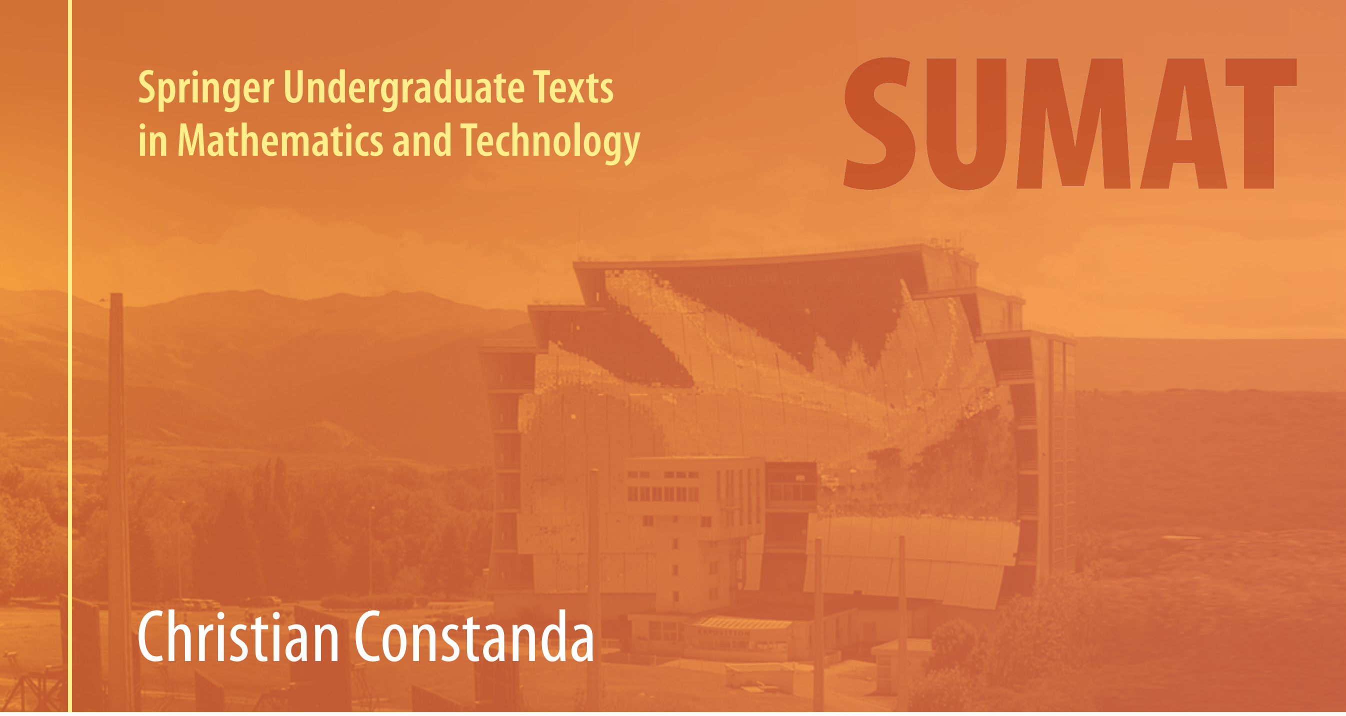

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SUMAT

Christian Constanda

Differential Equations

A Primer for Scientists and Engineers

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Differential Equations

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ISSN 1867-5506 ISSN 1867-5514 (electronic)
ISBN 978-1-4614-7296-4 ISBN 978-1-4614-7297-1 (eBook)
DOI 10.1007/978-1-4614-7297-1
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013937945

Mathematics Subject Classification (2010): 34-01

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For Lia

Preface

Arguably, one of the principles underpinning classroom success is that the instructor always knows best. Whether this mildly dictatorial premise is correct or not, it seems logical that performance can only improve if said instructor also pays attention to customer feedback. Students' opinion sometimes contains valuable points and, properly canvassed and interpreted, may exercise a positive influence on the quality of a course and the manner of its teaching.

When I polled my students about what they wanted from a textbook, their answers clustered around five main issues.

The book should be easy to follow without being excessively verbose. A crisp, concise, and to-the-point style is much preferred to long-winded explanations that tend to obscure the topic and make the reader lose the thread of the argument.

The book should not talk down to the readers. Students feel slighted when they are treated as if they have no basic knowledge of mathematics, and many regard the multicolored, heavily illustrated texts as better suited for inexperienced high schoolers than for second-year university undergraduates.

The book should keep the theory to a minimum. Lengthy and convoluted proofs should be dropped in favor of a wide variety of illustrative examples and practice exercises.

The book should not embed computational devices in the instruction process. Although born in the age of the computer, a majority of students candidly admit that they do not learn much from electronic number-crunching.

The book should be 'slim'. The size and weight of a 500-page volume tend to discourage potential readers and bode ill for its selling price.

In my view, a book that tries to be 'all things to all men' often ends up disappointing its intended audience, who might derive greater profit from a less ambitious but more focused text composed with a twist of pragmatism. The textbooks on differential equations currently on the market, while professionally written and very comprehensive, fail, I believe, on at least one of the above criteria; by contrast, this book attempts to comply with the entire set. To what extent it has succeeded is for the end user to decide. All I can say at this stage is that students in my institution and elsewhere, having adopted an earlier draft as prescribed text, declared themselves fully satisfied by it and agreed that every one of the goals on the above wish list had been met. The final version incorporates several additions and changes that answer some of their comments and a number of suggestions received from other colleagues involved in the teaching of the subject.

In earlier times, mathematical analysis was tackled from the outset with what is called the ε–δ methodology. Those times are now long gone. Today, with a few exceptions, all science and engineering students, including mathematics majors, start by going through Calculus I, II, and III, where they learn the mechanics of differentiation and integration but are not shown the proofs of some of the statements in which the formal technics are rooted, because they have not been exposed yet to the ϵ–δ language. Those who want to see these proofs enroll in Advanced Calculus. Consequently, the natural continuation of the primary Calculus sequence for all students is a differential equations course that teaches them solution techniques without the proofs of a number of fundamental theorems. The missing proofs are discussed later in an Advanced Differential Equations sequel (compulsory for mathematics majors and optional for the interested engineering students), where they are developed with the help of Advanced Calculus concepts. This book is intended for use with the first—elementary—differential equations course, taken by mathematics, physics, and engineering students alike.

Omitted proofs aside, every building block of every method described in this textbook is assembled with total rigor and accuracy.

The book is written in a style that uses words (sparingly) as a bonding agent between consecutive mathematical passages, keeping the author's presence in the background and allowing the mathematics to be the dominant voice. This should help the readers navigate the material quite comfortably on their own. After the first examples in each section or subsection are solved with full details, the solutions to the rest of them are presented more succinctly: every intermediate stage is explained, but incidental computation (integration by parts or by substitution, finding the roots of polynomial equations, etc.) is entrusted to the students, who have learned the basics of calculus and algebra and should thus be able to perform it routinely.

The contents, somewhat in excess of what can be covered during one semester, include all the classical topics expected to be found in a first course on ordinary differential equations. Numerical methods are off the ingredient list since, in my view, they fall under the jurisdiction of numerical analysis. Besides, students are already acquainted with such approximating procedures from Calculus II, where they are introduced to Euler's method. Graphs are used only occasionally, to offer help with less intuitive concepts (for instance, the stability of an equilibrium solution) and not to present a visual image of the solution of every example. If the students are interested in the latter, they can generate it themselves in the computer lab, where qualified guidance is normally provided.

The book formally splits the 'pure' and 'applied' sides of the subject by placing the investigation of selected mathematical models in separate chapters. Boundary value problems are touched upon briefly (for the benefit of the undergraduates who intend to go on to study partial differential equations) but without reference to Sturm–Liouville analysis.

Although only about 260 pages long, the book contains 232 worked examples and 810 exercises. There is no duplication among the examples: no two of them are of exactly the same kind, as they are intended to make the user understand how the methods are applied in a variety of circumstances. The exercises aim to cement this knowledge and are all suitable as homework; indeed, each and every one of them is part of my students' assignments.

Computer algebra software—specifically, *Mathematica*®—is employed in the book for only one purpose: to show how to verify quickly that the solutions obtained are correct. Since, in spite of its name, this package has not been created by mathematicians, it does not always do what a mathematician wants. In many other respects, it is a perfectly good instrument, which, it is hoped, will keep on improving so that when, say, version 54 is released, all existing deficiencies will have been eliminated. I take the view that to learn mathematics properly, one must use pencil and paper and solve problems by brain and hand alone. To encourage and facilitate this process, almost all the examples and exercises in the book have been constructed with integers and a few easily managed fractions as coefficients and constant terms.

Truth be told, it often seems that the aim of the average student in any course these days is to do just enough to pass it and earn the credits. This book provides such students with everything they need to reach their goal. The gifted ones, who are interested not only in the *how* but also in the *why* of mathematical methods and try hard to improve from a routinely achieved 95% on their tests to a full 100%, can use the book as a springboard for progress to more specialized sources (see the list under Further Reading) or for joining an advanced course where the theoretical aspects left out of the basic one are thoroughly investigated and explained.

And now, two side issues related to mathematics, though not necessarily to differential equations.

Scientists, and especially mathematicians, need in their work more symbols than the Latin alphabet has to offer. This forces them to borrow from other scripts, among which Greek is the runaway favorite. However, academics—even English-speaking ones—cannot agree on a common pronunciation of the Greek letters. My choice is to go to the source, so to speak, and simply follow the way of the Greeks themselves. If anyone else is tempted to try my solution, they can find details in Appendix D.

Many instructors would probably agree that one of the reasons why some students do not get the high grades they aspire to is a cocktail of annoying bad habits and incorrect algebra and calculus manipulation 'techniques' acquired (along with the misuse of the word "like") in elementary school. My book *Dude, Can You Count?* (Copernicus, Springer, 2009) systematically collects the most common of these bloopers and shows how any number of absurdities can be 'proved' if such errors are accepted as legitimate mathematical handling. *Dude* is recommended reading for my classroom attendees, who, I am pleased to report, now commit far fewer errors in their written presentations than they used to. Alas, the cure for the "like" affliction continues to elude me.

This book would not have seen the light of day without the special assistance that I received from Elizabeth Loew, my mathematics editor at Springer–New York. She monitored the evolution of the manuscript at every stage, offered advice and encouragement, and was particularly understanding over deadlines. I wish to express my gratitude to her for all the help she gave me during the completion of this project.

I am also indebted to my colleagues Peyton Cook and Kimberly Adams, who trawled the text for errors and misprints and made very useful remarks; to Geoffrey Price for useful discussions; and to Dale Doty, our departmental *Mathematica*® guru. (Readers interested in finding out more about this software are directed to the web site http://www.wolfram.com/mathematica/.)

Finally, I want to acknowledge my students for their interest in working through all the examples and exercises and for flagging up anything that caught their attention as being inaccurate or incomplete.

My wife, of course, receives the highest accolade for her inspiring professionalism, patience, and steadfast support.

Tulsa, OK, USA Christian Constanda

Contents

Acronyms

DE	Differential equation
IC	Initial condition
BC	Boundary condition
IVP	Initial value problem
BVP	Boundary value problem
GS	General solution
PS	Particular solution
FSS	Fundamental set of solutions

Chapter 1
Introduction

Mathematical modeling is one of the most important and powerful methods for studying phenomena occurring in our universe. Generally speaking, such a model is made up of one or several equations from which we aim to determine one or several unknown quantities of interest in terms of other, prescribed, quantities. The unknown quantities turn out in many cases to be functions of a set of variables. Since very often the physical or empirical laws governing evolutionary processes implicate the rates of change of these functions with respect to their variables, and since rates of change are represented in mathematics by derivatives, it is important for us to gain knowledge of how to solve equations where the unknown functions occur together with their derivatives.

Mathematical modeling consists broadly of three stages: the construction of the model in the form of a collection of equations, the solution of these equations, and the interpretation of the results from a practical point of view. In what follows we are concerned mostly with the second stage, although at times we briefly touch upon the other two as well. Furthermore, we restrict our attention to equations where the unknowns are functions of only one independent (real) variable. We also assume throughout that every equation we mention and study obeys the principle of physical unit consistency, and that the quantities involved have been scaled and non-dimensionalized according to some suitable criteria. Consequently, with very few exceptions, no explicit reference will be made to any physical units.

1.1 Calculus Prerequisites

Let f be a function of a variable x, defined on an interval J of the real line. We denote by $f(x)$ the value of f at x.

Leaving full mathematical rigor aside, we recall that f is said to have a *limit* α at a point x_0 in J if the values $f(x)$ get arbitrarily close to α as x gets arbitrarily close to x_0 from either side of x_0; in this case, we write

$$\lim_{x \to x_0} f(x) = \alpha.$$

We also say that f is *continuous at* x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0),$$

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_1,
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and that f is *differentiable at* x_0 if

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists. If this happens at every point in J, the function f is said to be *differentiable on* J and f' is called the (first-order) *derivative* of f. This process can be generalized to define higher-order derivatives. We denote the derivatives of f by f', f'', f''', $f^{(4)}$, $f^{(5)}$, and so on; alternatively, sometimes we use the more formal notation

$$\frac{df}{dx}, \quad \frac{d^2 f}{dx^2}, \quad \frac{d^3 f}{dx^3}, \quad \dots .$$

Since at some points in the book we have brief encounters with functions of several variables, it is useful to list in advance some of the properties of these functions that are relevant to their differentiation and integration. For simplicity and without loss of generality, we confine ourselves to the two-dimensional case.

Let f be a function of two variables x and y, defined in a region S (called the *domain* of f) in the Cartesian (x, y)-plane. We denote the value of f at a point $P(x, y)$ in S by $f(x, y)$ or $f(P)$.

(i) The function f is said to have a *limit* α at a point P_0 in its domain S if

$$\lim_{P\to P_0} f(P) = \alpha,$$

with P approaching P_0 on any path lying in S. The function f is called *continuous at* P_0 if

$$\lim_{P\to P_0} f(P) = f(P_0).$$

(ii) Suppose that we fix the value of y. Then f becomes a function f_1 that depends on x alone. If f_1 is differentiable, we call f_1' the *partial derivative of* f *with respect to* x and write

$$f_1' = f_x = \frac{\partial f}{\partial x}.$$

The other way around, when x is fixed, f becomes a function f_2 of y alone, which, if differentiable, defines the *partial derivative of* f *with respect to* y, denoted by

$$f_2' = f_y = \frac{\partial f}{\partial y}.$$

Formally, the definitions of these first-order partial derivatives are

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

(iii) We can repeat the above process starting with f_x and then f_y, and define the second-order partial derivatives of f, namely

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \qquad f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x},$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.$$

If the *mixed* second-order derivatives f_{xy} and f_{yx} are continuous in a disk (that is, in a finite region bounded by a circle) inside the domain S of f, then $f_{xy} = f_{yx}$ at every point in that disk. Since all the functions in our examples satisfy this continuity condition, we will not verify it explicitly.

(iv) The *differential* of the function f is

$$df = f_x\,dx + f_y\,dy.$$

(v) The chain rule of differentiation is also a logical extension of the same rule for functions of one variable. Thus, if $x = x(r,s)$, $y = y(r,s)$, and $f(x,y) = g(r,s)$, then

$$g_r = f_x x_r + f_y y_r, \quad g_s = f_x x_s + f_y y_s.$$

(vi) When we evaluate the indefinite integral of a two-variable function f with respect to one of its variables, the arbitrary constant of integration is a constant only as far as that variable is concerned, but may depend on the other variable. If, say, F_1 is an antiderivative of f with respect to x, then

$$\int f(x,y)\,dx = F_1(x,y) + C_1(y),$$

where C_1 is an arbitrary function of y. Symmetrically, if F_2 is an antiderivative of f with respect to y, then

$$\int f(x,y)\,dy = F_2(x,y) + C_2(x),$$

where C_2 is an arbitrary function of x.

1.1 Example. For

$$f(x,y) = 6x^2y - 4xy^3$$

we have

$$f_x = 12xy - 4y^3, \quad f_y = 6x^2 - 12xy^2,$$
$$f_{xx} = 12y, \quad f_{xy} = f_{yx} = 12x - 12y^2, \quad f_{yy} = -24xy,$$
$$\int f(x,y)\,dx = \int (6x^2y - 4xy^3)\,dx = 2x^3y - 2x^2y^3 + C_1(y),$$
$$\int f(x,y)\,dy = \int (6x^2y - 4xy^3)\,dy = 3x^2y^2 - xy^4 + C_2(x). \ ■$$

1.2 Remark. We use a number of different symbols for functions and their variables. Usually—but not always—a generic unknown function (to be determined) is denoted by y and its variable by t or x. ■

1.2 Differential Equations and Their Solutions

To avoid cumbersome language and notation, unless specific restrictions are mentioned, all mathematical expressions and relationships in what follows will be understood to be defined for the largest set of 'admissible' values of their variables and function components—in other words, at all the points on the real line where they can be evaluated according to context.

1.3 Definition. Roughly speaking, a *differential equation* (DE, for short) is an equation that contains an unknown function and one or more of its derivatives. ∎

Here are a few examples of DEs that occur in some simple mathematical models.

1.4 Example. (*Population growth*) If $P(t)$ is the size of a population at time $t > 0$ and $\beta(t)$ and $\delta(t)$ are the birth and death rates within the population, then

$$P' = \big[\beta(t) - \delta(t)\big]P. \ \blacksquare$$

1.5 Example. (*Radioactive decay*) The number $N(t)$ of atoms of a radioactive isotope present at time $t > 0$ satisfies the equation

$$N' = -\kappa N,$$

where $\kappa = \text{const} > 0$ is the constant rate of decay of the isotope. ∎

1.6 Example. (*Free fall in gravity*) If $v(t)$ is the velocity at time $t > 0$ of a material particle falling in a gravitational field, then

$$mv' = mg - \gamma v,$$

where the positive constants m, g, and γ are, respectively, the particle mass, the acceleration of gravity, and a coefficient characterizing the resistance of the ambient medium to motion. ∎

1.7 Example. (*Newton's law of cooling*) Let $T(t)$ be the temperature at time $t > 0$ of an object immersed in an outside medium of temperature θ. Then

$$T' = -k(T - \theta),$$

where $k = \text{const} > 0$ is the heat transfer coefficient of the object material. ∎

1.8 Example. (*RC electric circuit*) Consider an electric circuit with a source, a resistor, and a capacitor connected in series. We denote by $V(t)$ and $Q(t)$ the voltage generated by the source and the electric charge at time $t > 0$, and by R and C the (constant) resistance and capacitance. Then

$$RQ' + \frac{1}{C}\,Q = V(t). \ \blacksquare$$

1.9 Example. (*Compound interest*) If $S(t)$ is the sum of money present at time $t > 0$ in a savings account that pays interest (compounded continuously) at a rate of r, then

$$S' = rS. \ \blacksquare$$

1.10 Example. (*Loan repayment*) Suppose that a sum of money is borrowed from a bank at a (constant) interest rate of r. If m is the (constant) repayment per unit time, then the outstanding loan amount $A(t)$ at time $t > 0$ satisfies the differential equation

$$A' = rA - m$$

for $0 < t < n$, where n is the number of time units over which the loan is taken. ∎

1.11 Example. (*Equilibrium temperature in a rod*) The equilibrium distribution of temperature $u(x)$ in a heat-conducting rod of length l with an insulated lateral surface and an internal heat source proportional to the temperature is the solution of the DE

$$u'' + qu = 0$$

for $0 < x < l$, where q is a physical constant of the rod material. ■

Informally, we say that a function is a *solution* of a DE if, when replaced in the equation, satisfies it identically (that is, for all admissible values of the variable). A more rigorous definition of this concept will be given at the end of the chapter.

1.12 Example. Consider the equation

$$y'' - 3y' + 2y = 0.$$

If $y_1(t) = e^t$, then $y_1'(t) = e^t$ and $y_1''(t) = e^t$, so, for all real values of t,

$$y_1'' - 3y_1' + 2y_1 = e^t - 3e^t + 2e^t = 0,$$

which means that y_1 is a solution of the given DE. Also, for $y_2(t) = e^{2t}$ we have $y_2'(t) = 2e^{2t}$ and $y_2''(t) = 4e^{2t}$, so

$$y_2'' - 3y_2' + 2y_2 = 4e^{2t} - 6e^{2t} + 2e^{2t} = 0;$$

hence, y_2 is another solution of the equation. ■

1.13 Example. The functions defined by

$$y_1(t) = 2t^2 + \ln t, \quad y_2(t) = -t^{-1} + \ln t$$

are solutions of the equation

$$t^2y'' - 2y = -1 - 2\ln t$$

for $t > 0$ since

$$\begin{aligned} y_1'(t) &= 4t + t^{-1}, & y_1''(t) &= 4 - t^{-2}, \\ y_2'(t) &= t^{-2} + t^{-1}, & y_2''(t) &= -2t^{-3} - t^{-2} \end{aligned}$$

and so, for all $t > 0$,

$$\begin{aligned} t^2y_1'' - 2y_1 &= t^2(4 - t^{-2}) - 2(2t^2 + \ln t) = -1 - 2\ln t, \\ t^2y_2'' - 2y_2 &= t^2(-2t^{-3} - t^{-2}) - 2(-t^{-1} + \ln t) = -1 - 2\ln t. \end{aligned}$$ ■

1.14 Remark. Every solution $y = y(t)$ is represented graphically by a curve in the (t, y)-plane, which is called a *solution curve.* ■

Exercises

Verify that the function y is a solution of the given DE.

1 $y(t) = 5e^{-3t} + 2, \quad y' + 3y = 6.$

2 $y(t) = -2te^{t/2}, \quad 2y' - y = -4e^{t/2}.$

3 $y(t) = -4e^{2t}\cos(3t) + t^2 - 2, \quad y'' - 4y' + 13y = 13t^2 - 8t - 24.$

4 $y(t) = (2t - 1)e^{-3t/2} - 4e^{2t}, \quad 2y'' - y' - 6y = -14e^{-3t/2}.$

5 $y(t) = 2e^{-2t} - e^{-3t} + 8e^{-t/2}, \quad y''' + 4y'' + y' - 6y = -45e^{-t/2}.$

6 $y(t) = \cos(2t) - 3\sin(2t) + 2t, \quad y''' + y'' + 4y' + 4y = 8(t + 1).$

7 $y(t) = 2t^2 + 3t^{-3/2} - 2e^{t/2}, \quad 2t^2y'' + ty' - 6y = (12 - t - t^2)e^{t/2}.$

8 $y(t) = 1 - 2t^{-1} + t^{-2}(2\ln t - 1), \quad t^2y'' + 5ty' + 4y = 4 - 2t^{-1}.$

1.3 Initial and Boundary Conditions

Examples 1.12 and 1.13 show that a DE may have more than one solution. This seems to contradict our expectation that one set of physical data should produce one and only one effect. We therefore conclude that, to yield a unique solution, a DE must be accompanied by some additional restrictions.

To further clarify what was said in Remark 1.2, we normally denote the independent variable by t in DE problems where the solution—and, if necessary, some of its derivatives—are required to assume prescribed values at an admissible point t_0. Restrictions of this type are called *initial conditions* (ICs). They are appropriate for the models mentioned in Examples 1.4–1.9 with $t_0 = 0$.

The independent variable is denoted by x mostly when the DE is to be satisfied on a finite interval $a < x < b$, as in Example 1.11 with $a = 0$ and $b = l$, and the solution and/or some of its derivatives must assume prescribed values at the two end-points $x = a$ and $x = b$. These restrictions are called *boundary conditions* (BCs).

1.15 Definition. The solution obtained without supplementary conditions is termed the *general solution* (GS) of the given DE. Clearly, the GS includes all possible solutions of the equation and, therefore, contains a certain degree of arbitrariness. When ICs or BCs are applied to the GS, we arrive at a *particular solution* (PS). ■

1.16 Definition. A DE and its attending ICs (BCs) form an *initial value problem* (IVP) (*boundary value problem* (BVP)). ■

1.17 Example. For the simple DE

$$y' = 5 - 6t$$

we have, by direct integration,

$$y(t) = \int (5 - 6t)\,dt = 5t - 3t^2 + C,$$

where C is an arbitrary constant. This is the GS of the equation, valid for all real values of t. However, if we turn the DE into an IVP by adjoining, say, the IC $y(0) = -1$, then the GS yields $y(0) = C = -1$, and we get the PS

$$y(t) = 5t - 3t^2 - 1. \;■$$

1.18 Example. To obtain the GS of the DE

$$y'' = 6t + 8,$$

we need to integrate both sides twice; thus, in the first instance we have

$$y'(t) = \int (6t + 8)\,dt = 3t^2 + 8t + C_1, \quad C_1 = \text{const},$$

from which

$$y(t) = \int (3t^2 + 8t + C_1)\,dt = t^3 + 4t^2 + C_1 t + C_2, \quad C_2 = \text{const},$$

for all real values of t. Since the GS here contains two arbitrary constants, we will need two ICs to identify a PS. Suppose that

$$y(0) = 1, \quad y'(0) = -2;$$

then, replacing t by 0 in the expressions for y and y', we find that $C_1 = -2$ and $C_2 = 1$, so the PS corresponding to our choice of ICs is

$$y(t) = t^3 + 4t^2 - 2t + 1. \quad \blacksquare$$

1.19 Example. Let us rewrite the DE in Example 1.18 as

$$y'' = 6x + 8$$

and restrict it to the finite interval $0 < x < 1$. Obviously, the GS remains the same (with t replaced by x), but this time we determine the constants by prescribing BCs. If we assume that

$$y(0) = -3, \quad y(1) = 6,$$

we set $x = 0$ and then $x = 1$ in the GS and use these conditions to find that C_1 and C_2 satisfy

$$C_2 = -3, \quad C_1 + C_2 = 1;$$

hence, $C_1 = 4$, which means that the solution of our BVP is

$$y(x) = x^3 + 4x^2 + 4x - 3. \quad \blacksquare$$

1.20 Example. A material particle moves without friction along a straight line, and its acceleration at time $t > 0$ is $a(t) = e^{-t}$. If the particle starts moving from the point 1 with initial velocity 2, then its subsequent position $s(t)$ can be computed very easily by recalling that acceleration is the derivative of velocity, which, in turn, is the derivative of the function that indicates the position of the particle; that is,

$$v' = a, \quad s' = v.$$

Hence, we need to solve two simple IVPs: one to compute v and the other to compute s. In view of the additional information given in the problem, the first IVP is

$$v' = e^{-t}, \quad v(0) = 2.$$

Here, the DE has the GS

$$v(t) = \int e^{-t}\,dt = -e^{-t} + C.$$

Using the IC, we find that $C = 3$, so

$$v(t) = 3 - e^{-t}.$$

The second IVP now is

$$s' = 3 - e^{-t}, \quad s(0) = 1,$$

whose solution, obtained in a similar manner, is

$$s(t) = 3t + e^{-t}. \ \blacksquare$$

1.21 Example. A stone is thrown upward from the ground with an initial speed of 39.2. To describe its motion when the air resistance is negligible, we need to establish a formula that gives the position $h(t)$ at time $t > 0$ of a heavy object moving vertically under the influence of the force of gravity alone. If $g = 9.8$ is the acceleration of gravity and the object starts moving from a height h_0 with initial velocity v_0, and if we assume that the vertical axis points upward, then Newton's second law yields the IVP

$$h'' = -g, \quad h(0) = h_0, \quad h'(0) = v_0.$$

Integrating the DE twice and using the ICs, we easily find that

$$h(t) = -\tfrac{1}{2}\, gt^2 + v_0 t + h_0.$$

In our specific case, we have $g = 9.8$, $h_0 = 0$, and $v_0 = 39.2$, so

$$h(t) = -4.9t^2 + 39.2t.$$

If we now want, for example, to find the maximum height that the stone reaches above the ground, then we need to compute h at the moment when the stone's velocity is zero. Since

$$v(t) = h'(t) = -9.8t + 39.2,$$

we immediately see that v vanishes at $t = 4$, so

$$h_{\max} = h(4) = 78.4.$$

If, on the other hand, we want to know when the falling stone will hit the ground, then we need to determine the nonzero root t of the equation $h(t) = 0$, which, as can easily be seen, is $t = 8$. ■

Exercises

In 1–4, solve the given IVP.

1 $y'' = -2(6t + 1); \quad y(0) = 2, \quad y'(0) = 0.$

2 $y'' = -12e^{2t}; \quad y(0) = -3, \quad y'(0) = -6.$

3 $y'' = -2\sin t - t\cos t; \quad y(0) = 0, \quad y'(0) = 1.$

4 $y'' = 2t^{-3}; \quad y(1) = -1, \quad y'(1) = 1.$

In 5–8, solve the given BVP.

5 $y'' = 2, \quad 0 < x < 1; \quad y(0) = 2, \quad y(1) = 0.$

6 $y'' = 4\cos(2x)$, $0 < x < \pi$; $y(0) = -3$, $y(\pi) = \pi - 3$.

7 $y'' = -x^{-1} - x^{-2}$, $1 < x < e$; $y'(1) = 0$, $y(e) = 1 - e$.

8 $y'' = (2x - 3)e^{-x}$, $0 < x < 1$; $y(0) = 1$, $y'(1) = -e^{-1}$.

In 9–12, find the velocity $v(t)$ and position $s(t)$ at time $t > 0$ of a material particle that moves without friction along a straight line, when its acceleration, initial position, and initial velocity are as specified.

9 $a(t) = 2$, $s(0) = -4$, $v(0) = 0$.

10 $a(t) = -12\sin(2t)$, $s(0) = 0$, $v(0) = 6$.

11 $a(t) = 3(t + 4)^{-1/2}$, $s(0) = 1$, $v(0) = -1$.

12 $a(t) = (t + 3)e^t$, $s(0) = 1$, $v(0) = 2$.

In 13 and 14, solve the given problem.

13 A ball, thrown downward from the top of a building with an initial speed of 3.4, hits the ground with a speed of 23. How tall is the building if the acceleration of gravity is 9.8 and the air resistance is negligible?

14 A stone is thrown upward from the top of a tower, with an initial velocity of 98. Assuming that the height of the tower is 215.6, the acceleration of gravity is 9.8, and the air resistance is negligible, find the maximum height reached by the stone, the time when it passes the top of the tower on its way down, and the total time it has been traveling from launch until it hits the ground.

Answers to Odd-Numbered Exercises

1 $y(t) = 2 - t^2 - 2t^3$. **3** $y(t) = t\cos t$. **5** $y(x) = x^2 - 3x + 2$.

7 $y(x) = (1 - x)\ln x$. **9** $v(t) = 2t$, $s(t) = t^2 - 4$.

11 $v(t) = 6(t + 4)^{1/2} - 13$, $s(t) = 4(t + 4)^{3/2} - 13t - 31$. **13** 26.4.

1.4 Classification of Differential Equations

We recall that, in calculus, a function is a correspondence between one set of numbers (called *domain*) and another set of numbers (called *range*), which has the property that it associates each number in the domain with exactly one number in the range. If the domain and range consist not of numbers but of functions, then this type of correspondence is called an *operator*. The image of a number t under a function f is denoted, as we already said, by $f(t)$; the image of a function y under an operator L is usually denoted by Ly. In special circumstances, we may also write $L(y)$.

1.22 Example. We can define an operator D that associates each differentiable function with its derivative; that is,

$$Dy = y'.$$

Naturally, D is referred to as the operator of differentiation. For twice differentiable functions, we can iterate this operator and write

$$D(Dy) = D^2y = y''.$$

This may be extended in the obvious way to higher-order derivatives. ■

1.23 Remark. Taking the above comments into account, we can write a differential equation in the generic form

$$Ly = f, \tag{1.1}$$

where L is defined by the sequence of operations performed on the unknown function y on the left-hand side, and f is a given function. We will use form (1.1) only in non-specific situations; in particular cases, this form is rather cumbersome and will be avoided. ■

1.24 Example. The DE in the population growth model (see Example 1.4) can be written as $P' - (\beta - \delta)P = 0$. In the operator notation (1.1), this is

$$LP = DP - (\beta - \delta)P = [D - (\beta - \delta)]P = 0;$$

in other words, $L = D - (\beta - \delta)$ and $f = 0$. ■

1.25 Example. It is not difficult to see that form (1.1) for the DE

$$t^2y'' - 2y' = e^{-t}$$

is

$$Ly = (t^2D^2)y - (2D)y = (t^2D^2 - 2D)y = e^{-t},$$

so $L = t^2D^2 - 2D$ and $f(t) = e^{-t}$. ■

1.26 Remarks. (i) The notation in the preceding two example is not entirely rigorous. Thus, in the expression $D-(\beta-\delta)$ in Example 1.24, the first term is an operator and the second one is a function. However, we adopted this form because it is intuitively helpful.

(ii) A similar comment can be made about the term t^2D^2 in Example 1.25, where t^2 is a function and D^2 is an operator. In this context, it must be noted that t^2D^2 and D^2t^2 are not the same operator. When applied to a function y, the former yields

$$(t^2D^2)y = t^2(D^2y) = t^2y'',$$

whereas the latter generates the image

$$(D^2t^2)y = D^2(t^2y) = (t^2y)'' = 2y + 4ty' + t^2y''.$$

(iii) The rigorous definition of a mathematical operator is more general, abstract, and precise than the one given above, but it is beyond the scope of this book. ■

1.27 Definition. An operator L is called *linear* if for any two functions y_1 and y_2 in its domain and any two numbers c_1 and c_2 we have

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2. \tag{1.2}$$

Otherwise, L is called *nonlinear*. ■

1.28 Example. The differentiation operator D is linear because for any differentiable functions y_1 and y_2 and any constants c_1 and c_2,

$$D(c_1y_1 + c_2y_2) = (c_1y_1 + c_2y_2)' = c_1y_1' + c_2y_2' = c_1Dy_1 + c_2Dy_2. \ ■$$

1.29 Example. The operator $L = tD^2 - 3$ is also linear, since

$$\begin{aligned} L(c_1y_1 + c_2y_2) &= (tD^2)(c_1y_1 + c_2y_2) - 3(c_1y_1 + c_2y_2) \\ &= t(c_1y_1 + c_2y_2)'' - 3(c_1y_1 + c_2y_2) \\ &= t(c_1y_1'' + c_2y_2'') - 3(c_1y_1 + c_2y_2) \\ &= c_1(ty_1'' - 3y_1) + c_2(ty_2'' - 3y_2) = c_1Ly_1 + c_2Ly_2. \ \blacksquare \end{aligned}$$

1.30 Remark. By direct verification of property (1.2), it can be shown that, more generally:

(i) the operator D^n of differentiation of any order n is linear;
(ii) the operator of multiplication by a fixed function (in particular, a constant) is linear;
(iii) the sum of finitely many linear operators is a linear operator. ■

1.31 Example. According to the above remark, the operators written formally as

$$L = a(t)D + b(t), \quad L = D^2 + p(t)D + q(t)$$

with given functions a, b, p, and q, are linear. ■

1.32 Example. Let L be the operator defined by $Ly = yy'$. Then, taking, say, $y_1(t) = t$, $y_2(t) = t^2$, and $c_1 = c_2 = 1$, we have

$$\begin{aligned} L(c_1y_1 + c_2y_2) = L(t + t^2) = (t + t^2)(t + t^2)' = t + 3t^2 + 2t^3, \\ c_1Ly_1 + c_2Ly_2 = (t)(t)' + (t^2)(t^2)' = t + 2t^3, \end{aligned}$$

which shows that (1.2) does not hold for this particular choice of functions and numbers. Consequently, L is a nonlinear operator. ■

DEs can be placed into different categories according to certain relevant criteria. Here we list the most important ones, making reference to the generic form (1.1).

Number of independent variables. If the unknown is a function of a single independent variable, the DE is called an *ordinary differential equation*. If several independent variables are involved, then the DE is called a *partial differential equation*.

1.33 Example. The DE

$$ty'' - (t^2 - 1)y' + 2y = t\sin t$$

is an ordinary differential equation for the unknown function $y = y(t)$.

The DE

$$u_t - 3(x + t)u_{xx} = e^{2x-t}$$

is a partial differential equation for the unknown function $u = u(x, t)$. ■

Order. The order of a DE is the order of the highest derivative occurring in the expression Ly in (1.1).

1.34 Example. The equation

$$t^2y'' - 2y' + (t - \cos t)y = 3$$

is a second-order DE. ■

Linearity. If the differential operator L in (1.1) is linear (see Definition 1.27), then the DE is a *linear equation*; otherwise it is a *nonlinear equation*.

1.35 Example. The equation

$$ty'' - 3y = t^2 - 1$$

is linear since the operator $L = tD^2 - 3$ defined by its left-hand side was shown in Example 1.29 to be linear. On the other hand, the equation

$$y'' + yy' - 2ty = 0$$

is nonlinear: as seen in Example 1.32, the term yy' defines a nonlinear operator. ■

Nature of coefficients. If the coefficients of y and its derivatives in every term in Ly are constant, the DE is said to be an *equation with constant coefficients*. If at least one of these coefficients is a prescribed function, we have an *equation with variable coefficients*.

1.36 Example. The DE

$$3y'' - 2y' + 4y = 0$$

is an equation with constant coefficients, whereas the DE

$$y' - (2t + 1)y = 1 - e^t$$

is an equation with variable coefficients. ■

Homogeneity. If $f = 0$ in (1.1), the DE is called *homogeneous*; otherwise it is called *nonhomogeneous*.

1.37 Example. The DE

$$(t - 2)y' - ty^2 = 0$$

is homogenous; the DE

$$y''' - e^{-t}y' + \sin y = 4t - 3$$

is nonhomogeneous. ■

Of course, any equation can be classified by means of all these criteria at the same time.

1.38 Example. The DE

$$y'' - y' + 2y = 0$$

is a linear, homogeneous, second-order ordinary differential equation with constant coefficients. The linearity of the operator $D^2 - D + 2$ defined by the left-hand side is easily verified. ■

1.39 Example. The DE

$$y''' - y'y'' + 4ty = 3$$

is a nonlinear, nonhomogeneous, third-order ordinary differential equation with variable coefficients. The nonlinearity is caused by the second term on the left-hand side. ■

1.40 Example. The DE

$$u_t + (x^2 - t)u_x - xtu = e^x \sin t$$

is a linear, nonhomogeneous, first-order partial differential equation with variable coefficients. ■

1.41 Definition. An ordinary differential equation of order n is said to be in *normal form* if it is written as

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}), \tag{1.3}$$

where F is some function of $n+1$ variables. ■

1.42 Example. The equation

$$(t+1)y'' - 2ty' + 4y = t + 3$$

is not in normal form. To write it in normal form, we solve for y'':

$$y'' = \frac{1}{t+1}(2ty' - 4y + t + 3) = F(t, y, y'). \ ■$$

1.43 Definition. A function y defined on an open interval J (of the real line) is said to be a *solution* of the DE (1.3) on J if the derivatives y', y'', …, $y^{(n)}$ exist on J and (1.3) is satisfied at every point of J. ■

1.44 Remark. Sometimes a model is described by a *system* of DEs, which consists of several DEs for several unknown functions. ■

1.45 Example. The pair of equations

$$\begin{aligned} x_1' &= 3x_1 - 2x_2 + t, \\ x_2' &= -2x_1 + x_2 - e^t \end{aligned}$$

form a linear, nonhomogeneous, first-order system of ordinary DEs with constant coefficients for the unknown functions x_1 and x_2. ■

Exercises

Classify the given DE in terms of all the criteria listed in this section.

1 $y^{(4)} - ty'' + y^2 = 0$. **2** $y'' - 2y = \sin t$.

3 $y' - 2\sin y = t$. **4** $u_t - 2u_{xx} + (2xt+1)u = 0$.

5 $2y''' + ye^y = 0$. **6** $uu_x - 2u_{xx} + 3u_{xxyy} = 1$.

7 $tu_t - 4u_x - u = x$. **8** $y'y''' - t^2u = \cos(2t)$.

Answers to Odd-Numbered Exercises

1 Nonlinear, homogeneous, fourth-order ordinary DE with variable coefficients.

3 Nonlinear, nonhomogeneous, first-order ordinary DE with constant coefficients.

5 Nonlinear, homogeneous, third-order ordinary DE with constant coefficients.

7 Linear, nonhomogeneous, first-order partial DE with variable coefficients.

Chapter 2
First-Order Equations

Certain types of first-order equations can be solved by relatively simple methods. Since, as seen in Sect. 1.2, many mathematical models are constructed with such equations, it is important to get familiarized with their solution procedures.

2.1 Separable Equations

These are equations of the form

$$\frac{dy}{dx} = f(x)g(y), \tag{2.1}$$

where f and g are given functions.

We notice that if there is any value y_0 such that $g(y_0) = 0$, then $y = y_0$ is a solution of (2.1). Since this is a constant function (that is, independent of x), we call it an *equilibrium solution*.

To find all the other (non-constant) solutions of the equation, we now assume that $g(y) \neq 0$. Applying the definition of the differential of y and using (2.1), we have

$$dy = y'(x)\,dx = \frac{dy}{dx}\,dx = f(x)g(y)\,dx,$$

which, after division by $g(y)$, becomes

$$\frac{1}{g(y)}\,dy = f(x)\,dx.$$

Next, we integrate each side with respect to its variable and arrive at the equality

$$G(y) = F(x) + C, \tag{2.2}$$

where F and G are any antiderivatives of f and $1/g$, respectively, and C is an arbitrary constant. For each value of C, (2.2) provides a connection between y and x, which defines a function $y = y(x)$ implicitly.

We have shown that every solution of (2.1) also satisfies (2.2). To confirm that these two equations are fully equivalent, we must also verify that, conversely, any function $y = y(x)$ satisfying (2.2) also satisfies (2.1). This is easily done by differentiating both sides of (2.2) with respect to x. The derivative of the right-hand side is $f(x)$; on the left-hand side, by the chain rule and bearing in mind that $G(y) = G(y(x))$, we have

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_2,

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$$\frac{d}{dx}\,G(y(x)) = \frac{d}{dy}\,G(y)\frac{dy}{dx} = \frac{1}{g(y)}\,\frac{dy}{dx},$$

which, when equated to $f(x)$, yields equation (2.1).

In some cases, the solution $y = y(x)$ can be determined explicitly.

2.1 Remark. The above handling suggests that dy/dx could be treated formally as a ratio, but this would not be technically correct. ■

2.2 Example. Bringing the DE

$$y' + 8xy = 0$$

to the form

$$\frac{dy}{dx} = -8xy,$$

we see that it has the equilibrium solution $y = 0$. Then for $y \neq 0$,

$$\int \frac{dy}{y} = \int -8x\,dx,$$

from which

$$\ln|y| = -4x^2 + C,$$

where C is the amalgamation of the arbitrary constants of integration from both sides. Exponentiating, we get

$$|y| = e^{-4x^2+C} = e^C\,e^{-4x^2},$$

so

$$y(x) = \pm e^C e^{-4x^2} = C_1 e^{-4x^2}.$$

Here, as expected, C_1 is an arbitrary nonzero constant (it replaces $\pm e^C \neq 0$), which generates all the nonzero solutions y. However, if we allow C_1 to take the value 0 as well, then the above formula also captures the equilibrium solution $y = 0$ and, thus, becomes the GS of the given equation.

Verification with Mathematica®. The input

```
y=C1*E^(-4*x^2);
D[y,x]+8*x*y
```

evaluates the difference between the left-hand and right-hand sides of our DE for the function y computed above. This procedure will be followed in all similar situations. As expected, the output here is 0, which confirms that this function is indeed the GS of the given equation. ■

2.3 Example. In view of the properties of the exponential function, the DE in the IVP

$$y' + 4xe^{y-2x} = 0, \quad y(0) = 0$$

can be rewritten as

$$\frac{dy}{dx} = -4xe^{-2x}e^y,$$

and we see that, since $e^y \neq 0$ for any real value of y, the equation has no equilibrium solutions. After separating the variables, we arrive at

$$\int e^{-y}\,dy = -\int 4xe^{-2x}\,dx,$$

from which, using integration by parts (see Sect. B.2) on the right-hand side, we find that

$$-e^{-y} = 2xe^{-2x} - \int 2e^{-2x}\,dx = (2x+1)e^{-2x} + C.$$

We now change the signs of both sides, take logarithms, and produce the GS

$$y(x) = -\ln[-(2x+1)e^{-2x} - C].$$

The constant C is more easily computed if we apply the IC not to this explicit expression of y but to the equality immediately above it. The value is $C = -2$, so the solution of the IVP is

$$y(x) = -\ln[2-(2x+1)e^{-2x}].$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = -Log[2 - (2*x + 1)*E^(-2*x);
{D[y,x] + 4*x*E^(y - 2*x),y/. x→ 0}//Simplify
```

evaluates both the difference between the left-hand and right-hand sides (as in the preceding example) and the value of the computed function y at $x = 0$. Again, this type of verification will be performed for all IVPs and BVPs in the rest of the book with no further comment. Here, the output is, of course, $\{0, 0\}$. ■

2.4 Example. Form (2.1) for the DE of the IVP

$$xy' = y+2, \quad y(1) = -1$$

is

$$\frac{dy}{dx} = \frac{y+2}{x}.$$

Clearly, $y = -2$ is an equilibrium solution. For $y \neq -2$ and $x \neq 0$, we separate the variables and arrive at

$$\int \frac{dy}{y+2} = \int \frac{dx}{}x;$$

hence,

$$\ln|y+2| = \ln|x| + C,$$

from which, by exponentiation,

$$|y+2| = e^{\ln|x|+C} = e^C e^{\ln|x|} = e^C|x|.$$

This means that

$$y+2 = \pm e^C x = C_1 x, \quad C_1 = \text{const} \neq 0,$$

so

$$y(x) = C_1 x - 2.$$

To make this the GS, we need to allow C_1 to be zero as well, which includes the equilibrium solution $y = -2$ in the above equality. Applying the IC, we now find that $C_1 = 1$; therefore, the solution of the IVP is

$$y(x) = x - 2.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=x - 2;
{x*D[y,x] - y - 2,y/.x→ 1}//Simplify
```

generates the output $\{0, -1\}$. ■

2.5 Example. The DE in the IVP

$$2(x+1)yy' - y^2 = 2, \quad y(5) = 2,$$

rewritten in the form

$$\frac{dy}{dx} = \frac{y^2+2}{2(x+1)y},$$

can be seen to have no equilibrium solutions; hence, after separation, for $x \neq -1$ we have

$$\int \frac{2y\,dy}{y^2+2} = \int \frac{dx}{x+1},$$

so

$$\ln(y^2+2) = \ln|x+1| + C,$$

which, after simple algebraic manipulation, leads to

$$y^2 = C_1(x+1) - 2, \quad C_1 = \text{const} \neq 0.$$

Applying the IC, we obtain $y^2 = x - 1$, or $y = \pm(x-1)^{1/2}$. However, the function with the '$-$' sign must be rejected because it does not satisfy the IC. In conclusion, the solution to our IVP is

$$y(x) = (x-1)^{1/2}.$$

If the IC were $y(5) = -2$, then the solution would be

$$y(x) = -(x-1)^{1/2}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y= (x - 1) * (1/2);
{2* (x + 1) *y*D[y,x] - y^2 - 2,y/.x→ 5}//Simplify
```

generates the output $\{0, 2\}$. ■

2.6 Example. Treating the DE in the IVP

$$(5y^4 + 3y^2 + e^y)y' = \cos x, \quad y(0) = 0$$

in the same way, we arrive at

$$\int (5y^4 + 3y^2 + e^y)\,dy = \int \cos x\,dx;$$

consequently,

$$y^5 + y^3 + e^y = \sin x + C.$$

This equality describes the family of all the solution curves for the DE, representing its GS in implicit form. It cannot be solved explicitly for y.

The IC now yields $C = 1$, so the solution curve passing through the point $(0, 0)$ has equation

$$y^5 + y^3 + e^y = \sin x + 1.$$

Figure 2.1 shows the solution curves for $C = -2$, -1, 0, 1, 2. The heavier line (for $C = 1$) represents the solution of our IVP.

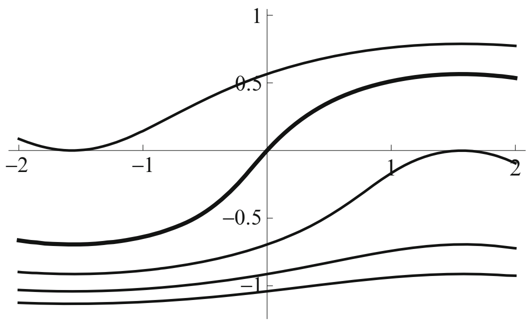

Fig. 2.1

VERIFICATION WITH MATHEMATICA®. The input

```
u=y[x]^5 + y[x]^3 + E^y[x] - Sin[x] - 1;
{(5*y[x]^4 + 3*y[x]^2 + E^y[x])*(Solve[D[u,x]==0,y'[x]])
        [[1,1,2]] - Cos[x], u/.{x→ 0, y→ 0}}//Simplify
```

generates the output $\{0, 0\}$, which shows that the function y defined implicitly above satisfies the DE and IC. ■

Exercises

Solve the given IVP.

1 $y' = -4xy^2$, $y(0) = 1$. **2** $y' = y\sin(2x)$, $y(\pi/4) = 1$.

3 $(1 + 2x)y' = 3 + y$, $y(0) = -2$. **4** $y' = 2x\sqrt{y}$, $y(0) = 1$.

5 $y' = 2x\sec y$, $y(0) = \pi/6$. **6** $(4 - x^2)y' = 3y$, $y(0) = 1$.

7 $(y^4 + 2y)y' = xe^{2x}$, $y(0) = -1$. **8** $y' = 2ye^{2x+1}$, $y(-1/2) = e^2$.

9 $y' = (x - 3)(y^2 + 1)$, $y(0) = 1$. **10** $(e^{-2y} + 4y)y' = 2x^2 + 1$, $y(0) = 0$.

Answers to Odd-Numbered Exercises

1 $y(x) = 1/(2x^2 + 1)$. **3** $y(x) = -3 + (1 + 2x)^{1/2}$.

5 $y(x) = \sin^{-1}(x^2 + 1/2\,)$. **7** $4y^5 + 20y^2 = 5(2x - 1)e^{2x} + 21$.

9 $y(x) = \tan[(2x^2 - 12x + \pi)/4]$.

2.2 Linear Equations

The standard form of this type of DE is

$$y' + p(t)y = q(t), \tag{2.3}$$

where p and q are prescribed functions. To solve the equation, we first multiply it by an unknown nonzero function $\mu(t)$, called an *integrating factor*. Omitting, for simplicity, the mention of the variable t, we have

$$\mu y' + \mu p y = \mu q. \tag{2.4}$$

We now choose μ so that the left-hand side in (2.4) is the derivative of the product μy; that is,

$$\mu y' + \mu p y = (\mu y)' = \mu y' + \mu' y.$$

Clearly, this occurs if

$$\mu' = \mu p.$$

The above separable equation yields, in the usual way,

$$\int \frac{d\mu}{\mu} = \int p\, dt.$$

Integrating, we arrive at

$$\ln|\mu| = \int p\, dt,$$

so, as in Example 2.4,

$$\mu = C \exp\left\{\int p\, dt\right\}, \quad C = \text{const} \neq 0.$$

Since we need just one such function, we may take $C = 1$ and thus consider the integrating factor

$$\mu(t) = \exp\left\{\int p(t)\, dt\right\}. \tag{2.5}$$

With this choice of μ, equation (2.4) becomes

$$(\mu y)' = \mu q; \tag{2.6}$$

hence,

$$\mu y = \int \mu q\, dt + C,$$

or

$$y(t) = \frac{1}{\mu(t)}\left\{\int \mu(t) q(t)\, dt + C\right\}. \tag{2.7}$$

2.7 Remarks. (i) Technically, C does not need to be inserted explicitly in (2.7) since the indefinite integral on the right-hand side produces an arbitrary constant, but it is good practice to have it in the formula for emphasis and to prevent its accidental omission when the integration is performed.

(ii) It should be obvious that the factor $1/\mu$ cannot be moved inside the integral to be canceled with the factor μ already there.

(iii) Points (i) and (ii) become moot if the equality $(\mu y)' = \mu q$ (see (2.6)) is integrated from some admissible value t_0 to a generic value t. Then

$$\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} \mu(\tau)q(\tau)\,d\tau,$$

from which we easily deduce that

$$y(t) = \frac{1}{\mu(t)}\left\{\int_{t_0}^{t} \mu(\tau)q(\tau)\,d\tau + \mu(t_0)y(t_0)\right\}. \tag{2.8}$$

In the case of an IVP, it is convenient to choose t_0 as the point where the IC is prescribed.

(iv) In (2.8) we used the 'dummy' variable τ in the integrand to avoid a clash with the upper limit t of the definite integral. ■

2.8 Example. Consider the IVP

$$y' - 3y = 6, \quad y(0) = -1,$$

where, by comparison to (2.3), we have $p(t) = -3$ and $q(t) = 6$. The GS of the DE is computed from (2.5) and (2.7). Thus,

$$\mu(t) = \exp\left\{\int -3\,dt\right\} = e^{-3t},$$

so

$$y(t) = e^{3t}\left\{\int 6e^{-3t}\,dt + C\right\} = e^{3t}(-2e^{-3t} + C) = Ce^{3t} - 2.$$

Applying the IC, we find that $C = 1$, which yields the IVP solution

$$y(t) = e^{3t} - 2.$$

Alternatively, we could use formula (2.8) with μ as determined above and $t_0 = 0$, to obtain directly

$$y(t) = e^{3t}\left\{\int_0^t 6e^{-3\tau}\,d\tau + \mu(0)y(0)\right\} = e^{3t}\left[-2e^{-3\tau}\Big|_0^t - 1\right] = e^{3t} - 2.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = E^(3*t) - 2;
{D[y,t] - 3*y - 6, y/.t→0}//Simplify
```

generates the output $\{0, -1\}$. ■

2.9 Example. The DE in the IVP

$$ty' + 4y = 6t^2, \quad y(1) = 4$$

is not in the standard form (2.3). Assuming that $t \neq 0$, we divide the equation by t and rewrite it as

$$y' + \frac{4}{t}\,y = 6t.$$

This shows that $p(t) = 4/t$ and $q(t) = 6t$, so, by (2.5),

$$\mu(t) = \exp\left\{\int \frac{4}{t}\,dt\right\} = e^{4\ln|t|} = e^{\ln(t^4)} = t^4.$$

Using (2.8) with $t_0 = 1$, we now find the solution of the IVP to be

$$y(t) = t^{-4}\left\{\int\limits_1^t 6\tau^5\,d\tau + \mu(1)y(1)\right\} = t^{-4}\big(\tau^6\big|_1^t + 4\big) = t^{-4}(t^6 + 3) = t^2 + 3t^{-4}.$$

Verification with Mathematica®. The input

```
y = t^2 + 3 * t^(-4);
{t * D[y,t] + 4 * y - 6 * t^2, y /. t → 1} // Simplify
```

generates the output $\{0, 4\}$. ■

2.10 Example. To bring the DE in the IVP

$$y' = (2 + y)\sin t, \quad y(\pi/2) = -3$$

to the standard form, we move the y-term to the left-hand side and write

$$y' - y\sin t = 2\sin t.$$

This shows that $p(t) = -\sin t$ and $q(t) = 2\sin t$; hence, by (2.5),

$$\mu(t) = \exp\left\{-\int \sin t\,dt\right\} = e^{\cos t},$$

and, by (2.8) with $t_0 = \pi/2$,

$$\begin{aligned} y(t) &= e^{-\cos t}\left\{2\int\limits_{\pi/2}^t e^{\cos\tau}\sin\tau\,d\tau + \mu(\pi/2)y(\pi/2)\right\} \\ &= e^{-\cos t}\left\{-2\int\limits_{\pi/2}^t e^{\cos\tau}\,d(\cos\tau) - 3\right\} = e^{-\cos t}\big(-2e^{\cos\tau}\big|_{\pi/2}^t - 3\big) \\ &= e^{-\cos t}(-2e^{\cos t} - 1) = -e^{-\cos t} - 2. \end{aligned}$$

Verification with Mathematica®. The input

```
y = -E^(-Cos[t]) - 2;
{D[y,t] - (2 + y) * Sin[t], y /. t → Pi/2} // Simplify
```

generates the output $\{0, -3\}$. ■

2.11 Example. Consider the IVP

$$(t^2 + 1)y' - ty = 2t(t^2 + 1)^2, \quad y(0) = \tfrac{2}{3}.$$

Proceeding as in Example 2.9, we start by rewriting the DE in the standard form

$$y' - \frac{t}{t^2+1}\,y = 2t(t^2+1).$$

Then, with $p(t) = -t/(t^2+1)$ and $q(t) = 2t(t^2+1)$, we have, first,

$$\begin{aligned}\mu(t) &= \exp\left\{-\int \frac{t}{t^2+1}\,dt\right\} = \exp\left\{-\frac{1}{2}\int \frac{d(t^2+1)}{t^2+1}\right\}\\ &= e^{-(1/2)\ln(t^2+1)} = e^{\ln[(t^2+1)^{-1/2}]} = (t^2+1)^{-1/2},\end{aligned}$$

followed by

$$\begin{aligned}y(t) &= (t^2+1)^{1/2}\left\{\int_0^t (\tau^2+1)^{-1/2} 2\tau(\tau^2+1)\,d\tau + \mu(0)y(0)\right\}\\ &= (t^2+1)^{1/2}\left\{\int_0^t (\tau^2+1)^{1/2}\,d(\tau^2+1) + \tfrac{2}{3}\right\}\\ &= (t^2+1)^{1/2}\left\{\tfrac{2}{3}\,(\tau^2+1)^{3/2}\Big|_0^t + \tfrac{2}{3}\right\} = \tfrac{2}{3}\,(t^2+1)^2.\end{aligned}$$

Verification with Mathematica®. The input

```
y = (2/3) * (t^2 + 1)^2;
{(t^2 + 1) * D[y,t] - t * y - 2 * t * (t^2 + 1)^2, y /. t → 0} // Simplify
```

generates the output $\{0,\ 2/3\}$. ■

2.12 Example. The standard form of the DE in the IVP

$$(t-1)y' + y = (t-1)e^t, \quad y(2) = 3$$

is

$$y' + \frac{1}{t-1}\,y = e^t,$$

so $p(t) = 1/(t-1)$ and $q(t) = e^t$. Consequently, for $t \neq 1$,

$$\mu(t) = \exp\left\{\int \frac{1}{t-1}\,dt\right\} = e^{\ln|t-1|} = |t-1| = \begin{cases} t-1, & t > 1,\\ -(t-1), & t < 1.\end{cases}$$

Since formula (2.8) uses the value of μ at $t_0 = 2 > 1$, we take $\mu(t) = t-1$ and, after integration by parts and simplification, obtain the solution

$$y(t) = \frac{1}{t-1}\left\{\int_2^t (\tau-1)e^\tau\,d\tau + \mu(2)y(2)\right\} = \frac{(t-2)e^t+3}{t-1}.$$

Verification with Mathematica®. The input

```
y = ((t - 2) * E^t + 3)/(t - 1);
{(t-1) * D[y,t] + y - (t - 1) * E^t, y /. t → 2} // Simplify
```

generates the output $\{0, 3\}$. ■

2.13 Remark. If we do not have an IC and want to find only the GS of the equation in Example 2.12, it does not matter if we take μ to be $t-1$ or $-(t-1)$ since μ has to be replaced in (2.6) and, in the latter case, the '$-$' sign would cancel out on both sides. ■

Exercises

Solve the given IVP.

1 $y' + 4y + 16 = 0, \quad y(0) = -2.$ **2** $y' + y = 4te^{-3t}, \quad y(0) = 3.$

3 $2ty' + y + 12t\sqrt{t} = 0, \quad y(1) = -1.$ **4** $t^2y' + 3ty = 4e^{2t}, \quad y(1) = e^2.$

5 $(t-2)y' + y = 8(t-2)\cos(2t), \quad y(\pi) = 2/(\pi - 2).$

6 $y' + y\cot t = 2\cos t, \quad y(\pi/2) = 1/2.$

7 $(t^2+2)y' + 2ty = 3t^2 - 4t, \quad y(0) = 3/2.$

8 $ty' + (2t-1)y = 9t^3e^t, \quad y(1) = 2e^{-2} + 2e.$

9 $(t^2-1)y' + 4y = 3(t+1)^2(t^2-1), \quad y(0) = 0.$

10 $(t^2+2t)y' + y = \sqrt{t}, \quad y(2) = 0.$

Answers to Odd-Numbered Exercises

1 $y = 2e^{-4t} - 4.$ **3** $y = (2 - 3t^2)t^{-1/2}.$

5 $y = [2\cos(2t) + 4(t-2)\sin(2t)]/(t-2).$ **7** $y = (t^3 - 2t^2 + 3)/(t^2+2).$

9 $y = (t^3 - 3t^2 + 3t)(t+1)^2/(t-1)^2.$

2.3 Homogeneous Polar Equations

These are DEs of the form

$$y'(x) = f\left(\frac{y}{x}\right), \tag{2.9}$$

where f is a given one-variable function. Making the substitution

$$y(x) = xv(x) \tag{2.10}$$

and using the fact that, by the product rule, $y' = v + xv'$, from (2.9) we see that the new unknown function v satisfies the DE

$$xv' + v = f(v),$$

or

$$\frac{dv}{dx} = \frac{f(v) - v}{x}.$$

This is a separable equation, so

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}, \tag{2.11}$$

which, with v replaced by y/x in the result, produces the GS of (2.9). Clearly, in this type of problem we must assume that $x \neq 0$.

2.14 Example. The DE in the IVP

$$xy' = x + 2y, \quad y(1) = 1$$

can be written as

$$y' = 1 + 2\,\frac{y}{x},$$

so $f(v) = 1 + 2v$. Then $f(v) - v = v + 1$ and, by (2.11),

$$\int \frac{dv}{v+1} = \int \frac{dx}{x},$$

which yields

$$\ln|v+1| = \ln|x| + C.$$

Exponentiating and simplifying, we find that

$$v + 1 = C_1 x;$$

hence, using (2.10), we obtain the GS

$$y = C_1 x^2 - x.$$

The constant is found from the IC; specifically, $C_1 = 2$, so the solution of the IVP is

$$y(x) = 2x^2 - x.$$

Verification with Mathematica®. The input

```
y=2*x^2 - x;
{x*D[y,x] - x - 2*y, y/. x→1}//Simplify
```

generates the output $\{0, 1\}$. ■

2.15 Example. Consider the IVP

$$(x^2 + 2xy)y' = 2(xy + y^2), \quad y(1) = 2.$$

Solving for y' and then dividing both numerator and denominator by x^2 brings the DE to the form

$$y' = \frac{2(xy + y^2)}{x^2 + 2xy} = \frac{2\,\dfrac{y}{x} + 2\left(\dfrac{y}{x}\right)^2}{1 + 2\,\dfrac{y}{x}} = \frac{2(v + v^2)}{1 + 2v} = f(v),$$

from which

$$f(v) - v = \frac{2(v + v^2)}{1 + 2v} - v = \frac{v}{1 + 2v}.$$

By (2.11), we have

$$\int \frac{1 + 2v}{v}\,dv = \int \left(2 + \frac{1}{v}\right) dv = \int \frac{dx}{x},$$

so

$$2v + \ln|v| = \ln|x| + C,$$

or, according to (2.10),

$$2\,\frac{y}{x} + \ln\left|\frac{y}{x^2}\right| = C.$$

Applying the IC, we get $C = 4 + \ln 2$; hence, the solution of the IVP is defined implicitly by the equality

$$2\,\frac{y}{x} + \ln\left|\frac{y}{2x^2}\right| = 4.$$

VERIFICATION WITH MATHEMATICA®. The input

```
u = 2 * y[x] / x + Log[y[x] / (2 * x^2)] - 4;
{(x^2 + 2 * x * y[x]) * (Solve[D[u,x]==0, y'[x]])[[1,1,2]]
        - 2 * (x * y[x] + y[x]^2), u /. {x → 1, y[x] → 2}} // Simplify
```

generates the output $\{0, 0\}$. ■

Exercises

Solve the given IVP, or find the GS of the DE if no IC is given.

1 $xy' = 3y - x, \quad y(1) = 1.$ **2** $x^2y' = xy + y^2.$

3 $3xy^2y' = x^3 + 3y^3.$ **4** $x^2y' - 2xy - y^2 = 0, \quad y(1) = 1.$

5 $(2x^2 - xy)y' = xy - y^2.$

6 $(2x^2 - 3xy)y' = x^2 + 2xy - 3y^2.$

Answers to Odd-Numbered Exercises

1 $y(x) = (x^3 + x)/2.$ **3** $y(x) = x(C + \ln|x|)^{1/3}.$ **5** $y(x) = x[\ln(y^2/|x|) + C].$

2.4 Bernoulli Equations

The general form of a Bernoulli equation is

$$y' + p(t)y = q(t)y^n, \quad n \neq 1. \tag{2.12}$$

Making the substitution

$$y(t) = (w(t))^{1/(1-n)} \tag{2.13}$$

and using the chain rule, we have

$$y' = \frac{1}{1-n}\,w^{1/(1-n)-1}w' = \frac{1}{1-n}\,w^{n/(1-n)}w',$$

so (2.12) becomes

$$\frac{1}{1-n}\, w^{n/(1-n)}w' + pw^{1/(1-n)} = qw^{n/(1-n)}.$$

Since $1/(1-n) - n/(1-n) = 1$, after division by $w^{n/(1-n)}$ and multiplication by $1-n$ this simplifies further to

$$w' + (1-n)pw = (1-n)q. \tag{2.14}$$

Equation (2.14) is linear and can be solved by the method described in Sect. 2.2. Once its solution w has been found, the GS y of (2.12) is given by (2.13).

2.16 Example. Comparing the DE in the IVP

$$y' + 3y + 6y^2 = 0, \quad y(0) = -1$$

to (2.12), we see that this is a Bernoulli equation with $p(t) = 3$, $q(t) = -6$, and $n = 2$. Substitution (2.13) in this case is $y = w^{-1}$; hence,

$$y' = -w^{-2}w', \quad w(0) = (y(0))^{-1} = -1,$$

so the IVP becomes

$$w' - 3w = 6, \quad w(0) = -1.$$

This problem was solved in Example 2.8, and its solution is

$$w(t) = e^{3t} - 2;$$

hence, the solution of the IVP for y is

$$y(t) = (w(t))^{-1} = \frac{1}{e^{3t} - 2}.$$

Verification with Mathematica®. The input

```
y = (E^(3*t) - 2)^(-1);
{D[y,t] + 3*y + 6*y^2, y/.t→0}//Simplify
```

generates the output $\{0, -1\}$. ■

2.17 Example. For the DE in the IVP

$$ty' + 8y = 12t^2\sqrt{y}, \quad y(1) = 16$$

we have $p(t) = 8/t$, $q(t) = 12t$, and $n = 1/2$; therefore, by (2.13), we substitute $y = w^2$ and, since $y' = 2ww'$, arrive at the new IVP

$$tw' + 4w = 6t^2, \quad w(1) = 4.$$

From Example 2.9 we see that $w(t) = t^2 + 3t^{-4}$, so

$$y(t) = (w(t))^2 = (t^2 + 3t^{-4})^2.$$

Verification with Mathematica®. The input

```
y = (t^2 + 3*t^(-4))^2;
Simplify[{t*D[y,t] + 8*y - 12*t^2*Sqrt[y], y/.t→1}, t>0]
```

generates the output $\{0, 16\}$. ■

Exercises

Solve the given IVP, or find the GS of the DE if no IC is given.

1 $y' + y = -y^3, \quad y(0) = 1.$ **2** $2ty' + 3y = 9y^{-1/3}, \quad y(1) = 1.$

3 $y' - y + 6e^t y^{3/2} = 0, \quad y(0) = 1/4.$ **4** $9y' + 2y = 30e^{-2t}y^{-1/2}, \quad y(0) = 4^{1/3}.$

5 $3ty' - y = 4t^2 y^{-2}, \quad y(1) = 6^{1/3}.$ **6** $5t^2 y' + 2ty = 2y^{-3/2}.$

Answers to Odd-Numbered Exercises

1 $y(t) = (2e^{2t} - 1)^{-1/2}.$ **3** $y(t) = e^{-2t}/4.$ **5** $y(t) = (4t^2 + 2t)^{1/3}.$

2.5 Riccati Equations

The general form of these DEs is

$$y' = q_0(t) + q_1(t)y + q_2(t)y^2, \tag{2.15}$$

where q_0, q_1, and q_2 are given functions, with $q_0, q_2 \neq 0$. After some analytic manipulation, we can rewrite (2.15) as

$$v'' + p_1(t)v' + p_2(t)v = 0. \tag{2.16}$$

This is a second-order DE whose coefficients p_1 and p_2 are combinations of q_0, q_1, q_2, and their derivatives. In general, the solution of (2.16) cannot be obtained be means of integrals. However, when we know a PS y_1 of (2.15), we are able to compute the GS of that equation by reducing it to a linear first-order DE by means of the substitution

$$y = y_1 + \frac{1}{w}. \tag{2.17}$$

In view of (2.15) and (2.17), we then have

$$y' = y_1' - \frac{w'}{w^2} = q_0 + q_1\left(y_1 + \frac{1}{w}\right) + q_2\left(y_1 + \frac{1}{w}\right)^2.$$

Since y_1 is a solution of (2.15), it follows that

$$q_0 + q_1 y_1 + q_2 y_1^2 - \frac{w'}{w^2} = q_0 + q_1 y_1 + \frac{q_1}{w} + q_2 y_1^2 + 2\frac{q_2 y_1}{w} + \frac{q_2}{w^2},$$

which, after a rearrangement of the terms, becomes

$$w' + (q_1 + 2q_2 y_1)w = -q_2. \tag{2.18}$$

Equation (2.18) is now solved by the method described in Sect. 2.2.

The matrix version of the Riccati equation occurs in optimal control. Its practical importance and the fact that it cannot be solved by means of integrals have led to the development of the so-called *qualitative theory* of differential equations.

2.18 Example. The DE in the IVP

$$y' = -1 - t^2 + 2(t^{-1} + t)y - y^2, \quad y(1) = \tfrac{10}{7}$$

is of the form (2.15) with $q_0(t) = -1 - t^2$, $q_1(t) = 2(t^{-1} + t)$, and $q_2(t) = -1$, and it is easy to check that $y_1(t) = t$ satisfies it; hence, according to (2.18),

$$w' + 2t^{-1}w = 1,$$

whose solution, constructed by means of (2.5) and (2.7), is

$$w(t) = \tfrac{1}{3}\,t + Ct^{-2}.$$

Next, by (2.17),

$$y(t) = t + \frac{3t^2}{t^3 + 3C}.$$

The constant C is determined from the IC as $C = 2$, so the solution of the IVP is

$$y(t) = \frac{t^4 + 3t^2 + 6t}{t^3 + 6}.$$

Verification with Mathematica®. The input

```
y = (t^4 + 3*t^2 + 6*t)/(t^3 + 6);
Simplify[{D[y,t] + 1 + t^2 - 2*(t^(-1) + t)*y + y^2, y/.t→ 1}]
```

generates the output $\{0,\, 10/7\}$. ■

2.19 Example. The IVP

$$y' = -\cos t + (2 - \tan t)y - (\sec t)y^2, \quad y(0) = 0$$

admits the PS $y_1(t) = \cos t$. Then substitution (2.17) is $y = \cos t + 1/w$, and the linear first-order equation (2.18) takes the form

$$w' - (\tan t)w = \sec t,$$

with GS

$$w(t) = \frac{t + C}{\cos t},$$

from which

$$y(t) = \left(1 + \frac{1}{t + C}\right)\cos t.$$

The IC now yields $C = -1$, so the solution of the IVP is

$$y(t) = \frac{t\cos t}{t - 1}.$$

Verification with Mathematica®. The input

```
y = (t + Cos[t])/(t - 1);
Simplify[{D[y,t] + Cos[t] - (2 - Tan[t])*y + Sec[t]*y^2,
        y/.t→ 0}]
```

generates the output $\{0,\, 0\}$. ■

Exercises

In the given IVP, verify that y_1 is a solution of the DE, use the substitution $y = y_1 + 1/w$ to reduce the DE to a linear first-order equation, then find the solution of the IVP.

1 $y' = t^{-2} + 3t^{-1} - (4t^{-1} + 3)y + 2y^2, \quad y(1) = 5/2; \quad y_1(t) = 1/t.$

2 $y' = -(t^2 + 6t + 4) + 2(t + 3)y - y^2, \quad y(0) = 9/5; \quad y_1(t) = t + 1.$

3 $y' = 3t^{-2} - t^{-4} + 2(t^{-1} - t^{-3})y - t^{-2}y^2, \quad y(1) = -1/2; \quad y_1(t) = -1/t.$

4 $y' = 4t - 4e^{-t^2} + (2t - 4e^{-t^2})y - e^{-t^2}y^2, \quad y(0) = -1; \quad y_1(t) = -2.$

5 $y' = 2 - 4\cos t + 4\sin t + (4\cos t - 4\sin t - 1)y + (\sin t - \cos t)y^2, \quad y(0) = 3; \quad y_1(t) = 2.$

6 $y' = 2t^2 - 1 - t\tan t + (4t - \tan t)y + 2y^2, \quad y(\pi/4) = -1/2 - \pi/4; \quad y_1(t) = -t.$

Answers to Odd-Numbered Exercises

1 $y(t) = (3t + 2)/(2t).$ **3** $y(t) = (t^3 - t - 1)/(t^2 + t).$

5 $y(t) = 2 + 1/(e^t + \sin t).$

2.6 Exact Equations

Consider an equation of the form

$$P(x, y) + Q(x, y)y' = 0, \tag{2.19}$$

where P and Q are given two-variable functions. Recalling that the differential of a function $y = y(x)$ is $dy = y'(x)\,dx$, we multiply (2.19) by dx and rewrite it as

$$P\,dx + Q\,dy = 0. \tag{2.20}$$

The DE (2.19) is called an *exact equation* when the left-hand side above is the differential of a function $f(x, y)$. If f is found, then (2.20) becomes

$$df(x, y) = 0,$$

with GS

$$f(x, y) = C, \quad C = \text{const}. \tag{2.21}$$

2.20 Remark. Suppose that such a function f exists; then (see item (iv) in Sect. 1.1)

$$df = f_x\,dx + f_y\,dy,$$

so, by comparison to (2.20), this happens if

$$f_x = P, \quad f_y = Q. \tag{2.22}$$

In view of the comment made in item (iii) in Sect. 1.1, we have $f_{xy} = f_{yx}$, which, by (2.22), translates as

$$P_y = Q_x. \tag{2.23}$$

Therefore, if a function of the desired type exists, then equality (2.23) must hold.

The other way around, it turns out that for coefficients P and Q continuously differentiable in an open disc in the (x, y)-plane, condition (2.23), if satisfied, guarantees the existence of a function f with the required property. Since in all our examples P and Q meet this degree of smoothness, we simply confine ourselves to checking that (2.23) holds and, when it does, determine f from (2.22). ■

2.21 Example. For the DE in the IVP

$$y^2 - 4xy^3 + 2 + (2xy - 6x^2y^2)y' = 0, \quad y(1) = 1$$

we have

$$P(x, y) = y^2 - 4xy^3 + 2, \quad Q(x, y) = 2xy - 6x^2y^2,$$

so

$$P_y = 2y - 12xy^2 = Q_x,$$

which means that the equation is exact. Then, according to Remark 2.20, there is a function $f = f(x, y)$ such that

$$\begin{aligned} f_x(x, y) &= P(x, y) = y^2 - 4xy^3 + 2, \\ f_y(x, y) &= Q(x, y) = 2xy - 6x^2y^2. \end{aligned} \tag{2.24}$$

Integrating, say, the first equation (2.24) with respect to x, we find that

$$\begin{aligned} f(x, y) &= \int f_x(x, y)\, dx = \int P(x, y)\, dx \\ &= \int (y^2 - 4xy^3 + 2)\, dx = xy^2 - 2x^2y^3 + 2x + g(y), \end{aligned}$$

where, as mentioned in item (vi) in Sect. 1.1, g is an arbitrary function of y. To find g, we use this expression of f in the second equation (2.24):

$$f_y(x, y) = 2xy - 6x^2y^2 + g'(y) = 2xy - 6x^2y^2;$$

consequently, $g'(y) = 0$, from which $g(y) = c = \text{const}$. Since, by (2.21), we equate f to an arbitrary constant, it follows that, without loss of generality, we may take $c = 0$. Therefore, the GS of the DE is defined implicitly by the equality

$$xy^2 - 2x^2y^3 + 2x = C.$$

Using the IC, we immediately see that $C = 1$; hence,

$$xy^2 - 2x^2y^3 + 2x = 1$$

is the equation of the solution curve for the given IVP.

Instead of integrating f_x, we could equally start by integrating f_y from the second equation (2.24); that is,

$$\begin{aligned} f(x, y) &= \int f_y(x, y)\, dy = \int Q(x, y)\, dy \\ &= \int (2xy - 6x^2y^2)\, dy = xy^2 - 2x^2y^3 + h(x), \end{aligned}$$

where h is a function of x to be found by means of the first equation (2.24). Using this expression of f in that equation, we have

$$f_x(x,y) = y^2 - 4xy^3 + h'(x) = y^2 - 4xy^3 + 2,$$

so $h'(x) = 2$, giving $h(x) = 2x$. (Just as before, and for the same reason, we suppress the integration constant.) This expression of h gives rise to the same function f as above.

VERIFICATION WITH MATHEMATICA®. The input

```
u=x*y[x]^2 - 2*x^2*y[x]^3 + 2*x - 1;
{y[x]^2 - 4*x*y[x]^3 + 2 + (2*x*y[x] - 6*x^2*y[x]^2)
        *(Solve[D[u,x]==0,y'[x]])[[1,1,2]],
        u/.{x→ 1, y[x] → 1}}//Simplify
```

generates the output $\{0, 0\}$, which confirms that the function y defined implicitly by the equation of the solution curve satisfies both the DE and the IC. ■

2.22 Example. The DE in the IVP

$$6xy^{-1} + 8x^{-3}y^3 + (4y - 3x^2y^{-2} - 12x^{-2}y^2)y' = 0, \quad y(1) = \tfrac{1}{2}$$

has $P(x,y) = 6xy^{-1} + 8x^{-3}y^3$ and $Q(x,y) = 4y - 3x^2y^{-2} - 12x^{-2}y^2$. Obviously, here we must have $x,\ y \neq 0$.

Since

$$P_y(x,y) = -6xy^{-2} + 24x^{-3}y^2 = Q_x(x,y),$$

it follows that this is an exact equation. The function f we are seeking, obtained as in Example 2.21, is

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\,dx = \int P(x,y)\,dx \\ &= \int (6xy^{-1} + 8x^{-3}y^3)\,dx = 3x^2y^{-1} - 4x^{-2}y^3 + g(y), \end{aligned}$$

with g determined from

$$f_y(x,y) = -3x^2y^{-2} - 12x^{-2}y^2 + g'(y) = 4y - 3x^2y^{-2} - 12x^{-2}y^2;$$

hence, $g'(y) = 4y$, so $g(y) = 2y^2$, which produces the GS of the DE in the implicit form

$$3x^2y^{-1} - 4x^{-2}y^3 + 2y^2 = C.$$

The IC now yields $C = 6$.

