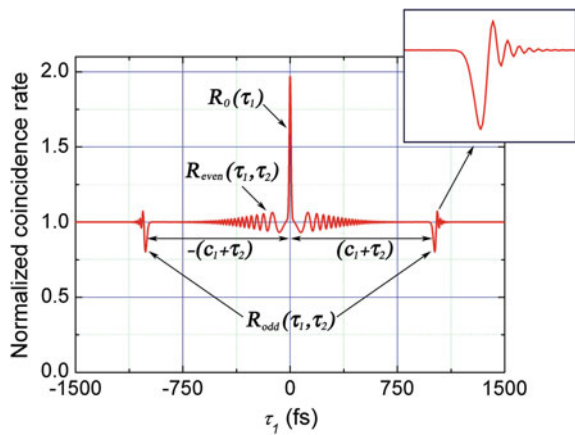
$$R_{\text{odd}}(\tau_1, \tau_2) = \int d\omega \Phi(\omega) \Phi^*(-\omega) \times [e^{-2i\omega(\tau_1+\tau_2)} e^{i\phi(\omega_0-\omega)} e^{-i\phi(\omega_0+\omega)} + e^{2i\omega(\tau_1+\tau_2)} e^{-i\phi(\omega_0-\omega)} e^{i\phi(\omega_0+\omega)}], \quad (3.36)$$

depends only on the combination  $\phi(\omega_0 - \omega) - \phi(\omega_0 + \omega)$ , and therefore displays even-order cancellation.

A plot of the resulting interferogram is shown in Fig. 3.4. The large peak at zero time delay is due to the  $\tau_2$ -independent source term  $R_0$ . The oscillations spreading symmetrically away from the center is due to  $\phi_{\text{even}}$ ; it is dispersively broadened by the group delay dispersion term,  $c_2$ .

The two symmetrically placed dips are located at  $\tau_1 = \pm(c_1 + \tau_2)$ , where  $c_1$  is the group velocity delay coefficient. The oscillating fringes on the dip are due to the higher odd-order terms. Note that by varying the second delay,  $\tau_2$ , the two dips may be moved to any desired location in the plot. Alternatively, by applying a known  $\tau_2$ , one may then read off the value of  $c_1$  by measuring the position of the dip. The precision of the measurement is determined by the width of the dip. By taking  $\tau_2$

**Fig. 3.4** Interferogram showing separation of even and odd dispersive orders. The central peak is due to the power spectrum of the down conversion source, the surrounding oscillations are from even order terms, and the two isolated dips originate from odd orders. (Figure reproduced from [13])



large, the dips can be moved completely away from the region of the even-order oscillation in order to locate the zero of the dip more accurately.

The effect can be interpreted physically as follows. Let  $\tau_c$  be the width of the incoming photon wave packet. Then there should be no single-photon interference if  $|\tau_2| \gg \tau_c$ ; similarly, there should be no two-photon interference if  $|\tau_1| \gg \tau_c$  since this would make the photon paths distinguishable by their arrival times. However, there is an exception to the latter statements: regardless of how large  $\tau_1$  and  $\tau_2$  are relative to  $\tau_c$ , interference can still occur if  $|\tau_2 - \tau_1| \ll \tau_c$ . This is because, analogous to the case of long-long interference in the Franson interferometer of Sect. 2.6, one photon can take the long path in the HOM stage and the other take the long path in the MZ stage. Then they arrive nearly simultaneously, making it impossible to determine which photon took which path. Because of this, destructive cancellation can occur, resulting in an HOM dip (Sect. 2.5). Further, by an argument similar to that given in Sect. 3.2, even order cancellation of the Steinberg–Kwiat–Chiao type occurs as  $\tau_1$  is varied; this results in the two odd-order dips. When  $\tau_1 < \tau_c$ , the two photons are distinguishable due to the large  $\tau_2$  delay, unless they both take the same path through the MZ interferometer; in this case the reflection-reflection and transmission-transmission amplitudes interfere. Then the total phase change of the two-photon wavefunction is the sum of the two phases,  $\phi(\omega_0 + \omega) + \phi(\omega_0 - \omega)$ , leading to odd-order cancellation. This interference pattern is broadened by the even-order dispersion terms that survive.

Variations of the same idea were carried out in [21, 22]; for example, in the latter paper switching between even-order and odd-order dispersion cancellation is achieved by rotating polarizers in front of the detectors: even orders cancel when the polarizers are orthogonal and odd orders cancel when they are parallel.

### 3.4 Aberration Cancellation in Interferometry

In the previous section, the focus was on dispersion, in which a distortion of the *temporal* profile of a wave results from a frequency-dependent refractive index. In this section, the *spatial* analog of this effect is examined, in which the transverse distribution of light is altered by a system whose optical response depends on the position of the light in the plane perpendicular to the propagation axis. The focus here is on aberration cancellation in interferometry; the extension to imaging systems is discussed after ghost imaging is introduced in Chap. 6.

#### 3.4.1 Optical Aberration

Consider light reflecting from or transmitting through an object, a lens, or some other optical element. In the process, the outgoing light may gain spatially-dependent phase shifts,  $\phi(\mathbf{x})$ , where  $\mathbf{x}$  is the two-dimensional position vector perpendicular to

the propagation axis (taken to be the  $z$ -axis). *Aberration* or *spatial dispersion* then occurs when these phases deviate from those that would be obtained via Gaussian or paraxial optics [23] with ideal optical elements. These deviations cause distortions of the outgoing wavefronts, and thus reduce the quality of images. A brief review of optical aberrations may be found in Appendix C.1.

The goal here is to cancel or manipulate some or all of the phase shifts introduced by passage through an optical device. These manipulations are collectively referred to as *aberration cancellation*, although the process is actually slightly more general than this name implies: the methods can be used to cancel *all* phases arising in a fixed plane, not just those leading to aberrations.

The phases  $\phi(\mathbf{x})$  are often expanded into a power series in  $|\mathbf{x}|$  and separated into even and odd orders,

$$\phi(\mathbf{x}) = \phi_{\text{even}}(\mathbf{x}) + \phi_{\text{odd}}(\mathbf{x}), \quad (3.37)$$

$$\phi_{\text{even}}(\mathbf{x}) = \sum_j a_{2j} P_{2j}(\theta) r^{2j}, \quad (3.38)$$

$$\phi_{\text{odd}}(\mathbf{x}) = \sum_j a_{2j+1} P_{2j+1}(\theta) r^{2j+1}. \quad (3.39)$$

Here,  $r = |\mathbf{x}|$  is the distance from the axis, while  $P_{2j}(\theta)$  and  $P_{2j+1}(\theta)$  are polynomials in  $\sin \theta$  and  $\cos \theta$ . Usually, the expansion is expressed in terms of Seidel or Zernike polynomials (see Appendix C.1 or [24–26]). For the present purposes, the details of the expansion are unimportant; the primary concern here is with the partition of the phase function into terms that are symmetric under reflection,  $\phi_{\text{even}}(\mathbf{x}) = \phi_{\text{even}}(-\mathbf{x})$ , or antisymmetric,  $\phi_{\text{odd}}(\mathbf{x}) = -\phi_{\text{odd}}(-\mathbf{x})$ .

Methods for canceling or manipulating the even- and odd-order parts of the phases by means of two-photon correlation is the focus of this section. We begin with a two-photon interferometric device [27, 28] that allows all even-order phase shifts introduced by an object to cancel from measured temporal correlation functions in a manner similar to the frequency-dispersion cancellation described in previous sections. The cancellation effect depends on the entanglement of the transverse spatial momenta in the resulting entangled photon pairs.

The apparatus and the even-order aberration cancellation effect are reviewed in the next subsection. It is also seen (Sect. 3.4.3) that it is possible in fact to use the apparatus to cancel *all* phase shifts, both even-order and odd-order arising from a sample when it is placed in a particular plane. The latter fact opens up the possibility of canceling sample-induced aberration in dynamic light scattering [29, 30], fluorescence correlation spectroscopy [31], or other temporal correlation-based experiments. Further discussion of the physical mechanisms responsible for aberration cancellation follows in Sect. 3.4.5.

### 3.4.2 Even-Order Aberration-Cancellation

Consider the setup [27, 28, 32] shown in Fig. 3.5. In the portion of the apparatus before the beam splitter, the two branches each consist of a  $4f$  Fourier transform system (see Appendix A) containing lenses of focal length  $f$  and a sample providing a modulation  $G_j(\mathbf{y})$  of the beam, where  $j = s, i$  labels the branch and  $\mathbf{y}$  is the transverse distance from the optical axis. The  $G_j$  represent the transmission or reflection profiles of objects or samples under study, and the goal is to cancel optical aberrations (or more generally, spatially-dependant phase shifts) introduced by the samples themselves. The special case where there is a sample only in the upper branch is covered by setting  $G_i = 1$ . The sample is in the plane that is simultaneously the back focal plane of the first lens and the front focal plane of the second. This is referred to as the  $\Pi$ -plane and is the location at which aberration cancellation occurs. Assume that the sample is of negligible thickness compared to all of the other distances involved in the apparatus and the illumination is type II SPDC. The polarizing beam splitter sends the horizontally polarized photon (the signal) into the upper branch and the vertically polarized photon (the idler) into the lower branch. The two beams are then recombined at a beam splitter and the coincidence rate is measured as the time delay  $\tau$  between the two arms is varied.

The state of the entangled photon pair is given by (Sect. 1.6.1)

$$|\Psi\rangle \approx \int dq d\nu \Phi(\mathbf{q}, \nu) \hat{a}_s^\dagger(\mathbf{q}, \omega_0 + \nu) \hat{a}_i^\dagger(-\mathbf{q}, \omega_0 - \nu) |0\rangle. \quad (3.40)$$

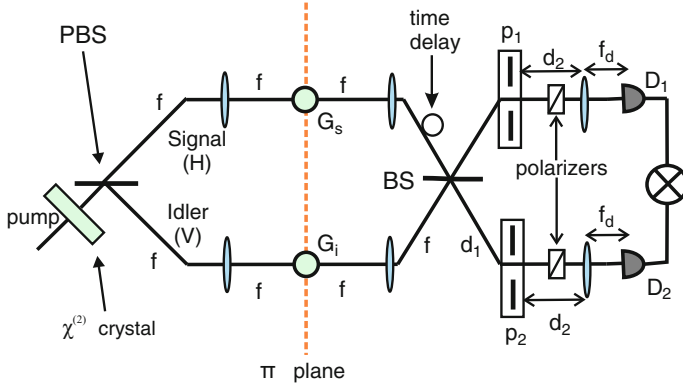


**Fig. 3.5** Schematic view of aberration-cancellation setup. (Distances and angles not necessarily drawn in correct proportions.) The polarizing beam splitter (PBS) directs the horizontally polarized signal into the upper branch, where it experiences modulation  $G_s$ , while the vertically polarized idler experiences modulation  $G_i$  in the lower branch.  $G_s$  and  $G_i$  are both located in the plane  $\Pi$ , halfway between the lenses of focal length  $f$ . The nonpolarizing beam splitter (BS) mixes the beams before they pass through apertures  $p_1$  and  $p_2$ , eventually reaching detectors  $D_1$  and  $D_2$ , which are connected by a coincidence circuit

Note that  $G_s$  and  $G_i$  need not be separate objects; they could represent the different actions produced by a single object on two orthogonal polarization states, in which case the first polarizing beam splitter can be dispensed with and the two branches would be merged into one until the final beam splitter.

The two bucket detectors  $D_1$  and  $D_2$  are connected in coincidence, and for generality, two adjustable irises with aperture functions  $p_1(\mathbf{x}_1)$  and  $p_2(\mathbf{x}_2)$  are added in front of the detectors. A lens of focal length  $f_d$  is placed one focal length in front of each detector. The distances from the Fourier plane of the main part of the apparatus to the aperture and from the aperture to the lens are  $d_1$  and  $d_2$ . In order to erase path information for the photons reaching each detector, a polarizer at  $45^\circ$  to the polarization directions of both incoming beams is placed in each path. The two polarizers are oriented orthogonal to each other.

The full transfer function (see Appendix A) for each branch is [28]

$$H_{j\alpha}(\mathbf{x}_\alpha, \mathbf{q}_j, \omega) = G_j\left(\frac{f}{k}\mathbf{q}_j\right)H_{D_\alpha}(\mathbf{x}_\alpha, \mathbf{q}_j, \omega), \quad (3.41)$$

where the transfer function of the detection stage is

$$H_{D_\alpha}(\mathbf{x}_\alpha, \mathbf{q}_j, \omega) = e^{ik(d_1+d_2+f_D)} e^{-\frac{ik}{2f_D}\left(\frac{d_2}{f_D}-1\right)\mathbf{x}_\alpha^2} e^{-\frac{id_1}{2k}\mathbf{q}_j^2} \tilde{\mathcal{P}}_\alpha\left(\frac{k}{f_D}\mathbf{x}_\alpha - \mathbf{q}_j\right). \quad (3.42)$$

$\tilde{\mathcal{P}}_\alpha(\mathbf{k})$  is the Fourier transform of the aperture function  $p_\alpha(\mathbf{x})$ ,

$$\tilde{\mathcal{P}}_\alpha\left(\frac{k}{f_D}\mathbf{x}_\alpha - \mathbf{q}_j\right) = \int d^2x' p_\alpha(\mathbf{x}') e^{-i\left(\frac{k}{f_D}\mathbf{x}_\alpha - \mathbf{q}_j\right) \cdot \mathbf{x}'}, \quad (3.43)$$

with  $\alpha = \{1, 2\}$  labelling the detector and  $j = \{s, i\}$  labelling the signal or idler branch. In these expressions,  $k$  is the longitudinal wavenumber,  $k = \sqrt{(\omega/c)^2 - q^2} \approx \frac{\omega}{c}$  for  $|\mathbf{q}| \ll k$ .

The nonpolarizing beam splitter mixes the incident beams, so each detector sees a superposition of signal and idler. The positive-frequency part of the field entering detector  $\alpha$  is given by

$$E_\alpha^{(+)}(\mathbf{x}_\alpha, t_\alpha) = \int dq d\omega e^{-i\omega t_\alpha} \left[ H_s(\mathbf{x}_\alpha, \mathbf{q}_s, \omega) \hat{a}_s(\mathbf{q}_s, \omega) + H_{is}(\mathbf{x}_\alpha, \mathbf{q}_i, \omega) \hat{a}_i(\mathbf{q}_i, \omega) \right]. \quad (3.44)$$

Using this field, the amplitude for coincidence detection is found

$$A(\mathbf{x}_1, \mathbf{x}_2, t_1, t_2) = \langle 0 | E_1^{(+)}(\mathbf{x}_1, t_1) E_2^{(+)}(\mathbf{x}_2, t_2) | \Psi \rangle, \quad (3.45)$$

and the coincidence rate as a function of time delay  $\tau$  is then

$$R(\tau) = \int d^2x_1 d^2x_2 dt_1 dt_2 |A(\mathbf{x}_1, \mathbf{x}_2, t_1, t_2)|^2. \quad (3.46)$$

As was shown in [33],  $R(\tau)$  is generically of the form of a modulated HOM dip (Sect. 2.5),

$$R(\tau) = R_0 \left[ 1 - \Lambda \left( 1 - \frac{2\tau}{DL} \right) W(\tau) \right]. \quad (3.47)$$

where  $\Lambda(x)$  is the triangular function:

$$\Lambda(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \quad (3.48)$$

The  $\tau$ -independent background term  $R_0$  and  $\tau$ -dependent modulation term  $W(\tau)$  are calculated [28] to be:

$$\begin{aligned} R_0 = & \int d^2q d^2q' \text{sinc}[ML\mathbf{e}_2 \cdot (\mathbf{q} - \mathbf{q}')] \\ & \times G_s^* \left( \frac{f\mathbf{q}}{k} \right) G_i^* \left( -\frac{f\mathbf{q}}{k} \right) G_s \left( \frac{f\mathbf{q}'}{k} \right) G_i \left( -\frac{f\mathbf{q}'}{k} \right) \\ & \times \tilde{\mathcal{P}}_1(\mathbf{q} - \mathbf{q}') \tilde{\mathcal{P}}_2(-\mathbf{q} + \mathbf{q}') e^{-\frac{iML}{2}\mathbf{e}_2 \cdot (\mathbf{q} - \mathbf{q}')} e^{\frac{2id_1}{k_p}(\mathbf{q}^2 - \mathbf{q}'^2)} \end{aligned} \quad (3.49)$$

$$\begin{aligned} W(\tau) = & \frac{1}{R_0} \int d^2q d^2q' \text{sinc} \left[ ML\mathbf{e}_2 \cdot (\mathbf{q} + \mathbf{q}') \Lambda \left( 1 - \frac{2\tau}{DL} \right) \right] \\ & \times G_s^* \left( \frac{f\mathbf{q}}{k} \right) G_i^* \left( -\frac{f\mathbf{q}}{k} \right) G_s \left( \frac{f\mathbf{q}'}{k} \right) G_i \left( -\frac{f\mathbf{q}'}{k} \right) \\ & \times \tilde{\mathcal{P}}_1(\mathbf{q} + \mathbf{q}') \tilde{\mathcal{P}}_2(-\mathbf{q} - \mathbf{q}') e^{-\frac{iM}{D}\tau \mathbf{e}_2 \cdot (\mathbf{q} - \mathbf{q}')} e^{\frac{2id_1}{k_p}(\mathbf{q}^2 - \mathbf{q}'^2)}, \end{aligned} \quad (3.50)$$

where  $k_p$  is the wavenumber of the pump.  $M$  is the spatial walk-off in the direction  $\mathbf{e}_2$  perpendicular to the interferometer plane.

Now taking the apertures be large, the  $\tilde{\mathcal{P}}_j$  become delta functions, thus reducing Eqs. 3.49 and 3.50 to:

$$R_0 = \int d^2q \left| G_s \left( \frac{f\mathbf{q}}{k} \right) G_i \left( -\frac{f\mathbf{q}}{k} \right) \right|^2 \quad (3.51)$$

$$\begin{aligned} W(\tau) = & \frac{1}{R_0} \int d^2q e^{-\frac{2iM\tau}{D}\mathbf{e}_2 \cdot \mathbf{q}} \\ & \times G_s^* \left( \frac{f\mathbf{q}}{k} \right) G_s \left( -\frac{f\mathbf{q}}{k} \right) G_i^* \left( -\frac{f\mathbf{q}}{k} \right) G_i \left( \frac{f\mathbf{q}}{k} \right). \end{aligned} \quad (3.52)$$

Set  $G_j(\mathbf{x}) = t_j(\mathbf{x})e^{i\phi_j(\mathbf{x})}$ , where  $t_j$  is real and the effects of aberrations are contained in the phase factor  $\phi_j$ . Disregarding the background term  $R_0$  for the moment, it is seen from the presence in Eq. 3.52 of the factors

$$G_s^* \left( \frac{f\mathbf{q}}{k} \right) G_s \left( -\frac{f\mathbf{q}}{k} \right) = t_s^* \left( \frac{f\mathbf{q}}{k} \right) t_s \left( -\frac{f\mathbf{q}}{k} \right) e^{-i[\phi_s(\frac{f\mathbf{q}}{k}) - \phi_s(-\frac{f\mathbf{q}}{k})]} \quad (3.53)$$

that even order aberration terms arising from sample 1 cancel from the modulation term; similarly for sample 2. This is the even-order cancellation effect of references [27] and [28].

In the large aperture limit, the lenses immediately in front of the detectors may be removed without altering the functioning of the apparatus. However, it is useful to include them because they lead to the presence of the Fourier transformed aperture functions  $\tilde{\mathcal{P}}_j$  in Eqs. 3.49 and 3.50, and these serve below as convenient bookkeeping devices in order to trace various terms back to their origins. If the lenses are removed, then Eq. 3.42 is replaced by

$$H_{D_\alpha}(\mathbf{x}_\alpha, \mathbf{q}_j, \omega) = e^{ik(d_1+d)} e^{\frac{-id_1\mathbf{q}_j^2}{2k}} \int p(\mathbf{x}') e^{\frac{ik}{2d}(\mathbf{x}'-\mathbf{x}_\alpha)^2} e^{i\mathbf{q}\cdot\mathbf{x}'} d^2x', \quad (3.54)$$

where  $d$  is the total aperture-to-detector distance, with corresponding changes in Eqs. 3.49 and 3.50. In the large-aperture limit, the coincidence rate of Eqs. 3.47, 3.51, and 3.52 remains unchanged.

### 3.4.3 Aberration Cancellation to All Orders

The background term  $R_0$  in Eq. 3.51 depends on  $G_s$  and  $G_i$  only through the squared modulus of each. Thus any phase changes introduced by  $G_s$  or  $G_i$  cancel completely; in particular, the background term  $R_0$  exhibits cancellation of aberrations of *all* orders, not just even orders. Although it has an important role when imaging is discussed in Chap. 6, in the current situation  $R_0$  plays no such role, simply being a constant and thus having no effect on the  $\tau$ -dependence of the correlation. However, the fact that all orders of aberration can be canceled in the background term raises the question as to whether it can be arranged for this to happen in the more important modulation term as well.

For a thin sample, the means for achieving this is evident from examining Eq. 3.52. Suppose that  $G_s(\mathbf{x}) = G_i(\mathbf{x})$ , as shown schematically in Fig. 3.6. This may happen in one of two ways: either two identical samples may be placed in the two arms, or else the two beams may both pass through the same non-birefringent sample. The second possibility is usually of more practical interest, since identical samples are normally not available. For  $G_s = G_i$ , Eqs. 3.49 and 3.50 give

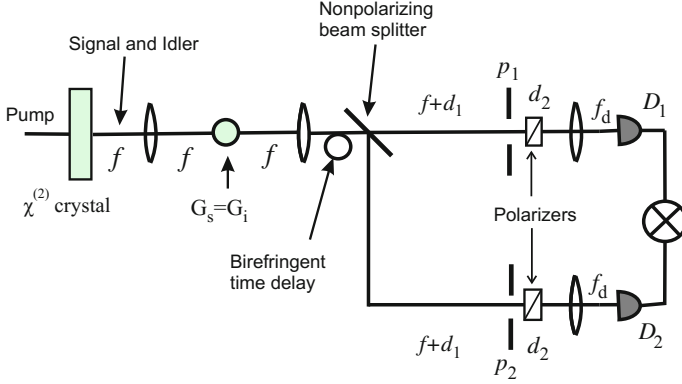


**Fig. 3.6** Schematic view of apparatus in Fig. 3.5, with  $G_s$  set equal to  $G_i$ . (Distances and angles not necessarily drawn in correct proportions.) Here  $G_s$  and  $G_i$  are being produced by a single object. The signal and idler are collinear. It is also possible for  $G_s$  and  $G_i$  to be produced by two identical but spatially separate objects interacting with noncollinear signal and idler

$$R_0 = \int d^2q \left| G_s \left( \frac{f\mathbf{q}}{k} \right) G_s \left( -\frac{f\mathbf{q}}{k} \right) \right|^2 \quad (3.55)$$

$$W(\tau) = \frac{1}{R_0} \int d^2q e^{-\frac{2iM\tau\epsilon_2 \cdot \mathbf{q}}{D}} \left| G_s \left( \frac{f\mathbf{q}}{k} \right) G_s \left( -\frac{f\mathbf{q}}{k} \right) \right|^2. \quad (3.56)$$

Setting  $G_s(\mathbf{x}) = t(\mathbf{x})e^{i\phi(\mathbf{x})}$ , it is seen that all phases now cancel from the  $\tau$ -modulated term,  $W$ . Thus, *all aberrations induced by the sample, of any order, completely cancel from the coincidence rate*. The sample affects only the amplitude of the probing light, not the spatial variation of phase in the transverse plane.

Any cancellation that occurs can hold exactly *only* for phases arising in the  $\Pi$ -plane of the Fourier transform system. The cancellation becomes more approximate as one moves away from this plane or inserts samples of finite thickness. An order of magnitude estimate of the distance  $|z|$  away from this plane one may move while maintaining approximate cancellation can readily be derived [32]. The result is

$$|z| \lesssim \frac{fr_s}{a}. \quad (3.57)$$

Taking for example the values  $r_s \sim 10^{-4}$  m,  $a \sim 1$  cm,  $f \sim 10$  cm, and  $\lambda \sim 10^{-7}$  m, this gives us an upper limit of about 1 mm.

### 3.4.4 Comparison with Dispersion Cancellation

The idea of aberration cancellation via entangled-photon interferometry arose in analogy to the similar dispersion cancellation effect [3, 4], discussed in Sect. 3.2. We know

from Sect. 3.3 that dispersion effects may be separated from each other in such a way that either even-order terms or odd-order terms may be canceled in each component [13], but that it is impossible to simultaneously cancel both sets of terms at once. Thus, it is somewhat surprising that in the case of spatial aberrations such a simultaneous cancellation should be possible. This is essentially due to the fact that all-order phase cancellation can only be accomplished for phases arising from a single transverse plane: the dispersive phase differences accumulate along the longitudinal path of the light, rather than from a single transverse plane. Alternatively, the impossibility of simultaneous even- and odd-order temporal dispersion cancellation can be seen by noting that if both the odd-order and even-order dispersion coefficients vanish simultaneously (including the zeroth-order term), then  $n(\omega)$  and  $\frac{dn}{d\omega}$  both vanish. In consequence, the phase and group velocities,  $v_p = \frac{c}{n(\omega)}$  and  $v_g = c \left[ n(\omega) + \omega \frac{dn(\omega)}{d\omega} \right]^{-1}$ , both diverge. This is in contradiction to special relativity, which requires a finite group velocity.

### 3.4.5 Physical Interpretation

A more physical understanding of the mechanism underlying the phase cancellation effect can be developed. Let  $\mathbf{q}$  and  $\mathbf{q}'$  be the ingoing and outgoing momenta in the upper branch at the beam splitter of Fig. 3.5. The ingoing and outgoing momenta for the lower branch are  $-\mathbf{q}$  and  $-\mathbf{q}'$ , as in Fig. 3.7a and b below.

Note first that the coincidence detection amplitude in transverse momentum space may be written in the form  $A(\mathbf{q}) = A_r(\mathbf{q}) + A_t(\mathbf{q})$ , where  $A_t$  and  $A_r$  represent the respective amplitudes for both photons to be transmitted at the beam splitter or for both to be reflected. The counting rate involves the integrated and squared amplitude; if the momenta  $\mathbf{q}$  and  $\mathbf{q}'$  were independent variables, this could be written as

$$\left| \int A(\mathbf{q}) d^2q \right|^2 = \int A(\mathbf{q}) A^*(\mathbf{q}') d^2q d^2q', \quad (3.58)$$

which contains terms  $A_r(\mathbf{q}) A_t^*(\mathbf{q}') + A_t(\mathbf{q}) A_r^*(\mathbf{q}')$  involving interference between reflection and transmission (see Fig. 3.7a), as well as non-interference terms  $A_r(\mathbf{q}) A_r(\mathbf{q}')^* + A_t(\mathbf{q}) A_t(\mathbf{q}')^*$  (Fig. 3.7b). However,  $\mathbf{q}$  and  $\mathbf{q}'$  are not independent; they are linked by momentum conservation at the down conversion source, which forces the requirement  $\mathbf{q}' = \pm \mathbf{q}$ . These constraints are explicitly enforced by the factors of  $\tilde{\mathcal{P}}_j$  in Eqs. 3.49 and 3.50, which become delta functions in the large aperture limit. The delta functions tie the amplitudes  $A_r$  and  $A_t$  together as shown in Fig. 3.7. Each line in the diagram represents a delta function.

Suppose again that  $G_j(\mathbf{x}) = t_j(\mathbf{x}) e^{i\phi_j(\mathbf{x})}$ . Since effects related to amplitude modulation are not of interest here, set  $t_j(\mathbf{x}) = 1$ . Examining Eqs. 3.49 and 3.50, it is then to be noted that even-order and odd-order aberration cancellation arise from different sources. Several possibilities occur:

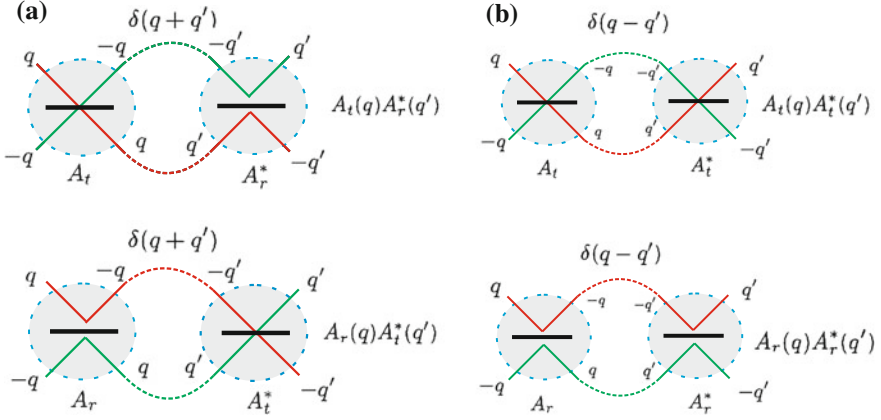


**Fig. 3.7** **a** Schematic representation of interference terms. In the squared amplitude  $\int dq dq' A(q)A^*(q')$ , the amplitude ( $A_r$ ) for both photons reflecting at the beam splitter interferes with the amplitude for both being transmitted ( $A_t$ ). For these terms,  $q = -q'$ , due to the delta function that connects the amplitudes. **b** Schematic representation of noninterference terms. In the *top part of the figure* the transmission portion of the amplitude  $A_t$  interacts only with itself, while in the *bottom part* the same is true of the reflection amplitude  $A_r$ . For these terms,  $q = q'$  (Figure reproduced from [32].)

(i) If  $\mathbf{q} = -\mathbf{q}'$ , then the  $G_j \left( \frac{f\mathbf{q}}{k} \right) G_j^* \left( \frac{f\mathbf{q}'}{k} \right)$  structure arising from the absolute square of the amplitude in the counting rates leads to even-order-canceled terms,

$$G_j \left( \frac{f\mathbf{q}}{k} \right) G_j^* \left( -\frac{f\mathbf{q}}{k} \right) = e^{i[\phi_j(\mathbf{q}) - \phi_j(-\mathbf{q})]}. \quad (3.59)$$

(ii) If  $G_s = G_i$ , then the  $G_s \left( \frac{f\mathbf{q}}{k} \right) G_i \left( -\frac{f\mathbf{q}}{k} \right)$  factors become

$$G_s \left( \frac{f\mathbf{q}}{k} \right) G_s \left( -\frac{f\mathbf{q}}{k} \right) = e^{i[\phi_2(\mathbf{q}) + \phi_2(-\mathbf{q})]} \quad (3.60)$$

then odd-order cancellation occurs. Note that different combinations of terms cause the cancellation in (i) and (ii).

(iii) If  $\mathbf{q} = \mathbf{q}'$ , a structure of the form

$$G_j \left( \frac{f\mathbf{q}}{k} \right) G_j^* \left( \frac{f\mathbf{q}}{k} \right) = e^{i[\phi_j(\mathbf{q}) - \phi_j(\mathbf{q})]} = 1 \quad (3.61)$$

occurs, which gives cancellation of all orders.

In (i) and (iii) the phase from a single arm of the interferometer cancels with itself, whereas (ii) is a cancellation between the two different (but identical) arms.

Cases (i) and (ii) both involve interference between the amplitudes  $A_r$  and  $A_t$  (shown schematically in Fig. 3.7), while (iii) comes from the non-interference ( $|A_r|^2$  or  $|A_t|^2$ ) terms of Fig. 3.7, and so occur even if only one of the two amplitudes  $A_r$  and  $A_t$  is present.

To summarize the main results of this section, if  $G_s$  and  $G_i$  are approximately confined to the  $z = 0$  plane, then the apparatus of Fig. 3.5 leads to the following effects:

- Even-order aberrations induced by the samples  $G_s$  and  $G_i$  cancel.
- If  $G_s = G_i$ , then *all* orders of aberration cancel.

In this chapter, correlated photon methods have been used to manipulate the effects of spatial and temporal aberration in a simple optical imaging system. Similar effects are revisited later (Chap. 6), with the appearance of aberration cancellation in two-photon imaging systems and the mitigation of the effect of turbulence on optical propagation.

## References

1. L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, London, 1960)
2. J.M. Lukens, A. Dezfolyiyan, C. Langrock, M.M. Fejer, D.E. Leaird, A.M. Weiner, *Phys. Rev. Lett.* **111**, 193603 (2013)
3. J.D. Franson, *Phys. Rev. A* **45**, 3126 (1996)
4. A.M. Steinberg, P.G. Kwiat, R.Y. Chiao, *Phys. Rev. Lett.* **68**, 2421 (1992)
5. A. Abouraddy, M.B. Nasr, B.E.A. Saleh, A.V. Sergienko, M.C. Teich, *Phys. Rev. A* **65**, 053817 (2002)
6. A.M. Steinberg, P.G. Kwiat, R.Y. Chiao, *Phys. Rev. A* **45**, 6659 (1992)
7. V. Giovannetti, S. Lloyd, L. Maccone, F.N.C. Wong, *Phys. Rev. Lett.* **87**, 117902 (2001)
8. E. Dauler, G. Jaeger, A. Muller, A. Migdall, *J. Res. Natl. Inst. Stand. Technol.* **104**, 1 (1999)
9. D. Branning, A.L. Migdall, A.V. Sergienko, *Phys. Rev. A* **62**, 063808 (2000)
10. A. Fraine, D.S. Simon, O. Minaeva, R. Egorov, A.V. Sergienko, *Opt. Exp.* **19**, 22820 (2011)
11. M.B. Nasr, B.E.A. Saleh, A.V. Sergienko, M.C. Teich, *Phys. Rev. Lett.* **91**, 083601 (2003)
12. M.B. Nasr, B.E.A. Saleh, A.V. Sergienko, M.C. Teich, *Opt. Express* **12**, 1353–1362 (2004)
13. O. Minaeva, C. Bonato, B.E.A. Saleh, D.S. Simon, A.V. Sergienko, *Phys. Rev. Lett.* **102**, 100504 (2009)
14. B.I. Erkmen, J.H. Shapiro, *Phys. Rev. A* **74**, 041601 (2006)
15. K.J. Resch, P. Puvanathanasan, J.S. Lunden, M.W. Mitchell, K. Bizheva, *Opt. Express* **15**, 8797 (2007)
16. P. Kaltenbaek, J. Lavoie, D.N. Biggerstaff, K.J. Resch, *Nat. Phys.* **4**, 864 (2008)
17. J. Lavoie, P. Kaltenbaek, K.J. Resch, *Opt. Express* **17**, 3818 (2009)
18. P. Kaltenbaek, J. Lavoie, K.J. Resch, *Phys. Rev. Lett.* **102**, 243601 (2009)
19. V. Torres-Company, H. Lajunen, A.T. Friberg, *New J. Phys.* **11**, 063041 (2009)
20. J.D. Franson, *Phys. Rev. A* **80**, 032119 (2009)
21. A. Barak, M. Segev, *Phys. Rev. A* **86**, 043838 (2012)
22. J. Qiu, J.S. Pan, G.Y. Xiang, Y.S. Zhang, S.S. Han, Y.Z. Gui, *J. Opt. Soc. Am. B* **32**, 907 (2015)
23. A. Lipson, S.G. Lipson, H. Lipson, *Optical Physics*, 4th edn. (Cambridge University Press, Cambridge, 2010)
24. M. Born, E. Wolf, *Principles of Optics*, 7th edn. (Cambridge University Press, Cambridge, 1999)
25. H.A. Buchdahl, *Optical Aberration Coefficients* (Dover, Mineola, 1968)

26. J.C. Wyant, K. Creath, *Basic Wavefront Aberration Theory for Optical Metrology, Applied Optics and Optical Engineering XI* (Academic Press, London, 1992)
27. C. Bonato, A.V. Sergienko, B.E.A. Saleh, S. Bonora, P. Villoresi, *Phys. Rev. Lett.* **101**, 233603 (2008)
28. C. Bonato, D.S. Simon, P. Villoresi, A.V. Sergienko, *Phys. Rev. A* **79**, 062304 (2009)
29. B.J. Berne, R. Pecora, *Dynamic Light Scattering; with Applications to Chemistry, Biology, and Physics* (Dover, Mineola, 2000)
30. R. Pecora, *Dynamic Light Scattering: Applications to Photon Correlation Spectroscopy* (Plenum Press, New York, 1985)
31. S. Maiti, U. Haupts, W.W. Webb, *Proc. Nat. Acad. Sci. USA* **94**, 11753 (1997)
32. D.S. Simon, A.V. Sergienko, *Phys. Rev. A* **80**, 053813 (2009)
33. M.H. Rubin, Y.H. Shih, D.N. Klyshko, A.V. Sergienko, *Phys. Rev. A* **50**, 5122 (1994)

## Chapter 4

# Quantum Metrology

Metrology is the study of precision measurement. In this chapter, a number of areas are introduced in which quantum approaches can be used for improved metrological applications in optics. In some cases, the result is the ability to make measurements with sensitivities and resolutions beyond the limits imposed by the laws of classical physics. In addition to providing such improvements directly to the measurements themselves, quantum methods can also aid indirectly, by providing absolute light sources and improvements in the ability to calibrate light detectors, as is seen in the next two sections.

Normally, measurements at low optical intensity are limited by the Poisson statistics of shot noise, which gives a sensitivity proportional to  $\frac{1}{\sqrt{N}}$  when probed with  $N$  photons (see Sect. 4.6). However, a number of schemes have been demonstrated [1–7] that use entangled photons to achieve sub-Poisson measurement statistics. This reduction in the joint fluctuations of entangled states is the key point that allows many of the applications of this chapter to work.

Besides those applications presented in this chapter, additional metrological applications appear at other points in this book, such as the supersensitive angular and rotational measurements of Chap. 7. Reviews of quantum methods in metrology and sensing can be found in [8–11].

As a point of terminology, recall that *sensitivity* is a measure of the smallest detail that can be reliably detected, whereas *resolution* is a measure of how well two nearby values can be distinguished. Both of these properties tend to be enhanced by quantum-based methods, although often at the cost of reduced data-collection rates.

### 4.1 Absolute Photon Sources

Lasers have intensity fluctuations that obey Poisson statistics. As a result, a laser light source with an average of  $\langle \hat{n} \rangle = N$  photons per unit time exhibits typical fluctuations of size  $\sqrt{N}$ . At high intensities, this leads to a high signal to noise

ratio,  $\sim \frac{N}{\sqrt{N}} = \sqrt{N}$ , but at the very low intensities relevant to many quantum optical applications, the fluctuations are a significant fraction of the signal size. Thus, any ability to correct for these fluctuations is of great interest.

The use of entangled-photon sources allows such an ability at low photon intensities. Although the photon number fluctuations still occur, the fluctuations can be determined, and therefore taken into account in measurements. The enabling fact is that in parametric down conversion, signal and idler photons are always created together. As a result, the idler photon can be detected and used to announce or herald the creation of the signal photon used in the measurement. Each time an idler photon is detected, an electronic gate can be arranged to open for a short time, allowing optical measurements to be carried out by the signal photon only during that interval, while if there is no idler detected, there is no measurement made in the signal branch, thus eliminating random noise uncorrelated with the idler. By means of the idler measurements, the number of optical quanta at the signal frequency traveling in given direction can be known exactly, providing an absolute photon source whose emissions are known precisely at each moment. Since most optical measurement techniques record the response of a system relative to the incident optical energy, such absolute sources open up the possibility of improved measurements in many areas.

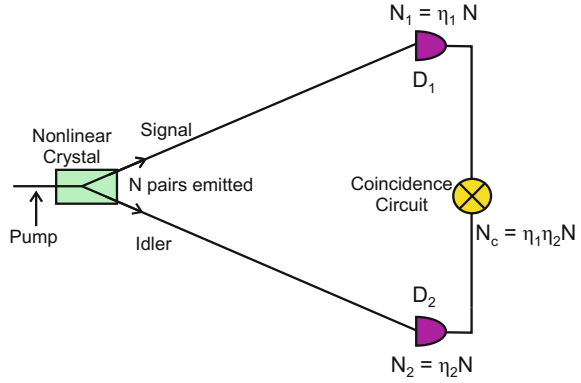
If the pump frequency is known and the frequency of the detected idler is known, then the signal frequency can be determined, allowing determination of the energy flux as well as the photon number. For typical pump frequencies used in down-conversion, the signal and idler are each typically in the visible or infrared, but in principle creating such an absolute source in any part of the electromagnetic spectrum is possible. Since the method works best at low intensities, it is well suited to the study of quantum effects in optics, or to applications at low light levels, such as measurement of retinal response to individual photons.

The discussion here is, of course, idealized in that it assumes that all the idler photons that could be detected are actually detected; in practice, detectors are not perfect so that some idler photons are missed. Consequently, there is still a degree of uncharacterized fluctuation due to the limited detector efficiency. The question of detector efficiency is therefore addressed in the next section.

## 4.2 Absolute Calibration of Photon-Counting Detectors

No detector has 100% efficiency. In particular, a given photon-counting detector typically registers only a fraction of the photons incident on its active surface. This detection fraction,  $\eta$ , is known as the *quantum efficiency* of the detector. Knowing the quantum efficiency of the detectors used in an experiment is necessary for extracting accurate event rates from the measured data. Although measuring the quantum efficiency normally requires having a calibrated photon source of known emission rate, the use of an entangled photon source allows absolute measurement of the detector's quantum efficiency *without* source calibration [12–16].

**Fig. 4.1** Optical setup for absolute calibration of a detector's quantum efficiency without a calibrated source. The efficiency of one detector is found by dividing the coincidence rate by the singles counting rate in the other detector:  $\eta_1 = N_c/N_2$  and  $\eta_2 = N_c/N_1$



A schematic illustrating this method is shown in Fig. 4.1 involving two detectors  $D_1$  and  $D_2$  with unknown quantum efficiencies  $\eta_1$  and  $\eta_2$ , respectively. Suppose that  $N$  down-conversion events occur during a measurement and assume that the detectors are situated symmetrically about the pump beam, so that whenever the signal reaches one detector one can be sure that the idler reaches the other. Then the singles rates, i.e. the number of counts in each of the two detectors separately, are

$$N_1 = \eta_1 N, \text{ and } N_2 = \eta_2 N. \quad (4.1)$$

The coincidence count during the same interval is

$$N_c = \eta_1 \eta_2 N. \quad (4.2)$$

Hence, by measuring both the singles and coincidence counts, the quantum efficiency of each detector is easily extracted:

$$\eta_1 = \frac{N_c}{N_2}, \quad \text{and} \quad \eta_2 = \frac{N_c}{N_1}. \quad (4.3)$$

It is in fact possible to accomplish the same result with just a single detector. In this method, both down-conversion photons are sent to the same detector, but with one of them delayed. This can be accomplished by using type-II down-conversion (so that the two outgoing photons have opposite polarization) and adding a birefringent delay line in the optical path. Now, rather than looking at correlations between two different detectors, the idea is to look at the temporal autocorrelation of a single detector. The autocorrelation that results from detection of a single pair peaks when the delay is zero and decay as the delay increases. The detector acts as its own trigger; the detection of one photon effectively opens the gate for detection of the other. A more detailed discussion of this technique can be found in [14].

### 4.3 Quantum Ellipsometry

Ellipsometry [17, 18] is a method widely used both in research and in industry to nondestructively determine the thickness and optical properties of thin films via interference of light reflecting from the film. Among other applications, it is used for quality control of thin semiconductor layers in integrated circuits. In the following, we briefly review classical ellipsometry, and then discuss its quantum analog.

#### 4.3.1 Classical Ellipsometry

When light reflects from a surface, its polarization may change. However, there are two polarization directions for which the reflected light have the same polarization as the incident light, namely, the eigendirections of the polarization for the system. Let the complex reflection coefficients of these two eigenpolarizations be  $r_1$  and  $r_2$ . Then the optical properties of the film may be characterized by the two parameters  $\tan \Psi$  and  $\Delta$  to be defined below. The goal of ellipsometry is to determine these two parameters (Fig. 4.2).

Recall that at the interface between two materials of refractive indices  $n_1$  and  $n_2$ , the Fresnel equations provide the reflection coefficients for light hitting the surface from medium 1:

$$r_{12}^p = \frac{n_2 \cos \phi_1 - n_1 \cos \phi_2}{n_2 \cos \phi_1 + n_1 \cos \phi_2} \quad (4.4)$$

$$r_{12}^s = \frac{n_1 \cos \phi_1 - n_2 \cos \phi_2}{n_2 \cos \phi_1 + n_1 \cos \phi_2}, \quad (4.5)$$

where  $\phi_1$  and  $\phi_2$  are, respectively, the incident and transmitted angles. The superscripts give the incoming polarization relative to the incident plane:  $p$  for polarization parallel to the plane of incidence and  $s$  for polarization orthogonal to the plane. The

**Fig. 4.2** Definitions of variables used in the Fresnel equations.  $p$  and  $s$  represent polarizations parallel and perpendicular to the incident plane, respectively

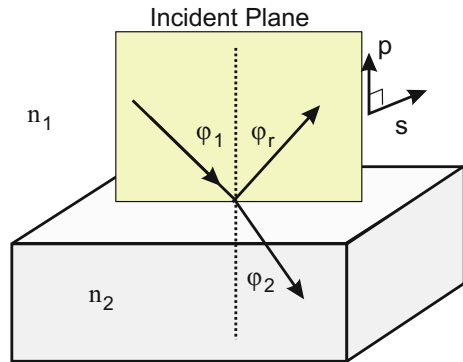
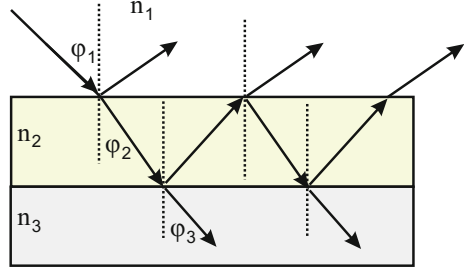


**Fig. 4.3** An incident light beam reflecting from a thin film (index  $n_2$ ) on a substrate (index  $n_3$ )



refractive indices are real for dielectric materials and complex for conductors. The transmission coefficients are obtained from  $|t^{p,s}|^2 + |r^{p,s}|^2 = 1$ .

For the case of a thin film, there are three materials with refractive indices  $n_1$  (the ambient medium),  $n_2$  (the film), and  $n_3$  (the substrate); see Fig. 4.3. By two uses of the Fresnel equations, the reflection coefficients are now

$$R^p = \frac{r_{12}^p + r_{23}^p e^{-2i\beta}}{1 + r_{12}^p r_{23}^p e^{-2i\beta}} \quad (4.6)$$

$$R^s = \frac{r_{12}^s + r_{23}^s e^{-2i\beta}}{1 + r_{12}^s r_{23}^p e^{-2i\beta}}, \quad (4.7)$$

where

$$\beta = \frac{2\pi d}{\lambda} n_2 \cos \phi_2. \quad (4.8)$$

The reflection probabilities are  $\mathcal{R}^{s,p} = |R^{s,p}|^2$ . Let  $\delta_1$  be the phase difference between the incoming s and p polarized components of the incident wave, with  $\delta_2$  being the corresponding phase difference for the outgoing reflected waves. Then the ratio of the reflected amplitudes for the two polarizations is

$$\rho = \frac{R^p}{R^s} \equiv \tan \Psi e^{i\Delta}, \quad (4.9)$$

where

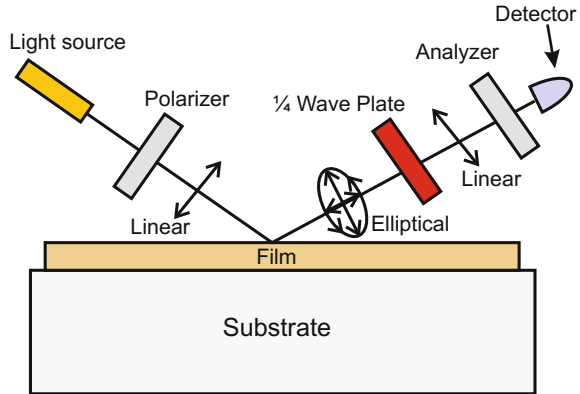
$$\tan \Psi = \frac{|R^p|}{|R^s|} \quad (4.10)$$

and

$$\Delta = \delta_1 - \delta_2. \quad (4.11)$$

Ellipsometric setups fall into two general types: null ellipsometers and interferometric ellipsometers. In *null ellipsometry* (Fig. 4.4), an unpolarized, monochromatic light source is used and a linear polarizer is inserted into the incident beam to prepare a linear polarization state. The reflected light is, in general, elliptically polarized and

**Fig. 4.4** Schematic of setup for null ellipsometry. An initially linearly polarized beam produces an elliptically polarized reflected beam. The quarter-wave plate converts it back to linear polarization, then the polarizing filter is rotated to find the direction perpendicular to this polarization



it is then converted back to linear polarization by a second linear polarizer (called the *analyzer*). Usually, the analyzer can be rotated by the user. By altering the polarization of the incident light, a state can be found in which the amplitude and phase of the outgoing light are made equal, so that the reflected light reduces to a linearly polarized state. At this point, the light is completely blocked when the analyzer is in the orthogonal direction. The goal is to find an incident polarization state for which this null output occurs; from this information,  $\psi$  and  $\Delta$  can be computed. One only need determine whether or not the output intensity is zero, rather than to determine the actual values of nonzero intensities. Therefore, calibrations of detector and source parameters are not needed. However, this method has the drawback that to extract the desired parameters from the data, it is necessary to have a well-characterized reference sample.

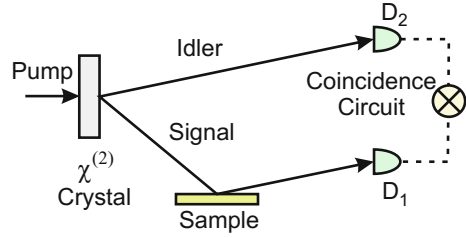
The alternate possibility is *interferometric ellipsometry*, in which the illuminating beam is split into two paths, one reflecting off the sample and the other not. When the beams are recombined, the resulting interference allows the desired information to be extracted. This, however, requires a more complex optical system, adding additional optical elements to be characterized, as well as stringent stability requirements.

Since both traditional ellipsometry techniques have drawbacks, it is worth examining whether a quantum version of ellipsometry may lead to improved results.

### 4.3.2 Quantum Ellipsometry

An illustration of the basic method of *quantum ellipsometry* (or *twin-photon ellipsometry*) [19–21] is shown in Fig. 4.5. The light source consists of noncollinear type-II SPDC in which the signal and idler photon have orthogonal polarizations. Any birefringence within the nonlinear crystal causes these two photons to have a time delay relative to one another that must be compensated for by inserting an additional piece

**Fig. 4.5** Schematic of setup for quantum ellipsometry, using an entangled down conversion source



of birefringent material of suitable length into each beam. This leads (after possible additional phase shifts in one beam) to the path- and polarization-entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|H\rangle_s |V\rangle_i + |V\rangle_s |H\rangle_i). \quad (4.12)$$

One of the two photons (the signal) is then reflected off the sample, while the other propagates freely. Afterward, the two photons are detected in coincidence.

The coincidence rate can be shown [20] to be proportional to

$$R_c = \tan \psi \cos^2 \theta_1 \sin^2 \theta_2 + \sin^2 \theta_1 \cos^2 \theta_2 + 2 \tan^{1/2} \psi \cos \Delta \cos \theta_1 \cos \theta_2 \sin \theta_1 \sin \theta_2, \quad (4.13)$$

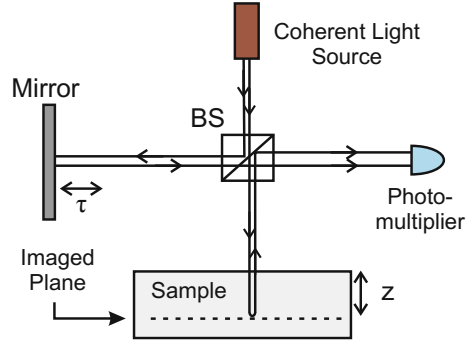
where  $\theta_1$  and  $\theta_2$  are the angles of two polarizers placed before detectors  $D_1$  and  $D_2$ . Measuring  $R_c$  for any two pairs  $(\theta_1, \theta_2)$  then leads to a pair of simultaneous equations to be solved for  $\psi$  and  $\Delta$ .

This quantum approach to ellipsometry has several advantages. In addition to a significant reduction in the relative error of estimates for the ellipsometric parameters, it also avoids the need for calibration with an external reference sample

## 4.4 Quantum Optical Coherence Tomography

Optical coherence tomography (OCT) [22] is of widespread use in biological and medical imaging; see, for example, [23–26]. It has a number of advantages: it is noninvasive and requires no staining or other sample preparation, and can be done at optical or near optical wavelengths that do no damage to the tissue; it is of high resolution, and most significantly, it allows subsurface imaging. Cross-sectional images may be obtained using this technique for planes at depths up to several millimeters before diffusive scattering begins blurring the image. An image cross-section is built up by scanning the beam over the sample; by varying the delay time, the resulting series of cross-sections can then be stacked to form a three dimensional image.

**Fig. 4.6** Schematic of setup for classical subsurface OCT. Moving the adjustable mirror changes the depth of the in-focus plane within the sample



#### 4.4.1 Classical OCT

A schematic of the setup for a classical OCT apparatus is shown in Fig. 4.6. A coherent light source enters a beam splitter. Half of the light (the reference beam) strikes a mirror, which reflects it to a photomultiplier. The other half (the sample beam) is directed to the sample, where it reflects back to the beam splitter and then to the detector. The two beams then interfere at the detector. The mirror can be moved, introducing a time delay. The choice of time delay picks out the sample plane to be imaged, at depth  $z = c\tau$ . Because of destructive interference, other planes are out of focus.

The effect of the passage and reflection of light through the sample is described by a transfer function,

$$H(\omega) = \int_0^\infty dz r(z, \omega) e^{2i\phi(z, \omega)}, \quad (4.14)$$

where  $r(z, \omega)$  is the amplitude for reflection at depth  $z$  of a frequency  $\omega$  plane wave and  $\phi$  is the phase accumulated each way between the surface of the material and depth  $z$ .

Suppose the light is peaked at frequency  $\omega_0$ , so that the detected frequency can be parameterized as  $\omega = \omega_0 + \Omega$ . The field at the detector is the sum of reference (r) and sample (s) arm fields, so the intensity is:

$$I = \langle |E_r + E_s|^2 \rangle = \frac{1}{2} (I_r + I_s) + \text{Re} \langle E_r^*(t + \tau) E_s(t) \rangle, \quad (4.15)$$

where  $I_r$  and  $I_s$  are the reference and sample arm intensities and the brackets are averages over the collection time of the detector. Up to constants, this can be written in the form

$$I(\tau) = \Gamma_0 + \Gamma(\tau), \quad (4.16)$$

where

$$\Gamma_0 = \int d\Omega [1 + (H(\omega_0 + \Omega))^2] S(\Omega) \quad (4.17)$$

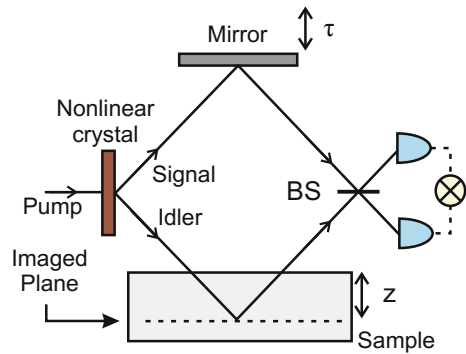
$$\Gamma(\tau) = \int d\Omega H(\omega_0 + \Omega) S(\Omega) e^{-i\Omega\tau}. \quad (4.18)$$

Here,  $S(\Omega) = |\Phi(\Omega)|^2$  is the spectral power density of the illumination source. The term  $\Gamma_0$  arises due to the incident and reflected fields interfering with themselves, while  $\Gamma(\tau)$  comes from the interference of the two fields with each other. Assuming that  $S(\Omega)$  is known, then by measuring these functions, the transfer function  $H$  can be extracted, and finally the reflection coefficient  $r(z, \omega)$  determined. Because the resolutions in the directions transverse and parallel to the beam are limited by the coherence length of the light, a light source with the largest possible spectrum is desirable. For very broadband sources, an axial resolution less than 1  $\mu\text{m}$  is possible.

#### 4.4.2 Quantum OCT

Quantum OCT (QOCT) [27–31] is a technique analogous to classical OCT, but which uses entangled photons in order to take advantage of dispersion cancellation (see Chap. 3). In the quantum version the light source is replaced by a source of frequency-entangled photon pairs from SPDC, and the interference takes place in an HOM-type interferometer instead of the Michelson-type interferometer of the classical setup. A schematic of the QOCT arrangement is shown in Fig. 4.7. This arrangement is insensitive to even-order dispersion and robust against photon loss. The light source is a nonlinear crystal which produces signal–idler pairs via spontaneous parametric down-conversion, whose spectral distribution is determined by the function  $\Phi(\omega)$  given in Chap. 1. These pairs have anticorrelated frequencies  $\omega_{\pm} = \omega_0 \pm \Omega$ , centered about  $\omega_0$ , which is half the pump frequency.

**Fig. 4.7** Schematic of setup for subsurface QOCT. The illumination is now provided by entangled photon pairs from parametric down conversion



The coincidence rate is, up to overall constants, of the form [27]

$$R_c(\tau) = \Lambda_0 + \text{Re}\Lambda(2\tau). \quad (4.19)$$

The self- and cross-interference terms are, respectively,

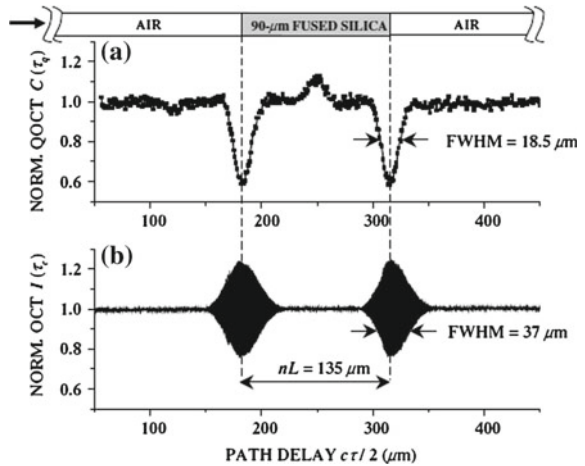
$$\Lambda_0 = \int d\Omega (H(\omega_0 + \Omega))^2 S(\Omega) \quad (4.20)$$

$$\Lambda(\tau) = \int d\Omega H(\omega_0 + \Omega) H^*(\omega_0 - \Omega) S(\Omega) e^{-i\Omega\tau}. \quad (4.21)$$

Several things are notable about these results. First, there is no constant background term (the factor of 1 in Eq. 4.17) in the self-interference integral, which increases the visibility of the interference in the quantum case. Second, the product of two transfer functions in the cross-interference term leads to the sample being probed by two frequencies,  $\omega_0 \pm \Omega$ , simultaneously. This, in turn, leads both the dispersion cancellation and to a factor-of-two resolution enhancement, signaled by the factor of 2 that appears in the argument of  $\Lambda$  in Eq. 4.19.

These features can be seen in the experimental data of Fig. 4.8 [28]. A piece of fused silica with air on both sides was used as the sample. As the delay was scanned, both classical and quantum OCT were able to locate the two ends of the silica. However, QOCT was able to determine the locations with twice the resolution: the HOM dips in the top figure are half the width of the interference structures on the bottom. The visibility of the quantum case was 45 %, close to the calculated value of 50 %, and nearly double the 28 % visibility of the classical case.

**Fig. 4.8** Experimental comparison of classical OCT and QOCT, reproduced from [28]. The quantum version (*top*) has dips that are both narrower and of higher visibility than in the classical version (*bottom*)



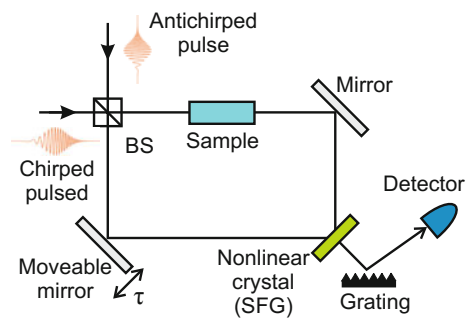
### 4.4.3 Mimicking Quantum OCT with Classical Light

One problem common to many quantum methods is that they involve low photon count rates and slow photon-counting detectors, leading to slow data collection. Therefore, it is advantageous to look for classical, “quantum inspired” methods that have still have many of the advantages of the true quantum method but can be used with strong light beams and more rapid detectors for more faster data collection. In the case of QOCT, such a quantum-inspired classical version has been found [32, 33] building on prior work that demonstrated dispersion cancelation in classical systems [34–36].

The system relies on the use of two oppositely chirped pulses; a pulse is called *chirped* if the frequency varies over the duration of the pulse. In this case, one pulse increases in frequency over time, while the frequency of the other pulse decreases at the same rate. This is accomplished by introducing a nonlinear time dependence into the phase  $\phi(t)$  of the pulse. The chirp rate is then given by  $c(t) = \frac{1}{2\pi} \frac{d^2\phi}{dt^2}$ . If the frequency increases over the duration of the pulse  $c(t) > 0$ , the pulse is said to be *up-chirped*, or simply *chirped*; if  $c(t) < 0$  then the pulse is *down-chirped* or *anti-chirped*. Chirping can be introduced by using variable pitch gratings or, more generally, by the use of systems with controlled amounts of frequency dispersion [37].

A schematic of the system is shown in Fig. 4.9. The arrangement is essentially an HOM interferometer, but is reversed in the sense that instead of a nonlinear process (parametric down-conversion) providing the input before the beam splitter, in this case the nonlinear process (sum frequency generation, SFG) occurs after the beam splitter. The input is a pair of optical pulses with opposite chirp and with central frequency  $\omega_0$ . The two pulses are combined on a beam splitter. In particular, the output from one beam splitter port goes through a sample, while the other port leads to a moveable mirror that introduces a time delay  $\tau$  and the two halves are combined through SFG in the nonlinear crystal. The output of the crystal contains components centered at the original frequency  $\omega_0$  and at twice that frequency,  $2\omega_0$ . Use of a grating separates off the  $\omega_0$  component, so that only the  $2\omega_0$  part is sent to a photodiode. No coincidence counting is needed, so just a single detector is required.

**Fig. 4.9** Schematic of arrangement for mimicking QOCT using oppositely chirped classical light pulses



Instead of interference between the two paths, this arrangement makes use of interference between pairs of frequencies on opposite sides of  $\omega_0$ . Because of the beam splitter, it is impossible to know which of the two frequencies is passing through the sample, allowing high interference visibility. The first experimental demonstration of the method [32] produced an HOM dip as  $\tau$  was scanned. This dip had high resolution (a width of about 20  $\mu\text{m}$ , FWHM) and high visibility (about 85 %). When a piece of material with even-order dispersion was added as a sample, the dip experienced negligible broadening, demonstrating dispersion cancelation.

## 4.5 Quantum Lithography and NOON States

A useful type of state for quantum metrology is the so-called *NOON state*. Let  $|n, m\rangle$  represent a state with  $n$  photons in one spatial mode (for example, in the top branch of a Mach–Zehnder interferometer) and  $m$  photons in a different spatial mode (the bottom branch). Then a NOON state is defined as

$$\frac{1}{\sqrt{2}} (|N, 0\rangle + |0, N\rangle). \quad (4.22)$$

In other words, it is an entangled  $N$ -photon state where all  $N$  of the photons are either in one mode or all are in the other.

NOON states first appeared in [38], and were independently rediscovered in the context of quantum lithography in [39]. The name “NOON” state was apparently first used in [40]. Two-photon NOON states are easily produced with a simple beam splitter, by means of the Hong–Ou–Mandel effect [41].  $N = 3$  [42],  $N = 4$  [43], and  $N = 5$  [44, 45] states can be made using parametric down-conversion followed by post-selection. The states become rapidly more difficult to produce as  $N$  increases, and they are extremely fragile in the presence of noise or loss.