VERIFICATION WITH MATHEMATICA®. The input

```
u=3*x^2*y[x]^(-1) - 4*x^(-2)*y[x]^3 + 2*y[x]^2-6;
{6*x*y[x]^(-1) + 8*x^(-3)*y[x]^3 + (4*y[x]
         - 3*x^2*y[x]^(-2) - 12*x^(-2)*y[x]^2)
        *(Solve[D[u,x]==0,y'[x]])[[1,1,2]],
        u/.{x→ 1, y[x] → 1/2}}//Simplify
```

generates the output $\{0, 0\}$. ■

2.23 Example. Consider the IVP

$$x\sin(2y) - 3x^2 + \big(y + x^2\cos(2y)\big)y' = 0, \quad y(1) = \pi.$$

Since, as seen from the left-hand side of the DE, we have $P(x,y) = x\sin(2y) - 3x^2$ and $Q(x,y) = y + x^2\cos(2y)$, we readily verify that

$$P_y(x,y) = 2x\cos(2y) = Q_x(x,y),$$

so this is an exact equation. Then, integrating, say, the y-derivative of the desired function f, we find that

$$\begin{aligned} f(x,y) &= \int f_y(x,y)\,dy = \int Q(x,y)\,dy \\ &= \int [y + x^2\cos(2y)]\,dy = \tfrac{1}{2}\,y^2 + \tfrac{1}{2}\,x^2\sin(2y) + g(x). \end{aligned}$$

The function g is determined by substituting this expression in the x-derivative of f; that is,

$$f_x(x,y) = x\sin(2y) + g'(x) = x\sin(2y) - 3x^2,$$

which yields $g'(x) = -3x^2$; therefore, $g(x) = -x^3$, and we obtain the function

$$f(x,y) = \tfrac{1}{2}\,y^2 + \tfrac{1}{2}\,x^2\sin(2y) - x^3.$$

Writing the GS of the DE as $f(x,y) = C$ and using the IC, we find that $C = \pi^2/2 - 1$, so the solution of the IVP is given in implicit form by

$$y^2 + x^2\sin(2y) - 2x^3 = \pi^2 - 2.$$

VERIFICATION WITH MATHEMATICA®. The input

```
u=y[x]^2 + x^2*Sin[2*y[x]] - 2*x^3 - Pi^2 + 2;
{x*Sin[2*y[x]] - 3*x^2 + (y[x] + x^2*Cos[2^y[x]])
        *(Solve[D[u,x]==0,y'[x]])[[1,1,2]],
        u/.{x→ 1,y[x] →Pi}}//Simplify
```

generates the output {0, 0}. ■

2.24 Example. The general procedure does not work for the IVP

$$2y^{-2} - 4xy + 1 + (2xy^{-1} - 6x^2)y' = 0, \quad y(1) = 1$$

because here, with $P(x,y) = 2y^{-2} - 4xy + 1$ and $Q(x,y) = 2xy^{-1} - 6x^2$, we have

$$P_y(x,y) = -4y^{-3} - 4x \neq Q_x(x,y) = 2y^{-1} - 12x,$$

so the DE is not exact. However, it may be possible to transform the equation into an exact one by using an *integrating factor* $\mu(x,y)$. Writing the DE as $P + Qy' = 0$ and multiplying it by μ, we arrive at $P_1 + Q_1y' = 0$, where $P_1 = P\mu$ and $Q_1 = Q\mu$. We now try to find a function μ such that $(P_1)_y = (Q_1)_x$; that is, $(P\mu)_y = (Q\mu)_x$, which leads to the partial differential equation

$$P\mu_y + P_y\mu = Q\mu_x + Q_x\mu. \tag{2.25}$$

Since, in general, this equation may be difficult to solve, we attempt to see if μ can be found in a simpler form, for example, as a one-variable function. In our case, let us try $\mu = \mu(y)$. Then (2.25) simplifies to

$$P\mu' = (Q_x - P_y)\mu,$$

and with our specific P and Q we get

$$\mu' = \frac{2y^{-1} - 12x + 4y^{-3} + 4x}{2y^{-2} - 4xy + 1}\,\mu = \frac{2(y^{-1} - 4x + 2y^{-3})}{y(2y^{-3} - 4x + y^{-1})}\,\mu = \frac{2}{y}\,\mu,$$

which is a separable equation with solution $\mu(y) = y^2$. Multiplying the DE in the given IVP by y^2, we arrive at the new problem

$$2 - 4xy^3 + y^2 + (2xy - 6x^2y^2)y' = 0, \quad y(1) = 1.$$

This IVP was solved in Example 2.21. ■

2.25 Remark. It should be pointed out that, in general, looking for an integrating factor of a certain form is a matter of trial and error, and that, unless some special feature of the equation gives us a clear hint, this type of search might prove unsuccessful. ■

Exercises

In 1–6, solve the given IVP, or find the GS of the DE if no IC is given.

1 $4xy + (2x^2 - 6y)y' = 0, \quad y(1) = 1.$

2 $12x^{-4} + 2xy^{-3} + 6x^2y + (2x^3 - 3x^2y^{-4})y' = 0, \quad y(1) = 2.$

3 $3x^2y^{-2} + x^{-2}y^2 - 2x^{-3} + (6y^{-4} - 2x^3y^{-3} - 2x^{-1}y)y' = 0, \quad y(1) = -1.$

4 $2x\sin y - y\cos x + (x^2\cos y - \sin x)y' = 0, \quad y(\pi/2) = \pi/2.$

5 $2e^{2x} - 2\sin(2x)\sin y + [2y^{-3} + \cos(2x)\cos y]y' = 0, \quad y(0) = \pi/2.$

6 $(x + y)^{-1} - x^{-2} - 4\cos(2x - y) + [(x + y)^{-1} + 2\cos(2x - y)]y' = 0, \quad y(\pi/2) = 0.$

In 7–10, find an integrating factor (of the indicated form) that makes the DE exact, then solve the given IVP or find the GS of the DE if no IC is given.

7 $y + 3y^{-1} + 2xy' = 0, \quad y(1) = 1, \quad \mu = \mu(y).$

8 $2xy + y^2 + xyy' = 0, \quad y(1) = 2, \quad \mu = \mu(y).$

9 $1 + 2x^2 - (x + 4xy)y' = 0, \quad \mu = \mu(x).$

10 $-2x^3 - y + (x + 2x^2)y' = 0, \quad y(1) = 1, \quad \mu = \mu(x).$

Answers to Odd-Numbered Exercises

1 $2x^2y - 3y^2 + 1 = 0.$ **3** $x^{-2} - 2y^{-3} + x^3y^{-2} - x^{-1}y^2 = 3.$

5 $\cos(2x)\sin y + e^{2x} - y^{-2} = 2 - 4/\pi^2.$ **7** $3x + xy^2 = 4.$

9 $x^2 + \ln|x| - y - 2y^2 = C.$

2.7 Existence and Uniqueness Theorems

Before attempting to solve an IVP or BVP for a mathematical model, it is essential to convince ourselves that the problem is uniquely solvable. This requirement is based on the reasonable expectation that, as mentioned at the beginning of Sect. 1.3, a physical system has one and only one response to a given set of admissible constraints.

Assertions that provide conditions under which a given problem has a unique solution are called *existence and uniqueness theorems*. We discuss the linear and nonlinear cases separately.

2.26 Theorem. *Let J be an open interval of the form $a < t < b$, let t_0 be a point in J, and consider the IVP*

$$y' + p(t)y = q(t), \quad y(t_0) = y_0, \tag{2.26}$$

where y_0 is a given initial value. If p and q are continuous on J, then the IVP (2.26) has a unique solution on J for any y_0. ■

2.27 Remark. In fact, we already know that the unique solution mentioned in Theorem 2.26 can be constructed by means of formulas (2.5) and (2.8). ■

2.28 Definition. The largest open interval J on which an IVP has a unique solution is called the *maximal interval of existence* for that solution. ■

2.29 Remarks. (i) Theorem 2.26 gives no indication as to what the maximal interval of existence for the solution might be. This needs to be determined by other means—for example, by computing the solution explicitly when such computation is possible.

(ii) If no specific mention is made of an interval associated with an IVP, we assume that this is the maximal interval of existence as defined above.

(iii) Many IVPs of the form (2.26) model physical processes in which the DE is meaningful only for $t > 0$. This would seem to create a problem when we try to apply Theorem 2.26 because an open interval of the form $0 < t < b$ does not contain the point $t_0 = 0$ where the IC is prescribed. A brief investigation, however, will easily convince us that, in fact, there is no inconsistency here. If the IVP in question is correctly formulated, we will find that the maximal interval of existence for the solution is larger than $0 < t < b$, extending to the left of the point $t_0 = 0$. The DE is formally restricted to the interval $0 < t < b$ simply because that is where it makes physical sense. ■

2.30 Example. In the IVP

$$y' - (t^2 + 1)y = \sin t, \quad y(1) = 2$$

we have

$$p(t) = -(t^2 + 1), \quad q(t) = \sin t.$$

Since both p and q are continuous on the interval $-\infty < t < \infty$, from Theorem 2.26 it follows that this IVP has a unique solution on the entire real line. ■

2.31 Example. Bringing the equation in the IVP

$$(t + 1)y' + y = e^{2t}, \quad y(2) = 3$$

to the standard form, we see that

$$p(t) = \frac{1}{t+1}, \quad q(t) = \frac{e^{2t}}{t+1}.$$

The functions p and q are continuous on each of the open intervals $-\infty < t < -1$ and $-1 < t < \infty$; they are not defined at $t = -1$. Since the IC is given at $t_0 = 2 > -1$, from Theorem 2.26 we conclude that the IVP has a unique solution in the interval $-1 < t < \infty$. ∎

2.32 Remark. If the IC in Example 2.31 were replaced by, say, $y(-5) = 2$, then, according to Theorem 2.26, the IVP would have a unique solution in the open interval $-\infty < t < -1$, which contains the point $t_0 = -5$. ∎

2.33 Example. To get a better understanding of the meaning and limitations of Theorem 2.26, consider the IVP

$$ty' - y = 0, \quad y(t_0) = y_0,$$

where $p(t) = -1/t$ and $q(t) = 0$. The function q is continuous everywhere, but p is continuous only for $t > 0$ or $t < 0$ since it is not defined at $t = 0$. Treating the DE as either a separable or linear equation, we find that its GS is

$$y(t) = Ct. \tag{2.27}$$

The IC now yields $y_0 = Ct_0$, which gives rise to three possibilities.

(i) If $t_0 \neq 0$, then, by Theorem 2.26, the IVP is guaranteed to have a unique solution

$$y(t) = \frac{y_0}{t_0}\, t \tag{2.28}$$

on any open interval containing t_0 but not containing 0; more specifically, on any interval of the form $0 < a < t < b$ (if $t_0 > 0$) or $a < t < b < 0$ (if $t_0 < 0$). However, direct verification shows that the function (2.28) satisfies the DE at every real value of t, so its maximal interval of existence is the entire real line.

(ii) If $t_0 = 0$ but $y_0 \neq 0$, then the IVP has no solution since the equality $y_0 = Ct_0 = 0$ is impossible for any value of C.

(iii) If $t_0 = y_0 = 0$, then the IVP has infinitely many solutions, given by (2.27) with any constant C, each of them existing on the entire real line.

The ‘anomalous’ cases (ii) and (iii) are explained by the fact that they prescribe the IC at the point where p is undefined, so Theorem 2.26 does not apply. ∎

We now turn our attention to the nonlinear case.

2.34 Theorem. *Consider the IVP*

$$y' = f(t, y), \quad y(t_0) = y_0, \tag{2.29}$$

where f is a given function such that f and f_y are continuous in an open rectangle

$$R = \{(t, y) : \ a < t < b, \ c < y < d\}. \tag{2.30}$$

If the point (t_0, y_0) lies in R, then the IVP (2.29) has a unique solution in some open interval J of the form $t_0 - h < t < t_0 + h$ contained in the interval $a < t < b$. ∎

2.35 Example. For the IVP

$$2(y-1)y'(t) = 2t+1, \quad y(2) = -1$$

we have

$$f(t,y) = \frac{2t+1}{2(y-1)}, \quad f_y(t,y) = -\frac{2t+1}{2(y-1)^2},$$

both continuous everywhere in the (t,y)-plane except on the line $y = 1$. By Theorem 2.34 applied in any rectangle R of the form (2.30) that contains the point $(2,-1)$ and does not intersect the line $y = 1$, the given IVP has a unique solution on some open interval J centered at $t_0 = 2$. Figure 2.2 shows such a rectangle and the arc of the actual solution curve lying in it. The open interval $2-h < t < 2+h$ indicated by the heavy line is the largest of the form mentioned in Theorem 2.34 for the chosen rectangle. But the solution exists on a larger interval than this, which can be determined by solving the IVP.

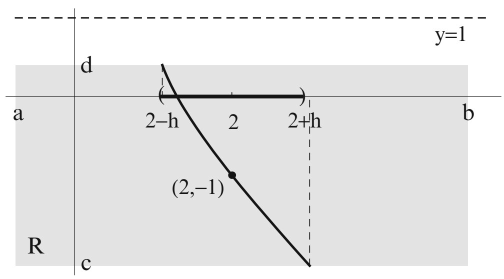

Fig. 2.2

We separate the variables in the DE and write

$$\int 2(y-1)\,dy = \int (2t+1)\,dt,$$

so

$$(y-1)^2 = t^2 + t + C. \tag{2.31}$$

The constant, computed from the IC, is $C = -2$. Replacing it in (2.31), we then find that

$$y(t) = 1 \pm (t^2+t-2)^{1/2}.$$

Of these two functions, however, only the one with the negative root satisfies the IC; therefore, the unique solution of the IVP is

$$y(t) = 1 - (t^2+t-2)^{1/2}. \tag{2.32}$$

We establish the maximal interval of existence for this solution by noticing that the square root in (2.32) is well defined only if

$$t^2+t-2 = (t-1)(t+2) \geq 0;$$

that is, for $t \leq -2$ or $t \geq 1$. Since $t_0 = 2$ satisfies the latter, we conclude that the maximal interval of existence is $1 < t < \infty$. ■

2.36 Example. The situation changes if the IC in Example 2.35 is replaced by $y(0) = 1$. Now the point $(t_0, y_0) = (0, 1)$ lies on the line $y = 1$ where f and f_y are undefined, so every rectangle R of the form (2.30) that contains this point will also contain a portion of the line $y = 1$ (see Fig. 2.3). Consequently, Theorem 2.34 cannot be applied.

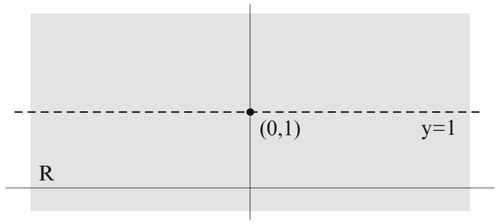

Fig. 2.3

To see what kind of 'pathology' attaches to the problem in this case, we note that the new IC leads to $C = 0$ in (2.31), which means that

$$y(t) = 1 \pm (t^2 + t)^{1/2}. \tag{2.33}$$

The square root is well defined for either $t \le -1$ or $t \ge 0$, and the two functions given by (2.33) are continuously differentiable and satisfy the DE in each of the open intervals $-\infty < t < -1$ and $0 < t < \infty$. The point $t_0 = 0$ is well outside the former, but is a *limit point* for the latter. Hence, we conclude that our IVP has a pair of distinct solutions, whose maximal interval of existence is $0 < t < \infty$. Both these solutions are right-continuous (though not right-differentiable) at 0 and comply with the IC in the sense that $y(0+) = 1$. The nonuniqueness issue we came across here is connected with the fact that the conditions in Theorem 2.34 are violated.

VERIFICATION WITH MATHEMATICA®. The input

```
{y1,y2}={1 + (t^2 + t)^(1/2),1 - (t^2 + t)^(1/2)};
Simplify[{2*({y1,y2} - 1)*D[{y1,y2},t] - 2*t - 1,
        {y1,y2}/.t→0},t>0]
```

generates the output $\{\{0, 0\}, \{1, 1\}\}$. ■

2.37 Remarks. (i) The conditions in Theorems 2.26 and 2.34 are *sufficient* but not *necessary*. In other words, if they are satisfied, the existence of a unique solution of the kind stipulated in these assertions is guaranteed. If they are not, then, as illustrated by Examples 2.33(ii), (iii) and 2.36, a more detailed analysis is needed to settle the question of solvability of the IVP.

The restrictions imposed on f in Theorem 2.34 can be relaxed to a certain extent. It is indeed possible to prove that the theorem remains valid for functions f subjected to somewhat less stringent requirements.

(ii) Theorems 2.26 and 2.34 imply that when the existence and uniqueness conditions are satisfied, the graphs of the solutions of a DE corresponding to distinct ICs do not intersect. For if two such graphs intersected, then the intersection point, used as an IC, would give rise to an IVP with two different solutions, which would contradict the statement of the appropriate theorem. Consequently, if y_1, y_2, and y_3 are the PSs of the same DE on an open interval J, generated by initial values y_{01}, y_{02}, and y_{03}, respectively, such that $y_{01} < y_{02} < y_{03}$, then $y_1(t) < y_2(t) < y_3(t)$ at all points t in J.

Figure 2.4 show the graphs of the solutions of the equation in Example 2.8 with $y_0 = -1,\ 0,\ 1$ at $t_0 = 0$. ■

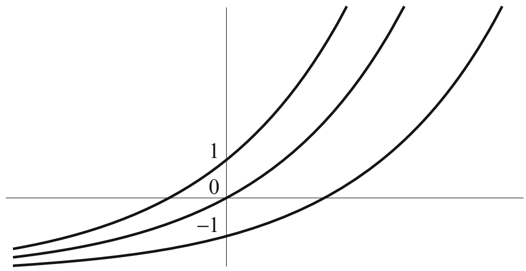

Fig. 2.4

Exercises

In 1–4, find the largest open interval on which the conditions of Theorem 2.26 are satisfied, without solving the IVP itself.

1 $(2t+1)y' - 2y = \sin t, \quad y(0) = -2.$

2 $(t^2 - 3t + 2)y' + ty = e^t, \quad y(3/2) = -1.$

3 $(2 - \ln t)y' + 3y = 5, \quad y(1) = e.$ **4** $y' + 3ty = 2\tan t, \quad y(\pi) = 1.$

In 5–8, indicate the regions in the (t, y)-plane where the conditions of Theorem 2.34 are not satisfied.

5 $(2y + t)y' = 2t + y.$ **6** $y' = t(y^2 - 1)^{1/2}.$

7 $(t^2 + y^2 - 9)y' = \ln t.$ **8** $y' = (t + 2)\tan(2y).$

In 9–12, solve the DE with each of the given ICs and find the maximum interval of existence of the solution.

9 $y' = 4ty^2; \quad y(0) = 2; \quad y(-1) = -2; \quad y(3) = -1.$

10 $y' = 8y^3; \quad y(0) = -1; \quad y(0) = 1.$

11 $(y - 2)y' = t; \quad y(2) = 3; \quad y(-2) = 1; \quad y(0) = 1; \quad y(1) = 2.$

12 $(y - 3)y' = 2t + 1; \quad y(-1) = 3; \quad y(-1) = 1; \quad y(-1) = 5; \quad y(2) = 1.$

Answers to Odd-Numbered Exercises

1 $-1/2 < t < \infty.$ **3** $0 < t < e^2.$ **5** $y = -t/2.$

7 $t^2 + y^2 = 9$ or $t \le 0.$

9 $y(t) = 2/(1 - 4t^2), \quad -1/2 < t < 1/2; \quad y(t) = 2/(3 - 4t^2), \quad -\infty < t < -\sqrt{3}/2;$
$y(t) = 1/(17 - 2t^2), \ \sqrt{17/2} < t < \infty.$

11 $y(t) = 2 + (t^2 - 3)^{1/2}, \quad \sqrt{3} < t < \infty; \quad y(t) = 2 - (t^2 - 3)^{1/2}, \quad -\infty < t < -\sqrt{3};$
$y(t) = 2 - (t^2 + 1)^{1/2}, \quad -\infty < t < \infty; \quad y(t) = 2 \pm (t^2 - 1)^{1/2}, \quad 1 < t < \infty.$

2.8 Direction Fields

Very often, a nonlinear first-order DE cannot be solved by means of integrals; therefore, to obtain information about the behavior of its solutions we must resort to qualitative analysis methods. One such technique is the sketching of so-called *direction fields*, based on the fact that the right-hand side of the equation $y' = f(t, y)$ is the slope of the tangent to the solution curve $y = y(t)$ at a generic point (t, y). Drawing short segments of the line with slope $f(t, y)$ at each node of a suitably chosen lattice in the (t, y)-plane, and examining the pattern formed by these segments, we can build up a useful pictorial image of the family of solution curves of the given DE.

2.38 Example. In Sect. 2.5 we mentioned the difficulty that arises when we try to solve a Riccati equation for which no PS is known beforehand. The method described above, applied to the equation

$$y' = e^{-2t} - 3 + (5 - 2e^{-2t})y + (e^{-2t} - 2)y^2$$

in the rectangle defined by $-1.5 \le t \le 2$ and $0.5 \le y \le 1.6$, yields the direction field shown in Fig. 2.5, where several solution curves are also graphed.

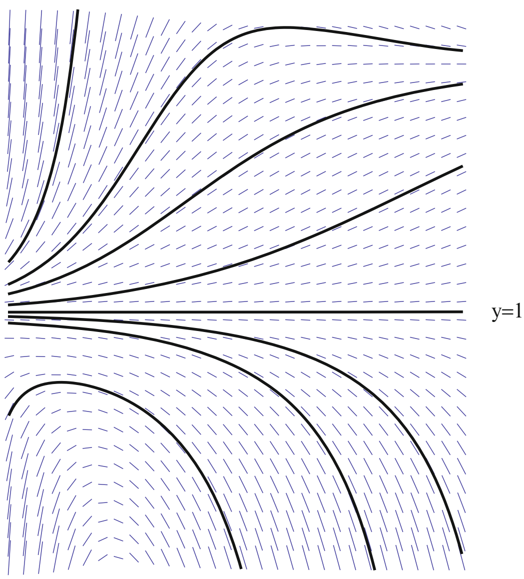

Fig. 2.5

Figure 2.5 suggests that $y(t) = 1$ might be a PS of the given DE, and direct verification confirms that this is indeed the case. Consequently, proceeding as in Sect. 2.5, we determine that the GS of our equation is

$$y(t) = 1 + \frac{1}{2 + e^{-2t} + Ce^{-t}}.$$

The solution curves in Fig. 2.5 correspond, from top to bottom, to $C = -2.78$, -1.05, 1, 10, 10,000, -30, -15, -6.

Of course, other equations may not benefit from this type of educated guess, however refined a point lattice is employed to generate their direction fields. ■

Chapter 3
Mathematical Models with First-Order Equations

In Sect. 1.2 we listed examples of DEs arising in some mathematical models. We now show how these equations are derived, and find their solutions under suitable ICs.

3.1 Models with Separable Equations

Population growth. Let $P(t)$ be the size of a population at time t, and let $\beta(t)$ and $\delta(t)$ be, respectively, the birth and death rates (that is, measured per unit of population per unit time) within the population. For a very short interval of time Δt, we may assume that β and δ are constant, so we approximate the change ΔP in the population during that interval by

$$\Delta P \approx (\beta - \delta) P \Delta t,$$

or, on division by Δt,

$$\frac{\Delta P}{\Delta t} \approx (\beta - \delta) P. \tag{3.1}$$

Clearly, this approximation improves when Δt decreases, becoming an equality in the limit as $\Delta t \to 0$. But the limit of the left-hand side in (3.1) as $\Delta t \to 0$ defines the derivative P' of P; therefore, we have

$$P' = (\beta - \delta) P, \tag{3.2}$$

which is a separable DE that yields

$$\frac{dP}{P} = (\beta - \delta)\, dt.$$

If the size of the population at $t = 0$ is $P(0) = P_0$, then, integrating the two sides above from P_0 to P and from 0 to t, respectively, we arrive at

$$\ln P - \ln P_0 = \int_0^t \big[\beta(\tau) - \delta(\tau)\big]\, d\tau.$$

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_3,
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We now use the fact that $\ln P - \ln P_0 = \ln(P/P_0)$ and take exponentials to obtain the solution of the IVP as

$$P(t) = P_0 \exp\left\{ \int_0^t \left[\beta(\tau) - \delta(\tau)\right] d\tau \right\}, \quad t > 0. \tag{3.3}$$

3.1 Remark. By (3.3), when β and δ are constant we have

$$P(t) = P_0 e^{(\beta-\delta)t},$$

so the population decreases asymptotically to zero if $\delta > \beta$ and increases exponentially without bound if $\beta > \delta$. The latter conclusion seems unreasonable because any limited environment has limited resources and can, therefore, sustain only a limited population. This leads us to the conclusion that the model governed by equation (3.2) is valid only for relatively small populations and for a finite time interval. A more refined model, with a wider range of validity, will be discussed in Sect. 3.3. ■

3.2 Example. Suppose that the annual birth and death rates in a population of initial size $P_0 = 100$ are $\beta(t) = 2t + 1$ and $\delta(t) = 4t + 4$. Then $\beta(t) - \delta(t) = -2t - 3$ and, by (3.3),

$$P(t) = 100 \exp\left\{ \int_0^t (-2\tau - 3)\, d\tau \right\} = 100 e^{-t^2-3t}.$$

Since $\beta(t) < \delta(t)$ for all $t > 0$, the population is in permanent decline. If we want to find out how long it takes for its size to decrease to, say, 40, we replace $P(t)$ by 40, take logarithms on both sides, and reduce the problem to the quadratic equation

$$t^2 + 3t - \ln \tfrac{5}{2} = 0.$$

This has two roots: $t = [-3 - (9 + 4\ln(5/2))^{1/2}]/2 \approx -3.28$, which must be discarded as physically unacceptable, and $t = [-3 + (9 + 4\ln(5/2))^{1/2}]/2 \approx 0.28$, which is the desired answer.

Verification with Mathematica®. The input

```
Round[100*E^(-0.28^2 - 3*0.28)]
```

generates the output 40. Since the computed result is an approximation, the command instructs the program to round the output to the nearest integer. ■

Radioactive decay. As in the preceding model, if $\kappa = \text{const} > 0$ is the rate of decay of a radioactive isotope (that is, the number of decaying atoms per unit of atom 'population' per unit time), then the approximate change ΔN in the number $N(t)$ of atoms during a very short time interval Δt is

$$\Delta N \approx -\kappa N \Delta t;$$

so, dividing by Δt and letting $\Delta t \to 0$, we obtain the DE

$$N' = -\kappa N.$$

After separating the variables, integrating, and using an IC of the form $N(0) = N_0$, we find the solution

$$N(t) = N_0 e^{-\kappa t}, \quad t > 0. \tag{3.4}$$

3.3 Example. An important characteristic of a radioactive substance is its so-called half-life, which is the length of time t^* it takes the substance to decay by half. Since $N(t^*) = N_0/2$, from (3.4) it follows that

$$\tfrac{1}{2}\,N_0 = N_0 e^{-\kappa t^*}.$$

Dividing both sides by N_0 and taking logarithms, we find that

$$t^* = \frac{\ln 2}{\kappa}. \;\blacksquare$$

Compound interest. A sum S_0 of money is deposited in a savings account that pays interest compounded continuously at a constant rate of r per unit of money per unit time. If $S(t)$ is the size of the deposit at time t, then, for a very short interval of time Δt, the approximate increment ΔS in S is

$$\Delta S \approx rS\Delta t.$$

Dividing by Δt and letting $\Delta t \to 0$, we arrive at the exact equality

$$S' = rS,$$

which is a separable equation. The procedure described in Sect. 2.1 with the IC $S(0) = S_0$ now yields

$$S(t) = S_0 e^{rt}, \quad t > 0. \tag{3.5}$$

3.4 Example. If we want to find out how long it takes an initial deposit to double in size at a rate of 6 %, we replace $S(t) = 2S_0$ and $r = 0.06$ in (3.5) and divide both sides by S_0 to find that

$$2 = e^{0.06t}.$$

Using logarithms to solve for t, we obtain $t \approx 11.55$.

Verification with Mathematica®. The input

```
S0 * Round[E^(0.06 * 11.55)]
```

generates the output $2S_0$. $\blacksquare$

3.5 Remarks. (i) It is obvious that the result in Example 3.4 is independent of the size of the initial deposit.

(ii) Suppose, for simplicity, that time is measured in years. If interest is not compounded continuously, but n times a year (for example, monthly, or weekly, or daily) at a constant annual rate of r, then it can be shown by mathematical induction that after t years the capital is

$$S(t) = S_0\left(1 + \frac{r}{n}\right)^{nt}.$$

We can easily verify that for any positive integer n, this value of $S(t)$ is smaller than the value given by (3.5), and that the right-hand side above tends to the right-hand side in (3.5) as $n \to \infty$.

Verification with Mathematica®. The input

```
Limit[S0 * (1 + r/n)^(n * t), n → ∞]
```

generates the output $S_0 e^{rt}$. $\blacksquare$

Exercises

Use the mathematical models discussed in this section to solve the given problem. In each case, assume that the conditions under which the models are constructed are satisfied.

1 The birth and death rates of two non-competing populations are $\beta_1 = 1/(t+1)$, $\delta_1 = 1/10$ and $\beta_2 = 2/(t+2)$, $\delta_2 = 1/10$, respectively. If their starting sizes are $P_{10} = 2$ and $P_{20} = 1$ units, find out how long it takes the second population to (i) become larger than the first one, and (ii) be twice as large as the first population.

2 For two non-competing populations, the birth and death rates and starting sizes are $\beta_1 = 1/20$, $\delta_1 = 2t/(t^2+1)$, $P_{10} = 1$ and $\beta_2 = 1/(t+1)$, $\delta_2 = a = \text{const}$, $P_{20} = 3$, respectively. Find the value of a if the two populations become equal in size after 5 time units.

3 The half-life of a radioactive isotope is 8. Compute the initial amount (number of atoms) of substance if the amount is 6,000 at time $t = 4$.

4 A radioactive substance with an initial size of 10 has a decaying coefficient of 1/16. At $t = 8$, the amount of substance is increased by 5. Find out how long it takes the increased amount of substance to decay to its original size.

5 A sum of money is invested at an annual interest rate of 3 %. Two years later, an equal sum is invested at an annual rate of 5 %. Determine after how many years the second investment starts being more profitable than the first one.

6 A bank pays an annual rate of interest of 5 %, and an annual fixed loyalty bonus of 1/100 of a money unit. Modify the compound interest model accordingly, then compute the value of one money unit after 5 years. Also, find out what interest rate would need to be paid by the bank to generate the same amount at that time if there was no loyalty bonus.

Answers to Odd-Numbered Exercises

1 $t = 2(1+\sqrt{2}\,) \approx 4.83$; $t = 2(3 + 2\sqrt{3}\,) \approx 12.93$.

3 $N_0 = 6{,}000\sqrt{2} \approx 8{,}485$. **5** $t = 3$ years.

3.2 Models with Linear Equations

Free fall in gravity. Let $m = \text{const} > 0$ be the mass of an object falling in a gravitational field, and let $y(t)$, $v(t)$, and $a(t)$, respectively, be its position, velocity, and acceleration at time t. Also, let g be the acceleration of gravity and $\gamma = \text{const} > 0$ a coefficient characterizing the resistance of the ambient medium to the motion of the object. Assuming that the vertical coordinate axis is directed downward and that resistance is proportional to the object's velocity, from Newton's second law we deduce that

$$ma = mg - \gamma v,$$

or, since $a = v'$,

$$v' + \frac{\gamma}{m}\, v = g. \tag{3.6}$$

This is a linear equation that we can solve by means of formulas (2.5) and (2.8). Thus, an integrating factor is

$$\mu(t) = \exp\left\{\int \frac{\gamma}{m}\,dt\right\} = e^{(\gamma/m)t},$$

so, with an IC of the form $v(0) = v_0$, we obtain the solution

$$\begin{aligned} v(t) &= e^{-(\gamma/m)t}\left\{\int_0^t e^{(\gamma/m)\tau} g\,d\tau + \mu(0)v(0)\right\} \\ &= e^{-(\gamma/m)t}\left\{\frac{mg}{\gamma}\,e^{(\gamma/m)\tau}\Big|_0^t + v_0\right\} \\ &= \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right)e^{-(\gamma/m)t}, \quad t > 0. \end{aligned} \tag{3.7}$$

Since $v = y'$, integrating the right-hand side above and using a second IC, of the form $y(0) = y_0$, we find that

$$y(t) = \frac{mg}{\gamma}\,t + \frac{m}{\gamma}\left(\frac{mg}{\gamma} - v_0\right)(e^{-(\gamma/m)t} - 1) + y_0, \quad t > 0. \tag{3.8}$$

If v is replaced by y' in (3.6), we see that the function y is a solution of the DE

$$y'' + \frac{\gamma}{m}\,y' = g.$$

This is a second-order equation, of a type investigated in Chap. 4.

3.6 Example. An object of mass $m = 10$ dropped into a liquid-filled reservoir reaches the bottom with velocity $v_b = 24.5$. If $g = 9.8$ and the motion resistance coefficient of the liquid is $\gamma = 2$, we can use (3.7) and (3.8) to compute the depth of the reservoir. First, we rewrite these two formulas in terms of the specific data given to us, which also include $y_0 = 0$ and $v_0 = 0$ (since the object is dropped, not thrown, from the top, where we choose the origin on the vertical downward-pointing axis); that is,

$$y(t) = 49t + 245(e^{-t/5} - 1), \quad v(t) = 49 - 49e^{-t/5}.$$

The latter allows us to find the time t_b when the object reaches the bottom. This is the root of the equation $v(t) = v_b$, which, after simplification, reduces to

$$1 - e^{-t/5} = \tfrac{1}{2}$$

and yields $t_b \approx 3.466$. Then the depth of the reservoir is given by the position $y(t_b)$ of the object at that time, namely

$$y(t_b) \approx 49 \cdot 3.466 + 245(e^{-3.466/5} - 1) \approx 47.33.$$

Verification with Mathematica®. The input

```
y = 49 * t + 245 * (E^(-t/5) - 1);
tb = Solve[{y == 47.33, t > 0}, t][[1, 1, 2]];
Round[D[y, t] /. t → tb, 0.5]
```

generates the output 24.5. The last command asks for the rounding of the result to the nearest half unit. ■

Newton's law of cooling. As mentioned in Example 1.7, the temperature $T(t)$ of a body immersed in a medium of (constant) temperature θ satisfies the DE

$$T' + kT = k\theta,$$

where $k = \text{const} > 0$ is the heat transfer coefficient. This is a linear equation, so, by (2.5), an integrating factor is

$$\mu(t) = \exp\left\{\int k\,dt\right\} = e^{kt};$$

hence, by (2.8) and with an IC of the form $T(0) = T_0$,

$$\begin{aligned} T(t) &= e^{-kt}\left\{\int_0^t k\tau e^{k\tau}\,d\tau + \mu(0)T(0)\right\} \\ &= e^{-kt}\{\theta(e^{kt} - 1) + T_0\} = \theta + (T_0 - \theta)e^{-kt}, \quad t > 0. \end{aligned} \tag{3.9}$$

3.7 Example. The temperature of a dead body found in a room at 6 a.m. is 32 °C. One hour later, its temperature has dropped to 30 °C. The room has been kept all the time at a constant temperature of 17 °C. Assuming that just before dying the person's temperature had the 'normal' value of 37 °C, we want to determine the time of death.

Let us measure time in hours from 6 a.m. With $T(0) = T_0 = 32$ and $\theta = 17$, formula (3.9) becomes

$$T(t) = 17 + 15e^{-kt}; \tag{3.10}$$

hence, since $T(1) = 30$, it follows that

$$30 = 17 + 15e^{-k},$$

which yields

$$k = \ln(15/13). \tag{3.11}$$

If t_d is the time of death, then, by (3.10),

$$T(t_d) = 37 = 17 + 15e^{-kt_d}. \tag{3.12}$$

The value of t_d is now computed from (3.12) with k given by (3.11):

$$t_d = -\frac{\ln(4/3)}{\ln(15/13)} \approx -2;$$

that is, death occurred at approximately 4 a.m.

VERIFICATION WITH MATHEMATICA®. The input

```
Round[N[17 + (32 - 17) * E^(-Log[15/13] * (-2))]]
```

generates the output 37, rounded to the nearest integer. ■

RC electric circuit. In a series RC circuit, the voltage across the resistor and the voltage across the capacitor are, respectively,

$$V_R = RI, \quad V_C = \frac{Q}{C},$$

where the positive constants R and C are the resistance and capacitance, respectively, I is the current, and Q is the charge. If V is the voltage of the source, then, since $I = Q'$ and since, by Kirchhoff's law, $V = V_R + V_C$, we have

$$V = RI + \frac{Q}{C} = RQ' + \frac{1}{C}\,Q,$$

or, what is the same,

$$Q' + \frac{1}{RC}\,Q = \frac{1}{R}\,V.$$

By (2.5), an integrating factor is

$$\mu(t) = \exp\left\{\int \frac{1}{RC}\,dt\right\} = e^{t/(RC)},$$

so, by (2.8) and with an IC of the form $Q(0) = Q_0$,

$$Q(t) = e^{-t/(RC)}\left\{\frac{1}{R}\int_0^t e^{\tau/(RC)}V(\tau)\,d\tau + Q_0\right\}. \tag{3.13}$$

3.8 Example. Suppose that $R = 20$, $C = 1/20$, $V(t) = \cos t$, and $Q_0 = -1/100$. By (3.13) and formula (B.1),

$$\begin{aligned} Q(t) &= e^{-t}\left\{\tfrac{1}{20}\int_0^t e^{\tau}\cos\tau\,d\tau - \tfrac{1}{100}\right\} = e^{-t}\left\{\tfrac{1}{20}\,\tfrac{1}{2}[(\cos t + \sin t)e^t - 1] - \tfrac{1}{100}\right\} \\ &= \tfrac{1}{200}\,(5\cos t + 5\sin t - 7e^{-t}). \end{aligned}$$

The current in the circuit is then computed as

$$I(t) = Q'(t) = \tfrac{1}{200}\,(5\cos t - 5\sin t + 7e^{-t}).$$

Verification with Mathematica®. The input

```
Q = (1/200) * (5 * Cos[t] + 5 * Sin[t] - 7 * E^(-t));
{D[Q,t] + Q - (1/20) * Cos[t], Q/. t→ 0} // Simplify
```

generates the output $\{0, -1/100\}$. ■

Loan repayment. Let $A(t)$ be the amount of outstanding loan at time t, let $r = \text{const}$ be the rate at which the money has been borrowed, and let $p = \text{const}$ be the repayment per unit time. Again considering a very short interval of time Δt, we see that the variation ΔA of A during this interval can be written as

$$\Delta A \approx rA\Delta t - p\Delta t,$$

which, when we divide by Δt and let $\Delta t \to 0$, becomes

$$A' - rA = -p.$$

By (2.5), an integrating factor for this linear equation is

$$\mu(t) = \exp\left\{-\int r\,dt\right\} = e^{-rt},$$

so, by (2.8) and with an IC of the form $A(0) = A_0$,

$$A(t) = e^{rt}\left\{ \int_0^t e^{-r\tau}(-p)\, d\tau + \mu(0)A(0) \right\}$$

$$= e^{rt}\left\{ \frac{p}{r}\,(e^{-rt} - 1) + A_0 \right\} = \frac{p}{r} + \left(A_0 - \frac{p}{r}\right)e^{rt}. \qquad (3.14)$$

3.9 Example. To be specific, let us compute the monthly repayment for a mortgage of \$100,000 taken out over 30 years at a fixed annual interest rate of 6 %. We replace $A_0 = 100{,}000$, $r = 0.06$, $t = 30$, and $A(30) = 0$ in (3.14), then solve for p to obtain

$$p = \frac{rA_0}{1 - e^{-rt}} = \frac{6{,}000}{1 - e^{-1.8}} \approx 7{,}188,$$

which, divided by 12, yields a monthly repayment of approximately \$600.

Verification with Mathematica®. The input

```
Round[7188/0.06 + (100000 - 7188/0.06) ∗ E^(0.06 ∗ 30), 100]
```

generates the output 0. The rounding is to the nearest 100. ■

Exercises

Use the mathematical models discussed in this section to solve the given problem. In each case, consider that the assumptions under which the models are constructed are satisfied.

1 An object of unit mass is dropped from a height of 20 above a liquid-filled reservoir of depth 50. If the acceleration of gravity is 9.8 and the resistance-to-motion coefficients of air and liquid are 1 and 4, respectively, compute the time taken by the object to hit the bottom of the reservoir.

2 An object of mass 4, projected vertically upward with a speed of 100 from the bottom of a liquid-filled reservoir, starts sinking back as soon as it reaches the surface. If the acceleration of gravity is 9.8 and the resistance-to-motion coefficient of the liquid is 1, find the depth of the reservoir.

3 An object is immersed in a medium kept at a constant temperature of 100. After one unit of time, the temperature of the object is 10. After another unit of time, its temperature is 15. The object starts melting when the temperature reaches 40. Find out (i) when the object starts melting, and (ii) the object's initial temperature.

4 An object with a heat exchange coefficient of 1/10 and an initial temperature of 10 is immersed in a medium of temperature $t - 3$. Given that the object starts freezing when the temperature reaches 0, find out how long it takes the object to start freezing.

5 An RC circuit has a resistance of 1, a capacitance of 2, an initial charge of 2/17, and a voltage $V(t) = \cos(2t)$. At time $t = \pi/4$, the resistance is doubled. Find the charge Q at time $t = \pi$.

6 An RC circuit has a resistance of 2, a capacitance of 1, and a voltage given by $V(t) = \cos(t/2)$. Find the value of the initial charge if the current at $t = \pi$ is $-1/4$.

7 A loan of \$200,000 is taken out over 20 years at an annual interest rate of 4 %. Compute the (constant) loan repayment rate and the total interest paid. Also, find out how much sooner the loan would be repaid and what savings in interest would be achieved if the loan repayment rate were increased by one half from the start.

8 A loan of \$1,000 is taken out over 15 years at a variable annual rate of interest of $(5/(t+1))\,\%$. Compute the (constant) repayment rate.

Answers to Odd-Numbered Exercises

1 $t \approx 22.7$.

3 $t = 1 + (\ln(3/2))/(\ln(18/17)) \approx 8.09$; $T = 80/17 \approx 4.7$.

5 $Q = 2/65 + (8/17)e^{-\pi/4} - (16/65)e^{-3\pi/16} \approx 0.108749$.

7 $r = 1{,}210.64$/month; \$90,554.60; 8.56 years earlier; \$41,332.30.

3.3 Autonomous Equations

These are equations of the form

$$y'(t) = f(y);$$

that is, where the function f on the right-hand side does not depend explicitly on t. Such equations are encountered in a variety of models, especially in population dynamics.

The simplest version of the population model discussed in Sect. 3.1 is described by an IVP that can be written as

$$y' = ry, \quad y(0) = y_0, \tag{3.15}$$

where the DE is an autonomous equation with $f(y) = ry$, $r = \text{const}$. But, as observed in Remark 3.1, this model is useful only for small populations and finite time intervals, becoming inadequate for large time when $r > 0$ and the environment has limited resources. To make the model realistic also in this case, we need to consider a more refined version of it.

Population with logistic growth. We replace the constant rate r in the function f by a variable one $u(y)$, which, for a physically plausible model, should have the following properties:

(i) $u(y) \approx r$ when y is small (so for a small population, the new model gives approximately the same results as (3.15));

(ii) $u(y)$ decreases as y increases (the rate of growth diminishes as the population gets larger);

(iii) $u(y) < 0$ when y is very large (the population is expected to decline as the environment resources are stretched too far).

The simplest function with these properties is of the form $u(y) = r - ay$, where $a = \text{const} > 0$. Therefore, the logistic equation is

$$y' = (r - ay)y,$$

or, with $B = r/a$,

$$y' = r\left(1 - \frac{y}{B}\right)y. \tag{3.16}$$

The roots of the algebraic equation $f(y) = 0$ are called *critical points.* Here, since $f(y) = r[1 - (y/B)]y$, they are 0 and B. The constant functions $y(t) = 0$ and $y(t) = B$ are the *equilibrium solutions* of the DE (3.16).

We notice that (3.16) is a separable equation. To compute its non-equilibrium solutions, first we separate the variables and write it as

$$\frac{B}{y(B-y)}\,dy = r\,dt,$$

then use partial fraction decomposition (see Sect. A.1) on the left-hand side, followed by integration:

$$\int\left(\frac{1}{y} + \frac{1}{B-y}\right)dy = r\int dt.$$

This leads to

$$\ln|y| - \ln|B-y| = rt + C,$$

or

$$\ln\left|\frac{y}{B-y}\right| = rt + C,$$

which, after exponentiation, becomes

$$\frac{y}{B-y} = \pm e^C e^{rt} = C_1 e^{rt}.$$

The constant C_1, determined from the generic initial condition $y(0) = y_0 > 0$, is $C_1 = y_0/(B - y_0)$. Replacing it in the above equality and solving for y, we arrive at

$$y(t) = \frac{y_0 B e^{rt}}{B - y_0 + y_0 e^{rt}}.$$

Finally, we multiply both numerator and denominator by e^{-rt} and bring the solution to the form

$$y(t) = \frac{y_0 B}{y_0 + (B - y_0)e^{-rt}}. \tag{3.17}$$

3.10 Remark. The denominator on the right-hand side in (3.17) is never equal to zero for any $t > 0$. This is clear when $0 < y_0 \le B$. If $y_0 > B$, then, solving the algebraic equation

$$y_0 + (B - y_0)e^{-rt} = 0$$

for t, we find that

$$t = -\frac{1}{r}\ln\frac{y_0}{y_0 - B},$$

which is a negative number because the argument of the logarithm is greater than 1. Hence, the function (3.17) is defined for all $t > 0$, so we can compute its limit when t increases without bound. Since $e^{-rt} \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} y(t) = B.$$

This shows that, in a certain sense, B is the size of the largest population that the environment can sustain long term, and for that reason it is called the *environmental carrying capacity.* ■

3.11 Definition. Roughly speaking, an equilibrium solution y_0 of a DE with time t as the independent variable is called *stable* if any other solution starting close to y_0 remains close to y_0 for all time, and it is called *asymptotically stable* if it is stable and any solution starting close to y_0 becomes arbitrarily close to y_0 as t increases. An equilibrium solution that is not stable is called *unstable.* ■

3.12 Remark. The explanation in Remark 3.10 shows that the equilibrium solution $y(t) = B$ is asymptotically stable. By contrast, the equilibrium solution $y(t) = 0$ is unstable since any solution starting near 0 at $t = 0$ moves away from it as t increases. ■

3.13 Example. Proceeding as above, we find that the critical points and equilibrium solutions of the DE

$$y' = f(y) = 400\left(1 - \frac{y}{100}\right)y = 400y - 4y^2 = 4y(100 - y)$$

are 0, 100 and $y(t) = 0$, $y(t) = 100$, respectively. Also, by (3.17), its GS is

$$y(t) = \frac{100y_0}{y_0 + (100 - y_0)e^{-400t}}. \tag{3.18}$$

According to Remark 3.10, this solution is defined for all $t > 0$, and we readily verify that $y(t) \to 100$ as $t \to \infty$. From the right-hand side of the DE we see that $y' > 0$ when $0 < y < 100$, and $y' < 0$ when $y > 100$. Differentiating both sides of the given equation with respect to t and replacing y' from the DE, we find that

$$y'' = \frac{df}{dy}\,y' = (400 - 8y) \cdot 4y(100 - y) = 32y(50 - y)(100 - y).$$

Consequently (see Sect. B.1), $y'' > 0$ when $0 < y < 50$ or $y > 100$, and $y'' < 0$ when $50 < y < 100$. These details together with the information they convey about the graph of y are summarized in Table 3.1.

Table 3.1

y	0	50	100
y'	+	+	−
y''	+	−	+
Graph of y	Ascending, concave up	Ascending, concave down	Descending, concave up

Taking, in turn, y_0 to be 25, 75, and 125, we obtain the PSs

$$y_1(t) = \frac{100}{1 + 3e^{-400t}}, \quad y_2(t) = \frac{300}{3 + e^{-400t}}, \quad y_3(t) = \frac{500}{5 - e^{-400t}}.$$

By Remark 2.37, $y_1(t) < y_2(t) < y_3(t)$ for all $t > 0$. The graphs of these solutions, sketched on the basis of this property and the details in Table 3.1, and the graphs of the equilibrium solutions, are shown in Fig. 3.1. These graphs confirm that, as expected, the equilibrium solution $y(t) = 100$ is asymptotically stable and that $y(t) = 0$ is unstable.

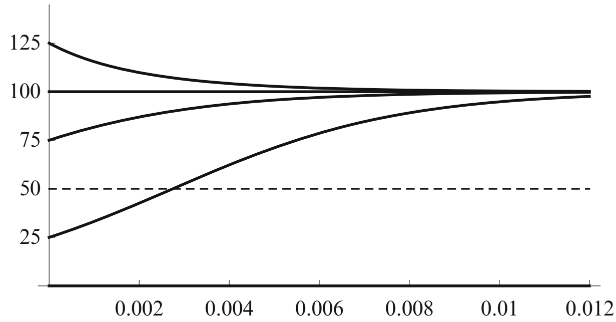

Fig. 3.1

The graph of any solution starting with a value between 0 and 50 at $t = 0$ changes its concavity from up to down when it crosses the dashed line $y = 50$.

VERIFICATION WITH MATHEMATICA®. The input

```
y1 = 100/(1 + 3*E^(-400*t));
y2 = 300/(3 + E^(-400*t));
y3 = 500/(5 - E^(-400*t));
y = {y1, y2, y3};
{D[y,t] - 4*y*(100 - y), y/.t → 0}//Simplify
```

generates the output {{0, 0, 0}, {25, 75, 125}}. ■

Logistic population with harvesting. The DE for this model is of the form

$$y' = r\left(1 - \frac{y}{B}\right)y - \alpha,$$

where $\alpha = \text{const} > 0$ is the rate of harvesting, which, obviously, contributes to the decline of the population. The analysis here proceeds along the same lines as in the preceding model.

3.14 Example. Consider the DE

$$y' = 28\left(1 - \frac{y}{7}\right)y - 40 = 4(7y - y^2 - 10) = 4(y-2)(5-y). \tag{3.19}$$

The critical points are 2 and 5, and the equilibrium solutions are $y(t) = 2$ and $y(t) = 5$. Investigating the signs of y' and

$$y'' = 4(7 - 2y)y' = 16(7-2y)(y-2)(5-y),$$

we gather the details about the graph of y in Table 3.2.

For $y \neq 2, 5$, separating the variables and integrating on both sides of the DE, we find that

$$\int \frac{dy}{(y-2)(5-y)} = \int \left\{\frac{1}{3(y-2)} - \frac{1}{3(y-5)}\right\} dy = 4\int dt;$$

hence, $\ln|(y-2)/(y-5)| = 12t + C$, or, after exponentiation,

$$\frac{y-2}{y-5} = C_1 e^{12t}.$$

Table 3.2

y	0	2	3.5	5
y'	$-$	$+$	$+$	$-$
y''	$-$	$+$	$-$	$+$
Graph of y	Descending, concave down	Ascending, concave up	Ascending, concave down	Descending, concave up

An IC of the form $y(0) = y_0$ now yields $C_1 = (y_0 - 2)/(y_0 - 5)$, so, after some simple algebra, in the end we obtain

$$y(t) = \frac{5(y_0 - 2) - 2(y_0 - 5)e^{-12t}}{y_0 - 2 - (y_0 - 5)e^{-12t}}. \tag{3.20}$$

As in Example 3.13, it is easy to check that for $y_0 > 2$, the denominator on the right-hand side in (3.20) does not vanish for any $t > 0$; therefore, we can compute

$$\lim_{t\to\infty} y(t) = 5.$$

For $0 < y_0 < 2$, the numerator of solution (3.20) is equal to zero at $t = t^*$, where, since $2(5 - y_0) > 5(2 - y_0)$,

$$t^* = \frac{1}{12} \ln \frac{2(5 - y_0)}{5(2 - y_0)} > 0.$$

This means that populations starting with a size less than 2 are not large enough to survive and die out in finite time.

The PSs for $y_0 = 1$, $y_0 = 3$, $y_0 = 4$, and $y_0 = 6$ are, respectively,

$$y_1(t) = \frac{-5 + 8e^{-12t}}{-1 + 4e^{-12t}}, \qquad y_2(t) = \frac{5 + 4e^{-12t}}{1 + 2e^{-12t}},$$
$$y_3(t) = \frac{10 + 2e^{-12t}}{2 + e^{-12t}}, \qquad y_4(t) = \frac{20 - 2e^{-12t}}{4 - e^{-12t}}.$$

Their graphs, sketched with Table 3.2, are shown in Fig. 3.2.

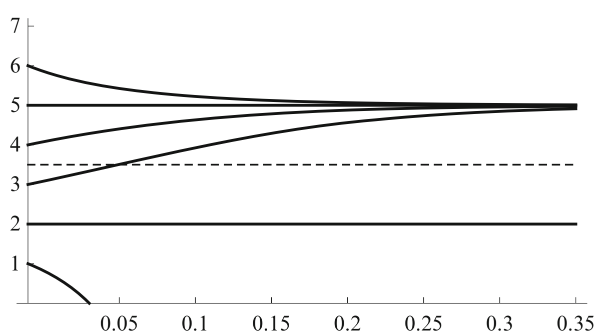

Fig. 3.2

The equilibrium solution $y(t) = 5$ is asymptotically stable, whereas $y(t) = 2$ is unstable. The graph of any solution starting with a value between 2 and 3.5 at $t = 0$ changes its concavity from up to down when it crosses the dashed line $y = 3.5$. Solution y_1 vanishes at $t \approx 0.04$.

VERIFICATION WITH MATHEMATICA®. The input

```
y1 = (-5 + 8 * E^(-12 * t))/(-1 + 4 * E^(-12 * t));
y2 = (5 + 4 * E^(-12 * t))/(1 + 2 * E^(-12 * t));
y3 = (10 + 2 * E^(-12 * t))/(2 + E^(-12 * t));
y4 = (20 - 2 * E^(-12 * t))/(4 - E^(-12 * t));
y = {y1, y2, y3, y4};
{D[y,t] - 4 * (y - 2) * (5 - y), y/. t → 0} // Simplify
```

generates the output $\{\{0, 0, 0, 0\}, \{1, 3, 4, 6\}\}$. ■

Population with a critical threshold. This model is governed by a DE of the form

$$y' = -r\left(1 - \frac{y}{B}\right)y, \tag{3.21}$$

where r, $B = \text{const} > 0$. Since (3.21) is the same as (3.16) with r replaced by $-r$, the solution of the IVP consisting of (3.21) and the IC $y(0) = y_0$ can be obtained from (3.17) by means of the same substitution:

$$y(t) = \frac{y_0 B}{y_0 + (B - y_0)e^{rt}}. \tag{3.22}$$

From this we see that when $y_0 < B$, the denominator does not vanish for any $t > 0$, so we can compute

$$\lim_{t\to\infty} y(t) = 0.$$

If, on the other hand, $y_0 > B$, then the denominator vanishes at $t = t^*$, where

$$t^* = \frac{1}{r} \ln \frac{y_0}{y_0 - B} > 0,$$

which implies that the solution becomes infinite as t approaches the value t^*. The number B in this model is called a *critical threshold*. A population that starts with a size above this value grows without bound in finite time; one that starts below this value becomes extinct asymptotically.

3.15 Example. The critical points and equilibrium solutions of the DE

$$y' = -40\left(1 - \frac{y}{20}\right)y = 2y^2 - 40y = 2y(y - 20)$$

are 0, 20 and $y(t) = 0$, $y(t) = 20$, respectively. Since

$$y'' = (4y - 40)y' = 8y(y - 10)(y - 20),$$

the analysis based on the signs of the derivatives is simple and its conclusions are listed in Table 3.3.

In this case, (3.22) yields

$$y(t) = \frac{20y_0}{y_0 - (y_0 - 20)e^{40t}},$$

Table 3.3

y	0	10	20	
y'	$-$	$-$	$+$	
y''	$+$	$-$	$+$	
Graph of y	Descending, concave up	Descending, concave down	Ascending, concave up	

which, for y_0 equal, in turn, to 5, 15, and 25, generates the PSs

$$y_1(t) = \frac{20}{1 + 3e^{40t}}, \quad y_2(t) = \frac{60}{3 + e^{40t}}, \quad y_3(t) = \frac{100}{5 - e^{40t}}.$$

Solution y_3 tends to infinity as t approaches the value $t^* = (\ln 5)/40 \approx 0.04$, where the denominator vanishes.

The graphs of these solutions, sketched according to Table 3.3, and the graphs of the equilibrium solutions, are shown in Fig. 3.3.

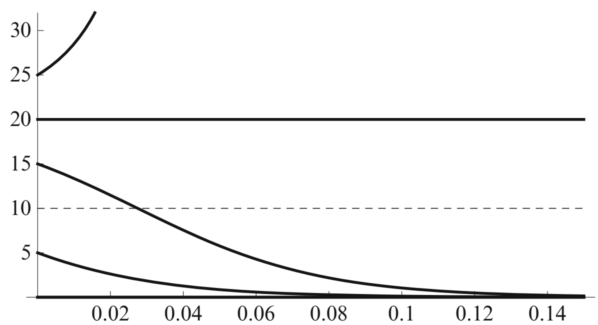

Fig. 3.3

It is clear that the equilibrium solution $y(t) = 0$ is asymptotically stable and that $y(t) = 20$ is unstable. The graph of any solution starting with a value between 10 and 20 at $t = 0$ changes its concavity from down to up when it crosses the dashed line $y = 10$.
Verification with Mathematica®. The input

```
y1 = 20/(1 + 3*E^(40*t));
y2 = 60/(3 + E^(40*t));
y3 = 100/(5 - E^(40*t));
y = {y1, y2, y3};
{D[y,t] - 2*y^2 + 40*y, y/.t→ 0}//Simplify
```

generates the output {{0, 0, 0}, {5, 15, 25}}. ■

3.16 Remark. As already mentioned, this model tells us that for $y_0 > B$ the solution increases without bound in finite time. Since this is not a realistic expectation, the model needs further adjustment, which can be done, for example, by considering the equation

$$y' = -r\left(1 - \frac{y}{B_1}\right)\left(1 - \frac{y}{B_2}\right)y, \quad r > 0, \quad 0 < B_1 < B_2.$$

We now have a population model with logistic growth (characterized by a carrying capacity B_2) and a threshold level B_1, where $y' < 0$ for large values of y, meaning that y remains bounded when y_0 is large. ■

Models of population dynamics are not the only ones that give rise to autonomous equations.

Chemical reaction. When two substances of concentrations c_1 and c_2 react to produce a new substance of concentration y, the DE governing the process is of the form

$$y' = a(c_1 - y)(c_2 - y),$$

where c_1, c_2, $a = \text{const} > 0$.

3.17 Example. Consider the DE

$$y' = 3(y^2 - 6y + 8) = 3(2 - y)(4 - y). \tag{3.23}$$

Its critical points and equilibrium solutions are 2, 4 and $y(t) = 2$, $y(t) = 4$, respectively. Here, we have

$$y'' = 6(y - 3)y' = 18(y - 2)(y - 3)(y - 4),$$

which helps provide the information given in Table 3.4.

Table 3.4

y	0	2	3	4
y'	+	−	−	+
y''	−	+	−	+
Graph of y	Ascending, concave down	Descending, concave up	Descending, concave down	Ascending, concave up

Proceeding as in Example 3.14, for $y \neq 2, 4$ we have

$$\int \frac{dy}{(2 - y)(4 - y)} = \int \left\{ \frac{1}{2(y - 4)} - \frac{1}{2(y - 2)} \right\} dy = 3 \int dt,$$

from which

$$\frac{y - 4}{y - 2} = Ce^{6t}.$$

We now use the generic IC $y(0) = y_0$ and then solve this equality for y to bring the solution of (3.23) to the form

$$y(t) = \frac{2(y_0 - 4) - 4(y_0 - 2)e^{-6t}}{y_0 - 4 - (y_0 - 2)e^{-6t}}. \tag{3.24}$$

It can be verified without difficulty that if $0 < y_0 < 4$, the denominator in (3.24) is never equal to zero; hence, we can evaluate

$$\lim_{t \to \infty} y(t) = 2.$$

When $y_0 > 4$, however, the denominator vanishes at $t = t^*$, where

$$t^* = \frac{1}{6} \ln \frac{y_0 - 2}{y_0 - 4} > 0.$$

Consequently, physically meaningful solutions obtained with such ICs do not exist for $t \geq t^*$.

The PSs corresponding to, say, y_0 equal to 1, 2.5, 3.5, and 5 are

$$y_1(t) = \frac{6 - 4e^{-6t}}{3 - e^{-6t}}, \qquad y_2(t) = \frac{6 + 4e^{-6t}}{3 + e^{-6t}},$$
$$y_3(t) = \frac{2 + 12e^{-6t}}{1 + 3e^{-6t}}, \qquad y_4(t) = \frac{2 - 12e^{-6t}}{1 - 3e^{-6t}}.$$

As expected, the first three functions approach the equilibrium solution $y(t) = 2$ as $t \to \infty$, whereas the fourth one becomes infinite at $t = t^* = (\ln 3)/6 \approx 0.18$.

The graphs of these functions, sketched in accordance with the details in Table 3.4, are shown in Fig. 3.4 and indicate that, of the two equilibrium solutions, $y(t) = 2$ is asymptotically stable and $y(t) = 4$ is unstable. The graph of any solution starting with a value between 3 and 4 at $t = 0$ changes its concavity from down to up when it crosses the dashed line $y = 3$.

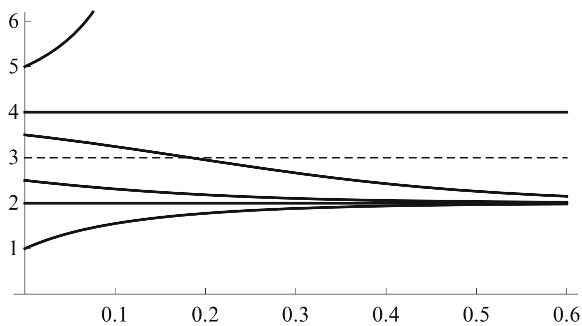

Fig. 3.4

Verification with Mathematica®. The input

```
y1 = (6 - 4 * E^(-6 * t)) / (3 - E^(-6 * t));
y2 = (6 + 4 * E^(-6 * t)) / (3 + E^(-6 * t));
y3 = (2 + 12 * E^(-6 * t)) / (1 + 3 * E^(-6 * t));
y4 = (2 - 12 * E^(-6 * t)) / (1 - 3 * E^(-6 * t));
y = {y1, y2, y3, y4};
{D[y,t] - 3 * (y^2 - 6 * y + 8), y /. t → 0} // Simplify
```

generates the output $\{\{0, 0, 0, 0\}, \{1, 2.5, 3.5, 5\}\}$. ■

3.18 Remark. Clearly, the methodology used in Examples 3.13–3.15 and 3.17 can be applied to any autonomous equation of the form $y' = ay^2 + by + c$ with constant coefficients a, b, and c, where the quadratic polynomial on the right-hand side has real roots. ■

Exercises

Find the critical points and equilibrium solutions of the given DE and follow the procedure set out in this section to solve the DE with each of the prescribed ICs. Sketch the graphs of the solutions obtained and comment on the stability/instability of the equilibrium solutions. Identify the model governed by the IVP, if any, and describe its main elements.

1 $y' = 300y - 2y^2$; $y(0) = 50$; $y(0) = 100$; $y(0) = 200$.

2 $y' = 240y - 3y^2$; $y(0) = 20$; $y(0) = 60$; $y(0) = 100$.

3 $y' = 15y - y^2/2$; $y(0) = 10$; $y(0) = 20$; $y(0) = 40$.

4 $y' = 5y - y^2/4$; $y(0) = 5$; $y(0) = 15$; $y(0) = 25$.

5 $y' = 8y - 2y^2 - 6$; $y(0) = 1/2$; $y(0) = 3/2$; $y(0) = 5/2$; $y(0) = 7/2$.

6 $y' = 30y - 3y^2 - 48$; $y(0) = 1$; $y(0) = 3$; $y(0) = 6$; $y(0) = 9$.

7 $y' = 2y^2 - 80y$; $y(0) = 10$; $y(0) = 30$; $y(0) = 50$.

8 $y' = 3y^2 - 180y$; $y(0) = 20$; $y(0) = 40$; $y(0) = 70$.

9 $y' = y^2/5 - 4y$; $y(0) = 5$; $y(0) = 15$; $y(0) = 25$.

10 $y' = y^2/20 - y/2$; $y(0) = 2$; $y(0) = 7$; $y(0) = 11$.

11 $y' = 2y^2 - 12y + 10$; $y(0) = 0$; $y(0) = 2$; $y(0) = 4$; $y(0) = 6$.

12 $y' = y^2 - 9y + 18$; $y(0) = 2$; $y(0) = 7/2$; $y(0) = 5$; $y(0) = 7$.

13 $y' = 2y - y^2 + 8$; $y(0) = -3$; $y(0) = -1$; $y(0) = 2$; $y(0) = 5$.

14 $y' = 2y - y^2 + 3$; $y(0) = -3/2$; $y(0) = 0$; $y(0) = 3/2$; $y(0) = 7/2$.

15 $y' = y^2 + y - 6$; $y(0) = -4$; $y(0) = -2$; $y(0) = 1$; $y(0) = 3$.

16 $y' = y^2 + 8y + 7$; $y(0) = -8$; $y(0) = -6$; $y(0) = -3$; $y(0) = 1$.

Answers to Odd-Numbered Exercises

1 0, 150; $y(t) = 0$ (unstable), $y(t) = 150$ (asymptotically stable);
$y = 150y_0/[y_0 - (y_0 - 150)e^{-300t}]$; population with logistic growth;
$r = 300$, $B = 150$.

3 0, 30; $y(t) = 0$ (unstable), $y(t) = 30$ (asymptotically stable);
$y = 30y_0/[y_0 - (y_0 - 30)e^{-15t}]$; population with logistic growth; $r = 15$, $B = 30$.

5 1, 3; $y(t) = 1$ (unstable), $y(t) = 3$ (asymptotically stable);
$y = [3(y_0 - 1) - (y_0 - 3)e^{-4t}]/[y_0 - 1 - (y_0 - 3)e^{-4t}]$;
population with logistic growth and harvesting; $r = 8$, $B = 4$, $\alpha = 6$.

7 0, 40; $y(t) = 0$ (asymptotically stable), $y(t) = 40$ (unstable);
$y = 40y_0/[y_0 - (y_0 - 40)e^{80t}]$; population with a critical threshold; $r = 80$, $T = 40$.

9 0, 20; $y(t) = 0$ (asymptotically stable), $y(t) = 20$ (unstable);
$y = 20y_0/[y_0 - (y_0 - 20)e^{4t}]$; population with a critical threshold; $r = 4$, $T = 20$.

11 1, 5; $y(t) = 1$ (asymptotically stable), $y(t) = 5$ (unstable);
$y = [y_0 - 5 - 5(y_0 - 1)e^{-8t}]/[y_0 - 5 - (y_0 - 1)e^{-8t}]$; chemical reaction;
$c_1 = 1$, $c_2 = 5$, $a = 2$.

13 $-2,\ 4;\ \ y(t)=-2$ (unstable), $y(t)=4$ (asymptotically stable);
$y=[4(y_0+2)+2(y_0-4)e^{-6t}]/[y_0+2-(y_0-4)e^{-6t}]$.

15 $-3,\ 2;\ \ y(t)=-3$ (asymptotically stable), $y(t)=2$ (unstable);
$y=[-3(y_0-2)-2(y_0+3)e^{-5t}]/[y_0-2-(y_0+3)e^{-5t}]$.

Chapter 4
Linear Second-Order Equations

A large number of mathematical models, particularly in the physical sciences and engineering, consist of IVPs or BVPs for second-order DEs. Among the latter, a very important role is played by linear equations. Even if the model is nonlinear, the study of its linearized version can provide valuable hints about the quantitative and qualitative behavior of the full model, and perhaps suggest a method that might lead to its complete solution.

4.1 Mathematical Models with Second-Order Equations

Below, we mention briefly a few typical examples of such models.

Free fall in gravity. As stated in Sect. 3.2, the position $y(t)$ of a heavy object falling in a gravitational field is a solution of the IVP

$$y'' + \frac{\gamma}{m}\, y' = g, \quad y(0) = y_0, \quad y'(0) = y_{10},$$

where m is the mass of the object and $\gamma = \text{const} > 0$ is a coefficient related to the resistance of the surrounding medium to motion.

RLC electric circuit. If an inductor (of inductance L) is added to the series circuit discussed in Sect. 3.2, and if the voltage generated by the source is constant, then the current $I(t)$ in the circuit satisfies the IVP

$$I'' + \frac{R}{L}\, I' + \frac{1}{LC}\, I = 0, \quad I(0) = I_0, \quad I'(0) = I_{10}.$$

Harmonic oscillator. A material point of mass m moving in a straight line is attached on one side to a spring characterized by an elastic coefficient k and on the other side to a shock absorber (damper) characterized by a damping coefficient γ. The point is also subjected to an additional external force $f(t)$. When the displacement of the point is small, the elastic force in the spring is proportional to the displacement, and the friction force in the damper is proportional to velocity. Since both these forces oppose the motion, it follows that, according to Newton's second law, the position $y(t)$ of the point at time t is the solution of the IVP

$$my'' + \gamma y' + ky = f(t), \quad y(0) = y_0, \quad y'(0) = y_{10},$$

where the numbers y_0 and y_{10} are the initial position and velocity of the point.

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_4,
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Steady-state convective heat. In this process, the distribution of temperature in a very thin, laterally insulated, uniform rod occupying the segment $0 \le x \le l$ on the x-axis is the solution of the BVP

$$ky'' - \alpha y' + f(x) = 0, \quad y(0) = y_0, \quad y(l) = y_l,$$

where k and α are positive coefficients related to the thermal properties of the rod material, f represents the heat sources and sinks inside the rod, and the numbers y_0 and y_l are the values of the temperature prescribed at the end-points. Of course, the BCs may have alternative forms, depending on the physical setup. For example, if the heat flux is prescribed at $x = l$ instead of the temperature, then the second BC is replaced by

$$y'(l) = y_l.$$

Temperature distribution in a circular annulus. If the heat sources and sinks inside a uniform annulus $a < r < b$ are described by a circularly symmetric and time-independent function $g(r)$, and the inner and outer boundary circles $r = a$ and $r = b$ are held at constant temperatures y_a and y_b, respectively, then the equilibrium temperature $y(r)$ in the annulus is the solution of the BVP

$$k(y'' + r^{-1}y') + g(r) = 0, \quad y(a) = y_a, \quad y(b) = y_b,$$

where k is the (constant) thermal diffusivity of the material.

Time-independent Schrödinger equation. The wave function $\psi(x)$ of a particle at a point x in a quantum field satisfies the DE

$$\hbar^2\psi''(x) - 2m[V(x) - E]\psi(x) = 0, \quad -\infty < x < \infty,$$

where m, V, and E are the mass, potential energy, and total energy of the particle, respectively, and $\hbar$ is the reduced Planck constant. Also, the solution ψ must not grow at infinity.

Motion of a pendulum. The position at time t of a pendulum of length l is identified by its angular displacement $\theta(t)$, which is a solution of the nonlinear IVP

$$l\theta'' + g\sin\theta = 0, \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_{10},$$

where g is the acceleration of gravity and we have assumed that the air resistance is negligible.

4.2 Algebra Prerequisites

Before discussing solution methods for DEs, we recall a few basic concepts of linear algebra and some related properties. The presentation of these concepts is restricted here to the form they assume in the context of second-order linear equations.

4.1 Definition. A 2×2 matrix is an array of four numbers a_{11}, a_{12}, a_{21}, and a_{22} (called *elements*, or *entries*), arranged in two rows and two columns:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The number defined by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

is called the *determinant* of A. ■

4.2 Remarks. (i) It is obvious that if we swap its rows (columns), the sign of the determinant changes.

(ii) It is also clear that if the elements of a row (column) have a common factor, then this factor can be taken out and multiplied by the value of the new, simplified determinant; in other words,

$$\begin{vmatrix} ca_{11} & ca_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} ca_{11} & a_{12} \\ ca_{21} & a_{22} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = c(a_{11}a_{22} - a_{21}a_{12}).$$

(iii) Determinants are very useful in algebraic calculations, for example, in the solution of a linear system by *Cramer's rule*. Specifically, this rule gives the solution of the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1, \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

as

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1a_{22} - b_2a_{12}}{\det(A)}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_2a_{11} - b_1a_{21}}{\det(A)}. \tag{4.1}$$

Obviously, Cramer's rule can be applied only if $\det(A) \neq 0$. ■

4.3 Example. By Cramer's rule, the solution of the system

$$\begin{aligned} 2x_1 - x_2 &= 4, \\ x_1 + 3x_2 &= -5 \end{aligned}$$

is

$$x_1 = \frac{\begin{vmatrix} 4 & -1 \\ -5 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{7}{7} = 1, \quad x_2 = \frac{\begin{vmatrix} 2 & 4 \\ 1 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{-14}{7} = -2. \ ■$$

4.4 Definition. Let f_1 and f_2 be two continuously differentiable functions on an interval J. The determinant

$$W[f_1, f_2](t) = \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix} = f_1(t)f_2'(t) - f_1'(t)f_2(t)$$

is called the *Wronskian* of f_1 and f_2. ■

4.5 Example. If $f_1(t) = 2t - 1$ and $f_2(t) = t^2$, then $f_1'(t) = 2$ and $f_2'(t) = 2t$, so

$$W[f_1, f_2](t) = \begin{vmatrix} 2t - 1 & t^2 \\ 2 & 2t \end{vmatrix} = 4t^2 - 2t - 2t^2 = 2t^2 - 2t. \ ■$$

4.6 Example. For $f_1(t) = e^{-t}$ and $f_2(t) = e^{3t}$ we have $f_1'(t) = -e^{-t}$ and $f_2'(t) = 3e^{3t}$; hence,

$$W[f_1, f_2](t) = \begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{vmatrix} = 3e^{-t}e^{3t} + e^{-t}e^{3t} = 4e^{2t}. \ \blacksquare$$

4.7 Example. The derivatives of $f_1(x) = \sin(ax)$ and $f_2(x) = \sin(a(x-c))$, where a, $c = \text{const} \neq 0$, are $f_1'(x) = a\cos(ax)$ and $f_2'(x) = a\cos(a(x-c))$, so

$$\begin{aligned} W[f_1, f_2](x) &= \begin{vmatrix} \sin(ax) & \sin(a(x-c)) \\ a\cos(ax) & a\cos(a(x-c)) \end{vmatrix} \\ &= a\sin(ax)\cos(a(x-c)) - a\cos(ax)\sin(a(x-c)). \end{aligned}$$

Using the trigonometric identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$, we can bring this to the simpler form

$$W[f_1, f_2](x) = a\sin(ax - a(x-c)) = a\sin(ac). \ \blacksquare$$

4.8 Example. If $f_1(x) = \sinh(ax)$ and $f_2(x) = \cosh(a(x-c))$, where a, $c = \text{const}$, $a \neq 0$, then $f_1'(x) = a\cosh(ax)$ and $f_2'(x) = a\sinh(a(x-c))$; therefore,

$$\begin{aligned} W[f_1, f_2](x) &= \begin{vmatrix} \sinh(ax) & \cosh(a(x-c)) \\ a\cosh(ax) & a\sinh(a(x-c)) \end{vmatrix} \\ &= a\sinh(ax)\sinh(a(x-c)) - a\cosh(ax)\cosh(a(x-c)) \\ &= -a\cosh(ax - a(x-c)) = -a\cosh(ac). \ \blacksquare \end{aligned}$$

In what follows, f_1 and f_2 are functions defined on the same interval J.

4.9 Definition. An expression of the form $c_1 f_1 + c_2 f_2$, where c_1 and c_2 are constants, is called a *linear combination* of f_1 and f_2. ■

4.10 Definition. The functions f_1 and f_2 are said to be *linearly dependent* on J if there are constants c_1 and c_2, not both zero, such that $c_1 f_1(t) + c_2 f_2(t) = 0$ for all t in J. If f_1 and f_2 are not linearly dependent, then they are called *linearly independent.* ■

4.11 Remarks. (i) Clearly, f_1 and f_2 are linearly dependent if and only if one of them is a constant multiple of the other.

(ii) It is not difficult to show that if f_1 and f_2 are linearly dependent, then $W[f_1, f_2](t) = 0$ for all t in J.

(iii) The *converse* of the statement in (ii) is not true: $W[f_1, f_2](t) = 0$ for all t in J does not imply that f_1 and f_2 are linearly dependent.

(iv) The *contrapositive* of the statement in (ii) is true: if $W[f_1, f_2](t_0) \neq 0$ for some t_0 in J, then f_1 and f_2 are linearly independent. ■

4.12 Example. Let $f_1(t) = \cos(at)$ and $f_2(t) = \sin(at)$, where $a = \text{const} \neq 0$. Since

$$W[f_1, t_2](t) = \begin{vmatrix} \cos(at) & \sin(at) \\ -a\sin(at) & a\cos(at) \end{vmatrix} = a\cos^2(at) + a\sin^2(at) = a \neq 0$$

for at least one value of t (in fact, for all real t), we conclude that f_1 and f_2 are linearly independent on the real line. ■

4.13 Example. The functions $f_1(x) = \sin(ax)$ and $f_2(x) = \sin(a(x-c))$ considered in Example 4.7 are linearly dependent on the real line if c is equal to any one of the numbers $k\pi/a$ with $k = \dots, -2, -1, 1, 2, \dots$. For then we have

$$\begin{aligned} f_2(x) &= \sin(a(x-c)) = \sin(ax - k\pi) = \sin(ax)\cos(k\pi) - \cos(ax)\sin(k\pi) \\ &= (-1)^k \sin(ax) = (-1)^k f_1(x) \quad \text{for all real } x; \end{aligned}$$

that is, f_2 is a multiple of f_1. In this case we also see that

$$W[f_1, f_2](x) = a\sin(ac) = a\sin(k\pi) = 0 \quad \text{for all real } x,$$

which confirms the assertion in Remark 4.11(ii). If c is ascribed any other value, then f_1 and f_2 are linearly independent on $-\infty < x < \infty$. ■

4.14 Example. According to Remark 4.11(iv), the functions $f_1(x) = \sinh(ax)$ and $f_2(x) = \cosh(a(x-c))$ considered in Example 4.8 are linearly independent for any $a = \text{const} \neq 0$ since their Wronskian is equal to $-a\cosh(ac)$, which vanishes only if $a = 0$. ■

4.15 Example. If $f_1(t) = t$, $W[f_1, f_2](t) = t^2 - 2$, and $f_2(0) = 2$, we can easily determine the function f_2. First, we write

$$W[f_1, f_2](t) = \begin{vmatrix} t & f_2 \\ 1 & f_2' \end{vmatrix} = tf_2' - f_2 = t^2 - 2,$$

then solve this linear equation by the method in Sect. 2.2 and find the GS

$$f_2(t) = t^2 + 2 + C.$$

The IC $f_2(0) = 2$ now yields $C = 0$, so

$$f_2(t) = t^2 + 2.$$

Verification with Mathematica®. The input

```
{f1,f2}={t,t^2 + 2};
{Det[{{f1,f2},{D[f1,t],D[f2,t]}}],f2/.t→ 0}
```

generates the output $\{t^2 - 2, 2\}$. ■

4.16 Theorem. *If f_1 and f_2 are linearly independent on J and*

$$a_1 f_1(t) + a_2 f_2(t) = b_1 f_1(t) + b_2 f_2(t) \quad \text{for all } t \text{ in } J,$$

then $a_1 = b_1$ and $a_2 = b_2$. ■

Proof. The above equality is equivalent to $(a_1 - b_1)f_1(t) + (a_2 - b_2)f_2(t) = 0$ for all values of t in J. Since f_1 and f_2 are linearly independent, from Definition 4.10 it follows that the latter equality holds only if the coefficients of the linear combination on the left-hand side are zero, which leads to $a_1 = b_1$ and $a_2 = b_2$. ■

4.17 Example. Suppose that

$$(2a - b)\cos t + (a + 3b)\sin t = 4\cos t - 5\sin t \quad \text{for all real } t.$$

In Example 4.12 it was shown that $\cos t$ and $\sin t$ are linearly independent functions; hence, by Theorem 4.16,

$$\begin{aligned} 2a - \ \ b &= 4, \\ a + 3b &= -5. \end{aligned}$$

This system was solved in Example 4.3, and its solution is $a = 1$ and $b = -2$.