If  $N$  photons are sent independently through a phase shifter in one arm of an interferometer, they each gain a phase factor of  $e^{i\phi}$ . However, if the entangled  $N$ -photon NOON state is sent through the same interferometer, the relative phase shift between the two terms of Eq. 4.22 is  $N$  times larger, giving a relative phase factor of  $e^{iN\phi}$ . As a result, phase shifts are magnified for NOON states, allowing the interferometer to measure phase values that are  $N$  times smaller than is possible with classical light. This increased phase sensitivity is discussed in more detail in the next section.

This phase supersensitivity is the basis for a proposed NOON microscope [46], which would allow sensitive measurements to be made to biological samples with low illumination to avoid damage to the sample. For  $N = 2$  and  $N = 3$ , proof-of-principle demonstrations of this have been made, in which the Heisenberg limit was approached. However, the above-mentioned sensitivity to loss and the difficulty of producing NOON states with large  $N$  seem to limit the practical application of this approach.

Instead of making subwavelength measurements, another possibility is to use the same NOON states to write subwavelength structures onto substrates. This is the idea behind quantum lithography [39, 47–49]. Optical lithography uses light to etch structures onto wafers of silicon or other materials, in order to make semiconductor chips for computers and other electronic devices. As computer chips get smaller, it is necessary to etch structures at progressively finer scales. Since the size of the structures that can be etched are limited by the wavelength of the illuminating light, this can only be done by going to shorter wavelengths.

However, a means of going to smaller size scales without reducing the wavelength was found in the late 1990s, when it was demonstrated that the use of two-photon absorption could double the resolution in optical lithography [50] at fixed wavelength. By requiring two-photon absorption, it is guaranteed that both photons interact with the material at the same point, instead of interacting at random points and washing out the interference pattern. Quantum lithography extends this idea by using  $N$ -photon NOON states, again relying on the fact that the  $N$ -photon entangled state oscillates  $N$  times faster than the single-photon state, or equivalently, that the effective wavelength of the system is reduced from  $\lambda$  to  $\frac{\lambda}{\sqrt{N}}$ . The method holds great promise, in principle, for use in fabricating more compact microchips and for other nanotechnology applications. However, practical application of the idea is again currently hampered by the difficulty of creating high  $N$  entangled states on demand. It is also necessary to use a material that allows strong  $N$ -photon absorption, which becomes rapidly harder for large  $N$ . Moreover, [51] has shown that the efficiency of the process is low for small values of  $N$ .

A related approach [52, 53] gave an  $N$ -fold enhancement in lithography using only classical illumination, suggesting that the improved resolution can ultimately be traced to the nonlinear  $N$ -fold absorption process in the material. For more on quantum lithography in general, see [54, 55].

## 4.6 Phase Measurements and Fundamental Measurement Limits

Phase measurements play an important part in the determination of many physical properties of systems. The precise measurements of small phase differences is the main goal of interferometry, and so here again we find that the use of entangled states can lead to major improvements in measurement outcomes. One notable application of entanglement-based phase measurements in a real-world setting is the use of two-photon NOON states for measurements of protein concentrations in microfluidic devices [56].

It is useful to ask what fundamental limit quantum mechanics places on our ability to make measurements [8, 57]. Phase measurements are a convenient area in which to investigate such questions, but complications arise due to the fact that there seems to be no Hermitian operator corresponding to the classical phase variable

in quantum mechanics. In his formulation of quantum electrodynamics, Dirac [58] assumed such a Hermitian operator,  $\hat{\phi}$  (or equivalently a unitary operator  $\widehat{e^{i\phi}}$ ), which was canonically conjugate to the photon number operator  $\hat{N}$ . The commutator  $[\hat{\phi}, \hat{N}]$  should then lead to an uncertainty relation which would define the ultimate physical limit on phase measurements. However, nobody has managed to construct such a Hermitian phase operator  $\hat{\phi}$  that gives the correct canonical commutation relations, and this task is now widely believed to be impossible. The origin of the problem is that, like all angular variables, the phase does not have a definite value, being defined only modulo  $2\pi$ , and this leads to inconsistencies in potential operator formulations [59]. A large literature exists investigating this problem (see, for example, [60–72]), and several possible ways to get around it have been proposed. The approach taken here makes use of a non-Hermitian operator, the Susskind-Glogower (SG) operator [60], that can play the role of  $\widehat{e^{i\phi}}$  and whose eigenstates have definite phase. The operator is defined by

$$\hat{S} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| = \hat{a}\hat{N}^{-1/2} = (\hat{N}+1)^{-1/2}\hat{a}, \quad (4.23)$$

where,  $\hat{a}$ ,  $\hat{N}$ , and  $|n\rangle$  are, respectively, the photon annihilation operator, the photon number operator, and the Fock state of photon number  $n$ . The eigenstates of  $\hat{S}$  are

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle, \quad (4.24)$$

and the corresponding eigenvalue relation is given by  $\hat{S}|\phi\rangle = e^{i\phi}|\phi\rangle$ , where  $\phi$  is the phase of the state, relative to some arbitrary fixed reference phase. An observable, Hermitian operator can then be constructed by adding  $\hat{S}$  to its Hermitian conjugate, thus forming the new operator

$$\hat{A} = \hat{S} + \hat{S}^\dagger = \sum_{n=0}^{\infty} (|n\rangle\langle n+1| + |n+1\rangle\langle n|). \quad (4.25)$$

The expectation value of  $\hat{A}$  measures the rate of single photon transitions within the apparatus and clearly has eigenvalues proportional to  $\cos \phi$ . Measurements of this phase-dependent observable can therefore be used to arrive at phase estimates. One may also define a second Hermitian operator  $\hat{B} = i(\hat{S} - \hat{S}^\dagger)$ , with eigenvalues proportional to  $\sin \phi$ ;  $A$  and  $B$  are commuting observables [73]. The net result is that although there is no Hermitian phase operator with all the desired properties, there are still states of well defined phase and this phase can be extracted.

Phase measurements can be made by means of an interferometer. To improve the resolution and sensitivity of the measurement, the input could involve  $N$  photons in some manner, rather than a single photon. This could be done in several ways, such using a coherent-state input (Poisson-distributed photon number fluctuations)

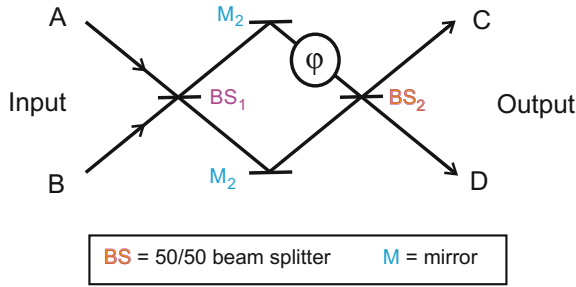


**Fig. 4.10** A Mach-Zehnder interferometer. Input at either of the input ports ( $A$  or  $B$ ) has two possible ways of reaching the final beam splitter,  $BS_2$ . Along the upper path, a phase shift of  $\phi$  is added. The second beam splitter makes it impossible to determine which path was followed simply from looking at the output at ports  $C$  and  $D$ . As a result, the amplitudes for the both possibilities must be present, leading to interference at the output ports

with mean amplitude  $\alpha = \sqrt{N}$ , a single Fock state (of fixed photon number  $N$ ), or  $N$  independent measurements on a stream of single-photon states. In each of these cases, the phase sensitivity—the minimum uncertainty in the measured phase value—scales for large average photon number  $N$  as  $\frac{1}{\sqrt{N}}$ . This is most easily seen by looking at the case of  $N$  measurements of consecutive single-photon inputs.

Suppose an optical state enters port  $A$  in the Mach-Zehnder interferometer of Fig. 4.10. Photons traveling along the upper path gain an extra phase shift of  $\phi$ , compared to those moving through the lower path. Any phase shifts due to reflection at the beam splitter can be absorbed into the definition of  $\phi$ . Denote the state leaving the first beam splitter by the number of photons in the *upper* branch, so that the state with a photon in the lower path is  $|0\rangle$  and the state leaving along the upper path is  $|1\rangle$ . The state arriving at the second beam splitter is then  $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$ . Because the photon number never exceeds 1 anywhere for this state, the operator  $\hat{A}$  can be truncated to

$$\hat{A}' = |0\rangle\langle 1| + |1\rangle\langle 0|. \quad (4.26)$$

$\hat{A}'$  picks out interference terms; when wedged between two states,  $\langle\psi_1|\hat{A}'|\psi_2\rangle$ , it links the  $|0\rangle$  component of one state to the  $|1\rangle$  component of the other. Taking a basis of the form

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (4.27)$$

$\hat{A}'$  is therefore simply given by the first Pauli spin matrix

$$\hat{A}' = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.28)$$

and the squared operator,  $\hat{A}'^2 = I$ , is the identity operator.

Taking the expectation value in state  $|\psi\rangle$ , it is found that

$$\langle \hat{A}' \rangle = \langle \psi | \hat{A}' | \psi \rangle = \cos \phi, \quad (4.29)$$

$$\langle \hat{A}'^2 \rangle = \langle \psi | \hat{A}'^2 | \psi \rangle = 1. \quad (4.30)$$

Therefore, the uncertainty in the measurement of  $\hat{A}'$  is

$$\Delta A' = \sqrt{\langle \hat{A}'^2 \rangle - \langle \hat{A}' \rangle^2} = \sqrt{1 - \cos^2 \phi} = |\sin \phi|. \quad (4.31)$$

If the experiment is repeated  $N$  times on an identical ensemble of states, standard statistical sampling theory tells us that the uncertainty is reduced by a factor of  $\sqrt{N}$ :

$$\Delta A' = \frac{|\sin \phi|}{\sqrt{N}}. \quad (4.32)$$

The best estimate of  $\phi$  is then provided by the value

$$\bar{\phi} = \cos^{-1} \frac{\langle \hat{A}' \rangle}{N}, \quad (4.33)$$

and the uncertainty in this phase estimate is given by

$$\Delta \phi = \frac{\Delta A'}{\left| \frac{d\langle \hat{A}' \rangle}{d\phi} \right|} = \frac{\sqrt{N} |\sin \phi|}{N |\sin \phi|} = \frac{1}{\sqrt{N}}, \quad (4.34)$$

the *standard quantum limit* or “shot noise” limit. In the case of single photon states being measured  $N$  times, this uncertainty can be viewed as being due to photonic shot noise. This is sometimes thought of as “sorting noise,” the Poisson-distributed random fluctuations of photon number in each arm of the interferometer due to the random choice taken by each photon when exiting the beam splitter. The independent random fluctuations of photon number in the coherent state beam have the same Poisson statistics as the sorting noise, so that use of a single coherent state pulse with mean photon number  $\langle \hat{N} \rangle = N$ ; leads to exactly the same form for the scaling of uncertainty with photon number.

The discussion of the standard quantum limit relies on the assumption that the  $N$  probing photons all acted independently. However, if the photons are entangled, so that the entire  $N$ -photon set is described by a single joint quantum state, then the joint state vector depends on the *sum* of the single-photon phases. In Fig. 4.10, let a quantum state with  $n_1$  photons in the top branch and  $n_2$  photons in the lower branch be denoted by  $|n_1, n_2\rangle$ . If one sends an entangled  $N$ -photon NOON state [39],

$$\frac{1}{\sqrt{2}} (|N, 0\rangle + |0, N\rangle), \quad (4.35)$$

into the system, the state reaching the second beam splitter is

$$|\psi_N\rangle = \frac{1}{\sqrt{2}} (|0, N\rangle + e^{iN\phi}|N, 0\rangle). \quad (4.36)$$

The phase shifts from the  $N$  entangled photons act collectively, giving a total shift of  $N\phi$ . The phase shift is thus amplified by a factor of  $N$  relative to the single-photon and separable  $N$ -photon cases, where each photon carried its own independent phase shift. An appropriate measurement operator to extract this total phase shift is now

$$\hat{B}_N = |0, N\rangle\langle N, 0| + |N, 0\rangle\langle 0, N|. \quad (4.37)$$

The expectation values are now given by

$$\langle \hat{B}_N \rangle = \langle \psi_N | \hat{B}_N | \psi_N \rangle = \cos N\phi, \quad (4.38)$$

$$\langle \hat{B}_N^2 \rangle = \langle \psi_N | \hat{B}_N^2 | \psi_N \rangle = 1, \quad (4.39)$$

which means that the measurement uncertainties are

$$\Delta B_N = \sin N\phi \quad \text{and} \quad \Delta\phi = \frac{1}{N}. \quad (4.40)$$

This entangled system therefore beats the standard quantum limit and reaches the *Heisenberg limit*, that is, the fundamental physical bound imposed by quantum mechanics [74] for states of fixed photon number. Phase microscopy has been demonstrated near the Heisenberg limit [46] by means of such a NOON state approach. In addition to the  $\frac{1}{\sqrt{N}}$  advantage in phase sensitivity over unentangled  $N$ -photon states, there is also an increase in oscillation frequency, indicated by the factor of  $N$  inside the cosine in Eq. 4.38. This leads to phase measurements with super-resolution, as well. Other entangled  $N$ -photon states, such as those of [75, 76] may be used instead of NOON states. Some of these give slightly better sensitivity or are slightly more robust to noise but, overall, they give qualitatively similar results to those derived from NOON states. However, NOON states and other similar entangled photon states are difficult to create for large  $N$ . Consequently, it is necessary to find ways to make such states more efficiently before such methods can move out of the lab and into real-world applications.

Instead of phase uncertainty, which is derived from the variance in some observable, as in Eq. 4.40, it is sometimes advantageous, especially in imaging applications, to quantify the optimal uncertainty via information theoretic means, such as by comparison to the Cramer–Rao bound [77, 78]. The quality of the estimate may be also be measured by means of mutual information [79, 80]. In a related vein, Bayesian analysis [81, 82] can be utilized to provide strategies for optimizing the estimation method.

The Heisenberg limit is strictly true only for states of fixed photon number,  $\Delta N = 0$ ; for such states the equality  $\langle \hat{N}^2 \rangle = \langle \hat{N} \rangle^2$  holds. However, for states of strongly

fluctuating photon number the quantities  $\langle \hat{N}^2 \rangle$  and  $\langle \hat{N} \rangle^2$  differ, and the lower limit of sensitivity can depend on both. This offers up the possibility that the lower limit for states of indefinite photon number may be lower than the Heisenberg limit. For states with large fluctuations, the Hofmann limit has been proposed [83], which states

$$\Delta\phi^2 \geq \frac{1}{\langle \hat{N}^2 \rangle}. \quad (4.41)$$

This is a weaker limit than the Heisenberg limit, due to the fact that  $\langle \hat{N}^2 \rangle \geq \langle \hat{N} \rangle^2$ . In the most general case, the maximum possible sensitivity depends on the form of the probe state used and the ultimate limit is then determined by the quantum Cramer–Rao bound set by the Fisher information of the probe state.

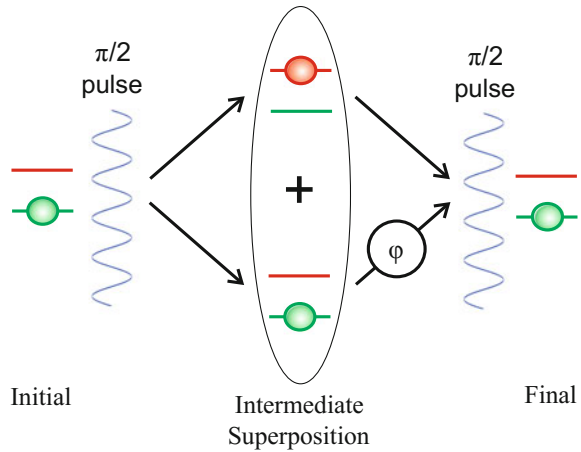
One means of exceeding the Heisenberg limit in some cases is by use of parity measurements. The parity operator is  $\hat{\Pi} = e^{i\pi\hat{N}}$ , with eigenvalues  $\pm 1$  depending on whether it acts on states with even or odd photon number. The parity can be measured by means of photon-number-resolving detectors, or by use of interferometric arrangements involving Kerr nonlinear media [84]. An alternate method makes use of the fact that the expectation value of the parity is related to the value of the Wigner function at the origin,  $\langle \hat{\Pi} \rangle = \frac{\pi}{2} W(0, 0)$  [85]. The value of  $W(0, 0)$  can then be found from quantities that can be measured via homodyne detection. It was demonstrated in [86] that parity measurements allow the phase of a two-mode squeezed vacuum state to be measured with a sensitivity that exceeds the Heisenberg limit and approaches the Hofmann limit.

Entangled states are not necessary to surpass the standard quantum limit. For example, squeezed states have been shown to be capable of achieving  $\frac{1}{N^{3/4}}$  phase sensitivity [87]. It has also been shown that it is possible to produce super-resolution using a post-selection scheme [88] without the use of entangled illumination states. Similarly, high phase sensitivity can also be achieved using non-entangled *twin-Fock states* [89, 90] of the form  $|N\rangle_A |N\rangle_B$  fed into an interferometer; however, the production of equal numbers of photons entering both ports simultaneously presents technical challenges for large  $N$ . Moreover, only entangled states currently appear capable of reaching the Heisenberg bound.

## 4.7 Frequency Measurements

The Mach–Zehnder interferometer, used to determine phase shifts, is conceptually equivalent to the *Ramsey interferometer*, which is used to measure atomic transition frequencies. This formal analogy between the Mach–Zehnder and Ramsey interferometers, as well as an analogy with quantum computer circuits, allows ideas from these three different areas to be translated directly into each other’s languages; these analogies have therefore been referred to as the quantum Rosetta stone [8, 40].

**Fig. 4.11** Schematic of a Ramsey interferometer. An optical pulse puts the ground state input into a superpositions of ground and excited states. After allowing each state to evolve freely for a fixed amount of time, their energy differences causes the two states in the superposition to differ in phase by an amount proportional to the transition frequency between them



A schematic of the Ramsey interferometer is shown in Fig. 4.11. Let the transition frequency between two energy levels  $E_e$  and  $E_0$  be given by  $\omega = \frac{E_e - E_0}{\hbar}$ . Then a  $\frac{\pi}{2}$  pulse is an optical pulse of duration  $T$  satisfying  $\omega T = \frac{\pi}{2}$ . Such a pulse is used to put an incoming atom into a superposition of the ground and excited states, of energies  $E_0$  and  $E_e$ ; this is analogous to the first beam splitter in the Mach-Zehnder case, which puts a photon into a superposition of two different paths. The two energy states gain different phase shifts under free propagation for a fixed time  $t$ , leading to a relative phase factor of  $e^{i\Delta E t} = e^{i(E_e - E_0)t}$  so that when they are returned to the same state again by a second pulse, they produce an interference pattern as the delay time between the two pulses is varied. The relative phase shift is  $\frac{\Delta E t}{\hbar} = \omega t$ , where  $\Delta E = E_e - E_0$  and  $\omega$  are the energy difference and emitted photon frequency of the levels. As a result, the interference pattern allows frequency differences to be determined the just as phase differences are determined in a Mach-Zehnder interferometer.

The formal equivalence between the Ramsey and Mach-Zehnder interferometers immediately implies that the use of entangled atomic states as input to the Ramsey interferometer should allow super-sensitive measurements of frequency, with the uncertainty scaling as  $\frac{1}{N}$  [91, 92]. By similar reasoning, entanglement can be used to produce Heisenberg-limited resolution in optical and atomic gyroscopes, which are also based on Mach-Zehnder interferometry [93].

## 4.8 Additional Applications in Metrology

A number of other metrological applications carried out with entangled photons have been proposed or carried out. A few of these are briefly mentioned here.

- Although, until recently, high quality infrared detection at low photon number has been difficult, entangled photon sources provide a means of detecting infrared photons using visible-light detectors (see [14]).
- The use of specialized quantum optical states (primarily squeezed states) has been an active area of research in enhancement of gravity-wave detection. Use of such methods has already improved the sensitivity of the Laser Interferometer Gravitational-Wave Observatory (LIGO) detector beyond the classical limit [94, 95]. The extreme sensitivity of the LIGO interferometer has recently been spectacularly demonstrated by the first successful detection of gravitational waves [96], a full century after their existence was first predicted.
- The use of entangled states can improve the measurement of group and phase velocity in a material, as well as the polarization mode dispersion. These methods are discussed in more detail in Chap. 5.
- Entangled photon states with nonzero orbital angular momentum allows the super-precise measurement of angular orientation and angular velocity. This is discussed in detail in Chap. 8.

## References

1. C.D. Nabor, R.M. Shelby, *Phys. Rev. A* **442**, 556 (1990)
2. P.R. Tapster, S.F. Seward, J.G. Rarity, *Phys. Rev. A* **44**, 3266 (1991)
3. P.H. Souto Ribeiro, C. Schwob, A. Maitre, C. Fabre, *Opt. Lett.* **22**, 1893 (1997)
4. O. Jedrkiewicz, Y.-K. Jiang, E. Brambilla, A. Gatti, M. Bache, L.A. Lugiato, P. Di Trapani, *Phys. Rev. Lett.* **93**, 243601 (2004)
5. M. Bondani, A. Allevi, G. Zambra, M.G.A. Paris, A. Andreoni, *Phys. Rev. A* **76**, 013833 (2007)
6. J.-L. Blanchet, F. Devaux, L. Furfaro, E. Lantz, *Phys. Rev. Lett.* **101**, 233604 (2008)
7. T. Iskhakov, M.V. Chekhova, G. Leuchs, *Phys. Rev. Lett.* **102**, 183602 (2009)
8. K.T. Kapale, L.D. Didomenico, H. Lee, P. Kok, J.P. Dowling, *Concepts Phys.* **2**, 225 (2005)
9. V. Giovannetti, S. Lloyd, L. Maccone, *Science* **306**, 1330 (2004)
10. V. Giovannetti, S. Lloyd, L. Maccone, *Nat. Photonics* **5**, 222 (2011)
11. D. Simon, *J. Sens.* Article ID 6051286 (2015)
12. G. Kitaeva, A.N. Penin, V.V. Fadeev, Y.A. Yanait, *Sov. Phys. Doklady* **24**, 564 (1979)
13. A. Migdall, *Phys. Today* **52**, 41 (1999)
14. A.V. Sergienko, in *Recent Advances in Metrology and Fundamental Constants: CXLVI Proceedings Int. School of Physics Enrico Fermi*, ed. by T.J. Quinn, S. Leschiutta, P. Tavella (IOS Press, Amsterdam, 2001), p. 715
15. D.N. Klyshko, *Sov. J. Quantum Electron.* **7**, 591 (1977)
16. A.V. Sergienko, G.S. Jaeger, *Cont. Phys.* **44**, 341 (2003)
17. H.G. Tompkins, *A User's Guide to Ellipsometry* (Academic Press, London, 1993)
18. R.M.A. Azzam, N.M. Bashara, *Ellipsometry and Polarized Light* (North-Holland, Amsterdam, 1977)
19. A.F. Abouraddy, K.C. Toussaint, A.V. Sergienko, B.E.A. Saleh, M.C. Teich, *Opt. Lett.* **26**, 1717 (2001)
20. A.F. Abouraddy, K.C. Toussaint, A.V. Sergienko, B.E.A. Saleh, M.C. Teich, *J. Opt. Soc. Am. B* **19**, 652 (2002)
21. K.C. Toussaint Jr., G. Di Giuseppe, K.J. Bycenski, A.V. Sergienko, B.E.A. Saleh, M.C. Teich, *Phys. Rev. A* **70**, 023801 (2004)
22. J.M. Schmitt, *IEEE J. Sel. Top. Quantum Electron.* **5**, 1205 (1999)

23. D. Huang, E.A. Swanson, C.P. Lin et al., *Science* **254**, 1178 (1991)
24. J. Welzel, *Skin Res. Technol.* **7**, 1 (2001)
25. J.G. Fujimoto, *Nat. Biotechnol.* **21**, 1361 (2003)
26. A.F. Fercher, W. Drexler, C.K. Hitzenberger, T. Lasser, *Rep. Prog. Phys.* **66**, 239 (2003)
27. A. Abouraddy, M.B. Nasr, B.E.A. Saleh, A.V. Sergienko, M.C. Teich, *Phys. Rev. A* **65**, 053817 (2002)
28. M.B. Nasr, B.E.A. Saleh, A.V. Sergienko, M.C. Teich, *Phys. Rev. Lett.* **91**, 083601 (2003)
29. M.B. Nasr, B.E.A. Saleh, A.V. Sergienko, M.C. Teich, *Opt. Exp.* **12**, 1353 (2004)
30. M.C. Teich, B.E. Saleh, F.N. Wong, J.H. Shapiro, *Quantum Inf. Process.* **11**, 903 (2012)
31. M.B. Nasr, D.P. Goode, N. Nguyen, G. Rong, L. Yang, B.M. Reinhard, B.E. Saleh, M.C. Teich, *Opt. Commun.* **282**, 1154 (2009)
32. R. Kaltenbaek, J. Lavoie, D.N. Biggerstaff, K.J. Resch, *Nat. Phys.* **4**, 864 (2008)
33. J. Lavoie, R. Kaltenbaek, K.J. Resch, *Opt. Exp.* **17**, 3818 (2009)
34. B.I. Erkmen, J.H. Shapiro, *Phys. Rev. A* **74**, 041601 (2006)
35. K. Banaszek, A.S. Radunsky, I.A. Walmsley, *Opt. Commun.* **269**, 152 (2007)
36. K.J. Resch, P. Puvanathan, J.S. Lundeen, M.W. Mitchell, K. Bizheva, *Opt. Exp.* **15**, 8797 (2007)
37. B.E.A. Saleh, M.C. Teich, *Fundamentals of Photonics*, 2nd edn. (Wiley-Interscience, Hoboken, 2007)
38. B.C. Sanders, *Phys. Rev. A* **40**, 2417 (1989)
39. A.N. Boto, P. Kok, D.S. Abrams, S.L. Braunstein, C.P. Williams, J.P. Dowling, *Phys. Rev. Lett.* **85**, 2733 (2000)
40. H. Lee, P. Kok, J.P. Dowling, *J. Mod. Opt.* **49**, 2325 (2002)
41. C.K. Hong, Z.Y. Ou, L. Mandel, *Phys. Rev. Lett.* **59**, 2044 (1987)
42. M.W. Mitchell, J.S. Lundeen, A.M. Steinberg, *Nature* **429**, 161 (2004)
43. P. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, A. Zeilinger, *Nature* **429**, 158 (2004)
44. I. Afek, O. Ambar, Y. Silberberg, *Science* **328**, 879 (2010)
45. Y. Israel, I. Afek, S. Rosen, O. Ambar, Y. Silberberg, *Phys. Rev. A* **85**, 022115 (2012)
46. Y. Israel, S. Rosen, Y. Silberberg, *Phys. Rev. Lett.* **112**, 103604 (2014)
47. M. D'Angelo, M. Chekhova, Y. Shih, *Phys. Rev. Lett.* **87**, 013602 (2001)
48. G. Björk, L.L. Sánchez-Soto, J. Söderholm, *Phys. Rev. Lett.* **86**, 4516 (2001)
49. G. Björk, L.L. Sánchez-Soto, J. Söderholm, *Phys. Rev. A* **64**, 013811 (2001)
50. E. Yablonovich, R.B. Vrijen, *Opt. Eng.* **38**, 334 (1999)
51. C. Kothe, G. Björk, S. Inoue, M. Bourennane, *New J. Phys.* **13**, 043028 (2011)
52. S.J. Bentley, R.W. Boyd, *Opt. Express* **12**, 5735 (2004)
53. R.W. Boyd, S.J. Bentley, *J. Mod. Opt.* **53**, 713 (2006)
54. P. Kok, S.L. Braunstein, J.P. Dowling, *J. Opt. B Quantum Semiclass. Opt.* **6**, S811 (2004)
55. R.W. Boyd, J.P. Dowling, *Quant. Inf. Proc.* **11**, 891 (2011)
56. A. Crespi, M. Lobino, J. Matthews, A. Politi, C. Neal, R. Ramponi, R. Osellame, J. O'Brien, *Appl. Phys. Lett.* **100**, 233704 (2012)
57. Z. Hradil, *Quant. Opt.* **4**, 93 (1992)
58. P.A.M. Dirac, *Proc. R. Soc. Lond. Ser. A* **114**, 243 (1927)
59. R. Peierls, *Surprises in Theoretical Physics* (Princeton University Press, Princeton, 1979)
60. L. Susskind, J. Glogower, *Physics* **1**, 49 (1964)
61. P. Caruthers, M.M. Nieto, *Rev. Mod. Phys.* **40**, 411 (1968)
62. J.-M. Levy-Leblond, *Ann. Phys. (N.Y.)* **101**, 319 (1976)
63. D.T. Pegg, S.M. Barnett, *Phys. Rev. A* **39**, 1665 (1989)
64. J.A. Vaccaro, D.T. Pegg, *Opt. Commun.* **70**, 529 (1989)
65. S.M. Barnett, S. Stenholm, D.T. Pegg, *Opt. Commun.* **73**, 314 (1989)
66. J.A. Vaccaro, D.T. Pegg, *J. Mod. Opt.* **37**, 17 (1990)
67. S.M. Barnett, D.T. Pegg, *Phys. Rev. A* **41**, 3427 (1990)
68. J.A. Vaccaro, D.T. Pegg, *Phys. Rev. A* **41**, 5156 (1990)
69. G.S. Summy, D.T. Pegg, *Opt. Commun.* **77**, 75 (1990)

70. S.M. Barnett, D.T. Pegg, Phys. Rev. A **42**, 6713 (1990)
71. D.T. Pegg, J.A. Vaccaro, S.M. Barnett, J. Mod. Opt. **37**, 1703 (1990)
72. J. Bergou, B.-G. Englert, Ann. Phys. (N.Y.) **209**, 479 (1991)
73. D.T. Pegg, S.M. Barnett, Europhys. Lett. **6**, 483 (1988)
74. A.S. Lane, S.L. Braunstein, C.M. Caves, Phys. Rev. A **47**, 1667 (1993)
75. B. Yurke, S.L. McCall, J.R. Klauder, Phys. Rev. A **33**, 4033 (1986)
76. S.D. Huver, C.F. Wildfeuer, J.P. Dowling, Phys. Rev. A **78**, 063828 (2008)
77. H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1946)
78. C.R. Rao, Bull. Calcutta Math. Soc. **37**, 81 (1945)
79. D.S. Simon, A.V. Sergienko, T.B. Bahder, Phys. Rev. A **78**, 053829 (2008)
80. T.B. Bahder, Phys. Rev. A **83**, 053601 (2011)
81. Z. Hradil, R. Myška, J. Peřina, M. Zawisky, Y. Hasegawa, H. Rauch, Phys. Rev. Lett. **76**, 4295 (1996)
82. L. Pezzé, A. Smerzi, G. Khoury, J.F. Hodelin, D. Bouwmeester, Phys. Rev. Lett. **99**, 223602 (2007)
83. H.F. Hofmann, Phys. Rev. A **79**, 033822 (2009)
84. C.C. Gerry, A. Benmoussa, R.A. Campos, Phys. Rev. A **72**, 053818 (2005)
85. A. Royer, Phys. Rev. A **15**, 449 (1977)
86. P.M. Anisimov, G.M. Raterman, A. Chiruvelli, W.N. Plick, S.D. Huver, H. Lee, J.P. Dowling, Phys. Rev. Lett. **104**, 103602 (2010)
87. C.M. Caves, Phys. Rev. D **23**, 1693 (1981)
88. K.J. Resch, K.L. Pregnell, R. Prevedel, A. Gilchrist, G.J. Pryde, J.L. O'Brien, A.G. White, Phys. Rev. Lett. **98**, 223601 (2007)
89. C.C. Gerry, J. Mimih, in *Advances in Photonics of Quantum Computing, Memory, and Communication IV*, ed. by Z. Ul Hasan, P.R. Hemmer, H. Lee, C.M. Santori, Proc. of SPIE, vol. 7948, 79480D (2011)
90. M.J. Holland, K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993)
91. J.J. Bollinger, W.M. Itano, D.J. Wineland, D.J. Heinzen, Phys. Rev. A **54**, R4649 (1996)
92. P. Bouyer, M.A. Kasevich, Phys. Rev. A **56**, R1083 (1997)
93. J.P. Dowling, Phys. Rev. A **57**, 4736 (1998)
94. J. Aasi et al., Nat. Photonics **7**, 613 (2013)
95. LIGO Scientific Collaboration, Nat. Phys. **7**, 962 (2011)
96. B.P. Abbott et al., LIGO Scientific Collaboration and Virgo Collaboration. Phys. Rev. Lett. **116**, 061102 (2016)

# Chapter 5

## Polarization Mode Dispersion

### 5.1 Classical Versus Quantum Measurement of Polarization-Dependent Dispersion

*Polarization mode dispersion* (PMD) is a polarization-dependent variation of a material's index of refraction. It leads to time delays and wavenumber differences between orthogonal polarization states of light at fixed wavelength. Classically, several methods have been developed for measuring PMD [1–8]. Many of these methods use interferometric measurements to determine optical delays. However, the accuracy of these approaches are limited by the stability of the interferometer, the signal-to-noise level of the detector, and by the need to keep track of the number of interference fringes.

The use of highly monochromatic laser sources creates additional problems, such as multiple reflections and strong irregular interference that further limit the method. Alternatively, white-light or low-coherence interferometry [7] may be used, giving resolution that is dependent on the spectral bandwidth of the light source. Achieving sub-femtosecond resolution in PMD measurement requires light sources with bandwidth in excess of 200 nm, which is technically difficult and often leads to spurious features appearing in measurements due to variations in the spectral shape over the full range. Further, the visibility of interference with such super-broadband light is reduced due to chromatic dispersion effects. Overall, even the most accurate classical methods for PMD measurement are currently limited to a few femtoseconds in resolution.

Many fiber optical communication networks now operate at data rates on the order of 100 Gigabytes per second. At these rates, precise characterization of polarization mode dispersion at the sub-femtosecond level becomes important. In the past, only the PMD of the fiber itself was large enough to be important; since this grows like the square root of the fiber length [9], the fiber PMD can be characterized as precisely as desired by taking a sufficiently long length for measurements. However, at the high data rates of present systems, even the PMD from small, discrete components such as optical switches and amplifiers must be taken into account. Although the effect of

each element is small, the large number of such elements in present-day networks adds up to a sizable contribution.

Characterizing the PMD properties of such small devices to the necessary precision has remained beyond the range of classical methods. However, quantum methods have been shown to meet this challenge. Quantum interferometry can exploit both phase and group velocity effects in the same measurement [10–13], a feat not possible in classical optics, and also take advantage of dispersion cancellation (Chap. 2). As a result, the use of interferometry with entangled photon states can provide an absolute value for polarization-dependent optical delay with high resolution, in contrast to classical methods which can only determine the delay up to an unknown integer multiple of the optical period.

The first uses [10, 11] of a quantum interferometer to measure polarization-dependent delays were carried out with a polarization analog of the HOM interferometer (Sect. 2.5), shown schematically in Fig. 5.1. The two polarization states are delayed by different amounts in the sample, leading to a relative phase shift. Although diagonal polarizers are placed before the detectors, the polarization state of the detected photon can be distinguished via the different detection times, so that no interference occurs. However, by varying the birefringent (polarization-dependent) delay  $\tau$ , the sample-induced differential delay can be compensated for. When the compensation is exact, the two polarization states become indistinguishable to the detectors, and destructive interference occurs, leading to an HOM-type dip. Because both photons are created simultaneously and take a common path through the sample, the interferometer is very stable and has high resolution. The first-order differential group delay due to PMD can be measured to a precision of at least 0.2 fs. Note that the even-order dispersion cancellation effect described in Chap. 2 occurs in this apparatus and aids in the measurement of odd-order effects, but prevents it from

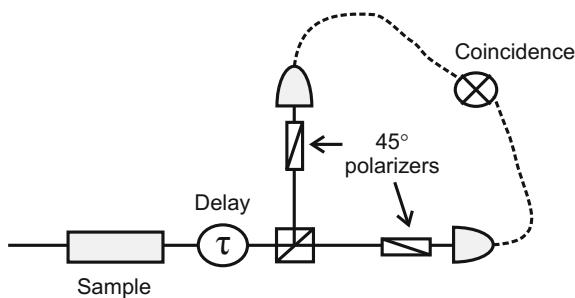


**Fig. 5.1** An HOM-like interferometer for measuring PMD [10, 11]. Type II SPDC is used as a light source, with the photons created in opposite polarization pairs. The sample is placed before the beam splitter and two diagonal polarizers are placed before the detectors in order to erase information about which polarization went which way. As the birefringent delay  $\tau$  is varied, destructive interference becomes complete when the delay exactly compensates the delay between the polarizations introduced by the sample. This is a special case of the Type A interferometer in the next figure

being used to measure even-order effects such as the second-order group velocity dispersion.

In [12], a modified version was presented, as shown schematically in Fig. 5.2. In the following sections, this arrangement is referred to as a *type A PMD interferometer*; the interferometer of Fig. 5.1 can be viewed as a special case. In the most general version, there are variable time delays both before and after the beam splitter (*pre-compensation* and *post-compensation*), and it is also useful to consider the possibility of placing samples either before or after the beam splitter. When placed before, even-order cancelation occurs, allowing high precision odd-order measurements. When the sample is placed after the beam splitter, *no* cancelation occurs and the even orders can be measured. If the pre-beamsplitter delay  $\tau_1$  is scanned, then the situation is the same as described in the previous paragraph, giving an HOM dip due to destructive interference at the point where the relative delay between the *H* and *V* photons is zero (Fig. 2.7). However, when the post-beam splitter delay  $\tau_2$  is scanned, a mechanism such as that of Sect. 2.6 [14] comes into play: the precompensation delay combined with a postcompensation delay of twice the size and opposite sign leaves a nonzero total delay between the two photon detections but still does not allow determination of the identity of the photon reaching each detector. As a result, the amplitudes for the two possibilities interfere. This interference can be constructive or destructive, giving oscillating fringes within the HOM-dip envelope. Adding an additional non-birefringent delay in one arm, the total delay between the two detectors can be canceled, bringing the interfering amplitudes back into coincidence.

In [13], the quantum PMD interferometer was modified once again to produce a further range of possible measurements; this variation is referred to below as a *type B PMD interferometer*. In the following sections, the use of an interferometric setup with entangled photons is described. This method allows measurements of the

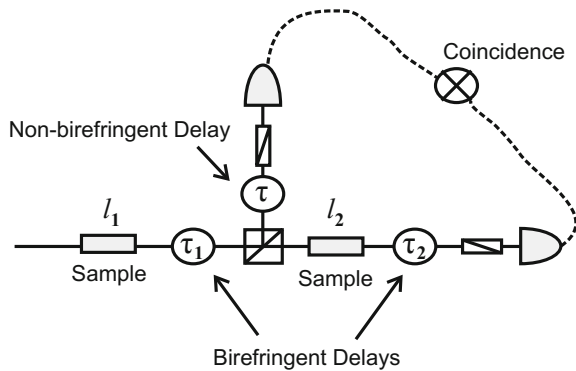


**Fig. 5.2** The type A interferometer, first used in [12]. Now, in addition to the effect occurring in the apparatus of Fig. 5.1, as the postcompensation delay is varied, interference fringes appear within the HOM envelope. These fringes are moved relative to the envelope by the PMD, in a sort of “quantum vernier” effect. If the sample is placed before the beam splitter, there is dispersion cancelation; there is no cancelation if the sample is placed after the beam splitter

PMD of small, discrete optical elements to sub-femtosecond precision, allowing both chromatic dispersion and PMD to be measured and for their effects to be separated.

Furthermore, the frequency anticorrelation in the entangled SPDC source allows even-order and odd-order parts of the PMD's frequency dependence to be isolated by making use of an effect first demonstrated in [15] and discussed further in Sect. 3.3. The availability of such a separation is often useful. For example, if there is too much pulse broadening (second-order dispersion), then accurate measurement of group velocity (first order dispersion) becomes very difficult, unless some mechanism is found to prevent the first-order group delay from being obscured by second-order broadening effects. The quantum interferometric method described below achieves exactly that: As a delay in one arm is scanned, the location of a dip in the coincidence rate may be used to find the group velocity. This location is found to be unaffected by the second-order broadening, as a result of the even-order dispersion cancelation (Sect. 3.4).

The separation method is an intrinsically two-photon coincidence-based quantum effect. Although, in principle, it is possible to mimic such an effect classically by symmetrically chirping and anti-chirping femtosecond laser pulses, this is impractical due to the narrow temporal duration of the wavepacket needed.

In addition, due to the entanglement in the quantum cases there is no need to know in advance the principal axis directions of the device or object being measured. Although the incoming photons are aligned along particular axes that are linked to a birefringent crystal orientation, their projections onto *any* rotated pair of orthogonal axes remain equally entangled (see Sect. 1.10), allowing the method to work without needing to align the axes of the source and the device under test.

Since the focus here is on measurements of compact optical elements, the principal polarization axes are assumed to remain constant over the length of the object and to be independent of frequency. Calculations can therefore be simplified by assuming that the dispersive contributions of the two polarization components remain independent of each other.

## 5.2 Chromatic Dispersion and Polarization Mode Dispersion

Consider a non-birefringent material, for which the index of refraction is independent of polarization. The frequency dependence of the wavenumber  $k = \frac{2\pi n(\lambda)}{\lambda}$  is given by some dispersion relation, which can be expanded in a power series about some fixed frequency value, as in Chap. 3. Let the frequency be  $\nu = \Omega_0 + \omega$ , where  $\omega$  is the deviation away from some central value  $\Omega_0$ . (In the case of down-conversion,  $\Omega_0$  is half the pump frequency, with anticorrelated signal and idler frequencies  $\Omega_0 \pm \omega$ .) As in Eq. (3.4), the wavenumber can be expanded about the central frequency,

$$k(\nu) = k(\Omega_0 \pm \omega) = k_0 \pm \alpha\omega + \beta\omega^2 \pm \gamma\omega^3 + \dots \quad (5.1)$$

for  $|\omega| \ll \Omega_0$ . The center of the wavepacket is at  $k_0 = k(\Omega_0)$ , and the coefficients  $\alpha, \beta, \dots$  characterize the chromatic dispersion. The first few are given explicitly by

$$\alpha = \left. \frac{dk(\omega')}{d\omega'} \right|_{\omega'=\Omega_0}, \quad (5.2)$$

$$\beta = \frac{1}{2!} \left. \frac{d^2k(\omega')}{d\omega'^2} \right|_{\omega'=\Omega_0}, \quad (5.3)$$

$$\gamma = \frac{1}{3!} \left. \frac{d^3k(\omega')}{d\omega'^3} \right|_{\omega'=\Omega_0}. \quad (5.4)$$

It is convenient to collect the individual terms of the expansion (5.1), lumping them into just two terms: one containing all terms with even powers of  $\omega$  and one with all odd power terms:

$$k(\Omega_0 + \omega) = k^E(\omega) + k^O(\omega), \quad (5.5)$$

where

$$k^E(\omega) = k_0 + \beta\omega^2 + \dots, \quad (5.6)$$

and

$$k^O(\omega) = \alpha\omega + \gamma\omega^3 + \dots \quad (5.7)$$

It should be emphasized here that the superscript *O* and *E* stand for odd- and even-order terms in the frequency expansion, *not* for the ordinary and extraordinary polarizations.

When PMD is present, the index of refraction varies with polarization, so there are now separate dispersion relations for the horizontal (*H*) and vertical (*V*) polarization states:

$$k_H(\Omega_0 \pm \omega) = k_{H0} \pm \alpha_H\omega + \beta_H\omega^2 + \dots = k_H^E(\omega) + k_H^O(\omega) \quad (5.8)$$

$$k_V(\Omega_0 \pm \omega) = k_{V0} \pm \alpha_V\omega + \beta_V\omega^2 + \dots = k_V^E(\omega) + k_V^O(\omega). \quad (5.9)$$

To describe the PMD, we must define quantities that measure the differences between the two polarization states, such as:

$$\Delta k_0 = k_{V0} - k_{H0}, \quad (5.10)$$

$$\Delta\alpha = \alpha_V - \alpha_H, \quad (5.11)$$

$$\Delta\beta = \beta_V - \beta_H. \quad (5.12)$$

Over length  $l$ ,  $\Delta k$  leads to a phase shift between the polarizations

$$\Delta\phi \equiv l\Delta k_0, \quad (5.13)$$

as well as group delay and group delay dispersion,  $l\Delta\alpha$  and  $l\Delta\beta$ , respectively. The quantity

$$\Delta k_0 = \frac{\Omega_0 \Delta n(\Omega_0)}{c} \quad (5.14)$$

serves to measure the difference in phase velocity between the two polarization modes. Note that the PMD coefficients ( $\Delta k_0$ ,  $\Delta\alpha$ ,  $\Delta\beta$ ) are in general dependent on the central frequency,  $\Omega_0$ .

When down-conversion is used as the illumination source, the down conversion time scale  $\tau_- = ((u_{\text{ord}})^{-1} - (u_{\text{ext}})^{-1})L$ , becomes relevant, where  $L$  is the thickness of the nonlinear crystal, while  $u_{\text{ord}}$ , and  $u_{\text{ext}}$  are the group velocities of the ordinary and extraordinary polarizations inside the crystal. For bulk crystal, the spectral distribution [16, 17] of the down-converted pairs is given by Eq. (1.77)

$$\Phi(\omega) = \text{sinc}\left(\frac{1}{2}\tau_-\omega\right). \quad (5.15)$$

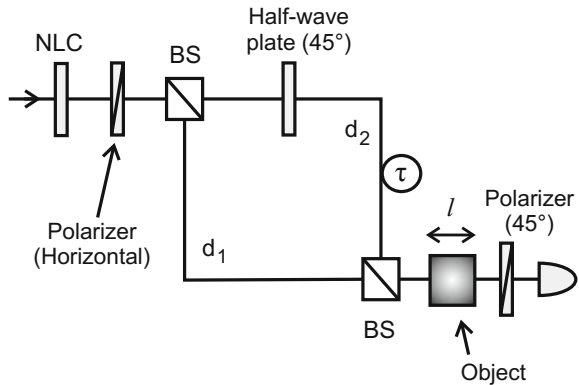
$\tau_-$  is inversely proportional to the spectral width of the source and determines the precision of PMD measurements. The optimal case of simultaneous high intensity and large bandwidth may be obtained by use of chirped crystals [13, 18].

### 5.3 Classical PMD Measurement

A classical measurement of PMD can be accomplished with a single-detector Mach-Zehnder interferometer such as the one shown schematically in Fig. 5.3 [7]. The illumination may be provided by any sufficiently broadband light source.

First, consider the action of the interferometer when there is no sample present. Let  $\hat{x}$  and  $\hat{y}$  be unit vectors in the horizontal and vertical directions, and let  $E_0(\omega)$

**Fig. 5.3** Classical white-light interferometer for measuring total PMD. The light source is any classical or quantum broadband source, and only a single detector is used. Here the source is taken to be SPDC in a non-linear crystal (NLC) in order to more directly compare the classical case the the quantum versions



be the frequency spectrum of the horizontally-polarized light at frequency  $\omega$  leaving the first polarizer. The full optical amplitude at the first beam splitter is then

$$\mathbf{E}_0 = \left( \int E_0(\omega) e^{-i\omega t} d\omega \right) \hat{\mathbf{x}}. \quad (5.16)$$

For light following the lower path, there is a phase accumulation proportional to the path length,  $d_1$ , so that the field after the second beam splitter is

$$\mathbf{E}_1 = \left( \int E_0(\omega) e^{ik(\omega)d_1 - i\omega t} d\omega \right) \hat{\mathbf{x}}, \quad (5.17)$$

where  $k(\omega) = \frac{\omega}{c}$  when no sample is present. For light following the upper path, there is a similar phase corresponding to path length  $d_1$ . However, there is also an additional travel distance  $\delta = c\tau$  due to a controllable time delay  $\tau$ . Moreover, the quarter-wave plate rotates the polarization by  $90^\circ$ . Hence, the contribution of this path to the field after the second beam splitter is

$$\mathbf{E}_2 = \left( \int E_0(\omega) e^{ik(\omega)(d_2 + \delta) - i\omega t} d\omega \right) \hat{\mathbf{y}}. \quad (5.18)$$

The total field after the second beam splitter is then the superposition of these two fields. This superposition then passes through a linear polarizer at  $45^\circ$ , making it impossible for the detector to distinguish the two polarization states. As a result, the horizontally and vertically polarized fields can interfere, with a detected intensity of the form

$$I = \int |E_0(\omega)|^2 \left| e^{ik(\omega)d_1} + e^{ik(\omega)(d_2 + \delta)} \right|^2 d\omega \quad (5.19)$$

$$= 2 \int |E_0(\omega)|^2 [1 + \cos(k(\omega)(\Delta d - \delta))] d\omega, \quad (5.20)$$

where  $\Delta d = d_1 - d_2$  is the path length difference between the two arms.

Now suppose a birefringent sample of length  $l$  is introduced between the last beam splitter and the final polarizer as shown in Fig. 5.3. This contributes an additional polarization-dependent phase shift  $k_P(\omega)l$  to light arriving from each arm, where  $P = \{H, V\}$  denotes the polarization. The two polarizations again interfere after the final BS, so that the detected intensity is now

$$I = 2 \int |E_0(\omega)|^2 \left[ 1 + \cos \left( \frac{\omega}{c} (\Delta d - \delta) - \Delta k(\omega)l \right) \right] d\omega, \quad (5.21)$$

where  $\Delta k(\omega) = k_H(\omega) - k_V(\omega)$  characterizes the PMD. Note that the first term inside the cosine arises entirely from non-polarization-dependent terms, so that  $k(\omega)$  can simply be written as  $k(\omega) \approx \frac{\omega}{c}$  for small  $\omega$  and  $\Delta\alpha$ . So, assume a type II

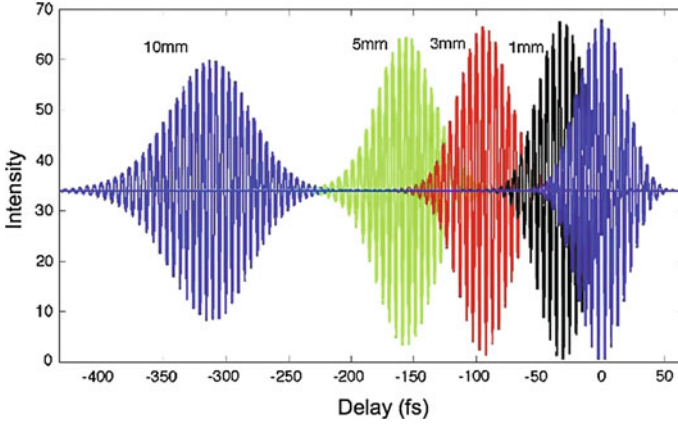


**Fig. 5.4** Interferograms produced by apparatus of Fig. 5.3 for samples of different thicknesses. For a fixed thickness, the size of the shift may be used as a measure of the difference in phase velocities of the two polarizations. A 200 nm bandwidth centered at 1550 nm was assumed, with a coherence length of  $x_c = \frac{\lambda_0^2}{\Delta\lambda} = 12 \mu\text{m}$  (Figure reproduced from [13])

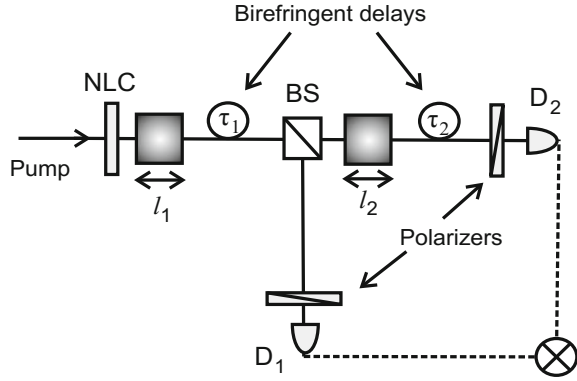
SPDC light source in order to better compare with the results of the quantum interferometers below. Then,  $E_0(\omega)$  is proportional to the down-conversion spectrum  $\Phi(\omega) = \text{sinc}(\frac{1}{2}\tau\omega)$ . Plotting Eq. (5.21) versus delay  $\delta$  leads to interferograms of the form shown in Fig. 5.4 [13].

The first-order contribution  $\Delta\alpha$  to the PMD shifts the envelope left or right, while the second-order PMD contribution  $\Delta\beta$  broadens the envelope. The maxima and minima of the interference pattern shift horizontally *within* the envelope due to the zero-order PMD term  $\Delta k_0$ . In principle the zero- and first-order contributions can be distinguished, since the shift of the envelope and the shift of the fringes relative to the envelope are distinguishable. However, the resolution is limited by the coherence length of the light and by the broadening caused by the second-order PMD,  $\Delta\beta$ , making such distinctions hard to detect in practice.

## 5.4 Type A Quantum Measurement

Moving now from classical to quantum interferometry, consider the possibility that entangled sources and coincidence counting may lead to improved measurements. In particular, the goals are to seek improved resolution in PMD measurement and to determine the even and odd orders of chromatic dispersion separately for each polarization. The methods here ultimately are based on the use of dispersion cancellation and HOM interferometry (Sects. 2.5, 3.4).

**Fig. 5.5** Type A setup for measuring PMD parameters  $\Delta\alpha \equiv \alpha_V - \alpha_H$  and  $\Delta\beta \equiv \beta_V - \beta_H$ . The dispersion is introduced by the two objects  $\mathcal{O}_1$  and  $\mathcal{O}_2$  of lengths  $l_1$  and  $l_2$ , and illumination is provided by type II parametric down conversion



Consider the interferometer [12] shown in Fig. 5.5, with type II down-conversion as the illumination source. Recall that in type II SPDC the signal and idler photons have opposite polarization ( $H$  and  $V$ ), in addition to anticorrelated frequencies  $\Omega_0 \pm \omega$ , where  $2\Omega_0$  is the pump frequency. Controllable birefringent time delays  $\tau_1$  and  $\tau_2$  are introduced before and after the beam splitter, leading to path length differences  $\delta_1$  and  $\delta_2$ , respectively. For generality, allow for the possibility of two objects  $\mathcal{O}_1$  and  $\mathcal{O}_2$  of lengths  $l_1$  and  $l_2$  being placed both before and after the beam splitter. Diagonal polarizers at angles  $\theta_1 = \theta_2 = \frac{\pi}{4}$  are also placed before the two detectors. Finally, assume that the beam splitter is 50/50 and nonpolarizing, so that information about which polarization state travels in which branch of the apparatus is unavailable, allowing interference to occur with maximum visibility.

Neglecting portions of the amplitude that play no role here, the biphoton state produced by SPDC is (see Sect. 1.6.1)

$$|\Psi\rangle = \int d\omega \Phi(\omega) \hat{a}_H^\dagger(\Omega_0 + \omega) \hat{a}_V^\dagger(\Omega_0 - \omega) |0\rangle. \quad (5.22)$$

The positive-frequency parts of the fields at detectors  $D_1$  and  $D_2$ , respectively, are given by

$$\hat{E}_1^{(+)}(t_1) = \int \frac{d\omega}{2} \left\{ \hat{a}_H(\omega_1) e^{ik_H(\omega_1)l_1} + \hat{a}_V(\omega_1) e^{i[k_V(\omega_1)l_1 + \omega_1\tau_1]} \right\} e^{-i\omega_1 t_1} \quad (5.23)$$

$$\hat{E}_2^{(+)}(t_2) = \int \frac{d\omega}{2} \left\{ \hat{a}_H(\omega_2) e^{ik_H(\omega_2)(l_1+l_2)} + \hat{a}_V(\omega_2) e^{i[k_V(\omega_2)(l_1+l_2) + \omega_2(\tau_1+\tau_2)]} \right\} e^{-i\omega_2(t_2+\tau)}. \quad (5.24)$$

Then the coincidence rate is

$$R_c(\tau_1, \tau_2) = \int_{-\infty}^{\infty} dt_1 dt_2 G^{(2)}(t_1, t_2), \quad (5.25)$$

$$= \int_{-\infty}^{\infty} dt_1 dt_2 \left| \langle 0 | E_1^{(+)}(t_1) E_2^{(+)}(t_2) | \Psi \rangle \right|^2. \quad (5.26)$$

where it has been assumed that the characteristic time of the detector is much larger than the down-conversion time scale  $\tau_-$  so that the integration limits can be taken to infinity.

Combining Eqs. (5.22)–(5.26), the coincidence rate is [16]

$$R_c(\tau_1, \tau_2) = R_0 \{1 + C M(\tau_1, \tau_2)\}, \quad (5.27)$$

where  $R_0$  is a delay-independent background term and  $C^{-1} = \int d\omega |\Phi(\omega)|^2 = \frac{2\pi}{\tau_-}$ . The delay-dependent modulation term can readily be shown [13] to be given by

$$M(\tau_1, \tau_2) = \int d\omega |\Phi(\omega)|^2 \quad (5.28)$$

$$\begin{aligned} & \times \cos \{[\Delta k(\omega) - \Delta k(-\omega)] l_1 + \Delta k(\omega) l_2 + 2\omega\tau_1 + (\Omega_0 + \omega)\tau_2\} \\ & = \int d\omega |\Phi(\omega)|^2 \cos [\Delta k^O(\omega)(2l_1 + l_2) + \omega(2\tau_1 + \tau_2)] \\ & \quad \times \cos [\Delta k^E(\omega) l_2 + \Omega_0 \tau_2]. \end{aligned} \quad (5.29)$$

It is seen that the even- and odd-order portions have separated into different cosine terms, which can be manipulated independently by altering the two delays. Note further that the even-order PMD from the first object,  $\mathcal{O}_1$ , does not appear in this result; there is no  $l_1$  or  $\tau_1$  dependence in the  $\Delta k^E$  term. It can be seen also that only the *differences* of the horizontal and vertical polarization parameters ( $\Delta\alpha$ ,  $\Delta\beta$ , etc.) appear in the coincidence rate, so the interferogram is insensitive to non-polarization-dependent dispersive effects.

When  $\Delta\beta$  and all higher order terms are negligible, the frequency integration can be done exactly, which leads to the result that [13]:

$$M(\tau_1, \tau_2) = \frac{2\pi}{\tau_-} \cos [\Delta k_0 l_2 + \Omega_0 \tau_2] \Lambda [f(\tau_1, \tau_2)] \quad (5.30)$$

and

$$R_c(\tau_1, \tau_2) = R_0 \{1 + \cos [\Delta k_0 l_2 + \Omega_0 \tau_2] \Lambda [f(\tau_1, \tau_2)]\}. \quad (5.31)$$

Here,

$$\Lambda(x) = \begin{cases} 1 - |x|, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 1 \end{cases} \quad (5.32)$$

is the unit triangle function, and the function  $f(\tau_1, \tau_2)$  is defined by

$$f(\tau_1, \tau_2) = \frac{(2l_1 + l_2)\Delta\alpha + (2\tau_1 + \tau_2)}{\tau_-}. \quad (5.33)$$

The result of Eq. (5.31) for the special case  $\tau_1 = 0$  was first obtained in [12].

By means of scans over the two delay times, various types of information about the sample can be obtained. For example, scanning over  $\tau_1$  with  $\tau_2 = 0$ , the result is a triangular HOM-like dip of the sort shown in Fig. 5.6. (Depending on the sign of the cosine term, this “dip” may actually be a peak.) The bottom of the dip is at  $\tau_1 = -\frac{\Delta\alpha}{2}(2l_1 + l_2)$ , where maximal cancelation occurs between the two arms of the apparatus. Thus, the location of the dip allows determination of the first-order PMD term,  $\Delta\alpha$ . Further, the depth of the dip is determined by the factor  $\cos(\Delta k_0 l_2)$ , so that by measuring this depth, the value of  $\Delta k_0$  to be determined as well: The dip is maximal when  $l_2 = \frac{\pi}{\Delta k_0}$ .

One could also scan over values of  $\tau_2$  while fixing  $\tau_1 = 0$ . In this case, interference fringes from the cosine term of Eq. (5.31) appear within the triangular envelope, similar to those of Fig. 5.4. As before, the value of  $\Delta\alpha$  can be determined from the distance the triangular envelope has been shifted horizontally. But now, the effect of nonzero  $\Delta k_0$  is to shift horizontally the interference fringes within the envelope. By measuring how far the minimum of the central fringe is shifted relative to the minimum of the envelope,  $\Delta k_0$  can be determined to a high level of accuracy. Figure 5.7 shows the effect of each order of dispersion on the  $\tau_2$  scan. Relative to the scan in the absence of dispersion (Fig. 5.7b), the presence of nonzero  $\Delta\alpha$  shifts the entire interference pattern (Fig. 5.7a), while nonzero  $\Delta k_0$  leaves the envelope unaffected and shifts the interference fringes within it (Fig. 5.7c) by distance  $\tau_2 = \frac{\Delta k_0 l_2}{\Omega_0}$ . This shifting of the fringes within the envelope, in what has been called the *quantum vernier effect*, makes it possible to evaluate relative phase delays of the two polarizations

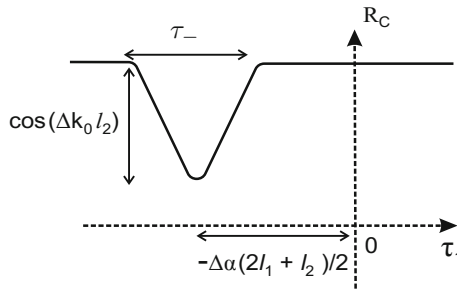


**Fig. 5.6** Scanning over  $\tau_1$  while keeping  $\tau_2 = 0$  leads to HOM-like dips in the coincidence rate,  $R_C$ . Measuring the horizontal location of the dip allows  $\Delta\alpha$  to be determined, while measuring its vertical depth determines  $\Delta k_0$ . Depending on the sign of the cosine term in Eq. (5.31), the “dip” may actually be a peak

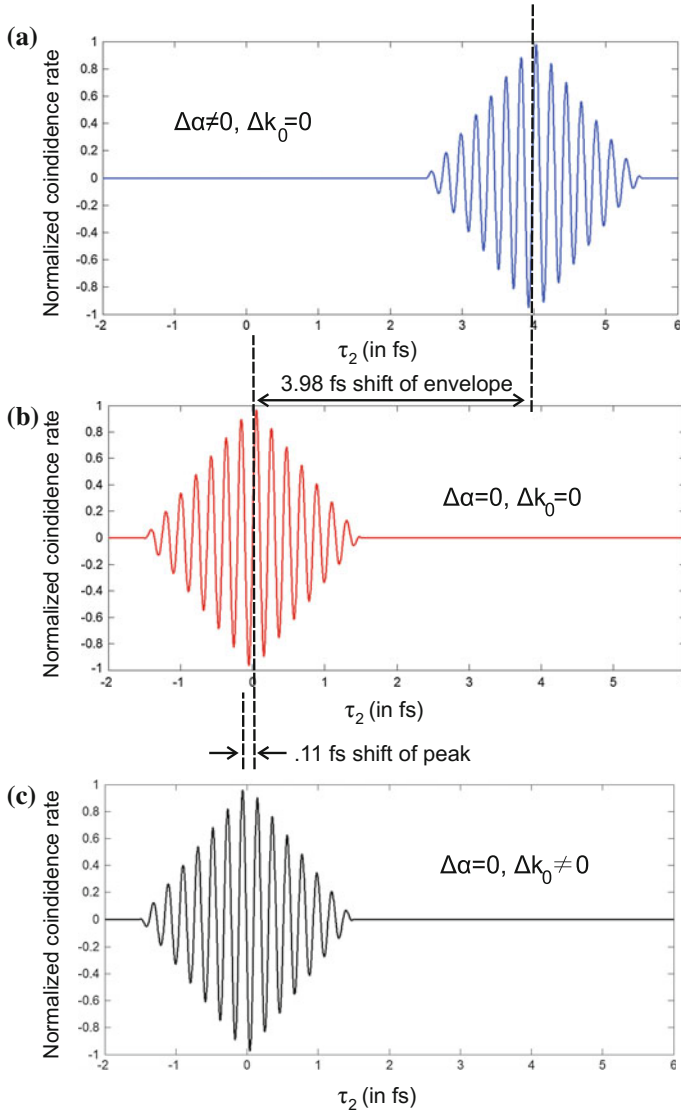


**Fig. 5.7** A more complex interference pattern is obtained by scanning over  $\tau_2$  while keeping  $\tau_1 = 0$ . Changing the value of  $\Delta\alpha$  causes a shift left or right of the envelope, as seen by comparing **a** and **b**. The size of the shift can be measured with accuracy on the order of 0.1 fs. Between **b** and **c**, a nonzero  $\Delta k_0$  shifts the locations of the peaks *within* the envelope. The size of the shift can be measured with accuracy on the order of 0.001 fs = 1 as (Figure reproduced from [13])

(due to the  $\Delta k_0$  term) to an accuracy on the order of attoseconds ( $10^{-18}$  s) [12]. Measurement of the envelope shift allows determination of group delays from  $\Delta\alpha$  to within 0.1 fs accuracy.

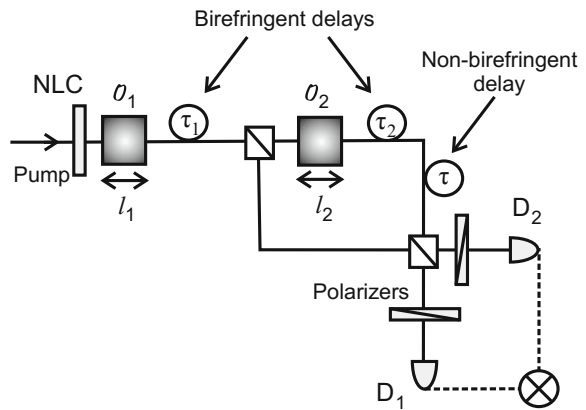
## 5.5 Type B Quantum Measurement

As was described more fully in Sect. 3.3, an interferometer can be constructed [15] in which even and odd portions of the chromatic dispersion separate into different parts of an interferogram, allowing them to be measured independently. A similar effect can be introduced into the interferometer of the last section, by simply adding an additional non-birefringent delay  $\tau$  in one arm (after the first beam splitter) and adding a second beam splitter to combine the output of the two arms before detection. This new setup is shown in Fig. 5.8. Once again, allow for the possibility of two birefringent samples  $\mathcal{O}_1$  and  $\mathcal{O}_2$  of lengths  $l_1$  and  $l_2$  before and after the first beam splitter. For the sake of definiteness, assume that the birefringent delays  $\tau_1$  and  $\tau_2$  act on the vertical (V) polarization and leave the horizontal (H) unaffected. Assume once again that the system is illuminated by type II SPDC beams.

In this arrangement, the even and odd orders of PMD once again play separate roles, with the odd part determining the locations of the triangular dip in the interferogram envelope, while the even part controls the depth of the dip. However, there now may be multiple dips. Their locations are determined by evaluating the different combinations of delays that for which it becomes impossible from the relative timing of detection events to identify which photon took which path. At these points of indistinguishability, destructive interference between paths becomes maximal.

The coincidence rate modulation term of Eqs. (5.27)–(5.29) then takes, in the general case, the form

**Fig. 5.8** Type B setup for separating even- and odd-order contributions to the PMD



$$\begin{aligned}
M(\tau_1, \tau_2, \tau) = & \int d\omega |\Phi(\omega)|^2 e^{-2i\omega\tau_1} e^{-2i\Delta k''(\omega)l_1} \\
& \times \left\{ 1 - e^{-i[(k_V(\Omega_0+\omega) - k_V(\Omega_0-\omega))l_2 + 2\omega(\tau+\tau_2)]} + e^{i[(k_H(\Omega_0+\omega) - k_V(\Omega_0+\omega))l_2 - (\Omega_0+\omega)\tau_2]} \right. \\
& + e^{i[(k_V(\Omega_0-\omega) - k_H(\Omega_0-\omega))l_2 + (\Omega_0-\omega)\tau_2]} - e^{i[(k_H(\Omega_0+\omega) - k_H(\Omega_0-\omega))l_2 + 2\omega\tau]} \\
& - e^{i[(k_H(\Omega_0+\omega) + k_V(\Omega_0-\omega))l_2 + 2\Omega_0\tau + (\Omega_0-\omega)\tau_2]} \\
& - e^{-i[(k_H(\Omega_0-\omega) + k_V(\Omega_0+\omega))l_2 + 2\Omega_0\tau + (\Omega_0+\omega)\tau_2]} \\
& \left. + e^{i[(k_H(\Omega_0+\omega) - k_H(\Omega_0-\omega) - k_V(\Omega_0+\omega) + k_V(\Omega_0-\omega))l_2 - 2\omega\tau_2]} \right\}.
\end{aligned} \tag{5.34}$$

Equation (5.34) yields a complicated interferogram as the delays are scanned. But for the special case where the quadratic and higher order terms are negligible in the dispersion relations, the modulation reduces to

$$\begin{aligned}
M(\tau_1, \tau_2, \tau) = & \frac{2\pi}{\tau_-} \left\{ \Lambda \left( \frac{2(\tau_1 + \Delta\alpha l_1)}{\tau_-} \right) \right. \\
& + 4\Lambda \left( \frac{2\tau_1 + \Delta\alpha(2l_1 + l_2) - \tau_2}{\tau_-} \right) \sin(k_{0V}l_2 + \Omega_0(\tau + \tau_2)) \sin(k_{0H}l_2 - \Omega_0\tau) \\
& - \Lambda \left( \frac{2(\tau_1 + \Delta\alpha l_1 + \alpha_V l_2 + \tau + \tau_2)}{\tau_-} \right) - \Lambda \left( \frac{2(\tau_1 + \Delta\alpha l_1 - \alpha_H l_2 - \tau)}{\tau_-} \right) \\
& \left. + \Lambda \left( \frac{2(\tau_1 + \Delta\alpha(l_1 + l_2) + \tau_2)}{\tau_-} \right) \right\},
\end{aligned} \tag{5.35}$$

where it has again been assumed that the spectral function of the source is  $\Phi(\omega) = \text{sinc}\left(\frac{\tau_-\omega}{2}\right)$ . The locations of the five triangle functions can be explained physically. Imagine that  $\tau$  and  $\tau_2$  are held fixed, while  $\tau_1$  is scanned. Define the relative delay between  $H$  and  $V$  photons that arises *before* the first beam splitter:

$$\Delta\tau_{\text{pre}} \equiv \tau_V - \tau_H = \Delta\alpha l_1 + \tau_1. \tag{5.36}$$

The delay *after* the first beam splitter,  $\Delta\tau_{\text{post}}$ , may cancel this pre-beam splitter delay in four distinct ways, leaving a total delay of zero between the two photons:

$$\Delta\tau \equiv \Delta\tau_{\text{pre}} + \Delta\tau_{\text{post}} = 0. \tag{5.37}$$

The possible values of  $\Delta\tau_{\text{post}}$  are enumerated in the table of Fig. 5.9 [13]. Setting  $\Delta\tau_{\text{post}}$  in the final column equal to  $\Delta\tau_{\text{pre}}$  gives a linear equation that can be solved for the values of  $\tau_1$  at which four dips in the coincidence rate occur. There is perfect path indistinguishability at these delays, since the two photons arrive at the detector simultaneously. This successfully predicts the centers of four of the five dips in Eq. (5.35).

The remaining triangle function, the second one in Eq. (5.35), leads to one further dip. Here the time delay between the two photons is nonzero, but there are two ways that the same *nonzero* delay can occur, once again making the paths indistin-

|     |  | V Delay                        | H Delay               | $\Delta\tau_{\text{post}}$     |
|-----|--|--------------------------------|-----------------------|--------------------------------|
| (a) |  | $\alpha_V l_2 + \tau_2 + \tau$ | 0                     | $\alpha_V l_2 + \tau_2 + \tau$ |
| (b) |  | 0                              | $\alpha_H l_2 + \tau$ | $-\alpha_H l_2 - \tau$         |
| (c) |  | 0                              | 0                     | 0                              |
| (d) |  | $\alpha_V l_2 + \tau_2 + \tau$ | $\alpha_H l_2 + \tau$ | $\Delta\alpha l_2 + \tau_2$    |

**Fig. 5.9** The four possible outcomes at the first beam splitter, with each of the photons following either the upper or lower path. The *second* and *third* columns give the post-BS delays of the vertical and horizontal photons, respectively, and the *final* column gives the difference (vertical delay minus horizontal). Interference occurs when the delay before the first beam splitter is adjusted to cancel  $\Delta\tau_{\text{post}}$ , thus erasing the path information (Table adapted from [13])

guishable; the situation is similar to that of the Franson interferometer in Sect. 2.6. If the vertically-polarized photon V follows the lower path after the first BS and the horizontally-polarized photon H takes the upper path, the total relative delay between the two photons is

$$\Delta\tau_{\text{total}} \equiv \tau_V - \tau_H = \Delta\alpha l_1 + \tau_1 = \Delta\tau_{\text{pre}}. \quad (5.38)$$

But when the two photons are interchanged (H on lower path, V on the upper), the relative delay is

$$\Delta\tau_{\text{total}} = \Delta\alpha (l_1 + l_2) + (\tau_1 + \tau_2). \quad (5.39)$$

If these are negatives of each other, so that  $\Delta\tau_{\text{post}} = -2\Delta\tau_{\text{pre}}$ , then  $\Delta\alpha (2l_1 + l_2) + (2\tau_1 + \tau_2) = 0$ . This condition leads to  $\Delta\tau_{\text{total}} = -\Delta\tau_{\text{pre}}$ .

So all five of the triangular spikes appearing in Eq. (5.35) have simple meanings. Explicitly, they are located at:

$$\tau_1 = -\Delta\alpha l_1, \quad (5.40)$$

$$\tau_1 = \frac{1}{2} [\tau_2 - \Delta\alpha (2l_1 + l_2)] \quad (5.41)$$

$$\tau_1 = -(\alpha_V l_2 + \Delta\alpha l_1 + \tau + \tau_2), \quad (5.42)$$

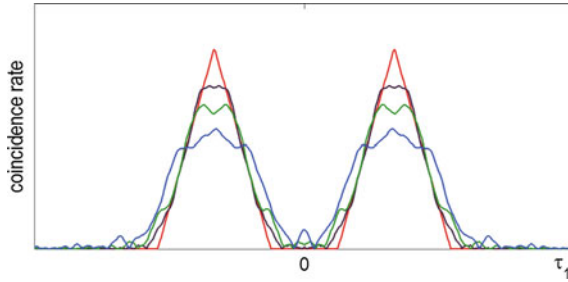


**Fig. 5.10** When  $\Delta\beta = 0$ , the plot of coincidence count versus  $\tau_1$  is sharply triangular (*top curve*), but when the quadratic dispersion term  $\Delta\beta$  is turned on, the peaks become flattened, with small periodic structures superimposed. The central position of each peak does not change. The *lower curves* are for progressively larger values of  $\Delta\beta$  (Reproduced from [13] )

$$\tau_1 = \alpha_H l_2 + \tau - \Delta\alpha l_1 \quad (5.43)$$

$$\tau_1 = \tau_2 - \Delta\alpha(l_1 + l_2). \quad (5.44)$$

If the sample is placed before the first beam splitter ( $l_1 \neq 0$ ,  $l_2 = 0$ ) it is clear from Eqs. (5.40)–(5.44) that by varying the delays and measuring the locations of the dips, it is possible to determine  $\Delta\alpha$ . However, if the sample is instead placed after the beam splitter ( $l_1 = 0$ ,  $l_2 \neq 0$ ), not only  $\Delta\alpha$  but also  $\alpha_V$  and  $\alpha_H$  can each be determined.

Turning the quadratic ( $\Delta\beta$ ) term back on, analytic expressions for  $M$  and  $R_c$  can no longer be obtained, and the interferogram must be plotted numerically. The dips are no longer simply triangle functions, rather, they are complex structures with oscillations superimposed on top flattened plateaux (Fig. 5.10). However, the conditions for indistinguishability do not change, so that of the dips or peaks remain centered on the same locations predicted by the linearized case. The net effect is simply a loss of the precision with which the peak centers can be determined.

### 5.5.1 Experimental Determination of PMD for Compact Devices

One potential problem with the methods presented above is that to have the large bandwidth needed for high resolution, the down conversion must occur in a thin nonlinear crystal. This is because the frequency range over which the phase-matching conditions can be approximately maintained is inversely proportional to the thickness of the crystal. But thin crystals produce very low intensities in down conversion. There is therefore a trade-off between resolution and data collection rate.

This problem is greatly mitigated by replacing the bulk crystal by a periodically-poled crystal. *Periodic-poling* [19] is a common technique in nonlinear optics, in which the direction of the optic axis reverses periodically along the axis. This allows the phase matching to be maintained over greater crystal thicknesses. The distance between flips is known as the poling period. Further improvement in bandwidth can be achieved by *chirping*, i.e. varying the poling period slowly over the length of the crystal.

Such a chirped periodically poled crystal has been used experimentally in the type A apparatus described above [20]. In this experiment, the PMD of a wavelength selective switch was measured. Such switches are used to route signals through fiber optical networks, and are an example of the type of discrete devices the PMD of which classical methods are unable to characterize with sufficient resolution. The experiment was able to determine the differential group delay down to a resolution of .1 fs, compared to resolutions  $>1$  fs that are the best to have been achieved with classical white light interferometry. With stronger entangled light sources, it is expected that the resolution that can be achieved with quantum interferometry could be improved by at least another factor of 10. The experiment of [20] seems to have been the first practical demonstration of a high resolution quantum metrology technique to characterize an industrial device. Progress toward achieving strong entangled photon sources with large ( $>125$  nm) bandwidths continues to be made as more refined methods for designing and producing chirped periodically poled crystals are developed [18].

## References

1. D. Andresciani, E. Curti, E. Matera, B. Daino, Opt. Lett. **12**, 844 (1987)
2. B. Costa, D. Mazzoni, M. Puleo, E. Vezzoni, IEEE J. Quantum Electron. **18**, 1509 (1982)
3. C.D. Poole, C.R. Giles, Opt. Lett. **13**, 155 (1987)
4. C.D. Poole, Opt. Lett. **14**, 523 (1989)
5. D. Derickson, *Fiber Optic Test and Measurement* (Prentice Hall, Upper Saddle River, 1998)
6. B. Bakhshi, J. Hansryd, P.A. Andrekson, J. Brentel, E. Kolltveit, B.K. Olsson, M. Karlsson, IEEE Photonics Technol. Lett. **11**, 593 (1999)
7. S. Diddams, J. Diels, J. Opt. Soc. Am. B **13**, 1120 (1996)
8. P. Williams, J. Opt. Fiber Commun. Rep. **1**, 84 (2004)
9. H. Kogelnik, R. Jopson. Polarization mode dispersion, in *Optical Fiber Telecommunications IVB: System and Impairments*, ed. by I. Kaminow, T. Li (Academic Press, London 2002), p. 725
10. A.V. Sergienko, Y.H. Shih, M.H. Rubin, J. Opt. Soc. Am. B **12**, 859 (1995)
11. E. Dauler, G. Jaeger, A. Muller, A. Migdall, J. Res. Natl. Inst. Stand. Technol. **104**, 1 (1999)
12. D. Branning, A.L. Migdall, A.V. Sergienko, Phys. Rev. A **62**, 063808 (2000)
13. A. Fraine, D.S. Simon, O. Minaeva, R. Egorov, A.V. Sergienko, Opt. Exp. **19**, 22820 (2011)
14. T.B. Pittman, D.V. Strekalov, A. Migdall, M.H. Rubin, A.V. Sergienko, Y.H. Shih, Phys. Rev. Lett. **77**, 1917 (1996)
15. O. Minaeva, C. Bonato, B.E.A. Saleh, D.S. Simon, A.V. Sergienko, Phys. Rev. Lett. **102**, 100504 (2009)
16. M.H. Rubin, D.N. Klyshko, Y.H. Shih, A.V. Sergienko, Phys. Rev. A **50**, 5122 (1994)
17. D.N. Klyshko, *Photons and Nonlinear Optics* (Gordon and Breach, Philadelphia, 1988)

18. A. Fraine, O. Minaeva, D.S. Simon, R. Egorov, A.V. Sergienko, *Opt. Lett.* **37**, 1910 (2012)
19. M. Houé, P.D. Townsend, *J. Phys. D Appl. Phys.* **28**, 1747 (1995)
20. A. Fraine, O. Minaeva, D.S. Simon, R. Egorov, A.V. Sergienko, *Opt. Exp.* **20**, 2025 (2012)

## Chapter 6

# Ghost Imaging and Related Topics

*Ghost imaging*, also known as *two-photon* or *correlated-photon imaging*, was first demonstrated in the 1990s and quickly gained widespread attention due to the fact that it exhibits a highly non-intuitive effect: an image can be formed by looking at correlations between two beams of light, neither of which is capable of forming an image by itself. In the early experiments, this was achieved by means of the non-local correlations present in systems of entangled photon pairs: one photon probes the object but is detected by a detector lacking significant spatial resolution, while the second photon never interacts with the object at all. Neither detector supplies sufficient information to form an image but the image can, nonetheless, be reconstructed by looking at the coincidence counts between the two detectors. This type of ghost imaging is often referred to as *quantum imaging*. After its introduction, a debate arose as to whether quantum entanglement was necessary for ghost imaging, that is, whether the effect could be mimicked classically. Although some aspects of this debate continue [1–7], it is now clear that the major ghost imaging effects *can* be accomplished classically, albeit with reduced visibility, using spatially correlated thermal light sources.

Correlated imaging has a number of advantages and unusual features. Importantly, two beams may be of different frequency so that the object can be probed at one wavelength but the image formed at a different wavelength. This has been used, among other things, to produce infrared images using visible-light detectors. Alternatively, high resolution images can be produced while using cheap, expendable detectors in dangerous or hard to reach environments (such as inside the human body), while keeping the more expensive high-quality cameras in a safe lab environment. It has also been shown that the quantum version of ghost imaging can produce higher visibilities and signal-to-noise ratios than conventional imaging systems can. In particular, one detected beam can be used to characterize photon number fluctuations in the illumination beam; these fluctuations can then be subtracted from the image formed by the other beam in order to improve imaging beyond the standard quantum limit for weak images [8].

In this chapter, we look at several variations on ghost imaging. In Sect. 6.1 we begin with quantum ghost imaging using entangled states. This is followed by a

discussion in Sect. 6.2 of ghost imaging with classical (non-entangled) light sources. It has been found that the use of two-photon imaging may cancel, or at least diminish, distorting effects caused by material in the intervening paths; applications of this to the cancellation or mitigation of aberration and turbulence in imaging is covered in Sects. 6.3 and 6.4. Computational and compressive ghost imaging have recently become active areas of research, to be briefly reviewed in Sect. 6.5, followed by quantum holography in Sect. 6.7. A number of other reviews of ghost imaging, from different points of view, can be consulted as well, including [9–12].

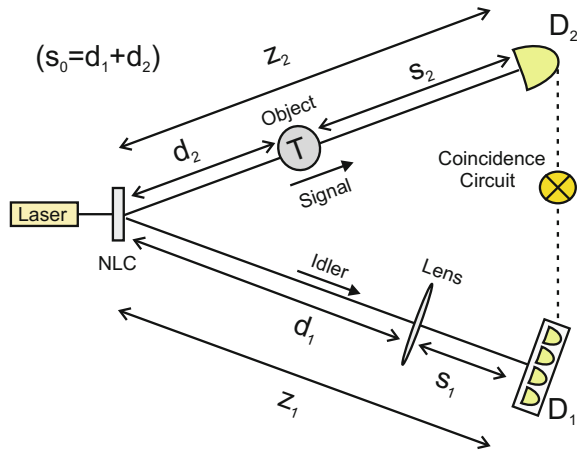
## 6.1 Quantum Ghost Imaging

### 6.1.1 Conceptual Introduction

In 1995, it was discovered [13] that if a double slit is placed in one of a pair of beams originating from down conversion, no interference pattern is formed in that beam, because each beam individually is incoherent. However, the interference effects reappeared when the coincidence detection rate *between* the two beams was measured. This is due to the fact that the coherence of the pump, although hidden, is still present, and can be retrieved by looking at both down converted photons together. The resulting two-photon interference and diffraction effects became known as *ghost interference* and *ghost diffraction* (see Chap. 2), the word “ghost” indicating the seemingly spooky “nonlocal” nature of the effect. A closely related effect was also demonstrated: *ghost imaging* [13, 14], in which spatially-entangled photon pairs generated by down conversion form images in a similar manner, exploiting nonlocal correlations.

A schematic diagram of the quantum ghost imaging apparatus is shown in Fig. 6.1. The use of down-conversion as a light source means that the flux of photons is

**Fig. 6.1** Ghost imaging with entangled photon pairs generated in a nonlinear crystal (NLC). Detector  $D_1$  is spatially resolving, but  $D_2$  is a single-pixel bucket detector. Solid lines represent the propagation paths of light, while dashed lines are electrical connections



very low, requiring the use of photon-counting detectors. Either non-collinear down conversion is used or collinear down conversion with a beam splitter to separate the signal and idler. A coincidence circuit records events in which detection occurs in both detectors within a short time window.

The signal is transmitted through the object to  $D_2$ , a single-pixel bucket detector that simply registers whether or not a photon is detected, without determining the location at which it arrived. This detector should be large enough to collect all the signal photons arriving at the right end of the apparatus.  $D_2$  registers whether the photons pass through the object or are blocked; but because it lacks the ability to resolve spatial properties no image can be obtained by using the information from this detector alone.

$D_1$ , on the other hand, is a high spatial-resolution detector—a CCD camera, an array of avalanche photodiodes, or a single small detector that can scan the imaging region. The lens in branch 1 has focal length  $f$ . Let  $d_1$  and  $d_2$  be the distances from the nonlinear crystal to the lens and from crystal to object, and let  $s_0 = d_1 + d_2$ . The distance  $s_1$  is that from the lens to detector  $D_1$ . The distances  $s_0$  and  $s_1$  satisfy the imaging condition  $\frac{1}{s_0} + \frac{1}{s_1} = \frac{1}{f}$ . Although this detector allows spatial structure to be recorded, the photons reaching  $D_2$  have not interacted with the object so that, once again, information from  $D_1$  alone does not enable reconstruction of the object image.

However, when the information from the two detectors is combined via coincidence counting, the image reappears as the coincidence rate is plotted versus position in  $D_1$ . The imaging process is therefore a highly nonlocal process; in fact one of the original motivations for this line of inquiry was to investigate the apparently nonlocal causal structure of quantum mechanics and the Einstein-Podolsky-Rosen (EPR) “paradox” [15] discussed in Sect. 1.9.

The imaging property of the apparatus is more clearly shown by displaying a schematic version drawn in the Klyshko “backward wave” picture [16, 17] (Fig. 6.2).

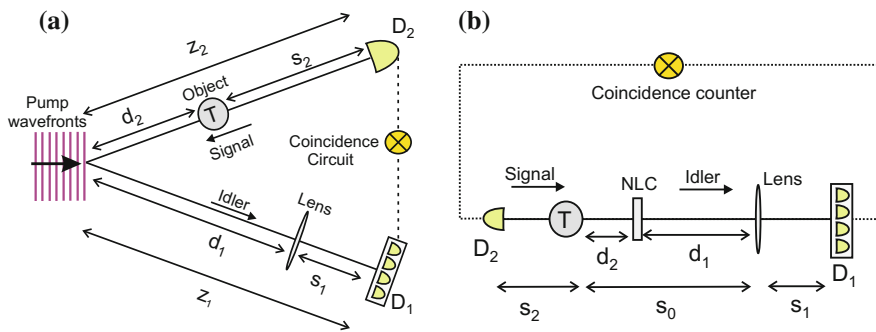


**Fig. 6.2** **a** Klyshko backward-wave picture, in which the signal and idler are treated as a single ray passing through the crystal from one detector to the other. The pump beam acts as a mirror off which this ray reflects; the curvature of the mirror is given by the curvature of the wavefronts in the pump beam. **b** One further view of the same system, with the crystal now treated as a lens, converting the signal into the idler. The focal length of the lens again is determined by the properties of the pump beam

Here, we view the signal and idler as a single photon passing through the crystal. The signal is viewed as traveling backward from the object, into the crystal, where it converts into the forward-moving idler, then travels onward to the detector  $D_1$ . In this view, the situation looks identical to a standard single lens imaging setup (such as in Fig. A.2 of Appendix A). The detector  $D_2$  acts like the light source in this view. Alternatively, we could fold the picture over, so the signal appears to reflect off the crystal in order to form the idler. In this latter version, the crystal acts as a mirror, and the pump determines the properties of that mirror. For the most part, we assume that the pump beam is approximately a plane wave, which means that the crystal acts as a planar mirror. However, if the pump beam is focused to a small spot in the crystal by a lens, then the crystal acts like a spherical mirror, with radius of curvature inversely proportional to the size of the spot; such a focused pump beam is used in Chap. 7.

The ghost imaging apparatus has improved resolution compared to a similar single lens apparatus and, in fact, can beat the usual diffraction limit by a factor of 2. Effectively, the resolution is determined by the shorter pump wavelength, rather than the longer signal or idler wavelengths of the detected photons. This fact has formed the basis for the process of *quantum lithography* [18, 19], in which two-photon imaging is used to write subdiffraction-sized structures onto a semiconductor surface (see Sect. 4.5). The idea has been extended to  $N$ -photon imaging with  $N > 2$ , although the prospects for this to become practical outside the research lab seem limited in the foreseeable future, due to the difficulty of reliably producing entangled states of more than two photons at a reasonable rate.

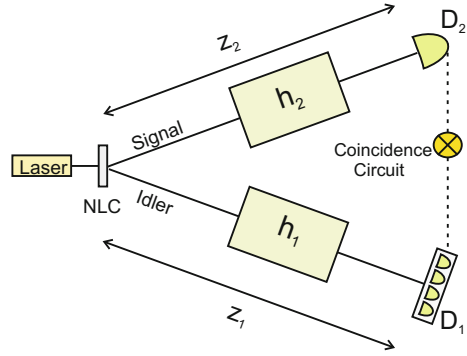
### 6.1.2 A Quantitative Discussion

For a quantitative treatment, assume a plane wave pump with negligible transverse momentum,  $\kappa_p \approx 0$ . Also assume for simplicity that the down conversion is type I, so that we may suppress polarization labels for the signal and idler photons. The crystal's outgoing two-photon state (see Sect. 1.6.1) can then be taken to be of the form

$$|\Psi\rangle = C_0 \int d^2\kappa_1 d^2\kappa_2 \delta^{(2)}(\kappa_1 + \kappa_2) \times \int d\omega_1 d\omega_2 \delta(\omega_1 + \omega_2 - \omega_p) \hat{a}^\dagger(\kappa_s, \omega_s) \hat{a}^\dagger(\kappa_i, \omega_i) |0\rangle, \quad (6.1)$$

where  $\kappa$  represents transverse momentum, and  $C_0$  is a constant. This also assumes perfect phase-matching in order to simplify the discussion; below it can be seen that, because of this, only the diagonal part of the source spatial coherence function (see Appendix A),  $K(\mathbf{r}_0, \mathbf{r}_0)$ , contributes; in the more realistic case of imperfect phase matching, off-diagonal pieces contribute and the coherence properties of the source become important.

**Fig. 6.3** A general two-photon detection system with a parametric down conversion light source.  $h_1$  and  $h_2$  represent the impulse response or propagation functions in the two branches



Begin by looking at the more general system of Fig. 6.3, allowing for arbitrary propagation functions in the two arms of the setup. The intensity–intensity correlation function for detections at points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  at the two detectors is

$$G^{(2)}(\mathbf{r}_1, z_1, t_1; \mathbf{r}_2, z_2, t_2) = \left| \langle 0 | \hat{E}_2^{(+)}(\mathbf{r}_2, z_2, t_2) \hat{E}_2^{(+)}(\mathbf{r}_1, z_1, t_1) | \Psi \rangle \right|^2 \quad (6.2)$$

$$\equiv |\psi(\mathbf{r}_1, z_1, t_1; \mathbf{r}_2, z_2, t_2)|^2. \quad (6.3)$$

The fields in the detection planes are obtained by propagation through each branch of the system from the source plane at the output of the nonlinear crystal,

$$\hat{E}_j^{(+)}(\mathbf{r}, z, t) = \int d\omega d^2\kappa d^2r_0 h_j(\omega, \mathbf{r}, \mathbf{r}_0) e^{-i\omega t} e^{-i\kappa \cdot \mathbf{r}_0} \hat{a}(\kappa, \omega), \quad (6.4)$$

where  $j = 1, 2$ .  $h_j$  is the propagation function in the  $j$ th branch, and  $\mathbf{r}_0$  is the initial location of the emitted photon in the crystal exit plane. So, using Eqs. (6.1)–(6.4), the two-photon detection amplitude becomes:

$$\begin{aligned} \psi(\mathbf{r}_1, z_1, t_1; \mathbf{r}_2, z_2, t_2) &= C_0 \int d^2\kappa_s d^2\kappa_i \delta^{(2)}(\kappa_i + \kappa_s) d\omega_i d\omega_s \delta(\omega_s + \omega_i - \omega_p) \\ &\quad \times \langle 0 | \hat{E}_2^{(+)}(\mathbf{r}_2, z_2, t_2) \hat{E}_1^{(+)}(\mathbf{r}_1, z_1, t_1) \hat{a}^\dagger(\kappa_s, \omega_s) \hat{a}^\dagger(\kappa_i, \omega_i) | 0 \rangle \end{aligned} \quad (6.5)$$

$$\begin{aligned} &= C_0 \int d^2\kappa_s d^2\kappa_i \delta^{(2)}(\kappa_i + \kappa_s) d\omega_i d\omega_s \delta(\omega_s + \omega_i - \omega_p) \\ &\quad \times \int d^2\kappa_1 d^2\kappa_2 d\omega_1 d\omega_2 d^2r_0 d^2r'_0 h_{z1}(\omega_1, \mathbf{r}_1, \mathbf{r}_0) h_{z2}(\omega_2, \mathbf{r}_2, \mathbf{r}'_0) \\ &\quad \times e^{-i(\omega_1 t_1 + \omega_2 t_2)} e^{-i(\kappa_1 \cdot \mathbf{r}_0 + \kappa_1 \cdot \mathbf{r}'_0)} \langle E_0(\mathbf{r}_0) E_0(\mathbf{r}'_0) \rangle \\ &\quad \times \langle 0 | \hat{a}(\kappa_1, \omega_1) \hat{a}(\kappa_2, \omega_2) \hat{a}^\dagger(\kappa_s, \omega_s) \hat{a}^\dagger(\kappa_i, \omega_i) | 0 \rangle. \end{aligned} \quad (6.6)$$

Define

$$K(\mathbf{r}_0, \mathbf{r}'_0) = \langle E_0(\mathbf{r}_0) E_0(\mathbf{r}'_0) \rangle. \quad (6.7)$$

Using the fact that

$$\begin{aligned} & \langle 0 | \hat{a}(\kappa_1, \omega_1) \hat{a}(\kappa_2, \omega_2) \hat{a}^\dagger(\kappa_s, \omega_s) \hat{a}^\dagger(\kappa_i, \omega_i) | 0 \rangle \\ &= \delta(\omega_1 - \omega_s) \delta(\omega_2 - \omega_i) \delta^{(2)}(\kappa_1 - \kappa_s) \delta^{(2)}(\kappa_2 - \kappa_i) \\ &+ \delta(\omega_1 - \omega_i) \delta(\omega_2 - \omega_s) \delta^{(2)}(\kappa_1 - \kappa_i) \delta^{(2)}(\kappa_2 - \kappa_s) \end{aligned} \quad (6.8)$$

and then making a relabeling of integration variables  $(\kappa_s, \omega_s) \leftrightarrow (\kappa_i, \omega_i)$  in the second term that arises, it is found that:

$$\begin{aligned} & \psi(\mathbf{r}_1, z_1, t_1; \mathbf{r}_2, z_2, t_2) \\ &= 2C_0 \int d^2\kappa_s d^2\kappa_i d\omega_s d^2r_0 d^2r'_0 K(\mathbf{r}_0, \mathbf{r}'_0) e^{-i(\omega_s t_1 + \omega_i t_2)} \\ & \quad \times e^{-i(\kappa_s \cdot \mathbf{r}_0 + \kappa_i \cdot \mathbf{r}'_0)} h_{z1}(\omega_s, \mathbf{r}_1, \mathbf{r}'_0) h_{z2}(\omega_s, \mathbf{r}_2, \mathbf{r}_0) \end{aligned} \quad (6.9)$$

$$\begin{aligned} &= 2C_0 \int d^2\kappa_s d\omega_s d^2r_0 d^2r'_0 K(\mathbf{r}_0, \mathbf{r}'_0) e^{-i\omega_p t_1} e^{-i\omega_s(t_1 - t_2)} \\ & \quad \times e^{-i\kappa_s \cdot (\mathbf{r}_0 - \mathbf{r}'_0)} h_{z1}(\omega_s, \mathbf{r}_1, \mathbf{r}'_0) h_{z2}(\omega_s, \mathbf{r}_2, \mathbf{r}_0) \end{aligned} \quad (6.10)$$

$$\begin{aligned} &= 2C_0 (2\pi)^3 \delta(t_1 - t_2) e^{-i\omega_p t_1} \int d\omega_s d^2r_0 d^2r'_0 \delta(r_0 - r'_0) \\ & \quad \times K(\mathbf{r}_0, \mathbf{r}'_0) h_{z1}(\omega_s, \mathbf{r}_1, \mathbf{r}'_0) h_{z2}(\omega_s, \mathbf{r}_2, \mathbf{r}_0) \end{aligned} \quad (6.11)$$

$$= 2C_0 (2\pi)^3 \delta(t_1 - t_2) e^{-i\omega_p t_1} \psi(\mathbf{r}_1, z_1; \mathbf{r}_2, z_2), \quad (6.12)$$

where, henceforth suppressing the time dependence and overall constants, we have defined the time-independent amplitude:

$$\psi(\mathbf{r}_1, z_1; \mathbf{r}_2, z_2) = \int d^2r_0 I_0(\mathbf{r}_0) h_{z1}(\mathbf{r}_1, \mathbf{r}'_0) h_{z2}(\mathbf{r}_2, \mathbf{r}_0) \quad (6.13)$$

It has also now been assumed for simplicity that  $\omega_s = \omega_i \equiv \omega$ , and so the frequency argument in the impulse response functions have been dropped. As mentioned above, the assumption of perfect phase matching implies that only the diagonal part of the coherence function,  $I_0(\mathbf{r}_0) \equiv K(\mathbf{r}_0, \mathbf{r}_0) = \langle E_0^2(\mathbf{r}_0) \rangle$  enters this expression.

Specializing now to the case of Fig. 6.1, take the signal branch to contain the object  $T(\mathbf{r})$ , with free propagations over distances  $d_2$  and  $s_2$  before and after the object, respectively. The idler branch contains a lens of focal length  $f$  with distances  $d_1$  and  $s_1$  before and after it, respectively. We then write the propagation function in the signal branch as

$$h_2(\mathbf{r}_2, \mathbf{r}_0, z_2) = \int d^2r h_{s_2}(\mathbf{r}_2, \mathbf{r}) T(\mathbf{r}) h_{d_2}(\mathbf{r}, \mathbf{r}_0), \quad (6.14)$$

so that

$$\psi(\mathbf{r}_1, z_1; \mathbf{r}_2, z_2) = \int d^2r d^2r_0 I_0(\mathbf{r}_0) h_{z_1}(\mathbf{r}_1, \mathbf{r}'_0) h_{s_2}(\mathbf{r}_2, \mathbf{r}) T(\mathbf{r}) h_{d_2}(\mathbf{r}, \mathbf{r}_0). \quad (6.15)$$

Following Abouraddy et al. [20], we define:

$$h_0(\mathbf{r}_1, \mathbf{r}) = \int d^2r_0 I_0(\mathbf{r}_0) h_{z_1}(\mathbf{r}_1, \mathbf{r}_0) h_{d_2}(\mathbf{r}, \mathbf{r}_0), \quad (6.16)$$

$$g(\mathbf{r}, \mathbf{r}') = \int d^2r_2 h_{s_2}^*(\mathbf{r}_2, \mathbf{r}) h_{s_2}(\mathbf{r}_2, \mathbf{r}'). \quad (6.17)$$

Then,

$$\psi(\mathbf{r}_1, z_1; \mathbf{r}_2, z_2) = \int d^2r h_0(\mathbf{r}_1, \mathbf{r}) T(\mathbf{r}) h_{s_2}(\mathbf{r}_2, \mathbf{r}) \quad (6.18)$$

$$\begin{aligned} G^{(2)}(\mathbf{r}_1, z_1; \mathbf{r}_2, z_2) &= \int d^2r d^2r' h_0(\mathbf{r}_1, \mathbf{r}) h_0^*(\mathbf{r}_1, \mathbf{r}') T(\mathbf{r}) T^*(\mathbf{r}') \\ &\quad \times h_{s_2}(\mathbf{r}_2, \mathbf{r}) h_{s_2}^*(\mathbf{r}_2, \mathbf{r}'). \end{aligned} \quad (6.19)$$

So the marginal coincidence rate obtained when the bucket detector  $D_2$  is integrated over is

$$R_c(\mathbf{r}_1) = \int d^2r_2 G^{(2)}(\mathbf{r}_1, z_1; \mathbf{r}_2, z_2) \quad (6.20)$$

$$= \int d^2r d^2r' h_0(\mathbf{r}_1, \mathbf{r}) h_0^*(\mathbf{r}_1, \mathbf{r}') T(\mathbf{r}) T^*(\mathbf{r}') g(\mathbf{r}, \mathbf{r}'), \quad (6.21)$$

where

$$g(\mathbf{r}, \mathbf{r}') = \int d^2r_2 h_{s_2}(\mathbf{r}_2, \mathbf{r}) h_{s_2}^*(\mathbf{r}_2, \mathbf{r}') \quad (6.22)$$

It is seen that  $h_0$  acts as a Klyshko-type response function, propagating light backward from the object to the crystal and then forward to detector  $D_1$ . The function  $g$  determines whether the system acts as a coherent or incoherent imager. If the object is followed by a  $2f$  or  $4f$  system (see Appendix), then  $g(\mathbf{r}, \mathbf{r}') = \delta^{(2)}(\mathbf{r} - \mathbf{r}')$ , so that the arrangement acts as an incoherent imaging system,

$$R_c(\mathbf{r}_1) = \int d^2r |h_0(\mathbf{r}_1, \mathbf{r}) T(\mathbf{r})|^2. \quad (6.23)$$

On the other hand, if  $g(\mathbf{r}, \mathbf{r}')$  factors,  $g(\mathbf{r}, \mathbf{r}') = f^*(\mathbf{r}) f(\mathbf{r}')$  for some function  $f$ , then the system performs coherent imaging,

$$R_c(\mathbf{r}_1) = \left| \int d^2r h_0(\mathbf{r}_1, \mathbf{r}) T(\mathbf{r}) \right|^2. \quad (6.24)$$

This latter case occurs when  $h_{s_2}$  is not a function of  $r_2$ , as happens if  $D_2$  is a point detector or is far from the focal plane (so that the intensity falling on it is approximately uniform). In either case (Eq. 6.23 or 6.24), the image then becomes imprinted on the spatial dependence of the coincidence rate and is in focus if the distances contained inside  $h_0$  satisfy the imaging condition  $\frac{1}{s_0} + \frac{1}{s_1} = \frac{1}{f}$ , where  $s_0 = d_1 + d_2$ .

## 6.2 Classical Ghost Imaging

In [21], it was shown that ghost imaging can be carried out with classically correlated beams in place of entangled photon pairs. The basic idea is shown in Fig. 6.4. A laser passes through a beam splitter, creating two identical copies of the beam. By rotating the beam splitter, the two copies move in a spatially anticorrelated manner. This experiment was the first indication that the essential element in ghost imaging is the spatial momentum correlation of the photons rather than entanglement.

In Refs. [22–25], the question was raised as to whether ghost imaging could be carried out with partially coherent thermal light from a classical source. This was successfully done in the experiments of Refs. [26, 27]. In [26], the experiment was done with low photon flux using photon counting detectors and coincidence counting. It was shown in [27] that a similar result could be obtained using higher flux (a bright light source) and a CCD array: The larger photocounts allowed the coincidence counting to be replaced by measuring correlations between the two photocurrents.

The apparatus for the pseudothermal light experiment is shown in Fig. 6.5. A laser passes through a rotating ground glass diffuser, then is directed to a beam splitter. Leaving the beam splitter, one copy of the beam passes through the object then reaches a bucket detector  $D_2$ , while the other copy travels unobstructed to a spatially

**Fig. 6.4** Classical ghost imaging with spatially correlated pairs of light beams

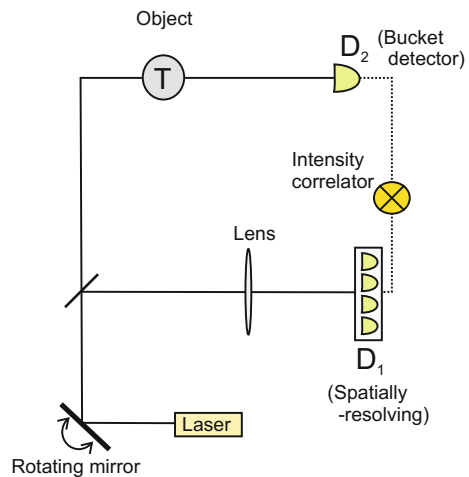
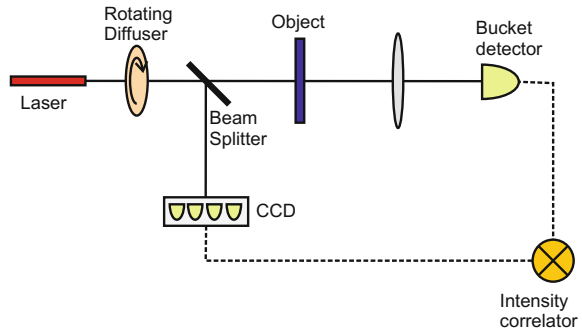


**Fig. 6.5** Classical ghost imaging with incoherent pseudothermal light



resolving detector  $D_1$ . The rotating diffuser causes random, time-varying fluctuations in the beam, but the use of the beam splitter assures that the fluctuations are strongly correlated between beams reaching the two detectors.

A Gaussian-state analysis has been carried out [12] that applies to both the classical and quantum cases; it was shown that the quantum case relies on phase-sensitive cross-correlations (see Sect. 1.10), whereas in the classical case both phase-sensitive and phase-insensitive correlations may be present. The absence of phase-insensitive correlations in the quantum case can be seen from Eq. 6.8, where replacing either of the annihilation operators with a creation operator (as would occur if one of the fields was replaced by its conjugate) causes the expectation value to vanish. Experimentally, the chief difference between the classical and quantum versions of ghost imaging is that the classical version has an unavoidable constant background that lowers the visibility. Classically, the visibility of interference patterns always remains below 71 %, whereas, in principle, it may reach perfect visibility in the quantum case.

A further variation on classical ghost imaging is *reflective standoff sensing* [28–30]. Here, the object is assumed to be reflective rather than transmissive, and may be at a large distance from the light source and detectors. Because the distances may be large there are photon losses, so that a strong classical light source, rather than a weak source of individual entangled photon pairs, is typically required. Compared to the case of transmissive ghost imaging, new features appear. For example, if the reflective surface is rough, speckle patterns arise from interference between light reflecting off different points in the object. Another interesting feature is that the image is insensitive to turbulence on the return path from the object to the bucket detector, although turbulence in the path to the spatially-resolving detector still affects the resolution. (See Sect. 6.4 below for more on turbulence and ghost imaging.)

Even when no entanglement is present, it has been shown [6] that split classical beams may still have nonzero quantum discord; there is some evidence that this discord, which is a weaker form of correlation than entanglement (see Sect. 1.5.5), may play a role in ghost imaging.

### 6.3 Aberration Cancellation in Imaging

The goal here [31] is to perform two-photon imaging using spatially correlated pairs of light beams in such a way that odd-order aberration effects introduced by an imaging system are canceled in a manner similar to the even-order interferometric cancellation described in Sect. 3.4. This can be done with either classically correlated beams or quantum entangled photon pairs. The key idea is to partially collapse the two arms of the ghost-imaging setup so that *both* of the output beams pass through the *same* optical system in an anticorrelated manner but with only one of the beams passing through the object. Here, the method is described in the context of a single-lens imaging system. The odd-order aberration cancellation effect in imaging has been observed experimentally [32] using a slightly different method, involving spatial inversion of one photon. Cancellation of odd-order aberrations arising due to turbulence have also been demonstrated [33].

#### 6.3.1 Odd-Order Aberration-Cancellation in Correlated-Photon Imaging

Many optical devices work by adding position-dependent phase shifts to a beam as it passes through the device. A lens, for example, produces focusing by adding to the beam a quadratic phase shift  $e^{-\frac{ik}{2f}\mathbf{x}^2}$ , where  $k$ ,  $f$  and  $\mathbf{x}$  are the wavenumber of the beam, the focal length of the lens, and the displacement of the given point in the beam from the axis of the lens. However, imperfections in the shape of the lens and variations in the index of refraction of the material from which it is made may lead to additional unwanted phase shifts  $e^{i\phi(\mathbf{x})}$ . The effects of these unwanted phases, or aberrations, which cause spatially-dependent distortions to the outgoing wavefronts, can be canceled in correlation experiments, as shown in Chap. 3.

The two-photon imaging setup of Fig. 6.1 can be altered in order to produce a similar aberration cancellation effect in imaging systems. Here the phases introduced by an optical system are split into parts that are odd and even under *spatial* inversion,  $\phi_{\text{even}}(-\mathbf{x}) = +\phi_{\text{even}}(\mathbf{x})$ ,  $\phi_{\text{odd}}(-\mathbf{x}) = -\phi_{\text{odd}}(\mathbf{x})$ . Because it is undesirable to cancel the even order phases introduced by lenses in the imaging system, the goal now is to cancel the odd part of the phases.

The two arms of the basic ghost imaging system (Fig. 6.1) can be partially merged, so that the photons in each pair separate from each other only *after* they have passed through the lens. In this manner, the two entangled photons arising from the same well-localized point  $\xi$  in the source emerge with opposite transverse momenta  $\pm\kappa$ , tending to pass through the lens from opposite sides of the axis.

Figure 6.6a shows one means of implementing this scheme, using entangled photon pairs produced via parametric down conversion. The field in the pump beam as it enters the crystal is  $E_p(\xi)$ . Let the distances satisfy the imaging condition

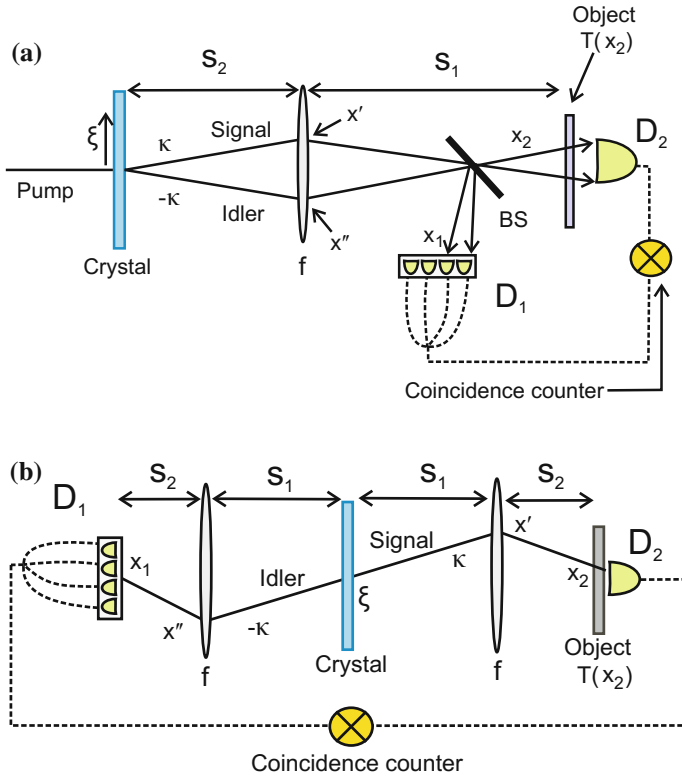


**Fig. 6.6** **a** Implementation of imaging with odd-order phase cancellation. This particular embodiment uses parametric down conversion as a source of correlated illumination. **b** Unfolded version of the same apparatus, with the signal and idler drawn as a single continuous ray, as in the Klyshko picture

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f}. \quad (6.25)$$

The lens have a pupil function which is taken as the usual quadratic phase plus an additional phase function describing aberrations:

$$p(\mathbf{x}) = e^{-i(\mathbf{x}^2/2f)} e^{i\phi(\mathbf{x})}. \quad (6.26)$$

The beam splitter sends half of the photons to a spatially-resolving detector  $D_1$ , the other half to a large bucket detector  $D_2$ . The object introduces a modulation  $T(\mathbf{x}_2)$  only in branch 2. The impulse response function for beam  $j$  ( $j = 1, 2$ ) to travel from transverse position  $\xi$  in the crystal plane to transverse position  $\mathbf{x}_j$  in the detection plane is then

$$h(\xi, \mathbf{x}_j) = H_j(\mathbf{x}_j) e^{ik[(\xi^2/z_1) + (\mathbf{x}_j^2/z_2)]/2} \int e^{-ik[(\xi/z_1) + (\mathbf{x}_j/z_2)] \cdot \mathbf{x}'} e^{i\phi(\mathbf{x}')} d^2x', \quad (6.27)$$

where the effect of the object enters through the function

$$H_j(\mathbf{x}_j) = \begin{cases} 1, & \text{for } j = 1 \\ T(\mathbf{x}_2), & \text{for } j = 2. \end{cases} \quad (6.28)$$

When the pump is approximated by a plane wave, there is perfect anti-correlation between the transverse components of the signal and the idler wave vectors causing the two photons to hit the lens at a pair of points arranged symmetrically about the point where the signal and idler rays meet in the crystal,  $\mathbf{x}' - \xi = \xi - \mathbf{x}''$ . If the crystal is sufficiently distant from the lens relative to the size of the crystal then these two points are approximately symmetric about the axis through the center of the lens.

With a thin crystal and narrow-band filters in the beams, the coincidence amplitude (or equivalently, the two-photon wavefunction) in the detection plane can be written (up to overall normalization) in the form [34]

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \int E_p(\xi) h_1(\xi, \mathbf{x}_1) h_2(\xi, \mathbf{x}_2) d^2\xi \quad (6.29)$$

$$= \int d^2\xi d^2x' d^2x'' E_p(\xi) T(\mathbf{x}_2) e^{(ik\xi^2/z_1)} e^{(ik/2z_2)(x_1^2 + x_2^2)} \\ \times e^{-ik\xi \cdot (\mathbf{x}' + \mathbf{x}'')/z_1} e^{-ik(\mathbf{x}_1 \cdot \mathbf{x}' + \mathbf{x}_2 \cdot \mathbf{x}'')} e^{i(\phi(\mathbf{x}') + \phi(\mathbf{x}''))}. \quad (6.30)$$

Defining

$$F(\xi) = E_p(\xi) e^{ik\xi^2/z_1}, \quad (6.31)$$

then the coincidence rate,

$$R(x_1) = \int d^2x_2 |\psi(\mathbf{x}_1, \mathbf{x}_2)|^2, \quad (6.32)$$

is expressed in terms of an amplitude of the form

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = T(\mathbf{x}_2) e^{ik(\mathbf{x}_2^2 + \mathbf{x}_1^2)/2z_2} \\ \times \int \tilde{F}\left(\frac{k(\mathbf{x}' + \mathbf{x}'')}{z_1}\right) e^{-ik(\mathbf{x}_1 \cdot \mathbf{x}' + \mathbf{x}_2 \cdot \mathbf{x}'')/z_2} e^{i(\phi(\mathbf{x}') + \phi(\mathbf{x}''))} d^2x' d^2x'' d^2\xi. \quad (6.33)$$

$E_p(\xi)$  may be set equal to a constant, since the pump is an approximate plane wave over the extent of the object. Also make the far-field assumption, so that the  $\xi^2$  term in the exponential may be neglected compared to the other terms. In that case, we find  $\tilde{F}\left(\frac{k}{z_1}(\mathbf{x}' + \mathbf{x}'')\right)$  is proportional to  $\delta^{(2)}(\mathbf{x}' + \mathbf{x}'')$  and the amplitude reduces to

$$\begin{aligned}
\psi(\mathbf{x}_1, \mathbf{x}_2) &= T(\mathbf{x}_2) e^{ik(\mathbf{x}_2^2 + \mathbf{x}_1^2)/(2z_2)} \int e^{-ik(\mathbf{x}_1 - \mathbf{x}_2) \cdot \mathbf{x}'/z_2} e^{2i\phi_{\text{even}}(\mathbf{x}')} d^2x' \\
&= T(\mathbf{x}_2) e^{ik(\mathbf{x}_2^2 + \mathbf{x}_1^2)/(2z_2)} \tilde{\Phi}_{\text{even}} \left( \frac{k(\mathbf{x}_1 - \mathbf{x}_2)}{z_2} \right), \tag{6.34}
\end{aligned}$$

where

$$\Phi_{\text{even}}(x) = e^{2i\phi_{\text{even}}(\mathbf{x})} \tag{6.35}$$

and the tilde denotes the Fourier transform. The coincidence rate then becomes:

$$R(\mathbf{x}_1) = \int d^2x_2 |\psi(\mathbf{x}_1, \mathbf{x}_2)|^2 \tag{6.36}$$

$$= \int d^2x_2 \left| T(\mathbf{x}_2) \tilde{\Phi}_{\text{even}} \left( \frac{k}{z_2} (\mathbf{x}_1 - \mathbf{x}_2) \right) \right|^2. \tag{6.37}$$

Equation (6.37) depends only on the even-order aberrations: all odd-order terms have canceled. If there is no aberration in the lens, this reduces to

$$R(\mathbf{x}_1) = |T(\mathbf{x}_1)|^2, \tag{6.38}$$

so the system clearly has unit magnification.

As the distances involved decrease or as the pump amplitude deviates from uniformity in the transverse plane,  $\tilde{F} \left( \frac{k}{z_1} (\mathbf{x}' + \mathbf{x}'') \right)$  begins to look less like a delta function, so aberration cancellation is exact only in the far field with a plane wave pump.

Up to constants, the image intensity  $I(x_1)$  output by an incoherently-illuminated single-lens imaging system such as that of Fig. 6.6b (including lens aberrations) is of the form

$$I(\mathbf{x}_1) = \int d^2\xi \left| T(\xi) E_p(\xi) \tilde{\Phi} \left( k \left( \frac{\xi}{z_1} + \frac{\mathbf{x}_1}{z_2} \right) \right) \right|^2, \tag{6.39}$$

where

$$\Phi(\mathbf{x}) = e^{i\phi(\mathbf{x})}. \tag{6.40}$$

For plane wave illumination ( $E_p(\xi)$  constant), it is apparent that in the aberration-free case both  $\tilde{\Phi}$  and  $\tilde{\Phi}_{\text{even}}$  become delta functions. Thus Eqs. (6.37) and (6.39) both lead to images of the form  $|T(M^{-1}\mathbf{x}_1)|^2$ , where  $M = +1$  for Eq. 6.37 and  $M = - \left( \frac{z_2}{z_1} \right)$  for Eq. 6.39.

The trade-off for odd-order cancellation is seen by considering the case where the aberrations are all of even order: the factor of 2 in the exponent of Eq. 6.35 doubles the effect of the even-order aberrations [compare Eqs. (6.35)–(6.40)]. The reason for this is clear: both the signal and the idler contribute to the image and both gain extra phases from the aberration. For the odd-order terms, the two phases cancel, whereas in the even-order terms the phases add constructively. So, odd-order aberrations are

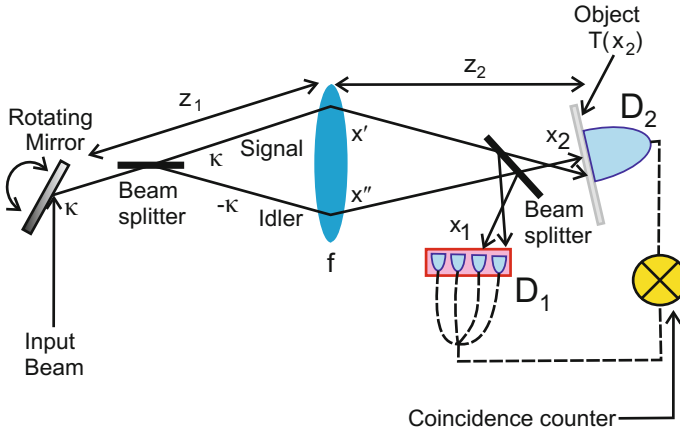


**Fig. 6.7** Implementation of aberration-canceled imaging using narrow classically correlated beams for illumination. The beam splitter produces two spatially anti-correlated copies of the illumination

eliminated at the expense of worsening any even-order aberrations. This method is therefore most beneficial when the odd order aberration is much larger than any of the even order terms present. It should also be noted that this effect can be taken advantage of, as a means to double the effective focusing power of a lens.

This setup has also reduced sensitivity to noise compared to standard single-detector imaging. The noise in the two arms of the apparatus tends to be uncorrelated, so it cancels out of the coincidence rate [31], leaving an improved signal-to-noise ratio.

The odd-order aberration cancellation occurs because the beams strike the lens in a spatially anticorrelated manner, leading to the structure  $\phi(\mathbf{x}') + \phi(\mathbf{x}'') = \phi(\mathbf{x}') + \phi(-\mathbf{x}')$  in the exponents. Quantum entanglement plays no essential role here. The same effect may be produced by any method that requires light to strike diametrically opposite points on the lens simultaneously. Figure 6.7 displays an apparatus with classical illumination that achieves the same effect. A narrow beam illuminates a rotating mirror or beam-steering modulator, which directs the light onto a beam splitter. The reflected and transmitted beams leave the beam splitter with opposite transverse momenta  $\pm \mathbf{q}$ , striking the lens on opposite sides of the axis. Over time, as the mirror moves, the area of the lens is filled. Since the impulse response function for the full system is just the product of those of the two individual beams, the same  $\phi(\mathbf{x}') + \phi(-\mathbf{x}')$  structure again appears, leading to odd-order aberration cancellation.

### 6.3.2 Two-Object Imaging

Variations on the optical systems described in this chapter and the last can be arranged to produce other useful effects. In this section, one such effect is described in which *two* objects can be placed in the apparatus, one in each arm, so that the image produced

is simply the point-by-point product of the images that would be generated by each of the two separately [35].

Consider the hybrid device of Fig. 6.8, which is essentially the ghost imaging analog of the aberration-canceling interferometer of Fig. 3.5, but with the final beam splitter removed and one of the bucket detectors ( $D_1$ ) replaced by a spatially resolving detector in order to allow imaging. The result is essentially a ghost imaging setup with a  $4f$  imaging system in each branch. We assume that the light source is parametric down conversion.

Taking the two detection apertures to be large and the illumination to be from down conversion, it can be shown [35] that the coincidence rate at location  $\mathbf{x}_1$  in  $D_1$  is of the form:

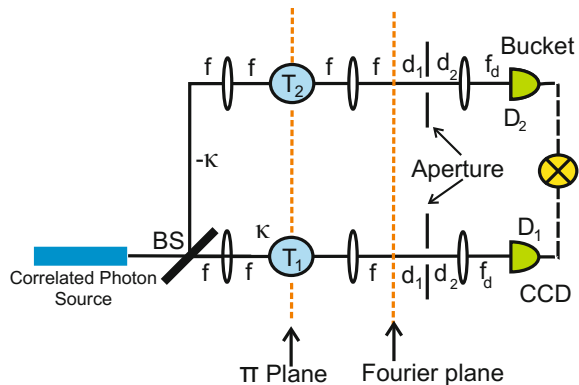
$$R(\mathbf{x}_1) = \mathcal{R}_0 \left| T_1 \left( \frac{f}{f_D} \mathbf{x}_1 \right) T_2 \left( -\frac{f}{f_D} \mathbf{x}_1 \right) \right|^2. \quad (6.41)$$

Only the modulus of each  $T_j$  enters into  $R(\mathbf{x})$  and the aberrations introduced by the object phases cancel to all orders. This is exact only in the Fourier plane, as was true for the interferometer case, but holds approximately as the object moves out of the plane up to a maximum distance on the order of  $\frac{f r_s}{a}$ , where  $f$  and  $a$  are the focal length and radius of the lens and  $r_s$  is the maximum radius of the object being viewed.

If  $T_2 = 1$ , then the result is an ordinary (non-ghost) image of  $|T_1|$ . On the other hand, if  $T_1 = 1$  then an inverted ghost image of  $|T_2|$  arises. In either case, the image is magnified by a factor of  $m = \frac{f_D}{f}$ . Note that, in contrast to the interferometry case, even and odd order phases both cancel, including in the general case  $T_1 \neq T_2$ . Again this holds only near the Fourier plane. Note that if both  $T_1$  and  $T_2$  are nontrivial objects, what is seen is their pointwise product, as mentioned at the beginning of the section.

This product structure may be used in a number of ways. For example, if  $T_2$  has a dim, low-transmissivity area that one wishes to view, but it is being obscured by a bright, high-transmissivity area nearby, one could use a mask for  $T_1$  which allows a view only of the twins of photons coming from the dim region of interest, blocking

**Fig. 6.8** Schematic of correlated-photon imaging setup with two objects. All orders of aberration cancel



photons that are partnered with light from other areas. More generally, various types of optical filters can be inserted in one arm to pick out desired properties of the object in the other arm.

One additional observation on applications of the product structure arises if the position-resolving detector in branch 1 is replaced by a bucket detector, thus introducing an integration over  $\mathbf{x}_1$ . All imaging ability is lost, but suppose that one of the objects, say, object 1 is displaced by some distance in the transverse plane. If the 2-dimensional displacement vector is  $\mathbf{r}$ , then equation (6.41) is replaced by

$$R(\mathbf{r}) = \mathcal{R}_0 \int \left| T_1 \left( \frac{f}{f_D} (\mathbf{x}_1 + \mathbf{r}) \right) T_2 \left( -\frac{f}{f_D} \mathbf{x}_1 \right) \right|^2 dx_1. \quad (6.42)$$

Thus, despite the fact that neither detector has spatial resolution, the system optically computes the spatial intensity correlation function

$$\langle I_1 (m^{-1}(\mathbf{x} + \mathbf{r})) I_2 (-m^{-1}\mathbf{x}) \rangle, \quad (6.43)$$

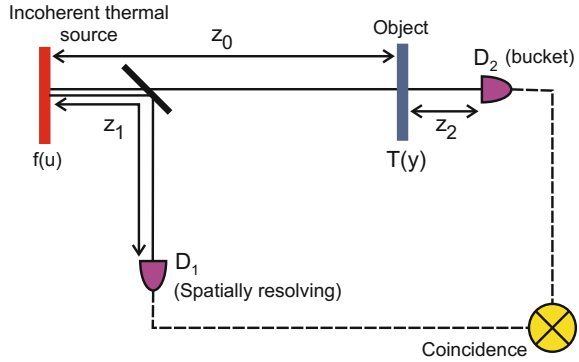
where  $m$  is the magnification. (The correlation here is actually between the object  $T_1(\mathbf{x})$  and the *inverted* object  $T_2(-\mathbf{x})$ , but an additional lens or mirror can be added to remove the inversion and cancel the minus sign in  $T_2$ .) The full correlation function can be found by scanning one object over the full range of relevant  $\mathbf{r}$  vectors. Taking one of the two objects to be unknown and the other to be some known template, this could provide a means of identifying the unknown object by quantifying its degree of similarity to the template. Note that the unknown object may be in a remote, inaccessible location, for example, a cell inside the body may be viewed through an endoscope and compared to a cell in the lab. As in the case of the temporal correlator studied with the interferometer of Refs. [36–38], the effect of object-induced aberrations (differences between phase shifts induced by the two samples) cancels out of the spatial correlator.