Verification with Mathematica®. The input

```
{a,b} = {1, - 2};
(2*a - b)*Cos[t] + (a + 3*b)*Sin[t] - 4*Cos[t] + 5*Sin[t]
        //Simplify
```

generates the output 0. ■

Exercises

In 1–10, compute the Wronskian of the given functions.

1 $f_1(t) = 2t - 1, \quad f_2(t) = te^{2t}.$ **2** $f_1(t) = \sin(2t), \quad f_2(t) = t\sin(2t).$

3 $f_1(t) = t\sin t, \quad f_2(t) = t\cos t.$ **4** $f_1(t) = e^{-t}, \quad f_2(t) = t^2e^{-t}.$

5 $f_1(t) = e^{2t}\sin t, \quad f_2(t) = e^{2t}\cos t.$

6 $f_1(t) = \cos(at), \quad f_2(t) = \cos(a(t - c)), \quad a,\ c = \text{const} \neq 0.$

7 $f_1(t) = \cos(at), \quad f_2(t) = \sin(a(t - c)), \quad a,\ c = \text{const}, \quad a \neq 0.$

8 $f_1(x) = \cosh(ax), \quad f_2(x) = \cosh(a(x - c)), \quad a,\ c = \text{const} \neq 0.$

9 $f_1(x) = \sinh(ax), \quad f_2(x) = \sinh(a(x - c)), \quad a,\ c = \text{const} \neq 0.$

10 $f_1(x) = \cosh(ax), \quad f_2(x) = \sinh(a(x - c)), \quad a,\ c = \text{const}, \quad a \neq 0.$

In 11–16, find the function f_2 when the function f_1, the Wronskian $W[f_1, f_2]$, and $f_2(0)$ are given.

11 $f_1(t) = 2t + 1, \quad W[f_1, f_2](t) = 2t^2 + 2t + 1, \quad f_2(0) = 0.$

12 $f_1(t) = 3t - 1, \quad W[f_1, f_2](t) = -3t^2 + 2t - 3, \quad f_2(0) = 1.$

13 $f_1(t) = t - 2, \quad W[f_1, f_2](t) = (3 - 2t)e^{-2t}, \quad f_2(0) = 1.$

14 $f_1(t) = 2t - 1, \quad W[f_1, f_2](t) = (2t^2 - t - 1)e^t, \quad f_2(0) = 0.$

15 $f_1(t) = t + 1, \quad W[f_1, f_2](t) = (1 + t)\cos t - \sin t, \quad f_2(0) = 0.$

16 $f_1(t) = 4t + 1, \quad W[f_1, f_2](t) = \cos t - (4t^2 + t)\sin t, \quad f_2(0) = 0.$

Answers to Odd-Numbered Exercises

1 $W[f_1, f_2](t) = (4t^2 - 2t - 1)e^{2t}.$ **3** $W[f_1, f_2](t) = -t^2.$

5 $W[f_1, f_2](t) = -e^{4t}.$ **7** $W[f_1, f_2](t) = a\cos(ac).$

9 $W[f_1, f_2](x) = a\sinh(ac).$ **11** $f_2(t) = t^2 + t.$

13 $f_2(t) = e^{-2t}.$ **15** $f_2(t) = \sin t.$

4.3 Homogeneous Equations

The general form of this type of linear second-order DE is

$$y'' + p(t)y' + q(t)y = 0, \tag{4.2}$$

where p and q are given functions. Sometimes we also consider the alternative form

$$a(t)y'' + b(t)y' + c(t)y = 0,$$

where a, b, and c are prescribed functions, with $a \neq 0$. These equations are accompanied in mathematical models by either ICs or BCs. Since the two cases exhibit a number of important differences, we discuss them separately.

4.3.1 Initial Value Problems

An IVP consists of equation (4.2) and ICs of the form

$$y(t_0) = y_0, \quad y'(t_0) = y_{10}. \tag{4.3}$$

In what follows, it is convenient to use the operator notation for the DE, as explained in Sect. 1.4. Thus, if we write formally $L = D^2 + pD + q$, then the IVP becomes

$$Ly = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_{10}. \tag{4.4}$$

4.18 Theorem (Existence and uniqueness). *If p and q are continuous on an open interval J that contains the point t_0, then the IVP (4.4) has a unique solution on J.* ■

4.19 Example. For the IVP

$$(t^2 - t - 6)y'' - (t + 1)y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

we have $p(t) = -(t+1)/(t^2-t-6)$ and $q(t) = 1/(t^2-t-6)$. Since $t^2-t-6 = 0$ at $t = -2$ and $t = 3$, it follows that p and q are continuous on $-\infty < t < -2$, $-2 < t < 3$, and $3 < t < \infty$; therefore, by Theorem 4.18, the IVP has a unique solution for $-2 < t < 3$, which is the interval containing the point $t_0 = 0$ where the ICs are prescribed. ■

4.20 Remarks. (i) From Theorem 4.18 it is obvious that if $y_0 = y_{10} = 0$, then the IVP (4.4) has only the zero solution.

(ii) The maximal interval of existence for the solution of (4.4) is defined exactly as in the case of an IVP for a first-order DE (see Definition 2.28). When the equation has constant coefficients, this interval is the entire real line.

(iii) The comments made in Remark 2.29 also apply to the IVP (4.4). ■

Below, we assume that p and q are continuous on some open interval J, whether J is specifically mentioned or not.

The next assertion is the *principle of superposition* (for homogeneous equations).

4.21 Theorem. *If y_1 and y_2 are such that $Ly_1 = Ly_2 = 0$, then $L(c_1y_1 + c_2y_2) = 0$ for any constants c_1 and c_2.* ■

Proof. Since L is a linear operator (see Example 1.31), it follows that

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2 = 0. \quad \blacksquare$$

We now turn our attention to finding the GS of the equation $Ly = 0$.

4.22 Theorem. *If y_1 and y_2 are solutions of $Ly = 0$ on an open interval J and $W[y_1, y_2](t_0) \neq 0$ at some point t_0 in J, then there are unique numbers c_1 and c_2 such that $c_1y_1 + c_2y_2$ is the solution of the IVP (4.4) on J.* $\blacksquare$

Proof. By Theorem 4.21, $y = c_1y_1 + c_2y_2$ satisfies $Ly = 0$. The ICs now yield

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_{10}, \end{aligned}$$

which is a linear algebraic system for the unknowns c_1 and c_2. Since the determinant of the system is

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = W[y_1, y_2](t_0) \neq 0,$$

we conclude that the system has a unique solution. This solution can be computed in a variety of ways, for example, by Cramer's rule (4.1). $\blacksquare$

4.23 Example. Consider the IVP

$$y'' - 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = -5.$$

We can readily check that $y_1(t) = e^{-t}$ and $y_2(t) = e^{3t}$ are solutions of the equation on the entire real line, and that $W[y_1, y_2](0) = 4 \neq 0$. By Theorem 4.22, the solution of the IVP is of the form $y = c_1y_1 + c_2y_2$. Applying the ICs, we arrive at the system

$$\begin{aligned} c_1 + \ \ c_2 &= 1, \\ -c_1 + 3c_2 &= -5, \end{aligned}$$

which has the unique solution $c_1 = 2$ and $c_2 = -1$. Consequently, the solution of the IVP is

$$y(t) = 2e^{-t} - e^{3t}. \quad \blacksquare$$

4.24 Theorem. *If y_1 and y_2 are two solutions of the equation $Ly = 0$ on an open interval J and there is a point t_0 in J such that $W[y_1, y_2](t_0) \neq 0$, then $c_1y_1 + c_2y_2$, where c_1 and c_2 are arbitrary constants, is the GS of the equation $Ly = 0$ on J.* $\blacksquare$

Proof. By Theorem 4.21, any function of the form $c_1y_1 + c_2y_2$ satisfies the equation $Ly = 0$. Conversely, let y be any solution of this equation, and let $y(t_0) = y_0$ and $y'(t_0) = y_{10}$. By Theorem 4.22, y can be written uniquely as $c_1y_1 + c_2y_2$, which proves the assertion. $\blacksquare$

4.25 Remark. It can be shown that if y_1 and y_2 are two solutions of (4.2) on an open interval J and t_0 is a point in J, then for any t in J,

$$W[y_1, y_2](t) = W[y_1, y_2](t_0) \exp\left\{ -\int_{t_0}^{t} p(\tau)\, d\tau \right\}.$$

This implies that the Wronskian of two solutions is either zero at every point in J or nonzero everywhere on J. $\blacksquare$

4.26 Definition. A pair of functions $\{y_1, y_2\}$ with the properties mentioned in Theorem 4.24 is called a *fundamental set of solutions* (FSS) for equation (4.2). ■

4.27 Example. Clearly, $\{e^{-t}, e^{3t}\}$ is an FSS for the DE in Example 4.23, whose GS can therefore be written as $y(t) = c_1 e^{-t} + c_2 e^{3t}$. ■

4.28 Remarks. (i) The conclusion in Theorem 4.24 is independent of any additional conditions accompanying the DE.

(ii) Combining the statements in Remarks 4.11(iv) and 4.25 and Theorem 4.22, we can show that two solutions of (4.2) form an FSS if and only if they are linearly independent. This means that we may equally define an FSS as being any pair of linearly independent solutions.

(iii) In practice, it is generally very difficult to find particular solutions of homogeneous linear equations with variable coefficients. However, as we see in Sect. 4.4, this task is relatively simple when the coefficients are constant. ■

Exercises

In 1–10, find the largest interval on which the conditions of Theorem 4.18 are satisfied, without solving the given IVP.

1 $ty'' + t^2 y' + 2y = 0, \quad y(1) = 2, \quad y'(1) = 0.$

2 $(1-2t)y'' - 2ty' + y\sin t = 0, \quad y(0) = 4, \quad y'(0) = -1.$

3 $(t^2 - 3t + 2)y'' + y' + (t-1)y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

4 $(t^2 - 2t + 1)y'' + ty' - y = 0, \quad y(2) = 1, \quad y'(2) = -1.$

5 $(t^2 - 4)y'' + 2y' - 3(t+2)y = 0, \quad y(0) = 1, \quad y'(0) = 1.$

6 $(t^2 + 3t - 4)y'' - y' + e^{-t}y = 0, \quad y(3) = 0, \quad y'(3) = 2.$

7 $y''\sin t + y' = 0, \quad y(3\pi/2) = 1, \quad y'(3\pi/2) = -2.$

8 $y''\cos(2t) - 3y = 0, \quad y(-\pi/2) = 2, \quad y'(-\pi/2) = 3.$

9 $(2 - e^t)y'' + ty' + 4ty = 0, \quad y(1) = 1, \quad y'(1) = -2.$

10 $(\tan t + 1)y'' + y'\sin t + y = 0, \quad y(\pi) = -1, \quad y'(\pi) = 4.$

In 11–16, use the formula in Remark 4.25 to compute the value at the indicated point t of the Wronskian of two solutions y_1, y_2 of the given DE when its value at another point is as prescribed.

11 $ty'' + 2y' + 3ty = 0, \quad t = 2, \quad W[y_1, y_2](1) = 4.$

12 $(t+1)y'' - y' + 2(t^2 + t)y = 0, \quad t = 3, \quad W[y_1, y_2](0) = 2.$

13 $(t^2 + 1)y'' - ty' + e^t y = 0, \quad t = 1, \quad W[y_1, y_2](0) = -1.$

14 $(t^2 + t)y'' + (2t+1)y' + y\sin(2t) = 0, \quad t = 4, \quad W[y_1, y_2](1) = 9.$

15 $y'' + y'\cot t + e^{-t}y = 0, \quad t = \pi/2, \quad W[y_1, y_2](\pi/6) = -2.$

16 $y''\sin t + y'\sec t + y = 0, \quad t = \pi/3, \quad W[y_1, y_2](\pi/4) = 1.$

Answers to Odd-Numbered Exercises

1 $t > 0.$ **3** $t < 1.$ **5** $-2 < t < 2.$ **7** $\pi < t < 2\pi.$ **9** $t > \ln 2.$

11 $W[y_1, y_2](2) = 1.$ **13** $W[y_1, y_2](1) = -\sqrt{2}.$ **15** $W[y_1, y_2](\pi/2) = -1.$

4.3.2 Boundary Value Problems

A BVP consists of equation (4.2) (with t usually replaced by x as the independent variable) on a finite open interval $a < x < b$, and BCs prescribed at the end-points $x = a$ and $x = b$ of this interval. Here we consider the following three types of BCs:

$$y(a) = \alpha, \;\; y(b) = \beta; \quad y(a) = \alpha, \;\; y'(b) = \beta; \quad y'(a) = \alpha, \;\; y(b) = \beta, \tag{4.5}$$

where α and β are given numbers.

It turns out that the different nature of conditions (4.5), compared to (4.3), makes BVPs less simple to analyze than IVPs when it comes to the existence and uniqueness of solutions. Even when we can construct the GS of the DE, the outcome for the corresponding BVP is not necessarily a unique solution: it could be that, or multiple solutions, or no solution at all, depending on the BCs attached to the equation.

4.29 Example. Consider the BVP

$$y'' + y = 0, \quad y(0) = 3, \quad y(\pi/2) = -2.$$

We easily verify that $y_1 = \cos x$ and $y_2 = \sin x$ are solutions of the DE. Since, as shown in Example 4.12 with $a = 1$, they are also linearly independent, from Remark 4.28(ii) it follows that they form an FSS. Hence, by Theorem 4.24,

$$y(x) = c_1 \cos x + c_2 \sin x$$

is the GS of the equation. Applying the BCs, we obtain $c_1 = 3$ and $c_2 = -2$, so the unique solution of the BVP is

$$y(x) = 3\cos x - 2\sin x.$$

If we now consider the same DE on the interval $0 < x < 2\pi$ with $y(0) = 3$, $y(2\pi) = 3$, then the GS and the new BCs yield $c_1 = 3$ but no specific value for c_2, so the modified BVP has infinitely many solutions of the form

$$y(x) = 3\cos x + c_2 \sin x,$$

where c_2 is an arbitrary constant.

Finally, if the interval and the BCs are changed to $0 < x < \pi$ and $y(0) = 3$, $y(\pi) = -2$, then we get $c_1 = 3$ and $c_1 = 2$, which is impossible; therefore, this third BVP has no solution. ■

The next assertion states conditions under which BVPs of interest to us are uniquely solvable.

4.30 Theorem. *If J is an open finite interval $a < x < b$ and p and q are continuous functions on J, with $q(x) < 0$ for all x in J, then the BVP consisting of* (4.2) *and any one of the sets of BCs in* (4.5) *has a unique solution on J.* ■

4.31 Remark. As happens with many existence and uniqueness theorems, the conditions in the above assertion are sufficient but not necessary. When they are not satisfied, we cannot draw any conclusion about the solvability of the BVP. In Example 4.29 we have $q(x) = 1 > 0$, which contravenes one of the conditions in Theorem 4.30 and, therefore, makes the theorem inapplicable. ■

Exercises

Verify that the given functions y_1 and y_2 form an FSS for the accompanying DE, then discuss the existence and uniqueness of solutions for the equation with each set of BCs on the appropriate interval. Compute the solutions where possible.

1 $4y'' + y = 0, \quad y_1(x) = \cos(x/2), \quad y_2(x) = \sin(x/2).$

(i) $y(0) = 2,\ y(\pi) = -4$; (ii) $y(0) = 2,\ y(2\pi) = -2$; (iii) $y(0) = 2,\ y(2\pi) = 3$.

2 $y'' - 4y' + 8y = 0, \quad y_1(x) = e^{2x}\cos(2x), \quad y_2(x) = e^{2x}\sin(2x).$

(i) $y'(0) = 2,\ y(\pi/8) = e^{\pi/4}/\sqrt{2}$; (ii) $y'(0) = -4,\ y(\pi/4) = -3e^{\pi/2}$;

(iii) $y'(0) = 2,\ y(\pi/8) = 1$.

3 $y'' + 2y' + 2y = 0, \quad y_1(x) = e^{-x}\cos x, \quad y_2(x) = e^{-x}\sin x.$

(i) $y(0) = 1,\ y'(\pi/4) = 2$; (ii) $y(0) = 3,\ y'(\pi/2) = -e^{-\pi/2}$;

(iii) $y(0) = 1,\ y'(\pi/4) = -\sqrt{2}\,e^{-\pi/4}$.

4 $4y'' - 8y' + 5y = 0, \quad y_1(x) = e^{x}\cos(x/2), \quad y_2(x) = e^{x}\sin(x/2).$

(i) $y'(0) = 2,\ y'(\pi) = 3e^{\pi}/2$; (ii) $y'(0) = -4,\ y'(2\pi) = 2$;

(iii) $y'(0) = -4,\ y'(2\pi) = 4e^{2\pi}$.

Answers to Odd-Numbered Exercises

1 (i) $y(x) = 2\cos(x/2) - 4\sin(x/2)$;

(ii) $y(x) = 2\cos(x/2) + c\sin(x/2), \quad c = \text{const arbitrary}$; (iii) no solution.

3 (i) No solution; (ii) $y(x) = e^{-x}(3\cos x - 2\sin x)$;

(iii) $y(x) = e^{-x}(\cos x + c\sin x), \quad c = \text{const arbitrary}$.

4.4 Homogeneous Equations with Constant Coefficients

As mentioned in Remark 4.28(iii), for a DE of the form

$$ay'' + by' + cy = 0, \quad a,\ b,\ c = \text{const}, \quad a \neq 0, \tag{4.6}$$

it is fairly easy to find an FSS. Since the left-hand side in (4.6) is a linear combination of y, y', and y'', we are looking for a function that keeps its original 'shape' when differentiated. A reasonable candidate is a solution of the form

$$y(t) = e^{rt}, \quad r = \text{const}.$$

Replacing this in (4.6), we arrive at the equality $(ar^2 + br + c)e^{rt} = 0$ for all real t, which holds if and only if

$$ar^2 + br + c = 0. \tag{4.7}$$

The quadratic equation (4.7) is called the *characteristic equation* corresponding to (4.6), and its left-hand side is called the *characteristic polynomial*. This equation has two *characteristic roots*, r_1 and r_2, that could be real and distinct, real and repeated, or complex conjugate. Obviously, the nature of the roots influences the form of the GS.

4.4.1 Real and Distinct Characteristic Roots

If $r_1 \neq r_2$, the Wronskian of the solutions

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$$

at any point t is

$$W[y_1, y_2](t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0. \tag{4.8}$$

By Theorem 4.24 and Definition 4.26, $\{y_1, y_2\}$ is an FSS for (4.6), whose GS can therefore be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{4.9}$$

4.32 Example. Consider the IVP

$$y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = 8.$$

The characteristic equation is $r^2 - r - 6 = 0$, with roots $r_1 = -2$ and $r_2 = 3$, so the GS of the DE is

$$y(t) = c_1 e^{-2t} + c_2 e^{3t}.$$

Then $y'(t) = -2c_1 e^{-2t} + 3c_2 e^{3t}$, and applying the BCs, we find that

$$\begin{aligned} c_1 \;\; + \; c_2 &= 1, \\ -2c_1 + 3c_2 &= 8, \end{aligned}$$

from which $c_1 = -1$ and $c_2 = 2$; hence, the solution of the IVP is

$$y(t) = -e^{-2t} + 2e^{3t}.$$

The graph of the solution for $t \geq 0$ is shown in Fig. 4.1.

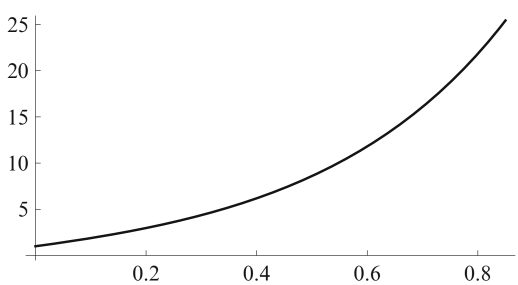

Fig. 4.1

VERIFICATION WITH MATHEMATICA®. The input

```
y= - E^(-2*t) + 2*E^(3*t);
{D[y,t,t] - D[y,t] - 6*y,{y,D[y,t]}/.t→ 0}//Simplify
```

generates the output $\{0, \{1, 8\}\}$. ■

4.33 Example. The characteristic equation for the DE in the BVP

$$y'' - y = 0, \quad y(0) = 2, \quad y'(1) = -\tfrac{1}{2}$$

is $r^2 - 1 = 0$, with roots $r_1 = -1$ and $r_2 = 1$, so its GS is

$$y(x) = c_1 e^{-x} + c_2 e^{x}.$$

Since $y'(x) = -c_1 e^{-x} + c_2 e^{x}$, from the BCs we find that

$$\begin{aligned} c_1 + \quad c_2 &= 2, \\ -e^{-1} c_1 + e c_2 &= -\tfrac{1}{2}; \end{aligned}$$

hence, $c_1 = (4e^2 + e)/(2e^2 + 2)$ and $c_2 = (4 - e)/(2e^2 + 2)$, which yields the solution

$$y(x) = \frac{4e^2 + e}{2e^2 + 2}\, e^{-x} + \frac{4 - e}{2e^2 + 2}\, e^{x}.$$

There is an alternative, perhaps neater, way to write the solution of this BVP. It is easily verified that $y_1(x) = \sinh x$ and $y_2(x) = \cosh(x - 1)$ satisfy the DE. Additionally, they satisfy $y_1(0) = 0$ and $y_2'(1) = 0$, which turns out to be very handy when we apply the BCs. Since, as seen from Example 4.8 with $a = c = 1$, the Wronskian of these two functions is nonzero, it follows that they form an FSS, so the GS of the DE can be expressed as

$$y(x) = c_1 \sinh x + c_2 \cosh(x - 1).$$

Then $y'(x) = c_1 \cosh x + c_2 \sinh(x - 1)$, so, using the BCs, we find that $c_2 \cosh 1 = 2$ and $c_1 \cosh 1 = -1/2$, from which $c_2 = 2 \operatorname{sech} 1$ and $c_1 = -(1/2) \operatorname{sech} 1$. Consequently, the solution of the BVP is

$$y(x) = (\operatorname{sech} 1)\big[-\tfrac{1}{2} \sinh x + 2 \cosh(x - 1)\big].$$

The graph of the solution for $t \geq 0$ is shown in Fig. 4.2.

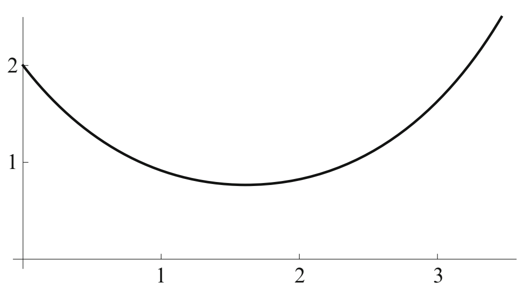

Fig. 4.2

VERIFICATION WITH MATHEMATICA®. The input

```
y = Sech[1] * (Sinh[x] + 2 * Cosh[x-1]);
{D[y,x,x] - y, {y/. x→ 0, D[y,x] /. x→ 1}} // Simplify
```

generates the output {0, {2, 1}}. ■

Exercises

In 1–8, find the solution $y(t)$ of the given IVP.

1 $y'' + y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 9.$

2 $y'' - 4y' + 3y = 0, \quad y(0) = 0, \quad y'(0) = -4.$

3 $2y'' + y' - y = 0, \quad y(0) = 4, \quad y'(0) = -5/2.$

4 $3y'' - 4y' - 4y = 0, \quad y(0) = 1, \quad y'(0) = -10/3.$

5 $y'' + 4y' = 0, \quad y(0) = -2, \quad y'(0) = -4.$

6 $3y'' - y' = 0, \quad y(0) = 2, \quad y'(0) = -2/3.$

7 $6y'' - y' - y = 0, \quad y(0) = 2, \quad y'(0) = -3/2.$

8 $4y'' + 4y' - 15y = 0, \quad y(0) = 2, \quad y'(0) = 11.$

In 9–16, find the solution $y(x)$ of the given BVP.

9 $y'' - 2y' - 3y = 0, \quad y(0) = 0, \quad y(1) = e^{-1} - e^3.$

10 $2y'' + 3y' - 2y = 0, \quad y'(-1) = e^{-1/2} - 2e^2, \quad y(1) = 2e^{1/2} + e^{-2}.$

11 $y'' - 2y' = 0, \quad y(0) = 1, \quad y'(\ln 2) = -16.$

12 $y'' - 2y' - 8y = 0, \quad y'(-\ln 2) = 63/4, \quad y'(0) = 0.$

13 $y'' - 4y = 0, \quad y(0) = 2, \quad y(1) = 3.$

14 $9y'' - y = 0, \quad y'(0) = 4/3, \quad y(1) = -2.$

15 $y'' - 16y = 0, \quad y(0) = -1, \quad y'(2) = 12.$

16 $9y'' - 4y = 0, \quad y'(0) = -2, \quad y'(3) = -1/3.$

Answers to Odd-Numbered Exercises

1 $y(t) = 3e^{2t} - e^{-3t}$. **3** $y(t) = 3e^{-t} + e^{t/2}$. **5** $y(t) = -3 + 4e^{-4t}$.

7 $y(t) = 3e^{-t/3} - e^{t/2}$. **9** $y(x) = e^{-x} - e^{3x}$. **11** $y(x) = 3 - 2e^{2x}$.

13 $y(x) = (\operatorname{csch} 2)[3\sinh(2x) - 2\sinh(2(x-1))]$.

15 $y(x) = (\operatorname{sech} 8)[3\sinh(4x) - \cosh(4(x-2))]$.

4.4.2 Repeated Characteristic Roots

When the characteristic equation has a repeated root $r_1 = r_2 = r_0$, we have only one purely exponential solution available, namely $y_1(t) = e^{r_0 t}$. To find a second one that is not simply a constant multiple of y_1, we try a function of the form $y_2(t) = u(t)e^{r_0 t}$, where u is determined by asking y_2 to satisfy the equation. Since

$$ar_0^2 + br_0 + c = 0 \quad \text{and} \quad r_0 = -\frac{b}{2a}$$

when r_0 is a repeated root, and since

$$\begin{aligned} y_2' &= u'e^{r_0 t} + r_0 u e^{r_0 t}, \\ y_2'' &= u''e^{r_0 t} + 2r_0 u' e^{r_0 t} + r_0^2 u e^{r_0 t}, \end{aligned}$$

we have

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(u'' + 2r_0 u' + r_0^2 u)e^{r_0 t} + b(u' + r_0 u)e^{r_0 t} + cue^{r_0 t} \\ &= [au'' + (2ar_0 + b)u' + (ar_0^2 + br_0 + c)u]e^{r_0 t} = au''e^{r_0 t}, \end{aligned}$$

which is zero if $u''(t) = 0$; that is, if $u(t) = \alpha_0 t + \alpha_1$, where α_0 and α_1 are arbitrary numbers. The constant term in u reproduces the solution y_1, so we discard it. Given that we need only one solution, we take $\alpha_0 = 1$ and end up with $y_2 = te^{r_0 t}$. Then

$$W[y_1, y_2](t) = \begin{vmatrix} e^{r_0 t} & te^{r_0 t} \\ r_0 e^{r_0 t} & (1 + r_0 t)e^{r_0 t} \end{vmatrix} = (1 + r_0 t)e^{2r_0 t} - r_0 t e^{2r_0 t} = e^{2r_0 t} \neq 0,$$

so $\{y_1, y_2\}$ is an FSS for the DE, yielding the GS

$$y(t) = c_1 e^{r_0 t} + c_2 t e^{r_0 t} = (c_1 + c_2 t)e^{r_0 t}. \tag{4.10}$$

4.34 Example. The characteristic equation for the IVP

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -8$$

is $r^2 + 4r + 4 = 0$, with roots $r_1 = r_2 = -2$. Consequently, the GS of the DE is

$$y(t) = (c_1 + c_2 t)e^{-2t}.$$

Using the ICs, we find that $c_1 = 1$ and $c_2 = -6$, which leads to the solution

$$y(t) = (1 - 6t)e^{-2t}.$$

The graph of the solution for $t \geq 0$ is shown in Fig. 4.3.

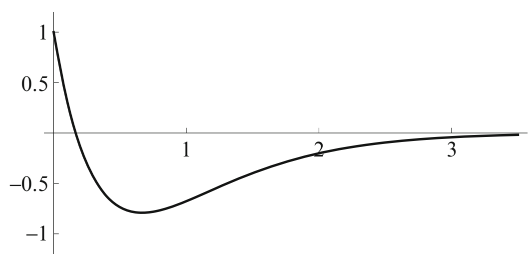

Fig. 4.3

VERIFICATION WITH MATHEMATICA®. The input

```
y = (1 - 2*t)*E^(-2*t);
{D[y,t,t] + 4*D[y,t] + 4*y, {y, D[y,t]}/. t→ 0}//Simplify
```

generates the output $\{0, \{1, -4\}\}$. ■

Reduction of order. The characteristic equation method does not work for a linear DE with variable coefficients. Sometimes, however, when by some means or other we manage to find a PS y_1 of such an equation, the technique described above may help us compute a second one, and, thus, the GS.

4.35 Example. It is easy to verify that the DE

$$2t^2y'' + ty' - 3y = 0$$

admits the solution $y_1 = t^{-1}$. To find a second, linearly independent, solution, we set $y_2(t) = u(t)y_1(t) = t^{-1}u(t)$. Then

$$y_2' = -t^{-2}u + t^{-1}u', \quad y_2'' = 2t^{-3}u - 2t^{-2}u' + t^{-1}u'',$$

which, replaced in the DE, lead to

$$2tu'' - 3u' = 0.$$

Setting first $u' = v$, we arrive at the separable equation $2tv' = 3v$, with solution $v(t) = u'(t) = t^{3/2} + C$. Since we need only one solution, we discard the constant and integrate again to obtain $u(t) = (2/5)t^{5/2}$. Hence, we may take $y_2(t) = u(t)y_1(t) = t^{3/2}$. (The coefficient 2/5 is unnecessary because y_2 is multiplied by an arbitrary constant in the GS.) The Wronskian of y_1 and y_2 is

$$W[y_1, y_2](t) = \begin{vmatrix} t^{-1} & t^{3/2} \\ -t^{-2} & \frac{3}{2}\,t^{1/2} \end{vmatrix} = \tfrac{3}{2}\,t^{-1/2} + t^{-1/2} = \tfrac{5}{2}\,t^{-1/2} \neq 0$$

for any $t \neq 0$, so $\{y_1, y_2\}$ is an FSS, which means that the GS of the given DE is

$$y(t) = c_1y_1(t) + c_2y_2(t) = c_1t^{-1} + c_2t^{3/2}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = c1*t^(-1) + c2*t^(3/2);
2*t^2*D[y,t,t] + t*D[y,t] - 3*y//Simplify
```

generates the output 0. ■

Exercises

In 1–4, find the solution $y(t)$ of the given IVP.

1 $y'' + 6y' + 9y = 0, \quad y(0) = -3, \quad y'(0) = 11.$

2 $4y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = -9/2.$

3 $25y'' - 20y' + 4y = 0, \quad y(0) = 10, \quad y'(0) = 9.$

4 $y'' - 8y' + 16y = 0, \quad y(0) = 3, \quad y'(0) = 8.$

In 5 and 6, find the solution $y(x)$ of the given BVP.

5 $9y'' - 6y' + y = 0, \quad y(0) = 1, \quad y(1) = 3e^{1/3}.$

6 $y'' + 10y' + 25y = 0, \quad y(0) = -2, \quad y((\ln 2)/5) = -1 + (\ln 2)/10.$

In 7–12, use the method of reduction of order and the solution provided for the given DE to find a second, linearly independent, solution of the equation.

7 $t^2y'' + 2ty' - 6y = 0, \quad y_1(t) = t^{-3}.$

8 $3t^2y'' - 5ty' + 4y = 0, \quad y_1(t) = t^2.$

9 $t^2y'' + 3ty' + y = 0 \ (t > 0), \quad y_1(t) = t^{-1}.$

10 $4t^2y'' + y = 0 \ (t > 0), \quad y_1(t) = t^{1/2}.$

11 $(t - 1)y'' - (t + 1)y' + 2y = 0, \quad y_1(t) = e^t.$

12 $(t^2 + t)y'' - ty' + y = 0 \ (t > 0), \quad y_1(t) = t.$

Answers to Odd-Numbered Exercises

1 $y(t) = (2t - 3)e^{-3t}$. **3** $y(t) = 5(t + 2)e^{2t/5}$. **5** $y(x) = (2x + 1)e^{x/3}$.

7 $y_2(t) = t^2$. **9** $y_2(t) = t^{-1}\ln t$. **11** $y_2(t) = -(t^2 + 1)$.

4.4.3 Complex Conjugate Characteristic Roots

We recall the Taylor series expansions

$$\begin{aligned} e^\alpha &= 1 + \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \frac{\alpha^5}{5!} + \cdots, \\ \cos\alpha &= 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots, \\ \sin\alpha &= \frac{\alpha}{1!} - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots. \end{aligned}$$

Replacing $\alpha = i\theta$ in the first one and taking the other two and the fact that $i^2 = -1$ into account, we now obtain

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + \frac{\theta}{1!}i - \frac{\theta^2}{2!} - \frac{\theta^3}{3!}i + \frac{\theta^4}{4!} + \frac{\theta^5}{5!}i - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta. \end{aligned} \tag{4.11}$$

This equality is known as *Euler's formula.*

Suppose that the characteristic equation has complex conjugate roots

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu, \quad \mu \neq 0,$$

and consider the complex-valued solutions y_1 and y_2 defined by

$$\begin{aligned} y_1(t) &= e^{r_1 t} = e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}[\cos(\mu t) + i \sin(\mu t)], \\ y_2(t) &= e^{r_2 t} = e^{(\lambda - i\mu)t} = e^{\lambda t} e^{-i\mu t} = e^{\lambda t}[\cos(\mu t) - i \sin(\mu t)]. \end{aligned} \tag{4.12}$$

By (4.8),

$$W[y_1, y_2](t) = 2i\mu\, e^{2\lambda t} \neq 0$$

at any point t, so these functions form an FSS. However, since we are dealing exclusively with real DEs and real ICs or BCs, we want to write the GS of the equation in terms of a fundamental set of real-valued solutions. This is easily done if we notice, from (4.12), that the functions

$$\begin{aligned} u(t) &= \frac{1}{2}\, y_1(t) + \frac{1}{2}\, y_2(t) = e^{\lambda t} \cos(\mu t), \\ v(t) &= \frac{1}{2i}\, y_1(t) - \frac{1}{2i}\, y_2(t) = e^{\lambda t} \sin(\mu t) \end{aligned}$$

are real-valued and, by Theorem 4.21, are solutions of the DE. Also, since

$$\begin{aligned} u'(t) &= e^{\lambda t}[\lambda \cos(\mu t) - \mu \sin(\mu t)], \\ v'(t) &= e^{\lambda t}[\lambda \sin(\mu t) + \mu \cos(\mu t)], \end{aligned}$$

we have

$$\begin{aligned} W[u, v](t) &= \begin{vmatrix} e^{\lambda t} \cos(\mu t) & e^{\lambda t} \sin(\mu t) \\ e^{\lambda t}[\lambda \cos(\mu t) - \mu \sin(\mu t)] & e^{\lambda t}[\lambda \sin(\mu t) + \mu \cos(\mu t)] \end{vmatrix} \\ &= e^{2\lambda t}[\lambda \cos(\mu t) \sin(\mu t) + \mu \cos^2(\mu t) \\ &\qquad - \lambda \cos(\mu t) \sin(\mu t) + \mu \sin^2(\mu t)] = \mu e^{2\lambda t} \neq 0, \end{aligned}$$

which shows that $\{u,\, v\}$ is an FSS. Consequently, the real GS of the DE is

$$y(t) = c_1 u(t) + c_2 v(t) = e^{\lambda t}[c_1 \cos(\mu t) + c_2 \sin(\mu t)], \tag{4.13}$$

with real arbitrary constants c_1 and c_2.

4.36 Example. Consider the IVP

$$9y'' - 6y' + 10y = 0, \quad y(0) = 2, \quad y'(0) = \tfrac{11}{3}.$$

The roots of the characteristic equation $9r^2 - 6r + 10 = 0$ are $r_1 = 1/3 + i$ and $r_2 = 1/3 - i$, so, by (4.13) with $\lambda = 1/3$ and $\mu = 1$, the GS of the DE is

$$y(t) = e^{t/3}(c_1 \cos t + c_2 \sin t).$$

The constants c_1 and c_2, determined from the ICs, are found to be $c_1 = 2$ and $c_2 = 3$; hence, the solution of the IVP is

$$y(t) = e^{t/3}(2 \cos t + 3 \sin t).$$

The graph of the solution for $t \geq 0$ is shown in Fig. 4.4.

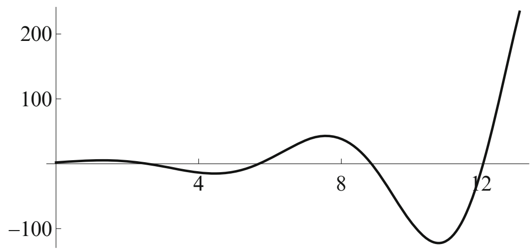

Fig. 4.4

Verification with Mathematica®. The input

```
y = E^(t/3) * (2 * Cos[t] + 3 * Sin[t]);
{9 * D[y,t,t] - 6 * D[y,t] + 10 * y, {y,D[y,t]}/. t → 0} // Simplify
```

generates the output {0, {2, 11/3}}. ■

Exercises

In 1–8, find the solution $y(t)$ of the given IVP.

1 $y'' + 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = -5.$

2 $y'' - 4y' + 13y = 0, \quad y(0) = 3, \quad y'(0) = 9.$

3 $y'' - 6y' + 10y = 0, \quad y(0) = 1/2, \quad y'(0) = -3/2.$

4 $y'' - 4y' + 20y = 0, \quad y(0) = 2/3, \quad y'(0) = -2/3.$

5 $4y'' - 4y' + 17y = 0, \quad y(0) = 2, \quad y'(0) = 13.$

6 $4y'' - 8y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = -1/2.$

7 $36y'' - 36y' + 13y = 0, \quad y(0) = 6, \quad y'(0) = 7.$

8 $9y'' - 6y' + 5y = 0, \quad y(0) = -3, \quad y'(0) = 3.$

In 9–16, discuss the existence of solutions $y(x)$ for the given DE with each set of BCs and compute the solutions where possible.

9 $y'' - 2y' + 5y = 0$, (i) $y(0) = 1, \quad y(\pi/4) = -2e^{\pi/4}$;
(ii) $y(0) = 1, \quad y(\pi/2) = -2e^{\pi/4}$; (iii) $y(0) = 1, \quad y(\pi) = e^{\pi}$.

10 $y'' - 4y' + 5y = 0$, (i) $y(0) = -2, \quad y(\pi/2) = -e^{\pi}$;
(ii) $y(0) = -2, \quad y(2\pi) = -2e^{4\pi}$; (iii) $y(0) = -2, \quad y(\pi) = -e^{\pi}$.

11 $y'' - 4y' + 8y = 0$, (i) $y'(0) = 8, \quad y(\pi/8) = -3$;
(ii) $y'(0) = 8, \quad y(\pi/4) = 3e^{\pi/2}$; (iii) $y'(0) = 8, \quad y(\pi/8) = 2^{3/2}e^{\pi/4}$.

12 $2y'' - 2y' + y = 0$, (i) $y'(0) = -1, \quad y(\pi/2) = -\sqrt{2}\, e^{\pi/4}$;
(ii) $y'(0) = -1, \quad y(\pi/2) = -2$; (iii) $y'(0) = -1, \quad y(\pi) = -4e^{\pi/2}$.

13 $y'' + 6y' + 18y = 0$, (i) $y(0) = -1, \quad y'(\pi/12) = 3\sqrt{2}\, e^{-\pi/4}$;
(ii) $y(0) = -1, \quad y'(\pi/6) = 9e^{-\pi/2}$; (iii) $y(0) = -1, \quad y'(\pi/12) = 1$.

14 $9y'' + 12y' + 8y = 0$, (i) $y(0) = 2$, $y'(3\pi/8) = 3$;
(ii) $y(0) = 2$, $y'(3\pi/8) = -(4\sqrt{2}/3)e^{-\pi/4}$; (iii) $y(0) = 2$, $y'(3\pi/4) = -e^{-\pi/2}$.

15 $9y'' - 12y' + 13y = 0$, (i) $y'(0) = 5/2$, $y'(\pi/2) = -(8/3)e^{\pi/3}$;
(ii) $y'(0) = 5/2$, $y'(\pi) = -(5/2)e^{2\pi/3}$; (iii) $y'(0) = 5/2$, $y'(\pi) = 1$.

16 $2y'' - 2y' + 5y = 0$, (i) $y'(0) = -1$, $y'(2\pi/3) = -2$;
(ii) $y'(0) = -1$, $y'(\pi/3) = -7e^{\pi/6}$; (iii) $y'(0) = -1$, $y'(2\pi/3) = -(5/2)e^{\pi/3}$.

Answers to Odd-Numbered Exercises

1 $y(t) = e^{-t}[\cos(2t) - 2\sin(2t)]$. **3** $y(t) = e^{3t}[(1/2)\cos t - 3\sin t]$.

5 $y(t) = e^{t/2}[2\cos(2t) + 6\sin(2t)]$. **7** $y(t) = e^{t/2}[6\cos(t/3) + 12\sin(t/3)]$.

9 (i) $y(x) = e^{x}[\cos(2x) - 2\sin(2x)]$; (ii) no solution;
(iii) $y(x) = e^{x}[\cos(2x) + c\sin(2x)]$, $c =$ const arbitrary.

11 (i) No solution; (ii) $y(x) = e^{2x}[\cos(2x) + 3\sin(2x)]$;
(iii) $y(x) = e^{2x}\{4\cos(2x) + c[\sin(2x) - \cos(2x)]\}$, $c =$ const arbitrary.

13 (i) $y(x) = e^{-3x}[-\cos(3x) + c\sin(3x)]$, $c =$ const arbitrary;
(ii) $y(x) = e^{3x}[-\cos(3x) - 2\sin(3x)]$; (iii) no solution.

15 (i) $y(x) = e^{2x/3}[3\cos x + (1/2)\sin x]$;
(ii) $y(x) = e^{2x/3}\{(15/4)\cos x + c[\sin x - (3/2)\cos x]\}$, $c =$ const arbitrary;
(iii) no solution.

4.5 Nonhomogeneous Equations

Using the operator notation introduced in Sect. 1.4, we can write the general linear, nonhomogeneous, second-order DE in the form

$$Ly = y'' + p(t)y' + q(t)y = f(t), \tag{4.14}$$

where the functions p, q, and f are prescribed.

4.37 Theorem. *The GS of equation* (4.14) *is*

$$y(t) = y_c(t) + y_p(t), \tag{4.15}$$

where y_c (called the complementary function*) is the GS of the associated homogeneous equation $Ly = 0$ and y_p is any PS of the full nonhomogeneous equation $Ly = f$.* ■

Proof. Let $\{y_1, y_2\}$ be an FSS for the associated homogeneous DE. Then, as we know from Sect. 4.3, $y_c = c_1y_1 + c_2y_2$ with arbitrary constants c_1 and c_2. Since L is a linear operator, it follows that for any numbers c_1 and c_2,

$$L(c_1y_1 + c_2y_2 + y_p) = L(c_1y_1 + c_2y_2) + Ly_p = 0 + f = f,$$

so any function of the form

$$y = c_1y_1 + c_2y_2 + y_p \tag{4.16}$$

is a solution of (4.14).

Now suppose that there is a solution $\tilde{y}$ of (4.14) that cannot be written in the form (4.16). As

$$L(\tilde{y} - y_p) = L\tilde{y} - Ly_p = f - f = 0,$$

we deduce that $\tilde{y} - y_p$ is a solution of the associated homogeneous equation, so there are $\tilde{c}_1$, $\tilde{c}_2 = \text{const}$ such that

$$\tilde{y} - y_p = \tilde{c}_1y_1 + \tilde{c}_2y_2;$$

hence,

$$\tilde{y} = \tilde{c}_1y_1 + \tilde{c}_2y_2 + y_p,$$

which contradicts our assumption.

In conclusion, every solution of (4.14) is of the form (4.16), confirming that (4.15) is the GS of the given nonhomogeneous equation. ■

Below, we restrict our attention to nonhomogeneous DEs with constant coefficients. Since we already know how to find y_c for such equations, we need to design a suitable procedure for computing y_p.

4.5.1 Method of Undetermined Coefficients: Simple Cases

This technique, based on educated guesses suggested by the nature of the nonhomogeneous term f, is best illustrated by examples.

4.38 Example. Consider the IVP

$$y'' - 2y' + y = 2t - 7, \quad y(0) = -5, \quad y'(0) = 1.$$

The roots of the characteristic equation $r^2 - 2r + 1 = 0$ are $r_1 = r_2 = 1$. Hence, by (4.10),

$$y_c = (c_1 + c_2t)e^t.$$

To find a PS y_p of the full equation, we notice that f is a first-degree polynomial, so we try y_p in the form of a first-degree polynomial with unknown coefficients; that is,

$$y_p(t) = \alpha_0t + \alpha_1.$$

Substituting in the DE, we arrive at

$$y_p'' - 2y_p' + y_p = \alpha_0t - 2\alpha_0 + \alpha_1 = 2t - 7.$$

We now equate the coefficients of t and the constant terms on both sides and obtain $\alpha_0 = 2$ and $-2\alpha_0 + \alpha_1 = -7$, from which $\alpha_1 = -3$; therefore, the GS of the DE is

$$y(t) = y_c(t) + y_p(t) = (c_1 + c_2t)e^t + 2t - 3.$$

The constants c_1 and c_2, determined from the ICs, are $c_1 = -2$ and $c_2 = 1$, so the solution of the IVP is

$$y(t) = (t - 2)e^t + 2t - 3.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = (t - 2) * E^t + 2 * t - 3;
{D[y,t,t] - 2 * D[y,t] + y - 2 * t + 7,{y,D[y,t]}/. t→ 0}//Simplify
```

generates the output $\{0, \{-5, 1\}\}$. ■

4.39 Example. The characteristic equation in the IVP

$$y'' + 3y' + 2y = -12e^{2t}, \quad y(0) = -2, \quad y'(0) = 2$$

is $r^2 + 3r + 2 = 0$, with roots $r_1 = -1$ and $r_2 = -2$; hence, by (4.9),

$$y_c(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

Since the right-hand side in the DE is an exponential, we try to find a function y_p of the same form; that is,

$$y_p(t) = \alpha e^{2t}.$$

From the DE we then obtain

$$y_p'' + 3y_p' + 2y = 4\alpha e^{2t} + 6\alpha e^{2t} + 2\alpha e^{2t} = 12\alpha e^{2t} = -12e^{2t},$$

so $\alpha = -1$, which yields the GS

$$y(t) = y_c(t) + y_p(t) = c_1 e^{-t} + c_2 e^{-2t} - e^{2t}.$$

Applying the ICs, we find that $c_1 = 2$ and $c_2 = -3$, so the solution of the IVP is

$$y(t) = 2e^{-t} - 3e^{-2t} - e^{2t}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = 2 * E^(-t) - 3 * E^(-2 * t) - E^(2 * t);
{D[y,t,t] + 3 * D[y,t] + 2 * y + 12 * E^(2 * t), {y,D[y,t]}/. t→ 0}
        //Simplify
```

generates the output $\{0, \{-2, 2\}\}$. ■

4.40 Example. In the IVP

$$y'' - 6y' + 8y = 40\sin(2t), \quad y(0) = 1, \quad y'(0) = -10,$$

the characteristic equation $r^2 - 6r + 8 = 0$ has roots $r_1 = 2$ and $r_2 = 4$, so, by (4.9),

$$y_c(t) = c_1 e^{2t} + c_2 e^{4t}.$$

The function f is a multiple of $\sin(2t)$. However, if we tried y_p in the form $\alpha\sin(2t)$, the DE would lead to

$$y_p'' - 6y_p' + 8y_p = 4\alpha\sin(2t) - 12\alpha\cos(2t) = 40\sin(2t),$$

and, matching the coefficients of the linearly independent functions $\sin(2t)$ and $\cos(2t)$ on both sides, we would get $\alpha = 10$ and $\alpha = 0$, which is impossible. To avoid this, we try y_p as a linear combination of both $\cos(2t)$ and $\sin(2t)$; that is,

$$y_p(t) = \alpha\cos(2t) + \beta\sin(2t).$$

Then, substituting in the DE, we find that

$$y_p'' - 6y_p' + 8y_p = (4\alpha - 12\beta)\cos(2t) + (12\alpha + 4\beta)\sin(2t) = 40\sin(2t),$$

and the matching of the coefficients now leads to the system

$$\begin{aligned} 4\alpha - 12\beta &= 0, \\ 12\alpha + \ \ 4\beta &= 40, \end{aligned}$$

from which $\alpha = 3$ and $\beta = 1$. Consequently, the GS of the DE is

$$y(t) = y_c(t) + y_p(t) = c_1 e^{2t} + c_2 e^{4t} + 3\cos(2t) + \sin(2t).$$

The constants c_1 and c_2 are determined from the ICs; they are $c_1 = 2$ and $c_2 = -4$, so the solution of the IVP is

$$y(t) = 2e^{2t} - 4e^{4t} + 3\cos(2t) + \sin(2t).$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=2*E^(2*t) - 4 E^(4*t) + 3*Cos[2*t] + Sin[2*t];
{D[y,t,t] - 6*D[y,t] + 8*y - 40*Sin[2*t],{y,D[y,t]}/.t→ 0}
        //Simplify
```

generates the output $\{0, \{1, -10\}\}$. ■

4.41 Example. Consider the IVP

$$y'' + 2y' + 2y = (5t - 11)e^t, \quad y(0) = -1, \quad y'(0) = -3.$$

The characteristic equation $r^2 + 2r + 2 = 0$ has roots $r_1 = -1 + i$ and $r_2 = -1 - i$, which means that, by (4.13),

$$y_c(t) = e^{-t}(c_1 \cos t + c_2 \sin t).$$

Given the form of f on the right-hand side of the DE, we try

$$y_p(t) = (\alpha_0 t + \alpha_1)e^t.$$

Then, differentiating this expression, replacing it and its derivatives in the equation, and sorting out the terms, we arrive at

$$y_p'' + 2y_p' + 2y_p = (5\alpha_0 t + 4\alpha_0 + 5\alpha_1)e^t = (5t - 11)e^t.$$

In Sect. 4.4.2 it was shown that te^t and e^t are linearly independent functions, so, equating their coefficients on both sides, we obtain $5\alpha_0 = 5$ and $4\alpha_0 + 5\alpha_1 = -11$, from which $\alpha_0 = 1$ and $\alpha_1 = -3$. Thus, the GS solution of the DE is

$$y(t) = y_c(t) + y_p(t) = e^{-t}(c_1 \cos t + c_2 \sin t) + (t - 3)e^t.$$

We now apply the ICs and find that $c_1 = 2$ and $c_2 = 1$, which means that the solution of the IVP is

$$y(t) = e^{-t}(2\cos t + \sin t) + (t - 3)e^t.$$

Verification with Mathematica®. The input

```
y = E^(-t) * (2 * Cos[t] + Sin[t]) + (t - 3) * E^t;
{D[y,t,t] + 2 * D[y,t] + 2 * y - (5 * t - 11) * E^t,
        {y,D[y,t]} /. t → 0} // Simplify
```

generates the output $\{0, \{-1, -3\}\}$. ■

4.42 Example. The IVP

$$y'' + 2y' + y = -24e^t \cos(2t), \quad y(0) = -2, \quad y'(0) = -3$$

is handled similarly. The roots of the characteristic equation are $r_1 = r_2 = -1$, so, by (4.10),

$$y_c(t) = (c_1 + c_2 t)e^{-t}.$$

Recalling the argument used in Example 4.40, we seek a PS of the form

$$y_p(t) = e^t[\alpha \cos(2t) + \beta \sin(2t)]$$

and, in the usual way, arrive at the equality

$$e^t[8\beta \cos(2t) - 8\alpha \sin(2t)] = -24e^t \cos(2t).$$

We know from Sect. 4.4.3 that functions such as $e^t \cos(2t)$ and $e^t \sin(2t)$ are linearly independent, so, equating their coefficients on both sides, we find that $\alpha = 0$ and $\beta = -3$. Consequently, the GS of the IVP is

$$y(t) = y_c(t) + y_p(t) = (c_1 + c_2 t)e^{-t} - 3e^t \sin(2t).$$

The ICs now produce the values $c_1 = -2$ and $c_2 = 1$, which means that the solution of the IVP is

$$y(t) = (t - 2)e^{-t} - 3e^t \sin(2t).$$

Verification with Mathematica®. The input

```
y = (t - 2) * E^( - t) - 3 * E^t * Sin[2 * t];
{D[y,t,t] + 2 * D[y,t] + y + 24 * E^t * Cos[2 * t],
        {y,D[y,t]} /. t → 0} // Simplify
```

generates the output $\{0, \{-2, -3\}\}$. ■

4.43 Example. The characteristic equation for the IVP

$$3y'' - 4y' + y = 50t \cos t, \quad y(0) = 0, \quad y'(0) = 0$$

is $3r^2 - 4r + 1 = 0$, with roots $r_1 = 1/3$ and $r_2 = 1$; therefore, by (4.9), we have

$$y_c(t) = c_1 e^{t/3} + c_2 e^t.$$

To find a PS, we note that the right-hand side of the DE is a product of a first-degree polynomial and a cosine. In this case, we try a function of the form

$$y_p(t) = (\alpha_0 t + \alpha_1) \cos t + (\beta_0 t + \beta_1) \sin t,$$

which, replaced in the equation, leads to the equality

$$[(-2\alpha_0 - 4\beta_0)t - 4\alpha_0 + 6\beta_0 - 2\alpha_1 - 4\beta_1]\cos t \\ + [(4\alpha_0 - 2\beta_0)t - 6\alpha_0 - 4\beta_0 + 4\alpha_1 - 2b_1]\sin t = 50t\cos t.$$

Using the Wronskian, we now convince ourselves that the functions $t\cos t$, $\cos t$, $t\sin t$, and $\sin t$ are linearly independent; hence, equating their coefficients on both sides above, we obtain the set of four equations

$$\begin{aligned} -2\alpha_0 - 4\beta_0 &= 50, \quad & -4\alpha_0 + 6\beta_0 - 2\alpha_1 - 4\beta_1 &= 0,\\ 4\alpha_0 - 2\beta_0 &= 0, \quad & -6\alpha_0 - 4\beta_0 + 4\alpha_1 - 2\beta_1 &= 0, \end{aligned}$$

from which $\alpha_0 = -5$, $\beta_0 = -10$, $\alpha_1 = -18$, and $\beta_1 = -1$, so the GS of the full DE is

$$y(t) = y_c(t) + y_p(t) = c_1 e^{t/3} + c_2 e^t - (5t + 18)\cos t - (10t + 1)\sin t.$$

Finally, using the ICs, we find that $c_1 = 18$ and $c_2 = 0$; therefore,

$$y(t) = 18e^{t/3} - (5t + 18)\cos t - (10t + 1)\sin t.$$

Verification with Mathematica®. The input

```
y=18*E^(t/3) - (5*t + 18)*Cos[t] - (10*t + 1)*Sin[t];
{3*D[y,t,t] - 4*D[y,t] + y - 50*t*Cos[t],{y,D[y,t]}/.t→ 0}
        //Simplify
```

generates the output {0, {0, 0}}. ■

4.44 Example. For the BVP

$$y'' + 4y' + 5y = -4\cos x - 4\sin x, \quad y'(0) = -3, \quad y(\pi) = -e^{-2\pi}$$

we have $r_1 = -2 + i$, $r_2 = -2 - i$, so, by (4.13),

$$y_c(x) = e^{-2x}(c_1\cos x + c_2\sin x).$$

Trying a PS of the form

$$y_p(x) = \alpha\cos x + \beta\sin x$$

in the DE, we find in the usual way that $\alpha = 0$ and $\beta = -1$, so the GS of the DE is

$$y(x) = y_c(x) + y_p(x) = e^{-2x}(c_1\cos x + c_2\sin x) - \sin x.$$

Since, as is easily seen,

$$y'(x) = e^{-2x}[(c_2 - 2c_1)\cos x - (c_1 + 2c_2)\sin x],$$

the BCs now yield $c_1 = 1$ and $c_2 = 0$, which means that

$$y(x) = e^{-2x}\cos x - \sin x.$$

Verification with Mathematica®. The input

```
y=E^(-2*x)*Cos[x] - Sin[x];
{D[y,x,x] + 4*D[y,x] + 5*y + 4*Cos[x] + 4*Sin[x],
        {D[y,x]/.x→ 0, y/.x→Pi}}//Simplify
```

generates the output $\{0, \{-3, -e^{-2\pi}\}\}$. ■

4.45 Example. The characteristic roots for the DE in the BVP

$$4y'' - y = 32 + x - 4x^2, \quad y(0) = 2, \quad y'(2) = \tfrac{29}{2}$$

are $r_1 = 1/2$ and $r_2 = -1/2$. By analogy with the comment made in Example 4.33, here it is more advantageous to write y_c as a combination of hyperbolic functions. In view of the given BCs, we choose the (admissible) form

$$y_c(x) = c_1 \sinh\left(\tfrac{1}{2}\,x\right) + c_2 \cosh\left(\tfrac{1}{2}\,(x-2)\right).$$

For a PS, we try a quadratic polynomial

$$y_p(x) = \alpha_0 x^2 + \alpha_1 x + \alpha_2,$$

which, when replaced in the equation, produces the coefficients $\alpha_0 = 4$, $\alpha_1 = -1$, and $\alpha_2 = 0$, so the GS is

$$y(x) = y_c(x) + y_p(x) = c_1 \sinh\left(\tfrac{1}{2}\,x\right) + c_2 \cosh\left(\tfrac{1}{2}\,(x-2)\right) + 4x^2 - x.$$

Using the fact that

$$y'(x) = \tfrac{1}{2}\,c_1 \cosh\left(\tfrac{1}{2}\,x\right) + \tfrac{1}{2}\,c_2 \sinh\left(\tfrac{1}{2}\,(x-2)\right) + 8x - 1,$$

we apply the BCs and arrive at the pair of simple equations

$$c_2 \cosh 1 = 2, \quad \tfrac{1}{2}\,c_1 \cosh 1 + 15 = \tfrac{29}{2},$$

from which $c_1 = -\operatorname{sech} 1$ and $c_2 = 2\operatorname{sech} 1$. Hence, the solution of the BVP is

$$y(x) = (\operatorname{sech} 1)\left[-\sinh\left(\tfrac{1}{2}\,x\right) + 2\cosh\left(\tfrac{1}{2}\,(x-2)\right)\right] + 4x^2 - x.$$

Verification with Mathematica®. The input

```
y = Sech[1] * (- Sinh[x/2] + 2 * Cosh[(x - 2)/2]) + 4 * x^2 - x;
{4 * D[y,x,x] - y - 32 - x + 4 * x^2, {y/. x→ 0, D[y,x] /. x→ 2}}
        // Simplify
```

generates the output {0, {2, 29/2}}. ■

Exercises

In 1–30, use the method of undetermined coefficients to find a particular solution of the DE and then solve the given IVP.

1 $y'' + 3y' + 2y = 8t + 8, \quad y(0) = -1, \quad y'(0) = 0.$

2 $y'' + y' - 6y = 14t - 6t^2, \quad y(0) = 4, \quad y'(0) = -9.$

3 $y'' - 8y' + 16y = 6(8t^2 - 8t + 1), \quad y(0) = -2, \quad y'(0) = -7.$

4 $y'' + 6y' + 9y = 9t - 39, \quad y(0) = -6, \quad y'(0) = 6.$

5 $y'' - 2y' + 2y = 2(t^2 - 5t + 6), \quad y(0) = 3, \quad y'(0) = 0.$

6 $y'' - 2y' + 5y = 5t^3 - 6t^2 - 4t + 4, \quad y(0) = 3/2, \quad y'(0) = -9/2.$

7 $y'' + 2y' - 8y = -16e^{-2t}, \quad y(0) = 3, \quad y'(0) = 4.$

8 $2y'' - 7y' + 3y = (7/2)e^{4t}, \quad y(0) = 3, \quad y'(0) = 7.$

9 $9y'' + 12y' + 4y = 2e^{-t}, \quad y(0) = -1, \quad y'(0) = 1.$

10 $y'' - 6y' + 9y = -25e^{-2t}, \quad y(0) = 1, \quad y'(0) = 10.$

11 $4y'' - 4y' + 5y = 58e^{3t}, \quad y(0) = 4, \quad y'(0) = 13/2.$

12 $4y'' - 16y' + 17y = (1/2)e^{2t}, \quad y(0) = 7/6, \quad y'(0) = 17/6.$

13 $y'' + 2y' - 3y = 8\cos(2t) - 14\sin(2t), \quad y(0) = 4, \quad y'(0) = -4.$

14 $y'' - 6y' + 8y = 85\cos t, \quad y(0) = 11, \quad y'(0) = 8.$

15 $y'' + 2y' + y = 25\sin(2t), \quad y(0) = 0, \quad y'(0) = 0.$

16 $9y'' + 6y' + y = (53/8)\cos(t/2) - \sin(t/2), \quad y(0) = -7/2, \quad y'(0) = 3.$

17 $y'' - 6y' + 10y = (89/6)\cos(t/3) + 3\sin(t/3), \quad y(0) = 1/2, \quad y'(0) = -2.$

18 $4y'' - 8y' + 5y = -17\cos t - 6\sin t, \quad y(0) = 1, \quad y'(0) = 5.$

19 $2y'' + y' - y = -18(t+1)e^{2t}, \quad y(0) = -1, \quad y'(0) = -4.$

20 $3y'' - 7y' + 2y = (25 - 12t)e^{-t}, \quad y(0) = 6, \quad y'(0) = 3.$

21 $y'' - 4y' + 4y = (t-4)e^{t}, \quad y(0) = -4, \quad y'(0) = -4.$

22 $y'' - 2y' + 5y = (65t - 56)e^{-2t}, \quad y(0) = -3, \quad y'(0) = 2.$

23 $4y'' - 5y' + y = 2e^{-t}(-13\cos t + 6\sin t), \quad y(0) = -3, \quad y'(0) = 2.$

24 $y'' - 2y' + y = -e^{2t}[3\cos(2t) + 4\sin(2t)], \quad y(0) = 1, \quad y'(0) = 5.$

25 $4y'' + 4y' + y = e^{-2t}(29\cos t + 2\sin t), \quad y(0) = 2, \quad y'(0) = -13/2.$

26 $4y'' - 4y' + 17y = -e^{t}[\cos(2t) - 8\sin(2t)], \quad y(0) = 0, \quad y'(0) = 15/2.$

27 $y'' - y = 4\cos(2t) - 5t\sin(2t), \quad y(0) = -1, \quad y'(0) = 1.$

28 $y'' - 2y' + y = 2[(2t-3)\cos t + (t+1)\sin t], \quad y(0) = -2, \quad y'(0) = -1.$

29 $y'' + y = 3(t+1)\cos(2t) + 4\sin(2t), \quad y(0) = 0, \quad y'(0) = -1.$

30 $y'' - 2y' + 2y = (8 - 7t)\cos(t/2) + 4(1-t)\sin(t/2), \quad y(0) = 0, \quad y'(0) = -2.$

In 31–38, use the method of undetermined coefficients to find a particular solution of the DE and then solve the given BVP.

31 $y'' + 4y' + 3y = 30e^{2x}, \quad y(0) = 3, \quad y(\ln 2) = 17/2.$

32 $y'' + 5y' + 6y = -(15/2)e^{-x/2}, \quad y(-\ln 2) = 4 - 2^{3/2}, \quad y'(0) = 2.$

33 $y'' - 4y = 4x, \quad y'(0) = 1, \quad y(1) = 0.$

34 $9y'' - 4y = 18(9 - 2x^2), \quad y'(0) = 2/3, \quad y'(3/2) = 85/3.$

35 $y'' - 4y' + 4y = (2\pi^2 - 8)\cos(\pi x) - 8\pi\sin(\pi x), \quad y(0) = 1, \quad y(3) = 2.$

36 $9y'' + 6y' + y = x^2 + 9x + 2, \quad y'(0) = 1/3, \quad y(2) = 0.$

37 $y'' - 4y' + 5y = -9\cos(2x) - 7\sin(2x), \quad y'(0) = -3, \quad y(\pi) = -1 + 2e^{2\pi}.$

38 $y'' + 2y' + 10y = 13e^{x}, \quad y'(0) = 3/2, \quad y'(\pi/3) = e^{\pi/3} - (1/2)e^{-\pi/3}.$

Answers to Odd-Numbered Exercises

1 $y_p(t) = 4t - 2, \quad y(t) = 3e^{-2t} - 2e^{-t} + 4t - 2.$

3 $y_p(t) = 3t^2$, $y(t) = (t-2)e^{4t} + 3t^2$.

5 $y_p(t) = t^2 - 3t + 2$, $y(t) = e^t(\cos t + 2\sin t) + t^2 - 3t + 2$.

7 $y_p(t) = 2e^{-2t}$, $y(t) = 2e^{2t} - e^{-4t} + 2e^{-2t}$.

9 $y_p(t) = 2e^{-t}$, $y(t) = (t-3)e^{-2t/3} + 2e^{-t}$.

11 $y_p(t) = 2e^{3t}$, $y(t) = e^{t/2}[2\cos t - (1/2)\sin t] + 2e^{3t}$.

13 $y_p(t) = 2\sin(2t)$, $y(t) = 3e^{-3t} + e^t + 2\sin(2t)$.

15 $y_p(t) = -4\cos(2t) - 3\sin(2t)$, $y(t) = (10t+4)e^{-t} - 4\cos(2t) - 3\sin(2t)$.

17 $y_p(t) = (3/2)\cos(t/3)$, $y(t) = e^{3t}(-\cos t + \sin t) + (3/2)\cos(t/3)$.

19 $y_p(t) = -2te^{2t}$, $y(t) = e^{-t} - 2e^{t/2} - 2te^{2t}$.

21 $y_p(t) = (t-2)e^t$, $y(t) = (t-2)(e^{2t} + e^t)$.

23 $y_p(t) = 2e^{-t}\sin t$, $y(t) = e^t - 4e^{t/4} + 2e^{-t}\sin t$.

25 $y_p(t) = e^{-2t}(\cos t - 2\sin t)$, $y(t) = (1-2t)e^{-t/2} + e^{-2t}(\cos t - 2\sin t)$.

27 $y_p(t) = t\sin(2t)$, $y(t) = -e^{-t} + t\sin(2t)$.

29 $y_p(t) = -(t+1)\cos(2t)$, $y(t) = \cos t - (t+1)\cos(2t)$.

31 $y_p(x) = 2e^{2x}$, $y(x) = e^{-x} + 2e^{2x}$.

33 $y_p(x) = -x$, $y(x) = (\operatorname{sech} 2)[\cosh(2x) + \sinh(2(x-1))] - x$.

35 $y_p(x) = -2\cos(\pi x)$, $y(x) = (3-x)e^{2x} - 2\cos(\pi x)$.

37 $y_p(x) = -\cos(2x) + \sin(2x)$, $y(x) = e^{2x}(-2\cos x - \sin x) - \cos(2x) + \sin(2x)$.

4.5.2 Method of Undetermined Coefficients: General Case

The technique described in the preceding subsection does not always work, even when the nonhomogeneous term looks quite simple.

4.46 Example. Consider the IVP

$$y'' + 3y' + 2y = 4e^{-t}, \quad y(0) = -1, \quad y'(0) = 8,$$

where, as we have seen in Example 4.39, $y_c(t) = c_1 e^{-t} + c_2 e^{-2t}$. The right-hand side in the above DE is again an exponential function, so we might be tempted to follow the procedure used in that example and seek a PS of the form $y_p(t) = \alpha e^{-t}$. But, since e^{-t} is a component of the complementary function—and, therefore, a solution of the associated homogeneous equation—such a choice would lead to the equality $0 = 4e^{-t}$, which is untrue. This means that our guess of y_p is incorrect and needs to be modified. ■

The next assertion sets out the generic form of a PS for nonhomogeneous terms of a certain structure.

4.47 Theorem. *If the nonhomogeneous term f in a linear DE with constant coefficients is of the form*

$$f(t) = e^{\lambda t}[P_n(t)\cos(\mu t) + Q_n(t)\sin(\mu t)], \tag{4.17}$$

where P_n and Q_n are polynomials of degree n and $\lambda + i\mu$ is a root of the characteristic equation of multiplicity s (that is, a root repeated s times), then the full equation has a PS of the form

$$y_p(t) = t^s e^{\lambda t}[A_n(t)\cos(\mu t) + B_n(t)\sin(\mu t)], \tag{4.18}$$

where A_n and B_n are polynomials of degree n. ■

4.48 Remark. Formula (4.18) remains valid if one of the polynomials P and Q is of degree n and the other one is of a lower degree. ■

4.49 Example. Applying Theorem 4.47, we can now solve the IVP in Example 4.46. The right-hand side of the DE in that IVP is as in (4.17), with $\lambda = -1$, $\mu = 0$, and $n = 0$ (since P_n is a constant). Here, $\lambda + i\mu = -1$ is a simple root of the characteristic equation, so $s = 1$. Then the PS suggested by (4.18) with $A_0(t) = \alpha$ is

$$y_p(t) = \alpha t e^{-t},$$

which, replaced in the DE, gives

$$y_p'' + 3y_p' + 2y_p = \alpha(-2e^{-t} + te^{-t}) + 3\alpha(e^{-t} - te^{-t}) + 2\alpha te^{-t} = \alpha e^{-t} = 4e^{-t}.$$

Thus, $\alpha = 4$, and the GS of the IVP is

$$y(t) = y_c(t) + y_p(t) = c_1e^{-t} + c_2e^{-2t} + 4te^{-t} = (4t + c_1)e^{-t} + c_2e^{-2t}.$$

When the ICs are implemented, it turns out that $c_1 = 2$ and $c_2 = -3$; consequently, the solution of the IVP is

$$y(t) = (4t + 2)e^{-t} - 3e^{-2t}.$$

Verification with Mathematica®. The input

```
y = (4 * t + 2) * E^(-t) - 3 * E^(-2 * t);
{D[y,t,t] + 3 * D[y,t] + 2 * y - 4 * E^(-t), {y,D[y,t]} /. t → 0}
        // Simplify
```

generates the output $\{0, \{-1, 8\}\}$. ■

4.50 Example. The characteristic equation for the DE

$$y'' - 4y' + 4y = (12t - 6)e^{2t}$$

is $r^2 - 4r + 4 = 0$, with roots $r_1 = r_2 = 2$, and the nonhomogeneous term is as in (4.17) with $\lambda = 2$, $\mu = 0$, and $n = 1$. Since $\lambda + i\mu = 2$ is a double root of the characteristic equation, it follows that $s = 2$, so, in accordance with (4.18), we seek a PS of the form

$$y_p(t) = t^2(\alpha_0 t + \alpha_1)e^{2t} = (\alpha_0 t^3 + \alpha_1 t^2)e^{2t}.$$

Differentiating and substituting in the DE, in the end we find that

$$y_p'' - 4y_p' + 4y_p = (6\alpha_0 t + 2\alpha_1)e^{2t} = (12t - 6)e^{2t},$$

so $\alpha_0 = 2$ and $\alpha_1 = -3$, which means that the GS of the DE is

$$y(t) = y_c(t) + y_p(t) = (c_1 + c_2 t)e^{2t} + (2t^3 - 3t^2)e^{2t} = (2t^3 - 3t^2 + c_2 t + c_1)e^{2t}.$$

Verification with Mathematica®. The input

```
y = 2 * t^3 - 3 * t^2 + c2 * t + c1) * E^(2 * t);
D[y,t,t] - 4 * D[y,t] + 4 * y - (12 * t - 6) * E^(2 * t) // Simplify
```

generates the output 0. ■

4.51 Example. The characteristic equation for the DE

$$y'' + 3y' = 12t + 1$$

is $r^2 + 3r = 0$, with roots $r_1 = 0$ and $r_2 = -3$; hence,

$$y_c(t) = c_1 + c_2 e^{-3t}.$$

The nonhomogeneous term is as in (4.17), with $\lambda = 0$, $\mu = 0$, and $n = 1$. The number $\lambda + i\mu = 0$ is a simple root of the characteristic equation, so $s = 1$. Then, using (4.18), we seek a PS of the form

$$y_p(t) = t(\alpha_0 t + \alpha_1) = \alpha_0 t^2 + \alpha_1 t.$$

Replaced in the equation, this yields

$$y_p'' + 3y_p' = 6\alpha_0 t + 2\alpha_0 + 3\alpha_1 = 12t + 1,$$

from which $\alpha_0 = 2$ and $\alpha_1 = -1$; hence, the GS of the given DE is

$$y(t) = y_c(t) + y_p(t) = c_1 + c_2 e^{-3t} + 2t^2 - t.$$

Verification with Mathematica®. The input

```
y = c1 + c2 * E^(-3 * t) + 2 * t^2 - t;
D[y,t,t] + 3 * D[y,t] - 12 * t - 1 // Simplify
```

generates the output 0. ■

4.52 Example. The characteristic equation for the DE

$$y'' + 4y = (16t + 4)\cos(2t)$$

is $r^2 + 4 = 0$, with roots $r_1 = 2i$ and $r_2 = -2i$. The right-hand side term is as in (4.17), with $\lambda = 0$, $\mu = 2$, and $n = 1$. Since $\lambda + i\mu = 2i$ is a simple characteristic root, it follows that $s = 1$ and, by (4.18), we seek a PS of the form

$$\begin{aligned} y_p(t) &= t[(\alpha_0 t + \alpha_1)\cos(2t) + (\beta_0 t + \beta_1)\sin(2t)] \\ &= (\alpha_0 t^2 + \alpha_1 t)\cos(2t) + (\beta_0 t^2 + \beta_1 t)\sin(2t). \end{aligned}$$

Performing the necessary differentiation and replacing in the DE, we find that

$$\begin{aligned} y_p'' + 4y_p &= (8\beta_0 t + 2\alpha_0 + 4\beta_1)\cos(2t) + (-8\alpha_0 t - 4\alpha_1 + 2\beta_0)\sin(2t) \\ &= (16t + 4)\cos(2t). \end{aligned}$$

Since $t\cos(2t)$, $\cos(2t)$, $t\sin(2t)$, and $\sin(2t)$ are linearly independent functions (see Example 4.43), we equate their coefficients on both sides and obtain the algebraic system

$$8\beta_0 = 16, \quad 2\alpha_0 + 4\beta_1 = 4, \quad -8\alpha_0 = 0, \quad -4\alpha_1 + 2\beta_0 = 0,$$

with solution $\alpha_0 = 0$, $\alpha_1 = 1$, $\beta_0 = 2$, and $\beta_1 = 1$; hence, the GS of the DE is

$$\begin{aligned} y(t) = y_c(t) + y_p(t) &= c_1\cos(2t) + c_2\sin(2t) + t\cos(2t) + (2t^2 + t)\sin(2t) \\ &= (t + c_1)\cos(2t) + (2t^2 + t + c_2)\sin(2t). \end{aligned}$$

Verification with Mathematica®. The input

```
y = (t + c1) * Cos[2 * t] + (2 * t^2 + t + c2) * Sin[2 * t];
D[y,t,t] + 4 * y - (16 * t + 4) * Cos[2 * t] // Simplify
```

generates the output 0. ■

The next assertion is the *principle of superposition* for nonhomogeneous linear equations, which is a generalization of Theorem 4.21.

4.53 Theorem. *If L is a linear operator and $Ly_1 = f_1$ and $Ly_2 = f_2$, then*

$$L(y_1 + y_2) = f_1 + f_2. \ \blacksquare$$

This theorem allows us to guess a suitable form for the PS of a linear nonhomogeneous DE when the right-hand side is a sum of functions of the types considered above.

4.54 Example. For the DE

$$y'' + 2y' + y = t^2 + 3t + 2 - te^{-t} - 4te^{-t}\cos(2t)$$

we have $r_1 = r_2 = -1$ and $f = f_1 + f_2 + f_3$, where

$$f_1(t) = t^2 + 3t + 2, \quad f_2(t) = -te^{-t}, \quad f_3(t) = -4te^{-t}\cos(2t).$$

By Theorem 4.53, we seek a PS of the form $y_p = y_{p1} + y_{p2} + y_{p3}$, with each term in y_p constructed according to formula (4.18) for the corresponding term in f. All the relevant details required by Theorem 4.47 are gathered in Table 4.1.

Table 4.1

Term in f	n	$\lambda + i\mu$	s	Term in y_p
$t^2 + 3t + 2$	2	0	0	$\alpha_0 t^2 + \alpha_1 t + \alpha_2$
$-te^{-t}$	1	-1	2	$t^2(\beta_0 t + \beta_1)e^{-t}$
$-4te^{-t}\cos(2t)$	1	$-1 + 2i$	0	$e^{-t}[(\gamma_0 t + \gamma_1)\cos(2t) + (\delta_0 t + \delta_1)\sin(2t)]$

Consequently, our guess for a PS of the full equation is a function of the form

$$\begin{aligned} y_p(t) = \alpha_0 t^2 + \alpha_1 t + \alpha_2 + (\beta_0 t^3 + \beta_1 t^2)e^{-t} \\ + e^{-t}[(\gamma_0 t + \gamma_1)\cos(2t) + (\delta_0 t + \delta_1)\sin(2t)]. \end{aligned}$$

A long but straightforward computation now yields

$$y_p(t) = t^2 - t + 2 - \tfrac{1}{6}t^3 e^{-t} + e^{-t}[t\cos(2t) - \sin(2t)].$$

Verification with Mathematica®. The input

```
y = t^2 - t + 2 - (1/6) * t^3 * E^(-t) + E^(-t) * (t * Cos[2 * t]
        - Sin[2 * t]);
D[y,t,t] + 2 * D[y,t] + y - t^2-3 * t - 2 + t * E^(-t)
        + 4 * t * E^(-t) * Cos[2 * t] // Simplify
```

generates the output 0. ■

Exercises

In 1–38, use the method of undetermined coefficients in conjunction with Theorem 4.47 to find a particular solution of the DE, and then solve the given IVP.