As with several of the other applications covered in this book, entanglement is not strictly necessary here: a classical light source arranged so that the two beams are anticorrelated gives a very similar result, and is much more practical for realistic applications.

## 6.4 Ghost Imaging and Turbulence

Given that aberration and dispersion can be mitigated under some circumstances by correlated two-photon methods, it is a natural question to ask whether the effects of turbulence can also be reduced or eliminated. In a short span of years, since about 2009, a large body of experimental and theoretical work has rapidly grown up to investigate the subject [39–51].

**Fig. 6.9** Schematic of ghost imaging setup with turbulence [39]



Consider the following setup investigated by Cheng [39] and shown in Fig. 6.9. The incident light comes from an incoherent thermal or pseudothermal source. After the beam splitter, the light in branch 1 (the reference branch) propagates to a spatially resolving detector  $D_1$ , while the light in branch 2 (the test branch) passes through an object described by transmissivity  $t(y)$  before traveling on to bucket detector  $D_2$ . The detectors are connected in coincidence. In this lensless ghost imaging setup, the distance from the source to the object is taken to equal the distance from the source to the reference detector,  $z_1 = z_0 \equiv z$ . The degree of turbulence is parameterized by a structure constant  $C_n^2$ , with dimensions of  $(distance)^{-2/3}$ . (See Appendix 3 for the definition of  $C_n^2$  and for a brief review of optical propagation through turbulence.)

The setup in [39] allows for the possibility of turbulence arbitrarily distributed throughout the length of all optical paths. The turbulence can be described by random phase functions of the form  $\phi_j(x_i, x_f)$ , where  $x_i$  and  $x_f$  are initial and final transverse positions (at the beginning and end of the turbulent regions). The phase fluctuations are assumed to obey [52]:

$$\langle e^{i(\phi_i(x, y) - \phi_i^*(x', y'))} \rangle = \exp \left\{ -\frac{1.376}{r_{0i}^2} \left[ (x - x')^2 + (x - x') \cdot (y - y') + (y - y')^2 \right] \right\}, \quad (6.44)$$

where  $i = 0, 1$  labels the reference and test arms and the angular brackets denote time average. The *Fried parameter* (see Appendix 3)  $r_{0i}$  is the coherence length of a spherical wave propagating through the  $i$ th medium, and is related to the structure parameter  $C_n^{2(i)}$  via

$$r_{0i} = (.423 C_n^{2(i)} k^2 z_i)^{-3/5}. \quad (6.45)$$

Here the factors of  $5/3$  that appear in the Kolmogorov theory of turbulence (Appendix C) have been approximated as  $\frac{5}{3} \approx 2$ . The spatial distribution of source intensity is also assumed to be Gaussian,  $I(u) = e^{-u^2/\rho_s^2}$ , where  $\rho_s$  is a measure of the transverse size of the source.

Given these assumptions, the point spread function (PSF) is found to be

$$h(y) = \frac{1}{\sqrt{\pi} R_z} e^{-y^2/R_z^2}, \quad (6.46)$$

where

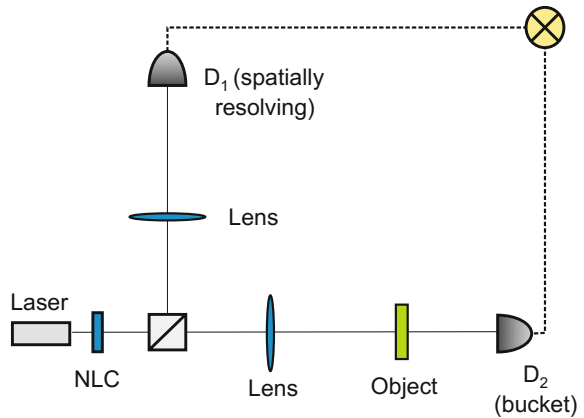
$$R_z = \frac{\lambda z}{\sqrt{2\pi}\rho_s} \sqrt{1 + \rho_s^2 (.55k^2 z)^{6/5} \left[ \left(C_n^{2(0)}\right)^{6/5} + \left(C_n^{2(1)}\right)^{6/5} \right]}. \quad (6.47)$$

As  $R_z$  increases, the PSF broadens and the turbulence causes greater image degradation.

The numerical analysis of [39] led to several conclusions in the case that the turbulence in the reference branch was set to zero ( $C_n^{2(1)} = 0$ ), leaving only turbulence in the object branch: (i) If  $R_z \rightarrow 0$ , then diffraction-limited imaging occurs, i.e. the coincidence rate as a function of position  $x_1$  in detector  $D_r$  is proportional to  $|t(x_1)|^2$  up to convolution with the turbulence-free propagation functions; (ii)  $R_z$  increases (and so the image degrades) when the distance  $z$  increases or when the level of turbulence (as measured by  $C_n^{2(0)}$ ) increases; (iii) For finite  $z$  and  $C_n^{2(1)}$ , the point spread function is narrower than would occur for standard imaging with a single detector at the same level of turbulence. The advantage diminishes with increasing turbulence; (iv) Increasing the transverse size of the source improves the image quality, at least up to a point.

Several theoretical analyses of similar setups have been carried out, including [41] (where the incoherent light source was replaced by a coherent source) and [40] (where the bucket detector  $D_t$  was replaced with a spatially resolving detector). In [40] the correlation function  $G^{(2)}(x_r, x_t)$  between the two detectors is computed, where  $x_r$  and  $x_t$  are the transverse locations in the respective detection planes. It was

**Fig. 6.10** Setup for the turbulent ghost imaging experiment of [45]



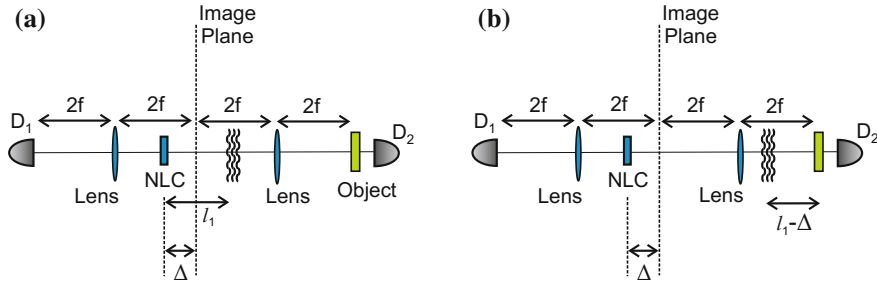


**Fig. 6.11** Klyshko-type view of the previous figure, with a thin layer of turbulence inserted. **a** The turbulence is between the crystal and the lens. This is referred to as case *i* in the text. **b** The turbulence is between the lens and the object (case *ii*)

shown for the diagonal case,  $x_t = x_r$ , that the image from object point  $\xi$  to detection point  $x_r$  is given by the simple form

$$h_g(x_r, \xi) = e^{-A(\xi - \frac{x_r}{M})^2}, \quad (6.48)$$

where  $A$  is an easily computed constant that depends on the wavelength, distances in the setup, size of the source, and degree of turbulence. Setting  $\xi = 0$  gives the PSF. As in [39], it is found that the PSF is narrower than for the standard imaging case, but that it still broadens with increasing level of turbulence.

The effect of turbulence on ghost imaging was studied experimentally in [45], with a more detailed theoretical analysis of the experiment in [46]. In this experiment (Figs. 6.10, 6.11), a quantum ghost imaging system contained an object with a transmission function of the form  $T(x) = e^{-\frac{x^2}{2w^2}}(1 + \cos k_0 x)$ , where  $w$  is the width of the illuminating beam and  $k_0 = \frac{2\pi}{\Lambda}$  for an object of spatial frequency  $\Lambda$ . A thin layer of turbulence was placed in the object arm of the apparatus, either between the crystal and the lens (case *i*) or between the lens and the object (case *ii*). The crystal was allowed to be moved out of the imaging plane by a distance  $\Delta$ . A distance parameter  $l_1$  was defined, such that  $l_1 - \Delta$  was the distance from the crystal to the turbulence in case *i* and the distance from the turbulence to the object in case *ii*. The turbulence was modeled in the quadratic structure function approximation by taking the random phases introduced by the turbulence to have a correlation function of the form

$$\langle e^{-i(\phi(x) - \phi(y))} \rangle = e^{-\frac{1}{2}\alpha|x-y|^2}, \quad (6.49)$$

where  $\alpha$  is a parameter characterizing the strength of the turbulence and is inversely proportional to the square of the Fried parameter:  $\alpha \approx 6.88/r_0^2$ . The resulting intensity–intensity correlation function turns out to be

$$G^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = e^{-\frac{k^2|\mathbf{x}_1 - \mathbf{x}_2|^2}{2\alpha(l_1 - \Delta)}}. \quad (6.50)$$

The image intensity upon integrating over detector 2, for  $l_1 - w \ll kw/\sqrt{2}$ , is proportional to

$$I(\mathbf{x}_1) = e^{-\frac{1}{2}\left(\frac{x_1}{w}\right)^2} (1 + V \cos(k_0 x_1)), \quad (6.51)$$

where the visibility is given by

$$V = (\text{constant}) \cdot e^{-\frac{\alpha(l_1 - \Delta)^2}{2(k/k_0)^2}}. \quad (6.52)$$

The image quality, as measured by the visibility, clearly decays in general as the turbulence strength  $\alpha$  increases or as the turbulent layer moves away from the detector or the imaging plane (increasing  $l_1 - \Delta$ ). Nonetheless, these results show how the effects of the turbulence can be mitigated: when the object is imaged into the turbulent plane, the turbulence has no effect; in other words, when  $l_1 = \Delta$ , the image visibility equals its turbulence-free value.

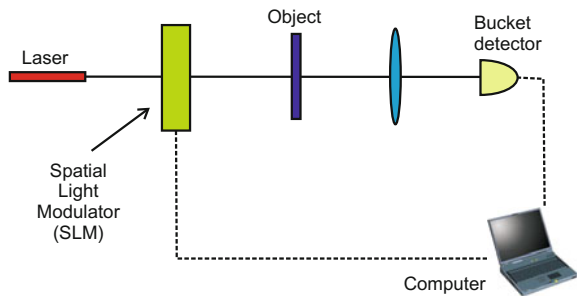
In [46], it is shown that the calculations done using a thin phase screen model of turbulence give theoretical results that match the experimental data. Qualitatively similar results occur if the turbulence is moved to other positions in the setup or if the turbulent layers are allowed to have finite thickness  $L$ . It is found that the field of view is independent of both  $\alpha$  and  $L$ . The principal effect of increasing  $L$  is to widen the point spread function.

It has also been demonstrated experimentally that turbulence-induced odd-order aberration effects can be canceled by means of entangled-photon methods [33].

## 6.5 Computational and Compressive Ghost Imaging

It has been pointed out that if the state of the input light is known and if the propagation function in the path not containing the object can be computed, then it is possible to carry out ghost-imaging with a single bucket detector. This so-called *computational ghost imaging* was proposed in [53] and carried out experimentally in [54]. The apparatus is shown schematically in Fig. 6.12. A laser beam is sent through a spatial light modulator, which imparts a random spatial structure to the light, rep-

**Fig. 6.12** Computational ghost imaging with a single bucket detector



represented by a spatially dependent phase shift,  $\phi_r(x, y)$ . The modulated beam passes through the object, then is collected by a lens and directed to a bucket detector. The reference branch in the usual ghost imaging apparatus is replaced by an electronic signal carrying information about  $\phi(x, y)$  to a computer. The computer then uses the free-space propagator to compute what *would have been* measured if the second detector had been present, and then computes the correlation function of this simulated intensity with the measured intensity in the bucket detector. The result is that something analogous to ghost imaging is carried out with a single detector.

One problem with both the computational ghost imaging mentioned above and with pseudothermal ghost imaging is that the collection times required for reconstructing the image are long; the problem is even worse for quantum ghost imaging. A recent suggestion for alleviating this problem is *compressive ghost imaging* [55]. A version of this approach for pseudothermal imaging is similar to that of Fig. 6.5. A rotating diffuser is placed before the beam splitter of a ghost imaging setup to simulate the spatial and temporal fluctuations in thermal light. A set of  $M$  measurements are made, each with a different realization of the diffuser speckle pattern. The various realizations are labeled by  $r = 1, 2, \dots, M$ . If the  $r$ th intensity pattern at the object is  $I_r(x, y)$ , then the measured intensity at the bucket detector is proportional to

$$B_r(x, y) = \int dx dy I_r(x, y) T(x, y), \quad (6.53)$$

where  $T(x, y)$  is the transmission function of the object. The transmissivity is then estimated by computing the correlation function

$$T_{GI}(x, y) = \frac{1}{M} \sum_{r=1}^M (B_r - \langle B_r \rangle) I_r(x, y). \quad (6.54)$$

Suppose that the average number of speckles needed to cover the object is  $N$ . Then the Nyquist theorem [56, 57] implies that the number of measurements needed to carry out the reconstruction is  $M \geq N$ . Since the speckle patterns have some overlap and are not orthogonal, in reality the number of measurements needed is actually  $M \gg N$  in order to achieve a good signal to noise ratio. However, *compressive sensing* or *compressive sampling* (CS) may be used to reduce the number of measurements required to a much smaller number,  $M < N$ . Compressive sensing makes use of the fact that most images are extremely sparse: if the image is expanded in some orthonormal basis, the vast majority of the coefficients are normally very small, so that dropping these terms from the expansion has no noticeable effect on the image. The CS algorithm [58–60] then consists of searching for the image which matches the measurements, while also minimizing the  $L_1$  norm. The result is that ghost images may be formed at video rates [55].

## 6.6 Quantum Illumination

Quantum illumination, first proposed in [61], further elaborated in [62–68], and experimentally demonstrated in [69, 70], is a method for determining the presence or absence of a low-reflectivity object through a surrounding thermal noise bath. As with the other methods described in this chapter, the essential idea is to use intensity–intensity correlations between two photons; the noise is uncorrelated between the two photon paths, so that it does not contribute to the correlation function.

The method uses pairs of entangled photons. One photon, the ancilla, is retained by the user. The other, the signal, is sent to the target region. If a photon returns from the target region, then it must be decided whether it is the original signal photon that has been reflected back to the source by the object, or if it is a thermal noise photon. An optimal (minimum error probability) quantum measurement (see Sect. 1.11) is applied to the pair formed by the ancilla and the returned photon, in which the state of this pair is compared to the initial state of the ancilla–signal pair. If the original signal photon has been lost, then the ancilla goes from its initial pure state into a mixed state, as the signal degrees of freedom get traced out. Consequently, the probability of the new two-photon state being able to deceive the experimenters decreases by a factor of  $d$ , where  $d = 2^m$  is the number signal and detector modes available and  $m$  is the number of e-bits of entanglement, where 1 e-bit is the amount of entanglement possessed by a Bell state. This means a corresponding increase in the signal-to-noise ratio (SNR) by a factor of  $d$ . The source of this increase can be traced back to an increase in the mutual information between the incident and reflected beams [71].

One thing that is especially notable about this effect is that as long as the initial state was entangled, the SNR improvement persists even after the entanglement has been lost through coupling to the environment. This means that the effect, though quantum in origin, may be robust enough to be used in real-world settings. The initial phase-sensitive cross-correlation between the signal and idler is stronger than allowed by classical physics. Even after the entanglement has decayed away, there is still a residual correlation which remains strong enough in the low-photon-number regime to exceed any correlations in a classical state with the same number of photons that may have been emitted by the object or its surroundings.

The work of [68] extends the idea from the visible to the microwave region, making it more suitable for use in remote sensing of distant objects and opening up the possibility of what the authors referred to as “quantum radar.”

A further application of quantum illumination occurs in quantum secure communication. Suppose that an eavesdropper (Eve) tries to listen in on a private communication between two participants, Alice and Bob. Such eavesdropping introduces errors in the communication. In most quantum cryptography scenarios, these eavesdropper-induced errors are beneficial because they help enforce security: detection of the excess errors signals to Alice and Bob that their communication has been compromised (see Chap. 9). Another possibility, though, is that some mechanism can be introduced that prevents errors between Alice and Bob, while decreasing the amount of information Eve can extract about the message by increasing her error

rate. A method for doing this has been presented [65, 67, 72, 73] in which the use of quantum illumination can lower the error rate for Alice and Bob by orders of magnitude relative to Eve's error rate. Unlike the quantum key distribution protocols discussed in Chap. 9, this method involves direct communication between Alice and Bob, with no need to generate a secret key. It has potential to allow secure communication at gigabyte/second rates over tens of kilometers even in the presence of photon losses.

## 6.7 Quantum Holography

As one further variation on quantum imaging, in this section we briefly describe the idea of quantum holography. Conventional holography relies on illuminating a detector simultaneously with two beams, a reference beam (often taken to be a plane wave) and an object beam carrying the image of the object to be photographed. The two beams are allowed to combine, with the result used to illuminate a photographic film or a digital camera. If the reference and object fields are  $E_r = A_r e^{i\phi_r}$  and  $E_o = A_o e^{i\phi_o}$ , then the transmittance of the film is proportional to

$$t \sim |E_r + E_o|^2 = I_r + I_o + 2 \operatorname{Re} (A_r A_o^*) \cos(\phi_r - \phi_o), \quad (6.55)$$

where the intensities  $I_r$  and  $I_o$  are the absolute squares of the fields. If this film is then illuminated by another copy of the reference wave, the transmitted field (divided by the reference amplitude) is then

$$E \sim t E_r \sim I_r + I_o(\mathbf{r}) + E_r (E_o(\mathbf{r}) + E_o^*(\mathbf{r})). \quad (6.56)$$

By extracting the third term, the image can be reconstructed. A variety of methods exist for extracting this term—see, for example, [74] for details.

Conventional holography as outlined above makes use of second-order interference in the two fields. Quantum holography [75, 76] instead makes use of interference that is fourth-order in the fields (second order in intensities) between an idler photon and two different potential signal photon paths. The setup is shown in Fig. 6.13. A nonlinear crystal produces signal-idler pairs. The idler proceeds directly to the spatially-resolving detector  $D_1$ . The signal travels toward the object, which is partially enclosed inside a chamber lined whose walls are photosensitive. The chamber walls act as a bucket detector, labeled  $D_2$ . Since the light striking the object cannot reach  $D_1$  and the detector on the chamber wall has no spatial resolution, it would seem that imaging is impossible. But as in ghost imaging the correlations between the signal and idler photon once again allow the image to be reconstructed.

Consider two possible paths for the signal: one path scatters off the object then arrives at point  $x_2$  on the chamber wall, while the other path misses the object and proceeds directly to the same point  $x_2$ . These two paths are indistinguishable with

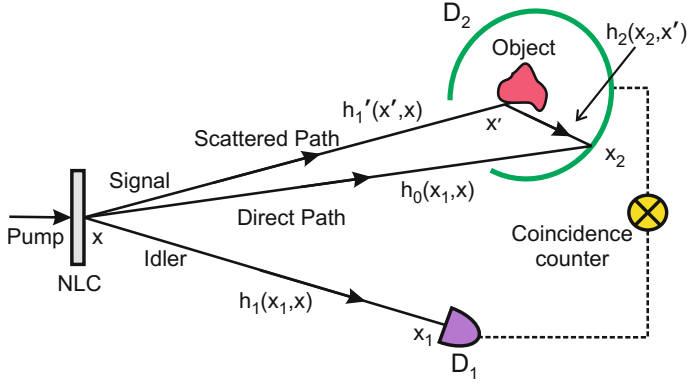


**Fig. 6.13** The setup for quantum holography. Down conversion in the nonlinear crystal (NLC) produces an idler photon that is detected by the spatially resolving detector  $D_1$  and a signal photon that enters a chamber containing the object. The walls of the chamber act as a bucket detector  $D_2$ , which is connected in coincidence with  $D_1$ . The signal could strike the wall directly or after scattering off the object. Interference occurs between the two possibilities

the given apparatus, and so may interfere. The scattered path and the direct path act, respectively, as the object and reference paths of conventional holography.

Consider a single small scattering particle at position  $\mathbf{x}'$  for simplicity; generalizing to an arbitrary finite object is straightforward. Suppose  $\psi(\mathbf{x})$  is the amplitude for down conversion at point  $\mathbf{x}$  of the crystal. The coincidence rate between points  $\mathbf{x}_1$  and  $\mathbf{x}_2$  on the two detectors is

$$R_c(\mathbf{x}_1, \mathbf{x}_2) \sim \left| \int d\mathbf{x} \psi(\mathbf{x}) h_1(\mathbf{x}_1, \mathbf{x}) h_2(\mathbf{x}_2, \mathbf{x}) \right|^2, \quad (6.57)$$

which looks like the intensity in a coherent imaging system. Integrating over the bucket detector,  $D_2$ , the marginal detection rate at detector  $D_1$  is

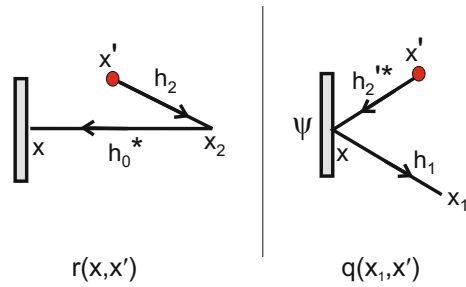
$$R_1(\mathbf{x}_1) = \int d\mathbf{x}_2 R_c(\mathbf{x}_1, \mathbf{x}_2), \quad (6.58)$$

which is the incoherent sum over all of the coherent images formed from all of the points of  $D_2$ . If a film is made with transparency proportional to  $R_1$ , it can be shown that

$$t(\mathbf{x}_1) \sim R_1^{(0)}(\mathbf{x}_1) + R_1^{(1)}(\mathbf{x}_1) + [\epsilon(\mathbf{x}') r(\mathbf{x}_1, \mathbf{x}') q(\mathbf{x}_1, \mathbf{x}') + c.c.], \quad (6.59)$$

where  $\epsilon$  is the scattering strength [76]. The propagation functions  $r(\mathbf{x}_1, \mathbf{x}')$  and  $q(\mathbf{x}_1, \mathbf{x}')$  are defined as follows (see Fig. 6.14). The function  $r$  describes Klyshko-type propagation (see Sect. 6.1) from the scatterer to the crystal plane by propagating forward to  $D_2$  then backward via the direct path to the crystal, along with backward

**Fig. 6.14** Definitions of the propagation functions  $q$  and  $r$



propagation from  $D_1$  to the crystal. The function  $q$  represents forward propagation from the crystal directly to the scatterer, along with forward propagation to  $D_1$ . The first two terms in Eq. 6.59 represent the marginal rate in the absence of the scatterer and the rate due to scattering alone. The terms in the bracket are interference terms between these two possibilities. These interference terms are analogous to the last two terms in Eq. 6.56. Illumination of the film by a copy of the unscattered signal wave then allows reconstruction of the function  $\epsilon(\mathbf{x})$ , which provides the desired characterization of the object. It is noteworthy that this image is formed without any photon ever having left the chamber that contains the object.

In summary, it is the two-photon second-order correlation that makes imaging possible, while it is the interference between two potential paths that gives the result a holographic character.

## 6.8 Additional Topics in Ghost Imaging

A number of recent developments have potential to help make the use of ghost imaging in practical applications more widespread. A few of these are briefly mentioned here.

- Ghost imaging relies on correlations between the momenta or positions of two photons. Attempts to perform ghost imaging using fiber optic systems immediately run into the problem that the range of positions and momenta carried by a fiber is restricted. This has provided a serious impediment to use of ghost imaging over long distances via optical fibers. However, ghost imaging over 50 km distances using optical fiber has recently been demonstrated [77]. This was accomplished by converting position differences into time differences, and then comparing arrival times of photons in frequency-correlated pairs. The frequency correlations and time differences are preserved by transmission over the fiber, allowing images to be reconstructed. The frequency-correlated photon pairs that are needed may be generated through four-wave mixing in third-order nonlinear materials [78, 79].

- LADAR (laser detection and ranging or laser radar), also known as LIDAR (light detection and ranging) is a remote sensing method for reconstructing three-dimensional images from the light reflected off an object (see [80] for a review).

A ghost-imaging-based version of LADAR [81–84] has been implemented using two correlated light beams. One is used as a reference, the other is reflected off the target. Three-dimensional object information is then reconstructed from analysis of the resulting correlation functions. In experiments done over distances of a few hundred meters to a kilometer, this has led to improved transverse and longitudinal resolution, as well as improved range accuracy, compared to standard LADAR methods.

- A storage device for ghost images has also been demonstrated [85]. It has been known for some time that optical images can be stored in atomic gas vapors by means of electromagnetically induced transparency [86–92]. More recently, it has been shown [85] that ghost images can also be stored in a gas of hot rubidium atoms. To retrieve the image, a thermal beam is split at a beam splitter, producing a reference beam and a probe beam sent through the vapor cell to a CCD. The image then reappears in the correlation function of the two beams, showing that the second-order correlations can survive the storage and retrieval processes.

- Magneto-optical imaging [93, 94] uses Kerr nonlinearities and the Faraday effect to determine the magnetic properties of material samples, for example mapping the size and shapes of magnetic domains. However, under some conditions, such as very low temperatures or high magnetic fields, standard methods become less useful. It has recently been shown [95] that approaches based on thermal light ghost imaging can retrieve useful magnetic domain imaging under these conditions. This promises to open a new and very useful range of applications for correlated imaging methods.

## References

1. R.E. Meyers, K.S. Deacon, Y. Shih, *Appl. Phys. Lett.* **98**, 111115 (2011)
2. J.H. Shapiro, [arXiv:1201.4513](https://arxiv.org/abs/1201.4513) (2012)
3. J.H. Shapiro, R.W. Boyd, *Quant. Inf. Proc.* **11**, 949 (2012)
4. Y. Shih, *Quant. Inf. Proc.* **11**, 995 (2012)
5. J.H. Shapiro, R.W. Boyd, *Quant. Inf. Proc.* **11**, 1003 (2012)
6. S. Ragy, G. Adesso, *Sci. Rep.* **2**, 651 (2012)
7. J.H. Shapiro, D. Venkatraman, F.N.C. Wong, *Sci. Rep.* **3**, 1849 (2013)
8. E. Brambilla, L. Caspani, O. Jedrkiewicz, L.A. Lugiato, A. Gatti, *Phys. Rev. A* **77**, 053807 (2008)
9. L. Basano, P. Ottonello, *Am. J. Phys.* **75**, 343 (2007)
10. A. Gatti, E. Brambilla, M. Bache, L.A. Lugiato, Ghost imaging, in *Quantum Imaging*, ed. by M.I. Kolobov (Springer, Berlin, 2007)
11. Y. Shih, [arXiv:0805.1166](https://arxiv.org/abs/0805.1166) [quant-ph](2008)
12. B.I. Erkmen, J.H. Shapiro, *Phys. Rev. A* **77**, 043809 (2008)
13. D.V. Strekalov, A.V. Sergienko, D.N. Klyshko, Y.H. Shih, *Phys. Rev. Lett.* **74**, 3600 (1995)
14. T.B. Pittman, Y.H. Shih, D.V. Strekalov, A.V. Sergienko, *Phys. Rev. A* **52**, R3429 (1995)
15. A. Einstein, B. Podolsky, N. Rosen, *Phys. Rev.* **47**, 777 (1935)
16. D.N. Klyshko, *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **83**, 1313 (1982); English edition: D.N. Klyshko, *Sov. J. Exp. Theor. Phys.* **56**, 753 (1982)
17. D.N. Klyshko, *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* **94**, 82 (1988); English edition: D.N. Klyshko, *Sov. J. Exp. Theor. Phys.* **67**, 1131 (1988)
18. A.N. Boto, P. Kok, D.S. Abrams, S.L. Braunstein, C.P. Williams, J.P. Dowling, *Phys. Rev. Lett.* **85**, 2733 (2000)

19. P. Kok, A.N. Boto, D.S. Abrams, C.P. Williams, S.L. Braunstein, J.P. Dowling, *Phys. Rev. A* **63**, 063407 (2001)
20. A.F. Abouraddy, B.E.A. Saleh, A.V. Sergienko, M.C. Teich, *J. Opt. Soc. Am. B* **19**, 1174 (2002)
21. R.S. Bennink, J. Bentley, R.W. Boyd, J.C. Howell, *Phys. Rev. Lett.* **92**, 033601 (2004)
22. A. Gatti, E. Brambilla, M. Bache, L.A. Lugiato, *Phys. Rev. A* **70**, 013802 (2004)
23. A. Gatti, E. Brambilla, M. Bache, L.A. Lugiato, *Phys. Rev. Lett.* **93**, 093602 (2004)
24. Y.J. Cai, S.Y. Zhu, *Phys. Rev. A* **71**, 056607 (2005)
25. Y.J. Cai, S.Y. Zhu, *Opt. Lett.* **29**, 2716 (2004)
26. A. Valencia, G. Scarcelli, M. D'Angelo, Y.H. Shih, *Phys. Rev. Lett.* **94**, 063601 (2005)
27. F. Ferri, D. Magatti, A. Gatti, M. Bache, E. Brambilla, L.A. Lugiato, *Phys. Rev. Lett.* **94**, 183602 (2005)
28. N.D. Hardy, J.H. Shapiro, *Phys. Rev. A* **84**, 063824 (2011)
29. N.D. Hardy, J.H. Shapiro, in *Quantum computation and imaging VIII*, eds. by R.E. Meyers, Y. Shih, K.S. Deacon, *Proc. SPIE*, vol. 7815 (2011), p. 78150L
30. B. Erkmen, JPL-IPN Progress Report 42-125 (2011). [http://ipnpr.jpl.nasa.gov/progress\\_report/42-185/185B.pdf](http://ipnpr.jpl.nasa.gov/progress_report/42-185/185B.pdf)
31. D.S. Simon, A.V. Sergienko, *Phys. Rev. A* **82**, 023819 (2010)
32. L.A.P. Filpi, M.V. da Cunha Pereira, C.H. Monken, *Opt. Exp.* **23**, 3841 (2015)
33. M.V. da Cunha Pereira, L.A.P. Filpi, C.H. Monken, *Phys. Rev. A* **88**, 053836 (2013)
34. A.F. Abouraddy, K.C. Toussaint, A.V. Sergienko, B.E.A. Saleh, M.C. Teich, *Opt. Lett.* **26**, 1717 (2001)
35. D.S. Simon, A.V. Sergienko, *J. Opt. Soc. Am. B* **28**, 247 (2011)
36. D.S. Simon, A.V. Sergienko, *Phys. Rev. A* **80**, 053813 (2009)
37. C. Bonato, A.V. Sergienko, B.E.A. Saleh, S. Bonora, P. Villoresi, *Phys. Rev. Lett.* **101**, 233603 (2008)
38. C. Bonato, D.S. Simon, P. Villoresi, A.V. Sergienko, *Phys. Rev. A* **79**, 062304 (2009)
39. J. Cheng, *Opt. Exp.* **17**, 7916 (2009)
40. P. Zhang, W. Gong, X. Shen, S. Han, [arXiv:1005.5011](https://arxiv.org/abs/1005.5011) (2010)
41. C. Li, T. Wang, J. Pu, W. Zhu, R. Rao, *Appl. Phys. B* **99**, 599 (2010)
42. R.E. Meyers, K.S. Deacon, in *Quantum communications and quantum imaging VIII*, eds. by R.E. Meyers, Y. Shih, K.S. Deacon, *Proceedings of the SPIE*, vol. 7815 (2010), p. 78150I
43. R.E. Meyers, K.S. Deacon, Y. Shih, in *Quantum communications and quantum imaging VIII*, eds. by R.E. Meyers, Y. Shih, K.S. Deacon, *Proceedings of the SPIE*, vol. 7815 (2010), p. 781506
44. R.E. Meyers, K.S. Deacon, Y. Shih, *J. Mod. Opt.* **54**, 2381 (2007)
45. P.B. Dixon, G. Howland, K.W.C. Chan, C. O'Sullivan-Hale, B. Rodenburg, N.C. Hardy, D.S. Simon, J.H. Shapiro, A.V. Sergienko, R.W. Boyd, J.C. Howell, *Phys. Rev. A* **83**, 051803(R) (2011)
46. K.W.C. Chan, D.S. Simon, A.V. Sergienko, N.C. Hardy, J.H. Shapiro, P. Ben, Dixon, G. Howland, J.C. Howell, J.H. Eberly, M.N. O'Sullivan, B. Rodenburg, R.W. Boyd, *Phys. Rev. A* **84**, 043807 (2011)
47. G.J. Turiaci, J.J.M. Varga, C. Iemmi, *Optik* **124**, 6212 (2013)
48. D. Shi, C. Fan, P. Zhang, H. Shen, J. Zhang, C. Qiao, Y. Wang, *Opt. Exp.* **21**, 2050 (2013)
49. M. Mirhosseini, B. Rodenburg, M. Malik, R.W. Boyd, *J. Mod. Opt.* **61**, 43 (2014)
50. M.V. da Cunha Pereira, L.A.P. Filpi, C.H. Monken, *Phys. Rev. A* **88**, 053836 (2013)
51. H. Shao, J. Wang, P. Xu, M. Yang, *J. Inf. Comp. Sci.* **11**, 2817 (2014)
52. M.C. Roggemann, B. Welsh, *Imaging Through Turbulence* (CRC Press, Boca Raton, 1996)
53. J.H. Shapiro, *Phys. Rev. A* **78**, 061802(R) (2008)
54. Y. Bromberg, O. Katz, Y. Silberberg, *Phys. Rev. A* **79**, 053840 (2009)
55. O. Katz, Y. Bromberg, Y. Silberberg, *Appl. Phys. Lett.* **95**, 113110 (2009)
56. H. Nyquist, *Trans. AIEE* **47**, 617 (1928)
57. C.E. Shannon, *Proc. Inst. Radio Eng.* **37**, 10 (1949)
58. E. Candes, J. Romberg, T. Tao, *IEEE Trans. Inf. Theory* **52**, 489 (2006)
59. E. Candes, T. Tao, *IEEE Trans. Inf. Theory* **52**, 5406 (2006)

60. D. Donoho, I.E.E.E. Trans, Inf. Theory **52**, 1289 (2006)
61. S. Lloyd, Science **321**, 1463 (2008)
62. S.-H. Tan, B.I. Erkmen, V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, S. Pirandola, J.H. Shapiro, Phys. Rev. Lett. **101**, 253601 (2008)
63. S. Guha, B.I. Erkmen, Phys. Rev. A **80**, 052310 (2009)
64. C. Weedbrook, S. Pirandola, J. Thompson, V. Vedral, M. Gu, [arXiv:1312.3332](https://arxiv.org/abs/1312.3332) (2013)
65. J.H. Shapiro, Phys. Rev. A **80**, 022320 (2009)
66. J.H. Shapiro, S. Lloyd, New. J. Phys. **11**, 063045 (2009)
67. Z. Zhang, M. Tengner, T. Zhong, F.N.C. Wong, J.H. Shapiro, Phys. Rev. Lett. **111**, 010501 (2013)
68. S. Barzanjeh, S. Guha, C. Weedbrook, D. Vitali, J.H. Shapiro, S. Pirandola, Phys. Rev. Lett. **114**, 080503 (2015)
69. E.D. Lopaeva, I. Ruo, Berchera, I.P. Degiovanni, S. Olivares, G. Brida, M. Genovese, Phys. Rev. Lett. **110**, 153603 (2013)
70. E.D. Lopaeva, I. Ruo Berchera, S. Olivares, G. Brida, I.P. Degiovanni, M. Genovese, Phys. Scr. **T160**, 014026 (2014)
71. S. Ragy, I. Ruo, Berchera, I.P. Degiovanni, S. Olivares, M.G.A. Paris, G. Adesso, M. Genovese, J. Opt. Soc. Am. B **31**, 2045 (2014)
72. J.H. Shapiro, Z. Zhang, F.N.C. Wong, Quantum Inf. Process. **13**, 21712193 (2014)
73. Q. Zhuang, Z. Zhang, J. Dove, F.N.C. Wong, J.H. Shapiro, [arXiv:1508.01471](https://arxiv.org/abs/1508.01471) [quant-ph] (2015)
74. P. Hariharan, *Optical Holography: Principles, Techniques and Applications* (Cambridge University Press, Cambridge, 1996)
75. A.V. Belanskii, D.N. Klyshko, Zh. Exsp. Teor. Fiz. **105**, 487 (1994); English edition: A.V. Belanskii, D.N. Klyshko, Sov. Phys. JETP **78**, 259 (1994)
76. A.F. Abouraddy, B.E.A. Saleh, A.V. Sergienko, M.C. Teich, Opt. Exp. **9**, 498 (2001)
77. S. Dong, W. Zhang, Y. Huang, J. Peng, [arXiv:1508.05248](https://arxiv.org/abs/1508.05248) [quant-ph] (2015)
78. E. Brainis, Phys. Rev. A **79**, (2009)
79. J.R. Cheng, W. Zhang, Q. Zhou, X. Feng, Y.D. Huang, J.D. Peng, Chin. Phys. Lett. **27**, 124208 (2010)
80. P.F. McManamon, Opt. Eng. **51**, 060901 (2012)
81. C. Zhao, W. Gong, M. Chen, E. Li, H. Wang, W. Xu, S. Han, Appl. Phys. Lett. **101**, 141123 (2012)
82. W. Gong, C. Zhao, J. Jiao, E. Li, M. Chen, H. Wang, W. Xu, S. Han, [arXiv:1301.5767](https://arxiv.org/abs/1301.5767) [quant-ph] (2013)
83. Y. Xu, Z. Yong, X. Lu, Y. Cheng-Hua, W. Qiang, L. Yue-Hao, Z. Yuan, Chin. Phys. B **24**, 124202 (2015)
84. H. Yu, E. Li, W. Gong, S. Han, Opt. Exp. **23**, 14541 (2015)
85. Y.-W. Cho, J.-E. Oh, Y.-H. Kim, Opt. Exp. **20**, 5809 (2012)
86. M. Fleischhauer, M.D. Lukin, Phys. Rev. Lett. **84**, 5094 (2000)
87. D.F. Phillips, A. Fleischhauer, A. Mair, R.L. Walsworth, M.D. Lukin, Phys. Rev. Lett. **86**, 783 (2001)
88. M.D. Eisaman, A. Andre, F. Massou, M. Fleischhauer, A.S. Zibrov, M.D. Lukin, Nature **438**, 837 (2005)
89. K. Honda, D. Akamatsu, M. Arikawa, Y. Yokoi, K. Akiba, S. Nagatsuka, T. Tanimura, A. Furusawa, M. Kozuma, Phys. Rev. Lett. **100**, 093601 (2008)
90. H. Tanji, S. Ghosh, J. Simon, B. Bloom, V. Vuletić, Phys. Rev. Lett. **103**, 043601 (2009)
91. Y.-W. Cho, Y.-H. Kim, Opt. Exp. **18**, 25786 (2010)
92. Y.-W. Cho, Y.-H. Kim, Phys. Rev. A **82**, 033830 (2010)
93. A. Hubert, R. Schaefer, *Magnetic Domains* (Springer, Berlin, 1998)
94. B.E. Argyle, J.G. Mccord, in Magnetic storage systems beyond 2000, ed. by G. Hadjipanayis. *NATO Science Series*, vol. 41 (Springer, Netherlands, 2001), p. 287
95. A. Meda, A. Caprile, A. Avella, I. Ruo Berchera, I.P. Degiovanni, A. Magni, M. Genovese, Appl. Phys. Lett. **106**, 262405 (2015)

# Chapter 7

## Quantum Microscopy

### 7.1 Introduction

In the previous chapter, several variations on correlated imaging systems were discussed. In this chapter, it is shown that imaging techniques using correlated light can also be used to improve the resolution in microscopy. There are several ways that correlated-photon methods can improve resolution. The first is by means of the cancelation of aberration and other distorting effects, as discussed in Chap. 3. A second improvement comes from using photon correlations and coincidence detection to reduce the intrinsic broadening of the point-spread function that occurs even in the absence of disturbing effects, due to diffraction introduced by the finite size of the microscope. Use of correlation to reduce such broadening leads to an improvement in determination of a photon's interaction point with the object. This second type of improvement is covered in the following sections.

There are several possible strategies for enhancing images in this way. In one approach, one photon is passed through the microscope and allowed to interact with the sample being imaged, being distorted by the interaction and by the imaging device itself, while a second, reference photon avoids the sample and optical system, remaining undistorted. In this way some of the information lost from the signal photon is recoverable using information that remains intact in the reference photon, such as information about the incident wavevector direction. An alternative approach is to arrange for the two photons to follow the same path, but to gain opposite phase distortions. In coincidence detection, these phases then cancel. In either approach, the goal is ultimately the same: to more precisely localize the point at which the photon interacted with the sample.

Let  $\mathbf{r}_o$  and  $\mathbf{r}_d$  represent the locations of the point in the object where a photon was emitted or scattered and the point in the detector where it was detected. Suppose the goal is to estimate  $\mathbf{r}_o$  based on a measurement of  $\mathbf{r}_d$ . Let  $\pi(\mathbf{r}_d|\mathbf{r}_o)$  be the conditional probability of a detection event at  $\mathbf{r}_d$ , given emission at  $\mathbf{r}_o$ . This conditional probability should be centered at the location of the image of  $\mathbf{r}_o$ , and the width of the distribution should increase as the resolution of the optical system degrades. In

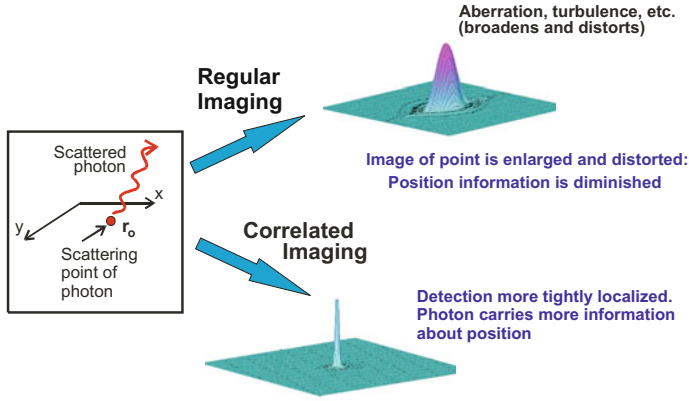


**Fig. 7.1** Aberration, turbulence, diffraction and other effects cause broadening and distortion of the spatial probability distribution for photon detection, causing loss of resolution and decrease of information transfer. Correlated imaging allows more information to be preserved, so that the distributions are more tightly localized and the information gain per detected photon is larger

fact,  $\pi(\mathbf{r}_d|\mathbf{r}_o)$  is essentially the point spread function, normalized to unit integral. The information gained about  $\mathbf{r}_o$  from each measurement can be quantified by using the *Fisher information* [1],

$$J(\mathbf{r}_d|\mathbf{r}_o) = E \left\{ \left( \frac{\partial \ln \pi(\mathbf{r}_d|\mathbf{r}_o)}{\partial \mathbf{r}_d} \right) \cdot \left( \frac{\partial \ln \pi(\mathbf{r}_d|\mathbf{r}_o)}{\partial \mathbf{r}_d} \right)^T \right\}. \quad (7.1)$$

Fisher information is a commonly used means of quantifying information, especially in imaging. In Eq. 7.1,  $\pi(\mathbf{r}_d|\mathbf{r}_o)$  is thought of as a matrix in the bases of  $\mathbf{r}_o$  and  $\mathbf{r}_d$ ,  $E$  represents expectation value over the  $\pi$  distribution and  $T$  denotes matrix transpose; the information is seen to increase as the gradients  $\partial \ln \pi(\mathbf{r}_d|\mathbf{r}_o)/\partial \mathbf{r}_d$  are large, i.e. when  $\pi(\mathbf{r}_d|\mathbf{r}_o)$  is tightly localized about  $\mathbf{r}_o$ . This is exactly what is achieved by means of photon correlations: We are able to more tightly localize the spatial distribution of detected photons despite the action of distorting effects such as turbulence and aberration (Fig. 7.1), which then allows for improved resolution. From an information theoretic approach, resolution limits arise from the Cramer-Rao bound [2, 3], which says that the variance in a measured variable must be greater than the inverse of the Fisher information.

The methods described in the following sections improve resolution by enhancing photon localization through the enforcement of correlations between pairs of photons. The difference between the various methods is largely due to the way the correlation is created, and the location where it is introduced: at the sample, detection, or illumination stages.

In the widely-used two-photon microscope [4, 5], spatial correlations are enforced in the sample: uncorrelated photons are inserted into the microscope at the source,

but the requirement that the two photons interact with the same fluorescent molecule effectively enforces spatial correlations among the photon pairs contributing to the detected signal. Because the two-photon transition rate increases with illumination intensity, a trade-off has to be made in many cases between the data collection rate and damage to the sample. *Entangled-photon fluorescence microscopy* [6] (EPM, Sect. 7.6) is a means of enhancing the interaction probability at the photon-sample interaction site without needing to increase the intensity of light incident on the sample. As an alternative, the *correlated confocal microscope* (Sect. 7.7) enforces correlations at the *detection stage*, through coincidence detection and post-selection. One further possibility, used by the *twin-photon microscope* (Sect. 7.8) is to use illumination from spontaneous parametric down conversion, so that correlation is produced at the source. In all of these methods, there is a potential reduction in the need for high intensities, reducing possible damage to the sample.

Historically, the idea of using entanglement to beat the diffraction limit originated in work on quantum lithography [7], and the first detailed proposal to use entangled photons to obtain subwavelength microscopic resolution seems to have been [8].

## 7.2 Resolution, Super-Resolution, and the Abbé Limit

As the resolution of an optical system improves, the Abbé diffraction limit [9] is eventually reached. Beyond this point, imaging the shape of smaller features in extended objects with an ordinary microscope is impossible because the high spatial-frequency components needed to reconstruct the shape do not propagate into the far field or diffract into the objective lens. It was long assumed on this basis that beyond a certain point it becomes meaningless to seek further improvements in the resolution of a microscope. However, the derivation of the Abbé limit makes use of two assumptions: (i) uniform illumination and (ii) linear response in the detection system. When these two assumptions are removed, it is possible to extend the range of spatial frequencies the system allows to pass, leading to super-resolution [10]. This fact has been exploited in a number of different types of fluorescence-based far-field microscopes in recent years (see [11] for an overview). In the quantum microscopes discussed in the following sections, at least one of these two assumptions are violated. For example, the twin-photon microscope violates both: there is nonuniform, spatially-structured illumination in the sample (the pump beam is focused to a spot) and nonlinear response in the detection system (since coincidence detection leads to an effectively quadratic response). The usual Abbé limit therefore fails to apply, allowing an expansion of the spatial frequency pass-band. The ultimate resolution then is limited by quantum mechanics and the uncertainty principle. (See Chap. 4 for more discussion of quantum mechanical measurement limits.)

The existence of sub-wavelength or super-resolution imaging can also be viewed as related to the existence of *super-oscillating functions* [12, 13]: a function is said to undergo super-oscillations if it oscillates faster than any of its Fourier components. In the optical case, this means that the measured correlation functions oscillate on

a finer spatial scale than the shortest photon wavelength present in the beam; as a result, sub-wavelength imaging becomes possible. Superoscillating functions were first discovered in the context of microwave engineering in the 1950s [14], before being rediscovered in the context of quantum mechanics [12] almost forty years later. For more information on the physical and optical aspects of super-oscillations, see [15–17].

Moreover, light can scatter even from very small particles, including those of extreme sub-wavelength size. Although the high spatial-frequency components needed to reconstruct the object's shape may be missing from the image, this scattered light may still be used to identify the presence of such particles and to partially localize their positions. The use of optical microscopes (especially polarization microscopes) for this is an active field of research: a number of papers have appeared, for example, which study the use of polarizing confocal and widefield microscopes to view subwavelength dielectric particles [18–22]. Limits on point source localization for quantum microscopes have been studied [23] from a Fisher information approach, using a quantum Cramer-Rao bound [24, 25]; roughly speaking, an  $N$ -photon quantum state can localize a particle to within a distance of  $\frac{\lambda}{\sqrt{N}}$ , where  $\lambda$  is the photon wavelength.

The correlation-based confocal microscopes described in this chapter use the combined action of pinholes, focused beams, entangled sources and coincidence detection to provide a very high degree of removal of stray light from regions outside the area of interest. In particular, multiply-scattered light does not contribute to the signal. Thus, the effect of the surrounding material is greatly reduced, leading to less blurring of the image. Coincidence methods also reduce the effect of dark current noise in the detectors, leading to improved signal-to-noise ratios. The main trade-off is that the reliance on parametric down-conversion and coincidence detection reduce the counting rate, leading to longer collection times than is required to form an image by standard confocal microscopy.

In the following sections, several approximations are used in order to keep the analysis simple. Specifically, scalar diffraction theory in the paraxial approximation, and infinitesimal pinholes are assumed. However, these approximations can be inappropriate when used in conjunction with the high numerical aperture lenses often found in confocal microscopes. So, in order to verify that the main points of the chapter are not much altered by the approximations used, a similar analysis has also been carried out with nonparaxial vector diffraction theory and pinholes of finite size taken into account. Using this more exact analysis it is shown that the resolution enhancements found in the simplified version presented in this chapter do indeed survive when the approximations are removed. Because the full vector calculations would make the treatment in the current chapter too cumbersome and would obscure the relatively simple physical ideas involved, this chapter uses a simplified paraxial scalar field treatment. A sample of the more exact approach using vector fields and finite pinholes for the case of the twin-photon microscope is therefore relegated to Appendix E.

### 7.3 The Standard Confocal Microscope

Before discussing correlated microscopy methods, we review here the standard confocal microscope [26, 27], whose development, starting in the late 1950s, revolutionized microscopy. The basic idea is that introduction of small source and detection pinholes leads to improved resolution and contrast, and to the ability to image thin optical sections of a sample in a noninvasive manner. As a result, the confocal microscope has become ubiquitous in modern biomedical optics research, where it is vital not only for imaging but also for dynamic light scattering [28–30], fluorescent correlation spectroscopy [31, 32] and other types of measurements. Because all of these experiments rely on minimizing the overlap of the images of the source and detector pinholes in the sample (i.e. on minimizing the *confocal volume*), a great deal of effort has gone into improving the resolution of confocal microscopes by reducing this volume. One common approach, which exploits the idea of correlated excitation, is two-photon microscopy (see Sect. 7.4), in which a pair of photons must be absorbed by a fluorescent molecule simultaneously; because this happens with appreciable probability only where the photon density is very high, only the central, most intense part of the confocal volume contributes, leading to a reduced *effective* confocal volume.

The basic setup of a standard confocal microscope is shown schematically in Fig. 7.2.<sup>1</sup> The two lenses are identical. In real setups, they are in fact usually the *same* lens, with reflection rather than transmission occurring at the sample.<sup>2</sup> This lens serves as the objective; it has focal length  $f$  and radius  $a$ , and serves to focus the light going in and out of the sample. The sample is represented at point  $\mathbf{y}$  by a function  $t(\mathbf{y})$ ; depending on the setup,  $t(\mathbf{y})$  represents either the transmittance or reflectance of the sample. Distances are chosen so that the imaging condition

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f} \quad (7.2)$$

is satisfied at the first lens; light entering the microscope through the source pinhole is focused to a small diffraction-limited three-dimensional ellipsoid centered at a point  $P$  in the sample, which constitutes the confocal volume. Any stray light not focused to this ellipsoid is blocked by the pinhole, thus providing an improvement in contrast between  $P$  and neighboring points. The distances at the second lens also satisfy the imaging condition, so the second lens performs the inverse of the operation carried out by the first one, mapping the illuminated ellipsoid in the sample back to a point at the detection plane. The pinhole in this plane blocks any light not coming from the immediate vicinity of  $P$ , providing further contrast. Together, the two pinholes serve to pass light from a small in-focus region in the sample and to block light from out-of-focus regions. Scanning the in-focus point over the sample, this results in a significant improvement in contrast over the conventional widefield microscope.

<sup>1</sup>For a more detailed review see [33, 34].

<sup>2</sup>In this chapter, for simplicity, only the transmission case is considered—most of the considerations are equally relevant to the reflection case.

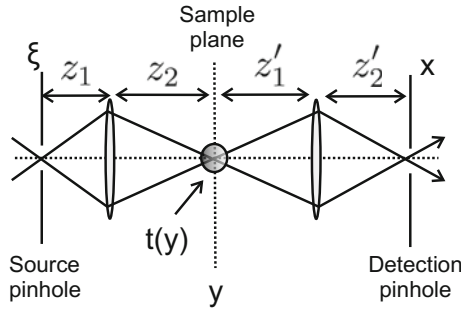


**Fig. 7.2** The standard confocal microscope. Double passage through the lenses and pinholes act to improve both resolution and contrast

A resolution enhancement also occurs from the double passage through the lens. Consider the impulse response function  $h(\mathbf{y})$  and transverse point-spread function (PSF) of the microscope. Let  $h_i(\xi, \mathbf{y})$  ( $i = 1, 2$ ) be the impulse response functions for the first and second lenses individually (including the free space propagation before and after the lens). Up to multiplication by overall constants, these are of the form (see Appendix A)

$$h_i(\xi, \mathbf{y}) = e^{\frac{ik}{2}\left(\frac{y^2}{z_2} + \frac{\xi^2}{z_1}\right)} \tilde{p}\left(k\left(\frac{\mathbf{y}}{z_2} + \frac{\xi}{z_1}\right)\right), \quad (7.3)$$

where  $\tilde{p}(\mathbf{q})$  is the Fourier transform of the aperture function  $p(\mathbf{x}')$  of the lens. The transverse and longitudinal momenta of the incoming photon are denoted by  $\mathbf{q}$  and  $k$ , respectively. Assume that a circular, aberration-free lens of radius  $a$  is used. Applying a pinhole at one end, one then finds that

$$h_i(\mathbf{y}) \equiv h_i(0, \mathbf{y}) = e^{\frac{iky^2}{2z_2}} \tilde{p}\left(\frac{k\mathbf{y}}{z_2}\right), \quad (7.4)$$

where

$$\tilde{p}(\mathbf{q}) = 2\pi a^2 \frac{J_1(aq)}{(aq)}, \quad (7.5)$$

and  $q \ll k$  is the magnitude of  $\mathbf{q}$  and  $J_1$  is the Bessel function of first order.

The amplitude impulse response for the microscope while focused at sample point  $\mathbf{y}$  is

$$h(\mathbf{y}) = \int d^2 y' h_1(\xi, \mathbf{y}') t(\mathbf{y}' + \mathbf{y}) h_2(\mathbf{y}', \mathbf{x}) \Big|_{\mathbf{x}=\xi=0} \quad (7.6)$$

$$= \int d^2 y' h_1(\mathbf{y}') t(\mathbf{y}' + \mathbf{y}) h_2(\mathbf{y}'). \quad (7.7)$$

Upon inserting a sample that is non-transmitting except at a single point  $t(\mathbf{y}) = \delta(\mathbf{y})$ , the impulse response becomes the *coherent spread function* (CSF) (see Appendix A)

$$CSF(\mathbf{y}) = h_1(-\mathbf{y})h_2(-\mathbf{y}), \quad (7.8)$$

with corresponding point-spread function

$$PSF(\mathbf{y}) = |h_1(-\mathbf{y})h_2(-\mathbf{y})|^2. \quad (7.9)$$

Taking the distances at the first and second lenses to be equal for simplicity ( $z'_i = z_i$  for  $i = 1, 2$ ), use of Eq. (7.5) leads to [33, 34]

$$PSF(\mathbf{y}) \propto \left[ \tilde{p}\left(\frac{k\mathbf{y}}{z_2}\right) \right]^4 \propto \left( \frac{J_1\left(\frac{kay}{z_2}\right)}{(kay/z_2)} \right)^4. \quad (7.10)$$

This should be compared to the widefield microscope, which has

$$PSF_{\text{wide}}(\mathbf{y}) \propto \left[ \tilde{p}\left(\frac{k\mathbf{y}}{z_2}\right) \right]^2 \propto \left( \frac{J_1\left(\frac{kay}{z_2}\right)}{(kay/z_2)} \right)^2. \quad (7.11)$$

The higher power in the confocal result leads to a further increase of sharpness and a resulting improvement in resolution. In general, confocal microscopes lead to point spread functions that are 28 % narrower than those of wide-field microscopes.

## 7.4 Two-Photon Excitation Microscopy

As discussed in Chap. 2, the 1950s-era work of Hanbury Brown and Twiss [35–37] in astronomy showed that the use of intensity correlations on the detection side could improve the resolution of radio and optical measurements in astronomy. In recent decades, intensity correlation and coincidence methods have become a staple of quantum optics, leading to a number of developments including quantum ellipsometry [38–40] (Chap. 4), correlated two-photon or ghost imaging [41] (Chap. 3) and aberration and dispersion cancelation [42–48] (Chap. 3). These effects were all originally discovered using entangled photon pairs from parametric down conversion, but many have since been reproduced using beams of light with classical spatial correlations; see for example [49–58]. Standard two-photon microscopy can be viewed as similar in spirit, but with the two-photon correlations enforced at the point of interaction between the photons and the sample, instead of at the detection stage. Because of the requirement of two-photon excitation the noise and background signals are reduced, while the use of infrared photons increases the penetration depth into the material and reduces the phototoxicity compared to standard confocal microscopy.

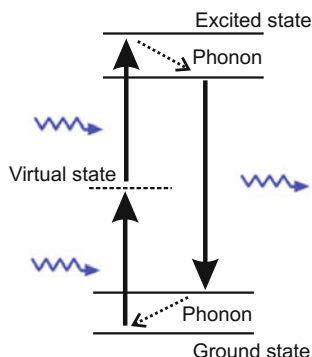


**Fig. 7.3** Schematic of two-photon excitation. Two photons are absorbed, so that the electron passes from the ground state to the excited state via a virtual intermediate state. A higher frequency fluorescent photon is then emitted during the electron's de-excitation, usually accompanied by two phonons from vibrational state transitions (the *dotted arrows*)

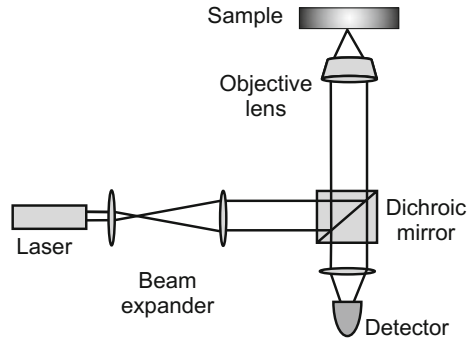
Two-photon fluorescence involves the excitation of a molecular energy level in a fluorescent dye by the near-simultaneous absorption of two low-energy photons, by way of passage through a virtual state intermediate in energy between the ground and excited states (Fig. 7.3). The excited state then emits a photon of higher energy. The concept of two-photon absorption goes back to the theoretical work of Göppert-Mayer [59] in the 1930s, and was observed experimentally in the early 1960s [60, 61]. The successful use of two-photon transitions in microscopy was first carried out in 1990 [62].

Typically, the two excitation photons are infrared, and the resulting fluorescent emission is visible. The absorption probability, which is very small at low intensities, increases roughly quadratically with illumination intensity. With a confocal arrangement, the region in which the intensity is sufficient to produce two-photon excitations with significant probability is a three-dimensional ellipsoid. So depth discrimination of  $\leq 10\ \mu\text{m}$  is possible, with regions before or behind the ellipsoid making essentially no contribution.

The basic arrangement is shown in Fig. 7.4. The initial infrared illumination is focused into a small spot in the sample, which then emits visible light back out. The visible output is separated from the illumination by a dichroic mirror (see Appendix A), which is then imaged onto the surface of a detector. The illuminated confocal volume (typically of femtoliter size) is then scanned over the surface of the sample to produce a two dimensional image. In this way, a series of two-dimensional slices can be formed and stacked together to arrive at a three-dimensional image. The point spread function is very highly localized in the longitudinal direction, allowing thin, high resolution slices to be constructed.

This approach has been of great utility in imaging biological systems. But a chief disadvantage is that the high intensities needed to produce the two-photon transitions can damage the tissue being examined: while often lower than confocal

**Fig. 7.4** Schematic of a standard two-photon microscope. Light is focused onto the sample, so that a small region has sufficient intensity to emit fluorescence from molecules excited by a two-photon transition



microscopy, the intensity is still significant in the vicinity of the focal point. This is mitigated somewhat by using pulsed, rather than continuous wave, illumination. Pulsing reduces the time-averaged intensity, and so decreases the damage; at the same time, it increases the peak illumination intensity, which maximizes the two-photon transition rate.

For more detailed review of two-photon fluorescent microscopy, see [63]. In Sect. 7.6, an entanglement-based variation on this system is described, which can further reduce the required intensities.

## 7.5 Correlations Versus Confocality

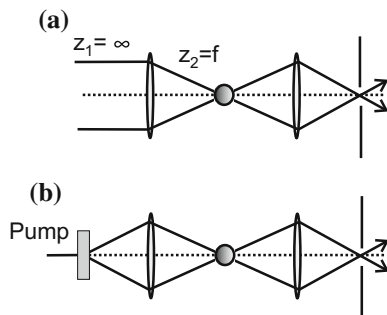
The goal in quantum microscopy is to introduce spatial correlations into the confocal microscope, but this leads immediately to a problem: the source pinhole is an approximate delta function in position space, so that its Fourier transform is constant in momentum space. As a result, regardless of the spectrum of transverse spatial momentum  $\mathbf{q}$  entering the pinhole, the spectrum leaving the pinhole is approximately flat, destroying all transverse momentum correlations. Therefore, it would seem that a choice must be made between losing either the correlations or advantages arising from the source pinhole.

However, this turns out to be a false choice, since the problem can be avoided in several ways. One solution involves removing the pinhole and using some other means to preserve the confocality. The source pinhole ensures that all light entering the microscope is focused to a single point, namely the on-axis point in the object plane, where the imaging condition

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f} \quad (7.12)$$

is satisfied. There are several ways that all entering light can be forced to pass through this point without destroying spatial correlations:

**Fig. 7.5** Two methods for making a confocal microscope without a source pinhole



- (1) One method often used in the standard confocal microscope is to collimate the illumination beam, so that rays strike the first lens parallel to the axis, focusing the beam to a point one focal length  $f$  away (Fig. 7.5a). Correlations are then introduced by arranging for pairs of narrow, well-localized beams of light to strike the lens at equal distances from the axis. This satisfies the imaging condition 7.2 with  $z_1 = \infty$  and  $z_2 = f$ .
- (2) Another method is to use entangled photon pairs produced by spontaneous parametric down conversion. These photons are produced at correlated angles, so that, if traced backward from the sample, they seem to emanate from the same point (Fig. 7.5b) analogously to a pinhole. The down conversion crystal has to be sufficiently far from the lens (or the pump beam narrow enough) for the cross-section of the pump beam to appear to be point-like. This technique is used in the twin-photon microscope discussed below.
- (3) A third method is to have a very narrow beam first reflect from a fixed point on a rotating mirror and then pass through a beam splitter. The resulting two beams then have anti-correlated directions, so both can be traced back to the same illuminated point on the mirror. The mirror, as it rotates, fills the entire lens aperture with light over time.

The latter two methods have been used to enforce spatial correlations in quantum [41] and classical [49] versions, respectively, of correlated-photon or “ghost” imaging (see Chap. 6). In all of these setups, we can think of the arrangement as providing an *effective* or *virtual pinhole*, to replace the physical pinhole.