1 $y'' - 5y' + 6y = 2e^{3t}, \quad y(0) = 1, \quad y'(0) = 4.$

2 $2y'' - 7y' + 3y = 20e^{t/2}, \quad y(0) = 1, \quad y'(0) = -6.$

3 $2y'' - y' - y = 6e^{-t/2}, \quad y(0) = 4, \quad y'(0) = -5/2.$

4 $3y'' + y' - 2y = -15e^{-t}, \quad y(0) = -2, \quad y'(0) = 5.$

5 $y'' - 2y' - 8y = 2(5 - 6t)e^{4t}, \quad y(0) = -3, \quad y'(0) = 8.$

6 $3y'' - 7y' + 2y = -2(5t + 2)e^{t/3}, \quad y(0) = -1, \quad y'(0) = -5/3.$

7 $2y'' - y' - 3y = (7 - 20t)e^{3t/2}, \quad y(0) = 3, \quad y'(0) = 10.$

8 $y'' - 6y' + 5y = 2(4t + 3)e^{5t}, \quad y(0) = -2, \quad y'(0) = -13.$

9 $y'' + 3y' = 6, \quad y(0) = 1, \quad y'(0) = -4.$

10 $2y'' - y' = -1, \quad y(0) = 2, \quad y'(0) = 0.$

11 $3y'' - 2y' = -6, \quad y(0) = -3/2, \quad y'(0) = 10/3.$

12 $y'' + 4y' = 2, \quad y(0) = -3, \quad y'(0) = 17/2.$

13 $y'' - y' = 7 - 6t, \quad y(0) = 4, \quad y'(0) = 3.$

14 $3y'' - y' = 2t - 8, \quad y(0) = 4, \quad y'(0) = 10/3.$

15 $4y'' + y' = 24 - 18t - 3t^2, \quad y(0) = 8, \quad y'(0) = -2.$

16 $2y'' - 3y' = -9t^2 + 18t - 10, \quad y(0) = 2, \quad y'(0) = 5.$

17 $y'' + 4y' + 4y = 6e^{-2t}, \quad y(0) = -2, \quad y'(0) = 5.$

18 $4y'' - 4y' + y = -16e^{t/2}, \quad y(0) = 1, \quad y'(0) = -3/2.$

19 $9y'' - 6y' + y = 9e^{t/3}, \quad y(0) = 3, \quad y'(0) = 0.$

20 $y'' - 8y' + 16y = (1/2)e^{4t}, \quad y(0) = 2, \quad y'(0) = 5.$

21 $16y'' - 24y' + 9y = 32(3t - 1)e^{3t/4}, \quad y(0) = 0, \quad y'(0) = -2.$

22 $4y'' + 12y' + 9y = 24(t - 1)e^{-3t/2}, \quad y(0) = -2, \quad y'(0) = 4.$

23 $y'' + 4y = 4\sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$

24 $4y'' + y = 4\sin(t/2) - 4\cos(t/2), \quad y(0) = 1, \quad y'(0) = -2.$

25 $y'' + 9y = 6[3\cos(3t) + \sin(3t)], \quad y(0) = 2, \quad y'(0) = 2.$

26 $9y'' + 4y = -12[\cos(2t/3) + 3\sin(2t/3)], \quad y(0) = 1, \quad y'(0) = 5/3.$

27 $y'' + y = -2\sin t + 2(1 - 2t)\cos t, \quad y(0) = 2, \quad y'(0) = 0.$

28 $y'' + 25y = 4\cos(5t) - 10(4t + 1)\sin(5t), \quad y(0) = 1, \quad y'(0) = -4.$

29 $9y'' + y = 6(3 - 2t)\cos(t/3) - 12t\sin(t/3), \quad y(0) = 2, \quad y'(0) = -8/3.$

30 $25y'' + 4y = 10(4t + 7)\cos(2t/5) + 10(9 - 4t)\sin(2t/5), \quad y(0) = 1, \quad y'(0) = 4.$

31 $y'' - 2y' + 2y = -4e^t\cos t, \quad y(0) = 3, \quad y'(0) = 3.$

32 $y'' + 2y' + 5y = 4e^{-t}[\cos(2t) + \sin(2t)], \quad y(0) = 0, \quad y'(0) = -5.$

33 $y'' - 4y' + 13y = 6e^{2t}\sin(3t), \quad y(0) = 1, \quad y'(0) = 10.$

34 $4y'' - 4y' + 5y = 8e^{t/2}(2\cos t + \sin t), \quad y(0) = 2, \quad y'(0) = -4.$

35 $y'' + 4y' + 8y = -2e^{-2t}[\cos(2t) + 2(1 - 2t)\sin(2t)], \quad y(0) = 2, \quad y'(0) = -11.$

36 $4y'' + 8y' + 5y = 8e^{-t}[8t\cos(t/2) + \sin(t/2)]$, $y(0) = 1$, $y'(0) = -1$.

37 $9y'' - 12y' + 13y = -18e^{2t/3}[(4t-1)\cos t + (2t+3)\sin t]$, $y(0) = 2$, $y'(0) = -11/3$.

38 $y'' - 6y' + 13y = 2e^{3t}[(8t+3)\cos(2t) + 2(3-2t)\sin(2t)]$, $y(0) = 0$, $y'(0) = -6$.

In 39–46, use Theorems 4.47 and 4.53 to set up the form of a particular solution for the given DE. Do not compute the undetermined coefficients.

39 $y'' - 4y' + 3y = t - 1 + te^t - 2t^2e^{-t} + 3\sin t$.

40 $6y'' - y' + 2y = (2t+1)e^t + 3e^{-t/2} - t\sin t + 4e^{-t}\cos(2t)$.

41 $3y'' - 2y' = t + 2 + 2t\cos(3t) + (t^2-1)e^{2t/3} - (1-2t)e^t$.

42 $y'' - 6y' = t^2 - 3e^{t/6} + (1-t^2)e^{6t} + 2te^t\sin t$.

43 $y'' - 6y' + 9y = 3 + (1-t)e^{-3t} + 2te^{3t} - e^{3t}\cos(2t)$.

44 $16y'' + 8y' + y = t + 4 + t^2e^{-t/4} + (1-2t^2)e^{t/4} - 2(t^2-1)e^{-t/4}\sin t$.

45 $y'' - 6y' + 10y = (2t-3)\sin t + t^2e^{3t} - 4e^t + 2e^{3t}\cos t$.

46 $9y'' - 18y' + 10y = 2t + te^{t/3} + 3e^t\cos t - (t-2)e^t\sin(t/3)$.

Answers to Odd-Numbered Exercises

1 $y_p(t) = 2te^{3t}$, $y(t) = e^{2t} + 2te^{3t}$.

3 $y_p(t) = -2te^{-t/2}$, $y(t) = (3-2t)e^{-t/2} + e^t$.

5 $y_p(t) = (2t-t^2)e^{4t}$, $y(t) = -3e^{-2t} + (2t-t^2)e^{4t}$.

7 $y_p(t) = (3t-2t^2)e^{3t/2}$, $y(t) = (4+3t-2t^2)e^{3t/2} - e^{-t}$.

9 $y_p(t) = 2t - 1$, $y(t) = 2e^{-3t} + 2t - 1$.

11 $y_p(t) = 3t - 2$, $y(t) = (1/2)e^{2t/3} + 3t - 2$.

13 $y_p(t) = 3t^2 - t$, $y(t) = 4e^t + 3t^2 - t$.

15 $y_p(t) = 3t^2 - t^3$, $y(t) = 8e^{-t/4} + 3t^2 - t^3$.

17 $y_p(t) = 3t^2e^{-2t}$, $y(t) = (3t^2 + t - 2)e^{-2t}$.

19 $y_p(t) = (1/2)t^2e^{t/3}$, $y(t) = [(1/2)t^2 - t + 3)e^{t/3}$.

21 $y_p(t) = (t^3 - t^2)e^{3t/4}$, $y(t) = (t^3 - t^2 - 2t)e^{3t/4}$.

23 $y_p(t) = -t\cos(2t)$, $y(t) = (1-t)\cos(2t) + \sin(2t)$.

25 $y_p(t) = -t[\cos(3t) - 3\sin(3t)]$, $y(t) = (2-t)\cos(3t) + (3t+1)\sin(3t)$.

27 $y_p(t) = (t-t^2)\sin t$, $y(t) = 2\cos t + (t-t^2)\sin t$.

29 $y_p(t) = (t^2 - 3t)\cos(t/3) - t^2\sin(t/3)$,
$y(t) = (t^2 - 3t + 2)\cos(t/3) + (1-t^2)\sin(t/3)$.

31 $y_p(t) = -2te^t\sin t$, $y(t) = e^t(3\cos t - 2t\sin t)$.

33 $y_p(t) = -te^{2t}\cos(3t)$, $y(t) = e^{2t}[(1-t)\cos(3t) + 3\sin(3t)]$.

35 $y_p(t) = (t-t^2)e^{-2t}\cos(2t)$, $y(t) = e^{-2t}[(2+t-t^2)\cos(2t) - 4\sin(2t)]$.

37 $y_p(t) = e^{2t/3}[(t^2+t)\cos t + (1-2t^2)\sin t]$,
$y(t) = e^{2t/3}[(t^2+t+2)\cos t - 2(1+t^2)\sin t]$.

39 $y_p(t) = \alpha_0 t + \alpha_1 + t(\beta_0 t + \beta_1)e^t + (\gamma_0 t^2 + \gamma_1 t + \gamma_2)e^{-t} + \delta\cos t + \varepsilon\sin t$.

41 $y_p(t) = t(\alpha_0 t + \alpha_1) + (\beta_0 t + \beta_1)\cos(3t) + (\gamma_0 t + \gamma_1)\sin(3t)$
$\qquad + t(\delta_0 t^2 + \delta_1 t + \delta_2)e^{2t/3} + (\varepsilon_0 t + \varepsilon_1)e^t.$

43 $y_p(t) = \alpha + (\beta_0 t + \beta_1)e^{-3t} + t^2(\gamma_0 t + \gamma_1)e^{3t} + e^{3t}[\delta\cos(2t) + \varepsilon\sin(2t)].$

45 $y_p(t) = (\alpha_0 t + \alpha_1)\cos t + (\beta_0 t + \beta_1)\sin t + (\gamma_0 t^2 + \gamma_1 t + \gamma_2)e^{3t} + \delta e^t$
$\qquad + te^{3t}(\varepsilon\cos t + \zeta\sin t).$

4.5.3 Method of Variation of Parameters

This technique is generally used to compute a PS of the full DE when the (simpler) method of undetermined coefficients does not work because the nonhomogeneous term f is not of the form (4.17).

Suppose that $\{y_1, y_2\}$ is an FSS for the differential equation $y'' + p(t)y' + q(t)y = f(t)$. We seek a PS of the form

$$y_p = u_1 y_1 + u_2 y_2, \tag{4.19}$$

where u_1 and u_2 are functions of t; then

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

Imposing the condition

$$u_1' y_1 + u_2' y_2 = 0, \tag{4.20}$$

we reduce the first derivative to the form

$$y_p' = u_1 y_1' + u_2 y_2',$$

from which

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''.$$

Replacing y_p, y_p', and y_p'' in the DE and taking into account that

$$y_1'' + py_1' + qy_1 = 0, \quad y_2'' + py_2' + qy_2 = 0,$$

we find that

$$\begin{aligned} y_p'' + py_p' + qy_p &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + p(u_1 y_1' + u_2 y_2') + q(u_1 y_1 + u_2 y_2) \\ &= u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) + u_1' y_1' + u_2' y_2' \\ &= u_1' y_1' + u_2' y_2' = f. \end{aligned}$$

This equation and (4.20) form the system

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f \end{aligned}$$

for the unknown functions u_1' and u_2'; hence, by Cramer's rule (see (4.1)),

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 f}{W[y_1, y_2]}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f}{W[y_1, y_2]}.$$

Integrating to find u_1 and u_2 and then replacing these functions in (4.19), we obtain the PS

$$y_p(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W[y_1,y_2](t)}\,dt + y_2(t)\int \frac{y_1(t)f(t)}{W[y_1,y_2](t)}\,dt. \tag{4.21}$$

4.55 Example. The characteristic equation for the DE

$$y'' + y = \tan t$$

is $r^2+1=0$, with roots $r_1=i$ and $r_2=-i$. Hence, $y_c(t)=c_1\cos t + c_2\sin t$, and we can apply (4.21) with $y_1(t)=\cos t$, $y_2(t)=\sin t$, and $f(t)=\tan t$. Since $W[y_1,y_2](t)=1$ (see Example 4.29), we have

$$\begin{aligned} y_p(t) &= -\cos t\int \sin t\tan t\,dt + \sin t\int \cos t\tan t\,dt \\ &= -\cos t\int \frac{\sin^2 t}{\cos t}\,dt + \sin t\int \sin t\,dt. \end{aligned}$$

The first integrand is rewritten as

$$\frac{\sin^2 t}{\cos t} = \frac{1-\cos^2 t}{\cos t} = \sec t - \cos t,$$

and, since

$$\begin{aligned} \int \sec t\,dt &= \int \frac{\sec t(\sec t + \tan t)}{\sec t + \tan t}\,dt = \int \frac{\sec^2 t + \sec t\tan t}{\sec t + \tan t}\,dt \\ &= \int \frac{(\sec t + \tan t)'}{\sec t + \tan t}\,dt = \ln|\sec t + \tan t| + C, \end{aligned}$$

we finally obtain

$$\begin{aligned} y_p(t) &= -(\cos t)(\ln|\sec t + \tan t| - \sin t) - \sin t\cos t \\ &= -(\cos t)\ln|\sec t + \tan t|, \end{aligned}$$

where, of course, no arbitrary constant of integration is necessary.

Verification with Mathematica®. The input

```
y= - Cos[t]*Log[Sec[t] + Tan[t]];
D[y,t,t] + y - Tan[t]//Simplify
```

generates the output 0. ■

4.56 Example. The characteristic equation for the DE

$$y'' - 4y' + 3y = 3t + 2$$

is $r^2-4r+3=0$, with roots $r_1=1$ and $r_2=3$, so an FSS for the corresponding homogeneous equation consists of $y_1(t)=e^t$ and $y_2(t)=e^{3t}$. Since

$$W[y_1,y_2] = \begin{vmatrix} e^t & e^{3t} \\ e^t & 3e^{3t} \end{vmatrix} = 3e^{4t} - e^{4t} = 2e^{4t},$$

from (4.21) with $f(t) = 3t + 2$ and integration by parts it follows that

$$\begin{aligned} y_p(t) &= -e^t \int \frac{(3t+2)e^{3t}}{2e^{4t}}\,dt + e^{3t} \int \frac{(3t+2)e^t}{2e^{4t}}\,dt \\ &= -\tfrac{1}{2}\,e^t \int (3t+2)e^{-t}\,dt + \tfrac{1}{2}\,e^{3t} \int (3t+2)e^{-3t}\,dt \\ &= \tfrac{1}{2}\,e^t(3t+5)e^{-t} - \tfrac{1}{2}\,e^{3t}(t+1)e^{-3t} = t+2. \end{aligned}$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = t + 2;
D[y,t,t] - 4*D[y,t] + 3*y - 3*t - 2//Simplify
```

generates the output 0. ■

4.57 Example. Applying the same treatment to the DE

$$y'' - 2y' + y = 4e^t,$$

we have $r_1 = r_2 = 1$, so $y_1(t) = e^t$, $y_2(t) = te^t$, $f(t) = 4e^t$, and $W[y_1, y_2](t) = e^{2t}$; then, by (4.21),

$$y_p(t) = -e^t \int \frac{4te^t}{e^{2t}}\,dt + te^t \int \frac{4e^t}{e^{2t}}\,dt = -2t^2e^t + 4t^2e^t = 2t^2e^t.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = 2*t^2*E^t;
D[y,t,t] - 2*D[y,t] + y - 4*E^t//Simplify
```

generates the output 0. ■

4.58 Remark. Obviously, if the DE is written as $a(t)y'' + b(t)y' + c(t)y = f(t)$ with $a \neq 0$, then division by the coefficient a brings it to the form considered at the beginning of this section so that formula (4.21) can be applied. ■

Exercises

In 1–26, use the method of variation of parameters to find a particular solution y_p of the given DE. Work from first principles, without applying formula (4.21) directly.

1 $y'' + 2y' - 3y = 9 - 9t$. **2** $y'' - y' - 6y = 4(3t - 4)$.

3 $y'' - 4y' + 4y = 4(5 + 2t)$. **4** $9y'' - 6y' + y = 19 - 3t$.

5 $y'' + y = -t^2 - 2$. **6** $4y'' + y = 2t^2 + t + 16$.

7 $2y'' - 5y' + 2y = 40e^{-2t}$. **8** $3y'' - 2y' - y = 4(t - 2)e^{-t}$.

9 $y'' - 2y' + 2y = e^t$. **10** $y'' + 2y' + 5y = 8e^{-t}$.

11 $25y'' - 20y' + 4y = -98e^{-t}$. **12** $y'' - 6y' + 9y = (5/4)(5t - 14)e^{t/2}$.

13 $2y'' - 3y' - 2y = -18e^{t/2}$. **14** $4y'' - y' - 3y = (22t + 19)e^{2t}$.

15 $y'' + 2y' + y = 2e^{-t}$. **16** $4y'' + 4y' + y = 8(3t - 2)e^{-t/2}$.

17 $y'' - 6y' + 10y = te^{3t}$. **18** $y'' + 2y' + 10y = 9(2 - t)e^{-t}$.

19 $y'' - 9y = 12e^{-3t}$. **20** $y'' - 2y' - 8y = 6e^{4t}$.

21 $y'' + 2y = 4$. **22** $3y'' - y' = 2(4 - t)$.

23 $y'' + 4y = 4\cos(2t)$. **24** $9y'' + 4y = -48\sin(2t/3)$.

25 $y'' - 2y' + y = t^{-3}e^t$. **26** $4y'' + 4y' + y = -8(t^{-2} + t^{-4})e^{-t/2}$.

In 27–30, verify that the given pair of functions form an FSS for the associated homogeneous DE, then, working from first principles, use the method of variation of parameters to find a particular solution y_p of the nonhomogeneous equation.

27 $(t + 1)y'' + ty' - y = (t + 1)^2, \quad y_1(t) = t, \quad y_2(t) = e^{-t}$.

28 $2ty'' - (t + 2)y' + y = -t^2, \quad y_1(t) = t + 2, \quad y_2(t) = e^{t/2}$.

29 $ty'' + (2t - 1)y' - 2y = 32t^3e^{2t}, \quad y_1(t) = 2t - 1, \quad y_2(t) = e^{-2t}$.

30 $(t - 2)y'' + (1 - t)y' + y = 2(t - 1)e^{-t}, \quad y_1(t) = t - 1, \quad y_2(t) = e^t$.

Answers to Odd-Numbered Exercises

1 $y_p(t) = 3t - 1$. **3** $y_p(t) = 2t + 3$. **5** $y_p(t) = -t^2$.

7 $y_p(t) = 2e^{-2t}$. **9** $y_p(t) = e^t$. **11** $y_p(t) = -2e^{-t}$.

13 $y_p(t) = 6e^{t/2}$. **15** $y_p(t) = t^2e^{-t}$. **17** $y_p(t) = te^{3t}$.

19 $y_p(t) = -(2t + 1/3)e^{-3t}$. **21** $y_p(t) = 2t - 1$.

23 $y_p(t) = (1/4)\cos(2t) + t\sin(2t)$. **25** $y_p(t) = (1/2)t^{-1}e^t$.

27 $y_p(t) = t^2 - t + 1$. **29** $y_p(t) = (2t - 1)^2e^{2t}$.

4.6 Cauchy–Euler Equations

The general form of this class of linear second-order DEs with variable coefficients is

$$at^2y'' + bty' + cy = 0, \tag{4.22}$$

where a, b, and c are constants.

Suppose that $t > 0$. To find an FSS for (4.22), we try a power-type solution; that is,

$$y(t) = t^r, \quad r = \text{const.}$$

Replacing in (4.22), we arrive at

$$[ar(r - 1) + br + c]t^r = 0,$$

which holds for all $t > 0$ if and only if the coefficient of t^r is zero:

$$ar^2 + (b - a)r + c = 0. \tag{4.23}$$

The quadratic equation (4.23) has two roots, r_1 and r_2, and it is clear that the form of the GS of (4.22) depends on the nature of these roots.

(i) If r_1 and r_2 are real and distinct, we construct the functions $y_1(t) = t^{r_1}$ and $y_2(t) = t^{r_2}$. Since their Wronskian

$$W[y_1, y_2](t) = \begin{vmatrix} t^{r_1} & t^{r_2} \\ r_1 t^{r_1-1} & r_2 t^{r_2-1} \end{vmatrix} = (r_2 - r_1)t^{r_1+r_2-1}$$

is nonzero for all $t > 0$, the pair $\{y_1, y_2\}$ is an FSS and we can write the GS of the DE as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 t^{r_1} + c_2 t^{r_2}.$$

4.59 Example. For the DE in the IVP

$$t^2 y'' - ty' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 7$$

we have $a = 1$, $b = -1$, and $c = -3$, so equation (4.23) is $r^2 - 2r - 3 = 0$, with roots $r_1 = 3$ and $r_2 = -1$. Hence, the GS of the DE is $y(t) = c_1 t^3 + c_2 t^{-1}$. Applying the ICs, we now obtain $c_1 = 2$ and $c_2 = -1$, which means that the solution of the IVP is

$$y(t) = 2t^3 - t^{-1},$$

with maximal interval of existence $0 < t < \infty$.

Verification with Mathematica®. The input

```
y=2*t^3 - 1/t;
{t^2*D[y,t,t] - t*D[y,t] - 3*y,{y,D[y,t]}/.t→ 1}//Simplify
```

generates the output $\{0, \{1, 7\}\}$. ■

(ii) If $r_1 = r_2 = r_0$, it is not difficult to verify (by direct replacement in the equation and use of the Wronskian) that $y_1(t) = t^{r_0}$ and $y_2(t) = t^{r_0} \ln t$ form an FSS. Then the GS of (4.22) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = (c_1 + c_2 \ln t)t^{r_0}.$$

4.60 Example. In the case of the IVP

$$t^2 y'' + 5ty' + 4y = 0, \quad y(1) = 2, \quad y'(1) = -5$$

we have $r^2 + 4r + 4 = 0$, so $r_1 = r_2 = -2$, which leads to the GS $y(t) = (c_1 + c_2 \ln t)t^{-2}$. Using the ICs, we find that $c_1 = 2$ and $c_2 = -1$; hence, the solution of the IVP is

$$y(t) = (2 - \ln t)t^{-2},$$

with maximal interval of existence $0 < t < \infty$.

Verification with Mathematica®. The input

```
y=(2 - Log[t])*t^(-2);
{t^2*D[y,t,t] + 5*t*D[y,t] + 3*y,{y,D[y,t]}/.t→ 1}
        //Simplify
```

generates the output $\{0, \{2, -5\}\}$. ■

(iii) If r_1 and r_2 are complex conjugate roots—that is, $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, where $\mu \neq 0$—we start with the complex FSS $\{t^{\lambda+i\mu}, t^{\lambda-i\mu}\}$ and then use the formula $a^b = e^{b \ln a}$ to form the real FSS $\{t^\lambda \cos(\mu \ln t), t^\lambda \sin(\mu \ln t)\}$. Hence, the real GS of the DE is

$$y(t) = t^\lambda [c_1 \cos(\mu \ln t) + c_2 \sin(\mu \ln t)].$$

4.61 Example. For the IVP

$$t^2y'' - 5ty' + 10y = 0, \quad y(1) = 1, \quad y'(1) = 0$$

we have $r^2 - 6r + 10 = 0$, so $r_1 = 3 + i$ and $r_2 = 3 - i$. The real GS is

$$y(t) = t^3[c_1 \cos(\ln t) + c_2 \sin(\ln t)],$$

with the two constants computed from the ICs as $c_1 = 1$ and $c_2 = -3$; therefore, the solution of the IVP is

$$y(t) = t^3[\cos(\ln t) - 3 \sin(\ln t)],$$

with maximal interval of existence $0 < t < \infty$.

VERIFICATION WITH MATHEMATICA®. The input

```
y = t^3 * (Cos[Log[t]] - 3 * Sin[Log[t]]);
{t^2 * D[y,t,t] - 5 * t * D[y,t] + 10 * y, {y,D[y,t]} /. t → 1}
      // Simplify
```

generates the output $\{0, \{1, 0\}\}$. ■

4.62 Remark. If $t < 0$, we substitute $t = -\tau$ and $y(t) = u(\tau)$ in the DE and bring it to the form

$$a\tau^2 u'' + b\tau u' + cu = 0,$$

which is the same as (4.22) with $\tau > 0$. ■

Exercises

Solve the given IVP.

1 $t^2y'' + ty' - 4y = 0, \quad y(1) = -2, \quad y'(1) = 8.$

2 $t^2y'' - 4ty' + 4y = 0, \quad y(-1) = 1, \quad y'(-1) = -10.$

3 $3t^2y'' - 2ty' - 2y = 0, \quad y(-1) = 5, \quad y'(-1) = -2/3.$

4 $t^2y'' + 5ty' + 3y = 0, \quad y(2) = -1, \quad y'(2) = -1/2.$

5 $6t^2y'' + 5ty' - y = 0, \quad y(1) = -2, \quad y'(1) = 7/3.$

6 $2t^2y'' - 7ty' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 11.$

7 $t^2y'' - ty' + y = 0, \quad y(1) = 1, \quad y'(1) = -1.$

8 $4t^2y'' + 8ty' + y = 0, \quad y(1) = 0, \quad y'(1) = 3.$

9 $t^2y'' - ty' + 5y = 0, \quad y(1) = 1, \quad y'(1) = 3.$

10 $t^2y'' + 5ty' + 5y = 0, \quad y(1) = 3, \quad y'(1) = -7.$

Answers to Odd-Numbered Exercises

1 $y(t) = t^2 - 3t^{-2}$. **3** $y(t) = t^2 - 4t^{-1/3}$. **5** $y(t) = 2t^{1/2} - 4t^{-1/3}$.

7 $y(t) = t(1 - 2\ln t)$. **9** $y(t) = t[\cos(2\ln t) + \sin(2\ln t)]$.

4.7 Nonlinear Equations

Consider an IVP of the form

$$y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y_{10}, \tag{4.24}$$

where the function f satisfies certain smoothness requirements guaranteeing the existence of a unique solution in some open interval J that contains the point t_0. As we would expect, no method is available for solving (4.24) analytically when f is a generic admissible function. In certain cases, however, this problem can be reduced to one for a first-order equation, which, depending on its structure, we may be able to solve by means of one of the techniques described in Chap. 2. We discuss two such specific cases.

Function f does not contain y explicitly. If $f = f(t, y')$, we substitute

$$y'(t) = u(t), \tag{4.25}$$

so (4.24) becomes

$$u' = f(t, u), \quad u(t_0) = y_{10}.$$

Once we find u, we integrate it and use the IC $y(t_0) = y_0$ to determine y.

4.63 Example. With (4.25), the IVP

$$2ty'y'' + 4(y')^2 = 5t + 8, \quad y(2) = \tfrac{16}{3}, \quad y'(2) = 2$$

changes to

$$2tuu' + 4u^2 = 5t + 8, \quad u(2) = 2,$$

or, on division of the DE by u,

$$2tu' + 4u = (5t + 8)u^{-1}, \quad u(2) = 2.$$

This is an IVP for a Bernoulli equation (see Sect. 2.4), so we set $u = w^{1/2}$ and arrive at the new IVP

$$tw' + 4w = 5t + 8, \quad w(2) = 4.$$

Since the DE above is linear, the procedure laid out in Sect. 2.2 yields the integrating factor

$$\mu(t) = \exp\left\{ \int \frac{4}{t}\, dt \right\} = t^4$$

and then the solution

$$w(t) = \frac{1}{t^4}\left\{ \int\limits_2^t \tau^3(5\tau + 8)\, d\tau + \mu(2)w(2) \right\} = t + 2,$$

from which

$$y' = u = w^{1/2} = (t + 2)^{1/2}.$$

Consequently, taking the IC $y(2) = 16/3$ into account, we find that the solution of the given IVP is

$$y(t) = \int (t + 2)^{1/2}\, dt = \tfrac{2}{3}\,(t + 2)^{3/2} + C = \tfrac{2}{3}\,(t + 2)^{3/2}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = (2/3) * (t + 2)^(3/2);
{2*t*D[y,t]*D[y,t,t] + 4*(D[y,t])^2 - 5*t - 8,
        {y,D[y,t]}/. t → 2}//Simplify
```

generates the output $\{0, \{16/3, 2\}\}$. ■

Function f does not contain t explicitly. If $f = f(y, y')$, we set

$$u = y' \tag{4.26}$$

and see that, by the chain rule of differentiation,

$$y'' = \frac{dy'}{dt} = \frac{du}{dy}\frac{dy}{dt} = y'\frac{du}{dy} = uu', \tag{4.27}$$

where u is now regarded as a function of y. Also, at $t = t_0$ we have

$$u(y_0) = y'(t_0) = y_{10}, \tag{4.28}$$

so the IVP (4.24) changes to

$$uu' = f(y, u), \quad u(y_0) = y_{10}.$$

Once u is found, we integrate it and use the IC $y(t_0) = y_0$ to determine y.

4.64 Example. Consider the IVP

$$yy'' = y'(y' + 2), \quad y(0) = 1, \quad y'(0) = -1.$$

Implementing (4.26)–(4.28), we arrive at the new IVP

$$yu' = u + 2, \quad u(1) = -1,$$

which can be solved by separation of variables. Thus,

$$\int \frac{du}{u+2} = \int \frac{dy}{y},$$

so $\ln|u + 2| = \ln|y| + C$, or, after exponentiation,

$$u = C_1 y - 2.$$

Since the IC gives $C_1 = 1$, we have

$$y' = u = y - 2.$$

A second separation of variables now leads to

$$\int \frac{dy}{y-2} = \int dt;$$

hence, $\ln|y - 2| = t + C$, or

$$y = 2 + C_1 e^t.$$

Using the first IC from the original IVP, we finally obtain the solution of that problem as

$$y(t) = 2 - e^t.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = 2 - E^t;
{y*D[y,t,t] - D[y,t]*(D[y,t] + 2),{y,D[y,t]}/.t → 0}
        //Simplify
```

generates the output $\{0, \{1, -1\}\}$. ■

Exercises

Solve the given IVP and state the maximal interval of existence for the solution.

1 $y'' = 2e^{-2t}(y')^2,\quad y(0) = -1,\quad y'(0) = 1.$

2 $2ty'y'' = 3(y')^2 + t^2,\quad y(1) = 4/15,\quad y'(1) = 1.$

3 $ty'' = (4t^2 - 9t)(y')^2 + 2y',\quad y(1) = -(1/3)\ln 2,\quad y'(1) = 1/2.$

4 $2t - 2 + (y')^2 + 2ty'y'' = 0,\quad y(1) = -2/3,\quad y'(1) = 1.$

5 $y^2y'' = y',\quad y(0) = 1,\quad y'(0) = -1.$

6 $3y'y'' = 16y,\quad y(0) = 1,\quad y'(0) = 2.$

7 $2yy'' = (y')^2 + 1,\quad y(0) = 5/2,\quad y'(0) = 2.$

8 $y'' = 2y(y')^3,\quad y(0) = \sqrt{2},\quad y'(0) = -1.$

Answers to Odd-Numbered Exercises

1 $y(t) = (1/2)e^{2t} - 3/2,\quad -\infty < t < \infty.$

3 $y(t) = (1/3)\ln(t/(3-t)),\quad 0 < t < 3.$

5 $y(t) = (1-2t)^{1/2},\quad -\infty < t < 1/2.$

7 $y(t) = (1/2)(t^2 + 4t + 5),\quad -\infty < t < \infty.$

Chapter 5
Mathematical Models with Second-Order Equations

In this chapter we illustrate the use of linear second-order equations with constant coefficients in the analysis of mechanical oscillations and electrical vibrations.

5.1 Free Mechanical Oscillations

We recall that the small oscillations of a mass–spring–damper system (see Sect. 4.1) are described by the IVP

$$my'' + \gamma y' + ky = f(t), \quad y(0) = y_0, \quad y'(0) = y_{10}, \tag{5.1}$$

where t is time, $\gamma = \text{const} > 0$ and $k = \text{const} > 0$ are the damping and elastic coefficients, respectively, m is the moving mass, and f is an external (driving) force. The system undergoes so-called free oscillations when $f = 0$.

5.1.1 Undamped Free Oscillations

As the label shows, oscillations of this type occur if $\gamma = 0$, in which case the IVP reduces to

$$my'' + ky = 0, \quad y(0) = y_0, \quad y'(0) = y_{10}.$$

Since $k/m > 0$, we can write $\omega_0^2 = k/m$, so the DE becomes

$$y'' + \omega_0^2 y = 0,$$

with GS

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

The constants are determined from the ICs as $c_1 = y_0$ and $c_2 = y_{10}/\omega_0$; therefore, the solution of the IVP is

$$y(t) = y_0 \cos(\omega_0 t) + \frac{y_{10}}{\omega_0} \sin(\omega_0 t). \tag{5.2}$$

The numbers ω_0 and $T = 2\pi/\omega_0$ are called, respectively, the *natural frequency* and the *period* of the oscillations.

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_5,
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5.1 Remark. Solution (5.2) can be changed to an alternative, more convenient form. It is easy to show that for generic nonzero coefficients a and b and argument θ, we can write

$$a\cos\theta + b\sin\theta = A\cos(\theta - \varphi), \tag{5.3}$$

with $A > 0$ and $0 \le \varphi < 2\pi$. Since, according to a well-known trigonometric formula, the right-hand side expands as $A(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$, it follows that

$$A\cos\varphi = a, \quad A\sin\varphi = b,$$

from which, first eliminating φ and then A, we readily find that

$$A = \sqrt{a^2 + b^2}, \quad \tan\varphi = \frac{b}{a}. \tag{5.4}$$

It should be noted that the value of $\tan\varphi$ in (5.4) is not enough to find the correct φ in the interval $0 \le \varphi < 2\pi$. For that, we also need to consider the signs of $\cos\varphi$ and $\sin\varphi$—that is, of a and b—and make any necessary adjustment. For example, $\tan\varphi = -1$ has two solutions between 0 and 2π, namely $\varphi = 3\pi/4$ and $\varphi = 7\pi/4$. However, the former has $\sin\varphi > 0$, $\cos\varphi < 0$, whereas the latter has $\sin\varphi < 0$, $\cos\varphi > 0$.

For function (5.2), we have $a = y_0$, $b = y_{10}/\omega_0$, and $\theta = \omega_0 t$, so, by (5.4),

$$A = \sqrt{y_0^2 + \frac{y_{10}^2}{\omega_0^2}} = \frac{1}{\omega_0}\sqrt{y_0^2\omega_0^2 + y_{10}^2}, \quad \tan\varphi = \frac{y_{10}}{y_0\omega_0}, \tag{5.5}$$

and (5.3) yields

$$y(t) = A\cos(\omega_0 t - \varphi). \tag{5.6}$$

In the context of oscillations, the numbers A and φ are referred to as the *amplitude* and *phase angle*, respectively. ■

The physical phenomenon described by this model is called *simple harmonic motion*.

5.2 Example. A unit mass in a mass–spring system with $k = 4$ and no external forces starts oscillating from the point $3/\sqrt{2}$ with initial velocity $3\sqrt{2}$. Consequently, the position $y(t)$ of the mass at time $t > 0$ is the solution of the IVP

$$y'' + 4y = 0, \quad y(0) = \frac{3}{\sqrt{2}}, \quad y'(0) = 3\sqrt{2}.$$

Here, $\omega_0 = 2$, $y_0 = 3/\sqrt{2}$, and $y_{10} = 3\sqrt{2}$; hence, by (5.5), we get $A = 3$ and $\tan\varphi = 1$. Given that $a = y_0 > 0$ and $b = y_{10}/\omega_0 > 0$, the angle φ has a value between 0 and $\pi/2$; that is, $\varphi = \tan^{-1} 1 = \pi/4$, and, by (5.2), (5.5), and (5.6), the solution of the IVP is

$$y(t) = \frac{3}{\sqrt{2}}\cos(2t) + \frac{3}{\sqrt{2}}\sin(2t) = 3\cos\left(2t - \frac{\pi}{4}\right).$$

If the ICs are changed to $y(0) = 3/\sqrt{2}$ and $y'(0) = -3\sqrt{2}$, then $a > 0$ and $b < 0$, so φ is an angle between $3\pi/2$ and 2π; specifically, $\varphi = 2\pi + \tan^{-1}(-1) = 2\pi - \pi/4 = 7\pi/4$. Since A obviously remains the same, the solution is

$$y(t) = 3\cos\left(2t - \frac{7\pi}{4}\right).$$

For the pair of ICs $y(0) = -3/\sqrt{2}$ and $y'(0) = 3\sqrt{2}$, we have $a < 0$ and $b > 0$, so φ is between $\pi/2$ and π; that is, $\varphi = \pi + \tan^{-1}(-1) = \pi - \pi/4 = 3\pi/4$, and

$$y(t) = 3\cos\left(2t - \frac{3\pi}{4}\right).$$

Finally, if $y(0) = -3/\sqrt{2}$ and $y'(0) = -3\sqrt{2}$, then $a < 0$ and $b < 0$, which indicates that φ is an angle between π and $3\pi/2$; more precisely, $\varphi = \pi + \tan^{-1}1 = \pi + \pi/4 = 5\pi/4$, so

$$y(t) = 3\cos\left(2t - \frac{5\pi}{4}\right).$$

Verification with Mathematica®. The input

```
y = 3 * Cos[2 * t - {Pi/4, 7 * Pi/4, 3 * Pi/4, 5 * Pi/4}];
{D[y,t,t] + 4 * y, {y, D[y,t]} /. t → 0} // Simplify
```

generates the output

$$\{\{0, 0, 0, 0\}, \{\{3/\sqrt{2}, 3/\sqrt{2}, -3/\sqrt{2}, -3/\sqrt{2}\}, \{3\sqrt{2}, -3\sqrt{2}, 3\sqrt{2}, -3\sqrt{2}\}\}\}. \quad \blacksquare$$

Exercises

1 In a system with $k = 1$, the position and velocity of a unit mass at $t = \pi/2$ are -3 and -2, respectively. Determine the position $y(t)$ of the mass at any time $t > 0$, its initial position and velocity, and the amplitude of the oscillations.

2 In a system with $k = \pi^2$, the position and velocity of a unit mass at $t = 3$ are -1 and -4π, respectively. Determine the position $y(t)$ of the mass at any time $t > 0$, its initial position and velocity, and the amplitude of the oscillations.

3 (i) Determine the position $y(t)$ at time $t > 0$ of a unit mass in a system with $k = 9$, which starts oscillating from rest at the point 2.

(ii) Find the times when the mass passes through the origin between $t = 0$ and $t = 3$.

4 (i) Determine the position $y(t)$ at any time $t > 0$ of a unit mass in a system with $k = 4$, which starts oscillating from the point 1 with initial velocity 2.

(ii) Compute the amplitude and phase angle of the oscillations and write $y(t)$ in the form (5.6).

(iii) Find the time of the next passage of the mass through its initial position.

5 An object of unit mass starts oscillating from the origin with initial velocity 12 and frequency 4. At $t = 3\pi/8$, the object's mass is increased four-fold. Determine the new oscillation frequency and the position $y(t)$ of the object at any time $t > 0$.

6 A unit mass starts oscillating from the point 1 with initial velocity $3\sqrt{3}$ and frequency 3. At the first passage of the mass through the origin, its velocity is increased by three units. Determine the position of the mass at any time $t > 0$ and express it in the form (5.6).

Answers to Odd-Numbered Exercises

1 $y(t) = 2\cos t - 3\sin t$; $y(0) = 2$, $y'(0) = -3$; $A = \sqrt{13}$.

3 (i) $y(t) = 2\cos(3t)$. (ii) $t = \pi/6$, $\pi/2$, $5\pi/6$.

5 $\omega = 2$; $y(t) = \begin{cases} 3\sin(4t), & 0 < t \le 3\pi/8, \\ (3/\sqrt{2}\,)[\cos(2t) - \sin(2t)], & t > 3\pi/8. \end{cases}$

5.1.2 Damped Free Oscillations

These are modeled by the IVP

$$my'' + \gamma y' + ky = 0, \quad y(0) = y_0, \quad y'(0) = y_{10}.$$

Introducing the *damping ratio* ζ by

$$\zeta = \frac{\gamma}{2m\omega_0} = \frac{\gamma}{2m\sqrt{k/m}} = \frac{\gamma}{2\sqrt{mk}} > 0,$$

we divide the DE by m and rewrite it in the form $y'' + (\gamma/m)y' + (k/m)y = 0$, or, what is the same,

$$y'' + 2\zeta\omega_0 y' + \omega_0^2 y = 0.$$

The characteristic roots are

$$r_1 = \omega_0\big(-\zeta + \sqrt{\zeta^2 - 1}\,\big), \quad r_2 = \omega_0\big(-\zeta - \sqrt{\zeta^2 - 1}\,\big),$$

and their nature depends on the sign of the quantity $\zeta^2 - 1$.

(i) If $\zeta > 1$, then r_1 and r_2 are real and distinct, which means that the GS of the DE is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Since, as is easily verified, $r_1 + r_2 = -2\zeta\omega_0 < 0$ and $r_1 r_2 = \omega_0^2 > 0$, both r_1 and r_2 are negative; hence, y decreases rapidly to zero regardless of the values of c_1 and c_2. This case is called *overdamping*. Using the ICs, we find that the solution of the IVP is

$$y(t) = \frac{y_0 r_2 - y_{10}}{r_2 - r_1}\, e^{r_1 t} + \frac{y_{10} - y_0 r_1}{r_2 - r_1}\, e^{r_2 t}. \tag{5.7}$$

5.3 Example. The position $y(t)$ at time t of a unit mass that starts oscillating from the point 2 with initial velocity -1 in a system with $\gamma = 5$, $k = 6$, and no external force, satisfies the IVP

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

The characteristic roots $r_1 = -3$, $r_2 = -2$ and the ICs $y_0 = 2$, $y_{10} = -1$, replaced in (5.7), yield the solution

$$y(t) = -3e^{-3t} + 5e^{-2t},$$

whose graph is shown in Fig. 5.1.

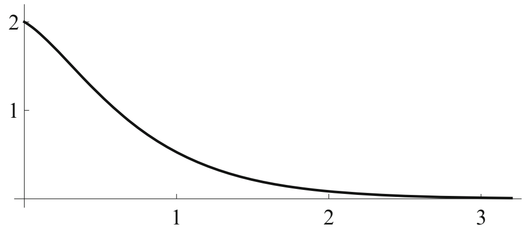

Fig. 5.1

Verification with Mathematica®. The input

```
y= - 3*E^(-3*t) + 5*E^(-2*t);
{D[y,t,t] + 5*D[y,t] + 6*y,{y,D[y,t]}/.t→ 0}//Simplify
```

generates the output $\{0, \{2, -1\}\}$. ■

(ii) If $\zeta = 1$, then $r_1 = r_2 = -\omega_0 < 0$, and the GS of the DE is

$$y(t) = (c_1 + c_2 t)e^{-\omega_0 t}.$$

Once again we see that, irrespective of the values of c_1 and c_2, y ultimately tends to 0 as $t \to \infty$, although its decay does not start as abruptly as in (i). This case is called *critical damping*. Applying the ICs, we obtain the solution of the IVP in the form

$$y(t) = [y_0 + (y_0\omega_0 + y_{10})t]e^{-\omega_0 t}. \tag{5.8}$$

Critical damping is the minimum damping that can be applied to the physical system without causing it to oscillate.

5.4 Example. A unit mass in a system with $\gamma = 4$, $k = 4$, and no external force starts oscillating from the point 1 with initial velocity 2. Then its position $y(t)$ at time $t > 0$ is the solution of the IVP

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

Since $r_1 = r_2 = -2$, $\omega_0 = 2$, $y_0 = 1$, and $y_{10} = 2$, from (5.8) it follows that

$$y(t) = (4t + 1)e^{-2t}.$$

The graph of this function is shown in Fig. 5.2.

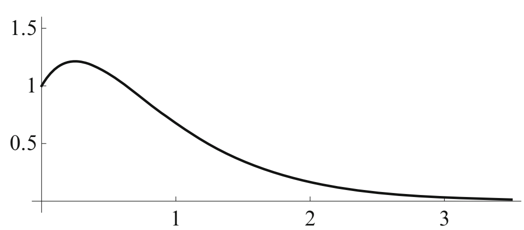

Fig. 5.2

Verification with Mathematica®. The input

```
y = (4*t + 1)*E^(-2*t);
{D[y,t,t] + 4*D[y,t] + 4*y, {y,D[y,t]}/. t→ 0}//Simplify
```

generates the output {0, {1, 2}}. ■

(iii) If $0 < \zeta < 1$, then the characteristic roots are

$$r_1 = \omega_0\big(-\zeta + i\sqrt{1-\zeta^2}\,\big), \quad r_2 = \omega_0\big(-\zeta - i\sqrt{1-\zeta^2}\,\big),$$

and the GS of the DE is

$$y(t) = e^{-\zeta\omega_0 t}[c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)], \tag{5.9}$$

where $\omega_1 = \omega_0\sqrt{1-\zeta^2}$. This is the *underdamping* case, where y still tends to 0 as $t \to \infty$ (because of the negative real part of the roots), but in an oscillatory manner. If the ICs are applied, we find that

$$c_1 = y_0, \quad c_2 = \frac{\zeta y_0\omega_0 + y_{10}}{\omega_1}. \tag{5.10}$$

5.5 Example. The position $y(t)$ at time $t > 0$ of a mass $m = 4$ that starts oscillating from the point 1/2 with initial velocity 7/4 in a system with $\gamma = 4$, $k = 17$, and no external force, is the solution of the IVP

$$4y'' + 4y' + 17y = 0, \quad y(0) = \tfrac{1}{2}, \quad y'(0) = \tfrac{7}{4}.$$

The characteristic roots of the DE are $r_1 = 1/2 + 2i$ and $r_2 = 1/2 - 2i$, and we have $\omega_0 = \sqrt{17}/2$, $\zeta = 1/\sqrt{17}$, $\omega_1 = 2$, $c_1 = 1/2$, and $c_2 = 1$; hence, by (5.9) and (5.10),

$$y(t) = e^{-t/2}\big[\tfrac{1}{2}\,\cos(2t) + \sin(2t)\big].$$

This solution is graphed in Fig. 5.3.

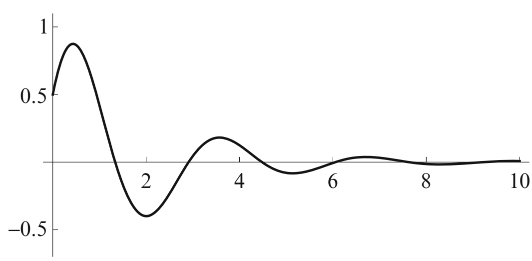

Fig. 5.3

Verification with Mathematica®. The input

```
y = E^(-t/2)*((1/2)*Cos[2*t] + Sin[2*t]);
{4*D[y,t,t] + 4*D[y,t] + 17*y, {y,D[y,t]}/. t→ 0}//Simplify
```

generates the output {0, {1/2, 7/4}}. ■

Exercises

1 In a system with $\gamma = 6$ and $k = 9$, the position and velocity of a unit mass at $t = \pi$ are $(\pi - 2)e^{-3\pi}$ and $(3\pi - 7)e^{-3\pi}$, respectively.

(i) Identify the type of damping.

(ii) Determine the position of the mass at any time $t > 0$ and find its initial position and velocity.

(iii) Find the velocity of the mass when it passes through the origin.

2 In a system with $\gamma = 2$ and $k = \pi^2 + 1$, the position and velocity of a unit mass at $t = 3/2$ are $e^{-3/2}$ and $(2\pi - 1)e^{-3/2}$, respectively.

(i) Identify the type of damping.

(ii) Determine the position of the mass at any time $t > 0$ and find its initial position and velocity.

(iii) Find the position and velocity of the mass at $t = 2$.

3 In a mass–spring system with $k = 1$, a unit mass starts oscillating from the origin with initial velocity 2. When it reaches the farthest position to the right for the first time, a damper with $\gamma = 5/2$ is attached to it.

(i) Determine the position $y(t)$ of the mass at any time $t > 0$.

(ii) Identify the type of damping after the damper is attached.

4 In a system with $m = 4$, $\gamma = 4$, and $k = 5$, the mass starts oscillating from the point 1 with initial velocity $-1/2$. When it reaches the origin for the first time, the damper is removed.

(i) Determine the position $y(t)$ of the mass at any time $t > 0$.

(ii) Identify the type of damping before the damper is removed.

5 (i) In a system with $\gamma = 4$ and $k = 3$, a unit mass starts oscillating from the point 2 with initial velocity -1. Identify the type of damping.

(ii) At $t = \ln(3/2)$, the damping coefficient γ is altered so that the system undergoes critical damping, and the velocity is changed to $-41\sqrt{3}/27$. Establish the IVP for the new model.

(iii) Determine the position $y(t)$ of the mass at any time $t > 0$.

6 (i) In a system with $\gamma = 2$ and $k = 2$, a unit mass starts oscillating from the point 1 with initial velocity -2. Identify the type of damping.

(ii) At $t = \pi/2$, the damping coefficient γ is changed so that the system undergoes critical damping. Establish the IVP for the new model.

(iii) Determine the position $y(t)$ of the mass at any time $t > 0$.

Answers to Odd-Numbered Exercises

1 (i) Critical damping. (ii) $y(t) = (2 - t)e^{-3t}$. (iii) $v = -e^{-6}$.

3 (i) $y(t) = \begin{cases} 2\sin t, & 0 < t \le \pi/2, \\ (8/3)e^{\pi/4 - t/2} - (2/3)e^{\pi - 2t}, & t > \pi/2. \end{cases}$ (ii) Overdamping.

5 (i) Overdamping.

(ii) $y'' + 2\sqrt{3}\,y' + 3y = 0, \quad y(\ln(3/2)) = 41/27, \quad y'(\ln(3/2)) = -41\sqrt{3}/27.$

(iii) $y(t) = \begin{cases} -(1/2)e^{-3t} + (5/2)e^{-t}, & 0 < t \le \ln(3/2), \\ (41/27)(3/2)^{\sqrt{3}} e^{-\sqrt{3}\,t}, & t > \ln(3/2). \end{cases}$

5.2 Forced Mechanical Oscillations

As we did for free oscillations, we discuss the undamped and damped cases separately.

5.2.1 Undamped Forced Oscillations

Here, the governing IVP is

$$my'' + ky = f_0 \cos(\omega t), \quad y(0) = y_0, \quad y'(0) = y_{10},$$

where ω is the frequency of an external driving force and $f_0 = \text{const}$. Using the notation $k/m = \omega_0^2$ introduced in Sect. 5.1.1, we divide the DE by m and rewrite it as

$$y'' + \omega_0^2 y = \frac{f_0}{m} \cos(\omega t). \tag{5.11}$$

The complementary function for this nonhomogeneous DE is

$$y_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

Clearly, the form of a convenient particular solution depends on whether ω coincides with ω_0 or not.

(i) If $\omega \ne \omega_0$, then, by Theorem 4.47 with $n = 0$, $\lambda = 0$, $\mu = \omega$, and $s = 0$, we seek a PS of the form $y_p(t) = \alpha \cos(\omega t) + \beta \sin(\omega t)$. Replacing in the DE and equating the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ on both sides, we find that $\alpha = f_0/(m(\omega_0^2 - \omega^2))$ and $\beta = 0$, so the GS of the equation is

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{f_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t). \tag{5.12}$$

The constants, determined from the ICs, are

$$c_1 = y_0 - \frac{f_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = \frac{y_{10}}{\omega_0}. \tag{5.13}$$

5.6 Example. In a mass–spring system with $k = 1$, a unit mass starts oscillating from rest at the origin and is acted upon by an external driving force of amplitude 1 and frequency $11/10$. The position $y(t)$ of the mass at time $t > 0$ is, therefore, the solution of the IVP

$$y'' + y = \cos\left(\tfrac{11}{10}\,t\right), \quad y(0) = 0, \quad y'(0) = 0.$$

Then $\omega_0 = 1$, $\omega = 11/10$, $f_0 = 1$, $y_0 = 0$, and $y_{10} = 0$, so, by (5.12), (5.13), and the trigonometric identity $\cos a - \cos b = 2\sin((a+b)/2)\sin((b-a)/2)$, we have

$$y(t) = \tfrac{100}{21}\cos t - \tfrac{100}{21}\cos\left(\tfrac{11}{10}\,t\right) = \left[\tfrac{200}{21}\sin\left(\tfrac{1}{20}\,t\right)\right]\sin\left(\tfrac{21}{20}\,t\right).$$

The function y represents an oscillation whose amplitude (given by the quantity between the square brackets) is itself oscillatory; this is known as *amplitude modulation*.

Figure 5.4 shows the graphs of the full function y (continuous line) and those of its amplitude and its reflection in the t-axis (dashed lines). A forcing frequency ω close to the natural frequency ω_0, as is the case here, produces so-called *beats*.

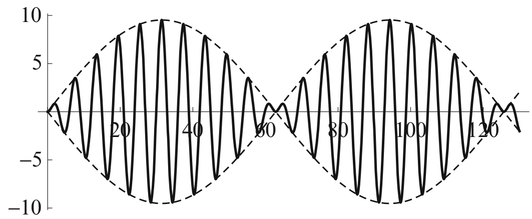

Fig. 5.4

VERIFICATION WITH MATHEMATICA®. The input

```
y=((200/21)*Sin[t/20])*Sin[(21/20)*t];
{D[y,t,t] + y - Cos[(11/10)*t],{y,D[y,t]}/.t→ 0}//Simplify
```

generates the output {0, {0, 0}}. ■

(ii) If $\omega = \omega_0$, then, by Theorem 4.47 with $n = 0$, $\lambda = 0$, $\mu = \omega_0$, and $s = 1$, we seek a PS of the form

$$y_p(t) = t[\alpha\cos(\omega_0 t) + \beta\sin(\omega_0 t)].$$

Replacing in (5.11) and identifying the coefficients of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ on both sides, we find that $\alpha = 0$ and $\beta = f_0/(2m\omega_0)$, so the GS of the DE is

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{f_0}{2m\omega_0}\,t\sin(\omega_0 t).$$

As the last term shows, this solution increases without bound as $t \to \infty$, a phenomenon known as *resonance*.

Exercises

1 In a system with $m = 4$, $k = 1$, and no external driving force, the mass starts oscillating from rest at the point 1. At $t = \pi$, a driving force $f(t) = 6\cos t$ begins to act on the mass. Determine the position of the mass at any time $t > 0$.

2 In a system with $k = 9\pi^2$ and no external driving force, a unit mass starts oscillating from the point 2 with initial velocity 3π. At $t = 1/6$, a driving force given by $f(t) = 8\pi^2\cos(\pi t)$ begins to act on the mass. Determine the position of the mass at any time $t > 0$.

3 In a system with $k = 1$ and an external driving force $f(t) = 3\cos(2t)$, a unit mass starts oscillating from rest at the point -2. At $t = \pi/2$, the driving force is removed. Determine the position of the mass at any time $t > 0$.

4 In a system with $m = 9$, $k = 4\pi^2$, and an external driving force $f(t) = 32\pi^2 \cos(2\pi t)$, the mass starts oscillating from the origin with velocity -2π. At $t = 3/4$, the driving force is removed. Determine the position of the mass at any time $t > 0$.

5 In a system with $k = 9$ and an external driving force $f(t) = 6\cos(3t)$, a unit mass starts oscillating from the origin with initial velocity 6. At $t = \pi/6$, the driving force is removed. Determine the position of the mass at any time $t > 0$.

6 In a system with $k = 4\pi^2$ and an external driving force $f(t) = 8\pi \cos(2\pi t)$, a unit mass starts oscillating from the point 1 with initial velocity -2π. At $t = 1$, the frequency of the driving force is halved. Determine the position of the mass at any time $t > 0$.

Answers to Odd-Numbered Exercises

1 $y(t) = \begin{cases} \cos(t/2), & 0 < t \le \pi, \\ \cos(t/2) - 2\sin(t/2) - 2\cos t, & t > \pi. \end{cases}$

3 $y(t) = \begin{cases} -\cos t - \cos(2t), & 0 < t \le \pi/2, \\ -\cos t + \sin t, & t > \pi/2. \end{cases}$

5 $y(t) = \begin{cases} (2 + t)\sin(3t), & 0 < t \le \pi/6, \\ -(1/3)\cos(3t) + (2 + \pi/6)\sin(3t), & t > \pi/6. \end{cases}$

5.2.2 Damped Forced Oscillations

By analogy with the unforced case, here the IVP is

$$y'' + 2\zeta\omega_0 y' + \omega_0^2 y = \frac{f_0}{m}\cos(\omega t), \quad y(0) = y_0, \quad y'(0) = y_{10}, \tag{5.14}$$

where $f_0 \cos(\omega t)$ is a periodic driving force. In this physical setting, the complementary function (with fully determined constants) is called the *transient solution*, and the particular solution is called the *steady state solution*, or *forced response*. Obviously, the form of the transient component depends on the type of damping. At the same time, the term on the right-hand side of the equation indicates that the forced response is a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$, which (see (5.6)) can be brought to the form $A\cos(\omega t - \varphi)$.

5.7 Example. The position $y(t)$ at time $t > 0$ of a unit mass in a system with $\gamma = 5$ and $k = 4$, which starts moving from the point 13 with initial velocity -95 and is acted upon by a periodic external force of amplitude 34 and frequency 1, is the solution of the IVP

$$y'' + 5y' + 4y = 34\cos t, \quad y(0) = 13, \quad y'(0) = -95.$$

The characteristic roots of the DE are $r_1 = -4$ and $r_2 = -1$. Applying the usual solution procedure, we find that

$$y(t) = 30e^{-4t} - 20e^{-t} + 3\cos t + 5\sin t.$$

The transient solution $30e^{-4t} - 20e^{-t}$ decays very rapidly as t increases. The forced response $3\cos t + 5\sin t$ has, as expected, an oscillatory nature and quickly becomes dominant. Performing the necessary computation, we find that its amplitude and phase angle are

$$A = \sqrt{3^2 + 5^2} = \sqrt{34}, \quad \varphi = \tan^{-1}(5/3) \approx 1.03;$$

hence, the forced response has the approximate expression $\sqrt{34}\cos(t - 1.03)$.

The graph of the solution is shown in Fig. 5.5.

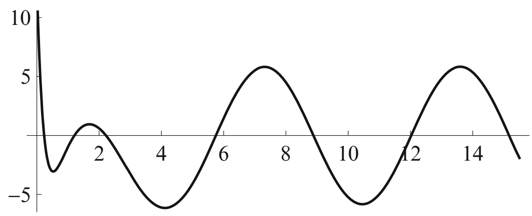

Fig. 5.5

Verification with Mathematica®. The input

```
y = 30 * E^(-4*t) - 20 * E^(-t) + Sqrt[34] * (Cos[t - ArcTan[5/3]]
        // TrigExpand;
{D[y,t,t] + 5 * D[y,t] + 4 * y - 34 * Cos[t], {y,D[y,t]} /. t → 0}
        // Simplify
```

generates the output $\{0, \{13, -95\}\}$. ■

Exercises

1 (i) In a system with $\gamma = 4$, $k = 3$, and an external driving force $f(t) = 20\cos t$, a unit mass starts oscillating from rest at the origin. Identify the type of damping in the transient solution.

(ii) At $t = \pi$, the driving force is removed and the velocity of the mass is increased by 6. Establish the IVP for the new model.

(iii) Determine the position $y(t)$ of the mass at any time $t > 0$.

2 (i) In a system with $\gamma = 2$, $k = 2$, and an external driving force $f(t) = 10\cos(2t)$, a unit mass starts oscillating from rest at the origin. Identify the type of damping in the transient solution.

(ii) At $t = \pi/2$, the driving force is removed. Establish the IVP for the new model.

(iii) Determine the position $y(t)$ of the mass at any time $t > 0$.

3 (i) In a system with $\gamma = 5$, $k = 4$, and an external driving force $f(t) = 65\cos(t/2)$, a unit mass starts oscillating from rest at the point 1. Identify the type of damping in the transient solution.

(ii) At $t = \pi$, the coefficient γ assumes the value that makes the damping critical, the frequency of the driving force is doubled and its amplitude is lowered to 25, and the velocity of the mass is changed to $(5/\pi - 10)e^{-4\pi} + 32e^{-\pi} - 26$. Establish the IVP for the new model.

(iii) Determine the position $y(t)$ of the mass at any time $t > 0$.

4 (i) In a system with $\gamma = 2$, $k = 1$, and an external driving force $f(t) = 2\cos t$, a unit mass starts oscillating from the point 1 with initial velocity 1. Verify that the damping in the transient solution is critical.

(ii) At $t = 2\pi$, the damping coefficient γ is doubled, the mass is increased so that the damping remains critical, the amplitude of the driving force is raised to 50, and the velocity of the mass is changed to $5 + (1/2 - \pi)e^{-2\pi}$. Establish the IVP for the new model.

(iii) Determine the position $y(t)$ of the mass at any time $t > 0$.

Answers to Odd-Numbered Exercises

1 (i) Overdamping.

(ii) $y'' + 4y' + 3y = 0$, $y(\pi) = 3e^{-3\pi} - 5e^{-\pi} - 2$, $y'(\pi) = -9e^{-3\pi} + 5e^{-\pi} + 2$.

(iii) $y(t) = \begin{cases} 3e^{-3t} - 5e^{-t} + 2\cos t + 4\sin t, & 0 < t \le \pi, \\ 3e^{-3t} - (2e^{\pi} + 5)e^{-t}, & t > \pi. \end{cases}$

3 (i) Overdamping.

(ii) $y'' + 4y' + 4y = 25\cos t$, $y(\pi) = 5e^{-4\pi} - 16e^{-\pi} + 8$,
$y'(\pi) = (5/\pi - 10)e^{-4\pi} + 32e^{-\pi} - 26$.

(iii) $y(t) = \begin{cases} 5e^{-4t} - 16e^{-t} + 12\cos(t/2) + 8\sin(t/2), & 0 < t \le \pi, \\ [11e^{2\pi} - 16e^{\pi} + (5/\pi)e^{-2\pi}t]e^{-2t} + 3\cos t + 4\sin t, & t > \pi. \end{cases}$

5.3 Electrical Vibrations

The IVP for a series RLC circuit can be written in two different, but very similar, ways. If the unknown function is the charge Q, then the problem is

$$LQ'' + RQ' + \frac{1}{C}Q = V(t), \quad Q(0) = Q_0, \quad Q'(0) = Q_{10}, \tag{5.15}$$

where, as mentioned earlier, R, L, C, and V denote, respectively, the resistance, inductance, capacitance, and voltage. Alternatively, we can make the current I the unknown function. In this case, since $I(t) = Q'(t)$, differentiating the DE in (5.15) term by term and adjoining appropriate ICs, we arrive at the IVP

$$LI'' + RI' + \frac{1}{C}I = V'(t), \quad I(0) = I_0, \quad I'(0) = I_{10}. \tag{5.16}$$

The equations in both (5.15) and (5.16) can be written in the form

$$Ly'' + Ry' + \frac{1}{C}y = E(t). \tag{5.17}$$

It is clear that (5.17) is analogous to the equation in (5.1). Comparing these two DEs, we can establish a direct correspondence between the various physical quantities occurring in the mathematical models of mechanical oscillations and electrical vibrations, as shown in Table 5.1.

Table 5.1

Mechanical oscillator	RLC circuit
m	L
γ	R
k	$\dfrac{1}{C}$
f	E
ω_0	$\dfrac{1}{\sqrt{LC}}$
ζ	$\dfrac{R}{2}\sqrt{\dfrac{C}{L}}$

The analysis of the RLC circuit model is therefore identical to that of the mass–spring–damper system, with the changes indicated above. Here, ω_0 is called the *resonance frequency*, and the case $\zeta = 1$, referred to as the *critically damped response*, represents the circuit response that decays in the shortest time without going into oscillation mode.

5.8 Example. A series circuit with $R = 2$, $L = 1$, $C = 1$, and $V(t) = (25/2)\sin(2t)$ is governed by the DE

$$I'' + 2I' + I = 25\cos(2t),$$

where the right-hand side (see (5.16)) is $V'(t)$. Taking a pair of ICs, for example, $I(0) = 7$ and $I'(0) = 28$, we obtain the solution

$$I(t) = (30t + 10)e^{-t} - 3\cos(2t) + 4\sin(2t).$$

The last two terms on the right-hand side can be rewritten in the form indicated in (5.3), with the amplitude and phase angle computed by means of (5.4) as $A = 5$ and $\varphi \approx 2.214$, respectively. Then the solution becomes

$$I(t) \approx (30t + 10)e^{-t} + 5\cos(2t - 2.214).$$

The graph of this function is shown in Fig. 5.6.

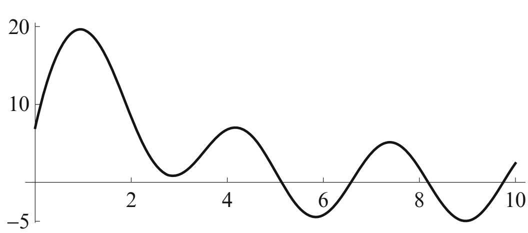

Fig. 5.6

Verification with Mathematica®. The input

```
y = (30*t + 10)*E^(-t) + 5*((Cos[a1 - a2] // TrigExpand)
      /. {a1 → 2*t, a2 → 2.214});
Eqn = D[y,t,t] + 2*D[y,t] + y - 25*Cos[2*t] // Simplify;
{Ccos,Csin} = Round[{Coefficient[Eqn,Cos[2*t]],
      Coefficient[Eqn,Sin[2*t]]}];
{Ccos*Cos[2*t] + Csin*Sin[2*t], Round[{y,D[y,t]}/. t → 0]}
```

generates the output $\{0, \{7, 28\}\}$. The input commands are rather convoluted since, our solution being an approximate one, we want the program to round the numerical coefficients to the nearest integer. ■

Exercises

1 (i) In an RLC circuit with $R = 6$, $L = 1$, $C = 1/8$, and $V(t) = 85\cos t$, the initial values of the charge and current are 8 and 4, respectively. Determine the charge $Q(t)$ and current $I(t)$ at any time $t > 0$.

(ii) Write the forced response for Q in the form (5.6) and find when it attains its maximum value for the first time.

2 (i) In an RLC circuit with $R = 4$, $L = 4$, $C = 1$, and $V(t) = 289\cos(2t)$, the initial values of the charge and current are -15 and 17, respectively. Determine the charge $Q(t)$ and then the current $I(t)$ at any time $t > 0$.

(ii) Write the forced response for Q in the form (5.6) and find when it attains its maximum value for the first time.

3 In an RLC circuit with $R = 0$, $L = 1$, $C = 1$, and $V(t) = 6\sin(t/2)$, the initial values of the current and its derivative are 4 and 1, respectively. At $t = \pi$, a resistance $R = 2$ is added to the circuit. Compute the current $I(t)$ at any time $t > 0$.

4 In an RLC circuit with $R = 0$, $L = 1$, $C = 1/(4\pi^2)$, and $V(t) = 3\pi\sin(\pi t)$, the initial values of the current and its derivative are 3 and 2π, respectively. At $t = 1/2$, a resistance $R = 5\pi$ is added to the circuit. Compute the current $I(t)$ at any time $t > 0$.

Answers to Odd-Numbered Exercises

1 (i) $Q(t) = e^{-2t} + 7\cos t + 6\sin t$; $I(t) = -2e^{-2t} + 6\cos t - 7\sin t$.

(ii) Forced response $\approx \sqrt{85}\,\cos(t - 0.7)$; $t \approx 0.7$.

3 $$I(t) = \begin{cases} \sin t + 4\cos(t/2), & 0 < t \le \pi, \\ (3/25)(35\pi - 16 - 35t)e^{\pi - t} + 12\cos(t/2 + 16\sin(t/2), & t > \pi. \end{cases}$$

Chapter 6
Higher-Order Linear Equations

Certain physical phenomena give rise to mathematical models that involve DEs of an order higher than two.

6.1 Modeling with Higher-Order Equations

We mention a couple of examples.

Generalized Airy equation. The motion of a particle in a triangular potential well is the solution of the IVP

$$y''' + aty' + by = 0, \quad y(0) = y_0, \quad y'(0) = y_{10}, \quad y''(0) = y_{20},$$

where a, b = const. This type of DE is also encountered in the study of a particular kind of waves in a rotating inhomogeneous plasma.

Harmonic oscillations of a beam. The stationary oscillations of a uniform, load-free beam of length l with both ends embedded in a rigid medium is modeled by the BVP

$$EIy^{(4)} - m\omega^2 y = 0, \quad y(0) = 0, \ y'(0) = 0, \ y(l) = 0, \ y'(l) = 0,$$

where $y(x)$ is the deflection of the beam at a generic point x in the open interval $0 < x < l$, the constants E, I, and m are the Young modulus of the material, the moment of inertia of the cross section, and the mass, and ω is the oscillation frequency. If the end-point $x = l$ is free, then the BCs at that point are $y''(l) = 0$ and $y'''(l) = 0$. If, on the other hand, the beam is simply supported at both end-points, the BCs are

$$y(0) = 0, \ y''(0) = 0, \ y(l) = 0, \ y''(l) = 0.$$

6.2 Algebra Prerequisites

All the concepts of linear algebra discussed in Sect. 4.2 can be extended to more general cases in the obvious way, although, as expected, in such generalizations the computation becomes more laborious.

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_6,
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6.2.1 Matrices and Determinants of Higher Order

An $n \times n$ matrix is a number array of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

For simplicity, we denote the generic element of such a matrix by a_{ij} and write $A = (a_{ij})$. The elements a_{11}, a_{22}, $\dots$, a_{nn} are said to form the *leading diagonal* of A.

6.1 Remark. The calculation of the determinant of an $n \times n$ matrix is more involved, and lengthier, than that of a 2×2 determinant. A method for evaluating $\det(A)$, called expansion in the elements of a row or column, consists of the following steps:

(1) Choose a row or a column of the given determinant—it does not matter which one because the result is always the same. If some of the entries are zero, it is advisable to choose a row or a column that contains as many zeros as possible, to simplify the calculation.

(2) For each element a_{ij} in that row/column, note the number i of the row and the number j of the column where the element is positioned, then multiply the element by $(-1)^{i+j}$; that is, form the product $(-1)^{i+j}a_{ij}$.

(3) For each element in the chosen row/column, delete the row and the column to which the element belongs and compute the $(n-1) \times (n-1)$ determinant d_{ij} formed by the remaining rows and columns.

(4) For each element in the chosen row/column, multiply the determinant computed in step (3) by the number obtained in step (2): $(-1)^{i+j}a_{ij}d_{ij}$.

(5) Sum up the numbers computed in step (4) for all the elements of the chosen row/column. ■

6.2 Example. The 3×3 determinant below is expanded in its second column:

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = (-1)^{1+2}\,0 \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} + (-1)^{2+2}(-1) \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} + (-1)^{3+2}(-2) \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$$
$$= -(8+1) + 2(4+3) = 5.$$

This column has been chosen because it contains a zero. If instead of the second column we choose, say, the third row, then, once again,

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = (-1)^{3+1}1 \begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} + (-1)^{3+2}(-2) \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + (-1)^{3+3}4 \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}$$
$$= -1 + 2(4+3) + 4(-2) = 5. \;\blacksquare$$

6.3 Remark. The properties listed in Remarks 4.2 remain valid in the case of an $n \times n$ determinant.

(i) Swapping any two rows (columns) changes the sign of the determinant.

(ii) If the elements of a row (column) have a common factor, then this factor can be taken out and multiplied by the value of the new determinant.

(iii) Cramer's rule for solving a linear algebraic system is adapted in the obvious way to the case of n equations in n unknowns. Thus, by analogy with formulas (4.1), for every $j = 1, \dots, n$, the value of x_j is given by the ratio of two $n \times n$ determinants: the one in the denominator (which must be nonzero) is formed from the system coefficients, and the one in the numerator is obtained by replacing the jth column in the former by the column of the terms on the right-hand side of the system. We will say more about the solution of such systems in Sect. 6.2.2. ■

Exercises

In 1–6, compute the given determinant by expanding it in a row or a column.

1 $\begin{vmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 1 & 0 & -2 \end{vmatrix}$. **2** $\begin{vmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{vmatrix}$.

3 $\begin{vmatrix} 1 & 3 & -2 \\ -1 & 4 & -1 \\ 2 & 1 & -3 \end{vmatrix}$. **4** $\begin{vmatrix} 2 & -3 & 2 \\ 1 & -2 & 1 \\ 4 & 3 & -1 \end{vmatrix}$.

5 $\begin{vmatrix} 2 & 1 & 0 & -1 \\ 1 & 3 & 1 & 2 \\ -1 & 4 & -2 & 0 \\ -2 & 1 & 0 & 3 \end{vmatrix}$. **6** $\begin{vmatrix} 0 & 3 & 1 & 1 \\ 2 & -1 & 0 & 2 \\ 1 & 5 & 1 & 3 \\ -2 & 1 & -3 & 4 \end{vmatrix}$.

In 7–10, solve the given equation for the parameter a.

7 $\begin{vmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ -4 & 0 & a \end{vmatrix} = -1.$ **8** $\begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ -1 & 5 & a \end{vmatrix} = 1.$

9 $\begin{vmatrix} a & 1 & -1 \\ 3 & 0 & 2 \\ -2 & -a & 7 \end{vmatrix} = 2.$ **10** $\begin{vmatrix} 2 & a & -1 \\ 0 & 1 & 3a \\ 3 & -2 & 4 \end{vmatrix} = 8.$

Answers to Odd-Numbered Exercises

1 -15. **3** -8. **5** -20. **7** $a = -1$. **9** $a = 3, -9/2$.

6.2.2 Systems of Linear Algebraic Equations

The general form of a system of n linear algebraic equations in n unknowns is

$$\begin{aligned} &a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ &\vdots \\ &a_{n1}x_1 + \cdots + a_{nn}x_n = b_n, \end{aligned} \tag{6.1}$$

where $A = (a_{ij})$ is the $n \times n$ matrix of the system coefficients, $x_1, \dots, x_n$ are the unknowns, and $b_1, \dots, b_n$ are given numbers.

6.4 Theorem. (i) *If* $\det(A) \neq 0$, *then system* (6.1) *has a unique solution.*

(ii) *If* $\det(A) = 0$, *then system* (6.1) *has either infinitely many solutions, or no solution.* ■

Systems with a unique solution can certainly be solved by means of Cramer's rule (see Remark 6.3(iii)), but that type of computation may take an inordinately long time. By contrast, the method known as *Gaussian elimination* is much faster and has the added advantage that it can also handle the case when $\det(A) = 0$. In this alternative technique, we write the system coefficients and the right-hand sides as a numerical array and then perform a succession of so-called *elementary row operations*, which are of three types:

(i) Multiplication of a row by a nonzero factor;

(ii) Swapping of two rows;

(iii) Replacement of a row by a linear combination of itself with another row.

These row operations correspond to exactly the same operations performed on the equations of the system. Selected to generate zeros in all the places below the leading diagonal of the matrix of coefficients, they produce equivalent versions of the given system in which the unknowns are eliminated one by one.

6.5 Example. Consider the system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -3,\\ x_1 + 2x_2 + 2x_3 &= 5,\\ 3x_1 - 2x_2 - x_3 &= -8. \end{aligned}$$

The initial array of the coefficients and right-hand sides mentioned above is

$$\begin{array}{rrr|r} 2 & -1 & 1 & -3\\ 1 & 2 & 2 & 5\\ 3 & -2 & -1 & -8 \end{array}$$

(For clarity, we have separated the system coefficients from the right-hand sides by a vertical line.) Let R_1, R_2, and R_3 be the rows of the array. In the next step, we use R_1 as a 'pivot' to make zeros below the number 2 in the first column, which is equivalent to eliminating x_1 from the second and third equations. The elementary row operations that achieve this are indicated to the right of the array:

$$\begin{array}{rrr|rl} 2 & -1 & 1 & -3 & \\ 1 & 2 & 2 & 5 & 2R_2 - R_1\\ 3 & -2 & -1 & -8 & 2R_3 - 3R_1 \end{array} \quad \Rightarrow \quad \begin{array}{rrr|r} 2 & -1 & 1 & -3\\ 0 & 5 & 3 & 13\\ 0 & -1 & -5 & -7 \end{array}$$

We now we use R_2 as a pivot to make a zero below 5 in the second column. This is equivalent to eliminating x_2 from the third equation:

$$\begin{array}{rrr|rl} 2 & -1 & 1 & -3 & \\ 0 & 5 & 3 & 13 & \\ 0 & -1 & -5 & -7 & 5R_3 + R_2 \end{array} \quad \Rightarrow \quad \begin{array}{rrr|r} 2 & -1 & 1 & -3\\ 0 & 5 & 3 & 13\\ 0 & 0 & -22 & -22 \end{array}$$

The final table means that the system has been brought to the equivalent form

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -3,\\ 5x_2 + 3x_3 &= 13,\\ -\,22x_3 &= -22, \end{aligned}$$

from which we find the unknowns in reverse order, by back-substitution: the third equation gives $x_3 = 1$, then the second yields $x_2 = (13 - 3x_3)/5 = 2$, and, finally, from the first one we obtain $x_1 = (-3 + x_2 - x_3)/2 = -1$.

VERIFICATION WITH MATHEMATICA®. The input

```
{2*x1 - x2 + x3 + 3, x1 + 2*x2 + 2*x3 - 5, 3*x1 - 2*x2 - x3 + 8}
        /. {x1 → −1, x2 → 2, x3 → 1}
```

generates the output {0, 0, 0}. ■

6.6 Remark. For the sake of neatness, we could have added a further step to the above scheme by replacing R_3 by $-R_3/22$, but this would not have improved the speed of the solution process. This is the same reason why we chose not to extend the work and convert the numbers above the leading diagonal to zeros as well. ■

6.7 Example. Trying to apply the same technique to the system

$$\begin{aligned} x_2 + 2x_3 &= 2, \\ x_1 - 2x_2 + 2x_3 &= -3, \\ 4x_1 + 2x_2 - 6x_3 &= -9, \end{aligned}$$

we notice that the initial system array

$$\left.\begin{array}{rrr} 0 & 1 & 2 \\ 1 & -2 & 2 \\ 4 & 2 & -6 \end{array}\right| \begin{array}{r} 2 \\ -3 \\ -9 \end{array}$$

does not allow us to start with R_1 as a pivot because of the 0 in the first column. This is easily remedied by swapping R_1 and R_2, which is an admissible row operation. The sequence of arrays in this case is

$$\left.\begin{array}{rrr} 0 & 1 & 2 \\ 1 & -2 & 2 \\ 4 & 2 & -6 \end{array}\right| \begin{array}{rl} 2 & R_2 \\ -3 & R_1 \\ -9 & \end{array} \quad \Rightarrow \quad \left.\begin{array}{rrr} 1 & -2 & 2 \\ 0 & 1 & 2 \\ 4 & 2 & -6 \end{array}\right| \begin{array}{rl} -3 & \\ 2 & \\ -9 & R_3 - 4R_1 \end{array}$$

$$\Rightarrow \left.\begin{array}{rrr} 1 & -2 & 2 \\ 0 & 1 & 2 \\ 0 & 10 & -14 \end{array}\right| \begin{array}{rl} -3 & \\ 2 & \\ 3 & R_3 - 10R_2 \end{array} \quad \Rightarrow \quad \left.\begin{array}{rrr} 1 & -2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -34 \end{array}\right| \begin{array}{r} -3 \\ 2 \\ -17 \end{array}$$

from which, proceeding as in Example 6.5, we find the solution $x_1 = -2$, $x_2 = 1$, and $x_3 = 1/2$.

VERIFICATION WITH MATHEMATICA®. The input

```
{x2 + 2*x3 - 2, x1 - 2*x2 + 2*x3 + 3, 4*x1 + 2*x2 - 6*x3 + 9}
        /. {x1 → −2, x2 → 1, x3 → 1/2}
```

generates the output {0, 0, 0}. ■

6.8 Example. For the system

$$\begin{aligned} x_1 - x_2 \qquad + 2x_4 &= 1, \\ -2x_2 + x_3 + x_4 &= -2, \\ 3x_1 \qquad + x_3 - x_4 &= 1, \\ -2x_1 + x_2 - 2x_3 + 2x_4 &= 2, \end{aligned}$$

we construct the sequence of arrays

$$\left.\begin{array}{rrrr|rl} 1 & -1 & 0 & 2 & 1 & \\ 0 & -2 & 1 & 1 & -2 & \\ 3 & 0 & 1 & -1 & 1 & R_3-3R_1 \\ -2 & 1 & -2 & 2 & 2 & R_4+2R_1 \end{array}\right. \Rightarrow \left.\begin{array}{rrrr|rl} 1 & -1 & 0 & 2 & 1 & \\ 0 & -2 & 1 & 1 & -2 & \\ 0 & 3 & 1 & -7 & -2 & 2R_3+3R_2 \\ 0 & -1 & -2 & 6 & 4 & 2R_4-R_2 \end{array}\right.$$

$$\Rightarrow \left.\begin{array}{rrrr|rl} 1 & -1 & 0 & 2 & 1 & \\ 0 & -2 & 1 & 1 & -2 & \\ 0 & 0 & 5 & -11 & -10 & \\ 0 & 0 & -5 & 11 & 10 & R_4+R_3 \end{array}\right. \Rightarrow \left.\begin{array}{rrrr|r} 1 & -1 & 0 & 2 & 1 \\ 0 & -2 & 1 & 1 & -2 \\ 0 & 0 & 5 & -11 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

The last array represents the equivalent three-equation system

$$\begin{aligned} x_1 - \quad x_2 \qquad\quad + \quad 2x_4 &= 1, \\ -\,2x_2 + \quad x_3 + \quad x_4 &= -2, \\ 5x_3 - 11x_4 &= -10 \end{aligned}$$

(the forth equation is $0 = 0$). Alternatively, moving the x_4-terms to the right-hand side, we can write

$$\begin{aligned} x_1 - \quad x_2 \qquad\quad &= \quad 1 - 2x_4, \\ -\,2x_2 + \quad x_3 &= -2 - x_4, \\ 5x_3 &= -10 + 11x_4, \end{aligned}$$

from which, in the usual way, we obtain

$$x_1 = 1 - \tfrac{2}{5}\,x_4, \quad x_2 = \tfrac{8}{5}\,x_4, \quad x_3 = -2 + \tfrac{11}{5}\,x_4.$$

If we write $x_4 = 5a$ (to avoid denominators), then

$$x_1 = 1 - 2a, \quad x_2 = 8a, \quad x_3 = -2 + 11a, \quad x_4 = 5a,$$

which shows that the given system has infinitely many solutions, one for each arbitrarily chosen real number a.

Verification with Mathematica®. The input

```
{x1 - x2 + 2*x4 - 1, -2*x2 + x3 + x4 + 2, 3*x1 + x3 - x4 - 1, -2*x1
        + x2 - 2 x3 + 2*x4 - 2} /. {x1 → 1 - 2*a, x2 → 8*a,
        x3 → -2 + 11*a, x4 → 5*a}
```

generates the output {0, 0, 0, 0}. ■

6.9 Example. Consider the system in Example 6.8 with the right-hand side in the fourth equation replaced by 0. It is easy to verify that the same chain of elementary row operations in this case yields the final array

$$\left.\begin{array}{rrrr|r} 1 & -1 & 0 & 2 & 1 \\ 0 & -2 & 1 & 1 & -2 \\ 0 & 0 & 5 & -11 & -10 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right.$$

The fourth equation of the equivalent system represented by this array is $0 = -4$, which is nonsensical, so the modified system is inconsistent. This means that the given system has no solution. ■

Exercises

Use Gaussian elimination to find the solutions (if any) of the given algebraic system.

1 $x_1 + 2x_2 - x_3 = 2, \quad 2x_1 - x_2 + 2x_3 = -2, \quad -2x_1 + 3x_2 + x_3 = 9.$

2 $2x_1 - x_2 + 3x_3 = 1, \quad x_1 - x_3 = 3, \quad 3x_1 + x_2 + x_3 = 5.$

3 $x_1 + x_2 - 2x_3 = 2, \quad 2x_1 + x_2 + 2x_3 = -2, \quad 4x_1 + x_2 + 10x_3 = -10.$

4 $2x_1 + x_3 = 1, \quad 3x_1 - x_2 + 2x_3 = -4, \quad 9x_1 - x_2 + 5x_3 = -1.$

5 $2x_2 - x_3 = 5, \quad x_1 - 3x_3 = 9, \quad -x_1 - 2x_2 + 4x_3 = 1.$

6 $3x_1 + x_2 - x_3 = -2, \quad 2x_1 - 3x_2 = 1, \quad -x_1 + 7x_2 - x_3 = 1.$

7 $3x_1 + x_2 + 2x_3 = -6, \quad -2x_1 - 3x_2 + x_3 = 11, \quad x_1 - 2x_2 = -1.$

8 $-2x_1 + 3x_2 + x_3 = 3, \quad x_1 - x_2 + 2x_3 = -2, \quad -5x_1 + 8x_2 + 5x_3 = 7.$

9 $x_1 - x_2 + x_3 + 2x_4 = -3, \quad 2x_1 + x_2 - 2x_3 + x_4 = 5,$
$x_1 - 2x_2 - x_4 = -6, \quad -x_1 + 3x_2 + x_3 + 2x_4 = 7.$

10 $2x_2 - x_3 + 3x_4 = -1, \quad x_1 - x_2 + x_3 - 2x_4 = 1,$
$-x_1 - 3x_3 + x_4 = -5, \quad -2x_1 + 4x_2 - x_3 + x_4 = -5.$

11 $x_1 - x_2 + x_4 = 1, \quad 2x_1 + x_2 + x_3 + 2x_4 = 1,$
$-2x_1 + x_3 + x_4 = 2, \quad 5x_1 + 3x_2 + x_3 + 2x_4 = -1.$

12 $2x_2 + x_3 = 0, \quad x_1 - x_3 + x_4 = -1,$
$-x_1 + 2x_2 + x_3 + 2x_4 = 1, \quad 2x_1 + 2x_2 - x_4 = -2.$

13 $2x_1 + x_2 - 2x_3 - x_4 = -7, \quad x_1 + 2x_2 - 3x_3 = -7,$
$-2x_1 - x_2 + 4x_3 + 2x_4 = 12, \quad -6x_1 - 2x_3 + 2x_4 = 0.$

14 $2x_1 - x_2 - x_3 + x_4 = 1, \quad -x_1 - 2x_2 + 3x_3 + 2x_4 = 0,$
$4x_1 + 3x_2 - 7x_3 - 3x_4 = 1, \quad 3x_1 - 4x_2 + x_3 + 4x_4 = -1.$

15 $3x_1 - x_2 + 2x_3 + x_4 = 8, \quad -x_1 - 2x_3 + 2x_4 = -5,$
$4x_1 + x_2 - 2x_3 = -14, \quad x_1 + x_2 + 3x_3 - x_4 = 8.$

16 $x_1 - x_2 + 2x_4 = -1, \quad x_1 - x_3 - 2x_4 = 1,$
$-x_1 - 2x_2 + 3x_3 + 10x_4 = -5, \quad 2x_1 - 3x_2 + x_3 + 8x_4 = -4.$

Answers to Odd-Numbered Exercises

1 $x_1 = -1, \quad x_2 = 2, \quad x_3 = 1.$

3 $x_1 = -4 - 4a, \quad x_2 = 6 + 6a, \quad x_3 = a.$

5 No solution. **7** $x_1 = -3, \quad x_2 = -1, \quad x_3 = 2.$

9 $x_1 = -1, \quad x_2 = 2, \quad x_3 = -2, \quad x_4 = 1.$

11 $x_1 = -2a, \quad x_2 = -1 + 3a, \quad x_3 = 2 - 9a, \quad x_4 = 5a.$

13 No solution. **15** $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = 1.$

6.2.3 Linear Independence and the Wronskian

These two concepts are also easily generalized.

6.10 Definition. We say that n functions f_1, f_2, ..., f_n defined on the same interval J are *linearly dependent* if there are constants c_1, c_2, ..., c_n, not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) = 0 \quad \text{for all } t \text{ in } J.$$

If these functions are not linearly dependent, then they are called *linearly independent.* ■

6.11 Definition. The Wronskian of n functions that have derivatives up to order $n-1$ inclusive is defined by

$$W[f_1, f_2, \dots, f_n](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}. \ ■$$

6.12 Remarks. (i) The assertions listed in Remarks 4.11(ii)–(iv) also hold for a set of n functions. In particular, if $f_1, \dots, f_n$ are linearly dependent on J, then $W[f_1, \dots, f_n](t) = 0$ for all t in J; on the other hand, if $W[f_1, \dots, f_n](t_0) \neq 0$ for some t_0 in J, then $f_1, \dots, f_n$ are linearly independent on J. Remark 4.11(i) is modified to state that $f_1, \dots, f_n$ are linearly dependent if and only if at least one of these functions is a linear combination of the others.

(ii) Theorem 4.16, generalized in the obvious way, remains valid.

(iii) It is useful to note that, for any positive integer n, the functions in each of the following sets are linearly independent on the real line:

$$1,\ t,\ t^2,\ \dots,\ t^n;$$

$$e^{at},\ te^{at},\ t^2e^{at},\ \dots,\ t^ne^{at},\quad a \neq 0;$$

$$\cos(at),\ \sin(at),\ t\cos(at),\ t\sin(at),\ \dots,\ t^n\cos(at),\ t^n\sin(at),\quad a \neq 0;$$

$$e^{at}\cos(bt),\ e^{at}\sin(bt),\ te^{at}\cos(bt),\ te^{at}\sin(bt),\ \dots,$$
$$t^ne^{at}\cos(bt),\ t^ne^{at}\sin(bt),\quad a,\ b \neq 0. \ ■$$

6.13 Example. Consider the functions

$$f_1(t) = 2t^2 + t, \quad f_2(t) = t^2 + 1, \quad f_3(t) = t - 2.$$

Simple algebra shows that

$$f_3(t) = f_1(t) - 2f_2(t),$$

which means that these functions are linearly dependent on the set of real numbers. Then, as expected (see Remark 6.12(i)), their Wronskian, computed by expansion in the third row, is

$$W[f_1, f_2, f_3](t) = \begin{vmatrix} 2t^2+t & t^2+1 & t-2 \\ 4t+1 & 2t & 1 \\ 4 & 2 & 0 \end{vmatrix} = 4\begin{vmatrix} t^2+1 & t-2 \\ 2t & 1 \end{vmatrix} - 2\begin{vmatrix} 2t^2+t & t-2 \\ 4t+1 & 1 \end{vmatrix}$$
$$= 4[t^2 + 1 - 2t(t-2)] - 2[2t^2 + t - (4t+1)(t-2)] = 0. \ ■$$

6.14 Example. Let $f_1(t) = e^{-t}$, $f_2(t) = e^t$, and $f_3(t) = te^t$. Using the common factor property mentioned in Remark 6.3(ii) and performing expansion in the first column, we find that

$$W[f_1, f_2, f_3](t) = \begin{vmatrix} e^{-t} & e^t & te^t \\ -e^{-t} & e^t & (t+1)e^t \\ e^{-t} & e^t & (t+2)e^t \end{vmatrix} = e^{-t}e^te^t \begin{vmatrix} 1 & 1 & t \\ -1 & 1 & t+1 \\ 1 & 1 & t+2 \end{vmatrix}$$

$$= e^t \left\{ \begin{vmatrix} 1 & t+1 \\ 1 & t+2 \end{vmatrix} + \begin{vmatrix} 1 & t \\ 1 & t+2 \end{vmatrix} + \begin{vmatrix} 1 & t \\ 1 & t+1 \end{vmatrix} \right\}$$

$$= e^t[(t+2) - (t+1) + (t+2) - t + (t+1) - t] = 4e^t.$$

Since the Wronskian is nonzero for all real values of t, from Remark 6.12(i) it follows that the three given functions are linearly independent on the real line. ■

6.15 Example. Suppose that we want to find numbers a, b, and c such that

$$(a+b)e^{-t} + (2a-b+c)e^t + (a-b-c)te^t = 5e^t$$

for all real values of t. According to Example 6.14, the functions e^{-t}, e^t, and te^t are linearly independent, so, by Remark 6.12(ii) and Theorem 4.16, we may identify their coefficients on both sides and arrive at the system

$$\begin{aligned} a + b \qquad &= 0, \\ 2a - b + c &= 5, \\ a - b - c &= 0, \end{aligned}$$

with the unique solution $a = 1$, $b = -1$, and $c = 2$.

Verification with Mathematica®. The input

```
((a + b) * E^(-t) + (2 * a - b + c) * E^t + (a - b - c) * t * E^t
          - 5 * E^t) /. {a→ 1, b→ −1, c→ 2} // Simplify
```

generates the output 0. ■

Exercises

In 1–4, compute the Wronskian of the given functions and state whether they are linearly dependent or independent on the real line.

1 $f_1(x) = x^2 + 3x, \quad f_2(x) = x^2 + 2, \quad f_3(x) = x^2 + 9x - 4.$

2 $f_1(x) = 2x + 3, \quad f_2(x) = 2x^2 - x + 1, \quad f_3(x) = x^2 - x.$

3 $f_1(t) = e^t, \quad f_2(t) = e^{2t}, \quad f_3(t) = te^{-t}.$

4 $f_1(t) = \cos t, \quad f_2(t) = \sin t, \quad f_3(t) = \cos(2t).$

In 5–8, verify that the given functions are linearly independent on the real line and then find the numbers a, b, and c for which the equality indicated in each case holds for all real values of the variable.

5 $f_1(t) = 1, \quad f_2(t) = t - 2, \quad f_3(t) = t^2 + t;$
$(2a + b + c)f_1(t) + (a - c)f_2(t) + (b + 2c)f_3(t) = 3t^2 + 2t + 5.$

6 $f_1(t) = t + 1, \quad f_2(t) = t^2 - 1, \quad f_3(t) = t^2;$
$(a - b - c)f_1(t) + (a + b - 2c)f_2(t) + (2a + b + c)f_3(t) = 3t^2 - t - 6.$

7 $f_1(x) = 1, \quad f_2(x) = e^x, \quad f_3(x) = xe^x;$
$(2a + 2b - c)f_1(x) + (a - b + c)f_2(x) + (a - 2c)f_3(x) = 1 + 6e^x + xe^x.$

8 $f_1(x) = 1, \quad f_2(x) = 2 - \cos x, \quad f_3(x) = \cos x - 3\sin x;$
$(a - 2b + c)f_1(x) + (3a + b - 2c)f_2(x) + (2a - b - c)f_3(x) = 5 + 3\cos x - 6\sin x.$

Answers to Odd-Numbered Exercises

1 $W[f_1, f_2, f_3](x) = 0$; linearly dependent.

3 $W[f_1, f_2, f_3](t) = (6t - 5)e^{2t}$; linearly independent.

5 $W[f_1, f_2, f_3](t) = 2; \quad a = 1, \ b = -1, \ c = 2.$

7 $W[f_1, f_2, f_3](x) = e^{2x}; \quad a = 3, \ b = -2, \ c = 1.$

6.3 Homogeneous Differential Equations

The general form of an IVP involving a linear DE of order n is

$$Ly = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = f(t), \tag{6.2}$$

$$y(t_0) = y_0, \ y'(t_0) = y_{10}, \ \ldots, \ y^{(n-1)}(t_0) = y_{n-1,0}, \tag{6.3}$$

where n is a positive integer, a_1, a_2, …, a_{n-1}, and f are given functions,

$$L = D^n + a_1(t)D^{n-1} + \cdots + a_{n-1}(t)D + a_n(t)$$

is the nth-order differential operator defined by the left-hand side of the equation, and y_0, y_{10}, …, $y_{n-1,0}$ are prescribed numbers.

6.16 Theorem (Existence and uniqueness). *If the functions a_1, a_2, …, a_n, and f are continuous on an open interval J and t_0 is a point in J, then the IVP* (6.2), (6.3) *has a unique solution on J.* ∎

6.17 Example. Consider the IVP

$$(t + 2)y''' - 2ty'' + y\sin t = te^t, \quad y(0) = -1, \ y'(0) = 2, \ y''(0) = 0.$$

To bring the DE to the form (6.2), we divide both sides by $t + 2$ and see that

$$a_1(t) = -\frac{2t}{t + 2}, \quad a_2(t) = 0, \quad a_3(t) = \frac{\sin t}{t + 2}, \quad f(t) = \frac{te^t}{t + 2}.$$

All these functions are continuous on the intervals $-\infty < t < -2$ and $-2 < t < \infty$. Since $t_0 = 0$ belongs to the latter, we conclude that, by Theorem 6.16, the given IVP has a unique solution on the open interval $-2 < t < \infty$. ∎

In the first instance, we consider IVPs for homogeneous DE of order n; that is, where equation (6.2) is of the form

$$Ly = 0. \tag{6.4}$$

6.18 Remark. The building blocks of the solution of the IVP (6.4), (6.3) coincide with those used for second-order equations. Thus, the generalizations to this case of Remarks 4.20, 4.25, and 4.28, Theorems 4.21, 4.22, and 4.24, and Definition 4.26 are obvious and immediate, and we will apply them without further explanations. ■

Below, we assume that all the functions $a_1, a_2, \dots, a_n$ are constant. The nature of the roots of the characteristic equation dictates the form of the GS just as it did in Chap. 4. Since here we use the same procedure to construct an FSS, we will omit the verification of the fact that the Wronskian of these solutions is nonzero.

6.19 Example. The characteristic equation for the DE

$$y^{(4)} - 5y'' + 4y = 0$$

is $r^4 - 5r^2 + 4 = 0$, which is easily solved if we notice that this is a quadratic in the variable r^2. Then we have $r^2 = 1$ and $r^2 = 4$, so the characteristic roots are 1, -1, 2, and -2. Since they are real and distinct, the GS of the DE is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}. \ ■$$

6.20 Example. The characteristic equation for the DE in the IVP

$$y''' + 3y'' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 7, \quad y''(0) = -14$$

is $r^3 + 3r^2 - 4 = 0$. To solve it, we notice that $r = 1$ is a root. Synthetic division (see Sect. A.2) now helps us find that the other two roots are both equal to -2, so the GS of the DE is

$$y(t) = c_1 e^t + (c_2 + c_3 t)e^{-2t}.$$

If we apply the ICs, we arrive at the linear system

$$\begin{aligned} c_1 + \ \ c_2 \qquad\quad &= 1, \\ c_1 - 2c_2 + \ \ c_3 &= 7, \\ c_1 + 4c_2 - 4c_3 &= -14, \end{aligned}$$

which yields $c_1 = 2$, $c_2 = -1$, and $c_3 = 3$; therefore, the solution of the IVP is

$$y(t) = 2e^t + (3t - 1)e^{-2t}.$$

Verification with Mathematica®. The input

```
y = 2 * E^t + (3 * t - 1) * E^(-2 * t);
{D[y,{t,3}] + 3 * D[y,t,t] - 4 * y, {y,D[y,t],D[y,t,t]}/. t → 0}
      // Simplify
```

generates the output $\{0, \{1, 7, -14\}\}$. ■

6.21 Remark. The algorithm of synthetic division mentioned above is a useful tool for determining whether a polynomial equation of degree $n \geq 3$ has integral (or, more generally, rational) roots. ■

6.22 Example. The characteristic equation for the DE

$$y^{(4)} - 2y''' - 3y'' + 4y' + 4y = 0$$

is $r^4 - 2r^3 - 3r^2 + 4r + 4 = 0$. Testing the divisors of the constant term with synthetic division, we find that $r_1 = r_2 = -1$ and $r_3 = r_4 = 2$; hence, the GS of the given DE is

$$y(t) = (c_1 + c_2 t)e^{-t} + (c_3 + c_4 t)e^{2t}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = (c1 + c2 * t) * E^(-t) + (c3 + c4 * t) * E^(2 * t);
D[y,{t,4}] - 2 * D[y,{t,3}] - 3 * D[y,t,t] + 4*D[y,t] + 4 * y
        // Simplify
```

generates the output 0. ■

6.23 Example. Consider the DE

$$2y''' - 9y'' + 14y' - 5y = 0,$$

whose characteristic equation is $2r^3 - 9r^2 + 14r - 5 = 0$. Here, synthetic division yields no integer roots, but identifies the fraction $r_1 = 1/2$ as a root, along with the complex conjugate roots $r_2 = 2 + i$ and $r_3 = 2 - i$. Thus, the general solution of the given DE is

$$y(t) = c_1 e^{t/2} + e^{2t}(c_2 \cos t + c_3 \sin t).$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = c1 * E^(t/2) + E^(2 * t) * (c2 * Cos[t] + c3 * Sin[t]);
2*D[y,{t,3}] - 9 * D[y,t,t] + 14*D[y,t] - 5 * y // Simplify
```

generates the output 0. ■

6.24 Example. The physical parameters E, I, and m of a horizontal beam of length π are such that $EI/m = 1$ (see Sect. 4.1). If the beam has one end embedded in a rigid wall and its other end starts moving from equilibrium with initial velocity 1, then the deflection $y(x)$ of the beam is the solution of the BVP

$$y^{(4)} - y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y(\pi) = 0, \quad y'(\pi) = 1.$$

The associated characteristic equation $r^4 - 1 = 0$ has roots 1, -1, i, and $-i$, so the GS of the DE can be written in the form

$$y(t) = c_1 \cosh t + c_2 \sinh t + c_3 \cos t + c_4 \sin t.$$

Since $y'(t) = c_1 \sinh t + c_2 \cosh t - c_3 \sin t + c_4 \cos t$, the BCs yield the algebraic system

$$\begin{aligned} c_1 \qquad\qquad\qquad + c_3 \qquad &= 0,\\ c_2 \qquad + c_4 &= 0,\\ (\cosh \pi)c_1 + (\sinh \pi)c_2 - c3 \qquad &= 0,\\ (\sinh \pi)c_1 + (\cosh \pi)c_2 \qquad - c4 &= 1. \end{aligned}$$

We could certainly solve the above system by Gaussian elimination, but this time it is easier to replace $c_3 = -c_1$ and $c_4 = -c_2$ from the first two equations into the last two and arrive at the new, simpler system

$$\begin{aligned} (1 + \cosh \pi)c_1 + \qquad (\sinh \pi)c_2 &= 0,\\ (\sinh \pi)c_1 + (1 + \cosh \pi)c_2 &= 1, \end{aligned}$$

with solution $c_1 = -\sinh\pi/(2(1+\cosh\pi))$ and $c_2 = 1/2$. Hence, the solution of the BVP is

$$y(x) = \frac{\sinh\pi}{2(1+\cosh\pi)}(\cos x - \cosh x) - \frac{1}{2}(\sin x - \sinh x).$$

The first coefficient on the right-hand side simplifies to $(\tanh(\pi/2))/2$.

VERIFICATION WITH MATHEMATICA®. The input

```
y = (Sinh[Pi]/(2 * (1 + Cosh[Pi]))) * (Cos[x] - Cosh[x])
- (1/2) * (Sin[x] - Sinh[x]);
{D[y,{x,4}] - y, {y,D[y,x]}/. x→ 0, {y,D[y,x]}/. x→Pi}
        // Simplify
```

generates the output {0, {0, 0}, {0, 1}}. ■

Exercises

In 1–8, write the form of the GS for the DE whose characteristic roots are as specified.

1 $r_1 = -2,\ r_2 = 0,\ r_3 = 4.$ **2** $r_1 = 1,\ r_2 = 3/2,\ r_3 = 3.$

3 $r_1 = r_2 = -1,\ r_3 = 1.$ **4** $r_1 = 2,\ r_2 = 3+i,\ r_3 = 3-i.$

5 $r_1 = r_2 = 1/2,\ r_3 = -1+2i,\ r_4 = -1-2i.$

6 $r_1 = r_2 = r_3 = 0,\ r_4 = 3.$

7 $r_1 = 2+3i,\ r_2 = 2-3i,\ r_3 = r_4 = -2,\ r_5 = 0.$

8 $r_1 = r_2 = r_3 = -1,\ r_4 = r_5 = 2,\ r_6 = i,\ r_7 = -i.$

In 9–16, at least one root of the characteristic equation is an integer. Use the synthetic division procedure described in Sect. A.2, if necessary, to compute the solution of the given IVP.

9 $y''' - 3y'' - 10y' = 0,\quad y(0) = 1,\ y'(0) = 2,\ y''(0) = -4.$

10 $y''' - y'' - 4y' + 4y = 0,\quad y(0) = 2,\ y'(0) = -5,\ y''(0) = 5.$

11 $y''' + y'' - 5y' + 3y = 0,\quad y(0) = 4,\ y'(0) = -5,\ y''(0) = 18.$

12 $y''' - y'' + 3y' + 5y = 0,\quad y(0) = 2,\ y'(0) = -4,\ y''(0) - 2.$

13 $4y''' - 4y'' - 3y' - 10y = 0,\quad y(0) = 2,\ y'(0) = -5,\ y''(0) = -12.$

14 $y''' + 6y'' + 12y' + 8y = 0,\quad y(0) = -3,\ y'(0) = 6,\ y''(0) = -10.$

15 $y^{(4)} - 6y''' + 9y'' + 6y' - 10y = 0,\quad y(0) = 3,\ y'(0) = 4,\ y''(0) = 11,\ y'''(0) = 9.$

16 $y^{(4)} - y''' - 2y'' = 0,\quad y(0) = 1,\ y'(0) = 0,\ y''(0) = 2,\ y'''(0) = -2.$

Answers to Odd-Numbered Exercises

1 $y(t) = c_1e^{-2t} + c_2 + c_3e^{4t}.$ **3** $y(t) = (c_1 + c_2t)e^{-t} + c_3e^{t}.$

5 $y(t) = (c_1 + c_2t)e^{t/2} + e^{-t}[c_3\cos(2t) + c_4\sin(2t)].$

7 $y(t) = e^{2t}[c_1\cos(3t) + c_2\sin(3t)] + (c_3t + c_4)e^{-2t} + c_5.$

9 $y(t) = 2 - e^{-2t}$. **11** $y(t) = (2-t)e^t + 2e^{-3t}$.

13 $y(t) = -2e^{2t} + e^{-t/2}(4\cos t + \sin t)$. **15** $y(t) = e^{-t} + e^{3t}(2\cos t - \sin t)$.

6.4 Nonhomogeneous Equations

These are DEs of the form (6.2) in which f is a nonzero function. The statement of Theorem 4.37 applies here as well, so the general solution of such an equation is written as $y(t) = y_c(t) + y_p(t)$, where y_c is the complementary function (the GS of the associated homogeneous equation) and y_p is any particular solution of the full nonhomogeneous equation. As in Chap. 4, the latter can be found by the method of undetermined coefficients or by that of variation of parameters.

6.4.1 Method of Undetermined Coefficients

We illustrate the application of this technique by discussing a few specific examples.

6.25 Example. Suppose that the characteristic roots and right-hand side of a DE are, respectively,

$$r_1 = r_2 = 0, \quad r_3 = -1, \quad r_4 = 3 + i, \quad r_5 = 3 - i,$$
$$f(t) = t^2 - 1 + te^{-t} + 2e^t + (t-2)e^{3t}\sin t.$$

Treating f as a sum of four individual terms and using Theorem 4.47, we enter all the necessary details in Table 6.1.

Table 6.1

Term in $f(t)$	n	$\lambda + i\mu$	s	Term in $y_p(t)$
$t^2 - 1$	2	0	2	$t^2(\alpha_0 t^2 + \alpha_1 t + \alpha_2)$
te^{-t}	1	-1	1	$t(\beta_0 t + \beta_1)e^{-t}$
$2e^t$	0	1	0	γe^t
$(t-2)e^{3t}\sin t$	1	$3+i$	1	$te^{3t}[(\delta_0 t + \delta_1)\cos t + (\varepsilon_0 t + \varepsilon_1)\sin t]$

The sum of the functions in the last column of the table is the general form of a PS for the given DE. ■

6.26 Example. Consider the IVP

$$y''' - y'' - 2y' = 4e^t, \quad y(0) = 0, \quad y'(0) = 2, \quad y''(0) = 0.$$

The characteristic equation $r^3 - r^2 - 2r = r(r^2 - r - 2) = 0$ has roots 0, -1, and 2, so

$$y_c(t) = c_1 + c_2 e^{-t} + c_3 e^{2t}.$$

Since 1 is not a characteristic root, we try a PS of the form $y_p(t) = \alpha e^t$. After replacing in the DE and equating the coefficients of e^t on both sides, we find that $\alpha = -2$. Hence, $y_p(t) = -2e^t$, and the GS of the DE is

$$y(t) = y_c(t) + y_p(t) = c_1 + c_2e^{-t} + c_3e^{2t} - 2e^t.$$

Using the ICs, we now arrive at the system

$$\begin{aligned} c_1 + c_2 + c_3 &= 2, \\ - c_2 + 2c_3 &= 4, \\ c_2 + 4c_3 &= 2, \end{aligned}$$

which is easily solved, either by Gaussian elimination, or, more simply in this case, by obtaining $c_2 = -2$ and $c_3 = 1$ from the last two equations and then $c_1 = 3$ from the first one. The solution of the IVP is, therefore,

$$y(t) = 3 - 2e^{-t} + e^{2t} - 2e^t.$$

Verification with Mathematica®. The input

```
y = 3 - 2*E^(-t) + E^(2*t) - 2*E^t;
{D[y,{t,3}] - D[y,t,t] - 2*D[y,t] - 4*E^t, {y,D[y,t],D[y,t,t]}
        /.t→ 0}//Simplify
```

generates the output $\{0, \{0, 2, 0\}\}$. ■

6.27 Example. The characteristic equation in the IVP

$$y''' - y'' - 5y' - 3y = -6t - 7, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 4$$

is $r^3 - r^2 - 5r - 3 = 0$. Following the procedure set out in Example 6.22, we find that the roots are -1, -1, and 3, so

$$y_c(t) = (c_1 + c_2t)e^{-t} + c_3e^{3t}.$$

Since 0 is not a root, we try

$$y_p(t) = \alpha_0t + \alpha_1$$

and from the DE obtain in the usual way the values $\alpha_0 = 2$ and $\alpha_1 = -1$. Consequently, the GS of the equation is

$$y(t) = y_c(t) + y_p(t) = (c_1 + c_2t)e^{-t} + c_3e^{3t} + 2t - 1.$$

The ICs now lead to the system

$$\begin{aligned} c_1 + c_3 &= 2, \\ -c_1 + c_2 + 3c_3 &= -3, \\ c_1 - 2c_2 + 9c_3 &= 4, \end{aligned}$$

with solution $c_1 = 2$, $c_2 = -1$, and $c_3 = 0$. Hence, the solution of the given IVP is

$$y(t) = (2 - t)e^{-t} + 2t - 1.$$

Verification with Mathematica®. The input

```
y = (2 - t)*E^(-t) + 2*t - 1;
{D[y,{t,3}] - D[y,t,t] - 5*D[y,t] - 3*y + 6*t + 7,
        {y,D[y,t],D[y,t,t]}/.t→ 0}//Simplify
```

generates the output $\{0, \{1, -1, 4\}\}$. ■

6.28 Example. Consider the IVP

$$y''' + 4y' = 8t - 4, \quad y(0) = -1, \quad y'(0) = -3, \quad y''(0) = 10.$$

The characteristic equation $r^3 + 4r = r(r^2 + 4) = 0$ has roots $2i$, $-2i$, and 0, so

$$y_c(t) = c_1 \cos(2t) + c_2 \sin(2t) + c_3.$$

Since 0 is a simple root (that is, of multiplicity 1) of the characteristic equation, from Theorem 4.47 with $n = 1$, $\lambda = \mu = 0$, and $s = 1$ it follows that we should try a PS of the form

$$y_p(t) = t(\alpha_0 t + \alpha_1) = \alpha_0 t^2 + \alpha_1 t.$$

Substituting in the DE and matching the coefficients of the first-degree polynomials on both sides, we find that $\alpha_0 = 1$ and $\alpha_1 = -1$, which leads to the GS

$$y(t) = y_c(t) + y_p(t) = c_1 \cos(2t) + c_2 \sin(2t) + c_3 + t^2 - t.$$

Then, applying the ICs, we get $c_1 = -2$, $c_2 = -1$, and $c_3 = 1$, which means that the solution of the IVP is

$$y(t) = -2\cos(2t) - \sin(2t) + t^2 - t + 1.$$

Verification with Mathematica®. The input

```
y= - 2*Cos[2*t] - Sin[2*t] + t^2 - t + 1;
{D[y,{t,3}] + 4*D[y,t] - 8*t + 4,{y,D[y,t],D[y,t,t]}
        /.t→ 0}//Simplify
```

generates the output $\{0, \{-1, -3, 10\}\}$. ■

6.29 Example. In the case of the IVP

$$y^{(4)} - 2y''' + 2y' - y = -12e^t, \quad y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 1, \quad y'''(0) = -11,$$

the roots of the characteristic equation are 1, 1, 1, and -1. This can be determined either by applying the method described in Example 6.22 or by noticing that the left-hand side of the characteristic equation can be factored in the form

$$\begin{aligned} r^4 - 2r^3 + 2r - 1 &= (r^4 - 1) - 2r(r^2 - 1) = (r^2 - 1)(r^2 + 1 - 2r) \\ &= (r+1)(r-1)(r-1)^2 = (r+1)(r-1)^3. \end{aligned}$$

Then

$$y_c(t) = (c_1 + c_2 t + c_3 t^2)e^t + c_4 e^{-t}$$

and, according to Theorem 4.47 with $r = 1$, $n = 1$, $\lambda = \mu = 0$, and $s = 3$, we try

$$y_p(t) = \alpha t^3 e^t.$$

After the necessary computations on the left-hand side of the DE, we find that $\alpha = -1$, so the GS of the DE is

$$y(t) = y_c(t) + y_p(t) = (c_1 + c_2 t + c_3 t^2 - t^3)e^t + c_4 e^{-t}.$$

To find the constants, we differentiate y three times and then use the ICs, which leads to the system

$$\begin{aligned}
c_1 \qquad\qquad\quad &+ c_4 = 3,\\
c_1 + \ \ c_2 \qquad\quad &- c_4 = 1,\\
c_1 + 2c_2 + 2c_3 &+ c_4 = 1,\\
c_1 + 3c_2 + 6c_3 &- c_4 = -5.
\end{aligned}$$

We could, of course, apply Gaussian elimination, but this time we prefer to simplify the procedure by making use of the peculiarities of the system's structure. Thus, replacing $c_1 + c_4 = 3$ from the first equation into the third and $c_1 - c_4 = 1 - c_2$ from the second equation into the fourth, after simplification we obtain the new system

$$\begin{aligned}
c_2 + \ \ c_3 &= -1,\\
c_2 + 3c_3 &= -3,
\end{aligned}$$

with solution $c_2 = 0$ and $c_3 = -1$. Then $c_1 = 2$ and $c_4 = 1$, so the solution of the IVP is

$$y(t) = (2 - t^2 - t^3)e^t + e^{-t}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=(2 - t^2 - t^3)*E^t + E^(-t);
{D[y,{t,4}] - 2*D[y,{t,3}] + 2*D[y,t] - y + 12*E^t,
{y,D[y,t],D[y,t,t],D[y,{t,3}]}/.t→ 0}//Simplify
```

generates the output $\{0,\ \{3,\ 1,\ 1,\ -11\}\}$. ■

Exercises

In 1–8, use Theorem 4.47 to write the form of a PS for the DE whose characteristic roots and right-hand side are as specified.

1 $r_1 = 1,\ r_2 = -2,\ r_3 = r_4 = 4;\quad f(t) = 2 - e^{2t} + te^t + (t-3)e^{4t}$.

2 $r_1 = 2,\ r_2 = r_3 = r_4 = -1,\ r_5 = 4;\quad f(t) = t + (2t+1)e^{-4t} + 3e^{2t} - (t+1)e^{-t}$.

3 $r_1 = 0,\ r_2 = r_3 = -3,\ r_4 = 2i,\ r_5 = -2i;\quad f(t) = -1 + t^2e^{-3t} + 2\cos t - 3\sin(2t)$.

4 $r_1 = r_2 = 0,\ r_3 = r_4 = r_5 = 1,\ r_6 = i/2,\ r_7 = -i/2;$
$f(t) = 2t - 1 + 4te^t + 3\cos(2t) + e^{t/2}\sin t$.

5 $r_1 = 0,\ r_2 = r_3 = 1/3,\ r_4 = 4i,\ r_5 = -4i;$
$f(t) = te^t + (t^2-1)e^{t/3} + e^t\sin t + t\cos(4t)$.

6 $r_1 = r_2 = 0,\ r_3 = r_4 = r_5 = -1,\ r_6 = i,\ r_7 = -i;$
$f(t) = t^3 - 2t + 4te^t + t^2e^{-t} + (1-t)\sin t$.

7 $r_1 = r_2 = r_3 = 0,\ r_4 = r_5 = 2,\ r_6 = 1+i,\ r_7 = 1-i;$
$f(t) = t + 2 + (3t^2-1)e^{2t} + te^t - 2e^{-t}\sin t$.

8 $r_1 = r_2 = r_3 = r_4 = 0,\ r_5 = 1,\ r_6 = 1+2i,\ r_7 = 1-2i;$
$f(t) = 2 + 3t^2e^t + (t-1)e^{2t} + t^2e^t\cos(2t)$.

In 9–20, at least one of the roots of the characteristic equation is an integer. Use the method of undetermined coefficients in conjunction with synthetic division, if necessary, to find a PS of the DE, then solve the given IVP.

9 $y''' - 2y'' - 8y' = -27e^t, \quad y(0) = 3,\ y'(0) = -7,\ y''(0) = -25.$

10 $y''' - (5/2)y'' + y' = 9e^{-t}, \quad y(0) = -3,\ y'(0) = 2,\ y''(0) = 1.$

11 $y''' - 2y'' - 7y' - 4y = 10e^{-t}, \quad y(0) = 4,\ y'(0) = 0,\ y''(0) = 19.$

12 $y''' - y'' - 8y' + 12y = 20e^{2t}, \quad y(0) = 1,\ y'(0) = 8,\ y''(0) = 7.$

13 $y''' - 2y' + 4y = 4t - 14, \quad y(0) = -3,\ y'(0) = 6,\ y''(0) = 0.$

14 $y''' + 3y'' + 7y' + 5y = -8e^{-t}, \quad y(0) = 3,\ y'(0) = -7,\ y''(0) = 11.$

15 $y^{(4)} - 2y''' + y'' = -2, \quad y(0) = -3,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = 5.$

16 $y^{(4)} - y''' + 4y'' - 4y' = -3(\cos t + \sin t),$
$y(0) = -1,\ y'(0) = 5,\ y''(0) = 6,\ y'''(0) = -7.$

17 $y^{(4)} + y''' - y'' + y' - 2y = -4\cos t + 12\sin t,$
$y(0) = 2,\ y'(0) = 5,\ y''(0) = -5,\ y'''(0) = 3.$

18 $y^{(4)} - 2y''' - 2y'' + 6y' + 5y = -20e^{-t},$
$y(0) = -1,\ y'(0) = 1,\ y''(0) = 1,\ y'''(0) = 29.$

19 $y^{(5)} - 2y^{(4)} + 5y''' = 48 - 120t,$
$y(0) = 0,\ y'(0) = -1,\ y''(0) = 0,\ y'''(0) = 2,\ y^{(4)}(0) = 0.$

20 $y^{(5)} - 2y^{(4)} - 3y''' + 4y'' + 4y' = -18e^{-t},$
$y(0) = 2,\ y'(0) = 1,\ y''(0) = 8,\ y'''(0) = -1,\ y^{(4)}(0) = 32.$

Answers to Odd-Numbered Exercises

1 $y_p(t) = \alpha + \beta e^{2t} + t(\gamma_1 + \gamma_2 t)e^t + t^2(\delta_1 + \delta_2 t)e^{4t}.$

3 $y_p(t) = \alpha t + t^2(\beta_1 + \beta_2 t + \beta_3 t^2)e^{-3t} + \gamma_1 \cos t + \gamma_2 \sin t + t[\delta_1 \cos(2t) + \delta_2 \sin(2t)].$

5 $y_p(t) = (\alpha_1 + \alpha_2 t)e^t + t^2(\beta_1 + \beta_2 t + \beta_3 t^2)e^{t/3} + e^t(\gamma_1 \cos t + \gamma_2 \sin t)$
$+ t[(\delta_1 + \delta_2 t)\cos(4t) + (\varepsilon_1 + \varepsilon_2 t)\sin(4t)].$

7 $y_p(t) = t^3(\alpha_1 + \alpha_2 t) + t^2(\beta_1 + \beta_2 t + \beta_3 t^2)e^{2t} + (\gamma_1 + \gamma_2 t)e^t + e^{-t}(\delta_1 \cos t + \delta_2 \sin t).$

9 $y(t) = 1 + e^{-2t} - 2e^{4t} + 3e^t.$ **11** $y(t) = (3 - t)e^{-t} + e^{4t} - t^2 e^{-t}.$

13 $y(t) = -e^{-2t} + e^t(\cos t + 2\sin t) + t - 3.$ **15** $y(t) = (2t - 1)e^t - t^2 - t - 2.$

17 $y(t) = e^t - e^{-2t} + (2t + 2)\cos t.$ **19** $y(t) = t + 2t^2 - t^4 - e^t \sin(2t).$

6.4.2 Method of Variation of Parameters

This solution technique for higher-order DEs is based on the same idea as for second-order equations.

6.30 Example. The characteristic equation for the DE

$$y''' - 2y'' - y' + 2y = 2t - 5$$

is $r^3 - 2r^2 - r + 2 = 0$, with roots 1, -1, and 2. Therefore, $\{e^t, e^{-t}, e^{2t}\}$ is an FSS, so we seek a PS of the form

$$y_p = u_1 e^t + u_2 e^{-t} + u_3 e^{2t},$$

where u_1, u_2, and u_3 are functions of t. Then

$$y_p' = u_1' e^t + u_1 e^t + u_2' e^{-t} - u_2 e^{-t} + u_3' e^{2t} + 2u_3 e^{2t},$$

and we impose the usual restriction on the functions u_1, u_2, and u_3, namely

$$u_1' e^t + u_2' e^{-t} + u_3' e^{2t} = 0. \tag{6.5}$$

The second-order derivative is now

$$y_p'' = u_1' e^t + u_1 e^t - u_2' e^{-t} + u_2 e^{-t} + 2u_3' e^{2t} + 4u_3 e^{2t},$$

and we impose the second restriction

$$u_1' e^t - u_2' e^{-t} + 2u_3' e^{2t} = 0. \tag{6.6}$$

In view of this, the third-order derivative is

$$y_p''' = u_1' e^t + u_1 e^t + u_2' e^{-t} - u_2 e^{-t} + 4u_3' e^{2t} + 8u_3 e^{2t}.$$

Replacing y_p', y_p'', and y_p''' in the DE and sorting out the terms, we arrive at the equality

$$u_1' e^t + u_2' e^{-t} + 4u_3' e^{2t} = 2t - 5. \tag{6.7}$$

Equations (6.5)–(6.7) form a linear system for $u_1' e^t$, $u_2' e^{-t}$, and $u_3' e^{2t}$, which, when solved by Gaussian elimination, gives rise to the arrays

$$\left.\begin{array}{rrr|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 4 & 2t-5 \end{array}\quad \begin{array}{c} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}\quad \Rightarrow \quad \begin{array}{rrr|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 3 & 2t-5 \end{array}\right.$$

and, hence, to the simpler system

$$\begin{aligned} u_1' e^t + \quad u_2' e^{-t} + \quad u_3' e^{2t} &= 0, \\ -\,2u_2' e^{-t} + \quad u_3' e^{2t} &= 0, \\ 3u_3' e^{2t} &= 2t - 5, \end{aligned}$$

with solution

$$u_1' e^t = \tfrac{1}{2}\,(5 - 2t), \quad u_2' e^{-t} = \tfrac{1}{6}\,(2t - 5), \quad u_3' e^{2t} = \tfrac{1}{3}\,(2t - 5).$$

Solving for u_1', u_2', and u_3', integrating by parts, and omitting the arbitrary constants of integration, we now obtain

$$u_1(t) = \tfrac{1}{2}\,(2t - 3)e^{-t}, \quad u_2(t) = \tfrac{1}{6}\,(2t - 7)e^{t}, \quad u_3(t) = \tfrac{1}{3}\,(2 - t)e^{-2t},$$

so

$$y_p(t) = \tfrac{1}{2}\,(2t - 3) + \tfrac{1}{6}\,(2t - 7) + \tfrac{1}{3}\,(2 - t) = t - 2.$$

Verification with Mathematica®. The input

```
y=t - 2;
D[y,{t,3}] - 2*D[y,t,t] - D[y,t] + 2*y - 2*t + 5//Simplify
```

generates the output 0. ■

6.31 Example. It is easily seen that the roots of the characteristic equation for the DE

$$y''' + y' = 2\sin t$$

are 0, i, and $-i$, so $\{1, \cos t, \sin t\}$ is an FSS. Then we seek a particular solution of the form

$$y_p = u_1 + u_2 \cos t + u_3 \sin t$$

and, proceeding as in Example 6.30, arrive at the system

$$\begin{aligned} u_1' + (\cos t)u_2' + (\sin t)u_3' &= 0, \\ -(\sin t)u_2' + (\cos t)u_3' &= 0, \\ -(\cos t)u_2' - (\sin t)u_3' &= 2\sin t. \end{aligned}$$

After computing u_2' and u_3' from the last two equations, u_1' is obtained from the first one; thus,

$$u_1'(t) = 2\sin t, \quad u_2'(t) = -2\sin t\cos t, \quad u_3'(t) = -2\sin^2 t = -1 + \cos(2t),$$

from which, by integration,

$$u_1(t) = -2\cos t, \quad u_2(t) = \cos^2 t, \quad u_3(t) = -t + \tfrac{1}{2}\sin(2t) = -t + \sin t\cos t,$$

yielding

$$\begin{aligned} y_p(t) &= -2\cos t + \cos^2 t\cos t + (-t + \sin t\cos t)\sin t \\ &= -2\cos t + \cos t(\cos^2 t + \sin^2 t) - t\sin t = -\cos t - t\sin t. \end{aligned}$$

Obviously, the term $-\cos t$ is not significant since it can be included in the complementary function, which is of the form

$$y_c(t) = c_1 + c_2\cos t + c_3\sin t.$$

Verification with Mathematica®. The input

```
y = - Cos[t] - t*Sin[t];
D[y,{t,3}] + D[y,t] - 2*Sin[t] // Simplify
```

generates the output 0. ■

Exercises

Use the method of variation of parameters to find a PS of the given DE.

1 $y''' - y' = -2$. **2** $y''' - 3y' + 2y = -4e^{-t}$.

3 $y''' + 3y'' + 3y' + y = 6e^{-t}$. **4** $y''' - 6y'' + 12y' - 8y = 6(4t - 1)e^{2t}$.

5 $y''' + 4y' = -8[\cos(2t) + \sin(2t)]$. **6** $y''' - 3y'' + 4y' - 2y = 30e^t\sin(2t)$.

Answers to Odd-Numbered Exercises

1 $y_p(t) = 2t$. **3** $y_p(t) = t^3e^{-t}$. **5** $y_p(t) = t[\cos(2t) + \sin(2t)]$.

Chapter 7
Systems of Differential Equations

As physical phenomena increase in complexity, their mathematical models require the use of more than one unknown function. This gives rise to systems of DEs.

7.1 Modeling with Systems of Equations

Here are a few simple illustrations of such models.

Parallel LRC circuit. If an inductor of inductance L, a resistor of resistance R, and a capacitor of capacitance C are connected in parallel, then the current $I(t)$ through the inductor and the voltage drop $V(t)$ across the capacitor are the solutions of the IVP

$$\begin{aligned} I' &= \frac{1}{L}\,V, & I(0) &= I_0,\\ V' &= -\frac{1}{C}\,I - \frac{1}{RC}\,V, & V(0) &= V_0. \end{aligned}$$

The (first-order) DEs in this IVP are linear and homogeneous.

Military combat. Suppose that two military forces of strengths $x_1(t)$ and $x_2(t)$, respectively, engage in battle, and that the rate at which the troops of each of them are put out of action is proportional to the troop strength of the enemy. If the corresponding proportionality constants are $a > 0$ and $b > 0$, then the evolution of the combat is described by the IVP

$$\begin{aligned} x_1' &= -ax_2, & x_1(0) &= x_{10},\\ x_2' &= -bx_1, & x_2(0) &= x_{20}, \end{aligned}$$

where x_{10} and x_{20} are the initial strengths of the two opposing forces. The above system consists of two linear, homogeneous, first-order DEs.

Solution mix. Consider two containers linked by two pipes. The containers hold volumes v_1 and v_2 of water in which initial amounts s_{10} and s_{20}, respectively, of salt have been dissolved. Additional salt solution is being poured into the containers at rates r_1 and r_2, with salt concentrations c_1 and c_2. Solution flows from the second container into the first one through one of the connecting pipes at a rate r_{21}, and from the first container into the second one through the other connecting pipe at a rate r_{12}. Also, solution is drained from the containers outside the system at rates r_{10} and r_{20}. If $s_1(t)$ and $s_2(t)$ are the amounts of salt in the two containers at time t and all the salt solutions involved are homogeneous, then the process is described by the IVP

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_7,

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$$s_1' = c_1 r_1 - \frac{r_{12} + r_{10}}{v_1}\, s_1 + \frac{r_{21}}{v_2}\, s_2, \quad s_1(0) = s_{10},$$
$$s_2' = c_2 r_2 + \frac{r_{12}}{v_1}\, s_1 - \frac{r_{21} + r_{20}}{v_2}\, s_2, \quad s_2(0) = s_{20},$$

where all the constants involved are nonnegative. The two first-order DEs in this system are linear and nonhomogeneous.

Predator–prey. Suppose that an ecological system consists of only two species: a prey population $p_1(t)$ and a predator population $p_2(t)$. Also, suppose that the net growth rate $\alpha > 0$ of prey in the absence of predator is proportional to the size of the prey population, the death rate $\delta > 0$ of predator in the absence of prey is proportional to the size of the predator population, the predation rate $\beta > 0$ is proportional to the rate of predator–prey encounters, and the birth rate $\gamma > 0$ of predator is proportional to the size of its food supply—in other words, of prey—and to the size of its own population. Under these assumptions, the evolution of the system is described by the IVP

$$\begin{aligned} p_1' &= \alpha p_1 - \beta p_1 p_2, \quad && p_1(0) = p_{10},\\ p_2' &= \gamma p_1 p_2 - \delta p_2, \quad && p_2(0) = p_{20}, \end{aligned}$$

where p_{10} and p_{20} are the initial sizes of the two populations. This system, which consists of two nonlinear, homogeneous, first-order DEs, also describes certain processes in chemistry and economics and is referred to generically as a consumer–resource model.

Contagious disease epidemic. In a given population, let $S(t)$ be the number of individuals susceptible (exposed) to some disease but not yet infected, $I(t)$ the number of those already infected, $R(t)$ the number of those recovered from the disease, α the rate of infection (assumed to be proportional to the number of infected individuals), β the recovery rate of those infected, and γ the reinfection rate of those recovered. The evolution of the disease in the population is modeled by the IVP

$$\begin{aligned} S' &= -\alpha I S + \gamma R, \quad && S(0) = S_0,\\ I' &= \alpha I S - \beta I, \quad && I(0) = I_0,\\ R' &= \beta I - \gamma R, \quad && R(0) = R_0, \end{aligned}$$

where S_0, I_0, and R_0 are the initial sizes of the corresponding population groups. All the first-order DEs above are homogeneous; the first two are nonlinear, whereas the third one is linear.

Coupled mechanical oscillators. Consider two collinear mass–spring systems (see Sect. 5.2.1) with spring constants k_1 and k_2, where the masses m_1 and m_2 are connected to each other by a third spring of constant k_3. Initially in equilibrium (that is, at rest) at points x_{10} and x_{20}, respectively, the two masses are acted upon by two external forces f_1 and f_2. If $x_1(t)$ and $x_2(t)$ are the displacements of the masses from their equilibrium positions at time t, then Newton's second law applied to each mass leads to the IVP

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_3(x_2 - x_1) + f_1, \quad && x_1(0) = x_{10}, \quad x_1'(0) = 0,\\ m_2 x_2'' &= -k_2 x_2 - k_3(x_2 - x_1) + f_2, \quad && x_2(0) = x_{20}, \quad x_2'(0) = 0. \end{aligned}$$

Both equations above are linear, nonhomogeneous, second-order DEs. Their ensemble can easily be rewritten as a system of four first-order equations. Indeed, if we set $x_1' = y_1$ and $x_2' = y_2$ and perform simple algebraic manipulations, we arrive at the equivalent IVP

$$\begin{aligned}
x_1' &= y_1, && x_1(0) = x_{10},\\
y_1' &= -\frac{k_1+k_3}{m_1}\,x_1 + \frac{k_3}{m_1}\,x_2 + \frac{1}{m_1}\,f_1, && y_1(0) = 0,\\
x_2' &= y_2, && x_2(0) = x_{20},\\
y_2' &= \frac{k_3}{m_2}\,x_1 - \frac{k_2+k_3}{m_2}\,x_2 + \frac{1}{m_2}\,f_2, && y_2(0) = 0.
\end{aligned}$$

7.2 Algebra Prerequisites

The algebraic concepts and methods mentioned in this section can be developed for general $m \times n$ matrices with either real or complex elements. However, we restrict our attention to square $n \times n$ real matrices since this is the only type that occurs in our treatment of systems of DEs.

7.2.1 Operations with Matrices

Let M_n be the set of $n \times n$ matrices introduced in Sect. 6.2.1. This set can be endowed with a structure of *vector space*, characterized by two operations: one 'external' (called multiplication of a matrix by a number, or scalar) and one 'internal' (called addition of matrices). Thus, if $A = (a_{ij})$ and $B = (b_{ij})$ are elements of M_n and λ is any number, then

$$\lambda A = (\lambda a_{ij}), \quad A + B = (a_{ij} + b_{ij}), \tag{7.1}$$

both of which also belong to M_n.

7.1 Example. Let

$$A = \begin{pmatrix} 2 & -1 & 3\\ 1 & 1 & -2\\ 4 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & -1\\ 2 & -2 & 1\\ 3 & 1 & -5 \end{pmatrix}, \quad \lambda = -2.$$

By (7.1), we have

$$\lambda A = \begin{pmatrix} -4 & 2 & -6\\ -2 & -2 & 4\\ -8 & 0 & -2 \end{pmatrix}, \quad A + B = \begin{pmatrix} 3 & 1 & 2\\ 3 & -1 & -1\\ 7 & 1 & -4 \end{pmatrix}. \ ■$$

Two particular matrices play a special role in matrix manipulation: the zero matrix 0 and the identity matrix I defined, respectively, by

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The use of the same symbol 0 for both the number zero and the zero matrix should create no ambiguity since its meaning will be clear from the context.

If $A = (a_{ij})$, then we write $-A = (-a_{ij})$.

7.2 Remarks. (i) As expected, two matrices in M_n are said to be equal if each element in one matrix is equal to the element at the same location in the other.

(ii) It is easily verified that for any matrices A, B, and C in M_n and any numbers λ and μ,

$$\begin{aligned}
&A+(B+C)=(A+B)+C,\\
&A+B=B+A,\\
&A+0=A,\\
&A+(-A)=0,\\
&\lambda(A+B)=\lambda A+\lambda B,\\
&(\lambda+\mu)A=\lambda A+\mu A.
\end{aligned}$$

These properties have special names in linear algebra but, for simplicity, we omit them. ■

On M_n we can also define multiplication of matrices. If $A=(a_{ij})$ and $B=(b_{ij})$, then

$$AB=(c_{ij})=\left(\sum_{k=1}^{n}a_{ik}b_{kj}\right). \tag{7.2}$$

We note that to form the matrix AB, we operate with one row of A and one column of B at a time. The generic element c_{ij} in AB consists of the sum of the products of each element in row i in A and the corresponding element in column j in B.

7.3 Example. Let A and B be the matrices defined in Example 7.1. According to formula (7.2), the first element in the matrix AB is

$$c_{11}=\sum_{k=1}^{3}a_{1k}b_{k1}=a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}=(2)(1)+(-1)(2)+(3)(3)=9.$$

The element in the second row and third column is

$$c_{23}=\sum_{k=1}^{3}a_{2k}b_{k3}=a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33}=(1)(-1)+(1)(1)+(-2)(-5)=10.$$

Calculating the remaining elements c_{ij} in a similar manner, we find that

$$AB=\begin{pmatrix}9&9&-18\\-3&-2&10\\7&9&-9\end{pmatrix}. \quad ■$$

7.4 Remarks. (i) Using definition (7.2), we readily verify that for any elements A, B, and C of M_n,

$$\begin{aligned}
&A(BC)=(AB)C,\\
&A(B+C)=AB+AC,\\
&(A+B)C=AC+BC,\\
&IA=AI=A.
\end{aligned}$$

(ii) In general, $AB\neq BA$. As an illustration, we see that for the matrices A and B in Example 7.3 we have

$$BA = \begin{pmatrix} 0 & 1 & -2 \\ 6 & -4 & 11 \\ -13 & -2 & 2 \end{pmatrix} \neq AB.$$

(iii) A $1 \times n$ matrix is usually called an *n-component row vector*. Similarly, an $n \times 1$ matrix is called an *n-component column vector*. In what follows we operate mainly with column vectors, which, to make the notation clearer, are denoted by bold letters.

(iv) As mentioned in Sect. 6.2.1, the elements $a_{11}, \dots, a_{nn}$ of an $n \times n$ matrix A are said to form the *leading diagonal* of A. ■

7.5 Definition. If $A = (a_{ij})$, then $A^{\mathrm{T}} = (a_{ji})$ is called the *transpose* of A. ■

7.6 Example. The transpose of the matrix

$$A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & -1 & 0 \\ 5 & 2 & -3 \end{pmatrix}$$

is

$$A^{\mathrm{T}} = \begin{pmatrix} -2 & 3 & 5 \\ 1 & -1 & 2 \\ 4 & 0 & -3 \end{pmatrix}.$$

The transpose of the column vector

$$\mathbf{v} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$$

is the row vector $\mathbf{v}^{\mathrm{T}} = (2,\ 5,\ -4)$. ■

In view of the definition of matrix multiplication, the linear algebraic system (6.1) can be written generically as

$$A\mathbf{x} = \mathbf{b}, \tag{7.3}$$

where $A = (a_{ij})$ is the $n \times n$ matrix of the system coefficients, $\mathbf{x}$ is the column vector of the unknowns $x_1, \dots, x_n$, and $\mathbf{b}$ is the column vector of the right-hand sides $b_1, \dots, b_n$.

7.7 Example. For the system

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= -1, \\ 2x_1 \phantom{{}- 2x_2} - x_3 &= 3, \\ x_1 + x_2 - 3x_3 &= 7 \end{aligned}$$

we have

$$A = \begin{pmatrix} 3 & -2 & 4 \\ 2 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix},$$

so, in matrix notation, the given system is

$$\begin{pmatrix} 3 & -2 & 4 \\ 2 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix}. \ ■$$

7.8 Definition. A square matrix A such that $\det(A) \neq 0$ is called *nonsingular*, or *invertible*. If $\det(A) = 0$, the matrix is called *singular*, or *non-invertible*. ■

Every nonsingular matrix A has a (unique) *inverse*; that is, there is a matrix A^{-1} of the same type as A that satisfies $AA^{-1} = A^{-1}A = I$.

The inverse matrix can be found by means of an extended version of Gaussian elimination. We place the identity matrix I to the right of A and then perform suitable elementary row operations simultaneously on both sides until the matrix on the left becomes I. It can be shown that the matrix we end up with on the right is A^{-1}.

7.9 Example. Applying the above procedure to the matrix

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 0 \\ 1 & 3 & 2 \end{pmatrix},$$

we generate the chain of arrays

$$\begin{array}{rrr|rrrl} 2 & 3 & -1 & 1 & 0 & 0 & \\ 1 & 2 & 0 & 0 & 1 & 0 & 2R_2 - R_1 \\ 1 & 3 & 2 & 0 & 0 & 1 & 2R_3 - R_1 \end{array}$$

$$\Rightarrow \begin{array}{rrr|rrrl} 2 & 3 & -1 & 1 & 0 & 0 & R_1 - 3R_2 \\ 0 & 1 & 1 & -1 & 2 & 0 & \\ 0 & 3 & 5 & -1 & 0 & 2 & R_3 - 3R_2 \end{array}$$

$$\Rightarrow \begin{array}{rrr|rrrl} 2 & 0 & -4 & 4 & -6 & 0 & R_1/2 + R_3 \\ 0 & 1 & 1 & -1 & 2 & 0 & R_2 - R_3/2 \\ 0 & 0 & 2 & 2 & -6 & 2 & R_3/2 \end{array}$$

$$\Rightarrow \begin{array}{rrr|rrr} 1 & 0 & 0 & 4 & -9 & 2 \\ 0 & 1 & 0 & -2 & 5 & -1 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}$$

from which we conclude that

$$A^{-1} = \begin{pmatrix} 4 & -9 & 2 \\ -2 & 5 & -1 \\ 1 & -3 & 1 \end{pmatrix}.$$

Verification with Mathematica®. The input

```
A={{w2,3,-1},{1,2,0},{1,3,2}};
InvA={{4,-9,2},{-2,5,-1},{1,-3,1}};
A.InvA
```

generates the output $\{\{1, 0, 0, \}, \{0, 1, 0\}, \{0, 0, 1\}\}$. ■

7.10 Remark. In the case of a nonsingular 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the above procedure reduces to swapping the elements on the leading diagonal, changing the sign of the other two elements, and dividing all the elements by the determinant of the matrix; that is,

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}. \quad ■$$

7.11 Example. If

$$A = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix},$$

then $\det(A) = 5$, so

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{pmatrix}.$$

Verification with Mathematica®. The input

```
A={{-2,3},{1,-4}};
InvA= - (1/5)*{{4,3},{1,2}};
A.InvA
```

generates the output $\{\{1, 0\}, \{0, 1\}\}$. ■

7.12 Remark. If A is nonsingular, then

$$A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x},$$

so, multiplying both sides of (7.3) on the left by A^{-1}, we obtain the solution as

$$\mathbf{x} = A^{-1}\mathbf{b}. \;■$$

7.13 Example. The matrix of coefficients for the system

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= -3, \\ x_1 + 2x_2 \phantom{{}+2x_3} &= -1, \\ x_1 + 3x_2 + 2x_3 &= 2 \end{aligned}$$

is the matrix A whose inverse A^{-1} was computed in Example 7.9. Hence, according to Remark 7.12, the solution of the given system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 4 & -9 & 2 \\ -2 & 5 & -1 \\ 1 & -3 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix};$$

that is, $x_1 = 1$, $x_2 = -1$, and $x_3 = 2$.
Verification with Mathematica®. The input

```
{2*x1 + 3*x2 - x3 + 3, x1 + 2*x2 + 1, x1 + 3*x2 + 2*x3-2}
        /. {x1→ 1, x2→ -1, x3→ 2}
```

generates the output $\{0, 0, 0\}$. ■

7.14 Remark. Sometimes we encounter matrix functions. These are matrices whose elements are functions of the same independent variable or variables, and, just as for single functions, we can talk about their continuity and differentiability, and their derivatives and integrals. Thus, if $A(t) = (f_{ij}(t))$ and each of the f_{ij} is differentiable, we write $A'(t) = (f'_{ij}(t))$. For any such $n \times n$ matrix functions A and B and any $n \times n$ constant matrix C,

$$\begin{aligned} (CA)' &= CA', \\ (A + B)' &= A' + B', \\ (AB)' &= A'B + AB'. \;■ \end{aligned}$$

Exercises

Use the given matrices A and B to compute $2A+B$, $3A-2B$, AB, BA, A^{-1} (if it exists), and B^{-1} (if it exists). Also, in each case verify that $A^2-B^2 \neq (A+B)(A-B)$.

1 $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -2 & 4 \end{pmatrix}.$ **2** $A = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}.$

3 $A = \begin{pmatrix} 2 & -4 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 3 \\ 5 & 2 \end{pmatrix}.$ **4** $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ -4 & 3 \end{pmatrix}.$

5 $A = \begin{pmatrix} 3 & 0 & -1 \\ -2 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 2 & -3 \\ -1 & 0 & 4 \end{pmatrix}.$

6 $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \\ -3 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -2 & -1 & 3 \end{pmatrix}.$

7 $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -3 \\ 4 & 1 & -7 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 2 & 1 \\ 0 & 1 & -1 \\ 4 & -3 & 0 \end{pmatrix}.$

8 $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & -2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 1 \\ 4 & -2 & 3 \\ 2 & 1 & -3 \end{pmatrix}.$

Answers to Odd-Numbered Exercises

1 $\begin{pmatrix} 3 & -2 \\ -8 & 8 \end{pmatrix}, \begin{pmatrix} 8 & -3 \\ -5 & -2 \end{pmatrix}, \begin{pmatrix} 0 & -4 \\ -1 & 8 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ -16 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}.$

3 $\begin{pmatrix} 3 & -5 \\ 9 & 4 \end{pmatrix}, \begin{pmatrix} 8 & -18 \\ -4 & -1 \end{pmatrix}, \begin{pmatrix} -22 & -2 \\ 3 & 8 \end{pmatrix}, \begin{pmatrix} 4 & 7 \\ 14 & -18 \end{pmatrix}, \frac{1}{10}\begin{pmatrix} 1 & 4 \\ -2 & 2 \end{pmatrix},$
$\frac{1}{17}\begin{pmatrix} -2 & 3 \\ 5 & 1 \end{pmatrix}.$

5 $\begin{pmatrix} 7 & -2 & -2 \\ -3 & 4 & 1 \\ -1 & 8 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 4 & -3 \\ -8 & -1 & 12 \\ 2 & 12 & -5 \end{pmatrix}, \begin{pmatrix} 4 & -6 & -4 \\ -3 & 6 & 5 \\ 3 & 8 & -8 \end{pmatrix}, \begin{pmatrix} 7 & -2 & -5 \\ -1 & -10 & 0 \\ -3 & 16 & 5 \end{pmatrix},$
$\frac{1}{13}\begin{pmatrix} 7 & 4 & -1 \\ -2 & -3 & 4 \\ 8 & 12 & -3 \end{pmatrix}, \frac{1}{10}\begin{pmatrix} 8 & 8 & 6 \\ -1 & 4 & 3 \\ 2 & 2 & 4 \end{pmatrix}.$

7 $\begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & -7 \\ 12 & -1 & -14 \end{pmatrix}, \begin{pmatrix} 10 & -7 & 1 \\ 0 & 1 & -7 \\ 4 & 9 & -21 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 3 \\ -12 & 10 & -1 \\ -36 & 30 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 5 & -15 \\ -4 & 0 & 4 \\ 8 & -7 & 13 \end{pmatrix},$
A is singular, $\frac{1}{6}\begin{pmatrix} 3 & 3 & 3 \\ 4 & 4 & 2 \\ 4 & -2 & 2 \end{pmatrix}.$

7.2.2 Linear Independence and the Wronskian

These concepts, introduced for scalar functions in Sects. 4.2 and 6.2.3, can be extended to vectors, with many similarities but also some differences.

7.15 Definition. We say that m vectors $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(m)}$ are *linearly dependent* if there are numbers $c_1, \dots, c_m$, not all zero, such that

$$c_1\mathbf{v}^{(1)} + \cdots + c_m\mathbf{v}^{(m)} = \mathbf{0}; \tag{7.4}$$

in other words, if at least one of these vectors is a linear combination of the others. Vectors that are not linearly dependent are called *linearly independent.* ■

For n vectors of n components each we have a simple test, based on (7.4) and Theorem 6.4, to establish their linear dependence or independence. Let $V = (\mathbf{v}^{(1)}\ \dots\ \mathbf{v}^{(n)})$ be the matrix whose columns consist of the components of these vectors.

7.16 Theorem. *The vectors* $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$ *are linearly independent if and only if* $\det(V) \neq 0$. ■

7.17 Remark. Theorem 7.16 implies that $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$ are linearly dependent if and only if $\det(V) = 0$. ■

7.18 Example. Consider the vectors

$$\mathbf{v}^{(1)} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{v}^{(2)} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}^{(3)} = \begin{pmatrix} -1 \\ 1 \\ a \end{pmatrix},$$

where a is a number. Since

$$\det(\mathbf{v}^{(1)}\ \mathbf{v}^{(2)}\ \mathbf{v}^{(3)}) = \begin{vmatrix} 4 & 2 & -1 \\ -2 & 0 & 1 \\ 3 & 1 & a \end{vmatrix} = 4 + 4a,$$

from Remark 7.17 and Theorem 7.16 it follows that the three given vectors are linearly dependent if $a = -1$ and linearly independent for all $a \neq -1$.

Verification with Mathematica®. The input

```
Det[{{4, 2, −1}, {−2, 0, 1}, {3, 1, −1}}]
```

generates the output 0. ■

7.19 Definition. Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ be vector functions of a variable t, defined in an open interval J. We say that these functions are *linearly dependent on* J if there are numbers $c_1, \dots, c_m$, not all zero, such that

$$c_1\mathbf{x}^{(1)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t) = \mathbf{0} \quad \text{for all } t \text{ in } J.$$

Otherwise, the vector functions are said to be *linearly independent on* J. ■

7.20 Definition. The *Wronskian* of n vector functions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ of n components each is

$$W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t) = \det(X(t)),$$

where $X = (\mathbf{x}^{(1)}\ \dots\ \mathbf{x}^{(n)})$ is the $n \times n$ matrix whose columns consist of the components of these vector functions. ■

7.21 Remarks. (i) By Theorem 7.16, if $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t_0) \neq 0$ at a point t_0 in J, then the vectors $\mathbf{x}^{(1)}(t_0), \dots, \mathbf{x}^{(n)}(t_0)$ are linearly independent. According to Definition 7.19, this also implies that the vector functions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent on J.

(ii) It is possible for a set of vector functions to be linearly dependent at some (or even all) points in J but linearly independent on J. ■

7.22 Example. If J is the set of real numbers and

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2t-3 \\ 2-t \\ -4 \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ t+1 \\ -2 \end{pmatrix}, \quad \mathbf{x}^{(3)}(t) = \begin{pmatrix} 1-t \\ 3 \\ -6 \end{pmatrix},$$

then

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}](t) = \begin{vmatrix} 2t-3 & 1 & 1-t \\ 2-t & t+1 & 3 \\ -4 & -2 & -6 \end{vmatrix} = 18(t^2 - t).$$

Since the Wronskian is zero only at $t = 0$ and $t = 1$, it follows that the given vector functions are linearly dependent at these two points but linearly independent on J. ■

Exercises

In 1–4, determine whether the given sets of vectors are linearly dependent or independent.

1 $\mathbf{v}^{(1)} = (2,\ 0,\ 1)^{\mathrm{T}}, \quad \mathbf{v}^{(2)} = (-1,\ -2,\ 1)^{\mathrm{T}}, \quad \mathbf{v}^{(3)} = (1,\ 3,\ -2)^{\mathrm{T}}.$

2 $\mathbf{v}^{(1)} = (1,\ 0,\ -3)^{\mathrm{T}}, \quad \mathbf{v}^{(2)} = (2,\ 1,\ 4)^{\mathrm{T}}, \quad \mathbf{v}^{(3)} = (1,\ 1,\ 7)^{\mathrm{T}}.$

3 $\mathbf{v}^{(1)} = (4,\ -1,\ 1)^{\mathrm{T}}, \quad \mathbf{v}^{(2)} = (-3,\ 2,\ -1)^{\mathrm{T}}, \quad \mathbf{v}^{(3)} = (3,\ -7,\ 2)^{\mathrm{T}}.$

4 $\mathbf{v}^{(1)} = (0,\ 2,\ -3)^{\mathrm{T}}, \quad \mathbf{v}^{(2)} = (1,\ 4,\ 3)^{\mathrm{T}}, \quad \mathbf{v}^{(3)} = (1,\ 6,\ -1)^{\mathrm{T}}.$

In 5–8, determine the (real) values of t where the given vector functions are linearly dependent and then state if the functions are linearly dependent or linearly independent on the set of real numbers.

5 $$\mathbf{f}^{(1)}(t) = \begin{pmatrix} t^2-2t \\ 1-t \\ 11 \end{pmatrix}, \quad \mathbf{f}^{(2)}(t) = \begin{pmatrix} 2t^2-1 \\ t \\ 2 \end{pmatrix}, \quad \mathbf{f}^{(3)}(t) = \begin{pmatrix} t+1 \\ 2t \\ -2 \end{pmatrix}.$$

6 $$\mathbf{f}^{(1)}(t) = \begin{pmatrix} te^t \\ e^{-t} \\ -2 \end{pmatrix}, \quad \mathbf{f}^{(2)}(t) = \begin{pmatrix} (1+t)e^t \\ 3e^{-t} \\ -1 \end{pmatrix}, \quad \mathbf{f}^{(3)}(t) = \begin{pmatrix} (2t-1)e^t \\ -2e^{-t} \\ -5 \end{pmatrix}.$$

7 $$\mathbf{f}^{(1)}(t) = \begin{pmatrix} t^2 \\ t-1 \\ 3 \end{pmatrix}, \quad \mathbf{f}^{(2)}(t) = \begin{pmatrix} t^2-t \\ 3t \\ 7 \end{pmatrix}, \quad \mathbf{f}^{(3)}(t) = \begin{pmatrix} 2t^2+t \\ -3 \\ 2 \end{pmatrix}.$$

8 $$\mathbf{f}^{(1)}(t) = \begin{pmatrix} \sin t \\ 2\cos t \\ -3 \end{pmatrix}, \quad \mathbf{f}^{(2)}(t) = \begin{pmatrix} 2\sin t \\ 3\cos t \\ -4 \end{pmatrix}, \quad \mathbf{f}^{(3)}(t) = \begin{pmatrix} \sin t \\ 0 \\ 1 \end{pmatrix}.$$

Answers to Odd-Numbered Exercises

1 Linearly independent. **3** Linearly dependent.

5 Linearly dependent only at $t = 0, -1, 31/34$. Linearly independent on the set of real numbers.

7 Linearly dependent for every real value of t. Linearly dependent on the set of real numbers.

7.2.3 Eigenvalues and Eigenvectors

Let $A = (a_{ij})$ be a numerical $n \times n$ matrix, let $\mathbf{v} = (v_j)$ be an $n \times 1$ vector, let r be a number, and consider the equality

$$A\mathbf{v} = r\mathbf{v}. \tag{7.5}$$

7.23 Definition. A number r for which there are nonzero vectors $\mathbf{v}$ satisfying (7.5) is called an *eigenvalue* of A, and those nonzero vectors $\mathbf{v}$ are called the *eigenvectors* of A associated with r. ■

7.24 Remarks. (i) Since $r\mathbf{v} = rI\mathbf{v}$, where I is the identity $n \times n$ matrix, we use the properties listed in Remarks 7.2(ii) and 7.4(i) to rewrite (7.5) in the equivalent form

$$(A - rI)\mathbf{v} = \mathbf{0}, \tag{7.6}$$

which is a homogeneous system of n equations for the n components of $\mathbf{v}$. By Theorem 6.4, if $\det(A - rI) \neq 0$, then system (7.6) (and, hence, (7.5)) has only the zero solution. Therefore, for nonzero solutions we must have

$$\det(A - rI) = 0. \tag{7.7}$$

This is a polynomial equation of degree n whose n roots are the eigenvalues of A. The roots can be real or complex, single or repeated, or any combination of these cases.

(ii) By analogy with the terminology used for single DEs, we may call (7.7) the *characteristic equation* of the matrix A and the polynomial on the left-hand side the *characteristic polynomial* of A. ■

7.25 Example. If

$$A = \begin{pmatrix} 5 & -4 \\ 8 & -7 \end{pmatrix},$$

then

$$\det(A - rI) = \begin{vmatrix} 5-r & -4 \\ 8 & -7-r \end{vmatrix} = (5-r)(-7-r) - (-4)8 = r^2 + 2r - 3.$$

By Remark 7.24(i), the eigenvalues of A are the roots of the equation $r^2 - 2r - 3 = 0$, which are $r_1 = 1$ and $r_2 = -3$.

To find the eigenvectors associated with the eigenvalue 1, we need to solve (7.6) with $r = 1$; in other words, the system $(A - I)\mathbf{v} = \mathbf{0}$, which, numerically, is

$$\begin{pmatrix} 4 & -4 \\ 8 & -8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\begin{aligned} 4v_1 - 4v_2 &= 0, \\ 8v_1 - 8v_2 &= 0. \end{aligned}$$

It is obvious that this system reduces to the single equation $v_1 - v_2 = 0$. Setting, say, $v_2 = a$, where a is any nonzero number, we have $v_1 = a$, so we obtain the one-parameter set of eigenvectors

$$\mathbf{v} = \begin{pmatrix} a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad a \neq 0.$$

Proceeding in a similar manner in the case of the second eigenvalue, we solve (7.6) with $r = -3$; that is, $(A + 3I)\mathbf{v} = \mathbf{0}$. As is easily verified, this reduces to the single equation $2v_1 - v_2 = 0$, whose general solution is the one-parameter set of eigenvectors

$$\mathbf{v} = \begin{pmatrix} b \\ 2b \end{pmatrix} = b \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad b \neq 0.$$

Verification with Mathematica®. The input

```
A = {{5,-4},{8,-7}};
Id = IdentityMatrix[2];
{r1,r2} = {1,-3};
{v1,v2} = {{a,a},{b,2*b}};
{(A - r1*Id).v1, (A - r2*Id).v2}
```

generates the output {{0,0}, {0,0}}. ■

7.26 Example. Let

$$A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ 2 & 0 & 3 \end{pmatrix}.$$

Then

$$\det(A - rI) = \begin{vmatrix} -r & 0 & -1 \\ -1 & 1-r & -1 \\ 2 & 0 & 3-r \end{vmatrix} = -r^3 + 4r^2 - 5r + 2,$$

so the eigenvalues of A are the roots of the equation $r^3 - 4r^2 + 5r - 2 = 0$. Trying synthetic division on this equation, we find that $r_1 = 2$ and $r_2 = r_3 = 1$.

The eigenvectors corresponding to $r = 2$ are the nonzero solutions of the system $(A - 2I)\mathbf{v} = \mathbf{0}$, or

$$\begin{aligned} -2v_1 \qquad\quad - v_3 &= 0, \\ -v_1 - v_2 - v_3 &= 0, \\ 2v_1 \qquad\quad + v_3 &= 0. \end{aligned}$$

Since the first equation is the same as the third one multiplied by -1, we may discard it and solve the subsystem consisting of the last two equations. Setting $v_1 = a$, where a is an arbitrary nonzero number, from the third equation we get $v_3 = -2a$ and then, from the second equation, $v_1 = a$. Thus, we obtain the one-parameter set of eigenvectors

$$\mathbf{v} = \begin{pmatrix} a \\ a \\ -2a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad a \neq 0.$$

Similarly, for the eigenvalue $r = 1$ we solve the system $(A - I)\mathbf{v} = \mathbf{0}$, which is

$$\begin{aligned} -v_1 - v_3 &= 0,\\ -v_1 - v_3 &= 0,\\ 2v_1 + 2v_3 &= 0. \end{aligned}$$

The equations of this system are multiples of $v_1 + v_3 = 0$; hence, $v_1 = -v_3$. We also see that v_2 can take any value, so we write $v_2 = b$ and $v_3 = c$, where b and c are arbitrary numbers, not both equal to zero. Since $v_1 = -c$, we obtain the two-parameter set of eigenvectors

$$\mathbf{v} = \begin{pmatrix} -c \\ b \\ c \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad b,\ c \text{ not both zero.}$$

We note that the two vectors on the right-hand side above are linearly independent.

Verification with Mathematica®. The input