Another possibility for maintaining correlations takes the light to be initially uncorrelated when it enters the microscope but then arrangements are made to select out pairs of photons which happen to be at the same spatial distance from the axis, effectively enforcing spatial correlation at the detection stage. This is the approach of correlation confocal microscopy, entangled photon microscopy, and twin-photon microscopy approaches, to be described below. A summary of where the correlation arises for several of the correlated-photon approaches described in this chapter is summarized in Table 7.1.

**Table 7.1** The approaches to microscopy in this chapter rely on using pairs of photons that are spatially correlated

| Microscope           | Section | Correlation point |
|----------------------|---------|-------------------|
| Two-photon           | 7.4     | Sample            |
| Entangled photon     | 7.6     | Source/sample     |
| Correlation confocal | 7.7     | Detection         |
| Twin-photon          | 7.8     | Source/detection  |

The correlations can be introduced at one of three places: the illumination source, the interaction point with the sample, or at the detection stage

7.6 Entangled-Photon Fluorescence Microscopy

The first quantum-based microscope to be described here takes the same approach as standard two-photon microscopy, but uses two photons that are entangled with each other. This approach is called *entangled-photon fluorescence microscopy (EPM)* [6] and uses signal-idler pairs from parametric down conversion in order to stimulate fluorescence. The schematic setup is shown in Fig. 7.6. As in standard two-photon microscopy, the two photons are required to stimulate a molecule to a higher energy state, then the fluorescence photons from decay to the ground state are observed. A narrow-band frequency filter blocks any remaining signal or idler photons from reaching the detector. The fluorescence is produced only in a small volume of the sample where the signal and idler volumes both pass simultaneously. Adjusting the positions of the mirrors allows the interaction region to be moved around within the sample.

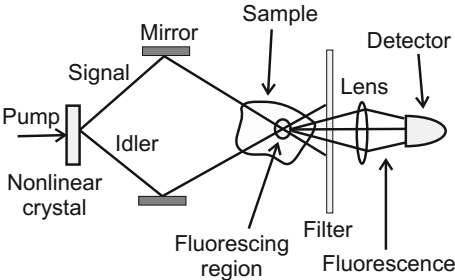


**Fig. 7.6** The entangled-photon fluorescence microscope. Two entangled photons simultaneously strike a molecule, leading to a two-photon excitation. The resulting fluorescence is then detected, while any unused signal and idler photons are removed by the filter

Because the usual two-photon microscope relies on chance for two photons to simultaneously interact with the same atom, it requires high intensities to make the chances of this sufficiently large to produce a measurable fluorescence output. These high intensities can cause damage to the sample, leading to effects like phototoxicity and photobleaching. The principle advantage of the EPM is that high intensities are not needed. The spatial entanglement of the SPDC output is used to guarantee that the two photons both arrive at the same location simultaneously (at least to a high level of approximation), thereby ensuring fluorescence without high illumination intensities. In other words, the optical system is used to transfer spatial correlation from the source to the sample. At the same time, the localization of the fluorescing atom's position can be determined with greater accuracy, due to the small effective confocal volume where the signal and idler cross.

## 7.7 The Correlation Confocal Microscope

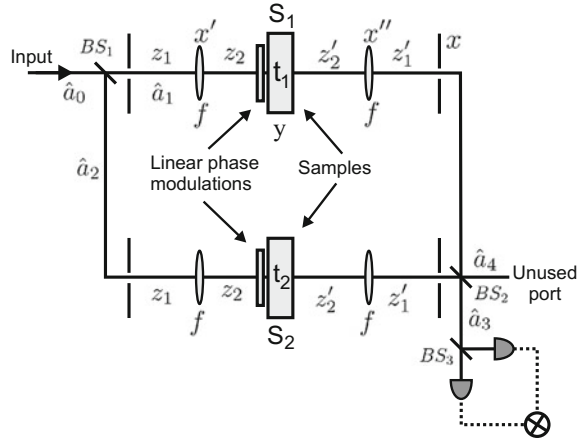
The correlation confocal microscope achieves a narrowed point spread function by means of interference effects. The photons must be tagged in some manner according to the location of their intersection with the sample, selecting the value of some photon property that is modulated in a position-dependant manner in the sample plane. One way to do this is to construct filters with narrow, position-dependent frequency pass bands: broadband illumination then leads to segregation of the light leaving the sample, with photons of different frequency leaving in different spatial regions or in different directions. The frequency of the detected light provides an indicator of position and can be used to enforce spatial correlation.

An apparatus in which the photons are tagged, not by frequency but by phase, is discussed in the next section. The two beams are combined at a beam splitter, converting the phase-tagging into correlation through interference. The arrangement amounts to two confocal microscopes fed into the branches of a Mach-Zehnder interferometer. The interferometer serves to compare the photon phases and to suppress pairs that differ by a large phase. A pair of detectors and a coincidence counter then measure the number of pairs that survive this comparison.

### 7.7.1 “Unfolded” Two-Sample Description

In order to clarify the operating principles of the correlation confocal microscope, the apparatus is here initially described for an “unfolded” version of the microscope in which the apparatus has two branches and two identical objects, one in each arm [64]. A reduction of this apparatus to a more useful version with a single arm and single sample is then described. The unfolded setup is shown schematically in Fig. 7.7. The locations in the transverse planes at the first lens, second lens, and image plane are represented respectively by  $\mathbf{x}'$ ,  $\mathbf{x}''$  and  $\mathbf{x}$ . In the figure, note that  $\mathbf{x}'$  and  $\mathbf{q}$  are related

**Fig. 7.7** The unfolded, two branch version of the correlation confocal microscope, is comprised of two confocal microscopes whose output is combined in a correlated manner



by

$$\frac{\mathbf{q}}{k} = \frac{\mathbf{x}'}{z_2}, \quad (7.13)$$

where  $k$  is the longitudinal component of momentum. Subscripts 1 and 2 on any variable other than  $z$  denote the branch (upper or lower) of the apparatus is being referenced. The quantities  $t_1$  and  $t_2$  represent the transmissivities of the samples  $S_1$  or  $S_2$  in the two arms. In the object plane,  $\mathbf{y}'$  denotes an arbitrary point in the plane, and  $\mathbf{y}$  are the point being viewed in the object. Equivalently,  $-\mathbf{y}$  is the displacement vector of the object during the scan. The arguments  $\mathbf{y}_1$  and  $\mathbf{y}_2$  of the sample functions  $t_1$  and  $t_2$  are partially linked by the spatial correlations imposed. This linkage leads directly to the transverse resolution enhancement.

In front of the sample in each branch, introduce a position-dependent phase shift

$$e^{i\phi_1(\mathbf{y})} = e^{i(c-\mathbf{y}\cdot\mathbf{b})}, \quad e^{i\phi_2(\mathbf{y})} = e^{-i(c-\mathbf{y}\cdot\mathbf{b})}, \quad (7.14)$$

for some constant  $c$  and some vector  $\mathbf{b}$ . Assume that  $\mathbf{b}$  is a radial vector of constant magnitude pointing out from the axis, so that the  $\phi_i(\mathbf{y})$  depend only on radial distance  $y = |\mathbf{y}|$  and are linear in  $y$ . The necessary phase shifts may be produced by a graded index material; for a material .1–1 mm thick, the gradient of index versus radial distance required is of the order of  $\frac{\Delta n}{\Delta r} \sim 10^{-4} - 10^{-3} \text{ mm}^{-1}$ . The refractive index gradient could also be achieved by inserting conical optical wedges in front of each sample, or by a thin film such as those designed for phase shifting masks [65, 66].

Uncorrelated photons are input at left, all in the same state, represented by the creation operator  $\hat{a}_0^\dagger$ . The first beam splitter ( $BS_1$ ) splits the beam between the upper arm (1) or lower arm (2) with equal probabilities. The creation operators in these branches are related to the initial state creation operator by

$$\hat{a}_0^\dagger = \frac{1}{\sqrt{2}} (\hat{a}_1^\dagger - i\hat{a}_2^\dagger). \quad (7.15)$$

The creation operators are multiplied by the impulse response functions for passage through the confocal microscope in each branch. By the methods of Appendix A, the creation operators for the upper ( $j = 1$ ) and lower ( $j = 2$ ) branch just before the second beam splitter are given by

$$\hat{b}_j^\dagger = \hat{a}_j^\dagger \int \tilde{p}^2 \left( \frac{k\mathbf{y}_j}{z_2} \right) t_j(\mathbf{y}_j + \mathbf{y}) e^{i\phi_j(\mathbf{y}_j)} e^{\frac{ik\mathbf{y}_j^2}{2z_2}} d^2y_j \quad (7.16)$$

Tracing the evolution through the system, it is found that, up to an overall constant, the two photons input at  $BS_1$  contributes an amplitude after  $BS_3$  of the form

$$A_3(\mathbf{y}) = \int d^2y' d^2y'' \tilde{p}^2 \left( \frac{k\mathbf{y}'}{z_2} \right) \tilde{p}^2 \left( \frac{k\mathbf{y}''}{z_2} \right) t(\mathbf{y}' + \mathbf{y}) t(\mathbf{y}'' + \mathbf{y}) e^{\frac{ik(\mathbf{y}'^2 + \mathbf{y}''^2)}{2z_2}} \times (\cos(\phi(\mathbf{y}') + \phi(\mathbf{y}'')) - \cos(\phi(\mathbf{y}') - \phi(\mathbf{y}'')) - i \sin(\phi(\mathbf{y}') - \phi(\mathbf{y}''))), \quad (7.17)$$

where it has now been assumed that  $\phi_1 = -\phi_2 \equiv \phi$ , and that the two objects or samples are identical ( $t_1 = t_2 \equiv t$ ). The expression in the square brackets ranges from  $-1$  when  $\phi(\mathbf{y}') = \phi(\mathbf{y}'') = \frac{\pi}{4}$  to zero at  $\phi(\mathbf{y}') = \phi(\mathbf{y}'') = 0$ . So by arranging for  $\phi(\mathbf{y})$  to drop from  $\frac{\pi}{4}$  at the axis to zero at the edge of the Airy disk, then values of  $\mathbf{y}'$  and  $\mathbf{y}''$  that fall inside the Airy disk but away from the axis are strongly suppressed, leading to resolution enhancement. In this case the constants defined in Eq. 7.14 are

$$c = \frac{\pi}{4} \quad \text{and} \quad \mathbf{b} = \frac{\pi}{4R_{\text{airy}}} \hat{r}. \quad (7.18)$$

When two photons simultaneously emerge from  $BS_2$  into arm 3, the final beam splitter  $BS_3$  routes them to two different detectors, so that a coincidence count may be measured. (50 % of the pairs are lost to the coincidence count, when both photons exit the beam splitter in the same direction.) Up to overall constants, the coincidence rate is

$$R_c(\mathbf{y}) = |A_3(\mathbf{y})|^2, \quad (7.19)$$

where  $A_3$  is given by equation (7.17). Since the coincidence rate is proportional to the *square* of the intensity, the appropriate quantity to compare with the usual intensity images of other microscopes is  $\sqrt{R_c} \sim |t(\mathbf{y})|^2$ .

The point spread function is of the form:

$$PSF(\mathbf{y}) \sim \sqrt{R_c} = [1 - \cos(2\phi(\mathbf{y}))] \tilde{p}^4 \left( \frac{k\mathbf{y}}{z_2} \right) \quad (7.20)$$

$$= \sin^2(\phi(\mathbf{y})) \tilde{p}^4 \left( \frac{k\mathbf{y}}{z_2} \right), \quad (7.21)$$

where  $y$  is the magnitude of  $\mathbf{y}$ . The factor  $\sin^2(\phi(y))$  is responsible for the improved resolution compared to the standard confocal microscope. The suppression of photon pairs with values of  $\mathbf{y}'$  and  $\mathbf{y}''$  that differ significantly from each other, when combined with the factors of  $\tilde{p}$ , reduces the contribution to the coincidence rate of points in the outer part of the Airy disk.

The improved resolution comes about because the factors involving the trigonometric functions in Eq. (7.17) provide a coupling between the integration variables  $\mathbf{y}'$  and  $\mathbf{y}''$ . This amounts to a spatial correlation between the photons detected from the two branches. Without this coupling, Eq. (7.17) would factor into two independent integrals, each of the form of a standard confocal microscope amplitude. Hence, without the correlation, nothing is gained that could not be obtained from simply using a standard confocal microscope and squaring the output intensity.

### 7.7.2 “Folded” Description: Reduction to One Sample

The unfolded setup in the previous section was described for pedagogical purposes, but is unrealistic, because it requires two identical copies of the sample in order to function. Here, the more realistic case is considered, showing how the apparatus can be altered without affecting the basic operating principles so that only a single copy of the sample is needed. The idea is to partially collapse the two branches of the previous implementation into a single branch, so that both beams go through the same sample. We assume here for simplicity that the transmission function  $t(\mathbf{y})$  is real.

As shown in Fig. 7.8, all the light now passes through a single sample, gaining a phase shift  $e^{+i\phi(\mathbf{y})}$ . After passing through a detection pinhole, it is split so that half the beam reaches a second beam splitter  $BS_2$  unaltered, while half is deflected to a phase conjugating mirror (see Appendix A). The mirror reverses the sign of the phase

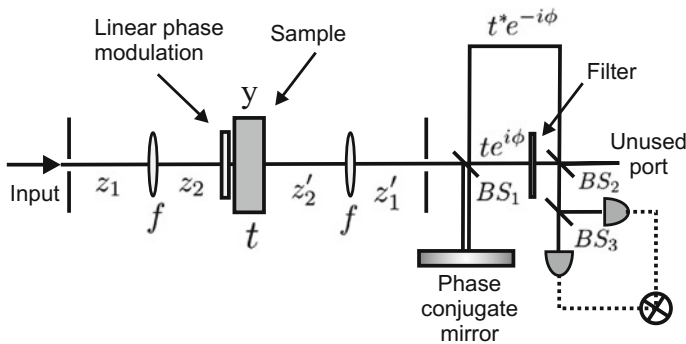


**Fig. 7.8** “Folded” version of the correlation confocal microscope, with a single branch and a single sample

before deflecting the beam back to  $BS_1$ . Because of the reality of  $t(\mathbf{y})$ , it is unaffected by the phase conjugation. The beams then recombine. From this point on, everything proceeds as in the unfolded version of the previous subsection. A neutral-density filter is placed between  $BS_1$  and  $BS_2$ , in order to equalize the intensity in the upper and lower branches.

By means of interfering the two paths through the microscope, the point spread function becomes substantially narrowed, leading to smaller confocal volume and improved resolution. However, there is a price to be paid for this. The data collection rate is slow due to the lost photons in the unused port, the low rate of down conversion, and the elimination of photons by the phase-correlation requirement. Also, it was assumed that the phase shifts at the sample vary linearly with radial distance; nonlinear radial variations can instead be used in order to provide further suppression of photon pairs falling far from the center of the axis, leading to further resolution improvements at the cost of further reducing the detection rate.

Although this setup allows improved resolution in imaging, there is no improvement over the standard confocal microscope for applications like fluorescence correlation spectroscopy (FCS) and dynamic light scattering (DLS). This is because the random phase shifts from the scattering or fluorescence events destroy the imposed phase correlations.

In the next section, another correlation-based microscope is described, in which the effective confocal volume is reduced by a different means.

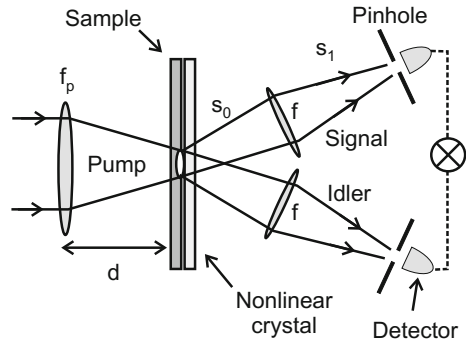
## 7.8 Twin-Photon Confocal Microscopy

The twin-photon confocal microscope achieves a narrowing of the point spread function and reduction of the confocal volume by restricting the detected photons to the overlap of *three* focal regions: those of the signal, idler and pump. The pump needs to be focused by a lens before the down conversion; this is in contrast to the other applications of down conversion in this book, where the pump beam is usually assumed to be an approximate plane wave over the region of interest.

### 7.8.1 Multiple Photons in Confocal Microscopy

Recall from Sect. 7.3 that confocal microscopy derives its benefits from restricting detection to the overlap of two in-focus regions. The goal of the twin-photon microscope is to go one step beyond confocal microscopy by requiring the overlap of two detection regions (the inverse images of the pinholes) and one focused illumination region (see Fig. 7.9) [67]. Only points in the overlap of all three regions are visible. If the third region is comparable to the size of the Airy disk of the objective, the overall area that can be resolved become smaller due to the combined drop-off of the detection and illumination probabilities. The illumination of the sample is assumed

**Fig. 7.9** Setup for the twin photon microscope. The signal and idler from SPDC are each passes through a pinhole before coincidence detection. The signal and idler portions act as two halves of a confocal microscope



to be provided by spontaneous parametric down conversion (SPDC). The two outgoing photons (signal and idler) each pass through an objective lens of focal length  $f$ , and then through pinholes to photon-counting detectors, with the signal and idler branches acting in a manner analogous to the two halves (before the sample and after the sample) of a standard confocal microscope. The detectors are connected in coincidence, with detection events recorded only when both of the photons survive the corresponding pinhole. This already is enough to produce the characteristic  $\tilde{p}^4$  behavior of a standard confocal microscope.

The pump is an approximate plane wave over an area of size much larger than the Airy disk of a microscope lens. However, the pump beam can be strongly focused to an approximate point-like form by means of a lens of focal length  $f_p$ . The effect of weak focusing of the pump beam in SPDC was studied in [68] where it was shown that interesting geometrical optics effects could be obtained with the pump beam acting like a spherical mirror and the signal and idler acting like incident and reflected beams from the mirror. The pump focusing effectively adds a third pinhole, so that the effective overlap region of the signal, idler and pump can be made sufficiently small to significantly enhance the transverse and longitudinal resolutions, relative to the standard confocal microscope. Further, there is an additional transverse resolution enhancement from the entanglement of the signal and idler, by the same mechanism as in quantum lithography (see Chap. 4). The result is that visible-light images can be produced with resolution that normally would be possible only in the ultraviolet. Although the twin-photon microscope makes intrinsic use of three photons (signal, idler and pump), the constraints introduced by energy and momentum conservation in SPDC give full knowledge of the properties of all three photons from the measurement of any two.

Assuming that the pump is focused to a small region, imagine scanning this illuminated region over the sample. During the scan only the illuminated region would undergo down conversion. In addition, the detection pinholes allow the detectors to see only two small regions. The region visible to the coincidence detection system is therefore the overlap of all three of these small volumes. The coincidence rate should therefore drop rapidly when moving away from the center of overlap in any direction, producing a very narrow point spread function due to the ability to tightly localize

the origin point of the two detected photons. The standard confocal microscope has improved localization ability over the widefield microscope due to the convolution of one illumination branch and one detection branch; the twin-photon microscope thus goes further, achieving additional localization via the convolution of one illumination branch with the product of *two* detection branches.

In contrast to standard two-photon microscopy, the twin-photon microscope can operate with a continuous wave laser source, rather than requiring pulsed lasers. In addition, parametric down conversion provides an entangled-photon source which guarantees that the two photons in a pair are always produced and detected simultaneously; in contrast, standard two-photon microscopy relies on the chance occurrence of two photons arriving simultaneously at the object of interest, for example, a molecule. As a result of these facts and of the improved signal-to-noise ratio induced by the coincidence detection, the twin-photon microscope can operate at lower illumination intensities than the standard two-photon microscope, leading to lower power requirements and less damage to the sample.

### 7.8.2 The Coincidence Rate and Point Spread Function

The qualitative arguments above can be verified by direct calculations. Consider the setup of Fig. 7.9 [67]. The pump beam of waist radius  $w_0$  and frequency  $\omega_p = \frac{2\pi c}{\lambda_p}$  is focused by a lens of focal length  $f_p$  to a small region at the face of a  $\chi^{(2)}$  nonlinear crystal, where a thin sample is placed having transmittance  $t(\mathbf{y})$ , where  $\mathbf{y}$  is the position in the plane transverse to the propagation direction ( $z$ ). Spontaneous parametric down conversion occurs inside the crystal, producing two beams, the signal (ordinary ray) and idler (extraordinary ray), of respective frequencies  $\omega_o$  and  $\omega_e$ .

With no sample in place, the pump field inside the crystal would be given by [68]

$$E_p(\mathbf{r}_\perp, z, t) = \int d^2k_\perp e^{-i\omega_p t} e^{-i(k_p z + \mathbf{k}_\perp \cdot \mathbf{r}_\perp)} \tilde{E}_p(\mathbf{k}_\perp). \quad (7.22)$$

$k_p$  and  $\mathbf{k}_\perp$  are the pump momentum in the longitudinal and transverse directions, and  $\mathbf{r}_\perp$  is the position within the pump beam in the transverse direction.  $\tilde{E}_p$  is the Fourier transform of the pump field inside the crystal, which is given by [68]

$$\tilde{E}_p(\mathbf{k}_\perp) = e^{-ik_\perp^2 \sigma_p^2 / 2}, \quad (7.23)$$

with

$$\sigma_p^2 \approx \frac{c}{\omega_p} \left[ d - f_p - i \left( \frac{\lambda_p}{\pi w_0^2} \right) f_p^2 \right]. \quad (7.24)$$

The pump field inside the crystal reduces approximately to [68]:

$$E_p(\mathbf{r}_\perp, z, t) = E_p'' e^{-i(\omega_p t - k_p z)} e^{i r_\perp^2 / 2\sigma_p^2}, \quad (7.25)$$

where all overall constants have been combined into  $E_p''$ , and  $k_p$  is the pump wavenumber. The sample multiplies this at each point by  $t(\mathbf{r}_\perp)$ .

The two outgoing beams each pass through half of a confocal microscope arrangement: after propagating distance  $s_0$ , each beam passes through the objective lens, then propagates another distance  $s_1$  to reach a pinhole. Distances  $s_0$  and  $s_1$  satisfy the imaging condition

$$\frac{1}{s_0} + \frac{1}{s_1} = \frac{1}{f}. \quad (7.26)$$

If the positive frequency parts of the outgoing fields with transverse momenta  $k'_j$  ( $j = e, o$ ) are

$$\hat{E}_j^{(+)}(\mathbf{r}_{j\perp}, t) = E_j(\mathbf{r}_{j\perp}, t) \hat{a}_{k'_j}, \quad (7.27)$$

then the resulting field detected at longitudinal distance  $s_0 + s_1$  away due to field  $j$  at transverse distance  $\mathbf{r}_{j\perp}$  in the crystal is given by

$$\hat{E}_j^{(+),D}(\mathbf{r}_{j\perp}, t_j) = \int d^3 k'_j h_{k'_j}(\mathbf{r}_{j\perp}, t_j) \hat{E}_j^{(+)}(\mathbf{r}_{j\perp}, t_j), \quad (7.28)$$

where  $h_{k'_j}$  is the amplitude for a ray launched from  $\mathbf{r}_\perp$  with momentum  $k'_j$  to propagate through the lenses, survive the pinhole and reach the detector. For the confocal system shown, this propagation factor is given up to overall normalization by

$$h_{k'_j}(\xi) = e^{-i(k/s_0)\xi^2} e^{-i\mathbf{k}_\perp \cdot \xi} p\left(\xi + \frac{s_0}{k} \mathbf{k}_\perp\right), \quad (7.29)$$

where  $p(\mathbf{x})$  is the aperture function. The contributions to the fields at the detectors due to the down conversion fields at  $\mathbf{r}_{j\perp}$  are

$$E_j^{(+),D}(\mathbf{r}_{j\perp}, t_j) = \int d^3 k'_j e^{-i(k'_j/s_0)r_{j\perp}^2} e^{-i\mathbf{k}'_{j\perp} \cdot \mathbf{r}_{j\perp}} e^{-i\omega_j T_j} p\left(\mathbf{r}_{j\perp} + \frac{s_0}{k'_j} \mathbf{k}'_{j\perp}\right) E_j \hat{a}_{k'_j}, \quad (7.30)$$

where  $T_l = t_j - \frac{s_0 + s_1}{c}$ , with coincidence amplitude

$$A(T_1, T_2) = \int d^2 r_\perp \langle 0 | E_e^{(+),D}(\mathbf{r}_\perp) E_o^{(+),D}(\mathbf{r}_\perp) | \psi(\mathbf{r}_\perp, \mathbf{y}) \rangle. \quad (7.31)$$

Assuming parametric downconversion from a thin crystal, the coincidence amplitude becomes (see [67] for details)

$$A(T_1, T_2, \mathbf{y}) = C_1 \Pi(T_{12}) \int d^2 k_{e\perp} d^2 k_{o\perp} d^2 r_{\perp} e^{-2i(\mathbf{k}_{e\perp} + \mathbf{k}_{o\perp}) \cdot \mathbf{r}_{\perp}} \quad (7.32)$$

$$\times e^{-ir_{\perp}^2 \left( \frac{\omega_p}{c s_0} - 1/2\sigma_p^2 \right)} t(\mathbf{r}_{\perp} + \mathbf{y}) p(\mathbf{r}_{\perp} + \frac{s_0}{k_e} \mathbf{k}_{e\perp}) p(\mathbf{r}_{\perp} + \frac{s_0}{k_o} \mathbf{k}_{o\perp}),$$

where  $\omega_p = \Omega_o + \Omega_e$ . In this expression,  $C_1$  is a constant,  $T_{12} = T_1 - T_2$  is the difference between detection times in the two detectors, and

$$D = \frac{1}{u_o} - \frac{1}{u_e} \quad (7.33)$$

is the difference between inverse group velocities of the ordinary and extraordinary waves.  $\Pi(T_{12})$  is the unit step function, only nonzero for  $0 < T_{12} < DL$ . This step function arises from the integrations over frequency and longitudinal origin position for the signal-idler pairs in the crystal,

$$\Pi(T_{12}) = \int_{-\infty}^{\infty} d\nu \int_0^L dz e^{i\nu(T_{12} - Dz)}, \quad (7.34)$$

and reflects the fact that both photons must be created simultaneously within the crystal and cannot be separated from each other due to dispersion by a time longer than the time needed to cross the crystal: so there can be no coincidence signal if the time difference is longer than this.

Assuming  $d \approx f_p$ , define parameter  $r_0$ , define

$$\sigma_p^2 \approx -i \left( \frac{c \lambda_p f_p^2}{\pi \omega_p w_0^2} \right) \equiv -ir_0^2, \quad (7.35)$$

and then carry out the  $\mathbf{k}_{\perp}$  integrations. The result is an amplitude of the form

$$A(T_1, T_2, y) = C_2 \Pi(T_{12}) \int d^2 r_{\perp} e^{-r_{\perp}^2/2\eta_0^2} t(\mathbf{r}_{\perp} + \mathbf{y}) \tilde{p} \left( \frac{2\omega_o}{s_0 c} r_{\perp} \right) \tilde{p} \left( \frac{2\omega_e}{s_0 c} r_{\perp} \right), \quad (7.36)$$

with

$$\frac{1}{\eta_0^2} \equiv \frac{1}{r_0^2} - \frac{2i\omega_p}{s_0 c}. \quad (7.37)$$

The lateral (transverse) PSF,

$$\text{PSF}(\mathbf{y}) = \left| \tilde{p} \left( -\frac{2\Omega_o}{s_0 c} y \right) \tilde{p} \left( -\frac{2\Omega_e}{s_0 c} y \right) \right|^2 e^{-y^2/r_0^2} \quad (7.38)$$

is significantly narrowed relative to the standard confocal microscope, due to both the exponential factor  $e^{-y^2/r_0^2}$  and the factors of 2 inside each  $\tilde{p}$ . The exponential factor results from the focusing by the pump lens, while the factors of 2 arise in the

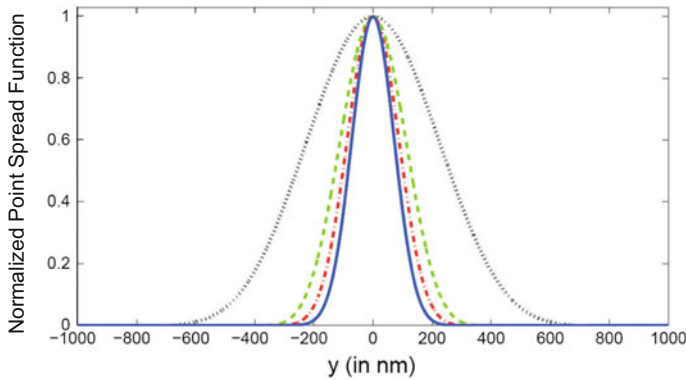


**Fig. 7.10** Comparison of point spread functions for the standard confocal microscope (the outermost, *dotted curve*) and the twin-photon confocal microscope with pump radii of 1 mm (*dashed*), 8 mm (*dash-dot*), and 12 mm (*solid*). The curves are plotted for a 351 nm pump and 2 cm focal length lenses of numerical aperture  $NA = 1/\sqrt{2} = 0.707$  (Figure reproduced from [67].)

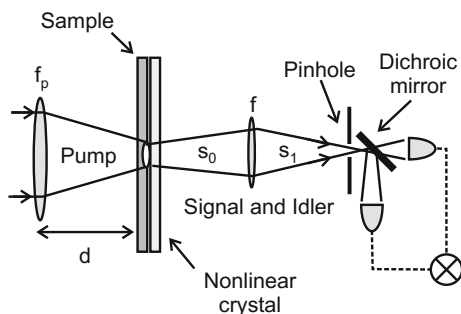
following way. Both signal and idler exhibit phase shifts of the form  $e^{-i\mathbf{k}_\perp \cdot \mathbf{r}_\perp}$ . The pump photon thus contributes an additional phase factor equal in size to the sum of the phases gained by the signal and idler. So the phase is doubled, and the argument of the Fourier transformed pupil function is also doubled. Integrating over  $\mathbf{r}_\perp$  then gives the Fourier transform.

In the degenerate case,  $\Omega_o = \Omega_e = \omega_p/2$ , the meaning of the factors of 2 becomes clear. The resolution is determined by the properties of the pump, which has twice the frequency of the signal and idler. For an ultraviolet pump, with visible signal and idler, the result is that visible-light images can be formed with UV-level resolution, but without the sample damage that may occur with UV illumination.

Figure 7.10 compares the calculated PSF of the twin-photon microscope to that of the standard confocal microscope for several values of  $w_0$ . The PSF decreases significantly in width compared to the standard confocal microscope; the curves shown are narrower by 50, 61 and 68 % respectively, for pump radii of 1 mm, 8 mm, and 12 mm. For the extreme case where the beam completely fills the focusing lens (not shown on the plot), the maximum resolution improvement over the standard confocal microscope is 77.3 %. For small beam radius, the PSF becomes roughly constant as  $w_0$  decreases below a certain point.

Because it is essential that a large angular range of light be collected by the lenses, the down converted photons must be arranged to emerge from the crystal with a large angular spread. In noncollinear down conversion, a large angular spread means a large signal and idler frequency bandwidth. In the twin-photon microscope, the angular spread is significant, because the spread in frequency is inversely proportional to the thickness of the crystal, which we are assuming to be thin. Further, the tight focusing of the pump beam causes an additional increase in this spread, since narrowing the pump in position space broadens the spread of transverse momentum vectors for both the ingoing and outgoing beams. In this sense, focusing of the pump has an effect

**Fig. 7.11** Alternate version of twin photon microscope with single lens. The beam splitter after the pinhole separates the twin photons for coincidence detection



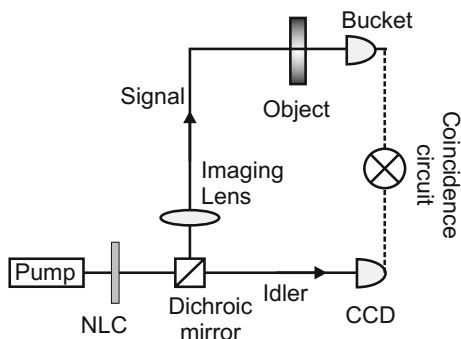
similar to that of placing a pinhole in the beam, diffracting incident plane waves to large angles.

The apparatus can be simplified as shown in Fig. 7.11. Both outgoing beams are sent through the *same* lens and separated after the pinhole by a dichroic mirror (or by a polarizing beam splitter in the case of type II down conversion). This reduces the number of lenses required, making alignment significantly easier.

## 7.9 Two-Frequency Quantum Microscope

Two-color ghost imaging [69–72] has also been shown to enhance image resolution. In [73], this has been adapted to produce another type of high contrast quantum microscopy. This approach has been proposed for use in biological samples, where it would avoid tissue damage, since it avoids the need for high intensities or high (ultraviolet) frequencies. The setup is shown schematically in Fig. 7.12. A ghost imaging-type arrangement is used with a nonlinear crystal (NLC) feeding signal-idler pairs into the apparatus. A coincidence measurement is made between a bucket detector and a spatially resolving CCD camera. The coincidence rate as a function

**Fig. 7.12** Two-color quantum microscope. The resolution is enhanced by the ratio of the signal and idler frequencies



of position in the CCD can then be displayed on a screen to produce an image. A dichroic mirror is used to send long wavelength ( $\lambda_s$ ) signal photons into the upper branch and short wavelength idlers ( $\lambda_i$ ) into the lower branch. The resolution is then enhanced by a factor of  $\frac{\lambda_s}{\lambda_i}$ , via a similar mechanism to that which leads to resolution enhancement in the two-crystal imaging experiment of Sect. 2.7.

## 7.10 Related Methods

There are other methods appearing in the literature under the name of “quantum microscopy” that differ from those presented in this chapter.

After the proposal [7] that entanglement could be used to beat the diffraction limit, the first detailed proposals to use entangled photons and coincidence counting in microscopy appeared in [8]. In fact, two methods were actually proposed. The first was a form of spectral microscopy, where path- and frequency-entangled photons from the cascade decay of two atoms produced enhancement of energy level measurements, with resolution lower than the limit imposed by the natural line widths of the atomic levels. The second proposal was spatial microscopy, with path-entangled photons emitted by cascades from an array of atoms, to produce sub-wavelength resolution, greatly enhanced compared to a similar classical grating.

In reference [74], the idea of *quantum differential interference contrast microscopy* was proposed. Classical differential interference microscopy is a well-known method [34] for imaging through biological samples and other opaque materials. In the classical approach, the signal to noise ratio is limited by the standard quantum limit (Chap. 4). The quantum version uses entangled-photon illumination in a confocal arrangement to suppress quantum noise, thereby surpassing the standard quantum limit. The result is a signal-to-noise enhancement of  $\sqrt{N}$ , where  $N$  is the number of photons entangled. To date, the principle has been demonstrated experimentally for  $N = 2$  [74].

A further approach that has been referred to as quantum microscopy is the idea of using squeezed states, instead of entanglement, to enhance the signal-to-noise ratio, thereby allowing enhanced particle-tracking ability. These methods [75–77] can be used to produce optical tweezers or to study molecular dynamics with enhanced resolution due to the suppression of quantum noise and are well-suited to applications in biological systems.

## References

1. T.M. Cover, J.A. Thomas, *Elements of Information Theory* (Wiley, New York, 1999)
2. H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1946)
3. C.R. Rao, Bull. Calcutta Math. Soc. **37**, 81 (1945)
4. W. Denk, J. Strickler, W. Webb, Science **248**(4951), 73 (1990)
5. W.R. Zipfel, R.M. Williams, W.W. Webb, Nat. Biotechnol. **21**, 1369 (2003)

6. M.C. Teich, B.E. Saleh, *Česk. Čas. Fyz.* **47**, 3 (1997)
7. A.N. Boto, P. Kok, D.S. Abrams, S.L. Braunstein, C.P. Williams, J.P. Dowling, *Phys. Rev. Lett.* **85**, 2733 (2000)
8. A. Muthukrishnan, M.O. Scully, M.S. Zubairy, *J. Opt. B: Quant. Semiclass. Opt.* **6**, S575 (2004)
9. E. Hecht, *Optics*, 4th edn. (Addison-Wesley, San Francisco, 2002)
10. W. Lukosz, *J. Opt. Soc. Am.* **56**, 1463 (1966)
11. R. Heintzmann, M.G.L. Gustafsson, *Nat. Photonics* **3**, 362 (2009)
12. Y. Aharonov, J. Anandan, S. Popescu, L. Vaidman, *Phys. Rev. Lett.* **64**, 2965 (1990)
13. M.V. Berry, in *Proceedings of the International Conference on Fundamental Aspects of Quantum Theory, Columbia, SC, USA, 10–12 Dec. 1992*, eds. by J.S. Anandan, J.L. Safko (World Scientific, Singapore, 1995)
14. G. Torraldo di Francia, *Nuovo Cimento Suppl.* **9**, 426 (1952)
15. M.S. Calder, A. Kempf, *J. Math. Phys.* **46**, 012101 (2005)
16. A. Kempf, *J. Math. Phys.* **41**, 2360 (2000)
17. E.T.F. Rogers, N.I. Zheludev, *J. Opt.* **15**, 094008 (2013)
18. P. Török, T. Wilson, *Opt. Commun.* **137**, 127 (1997)
19. T. Wilson, R. Juškaitis, P. Higdon, *Opt. Commun.* **141**, 298 (1997)
20. P. Török, P.D. Higdon, T. Wilson, *Opt. Commun.* **148**, 300 (1998)
21. P. Török, P.D. Higdon, T. Wilson, *J. Mod. Opt.* **45**, 1681 (1998)
22. C.J.R. Sheppard, J.F. Aguilar, *Opt. Commun.* **180**, 1 (2000)
23. M. Tdsang, *Optica* **2**, 646 (2015)
24. C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, London, 1976)
25. A.S. Holevo, *Statistical Structure of Quantum Theory* (Springer, Berlin, 2001)
26. M. Minsky, U.S. Patent # 3013467, *Microscopy Apparatus* (1957)
27. M. Minsky, *Scanning* **10**, 128 (1988)
28. R. Pecora, *Dynamic Light Scattering: Applications of Photon Correlation Spectroscopy* (Plenum Press, NY, 1985)
29. K.S. Schmitz, *An Introduction to Dynamic Light Scattering By Macromolecules* (Academic Press, London, 1990)
30. B.J. Berne, R.J. Pecora, *Dynamic Light Scattering with Applications to Chemistry, Biology, and Physics* (Dover, Mineola, 2000); Reprint of 1976 Wiley edition
31. D. Magde, E. Elson, W.W. Webb, *Phys. Rev. Lett.* **29**, 705 (1972)
32. W.W. Webb, *Appl. Opt.* **40**, 3969 (2001)
33. R.H. Webb, *Rep. Prog. Phys.* **59**, 427 (1996)
34. J.C. Mertz, *Introduction to Optical Microscopy* (Roberts and Company, Greenwood Village, 2009)
35. R. Hanbury Brown, R.Q. Twiss, *Nature* **178**, 104 (1956)
36. R. Hanbury Brown, R.Q. Twiss, *Proc. R. Soc. Lond. A* **242**, 300 (1957)
37. R. Hanbury Brown, R.Q. Twiss, *Proc. R. Soc. Lond. A* **243**, 291 (1958)
38. A.F. Abouraddy, K.C. Toussaint, A.V. Sergienko, B.E.A. Saleh, M.C. Teich, *Opt. Lett.* **26**, 1717 (2001)
39. A.F. Abouraddy, K.C. Toussaint, A.V. Sergienko, B.E.A. Saleh, M.C. Teich, *J. Opt. Soc. Am. B* **19**, 656 (2002)
40. K.C. Toussaint, G. Di Giuseppe, K.J. Bycenski, A.V. Sergienko, B.E.A. Saleh, M.C. Teich, *Phys. Rev. A* **70**, 023801 (2004)
41. T.B. Pittman, D.V. Strekalov, M.H. Migdall, M.H. Rubin, A.V. Sergienko, Y.H. Shih, *Phys. Rev. Lett.* **77**, 1917 (1996)
42. J.D. Franson, *Phys. Rev. A* **45**, 3126 (1992)
43. A.M. Steinberg, P.G. Kwiat, R.Y. Chiao, *Phys. Rev. Lett.* **68**, 2421 (1992)
44. A.M. Steinberg, P.G. Kwiat, R.Y. Chiao, *Phys. Rev. Lett.* **71**, 708 (1993)
45. O. Minaeva, C. Bonato, B.E.A. Saleh, D.S. Simon, A.V. Sergienko, *Phys. Rev. Lett.* **102**, 100504 (2009)
46. C. Bonato, A.V. Sergienko, B.E.A. Saleh, S. Bonora, P. Villoresi, *Phys. Rev. Lett.* **101**, 233603 (2008)

47. C. Bonato, D.S. Simon, P. Villoresi, A.V. Sergienko, Phys. Rev. A **79**, 062304 (2009)
48. D.S. Simon, A.V. Sergienko, Phys. Rev. A **80**, 053813 (2009)
49. R.S. Bennink, S.J. Bentley, R.W. Boyd, Phys. Rev. Lett. **89**, 113601 (2002)
50. R.S. Bennink, J. Bentley, R.W. Boyd, J.C. Howell, Phys. Rev. Lett. **92**, 033601 (2004)
51. A. Gatti, E. Brambilla, M. Bache, L.A. Lugiato, Phys. Rev. A **70**, 013802 (2004)
52. A. Gatti, E. Brambilla, M. Bache, L.A. Lugiato, Phys. Rev. Lett. **93**, 093602 (2004)
53. Y.J. Cai, S.Y. Zhu, Phys. Rev. A **71**, 056607 (2005)
54. Y.J. Cai, S.Y. Zhu, Opt. Lett. **29**, 2716 (2004)
55. G. Scarcelli, V. Berardi, Y.H. Shih, Phys. Rev. Lett. **96**, 063602 (2006)
56. A. Valencia, G. Scarcelli, M. D'Angelo, Y.H. Shih, Phys. Rev. Lett. **94**, 063601 (2005)
57. F. Ferri, D. Magatti, A. Gatti, M. Bache, E. Brambilla, L.A. Lugiato, Phys. Rev. Lett. **94**, 183602 (2005)
58. D. Zhang, Y.H. Zhai, L.A. Wu, X.H. Chen, Opt. Lett. **30**, 2354 (2005)
59. M. Göppert-Mayer, Ann. Phys. **5**, 273 (1931)
60. P.A. Franken, A.E. Hill, C.W. Peters, G. Weinreich, Phys. Rev. Lett. **7**, 118 (1961)
61. W. Kaiser, C.G.B. Garrett, Phys. Rev. Lett. **7**, 229 (1961)
62. W. Denk, J.H. Strickler, W.W. Webb, Science **248**, 73 (1990)
63. P.T.C. So, C.Y. Dong, B.R. Masters, K.M. Berland, Ann. Rev. Biomed. Eng. **2**, 399 (2000)
64. D.S. Simon, A.V. Sergienko, Opt. Expr. **18**, 9765 (2010)
65. P.F. Carcia, R.H. French, M.H. Reilly, M.F. Lemon, D.J. Jones, Appl. Phys. Lett. **70**, 2371 (1997)
66. P.F. Carcia, G. Hughes, R.H. French, C. Torardi, G. Reynolds, L. Dieu, Vac. Thin Film **1** (1999)
67. D.S. Simon, A.V. Sergienko, Opt. Exp. **21**, 22147 (2010)
68. T.B. Pittman, D.V. Strekalov, D.N. Klyshko, M.H. Rubin, A.V. Sergienko, Y.H. Shih, Phys. Rev. A **53**, 2804 (1996)
69. S. Karmakar, Y. Shih, Phys. Rev. A **81**, 033845 (2010)
70. S. Karmakar, Y. Shih, Proc. SPIE **7702**, 770204 (2010)
71. M.H. Rubin, Y. Shih, Phys. Rev. A **78**, 033836 (2008)
72. K.W.C. Chan, M.N. OSullivan, R.W. Boyd, Phys. Rev. A **79**, 033808 (2009)
73. S. Karmakar, R.E. Meyers, Y. Shih, J. Biomed. Opt. **20**, 016008 (2015)
74. T. Ono, R. Okamoto, S. Takeuchi, Nat. Commun. **4**, 2426 (2013)
75. M.A. Taylor, J. Janousek, V. Daria, J. Knittel, B. Hage, H.-A. Bachor, W.P. Bowen, Nat. Photonics **7**, 229 (2013)
76. M.A. Taylor, W.P. Bowen, [arXiv:1409.0950](https://arxiv.org/abs/1409.0950) (2014)
77. M.A. Taylor, *Quantum Microscopy of Biological Systems* (Springer, Berlin, 2015)

## Chapter 8

# Correlated and Entangled Orbital Angular Momentum

In this chapter, the study of optical orbital angular momentum (OAM) is introduced, a topic whose literature has grown rapidly since the early 1990s, as a result of both the intrinsic scientific interest of the subject and of the potential applications in areas ranging from high capacity quantum cryptography to the creation of optical spanners for the manipulation of nanostructures. It is be shown that the use of correlated photon methods leads to improved results and new applications in this area as well.

### 8.1 Orbital Angular Momentum in Optics

It was realized early on in the history of quantum mechanics that circularly polarized photons carry angular momentum, quantized in multiples of  $\hbar$ , and that this angular momentum can be transferred to matter [1–4]. This polarization-related angular momentum came to be recognized as due to the photon's intrinsic spin: left- and right-circularly polarized photons have spin quantum number  $s_z = \pm 1$  along the propagation axis, while linearly polarized photons are equal superpositions of the two spin states, with average spin  $\langle s_z \rangle = 0$ .

It took much longer to appreciate that, in addition to the intrinsic spin angular momentum  $\hat{S}$ , an individual photon can also carry orbital angular momentum (OAM)  $\hat{L}$  about its propagation axis [5]. This OAM is due to the possibility of the photon state having nontrivial spatial structure. This sparked a surge of investigations into optical OAM, quickly leading to simple methods for reliably producing, manipulating, and detecting it. There are a number of excellent reviews of the subject, including [6–8], for the reader to consult as further reading.

The key observation is that, if an approximate plane wave is given an azimuthally-dependent phase shift of the form  $e^{il\phi}$ , where  $\phi$  is the angle about the propagation axis,  $z$ , the resulting wave has angular momentum about the  $z$ -axis given by  $L_z = l\hbar$ . The single-valuedness of the field under  $\phi \rightarrow \phi + 2\pi$  forces the *topological charge*  $l$  to be quantized with integer values. This phase factor has the effect of tilting

the wavefronts by an increasing amount as the axis is circumnavigated, so that the wavefronts have a corkscrew shape, as in Fig. 8.1. The Poynting vector,  $\mathbf{W} = \mathbf{E} \times \mathbf{H}$ , must be perpendicular to the wavefront, so it is at an angle to the propagation axis. The vector  $\mathbf{W}$ , therefore, rotates about the axis as the wave propagates, corresponding to a nonzero orbital angular momentum. The separation of angular momentum into spin and orbital parts can be gauge-dependent for the most general case [9–15], but it is simple and unambiguous for propagating fields in the paraxial case, where only the region close to the propagation axis need be considered. In the following, the paraxial case is always assumed.

A number of different optical beam modes can carry OAM, including higher-order Bessel modes. Here, we focus on Laguerre–Gauss (LG) modes. Let  $|l, p\rangle$  and  $|\mathbf{r}\rangle = |r, z, \phi\rangle$  be, respectively, OAM and position eigenstates. Then the LG wavefunction with OAM  $l\hbar$  and with  $p$  radial nodes is [16]

$$u_{lp}(r, z, \phi) = \langle \mathbf{r} | l, p \rangle \quad (8.1)$$

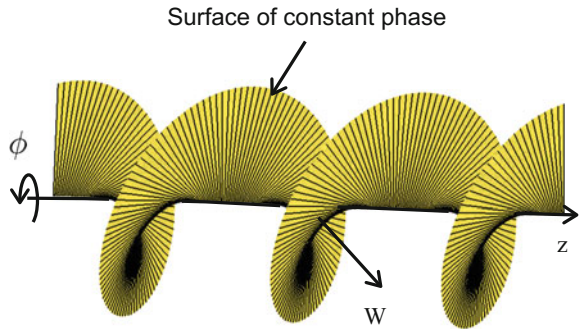
$$= \frac{C_p^{|l|}}{w(z)} \left( \frac{\sqrt{2}r}{w(z)} \right)^{|l|} e^{-r^2/w^2(z)} L_p^{|l|} \left( \frac{2r^2}{w^2(z)} \right) \times e^{-ikr^2 z / (2(z^2 + z_R^2))} e^{-i\phi l + i(2p + |l| + 1) \arctan(z/z_R)}, \quad (8.2)$$

with normalization  $C_p^{|l|} = \sqrt{\frac{2p!}{\pi(p+|l|)!}}$  and beam radius  $w(z) = w_0 \sqrt{1 + \frac{z}{z_R}}$  at  $z$ .

$L_p^\alpha(x)$  are the associated Laguerre polynomials [17], and  $z_R = \frac{\pi w_0^2}{\lambda}$  is the Rayleigh range and the arctangent term is the Gouy phase. The index  $p$  characterizes the radial structure of the mode. States with different  $p$  values can be related by a type of ladder operator and can be shown to be eigenstates of a differential operator [18–20].

There are several ways to generate optical OAM states, one of the most common being the use of *spiral phase plates*—plates whose optical thickness varies azimuthally according to  $\Delta z = \frac{l\phi}{k(n-1)}$  [21], where  $n$  is the refractive index, and the surrounding medium is assumed to have index 1.0. Another method is by the use of forked diffraction gratings [22], which convert Gaussian modes into OAM modes in first-order diffraction; these gratings can be in the form of computer generated

**Fig. 8.1** An optical wavefront with nonzero orbital angular momentum. Surfaces of constant phase are corkscrew-shaped. The Poynting vector,  $\mathbf{W}$ , must be everywhere perpendicular to the wavefront, so that it rotates as the wave propagates along the  $z$ -axis



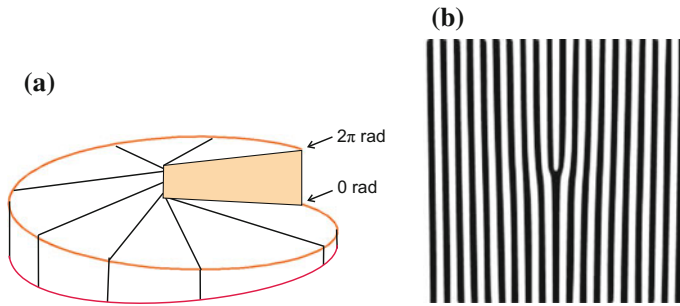


**Fig. 8.2** Two ways to generate OAM states. **a** A spiral wave plate has a thickness that increases linearly with azimuthal angle  $\phi$ , so that the phase factor accumulated by light passing through it is  $e^{il\phi}$ . The thickness is arranged to increase linearly with angle  $\phi$ . **b** Forked hologram. The diffraction pattern has maxima with OAM determined by the form of the dislocation at the center

holograms or they can be programmed on the surface of a spatial light modulator (SLM) (Fig. 8.2).

Measurement of the OAM of a beam can be accomplished several ways. One common method is to use an interferometric arrangement that sorts different  $l$  values into different outgoing spatial modes [23–25]. An example of an interferometric sorter is shown in Fig. 8.3 [23]. The basic unit consists of a Mach–Zehnder interferometer with a Dove prism (“Appendix A”) in each arm. One prism is oriented at angle  $\alpha/2$  relative to the other. When  $\alpha = \frac{\pi}{2}$  the net result is that this sorter separates even and odd values of  $l$  into different output ports of the final beam splitter. By concatenating several layers of such units with progressively smaller values of  $\alpha$  and introducing

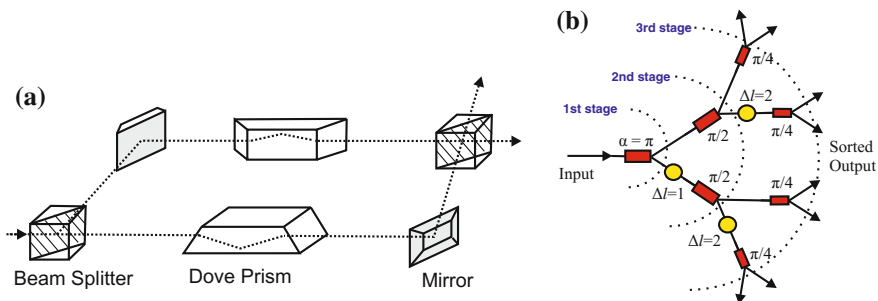


**Fig. 8.3** An interferometric OAM sorter [23]. **a** Two Dove prisms in a Mach–Zehnder interferometer. One prism is rotated by an angle  $\frac{\alpha}{2}$  relative to the other. For  $\alpha = \pi$ , this arrangement sends even values of OAM to one output port and odd values to the other. **b** Several of the basic interferometric units of **a** (the rectangles) arranged in stages with the values of  $\alpha$  shown. The circles are OAM shifters, altering  $l$  by the amounts shown. The three-stage version shown here can sort the values  $l = 1$  to  $l = 8$  into the 8 output ports. An  $n$ -layer version can sort values up to  $l = 2^n$ . This can be viewed a special case of a general interferometric method for detecting arbitrary linear combinations of OAM states [33]

appropriate OAM shifts  $\Delta l$  into one outgoing beam of each layer, all values of  $l$  can eventually be sorted out. A sequence of  $n$  layers can sort values up to  $l = 2^n$ . By a similar strategy, the addition of units with quarter-wave plates instead of Dove prisms allow the values of both OAM and spin to be sorted simultaneously [24]. It is important to note for quantum applications that (i) all of these sortings can be done at the single photon level, and that (ii) OAM can be sorted without collapsing the spin wavefunction, and vice-versa.

Other possible methods for sorting OAM values include the use of q-plates [26], polarizing Sagnac interferometers [27], pinhole arrangements followed by Fourier-transforming lenses [28] and use of specialized refractive elements [29–31]. Recently a simultaneous sorter for spin and orbital angular momentum that makes use of Pancharatnam–Berry phase has been proposed [32].

## 8.2 Entangled OAM in Parametric Down Conversion

Consider the class of biphoton state produced by spontaneous parametric down conversion (SPDC, see Chap. 1). Laguerre–Gauss functions form a complete set, so the state can be expanded in the space of orbital angular momenta of the outgoing signal-idler pair. Consider a pump beam of definite OAM  $l_0$ , having spatial profile  $E(\mathbf{r}) = u_{l_0 p_0}(\mathbf{r})$ . Imagine it encounters a  $\chi^2$  nonlinear crystal, and produces outgoing signal and idler beams. For fixed beam waist, the range of OAM values produced by the crystal is inversely proportional to the square root of the crystal thickness  $L$  [34] through down conversion. In order to produce a broad OAM bandwidth, assume the crystal is relatively thin; the price of this is that the down conversion rate is lower than for a thicker crystal. Also assume that the crystal is located at the beam waist ( $z = 0$ ). The output is an entangled state [35], with a superposition of terms of form  $u_{l'_1, p'_1} u_{l'_2, p'_2}$ . Angular momentum conservation requires  $l_0 = l'_1 + l'_2$  [36]. In contrast, the  $p'_1, p'_2$  values are unconstrained, although the amplitudes drop rapidly with increasing  $p'$  values [see Eq. (8.5) below]. The output of the crystal may be expanded as a superposition of signal and idler LG states:

$$|\Psi\rangle = \sum_{l'_1, l'_2=-\infty}^{\infty} \sum_{p'_1, p'_2=0}^{\infty} C_{p'_1 p'_2}^{l'_1, l'_2} |l'_1, p'_1; l'_2, p'_2\rangle \delta(l_0 - l'_1 - l'_2), \quad (8.3)$$

where the coupling coefficients are given by

$$C_{p'_1 p'_2}^{l'_1, l'_2} = \int d^2 r \Phi(\mathbf{r}) [u_{l'_1 p'_1}(\mathbf{r}) u_{l'_2 p'_2}(\mathbf{r})]^*. \quad (8.4)$$

Explicitly, for the case of a pump beam with  $l = p = 0$ , the coefficients are [34, 37]

$$C_{p_1, p_2}^{l, -l} = \sum_{m=0}^{p_1} \sum_{n=0}^{p_2} \left(\frac{2}{3}\right)^{m+n+l} (-1)^{m+n} \times \frac{\sqrt{p_1! p_2! (l+p_1)! (l+p_2)! (l+m+n)!}}{(p_1-m)! (p_2-n)! (l+m)! (l+n)! m! n!}. \quad (8.5)$$

More general expressions for the outgoing entangled wavefunction for nonzero OAM can be found in [34, 37–39].

The outgoing signal-idler pairs are entangled in OAM [35] and have been used to demonstrate violations of Bell inequalities [40, 41]. The degree of entanglement can be quantified by the Schmidt number (Sect. 1.5.2). Most experiments couple the signal and idler into detectors using fibers that only propagate  $p = 0$  modes, so henceforth we assume that  $p_1 = p_2 = 0$ . We also take the pump to have  $l_0 = 0$ , so that the OAM values just after the crystal are equal and opposite:  $l'_1 = -l'_2 \equiv l$ . Then Eq. (8.3) reduces to

$$|\Psi\rangle = \sum_{l=0}^{\infty} C_{00}^{l, -l} |l, 0\rangle | -l, 0\rangle \equiv \sum_{l=0}^{\infty} \sqrt{\lambda_l} |l\rangle | -l\rangle, \quad (8.6)$$

or

$$\psi(\mathbf{r}) = \sum_{l=0}^{\infty} \sqrt{\lambda_l} u_{l0}(\mathbf{r}) u_{-l,0}(\mathbf{r}). \quad (8.7)$$

From this, the state is already in Schmidt form with  $\lambda_l = \left(C_{00}^{l, -l}\right)^2$ , so that the effective Schmidt number and von Neumann entropy can then be readily calculated.

Quantitative estimates of the entanglement of OAM modes from parametric down conversion have been studied both theoretically and experimentally [42–46]. It is found that the full Schmidt number for the system can be written in the form  $K = \frac{1}{4} \left(b\sigma + \frac{1}{b\sigma}\right)^2$ , where  $b^{-1} = \sqrt{\frac{8\pi}{L\lambda_p}}$  is the phase matching width for a pump beam of waist  $\sigma$  and wavelength  $\lambda_p$  passing through a nonlinear crystal of thickness  $L$ . This total Schmidt number can be very large; for example  $K \approx 350$  for the setup of [46].

However, Schmidt numbers can also be defined for the radial and angular modes separately. Taking  $p = 0$  and no pump OAM, we obtain an expansion of the form of Eq. (8.6), which gives the azimuthal Schmidt number  $K_{az} = 1 / \sum_l \lambda_l^2$ . For large  $K$ , this is approximately given by  $K_{az} \approx 2\sqrt{K}$  [42, 43].  $K_{az}$  depends strongly on the phase-mismatch of the crystal [44].

Similarly, a radial Schmidt mode  $K_r$  can be defined by holding  $l$  constant and summing over  $p$  values; for large  $K$  it is found that  $K_r \approx \sqrt{K}$ . The sum  $K_{az} + K_r$  is typically much smaller than  $K$ , so most of the entanglement comes from radial-azimuthal cross-correlations, with  $p$  and  $l$  values jointly entangled.

### 8.3 Supersensitive Angular Measurement

In Sect. 4.6, it was shown that phase measurements could be made with optical OAM states that exceed the standard quantum limit. Because the phase and angle are proportional to each other for Laguerre–Gauss states and for other states with nonzero OAM, it is natural to ask if a similar improvement can be made in resolution of *angular displacements*.

#### 8.3.1 Angular and Rotational Measurements

In order to illustrate angular measurements, consider the setup shown in Fig. 8.4 [47]. The Dove prism (see Appendix A) is rotated by some unknown angle  $\theta$  and the goal is to measure  $\theta$  as precisely as possible. As input, we take a pair of entangled photons, with signal and idler of topological quantum numbers  $+l$  and  $-l$ , respectively. The two-photon state at any point in the system can be written in the form

$$\begin{pmatrix} U, +l \\ U, -l \\ L, +l \\ L, -l \end{pmatrix}, \quad (8.8)$$

where the four entries represent the amplitudes for being in the upper (U) or lower (L) branch, with OAM  $+l$  or  $-l$ . In this basis, the beam splitter, mirror, and Dove prism act according to the matrices:

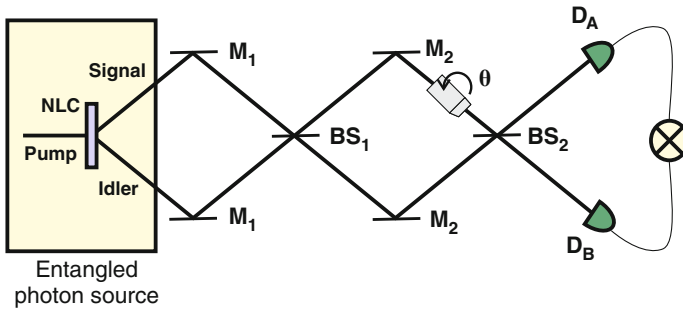


**Fig. 8.4** Schematic of setup for super-resolution angular displacement measurements using entangled OAM states. The goal is to measure the rotation angle  $\theta$  of a Dove prism placed in one arm

$$M_{bs} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \\ 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \end{pmatrix}, \quad M_m = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (8.9)$$

$$M_{dp}(\theta) = \begin{pmatrix} 0 & -e^{2il\theta} & 0 & 0 \\ -e^{-2il\theta} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (8.10)$$

$M_m$  represents the action of the *pair* of mirrors in opposite arms at the same longitudinal point. Note that the mirrors invert transverse spatial structures, and so interchange  $+l$  and  $-l$ , as well as adding a  $\pi$  phase shift. The Dove prism also adds a  $\pi$  phase change due to the total internal reflection, as well as an additional OAM-dependent phase change  $2l\theta$  when the prism is rotated by  $\theta$ .

The setup of Fig. 8.4 acts on the two-photon input state according to  $|\psi\rangle_{out} = M_{bs2}M_{dp}(\theta)M_{m2}M_{bs1}M_{m1}|\psi\rangle_{in} \equiv M|\psi\rangle_{in}$ . The full transformation matrix implemented by the interferometer is then

$$M = \frac{1}{2} \begin{pmatrix} 1 & -e^{-2il\theta} & i & ie^{-2il\theta} \\ -e^{2il\theta} & 1 & ie^{2il\theta} & i \\ ie^{2il\theta} & ie^{2il\theta} & -1 & \\ i & ie^{-2il\theta} & -1 & e^{-2il\theta} \end{pmatrix} \quad (8.11)$$

Imagine taking the input to be of the form of Eq. (8.6), but filter the crystal output in order to keep only the  $\pm l$  terms for some fixed value of  $l$ :

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}} (|l\rangle_s | -l\rangle_i + | -l\rangle_s |l\rangle_i), \quad (8.12)$$

where  $s$  and  $i$  denote signal and idler, which we take to enter the interferometer in the upper and lower branches respectively. Writing each single photon state in the form of a creation operator acting on a vacuum state,  $a^\dagger|vac\rangle$ , it is straightforward to apply the adjoint of the matrix  $M$  to write the input operators at the left end of the system in terms of the output operators on the right end, thus arriving at the output state,  $|\psi_{out}\rangle$ . The two-photon coincidence rate for detecting  $+l$  in detector  $D_A$  and  $-l$  in detector  $D_B$  can be thought of as measuring the operator

$$\hat{A} = | +l\rangle_A | -l\rangle_B {}_A\langle +l | {}_B\langle -l | + | -l\rangle_A | +l\rangle_B {}_A\langle -l | {}_B\langle +l|. \quad (8.13)$$

It is then found that the expectation values of  $\hat{A}$  and  $\hat{A}^2$  are

$$\langle \hat{A} \rangle = \langle \psi_{out} | \hat{A} | \psi_{out} \rangle = \cos^2(2l\theta), \quad (8.14)$$

$$\langle \hat{A}^2 \rangle = \langle \psi_{out} | \hat{A}^2 | \psi_{out} \rangle = \cos^2(2l\theta), \quad (8.15)$$

so that the uncertainty in  $\hat{A}$  is

$$\Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2} = \cos(2l\theta) \sin(2l\theta) = \frac{1}{2} \sin(4l\theta). \quad (8.16)$$

The cosine oscillates rapidly with  $\theta$  for large  $l$ , giving super-resolution of the angle.  $\hat{A}$  now provides a  $\theta$ -dependent observable from which an estimate of the angular displacement can be extracted. The resulting estimate has uncertainty

$$\Delta\theta = \frac{\Delta\hat{A}}{|\frac{\partial\langle\hat{A}\rangle}{\partial\theta}|}. \quad (8.17)$$

This whole procedure can be generalized to the case of  $N$  entangled photons instead of two. In an identical manner it can be shown that the phase sensitivity is now given by

$$\Delta\theta = \frac{1}{2Nl}. \quad (8.18)$$

This should be compared to the case when an ensemble of  $N$  independent photons are measured with the same apparatus:

$$\Delta\theta_{SQL} = \frac{1}{2\sqrt{N}l}. \quad (8.19)$$

As was the case with phase measurements, it is seen once again that entanglement allows the sensitivity to be reduced by a factor of  $\sqrt{N}$  beyond the case standard quantum limit, but further improvements can be made to the angular sensitivity by using states of higher angular momentum.