```
A = {{0,0,-1},{−1,1,−1},{2,0,3}};
Id = IdentityMatrix[3];
{r1,r2} = {2,1};
{v1,v2} = {{a,a,-2*a},{-c,b,c}};
{(A - r1*Id).v1, (A - r2*Id).v2}
```

generates the output $\{\{0,0,0\},\{0,0,0\}\}$. ■

7.27 Remarks. (i) If m of the roots of the equation $\det(A - rI) = 0$ are equal to r, we say that r is an eigenvalue of *algebraic multiplicity* m. For convenience, when $m = 1$ we call r a *simple* eigenvalue, and when $m = 2$ we call it a *double* eigenvalue.

(ii) It can be shown that for each eigenvalue r of algebraic multiplicity m, there are k, $0 < k \le m$, linearly independent associated eigenvectors. Obviously, each simple eigenvalue has only one such associated eigenvector. The number k is called the *geometric multiplicity* of r.

(iii) The *deficiency* of an eigenvalue r is the difference $m - k$ between its algebraic and geometric multiplicities. Normally, k is also the number of arbitrary constants in the general expression of the eigenvectors associated with r. From what we said in (ii) above, it follows that a simple eigenvalue is non-deficient (in other words, its deficiency is zero). Both eigenvalues in Example 7.26 are non-deficient: 1 is a simple eigenvalue with one linearly independent eigenvector (that is, $m = k = 1$) and 2 is a double eigenvalue with two linearly independent eigenvectors ($m = k = 2$).

(iv) Usually we need to operate with just one selection of linearly independent eigenvectors associated with an eigenvalue. The simplest selection can be made from the general formula of the eigenvectors by setting, in turn, each arbitrary constant equal to some convenient value (normally 1) and the rest of them equal to 0. Thus, in Example 7.26 we can take the representatives

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \text{ for } r = 2; \quad \mathbf{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{v}^{(3)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ for } r = 1.$$

We adopt this type of choice throughout the chapter and refer to the eigenvectors selected in this way as *basis eigenvectors*. ■

7.28 Theorem. *Eigenvectors associated with distinct eigenvalues are linearly independent.* ■

Proof. For brevity, we show this for $n = 2$, but the technique is applicable in exactly the same way for any n.

Let A be a 2×2 matrix with eigenvalues $r_1 \neq r_2$, let $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ be eigenvectors associated with r_1 and r_2, respectively, and suppose that $c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)} = \mathbf{0}$. Multiplying this equality on the left by $A - r_1 I$ and taking into account that $(A - r_1 I)\mathbf{v}^{(1)} = \mathbf{0}$ and $(A - r_2 I)\mathbf{v}^{(2)} = \mathbf{0}$, we see that

$$\begin{aligned}\mathbf{0} &= c_1(A - r_1 I)\mathbf{v}^{(1)} + c_2(A - r_1 I)\mathbf{v}^{(2)} = c_2(A\mathbf{v}^{(2)} - r_1\mathbf{v}^{(2)})\\ &= c_2(r_2\mathbf{v}^{(2)} - r_1\mathbf{v}^{(2)}) = c_2(r_1 - r_2)\mathbf{v}^{(2)}.\end{aligned}$$

Since $r_1 \neq r_2$ and $\mathbf{v}^{(2)} \neq \mathbf{0}$ (because it is an eigenvector), it follows that $c_2 = 0$. Repeating the procedure but starting with $A - r_2 I$, we also arrive at $c_1 = 0$, so we conclude that $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are linearly independent. ■

Exercises

Compute the eigenvalues and corresponding basis eigenvectors of the given matrix A.

1 $A = \begin{pmatrix} 3 & -12 \\ 2 & -7 \end{pmatrix}$. **2** $A = \begin{pmatrix} -7 & -9 \\ 6 & 8 \end{pmatrix}$. **3** $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$.

4 $A = \begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix}$. **5** $A = \begin{pmatrix} -2 & 4 & -2 \\ -3 & 5 & -3 \\ -2 & 2 & -2 \end{pmatrix}$. **6** $A = \begin{pmatrix} 3 & -2 & -2 \\ 1 & 0 & -2 \\ 3 & -3 & -1 \end{pmatrix}$.

7 $A = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$. **8** $A = \begin{pmatrix} 5/2 & -9 \\ 1 & -7/2 \end{pmatrix}$. **9** $A = \begin{pmatrix} 0 & -1 & 2 \\ -2 & -1 & 4 \\ -1 & -1 & 3 \end{pmatrix}$.

10 $A = \begin{pmatrix} -1/2 & 1/2 & -1 \\ -1 & 1 & -1 \\ 1 & -1/2 & 3/2 \end{pmatrix}$. **11** $A = \begin{pmatrix} 4 & 6 & -5 \\ -2 & -3 & 2 \\ 1 & 2 & -2 \end{pmatrix}$.

12 $A = \begin{pmatrix} -1 & 1 & 0 \\ 4 & -1 & -4 \\ -4 & 2 & 3 \end{pmatrix}$. **13** $A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}$. **14** $A = \begin{pmatrix} 3 & -2 & 6 \\ 2 & -2 & 3 \\ -2 & 1 & -4 \end{pmatrix}$.

15 $A = \begin{pmatrix} -3 & 2 & -1 \\ -3 & 3 & -2 \\ -2 & 4 & -3 \end{pmatrix}$. **16** $A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}$.

Answers to Odd-Numbered Exercises

1 $r = -3, -1$; $\mathbf{v}^{(1)} = (2, 1)^{\mathrm{T}}$, $\mathbf{v}^{(2)} = (3, 1)^{\mathrm{T}}$.

3 $r = i, -i$; $\mathbf{v}^{(1)} = (1 + i, 1)^{\mathrm{T}}$, $\mathbf{v}^{(2)} = (1 - i, 1)^{\mathrm{T}}$.

5 $r = 0, 2, -1$; $\mathbf{v}^{(1)} = (-1, 0, 1)^{\mathrm{T}}$, $\mathbf{v}^{(2)} = (1, 1, 0)^{\mathrm{T}}$, $\mathbf{v}^{(3)} = (0, 1, 2)^{\mathrm{T}}$.

7 $r = 1, 1$; $\mathbf{v} = (1, 2)^{\mathrm{T}}$.

9 $r = 0, 1, 1;\ \ \mathbf{v}^{(1)} = (1, 2, 1)^{\mathrm{T}},\ \ \mathbf{v}^{(2)} = (-1, 1, 0)^{\mathrm{T}},\ \ \mathbf{v}^{(3)} = (0, 2, 1)^{\mathrm{T}}.$

11 $r = 1, -1, -1;\ \ \mathbf{v}^{(1)} = (2, -1, 0)^{\mathrm{T}},\ \ \mathbf{v}^{(2)} = (1, 0, 1)^{\mathrm{T}}.$

13 $r = 1, 1, 1;\ \ \mathbf{v}^{(1)} = (1, 1, 0)^{\mathrm{T}},\ \ \mathbf{v}^{(2)} = (-2, 0, 1)^{\mathrm{T}}.$

15 $r = -1, -1, -1;\ \ \mathbf{v} = (0, 1, 2)^{\mathrm{T}}.$

7.3 Systems of First-Order Differential Equations

A general system of n first-order DEs for n unknown functions $x_1, \dots, x_n$ of one independent variable t can be written as

$$\begin{aligned} x_1' &= F_1(t, x_1, \dots, x_n), \\ &\vdots \\ x_n' &= F_n(t, x_1, \dots, x_n), \end{aligned} \tag{7.8}$$

where $F_1, \dots, F_n$ are given functions of $n + 1$ variables. In an IVP, such a system is accompanied by ICs of the form

$$x_1(t_0) = x_{10},\ \dots,\ x_n(t_0) = x_{n0}, \tag{7.9}$$

where $x_{10}, \dots, x_{n0}$ are prescribed numbers.

7.29 Definition. A *solution* of system (7.8) on an open interval J is a collection of functions $x_1, \dots, x_n$ that are differentiable on J and satisfy the system at every point t in J. A solution of the IVP (7.8), (7.9) is any solution of (7.8) that also satisfies the ICs (7.9) at a specified point t_0 in J. ■

Naturally, we are interested to know under what conditions the given IVP has a unique solution. The answer is given by the next assertion.

7.30 Theorem (Existence and uniqueness). *If all the functions F_i and their partial derivatives $F_{i,j}$, $i, j = 1, \dots, n$, are continuous in an $(n+1)$-dimensional 'rectangular' domain*

$$R = \{\alpha < t < \beta,\ \ \alpha_1 < x_1 < \beta_1,\ \ \dots,\ \ \alpha_n < x_n < \beta_n\}$$

and $(t_0, x_{10}, \dots, x_{n0})$ is a point in R, then there is a number $h > 0$ such that the IVP (7.8), (7.9) has a unique solution $\{x_1, \dots, x_n\}$ in the open interval $t_0 - h < t < t_0 + h$. ■

We point out that the restrictions on the F_i above are sufficient conditions, which may be relaxed to a certain extent without invalidating the statement of the theorem.

7.31 Definition. We say that system (7.8) is *linear/homogeneous/has constant coefficients* if all its equations are linear/homogeneous/have constant coefficients. For simplicity, we refer to a system of first-order DEs as a first-order system. ■

The general form of a linear, nonhomogeneous, first-order system of n DEs for unknown functions $x_1, \dots, x_n$ in one variable t is

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + \dots + a_{1n}(t)x_n + f_1(t), \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + \dots + a_{nn}(t)x_n + f_n(t). \end{aligned} \tag{7.10}$$

For system (7.10), Theorem 7.30 becomes simpler.

7.32 Theorem (Existence and uniqueness). *If all a_{ij} and f_i, $i, j = 1, \dots, n$, are continuous on an open interval J and t_0 is a point in J, then the IVP (7.10), (7.9) has a unique solution on J.* ■

7.33 Remark. As in Definition 2.28, in what follows we take J to be the largest open interval (maximal interval of existence) on which the unique solution of the IVP can be defined. For our specific examples, J will normally be the entire real line. ■

Sometimes it is possible to change an IVP for a first-order linear system into an equivalent IVP for a single higher-order equation.

7.34 Example. Consider the IVP

$$\begin{aligned} x_1' &= x_1 + x_2, \quad & x_1(0) &= -2,\\ x_2' &= 4x_1 + x_2, \quad & x_2(0) &= -4. \end{aligned}$$

Differentiating the first equation and then replacing x_2' from the second equation, we find that

$$x_1'' = x_1' + x_2' = x_1' + (4x_1 + x_2).$$

Since, from the first equation, $x_2 = x_1' - x_1$, it follows that

$$x_1'' = x_1' + 4x_1 + (x_1' - x_1) = 2x_1' + 3x_1,$$

which is a second-order DE for x_1, with GS

$$x_1(t) = c_1 e^{-t} + c_2 e^{3t}.$$

We are already given that $x_1(0) = 2$; the second IC for x_1 is obtained from the first equation:

$$x_1'(0) = x_1(0) + x_2(0) = -2 - 4 = -6.$$

Applying the ICs to the GS, we arrive at

$$x_1(t) = -2e^{3t}.$$

Finally, from the first equation we have

$$x_2(t) = x_1'(t) - x_1(t) = -6e^{3t} + 2e^{3t} = -4e^{3t}.$$

Verification with Mathematica®. The input

```
{x1,x2}={-2*E^(3*t),-4*E^(3*t)};
{{D[x1,t] - x1 - x2,D[x2,t] - 4*x1 - x2},{x1,x2}/.t→ 0}
```

generates the output $\{\{0,0\},\{-2,-4\}\}$. ■

In matrix form, an IVP for a general linear first-order system of n DEs in n unknowns can be written as

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t), \tag{7.11}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{7.12}$$

where A is the $n \times n$ coefficient matrix, $\mathbf{x}$ is the column vector of the unknowns, $\mathbf{f}$ is the column vector of the nonhomogeneous terms, and $\mathbf{x}_0$ is the column vector of the ICs.

When the system is homogeneous, (7.11) reduces to

$$\mathbf{x}' = A(t)\mathbf{x}. \tag{7.13}$$

The principle of superposition mentioned in Sect. 4.3.1 also applies to homogeneous linear systems.

7.35 Theorem. *If $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are solutions of system (7.13) in J, then so is the linear combination $c_1\mathbf{x}^{(1)} + \dots + c_n\mathbf{x}^{(n)}$ for any constants $c_1, \dots, c_n$.* ■

7.36 Theorem. *If $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent solutions of (7.13) in J, then for any solution $\mathbf{x}$ of (7.13) there are unique numbers $c_1, \dots, c_n$ such that*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + \dots + c_n\mathbf{x}^{(n)}(t) \quad \textit{for all } t \textit{ in } J. \; ■ \tag{7.14}$$

7.37 Remark. According to Theorems 7.35 and 7.36, formula (7.14) with arbitrary constants $c_1, \dots, c_n$ represents the GS of system (7.13). ■

7.38 Definition. As in the case of a single DE of order n (see Remark 4.28(ii) for $n = 2$), a set of n linearly independent solutions of (7.13) is called a *fundamental set of solutions* (FSS). ■

7.39 Remark. The property of the Wronskian stated in Remark 4.25 carries over unchanged to homogeneous systems; that is, if $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are solutions of (7.13) on J, then their Wronskian is either zero at every point of J or nonzero everywhere on J. ■

Exercises

In 1–6, use the method of reduction to a second-order equation to solve the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ for the given matrix A and vector $\mathbf{x}_0$.

1 $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$ **2** $A = \begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$

3 $A = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$ **4** $A = \begin{pmatrix} -5/2 & 9 \\ -1 & 7/2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

5 $A = \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$ **6** $A = \begin{pmatrix} 4 & 1 \\ -10 & -2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

In 7–10, use the method of reduction to a second-order equation to solve the IVP $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ for the given matrix A, vector function $\mathbf{f}(t)$, and vector $\mathbf{x}_0$.

7 $A = \begin{pmatrix} 2 & -6 \\ 3 & -7 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 2t \\ t-2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

8 $A = \begin{pmatrix} 2 & 3/2 \\ -3 & -5/2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 8e^{2t} \\ 6(t-3)e^{2t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$

9 $A = \begin{pmatrix} -8 & -9 \\ 4 & 4 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 17\cos t - 8\sin t \\ -8\cos t + 2\sin t \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 7 \\ -3 \end{pmatrix}.$

10 $A = \begin{pmatrix} -4/3 & 1 \\ -1 & 2/3 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 6t + e^{-t} \\ 4t - 6 + 3e^{-t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$

Answers to Odd-Numbered Exercises

1 $\mathbf{x}(t) = \begin{pmatrix} 4e^{2t} - 3e^{3t} \\ 2e^{2t} - 3e^{3t} \end{pmatrix}$. **3** $\mathbf{x}(t) = \begin{pmatrix} (2-8t)e^{t} \\ (1+4t)e^{t} \end{pmatrix}$.

5 $\mathbf{x}(t) = \begin{pmatrix} 2\cos t - 5\sin t \\ -\cos t - 12\sin t \end{pmatrix}$. **7** $\mathbf{x}(t) = \begin{pmatrix} 4e^{-t} - 3e^{-4t} + 2t + 1 \\ 2e^{-t} - 3e^{-4t} + t \end{pmatrix}$.

9 $\mathbf{x}(t) = \begin{pmatrix} (3t+7)e^{-2t} - \sin t \\ -(2t+5)e^{-2t} + 2\cos t \end{pmatrix}$.

7.4 Homogeneous Linear Systems with Constant Coefficients

It is useful to have a quick look at this type of system for $n = 1$, which is, of course, the single DE

$$x' = ax, \quad a = \text{const} \neq 0. \tag{7.15}$$

As we know, the GS of (7.15) is

$$x(t) = Ce^{at}, \tag{7.16}$$

where C is an arbitrary constant. Among all the solutions (7.16) there is one, and only one, which is time-independent, namely $x = 0$; we call it an *equilibrium solution.*

If $a < 0$, then

$$\lim_{t\to-\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} x(t) = 0.$$

On the other hand, if $a > 0$, then

$$\lim_{t\to-\infty} x(t) = 0, \quad \lim_{t\to\infty} |x(t)| = \infty.$$

Recalling Definition 3.11, this means that the equilibrium solution $x = 0$ of equation (7.15) is asymptotically stable if $a < 0$ and unstable if $a > 0$.

Now consider a general homogeneous linear system of DEs of the form

$$\mathbf{x}' = A\mathbf{x}, \tag{7.17}$$

where A is a constant nonsingular $n \times n$ matrix. The equilibrium (or time-independent) solutions of (7.17) are obtained by setting $\mathbf{x}' = \mathbf{0}$; that is, by solving $A\mathbf{x} = \mathbf{0}$. Since $\det(A) \neq 0$, this homogeneous algebraic system has only the zero solution. In other words, just as for equation (7.15), the only equilibrium solution here is $\mathbf{x} = \mathbf{0}$. In the case of (7.15), the stability or instability of the equilibrium solution depends on the sign of the number a. In the case of system (7.17), that property depends on the sign of the eigenvalues of the matrix A. Later in this section we will have a full discussion of several specific examples for $n = 2$.

By analogy with single DEs, we seek solutions of (7.17) of the form

$$\mathbf{x}(t) = \mathbf{v}e^{rt}, \tag{7.18}$$

where $\mathbf{v}$ is an unknown constant, nonzero, n-component column vector and r is an unknown number. Differentiating (7.18), we have $\mathbf{x}' = r\mathbf{v}e^{rt}$, which, substituted in (7.17), yields the equality $r\mathbf{v}e^{rt} = A\mathbf{v}e^{rt}$. Since the coefficients of e^{rt} on both sides must be the same, it follows that $A\mathbf{v} = r\mathbf{v}$, so, by Definition 7.23, r is an eigenvalue of A and $\mathbf{v}$ is an eigenvector associated with r. Once we have computed the eigenvalues and eigenvectors of A—the former as the roots of the equation $\det(A - rI) = 0$ and the latter as the nonzero solutions of (7.6) (see Remark 7.24)—we use them in (7.18) to generate solutions of system (7.17). However, in view of Theorem 7.36 and Remark 7.37, to write the GS of the system we need to construct n linearly independent solutions. Below, we examine in turn each of the several situations that may occur in this type of construction.

Solutions of the form (7.18) have an important property.

7.40 Theorem. *If the eigenvalues of A are distinct, then the solutions of system* (7.17) *obtained from* (7.18) *are linearly independent.* ■

Proof. Let $r_1, \dots, r_n$ be the (real or complex) distinct eigenvalues of A, let $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$ be corresponding associated eigenvectors, and let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ be the solutions constructed from these elements according to (7.18). Given that $\mathbf{x}^{(i)}(0) = \mathbf{v}^{(i)}$, $i = 1, \dots, n$, we have

$$W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](0) = \det(\mathbf{v}^{(1)} \ \dots \ \mathbf{v}^{(n)}),$$

and since, by Theorem 7.28, $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$ are linearly independent, from Theorem 7.16 it follows that the above determinant is nonzero. Hence, by Remark 7.21(i), $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent. ■

7.4.1 Real and Distinct Eigenvalues

For simplicity, we restrict our attention to systems in two or three unknown functions and discuss several specific illustrations.

In the two-dimensional case, we may regard the components $x_1(t)$ and $x_2(t)$ of the GS $\mathbf{x}(t)$ as the coordinates of a point moving in the (x_1, x_2)-plane, called the *phase plane*. The curves traced by this point for various values of the arbitrary constants c_1 and c_2 are graphical representations (called *trajectories*) of the solutions, and form what is known as the *phase portrait* of the system. If the eigenvectors are real, a line through the origin and parallel to an eigenvector is called an *eigenline*.

7.41 Example. For the IVP

$$\begin{aligned} x_1' &= 3x_1 + 4x_2, \quad & x_1(0) &= 2,\\ x_2' &= 3x_1 + 2x_2, \quad & x_2(0) &= 5 \end{aligned}$$

we have

$$\det(A - rI) = \begin{vmatrix} 3-r & 4 \\ 3 & 2-r \end{vmatrix} = (3-r)(2-r) - 12 = r^2 - 5r - 6.$$

Setting $r^2 - 5r - 6 = 0$, we find the eigenvalues $r_1 = -1$ and $r_2 = 6$. For $r = -1$, the system $(A + I)\mathbf{v} = \mathbf{0}$ reduces to the equation $v_1 + v_2 = 0$, so an associated basis eigenvector (see Remark 7.27(iii)) is

$$\mathbf{v}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Similarly, for $r = 6$ the system $(A - 6I)\mathbf{v} = \mathbf{0}$ reduces to the single equation $3v_1 - 4v_2 = 0$, which yields the basis eigenvector

$$\mathbf{v}^{(2)} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Using r_1, $\mathbf{v}^{(1)}$ and r_2, $\mathbf{v}^{(2)}$, we now construct the two solutions

$$\mathbf{x}^{(1)}(t) = \mathbf{v}^{(1)} e^{r_1 t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \mathbf{v}^{(2)} e^{r_2 t} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t}.$$

Since the eigenvalues are distinct, from Theorem 7.40 it follows that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent, so the GS of the system is

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t} = \begin{pmatrix} -c_1 e^{-t} + 4c_2 e^{6t} \\ c_1 e^{-t} + 3c_2 e^{6t} \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

To sketch the phase portrait of this system, suppose first that $c_1 \neq 0$, $c_2 = 0$. Then

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} = c_1 \mathbf{v}^{(1)} e^{-t},$$

and we can see that, as t increases from $-\infty$ to ∞, the point (x_1, x_2) moves along the first eigenline ($x_1 + x_2 = 0$), asymptotically approaching the origin from the second quadrant (if $c_1 > 0$) or from the fourth quadrant (if $c_1 < 0$). If we now take $c_1 = 0$, $c_2 \neq 0$, we have

$$\mathbf{x}(t) = c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t} = c_2 \mathbf{v}^{(2)} e^{6t},$$

so the point moves along the second eigenline ($3x_1 - 4x_2 = 0$), away from the origin, in the first quadrant (if $c_2 > 0$) or in the third quadrant (if $c_2 < 0$).

Next, consider the full solution $\mathbf{x}(t)$. As t starts increasing from $-\infty$, the first term is the dominant one (since $\lim\limits_{t\to-\infty} e^{-t} = \infty$, $\lim\limits_{t\to-\infty} e^{6t} = 0$), so the point begins its motion at 'infinity' on a curve that is asymptotically parallel to the first eigenline. As t increases to ∞, the second term becomes dominant (since $\lim\limits_{t\to\infty} e^{6t} = \infty$, $\lim\limits_{t\to\infty} e^{-t} = 0$), so the point swings and goes back to 'infinity', its trajectory ending up asymptotically parallel to the second eigenline. A few of the system trajectories and the direction of motion on them are shown in Fig. 7.1.

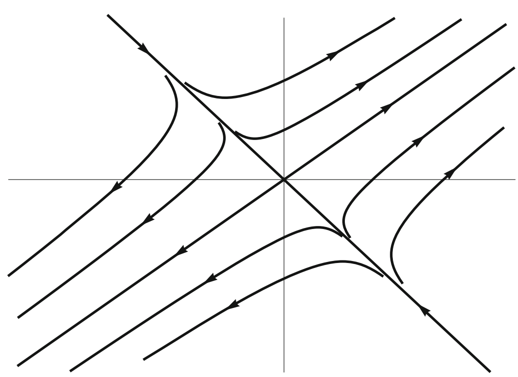

Fig. 7.1

As we can see, the point approaches the origin indefinitely on two trajectories but moves away from it on any other, which means that the equilibrium solution $\mathbf{x} = \mathbf{0}$ is unstable. In this case, the origin is called a *saddle point*. This situation always arises when the eigenvalues of A are of opposite signs.

To find the (unique) solution of the IVP, we apply the ICs to the GS and arrive at the system

$$c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix},$$

or, in terms of components,

$$\begin{aligned} -c_1 + 4c_2 &= 2, \\ c_1 + 3c_2 &= 5, \end{aligned}$$

from which $c_1 = 2$ and $c_2 = 1$; hence, the solution of the given IVP is

$$\mathbf{x}(t) = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t} = \begin{pmatrix} -2e^{-t} + 4e^{6t} \\ 2e^{-t} + 3e^{6t} \end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2}={-2*E^(-t) + 4*E^(6*t),2*E^(-t) + 3*E^(6*t)};
{{D[x1,t] - 3*x1 - 4*x2,D[x2,t] - 3*x1 - 2*x2},{x1,x2}/.t→ 0}
```

generates the output $\{\{0, 0\}, \{2, 5\}\}$. ■

7.42 Remark. It is obvious that if we choose a different pair of basis eigenvectors, the general solution will have a different look. Thus, if in the preceding example we took

$$\mathbf{v}^{(1)} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \mathbf{v}^{(2)} = \begin{pmatrix} 8 \\ 6 \end{pmatrix},$$

then the GS would be written as

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 8 \\ 6 \end{pmatrix} e^{6t} = \begin{pmatrix} 2c_1e^{-t} + 8c_2e^{6t} \\ -2c_1e^{-t} + 6c_2e^{6t} \end{pmatrix}.$$

The solution of the IVP, however, does not change since the ICs now lead to the system

$$\begin{aligned} 2c_1 + 8c_2 &= 2, \\ -2c_1 + 6c_2 &= 5, \end{aligned}$$

which yields $c_1 = -1$ and $c_2 = 1/2$. When replaced in the new form of the GS, these two values give rise to the same solution $\mathbf{x}(t)$ as before. ■

7.43 Example. For the IVP

$$\begin{aligned} x_1' &= \tfrac{7}{3}\, x_1 + \tfrac{2}{3}\, x_2, \quad & x_1(0) &= 4, \\ x_2' &= \tfrac{4}{3}\, x_1 + \tfrac{5}{3}\, x_2, \quad & x_2(0) &= 1 \end{aligned}$$

we have

$$\det(A - rI) = \begin{vmatrix} \frac{7}{3} - r & \frac{2}{3} \\ \frac{4}{3} & \frac{5}{3} - r \end{vmatrix} = \big(\tfrac{7}{3} - r\big)\big(\tfrac{5}{3} - r\big) - \tfrac{8}{9} = r^2 - 4r + 3.$$

The eigenvalues of A are the roots of $r^2 - 4r + 3 = 0$; that is, $r_1 = 1$ and $r_2 = 3$. Computing associated basis eigenvectors as in Example 7.41, we find that

$$\mathbf{v}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so the GS of the system is

$$\mathbf{x}(t) = c_1\mathbf{v}^{(1)}e^t + c_2\mathbf{v}^{(2)}e^{3t} = c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} = \begin{pmatrix} -c_1e^t + c_2e^{3t} \\ 2c_1e^t + c_2e^{3t} \end{pmatrix}.$$

Because both eigenvalues are positive, we easily deduce that on each eigenline, the point $\mathbf{x}$ moves away from the origin as t increases from $-\infty$ to ∞. The same argument shows that this is also true for the point on any other trajectory (when both c_1 and c_2 are nonzero). However, we note that as $t \to -\infty$, the first term in the general solution is the dominant one (it tends to $\mathbf{0}$ more slowly than the second one), whereas as $t \to \infty$, the second term is dominant (the absolute values of its components tend to infinity faster than those of the first term). Consequently, trajectories start (asymptotically) at the origin parallel to the first eigenline ($2x_1 + x_2 = 0$) and head toward 'infinity' asymptotically parallel to the second eigenline ($x_1 - x_2 = 0$). A few trajectories of the system and the direction of motion on them are shown in Fig. 7.2.

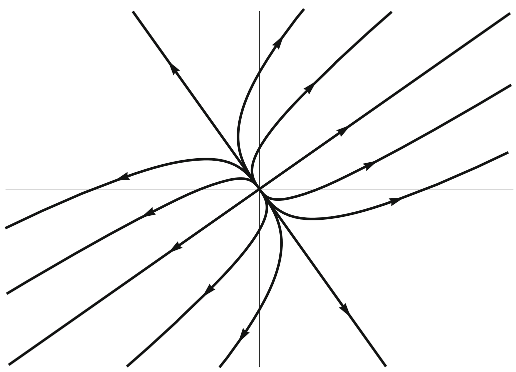

Fig. 7.2

Clearly, the equilibrium solution $\mathbf{x} = \mathbf{0}$ is unstable. In this case, the origin is called an *unstable node*. If both eigenvalues were negative, we would have a *stable node*, with the trajectories looking as in Fig. 7.2 but with the direction of motion toward the origin.

Using the IC, we readily find that $c_1 = -1$ and $c_2 = 3$, so the solution of the given IVP is

$$\mathbf{x}(t) = -\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^t + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} = \begin{pmatrix} e^t + 3e^{3t} \\ -2e^t + 3e^{3t} \end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2}={E^t + 3*E^(3*t),-2*E^t + 3*E^(3*t)};
{{D[x1,t] - (7/3)*x1 - (2/3)*x2,D[x2,t] - (4/3)*x1 - (5/3)*x2},
      {x1,x2}/.t→ 0}
```

generates the output {{0, 0}, {4, 1}}. ■

7.44 Example. Consider the IVP

$$\begin{aligned} x_1' &= 5x_1 \qquad\quad - 6x_3, \quad & x_1(0) &= -2,\\ x_2' &= 2x_1 - \ x_2 - 2x_3, \quad & x_2(0) &= -1,\\ x_3' &= 4x_1 - 2x_2 - 4x_3, \quad & x_3(0) &= -1. \end{aligned}$$

Expanding the determinant, say, in the first row, we see that

$$\det(A - rI) = \begin{vmatrix} 5-r & 0 & -6\\ 2 & -1-r & -2\\ 4 & -2 & -4-r \end{vmatrix} = r - r^3,$$

so the eigenvalues of A are $r_1 = 0$, $r_2 = -1$, and $r_3 = 1$. The corresponding algebraic systems $(A - rI)\mathbf{v} = \mathbf{0}$ for these values of r reduce, respectively, to the pairs of equations

$$\begin{aligned} 5v_1 \qquad -6v_3 &= 0, & v_1 \qquad\quad - v_3 &= 0, & 2v_1 \qquad -3v_3 &= 0,\\ 2v_1 - v_2 - 2v_3 &= 0, & 4v_1 - 2v_2 - 3v_3 &= 0, & 2v_2 - v_3 &= 0 \end{aligned}$$

and yield associated basis eigenvectors

$$\mathbf{v}^{(1)} = \begin{pmatrix}6\\2\\5\end{pmatrix}, \quad \mathbf{v}^{(2)} = \begin{pmatrix}2\\1\\2\end{pmatrix}, \quad \mathbf{v}^{(3)} = \begin{pmatrix}3\\1\\2\end{pmatrix}.$$

Therefore, the GS of the system is

$$\mathbf{x}(t) = c_1\begin{pmatrix}6\\2\\5\end{pmatrix} + c_2\begin{pmatrix}2\\1\\2\end{pmatrix}e^{-t} + c_3\begin{pmatrix}3\\1\\2\end{pmatrix}e^{t}.$$

Using the ICs, in terms of components we have

$$\begin{aligned} 6c_1 + 2c_2 + 3c_3 &= -2,\\ 2c_1 + \ c_2 + \ c_3 &= -1,\\ 5c_1 + 2c_2 + 2c_3 &= -1. \end{aligned}$$

We can solve this system by Gaussian elimination and arrive at the values $c_1 = 1$, $c_2 = -1$, and $c_3 = -2$, so the solution of the IVP is

$$\mathbf{x}(t) = \begin{pmatrix}6\\2\\5\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix}e^{-t} - 2\begin{pmatrix}3\\1\\2\end{pmatrix}e^{t} = \begin{pmatrix}6 - 2e^{-t} - 6e^{t}\\ 2 - \ e^{-t} - 2e^{t}\\ 5 - 2e^{-t} - 4e^{t}\end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2,x3}={6 - 2*E^(-t) - 6*E^t, 2 - E^(-t) - 2*E^t,
        5 - 2*E^(-t) - 4*E^t};
{{D[x1,t] - 5*x1 + 6*x3, D[x2,t] - 2*x1 + x2 + 2*x3,
        D[x3,t] - 4*x1 + 2*x2 + 4*x3}, {x1,x2,x3}/. t→ 0}
```

generates the output $\{\{0, 0, 0\}, \{-2, -1, -1\}\}$. ■

Exercises

In 1–10, use the eigenvalue-eigenvector method to solve the IVP $\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$ for the given matrix A and vector $\mathbf{x}_0$, and sketch the phase portrait of the general solution of the system.

1 $A = \begin{pmatrix} -8 & 18 \\ -3 & 7 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}.$ **2** $A = \begin{pmatrix} -7 & 8 \\ -6 & 7 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$

3 $A = \begin{pmatrix} -7 & 10 \\ -5 & 8 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -5 \\ -2 \end{pmatrix}.$ **4** $A = \begin{pmatrix} 0 & -4 \\ 2 & 6 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$

5 $A = \begin{pmatrix} -5/2 & 3 \\ -3/2 & 2 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$ **6** $A = \begin{pmatrix} 5/2 & 6 \\ -1 & -5/2 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 9 \\ -4 \end{pmatrix}.$

7 $A = \begin{pmatrix} 11 & 18 \\ -3 & -4 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$ **8** $A = \begin{pmatrix} -1/2 & -1 \\ 2 & 5/2 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 6 \\ -10 \end{pmatrix}.$

9 $A = \begin{pmatrix} 5/3 & -4/3 \\ 2/3 & -1/3 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -4 \\ -1 \end{pmatrix}.$ **10** $A = \begin{pmatrix} -11/2 & 5/2 \\ -15 & 7 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 7/2 \\ 10 \end{pmatrix}.$

In 11–16, use the eigenvalue–eigenvector method to solve the IVP $\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$ for the given matrix A and vector $\mathbf{x}_0$.

11 $A = \begin{pmatrix} 5 & -4 & -3 \\ 1 & 0 & -1 \\ 4 & -4 & -2 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$ **12** $A = \begin{pmatrix} 0 & -1 & 1 \\ 3 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}.$

13 $A = \begin{pmatrix} 0 & 4 & -2 \\ -1 & 2 & 1 \\ -1 & 4 & -1 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$ **14** $A = \begin{pmatrix} -4 & -7 & 6 \\ 1 & 4 & -2 \\ -2 & -2 & 3 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}.$

15 $A = \begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -1 \\ 4 & 4 & 3 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}.$ **16** $A = \begin{pmatrix} 1 & 1 & -1 \\ -4 & -1 & 4 \\ -2 & 1 & 2 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}.$

Answers to Odd-Numbered Exercises

1 $\mathbf{x}(t) = \begin{pmatrix} -3e^{-2t} \\ -e^{-2t} \end{pmatrix}.$ **3** $\mathbf{x}(t) = \begin{pmatrix} -6e^{-2t} + e^{3t} \\ -3e^{-2t} + e^{3t} \end{pmatrix}.$

5 $\mathbf{x}(t) = \begin{pmatrix} -2e^{-t} + 3e^{t/2} \\ -e^{-t} + 3e^{t/2} \end{pmatrix}.$ **7** $\mathbf{x}(t) = \begin{pmatrix} -6e^{2t} + 3e^{5t} \\ 3e^{2t} - e^{5t} \end{pmatrix}.$

9 $\mathbf{x}(t) = \begin{pmatrix} 2e^{t/3} - 6e^{t} \\ 2e^{t/3} - 3e^{t} \end{pmatrix}.$ **11** $\mathbf{x}(t) = \begin{pmatrix} 2e^{t} - e^{2t} \\ 2e^{t} \\ -e^{2t} \end{pmatrix}.$

13 $\mathbf{x}(t) = \begin{pmatrix} 3e^{-2t} - 2e^{t} + e^{2t} \\ -e^{t} + e^{2t} \\ 3e^{-2t} - e^{t} + e^{2t} \end{pmatrix},$ **15** $\mathbf{x}(t) = \begin{pmatrix} -2 - e^{t} \\ 2 + e^{-t} \\ -e^{-t} + 2e^{t} \end{pmatrix}.$

7.4.2 Complex Conjugate Eigenvalues

It is obvious that if the (real) coefficient matrix A has a complex eigenvalue r, then its associated eigenvectors $\mathbf{v}$, which are the nonzero solutions of the algebraic system $(A-rI)\mathbf{v}=\mathbf{0}$, are complex as well. Taking the conjugate of each term in the system, we see that $(A-\bar{r}I)\bar{\mathbf{v}}=\mathbf{0}$, which means that $\bar{r}$ is also an eigenvalue of A with associated eigenvectors $\bar{\mathbf{v}}$; therefore, we can set up two complex solutions for system (7.17), of the form $\mathbf{v}e^{rt}$ and $\bar{\mathbf{v}}e^{\bar{r}t}$. However, since A is a real matrix, we would like to have two real solutions instead of the above complex conjugate pair. These solutions, constructed on the basis of the argument used in Sect. 4.4.3, are

$$\mathbf{x}^{(1)}(t)=\mathrm{Re}(\mathbf{v}e^{rt}),\qquad \mathbf{x}^{(2)}(t)=\mathrm{Im}(\mathbf{v}e^{rt}). \tag{7.19}$$

7.45 Remark. Since the eigenvalues r and $\bar{r}$ are distinct, from Theorem 7.40 it follows that the complex solutions $\mathbf{v}e^{rt}$ and $\bar{\mathbf{v}}e^{\bar{r}t}$ are linearly independent, and it is not difficult to verify that so are the real solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ defined by (7.19). ■

7.46 Example. For the IVP

$$\begin{aligned} x_1' &= -\ x_1 + x_2, \qquad x_1(0)=3,\\ x_2' &= -2x_1 + x_2, \qquad x_2(0)=1 \end{aligned}$$

we have

$$\det(A-rI)=\begin{vmatrix} -1-r & 1\\ -2 & 1-r \end{vmatrix}=(-1-r)(1-r)+2=r^2+1,$$

so the eigenvalues of A are the roots of the equation $r^2+1=0$; that is, $r_1=i$ and $r_2=-i$. Solving the algebraic system $(A-iI)\mathbf{v}=\mathbf{0}$, we find the basis eigenvector

$$\mathbf{v}=\begin{pmatrix}1\\1+i\end{pmatrix}.$$

Then, using (7.19) and Euler's formula, we construct the real solutions

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= \mathrm{Re}\left\{\begin{pmatrix}1\\1+i\end{pmatrix}e^{it}\right\}=\mathrm{Re}\left\{\begin{pmatrix}1\\1+i\end{pmatrix}(\cos t+i\sin t)\right\}\\ &= \begin{pmatrix}\cos t\\ \cos t-\sin t\end{pmatrix},\\ \mathbf{x}^{(2)}(t) &= \mathrm{Im}\left\{\begin{pmatrix}1\\1+i\end{pmatrix}e^{it}\right\}=\mathrm{Im}\left\{\begin{pmatrix}1\\1+i\end{pmatrix}(\cos t+i\sin t)\right\}\\ &= \begin{pmatrix}\sin t\\ \cos t+\sin t\end{pmatrix}. \end{aligned}$$

According to Remark 7.45, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent, so we can write the real GS of the system as

$$\mathbf{x}(t)=c_1\mathbf{x}^{(1)}(t)+c_2\mathbf{x}^{(2)}(t)=c_1\begin{pmatrix}\cos t\\ \cos t-\sin t\end{pmatrix}+c_2\begin{pmatrix}\sin t\\ \cos t+\sin t\end{pmatrix}.$$

Since both x_1 and x_2 are periodic functions, the trajectories are closed curves around the origin. Therefore, the equilibrium solution $\mathbf{x} = \mathbf{0}$ is stable and the origin is called a *center*. A few trajectories and the direction of motion on them are shown in Fig. 7.3.

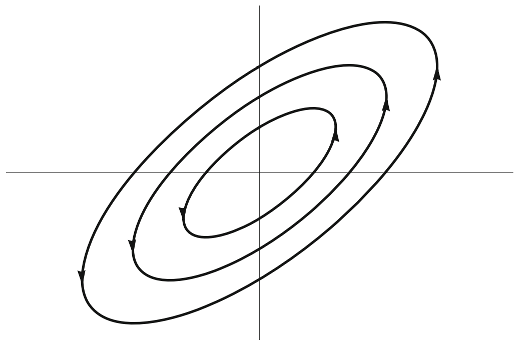

Fig. 7.3

Applying the ICs to the GS, we find that $c_1 = 3$ and $c_2 = -2$, so the solution of the given IVP is

$$\mathbf{x}(t) = 3\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} - 2\begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix} = \begin{pmatrix} 3\cos t - 2\sin t \\ \cos t - 5\sin t \end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2}={3*Cos[t] - 2*Sin[t],Cos[t] - 5*Sin[t]};
{{D[x1,t] + x1 - x2,D[x2,t] + 2x1 - x2},{x1,x2}/.t→ 0}
```

generates the output {{0, 0}, {3, 1}}. ■

7.47 Example. Consider the system

$$\begin{aligned} x_1' &= -x_1 - 2x_2, \quad & x_1(0) = 2, \\ x_2' &= \;\;2x_1 - \;\;x_2, \quad & x_2(0) = 5. \end{aligned}$$

Since

$$\det(A - rI) = \begin{vmatrix} -1-r & -2 \\ 2 & -1-r \end{vmatrix} = (-1-r)(-1-r) + 4 = r^2 + 2r + 5,$$

the eigenvalues of A are the roots of the quadratic equation $r^2 + 2r + 5 = 0$; that is, $r_1 = -1 + 2i$ and $r_2 = -1 - 2i$. Solving the system $(A - (-1+2i)I)\mathbf{v} = \mathbf{0}$, we obtain the basis eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

By (7.19), two real solutions of the given system are

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= \mathrm{Re}(\mathbf{v}e^{(-1+2i)t}) = \mathrm{Re}\left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-t}[\cos(2t) + i\sin(2t)] \right\} = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} e^{-t}, \\ \mathbf{x}^{(2)}(t) &= \mathrm{Im}(\mathbf{v}e^{(-1+2i)t}) = \mathrm{Im}\left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-t}[\cos(2t) + i\sin(2t)] \right\} = \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix} e^{-t}. \end{aligned}$$

As in Example 7.46, from Theorem 7.40 and Remark 7.45 we conclude that these two solutions are linearly independent, so the real GS of the system is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = \left\{ c_1 \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix} \right\} e^{-t}.$$

The periodic factor in the solution indicates that, as t increases from $-\infty$ to ∞, the trajectories keep going around the origin. The exponential factor shows that they are getting arbitrarily close to the origin, which means that the equilibrium solution $\mathbf{x} = \mathbf{0}$ is asymptotically stable. The origin is called a *stable spiral point*. A few trajectories and the direction of motion on them are shown in Fig. 7.4. If the real part of the eigenvalues were positive, the motion on each trajectory would be away from the origin, which means that the origin would be an *unstable spiral point*.

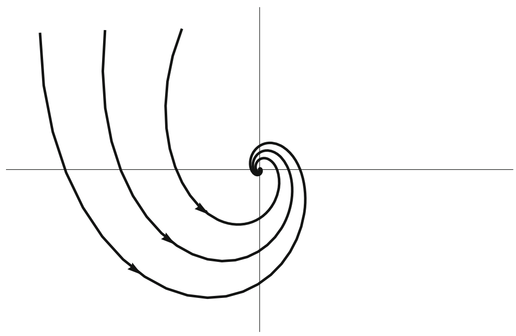

Fig. 7.4

The ICs yield $c_1 = 2$ and $c_2 = -5$, so the solution of the IVP is

$$\mathbf{x}(t) = \begin{pmatrix} 2\cos(2t) - 5\sin(2t) \\ 5\cos(2t) + 2\sin(2t) \end{pmatrix} e^{-t}.$$

Verification with Mathematica®. The input

```
{x1,x2}={E^(-t)*(2*Cos[2*t] - 5*Sin[2*t]),
        E^(-t)*(5*Cos[2*t] + 2*Sin[2*t])};
{{D[x1,t] + x1 + 2*x2,D[x2,t] - 2*x1 + x2},{x1,x2}/.t→ 0}
```

generates the output $\{\{0, 0\}, \{2, 5\}\}$. ■

7.48 Example. For the system in the IVP

$$\begin{aligned} x_1' &= 2x_1 + x_2 - x_3, & x_1(0) &= -2, \\ x_2' &= -4x_1 - 3x_2 - x_3, & x_2(0) &= 3, \\ x_3' &= 4x_1 + 4x_2 + 2x_3, & x_3(0) &= -2 \end{aligned}$$

we have, after expansion of the determinant in any of its rows or columns,

$$\det(A - rI) = \begin{vmatrix} 2-r & 1 & -1 \\ -4 & -3-r & -1 \\ 4 & 4 & 2-r \end{vmatrix} = 4 - 4r + r^2 - r^3,$$

so the eigenvalues of A are the roots of the equation $r^3 - r^2 + 4r - 4 = 0$. Using the synthetic division algorithm or taking advantage of the symmetry of the coefficients, we find that $r_1 = 1$, $r_2 = 2i$, and $r_3 = -2i$. A basis eigenvector associated with $r = 1$ is found by solving the algebraic system $(A - I)\mathbf{v} = \mathbf{0}$, and gives rise to a first solution of the DE system as

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^t.$$

A basis eigenvector associated with $r = 2i$ is a solution of $(A - 2iI)\mathbf{v} = \mathbf{0}$; specifically,

$$\mathbf{v} = \begin{pmatrix} 1+i \\ -2 \\ 2 \end{pmatrix} e^{2i},$$

so, by (7.19), two other—this time, real—solutions of the system are

$$\mathbf{x}^{(2)}(t) = \mathrm{Re}(\mathbf{v}e^{2it}) = \begin{pmatrix} \cos(2t) - \sin(2t) \\ -2\cos(2t) \\ 2\cos(2t) \end{pmatrix},$$

$$\mathbf{x}^{(3)}(t) = \mathrm{Im}(\mathbf{v}e^{2it}) = \begin{pmatrix} \cos(2t) + \sin(2t) \\ -2\sin(2t) \\ 2\sin(2t) \end{pmatrix}.$$

Since, by direct computation, $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}](t) = 2e^t \neq 0$, it follows that the three solutions constructed above are linearly independent, which means that the real GS of the system can be written in the form

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos(2t) - \sin(2t) \\ -2\cos(2t) \\ 2\cos(2t) \end{pmatrix} + c_3 \begin{pmatrix} \cos(2t) + \sin(2t) \\ -2\sin(2t) \\ 2\sin(2t) \end{pmatrix}.$$

Applying the ICs, we find that $c_1 = 1$, $c_2 = -1$, and $c_3 = 0$. Hence, the solution of the IVP is

$$\mathbf{x}(t) = \begin{pmatrix} -e^t - \cos(2t) + \sin(2t) \\ e^t + 2\cos(2t) \\ -2\cos(2t) \end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2,x3}={ - E^t - Cos[2*t] + Sin[2*t], E^t + 2*Cos[2*t],
        -2*Cos[2*t]};
{{D[x1,t] - 2*x1 - x2 + x3, D[x2,t] + 4*x1 + 3*x2 + x3,
        D[x3,t] - 4*x1 - 4*x2 - 2*x3}, {x1,x2,x3}/. t→ 0}
```

generates the output $\{\{0, 0, 0\}, \{-2, 3, -2\}\}$. ∎

Exercises

In 1–6, use the eigenvalue-eigenvector method to solve the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ for the given matrix A and vector $\mathbf{x}_0$, and sketch the phase portrait of the GS of the system.

1 $A = \begin{pmatrix} -4 & 2 \\ -10 & 4 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$ **2** $A = \begin{pmatrix} -1/2 & 1/4 \\ -2 & 1/2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}.$

3 $A = \begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$ **4** $A = \begin{pmatrix} 3/2 & 1 \\ -5/4 & 5/2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

5 $A = \begin{pmatrix} 0 & -1/2 \\ 5/2 & 2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$ **6** $A = \begin{pmatrix} -2 & -2 \\ 13/2 & 4 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -4 \\ 6 \end{pmatrix}.$

In 7–10, use the eigenvalue–eigenvector method to solve the IVP $\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$ for the given matrix A and vector $\mathbf{x}_0$.

7 $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -4 & 6 \\ 1 & -2 & 3 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$ **8** $A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}.$

9 $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}.$ **10** $A = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$

Answers to Odd-Numbered Exercises

1 $\mathbf{x}(t) = \begin{pmatrix} 2\cos(2t) - \sin(2t) \\ 3\cos(2t) - 4\sin(2t) \end{pmatrix}.$ **3** $\mathbf{x}(t) = \begin{pmatrix} -\cos t + 2\sin t \\ 5\sin t \end{pmatrix} e^t.$

5 $\mathbf{x}(t) = \begin{pmatrix} 2\cos(t/2) - 3\sin(t/2) \\ -\cos(t/2) + 8\sin(t/2) \end{pmatrix} e^t.$ **7** $\mathbf{x}(t) = \begin{pmatrix} 4 - \cos t - \sin t \\ 2 - 2\sin t \\ -\sin t \end{pmatrix}.$

9 $\mathbf{x}(t) = \begin{pmatrix} -e^t \sin t \\ 2e^{-t} + e^t \cos t \\ -e^t \cos t \end{pmatrix}.$

7.4.3 Repeated Eigenvalues

If all the eigenvalues of the coefficient matrix A are non-deficient, then the GS of system (7.17) is written, as before, in the form

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) = c_1\mathbf{v}^{(1)}e^{r_1 t} + \cdots + c_n\mathbf{v}^{(n)}e^{r_n t}, \tag{7.20}$$

where the eigenvalues are repeated according to algebraic multiplicity and $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$ are linearly independent basis eigenvectors associated with the eigenvalues.

7.49 Example. Consider the system

$$\begin{aligned} x_1' &= -2x_1 - 2x_2 + 2x_3, & x_1(0) &= 0, \\ x_2' &= -4x_1 \qquad\quad + 2x_3, & x_2(0) &= 4, \\ x_3' &= -8x_1 - 4x_2 + 6x_3, & x_3(0) &= 3. \end{aligned}$$

Since

$$\det(A - rI) = \begin{vmatrix} -2-r & -2 & 2 \\ -4 & -r & 2 \\ -8 & -4 & 6-r \end{vmatrix} = -r^3 + 4r^2 - 4r = -r(r-2)^2,$$

the eigenvalues are $r_1 = 0$ and $r_2 = r_3 = 2$. Computing the solutions of the algebraic systems $A\mathbf{v} = \mathbf{0}$ and $(A - 2I)\mathbf{v} = \mathbf{0}$, respectively, we obtain the basis eigenvectors

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ for } r = 0; \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ for } r = 2.$$

These eigenvectors are linearly independent, so, by (7.20), the GS of the system is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{2t}.$$

Applying the ICs, we find that $c_1 = 1$, $c_2 = -1$, and $c_3 = 2$; hence, the solution of the given IVP is

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t} + 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} 1 - e^{2t} \\ 1 + 3e^{2t} \\ 2 + e^{2t} \end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2,x3}={1 - E^(2*t), 1 + 3*E^(2*t), 2 + E^(2*t)};
{{D[x1,t] + 2*x1 + 2*x2 - 2*x3, D[x2,t] + 4*x1 - 2*x3,
      D[x3,t] + 8*x1 + 4*x2 - 6*x3}, {x1,x2,x3}/. t→ 0}
```

generates the output {{0, 0, 0}, {0, 4, 3}}. ■

7.50 Remark. In the case of deficient eigenvalues, we do not have enough basis eigenvectors for a full FSS and must construct the missing solutions in some other way. Suppose that the coefficient matrix A has a double eigenvalue r_0 with deficiency 1, and let $\mathbf{v}$ be a (single) basis eigenvector associated with r_0; that is, $A\mathbf{v} = r_0\mathbf{v}$. This yields the solution $\mathbf{x}^{(1)}(t) = \mathbf{v}e^{r_0 t}$ for system (7.17). By analogy with the case of a double root of the characteristic equation for a DE, we seek a second solution of the system of the form

$$\mathbf{x} = (\mathbf{u} + \mathbf{v}t)e^{r_0 t}, \tag{7.21}$$

where $\mathbf{u}$ is a constant column vector. Then

$$\mathbf{x}' = [\mathbf{v} + r_0(\mathbf{u} + \mathbf{v}t)]e^{r_0 t},$$

which, replaced in (7.17), leads to the equality

$$(\mathbf{v} + r_0\mathbf{u} + r_0\mathbf{v}t)e^{r_0 t} = A((\mathbf{u} + \mathbf{v}t)e^{r_0 t}) = (A\mathbf{u} + A\mathbf{v}t)e^{r_0 t}.$$

Since $A\mathbf{v} = r_0\mathbf{v}$, this reduces to

$$(\mathbf{v} + r_0\mathbf{u})e^{r_0 t} = A\mathbf{u}e^{r_0 t}.$$

Equating the coefficients of $e^{r_0 t}$ on both sides and writing $r_0\mathbf{u} = r_0 I\mathbf{u}$, we see that $\mathbf{u}$ satisfies the algebraic system

$$(A - r_0 I)\mathbf{u} = \mathbf{v}. \tag{7.22}$$

We notice that the left-hand side of this system is the same as that of the system satisfied by $\mathbf{v}$. The vector $\mathbf{u}$ is called a *generalized eigenvector*. ■

7.51 Example. For the system in the IVP

$$\begin{aligned} x_1' &= -\tfrac{1}{2}\,x_1 - \tfrac{1}{2}\,x_2, \quad & x_1(0) &= 0,\\ x_2' &= \tfrac{1}{2}\,x_1 - \tfrac{3}{2}\,x_2, \quad & x_2(0) &= 2 \end{aligned}$$

we have

$$\det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} - r \end{vmatrix} = r^2 + 2r + 1 = (r+1)^2,$$

which yields the eigenvalues $r_1 = r_2 = -1$. The algebraic system $(A + I)\mathbf{v} = \mathbf{0}$ reduces to the single equation $x_1 - x_2 = 0$, so we obtain only one basis eigenvector, namely

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which generates the solution

$$\mathbf{x}^{(1)}(t) = \mathbf{v}e^{-t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Writing out (7.22), we see that it reduces to the single equation

$$\tfrac{1}{2}\,u_1 - \tfrac{1}{2}\,u_2 = 1.$$

Since we need only one such vector $\mathbf{u}$, we can choose, say, $u_2 = 0$ and obtain the generalized eigenvector

$$\mathbf{u} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Then the solution (7.21) is

$$\mathbf{x}^{(2)}(t) = (\mathbf{u} + \mathbf{v}t)e^{-t} = \begin{pmatrix} 2+t \\ t \end{pmatrix} e^{-t}.$$

The Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ computed at $t = 0$ is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](0) = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2 \neq 0,$$

so, by Remark 7.21(i), $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent. Therefore, the GS of our system is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2+t \\ t \end{pmatrix} e^{-t} = \begin{pmatrix} c_1 + c_2(2+t) \\ c_1 + c_2 t \end{pmatrix} e^{-t}.$$

Given that e^{-t} decays to zero much faster than t increases to infinity, the point (x_1, x_2) approaches the origin asymptotically on each trajectory. Also, rewriting the GS in the form

$$\mathbf{x}(t) = (c_1 + c_2 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t},$$

we note that the first term is the dominant one when t increases without bound in absolute value, so all trajectories are asymptotically parallel to the eigenline $(x_1 - x_2 = 0)$ both as $t \to -\infty$ and as $t \to \infty$.

A few trajectories of the system and the direction of motion on them are shown in Fig. 7.5.

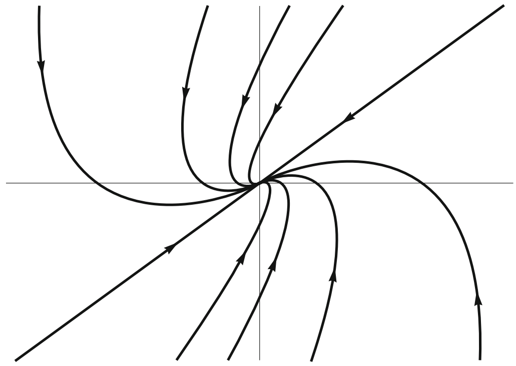

Fig. 7.5

The equilibrium solution $\mathbf{x} = \mathbf{0}$ is asymptotically stable, and the origin is called a *stable degenerate node*. If the (repeated) eigenvalue were positive, we would have an *unstable degenerate node*, with the trajectories looking just as in Fig. 7.5 but with the direction of motion on them away from the origin.

Using the ICs, we now find that $c_1 = 2$ and $c_1 = -1$, so the solution of the IVP is

$$\mathbf{x}(t) = 2\begin{pmatrix}1\\1\end{pmatrix} e^{-t} - \begin{pmatrix}2+t\\t\end{pmatrix} e^{-t} = \begin{pmatrix}-t\\2-t\end{pmatrix} e^{-t}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
{x1,x2}={-t*E^(-t),(2 - t)*E^(-t)};
{{D[x1,t] + (1/2)*x1 + (1/2)*x2,D[x2,t] - (1/2)*x1 + (3/2)*x2},
        {x1,x2}/.t→ 0}
```

generates the output $\{\{0, 0\}, \{0, 2\}\}$. ■

7.52 Remark. The various cases occurring in the discussion of the nature of the origin as an equilibrium solution are summarized in Table 7.1. ■

Table 7.1

Nature of eigenvalues	Nature of equilibrium solution $\mathbf{x} = \mathbf{0}$
Opposite signs	Saddle point (unstable)
Both negative	(Asymptotically) stable node
Both positive	Unstable node
Equal and negative	(Asymptotically) stable degenerate node
Equal and positive	Unstable degenerate node
Complex conjugate, zero real part	Center (stable)
Complex conjugate, negative real part	(Asymptotically) stable spiral point
Complex conjugate, positive real part	Unstable spiral point

7.53 Remark. In the case of a system that consists of at least three DEs, the method for computing a generalized eigenvector needs to be modified slightly if there is more than one basis eigenvector for a deficient eigenvalue. Again, suppose that the coefficient matrix A has an eigenvalue r_0 with deficiency 1 but two associated basis eigenvectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$, which give rise to the solutions $\mathbf{x}^{(1)}(t) = \mathbf{v}^{(1)}e^{r_0t}$ and $\mathbf{x}^{(2)}(t) = \mathbf{v}^{(2)}e^{r_0t}$ for (7.17). We seek a third solution of that system in the form

$$\mathbf{x}(t) = [\mathbf{u} + (a_1\mathbf{v}^{(1)} + a_2\mathbf{v}^{(2)})t]e^{r_0t}, \tag{7.23}$$

where a_1 and a_2 are nonzero constants chosen later in the procedure to ensure computational consistency. Then $\mathbf{x}' = \{a_1\mathbf{v}^{(1)} + a_2\mathbf{v}^{(2)} + r_0[\mathbf{u} + (a_1\mathbf{v}^{(1)} + a_2\mathbf{v}^{(2)})t]\}e^{r_0t}$, which, replaced in (7.17), yields

$$\begin{aligned}&[a_1\mathbf{v}^{(1)} + a_2\mathbf{v}^{(2)} + r_0\mathbf{u} + a_1r_0\mathbf{v}^{(1)}t + a_2r_0\mathbf{v}^{(2)}t]e^{r_0t}\\ &\qquad = A[\mathbf{u} + (a_1\mathbf{v}^{(1)} + a_2\mathbf{v}^{(2)})t]e^{r_0t} = [A\mathbf{u} + a_1A\mathbf{v}^{(1)}t + a_2A\mathbf{v}^{(2)}t]e^{r_0t}.\end{aligned}$$

Recalling that $A\mathbf{v}^{(1)} = r_0\mathbf{v}^{(1)}$ and $A\mathbf{v}^{(2)} = r_0\mathbf{v}^{(2)}$, canceling the like terms, writing $r_0\mathbf{u} = r_0I\mathbf{u}$, and equating the coefficients of e^{r_0t} on both sides, we conclude that the generalized eigenvector $\mathbf{u}$ is a solution of the algebraic system

$$(A - r_0I)\mathbf{u} = a_1\mathbf{v}^{(1)} + a_2\mathbf{v}^{(2)}. \;\blacksquare \tag{7.24}$$

7.54 Example. For the system

$$\begin{aligned} x_1' &= -2x_1 - 9x_2, & x_1(0) &= -1,\\ x_2' &= x_1 + 4x_2, & x_2(0) &= 1,\\ x_3' &= x_1 + 3x_2 + x_3, & x_3(0) &= 2\end{aligned}$$

we have

$$\det(A - rI) = \begin{vmatrix} -2-r & -9 & 0\\ 1 & 4-r & 0\\ 1 & 3 & 1-r\end{vmatrix} = (1-r)(r^2 - 2r + 1) = (1-r)^3,$$

so the eigenvalues of the matrix A are $r_1 = r_2 = r_3 = 1$. Since the algebraic system $(A - I)\mathbf{v} = \mathbf{0}$ reduces to the single equation $v_1 + 3v_2 = 0$, it follows that v_3 is arbitrary and that, consequently, two (linearly independent) basis eigenvectors are

$$\mathbf{v}^{(1)} = \begin{pmatrix} -3\\ 1\\ 0\end{pmatrix}, \quad \mathbf{v}^{(2)} = \begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}.$$

In this case, (7.24) with $r_0 = 1$ yields the three equations

$$\begin{aligned} -3u_1 - 9u_2 &= -3a_1,\\ u_1 + 3u_2 &= a_1,\\ u_1 + 3u_2 &= a_2,\end{aligned}$$

which are consistent if $a_1 = a_2$. We notice that u_3 remains arbitrary. Setting, say, $a_1 = a_2 = 1$ and $u_2 = u_3 = 0$, we obtain the generalized eigenvector

$$\mathbf{u} = \begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}.$$

Hence, using the eigenvectors, **u**, and (7.23), we construct the three solutions

$$\mathbf{x}^{(1)}(t) = \mathbf{v}^{(1)}e^t = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} e^t, \quad \mathbf{x}^{(2)}(t) = \mathbf{v}^{(2)}e^t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t,$$

$$\mathbf{x}^{(3)}(t) = [\mathbf{u} + (\mathbf{v}^{(1)} + \mathbf{v}^{(2)})t]e^t = \begin{pmatrix} 1-3t \\ t \\ t \end{pmatrix} e^t.$$

Since the Wronskian of these solutions at $t = 0$ is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}](0) = \begin{vmatrix} -3 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0,$$

from Remark 7.21(i) it follows that they are linearly independent, so the GS of our system is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + c_3\mathbf{x}^{(3)}(t) = \begin{pmatrix} -3c_1 + c_3(1-3t) \\ c_1 + c_3 t \\ c_2 + c_3 t \end{pmatrix} e^t.$$

Applying the ICs leads to the values $c_1 = 1$, $c_2 = 2$, and $c_3 = 2$, which means that the solution of the given IVP is

$$\mathbf{x}(t) = \begin{pmatrix} -1-6t \\ 1+2t \\ 2+2t \end{pmatrix} e^t.$$

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```
{x1,x2,x3}={(-1-6*t)*E^t, (1+2*t)*E^t, (2+2*t)*E^t};
{{D[x1,t] + 2*x1 + 9*x2, D[x2,t] - x1 - 4*x2,
      D[x3,t] - x1 - 3*x2 - x3}, {x1,x2,x3} /. t→ 0}
```

generates the output $\{\{0, 0, 0\}, \{-1, 1, 2\}\}$. ■

7.55 Remark. If the deficiency of an eigenvalue is $m > 1$, we need to construct a chain of m generalized eigenvectors to have a full FSS for the homogeneous system (7.17). For definiteness, suppose that the coefficient matrix A has an eigenvalue r_0 with deficiency 2 and one basis eigenvector **v**, which gives rise to a solution $\mathbf{x}^{(1)}(t) = \mathbf{v}e^{r_0 t}$. As in Remark 7.50, we can use (7.21) and (7.22) to produce a first generalized eigenvector **u** and, thus, a second solution $\mathbf{x}^{(2)}$. To obtain a third solution of (7.17), we seek it in the form

$$\mathbf{x}(t) = \big(\mathbf{w} + \mathbf{u}t + \tfrac{1}{2}\,\mathbf{v}t^2\big)e^{r_0 t}, \tag{7.25}$$

where **w** is a second generalized eigenvector. Then, replacing **x** in (7.17) and recalling that $A\mathbf{v} = r_0\mathbf{v}$ and $A\mathbf{u} = r_0\mathbf{u} + \mathbf{v}$, we easily find that **w** is a solution of the algebraic system

$$(A - r_0 I)\mathbf{w} = \mathbf{u}. \ ■ \tag{7.26}$$

7.56 Example. Consider the system

$$\begin{aligned} x_1' &= x_1 - x_2 + 2x_3, & x_1(0) &= 6, \\ x_2' &= 6x_1 - 4x_2 + 5x_3, & x_2(0) &= 3, \\ x_3' &= 2x_1 - x_2, & x_3(0) &= -5. \end{aligned}$$

Here we have

$$\det(A-rI)=\begin{vmatrix}1-r & -1 & 2\\ 6 & -4-r & 5\\ 2 & -1 & -r\end{vmatrix}=(-1-r)(r^2+2r+1)=-(r+1)^3,$$

so the eigenvalues are $r_1=r_2=r_3=-1$. Solving the algebraic system $(A+I)\mathbf{v}=\mathbf{0}$, we find that this triple eigenvalue has geometric multiplicity 1, and we choose the associated basis eigenvector

$$\mathbf{v}=\begin{pmatrix}1\\2\\0\end{pmatrix}.$$

By (7.22), the first generalized eigenvector, $\mathbf{u}$, satisfies $(A+I)\mathbf{u}=\mathbf{v}$. On components, this is equivalent to

$$\begin{aligned}2u_1-\ \ u_2+2u_3&=1,\\ 6u_1-3u_2+5u_3&=2,\\ 2u_1-\ \ u_2+\ \ u_3&=0,\end{aligned}$$

which Gaussian elimination reduces to the equations $2u_1-u_2+2u_3=1$ and $u_3=1$. Since one of u_1 and u_2 may be chosen arbitrarily, we take, say, $u_1=0$, and obtain

$$\mathbf{u}=\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Then, by (7.26),

$$\begin{aligned}2w_1-\ \ w_2+2w_3&=0,\\ 6w_1-3w_2+5w_3&=1,\\ 2w_1-\ \ w_2+\ \ w_3&=1.\end{aligned}$$

Applying Gaussian elimination once more, we reduce this system to the pair of equations $2w_1-w_2+2w_3=0$ and $w_3=-1$, and, with the choice $w_2=0$, get the second generalized eigenvector

$$\mathbf{w}=\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

According to (7.21) and (7.25), we can now construct for the system the three solutions

$$\mathbf{x}^{(1)}(t)=\mathbf{v}e^{-t}=\begin{pmatrix}1\\2\\0\end{pmatrix}e^{-t},\quad \mathbf{x}^{(2)}(t)=(\mathbf{u}+\mathbf{v}t)e^{-t}=\begin{pmatrix}t\\1+2t\\1\end{pmatrix}e^{-t},$$

$$\mathbf{x}^{(3)}(t)=\big(\mathbf{w}+\mathbf{u}t+\tfrac{1}{2}\,\mathbf{v}t^2\big)e^{-t}=\begin{pmatrix}1+\frac{1}{2}\,t^2\\ t+t^2\\ -1+t\end{pmatrix}e^{-t}.$$

Since their Wronskian at $t=0$ is

$$W[\mathbf{x}^{(1)},\mathbf{x}^{(2)},\mathbf{x}^{(3)}](0)=\begin{vmatrix}1 & 0 & 1\\ 2 & 1 & 0\\ 0 & 1 & -1\end{vmatrix}=1\neq 0,$$

from Remarks 7.39 and 7.21(ii) it follows that they are linearly independent, so the GS of the system is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + c_3\mathbf{x}^{(3)}(t) = \begin{pmatrix} c_1 + c_2 t + c_3\bigl(1 + \frac{1}{2}t^2\bigr) \\ 2c_1 + c_2(1+2t) + c_3(t+t^2) \\ c_2 + c_3(t-1) \end{pmatrix} e^{-t}.$$

To compute the solution of the given IVP, we apply the ICs and find that $c_1 = 2$, $c_2 = -1$, and $c_3 = 4$; hence,

$$\mathbf{x}(t) = \begin{pmatrix} 2t^2 - t + 6 \\ 4t^2 + 2t + 3 \\ 4t - 5 \end{pmatrix} e^{-t}.$$

Verification with Mathematica®. The input

```
{x1,x2,x3}={(2*t^2 - t + 6)*E^(-t), (4*t^2 + 2*t + 3)*E^(-t),
      (4*t - 5)*E^(-t)};
{{D[x1,t] - x1 + x2 - 2*x3, D[x2,t] - 6*x1 + 4*x2 - 5*x3,
      D[x3,t] - 2*x1 + x2}, {x1,x2,x3}/.t→ 0}
```

generates the output $\{\{0, 0, 0\}, \{6, 3, -5\}\}$. ■

Exercises

In 1–6, use the eigenvalue–eigenvector method to solve the IVP $\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$ for the given matrix A and vector $\mathbf{x}_0$, and sketch the phase portrait of the general solution of the system.

1 $A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. **2** $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

3 $A = \begin{pmatrix} -2/3 & -1 \\ 1 & 4/3 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$. **4** $A = \begin{pmatrix} 3/2 & -1 \\ 4 & -5/2 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

5 $A = \begin{pmatrix} 1 & -9 \\ 1 & -5 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. **6** $A = \begin{pmatrix} 2 & -1/2 \\ 1/2 & 1 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

In 7–16, use the eigenvalue–eigenvector method to solve the IVP $\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$ for the given matrix A and vector $\mathbf{x}_0$.

7 $A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. **8** $A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$.

9 $A = \begin{pmatrix} 2 & -3 & -3 \\ -3 & 2 & 3 \\ 3 & -3 & -4 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$. **10** $A = \begin{pmatrix} 1/4 & -1/4 & 1/2 \\ 1/4 & -1/4 & 1/2 \\ -1/4 & 1/4 & -1/2 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$.

11 $A = \begin{pmatrix} -2 & -3 & 4 \\ 1 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$. **12** $A = \begin{pmatrix} 1 & -1 & -2 \\ 1 & -2 & -1 \\ 0 & 1 & -1 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$.

13 $A = \begin{pmatrix} -2 & 1 & -3 \\ 2 & -3 & 6 \\ 1 & -1 & 2 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$. **14** $A = \begin{pmatrix} 3 & 1 & 1 \\ -6 & -2 & -3 \\ 2 & 1 & 2 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$.

15 $A = \begin{pmatrix} -2 & 5 & 6 \\ -1 & 2 & 2 \\ -1 & 2 & 3 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. **16** $A = \begin{pmatrix} -3 & 2 & -3 \\ -1 & 0 & -2 \\ 1 & -1 & 0 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$.

Answers to Odd-Numbered Exercises

1 $\mathbf{x}(t) = \begin{pmatrix} 4t \\ 1-2t \end{pmatrix} e^{-t}$. **3** $\mathbf{x}(t) = \begin{pmatrix} -t-3 \\ t+4 \end{pmatrix} e^{t/3}$.

5 $\mathbf{x}(t) = \begin{pmatrix} -9t \\ 1-3t \end{pmatrix} e^{-2t}$. **7** $\mathbf{x}(t) = \begin{pmatrix} 1+e^t \\ 1-e^t \\ 1+2e^t \end{pmatrix}$.

9 $\mathbf{x}(t) = \begin{pmatrix} 2e^{-t}+2e^{2t} \\ e^{-t}-2e^{2t} \\ e^{-t}+2e^{2t} \end{pmatrix}$. **11** $\mathbf{x}(t) = \begin{pmatrix} 4+(1-t)e^t \\ (t-2)e^t \\ 2-e^t \end{pmatrix}$.

13 $\mathbf{x}(t) = \begin{pmatrix} t-1 \\ 3-2t \\ 1-t \end{pmatrix} e^{-t}$. **15** $\mathbf{x}(t) = \begin{pmatrix} 1-4t-2t^2 \\ 1-2t \\ -1-t-t^2 \end{pmatrix} e^{t}$.

7.5 Other Features of Homogeneous Linear Systems

Fundamental matrices. Consider the general homogeneous linear system (7.13); that is,

$$\mathbf{x}' = A(t)\mathbf{x}, \tag{7.27}$$

let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ be an FSS for it, and construct the matrix

$$X(t) = (\mathbf{x}^{(1)}(t) \ \dots \ \mathbf{x}^{(n)}(t)),$$

whose columns are the components of the vector functions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$.

7.57 Definition. The matrix function X is called a *fundamental matrix* for system (7.27). ■

7.58 Remarks. (i) Obviously, there are infinitely many fundamental matrices for any given system.

(ii) Since each column of a fundamental matrix X is a solution of (7.27), it follows that X satisfies the matrix equation $X' = AX$.

(iii) A fundamental matrix X allows us to construct an alternative form of the solution of an IVP for system (7.27). Let $\mathbf{c}$ be a column vector of constant components $c_1, \dots, c_n$. Denoting by $x_i^{(j)}$ the ith component of $\mathbf{x}^{(j)}$, we can write the GS of (7.27) as

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)} = c_1 \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_n^{(1)} \end{pmatrix} + \cdots + c_n \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_n^{(n)} \end{pmatrix}$$
$$= \begin{pmatrix} c_1x_1^{(1)} + \cdots + c_nx_1^{(n)} \\ \vdots \\ c_1x_n^{(1)} + \cdots + c_nx_n^{(n)} \end{pmatrix} = \begin{pmatrix} x_1^{(1)} & \dots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \dots & x_n^{(n)} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = X\mathbf{c}. \tag{7.28}$$

The columns of $X(t)$ are linearly independent vector functions, so, according to Definition 7.19, there is at least one point t where $\det(X(t)) \neq 0$ and where, therefore, $X(t)$ is invertible. If we are also given an IC of the form $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is a prescribed constant vector and t_0 is such that $\det(X(t_0)) \neq 0$, then, in view of (7.28), we have $X(t_0)\mathbf{c} = \mathbf{x}_0$, from which $\mathbf{c} = X^{-1}(t_0)\mathbf{x}_0$. Replacing in (7.28), we conclude that the solution of the IVP is

$$\mathbf{x}(t) = X(t)(X^{-1}(t_0)\mathbf{x}_0) = (X(t)X^{-1}(t_0))\mathbf{x}_0. \ \blacksquare \tag{7.29}$$

7.59 Example. The IVP

$$\begin{aligned} x_1' &= 3x_1 + 4x_2, \quad & x_1(0) = 2, \\ x_2' &= 3x_1 + 2x_2, \quad & x_2(0) = 5 \end{aligned}$$

was discussed in Example 7.41, where we constructed for it the two linearly independent solutions

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t}.$$

Using these solutions, we have

$$X(t) = (\mathbf{x}^{(1)}(t) \;\; \mathbf{x}^{(2)}(t)) = \begin{pmatrix} -e^{-t} & 4e^{6t} \\ e^{-t} & 3e^{6t} \end{pmatrix},$$
$$X(0) = \begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix}, \quad X^{-1}(0) = \begin{pmatrix} -\frac{3}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix},$$

so

$$X(t)X^{-1}(0) = \begin{pmatrix} -e^{-t} & 4e^{6t} \\ e^{-t} & 3e^{6t} \end{pmatrix} \begin{pmatrix} -\frac{3}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 3e^{-t} + 4e^{6t} & -4e^{-t} + 4e^{6t} \\ -3e^{-t} + 3e^{6t} & 4e^{-t} + 3e^{6t} \end{pmatrix};$$

hence, by (7.29) with $t_0 = 0$, the solution of the given IVP is

$$\mathbf{x}(t) = (X(t)X^{-1}(0))\mathbf{x}_0 = \frac{1}{7} \begin{pmatrix} 3e^{-t} + 4e^{6t} & -4e^{-t} + 4e^{6t} \\ -3e^{-t} + 3e^{6t} & 4e^{-t} + 3e^{6t} \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2e^{-t} + 4e^{6t} \\ 2e^{-t} + 3e^{6t} \end{pmatrix},$$

confirming the earlier result. ■

Exponential matrix. As mentioned at the beginning of Sect. 7.4, the solution of the scalar IVP $x' = ax$, $x(0) = x_0$, $a = \text{const}$, is $x = x_0e^{at}$. It can be shown that a similar formula exists for the solution of the corresponding IVP in the case of systems of the form

$$\mathbf{x}' = A\mathbf{x}, \tag{7.30}$$
$$\mathbf{x}(0) = \mathbf{x}_0, \tag{7.31}$$

where A is a constant $n \times n$ matrix. To construct this formula, we need a definition for e^{At}, which can easily be formulated if we make use of series expansions. The Taylor series

$$e^{at} = \sum_{n=0}^{\infty} \frac{1}{n!}\, a^n t^n = 1 + \frac{1}{1!}\, at + \frac{1}{2!}\, a^2 t^2 + \cdots \tag{7.32}$$

is convergent for any real values of a and t. Since the power A^n of an $n \times n$ matrix A can be computed for any positive integer n (by multiplying A by itself n times) and the result is always an $n \times n$ matrix, it follows that the right-hand side in (7.32) remains meaningful if a is replaced by A, and we write formally

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}\, A^n t^n = I + \frac{1}{1!}\, At + \frac{1}{2!}\, A^2 t^2 + \cdots, \tag{7.33}$$

where we have set $A^0 = I$. It turns out that, just like (7.32), series (7.33) converges for any constant $n \times n$ matrix A and any value of t. Its sum, which we have denoted by e^{At}, is referred to as an *exponential matrix*.

7.60 Remark. The matrix function e^{At} has many of the properties of the scalar exponential function e^{at}. Thus,

(i) $e^0 = I$, where 0 is the zero matrix;
(ii) If $AB = BA$, then $e^{At+Bt} = e^{At} e^{Bt}$;
(iii) $\big(e^{At}\big)' = Ae^{At} = e^{At}A$.

These properties are proved directly from definition (7.33). ■

7.61 Remarks. (i) The vector function $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ is the solution of the IVP (7.30), (7.31) since, by properties (i) and (iii) in Remark 7.60,

$$\mathbf{x}'(t) = Ae^{At}\mathbf{x}_0 = A\mathbf{x}(t), \quad \mathbf{x}(0) = I\mathbf{x}_0 = \mathbf{x}_0.$$

But the same solution is also given by (7.29), so

$$e^{At} = X(t)X^{-1}(0),$$

where X is any fundamental matrix for the system.

(ii) If the ICs are given at $t = t_0$, then it is easily verified that the solution of the new IVP is

$$\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x}_0. \ ■$$

7.62 Example. As can be seen from the solution of Example 7.59 in conjunction with Remark 7.61(i), for the matrix

$$A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$$

we have

$$e^{At} = X(t)X^{-1}(0) = \frac{1}{7}\begin{pmatrix} 3e^{-t} + 4e^{6t} & -4e^{-t} + 4e^{6t} \\ -3e^{-t} + 3e^{6t} & 4e^{-t} + 3e^{6t} \end{pmatrix}. \ ■$$

Diagonalization. The IVP (7.30), (7.31) would be much easier to solve if the equations of the system were decoupled; that is, if they were individual (first-order) DEs for each of the unknown components of the vector function $\mathbf{x}$. As it happens, there is a procedure that, under certain conditions, allows us to reduce the problem to an equivalent one of just this form.

Let $r_1, \dots, r_n$ be the (real) eigenvalues of A (some may be repeated according to multiplicity), and suppose that there are n linearly independent basis eigenvectors $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$ associated with them. In the usual notation, consider the matrix $V = (\mathbf{v}^{(1)} \ \dots \ \mathbf{v}^{(n)})$ whose columns consist of the components of these eigenvectors. Then

$$\begin{aligned} AV &= (A\mathbf{v}^{(1)} \ \dots \ A\mathbf{v}^{(n)}) = (r_1\mathbf{v}^{(1)} \ \dots \ r_n\mathbf{v}^{(n)}) \\ &= \begin{pmatrix} r_1 v_1^{(1)} & \dots & r_n v_1^{(n)} \\ \vdots & \ddots & \vdots \\ r_1 v_n^{(1)} & \dots & r_n v_n^{(n)} \end{pmatrix} = \begin{pmatrix} v_1^{(1)} & \dots & v_1^{(n)} \\ \vdots & \ddots & \vdots \\ v_n^{(1)} & \dots & v_n^{(n)} \end{pmatrix} \begin{pmatrix} r_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_n \end{pmatrix} = VR, \end{aligned}$$

where R is the $n \times n$ matrix with $r_1, \dots, r_n$ on its leading diagonal and zeros elsewhere. Since the eigenvectors are linearly independent, from Theorem 7.16 it follows that $\det V \neq 0$, so V is invertible. Multiplying the above equality by V^{-1} on the left yields

$$R = V^{-1}AV. \tag{7.34}$$

If we now make the substitution

$$\mathbf{x} = V\mathbf{y} \tag{7.35}$$

in (7.30) and (7.31), we arrive at the equalities $V\mathbf{y}' = AV\mathbf{y}$ and $V\mathbf{y}(0) = \mathbf{x}_0$, which, on multiplication on the left by V^{-1} and in view of (7.34), reduce to

$$\mathbf{y}' = (V^{-1}AV)\mathbf{y} = R\mathbf{y}, \tag{7.36}$$

$$\mathbf{y}(0) = V^{-1}\mathbf{x}_0. \tag{7.37}$$

The equations in (7.36) are $y_1' = r_1 y_1, \dots, y_n' = r_n y_n$. Solving each of them with its appropriate IC supplied by (7.37), we then use (7.35) to obtain the solution $\mathbf{x}$ of the IVP (7.30), (7.31).

7.63 Example. The IVP

$$\begin{aligned} x_1' &= 3x_1 + 4x_2, \quad & x_1(0) &= 2, \\ x_2' &= 3x_1 + 2x_2, \quad & x_2(0) &= 5 \end{aligned}$$

was solved in Example 7.41 (see also Example 7.59). Using the eigenvalues and eigenvectors of its associated matrix A, we have

$$V = \begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix}, \quad V^{-1} = \frac{1}{7}\begin{pmatrix} -3 & 4 \\ 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}.$$

Thus, the decoupled IVP (7.36), (7.37) is

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -3 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

or, explicitly,

$$\begin{aligned} y_1' &= -y_1, \quad & y_1(0) &= 2, \\ y_2' &= \ 6y_2, \quad & y_2(0) &= 1. \end{aligned}$$

Solving these two individual scalar IVPs, we find that $y_1(t) = 2e^{-t}$ and $y_2(t) = e^{6t}$, so, by (7.35), the solution of the original IVP is

$$\mathbf{x}(t) = \begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2e^{-t} \\ e^{6t} \end{pmatrix} = \begin{pmatrix} -2e^{-t} + 4e^{6t} \\ 2e^{-t} + 3e^{6t} \end{pmatrix},$$

confirming the earlier result. ■

Exercises

In 1–10, compute the exponential matrix e^{At} for the IVP $\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$ with the given matrix A and vector $\mathbf{x}_0$, and use it to solve the IVP.

1 $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$ **2** $A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$

3 $A = \begin{pmatrix} 1 & -2 \\ 1/2 & 1 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$ **4** $A = \begin{pmatrix} -1 & 1 \\ -5 & 1 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$

5 $A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$ **6** $A = \begin{pmatrix} -5/3 & -4 \\ 1 & 7/3 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

7 $A = \begin{pmatrix} -4 & 12 & -6 \\ 1 & -3 & 1 \\ 6 & -18 & 8 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$ **8** $A = \begin{pmatrix} 5 & -4 & 4 \\ -1 & 2 & -2 \\ -3 & 3 & -3 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$

9 $A = \begin{pmatrix} -2 & -2 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$ **10** $A = \begin{pmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$

In 11–16, use the diagonalization method to solve the IVP $\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$ with the given matrix A and vector $\mathbf{x}_0$.

11 $A = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$ **12** $A = \begin{pmatrix} 8 & -10 \\ 5 & -7 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

13 $A = \begin{pmatrix} -1 & -3 \\ 3/2 & 7/2 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$ **14** $A = \begin{pmatrix} -2 & 1/2 \\ -3 & 1/2 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

15 $A = \begin{pmatrix} -1/2 & -1/2 \\ 1 & 1/2 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$ **16** $A = \begin{pmatrix} -4 & 4 \\ -5 & 4 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$

Answers to Odd-Numbered Exercises

1 $e^{At} = \begin{pmatrix} -e^{-t} + 2e^{t} & -2e^{-t} + 2e^{t} \\ e^{-t} - e^{t} & 2e^{-t} - e^{t} \end{pmatrix},\quad \mathbf{x}(t) = \begin{pmatrix} -4e^{-t} + 2e^{t} \\ 4e^{-t} - e^{t} \end{pmatrix}.$

3 $e^{At} = \begin{pmatrix} \cos t & -2\sin t \\ (1/2)\sin t & \cos t \end{pmatrix} e^{t},\quad \mathbf{x}(t) = \begin{pmatrix} -2\cos t - 2\sin t \\ \cos t - \sin t \end{pmatrix} e^{t}.$

5 $e^{At} = \begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix} e^{-2t},\quad \mathbf{x}(t) = \begin{pmatrix} t-1 \\ t \end{pmatrix} e^{-2t}.$

7 $e^{At} = \begin{pmatrix} 3 - 2e^{2t} & -6 + 6e^{2t} & 3 - 3e^{2t} \\ 1 - e^{-t} & -2 + 3e^{-t} & 1 - e^{-t} \\ -2e^{-t} + 2e^{2t} & 6e^{-t} - 6e^{2t} & -2e^{-t} + 3e^{2t} \end{pmatrix},\quad \mathbf{x}(t) = \begin{pmatrix} -3 + 3e^{2t} \\ -1 + 2e^{-t} \\ 4e^{-t} - 3e^{2t} \end{pmatrix}.$

9 $\mathbf{x}(t) = \begin{pmatrix} 2e^{-t} - \cos t + 2\sin t \\ -e^{-t} + \cos t - 2\sin t \\ 2\cos t + \sin t \end{pmatrix}.$ **11** $\mathbf{x}(t) = \begin{pmatrix} 3e^{-2t} - e^{2t} \\ -3e^{-2t} + 2e^{2t} \end{pmatrix}.$

13 $\mathbf{x}(t) = \begin{pmatrix} 2e^{t/2} + e^{2t} \\ -e^{t/2} - e^{2t} \end{pmatrix}.$ **15** $\mathbf{x}(t) = \begin{pmatrix} -\sin(t/2) \\ \cos(t/2) + \sin(t/2) \end{pmatrix}.$

7.6 Nonhomogeneous Linear Systems

Owing to its linear nature, it is easy to see that, as in the case of single nonhomogeneous linear DEs, the GS $\mathbf{x}$ of a system of the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t) \tag{7.38}$$

is written as the sum of the complementary (vector) function $\mathbf{x}_c$ (the GS of the associated homogeneous system) and a particular solution $\mathbf{x}_p$ of the nonhomogeneous system; that is,

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t). \tag{7.39}$$

Since we have shown how $\mathbf{x}_c$ is computed, we now turn our attention to methods for finding $\mathbf{x}_p$ in the case of systems (7.38) with constant coefficients.

7.64 Remark. It is important to stress that any IC attached to (7.38), say,

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{7.40}$$

must be applied to the full GS (7.39) and not to $\mathbf{x}_c$. ■

Diagonalization. This technique, described in Sect. 7.5 for the homogeneous case, carries over to nonhomogeneous systems with the obvious modifications. Once again, we assume that there are n linearly independent eigenvectors $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$ associated with the (real) eigenvalues $r_1, \dots, r_n$ of the (constant) matrix A, and construct the matrix $V = (\mathbf{v}^{(1)} \ \ \dots \ \ \mathbf{v}^{(n)})$. Then, making the substitution $\mathbf{x} = V\mathbf{y}$ in (7.38) and (7.40), we arrive at the equivalent decoupled IVP

$$\mathbf{y}' = R\,\mathbf{y} + V^{-1}\mathbf{f}, \tag{7.41}$$

$$\mathbf{y}(t_0) = V^{-1}\mathbf{x}_0, \tag{7.42}$$

where R is the diagonal matrix of the eigenvalues defined earlier (see also (7.34)).

7.65 Example. Consider the IVP

$$\begin{aligned} x_1' &= \tfrac{7}{3}\,x_1 + \tfrac{2}{3}\,x_2 - 1, & x_1(0) &= 2, \\ x_2' &= \tfrac{4}{3}\,x_1 + \tfrac{5}{3}\,x_2 + 9t - 10, & x_2(0) &= 5. \end{aligned}$$

The corresponding homogeneous system was discussed in Example 7.43, where it was shown that the eigenvalues and basis eigenvectors of the matrix A are, respectively, $r_1 = 1$ and $r_2 = 3$ and

$$\mathbf{v}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so

$$V = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \quad V^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Therefore, the decoupled IVP (7.41), (7.42) is

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 9t - 10 \end{pmatrix},$$
$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix},$$

or, on components,

$$y_1' = y_1 + 3t - 3, \qquad y_1(0) = 1,$$
$$y_2' = 3y_2 + 3t - 4, \qquad y_2(0) = 3.$$

Using, for example, the method of undetermined coefficients, we find that the solutions of these two individual scalar IVPs are $y_1(t) = e^t - 3t$ and $y_2(t) = 2e^{3t} + 1 - t$. Hence, the solution of the given IVP is

$$\mathbf{x}(t) = V\,\mathbf{y}(t) = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^t - 3t \\ 2e^{3t} + 1 - t \end{pmatrix} = \begin{pmatrix} -e^t + 2e^{3t} + 2t + 1 \\ 2e^t + 2e^{3t} - 7t + 1 \end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2}={ - E^t + 2*E^(3*t) + 2*t + 1, 2*E^t + 2*E^(3*t)
        - 7*t + 1};
{{D[x1,t] - (7/3)*x1 - (2/3)*x2 + 1, D[x2,t] - (4/3)*x1
        - (5/3)*x2 - 9*t + 10}, {x1,x2}/. t→ 0}
```

generates the output $\{\{0, 0\}, \{2, 5\}\}$. ■

Undetermined coefficients. In essence, this method works in the same way as for single DEs, with some adjustments, one of which is illustrated below.