### 8.3.2 Rotational Measurements

Not only the size of the angular displacement, but also the rate of rotation can be measured with the help of optical OAM [48–52]. Imagine two photons of opposite OAM  $\pm l$  reflecting off the surface of a rotating object. Let  $\omega$  be the photon frequency and  $\Omega$  be the rotational frequency of the object. The two reflected photons experience equal and opposite angular Doppler shifts:  $\omega \rightarrow \omega \pm l\Omega$ . Interfering the reflected photons, the resulting intensity pattern produces a beat frequency of  $2l\Omega$ . For sufficiently large  $l$ , then precise measurements can be made of very small values of  $\Omega$ .

## 8.4 Edge Contrast Enhancement

OAM-based methods have been used in microscopy as a means of enhancing both resolution and contrast when imaging phase objects. In particular, the use of spiral phase filters (filters with transmission profile  $\sim e^{i\phi}$ ) can increase the edge contrast of images by orders of magnitude [53–55]. Effectively, the spiral phase factor introduces a factor  $\frac{1}{r^2} e^{i\phi}$  that is convolved with the object transmission function [56]. The  $\frac{1}{r^2}$  factor causes the convolution to smear the object over a very small disc; due to the exponential phase, points on opposite sides of the disk gain a relative phase shift of  $e^{i\pi} = -1$ . So, there is cancelation in regions where the phase and amplitude due to the object are constant. The result is a large increase in contrast at the edges of an object, or in other regions where the amplitude or phase changes suddenly.

A similar idea has been used in ghost imaging, where large edge-contrast enhancement has also been demonstrated [41]. An object is placed in one arm of the apparatus and a spiral filter in the other (in the experiment, both the object and the filter were written onto spatial light modulators). The object can be decomposed into a sum of modes of different OAM. In the absence of an object and assuming a pump with no angular momentum, the signal and idler modes arriving at the detectors should obey  $l_s + l_i = 0$ . Suppose that the object has a large component at mode  $l = l_{\text{obj}}$ . Then with the object and the filter in place, the coincidence count should be large when the filter's value  $l_{\text{ref}}$  obeys  $l_{\text{ref}} = -l_{\text{obj}}$ . So then, in regions where the phase is constant ( $l_{\text{obj}} = 0$ ), the count should be large at  $l_{\text{ref}} = 0$ , while, in regions where the phase changes suddenly, the  $l_{\text{ref}} = 0$  counts should be small and the  $l_{\text{ref}} \neq 0$  coincidence counts should be high. As a result, when we set  $l_{\text{ref}} = 0$  there are high-contrast *dark* edges outlining the object, and when  $l_{\text{ref}} \neq 0$  there should be high-contrast *bright* lines at the edges. Note that, in this case all that is needed is strong anti-correlation between the OAM of the two beams, not entanglement.

## 8.5 Spiral Imaging, Symmetry Detection, and Object Recognition

Because Laguerre–Gauss modes form a complete set, it is natural to ask if it is possible to construct an image entirely from the OAM intensity spectrum of light reflecting off or transmitting through the object. The answer is that it is *almost* possible. To reconstruct the full image one additional piece of information is needed in addition to the spectrum, namely the relative phases with which to add the various LG modes.

*Digital spiral imaging* was proposed [57, 58] as a form of angular momentum spectroscopy. Object properties are reconstructed based on how the OAM spectrum of an illuminating light beam is altered by passage through the object (Fig. 8.5). The input and output light may be expanded in LG functions, with the object acting by transforming the coefficients of the ingoing expansion into those of the outgoing expansion. Information about the transmission profiles of both phase and amplitude

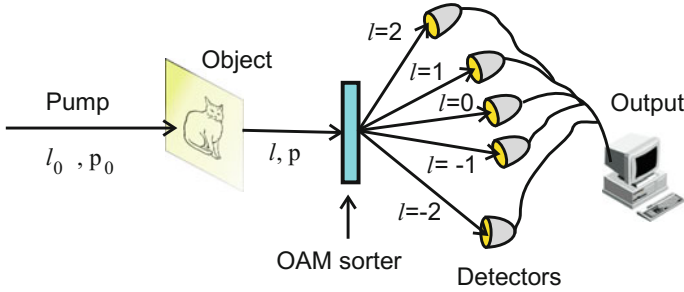


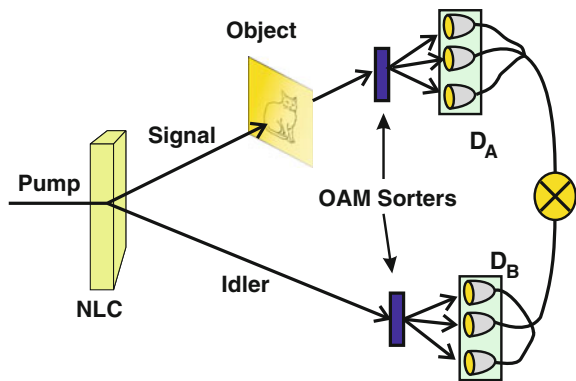
**Fig. 8.5** A schematic setup for digital spiral imaging. The presence of an object in the light beam alters the distribution of angular momentum values in the outgoing light. The effect on the OAM spectrum can be used to gain information about the object

objects may be retrieved [34, 57]. Only intensities are measured, so that, although a great deal of information can be obtained about the object, the lack of phase information prevents full image reconstruction. One way to extract the necessary phase is to use pairs of beams in some sort of interferometric arrangement. This leads naturally to the idea of using pairs of photons with entangled OAM states.

The addition of entangled photon pairs opens up several new possibilities [59–61]. First consider the setup shown schematically in Fig. 8.6. The two-photon input state is assumed to be of the form of Eq. (8.6). When no object is present, the photons are perfectly anticorrelated in OAM, so that if the OAM of the two detected photons are plotted against each other, all coincidence events lie on the diagonal of Fig. 8.7.

Now inserting an object in one arm, one photon passes through the object, while the other is unaffected by it. The two photons are then detected in coincidence. Denoting OAM eigenstates by  $|l\rangle$ , the action of the object is to alter the angular momentum of the first beam through an operator  $\hat{A}$  with matrix elements

**Fig. 8.6** Schematic setup for correlated-photon measurement of the OAM spectrum of an object



$$A_{kl} = \langle k | \hat{A} | l \rangle = \sqrt{\frac{1}{\pi^2 |k|! |l|!}} \int d\rho d\phi \rho^{|k|+|l|+1} e^{-\rho^2} e^{-i\phi(k-l)} A(\rho, \phi), \quad (8.20)$$

where  $\rho = \sqrt{2}r/w_0$  and  $A(\rho, \phi)$  is the object transmissivity function in polar coordinates. Denoting  $m \equiv k - l$ , the matrix elements may be written

$$A_{kl} = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-im\phi} R_{kl}(\phi), \quad (8.21)$$

where

$$R_{kl}(\phi) = \frac{2}{\sqrt{|k|! |l|!}} \int_0^\infty d\rho \rho^{|k|+|l|+1} e^{-\rho^2} A(\rho, \phi). \quad (8.22)$$

It can be seen that  $R_{kl}(\phi)$  carries information about the azimuthal structure of the object and that  $A_{kl}$  is the  $m$ -th coefficient of the Fourier series of  $R_{kl}(\phi)$ . Let  $l_o$  and  $l_r$  be the measured OAM in the object and reference beam, respectively. Then, in the absence of an object there should be coincidences only when  $l_r + l_o = 0$ , so that a histogram of coincidence probability versus  $l_r$  and  $l_o$  should have nonzero elements only along the descending diagonal (Fig. 8.7). However, when the object is present, the OAM no longer need be conserved; the interaction with the object can introduce OAM values  $m = l_o + l_r \neq 0$ . The spectrum of these values is dependent on the rotational symmetries of the object and introduce off-diagonal elements to the coincidence plots (Fig. 8.8 [60]). For simple objects with a single rotational symmetry, the symmetry can be read directly off the plots: an  $n$ -fold rotational symmetry introduces new diagonal bands shifted up and down on the plot by  $\pm n$  units. For more complicated objects, the spectrum is more complicated, as in Fig. 8.9 [61], but is still unique to the object. The rotational symmetries of an unknown object can therefore be detected experimentally in an efficient manner, and then can

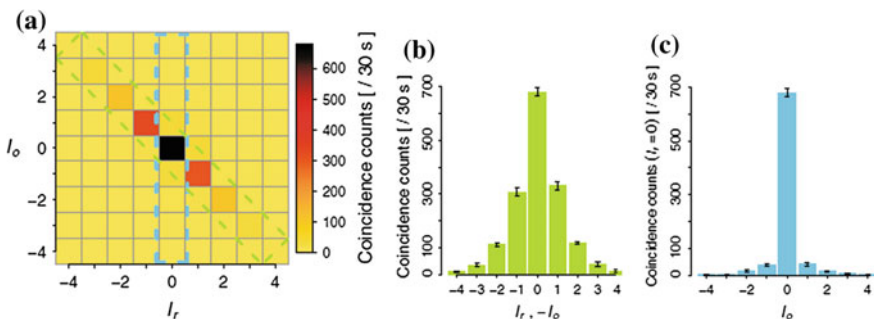


**Fig. 8.7** Plotting the OAM of two photons produced by SPDC, the coincidence lies on the diagonal (a), since the photons are created in pairs of opposite OAM. Plotting the coincidences along the diagonal (b) or along one row or column (c), it is seen that the absolute value of  $l$  decreases rapidly away from zero (figure reproduced from [60])

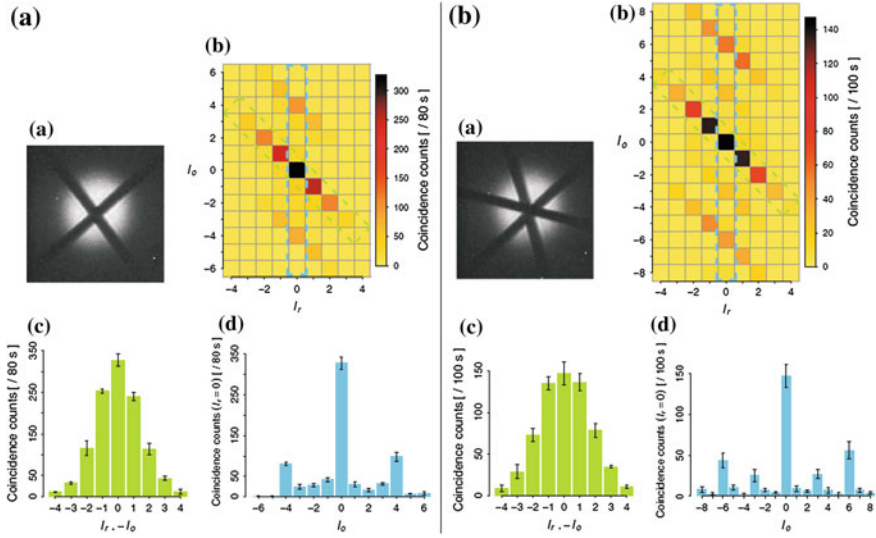


**Fig. 8.8** The addition of an object in one arm of the apparatus causes off-diagonal elements to appear in the joint-OAM coincidence plots. If the object has a well defined rotational, the joint-OAM plot has a corresponding translational symmetry: an  $n$ -fold rotational symmetry causes new diagonals to appear, shifted  $n$  spaces up and down from the main diagonal (figure reproduced from [60])

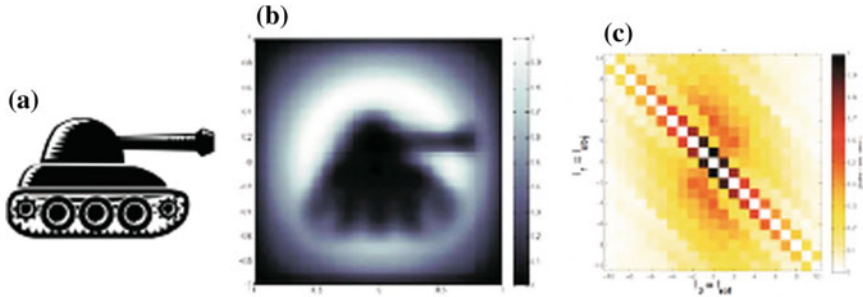
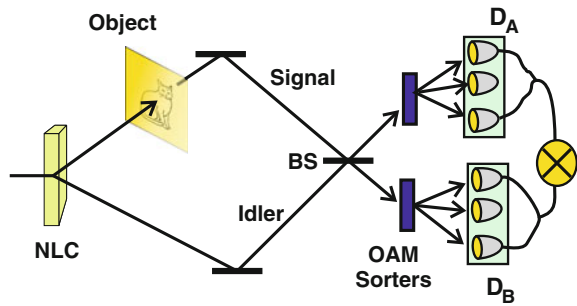


**Fig. 8.9** The addition of more complex objects to the apparatus results in more complex joint-OAM spectra, reflecting the approximate rotational symmetries of different portions of the object. Each shape produces its own unique spectrum, which is invariant under rotations of the object (figure reproduced from [61])

be checked the known symmetries of some other object for quality control, object identification, or other purposes.

Further, if a beam splitter is added to mix the two beams of the previous apparatus before the coincidence detection (Fig. 8.10), there is now a means of interferometrically determining the phases as well as the magnitudes of the amplitudes for each OAM component [59, 61]. As a result, it is now possible to reconstruct the full image of the object. This means of image reconstruction purely by measurement of

**Fig. 8.10** Schematic setup for correlated spiral imaging, the construction of an image from coincidence measurement of the OAM spectrum



the OAM content of entangled or classically correlated beams is called *correlated spiral imaging*.

The two variations here both involve detecting elements of rotational symmetry in the object. Beams with other types of symmetry can be used in an analogous manner to detect other types of symmetry. For example, entangled Hermite–Gauss beams can be used to detect translational symmetries.

Although this correlated-OAM method is too cumbersome to be of use in most everyday imaging or object identification applications, it could have significant advantages in some specialized cases. One advantage becomes clear, for example, when it is noted that the OAM spectrum is invariant under rotation of the object, aside from a trivial overall phase shift. As a result, there is no need to know the spatial orientation of an object to identify its spectrum, and even rapid rotation of the object results in no loss of accuracy; this approach can be used, for example, to identify abnormally shaped cells (sickle cells or cancerous cells) in the blood stream by a rapid identification of their rotational symmetry spectrum. Similarly, imaging or detecting defects in a rapidly moving object like a rotor blade can be made with slow (and therefore cheap) optical devices, reducing the need for more expensive high-speed optics.

## References

1. J.H. Poynting, Proc. R. Soc. Lond. A **82**, 560 (1909)
2. R.A. Beth, Phys. Rev. **48**, 471 (1935)
3. R.A. Beth, Phys. Rev. **50**, 115 (1936)
4. N. Carrara, Nuovo Cimento **6**, 50 (1949)
5. L. Allen, M.W. Beijersbergen, R.J.C. Spreeuw, J.P. Woerdman, Phys. Rev. A **45**, 8185 (1992)
6. A.M. Yao, M.J. Padgett, Adv. Opt. Photonics **3**, 161 (2011)
7. J.P. Torres, L. Torner (eds.), *Twisted Photons: Applications of Light with Orbital Angular Momentum* (Wiley, Hoboken, 2011)
8. S. Franke-Arnold, L. Allen, M. Padgett, Laser Photonics Rev. **2**, 299 (2008)
9. S. Barnett, L. Allen, Opt. Commun. **111**, 670 (1994)
10. S.J. van Enk, G. Nienhuis, J. Mod. Opt. **41**, 963 (1994)
11. S.J. van Enk, G. Nienhuis, Europhys. Lett. **25**, 497 (1994)

12. S.M. Barnett, J. Opt. B Quantum Semiclass. Opt. **4**, 7 (2002)
13. Y. Zhao, J.S. Edgar, G.D.M. Jeffries, D. McGloin, D.T. Chiu, Phys. Rev. Lett. **99**, 073901 (2007)
14. T.A. Nieminen, A.B. Stilgoe, N.R. Hechenberg, H. Rubinstein-Dunlop, J. Opt. A Pure Appl. Opt. **10**, 115005 (2008)
15. E. Santamoto, Fortschr. Phys. **52**, 1141 (2004)
16. L. Allen, M. Padgett, M. Babiker, Prog. Opt. **39**, 291 (1999)
17. G. Arfken, H. Weber, *Mathematical Methods for Physicists* (Academic Press, London, 2000)
18. E. Karimi, E. Santamoto, Opt. Lett. **37**, 2484 (2012)
19. E. Karimi, R.W. Boyd, H. de Guise, J. Řeháček, Z. Hradil, A. Aiello, G. Leuchs, L.L. Sánchez-Soto, Phys. Rev. A **89**, 063813 (2014)
20. W.N. Plick, R. Lapkiewicz, S. Ramelow, A. Zeilinger, [arXiv:1306.5617](https://arxiv.org/abs/1306.5617) [quant-ph] (2013)
21. M.W. Beijersbergen, R. Coerwinkel, M. Kristensen, J.P. Woerdman, Opt. Commun. **112**, 321 (1994)
22. V.Y. Bazhenov, M.V. Vasnetsov, M.S. Soskin, JETP Lett. **52**, 429 (1990)
23. J. Leach, M.J. Padgett, S.M. Barnett, S. Franke-Arnold, J. Courtial, Phys. Rev. Lett. **88**, 257901 (2002)
24. J. Leach, J. Courtial, K. Skeldon, S.M. Barnett, S. Franke-Arnold, M.J. Padgett, Phys. Rev. Lett. **92**, 013601 (2004)
25. C. Gao, X. Qi, Y. Liu, J. Xin, L. Wang, Opt. Commun. **284**, 48 (2011)
26. E. Karimi, B. Piccilillo, E. Nagali, L. Marrucci, E. Santamoto, Appl. Phys. Lett. **94**, 231124 (2009)
27. S. Slussarenko, V. D'Ambrosio, B. Piccirillo, L. Marrucci, E. Santamoto, Opt. Express **18**, 27205 (2010)
28. C.S. Guo, S.J. Yue, G.X. Wei, Appl. Phys. Lett. **94**, 231104 (2009)
29. M.J. Padgett, L. Allen, J. Opt. B Quantum Semiclass. Opt. **4**, S17 (2002)
30. G.C.G. Berkhout, M.P.J. Lavery, J. Courtial, M.W. Beijersbergen, M.J. Padgett, Phys. Rev. Lett. **105**, 153601 (2010)
31. M.P.J. Lavery, D.J. Robertson, G.C.G. Berkhout, G.D. Love, M.J. Padgett, J. Court. Opt. Exp. **20**, 2110 (2012)
32. G.F. Walsh, Opt. Express **24**, 6689 (2016)
33. D.S. Simon, C.A. Fitzpatrick, A.V. Sergienko, Phys. Rev. A **91**, 043806 (2015)
34. J.P. Torres, A. Alexandrescu, L. Torner, Phys. Rev. A **68**, 050301(R) (2003)
35. A. Mair, A. Vaziri, G. Weihs, A. Zeilinger, Nature **412**, 313 (2001)
36. S.P. Walborn, A.N. Oliveira, R.S. Thebaldi, C.H. Monken, Phys. Rev. A **69**, 023811 (2004)
37. X.F. Ren, G.P. Guo, B. Yu, J. Li, G.C. Guo, J. Opt. B Quantum Semiclass. Opt. **6**, 243 (2004)
38. G.A. Barbosa, Phys. Rev. A **80**, 063833 (2009)
39. F.M. Miatto, A.M. Yao, S.M. Barnett, Phys. Rev. A **83**, 033816 (2011)
40. A. Vaziri, G. Weihs, A. Zeilinger, Phys. Rev. Lett. **89**, 240401 (2002)
41. B. Jack, J. Leach, S. Franke-Arnold, M. Ritsche-Marte, S.M. Barnett, M.J. Padgett, Phys. Rev. Lett. **103**, 083602 (2009)
42. M.P. van Exter, A. Aiello, S.S.R. Oemrawsingh, G. Nienhuis, J.P. Woerdman, Phys. Rev. A **74**, 012309 (2006)
43. M.P. van Exter, P.S.K. Lee, S. Doesburg, J.P. Woerdman, Opt. Express **15**, 6431 (2007)
44. H. Di Lorenzo Pires, H.C.B. Florijn, M.P. van Exter, Phys. Rev. Lett. **104**, 020505 (2010)
45. S.S. Straupe, D.P. Ivanov, A.A. Kalinkin, I.B. Bobrov, S.P. Kulik, Phys. Rev. A **83**, 060302 (2011)
46. V.D. Salakhutdinov, E.R. Eliel, W. Löffler, Phys. Rev. Lett. **108**, 173604 (2012)
47. A.K. Jha, G.S. Agarwal, R.W. Boyd, Phys. Rev. A **83**, 053829 (2011)
48. M.V. Vasnetsov, J.P. Torres, D.V. Petrov, L. Torner, Opt. Lett. **28**, 2285 (2003)
49. M.P.J. Lavery, F.C. Speirits, S.M. Barnett, M.J. Padgett, Science **341**, 537 (2013)
50. C. Rosales-Guzman, N. Hermosa, A. Belmonte, J. P. Torres, Sci. Rep. **3**, article ID 2815 (2013)
51. M.P.J. Lavery, S.M. Barnett, F.C. Speirits, M.J. Padgett, Optica **1**, 1 (2014)
52. M. Padgett, Phys. Today **67**, 58 (2014)

53. S. Führhapter, A. Jesacher, S. Bernet, M. Ritsch-Marte, *Opt. Exp.* **13**, 689 (2005)
54. S. Führhapter, A. Jesacher, C. Maurer, S. Bernet, M. Ritsch-Marte, *Adv. Imaging Electron Phys.* **146**, 1 (2007)
55. C. Maurer, A. Jesacher, S. Führhapter, S. Bernet, M. Ritsch-Marte, *J. Microsc.* **230**, 134 (2008)
56. K.G. Larkin, D.J. Bone, M.A. Oldfield, *J. Opt. Soc. Am. A* **18**, 1862 (2001)
57. L. Torner, J.P. Torres, S. Carrasco, *Opt. Exp.* **13**, 873 (2005)
58. G. Molina-Terriza, L. Rebane, J.P. Torres, L. Torner, S. Carrasco, *J. Eur. Opt. Soc.* **2**, 07014 (2007)
59. D.S. Simon, A.V. Sergienko, *Phys. Rev. A* **85**, 043825 (2012)
60. N. Uribe-Patarroyo, A.M. Fraine, D.S. Simon, O.M. Minaeva, A.V. Sergienko, *Phys. Rev. Lett.* **110**, 043601 (2013)
61. C.A. Fitzpatrick, D.S. Simon, A.V. Sergienko, *Int. J. Quantum Inf.* **12**, 1560013 (2015)

# Chapter 9

## Quantum Communication and Cryptography

### 9.1 Introduction

Many of the applications covered in this book involve attempts to mitigate disturbances to an optical signal. Such disturbances may come from the propagation medium (dispersion and turbulence), from the optical instruments being used (aberration) or from random noise in the light source or detectors. However, there are disturbances that are inherent to any quantum system, which can not be eliminated or reduced beyond a fundamental limit; these stem from the Heisenberg uncertainty relations and from the inability to distinguish non-orthogonal states. Because these fundamental quantum disturbances cannot be removed, a natural question to ask is whether they can be harnessed to achieve useful effects. The most successful area for this approach is *quantum key distribution* (QKD), also known as *quantum cryptography*, which takes advantage of the fact that if an eavesdropper (traditionally known as Eve) tries to read a secret communication sent over a quantum channel, she inevitably introduces disturbances to the signal. These disturbances produce measurable errors in the signal which then tip off the legitimate users of the system (Alice and Bob) to the tampering. When the error rates exceed those expected from normal physical noise sources in the system alone, the users avoid possible eavesdropping by shutting down the channel and switching to a different secured key distribution channel to continue their cryptographic task. As a result, in principle, QKD enables unconditional communication security, through the combination of this secure key distribution with one-time-pad cryptography. The catch, of course, is that implementation of such a system in practice can be difficult; in currently realizable systems, there is always a trade-off between having unconditional security at very low data transmission rates (using single photons or entangled photon pairs) or having high transmission rates at lower levels of security (using coherent states).

In this chapter, we outline the ideas underlying the basic principles of QKD and examine some representative methods that have been used to implement those principles in practical optical systems. A review of basic quantum cryptographic principles in the full theoretical context of quantum information theory can be found

in [1]. Proofs of security in QKD tend to be complicated and would take us far from the main line of topics in this book, so we only give very brief discussions of the subject; for more detailed discussions of security aspects, see [2] and references therein.

## 9.2 Basic Principles

Alice and Bob, who may be widely separated in space, wish to generate a shared secret key for encrypting and decrypting messages. Their problem can be broken into two parts. (i) *The encryption problem*: finding an encryption method that cannot be broken by anyone without access to the encryption key. (ii) *The distribution problem*: agreeing on such a key over a long distance without an eavesdropper obtaining enough information to reconstruct the key. The solution to the first part is well known. The only truly unbreakable code is the *one-time pad* or *Vernam cipher*, in which the secret key  $k$  is a random string of binary digits intended to be used just once, then discarded. Let  $m$  represent the text to be encoded; it is assumed to be in the form of a binary string. The encoded message is given by the new binary string  $s = m \oplus k$ , where  $\oplus$  is base-two addition. To decode the message, Bob simply adds the same key to the encoded message because  $k \oplus k = 0$  from which it follows that  $s \oplus k = m \oplus (k \oplus k) = m$ . If the key is truly random, then  $s$  is an equally random string of 0s and 1s. So there are no patterns in  $s$  that can be used to break the code. The key has the maximum possible entropy and therefore no useful information can be extracted from it. However, if the same key is used more than once, correlations between the different messages encoded with that single key allow information about the key to be extracted, which in turn allows information about the encoded messages to be revealed. It is, therefore, essential that no key is used more than once.

The Vernam cipher solves the problem of unbreakable encryption of messages once the participants share a random key, but does not solve the problem of safely *distributing* the key among the legitimate users. Because Alice and Bob must use the *same* key, Eve may be able to intercept the passing of the key from one to the other, thus destroying the security of the message encrypted with it. In classical physics, there is no foolproof means for completely secure key distribution but quantum mechanics allows for such means via QKD. Indeed, the goal of QKD is to generate one-time encryption keys shared between two widely-separated legitimate users in such a way that the laws of quantum mechanics prevent illegitimate eavesdroppers from obtaining such keys without being detectable. The eavesdropping itself is not preventable, but if it occurs it is revealed to the legitimate agents. In this way, the eavesdropping can be rendered ineffective, because if Eve's tampering is detected, then Alice and Bob know their communication line has been compromised and so they switch to another key distribution channel.

Alice and Bob each need to be sure that they are distributing key material with each other, not with impostors. So, it is usually assumed that they already share a small amount of secret information between them initially, in order to authenticate each

others identities, before proceeding with the key generation and message encryption. The key distribution procedure is, thus, more precisely thought of as a *key expansion protocol*, in which an initially shared, short authentication key grows into a much longer encryption key.

QKD is made possible by the existence of noncommuting operators in quantum mechanics. Let  $\hat{O}$  and  $\hat{O}'$  be a pair of Hermitian operators and suppose that they do not commute:  $[\hat{O}, \hat{O}'] \neq 0$ . These two operators cannot be simultaneously diagonalized, and do not share common eigenstates. Any measurement of one operator ( $\hat{O}$ , let us say) projects the system onto an  $\hat{O}$  eigenstate. Since this state is not an eigenstate of  $\hat{O}'$ , a subsequent measurement of  $\hat{O}'$  leads to random outcomes, in particular, when the eigenbasis of  $\hat{O}'$  is unbiased relative to that of  $\hat{O}$ . If Alice and Bob both make measurements of  $\hat{O}$ , they expect to obtain perfectly correlated results. However, if Eve makes such a measurement of the noncommuting variable  $\hat{O}'$  between their measurements, the correlation between their outcomes should be reduced or even completely destroyed. Suppose that Alice and Bob randomly switch between the two operators. After all the measurements are made, they can compare their choice of operators (*without* sharing the results of their measurements), and throw out those trials on which their operator choices failed to match. Eve is then left with a dilemma because she can't know which operator they used on a given trial until after the measurements are made: the best she can do is guess. If she makes any measurements, she therefore inevitably introduces discrepancies between the results obtained by Alice and Bob, which they can easily spot by comparing a random sample of their measurement results.

A different, but equivalent view, is that the security of QKD is provided by the inability to distinguish between non-orthogonal states in quantum mechanics (see Sect. 1.11). This becomes apparent when the operators in question are chosen to be projection operators onto non-orthogonal pure states, as is to be done in the next section. It is straightforward to show that if the eigenvectors are orthogonal then the projectors obey  $[\hat{A}, \hat{B}] = 0$ , while conversely if the projectors commute then the eigenvectors must be either parallel or orthogonal to each other.

QKD schemes fall into two categories. In one, Alice prepares a state, makes a measurement on it and then sends it to Bob. In the other, a third party creates an entangled pair of states, sending half of the state to Alice and the other half to Bob. In either case, the values measured by Alice and Bob for the relevant operators are either correlated or anticorrelated. In the second category, it is even possible that the third party providing the entangled state is the eavesdropper herself, but the communications between Alice and Bob can nevertheless be made secure. The optimal choice of operators  $\hat{O}$  and  $\hat{O}'$  for achieving this is one in which the application of one operator makes all possible eigenvalues of the other one equally likely—such operators are called *mutually unbiased*, *conjugate*, or *incompatible*.

For the sake of definiteness, assume that the states are prepared by Alice. Then, the secure key distribution procedure consists schematically of the following steps:

- (i) Alice generates a sequence of states.
- (ii) For each state, she randomly chooses  $\hat{O}$  or  $\hat{O}'$  and makes a measurement.

- (iii) After Alice transmits her state to Bob, Bob then randomly chooses  $\hat{O}$  or  $\hat{O}'$  for each state and also makes a measurement.
- (iv) Alice and Bob then communicate over a classical channel, which could be completely public and unsecured. They tell each other which measurement operator they chose for each state, but *not* the result of the measurement.
- (v) Alice and Bob then carry out the *sifting* process, in which those values for which they made the same choice are kept and the remainder are discarded.
- (vi) Randomly selecting a subset of the sifted trials to subject to a security check, Alice and Bob compare the values they obtained on these trials. They look to see if this security subset has the expected correlation (or anticorrelation). Unexpected drops in correlation are interpreted as a signal of eavesdropper activity.
- (vii) If the security trials have the expected level of correlation, then Alice and Bob can be confident that no eavesdropping occurred. They can therefore use the values they measured on the remaining trials (those left after sifting and security trials) as bits of the one-time key. Although they have not communicated the values to each other, the fact that they measured the same operator on these correlated or anticorrelated states guarantees that each can deduce the other's value from their own.

In each trial, Eve must guess which operator to measure. She has only a 50 % probability per trial of guessing correctly. Suppose on a given trial Alice measures  $\hat{O}$ . Then, if Eve also measures  $\hat{O}$ , she obtains the same eigenvalue  $o_A$  as Alice, and so can generate a copy of the state to send to Bob. Bob, if he also measures  $\hat{O}$ , determines the same value  $o_A$ , and so the tampering remains undetected. However, when Eve measures the wrong operator  $\hat{O}'$ , she (with some probability  $p$ ) measures the correct value  $o_A$  but also sometimes measures an incorrect value  $o'_A$  with probability  $1 - p$ . She then sends on a state with the wrong eigenvalue. This shows up during the security check: Instead of Alice and Bob agreeing 100 % of the time on the sifted set, they find that they now only agree on a fraction  $\frac{p+1}{2}$  of the trials. This drop in correlation immediately signals the presence of an eavesdropper. The strategy used here by Eve is called an *intercept-resend attack*. It is the simplest type of eavesdropping attack. More sophisticated attacks may require additional safeguards [3–7].

To make the protocol as safe as possible, the overlap between each eigenstate of one operator with all of the eigenstates of the other should be as uniform as possible, i.e. the optimal choice is a set of mutually unbiased operators. In the most common case, the different operators represent projections onto polarization states measured along the axes of different bases, for example,  $\hat{O}_j = |\psi_j\rangle\langle\psi_j|$  and  $\hat{O}'_k = |\psi'_k\rangle\langle\psi'_k|$ . In that case, the eigenvectors represent the basis vectors themselves. For the case of two incompatible bases, the best possible choice is  $\langle\psi_i|\psi'_j\rangle = \frac{1}{\sqrt{2}}$  for all  $i \in \{1, 2\}$  and  $j \in \{1, 2\}$  (so that  $p = 1 - p = \frac{1}{2}$ ). Here,  $|\psi_i\rangle$  and  $|\psi'_j\rangle$  are respectively the eigenvectors of  $\hat{O}$  and  $\hat{O}'$ . More generally, we may use  $m$  incompatible operators  $\hat{O}_1, \dots, \hat{O}_m$ , such that

$$\langle\psi_i^{(\mu)}|\psi_j^{(\nu)'}\rangle = \frac{1}{\sqrt{m}}, \quad (9.1)$$

for all  $i, j \in \{1, 2\}$  and all  $\mu, \nu \in \{1, 2, \dots, m\}$  where the superscripts  $\mu, \nu$  label the operator, whereas subscripts  $i, j$  label the states within the set of eigenstates of each operator. Bases satisfying the conditions Eq. (9.1) are called *mutually unbiased* or *conjugate* bases. Mutually unbiased bases are such that measuring in one basis gives no information about the value in the other basis: A measurement in one basis completely randomizes values in the conjugate basis, leading to a uniform probability distribution. For a review of mutually unbiased bases and their construction see [8].

In reality, even with perfectly correlated states and no eavesdropping, some disagreement normally exists between Alice's and Bob's measurements. This is because detector noise, disturbances in the propagation path, and other real-world sources of error are inevitably present. The result is that Alice and Bob never see perfectly correlated values and they must set a nonzero threshold for the error rate, above which they conclude that there is eavesdropping and below which they consider their communications to be safe. The appropriate choice of threshold is determined by the expected level of noise in their system. In the remainder of this chapter, these complications are ignored and the idealized noiseless case is assumed for simplicity. For a detailed discussion of QKD security in the realistic case of noisy systems, see [9].

An alternate way to understand the security of QKD [10] is via the no-cloning theorem, which states that an unknown quantum state cannot be copied faithfully. This prevents Eve from simply cloning the state while it is on the way to Bob, and then making a measurement on the copy once she has discovered Alice and Bob's measurement bases. One way to view the reduced security of strong-light coherent state protocols is to observe that in this case, although Eve cannot make a perfect clone of a transmitted state, she can steal a small portion of the beam in order to probabilistically make imperfect clones, whose fidelity to the original is proportional to the amount of beam she steals. In this way, she can obtain partial information about the key.

## 9.3 Some Discrete QKD Protocols

### 9.3.1 The BB84 and E91 Protocols

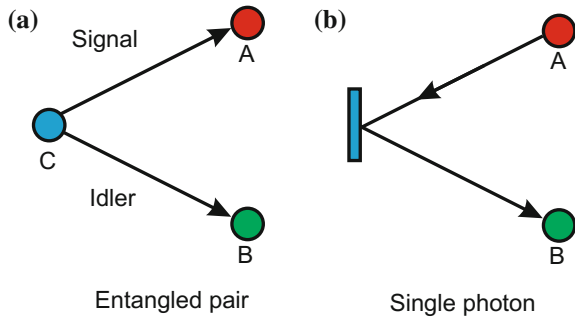
Consider in more detail the first successful QKD method, that of Bennett and Brassard [11], known as the *BB84 protocol*. Take the states to be polarization states of a photon, and the operator  $\hat{O}$  to be the polarization operator in a coordinate system defined by a pair of perpendicular axes, the horizontal (H) and vertical (V) axes:  $\hat{O} = |V\rangle\langle V|$ . Let the value of the key bit be 0 if the polarization is horizontal, and 1 if it is vertical, with corresponding eigenvectors  $|0\rangle = |H\rangle$  and  $|1\rangle = |V\rangle$ . Then  $\hat{O}|0\rangle = 0 \cdot |0\rangle$  and  $\hat{O}|1\rangle = 1 \cdot |1\rangle$ . The second operator  $\hat{O}'$ , incompatible with  $\hat{O}$ , is the polarization operator in a system defined by two axes ( $| \nearrow \rangle$  and  $| \searrow \rangle$ ) at  $\pm 45^\circ$  to the horizontal;  $\hat{O}' = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ . We denote the eigenvectors  $|0'\rangle = | \nearrow \rangle$  and  $|1'\rangle = | \searrow \rangle$ . These two bases, discussed in Chap. 1, are clearly mutually unbiased.

To generate a secure key, Alice randomly selects one of the two bases for each photon and makes a measurement of the polarization in that basis. She then sends the photon on to Bob, who similarly makes a random choice among the two bases and measures the polarization. If they happen both to have chosen the same basis, then they should always measure the same value for polarization. However, if they make different choices, then (due to the incompatibility of the bases), the result of Bob's measurement should be, on the basis of the principles of quantum theory, completely random. Alice and Bob select a random subset  $\mathcal{S}$  of photons to use for a security check and tell each other (over a classical and potentially public channel) both their basis choices and the results of their measurements. The trials on which they used different bases are discarded, as their results on these are uncorrelated. For those retained, the two compare the corresponding measurement outcomes. Assuming ideal conditions (negligible noise, perfect detectors, etc.), in principle these measurements match 100% of the time if there is no eavesdropping but only  $(100 - \frac{\eta}{4})\%$  of the time if Eve has intercepted and resent a fraction  $\eta$  of the photons. The presence of any eavesdropping is therefore immediately detectable, unless the eavesdropping rate  $\eta$  is so small that Eve cannot, in any case, obtain significant information. If no eavesdropping has been detected, then for the remaining photons (those not in  $\mathcal{S}$ ) classical information is again exchanged between Alice and Bob, but only concerning the choice of bases, *not* the actual polarization values in those bases. The photons for which the choices disagreed are again discarded. For the remainder, the polarizations are guaranteed to match; these polarizations then form a random sequence which is then used as the cryptographic key.

The *Ekert or E91 protocol* [12] is another QKD protocol closely related to the BB84. In it, a third party, called Charlie, generates a pair of polarization-entangled photons, via spontaneous parametric down conversion (SPDC). Charlie sends one photon from each pair to Alice and one to Bob. In the absence of eavesdropping, Alice and Bob receive, in the absence of external interference, perfectly anticorrelated polarizations whenever they measure in the same basis. When Eve intervenes on the system she reduces or destroys the expected correlation beyond that known and anticipated from knowledge of the natural environmental, in exactly the same manner as in the BB84 protocol. Indeed, the two protocols can be seen as formally equivalent to each other, for example, by means of the Klyshko picture (see Sect. 6.1): the BB84 protocol is simply the unfolded version of the Ekert protocol (Fig. 9.1).

In the Ekert version, the comparison of measurements in randomly chosen bases can be accomplished by measurement of a Bell operator, that is, the operator corresponding to the left-hand side of the Bell inequality (Sect. 1.9). If the Bell inequalities are violated, then the photons are guaranteed to remain entangled; in this case the communication is considered secure. However, if there has been eavesdropping, that reduces the degree of entanglement, causing a drop in the anticorrelations to classical levels. The lack of Bell inequality violations is, therefore, assumed to result from eavesdropping. In place of measuring the error rate in polarization comparisons, the corresponding quantity to be measured is the visibility of interference in the two-photon coincidence rate, as the angles of the measurement bases are rotated.

**Fig. 9.1** The Ekert protocol **a** is essentially equivalent to the BB84 protocol **b** when viewed via the Klyshko picture. The signal photon traveling from *C* to *A* can be viewed as moving backward from *A* to *C*, where it is then reflected forward to *B*. In the entangled case, the pump beam plays the role of the mirror



### 9.3.2 B92 Two-State Protocol

Bennett's B92 [13] protocol is a variation on BB84 in which Alice uses just a *single* basis. To accomplish this, the states of this basis are nonorthogonal. For example, the basis used by Alice can be taken to be

$$|0\rangle = |\rightarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = |\nearrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (9.2)$$

where the column vector forms are in the horizontal-vertical (*HV*) basis. Note that because the  $|0\rangle$  and  $|1\rangle$  states are not orthogonal to each other, they can not be unambiguously distinguished, except by introducing additional loss (see Sect. 1.11).

Bob still makes measurements in two bases, the same as in BB84. He switches randomly between the *HV* basis and the diagonal *D* basis ( $|\nearrow\rangle$  and  $|\searrow\rangle$ ). The protocol consists of the following steps.

- (1) On each trial, Alice randomly chooses either the  $|\rightarrow\rangle$  or  $|\nearrow\rangle$  state to transmit to Bob.
- (2) Bob measures in either the *HV* or *D* basis. The possible outcomes for his measurement are given in Table 9.1. Depending on the state Alice sent and the basis Bob used, either Bob unambiguously knows the state Alice sent, or he is unable to determine it. Bob discards the trials on which he can not determine the transmitted state.
- (3) Bob tells Alice (over a possibly public channel) which are discarded.
- (4) Alice and Bob agree on a subset of trials to use for a security check.
- (5) If the security subset survives the check, then the remaining trials form the key.

At the end of this procedure, Alice and Bob should share a common key. In this protocol, the inability to unambiguously distinguish between the two vectors of a single nonorthogonal basis plays the same role that the inability to unambiguously determine vectors created and detected in two *different*, mutually unbiased, nonorthogonal bases did in the BB84 protocol.

**Table 9.1** Bob measures his received state in one of two randomly chosen bases. If he measures states  $|\uparrow\rangle$  or  $|\searrow\rangle$  then he can determine which state was sent, because each of these is orthogonal to one of Alice's possible states. If he measures  $|\rightarrow\rangle$  or  $|\nearrow\rangle$ , then he can't determine the transmitted state because his measured state is nonorthogonal to both

| Bob's basis | Bob's result          | Bob's conclusion                                                            |
|-------------|-----------------------|-----------------------------------------------------------------------------|
| $HV$        | $ \uparrow\rangle$    | A sent $ \nearrow\rangle =  1\rangle$                                       |
| $HV$        | $ \rightarrow\rangle$ | Indeterminate: A sent either $ \rightarrow\rangle$ or $ \nearrow\rangle$    |
| $D$         | $ \searrow\rangle$    | A sent $ \rightarrow\rangle =  0\rangle$                                    |
| $D$         | $ \nearrow\rangle$    | Indeterminate: A sent either or $ \rightarrow\rangle$ or $ \nearrow\rangle$ |

In a variation, Bob can randomly measure one of three detection operators,

$$\Pi_0 = |0\rangle\langle 0| \quad (9.3)$$

$$\Pi_1 = |1\rangle\langle 1| \quad (9.4)$$

$$\Pi_? = 1 - \Pi_0 - \Pi_1, \quad (9.5)$$

and use unambiguous state determination (see Sect. 1.11). Once again, Bob sometimes fails to determine the state, but when he succeeds he can be perfectly certain that he obtained the correct bit.

The procedure is insecure in the presence of loss, since Eve can intercept the signal, making the same type of measurements as Bob. Any inconclusive measurements, she then discards. When she gets a definite result, she sends on a new photon in the same state, and then she covers her interference by sending additional photons of her own to eliminate the loss of intensity caused by those that she discarded. To make the protocol secure again, Alice and Bob can implement it with a pair of pulses: one dim and one bright, with a phase difference between them [13]. Bob monitors both pulses and the interference between them. Eve can't remove the strong pulse without making the drop in intensity obvious, and if she blocks the weak pulse then the expected interference between the two pulses does not appear. Fabricating false pulses to replace those she took introduces noticeable errors due to the trials on which her result was inconclusive. This approach enables security to be maintained even in the presence of strong attenuation.

### 9.3.3 Six-State Protocol

Instead of the four states of BB84, a six-state protocol [14, 15] can be implemented in a similar manner, using the states

$$|0\rangle, |1\rangle, |\nearrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |\nwarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \quad (9.6)$$

$$|r\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \quad |l\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \quad (9.7)$$

The greater symmetry of these states on the Poincaré–Bloch sphere simplifies the security analysis. The six-state protocol lowers the chances that Alice and Bob use the same basis, so that more trials are lost in reconciliation. However, Eve’s probability of guessing the correct basis during an intercept-resend attack is lowered from  $\frac{1}{2}$  to  $\frac{1}{3}$  so that the error rate she introduces into Alice and Bob’s result are greater relative to the amount of key information she gains.

### 9.3.4 Decoy State and SARG04 Protocols

One problem with single-photon-based QKD protocols is that true single photon states are difficult to produce in practice. The usual method is to approximate them by attenuating a weak coherent state pulse to the point that most pulses contain only a single photon. In general there are still multiphoton pulses produced occasionally, a weakness which can be exploited by an eavesdropper. In a *photon-number-splitting (PNS) attack* [16–21], Eve steals photons from these multiphoton pulses to gain partial information about the key; she sends the remainder of the pulse on to Bob, in an attempt to introduce minimal noise into Alice and Bob’s results. In this way, she gains more information for fixed error rate than would be possible for idealized single-photon transmission.

One way to counteract PNS attacks is to use decoy states. The idea is to intersperse two sequences of pulses, of different mean photon numbers. One set serves to generate the key, the other is a set of decoy pulses used to bolster security. Eve has no way of knowing whether a given pulse is a decoy or a key-generating pulse. By comparing the pulse detection probability versus photon number, Eve’s action appear as a change in the probability distribution. The result is that by using decoy states the same security level can be maintained over longer distances, as stronger pulses may be used.

An alternate approach to defeating PNS attacks is the *SARG04 protocol* [22–24], where the classical bits are encoded into pairs of nonorthogonal states, with one pair representing 0 and a different pair representing 1. Alice does not reveal the basis she used, but instead tells Bob which nonorthogonal pair she used. Then Bob can deduce the correct bit if he detects a state perpendicular to one of the two states in the corresponding pair. The four states used (two pairs) can be the same states as in the BB84 protocol, so that the only change necessary from BB84 is the classical communication.

The SARG04 provides greater security against PNS attacks for the following reason. In the BB84 protocol, all Eve needs to obtain complete knowledge of a bit value is to steal one photon from a multiphoton pulse and store it until she knows the

basis in which to measure it. However in the SARG04 protocol, if she steals a photon from a two-photon pulse, she can only learn that it is in one of two non-orthogonal states. The protocol has been shown to be secure against one- and two-photon attacks [25]. In order to obtain a definite value, she must attack the three-photon pulses (which are rarer), steal two photons to measure, and perform unambiguous state discrimination (Sect. 1.11), which only succeeds 50 % of the time. As a result, Eve must exert significantly greater effort to obtain much less information per pulse.

## 9.4 Continuous Variable QKD Schemes

The protocols in the previous section depend on the transmission and detection of single photons or of entangled pairs of photons. There are two disadvantages to those low intensity approaches: (i) expensive single-photon detectors are required and (ii) losses in the channel severely affect the efficiency of the transmission, resulting in low key generation and data transmission rates. One approach designed to get around these problems is to use strong light beams, that is, coherent or squeezed states. These more intense beams achieve much higher key generation rates, are less strongly affected by losses, and measurements can be carried out with less specialized detectors, making use of homodyne detection (see “Appendix B.2”). In contrast to the single-particle case, there is a detection made for every pulse even in the presence of loss, so efficiency is much greater. On the other hand, the difficulty with these stronger beams is that security is greatly reduced: An eavesdropper can steal a tiny portion of the amplitude and read the data while introducing only very small error rates. It requires considerably ingenuity and care, therefore, to arrange key generation schemes that have security levels similar to the single photon schemes.

Whereas the single-photon protocols make use of discrete variables (polarization, discrete time bins, discrete orbital angular momentum values), these strong-beam protocols often make use of continuous variables, such as the field quadratures  $X$  and  $P$  (see “Appendix B”). Eavesdropper detection here relies on the fact that if Eve measures one quadrature, then the variance of the opposite quadrature must increase to accord with the Heisenberg relation,  $\Delta X \Delta P \geq \frac{\hbar}{4}$ .

Early continuous-variable QKD protocols, cf. e.g. [26–36], were highly susceptible to loss, but this problem has been largely overcome [37]. Noise, though, has a significant effect. Security in continuous variable schemes relies on the fact that attempts to measure or clone a pulse add noise to the quadratures. Exact cloning cannot be done, but approximate cloning can be carried out. The higher the fidelity of the cloned state, the more effect it has on the measured quadratures. It can be shown that the optimal approximate cloner is a Gaussian cloning machine that adds an equal amount of noise—twice the amount of the shot noise [38]—to both quadratures, leading to a cloning fidelity of  $\frac{2}{3}$  for all coherent states [39]. Such a cloner is shown in Fig. 9.2.

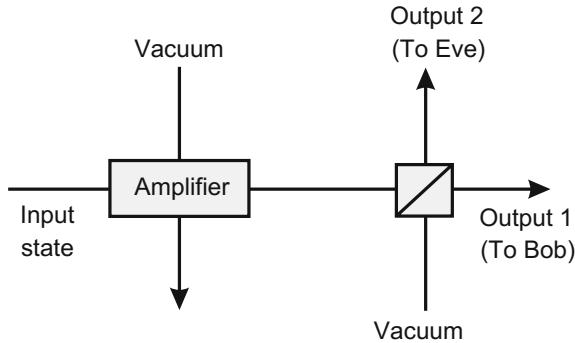


**Fig. 9.2** An optimal quantum cloner [39–41] that can be used to attempt eavesdropping on continuous variable communication. The first stage is a phase-insensitive amplifier used to increase the amplitude and mask the fact that amplitude was stolen at the second output of the 50/50 beam splitter

The earliest continuous variable protocols [33, 34] started with a coherent state, encoding a bit value into it by squeezing and displacing either the  $x$  quadrature or the  $p$  quadrature. This protocol then proceeds as a direct analog of the discrete BB84 protocol. Unfortunately, the need for squeezed states limits the practicality of the procedure. A more practical approach exists [35], using only unsqueezed coherent states. The  $x$  and  $p$  quadratures are both modulated by a Gaussian. The received state is then a thermal state formed from a mixture of coherent states. Bob randomly chooses to measure one quadrature, and he informs Alice of his choice. Alice, who knows the value of the quadrature, now has results correlated with Bob's. Any attempt at eavesdropping adds noise, thus decreasing the correlation; the transmission is considered to be secure if the added noise during transmission is less than that which would be provided by an optimal cloning machine. The noise has two parts: that added by losses during transmission over a finite distance ( $N_L$ ) and that due to eavesdropping or other disturbances (the excess loss,  $\epsilon$ ). The loss-induced noise is given by  $N_L = \frac{1-T}{T}$ , where  $T$  is the fraction of the amplitude that survives the transmission. The requirement that the total be less than one,  $N_L + \epsilon < 1$ , can be rephrased as  $T > \frac{1}{(2-\epsilon)}$ , so that security requires at least half of the transmitted states to reach the receiver; this is known as the *3 dB limit*. This three decibel limit can be evaded by use either of reverse reconciliation [42] or by a postselection process [37, 43–45].

A number of other variations on continuous-variable schemes have been proposed or carried out, including schemes that use simultaneous measurements of both quadratures [46, 47] and entanglement of coherent states with individual photons [48–50] through nonlinear Kerr interactions, just to list a few. Reviews of continuous variable QKD and information processing may be found in [51, 52]. Related developments include continuous variable quantum erasure [53], cloning [54], teleportation [55].

## 9.5 Other Protocols

Many other QKD protocols have been introduced over the years, all relying in some manner on the inability of an eavesdropper to distinguish between nonorthogonal states. Here, we simply list a few of these, with references to detailed discussions.

Many protocols make use of Hilbert spaces of dimension greater than 2, or in other words they make use of qudits instead of qubits. Increasing the dimension of the effective Hilbert space not only increases coding capacity, but also allows use of higher-dimensional nonorthogonal bases in security checks, which increases eavesdropper-induced error rates and makes it easier to reveal the eavesdropper [56–61].

Some early examples are [56–58]. Many theoretical and experimental papers have been published on schemes for higher dimensional QKD using optical orbital angular momentum (OAM) or some combination of OAM and polarization; these include [61–73]. High-dimensional QKD has also been carried out in the spatial domain by encoding information in the transverse linear momentum and position bases [74, 75], as well as by means of velocity dispersion [76]. Further approaches make use of time bins (Sect. 1.4.2) or time-energy entanglement [77–81] generated in Franson interferometers (Sect. 3.2.3). The time bin approach is well-suited to communication in either fiber or free space, and is undisturbed by turbulence. The distances over which time bins may be used are limited by the bandwidth of the signal and the dispersion of the medium, since dispersion causes wave-packets in different bins to overlap; however, this may be mitigated to some extent by dispersion-cancellation methods (Chap. 3).

## 9.6 Security

Currently available optical sources are imperfect. They fail, for example, to deterministically produce single-photon quantum states on demand. Similarly, current detectors have less than perfect efficiency. In the absence of a perfect authentication scheme, there is also the possibility that one of the participants is communicating with an imposter. Therefore, in a world of such real life imperfections, there is never be perfect security. Nonetheless, it is often useful to attempt proving the security of a given protocol in an idealized world of perfect sources and detectors. Such proofs in QKD attempt to demonstrate unconditional security in the face of all conceivable attacks within the bounds of currently known physical law.

Security proofs for QKD are generally applicable only to particular protocols and a particular limited set of possible attack methods, but remain significant because the options of Eve in the real world are also limited. As a general guideline, it is usually safe to assume that the protocol is secure if the amount of mutual information (see Chap. 1) that Alice and Bob share about the key is greater than the maximum mutual information Eve can share with either Alice or Bob.

Eavesdropping attacks can generally be divided into three categories: (i) *Individual (or incoherent) attacks*: In an individual attack, Eve probes each transmitted photon separately, making independent measurements on each. (ii) *Collective (or joint) attacks*: Here, each photon is again probed separately, but then joint measurements are allowed among these multiple probes, which often allows extra information to be extracted. (iii) *Coherent attacks*: The most general attack an eavesdropper can carry out is to probe all photons simultaneously and coherently measure the full multiphoton quantum state of the system. This type of measurement can introduce new correlations between the photons that were not present before the measurement.

It is still unclear whether coherent attacks are more general than collective attacks. It is believed, but not convincingly proven, that any system safe against collective attacks is also secure against individual and coherent attacks as well. For more detail on security proofs in QKD, see the reviews [2, 9, 82].

## 9.7 Related Topics

### 9.7.1 *Quantum Bit Commitment and Quantum Oblivious Transfer*

Quantum key distribution assumes that the two participants, Alice and Bob, trust each other. But in the real world, it is often the case that people who do not completely trust each other must find ways to work together. Quantum mechanics may be of help in these situations as well, in the form of *quantum bit commitment*.

The idea is that Alice commits a bit to be a certain value; but she doesn't want to reveal the value to Bob until some later time. In the meantime, Bob wants to be sure that Alice cannot change the value after she commits to it. In a metaphoric sense, the information is to be placed in a safe, where it cannot be seen by one party and cannot be tampered with by either.

It is known that classical bit commitment can not be made unconditionally secure, so the idea of constructing a quantum version arose in order to increase the security. Quantum bit commitment was first discussed by Bennett and Brassard [11]. It was later proven that quantum bit commitment schemes cannot be made unconditionally secure using only nonrelativistic quantum mechanics [83–91], since they are vulnerable to EPR-based attacks, but it was further demonstrated [92, 93] that by combining quantum mechanics and special relativity, quantum bit commitment could be made to work. The basic idea is that for each bit to be committed to the safe box, Bob sends Alice a collection of  $N$  entangled photon pairs. These photons are randomly polarized in one of four directions ( $|\uparrow\rangle$ ,  $|\rightarrow\rangle$ ,  $|\nearrow\rangle$ , or  $|\searrow\rangle$ ), forming a pair of mutually unbiased bases. Bob knows the polarizations, but he does not tell them to Alice. When Alice receives these photons, she passes them through a polarizer. If the value of her bit is 0, she randomly sends them either through a polarizer in the  $\uparrow$  or  $\rightarrow$  directions, while for a bit value of 1 she randomly chooses either  $\nearrow$  or  $\searrow$ . She then

sends the photons to a pair of widely separated agents who she trusts, one photon to each agent. To unveil the bit, those two agents then reveal them to two agents employed by Bob, who verify that they receive the same bit at the same moment. They also verify that the polarization distribution is consistent with the polarizations sent by Bob. Alice and her agents are prevented from cheating (by making changes to the sent bit) due to the fact that if the distance between the agents is sufficiently large there is no possibility of information exchange between Alice or her agents within the short time interval in which Bob's two agents verify the information. The size of this interval is determined by the resolution of the clocks carried by Bob's agents. Bob has no knowledge of the bit value until Alice gives her agents the go-ahead to unveil the bits.

The scheme of [93] was implemented experimentally in [94], with the agents separated by 20 km. This introduced a  $30\mu\text{s}$  delay, which was sufficient to ensure that the probability of cheating was below 5%. A variation on the same procedure was carried out in [95], with the agents in labs in Singapore and Geneva. These are separated by a 9354 km straight-line distance through the earth, introducing a 15 ms time delay, sufficient to reduce the cheating probability to about 1 in 18 million.

In *oblivious transfer protocols* [96], Alice has some stored information and Bob wants to try to access it. He attempts to read the information and has some probability  $\alpha$  of success. Alice is unable to determine whether or not Bob's attempts at accessing the data are successful. In the related *one-of-two oblivious transfer*, Alice sends two messages and Bob has a choice of reading one or the other of the messages (but not both), and Alice cannot determine which message was read.

Oblivious transfer protocols and bit commitment are closely related [97]. Classical oblivious transfer protocols are used for applications that include secure multiparty communication, and for anonymous access of stored information. Oblivious transfer protocols were first discussed in [98], and introduced into classical cryptography in [96], but are not secure classically. Quantum versions [99–101] of oblivious transfer protocols have been developed, and can be implemented if a secure bit commitment scheme is in place [102, 103]. Oblivious transfer is sufficient for the existence of secure two-party computation [104, 105], in which two parties can jointly compute a two-argument function, without allowing either party to determine the input data used by the other [106]. A quantum one-of-two oblivious transfer scheme has been implemented experimentally [107], in which a 1360 bit string was successfully transferred under noisy conditions.

### 9.7.2 Quantum Secret Sharing

Suppose Alice wants to communicate some information to Bob and Charlie. However, she wants to make sure that the two of them cooperate, or at least that neither of them can act upon the information in advance of the other. For example, she might suspect that one of them is dishonest and desire to guarantee that each keeps an eye on the other to prevent misuse of the information. Or maybe the information is about

a financial stock, and she wants to make sure that neither of them has a head start on the trading floor. *Quantum secret sharing* is about making sure that two participants can only act on information simultaneously by requiring them to cooperate in order for the information to be unlocked, and to do so in such a way that other, unauthorized parties cannot intercept the information first.

This can be done classically through the following scheme: Alice generates a random key  $k$ , and then encodes the binary message  $m$  to obtain an encrypted message  $c = m \oplus k$  through binary addition. If the key is truly random, then  $c$  is also be a random string of digits from which no information can be extracted. So all Alice has to do is send  $k$  to Bob and  $c$  to Charlie. Neither of them alone can extract the message from the string of random digits that they receive, but if they cooperate they can reconstruct the message simply by adding their random strings together:

$$c \oplus k = (m \oplus k) \oplus k = m. \quad (9.8)$$

The problem, as in quantum key distribution, is the possibility of eavesdropping.

There are at least two potential strategies for achieving quantum secret sharing. One is to simply mimic the security procedures of standard QKD protocols, such as BB84 (Sect. 9.3.1) to send  $k$  and  $c$  to the relevant parties. A different approach [108] makes use of GHZ states (Sect. 1.2). In this latter approach, the state is altered in such a way that when one particle each is sent to Bob and Charlie, the two can together reconstruct the required information. This involves a teleportation-like scheme, in which Alice has to classically transmit information to Bob and Charlie, while Bob and Charlie each need to send some information to each other; only after these information transfers are complete can the two at the receiving end reconstruct the state. See [108] for details. The advantage of this approach is that it can be generalized to larger numbers of participants.

### 9.7.3 *Quantum Data Locking and Quantum Enigma Machines*

According to a classical theorem proven by Shannon [109], information-theoretic security in cryptography is only possible if the secret key being used for the encryption is at least as long as the message itself. This picture changes drastically in quantum mechanics, however, where a short key can provide provable security for a much longer message via a phenomenon known as *quantum data locking* [110–113].

To understand quantum data locking, consider a set of  $M$  mutually unbiased bases, each spanning an  $N$ -dimensional space. Denote the basis vectors by  $\psi_j^{(k)}$ , where  $j = 1, 2, \dots, N$  specifies the particular vector in the basis and  $k = 1, 2, \dots, M$  specifies the basis itself. Then

$$\langle \psi_i^{(k)} | \phi_j^{(l)} \rangle = \frac{1}{\sqrt{M}}, \quad (9.9)$$

independent of the values of  $i, j, k, l$  (as long as  $k \neq l$ ).

If Alice sends to Bob a single state encoded in one of these bases, then if Bob knows the encoding basis he can extract  $\log_2(N)$  bits of information. However if he doesn't know the basis, then most of the time he gains no information, because measurement in the wrong basis gives him completely random results. Therefore, if Alice tells him the basis (which requires sending him  $\log_2(M)$  bits of information) then that allows Bob to access all  $\log_2(N)$  bits of information in the original transmission. So, if  $M \ll N$ , then a small amount of additional information allows Bob to access a much larger amount of information that was previously inaccessible to him. This can be rephrased by specifying a set of unitary operators  $U$  instead of bases: a set of  $M$  matrices  $U_j$  give the rotations of the bases from some standard reference bases.

The quantum enigma procedure [114, 115] is a way of doing this. Alice and Bob randomly (and possibly publicly) choose a set of  $N \times N$  unitary operators  $U_k$ , where  $k$  (the *seed*) takes on one of  $M$  possible values. The  $U_k$  operators rotate between a set of mutually unbiased bases. It has been shown [111] that if the  $N \times N$  unitary matrices are chosen randomly then Eve's accessible information can be made arbitrarily small using a key of size on the order of  $M \sim \log(N) \ll N$ . Alice encodes the message into a quantum state  $|j\rangle$ , then applies a unitary operator to it, arriving at the encrypted state:

$$|j\rangle_k \equiv U_k |j\rangle. \quad (9.10)$$

Bob knows the value of  $k$  corresponding to the matrix, so he can apply the inverse and decode it:

$$U_k^\dagger |j\rangle_k = |j\rangle. \quad (9.11)$$

Eve, assuming she hasn't intercepted the seed value  $k$ , must guess a unitary matrix to apply; with nearly 100% probability she applies the wrong matrix, completely randomizing the message. So, by sharing the seed (containing  $\log_2 M$  bits of information), Bob can unlock a much longer message of length  $N$  (containing  $\log_2 N$  bits of information). The accessible information  $I_{\text{acc}}$  is defined to be the maximum classical information that can be extracted by local measurements on a bipartite system. For long message lengths, the accessible information available from Alice's encrypted message is much larger for a legitimate agent who has access to the key, than for an eavesdropper who does not [114].

One way possible way to use the quantum enigma machine is in combination with QKD methods: some standard QKD protocol, such as BB84, is used to generate a short, secure key that can provide Alice and Bob's shared seed  $k$ , which is then used to encrypt and unlock a much longer message. This is one potential way that the problem of low bit-transmission rates in single photon-based QKD methods can be overcome.

### 9.7.4 Quantum Burglar Alarm

One further application related to the topics in this chapter is the recently proposed *quantum burglar alarm* or *quantum seal* [116]. The goal is to be able to secure any type of physical system (an information storage site, a nuclear materials site, etc.) from tampering. A stream of entangled photon pairs is generated, with one pair sent to a detector in a secure location and the other sent through a fiber to the location of the site that is to be protected. If the fiber is broken, the burglar is unable to clone the unknown state that was passing through it without introducing errors that can then be detected by comparison with the reference photon, similar to the strategy used to secure QKD. A proof of principle experiment of this method has been carried out [116], where it is also shown that the HOM effect can be used to thwart unauthorized attempts at modifying the seal. Analysis of the experiment showed a probability of  $10^{-5}$  of an inauthentic signal being accepted as authentic, with a false-alarm rate of one in  $10^{-9}$  for a 10s sampling interval. This demonstrates security against simple intercept-resend attacks, where the signal in the broken trip wire is measured and reproduced by the intruder. However, the quantum burglar alarm may still be vulnerable to more sophisticated teleportation-based attacks, where the state in the wire is transferred to a new photon.

## References

1. G. Jaeger, *Quantum Information: An Overview* (Springer, New York, 2006)
2. H.-K. Lo, M. Curty, K. Tamaki, Nat. Photonics **8**, 595 (2014)
3. G. Brassard, N. Lutkenhaus, T. Mor, B.C. Sanders, Phys. Rev. Lett. **85**, 1330 (2000)
4. A.V. Vakhitov, V. Makarov, D.R. Hjelme, J. Mod. Opt. **48**, 2023 (2001)
5. V. Makarov, D.R. Hjelme, J. Mod. Opt. **52**, 691 (2005)
6. C.-H.F. Fung, B. Qi, K. Tamaki, H.-K. Lo, Phys. Rev. A **75**, 032314 (2007)
7. B. Qi, C.-H.F. Fung, H.-K. Lo, X. Ma, Quantum Inf. Comput. **7**, 43 (2007)
8. T. Durt, B.G. Englert, I. Bengtsson, K. Życzkowski, Int. J. Quantum Inf. **8**, 535 (2010)
9. V. Scarani, H. Bechmann-Pasquinucci, N.J. Cerf, M. Dušek, N. Lütkenhaus, M. Peev, Rev. Mod. Phys. **81**, 1301 (2009)
10. W.K. Wootters, W.H. Zurek, Nature **299**, 802 (1982)
11. C.H. Bennett, G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing*, (Bangalore, 1984), p. 175
12. A.K. Ekert, Phys. Rev. Lett. **67**, 661 (1991)
13. C.H. Bennett, Phys. Rev. Lett. **68**, 3121 (1992)
14. D. Bruß, Phys. Rev. Lett. **81**, 3018 (1998)
15. D.G. Enzer, P.G. Hadley, R.J. Hughes, C.G. Pererson, P.G. Kwiat, New J. Phys. **4**, 45 (2002)
16. W.Y. Hwang, Phys. Rev. Lett. **91**, 057901 (2003)
17. X.B. Wang, [arXiv: quant-ph/0410075](https://arxiv.org/abs/quant-ph/0410075) (2004)
18. X.B. Wang, [arXiv: quant-ph/0411047](https://arxiv.org/abs/quant-ph/0411047) (2004)
19. H.K. Lo, X. Ma, K. Chen, Phys. Rev. Lett. **94**, 230504 (2005)
20. X. Ma, [arXiv: quant-ph/0503057](https://arxiv.org/abs/quant-ph/0503057) (2004)
21. D. Gottesman, H.-K. Lo, N. Lütkenhaus, J. Preskill, Quantum Inf. Comput. **4**, 325 (2004)
22. V. Scarani, A. Acín, G. Ribordy, N. Gisin, Phys. Rev. Lett. **92**, 057901 (2004)
23. C. Brianciard, N. Gisin, B. Kraus, V. Scarani, Phys. Rev. A **72**, 032301 (2005)

24. N. Lütkenhaus, Phys. Rev. A **61**, 052304 (2000)
25. C.-H. Fung, K. Tamaki, H.-K. Lo, Phys. Rev. A **73**, 012337 (2006)
26. T.C. Ralph, Phys. Rev. A **61**, 010303(R) (1999)
27. M. Hillery, Phys. Rev. A **61**, 022309 (2000)
28. T.C. Ralph, Phys. Rev. A **62**, 062306 (2000)
29. M.D. Reid, Phys. Rev. A **62**, 062308 (2000)
30. K. Bencheikh, T. Smyul, A. Jankovic, J.A. Levenson, J. Mod. Opt. **48**, 1903 (2001)
31. A.C. Funk, M.G. Raymer, Phys. Rev. A **65**, 042307 (2002)
32. Ch. Silberhorn, N. Korolkova, G. Leuchs, Phys. Rev. Lett. **88**, 167902 (2002)
33. D. Gottesman, J. Preskill, Phys. Rev. A **63**, 022309 (2001)
34. N.J. Cerf, M. Levy, G. Van Assche, Phys. Rev. A **63**, 052311 (2001)
35. F. Grosshans, P. Grangier, Phys. Rev. Lett. **88**, 057902 (2002)
36. F. Grosshans, G. Van Assche, J. Wenger, R. Brouri, N.J. Cerf, P. Grangier, Nature **421**, 238 (2003)
37. C. Silberhorn, T.C. Ralph, N. Lütkenhaus, G. Leuchs, Phys. Rev. Lett. **89**, 167901 (2002)
38. N.J. Cerf, S. Iblisdir, Phys. Rev. A **62**, 040301(R) (2000)
39. N.J. Cerf, A. Ipe, X. Rottenberg, Phys. Rev. Lett. **85**, 1754 (2000)
40. S.L. Braunstein, N.J. Cerf, S. Iblisdir, P. van Loock, S. Massar, Phys. Rev. Lett. **86**, 4938 (2001)
41. J. Fiurasek, Phys. Rev. Lett. **86**, 4942 (2001)
42. F. Grosshans, P. Grangier, [arXiv: quant-ph/0204127](https://arxiv.org/abs/quant-ph/0204127) (2002)
43. S. Lorenz, N. Korolkova, G. Leuchs, Appl. Phys. B **79**, 273 (2004)
44. R. Namiki, T. Hirano, Phys. Rev. A **67**, 022308 (2003)
45. T. Hirano, H. Yamanaka, M. Ashikaga, I. Konishi, R. Namiki, Phys. Rev. A **68**, 042331 (2003)
46. C. Weedbrook, A.M. Lance, W.P. Bowen, T. Symul, T.C. Ralph, P.K. Lam, Phys. Rev. Lett. **93**, 170504 (2004)
47. C. Weedbrook, A.M. Lance, W.P. Bowen, T. Symul, T.C. Ralph, P.K. Lam, Phys. Rev. A **73**, 022316 (2006)
48. B.T. Kirby, J.D. Franson, Phys. Rev. A **87**, 053822 (2013)
49. B.T. Kirby, J.D. Franson, Phys. Rev. A **89**, 033861 (2014)
50. D.S. Simon, G. Jaeger, A.V. Sergienko, Phys. Rev. A **89**, 012315 (2014)
51. S.L. Braunstein, P. van Loock, Rev. Mod. Phys. **77**, 513 (2005)
52. C. Weedbrook, S. Pirandola, R. Garcia-Petrón, N.J. Cerf, T.C. Ralph, J.H. Shapiro, S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012)
53. U.L. Andersen, O. Glöckl, S. Lorenz, G. Leuchs, R. Filip, Phys. Rev. Lett. **93**, 100403 (2004)
54. U.L. Andersen, V. Josse, G. Leuchs, Phys. Rev. Lett. **94**, 240503 (2005)
55. A. Furusawa, J.L. Sorensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble, E.S. Polzik, Science **282**, 706 (1998)
56. H. Bechmann-Pasquinucci, W. Tittel, Phys. Rev. A **61**, 062308–1 (2000)
57. H. Bechmann-Pasquinucci, A. Peres, Phys. Rev. Lett. **85**, 3313 (2000)
58. M. Bourennane, A. Karlsson, G. Björn, Phys. Rev. A **64**, 012306 (2001)
59. M. Bourennane, A. Karlsson, G. Bjork, N. Gisin, N.J. Cerf, J. Phys. A Math. Gen. **35**, 10065 (2002)
60. N.J. Cerf, M. Bourennane, A. Karlsson, N. Gisin, Phys. Rev. Lett. **88**, 127902 (2002)
61. M. Mafu, A. Dudley, S. Goyal, D. Giovannini, M. McLaren, M.J. Padgett, T. Konrad, F. Petruccione, N. Lütkenhaus, A. Forbes, Phys. Rev. A **88**, 032305 (2013)
62. A. Vaziri, G. Weihs, A. Zeilinger, Phys. Rev. Lett. **89**, 240401 (2002)
63. S. Groblacher, T. Jennewein, A. Vaziri, G. Weihs, A. Zeilinger, New J. Phys. **8**, 1 (2006)
64. D.S. Simon, N. Lawrence, J. Trevino, L. dal Negro, A.V. Sergienko, Phys. Rev. A **87**, 032312 (2013)
65. D.S. Simon, A.V. Sergienko, New J. Phys. **16**, 063052 (2014)
66. M. Krenn, R. Fickler, M. Fink, J. Handsteiner, M. Malik, T. Scheidl, R. Ursin, A. Zeilinger, New J. Phys. **16**, 113028 (2014)

67. M. Mirhosseini, O.S. Magaña-Loaiza, M.N. O'Sullivan, B. Rodenburg, M. Malik, M.P.J. Lavery, M.J. Padgett, D.J. Gauthier, R.W. Boyd, *New J. Phys.* **17**, 033033 (2015)
68. F.M. Spedalieri, *Opt. Commun.* **260**, 340 (2006)
69. L. Aolita, S.P. Walborn, *Phys. Rev. Lett.* **98**, 100501 (2007)
70. M. Stutz, S. Groblacher, T. Jennewein, A. Zeilinger, *Appl. Phys. Lett.* **90**, 261114 (2007)
71. C.E.R. Souza, C.V.S. Borges, A.Z. Khoury, J.A.O. Huguenin, L. Aolita, S.P. Walborn, *Phys. Rev. A* **77**, 032345 (2008)
72. J.T. Barreiro, T.-C. Wei, P.G. Kwiat, *Nat. Phys.* **4**, 282 (2008)
73. W. Boyd, A. Jha, M. Malik, C. OSullivan, B. Rodenburg, D.J. Gauthier, in *Advances in Photonics of Quantum Computing, Memory, and Communication IV*, eds. by Z. Ul Hasan, P.R. Hemmer, H. Lee, C.M. Santori, Proceedings of SPIE, vol. 7948, (2011), p. 79480L
74. S. Walborn, D. Lemelle, M. Almeida, P. Ribeiro, *Phys. Rev. Lett.* **96**, 090501 (2006)
75. S. Etcheverry, G. Cañas, E.S. Gómez, W.A.T. Nogueira, C. Saavedra, G.B. Xavier, G. Lima, *Sci. Rep.* **3**, 2316 (2013)
76. J. Mower, Z. Zhang, P. Desjardins, C. Lee, J.H. Shapiro, D. Englund, *Phys. Rev. A* **87**, 062322 (2013)
77. W. Tittel, J. Brendel, H. Zbinden, N. Gisin, *Phys. Rev. Lett.* **84**, 4737 (2000)
78. P.G. Kwiat, A.M. Steinberg, R.Y. Chiao, *Phys. Rev. A* **47**, 2472 (1993)
79. J. Brendel, N. Gisin, W. Tittel, H. Zbinden, *Phys. Rev. Lett.* **82**, 2594 (1999)
80. I. Ali-Khan, C.J. Broadbent, J.C. Howell, *Phys. Rev. Lett.* **98**, 060503 (2007)
81. T. Brougham, S.M. Barnett, K.T. McCusker, P.G. Kwiat, D.J. Gauthier, *J. Phys. B At. Mol. Opt. Phys.* **46**, 104010 (2013)
82. G. Brassard, N. Lütkenhaus, T. Mor, B.C. Sanders, *Phys. Rev. Lett.* **85**, 1330 (2000)
83. G. Chiribella, G.M. D'Ariano, P. Perinotti, D. Schlingemann, R. Werner, *Phys. Rev. Lett.* **A 377**, 1076 (2013)
84. G. Brassard, C. Crépeau, D. Mayers, L. Salvail, [arXiv: quant-ph/9712023](https://arxiv.org/abs/quant-ph/9712023) (1997)
85. H.-K. Lo, H. Chau, in *Proceedings of the Fourth Workshop on Physics and Computation* (New England Complex System Inst., Boston, 1996), p. 76
86. D. Mayers, in *Proceedings of the Fourth Workshop on Physics and Computation* (New England Complex System Inst., Boston, 1996), p. 226
87. D. Mayers, *Phys. Rev. Lett.* **78**, 3414 (1997)
88. H.-K. Lo, H.F. Chau, *Phys. Rev. Lett.* **78**, 3410 (1997)
89. H.-K. Lo, H.F. Chau, *Phys. D* **120**, 177 (1998)
90. A. Kitaev, D. Mayers, J. Preskill, *Phys. Rev. A* **69**, 052326 (2004)
91. G.M. D'Ariano, D. Kretschmann, D. Schlingemann, R.F. Werner, *Phys. Rev. A* **76**, 032328 (2007)
92. A. Kent, *New J. Phys.* **13**, 113015 (2011)
93. A. Kent, *Phys. Rev. Lett.* **109**, 130501 (2012)
94. Y. Liu et al., [arXiv: quant-ph/1306.4413](https://arxiv.org/abs/quant-ph/1306.4413) (2013)
95. T. Lunghi et al., *Phys. Rev. Lett.* **111**, 180504 (2013)
96. M. Rabin, *Technical Report, TR-81* (Aiken Comp. Lab., Harvard University, 1981)
97. G. Brassard, D. Chaum, C. Crépeau, *J. Comput. Syst. Sci.* **37**, 156 (1988)
98. S. Wiesner, *SIGACT News* **15**, 78 (1983)
99. G. He, Z. Wang, *Phys. Rev. A* **73**, 012331 (2006)
100. G. He, Z. Wang, *Phys. Rev. A* **73**, 044304 (2006)
101. D. Fattal, M. Fiorentino, R. G. Beausoleil, in *Quantum Communications and Quantum Imaging VI*, eds. by R.E. Meyers, Y. Shih, K.S. Deacon, Proceedings of SPIE, vol. 7092 (2008), p. 70920X
102. A. C.-C. Yao, in *Proceedings of 20th ACM STOC ACM*, (New York, 1995), p. 67
103. C. Crépeau, *J. Mod. Opt.* **41**, 2445 (1994)
104. C. Crépeau, J. van de Graaf, A. Tapp, in *CRYPTO 95: Proceedings of the 15th Annual International Cryptology Conference on Advances in Cryptology* (Springer, Berlin, 1995), p. 110
105. J. Kilian, in *Proceedings of 20th ACM STOC ACM*, (New York, 1988), p. 20

- 106. A.C.-C. Yao, in *Proceedings of 23th Annual IEEE Symposium on Foundations in Computer Science*, (IEEE Press, 1982), p. 160
- 107. C. Erven, N. Ng, N. Gïgov, R. Laflamme, S. Wehner, G. Weihs, *Nat. Commun.* **5**, 3418 (2014)
- 108. M. Hillery, V. Buřek, A. Berthiaumme, [arXiv: quant-ph/9806063](#) (1998)
- 109. C. Shannon, *Bell Syst. Tech. J.* **28**, 656 (1949)
- 110. D.P. DiVincenzo, M. Horodecki, D.W. Leung, J.A. Smolin, B.M. Terhal, *Phys. Rev. Lett.* **92**, 067902 (2004)
- 111. P. Hayden, D. Leung, P.W. Shor, A. Winter, *Commun. Math. Phys.* **250**, 371 (2004)
- 112. F. Dupuis, J. Florjanczyk, P. Hayden, and D. Leung, *Locking Classical Information*, [arXiv: 1011.1612](#) (2012)
- 113. O. Fawzi, P. Hayden, P. Sen, *J. ACM* **60**, 44 (2013)
- 114. S. Lloyd, *Quantum Engima Machines*, [arXiv: 1307.0380](#) (2013)
- 115. S. Guha, P. Hayden, H. Krovï, S. Lloyd, C. Lupo, J.H. Shapiro, M. Takeoka, M.M. Wilde, *Phys. Rev. X* **4**, 011016 (2014)
- 116. B.P. Williams, K.A. Britt, T.S. Humble, *Phys. Rev. Appl.* **5**, 014001 (2016)

# Appendix A

## Review of Optics

In this “Appendix”, some topics are reviewed that are related to optics and that are useful for the discussions in the main text. In particular, the propagation of optical waves, beam splitters, and the principal facts concerning laser beam optics are outlined. For more details, see Refs. [1–6].

### A.1 Conventions

We first define the conventions used here and throughout the book for Fourier transforms and Dirac delta functions. Define the two-dimensional Fourier transform and its inverse according to:

$$f(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \int d^2k \tilde{f}(\mathbf{k}) e^{+i\mathbf{k}\cdot\mathbf{r}}, \quad (\text{A.1})$$

$$\tilde{f}(\mathbf{k}) = \int d^2r f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}. \quad (\text{A.2})$$

Some authors, especially in optics texts, prefer to use *spatial frequencies*  $\nu_j = \frac{k_j}{2\pi}$  ( $j = x, y$ ) instead of the wavevector  $\mathbf{k}$ ; in that case the definitions become:

$$f(\mathbf{r}) = \int d^2\nu \tilde{f}(\boldsymbol{\nu}) e^{+2\pi i \boldsymbol{\nu}\cdot\mathbf{r}}, \quad (\text{A.3})$$

$$\tilde{f}(\boldsymbol{\nu}) = \int d^2r f(\mathbf{r}) e^{-2\pi i \boldsymbol{\nu}\cdot\mathbf{r}}. \quad (\text{A.4})$$

Whenever integration limits are not specified, it is assumed that they run from  $-\infty$  to  $+\infty$  for each cartesian component.

The two-dimensional *Dirac delta function* is then represented via a Fourier integral as

$$\delta(\mathbf{r} - \mathbf{a}) = \left(\frac{1}{2\pi}\right)^2 \int d^2k e^{+ik \cdot (\mathbf{r} - \mathbf{a})} = \int d^2\nu e^{+2\pi i \nu \cdot (\mathbf{r} - \mathbf{a})}. \quad (\text{A.5})$$

The delta function obeys the sifting property,

$$\int f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{a}) d^2r = f(\mathbf{a}), \quad (\text{A.6})$$

and the scaling property,

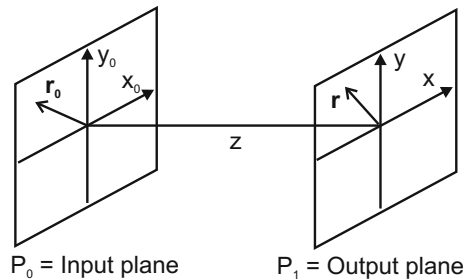
$$\delta(c\mathbf{r}) = \frac{1}{|c|} \delta(\mathbf{r}). \quad (\text{A.7})$$

More generally, the change of variable formula for the delta function is  $\delta(f(\mathbf{r})) = \frac{\delta(\mathbf{r})}{|\nabla f(\mathbf{r})|}$ .

## A.2 Propagation of Optical Fields

Consider a complex optical field propagating distance  $z$  from initial plane  $P_0$  to final plane  $P_1$  (Fig. A.1). The electric field in  $P_0$  is given by  $\mathbf{E}_0(x_0, y_0)$ , where  $x_0$  and  $y_0$  are the transverse coordinates in the initial plane. Similarly, the field in the final plane,  $P_1$ , is  $\mathbf{E}(x, y)$ . Defining transverse position vectors in the two planes,  $\mathbf{r}_0 = x_0\hat{i} + y_0\hat{j}$  and  $\mathbf{r} = x\hat{i} + y\hat{j}$ , the field may be written more compactly as  $\mathbf{E}_0(\mathbf{r}_0)$  and  $\mathbf{E}(\mathbf{r})$ . Unless otherwise stated, throughout this book attention is restricted to the paraxial case (small propagation angles from the axis), in which one may ignore the vector nature of the field and treat each component separately. This is known as the *scalar optics approximation*. The paraxial case also allows all inclination or obliquity factors [2] to be treated as constants. In the following, we consider a single component of the field, which is simply denoted by  $E(\mathbf{r})$ . Let  $\tilde{E}(\mathbf{k})$  be the Fourier transform of  $E(\mathbf{r})$ .

**Fig. A.1** Free-space propagation of optical disturbance from the initial plane to the final plane



The treatment of optical propagation begins with the Huygens–Fresnel principle, which says that each point of an object or aperture acts as a source of spherical outgoing waves of the form  $\sim \frac{1}{r} e^{-ikr}$ . The general relation between the fields in the initial and final planes can be written as

$$E(\mathbf{r}) = \int_{P_0} d^2r_0 h(\mathbf{r}, \mathbf{r}_0) E_0(\mathbf{r}_0), \quad (\text{A.8})$$

or equivalently in wave-vector space as

$$\tilde{E}(\mathbf{k}_\perp) = \mathcal{H}(\mathbf{k}_\perp) \tilde{E}_0(\mathbf{k}_\perp), \quad (\text{A.9})$$

where  $\mathbf{k}_\perp = k_x \hat{i} + k_y \hat{j}$  is the transverse part of the wave-vector. The *impulse response function*  $h(\mathbf{r}, \mathbf{r}_0)$  and the *transfer function*  $\mathcal{H}(\mathbf{k}_\perp)$  comprise a Fourier transform pair,

$$h(\mathbf{r}, \mathbf{r}_0) = h(\mathbf{r} - \mathbf{r}_0) = \left( \frac{1}{2\pi} \right)^2 \int d^2k_\perp \mathcal{H}(\mathbf{k}_\perp) e^{+i\mathbf{k}_\perp \cdot (\mathbf{r} - \mathbf{r}_0)}, \quad (\text{A.10})$$

where systems for which the first equality holds are called *shift-invariant* or *isoplanatic*. In situations where multiple propagators occur, each possibly involving different frequencies  $\omega$  and different propagation distances  $z$ , the notation may be augmented in order to distinguish between them, denoting the impulse response functions as  $h(\omega, \mathbf{r}, \mathbf{r}_0, z)$ ; here we suppress the dependence on everything but the initial and final points in order to keep notation simple.

For free space propagation, the transfer function is simply

$$\mathcal{H}(\mathbf{k}_\perp) = e^{ik \cdot \mathbf{R}}, \quad (\text{A.11})$$

where  $\mathbf{R} = (\mathbf{r}, z) = (x, y, z)$  is the full three-dimensional displacement vector.

In the Fresnel case ( $\frac{1}{\lambda^2} \gg \nu_x^2 + \nu_y^2$ ), a binomial approximation may be applied to Eq. (A.14):

$$k_z \approx k - \frac{1}{2k} (k_x^2 + k_y^2). \quad (\text{A.12})$$

In this case, the Fresnel transfer function is

$$\mathcal{H}(\mathbf{k}) = e^{-\frac{iz}{2k} (k_x^2 + k_y^2)} e^{ik_z z}, \quad (\text{A.13})$$

$k_z$  is viewed as a function of the transverse components and the wavelength,  $\lambda$ :

$$k_z = \sqrt{k^2 - \mathbf{k}_\perp^2} = \sqrt{k^2 - k_x^2 - k_y^2}, \text{ with } k = \frac{2\pi}{\lambda}. \quad (\text{A.14})$$

In position space, the exact impulse response function in three dimensions is

$$h(\mathbf{R}, \mathbf{R}_0) = \frac{1}{|\mathbf{R} - \mathbf{R}_0|} e^{-ik|\mathbf{R} - \mathbf{R}_0|}, \quad (\text{A.15})$$

where  $\mathbf{R}$  and  $\mathbf{R}_0$  are three-dimensional position vectors. Restricting this to a two dimensional plane at distance  $z$  from the source, assume that the distance is sufficiently large to approximate  $|\mathbf{R} - \mathbf{R}_0| \approx z$  in the denominator. In the exponent apply a binomial expansion:

$$|\mathbf{R} - \mathbf{R}_0| = \sqrt{z^2 + |\mathbf{r} - \mathbf{r}_0|^2} = z \sqrt{1 + \frac{|\mathbf{r} - \mathbf{r}_0|^2}{z^2}} \quad (\text{A.16})$$

$$\approx z + \frac{|\mathbf{r} - \mathbf{r}_0|^2}{2z} - \frac{|\mathbf{r} - \mathbf{r}_0|^4}{8z^3} + \dots \quad (\text{A.17})$$

$$= z + \frac{|\mathbf{r}|^2}{2z} - \frac{\mathbf{r} \cdot \mathbf{r}_0}{z} + \frac{|\mathbf{r}_0|^2}{2z} - \dots \quad (\text{A.18})$$

The Fresnel response function is obtained by keeping only the first four terms in Eq. (A.18) (equivalently, the first two terms in Eq. (A.17)), so that

$$h(\mathbf{r}, \mathbf{r}_0) = \frac{1}{i\lambda z} e^{-\frac{ik}{2z}|\mathbf{r} - \mathbf{r}_0|^2} e^{-ikz}. \quad (\text{A.19})$$

(Note that  $k_z$  is also being approximated by  $k$  because the angles are small.)

The far-field or Fraunhofer limit occurs when the wavefronts throughout the system are approximately planar over distances of the order of the largest aperture or object in their path. More quantitatively, this occurs when  $R \gg \frac{a}{\lambda^2}$ , where  $a$  is the largest aperture or object size, and  $R$  is the smallest propagation distance between elements of the system (apertures, sources, objects, etc.). In this case, the term  $|\mathbf{r}_0|^2$  can be dropped from the exponent in Eq. (A.19):

$$h(\mathbf{r}, \mathbf{r}_0) = \frac{1}{i\lambda z} e^{-ik\left(z + \frac{|\mathbf{r}|^2}{2z}\right)} e^{\frac{ik}{z}\mathbf{r} \cdot \mathbf{r}_0}. \quad (\text{A.20})$$

This is equivalent to keeping only the first three terms in Eq. (A.18). The free propagation simply leads in the Fraunhofer approximation to a Fourier transform of the incident field with respect to a rescaled position variable.

$$E(\mathbf{r}) = \frac{1}{i\lambda z} e^{-ik\left(z + \frac{|\mathbf{r}|^2}{2z}\right)} \tilde{E}\left(-\frac{k\mathbf{r}}{\lambda z}\right). \quad (\text{A.21})$$

If an object is placed in a beam path, then it affects the field by inserting an additional factor  $T(\mathbf{r})$  or  $\tilde{T}(\mathbf{k})$  into the propagation functions  $h(\mathbf{r}, \mathbf{r}_0)$  or  $\mathcal{H}(\mathbf{k})$ , respectively, where  $T(\mathbf{r})$  (which may be complex) is the *transmissivity* of the object. For example, consider a lens of focal length  $f$ . It introduces a factor of

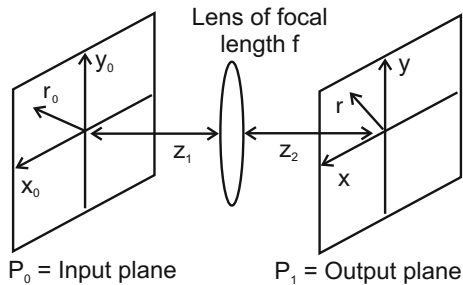


**Fig. A.2** An input plane is propagated to an output plane through a lens of focal length  $f$ . The input plane has coordinates  $\mathbf{r}_0 = (x_0, y_0)$ , with  $\mathbf{r} = (x, y)$  in the output plane. If the distances of the two planes from the lens satisfy the imaging condition,  $\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}$ , an image of the input plane is formed in the output plane

$$T(\mathbf{r}) = p(\mathbf{r}) e^{+ \frac{ik}{2f} \mathbf{r}^2} \quad (\text{A.22})$$

in position space, where  $p(\mathbf{r})$  is the *aperture function* of the lens. In the simplest case,  $p(\mathbf{r}) = 1$  for  $|\mathbf{r}| \leq R$  and  $p(\mathbf{r}) = 0$  for  $|\mathbf{r}| > R$ , where  $R$  is the radius of the lens. However, in realistic cases,  $p(\mathbf{x})$  may be a more complicated function.

For the situation shown in Fig. A.2, the full Fresnel propagation kernel from  $\mathbf{r}_0$  to  $\mathbf{r}$  is therefore

$$h(\mathbf{r}, \mathbf{r}_0) = \int \left[ \frac{1}{i\lambda z_1} e^{-\frac{ik}{2z_1} |\mathbf{r}' - \mathbf{r}_0|^2} e^{-ikz_1} \right] \left[ p(\mathbf{r}') e^{+ \frac{ik}{2f} \mathbf{r}'^2} \right] \left[ \frac{1}{i\lambda z_2} e^{-\frac{ik}{2z_2} |\mathbf{r} - \mathbf{r}'|^2} e^{-ikz_2} \right] d^2 r', \quad (\text{A.23})$$

where the factors in the three brackets correspond respectively to propagation before the lens, the effect of the lens itself, and propagation after the lens. This may be more conveniently written as

$$h(\mathbf{r}, \mathbf{r}_0) = C(z_1, z_2) e^{-\frac{ik}{2} \left( \frac{|\mathbf{r}_0|^2}{z_1} + \frac{|\mathbf{r}|^2}{z_2} \right)} \int e^{-\frac{ik}{2} D |\mathbf{r}'|^2} e^{ik\mathbf{r}' \cdot \left( \frac{\mathbf{r}_0}{z_1} + \frac{\mathbf{r}}{z_2} \right)} p(\mathbf{r}') d^2 r', \quad (\text{A.24})$$

where  $C(z_1, z_2) = \left( \frac{1}{i\lambda z_1} \right) \left( \frac{1}{i\lambda z_2} \right) e^{-ik(z_1 + z_2)}$  and  $D = \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f}$ . The system is referred to as *in-focus* if  $D = 0$ . Consider some special cases arising from Eq. (A.24).

**(i) Lenses of finite diameter.** Consider the in-focus far-field case ( $z_1 \gg |\mathbf{r}_0|$  and  $z_2 \gg |\mathbf{r}|$ ) for a lens with arbitrary aperture function  $p(\mathbf{r})$ . The quadratic phase factors can be dropped so that the position-space propagator is

$$h(\mathbf{r}, \mathbf{r}_0) = C \int e^{ik\mathbf{r}' \cdot \left( \frac{\mathbf{r}_0}{z_1} + \frac{\mathbf{r}}{z_2} \right)} p(\mathbf{r}') d^2 r'. \quad (\text{A.25})$$

Suppose we wish to image a point source,  $S(\mathbf{r}_0) = \delta^{(2)}(\mathbf{r}_0)$ . The amplitude of the point source in the image plane is known as the *coherent spread function (CSF)* and is given by

$$\text{CSF}(\mathbf{r}) = E(\mathbf{r}) = \int h(\mathbf{r}, \mathbf{r}_0) \delta^{(2)}(\mathbf{r}_0) d^2 r_0 \quad (\text{A.26})$$

$$= C \int p(\mathbf{r}') e^{\frac{ik}{z_2} \mathbf{r} \cdot \mathbf{r}'} d^2 r' = C \tilde{p} \left( -\frac{k\mathbf{r}}{z_2} \right). \quad (\text{A.27})$$

The CSF is therefore proportional to the Fourier transform of the aperture function. For coherent illumination, the image of a point object is given by  $|\text{CSF}(\mathbf{r})|^2$ . The CSF for any optical device is given by the image amplitude in the object plane of a point source or point object. More generally, for coherent illumination with an incident amplitude  $E(\mathbf{r}_0)$  impinging on the optical device, the output amplitude in the image plane is

$$E(\mathbf{r}) = \int \text{CSF}(\mathbf{r} - \mathbf{r}_0) E(\mathbf{r}_0) d^2 r_0. \quad (\text{A.28})$$

For incoherent illumination, the intensities (rather than the amplitudes) must be integrated over the object plane. In that case, the intensity in the output plane is related to the intensity at the input by

$$I(\mathbf{r}) = \int \text{PSF}(\mathbf{r} - \mathbf{r}_0) I(\mathbf{r}_0) d^2 r_0, \quad (\text{A.29})$$

where the *point spread function* (PSF) is given by

$$\text{PSF}(\mathbf{r}) = |\text{CSF}(\mathbf{r})|^2. \quad (\text{A.30})$$

$I(\mathbf{r}) = \langle |E(\mathbf{r})|^2 \rangle$ , so, for the case of incoherent illumination,  $\langle E_0^*(\mathbf{x}) E_0(\mathbf{y}) \rangle \propto \delta(\mathbf{x} - \mathbf{y})$ , Eq. (A.30) is simply the square of Eq. (A.28).

Now let us specialize to the case where the lens radius  $R$  is finite, and that the aperture function (assumed to be circularly symmetric) vanishes for  $r < R$  and is equals 1.0 for  $r > R$ . In this case, by going to radial coordinates it is found that the Fourier transformed aperture function is

$$\tilde{p} \left( \frac{k\mathbf{r}}{z_2} \right) = \int_0^{2\pi} \int_0^R e^{+\frac{ik}{z_2} r r' \cos \theta} r' dr' d\theta \quad (\text{A.31})$$

$$= 2\pi R^2 \left| \frac{J_1 \left( \frac{krR}{z_2} \right)}{\left( \frac{krR}{z_2} \right)} \right|, \quad (\text{A.32})$$

where  $J_1$  is a Bessel function of the first kind. Because of the finite size of the lens, diffraction limits the resolution possible with the lens, smearing the image by modulating it with a series of concentric spherical rings arising from the Bessel function.

In the remaining examples we assume a lens much larger than the beam,  $R \approx \infty$ , so that  $p(\mathbf{x})$  is identically equal to one. The integrals of Eq. (A.24) are then Gaussian

integrations that can be performed exactly. We present three useful applications of these equations.

**(ii) Imaging.** Suppose that the distances obey  $\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}$ . Then the coefficient  $D$  of the quadratic phases in the Gaussian integral of Eq. (A.24) vanishes, so that the lens can simulate the far-field case even at short distances. This produces a magnified image in the output plane of the field in the input plane:

$$h(\mathbf{r}, \mathbf{r}_0) = \frac{1}{i\lambda f} \delta\left(\mathbf{r} + \frac{z_2}{z_1} \mathbf{r}_0\right), \quad (\text{A.33})$$

so that

$$E(\mathbf{r}) = E_0\left(-\frac{z_2}{z_1} \mathbf{r}\right). \quad (\text{A.34})$$

The image is therefore an inverted copy of the object, with magnification  $-\frac{z_1}{z_2}$ .

**(iii)  $2f$  system.** Suppose that  $z_1 = z_2 = f$ . Carrying out the integrations in Eq. (A.24) leads to

$$h_{2f}(\mathbf{r}, \mathbf{r}_0) = \frac{1}{i\lambda f} e^{-2ikf} e^{-\frac{ik}{f} \mathbf{r}_0 \cdot \mathbf{r}}, \quad (\text{A.35})$$

resulting in an output field of the form:

$$E(\mathbf{r}) = \frac{1}{i\lambda f} e^{-2ikf} \tilde{E}_0\left(\frac{k\mathbf{r}}{f}\right). \quad (\text{A.36})$$

The  $2f$  system therefore carries out an optical Fourier transform system; the field in the output plane (*the Fourier plane*) describes the Fourier spectrum of the incident field.

**(iv)  $4f$  system.** The  $4f$  system consists of two sequential  $2f$  systems, possibly with an additional modulator in the intermediate Fourier plane between them (Fig. A.3). With nothing in the central plane, the  $4f$  system simply carries out two successive Fourier transforms, which reproduces an inverted copy of the object: an object  $E_0(\mathbf{r}_0)$  produces an inverted, unit magnification image:

$$E(\mathbf{r}) = -ie^{-4ikf} E_0(-\mathbf{r}). \quad (\text{A.37})$$

The impulse response function is

$$h_{4f}(\mathbf{r}, \mathbf{r}_0) = \frac{1}{i\lambda f} e^{-4ikf} \tilde{T}\left(-\frac{k}{f}(\mathbf{r}_0 + \mathbf{r})\right). \quad (\text{A.38})$$

On the other hand, a filter can be placed between the two  $2f$  systems, which can selectively remove particular components of the Fourier spectrum in order to carry out various types of image analysis or processing. For example, low spatial-frequency noise may be removed to improve spatial resolution of the image.

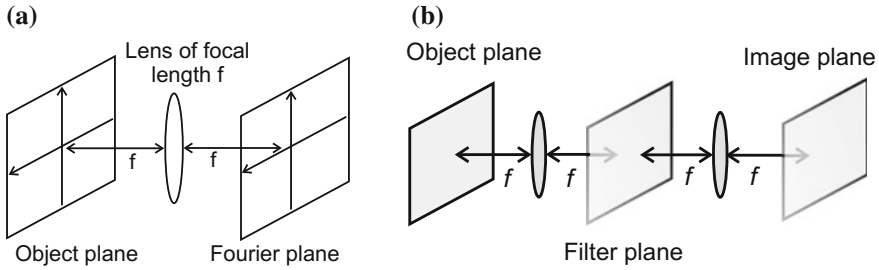


**Fig. A.3** **a** The  $2f$  system images the Fourier transform of the object onto the output Fourier plane. **b** With the intermediate plane empty, the  $4f$  system performs two successive Fourier transform, producing an inverted, unit magnification of the object. Image processing can be performed by inserting a filter in the intermediate plane to remove undesired spatial frequencies

**Source properties and coherence.** In addition to the propagation of light, it is necessary to consider the properties of the light in the source plane. Consider two points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  in the source plane and define the spatial correlation function or coherence function,

$$G(\mathbf{r}_1, \mathbf{r}_2) = \langle E_0(\mathbf{r}_1) E_0(\mathbf{r}_2) \rangle. \quad (\text{A.39})$$

The brackets denote time averaging. In general, if the beam is of Gaussian amplitude profile, the first-order correlation function can often be written in the form

$$G(\mathbf{r}_1, \mathbf{r}_2) = I_0 e^{-|\mathbf{r}_1 - \mathbf{r}_2|^2 / 2\sigma_c^2} e^{-|\mathbf{r}_1 + \mathbf{r}_2|^2 / 2\sigma_0^2}, \quad (\text{A.40})$$

where  $\sigma_c$  is the coherence width of the beam and  $\sigma_0$  is the width of the Gaussian beam profile.  $I_0$  is a constant with dimensions of intensity. The source-plane correlation function plays the role of the two-photon source field in ghost imaging experiments (Chap. 6).

If the beam width is large, then one may approximate

$$G(\mathbf{r}_1, \mathbf{r}_2) \approx I_0 e^{-|\mathbf{r}_1 - \mathbf{r}_2|^2 / 2\sigma_c^2}, \quad (\text{A.41})$$

near its center. Two limiting cases for the coherence width may be distinguished. For a coherent source,  $\sigma_0 \rightarrow \infty$ , the coherence function is equal to  $I_0$ . At the opposite extreme,  $\sigma_0 \rightarrow 0$  for an incoherent source; in this case the coherence function is proportional to  $\delta^{(2)}(\mathbf{r}_1 - \mathbf{r}_2)$ .

In general, the intensity at point  $\mathbf{r}$  in the detector is  $I(\mathbf{r}) \sim \langle E^*(\mathbf{r}) E(\mathbf{r}) \rangle$ , up to overall constants. For the two extreme cases, we find

$$I \sim \left| \int E(\mathbf{r}_0) h(\mathbf{r}, \mathbf{r}_0) d^2 r \right|^2, \text{ for coherent imaging,} \quad (\text{A.42})$$

$$I \sim \int |E(\mathbf{r}_0) h(\mathbf{r}, \mathbf{r}_0)|^2 d^2 r, \text{ for incoherent imaging.} \quad (\text{A.43})$$

### A.3 Beam Splitters

Classically, a *beam splitter* (BS) divides the amplitude of an incoming beam into two spatially separated portions, or mixes two separate incoming beams into a combined outgoing beam. A beam splitter is a passive linear four-port device, consisting of two incoming and two outgoing ports. Quantum mechanically, even if no incident light is applied to one of the input ports the existence of this second port can have effects, since vacuum fluctuations can enter it and be mixed with the light entering the other port. This has the effect of increasing the noise and the quadrature variances of the outgoing light.

Consider the beam splitter shown in Fig. A.4, where  $\hat{a}_0$  and  $\hat{a}_1$  are the annihilation operators for states arriving at the input ports, and  $\hat{a}_2$  and  $\hat{a}_3$  are the corresponding operators at the output ports. We may define the reflection and transmission coefficients associated with each output port ( $r_2, r_3, t_2, t_3$ ). Then the output operators are linear combinations of the input operators:

$$\hat{a}_2 = r_2 \hat{a}_1 + t_2 \hat{a}_0 \quad (\text{A.44})$$

$$\hat{a}_3 = t_3 \hat{a}_1 + r_3 \hat{a}_0, \quad (\text{A.45})$$

or, more conveniently:

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = B \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}, \quad B = \begin{pmatrix} t_2 & r_2 \\ r_3 & t_3 \end{pmatrix}. \quad (\text{A.46})$$

The entries in the beam splitter matrix  $B$  must satisfy several constraints. First, the matrix must be unitary in order to conserve total probability of finding a photon:  $B^\dagger = B$ . Second, the annihilation operators must obey canonical commutation relations:

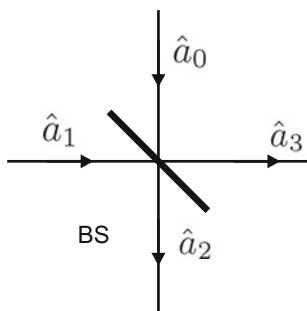


**Fig. A.4** Schematic of a beam splitter (BS). The annihilation operators for states entering the input ports are  $\hat{a}_0$  and  $\hat{a}_1$ . Similarly,  $\hat{a}_2$  and  $\hat{a}_3$  represent the outgoing states. In the case of a 50/50 beam splitter (equal probability for reflection and transmission), the transmitted light undergoes no phase shift, but the reflected light has a phase shift of  $\frac{\pi}{2}$

$$\left[ \hat{a}_i, \hat{a}_j^\dagger \right] = \delta_{ij}, \quad \left[ \hat{a}_i, \hat{a}_j \right] = \left[ \hat{a}_i^\dagger, \hat{a}_j^\dagger \right] = 0, \quad (\text{A.47})$$

for  $j = 2, 3$ . Together, these constraints imply the relations:

$$|r_2| = |r_3|, \quad |t_2| = |t_3| \quad (\text{A.48})$$

$$r_2^* t_3 + r_3 t_2^* = r_2^* t_2 + r_3 t_3^* = 0. \quad (\text{A.49})$$

For the symmetric case of a 50/50 beam splitter, where  $|r_i|^2 = |t_i|^2 = \frac{1}{2}$  for  $i = 2, 3$ , there are several solutions that satisfy these constraints, including

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (\text{A.50})$$

and

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (\text{A.51})$$

Throughout this book we assume that all beam splitters are described by Eq. (A.51).

For the latter matrix, the reflected beam is phase shifted by  $\frac{\pi}{2}$  relative to the transmitted beam. This is true regardless of which port the incident beam enters. In the more general case of arbitrary reflectance, the latter may not be true. However the phase shifts of the two reflected beams must still add up to  $\pi$  [7]. The  $\frac{\pi}{2}$  phase shift is correct for beam splitters made from a single dielectric layer between two pieces of glass; multilayer beam splitters can be designed with other phase shifts [8].

In the Heisenberg picture of quantum mechanics, the beam splitter can also be described by the action of a unitary operator,

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = U^\dagger \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix} U, \quad (\text{A.52})$$

where

$$U = e^{\frac{i\pi}{4} (\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_0 \hat{a}_1^\dagger)}. \quad (\text{A.53})$$

The descriptions in terms of  $U$  and  $B$  are readily shown to be equivalent by an application of the Baker–Campbell–Hausdorff formula.

The beam splitters described above are non-polarizing: orthogonal polarization directions are acted on in an identical manner by the beam splitter, so that if the incident light is unpolarized, then both of the outgoing beams are unpolarized as well. In contrast, a *polarizing beam splitter* (PBS) ideally transmits all of the incident light that is polarized along some axis and reflect all light polarized perpendicular to that axis. The PBS therefore separates unpolarized incident light into two polarized beams, with different polarizations spatially separated from each other.

A *dichroic mirror* is a form of beam splitter in which wavelengths below a certain cutoff are reflected, while those above the cutoff are transmitted. It therefore provides

a simple means of separating different wavelength or frequency ranges into different spatial paths; dichroic mirrors are used for example in the interferometer of Sect. 2.7 and the microscope of Sect. 7.9.

By combining multiple beam splitters and mirrors, it is possible to creating multiport generalizations of the beam splitter [9, 10]. These multiports take  $n$  inputs to  $n$  outputs, and are useful in a variety of applications.

## A.4 Dove Prisms

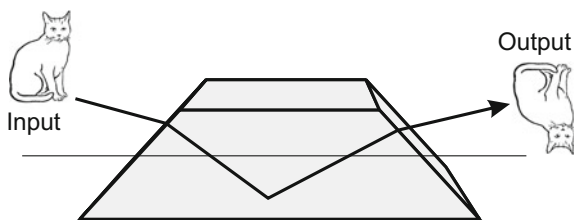
Another linear optical device that is used in Chap. 8 is the *Dove prism*. The prism, shown in Fig. A.5, is designed so that total internal reflection occurs on the bottom surface. The result is that images are inverted in the vertical direction, without any corresponding horizontal inversion. One interesting property of these prisms is that if they are rotated by an angle  $\theta$  about the propagation axis, the image is rotated by  $2\theta$ . Polarization of the incoming light is also rotated in the same manner [11].

In contrast, for well-collimated light with a well-defined value of orbital angular momentum,  $L = l\hbar$ , the Dove prism reverses the angular momentum direction, while maintaining its absolute value:  $l \rightarrow -l$ . However, this is not necessarily true if the beam is tightly focused. Tight focusing can turn a state of definite angular momentum into a superposition of multiple OAM values; see [12] for a quantitative discussion of this effect.

## A.5 Phase Conjugate Mirrors

One further optical device used in this text (Chap. 7) is the *phase conjugate mirror*. It was realized in the 1970s [13] that when a right-moving optical field  $\mathbf{E}(\mathbf{x}) = E_0(x, y) e^{i(\omega t - kz)}$  is sent into a nonlinear material, four wave mixing can lead to production of a field  $\mathbf{E}_c(\mathbf{x})$  that moves toward the left and whose spatial profile is proportional to the complex conjugate of the original:  $\mathbf{E}_c(\mathbf{x}) = K E_1 E_2 E_0^*(x, y) e^{i(\omega t + kz)}$ , where  $E_1$  and  $E_2$  are the complex amplitudes of two counter-propagating pump fields

**Fig. A.5** Dove prism. Total internal reflection at the horizontal surface causes images to be inverted vertically



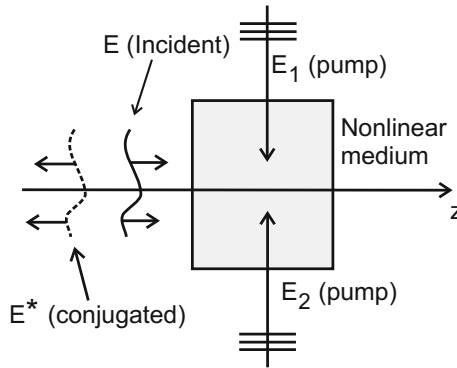


**Fig. A.6** A phase conjugate mirror. The pump beams  $E_1$  and  $E_2$  undergo four-wave mixing with the incident beam  $E$ , to produce an outgoing beam proportional to the complex conjugate of  $E$

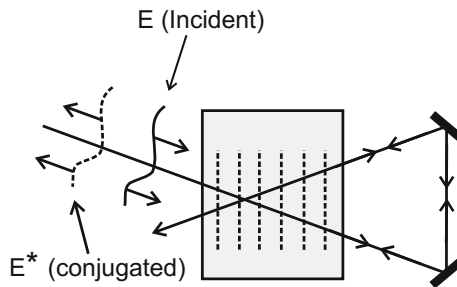


**Fig. A.7** One arrangement that can lead to self-pumped phase conjugation, in which the incident beam that is to be conjugated passes through the nonlinear medium multiple times, so that it can also act as the pump beams as well. Part of the incident beam passes out the other side of the material, where a pair of mirrors in a ring cavity arrangement directs it back in. Scattering off a virtual diffraction grating formed in the material allows for reversal of direction about the rings, leading to counterpropagating waves inside the material

(Fig. A.6) that drive the nonlinear interaction, and  $K$  is a constant determined by the nonlinear coupling constant of the material.

Various self-pumping arrangements can be constructed, in which the two pump fields are provided by the input beam itself [14, 15], by having it pass through the medium multiple times, as in Fig. A.7. In these schemes, interference between the waves creates virtual holographic gratings inside the material, leading to nonlinear backscattering.

Phase conjugate mirrors have a number of unusual features. For example, the reflected beam is always directed back along the incident direction, regardless of the angle at which the incident beam hit the mirror. One important use of phase conjugate mirrors is to compensate for aberrations and other distortions. Consider a beam passing through an aberrative medium and then reflecting back off a phase conjugate mirror so that the conjugate beam then passes through the medium. The

phase distortions added during the two passes are negatives of each other; if  $E(\mathbf{x})$  gains a phase factor  $e^{i\phi(\mathbf{x})}$ , then  $E_c(\mathbf{x})$  gains phase factor  $e^{-i\phi(\mathbf{x})}$ . As a result the total phase gained on the round trip is zero, and the outgoing beam is undistorted.

Recently, a method of constructing phase conjugate mirrors without the use of nonlinear optics has been demonstrated [16]. In this approach, the phase of the incident wavefront is calculated by means of digital holographic recording, then the conjugated phase is imprinted onto the reflected beam using a spatial light modulator.

For more detail on nonlinear phase conjugation, see [1, 17–19]. Since the late 1990s a great deal of work has been done on creating optical phase conjugation in fibers by means of stimulated Brillouin scattering, starting with the papers [20–23]; for further discussion, see [24].

## A.6 Laser Beam Optics

The beam from a laser typically has a Gaussian intensity profile in the plane transverse to propagation. Suppose that the beam propagates along the  $z$ -axis, with coordinate vector  $\mathbf{r}$  in the plane perpendicular to the axis. Let  $w_0$  be the beam radius at its narrowest point (the *beam waist*), and take  $z = 0$  to be at the location of the waist. The maximum intensity  $I_0$  occurs on-axis at  $z = 0$ . Further, define the *Rayleigh range*  $z_0$  to be the distance from the waist at which the beam cross-section is doubled and the intensity halved ( $I(z_0) = \frac{1}{2}I_0$ .) The *depth of focus* or *confocal parameter* is the distance  $b = 2z_0$  over which the intensity varies by a factor of no more than 2. The Rayleigh range and the waist radius are related by  $w_0 = \sqrt{\lambda z_0 / \pi}$ , implying that a highly focused beam (small  $w_0$ ) rapidly diverges (small  $z_0$ ) as it moves away from the waist.

Then the amplitude  $E$  and intensity  $I$  of the beam can be expressed in the forms

$$E(\mathbf{r}, z) = \sqrt{I_0} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{ikz + \frac{ikr^2}{2R(z)} - i\zeta(z)}, \quad (\text{A.54})$$

$$I(\mathbf{r}, z) = |E(\mathbf{r}, z)|^2 = I_0 \left( \frac{w_0}{w(z)} \right)^2 e^{-\frac{2r^2}{w^2(z)}}. \quad (\text{A.55})$$

Here,

$$w(z) = w_0 \left[ 1 + \left( \frac{z}{z_0} \right)^2 \right]^{1/2} \quad (\text{A.56})$$

and

$$R(z) = z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right] \quad (\text{A.57})$$

are respectively the beam radius and the radius of curvature of the wavefront at distance  $z$  from the waist. The factor

$$\zeta(z) = \tan^{-1} \left( \frac{z}{z_0} \right) \quad (\text{A.58})$$

is the Guoy phase, which varies from  $-\frac{\pi}{2}$  to  $+\frac{\pi}{2}$  as the beam propagates from  $z = -\infty$  to  $z = +\infty$ . The power carried by the beam is given by  $P = \frac{\pi}{2} I_0 w_0^2$ .

For more detail on Gaussian beams see [1] or [25]. For a review of other types of beams, such as Laguerre–Gauss, Hermite–Gauss, and Bessel beams, see [26].

## References

1. B.E.A. Saleh, M.C. Teich, *Fundamentals of Photonics, 2nd Ed.* (John Wiley & Sons, Inc., Hoboken, 2007)
2. J.W. Goodman, *Fourier Optics* (Roberts and Company Publishers, Greenwood Village, CO, 2005)
3. E. Hecht, *Optics*, 4th edn. (Addison-Wesley, San Francisco, 2002)
4. M. Born, E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, 7th edn. (Cambridge University Press, Cambridge, 1999)
5. G.O. Reynolds, J.B. De Velis, G.B. Parrent, B.J. Thompson, *The New Physical Optics Notebook: Tutorials in Fourier Optics* (SPIE, Bellington, WA and American Institute of Physics, N.Y., 2000)
6. A. Lipson, S. Lipson, H. Lipson, *Optical Physics, 4th ed.* (Cambridge University Press, Cambridge, 2010)
7. A. Zeilinger, Am. J. Phys. **49** 882, (1981)
8. M.W. Hamilton, Am. J. Phys. **68**, 186 (2000)
9. M. Reck, A. Zeilinger, H. J. Bernstein, P. Bertani, Phys. Rev. Lett. **73**, 58 (1994)
10. D.S. Simon, C.A. Fitzpatrick, A.V. Sergienko, Phys. Rev. A **93**, 043845 (2016)
11. M.J. Padgett, J.P. Lesso, J. Mod. Opt. **46**, 175, (1999)
12. N. Gonzalez, G. Molina-Terriza, J.P. Torres, Opt. Express **14** 9093 (2006)
13. B.Y. Zeldovich, V.I. Popovichev, V.V. Ragulskiy, F.S. Faizullov, Zh. Eksp. Teor. Fiz. Pisma Red. **15**, 160 (1972) [Sov. Phys. JETP **15**, 109 (1972)]
14. J. Feinberg, Opt. Lett. **7**, 486 (1982)
15. M. Cronin-Golomb, B. Fischer, J.O. White, A. Yariv, Appl. Phys. Lett. **41**, 689 (1982)
16. I.N. Papadopoulos, S. Farahi, C. Moser, D. Psaltis, Opt. Exp. **20**, 10583 (2012)
17. J.P. Huignard, A. Brignon, in *Phase Conjugate Laser Optics*, ed. by Arnaud Brignon and Jean-Pierre Huignard (John Wiley & Sons, Inc., 2004)
18. B.Y. Zeldovich, N.F. Pilipetsky, V.U. Shkunov, *Principles of Phase Conjugation* (Springer-Verlag, Berlin, 1985)
19. G.S. He, Prog. Quant. Electr. **26**, 131 (2002)
20. H.J. Eichler, J. Kunde, B. Liu, Opt. Commun. **139**, 327 (1997)
21. H.J. Eichler, J. Kunde, B. Liu, Opt. Lett. **44**, 495 (1997)
22. A. Heuer, R. Menzel, Opt. Lett. **23**, 384 (1998)

23. H. Joshida, H. Fujita, M. Nakatsuka, K. Joshida, Jpn. J. Appl. Phys. **38**, L521 (1999)
24. G.P. Agrawal, *Nonlinear Fiber Optics*, 2nd edn. (Academic Press, New York, 1995)
25. A.E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986)
26. F. Pampaloni, J. Enderlein, [arXiv:physics/0410021](https://arxiv.org/abs/physics/0410021) (2004)

# Appendix B

## Optical Fields in Quantum Mechanics

### B.1 Fields and Quadratures

In quantum mechanics, the electric field is an operator, often written as

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) + \hat{\mathbf{E}}^{(-)}(\mathbf{r}, t), \quad (\text{B.1})$$

where the positive frequency part,  $\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)$  oscillates as  $e^{-i\omega t}$  and contains all annihilation operators. Normalized to volume  $V$ , it takes the form

$$\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) = i \sum_{\mathbf{k}, s} \left( \frac{\hbar \omega_{\mathbf{k}}}{2\epsilon_0 V} \right)^{1/2} \mathbf{e}_s(\mathbf{k}) \hat{a}_s(\mathbf{k}) e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})}, \quad (\text{B.2})$$

where  $\mathbf{e}_s(\mathbf{k})$  is a polarization unit vector, and the sum is over the optical modes included. The polarization vectors  $\mathbf{e}_s(\mathbf{k})$  for  $s = 1, 2, 3$  form an orthonormal set,  $\mathbf{e}_s(\mathbf{k}) \cdot \mathbf{e}_{s'}(\mathbf{k}) = \delta_{ss'}$ , and are transverse to the photon momentum vector,  $\mathbf{e}_s(\mathbf{k}) \cdot \mathbf{k} = 0$ . Here and in the following, all expressions are written in the radiation or Coulomb gauge, which has vanishing scalar potential ( $\Phi = 0$ ) and transverse vector potential ( $\nabla \cdot \mathbf{A} = 0$ ).

The negative frequency part, which contains the creation operators and a time dependence of the form  $e^{+i\omega t}$  is simply the Hermitian conjugate of the positive frequency part,  $\hat{\mathbf{E}}^{(-)}(\mathbf{r}, t) = [\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)]^\dagger$ . Despite the terminology, the frequencies  $\omega_{\mathbf{k}}$  are always taken to be positive. The creation and annihilation operators obey canonical commutation relations,

$$[\hat{a}_s(\mathbf{k}), \hat{a}_{s'}(\mathbf{k}')^\dagger] = \delta_{s,s'} \delta_{\mathbf{k}', \mathbf{k}}. \quad (\text{B.3})$$

Creation operators add a photon of the corresponding mode

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad (\text{B.4})$$

while annihilation operators remove a photon,

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle, \quad (\text{B.5})$$

where the momentum and polarization labels have been suppressed for simplicity. Here, the *Fock states*,  $|n\rangle$ , are states of definite photon number. There exists a lowest energy state (a state with no photons, referred to as the *vacuum state*), which is annihilated by all  $\hat{a}_s(\mathbf{k})$ :

$$\hat{a}_s(\mathbf{k})|0\rangle = 0. \quad (\text{B.6})$$

The expression of Eq. (B.2) assumes a discrete set of modes confined to a finite volume. The continuous-mode version of this expression is

$$\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) = i \sum_s \int d^3k \sqrt{\frac{\omega_k}{(2\pi)^3}} \mathbf{e}_s(\mathbf{k}) \hat{a}_s(\mathbf{k}) e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{r})}, \quad (\text{B.7})$$

which follows via  $\mathbf{E} = -\frac{d\mathbf{A}}{dt}$  from the Coulomb gauge vector potential

$$\mathbf{A}(\mathbf{r}, t) = \sum_s \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} \mathbf{e}_s(\mathbf{k}) [\hat{a}_s(\mathbf{k}) e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} + \hat{a}_s^\dagger(\mathbf{k}) e^{+i(\omega_k t - \mathbf{k} \cdot \mathbf{r})}]. \quad (\text{B.8})$$

The Fock states are states of definite photon number. Equivalently, they are eigenstates of the *photon number operator*,  $\hat{N} = \hat{a}^\dagger \hat{a}$ :  $\hat{N}|n\rangle = n|n\rangle$ . The phase of such a state is indeterminate. At the opposite extreme are the *coherent states*, which have well-defined phase but no definite photon number. Coherent states are superpositions of all possible Fock states, with Poisson-distributed amplitudes:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (\text{B.9})$$

where the complex number  $\alpha$  is the amplitude of the coherent state. These states are eigenstates of the annihilation operator:  $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ . The photon number operator has mean value  $\langle \hat{N} \rangle = |\alpha|^2$  in a coherent state  $|\alpha\rangle$ , with uncertainty  $\Delta N = \langle \hat{N} \rangle^{1/2} = |\alpha|$ . The uncertainty of the phase is  $\Delta\phi = \frac{1}{2|\alpha|}$ . It should be noted that coherent states with different amplitudes are not orthogonal to each other; rather they obey

$$\langle \alpha | \beta \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^* \beta}. \quad (\text{B.10})$$

It is often useful to define the position and momentum quadratures

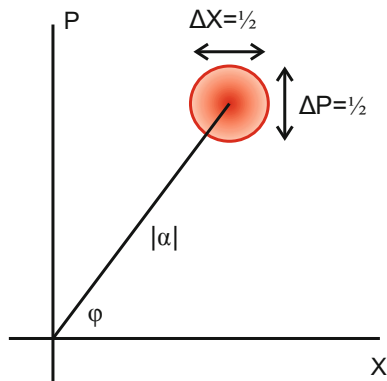


**Fig. B.1** Phase space probability amplitude for a coherent state of amplitude  $\alpha$ . The width of the amplitude is symmetric about the maximum, and occupies minimal phase space area. These states are essentially displaced vacuum states: the vacuum state is identical, but centered at the origin. Squeezed states are elliptical, rather than rotationally symmetric about the maximum, but still have minimal area

$$\hat{X} = \frac{1}{2} (\hat{a}^\dagger + \hat{a}) \quad (\text{B.11})$$

$$\hat{P} = \frac{i}{2} (\hat{a}^\dagger - \hat{a}). \quad (\text{B.12})$$

Inverting these equations, we may also write:

$$\hat{a} = \hat{X} + i\hat{P} \quad (\text{B.13})$$

$$\hat{a}^\dagger = \hat{X} - i\hat{P}. \quad (\text{B.14})$$

The quadratures have uncertainties  $\Delta X = \Delta P = \frac{1}{2}$  for both vacuum states and coherent states, while for Fock states the variances are given by  $\Delta X^2 = \Delta P^2 = \frac{1}{2} \left( \langle \hat{N} \rangle + \frac{1}{2} \right)$ .

Coherent states (Fig. B.1) are in a sense the most classical possible quantum state of light, having equal quadratures and having the minimum possible uncertainty product:  $\Delta X \Delta P = \frac{1}{4}$ . (Note that the value on the right hand side depends on the normalization used to define the quadrature operators; it is different in various references due to different choices of conventions.) In a coherent state, the photon number operator has variance equal to its expectation value,  $\Delta N^2 = \langle \hat{N} \rangle = |\alpha|^2$ , with phase uncertainty  $\Delta \phi = \frac{1}{2|\alpha|}$ .

States which have the same minimal uncertainty product as the coherent states but which have  $\Delta X \neq \Delta P$  are called *squeezed states*. Squeezed states have a number of applications and are the subject of a large literature (see for example [1–6]).

More detail about the quantum treatment of electromagnetic fields may be found in any text on quantum optics, such as [3–6].