7.66 Example. In the IVP

$$x_1' = \tfrac{7}{3}\,x_1 + \tfrac{2}{3}\,x_2 - 9t + 17, \qquad x_1(0) = -7,$$
$$x_2' = \tfrac{4}{3}\,x_1 + \tfrac{5}{3}\,x_2 - 1, \qquad x_2(0) = 5,$$

the matrix A is the same as that of the system in Example 7.65. Therefore, using its eigenvalues and eigenvectors already computed there, we obtain the complementary function

$$\mathbf{x}_c(t) = c_1\mathbf{v}^{(1)}e^{r_1 t} + c_2\mathbf{v}^{(2)}e^{r_2 t} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

Since the nonhomogeneous term

$$\mathbf{f}(t) = \begin{pmatrix} -9 \\ 0 \end{pmatrix} t + \begin{pmatrix} 17 \\ -1 \end{pmatrix}$$

is a first-degree polynomial in t with constant vector coefficients and 0 is not an eigenvalue of A, we seek (see Theorem 4.47) a particular solution of the form $\mathbf{x}_p = \mathbf{a}t + \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. Replacing in the system, we arrive at the equality $\mathbf{a} = A(\mathbf{a}t + \mathbf{b}) + \mathbf{f}$, from which, equating the coefficients of t and the constant terms on both sides, we find that

$$A\mathbf{a} + \begin{pmatrix} -9 \\ 0 \end{pmatrix} = 0, \quad \mathbf{a} = A\mathbf{b} + \begin{pmatrix} 17 \\ -1 \end{pmatrix}.$$

It is easily seen that A is invertible and that

$$A^{-1} = \frac{1}{9}\begin{pmatrix} 5 & -2 \\ -4 & 7 \end{pmatrix};$$

hence,

$$\mathbf{a} = A^{-1}\begin{pmatrix} 9 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 5 & -2 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} 9 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix},$$
$$\mathbf{b} = A^{-1}\left[\mathbf{a} - \begin{pmatrix} 17 \\ -1 \end{pmatrix}\right] = \frac{1}{9}\begin{pmatrix} 5 & -2 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} -12 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix},$$

so

$$\mathbf{x}_p = \begin{pmatrix} 5t - 6 \\ 3 - 4t \end{pmatrix}.$$

Then the GS of the system is

$$\mathbf{x}(t) = c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + \begin{pmatrix} 5t - 6 \\ 3 - 4t \end{pmatrix}.$$

Applying the ICs, we now find that $c_1 = 1$ and $c_2 = 0$, which means that the solution of the IVP is

$$\mathbf{x}(t) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}e^t + \begin{pmatrix} 5t - 6 \\ 3 - 4t \end{pmatrix} = \begin{pmatrix} -e^t + 5t - 6 \\ 2e^t + 3 - 4t \end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2}={ - E^t + 5*t - 6, 2*E^t + 3 - 4*t};
{{D[x1,t] - (7/3)*x1 - (2/3)*x2 + 9*t - 17, D[x2,t] - (4/3)*x1
        - (5/3)*x2 + 1}, {x1,x2}/. t→ 0}
```

generates the output $\{\{0, 0\}, \{-7, 5\}\}$. ■

7.67 Example. Consider the IVP

$$x_1' = \tfrac{7}{3}x_1 + \tfrac{2}{3}x_2 - 5e^t, \quad x_1(0) = 1,$$
$$x_2' = \tfrac{4}{3}x_1 + \tfrac{5}{3}x_2 - 8e^t, \quad x_2(0) = 4.$$

The matrix A is the same as that in the preceding example, so the complementary function $\mathbf{x}_c$ is unchanged:

$$\mathbf{x}_c(t) = c_1\mathbf{v}^{(1)}e^{r_1 t} + c_2\mathbf{v}^{(2)}e^{r_2 t} = c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}.$$

We notice that the exponential function e^t in the nonhomogeneous terms also occurs in $\mathbf{x}_c$, and that the eigenvalue $r = 1$ has algebraic multiplicity 1. In the case of a single DE, for a characteristic root $r = 1$ of multiplicity 1 and a similar nonhomogeneous term, it would suffice to try a PS of the form ate^t, $a = \text{const}$. For a system, however, the corresponding form $\mathbf{a}te^t$, where $\mathbf{a}$ is a constant vector, does not work. Instead, here we need to try

$$\mathbf{x}_p(t) = (\mathbf{a}t + \mathbf{b})e^t,$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors of components a_1, a_2 and b_1, b_2, respectively. Substituting in (7.38) and denoting the column vector of components -5, -8 by $\mathbf{q}$, we have

$$\mathbf{a}e^t + (\mathbf{a}t + \mathbf{b})e^t = A(\mathbf{a}t + \mathbf{b})e^t + \mathbf{q}e^t.$$

The functions te^t and e^t are linearly independent (see Sect. 4.4.2), so from Theorem 4.16 it follows that we can match their coefficients on both sides and arrive at the algebraic system

$$\begin{aligned}\mathbf{a} &= A\mathbf{a},\\ \mathbf{a}+\mathbf{b} &= A\mathbf{b}+\mathbf{q},\end{aligned}$$

or, what is the same,

$$\begin{aligned}(A-I)\mathbf{a} &= \mathbf{0},\\ (A-I)\mathbf{b} &= \mathbf{a}-\mathbf{q}.\end{aligned}$$

The first equality shows that $\mathbf{a}$ is an eigenvector of A associated with the eigenvalue $r=1$, so its components are $a_1=-p$ and $a_2=2p$ for some number p. Then the second equality yields the pair of equations

$$\begin{aligned}\tfrac{4}{3}\,b_1+\tfrac{2}{3}\,b_2 &= a_1+5=-p+5,\\ \tfrac{4}{3}\,b_1+\tfrac{2}{3}\,b_2 &= a_2+8=2p+8,\end{aligned}$$

which are consistent if and only if $-p+5=2p+8$; that is, $p=-1$. In this case, $a_1=1$ and $a_2=-2$, and the above system reduces to the single equation $2b_1+b_2=9$. If we choose, say, $b_1=4$ and $b_2=1$, then the GS of the given nonhomogeneous system can be written as

$$\mathbf{x}(t)=\mathbf{x}_c(t)+\mathbf{x}_p(t)=c_1\begin{pmatrix}-1\\ 2\end{pmatrix}e^t+c_2\begin{pmatrix}1\\ 1\end{pmatrix}e^{3t}+\left[\begin{pmatrix}1\\ -2\end{pmatrix}t+\begin{pmatrix}4\\ 1\end{pmatrix}\right]e^t.$$

Applying the ICs, we find that $c_1=2$ and $c_2=-1$, so the solution of the IVP is

$$\mathbf{x}(t)=2\begin{pmatrix}-1\\ 2\end{pmatrix}e^t-\begin{pmatrix}1\\ 1\end{pmatrix}e^{3t}+\left[\begin{pmatrix}1\\ -2\end{pmatrix}t+\begin{pmatrix}4\\ 1\end{pmatrix}\right]e^t=\begin{pmatrix}(t+2)e^t-e^{3t}\\ (5-2t)e^t-e^{3t}\end{pmatrix}.$$

Verification with Mathematica®. The input

```
{x1,x2}={(t + 2)*E^t - E^(3*t),(5 - 2*t)*E^t - E^(3*t)};
{{D[x1,t] - (7/3)*x1 - (2/3)*x2 + 5*E^t,D[x2,t] - (4/3)*x1
      - (5/3)*x2 + 8*E^t},{x1,x2}/.t→ 0}
```

generates the output $\{\{0, 0\}, \{1, 4\}\}$. ■

7.68 Remark. Just as in the case of single DEs, if the nonhomogeneous term $\mathbf{f}$ is a sum of several terms, then, by the principle of superposition (formulated for nonhomogeneous linear systems), $\mathbf{x}_p$ is the sum of the individual PSs computed for each of those terms. ■

7.69 Example. Suppose that the eigenvalues of the 2×2 coefficient matrix A are $r_1=1$ and $r_2=-1$, and that the nonhomogeneous term is

$$\mathbf{f}(t)=\begin{pmatrix}2t-3-4e^t\\ 1+2e^{2t}-5e^{-t}\end{pmatrix}.$$

This term is a linear combination of a first degree polynomial and the exponential functions e^t, e^{2t}, and e^{-t}, with constant vector coefficients. Since 0 and 2 are not eigenvalues of A and both 1 and -1 are eigenvalues of algebraic multiplicity 1, we seek a PS of the form

$$\mathbf{x}_p(t)=\mathbf{a}t+\mathbf{b}+(\mathbf{c}t+\mathbf{d})e^t+\mathbf{e}e^{2t}+(\mathbf{g}t+\mathbf{h})e^{-t},$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{g}$, and $\mathbf{h}$ are constant vectors. ■

Variation of parameters. Consider the homogeneous system $\mathbf{x}' = A\mathbf{x}$ in an open interval J, and let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be two solutions of this system such that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent for all t in J. According to (7.27), the complementary function for the nonhomogeneous system (7.38) is then written in the form

$$\mathbf{x}_c(t) = X(t)\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector and $X = (\mathbf{x}^{(1)} \;\; \mathbf{x}^{(2)})$. By analogy with the case of a single DE, we seek a PS of (7.38) of the form

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t), \tag{7.43}$$

where $\mathbf{u}$ is a vector function to be determined. Replacing in the system, we find that

$$X'\mathbf{u} + X\mathbf{u}' = A(X\mathbf{u}) + \mathbf{f} = (AX)\mathbf{u} + \mathbf{f}.$$

As noted in Remark 7.58(i), we have $X' = AX$, so the above equality reduces to

$$X\mathbf{u}' = \mathbf{f}. \tag{7.44}$$

The linear independence of the columns of X also implies that $\det X(t) \neq 0$ for all admissible t. Consequently, $X(t)$ is invertible, and we can multiply both sides of (7.44) on the left by $X^{-1}(t)$ to obtain

$$\mathbf{u}'(t) = X^{-1}(t)\mathbf{f}(t). \tag{7.45}$$

Hence, a PS of the nonhomogeneous system is given by (7.43) with $\mathbf{u}$ computed by integration from (7.45).

7.70 Example. The IVP

$$\begin{aligned} x_1' &= \tfrac{7}{3}\,x_1 + \tfrac{2}{3}\,x_2 - 5e^t, \qquad x_1(0) = 1,\\ x_2' &= \tfrac{4}{3}\,x_1 + \tfrac{5}{3}\,x_2 - 8e^t, \qquad x_2(0) = 4 \end{aligned}$$

was discussed in Example 7.67. From the eigenvalues and eigenvectors used there, we can construct the fundamental matrix

$$X(t) = \{\mathbf{v}^{(1)}e^t \;\; \mathbf{v}^{(2)}e^{3t}\} = \begin{pmatrix} -e^t & e^{3t} \\ 2e^t & e^{3t} \end{pmatrix}.$$

Then

$$X^{-1}(t) = \frac{1}{3}\begin{pmatrix} -e^{-t} & e^{-t} \\ 2e^{-3t} & e^{-3t} \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} -5e^t \\ -8e^t \end{pmatrix},$$

so, by (7.45),

$$\mathbf{u}'(t) = \frac{1}{3}\begin{pmatrix} -e^{-t} & e^{-t} \\ 2e^{-3t} & e^{-3t} \end{pmatrix}\begin{pmatrix} -5e^t \\ -8e^t \end{pmatrix} = \begin{pmatrix} -1 \\ -6e^{-2t} \end{pmatrix}.$$

Integrating (and setting the integration constant equal to zero since we need only one PS), we find that

$$\mathbf{u}(t) = \begin{pmatrix} -t \\ 3e^{-2t} \end{pmatrix};$$

hence, by (7.43),

$$\mathbf{x}_p(t) = \begin{pmatrix} -e^t & e^{3t} \\ 2e^t & e^{3t} \end{pmatrix} \begin{pmatrix} -t \\ 3e^{-2t} \end{pmatrix} = \begin{pmatrix} t+3 \\ 3-2t \end{pmatrix} e^t.$$

Consequently, the GS of the nonhomogeneous system (with the complementary function $\mathbf{x}_c$ as mentioned in Example 7.67) is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} t+3 \\ 3-2t \end{pmatrix} e^t.$$

The constants c_1 and c_2, determined from the ICs, are $c_1 = 1$ and $c_2 = -1$, which means that the solution of the given IVP is the same one we computed earlier, namely

$$\mathbf{x}(t) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t - \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} t+3 \\ 3-2t \end{pmatrix} e^t = \begin{pmatrix} (t+2)e^t - e^{3t} \\ (5-2t)e^t - e^{3t} \end{pmatrix}. \quad \blacksquare$$

Exercises

In 1–16, compute the solution of the IVP $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ with the given matrix A, vector function $\mathbf{f}$, and vector $\mathbf{x}_0$, by each of (a) the method of undetermined coefficients, (b) diagonalization, and (c) variation of parameters.

1 $A = \begin{pmatrix} -7 & -8 \\ 4 & 5 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 8t+9 \\ -5t-5 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

2 $A = \begin{pmatrix} -1 & -2 \\ 6 & 6 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} t-4 \\ 18-6t \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$

3 $A = \begin{pmatrix} 9 & -10 \\ 5 & -6 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 9+3e^{2t} \\ 5+3e^{2t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$

4 $A = \begin{pmatrix} -7/2 & -3 \\ 9/2 & 4 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 3+3e^{-t} \\ -4-5e^{-t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$

5 $A = \begin{pmatrix} -1/3 & -5/3 \\ -10/3 & 4/3 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -2 \\ -5 \end{pmatrix}.$

6 $A = \begin{pmatrix} -5 & -8 \\ 2 & 5 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 5+12e^{-3t} \\ -2-9e^{-3t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -7 \\ 3 \end{pmatrix}.$

7 $A = \begin{pmatrix} 4 & 0 & -6 \\ -2 & 0 & 4 \\ 3 & 0 & -5 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} -6-5e^{-t} \\ 4+2e^{-t} \\ -5-3e^{-t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}.$

8 $A = \begin{pmatrix} 1 & -2 & -1 \\ 4 & -5 & -4 \\ -6 & 6 & 6 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} t-1 \\ t-4 \\ 6 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$

9 $A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} -2 \\ 2t+5 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$

10 $A = \begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 1-6t \\ -7t \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$

11 $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} -12e^{-t} \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$

12 $A = \begin{pmatrix} 11/8 & -5/8 \\ -15/8 & 1/8 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 5/8 + e^t \\ -1/8 - e^t \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$.

13 $A = \begin{pmatrix} 9/10 & -1/5 \\ -1/5 & 3/5 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} (5/2)e^t \\ 0 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

14 $A = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 7e^{-2t} \\ -10e^{-2t} \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$.

15 $A = \begin{pmatrix} -2/3 & 2/3 & 2/3 \\ -1/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 0 \\ 5e^{2t} \\ 4e^{2t} \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

16 $A = \begin{pmatrix} 6 & 8 & -8 \\ -3 & -5 & 6 \\ 1 & 1 & 0 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 2t - 1 \\ -2t \\ -1 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$.

In 17–26, compute the solution of the IVP $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ with the given matrix A, vector function $\mathbf{f}$, and vector $\mathbf{x}_0$, by each of (a) the method of undetermined coefficients and (b) variation of parameters.

17 $A = \begin{pmatrix} -1 & -1/2 \\ 5/2 & 1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 1 \\ -2 - 5e^t \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$.

18 $A = \begin{pmatrix} 1 & 1/3 \\ -3 & 1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} -10e^{-2t} \\ 0 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 12 \end{pmatrix}$.

19 $A = \begin{pmatrix} -5 & 1 \\ -9 & 1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 4 \\ 8 + 9e^t \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$.

20 $A = \begin{pmatrix} -3/2 & 4 \\ -1 & 5/2 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} -5 - 2e^{-t} \\ -3 + 5e^{-t} \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$.

21 $A = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} t + 1 \\ t + 2 \\ 0 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$.

22 $A = \begin{pmatrix} 0 & 3 & -1 \\ 1 & -2 & 1 \\ -2 & -4 & -1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} -3 \\ 2 - 2e^{-t} \\ 4 + 2e^{-t} \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

23 $A = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

24 $A = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} -1 \\ t + 1 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

25 $A = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

26 $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 1 \\ 3t - 3 \end{pmatrix}$, $\mathbf{x}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Answers to Odd-Numbered Exercises

1 $\mathbf{x}(t) = \begin{pmatrix} e^t - 1 \\ -e^t + t + 2 \end{pmatrix}$. **3** $\mathbf{x}(t) = \begin{pmatrix} 2e^{-t} - 2e^{4t} - 1 + e^{2t} \\ 2e^{-t} - e^{4t} + e^{2t} \end{pmatrix}$.

5 $\mathbf{x}(t) = \begin{pmatrix} -3e^{-2t} + (t+1)e^{3t} \\ -3e^{-2t} - 2(t+1)e^{3t} \end{pmatrix}$. **7** $\mathbf{x}(t) = \begin{pmatrix} -2e^{-2t} + e^{-t} + 4e^{t} \\ 2e^{-2t} \\ -2e^{-2t} - 1 + 2e^{t} \end{pmatrix}$.

9 $\mathbf{x}(t) = \begin{pmatrix} e^{-t} + t + 2 \\ 2e^{-t} + t - 1 \end{pmatrix}$. **11** $\mathbf{x}(t) = \begin{pmatrix} 6e^{3t} + e^{2t} + 2e^{-t} \\ 3e^{3t} + e^{2t} - e^{-t} \end{pmatrix}$.

13 $\mathbf{x}(t) = \begin{pmatrix} (2t-1)e^{t} - e^{t/2} \\ (3-t)e^{t} - 2e^{t/2} \end{pmatrix}$. **15** $\mathbf{x}(t) = \begin{pmatrix} -2e^{-t} + e^{2t} \\ 2e^{t} + 2e^{2t} \\ e^{-t} - 2e^{t} + 2e^{2t} \end{pmatrix}$.

17 $\mathbf{x}(t) = \begin{pmatrix} 2\sin(t/2) + 2e^{t} \\ -2\cos(t/2) - 4\sin(t/2) + 2 - 8e^{t} \end{pmatrix}$.

19 $\mathbf{x}(t) = \begin{pmatrix} (2t-1)e^{-2t} + e^{t} + 1 \\ (6t-1)e^{-2t} + 6e^{t} + 1 \end{pmatrix}$. **21** $\mathbf{x}(t) = \begin{pmatrix} -\sin t + 4 + 2t \\ -\sin t + 2 + t \\ -\cos t + 2 \end{pmatrix}$.

23 $\mathbf{x}(t) = \begin{pmatrix} 4\cos(2t) - 2\sin(2t) + 3 \\ 2\cos(2t) + 2 \end{pmatrix}$. **25** $\mathbf{x}(t) = \begin{pmatrix} (4t+2)e^{-t} + 1 \\ -(2t+2)e^{-t} - 2 \end{pmatrix}$.

Chapter 8
The Laplace Transformation

The purpose of an analytic transformation is to change a more complicated problem into a simpler one. The Laplace transformation, which is applied chiefly with respect to the time variable, maps an IVP onto an algebraic equation or system. Once the latter is solved, its solution is fed into the inverse transformation to yield the solution of the original IVP.

8.1 Definition and Basic Properties

Let f be a function of one independent variable t on the interval $0 \le t < \infty$.

8.1 Definition. The function

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt, \tag{8.1}$$

if it exists, is called the *Laplace transform* of f. The variable s of F is called the *transformation parameter*. The operation $\mathcal{L}$ that produces F from the given function f is called the *Laplace transformation*. ■

8.2 Remarks. (i) The right-hand side in (8.1) must be understood as an improper integral; that is,

$$F(s) = \lim_{p\to\infty} \int_0^p e^{-st} f(t)\,dt. \tag{8.2}$$

(ii) It is clear from the above definition that we must distinguish between the terms 'transformation' and 'transform': the former is a mathematical operator, the latter is a function.

(iii) In general, the transformation parameter s is a complex number, but for the discussion in this chapter it suffices to consider it real. ■

We now indicate a class of functions for which the improper integral (8.1) exists.

8.3 Definition. A function f is said to be *piecewise continuous* on $0 \le t \le p$ if it is continuous everywhere on this interval except at a finite number of points, where it may

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_8,
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have finite discontinuities. In other words, if $t = t_0$ is a point of discontinuity for f, then the one-sided limits $f(t_0-)$ and $f(t_0+)$ exist and either $f(t_0-) = f(t_0+) \neq f(t_0)$ or $f(t_0-) \neq f(t_0+)$. ■

8.4 Example. The function defined by

$$f(t) = \begin{cases} t-5, & 0 \le t < 1, \\ 4-2t, & 1 \le t \le 3 \end{cases}$$

is continuous on the intervals $0 \le t < 1$ and $1 \le t \le 3$. At the point of discontinuity $t = 1$, its one-sided limits are $f(1-) = -4$ and $f(1+) = 2$, so f is piecewise continuous on $0 \le t \le 3$ (see Fig. 8.1). ■

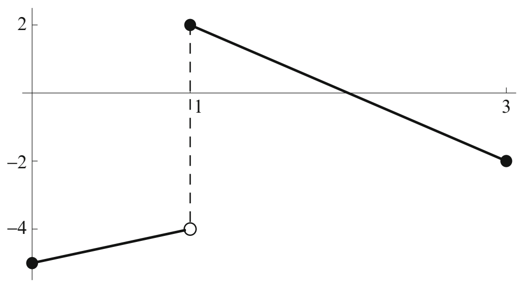

Fig. 8.1

8.5 Example. The function defined by

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 1, \\ 5-4t, & 1 \le t \le 2, \\ \dfrac{1}{t-2} - 3, & 2 < t \le 3 \end{cases}$$

is not piecewise continuous on $0 \le t \le 3$. Although it is continuous on each of the intervals $0 \le t < 1$, $1 \le t \le 2$, and $2 < t \le 3$, its one-sided limit $f(2+)$ does not exist (see Fig. 8.2). ■

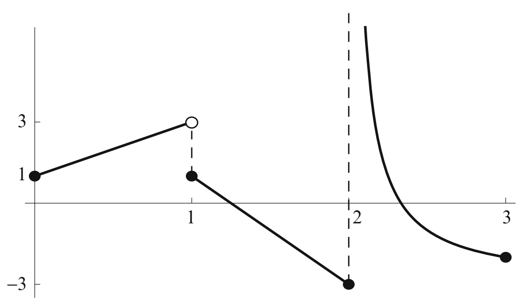

Fig. 8.2

8.6 Theorem. *If f is piecewise continuous on $0 \le t \le p$ for any $p > 0$ and there are constants $M \ge 0$, $k \ge 0$, and γ such that*

$$|f(t)| \le Me^{\gamma t} \text{ for all } t > k, \tag{8.3}$$

then the Laplace transform F of f exists for all $s > \gamma$. Furthermore, $F(s) \to 0$ as $s \to \infty$. ■

8.7 Remarks. (i) Since all the functions occurring in the rest of this chapter satisfy the conditions of Theorem 8.6, they will not be verified explicitly. Also, to keep the presentation simple, the constants M, k, and γ in (8.3) will be mentioned (if required) only in mathematical arguments of a general nature but not in specific cases.

(ii) The inverse Laplace transformation is the operator $\mathcal{L}^{-1}$ defined by

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{\beta\to\infty} \int\limits_{\alpha-i\beta}^{\alpha+i\beta} e^{st}F(s)\,ds,$$

where α and β are real numbers. This formula involves integration in the complex domain, which is beyond the scope of this book, so we do not use it. Instead, we find the original function f corresponding to a transform F by means of the table in Appendix C.

(iii) For practical purposes, the Laplace transformation is *bijective*; in other words, each admissible function has a unique transform, and each transform has a unique inverse transform (or *original*).

(iv) As is easily verified, both $\mathcal{L}$ and $\mathcal{L}^{-1}$ are linear operators; that is, for any functions f_1 and f_2 whose transforms F_1 and F_2 exist for $s > \gamma$, and any constants c_1 and c_2,

$$\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\},$$
$$\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s)\} = c_1\mathcal{L}^{-1}\{F_1(s)\} + c_2\mathcal{L}^{-1}\{F_2(s)\}. \ ■$$

8.8 Example. By (8.2), the Laplace transform of the constant function $f(t) = 1$ is

$$F(s) = \mathcal{L}\{1\} = \lim_{p\to\infty} \int\limits_0^p e^{-st}\,dt = \lim_{p\to\infty} \left[-\frac{1}{s}\,e^{-st} \right]_0^p = \lim_{p\to\infty} \left(\frac{1}{s} - \frac{1}{s}\,e^{-ps} \right) = \frac{1}{s},$$

which exists for all $s > 0$. This implies that we can also write $\mathcal{L}^{-1}\{1/s\} = 1$.

VERIFICATION WITH MATHEMATICA®. The input

```
F = 1/s;
InverseLaplaceTransform[F,s,t]
```

generates the output 1. ■

8.9 Example. The transform of the function $f(t) = e^{at}$, $a = \text{const}$, is

$$F(s) = \mathcal{L}\{e^{at}\} = \lim_{p\to\infty}\int_0^p e^{-st}e^{at}\,dt = \lim_{p\to\infty}\int_0^p e^{-(s-a)t}\,dt$$
$$= \lim_{p\to\infty}\left[-\frac{1}{s-a}\,e^{-(s-a)t}\right]_0^p = \lim_{p\to\infty}\left[-\frac{1}{s-a}\left(e^{-(s-a)p}-1\right)\right].$$

The limit on the right-hand side exists if and only if $s > a$ (which makes the coefficient of p negative). In this case, $e^{-(s-a)p} \to 0$ as $p \to \infty$, so

$$F(s) = \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a. \tag{8.4}$$

Clearly, we can also say that, the other way around, $\mathcal{L}^{-1}\{1/(s-a)\} = e^{at}$.
Verification with Mathematica®. The input

```
F=1/(s - a);
InverseLaplaceTransform[F,s,t]
```

generates the output e^{at}. ■

8.10 Example. By the linearity of the Laplace transformation and the result in Example 8.9,

$$\mathcal{L}\{\sinh(at)\} = \mathcal{L}\{\tfrac{1}{2}(e^{at}-e^{-at})\} = \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2-a^2}$$

for $s > |a|$. This also means that $\mathcal{L}^{-1}\{a/(s^2-a^2)\} = \sinh(at)$.
Verification with Mathematica®. The input

```
F=a/(s^2 - a^2);
InverseLaplaceTransform[F,s,t]//ExpToTrig//Simplify
```

generates the output $\sinh(at)$. ■

8.11 Remark. If a in formula (8.4) is a complex number—that is, $a = a_1 + ia_2$—then, by Euler's formula,

$$e^{-(s-a)p} = e^{-(s-a_1)p+ia_2p} = e^{-(s-a_1)p}e^{ia_2p}$$
$$= e^{-(s-a_1)p}[\cos(a_2p) + i\sin(a_2p)].$$

Since the factor multiplying the exponential is bounded, the left-hand side above tends to zero as $p \to \infty$ when $s > a_1$. Consequently, formula (8.4) remains valid if a is a complex number, and the transform of e^{at} in this case exists for $s > \mathrm{Re}(a)$. ■

8.12 Example. From Euler's formula applied to e^{iat} and e^{-iat} we readily find that $\cos(at) = (e^{iat} + e^{-iat})/2$. The real parts of the coefficients of t in both terms are zero, so, by (8.4) and Remark 8.11,

$$\mathcal{L}\{\cos(at)\} = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{s}{s^2+a^2}, \quad s > 0.$$

At the same time, we can write $\mathcal{L}^{-1}\{s/(s^2+a^2)\} = \cos(at)$.
Verification with Mathematica®. The input

```
F=s/(s^2 + a^2);
InverseLaplaceTransform[F,s,t]
```

generates the output $\cos(at)$. ■

8.13 Example. Using formulas 6–8 in Appendix C and the linearity of the Laplace transformation, we have

$$\begin{aligned}\mathcal{L}\{2e^{-3t} - 5t + \sin(4t)\} &= 2\mathcal{L}\{e^{-3t}\} - 5\mathcal{L}\{t\} + \mathcal{L}\{\sin(4t)\}\\ &= \frac{2}{s+3} - \frac{5}{s^2} + \frac{4}{s^2+16}, \quad s > 0.\end{aligned}$$

VERIFICATION WITH MATHEMATICA®. The input

```
F = 2/(s + 3) - 5/s^2 + 4/(s^2 + 16);
InverseLaplaceTransform[F,s,t]
```

generates the output $2e^{-3t} - 5t + \sin(4t)$. ■

8.14 Example. If $F(s) = 4/s^3$, then from formula 6 in Appendix C it follows that

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = 2t^2.$$

VERIFICATION WITH MATHEMATICA®. The input

```
f = 2*t^2;
LaplaceTransform[f,t,s]
```

generates the output $4/s^3$. ■

8.15 Example. The computation of the inverse transform of $F(s) = (3s+5)/(s^2+4)$ requires some simple algebraic manipulation. Thus,

$$\begin{aligned}f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{3s+5}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3s}{s^2+4} + \frac{5}{s^2+4}\right\}\\ &= 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{5}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\},\end{aligned}$$

from which, by formulas 8 and 9 in Appendix C,

$$f(t) = 3\cos(2t) + \tfrac{5}{2}\sin(2t).$$

VERIFICATION WITH MATHEMATICA®. The input

```
f = 3*Cos[2*t] + (5/2)*Sin[2*t];
LaplaceTransform[f,t,s] // Simplify
```

generates the output $(3s+5)/(s^2+4)$. ■

8.16 Example. Some preliminary algebraic work is also necessary to find the inverse transform of the function $F(s) = (s+7)/(2+s-s^2)$. Since $2+s-s^2 = (s+1)(2-s)$, we set up the partial fraction decomposition (see Sect. A.1)

$$\frac{s+7}{(s+1)(2-s)} = \frac{A}{s+1} + \frac{B}{2-s} = \frac{2}{s+1} - \frac{3}{s-2}$$

and then use the linearity of $\mathcal{L}^{-1}$ and formula 7 in Appendix C to deduce that

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s+7}{2+s-s^2}\right\} = 2e^{-t} - 3e^{2t}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
f=2*E^(-t) - 3*E^(2*t);
LaplaceTransform[f,t,s]//Simplify
```

generates the output $(s+7)/(2+s-s^2)$. ■

8.17 Remark. In general, $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$. ■

8.18 Definition. Let f and g be functions defined for $t \geq 0$. The function $f * g$ defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau, \quad t > 0 \tag{8.5}$$

is called the *convolution* of f and g. ■

8.19 Remark. It is not difficult to prove that

$$\begin{aligned}
&f * g = g * f;\\
&(f * g) * h = f * (g * h);\\
&f * (g + h) = f * g + f * h;\\
&f * 0 = 0 * f = 0,
\end{aligned}$$

where 0 is the zero function. Since these properties are also satisfied by multiplication of functions, the convolution operation is sometimes referred to as the *product of convolution*. ■

8.20 Theorem. *If the Laplace transforms F_1 and F_2 of f_1 and f_2 exist for $s > a \geq 0$, then*

$$\mathcal{L}\{(f_1 * f_2)(t)\} = \mathcal{L}\{f_1(t)\}\mathcal{L}\{f_2(t)\} = F_1(s)F_2(s). \ ■$$

8.21 Remark. The formula in Theorem 8.20 has the equivalent alternative

$$\mathcal{L}^{-1}\{F_1(s)F_2(s)\} = (f_1 * f_2)(t), \tag{8.6}$$

which is particularly useful in the computation of some inverse transforms. ■

8.22 Example. Let F be the transform defined by

$$F(s) = \frac{1}{s^4 + s^2} = \frac{1}{s^2}\,\frac{1}{s^2+1} = F_1(s)F_2(s).$$

Then

$$\begin{aligned}
f_1(t) &= \mathcal{L}^{-1}\{F_1(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t,\\
f_2(t) &= \mathcal{L}^{-1}\{F_2(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t,
\end{aligned}$$

so, by (8.6) and integration by parts,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)F_2(s)\} = (f_1 * f_2)(t)\\
&= \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau = \int_0^t (t-\tau)\sin\tau\,d\tau = t - \sin t.
\end{aligned}$$

VERIFICATION WITH MATHEMATICA®. The input

```
f = t - Sin[t];
LaplaceTransform[f,t,s] // Simplify
```

generates the output $1/(s^4 + s^2)$. ■

Exercises

In 1–6, use the definition of the Laplace transformation to compute the transform F of the given function f.

1 $f(t) = \begin{cases} 2, & 0 \le t < 2, \\ -1, & t \ge 2. \end{cases}$ **2** $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 3, & t \ge 1. \end{cases}$

3 $f(t) = \begin{cases} t, & 0 \le t < 1, \\ -1, & t \ge 1. \end{cases}$ **4** $f(t) = \begin{cases} 2, & 0 \le t < 3, \\ 1-t, & t \ge 3. \end{cases}$

5 $f(t) = \begin{cases} 0, & 0 \le t < 1, \\ e^{2t}, & t \ge 1. \end{cases}$ **6** $f(t) = \begin{cases} e^{-t}, & 0 \le t < 2, \\ 1, & t \ge 2. \end{cases}$

In 7–10, use the formulas in Appendix C to compute the transform F of the given function f.

7 $f(t) = 3t - 2 + \sin(4t)$. **8** $f(t) = 1 - 2t^3 + 5e^{-2t}$.

9 $f(t) = 3e^{2t} - 2\cos(3t)$. **10** $f(t) = 2\sin(\pi t) + 3t^2 - 4e^{t/2}$.

In 11–24, use partial fraction decomposition and the table in Appendix C to compute the inverse transform f of the given Laplace transform F.

11 $F(s) = \dfrac{2}{s^2 - 2s}$. **12** $F(s) = \dfrac{3s+2}{s^2+s}$. **13** $F(s) = \dfrac{7-s}{s^2+s-2}$.

14 $F(s) = \dfrac{6s+3}{2s^2+s-1}$. **15** $F(s) = \dfrac{4s^2-4s-2}{s^3-s}$.

16 $F(s) = \dfrac{2s^2+3s-3}{s^3-2s^2-3s}$. **17** $F(s) = \dfrac{2s^2-s-2}{s^3+s^2}$.

18 $F(s) = \dfrac{3s^2-5s-2}{2s^2-s^3}$. **19** $F(s) = \dfrac{s^2+2}{s^3+s}$. **20** $F(s) = \dfrac{s^2+4s+4}{s^3+4s}$.

21 $F(s) = \dfrac{5s^2-s+2}{s^3+s}$. **22** $F(s) = \dfrac{4s+1}{4s^3+s}$.

23 $F(s) = \dfrac{1+2s-s^2}{s^3-s^2+s-1}$. **24** $F(s) = \dfrac{2s^2+2s+10}{s^3+s^2+4s+4}$.

In 25–30, use the definition of convolution and the table in Appendix C to compute the inverse transform f of the given Laplace transform F.

25 $F(s) = \dfrac{4}{s(s+4)}$. **26** $F(s) = \dfrac{3}{(s-1)(s+2)}$.

27 $F(s) = \dfrac{5}{(s-2)(s+3)}$. **28** $F(s) = \dfrac{1}{s^2(s+1)}$.

29 $F(s) = \dfrac{4}{s(s^2+4)}$. **30** $F(s) = \dfrac{8}{s^2(s^2-4)}$.

Answers to Odd-Numbered Exercises

1 $F(s) = \dfrac{2-3e^{-2s}}{s}$. **3** $F(s) = \dfrac{1-e^{-s}}{s^2} - \dfrac{2e^{-s}}{s}$. **5** $F(s) = \dfrac{e^{6-3s}}{s-2}$.

7 $F(s) = \dfrac{3}{s^2} - \dfrac{2}{s} + \dfrac{4}{s^2+16}$. **9** $F(s) = \dfrac{3}{s-2} - \dfrac{2s}{s^2+9}$.

11 $f(t) = e^{2t} - 1$. **13** $f(t) = 2e^t - 3e^{-2t}$. **15** $f(t) = 2 - e^t + 3e^{-t}$.

17 $f(t) = 1 - 2t + e^{-t}$. **19** $f(t) = 2 - \cos t$. **21** $f(t) = 2 + 3\cos t - \sin t$.

23 $f(t) = e^t - 2\cos t$. **25** $f(t) = 1 - e^{-4t}$. **27** $f(t) = e^{2t} - e^{-3t}$.

29 $f(t) = 1 - \cos(2t)$.

8.2 Further Properties

Below, we consider the Laplace transforms of some types of functions that occur frequently in the solution of IVPs arising from certain mathematical models.

8.23 Definition. The function H defined for $t \ge 0$ by

$$H(t-a) = \begin{cases} 0, & 0 \le t < a, \\ 1, & t \ge a, \end{cases}$$

where $a = \text{const} \ge 0$, is called the *Heaviside function* or (because of the look of its graph—see Fig. 8.3) the *unit step function*. ■

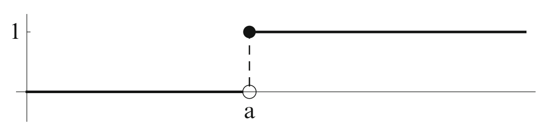

Fig. 8.3

8.24 Remark. The Heaviside function can be defined on any interval, including the entire real line. We restricted its definition to $t \ge 0$ because in this chapter we are concerned exclusively with IVPs where t is nonnegative. ■

8.25 Example. The Heaviside function can be used to write piecewise continuous functions in a more compact form. If f is defined by

$$f(t) = \begin{cases} f_1(t), & 0 \le t < a, \\ f_2(t), & t \ge a, \end{cases}$$

then, in view of Definition 8.23,

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]H(t-a).$$

Thus, for the function in Example 8.4 we have $a = 1$, $f_1(t) = t-5$, and $f_2(t) = 4-2t$, so

$$f(t) = t - 5 + (9 - 3t)H(t - 1), \quad 0 \le t \le 3. \ \blacksquare$$

8.26 Example. The other way around, a function defined with the help of the Heaviside function can easily be written explicitly in terms of its segments:

$$1 - H(t-2) = \begin{cases} 1, & 0 \le t < 2, \\ 0, & t \ge 2, \end{cases} \qquad H(t-1) - H(t-2) = \begin{cases} 0, & 0 \le t < 1, \\ 1, & 1 \le t < 2, \\ 0, & t \ge 2. \ \blacksquare \end{cases}$$

8.27 Theorem. *If f is a function that has a Laplace transform F defined for $s > a \ge 0$, then*

$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s), \quad s > a. \ \blacksquare \tag{8.7}$$

Proof. This is computed directly from the definition:

$$\mathcal{L}\{f(t-a)H((t-a)\} = \int_0^\infty e^{-st} f(t-a)H(t-a)\, dt = \int_a^\infty e^{-st} f(t-a)\, dt.$$

With the substitution $\tau = t - a$, the right-hand side now becomes

$$\int_0^\infty e^{-s(\tau+a)} f(\tau)\, d\tau = e^{-as} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau = e^{-as}F(s). \ \blacksquare$$

8.28 Example. For the function $(t-2)^3 H(t-2)$ we have $f(t) = t^3$ and $a = 2$. By formula 6 in Appendix C, $F(s) = \mathcal{L}\{f(t)\} = 6/s^4$, so, replacing in (8.7), we find that

$$\mathcal{L}\{(t-2)^3 H(t-2)\} = \frac{6}{s^4}\, e^{-2s}.$$

Verification with Mathematica®. The input

```
F = (6/s^4) * E^(-2 * s);
InverseLaplaceTransform[F,s,t] // Simplify
```

generates the output $(t-2)^3 H(t-2)$. $\blacksquare$

8.29 Example. To compute the inverse transform of $e^{-3s}/(s^2+4)$, we compare this expression to the right-hand side in (8.7) and see that $a = 3$ and $F(s) = 1/(s^2+4)$. Therefore, by formula 8 in Appendix C,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \tfrac{1}{2}\sin(2t),$$

so, according to (8.7), the desired inverse transform is

$$f(t-3)H(t-3) = \tfrac{1}{2}\sin(2(t-3))H(t-3) = \begin{cases} 0, & 0 \le t < 3, \\ \tfrac{1}{2}\sin(2(t-3)), & t \ge 3. \end{cases}$$

Verification with Mathematica®. The input

```
f = (1/2) * Sin[2 * (t - 3)] * HeavisideTheta[t - 3];
LaplaceTransform[f,t,s] // Simplify
```

generates the output $e^{-3s}/(s^2+4)$. ■

8.30 Theorem. *If f has a Laplace transform F defined for $s > \gamma \geq 0$ and $a =$ const, then*

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a), \quad s > \gamma + a. \; ■ \tag{8.8}$$

Proof. From the definition of the Laplace transformation,

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt,$$

and we notice that the right-hand side is the transform F evaluated at $s-a$; that is, $F(s-a)$, defined for $s-a > \gamma$ or, what is the same, $s > \gamma + a$. ■

8.31 Example. The Laplace transform of the function $e^{2t}\cos(3t)$ is computed by means of (8.8) with $a = 2$ and $f(t) = \cos(3t)$. By formula 9 in Appendix C, $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\cos(3t)\} = s/(s^2+9)$, so

$$\mathcal{L}\{e^{2t}\cos(3t)\} = \frac{s-2}{(s-2)^2+9} = \frac{s-2}{s^2-4s+13}.$$

Verification with Mathematica®. The input

```
F = (s - 2) / (s^2 - 4 * s + 13);
InverseLaplaceTransform[F,s,t] // FullSimplify
```

generates the output $e^{2t}\cos(3t)$. ■

8.32 Example. To find the inverse transform of $(2s-1)/(s^2+6s+13)$, we complete the square in the denominator and write

$$\begin{aligned}\frac{2s-1}{s^2+6s+13} &= \frac{2s-1}{(s+3)^2+4} = \frac{2(s+3)-7}{(s+3)^2+4}\\ &= 2\frac{s+3}{(s+3)^2+2^2} - \frac{7}{2}\,\frac{2}{(s+3)^2+2^2}.\end{aligned}$$

The fractions on the right-hand side are of the form $F_1(s-a)$ and $F_2(s-a)$, where

$$F_1(s) = \frac{s}{s^2+2^2}, \quad F_2(s) = \frac{2}{s^2+2^2}, \quad a = -3.$$

Then, by formulas 7 and 8 in Appendix C,

$$f_1(t) = \mathcal{L}^{-1}\{F_1(s)\} = \cos(2t), \quad f_2(t) = \mathcal{L}^{-1}\{F_2(s)\} = \sin(2t),$$

so, by the linearity of $\mathcal{L}^{-1}$ and (8.8) with $a = -3$,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+6s+13}\right\} &= 2e^{-3t} f_1(t) - \tfrac{7}{2}\, e^{-3t} f_2(t)\\ &= 2e^{-3t}\cos(2t) - \tfrac{7}{2}\, e^{-3t}\sin(2t).\end{aligned}$$

Verification with Mathematica®. The input

```
f = 2 * E^(-3 * t) * Cos[2 * t] - (7/2) * E^(-3 * t) * Sin[2 * t];
LaplaceTransform[f,t,s] // Simplify
```

generates the output $(2s - 1)/(s^2 + 6s + 13)$. ∎

8.33 Example. Consider the transform $F(s) = (3s^2 - 8s + 15)/(s^3 - 2s^2 + 5s)$. Since $s^3 - 2s^2 + 5s = s(s^2 - 2s + 5)$ and the second factor is an irreducible quadratic polynomial (see Sect. A.1), partial fraction decomposition yields

$$\frac{3s^2 - 8s + 15}{s(s^2 - 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 5} = \frac{3}{s} - \frac{2}{s^2 - 2s + 5} = \frac{3}{s} - \frac{2}{(s-1)^2 + 2^2},$$

so, by formula 5 in Appendix C and (8.8),

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2 + 2^2}\right\} = 3 - e^t \sin(2t).$$

Verification with Mathematica®. The input

```
f = 3 - E^t * Sin[2 * t];
LaplaceTransform[f,t,s] // Together
```

generates the output $(3s^2 - 8s + 15)/(s^3 - 2s^2 + 5s)$. ∎

8.34 Theorem. *If f, f', ..., $f^{(n-1)}$ are continuous on any interval $0 \le t \le p$ and satisfy (8.3) with the same constants M, k, and γ, and if $f^{(n)}$ is piecewise continuous on $0 \le t \le p$, then the Laplace transform of $f^{(n)}$ exists for $s > \gamma$ and*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0). \ ∎ \tag{8.9}$$

Proof. First we verify (8.9) in the 'base' case $n = 1$, assuming, for simplicity, that f' is continuous on $0 \le t \le p$ for any $p > 0$. (If f' is only piecewise continuous, the proof remains essentially the same, with minor modifications to accommodate the discontinuities of f'.)

Using (8.7) and integration by parts, we see that

$$\mathcal{L}\{f'(t)\} = \lim_{p\to\infty} \int_0^p e^{-st} f'(t)\, dt = \lim_{p\to\infty} \left\{ e^{-st} f(t)\Big|_0^p - \int_0^p -se^{-st} f(t)\, dt \right\}.$$

Since, according to (8.3), $f(p) \le Me^{\gamma p}$ for $p > k$, it follows that if $s > \gamma$, then

$$e^{-sp} f(p) \le Me^{(\gamma - s)p} \to 0 \quad \text{as } p \to \infty.$$

Hence,

$$\mathcal{L}\{f'(t)\} = -f(0) + s\int_0^\infty e^{-st} f(t)\, dt = sF(s) - f(0), \tag{8.10}$$

which is (8.9) for $n = 1$.

Under the appropriate assumptions on f'', we now have

$$\begin{aligned} \mathcal{L}\{f''(t)\} &= \mathcal{L}\{(f'(t))'\} = s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) = s^2 F(s) - sf(0) - f'(0), \end{aligned} \tag{8.11}$$

which is (8.9) for $n = 2$. Continuing in this fashion, after n steps we arrive at the general formula (8.9). ∎

8.35 Remark. Given that (8.9) makes use of the values of f and its derivatives at $t = 0$, all the IVPs that we solve in the rest of this chapter will have their ICs prescribed at $t_0 = 0$. ■

Exercises

In 1–16, use the table in Appendix C to compute the transform F of the given function f.

1 $f(t) = (t-4)H(t-4)$. **2** $f(t) = \sin(2(t-1))H(t-1)$.

3 $f(t) = e^{3-t}H(t-3)$. **4** $f(t) = (t^2 - 2t + 1)H(t-1)$.

5 $f(t) = (2t-1)H(t-2)$. **6** $f(t) = (t^2 - 8t + 10)H(t-3)$.

7 $f(t) = e^{2t}H(t-1)$. **8** $f(t) = \cos(t - 3\pi/2)H(t-\pi)$.

9 $f(t) = (t+2)e^{4t}$. **10** $f(t) = (3t-1)e^{-2t}$.

11 $f(t) = (t^2 - 2t)e^{-t}$. **12** $f(t) = (t^2 - t + 1)e^{t/2}$.

13 $f(t) = [2 - 3\sin(2t)]e^{-t}$. **14** $f(t) = [3t - 2 + 2\cos(3t)]e^{2t}$.

15 $f(t) = [\cos(2t) - 3\sin(2t)]e^{t}$. **16** $f(t) = [2\cos(4t) + \sin(4t)]e^{-3t}$.

In 17–30, use the table in Appendix C to compute the inverse transform f of the given Laplace transform F.

17 $F(s) = \dfrac{1-2s}{s^2}\, e^{-s}$. **18** $F(s) = \dfrac{8s+3}{s^2}\, e^{-2s}$. **19** $F(s) = -\dfrac{\pi}{s^2+\pi^2}\, e^{-s}$.

20 $F(s) = \dfrac{1}{s+2}\, e^{-6-s}$. **21** $F(s) = \dfrac{s+1}{s^2}\, e^{-s} + \dfrac{1}{s-1}\, e^{2-2s}$.

22 $F(s) = \dfrac{5s+2}{s^2}\, e^{-2s} - \dfrac{3}{s+1}\, e^{-1-s}$. **23** $F(s) = -\dfrac{s}{(s+2)^2}$.

24 $F(s) = \dfrac{4s-13}{(s-3)^2}$. **25** $F(s) = \dfrac{s+3}{(s+1)^3}$. **26** $F(s) = \dfrac{2+2s-s^2}{(s-2)^3}$.

27 $F(s) = \dfrac{12s+22}{4s^2+4s+17}$. **28** $F(s) = -\dfrac{4s+2}{4s^2+8s+5}$.

29 $F(s) = \dfrac{1}{(s-2)^2}\, e^{-s}$. **30** $F(s) = \dfrac{2s+3}{(s+1)^2}\, e^{-2-2s}$.

Answers to Odd-Numbered Exercises

1 $F(s) = \dfrac{1}{s^2}\, e^{-4s}$. **3** $F(s) = \dfrac{1}{s+1}\, e^{-3s}$.

5 $F(s) = \dfrac{3s+2}{s^2}\, e^{-2s}$. **7** $F(s) = \dfrac{1}{s-2}\, e^{2-s}$.

9 $F(s) = \dfrac{2s-7}{(s-4)^2}$. **11** $F(s) = -\dfrac{2s}{(s+1)^3}$.

13 $F(s) = \dfrac{2}{s+1} - \dfrac{6}{s^2+2s+5}$. **15** $F(s) = \dfrac{s-7}{s^2-2s+5}$.

17 $f(t) = (t-3)H(t-1)$. **19** $f(t) = -\sin(\pi(t-1))H(t-1)$.

21 $f(t) = tH(t-1) + e^t H(t-2)$. **23** $f(t) = (2t-1)e^{-2t}$.

25 $f(t) = (t^2+t)e^{-t}$. **27** $f(t) = e^{-t/2}[3\cos(2t) + 2\sin(2t)]$.

29 $f(t) = (t-1)e^{2(t-1)}H(t-1)$.

8.3 Solution of IVPs for Single Equations

In view of Theorem 8.34 and Remark 8.7(iii), the Laplace transformation can be used to good advantage in the solution of IVPs for linear DEs with constant coefficients. The strategy consists in applying $\mathcal{L}$ to reduce the differential problem given in the t-domain to an algebraic one in the s-domain, solving the latter, and then applying $\mathcal{L}^{-1}$ to find the solution of the original problem.

8.3.1 Continuous Forcing Terms

Borrowing the terminology used in the mechanical model represented by certain second-order equations, we refer to any nonhomogeneous terms in a DE as ‘forcing’ terms. Homogeneous equations can also be deemed to have continuous forcing terms since their right-hand sides are zero.

8.36 Example. Consider the IVP

$$y'' - 3y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Writing $Y(s) = \mathcal{L}\{y(t)\}$ and applying (8.10) and (8.11), we transform the IVP into the algebraic equation

$$[s^2Y - sy(0) - y'(0)] - 3[sY - y(0)] - 2Y = 0;$$

that is,

$$(s^2 - 3s + 2)Y - 1 = 0,$$

with solution

$$Y(s) = \frac{1}{s^2 - 3s + 2}.$$

To find the solution of the given IVP, we now use partial fraction decomposition and the table in Appendix C; thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{1}{s-1}\right\} = e^{2t} - e^t. \end{aligned}$$

Verification with Mathematica®. The input

```
y = E^(2*t) - E^t;
{D[y,t,t] - 3*D[y,t] + 2*y, {y,D[y,t]}/.t→0}//Simplify
```

generates the output $\{0, \{0, 1\}\}$. ■

8.37 Remark. We notice that the coefficient of Y in the above example is the characteristic polynomial associated with the given DE. This is always the case for an equation with constant coefficients. ■

8.38 Example. With the same notation and applying $\mathcal{L}$ on both sides of the DE, we change the IVP

$$y'' + 3y' = \sin t - 3\cos t, \quad y(0) = 1, \quad y'(0) = 5$$

to

$$[s^2Y - sy(0) - y'(0)] + 3[sY - y(0)] = \mathcal{L}\{\sin t\} - 3\mathcal{L}\{\cos t\},$$

or, with the given ICs,

$$(s^2 + 3s)Y - s - 8 = \frac{1}{s^2+1} - 3\frac{s}{s^2+1}.$$

After simple algebra (involving partial fraction decomposition), this yields

$$\begin{aligned} Y(s) &= \frac{1}{s^2+3s}\left(\frac{1-3s}{s^2+1} + s + 8\right) = \frac{s^3 + 8s^2 - 2s + 9}{s(s+3)(s^2+1)} \\ &= \frac{A}{s} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1} = \frac{3}{s} - \frac{2}{s+3} - \frac{1}{s^2+1}. \end{aligned}$$

Then the solution of the IVP is

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= 3 - 2e^{-3t} - \sin t. \end{aligned}$$

Verification with Mathematica®. The input

```
y=3 - 2*E^(-3*t) - Sin[t];
{D[y,t,t] + 3*D[y,t] - Sin[t] + 3*Cos[t],{y,D[y,t]}/.t→ 0}
        //Simplify
```

generates the output {0, {1, 5}}. ■

8.39 Example. The same technique changes the IVP

$$y'' - 2y' + 5y = -5, \quad y(0) = 0, \quad y'(0) = -3$$

to

$$[s^2Y - sy(0) - y'(0)] - 2[sY - y(0)] + 5Y = -5\mathcal{L}\{1\},$$

or, after the ICs are implemented,

$$(s^2 - 2s + 5)Y + 3 = -\frac{5}{s}.$$

Therefore,

$$Y(s) = \frac{-3s-5}{s(s^2-2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2-2s+5} = -\frac{1}{s} + \frac{s-5}{s^2-2s+5},$$

which, since $s^2 - 2s + 5 = (s-1)^2 + 4$, can be written as

$$Y(s) = -\frac{1}{s} + \frac{(s-1)-4}{(s-1)^2+4} = -\frac{1}{s} + \frac{s-1}{(s-1)^2+2^2} - 2\frac{2}{(s-1)^2+2^2}.$$

By Theorem 8.30 with $a = 1$ and formulas 5, 8, and 9 in Appendix C, we see that the solution of the given IVP is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -1 + e^t[\cos(2t) - 2\sin(2t)].$$

Verification with Mathematica®. The input

```
y= - 1 + E^t * (Cos[2*t] - 2*Sin[2*t]);
{D[y,t,t] - 2*D[y,t] + 5*y + 5, {y,D[y,t]}/. t→ 0}//Simplify
```

generates the output $\{0, \{0, -3\}\}$. ∎

8.40 Example. After application of the Laplace transformation, the IVP

$$y'' - 2y' + y = 2t - 1, \quad y(0) = 1, \quad y'(0) = 5$$

reduces to

$$(s^2 - 2s + 1)Y - s - 3 = \frac{2}{s^2} - \frac{1}{s},$$

which, since $s^2 - 2s + 1 = (s-1)^2$, in final analysis leads to

$$\begin{aligned} Y(s) &= \frac{s^3 + 3s^2 - s + 2}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} \\ &= \frac{3}{s} + \frac{2}{s^2} - \frac{2}{s-1} + \frac{5}{(s-1)^2}. \end{aligned}$$

The formulas in Appendix C now yield

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 3 + 2t - 2e^t + 5te^t = (5t-2)e^t + 2t + 3.$$

Verification with Mathematica®. The input

```
y= (5*t - 2) *E^t + 2*t + 3;
{D[y,t,t] - 2*D[y,t] + y - 2*t + 1, {y,D[y,t]}/. t→  0}//Simplify
```

generates the output $\{0, \{1, 5\}\}$. ∎

8.41 Example. Applying $\mathcal{L}$ to the IVP

$$y''' - 2y'' - y' + 2y = -6, \quad y(0) = -1, \quad y'(0) = 6, \quad y''(0) = 8$$

and sorting out the terms, we arrive at

$$(s^3 - 2s^2 - s + 2)Y + s^2 - 8s + 3 = -\frac{6}{s},$$

from which

$$Y(s) = \frac{-s^3 + 8s^2 - 3s - 6}{s(s^3 - 2s^2 - s + 2)}.$$

We now use synthetic division or the symmetry of the coefficients to obtain

$$s^3 - 2s^2 - s + 2 = (s-1)(s+1)(s-2),$$

so, by partial fraction decomposition,

$$Y(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2} = -\frac{3}{s} + \frac{1}{s-1} - \frac{1}{s+1} + \frac{2}{s-2}.$$

Hence, the solution of the given IVP is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -3 + e^t - e^{-t} + 2e^{2t}.$$

Verification with Mathematica®. The input

```
y= - 3 + E^t - E^(-t) + 2*E^(2*t);
{D[y,t,3] - 2*D[y,t,t] - D[y,t] + 2*y + 6, {y,D[y,t],D[y,t,t]}
      /.t→ 0}//Simplify
```

generates the output $\{0, \{-1, 6, 8\}\}$. ■

8.42 Example. Consider the IVP

$$y'' - 6y' + 8y = f(t), \quad y(0) = 2, \quad y'(0) = 1,$$

where f is an unspecified function with Laplace transform F. Applying $\mathcal{L}$ and performing the necessary algebra, we arrive at

$$(s^2 - 6s + 8)Y = 2s - 11 + F(s).$$

Since $s^2 - 6s + 8 = (s-2)(s-4)$, partial fraction decomposition now yields

$$Y(s) = \frac{7}{2(s-2)} - \frac{3}{2(s-4)} + \left\{\frac{1}{2(s-4)} - \frac{1}{2(s-2)}\right\}F(s).$$

The last term on the right-hand side can be written as $F(s)G(s)$, where

$$G(s) = \frac{1}{2(s-4)} - \frac{1}{2(s-2)},$$

so

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \tfrac{1}{2}\,e^{4t} - \tfrac{1}{2}\,e^{2t}.$$

Then, by (8.6) and (8.5), the inverse transform of that term is

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$= \tfrac{1}{2}\int_0^t f(t-\tau)(e^{4\tau} - e^{2\tau})\,d\tau.$$

Hence, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \tfrac{7}{2}\,e^{2t} - \tfrac{3}{2}\,e^{4t} + \tfrac{1}{2}\int_0^t f(t-\tau)(e^{4\tau} - e^{2\tau})\,d\tau.$$

If, say, $f(t) = 8t + 2$, then, integrating by parts, we have

$$\frac{1}{2}\int_0^t f(t-\tau)(e^{4\tau} - e^{2\tau})\,d\tau = \int_0^t (e^{4\tau} - 2^{2\tau})[1 + 4(t-\tau)]\,d\tau$$
$$= \tfrac{1}{2}\,e^{4t} - \tfrac{3}{2}\,e^{2t} + t + 1,$$

and so,

$$y(t) = \tfrac{7}{2}\,e^{2t} - \tfrac{3}{2}\,e^{4t} + \tfrac{1}{2}\,e^{4t} - \tfrac{3}{2}\,e^{2t} + t + 1 = 2e^{2t} - e^{4t} + t + 1.$$

Verification with Mathematica®. The input

```
y=2*E^(2*t) - E^(4*t) + t + 1;
{D[y,t,t] - 6*D[y,t] + 8*y - 8*t - 2,{y,D[y,t]}/.t→ 0}
        //Simplify
```

generates the output {0, {2, 1}}. ■

Exercises

Use the Laplace transformation method to solve the given IVP.

1 $y'' + 4y' + 3y = -3,\quad y(0) = -2,\quad y'(0) = 5.$

2 $2y'' - 3y' + y = 2,\quad y(0) = 7,\quad y'(0) = 3.$

3 $y'' + 2y' + y = 4,\quad y(0) = 2,\quad y'(0) = 3.$

4 $4y'' - 4y' + y = -3,\quad y(0) = -5,\quad y'(0) = 1.$

5 $y'' + y = 1,\quad y(0) = 4,\quad y'(0) = -1.$

6 $y'' + 4y = -8,\quad y(0) = -1,\quad y'(0) = 4.$

7 $y'' - 5y' + 6y = 6t - 5,\quad y(0) = -3,\quad y'(0) = -7.$

8 $2y'' + y' - 6y = 7 - 6t,\quad y(0) = 0,\quad y'(0) = -8.$

9 $y'' - 6y' + 9y = 24 - 9t,\quad y(0) = 2,\quad y'(0) = 0.$

10 $y'' - 2y' + 2y = 4 - 2t,\quad y(0) = 3,\quad y'(0) = 2.$

11 $y'' - 4y' + 5y = 10t - 13,\quad y(0) = 1,\quad y'(0) = 3.$

12 $y'' + 4y' = -3e^{-t},\quad y(0) = -1,\quad y'(0) = -5.$

13 $y'' + 4y' - 5y = -14e^{2t},\quad y(0) = -3,\quad y'(0) = 7.$

14 $y'' + 8y' + 16y = -75e^{t},\quad y(0) = -1,\quad y'(0) = -11.$

15 $9y'' - 12y' + 4y = 50e^{-t},\quad y(0) = -1,\quad y'(0) = -2.$

16 $4y'' - 4y' + 5y = 13e^{-t},\quad y(0) = 3,\quad y'(0) = -2.$

17 $4y'' + 8y' + 5y = -37e^{2t},\quad y(0) = -5,\quad y'(0) = 3.$

18 $y'' + 2y' - 8y = -2\cos t + 9\sin t,\quad y(0) = 2,\quad y'(0) = 9.$

19 $y'' - 2y' = -12\cos(2t) + 4\sin(2t),\quad y(0) = 4,\quad y'(0) = 0.$

20 $9y'' - 6y' + y = 4(3\cos t + 4\sin t),\quad y(0) = 0,\quad y'(0) = -1.$

21 $y'' + 6y' + 9y = 7\cos(2t) + 17\sin(2t), \quad y(0) = -3, \quad y'(0) = 8.$

22 $y'' - 2y' - 3y = -3t - 8 - 12e^t, \quad y(0) = 7, \quad y'(0) = 2.$

23 $y'' + 3y' - 10y = 30 - 12e^{-t}, \quad y(0) = -2, \quad y'(0) = 6.$

24 $y'' + 4y' + 4y = 2(e^{-t} - 8), \quad y(0) = -3, \quad y'(0) = 1.$

25 $y'' + y' = 3, \quad y(0) = 2, \quad y'(0) = 2.$

26 $2y'' - y' - y = 3e^t, \quad y(0) = 2, \quad y'(0) = -3.$

27 $y''' - 4y' = 8, \quad y(0) = 2, \quad y'(0) = -8, \quad y''(0) = -4.$

28 $y''' + 2y'' - 3y' = -3, \quad y(0) = 3, \quad y'(0) = 6, \quad y''(0) = -7.$

29 $y''' - 2y'' + y' = 2e^{2t}, \quad y(0) = 2, \quad y'(0) = 2, \quad y''(0) = 5.$

30 $y''' - 4y'' + 4y' = -9e^{-t}, \quad y(0) = 4, \quad y'(0) = 2, \quad y''(0) = 5.$

31 $y''' - y'' - 4y' + 4y = 4, \quad y(0) = 0, \quad y'(0) = 5, \quad y''(0) = -7.$

32 $y''' - 3y'' + 4y = 2e^t, \quad y(0) = 2, \quad y'(0) = -2, \quad y''(0) = 3.$

Answers to Odd-Numbered Exercises

1 $y(t) = e^{-t} - 2e^{-3t} - 1.$ **3** $y(t) = (t-2)e^{-t} + 4.$

5 $y(t) = 3\cos t - \sin t + 1.$ **7** $y(t) = t - e^{2t} - 2e^{3t}.$

9 $y(t) = te^{3t} + 2 - t.$ **11** $y(t) = e^{2t}(2\cos t - 3\sin t) + 2t - 1.$

13 $y(t) = e^t - 2e^{-5t} - 2e^{2t}.$ **15** $y(t) = (2t-3)e^{2t/3} + 2e^{-t}.$

17 $y(t) = 2e^{-t}[\sin(t/2) - 2\cos(t/2)] - e^{2t}.$

19 $y(t) = 3 - e^{2t} + 2\cos(2t) + \sin(2t).$

21 $y(t) = -2e^{-3t} - \cos(2t) + \sin(2t).$ **23** $y(t) = e^{2t} - e^{-5t} + e^{-t} - 3.$

25 $y(t) = 3t + 1 + e^{-t}.$ **27** $y(t) = e^{-2t} - 2e^{2t} - 2t + 3.$

29 $y(t) = 2 + (t-1)e^t + e^{2t}.$ **31** $y(t) = e^t - 2e^{-2t} + 1.$

8.3.2 Piecewise Continuous Forcing Terms

The Laplace transformation method is particularly useful when the right-hand side of the DE is a discontinuous function, since in such cases it is very difficult to find a PS of the nonhomogeneous equation by the methods discussed in Sect. 4.5.

8.43 Example. Consider the IVP

$$\begin{aligned} &y'' + y = g(t), \\ &y(0) = 0, \quad y'(0) = 0, \end{aligned} \qquad g(t) = \begin{cases} 1, & 0 < t < 1, \\ 0, & t \geq 1. \end{cases}$$

Noticing that $g(t) = 1 - H(t-1)$ and recalling Theorem 8.27 (with $f(t) = 1$ and, therefore, $F(s) = 1/s$), we apply $\mathcal{L}$ to every term in the equation and arrive at

$$[s^2Y - sy(0) - y'(0)] + Y = (s^2+1)Y = \mathcal{L}\{1 - H(t-1)\} = \frac{1}{s} - \frac{1}{s}\,e^{-s};$$

hence,

$$Y(s) = \frac{1}{s(s^2+1)}\,(1 - e^{-s}).$$

Since, by partial fraction decomposition,

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1},$$

it follows that

$$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)(1 - e^{-s}),$$

so, by (8.7),

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1} - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s}\right\} \\ &= 1 - \cos t - [1 - \cos(t-1)]H(t-1). \end{aligned}$$

Verification with Mathematica®. The input

```
y = 1 - Cos[t] - (1 - Cos[t - 1]) * HeavisideTheta[t - 1];
{D[y,t,t] + y - 1 + HeavisideTheta[t - 1], {y,D[y,t]} /. t → 0}
        // Simplify
```

generates the output $\{0, \{0, 0\}\}$. ■

8.44 Example. Consider the IVP

$$\begin{aligned} &y'' + 5y' + 4y = g(t), \\ &y(0) = 0, \quad y'(0) = 0, \end{aligned} \qquad g(t) = \begin{cases} 12, & 1 \le t < 2, \\ 0, & 0 < t < 1, \ \ t \ge 2. \end{cases}$$

The right-hand side of the DE can be written in the form $g(t) = 12[H(t-1) - H(t-2)]$, so, proceeding as in Example 8.43, we find that

$$(s^2 + 5s + 4)Y = \mathcal{L}\{H(t-1)\} - \mathcal{L}\{H(t-2)\} = \frac{12}{s}\,(e^{-s} - e^{-2s});$$

therefore,

$$Y(s) = \frac{12}{s(s^2+5s+4)}\,(e^{-s} - e^{-2s}).$$

Since $s^2 + 5s + 4 = (s+1)(s+4)$, partial fraction decomposition gives

$$\frac{12}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4} = \frac{3}{s} - \frac{4}{s+1} + \frac{1}{s+4},$$

from which

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{3}{s} - \frac{4}{s+1} + \frac{1}{s+4}\right)(e^{-s} - e^{-2s})\right\},$$

so, by (8.7),

$$\begin{aligned} y(t) &= [3 - 4e^{-(t-1)} + e^{-4(t-1)}]H(t-1) \\ &\qquad - [3 - 4e^{-(t-2)} + e^{-4(t-2)}]H(t-2). \end{aligned}$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = (3 - 4 * E^(-(t - 1)) + E^(-4 * (t - 1))) * HeavisideTheta[t - 1]
        - (3 - 4 * E^(-(t - 2)) + E^(-4 * (t - 2)))
        * HeavisideTheta[t - 2];
{D[y,t,t] + 5 * D[y,t] + 4 * y - 12 * (HeavisideTheta[t - 1]
        - HeavisideTheta[t - 2]), {y,D[y,t]} /. t → 0} // Simplify
```

generates the output $\{0, \{0, 0\}\}$. ■

8.45 Example. The forcing term in the IVP

$$\begin{aligned} &y'' + 4y' + 4y = g(t), \\ &y(0) = 0, \quad y'(0) = 1, \end{aligned} \qquad g(t) = \begin{cases} t, & 1 \le t < 3, \\ 0, & 0 < t < 1, \ \ t \ge 3 \end{cases}$$

can be written as

$$\begin{aligned} g(t) &= t[H(t-1) - H(t-3)] \\ &= (t-1)H(t-1) - (t-3)H(t-3) + H(t-1) - 3H(t-3), \end{aligned}$$

so, by Theorem 8.27 with $f(t) = t$ and then $f(t) = 1$,

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2}\,e^{-s} - \frac{1}{s^2}\,e^{-3s} + \frac{1}{s}\,e^{-s} - \frac{3}{s}\,e^{-3s} = \frac{s+1}{s^2}\,e^{-s} - \frac{3s+1}{s^2}\,e^{-3s}.$$

After application of $\mathcal{L}$, the left-hand side of the DE becomes

$$\begin{aligned} &[s^2Y - sy(0) - y'(0)] + 4[sY - y(0)] + 4Y \\ &\qquad\qquad = (s^2 + 4s + 4)Y - 1 = (s+2)^2Y - 1; \end{aligned}$$

therefore, equating the transforms of the two sides, solving for Y, and using partial fraction decomposition, we arrive at

$$\begin{aligned} Y(s) &= \frac{1}{(s+2)^2} + \frac{s+1}{s^2(s+2)^2}\,e^{-s} - \frac{3s+1}{s^2(s+2)^2}\,e^{-3s} \\ &= \frac{1}{(s+2)^2} + \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{(s+2)^2}\right]e^{-s} \\ &\qquad - \frac{1}{4}\left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+2} - \frac{5}{(s+2)^2}\right]e^{-3s}. \end{aligned}$$

Then, by Theorem 8.27 in conjunction with Theorem 8.30,

$$\begin{aligned} y(t) &= te^{-2t} + \tfrac{1}{4}\,[1 - e^{-2(t-1)}](t-1)H(t-1) \\ &\qquad - \tfrac{1}{4}\,[2 - 2e^{-2(t-3)} + (t-3) - 5(t-3)e^{-2(t-3)}]H(t-3). \end{aligned}$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = t * E^(-2 * t) + (1/4) * (1 - E^(-2 * (t - 1))) * (t - 1)
        * HeavisideTheta[t - 1] - (1/4) * (2 - 2 * E^( - 2* (t - 3))
        + t - 3 - 5 * (t - 3) * E^(-2 * (t - 3))) * HeavisideTheta[t - 3];
{D[y,t,t] + 4 * D[y,t] + 4 * y - t * (HeavisideTheta[t - 1]
        - HeavisideTheta[t - 3]), {y,D[y,t]} /. t → 0} // Simplify
```

generates the output $\{0, \{0, 1\}\}$. ■

Exercises

Use the Laplace transformation method to solve the given IVP.