## B.2 Homodyne Measurement of Field Quadratures

Recall that the quadrature operators are related to the annihilation operators by

$$\hat{X} = \frac{1}{2} (\hat{a}^\dagger + \hat{a}) \quad (\text{B.15})$$

$$\hat{P} = \frac{i}{2} (\hat{a}^\dagger - \hat{a}). \quad (\text{B.16})$$

The goal of *homodyne detection* is to measure these two quadratures, or any other pair of quadratures related to them by a phase rotation. Quadrature measurements are needed for applications such as continuous variable quantum key distribution (Sect. 9.4).

Consider the setup shown in Fig. B.2. At input port *A* is the state whose quadratures are to be measured. At port *B*, a reference beam (the *local oscillator*) is inserted. The reference beam is taken to be a coherent state,  $|\beta_0\rangle$ . Before entering the interferometer, this beam is sent through a phase shifter to produce the state  $|\beta\rangle = |\beta_0 e^{i\theta}\rangle$ . Let  $\hat{a}$  and  $\hat{b}$  be the annihilation operators corresponding to the states  $|\alpha\rangle$  and  $|\beta\rangle$ . Since  $|\beta\rangle$  is a coherent state, it follows that  $\hat{b}|\beta\rangle = \beta_0 e^{i\theta}|\beta\rangle$  at port *B*. The beam splitter is assumed to be 50/50, so that the situation being discussed here is referred to as *balanced* homodyne detection. (Also it is assumed that the signal and local oscillator have the same frequency; if the frequencies are different the measurement is referred to as *heterodyne detection*. This terminology originated in radio engineering.)

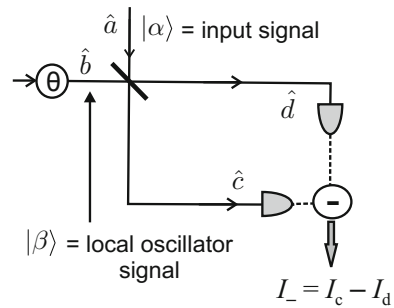
The currents generated in the two detectors are proportional to the average number of incoming photons in the corresponding input channels:

$$I_c = \eta \langle \hat{c}^\dagger \hat{c} \rangle \quad (\text{B.17})$$

$$I_d = \eta \langle \hat{d}^\dagger \hat{d} \rangle, \quad (\text{B.18})$$

where  $\eta$  is some constant dependent on the properties of the detectors, and  $\hat{c}$  and  $\hat{d}$  are the annihilation operators leaving the beam splitter. Rather than recording these two currents directly, their difference is recorded:

**Fig. B.2** Schematic of setup for homodyne measurement. The output signal is the difference of the output currents of the two detectors



$$I_- = I_c - I_d. \quad (\text{B.19})$$

Since the annihilation operators at the beam splitter outputs are related to those at the inputs by the relations

$$\hat{c} = \frac{1}{\sqrt{2}} (\hat{a} + i\hat{b}) \quad (\text{B.20})$$

$$\hat{d} = \frac{1}{\sqrt{2}} (\hat{b} + i\hat{a}), \quad (\text{B.21})$$

the difference current may therefore be expressed in terms of the input in port A:

$$I_- \equiv I_c - I_d \quad (\text{B.22})$$

$$\begin{aligned} &= \eta \left( \langle \hat{c}^\dagger \hat{c} \rangle - \langle \hat{d}^\dagger \hat{d} \rangle \right) \\ &= \frac{\eta}{2} \left[ \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle + i \langle \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} \rangle - \langle \hat{b}^\dagger \hat{b} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle - i \langle \hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b} \rangle \right] \\ &= i\eta \langle \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} \rangle \end{aligned} \quad (\text{B.23})$$

Given the assumptions made above about the incoming states, it is then the case that:

$$I_- = i\eta \left[ \beta_0 \langle \hat{a}^\dagger \rangle e^{i\theta} - \beta_0^* \langle \hat{a} \rangle e^{-i\theta} \right] \quad (\text{B.24})$$

$$= \eta\beta_0 \left[ \langle \hat{a}^\dagger \rangle e^{i\phi} + \langle \hat{a} \rangle e^{-i\phi} \right], \quad (\text{B.25})$$

where  $\phi = \theta + \frac{\pi}{2}$ .

Thus, for  $\phi = 0$  ( $\theta = -\frac{\pi}{2}$ ) the output current measures the  $X$  quadrature,

$$I_- = \eta\beta_0 \langle \hat{X} \rangle, \quad (\text{B.26})$$

while for  $\phi = \frac{\pi}{2}$  ( $\theta = 0$ ), the  $P$  quadrature is measured:

$$I_- = i\eta\beta_0 \langle \hat{P} \rangle. \quad (\text{B.27})$$

More generally, by varying the local oscillator phase, any quadrature on the unit circle in the  $X - P$  plane can be measured; for angle  $\theta$ ,

$$I_-(\phi) = \eta\beta_0 \left( \langle \hat{X} \rangle \cos \phi + i \langle \hat{P} \rangle \sin \phi \right). \quad (\text{B.28})$$

The ability to measure quadratures along different directions in the phase plane is one of the key tools used in continuous variable quantum key distribution (Sect. 9.4).

## References

1. D. Walls, Squeezed states of light, *Nature* **306**, 141 (1983)
2. M.C. Teich, B.E.A. Saleh, *Quantum Opt.* **I**, 152 (1989)
3. M. Fox, *Quantum Optics: An Introduction* (Oxford University Press, Oxford, 2006)
4. C.G. Gerry, P.L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005)
5. R. Loudon, *The Quantum Theory of Light*, 3rd edn. (Oxford University Press, Oxford, 2000)
6. M.O. Scully, M.S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997)

## Appendix C

# Optical Effects of Aberration and Turbulence

Much of this book deals with using correlation or entanglement between multiple light beams in order to mitigate distortions of optical measurements or images due to the effects of dispersion, aberration, and turbulence in the light path. These three types of distorting effects are similar in a number of ways: for example, aberration is essentially a direct spatial analog of temporal dispersion, while the effect of turbulence is to introduce an amount of aberration that fluctuates randomly in time and space. Dispersion, being simpler than the other two effects, is discussed in Chap. 3. Here we give brief reviews of the more complicated topics of static aberration and turbulence.

### C.1 Aberration

Aberration or spatial dispersion [1–4] occurs when light passing through or reflecting off an object gains unwanted phase-shifts that vary in the transverse spatial direction (orthogonal to the propagation axis). These phase shifts are “unwanted” in the sense that they differ from those obtained from Gaussian optics and cause distortions of the outgoing wavefronts. All optical systems introduce some level of aberration, leading to blurring of images.

Suppose that a lens or other optical device has an ideal (undistorted) pupil function  $p_0(\mathbf{x})$ , where  $\mathbf{x}$  is the transverse position. Mathematically, we can represent the aberrations by pure imaginary exponentials  $e^{i\phi(\mathbf{x})}$  representing phase shifts added to the wavefronts. The aberrated pupil function then becomes  $p(\mathbf{x}) = p_0(\mathbf{x})e^{i\phi(\mathbf{x})}$ . Assuming a circular aperture of radius  $R$ , define a normalized radial coordinate  $r = \frac{|\mathbf{x}|}{R}$ . Then one common way to parameterize the phase shifts is in terms of the *Seidel aberrations* which describe terms up to third order in  $x$ :

$$\phi(\mathbf{x}) = \Phi_p(\mathbf{x}) + \Phi_t(\mathbf{x}) + \Phi_d(\mathbf{x}) + \Phi_a(\mathbf{x}) + \Phi_c(\mathbf{x}) + \Phi_s(\mathbf{x}) + \cdots, \quad (\text{C.1})$$

where the neglected terms are order  $r^4$  or higher in the radial parameter. The individual terms in this expansion have the form

$$\begin{aligned}
 \Phi_p(\mathbf{x}) &= A_0 && \text{(Piston)} \\
 \Phi_t(\mathbf{x}) &= A_1 r^2 \cos \theta + A_2 r^2 \sin \theta && \text{(Tilt)} \\
 \Phi_d(\mathbf{x}) &= A_3 r^2 && \text{(Defocus)} \\
 \Phi_a(\mathbf{x}) &= A_4 r^2 \cos 2\theta + A_5 r^2 \sin 2\theta && \text{(Astigmatism)} \\
 \Phi_c(\mathbf{x}) &= A_6 r^3 \cos \theta + A_7 r^3 \sin \theta && \text{(Coma)} \\
 \Phi_s(\mathbf{x}) &= A_8 r^4 && \text{(Spherical Aberration)}
 \end{aligned}$$

*Piston* is the mean value of the phase over the pupil, describing an overall phase shift of all the rays. *Tilt* is a deviation of the propagation direction of the outgoing rays. The other terms describe the angular dependence of the focal length about the optical axis and the dependence of focal length on the distance of the incident ray from the axis. These are all monochromatic aberrations; in addition, there can also be chromatic aberrations, in which waves of different frequencies focus at different distances.

An alternative description of aberrations is via the *Zernike polynomials*, which form a complete, orthonormal set over the disk-shaped aperture. The aberrative phase may then be expanded in terms of these polynomials:

$$\phi(\mathbf{x}) = \sum_{nm} a_{nm} Z_n^m(r, \theta). \quad (\text{C.2})$$

Although there are a number of different notational conventions in the literature, one way to describe these polynomials is to write them in the form

$$Z_n^m(r, \theta) = \sqrt{2(n+1)} R_n^m(r) G^m(\theta). \quad (\text{C.3})$$

The radial functions are

$$R_n^m(r) = \sum_{s=0}^{(n-m)/2} \frac{(-1)^s (n-s)!}{s! \left(\frac{n+m}{2} - s\right)! \left(\frac{n-m}{2} - s\right)!}, \quad (\text{C.4})$$

for  $m < n$ . The angular functions are either even or odd functions of  $\theta$ , where the even functions are

$$G^m(\theta) = \cos m\theta \quad (\text{C.5})$$

and the odd functions are

$$G^{-m}(\theta) = \sin m\theta. \quad (\text{C.6})$$

Sometimes instead of the pair of indices  $m$  and  $n$ , the polynomials are labelled in the *Noll notation* [5] by a single index  $i$ .

The orthonormality relations for the various polynomials defined above are:

$$\int_0^1 R_n^m(r) R_p^m(r) r dr = \frac{\delta_{np}}{2n+1} \quad (\text{C.7})$$

$$\int_0^{2\pi} G^m(\theta) G^q(\theta) d\theta = \pi \delta_{mq} \quad (\text{C.8})$$

$$\int_0^{2\pi} \int_0^1 Z_n^m(r, \theta) Z_{n'}^{m'}(r, \theta) r dr d\theta = \pi \delta_{mm'} \delta_{nn'}. \quad (\text{C.9})$$

Using these relations, the expansion coefficients in eg. C.2 are seen to be given by

$$a_{mn} = \frac{\int_0^{2\pi} \int_0^1 \phi(r, \theta) Z_n^m(r, \theta) r dr d\theta}{\int_0^{2\pi} \int_0^1 [Z_n^m(r, \theta)]^2 r dr d\theta}. \quad (\text{C.10})$$

A convenient figure of merit to describe the degree of aberration in a system is the *Strehl ratio*,

$$\mathcal{S} = \left| \frac{\int p(\mathbf{x}) d^2x}{\int p_0(\mathbf{x}) d^2x} \right|^2 = |\langle e^{i\phi} \rangle|^2. \quad (\text{C.11})$$

The average in the last expression is taken over the area of the pupil. For an ideal, aberration-free system  $\mathcal{S} = 1$ , while the presence of aberration causes  $\mathcal{S} < 1$ .

## C.2 Turbulence

In the atmosphere, constantly changing temperature and pressure gradients lead to an index of refraction  $n(\mathbf{x}, t)$  that fluctuates with both position and time. The fluctuations typically occur on a millisecond time scale, and lead to ever-changing speckle patterns on images obtained from short-term exposures. Over longer exposures, the changing speckle patterns are time-averaged so that the principal effect of turbulence is to broaden the point spread function, leading to reduced resolution. The fluctuating index generally leads to phase fluctuations of the optical field, but since differential phase shifts cause bending of light rays this can also result in amplitude fluctuations (*scintillation* or *twinkling*) over sufficiently long distances.

The turbulence causes the appearance of eddies with varying temperatures and refractive indices, ranging in size from the *outer scale*  $L_0$  (roughly proportional to height at low altitudes, and typically tens to hundreds of meters higher up) down to the *inner scale*  $l_0$  (typically millimeter or centimeter in size, increasing with height). Eddies at smaller scales tend to be damped out by the viscosity of the atmosphere. The range of lengths from  $l_0$  to  $L_0$  is known as the *inertial subrange*. Typically, the ratio of the inner and outer scales is determined by the Reynolds number of air,  $R_e$ :  $\frac{L_0}{l_0} = R_e^{3/4}$ .

Let  $\delta n = n - 1$  be the deviation of the refractive index from its average value. The effect of propagation through a turbulent medium is to multiply the amplitude of the optical field by some factor  $e^{\chi(x)+i\phi(x)}$ , where  $\chi$  and  $\phi$  are randomly varying functions describing amplitude and phase fluctuations. We ignore possible amplitude fluctuations and focus on the dominant phase fluctuations,  $\Delta\phi$ . These are related to the refractive index fluctuations by integrating along the light beam's trajectory:  $\Delta\phi = \int k \delta n(z) dz$ . Here,  $k$  is the wavenumber, and  $z$  axis is taken to be along the direction of propagation of the light. A convenient measure of the turbulence strength is the averaged squared fluctuations  $C_n^2 = \langle \Delta n^2 \rangle$ , where  $\Delta n$  is taken to be the difference in  $n$  of a pair of points separated by one meter and the brackets denote time averaging. Typical atmospheric values of  $C_n^2$  range from  $10^{-17} \text{ m}^{-2/3}$  (weak turbulence) to  $10^{-13} \text{ m}^{-2/3}$  (strong turbulence), with values increasing at lower altitudes.

Because the time required to cross a turbulent eddy is typically much smaller than the time for refractive index fluctuations inside the eddy ( $\sim 100 \mu\text{s}$ ), the *frozen turbulence model* is often used, in which each eddy is taken to be of fixed (but random)  $n$ . The spatial and temporal variations are then entirely due to convective air motions, as these frozen eddies are randomly blown across the optical path.

The power spectrum density (the Fourier transform of the correlation function) for refractive index fluctuations over size scales within the inertial range is often modeled by the homogeneous, isotropic Kolmogorov distribution [6],

$$\Phi_n^K(\kappa) = .033 C_n^2 \kappa^{-11/3}, \quad (\text{C.12})$$

where  $\kappa$  is the spatial frequency of the eddies. To avoid the pole at  $\kappa = 0$ , the von Karman distribution is sometimes used instead:

$$\Phi_n^V(\kappa) = .033 C_n^2 \left( \kappa^2 + \left( \frac{2\pi}{L_0} \right)^2 \right)^{-11/6} e^{-(l_0 \kappa / 5.92)^2}. \quad (\text{C.13})$$

Both of these distributions match the empirical data very well.

For any stationary function  $f(\mathbf{x}_1, \mathbf{x}_2) = f(\mathbf{x}_2 - \mathbf{x}_1)$  with mean  $\bar{f}$ , we may define a correlation function,

$$\Gamma_f(\mathbf{x}) = \langle (f(\mathbf{x} + \mathbf{x}_0) - \bar{f}) (f(\mathbf{x}_0) - \bar{f})^* \rangle \quad (\text{C.14})$$

and a structure function,

$$D_f(\mathbf{x}) = \langle (f(\mathbf{x} + \mathbf{x}_0) - f(\mathbf{x}_0))^2 \rangle = 2 [\Gamma_f(0) - \Gamma_f(\mathbf{x})]. \quad (\text{C.15})$$

For the refractive index, the structure function is

$$D_n(\mathbf{r}) = \langle [n(0) - n(\mathbf{r})]^2 \rangle \quad (\text{C.16})$$

$$= 2 \int d^2\kappa (1 - \cos \kappa \cdot (\mathbf{r})) \Phi_n(\kappa) \quad (\text{C.17})$$

$$= 8\pi \int d^2\kappa \Phi_n(\kappa) \left[ 1 - \frac{\sin(\kappa r)}{\kappa r} \right] \quad (\text{C.18})$$

$$= C_n^2 r^{2/3}, \quad (\text{C.19})$$

for  $l_0 < r < L_0$ , where isotropic fluctuations were assumed to get the third equality and the Kolmogorov distribution was used for the fourth.

For the random phases, we may write

$$\phi(\mathbf{r}) = \int_0^L \frac{d\phi}{dz} dz = k \int_0^L n(\mathbf{r}, z) dz, \quad (\text{C.20})$$

for a ray passing through distance  $L$  of turbulent atmosphere. The correlation function for the phase (assuming negligible amplitude fluctuations) is

$$\Gamma_\phi(\mathbf{r}) = \left\langle \left( e^{i\phi(0)} \right) \left( e^{i\phi(\mathbf{r})} \right)^* \right\rangle \quad (\text{C.21})$$

$$= \left\langle e^{i(\phi(0) - \phi(\mathbf{r}))} \right\rangle, \quad (\text{C.22})$$

since  $\langle e^{i\phi(\mathbf{r})} \rangle = 0$ . Expanding the exponential in a Taylor series, noting that the linear term in the expansion vanishes inside the average, and then resumming the remaining terms to get a new exponential, it is readily found that

$$\Gamma_\phi(\mathbf{r}) = e^{-\frac{1}{2} D_\phi(r)}, \quad (\text{C.23})$$

where the phase structure function is

$$D_\phi(r) = \langle (\phi(0) - \phi(\mathbf{r}))^2 \rangle \quad (\text{C.24})$$

$$= k^2 \left\langle \left[ \int_0^L dz (n(0, z) - n(\mathbf{r}, z)) \right]^2 \right\rangle \quad (\text{C.25})$$

$$= Lk^2 \int \left[ D_n(\sqrt{r^2 + z^2}) - D_n(z) \right] dz. \quad (\text{C.26})$$

(Equations (C.14) and (C.15) were used to reach the last line.) Using Eq. (C.19) and approximating by taking the integration limit to infinity, the integral can be done, leading to the results that

$$D_\phi(r) = 2.91 k^2 L C_n^2 r^{5/3} \quad (\text{C.27})$$

$$\Gamma_\phi(r) = e^{-\frac{1}{2} D_\phi(r)} = \exp \left\{ -1.46 k^2 L C_n^2 r^{5/3} \right\}. \quad (\text{C.28})$$

More generally, if  $C_n^2(z)$  is varying with height and the light is propagating at angle  $\gamma$  from the zenith, the factors  $LC_n^2$  in the previous expressions can be replaced by  $(\cos \gamma)^{-1} \int_0^L C_n^2(z) dz$ . For wavelengths in the vicinity of  $\lambda \sim .5 \mu\text{m}$ , we have

$$D_\phi(r) \approx 6.88 \left( \frac{r}{r_0} \right)^{5/3}, \quad (\text{C.29})$$

where

$$r_0 = \left[ .423 (\cos \gamma)^{-1} k^2 \int_0^L C_n^2(z) dz \right]^{-3/5} \quad (\text{C.30})$$

is the *Fried parameter*.  $r_0$  is the diameter of the circular area over which the root-mean-square wavefront aberration due to passage through the atmosphere is equal to 1 radian.

It is common to make a further simplification by approximating  $\frac{5}{3} \approx 2$  in Eq.(C.29), which makes the correlation function of Eq.(C.23) Gaussian. This approximation is used in Chap. 6, where the structure function is written as  $D_\phi(r) = \alpha r^{5/3} \approx \alpha r^2$ . The turbulence strength is given by  $\alpha = 6.88 r_0^{-5/3} \approx 6.88 r_0^{-2}$ .

One useful result that follows from the above considerations is that

$$\langle e^{\phi(r-r_0) - \phi(r'-r'_0)} \rangle = e^{-1.376[(r-r')^2 + (r_0-r'_0)^2 + (r-r') \cdot (r_0-r'_0)]/r_0^2}. \quad (\text{C.31})$$

This relation is often needed when calculating output correlation functions after propagation.

When the propagation distances are sufficiently long or when the viewing aperture is smaller than  $r_0$ , then amplitude fluctuations must also be considered. Similar methods lead to a log-amplitude structure function of the form

$$D_\chi(r) = 2.91 k^2 r^{5/3} \int_0^L \eta(z) C_n^2(z) dz, \quad (\text{C.32})$$

where  $\eta(z) = 1$  for incoming plane waves and  $\eta(z) = \left( \frac{L-z}{L} \right)^{5/3}$  for spherical waves.

More detailed information about the theory of optical propagation through a turbulent medium may be found in [7–10]. For a good introduction to numerical modeling of optical propagation through a turbulent atmosphere, see [11].

## References

1. M. Born, E. Wolf, *Principles of Optics*, 7th edn. (Cambridge University Press, Cambridge, 1999)
2. H.A. Buchdahl, *Optical Aberration Coefficients* (Dover Publications, Mineola, N.Y., 1968)
3. J.C. Wyant, K. Creath, *Applied Optics and Optical Engineering XI*, ed. by R.R. Shannon and J.C. Wyant (Academic Press, New York, 1992), p. 11

4. V.N. Mahajan, *Optical Imaging and Aberrations* (SPIE Press, Bellingham, Washington (1998))
5. R.J. Noll, J. Opt. Soc. Amer. **66**, 207 (1976)
6. A.N. Kolmogorov, Dokl. Acad. Nauk. USSR **30**, 301 (1941)
7. V.I. Tatarskii, *Wave Propagation in a Turbulent Medium* (Dover Publications, Mineola, N.Y., 1967)
8. M.C. Roggemann and B. M. Welsh, *Imaging Through Turbulence* (CRC Press, Boca Raton, 1996)
9. A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Wiley-IEEE Press, New York, 1999)
10. L.C. Andrews, R.L. Phillips, *Laser Beam Propagation through Random Media*, 2nd edn., (SPIE Publications, Bellingham, WA, 2005)
11. J.D. Schmidt, *Numerical Simulation of Optical Wave Propagation with Examples in Matlab* (SPIE Press, Bellingham, WA, 2010)

## Appendix D

# Phase Matching in Spontaneous Parametric Down Conversion

*Spontaneous parametric down conversion* (SPDC), also known as *parametric fluorescence*, is the conversion of a single high-frequency photon into two outgoing photons of lower frequency. The process is mediated by nonlinear interactions with a crystal lattice. SPDC is a staple of quantum optics, as it provides a convenient and reliable way to produce entangled photon pairs with a great deal of control over the properties of the outgoing photons. The photons in each pair are generated at the same point within a time span of less than 100 fs [1], with entanglement in momentum, energy, and polarization. In fact, the photons are *hyperentangled*; i.e. the momentum, for example, of one photon is entangled not only with the momentum of the other, but also with the polarization and energy of the other. As a result, down conversion has been used to probe the foundations of quantum mechanics, for example in experiments verifying violation of the Bell inequalities by quantum mechanics [2, 3]. The strong correlations that accompany entanglement have led to a number of ideas for improved methods in optical metrology, such as dispersion and aberration cancelation, in which parametric down conversion has played a central role. One fact that is important in many of these applications is that, although each of the down converted beams is of low spatial and temporal coherence, the much higher coherence of the initial pump beam remains hidden but intact, in the entangled system. The pump coherence reemerges in the second order correlation properties as measured by coincidence experiments.

The first theoretical discussion of the possibility of spontaneous parametric down conversion was due to Klyshko [4]. Experimental verification of the existence of the effect rapidly followed, with near-simultaneous experiments carried out by several groups [5–7].

## D.1 Basics of Down Conversion

Consider the situation shown in Fig. D.1. A pump beam from a continuous wave laser enters a  $\chi^{(2)}$  nonlinear crystal, producing outgoing signal and idler beams. Commonly used crystals include BBO ( $\beta$ -barium borate) and KTP (potassium titanyl phosphate,  $\text{KTiOPO}_4$ ). The process is classified as either type I or type II, depending on the polarization states of the three photons. In *type I down conversion*, the signal and idler in a given pair have the same polarization state; both are polarized orthogonally to the parent photon in the pump beam. In the case of *type II down conversion*, the signal and idler are polarized orthogonally to each other. Occasionally, *type 0* down conversion, in which all three photons have the same polarization, is also used.

The down conversion process can also be classified as collinear or noncollinear. In *collinear down conversion*, the signal and idler propagate in the same direction as the original pump photon. In *non-collinear down conversion*, the signal and idler are emitted at a nonzero angle from the pump direction, on opposite sides of the pump; the two daughter photons have equal but opposite momentum in the plane transverse to the pump momentum. In addition, the process may be referred to as *degenerate* or *nondegenerate*, depending on whether or not the signal and idler emerge with the same energy and frequency.

While the pump beam always has a finite energy and momentum spread, this spread is often narrow enough so that the pump can be treated as a monochromatic plane wave, with frequency  $\omega_p = 2\omega_0$ . Let  $L$  be the thickness of the crystal in the direction of propagation ( $z$ ) and  $w$  be the width in each of the transverse directions ( $x$  and  $y$ ). Assuming that the transverse dimensions of the crystal are much larger than the beam size, then  $w$  can effectively be treated as infinite. For the moment, also assume that  $L$  is very large,  $L \rightarrow \infty$ . Then the frequencies  $\omega_s$  and  $\omega_i$  of the signal and idler both have a finite spread, but energy conservation requires that they add up to the incoming pump frequency:

$$\omega_s + \omega_i = 2\omega_0 = \omega_p. \quad (\text{D.1})$$

**Fig. D.1** Spontaneous parametric down conversion in a nonlinear crystal. The incoming pump interacts with the crystal lattice to produce the outgoing signal and idler beams

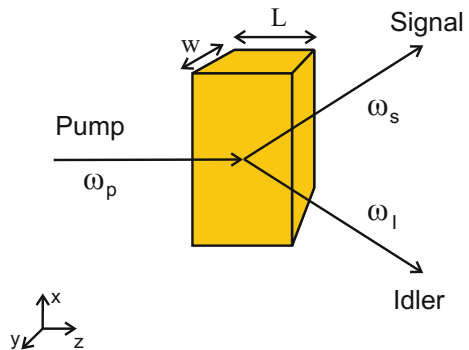
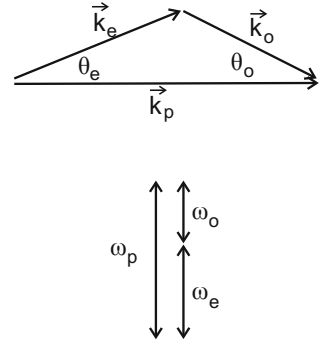


**Fig. D.2** The phase matching conditions simply express energy and momentum conservation. Equivalently, the wavevector of the pump must equal the sum of the wavevectors of the signal and idler. The same applies to the angular frequency



Similarly, if the pump beam is approximated by a plane wave of wavevector  $\mathbf{k}_p$  throughout the region in which down conversion is taking place, then the signal and idler wavevectors  $\mathbf{k}_s$  and  $\mathbf{k}_i$  must satisfy

$$\mathbf{k}_s + \mathbf{k}_i = \mathbf{k}_p \quad (\text{D.2})$$

as a result of momentum conservation; see Fig. D.2. Equations (D.1) and (D.2) are the *phase-matching conditions* for the down conversion process, and are examined in more detail in the next section. The phase-matching conditions are represented by delta functions of the form

$$\delta(\omega_s + \omega_i - 2\omega_0) \cdot \delta^{(3)}(\mathbf{k}_s + \mathbf{k}_i - \mathbf{k}_p) \quad (\text{D.3})$$

that appears in the wavefunction of the outgoing quantum state. Note that the energy and momentum of each photon has a finite spread and is undetermined until measurement, but that once the energy and momentum of one photon are measured the energy and momentum of the other photon are completely fixed. The same applies to the photon polarizations. This ability to determine the properties of one photon by making measurements on the other is one manifestation of the nonlocal entanglement of the two-photon state. Because the two photons are intrinsically linked in such an intimate manner, it is common to refer to the pair as a single *biphoton*, described by a single state vector. We denote the corresponding biphoton state as  $|\Psi\rangle$ .

For a slightly more refined approximation, the delta functions in Eq. (D.3) can be replaced by Gaussians, or more accurately still by sinc functions (see Chap. 1). Keeping the frequency delta function, but allowing a finite momentum mismatch, the output state is of the form discussed in Chap. 1,

$$|\Psi\rangle = C' \int d^3k_e d^3k_o \delta(\omega_p - \omega_e(\mathbf{k}_e) - \omega_o(\mathbf{k}_o)) \Phi(\Delta) \hat{a}_{k_o}^\dagger \hat{a}_{k_e}^\dagger |0\rangle, \quad (\text{D.4})$$

where

$$\Phi(\Delta) = e^{i\Delta L/2} \text{sinc}\left(\frac{L\Delta}{2}\right) \quad (\text{D.5})$$

and the longitudinal momentum mismatch is defined  $\Delta = \mathbf{k}_{sz} + \mathbf{k}_{iz} - \mathbf{k}_{pz}$ .

## D.2 Phase Matching

The discussion of phase matching in the remainder of this Appendix generally follows the treatment given in [8].

The starting point for understanding the behavior of the outgoing photons in SPDC and of designing an experimental setup with the desired properties is always the energy-momentum conservation relations, or phase matching conditions:

$$\mathbf{k}_p = \mathbf{k}_s + \mathbf{k}_i \quad (\text{D.6})$$

$$\omega_p = \omega_s + \omega_i. \quad (\text{D.7})$$

In this section, the use of these relations are examined in more detail for type I and type II down conversion. For simplicity, the focus is on the planar case, where the signal, idler, pump, and the optic axis of the crystal all lie in a single plane. The nonplanar case is briefly commented on at the end of the section.

In the following, it is more useful to write the frequency-matching condition [D.7](#) in terms of wavelength instead of frequency,

$$\frac{n_p}{\lambda_p} = \frac{n_s}{\lambda_s} + \frac{n_i}{\lambda_i}. \quad (\text{D.8})$$

Further, recall that the effective refractive index for the ordinary (o) polarization in a birefringent material is independent of direction, but that of the extraordinary (e) polarization depends on the propagation angle  $\phi_e$  of the photon from the optic axis,

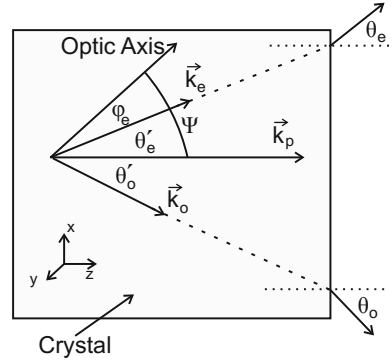
$$n_{\text{eff}}(\lambda_e, \phi_e) = \left[ \frac{\cos^2 \phi_e}{n_o^2(\lambda_e)} + \frac{\cos^2 \phi_o}{n_o^2(\lambda_o)} \right]^{-1/2}, \quad (\text{D.9})$$

where  $n_e(\lambda_e)$  and  $n_o(\lambda_o)$  are given by the Sellmeier equations of the material [9, 10]. For reference, the Sellmeier equations for BBO are

$$n_o(\lambda_o) = \sqrt{2.3759 + \frac{.01878}{\lambda_o^2 - .01822} - .01354\lambda_o^2} \quad (\text{D.10})$$

$$n_e(\lambda_e) = \sqrt{2.3753 + \frac{.01224}{\lambda_e^2 - .01667} - .01516\lambda_e^2}, \quad (\text{D.11})$$

**Fig. D.3** Definitions of the angles used to describe the photon directions and the phase matching conditions



and for KTP

$$n_x^2 = 3.0065 + 0.03901/(\lambda^2 - 0.04251) - 0.01327\lambda^2 \quad (\text{D.12})$$

$$n_y^2 = 3.0333 + 0.04154/(\lambda^2 - 0.04547) - 0.01408\lambda^2 \quad (\text{D.13})$$

$$n_z^2 = 3.0065 + 0.05694/(\lambda^2 - 0.05658) - 0.01682\lambda^2, \quad (\text{D.14})$$

with  $\lambda$  in  $\mu\text{m}$ .

### D.2.1 Type II Planar Phase-Matching

For type II down conversion, the outgoing photons have opposite polarization; if the signal is taken to be o, then the idler is e. The incoming pump photon is also e. The geometry of the situation is shown in Fig. D.3, which defines the angles to be used. The optic axis, is at angle  $\Psi$  to the propagation direction of the pump (the  $z$ -axis). The magnitudes of the photon wavevectors are given by:

$$k_p(\lambda_p, \Psi) = \frac{2\pi n_{\text{eff}}(\lambda_p, \Psi)}{\lambda_p} \quad (\text{D.15})$$

$$k_e(\lambda_e, \phi_e) = \frac{2\pi n_{\text{eff}}(\lambda_e, \phi_e)}{\lambda_e} \quad (\text{D.16})$$

$$k_o(\lambda_o) = \frac{2\pi n_o(\lambda_o)}{\lambda_o}. \quad (\text{D.17})$$

Writing the components of Eq. (D.6) parallel and perpendicular to the  $z$  direction yields

$$k_e \cos \theta'_e + k_o \cos \theta'_o = k_p \quad (\text{D.18})$$

$$k_e \sin \theta'_e = k_o \sin \theta'_o. \quad (\text{D.19})$$

The latter equation implies

$$k_o(\lambda_o) \cos \theta'_o = \sqrt{k_o^2(\lambda_o) - k_e^2(\lambda_e, \phi_e) \sin^2 \theta'_e}. \quad (\text{D.20})$$

Meanwhile, combining Eqs. (D.16), (D.17), (D.18), and D.20 leads to

$$\begin{aligned} \frac{2\pi}{\lambda_p} n_{\text{eff}}(\lambda_p, \Psi) &= \frac{2\pi n_{\text{eff}}(\lambda_e, \phi_e)}{\lambda_e} \cos \theta'_e \\ &+ \sqrt{\left(\frac{2\pi n_o(\lambda_o)}{\lambda_o}\right)^2 - \left(\frac{2\pi n_{\text{eff}}(\lambda_e, \phi_e)}{\lambda_e}\right)^2 \sin^2 \theta'_e}. \end{aligned} \quad (\text{D.21})$$

Note also from Fig. D.3 that

$$\Psi = \theta'_e + \phi_e. \quad (\text{D.22})$$

Making use of Eq. (D.8), the latter two equations may be solved (numerically) to find the signal and idler directions  $\theta_e$  and  $\theta_o$  inside the crystal in terms of the pump direction  $\Psi$  and the wavelengths. Application of Snell's law then gives the angles after leaving the crystal,  $\theta_e, \theta_o$ :

$$\theta_e = \sin^{-1} \left( n_{\text{eff}}(\lambda_e, \phi_e) \sin \theta'_e \right), \quad \theta_o = \sin^{-1} \left( \frac{\lambda_o}{\lambda_e} n_{\text{eff}}(\lambda_e, \phi_e) \sin \theta'_e \right). \quad (\text{D.23})$$

Once the angles  $\theta'_e$  and  $\theta'_o$  inside the crystal are determined, the exit angles  $\theta_e$  and  $\theta_o$  into the air are then obtained by Snell's law.

As an example, these equations have been solved and plotted at  $\Psi = 40^\circ$  in Figs. D.4 and D.5. In Fig. D.4 the outgoing angles of the two photons in the  $x$  and  $y$ -planes are plotted, while in Fig. D.5 the angles in a fixed plane ( $y = 0$ ) are plotted

**Fig. D.4** Outgoing angles for the two photons for type II down conversion at  $\Psi = 40^\circ$ . Each circle corresponds to a different signal or idler wavelength. The upper set of circles correspond to the e-polarized photons, while the bottom set are o-polarized. At the overlaps of the circles, the signal and idler are indistinguishable

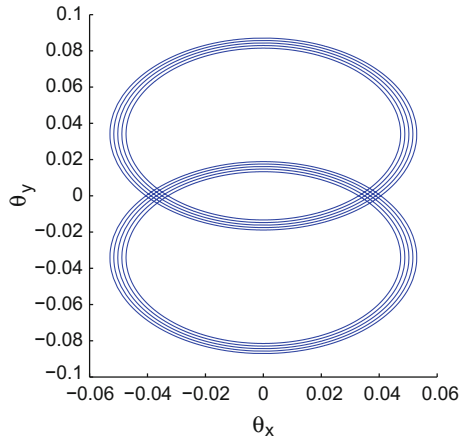
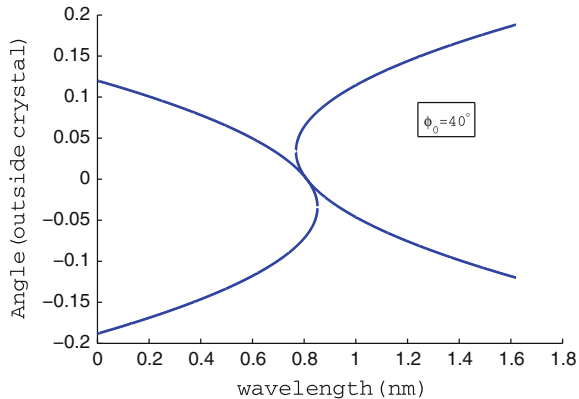


**Fig. D.5** Example of a *tuning curve* for SPDC. Outgoing angle versus wavelength ( $\lambda_o$  or  $\lambda_e$ ) in a fixed plane for the two photons for type II down conversion at  $\Psi = 40^\circ$



versus signal and idler wavelength. The latter type of plot is known as a *tuning curve*. Of particular note is the fact that the curves in the latter figure touch at  $0^\circ$ ; at this point, the wavelengths of the two photons are equal, so that the photons may be treated as indistinguishable in any experiment where the polarizations are not measurable. Indistinguishability of the outgoing photons is a requirement for many of the interference effects explored in this book.

### D.2.2 Type I Planar Phase-Matching

The difference between type I and type II is simply that in type I the outgoing photons have the same polarization, and both have polarization opposite to the pump beam. For the sake of specificity, suppose that the pump is e-polarized and the signal and idler are o-polarized. Then Eqs. (D.15)–(D.17) are now replaced by

$$k_p(\lambda_p, \Psi) = \frac{2\pi n_{\text{eff}}(\lambda_p, \Psi)}{\lambda_p} \quad (\text{D.24})$$

$$k_i(\lambda_i) = \frac{2\pi n_o(\lambda_i)}{\lambda_i} \quad (\text{D.25})$$

$$k_s(\lambda_s) = \frac{2\pi n_s(\lambda_s)}{\lambda_s}. \quad (\text{D.26})$$

All other equations of the previous subsection remain the same up to Eq. (D.19), except for the replacement of the labels *o* and *e* by *s* and *i* everywhere to distinguish between the two outgoing photons. The net result is the signal, idler, and pump wavevectors for type I phase-matching are related by



of Eq. (D.28) then allows one to find  $k_{\perp}$ . Application of Snell's law then gives the angles of the two photons after exiting the material,

$$\psi_e = \sin^{-1} \left( \frac{\lambda_e k_{\perp}(\lambda_e, \theta'_e)}{2\pi} \right), \quad (\text{D.31})$$

$$\psi_o = \sin^{-1} \left( \frac{\lambda_o k_{\perp}(\lambda_o, \theta'_e)}{2\pi} \right). \quad (\text{D.32})$$

The result, when plotted in the  $x$   $y$  plane gives a pair of rings for the signal and idler.

## References

1. C.K. Hong, Z.Y. Ou, L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987)
2. W. Tittel, J. Brendel, B. Gisin, T. Herzog, H. Zbinden, N. Gisin, Phys. Rev. A **57**, 3229 (1998)
3. J.G. Rarity, P.R. Tapster, Phys. Rev. Lett. **64**, 2495 (1990)
4. D.N. Klyshko, JETP **6**, 23 (1967) [Pis'ma v Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki, **6**, 490 (1967)]
5. S.A. Akhmanov, V.V. Fadeev, R.V. Khokhlov, O.N. Chunaev, JETP **6**, 85 (1967) [Pis'ma v Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki, **6**, 575 (1967)]
6. S.E. Harris, M.K. Osham, R.L. Byer, Phys. Rev. Lett. **18**, 732 (1967)
7. D. Magde, H. Mahr, Phys. Rev. Lett. **18**, 905 (1967)
8. Y. Shih, Rep. Prog. Phys. **66**, 1009 (2003)
9. R.W. Boyd, *Nonlinear Optics*, 3rd Ed. (Academic Press, London, 2008)
10. B.E.A. Saleh, M.C. Teich, *Fundamentals of Photonics*, 2nd edn. (John Wiley and Sons, Hoboken, 2007)

## Appendix E

# Vectorial Scattering Analysis of the Twin-Photon Microscope

In Chap. 7, in order to better get across the main ideas, the treatment of the correlation confocal and twin-photon microscopes were simplified by using scalar optics and the paraxial approximation, as well as by taking the size of the pinhole to be infinitesimal. These approximations are not always applicable, especially when used in conjunction with high numerical aperture (NA) objectives. In this Appendix, which is more technical than the rest of this book, it is shown that these simplifications are not necessary; the resolution enhancement survives even when a more exact treatment is applied. Rather than treat the general case, which would be excessively long, here one possible case is treated in order to make the point, namely that of the twin-photon confocal microscope (Sect. 7.8). The vectorial treatment of microscopic imaging was pioneered in [1] and has been more thoroughly worked out in a number of more recent papers, including [2–6]. Many of these papers have been devoted to the problem of detecting small (subwavelength-sized) dielectric scatterers using widefield or confocal microscopes, so we focus on the same problem here, and examine the image produced by such a pointlike particle as the region of the particle is scanned through the twin-photon microscope. It is seen how the image varies as the pinhole size is changed, and allow comparison to the corresponding results for the standard confocal microscope.

The situation is essentially the same as in Fig. 7.9, except that the object is now a pointlike scatterer and that the polarizations of the fields at each point in the apparatus must be considered. Also, rather than using a single lens in each arm of the apparatus, a double lens configuration is now used in order to parallel more closely the formalism of the papers cited above (see Fig. E.1). Since the electric and magnetic fields obey essentially the same equations, only the electric fields are discussed here. The incoming light is assumed to be polarized in the horizontal ( $x$ ) direction. Any vertically-polarized light present between the scatterer and the crystal must therefore have come from scattering off the particle. Assuming type I down conversion, this  $y$ -polarized light then produces  $x$ -polarized signal and idler beams after the crystal. By placing horizontal polarizers before the detectors, any light arising from down conversion of unscattered light is therefore weeded out. Narrow band filters or prisms remove any light from the pump that did not participate in the down conversion. Also

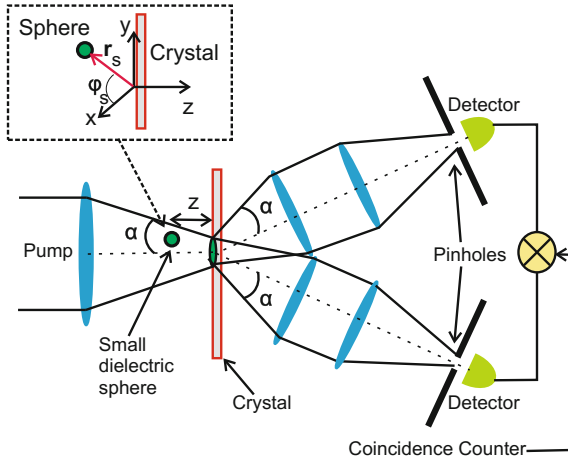


**Fig. E.1** Schematic depiction of the use of the twin-photon microscope to view a small dielectric sphere of radius much smaller than the wavelength. The acceptance half-angles  $\alpha$  of all of the lenses are assumed to be the same. The *inset* gives the coordinates of the sphere:  $r_s$  is the distance of the sphere from the origin and  $\phi_s$  is the angle from the  $x$ -axis. The origin is on the optical axis, at the edge of the crystal.  $z (< 0)$  is the distance of the *sphere* from the crystal. Although the two outgoing arms are drawn separated for clarity, in practice they would coincide as in Fig. 7.11

assume a horizontal polarizer between the crystal and the lenses; this it is not strictly needed, but it simplifies the calculation by cutting down the number of terms that must be considered. We assume all lenses have the same acceptance half-angle  $\alpha$ .

Now the formalism of [2–6] may be applied. The incoming focused pump field induces a dipole moment in the scatterer, given by [2, 4]

$$p_x = I_0 + I_2 \cos 2\phi_s \quad (\text{E.1})$$

$$p_y = I_2 \sin 2\phi_s \quad (\text{E.2})$$

$$p_z = -2i I_1 \cos \phi_s, \quad (\text{E.3})$$

where

$$I_0 = \int_0^\alpha (\cos \theta)^{1/2} \sin \theta (1 + \cos \theta) J_0(kr_s \sin \theta) e^{ikz_s \cos \theta} d\theta \quad (\text{E.4})$$

$$I_1 = \int_0^\alpha (\cos \theta)^{1/2} \sin^2 \theta J_1(kr_s \sin \theta) e^{ikz_s \cos \theta} d\theta \quad (\text{E.5})$$

$$I_2 = \int_0^\alpha (\cos \theta)^{1/2} \sin \theta (1 - \cos \theta) J_2(kr_s \sin \theta) e^{ikz_s \cos \theta} d\theta \quad (\text{E.6})$$

Here  $\phi_s$  and  $r_s$  are the angle from the  $x$ -axis (the initial polarization direction) and the distance from the optical axis of the scatterer.  $\phi_s$  and  $r_s$  vary as the sample is scanned. Since in this “Appendix” the desire is to keep the treatment as simple as

possible, we consider only one-dimensional scans along the axes; i.e.  $\phi_s$  is kept fixed at a value of either  $0^\circ$  or  $90^\circ$ . Then the goal is to find the detected coincidence rate as a function of  $r_s$ .

Taking  $\hat{\mathbf{r}}$  to be a unit vector in the scattering direction, the strength of the scattered field dipole in the  $\hat{\mathbf{r}}$  direction is

$$\begin{aligned} \mathbf{E}_{dp} &= -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}) \\ &= - \begin{pmatrix} p_x (\sin^2 \theta_1 \cos^2 \phi - 1) + p_y \sin^2 \theta_1 \sin \phi \cos \phi + p_z \sin \theta_1 \cos \theta_1 \cos \phi \\ p_x \sin^2 \theta_1 \cos \phi \sin \phi + p_y (\sin^2 \theta_1 \sin^2 \phi - 1) + p_z \cos \theta_1 \sin \theta_1 \sin \phi \\ p_x \sin \theta_1 \cos \theta_1 \cos \phi + p_y \sin \theta_1 \cos \theta_1 \sin \phi - 1 - p_z \sin^2 \theta_1 \end{pmatrix}. \end{aligned} \quad (\text{E.7})$$

This gives the angular distribution of the scattered radiation just after leaving the particle.

Let

$$\mathbf{R} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{L}_n = \begin{pmatrix} \cos \theta_n & 0 & \sin \theta_n \\ 0 & 1 & 0 \\ -\sin \theta_n & 0 & \cos \theta_n \end{pmatrix} \quad (\text{E.8})$$

represent rotation by angle  $\phi$  about the  $z$  axis and the effect of the  $n$ th lens on the rays, respectively. Given a field strength  $E_{in}$  in the down conversion region, the field strength in the detector plane is

$$\mathbf{E}_{out} = \mathbf{R}^{-1} \mathbf{L}_2 \mathbf{L}_1^{-1} \mathbf{R} \cdot \mathbf{E}_{in}. \quad (\text{E.9})$$

(The rotation matrices are needed to enforce the requirement that the field and momentum must remain orthogonal as the propagation direction changes.) With the  $x$ -polarizer in place before the detector, only the  $x$ -component of  $\mathbf{E}_{out}$  needs to be considered:

$$\begin{aligned} E_{out,x} &= [\cos^2 \phi \cos(\theta_1 - \theta_2) + \sin^2 \phi] E_{in,x} + \sin \phi \cos \phi [\cos(\theta_1 - \theta_2) - 1] E_{in,y} \\ &\quad + \cos \phi \sin(\theta_2 - \theta_1) E_{in,z}. \end{aligned} \quad (\text{E.10})$$

Further, with the polarizer just after the crystal only the  $x$ -component of  $\mathbf{E}_{in}$  contributes to the right hand side of the previous equation:

$$E_{out,x} = (\cos^2 \phi \cos(\theta_1 - \theta_2) + \sin^2 \phi) E_{in,x} \equiv H_{xx}(\theta_1, \theta_2, \phi) E_{in,x}(\theta_1, \phi), \quad (\text{E.11})$$

where  $H_{xx}(\theta_1, \theta_2, \phi)$  is the propagator of the  $x$ -component from the crystal to the detection plane.  $\theta_1$  and  $\theta_2$  are respectively the angles from the axis at which a given ray enters the first lens of the detection arm and the angle at which the ray leaves the second lens. Both rays are at azimuthal angle  $\phi$ . In this and all of the following expressions,  $\theta_1$  is to be viewed as a function of  $\theta_2$  by means of the Abbé sine condition,  $\sin \theta_1 = \beta \sin \theta_2$ , where  $\beta$  is the magnification of the imaging system.

The biphoton wavefunction at points  $\mathbf{r}_{ps}$  and  $\mathbf{r}_{pi}$  in the signal and idler detector planes is given by using  $H_x$  to propagate  $\mathbf{e}_{dp,y}$  at each point in the crystal forward in

each branch and integrating over the crystal. If the photons of the pump beam enter the crystal at angles  $(\theta_0, \phi_0)$ , then the signal and idler are on opposite sides of this direction, with respective angles  $(\theta_1, \phi) = (\theta_0 \pm \delta\theta, \phi_0 \pm \delta\phi)$ . Since the angular spread in  $\theta_0$  due to the focusing of the pump is much larger than the maximum spread  $\delta\theta$  due to the down conversion, and since the imaging systems are symmetric in the  $\phi$  direction, therefore taking  $(\theta_1, \phi) \approx (\theta_0, \phi)$  should be a good approximation. (By appropriately choosing the down conversion parameters or by inserting narrow band spectral filters into the apparatus, the spreads  $\delta\theta$  and  $\delta\phi$  can always be made smaller if necessary.) In that case, both the signal and idler will reach the corresponding detector plane at the same radius  $r_p$  and angle  $\phi_p$ , reducing the integrations over the two pinhole areas to just a single integration. In each branch, the field is propagated forward by factor  $H_{xx}(\theta_1, \theta_2, \phi)$ , then the result is multiplied by the factors [2, 4]

$$\sin \theta_2 \sqrt{\cos \theta_1} e^{ikr \cos(\phi - \phi_p) \sin \theta_2} \quad (\text{E.12})$$

and integrated over the input plane, with the  $\sqrt{\cos \theta_1}$  factor to account for the use of aplanatic lenses. Since *two* beams are being propagated, one through each outgoing branch, these factors are squared, with a copy for each beam. The result is that the coincidence amplitude due to photons arriving at coordinates  $(r_p, \phi_p)$  in the detection plane is

$$A(r_p, \phi_p) = \int E_{dp,x}(\theta_1, \phi) [H_{xx}(\theta_1, \theta_2, \phi)]^2 \sin^2 \theta_2 \cos \theta_1 e^{2ikr \cos(\phi - \phi_p) \sin \theta_2} d\theta_2 d\phi. \quad (\text{E.13})$$

Ignoring overall constants, the amplitude is then

$$A(r_p, \phi_p) = \int \{p_x \Lambda_x(\theta_2, \phi) - p_y \sin^2 \theta_1 \Lambda_y(\theta_2, \phi) - p_z \sin \theta_1 \cos \theta_1 \Lambda_z(\theta_2, \phi)\} \\ \times \sin^2 \theta_2 \cos \theta_1 e^{ikr \cos(\phi - \phi_p) \sin \theta_2} d\theta_2 d\phi, \quad (\text{E.14})$$

where we have defined

$$\begin{aligned} \Lambda_z(\theta_2, \phi) &= \cos^2(\theta_1 - \theta_2) \cos^5 \phi + 2 \cos(\theta_1 - \theta_2) \sin^2 \phi \cos^3 \phi + \sin^4 \phi \cos \phi, \\ \Lambda_y(\theta_2, \phi) &= \cos^2(\theta_1 - \theta_2) \sin \phi \cos^5 \phi + \sin^5 \phi \cos \phi + 2 \cos(\theta_1 - \theta_2) \sin^3 \phi \cos^3 \phi \\ \Lambda_x(\theta_2, \phi) &= \cos^2(\theta_1 - \theta_2) (\cos^4 \phi - \sin^2 \theta_1 \cos^6 \phi) + \sin^4 \phi - \sin^2 \theta_1 \sin^4 \phi \cos^2 \phi \\ &\quad + 2 \cos(\theta_1 - \theta_2) (\sin^2 \phi \cos^2 \phi - \sin^2 \theta_1 \sin^2 \phi \cos^4 \phi) \end{aligned}$$

The integrations over  $\phi$  may be carried out to find that:

$$\int \Lambda_z(\theta_2, \phi) e^{2ikr \cos(\phi-\phi_p) \sin \theta_2} d\phi = \frac{i\pi}{8} \left\{ (K(\theta_2) - 1)^2 L_5(\theta_2, r_p, \phi_p) \right. \\ \left. - 5 \left[ (K(\theta_2) - 1)^2 + 8(K(\theta_2) - 1) \right] L_3(\theta_2, r_p, \phi_p) \right. \\ \left. + 2 \left[ 5(K(\theta_2) - 1)^2 + 12(K(\theta_2) - 1) + 8 \right] L_1(\theta_2, r_p, \phi_p) \right\} \quad (\text{E.15})$$

$$\int \Lambda_y(\theta_2, \phi) e^{2ikr \cos(\phi-\phi_p) \sin \theta_2} d\phi = -\frac{\pi}{16} \left\{ \left[ (K^2(\theta_2) + 1) - 2K(\theta_2) \right] L_6(\theta_2, r_p, \phi_p) \right. \\ \left. + 4 \left[ 1 - K^2(\theta_2) \right] L_4(\theta_2, r_p, \phi_p) + [5(K(\theta_2) + 1) + 3K(\theta_2)] L_2(\theta_2, r_p, \phi_p) \right\} \\ \int \Lambda_x(\theta_2, \phi) e^{2ikr \cos(\phi-\phi_p) \sin \theta_2} d\phi = \frac{\pi}{16} \left\{ \sin^2 \theta_1 (K(\theta_2) - 1)^2 L_6(\theta_2, r_p, \phi_p) \right. \\ \left. + 2 \left[ 2 + \sin^2 \theta_1 K^2(\theta_2)(2 - 3 \sin^2 \theta_1) + 2K(\theta_2)(\sin^2 \theta_1 - 2) \right] L_4(\theta_2, r_p, \phi_p) \right. \\ \left. + \left[ 15 + \cos^2 \theta_1 + K^2(\theta_2)(15 \sin^2 \theta_1 - 16) - 2K(\theta_2) \sin^2 \theta_1 \right] L_2(\theta_2, r_p, \phi_p) \right. \\ \left. + \left[ 10 + 2 \cos^2 \theta_1 + 2K^2(\theta_2)(6 - 5 \sin^2 \theta_1) + 4K(\theta_2)(2 - \sin^2 \theta_1) \right] L_0(\theta_2, r_p, \phi_p) \right\}, \quad (\text{E.16})$$

where

$$K(\theta_2) \equiv \cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \beta \sin^2 \theta_2, \quad (\text{E.17})$$

and

$$L_n(\theta_2, r_p, \phi_p) = J_n(2kr_p \sin \theta_2) \cos(n\phi_p). \quad (\text{E.18})$$

( $J_n$  is the Bessel function of  $n$ th order.) Again, the goal here is not to treat the most general case, but simply to get a rough idea of the form of the exact results by looking at cases where the calculation is relatively simple. So, specialize now to the case of unit magnification, in which the results above simplify considerably. Setting  $\beta = 1$ , we find that  $\sin \theta_1 = \sin \theta_2$ ,  $\cos \theta_1 = \cos \theta_2$ , and  $K(\theta_2) = 1$ . So:

$$\int \Lambda_z(\theta_2, \phi) e^{2ikr \cos(\phi-\phi_p) \sin \theta_2} d\phi = 2\pi i L_1(\theta_2, r_p, \phi_p) \quad (\text{E.19})$$

$$\int \Lambda_y(\theta_2, \phi) e^{2ikr \cos(\phi-\phi_p) \sin \theta_2} d\phi = -\frac{13}{16} \pi L_2(\theta_2, r_p, \phi_p) \quad (\text{E.20})$$

$$\int \Lambda_x(\theta_2, \phi) e^{2ikr \cos(\phi-\phi_p) \sin \theta_2} d\phi = \frac{\pi}{4} \left[ 3 \sin^2 \theta_2 L_2(\theta_2, r_p, \phi_p) \right. \\ \left. + 4(2 - \sin^2 \theta_2) L_0(\theta_2, r_p, \phi_p) \right]. \quad (\text{E.21})$$

Thus, the amplitude becomes

$$A(r_p, \phi_p) = \int \left\{ (12p_x + 13p_y) \sin^2 \theta_2 L_2(\theta_2, r_p, \phi_p) + 16(2 - \sin^2 \theta_2) L_0(\theta_2, r_p, \phi_p) \right. \\ \left. - 32ip_z \sin \theta_2 \cos \theta_2 L_1(\theta_2, r_p, \phi_p) \right\} \sin^2 \theta_2 \cos \theta_2 d\theta_2. \quad (\text{E.22})$$

The form of the integrations in the latter result may be simplified further by making the change of variable  $\zeta = \sin \theta_2$ , if desired. In any event, the integrations are carried

out numerically, and then the coincidence rate is determined by taking the absolute square and doing a further integral over the final pinholes:

$$R_c(\mathbf{r}_s) = \int |A(\mathbf{r}_p - \beta \mathbf{r}_s)|^2 d^2 r_p, \quad (\text{E.23})$$

where the shift  $\mathbf{r}_p \rightarrow \mathbf{r}_p - \beta \mathbf{r}_s$  accounts for the scanning of the sample. The results for the standard confocal microscope may found in the same manner; the principle difference is that in Eq. (E.13) the propagator  $H_{xx}$  and the factors from Eq. (E.12) are not squared.

These results are plotted in Figs. E.2 and E.3. In each plot, the lenses are taken to be in air ( $n = 1$ ), with acceptance half-angle  $\alpha \approx \frac{\pi}{2}$  and  $NA \approx 1$ , i.e.  $\alpha$  and  $NA$  approaching their maximum values. The pump wavelength is 351 nm, with equal signal and idler wavelengths of (351 nm)/2. All scans are either along the  $x$ -axis ( $\phi_s = 0$ ), or along the  $y$ -axis with ( $\phi_s = \pi/2$ ), and curves for the standard confocal microscope are taken at the signal/idler wavelength except where stated otherwise.  $z$  gives the distance of the scatterer from the crystal, and is taken to be negative. Figure E.2 shows the curves computed from equations above for the twin photon microscope and standard confocal microscope, for fixed values of  $z$  and  $\phi_s$ , and for two values of the radius of the final pinhole. It is seen that the twin photon microscope has a narrower peak in each case than the standard confocal microscope, and that as the pinhole shrinks the results converge back to those of the simplified treatment of the main text. For both values of  $R$  plotted, the full-width at half-maximum is narrower for the twin-photon microscope by a factor in the vicinity of 50 %.

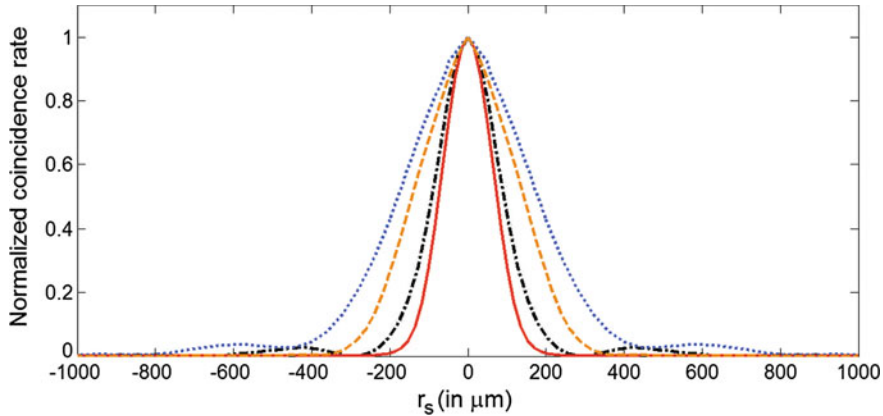


**Fig. E.2** The images of a small dielectric particle in the standard confocal and twin-photon confocal microscope for different pinhole radii,  $R$ . From the inside out, the curves correspond to twin-photon with  $R = .1 \mu\text{m}$  (solid) and  $R = 1 \mu\text{m}$  (dot-dash), and standard confocal for  $R = .1 \mu\text{m}$  (dash) and  $R = 1 \mu\text{m}$  (dotted). As  $R \rightarrow 0$ , the curves approximate the PSF curves obtained from the simplified treatment of the main text. The scans here are all along the  $y$ -axis at  $z = 0$ , with  $\phi_s = \frac{\pi}{2}$

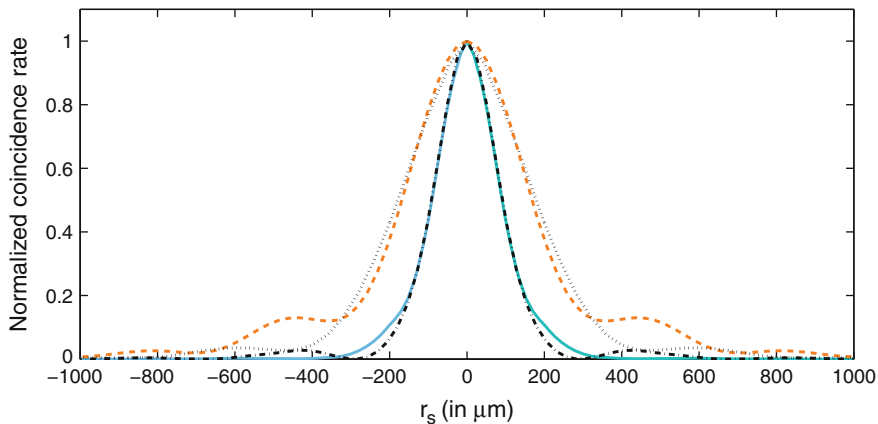


**Fig. E.3** The images of a small dielectric particle in the standard confocal and twin-photon confocal microscope for fixed pinhole radii, but scanned along different axes. The scan is along the  $x$ -axis for  $\phi_s = 0$  and along the  $y$ -axis for  $\phi_s = \frac{\pi}{2}$ . The curves correspond to twin-photon for  $\phi_s = 0$  (solid) and  $\phi_s = \frac{\pi}{2}$  (dot-dash), and to standard confocal for  $\phi_s = 0$  (dotted) and  $\phi_s = \frac{\pi}{2}$  (dashed). The plots are all for  $R = 1 \mu\text{m}$  and  $z = 0$

In Fig. E.3, the twin-photon microscope gives similar results for scans along the  $x$ - and  $y$ -axis, giving main peaks of the same widths, but with differences appearing in the tails. Both are significantly narrower than the curve for the standard confocal at the signal wavelength. As the scatterer moves away from the crystal, the central curve maintains its shape, but the tails of the curve begin to rise, reducing the contrast of the spot's center.

In short, it is seen from Figs. E.2 and E.3 that even with the approximations of Chap. 7 removed, the twin-photon microscope still demonstrates significantly improved lateral resolution compared to a standard confocal microscope at the same output wavelength, thus justifying the simplified treatment given in the main text.

## References

1. B. Richards, E. Wolf, Proc. R. Soc. Lond. A, **253**, 358 (1954)
2. T. Wilson, R. Juškaitis, P. Higdon, Opt. Commun. **141**, 298(1997)
3. P. Török, T. Wilson, Opt. Commun. **137**, 127 (1997)
4. P. Török, P.D. Higdon, T. Wilson, Opt. Commun. **148**, 300 (1998)
5. P. Török, P.D. Higdon, T. Wilson, J. Mod. Opt. **45**, 1681 (1998)
6. C.J.R. Sheppard, J.F. Aguilar, Opt. Commun. **180**, 1 (2000)

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