1 $y'' - y' = H(t-1),\quad y(0) = 0,\quad y'(0) = 0.$

2 $y'' + 2y' = -4H(t-2),\quad y(0) = 0,\quad y'(0) = 0.$

3 $2y'' - y' = 1 - H(t-1),\quad y(0) = 0,\quad y'(0) = 0.$

4 $y'' - 4y' = 16[2 - H(t-2)],\quad y(0) = 0,\quad y'(0) = 0.$

5 $y'' + y' - 2y = 6H(t-1/2),\quad y(0) = 0,\quad y'(0) = 0.$

6 $y'' - 4y' + 3y = 6[1 - 2H(t-3)],\quad y(0) = 0,\quad y'(0) = 0.$

7 $y'' - 2y' + y = H(t-2),\quad y(0) = 0,\quad y'(0) = 0.$

8 $y'' - 6y' + 9y = 9[1 - H(t-1)],\quad y(0) = 0,\quad y'(0) = 0.$

9 $y'' - 4y' + 5y = 5H(t-1),\quad y(0) = 0,\quad y'(0) = 0.$

10 $y'' + 4y = 4[2 - H(t-2)],\quad y(0) = 0,\quad y'(0) = 0.$

11 $y'' - 3y' + 2y = 4tH(t-1),\quad y(0) = 0,\quad y'(0) = 0.$

12 $y'' + y' - 6y = 30H(t-2),\quad y(0) = 5,\quad y'(0) = 0.$

13 $2y'' - 3y' - 2y = 10H(t-3),\quad y(0) = 0,\quad y'(0) = 5.$

14 $y'' + 3y' = 18H(t-1/2),\quad y(0) = 3,\quad y'(0) = -3.$

15 $y'' + 2y' + y = tH(t-4),\quad y(0) = 2,\quad y'(0) = 1.$

16 $4y'' + y = 1 - 5e^{1-t}H(t-1),\quad y(0) = 4,\quad y'(0) = 0.$

17 $3y'' + y' = H(t-1) - H(t-2),\quad y(0) = 0,\quad y'(0) = 0.$

18 $y'' - 5y' + 4y = 12[H(t-2) + H(t-3)],\quad y(0) = 3,\quad y'(0) = 0.$

19 $y'' + 9y = 9[3tH(t-1) - H(t-2)],\quad y(0) = 0,\quad y'(0) = 0.$

20 $y'' - 4y' + 4y = e^{t-1}H(t-1) - 4H(t-3),\quad y(0) = 0,\quad y'(0) = 1.$

Answers to Odd-Numbered Exercises

1 $y(y) = (e^{t-1} - t)H(t-1).$

3 $y(t) = 2e^{t/2} - t - 2 + [t + 1 - 2e^{(t-1)/2}]H(t-1).$

5 $y(t) = [e^{-2(t-1/2)} + 2e^{t-1/2} - 3]H(t-1/2).$

7 $y(t) = [1 + (t-3)e^{t-2}]H(t-2).$

9 $y(t) = \{1 - e^{2(t-1)}[\cos(t-1) - 2\sin(t-1)]\}H(t-1).$

11 $y(t) = [3e^{2(t-1)} - 8e^{t-1} + 2t + 3]H(t-1).$

13 $y(t) = 2e^{2t} - 2e^{-t/2} + [e^{2(t-3)} + 4e^{(t-3)/2} - 5]H(t-3).$

15 $y(t) = (3t+2)e^{-t} + [t - 2 + (10-3t)e^{4-t}]H(t-4).$

17 $y(t) = [t - 4 + 3e^{(1-t)/3}]H(t-1) + [5 - t - 3e^{(2-t)/3}]H(t-2).$

19 $$y(t) = [3t - 3\cos(3(t-1)) - \sin(3(t-1))]H(t-1) + [\cos(3(t-2)) - 1]H(t-2).$$

8.3.3 Forcing Terms with the Dirac Delta

Sometimes, mathematical modeling is required to take into account external influences that act either at a single specific point, or at a single moment of time, or both. Since such action cannot be described by a continuous function or even a Heaviside-type function, we need to introduce a new, special tool for it.

8.46 Example. Suppose that a spring–mass–damper system is subjected to an external force g that acts on the mass only at $t = t_0$, when it produces a unit impulse. Physically, it is unrealistic to consider that the action happens instantaneously, however fast it is, so it makes sense to assume that, in fact, it occurs over a short time interval, say, $t_0 - \varepsilon < t < t_0 + \varepsilon$, where ε is a very small positive number. Given the brief duration of its action, we may assume that the applied force is piecewise constant, as defined by

$$g(t) = \begin{cases} g_0, & t_0 - \varepsilon < t < t_0 + \varepsilon, \\ 0, & t \le t_0 - \varepsilon, \ \ t \ge t_0 + \varepsilon, \end{cases} \qquad g_0 = \text{const}.$$

The impulse generated by this force is

$$I(t_0, \varepsilon) = \int_{-\infty}^{\infty} g(t)\, dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} g_0\, dt = 2\varepsilon g_0.$$

Hence, the force produces a unit impulse if $g_0 = 1/(2\varepsilon)$. We denote the corresponding function by $g_{t_0,\varepsilon}$:

$$g_{t_0,\varepsilon}(t) = \begin{cases} 1/(2\varepsilon), & t_0 - \varepsilon < t < t_0 + \varepsilon, \\ 0, & t \le t_0 - \varepsilon \text{ or } t \ge t_0 + \varepsilon. \end{cases}$$

The heavy lines in Fig. 8.4 are the nonzero segments of the graphs of $g_{t_0,\varepsilon}$, $g_{t_0,\varepsilon/2}$, $g_{t_0,\varepsilon/4}$, and $g_{t_0,\varepsilon/8}$ for some $t_0 > 0$ and $\varepsilon > 0$.

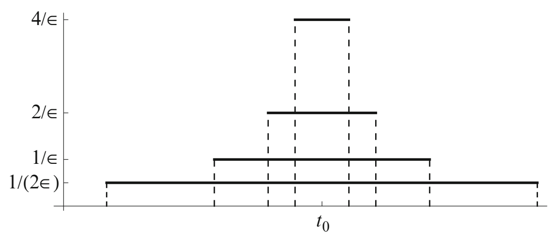

Fig. 8.4

The area of the rectangle formed by each of these lines with the horizontal axis and the appropriate vertical dashed lines is equal to the value of the impulse of the force represented by that particular function. The 'ideal' mathematical situation, where the force is assumed to act at the single moment t_0, is now achieved by shrinking ε to zero. We notice that, as ε decreases toward 0, the base of these rectangles decreases to 0 whereas their height increases to infinity. But for any nonzero value of ε, however small, the area of the rectangle—that is, the impulse—remains equal to 1. This peculiarity can be resolved by the introduction of a new mathematical concept. ■

8.47 Definition. The quantity

$$\delta(t-t_0) = \lim_{\varepsilon \to 0} g_{t_0,\varepsilon}(t)$$

is called the *Dirac delta*. ■

8.48 Remark. It is obvious that $\delta(t-t_0)$ is characterized by the following two properties:

(i) $\delta(t-t_0) = 0$ for $t \neq t_0$;

(ii) $\int\limits_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$.

Clearly, the limiting process by means of which the Dirac delta is defined does not ascribe it a finite value at $t = t_0$. (If it did, then the integral in (ii) would be zero.) Consequently, δ is not a function. In mathematics this object is called a *distribution*, or *generalized function*. ■

8.49 Theorem. *If f is a continuous function on the real line, then*

$$\int\limits_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0). \; ■ \tag{8.12}$$

Proof. By the mean value theorem for integrals, there is at least one point t^* in the interval $t_0 - \varepsilon \le t \le t_0 + \varepsilon$ such that

$$\int\limits_{-\infty}^{\infty} f(t)\,g_{t_0,\varepsilon}(t)\,dt = \int\limits_{t_0-\varepsilon}^{t_0+\varepsilon} \frac{1}{2\varepsilon}\,f(t)\,dt = \frac{1}{2\varepsilon}\int\limits_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\,dt = \frac{1}{2\varepsilon}\cdot 2\varepsilon f(t^*) = f(t^*).$$

Letting $\varepsilon \to 0$, we obtain (8.12). ■

8.50 Example. If $t_0 \ge 0$, then, by the definitions of $\mathcal{L}$ and δ and Theorem 8.49 with $f(t) = e^{-st}$,

$$\mathcal{L}\{\delta(t-t_0)\} = \int\limits_0^{\infty} e^{-st}\delta(t-t_0)\,dt = \int\limits_{-\infty}^{\infty} e^{-st}\delta(t-t_0)\,dt = e^{-st_0}, \quad s > 0.$$

In particular, $\mathcal{L}\{\delta(t)\} = 1$. ■

8.51 Example. Let g be a continuous function. A direct application of Theorem 8.49 with $f(t) = e^{-st}g(t)$ yields

$$\mathcal{L}\{g(t)\delta(t-t_0)\} = \int\limits_0^{\infty} e^{-st}g(t)\delta(t-t_0)\,dt = \int\limits_{-\infty}^{\infty} e^{-st}g(t)\delta(t-t_0)\,dt = g(t_0)e^{-st_0}. \; ■$$

8.52 Example. Let $\alpha = \text{const} > 0$. To compute $\mathcal{L}\{\delta(\alpha(t-t_0))\}$, we use the definition of $\mathcal{L}$ and the substitution $\alpha(t-t_0) = \tau$ (or, what is the same, $t = \tau/\alpha + t_0$):

$$\mathcal{L}\{\delta(\alpha(t-t_0))\} = \int\limits_{-\infty}^{\infty} e^{-st}\delta(\alpha(t-t_0))\,dt = \int\limits_{-\infty}^{\infty} e^{-s(\tau/\alpha+t_0)}\delta(\tau)\cdot\frac{1}{\alpha}\,d\tau;$$

so, by (8.12),

$$\mathcal{L}\{\delta(\alpha(t-t_0))\} = \frac{1}{\alpha}\,e^{-st_0}. \; ■$$

8.53 Example. Consider the IVP

$$y'' + \pi^2 y = \delta(t-1), \quad y(0) = 1, \quad y'(0) = 0.$$

Applying $\mathcal{L}$ to all the terms in the DE, we change the IVP to

$$[s^2 Y - sy(0) - y'(0)] + \pi^2 Y = e^{-s},$$

from which, after using the ICs, we obtain

$$Y(s) = \frac{s + e^{-s}}{s^2 + \pi^2} = \frac{s}{s^2 + \pi^2} + \frac{1}{\pi} \frac{\pi}{s^2 + \pi^2} e^{-s}.$$

Then the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \cos(\pi t) + \frac{1}{\pi} \sin(\pi(t-1))H(t-1).$$

Writing $\sin(\pi(t-1)) = \sin(\pi t - \pi) = \sin(\pi t)\cos\pi - \cos(\pi t)\sin\pi = -\sin(\pi t)$, we simplify this to

$$y(t) = \cos(\pi t) - \frac{1}{\pi}\sin(\pi t)H(t-1) = \begin{cases} \cos(\pi t), & 0 \le t < 1, \\ \cos(\pi t) - (1/\pi)\sin(\pi t), & t \ge 1. \end{cases}$$

Verification with Mathematica®. The input

```
y = Cos[Pi*t] - (1/Pi)*Sin[Pi*t]*HeavisideTheta[t - 1];
{D[y,t,t] + Pi^2*y - DiracDelta[t - 1], {y,D[y,t]}/. t → 0}
        // Simplify
```

generates the output $\{0, \{1, 0\}\}$. ■

8.54 Example. From formula 14 in Appendix C with $f(t) = 2t + 1$ and $a = 1/2$ it follows that $\mathcal{L}\{(2t+1)\,\delta(t-1/2)\} = 2e^{-s/2}$, so the Laplace transformation changes the IVP

$$y'' - 4y' + 3y = (2t+1)\delta\big(t - \tfrac{1}{2}\big), \quad y(0) = 0, \quad y'(0) = 2$$

into the algebraic problem

$$(s^2 - 4s + 3)Y - 2 = 2e^{-s/2};$$

hence,

$$Y(s) = \frac{2}{s^2 - 4s + 3}(1 + e^{-s/2}) = \left(\frac{1}{s-3} - \frac{1}{s-1}\right)(1 + e^{-s/2}).$$

By formula 2 in Appendix C, we now obtain

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{3t} - e^t + [e^{3(t-1/2)} - e^{t-1/2}]H\big(t - \tfrac{1}{2}\big).$$

Verification with Mathematica®. The input

```
y = E^(3*t) - E^t + (E^(3*(t - 1/2)) - E^(t - 1/2))
        *HeavisideTheta[t - 1/2];
{D[y,t,t] - 4*D[y,t] + 3*y - (2*t + 1)*DiracDelta[t - 1/2],
        {y,D[y,t]}/. t → 0} // Simplify
```

generates the output $\{0, \{0, 2\}\}$. ■

8.55 Example. Applying the Laplace transformation to the IVP

$$y'' + y' = \delta(t-1) - \delta(t-2), \quad y(0) = 0, \quad y'(0) = 0,$$

we arrive at

$$(s^2 + s)Y = e^{-s} - e^{-2s};$$

therefore,

$$Y(s) = \frac{1}{s^2+s}\,(e^{-s} - e^{-2s}) = \left(\frac{1}{s} - \frac{1}{s+1}\right)(e^{-s} - e^{-2s}),$$

which yields the solution

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = [1 - e^{-(t-1)}]H(t-1) - [1 - e^{-(t-2)}]H(t-2).$$

Explicitly, this is

$$y(t) = \begin{cases} 0, & 0 \le t < 1, \\ 1 - e^{-(t-1)}, & 1 \le t < 2, \\ -e^{-(t-1)} + e^{-(t-2)}, & t \ge 2. \end{cases}$$

Verification with Mathematica®. The input

```
y = (1 - E^(-(t - 1)))*HeavisideTheta[t - 1] - (1 - E^(-(t - 2)))
        *HeavisideTheta[t - 2];
{D[y,t,t] + D[y,t] - DiracDelta[t - 1] + DiracDelta[t - 2],
        {y,D[y,t]}/.t→ 0}//Simplify
```

generates the output $\{0, \{0, 0\}\}$. ■

Exercises

Use the Laplace transformation method to solve the given IVP.

1 $y'' - 2y' = 2\delta(t-2)$, $y(0) = 0$, $y'(0) = 0$.

2 $2y'' + y' = \delta(t-1)$, $y(0) = 1$, $y'(0) = 0$.

3 $y'' - y' = 1 - \delta(t-1)$, $y(0) = 0$, $y'(0) = 0$.

4 $y'' + y' = 2\delta(t-1) - \delta(t-3)$, $y(0) = 0$, $y'(0) = 1$.

5 $y'' + 4y = \delta(t-3)$, $y(0) = -1$, $y'(0) = 0$.

6 $y'' + y = 2 - \delta(t-2)$, $y(0) = 2$, $y'(0) = 1$.

7 $y'' - y = t - \delta(t-1)$, $y(0) = 0$, $y'(0) = 0$.

8 $y'' - 4y = e^{-t} + 2\delta(t-2)$, $y(0) = -1$, $y'(0) = 1$.

9 $y'' + 9y = (t+1)\delta(t-2)$, $y(0) = 2$, $y'(0) = 9$.

10 $9y'' - 4y = e^t\delta(t-2)$, $y(0) = 1$, $y'(0) = 0$.

11 $y'' + 3y' + 2y = \delta(t-1)$, $y(0) = 0$, $y'(0) = 0$.

12 $y'' - y' - 2y = t - 1 + \delta(t-2)$, $y(0) = 0$, $y'(0) = -1$.

13 $y'' + 2y' + y = 3 - \delta(t-1/2)$, $y(0) = 2$, $y'(0) = 0$.

14 $y'' - 4y' + 4y = 2\delta(t-1), \quad y(0) = 0, \quad y'(0) = 0.$

15 $y'' - 2y' - 3y = 4(\cos t)\delta(t-\pi), \quad y(0) = 0, \quad y'(0) = 0.$

16 $y'' - 2y' + 2y = (\sin t)\delta(t-\pi/2), \quad y(0) = 0, \quad y'(0) = 1.$

17 $y'' + y' = 2\delta(t-1) - \delta(t-2), \quad y(0) = 1, \quad y'(0) = 0.$

18 $y'' + y = \delta(t-\pi/2) + \delta(t-\pi), \quad y(0) = 0, \quad y'(0) = 1.$

19 $y'' - 4y = \delta(t-1/2) + \delta(t-1), \quad y(0) = 1, \quad y'(0) = -1.$

20 $y'' - y' - 6y = (t+1)\delta(t-2) - \delta(t-3), \quad y(0) = 0, \quad y'(0) = 0.$

Answers to Odd-Numbered Exercises

1 $y(t) = [e^{2(t-2)} - 1]H(t-2).$ **3** $y(t) = e^t - t - 1 + (1 - e^{t-1})H(t-1).$

5 $y(t) = -\cos(2t) + (1/2)\sin(2(t-3))H(t-3).$

7 $y(t) = \sinh t - t - \sinh(t-1)H(t-1).$

9 $y(t) = 2\cos(3t) + 3\sin(3t) + \sin(3(t-2))H(t-2).$

11 $y(t) = [e^{1-t} - e^{2(1-t)}]H(t-1).$

13 $y(t) = 3 - (t+1)e^{-t} - (t-1/2)e^{1/2-t}H(t-1/2).$

15 $y(t) = [e^{\pi-t} - e^{3(t-\pi)}]H(t-\pi).$

17 $y(t) = 1 + 2(1 - e^{1-t})H(t-1) - (1 - e^{2-t})H(t-2).$

19 $y(t) = \cosh(2t) - (1/2)\sinh(2t) + (1/2)\sinh(2t-1)H(t-1/2)$
$\qquad + (1/2)\sinh(2(t-1))H(t-1).$

8.3.4 Equations with Variable Coefficients

In special circumstances, the Laplace transformation can also be used to solve IVPs for DEs with nonconstant coefficients. Owing to the difficulties that arise in such cases, however, the scope of the method here is very limited.

8.56 Example. Consider the IVP

$$y'' + ty' - 2y = 4, \quad y(0) = 0, \quad y'(0) = 0.$$

As usual, let $\mathcal{L}\{y(t)\} = Y(s)$. By formula 16 (with $n = 1$) in Appendix C,

$$\mathcal{L}\{ty'(t)\} = -\frac{d}{ds}\,\mathcal{L}\{y'\}(s) = -[sY(s) - y(0)]' = -[sY(s)]' = -Y - sY'.$$

Consequently, the transformed problem is

$$[s^2Y - sy(0) - y'(0)] - Y - sY' - 2Y = \frac{4}{s},$$

or, in view of the ICs,

$$sY' + (3 - s^2)Y = -\frac{4}{s}.$$

Division by s brings this linear first-order DE to the form

$$Y' + \left(\frac{3}{s} - s\right)Y = -\frac{4}{s^2},$$

for which, by (2.5), an integrating factor is

$$\mu(s) = \exp\left\{\int\left(\frac{3}{s} - s\right)ds\right\} = e^{3\ln s - s^2/2} = e^{\ln(s^3)}e^{-s^2/2} = s^3e^{-s^2/2};$$

hence, by (2.7),

$$\begin{aligned} Y(s) &= \frac{1}{s^3}\,e^{s^2/2}\int s^3e^{-s^2/2}\left(-\frac{4}{s^2}\right)ds = \frac{1}{s^3}\,e^{s^2/2}\int -4se^{-s^2/2}\,ds \\ &= \frac{1}{s^3}\,e^{s^2/2}\left(4e^{-s^2/2} + C\right) = \frac{4}{s^3} + \frac{C}{s^3}\,e^{s^2/2}. \end{aligned}$$

Since we have used the Laplace transform of y, we have tacitly assumed that y is an admissible function, which, in this context, means that it satisfies the conditions in Theorem 8.6. Then, according to that theorem, the transform Y of y must tend to zero as $s \to \infty$. But this is not the case with the second term in the above expression of Y, which, if $C \neq 0$, increases without bound in absolute value as s increases. Consequently, Y cannot be the transform of what we called an admissible function unless $C = 0$. Therefore, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\} = 2t^2.$$

Verification with Mathematica®. The input

```
y = 2 * t^2;
{D[y,t,t] + t * D[y,t] - 2 * y - 4, {y,D[y,t]} /. t→ 0} // Simplify
```

generates the output $\{0, \{0, 0\}\}$. ■

8.57 Example. We apply the same procedure to the IVP

$$ty'' + 2ty' - 2y = -2, \quad y(0) = 1, \quad y'(0) = 2.$$

As seen in the preceding example, $\mathcal{L}\{ty'(t)\} = -Y - sY'$. Similarly,

$$\begin{aligned} \mathcal{L}\{ty''(t)\} &= -\frac{d}{ds}\,(\mathcal{L}\{y''\})(s) = -[s^2Y - sy(0) - y'(0)]' \\ &= -(s^2Y - s - 2)' = -2sY - s^2Y' + 1, \end{aligned}$$

so, transforming the equation and gathering the like terms together, we arrive at

$$(s^2 + 2s)Y' + (2s + 4)Y = 1 + \frac{2}{s} = \frac{s+2}{s}.$$

After division by $s(s + 2)$, this becomes

$$Y' + \frac{2}{s}\,Y = \frac{1}{s^2},$$

for which we construct the integrating factor

$$\mu(s) = \exp\left\{\int \frac{2}{s}\, ds\right\} = e^{2\ln s} = e^{\ln(s^2)} = s^2$$

and, thus, the GS

$$Y(s) = \frac{1}{s^2} \int s^2 \cdot \frac{1}{s^2}\, ds = \frac{1}{s^2}\,(s + C) = \frac{1}{s} + \frac{C}{s^2}.$$

Since this transform tends to zero as $s \to \infty$, we invert it and obtain

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{C}{s^2}\right\} = 1 + Ct.$$

To find C, we apply the second IC, which yields $C = 2$; therefore, the solution of the IVP is

$$y(t) = 2t + 1.$$

Verification with Mathematica®. The input

```
y = 2 * t + 1;
{t * D[y,t,t] + 2 * t * D[y,t] - 2 * y + 2, {y,D[y,t]} /. t → 0}
        // Simplify
```

generates the output {0, {1, 2}}. ■

Exercises

Use the Laplace transformation method to solve the given IVP.

1 $y'' - 2ty' + 4y = -2, \quad y(0) = 0, \quad y'(0) = 0.$

2 $2y'' + ty' - 2y = 2, \quad y(0) = 1, \quad y'(0) = 0.$

3 $y'' + ty' - y = 1, \quad y(0) = -1, \quad y'(0) = 3.$

4 $3y'' + 2ty' - 2y = 6, \quad y(0) = -3, \quad y'(0) = 2.$

5 $ty'' - 2ty' + 4y = 0, \quad y(0) = 0, \quad y'(0) = -1.$

6 $ty'' - 4ty' + 8y = 8, \quad y(0) = 1, \quad y'(0) = -1.$

7 $2ty'' + ty' - y = -3, \quad y(0) = 3, \quad y'(0) = -1.$

8 $ty'' - 3ty' + 3y = 15, \quad y(0) = 5, \quad y'(0) = 2.$

Answers to Odd-Numbered Exercises

1 $y(t) = -t^2$. **3** $y(t) = 3t - 1$. **5** $y(t) = t^2 - t$. **7** $y(t) = 3 - t$.

8.4 Solution of IVPs for Systems

The solution strategy for solving systems of linear first-order DEs with constant coefficients is the same as in the case of single equations.

8.58 Example. Consider the IVP

$$\begin{aligned} x_1' &= 2x_1 + \ \ x_2, \quad & x_1(0) &= 1,\\ x_2' &= \ \ x_1 + 2x_2, \quad & x_2(0) &= 0. \end{aligned}$$

Setting $X_1(s) = \mathcal{L}\{x_1(t)\}$ and $X_2(s) = \mathcal{L}\{x_2(t)\}$ and applying the ICs, we have

$$\begin{aligned} \mathcal{L}\{x_1'(t)\} &= sX_1(s) - x_1(0) = sX_1 - 1,\\ \mathcal{L}\{x_2'(t)\} &= sX_2(s) - x_2(0) = sX_2, \end{aligned}$$

so the given IVP reduces to the algebraic system

$$\begin{aligned} (s-2)X_1 - \qquad\quad X_2 &= 1,\\ -X_1 + (s-2)X_2 &= 0. \end{aligned}$$

Using, for example, Cramer's rule (see Remark 4.2(iii)), we find that

$$X_1(s) = \frac{\begin{vmatrix} 1 & -1\\ 0 & s-2 \end{vmatrix}}{\begin{vmatrix} s-2 & -1\\ -1 & s-2 \end{vmatrix}} = \frac{s-2}{s^2-4s+3} = \frac{s-2}{(s-1)(s-3)},$$

$$X_2(s) = \frac{\begin{vmatrix} s-2 & 1\\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s-2 & -1\\ -1 & s-2 \end{vmatrix}} = \frac{1}{s^2-4s+3} = \frac{1}{(s-1)(s-3)},$$

which, after partial fraction decomposition, become

$$X_1(s) = \frac{1}{2}\,\frac{1}{s-1} + \frac{1}{2}\,\frac{1}{s-3}, \quad X_2(s) = -\frac{1}{2}\,\frac{1}{s-1} + \frac{1}{2}\,\frac{1}{s-3}.$$

The solution of the IVP is now obtained by means of the inverse Laplace transformation:

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1}\{X_1(s)\} = \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = \tfrac{1}{2}\,e^t + \tfrac{1}{2}\,e^{3t},\\ x_2(t) &= \mathcal{L}^{-1}\{X_2(s)\} = -\frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = -\tfrac{1}{2}\,e^t + \tfrac{1}{2}\,e^{3t}. \end{aligned}$$

Verification with Mathematica®. The input

```
{x1,x2}={(1/2)*E^t + (1/2)*E^(3*t),-(1/2)*E^t
        + (1/2)*E^(3*t)};
{{D[x1,t] - 2*x1 - x2,D[x2,t] - x1 - 2*x2},
        {x1,x2}/.t→ 0}//Simplify
```

generates the output {{0, 0}, {1, 0}}. ■

8.59 Example. With the same notation as above, we use $\mathcal{L}$ and reduce the IVP

$$\begin{aligned} x_1' &= -11x_1 + 8x_2 - 10e^{-t}, \quad x_1(0) = 3,\\ x_2' &= -12x_1 + 9x_2 - 12e^{-t}, \quad x_2(0) = 5 \end{aligned}$$

to the algebraic problem

$$\begin{aligned} sX_1 - 3 &= -11X_1 + 8X_2 - \frac{10}{s+1},\\ sX_2 - 5 &= -12X_1 + 9X_2 - \frac{12}{s+1}, \end{aligned}$$

or, in simplified form,

$$\begin{aligned} (s+11)X_1 - \quad 8X_2 &= \frac{3s-7}{s+1},\\ 12X_1 + (s-9)X_2 &= \frac{5s-7}{s+1}. \end{aligned}$$

Performing the necessary algebra, from this we obtain

$$\begin{aligned} X_1(s) &= \frac{3s^2+6s+7}{(s+1)(s^2+2s-3)} = \frac{3s^2+6s+7}{(s+1)(s-1)(s+3)} = -\frac{1}{s+1} + \frac{2}{s-1} + \frac{2}{s+3},\\ X_2(s) &= \frac{5s+7}{s^2+2s-3} = \frac{5s+7}{(s-1)(s+3)} = \frac{3}{s-1} + \frac{2}{s+3}. \end{aligned}$$

Hence, the solution of the given IVP is

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1}\{X_1(s)\} = -e^{-t} + 2e^{t} + 2e^{-3t},\\ x_2(t) &= \mathcal{L}^{-1}\{X_2(s)\} = 3e^{t} + 2e^{-3t}. \end{aligned}$$

Verification with Mathematica®. The input

```
{x1,x2}={ - E^(-t) + 2*E^t + 2*E^(-3*t),3*E^t
        + 2*E^(-3*t)};
{{D[x1,t] + 11*x1 - 8*x2 + 10*E^(-t),D[x2,t] + 12*x1 - 9*x2
        + 12*E^(-t)},{x1,x2}/.t→ 0}//Simplify
```

generates the output {{0, 0}, {3, 5}}. ■

8.60 Example. Applying $\mathcal{L}$ to the IVP

$$\begin{aligned} x_1' &= -7x_1 - 9x_2 - 13, \quad x_1(0) = 9,\\ x_2' &= \quad 4x_1 + 5x_2 + 7, \quad x_2(0) = -8, \end{aligned}$$

we arrive at the algebraic system

$$\begin{aligned} (s+7)X_1 + \quad 9X_2 &= \frac{9s-13}{s},\\ -4X_1 + (s-5)X_2 &= \frac{7-8s}{s}, \end{aligned}$$

with solution

$$\begin{aligned} X_1(s) &= \frac{9s^2+14s+2}{s(s^2+2s+1)} = \frac{9s^2+14s+2}{s(s+1)^2} = \frac{2}{s} + \frac{7}{s+1} + \frac{3}{(s+1)^2},\\ X_2(s) &= \frac{-8s^2-13s-3}{s(s^2+2s+1)} = \frac{-8s^2-13s-3}{s(s+1)^2} = -\frac{3}{s} - \frac{5}{s+1} - \frac{2}{(s+1)^2}. \end{aligned}$$

Then the solution of the given IVP is

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1}\{X_1(s)\} = 2 + 7e^{-t} + 3te^{-t} = 2 + (3t+7)e^{-t}, \\ x_2(t) &= \mathcal{L}^{-1}\{X_2(s)\} = -3 - 5e^{-t} - 2te^{-t} = -3 - (2t+5)e^{-t}. \end{aligned}$$

Verification with Mathematica®. The input

```
{x1,x2}={2 + (3*t + 7)*E^(-t), - 3 - (2*t + 5)*E^(-t)};
{{D[x1,t] + 7*x1 + 9*x2 + 13, D[x2,t] - 4*x1 - 5*x2 - 7},
        {x1,x2}/.t→ 0}//Simplify
```

generates the output {{0, 0}, {9, −8}}. ■

8.61 Example. The same technique applied to the IVP

$$\begin{aligned} x_1' &= 2x_1 - x_2 + x_3, & x_1(0) &= 1, \\ x_2' &= 2x_1 - x_2 + 2x_3, & x_2(0) &= 1, \\ x_3 &= x_1 - x_2, & x_3(0) &= 2 \end{aligned}$$

leads to

$$\begin{aligned} (s-2)X_1 + X_2 - X_3 &= 1, \\ -2X_1 + (s+1)X_2 - 2X_3 &= 1, \\ -X_1 + X_2 + sX_3 &= 2. \end{aligned}$$

Using Gaussian elimination to solve this algebraic system, synthetic division or the symmetry of the coefficients to factor out the denominator, and partial fraction decomposition, after a rather long but straightforward calculation we find that

$$\begin{aligned} X_1 &= \frac{s^2+2s-1}{s^3-s^2+s-1} = \frac{s^2+2s-1}{(s-1)(s^2+1)} = \frac{1}{s-1} + \frac{2}{s^2+1}, \\ X_2 &= \frac{s^2+4s-3}{s^3-s^2+s-1} = \frac{s^2+4s-3}{(s-1)(s^2+1)} = \frac{1}{s-1} + \frac{4}{s^2+1}, \\ X_3 &= \frac{2s}{s^2+1}, \end{aligned}$$

which yields

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1}\{X_1(s)\} = e^t + 2\sin t, \\ x_2(t) &= \mathcal{L}^{-1}\{X_2(s)\} = e^t + 4\sin t, \\ x_3(t) &= \mathcal{L}^{-1}\{X_3(s)\} = 2\cos t. \end{aligned}$$

Verification with Mathematica®. The input

```
{x1,x2,x3}={E^t + 2*Sin[t], E^t + 4*Sin[t], 2*Cos[t]};
{{D[x1,t] - 2*x1 + x2 - x3, D[x2,t] - 2*x1 + x2 - 2*x3,
        D[x3,t] - x1 + x2}, {x1,x2,x3}/.t→ 0}//Simplify
```

generates the output {{0, 0, 0}, {1, 1, 2}}. ■

Exercises

Use the Laplace transformation method to solve the IVP $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, $\mathbf{x}(0) = \mathbf{x}_0$, with the given matrix A, vector function $\mathbf{f}$, and constant vector $\mathbf{x}_0$.

1 $A = \begin{pmatrix} 7 & -4 \\ 8 & -5 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$

2 $A = \begin{pmatrix} 7 & 10 \\ -5 & -8 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$

3 $A = \begin{pmatrix} -4 & 2 \\ -10 & 4 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

4 $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$

5 $A = \begin{pmatrix} 5 & -4 \\ 9 & -7 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$

6 $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

7 $A = \begin{pmatrix} -8 & 18 \\ -3 & 7 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

8 $A = \begin{pmatrix} 8 & -3 \\ 9 & -4 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 6 \\ 15 \end{pmatrix}.$

9 $A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 16 \\ -6 \end{pmatrix}.$

10 $A = \begin{pmatrix} -1 & -1/2 \\ 5/2 & 1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

11 $A = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} -4 \\ -2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

12 $A = \begin{pmatrix} -1 & -2 \\ 1 & -4 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} t-1 \\ -t-4 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$

13 $A = \begin{pmatrix} 3/2 & 1 \\ -1/2 & 0 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 3 \\ t-2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

14 $A = \begin{pmatrix} -1 & -9 \\ 1 & 5 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 11-4t \\ -3 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}.$

15 $A = \begin{pmatrix} 1/2 & -1 \\ 1 & 1/2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 5 \\ -5t-1 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$

16 $A = \begin{pmatrix} 7 & 3 \\ -6 & -2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} -2e^{-t} \\ 4e^{-t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 5 \\ -7 \end{pmatrix}.$

17 $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 2e^{-t} \\ 10e^{-t} \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$

18 $A = \begin{pmatrix} 2 & 1/2 \\ -9/2 & -1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 6 \\ -14 \end{pmatrix}.$

19 $A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$

20 $A = \begin{pmatrix} -3 & -6 \\ 4 & 7 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 4\cos t \\ -6\cos t \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

21 $A = \begin{pmatrix} -5 & 6 & -3 \\ -1 & 2 & -1 \\ 4 & -4 & 2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}.$

22 $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}.$

23 $A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}.$

24 $A = \begin{pmatrix} -3 & -2 & 2 \\ 2 & 1 & -1 \\ -2 & -2 & 2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}.$

25 $A = \begin{pmatrix} -1/2 & -3/2 & -1/2 \\ 1/2 & 3/2 & 1/2 \\ -3 & -5 & 1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -4 \\ 3 \\ -5 \end{pmatrix}.$

26 $A = \begin{pmatrix} 3 & -2 & -2 \\ 0 & 2 & 1 \\ 1 & -2 & -1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix}.$

Answers to Odd-Numbered Exercises

1 $\mathbf{x}(t) = \begin{pmatrix} 2e^{-t} - e^{3t} \\ 4e^{-t} - e^{3t} \end{pmatrix}.$ **3** $\mathbf{x}(t) = \begin{pmatrix} 2\cos(2t) - \sin(2t) \\ 3\cos(2t) - 4\sin(2t) \end{pmatrix}.$

5 $\mathbf{x}(t) = \begin{pmatrix} 1 - 6t \\ 3 - 9t \end{pmatrix} e^{-t}.$ **7** $\mathbf{x}(t) = \begin{pmatrix} 2e^{t} - 3e^{-2t} + 2 \\ e^{t} - e^{-2t} + 1 \end{pmatrix}.$

9 $\mathbf{x}(t) = \begin{pmatrix} (6t + 13)e^{-t} + 3 \\ -(3t + 5)e^{-t} - 1 \end{pmatrix}.$ **11** $\mathbf{x}(t) = \begin{pmatrix} e^{3t}(\cos t - 2\sin t) + 1 \\ -e^{3t}(2\cos t + \sin t) + 1 \end{pmatrix}.$

13 $\mathbf{x}(t) = \begin{pmatrix} 2e^{t} - 3e^{t/2} + 2t + 2 \\ -e^{t} + 3e^{t/2} - 3t - 4 \end{pmatrix}.$ **15** $\mathbf{x}(t) = \begin{pmatrix} -e^{t/2}(2\cos t + \sin t) + 4t + 2 \\ e^{t/2}(\cos t - 2\sin t) + 2t + 2 \end{pmatrix}.$

17 $\mathbf{x}(t) = \begin{pmatrix} (1 - t)e^{t} + e^{-t} \\ -te^{t} - 3e^{-t} \end{pmatrix}.$ **19** $\mathbf{x}(t) = \begin{pmatrix} 2\cos(2t) - 2\sin(2t) + 2 \\ -2\sin(2t) + 1 \end{pmatrix} e^{t}.$

21 $\mathbf{x}(t) = \begin{pmatrix} e^{t} - e^{-2t} \\ 2 + e^{t} \\ 4 + e^{-2t} \end{pmatrix}.$ **23** $\mathbf{x}(t) = \begin{pmatrix} (t + 4)e^{-t} + 1 \\ e^{-t} + 1 \\ (t + 2)e^{-t} \end{pmatrix}.$

25 $\mathbf{x}(t) = \begin{pmatrix} e^{t}(\sin t - \cos t) - 3 \\ e^{t}(\cos t - \sin t) + 2 \\ -2e^{t}(\cos t + \sin t) - 3 \end{pmatrix}.$

Chapter 9
Series Solutions

Owing to the complicated structure of some DEs, it is not always possible to obtain the exact solution of an IVP. In such situations, we need to resort to methods that produce an approximate solution, which is usually constructed in the form of an infinite series. In what follows we illustrate a procedure of this type, based on series expansions for functions of a real variable.

9.1 Power Series

9.1 Definition. A *power series* is an expression of the form

$$\sum_{n=0}^{\infty} a_n (t - t_0)^n, \tag{9.1}$$

where t_0 is a given real number and the a_n, $n = 0, 1, 2, \dots$, are constant real coefficients. Series (9.1) is said to be *convergent* at t if

$$s(t) = \lim_{m\to\infty} s_m(t) = \lim_{m\to\infty} \sum_{n=0}^{m} a_n (t - t_0)^n \text{ exists;}$$

$s_m(t)$ is called the *sequence of partial sums* and $s(t)$ is the *sum* of the series.

Series (9.1) is said to be *absolutely convergent* at t if the series $\sum\limits_{n=0}^{\infty} |a_n||t - t_0|^n$ is convergent.

A series that is not convergent is called *divergent*.

The *radius of convergence* of series (9.1) is a nonnegative number ρ such that the series is absolutely convergent for all t satisfying $|t - t_0| < \rho$ (that is, $t_0 - \rho < t < t_0 + \rho$) and divergent for all t satisfying $|t - t_0| > \rho$ (that is, $t < t_0 - \rho$ or $t > t_0 + \rho$). ■

9.2 Remarks. (i) The radius of convergence may be a finite number (including 0) or infinite, and its definition is independent of the behavior (convergence or divergence) of the series at the points $t = t_0 - \rho$ and $t = t_0 + \rho$.

(ii) Power series with the same radius of convergence ρ can be added, subtracted, multiplied, and differentiated term by term, and the result is always a series with radius of convergence ρ.

(iii) Two power series are equal if they have the same coefficients a_n. In particular, a series is equal to zero if all its coefficients are 0. ■

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_9,
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9.3 Example. Let f be an infinitely differentiable function, and let t_0 be a point in its domain. The Taylor series of f about t_0, namely

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}(t-t_0)^n, \tag{9.2}$$

where $f^{(n)}(t_0)$ is the derivative of order n of f at t_0, is a power series. ■

9.4 Definition. If the Taylor series (9.2) converges to $f(t)$ at all points t in some open interval containing t_0, then the function f is said to be *analytic* at t_0. ■

9.5 Remarks. (i) The sum $f+g$ and product fg of two analytic functions at t_0 are also analytic at t_0. The same is true for the quotient f/g provided that $g(t_0) \neq 0$.

(ii) Polynomials are terminating power series; that is, they are power series with finitely many terms, and, therefore, an infinite radius of convergence. The ratio of two polynomials is analytic at t_0 if t_0 is not a zero of the denominator. ■

In what follows we investigate equations of the form

$$P(t)y'' + Q(t)y' + R(t)y = 0, \tag{9.3}$$

where P, Q, and R are given analytic functions.

9.6 Definition. A point t_0 such that $P(t_0) \neq 0$ is called an *ordinary point* for (9.3). If $P(t_0) = 0$, then t_0 is called a *singular point.* ■

9.7 Remarks. (i) The above definition implies that if t_0 is an ordinary point, then the functions Q/P and R/P are analytic at t_0 and, thus, can be expanded in power series around this point.

(ii) To simplify the computation, in the examples discussed below we confine ourselves almost exclusively to the case where P, Q, and R are polynomials. ■

9.2 Series Solution Near an Ordinary Point

In the neighborhood of an ordinary point t_0, we may consider a solution of (9.3) of the form (9.1), where the coefficients a_n are determined from a recurrence relation obtained by replacing series (9.1) in the equation.

9.8 Theorem. *If t_0 is an ordinary point for equation (9.3), then the general solution of the equation can be written as*

$$y = c_1 y_1 + c_2 y_2,$$

where c_1 and c_2 are arbitrary constants and y_1 and y_2 are linearly independent series solutions of the form (9.1). The radius of convergence of y_1 and y_2 is at least as large as that of the series for Q/P and R/P. ■

9.9 Example. For the equation

$$y'' + y = 0$$

we have $P(t) = 1$, $Q(t) = 0$, and $R(t) = 1$, so $Q(t)/P(t) = 0$ and $R(t)/P(t) = 1$. It is trivial to see that the radius of convergence of the series for both Q/P and R/P is infinite, and that $t = 0$ is an ordinary point for the equation. This allows us to seek the solution of the DE in the neighborhood of 0 in the form (9.1) with $t_0 = 0$. Substituting the series in the equation, we arrive at

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 0. \tag{9.4}$$

To write both terms as a single infinite sum, we change the index of summation in the first term by setting $m = n - 2$, so the summation is now over m from -2 to ∞; that is, the first term becomes

$$\sum_{m=-2}^{\infty} (m+2)(m+1)a_{m+2}t^m.$$

Noting that the summand above vanishes for $m = -2$ and $m = -1$, we start the summation from 0 and, replacing m by n, we combine the two terms in (9.4) to obtain

$$\sum_{n=0}^{\infty} [(n+1)(n+2)a_{n+2} + a_n]t^n = 0.$$

Hence, by Remark 9.2(iii), the coefficient of t^n must be 0 for all nonnegative integers n, which yields the recurrence relation

$$a_{n+2} = -\frac{1}{(n+1)(n+2)}\, a_n, \quad n = 0, 1, \dots .$$

Taking, in turn, $n = 0, 1, \dots$, we compute all the coefficients a_n in terms of a_0 and a_1, which remain arbitrary; specifically,

$$\begin{aligned}
a_2 &= -\frac{1}{1 \cdot 2}\, a_0 = -\frac{1}{2!}\, a_0,\\
a_3 &= -\frac{1}{2 \cdot 3}\, a_1 = -\frac{1}{3!}\, a_1,\\
a_4 &= -\frac{1}{3 \cdot 4}\, a_2 = \frac{1}{2! \cdot 3 \cdot 4}\, a_0 = \frac{1}{4!}\, a_0,\\
a_5 &= -\frac{1}{4 \cdot 5}\, a_3 = \frac{1}{3! \cdot 4 \cdot 5}\, a_1 = \frac{1}{5!}\, a_1,\\
&\ \vdots
\end{aligned}$$

We now replace these coefficients in (9.1), gather the like terms together, and write the solution of the given equation as

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

where

$$\begin{aligned}
y_1(t) &= 1 - \tfrac{1}{2!}\, t^2 + \tfrac{1}{4!}\, t^4 - \cdots ,\\
y_2(t) &= \tfrac{1}{1!}\, t - \tfrac{1}{3!}\, t^3 + \tfrac{1}{5!}\, t^5 - \cdots ,
\end{aligned}$$

and the arbitrary constants c_1 and c_2 stand for a_0 and a_1, respectively. Since, clearly, y_1 and y_2 are solutions of the equation and since they are linearly independent (one is not a multiple of the other), this formula represents the GS of the DE. In fact, we recall that the power series for y_1 and y_2 are the Taylor series about $t = 0$ of $\cos t$ and $\sin t$. This confirms what we already knew from Sect. 4.3 about the form of the GS of our equation.

We also mention that, by Theorem 9.8, the series for y_1 and y_2 have an infinite radius of convergence; that is, they converge for all real values of t.

VERIFICATION WITH MATHEMATICA®. The input

```
y1 = (1 - (1/2!) * t^2 + (1/4!) * t^4;
y2 = (1/1!) * t - (1/3!) * t^3 + (1/5!) * t^5;
y = c1 * y1 + c2 * y2;
rem = D[y,t,t] + y;
rem // Expand
```

generates the output $c_1t^4/24+c_2t^5/120$, which is the remainder when the GS is replaced in the equation. ■

9.10 Remark. A 'rule of thumb' for determining the expected order of magnitude of the remainder when a truncated series solution is substituted in the given DE is based on considering the lowest power of t that occurs on the left-hand side of the equation after the replacement. If that power is N, then we anticipate the lowest power of t in the remainder to be no less than $N + 1$. In the above example, N for y_1 and y_2 is 2 and 3, respectively, and the corresponding remainders confirm our prediction. The rule is invalid for a solution represented by a terminating series, where the remainder is obviously zero.

This type of comment will be omitted in the rest of the examples. ■

9.11 Example. The series solution method does not always produce the general solution of a DE with constant coefficients in the form supplied by the characteristic equation technique. For the equation

$$y'' - 3y' + 2y = 0,$$

the latter leads to

$$y(t) = c_1e^t + c_2e^{2t}.$$

On the other hand, substituting series (9.1) in the equation and manipulating the summation index as we did in Example 9.9, we arrive at the equality

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n]t^n = 0,$$

from which we obtain the recurrence relation

$$a_{n+2} = \frac{1}{(n+1)(n+2)}\,[3(n+1)a_{n+1} - 2a_n], \quad n = 0, 1, \dots.$$

Hence, with arbitrary coefficients a_0 and a_1, we get the GS

$$y(t) = a_0\bigl(1 - t^2 - t^3 - \tfrac{7}{12}\,t^4 - \tfrac{1}{4}\,t^5 + \cdots\bigr) \\ + a_1\bigl(t + \tfrac{3}{2}\,t^2 + \tfrac{7}{6}\,t^3 + \tfrac{5}{8}\,t^4 + \tfrac{31}{120}\,t^5 + \cdots\bigr).$$

A quick check shows that the series multiplied by a_0 and a_1 are not the Taylor expansions around $t = 0$ of e^t and e^{2t}. Further checking shows that they are, in fact, the expansions of $2e^t - e^{2t}$ and $e^{2t} - e^t$. But that is perfectly acceptable since, as is easily verified by means of their Wronskian, these two functions also form an FSS for our equation and are, therefore, a legitimate alternative choice for writing its GS. We easily deduce from the (constant) coefficients of the equation that both series occurring in y converge for all real t.

Verification with Mathematica®. The input

```
y1 = 1 - t^2 - t^3 - (7/12)*t^4 - (1/4)*t^5;
y2 = t + (3/2)*t^2 + (7/6)*t^3 + (5/8)*t^4 + (31/120)*t^5;
y = c1*y1 + c2*y2;
rem = D[y,t,t] - 3*D[y,t] + 2*y;
rem // Expand
```

generates the output $c_1(31t^4/12 - t^5/2) + c_2(-21t^4/8 + 31t^5/60)$. ■

9.12 Example. It is obvious that $t = 0$ is an ordinary point for the DE

$$(t+2)y'' + (t-1)y' + 2ty = 0.$$

Here, we have $Q(t)/P(t) = (t-1)/(t+2)$ and $R(t)/P(t) = 2t/(t+2)$. Since these functions are undefined at $t = -2$, it follows that their power series in the neighborhood of $t = 0$ converge for $|t| < 2$; in other words, their radius of convergence is $\rho = 2$.

Replacing y by (9.1) in the equation, we find that

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-1} + 2\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} na_n t^n - \sum_{n=0}^{\infty} na_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^{n+1} = 0.$$

To even out the various powers of t in the sums above, we set $m = n-1$ in the first and fourth terms, $m = n-2$ in the second, and $m = n+1$ in the fifth, then change m back to n; thus,

$$\sum_{n=-1}^{\infty} (n+1)na_{n+1}t^n + 2\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty} na_n t^n - \sum_{n=-1}^{\infty} (n+1)a_{n+1}t^n + 2\sum_{n=1}^{\infty} a_{n-1}t^n = 0.$$

We notice that the terms with $n = -2$ and $n = -1$ vanish, and that the terms with $n = 0$ are $4a_2 - a_1$. Therefore, using only one summation symbol for $n \geq 1$, we can write

$$4a_2 - a_1 + \sum_{n=1}^{\infty}[2(n+2)(n+1)a_{n+2} + (n^2-1)a_{n+1} + na_n + 2a_{n-1}]t^n = 0,$$

from which

$$4a_2 - a_1 = 0,$$
$$2(n+2)(n+1)a_{n+2} + (n^2-1)a_{n+1} + na_n + 2a_{n-1} = 0, \quad n = 1, 2, \dots.$$

Consequently, $a_2 = a_1/4$ and

$$a_{n+2} = -\frac{1}{2(n+2)(n+1)}\,[(n^2-1)a_{n+1} + na_n + 2a_{n-1}], \quad n = 1, 2, \dots,$$

yielding the GS

$$y(t) = c_1\bigl(1 - \tfrac{1}{6}\,t^3 + \tfrac{1}{48}\,t^4 + \tfrac{1}{120}\,t^5 + \tfrac{1}{480}\,t^6 + \cdots\bigr) \\ + c_2\bigl(t + \tfrac{1}{4}\,t^2 - \tfrac{1}{12}\,t^3 - \tfrac{3}{32}\,t^4 + \tfrac{1}{80}\,t^5 + \cdots\bigr), \qquad (9.5)$$

where the constants c_1 and c_2 have replaced a_0 and a_1 in the series.

According to Theorem 9.8, the two series multiplied by c_1 and c_2 converge for at least all t satisfying $|t| < 2$.

VERIFICATION WITH MATHEMATICA®. The input

```
y1 = 1 - (1/6) * t^3 + (1/48) * t^4 + (1/120) * t^5 + (1/480) * t^6;
y2 = t + (1/4) * t^2 - (1/12) * t^3 - (3/32) * t^4 + (1/80) * t^5;
y = c1 * y1 + c2 * y2;
rem = (t+2) * D[y,t,t] + (t-1) * D[y,t] + 2 * t * y;
rem // Expand
```

generates the output $c_1(2t^5/15 + 7t^6/240 + t^7/240) + c_2(-17t^4/48 - t^5/8 + t^6/40)$. ■

9.13 Example. Consider the IVP

$$y'' - ty' + (t+1)y = 0, \quad y(0) = -1, \quad y'(0) = 3.$$

Clearly, $t = 0$ is an ordinary point for the DE, and the terminating series $Q(t)/P(t) = -t$ and $R(t)/P(t) = t + 1$ have an infinite radius of convergence. We follow the usual procedure and establish that

$$2a_2 + a_0 + \sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+2} - (n-1)a_n + a_{n-1}]t^n = 0,$$

from which, computing explicitly the first few terms, we construct the GS

$$y(t) = c_1\bigl(1 - \tfrac{1}{2}\,t^2 - \tfrac{1}{6}\,t^3 - \tfrac{1}{24}\,t^4 + \tfrac{1}{120}\,t^5 + \cdots\bigr) \\ + c_2\bigl(t - \tfrac{1}{12}\,t^4 - \tfrac{1}{120}\,t^6 + \tfrac{1}{504}\,t^7 - \tfrac{1}{1{,}344}\,t^8 + \cdots\bigr).$$

If we apply the ICs, we see that $c_1 = -1$ and $c_2 = 3$; hence, replacing these coefficients in the GS and taking care of the like terms, we find that the series solution of the IVP is

$$y(t) = -1 + 3t + \tfrac{1}{2}\,t^2 + \tfrac{1}{6}\,t^3 - \tfrac{5}{24}\,t^4 - \tfrac{1}{120}\,t^5 + \cdots.$$

This series converges for all real values of t.

Given the analyticity of the coefficients of the DE at $t = 0$, we could also construct this solution directly from the equation itself and the ICs. Thus, solving the equation for y'', we have

$$y'' = ty' - (t+1)y,$$

from which, by repeated differentiation, we obtain

$$\begin{aligned} y''' &= ty'' - ty' - y, \\ y^{(4)} &= ty''' - (t-1)y'' - 2y', \\ y^{(5)} &= ty^{(4)} - (t-2)y''' - 3y'', \\ &\vdots \end{aligned}$$

Hence, setting $t = 0$ in all these derivatives, we find that

$$y''(0) = 1, \quad y'''(0) = 1, \quad y^{(4)}(0) = -5, \quad y^{(5)}(0) = -1, \quad \dots .$$

Replaced in the Taylor series for y around $t = 0$, these values yield

$$\begin{aligned} y(t) &= y(0) + \frac{y'(0)}{1!}\,t + \frac{y''(0)}{2!}\,t^2 + \frac{y'''(0)}{3!}\,t^3 + \frac{y^{(4)}(0)}{4!}\,t^4 + \frac{y^{(5)}(0)}{5!}\,t^5 + \cdots \\ &= -1 + 3t + \tfrac{1}{2}\,t^2 + \tfrac{1}{6}\,t^3 - \tfrac{5}{24}\,t^4 - \tfrac{1}{120}\,t^5 + \cdots, \end{aligned}$$

which coincides with the solution constructed earlier.

Verification with Mathematica®. The input

```
y= - 1 + 3*t + (1/2)*t^2 + (1/6)*t^3 - (5/24)*t^4
        - (1/120)*t^5;
rem=D[y,t,t] - t*D[y,t] + (t + 1)*y;
{rem//Expand,{y,D[y,t]}/.t → 0}
```

generates the output $\{19t^4/24 - 7t^5/40 - t^6/120, \{-1, 3\}\}$. ■

9.14 Example. The point $t = 0$ is obviously an ordinary point for the DE

$$(t^2 + 1)y'' + ty' - 4y = 0,$$

where, since $t^2 + 1 \neq 0$ for all real t, the power series for $Q(t)/P(t) = t/(t^2 + 1)$ and $R(t)/P(t) = -4/(t^2 + 1)$ have an infinite radius of convergence (see Remark 9.5(ii)). This implies that the solution series to be constructed will converge at all points t.

Replacing (9.1) with $t_0 = 0$ in the equation and adjusting the summation index as necessary, we arrive at

$$2a_2 - 4a_0 + \sum_{n=1}^{\infty} (n + 2)[(n + 1)a_{n+2} + (n - 2)a_n]t^n = 0,$$

from which

$$\begin{aligned} a_2 &= 2a_0, \\ a_{n+2} &= -\frac{n - 2}{n + 1}\,a_n, \quad n = 1, 2, \dots . \end{aligned}$$

The computation of the first few coefficients now yields the general solution

$$y(t) = c_1(1 + 2t^2) + c_2\bigl(t + \tfrac{1}{2}\,t^3 - \tfrac{1}{8}\,t^5 + \tfrac{1}{16}\,t^7 - \tfrac{5}{128}\,t^9 + \cdots\bigr),$$

where the first term is a terminating series.

Verification with Mathematica®. The input

```
y1=1 + 2*t^2;
y2=t + (1/2)*t^3 - (1/8)*t^5 + (1/16)*t^7 - (5/128)*t^9;
y=c1*y1 + c2*y2;
rem= (t^2 + 1)*D[y,t,t] + t*D[y,t] - 4*y;
rem//Expand
```

generates the output $-385c_2t^9/128$. ■

Exercises

Construct the GS of the given DE from two linearly independent series solutions around $t = 0$. In each case, compute the first five nonzero terms of the corresponding series and indicate the interval on which the series are guaranteed to converge.

1 $y'' + y' - y = 0.$

2 $y'' + 2y' + 2y = 0.$

3 $y'' + 4y' + 3y = 0.$

4 $2y'' - y' - y = 0.$

5 $2y'' - y' + y = 0.$

6 $4y'' - 4y' + y = 0.$

7 $y'' + (1 - t)y' + 2y = 0.$

8 $y'' + 2y' + (2 - t)y = 0.$

9 $y'' + ty' + (2t + 1)y = 0.$

10 $y'' + (1 - t)y' + (t^2 - 1)y = 0.$

11 $(t + 1)y'' - 2y' + 2ty = 0.$

12 $(2t - 1)y'' + 2ty' - y = 0.$

13 $(t - 2)y'' + (t - 1)y' + y = 0.$

14 $(t + 2)y'' + (t - t^2)y' - 2y = 0.$

15 $(t^2 + 1)y'' - y' + (t - 1)y = 0.$

16 $(2t + 1)y'' + ty' + (1 - t)y = 0.$

17 $(t - 1)y'' + (t^2 + 1)y' + (t^2 + t)y = 0.$

18 $(1 - t^2)y'' + (2 - t)y' - t^2y = 0.$

19 $(t^2 + t + 1)y'' + (t^2 - t)y' + (t^2 - 2)y = 0.$

20 $(t^2 + t + 2)y'' + (t^2 - t + 1)y' + (2 + t - t^2)y = 0.$

Answers to Odd-Numbered Exercises

1 $y(t) = c_1\bigl(1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{40}t^5 + \cdots\bigr)$
$+ c_2\bigl(t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{8}t^4 + \frac{1}{24}t^5 - \cdots\bigr); \quad -\infty < t < \infty.$

3 $y(t) = c_1\bigl(1 - \frac{3}{2}t^2 + 2t^3 - \frac{13}{8}t^4 + t^5 - \cdots\bigr)$
$+ c_2\bigl(t - 2t^2 + \frac{13}{6}t^3 - \frac{5}{3}t^4 + \frac{121}{120}t^5 - \cdots\bigr); \quad -\infty < t < \infty.$

5 $y(t) = c_1\bigl(1 - \frac{1}{4}t^2 - \frac{1}{24}t^3 + \frac{1}{192}t^4 + \frac{1}{640}t^5 + \cdots\bigr)$
$+ c_2\bigl(t + \frac{1}{4}t^2 - \frac{1}{24}t^3 - \frac{1}{64}t^4 - \frac{1}{1{,}920}t^5 - \cdots\bigr); \quad -\infty < t < \infty.$

7 $y(t) = c_1\bigl(1 - t^2 + \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{30}t^5 - \cdots\bigr)$
$+ c_2\bigl(t - \frac{1}{2}t^2\bigr); \quad -\infty < t < \infty.$

9 $y(t) = c_1\bigl(1 - \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{8}t^4 + \frac{7}{60}t^5 + \cdots\bigr)$
$+ c_2\bigl(t - \frac{1}{3}t^3 - \frac{1}{6}t^4 + \frac{1}{15}t^5 + \frac{1}{20}t^6 + \cdots\bigr); \quad -\infty < t < \infty.$

11 $y(t) = c_1\big(1 - \frac{1}{3}\,t^3 + \frac{1}{45}\,t^6 - \frac{1}{105}\,t^7 + \frac{1}{210}\,t^8 - \cdots\big)$
$\qquad + c_2\big(t + t^2 + \frac{1}{3}\,t^3 - \frac{1}{6}\,t^4 - \frac{1}{15}\,t^5 + \cdots\big);\quad -1 < t < 1.$

13 $y(t) = c_1\big(1 + \frac{1}{4}\,t^2 + \frac{1}{32}\,t^4 + \frac{1}{160}\,t^5 + \frac{1}{240}\,t^6 + \cdots\big)$
$\qquad + c_2\big(t - \frac{1}{4}\,t^2 + \frac{1}{6}\,t^3 - \frac{1}{96}\,t^4 + \frac{7}{480}\,t^5 + \cdots\big);\quad -2 < t < 2.$

15 $y(t) = c_1\big(1 - \frac{1}{2}\,t^2 + \frac{1}{8}\,t^4 - \frac{13}{240}\,t^6 - \frac{1}{210}\,t^7 + \cdots\big)$
$\qquad + c_2\big(t + \frac{1}{2}\,t^2 - \frac{1}{24}\,t^4 + \frac{1}{60}\,t^5 + \frac{1}{48}\,t^6 + \cdots\big);\quad -\infty < t < \infty.$

17 $y(t) = c_1\big(1 + \frac{1}{6}\,t^3 + \frac{5}{24}\,t^4 + \frac{1}{6}\,t^5 + \frac{29}{180}\,t^6 + \cdots\big)$
$\qquad + c_2\big(t + \frac{1}{2}\,t^2 + \frac{1}{3}\,t^3 + \frac{5}{12}\,t^4 + \frac{11}{24}\,t^5 + \cdots\big);\quad -1 < t < 1.$

19 $y(t) = c_1\big(1 - t^2 + \frac{1}{3}\,t^3 + \frac{1}{12}\,t^4 - \frac{1}{30}\,t^5 + \cdots\big)$
$\qquad + c_2\big(t - \frac{1}{6}\,t^3 + \frac{11}{120}\,t^5 - \frac{2}{45}\,t^6 - \frac{47}{5{,}040}\,t^7 + \cdots\big);\quad -\infty < t < \infty.$

9.3 Singular Points

Singular points occur, as a rule, in DEs that model phenomena with a certain lack of smoothness in their geometric or physical makeup. Although their number is small in any specific case, they pose a significant challenge since solutions very often exhibit an unusual behavior in their vicinity, which requires special handling.

9.15 Definition. We say that t_0 is a *regular singular point* of equation (9.3) if $P(t_0) = 0$ and

$$\lim_{t\to t_0}(t - t_0)\,\frac{Q(t)}{P(t)} < \infty, \qquad \lim_{t\to t_0}(t - t_0)^2\,\frac{R(t)}{P(t)} < \infty. \tag{9.6}$$

A singular point that does not satisfy (9.6) is called an *irregular singular point.* ■

9.16 Remark. Employing an argument based on the concept of removable singularity, we may say that, generally speaking, conditions (9.6) are equivalent to stating that the functions $(t - t_0)Q(t)/P(t)$ and $(t - t_0)^2R(t)/P(t)$ are analytic at t_0. ■

9.17 Example. The singular points of the DE

$$(t^2 - 4t + 3)y'' + 8ty' + (5t - 1)y = 0$$

are the roots of the equation $P(t) = t^2 - 4t + 3 = 0$; that is, $t = 1$ and $t = 3$. Since $t^2 - 4t + 3 = (t - 1)(t - 3)$, at $t = 1$ conditions (9.6) yield

$$\lim_{t\to 1}(t - 1)\frac{8t}{(t - 1)(t - 3)} = \lim_{t\to 1}\frac{8t}{t - 3} = -4 < \infty,$$
$$\lim_{t\to 1}(t - 1)^2\frac{5t - 1}{(t - 1)(t - 3)} = \lim_{t\to 1}\frac{(t - 1)(5t - 1)}{t - 3} = 0 < \infty,$$

so $t = 1$ is a regular singular point.

A similar analysis shows that $t = 3$ is also a regular singular point. ■

9.18 Example. The singular points of the equation

$$t(t + 1)^2y'' + (t + 2)y' + 2ty = 0$$

are $t = 0$ and $t = -1$. For $t = 0$,

$$\lim_{t \to 0} t \frac{Q(t)}{P(t)} = \lim_{t \to 0} t \frac{t+2}{t(t+1)^2} = \lim_{t \to 0} \frac{t+2}{(t+1)^2} = 2 < \infty,$$
$$\lim_{t \to 0} t^2 \frac{R(t)}{P(t)} = \lim_{t \to 0} t^2 \frac{2t}{t(t+1)^2} = \lim_{t \to 0} \frac{2t^2}{(t+1)^2} = 0 < \infty;$$

hence, $t = 0$ is a regular singular point. However, in the case of $t = -1$,

$$(t+1) \frac{Q(t)}{P(t)} = (t+1) \frac{t+2}{t(t+1)^2} = \frac{t+2}{t(t+1)},$$

which becomes infinite as $t \to -1$. This means that $t = -1$ is an irregular singular point for the given DE. ■

9.19 Remarks. (i) The behavior of the solution of a DE in the neighborhood of an irregular singular point is difficult to analyze. In the rest of this chapter, we restrict our attention to the investigation of regular singular points.

(ii) Since frequently the solution is not analytic at a singular point, it cannot be sought as a power series of the form (9.1). ■

9.20 Example. It is easily checked that $t = 0$ is a regular singular point for the DE

$$t^2 y'' - 6y = 0.$$

This is a Cauchy–Euler equation (see Sect. 4.6) that has the linearly independent solutions $y_1(t) = t^3$ and $y_2(t) = t^{-2}$, and it is obvious that y_2 does not fit the form (9.1). ■

9.21 Remark. If $t = t_0$ is a singular point for (9.3), it can always be transferred to the origin by the simple change of variable $s = t - t_0$. Consequently, without loss of generality, in what follows we assume from the start that the singular point of interest is $t = 0$. ■

9.22 Example. The DE

$$(t-1)y'' + 2ty' - (1 - 2t^2)y = 0$$

has a singular point at $t = 1$. We set $s = t - 1$ and $y(t) = y(s+1) = x(s)$, and see that $y'(t) = x'(s)$ and $y''(t) = x''(s)$, so the equation is rewritten as

$$sx'' + 2(s+1)x' + (2s^2 + 4s + 1)x = 0,$$

which has a singular point at $s = 0$. ■

Exercises

In 1–4, find the singular points of the given DE, use a suitable substitution to transfer each such point to the origin, and write out the transformed equation.

1 $(t+2)y'' + ty' - 3y = 0$.

2 $(2t-1)y'' + y' - (t+2)y = 0$.

3 $(t^2-9)y''+(2t+1)y'+ty=0.$

4 $(3t^2-2t-1)y''+(1-t)y'+(2t+1)y=0.$

In 5–14, find the singular points of the given DE and check whether they are regular or irregular.

5 $ty''+(t^2+1)y'+(t-2)y=0.$

6 $(t-1)y''+(3t+2)y'+2y=0.$

7 $(2t-1)^3y''+(2t-1)^2y'+ty=0.$

8 $t^2y''+4y'+(t+1)y=0.$

9 $(t^3+2t^2)y''+2y'-(t+4)y=0.$

10 $(t^2-t)y''+(t+2)y'-y=0.$

11 $(t^2-1)^2y''-(t^2-t-2)y'+(1-2t)y=0.$

12 $t^2(2t+1)^3y''+(2t+1)^2y'+4y=0.$

13 $t^2y''+(e^t-1)y'+2y=0.$

14 $t^3y''-2t^2y'+y\sin t=0.$

Answers to Odd-Numbered Exercises

1 $t=-2;\quad s=t+2;\quad sx''+(s-2)x'-3x=0.$

3 $t_1=3;\quad s=t-3;\quad (s^2+6s)x''+(2s+7)x'+(s+3)x=0;$
$t_2=-3;\quad s=t+3;\quad (s^2-6s)x''+(2s-5)x'+(s-3)x=0.$

5 $t=0$: regular singular point.

7 $t=1/2$: irregular singular point.

9 $t=0$: irregular singular point; $t=-2$: regular singular point.

11 $t=1$: irregular singular point; $t=-1$: regular singular point.

13 $t=0$: regular singular point.

9.4 Solution Near a Regular Singular Point

Assuming, as mentioned in Remark 9.21, that the singular point of interest is $t=0$, we multiply equation (9.3) by $t^2/P(t)$ and bring it to the form

$$t^2y''+t^2\,\frac{Q(t)}{P(t)}\,y'+t^2\,\frac{R(t)}{P(t)}\,y=0. \tag{9.7}$$

By Definition 9.15 and Remark 9.16, the functions $tQ(t)/P(t)$ and $t^2R(t)/P(t)$ are analytic at $t=0$; that is, there are power series such that

$$t\,\frac{Q(t)}{P(t)}=\alpha_0+\alpha_1t+\cdots\,,\quad t^2\,\frac{R(t)}{P(t)}=\beta_0+\beta_1t+\cdots\,,$$

which, replaced in (9.7), lead to

$$t^2y'' + t(\alpha_0 + \alpha_1 t + \cdots)y' + (\beta_0 + \beta_1 t + \cdots)y = 0.$$

The coefficients in the above equation are of the Cauchy–Euler type multiplied by power series, so it seems natural that we should try a solution of the Cauchy–Euler form multiplied by a power series; that is,

$$y = t^r(a_0 + a_1 t + \cdots) = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}, \quad a_0 \neq 0. \tag{9.8}$$

As illustrated by the examples below, when we replace this series in the DE and equate the coefficient of the lowest power of t on the left-hand side to zero, we arrive at a quadratic equation in r, called the *indicial equation*. Its roots r_1 and r_2 are the so-called *exponents at the singularity*.

9.23 Remarks. (i) The technique that we just described is known as the *method of Frobenius*, and the solutions generated with it are called *Frobenius solutions*.

(ii) Although the case of complex conjugate roots can be handled more or less in the same way as that of real roots, we will not consider it here.

(iii) Without loss of generality, we construct solutions defined for $t > 0$. A problem where $t < 0$ can be reduced to this case by means of the substitution $s = -t$.

(iv) In what follows, we accept without formal checking that $t = 0$ is a regular singular point for the DEs under consideration. ∎

The solutions are constructed differently when r_1 and r_2 are distinct and do not differ by an integer, when $r_1 = r_2$, and when r_1 and r_2 differ by a nonzero integer. We examine each of these situations separately.

9.4.1 Distinct Roots That Do Not Differ by an Integer

For definiteness, we assume that $r_1 > r_2$.

9.24 Theorem. *Suppose that $t = 0$ is a regular singular point for equation (9.3), and let $\rho > 0$ be the minimum of the radii of convergence of the power series for the analytic functions $tQ(t)/P(t)$ and $t^2R(t)/P(t)$ around $t = 0$. If $r_1 - r_2$ is not a positive integer, then equation (9.3) has, for $t > 0$, two linearly independent Frobenius solutions of the form (9.8) with $r = r_1$ and $r = r_2$, respectively. The radii of convergence of the power series involved in the construction of these two solutions are at least ρ, and their coefficients are determined by direct substitution of (9.8) in equation (9.3).* ∎

9.25 Example. Consider the equation

$$2t^2y'' + (t^2 - t)y' + y = 0,$$

where, for $t \neq 0$,

$$t\frac{Q(t)}{P(t)} = \frac{t(t^2 - t)}{2t^2} = \frac{1}{2}(t - 1), \quad t^2\frac{R(t)}{P(t)} = \frac{t^2}{2t^2} = \frac{1}{2}$$

are polynomials. Hence, the power series occurring in our construction will converge for all real values of t.

Replacing y by (9.8) and performing all necessary differentiation, we arrive at

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n t^{n+r+1} - \sum_{n=0}^{\infty}(n+r)a_n t^{n+r} + \sum_{n=0}^{\infty} a_n t^{n+r} = 0.$$

To make the powers of t in the various summands above equal to $n+r$, we substitute $m = n+1$ in the second term and then change m back to n; thus,

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1} t^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n t^{n+r} + \sum_{n=0}^{\infty} a_n t^{n+r} = 0.$$

We now write out separately the term for $n = 0$ and, after some simple algebra, group the rest of the terms into one sum for $n \geq 1$:

$$(2r^2 - 3r + 1)a_0 t^r + \sum_{n=1}^{\infty}(n+r-1)[(2n+2r-1)a_n + a_{n-1}]t^{n+r} = 0.$$

Given that $a_0 \neq 0$, from this we conclude that

$$2r^2 - 3r + 1 = 0,$$
$$(n+r-1)[(2n+2r-1)a_n + a_{n-1}] = 0, \quad n = 1, 2, \ldots.$$

The top equality is the indicial equation, with roots (exponents at the singularity) $r_1 = 1$ and $r_2 = 1/2$. Since $n + r - 1 \neq 0$ for $n \geq 1$ and either of these two values of r, the bottom equality yields the recurrence relation

$$a_n = -\frac{1}{2n+2r-1}\, a_{n-1}, \quad n = 1, 2, \ldots.$$

For $r = 1$, this reduces to

$$a_n = -\frac{1}{2n+1}\, a_{n-1}, \quad n = 1, 2, \ldots,$$

and for $r = 1/2$,

$$a_n = -\frac{1}{2n}\, a_{n-1}, \quad n = 1, 2, \ldots.$$

Computing the first few coefficients a_n in each case and taking, for simplicity, $a_0 = 1$ in both, we obtain the Frobenius solutions

$$y_1(t) = t\big(1 - \tfrac{1}{3}\,t + \tfrac{1}{15}\,t^2 - \tfrac{1}{105}\,t^3 + \tfrac{1}{945}\,t^4 - \cdots\big),$$
$$y_2(t) = t^{1/2}\big(1 - \tfrac{1}{2}\,t + \tfrac{1}{8}\,t^2 - \tfrac{1}{48}\,t^3 + \tfrac{1}{384}\,t^4 - \cdots\big).$$

We see that y_1 and y_2 are linearly independent, so the GS of the given equation is $y = c_1y_1 + c_2y_2$.

VERIFICATION WITH MATHEMATICA®. The input

```
y1=t*(1 - (1/3)*t + (1/15)*t^2 - (1/105)*t^3 + (1/945)*t^4);
y2=t^(1/2)*(1 - (1/2)*t + (1/8)*t^2 - (1/48)*t^3
        + (1/384)*t^4);
y=c1*y1 + c2*y2;
rem=2*t^2*D[y,t,t] + (t^2 - t)*D[y,t] + y;
rem//Expand
```

generates the output $c_1t^6/189 + 3c_2t^{11/2}/256$. ■

9.26 Example. For the DE

$$3ty'' + 5y' + (t-5)y = 0$$

we have $tQ(t)/P(t) = 5/3$ and $t^2R(t)/P(t) = (t^2 - 5t)/3$, which is a trivial case of power series with an infinite radius of convergence.

The direct replacement of (9.8) in the equation leads to

$$3\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r-1} + 5\sum_{n=0}^{\infty}(n+r)a_nt^{n+r-1} + \sum_{n=0}^{\infty}a_nt^{n+r+1} - 5\sum_{n=0}^{\infty}a_nt^{n+r} = 0.$$

We make the powers of t equal to $n+r$ in all the summands by setting $m = n-1$ in the first and second terms and $m = n+1$ in the third one, and then switching from m to n. Adjusting the starting value of n accordingly, we arrive at

$$3\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}t^{n+r} + 5\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}t^{n+r} + \sum_{n=1}^{\infty}a_{n-1}t^{n+r} - 5\sum_{n=0}^{\infty}a_nt^{n+r} = 0.$$

Here, we need to write separately the terms for $n = -1$ and $n = 0$, combining the rest of them for $n \geq 1$; thus, after some algebraic simplification, we get

$$(3r^2 + 2r)a_0t^{-1+r} + [(r+1)(3r+5)a_1 - 5a_0]t^r + \sum_{n=1}^{\infty}[(n+r+1)(3n+3r+5)a_{n+1} - 5a_n + a_{n-1}]t^{n+r} = 0,$$

from which (since $a_0 \neq 0$)

$$3r^2 + 2r = 0,$$
$$(r+1)(3r+5)a_1 - 5a_0 = 0,$$
$$(n+r+1)(3n+3r+5)a_{n+1} - 5a_n + a_{n-1} = 0, \quad n = 1, 2, \dots.$$

The roots of the indicial equation (the top equality) are $r_1 = 0$ and $r_2 = -2/3$; the other two equalities yield, respectively,

$$a_1 = \frac{5}{(r+1)(3r+5)}\,a_0,$$
$$a_{n+1} = \frac{1}{(n+r+1)(3n+3r+5)}\,(5a_n - a_{n-1}), \quad n = 1, 2, \dots.$$

For $r=0$, these formulas become

$$a_1 = a_0,$$
$$a_{n+1} = \frac{1}{(n+1)(3n+5)}\,(5a_n - a_{n-1}), \quad n = 1, 2, \dots,$$

and for $r=-2/3$,

$$a_1 = 5a_0,$$
$$a_{n+1} = \frac{1}{(n+1)(3n+1)}\,(5a_n - a_{n-1}), \quad n = 1, 2, \dots.$$

Computing the first few terms (with $a_0 = 1$, as usual) in each case, we obtain the GS

$$y(t) = c_1\big(1 + t + \tfrac{1}{4}\,t^2 + \tfrac{1}{132}\,t^3 - \tfrac{1}{264}\,t^4 + \cdots\big) + c_2t^{-2/3}\big(1 + 5t + 3t^2 + \tfrac{10}{21}\,t^3 - \tfrac{13}{840}\,t^4 + \cdots\big).$$

Both power series above converge for all real values of t.

Verification with Mathematica®. The input

```
y1=1+t+(1/4)*t^2 + (1/132)*t^3 - (1/264)*t^4;
y2=t^(-2/3)*(1 + 5*t + 3*t^2 + (10/21)*t^3 - (13/840)*t^4);
y=c1*y1 + c2*y2;
rem=3*t*D[y,t,t] + 5*D[y,t] + (t - 5)*y;
rem//Expand
```

generates the output $c_1(7t^4/264 - t^5/264) + c_2(31t^{10/3}/56 - 13t^{13/3}/840)$. ■

Exercises

Find the singularity exponents of the given DE at the regular singular point $t=0$ and, in each case, use (9.8) to construct two (linearly independent) Frobenius solutions for $t>0$. Compute the first five nonzero terms in each of these solutions and determine the minimal interval of convergence for the power series occurring in them.

1 $2t^2y'' + (2t^2 + t)y' + (3t - 1)y = 0.$ **2** $3t^2y'' - ty + (t^2 + 5t + 1)y = 0.$

3 $2ty'' + y' + (t - 2)y = 0.$ **4** $3ty'' + y' + (t + 1)y = 0.$

5 $3t^2y'' + (t^2 + 5t)y' - (t + 1)y = 0.$

6 $2t^2y'' + (2t^2 - t)y' + (3t - 2)y = 0.$ **7** $2ty'' + (t + 3)y' + ty = 0.$

8 $4ty'' + (3 - 2t)y' + (t - 1)y = 0.$ **9** $2t^2y'' - 3ty' + (2t^2 + t + 3)y = 0.$

10 $6t^2y'' + (2t^2 + 5t)y' + (3t - 2)y = 0.$

11 $(2t - t^2)y'' + (3t + 5)y' + (t + 5)y = 0.$

12 $3ty'' + (2 - t)y' + (1 - t^2)y = 0.$ **13** $2t^2y'' - 3ty' + (t + 2)y = 0.$

14 $2t^2y'' + (t^2 + 3t)y' + (2t - 1)y = 0.$ **15** $2ty'' + (t^2 + 1)y' + (t^2 - 1)y = 0.$

16 $3(t^2 + t)y'' + (t^2 - t + 4)y' + (t^2 + 4)y = 0.$

Answers to Odd-Numbered Exercises

1 $r_1 = 1,\ \ r_2 = -\frac{1}{2};\ \ y_1(t) = t\big(1 - t + \frac{1}{2}\,t^2 - \frac{1}{6}\,t^3 + \frac{1}{24}\,t^4 - \cdots\big);$
$y_2(t) = t^{-1/2}\big(1 + 2t - 4t^2 + \frac{8}{3}\,t^3 - \frac{16}{15}\,t^4 + \cdots\big);\ \ -\infty < t < \infty.$

3 $r_1 = \frac{1}{2},\ \ r_2 = 0;\ \ y_1(t) = t^{1/2}\big(1 + \frac{2}{3}\,t + \frac{1}{30}\,t^2 - \frac{1}{35}\,t^3 - \frac{19}{7{,}560}\,t^4 + \cdots\big);$
$y_2(t) = 1 + 2t + \frac{1}{2}\,t^2 - \frac{1}{15}\,t^3 - \frac{19}{840}\,t^4 + \cdots;\ \ -\infty < t < \infty.$

5 $r_1 = \frac{1}{3},\ \ r_2 = -1;\ \ y_1(t) = t^{1/3}\big(1 + \frac{2}{21}\,t - \frac{1}{630}\,t^2 + \frac{2}{36{,}855}\,t^3 - \frac{1}{505{,}440}\,t^4 + \cdots\big);$
$y_2(t) = t^{-1} - 2 - \frac{1}{2}\,t;\ \ -\infty < t < \infty.$

7 $r_1 = 0,\ \ r_2 = -\frac{1}{2};\ \ y_1(t) = 1 - \frac{1}{10}\,t^2 + \frac{1}{105}\,t^3 + \frac{1}{504}\,t^4 - \frac{1}{3{,}150}\,t^5 + \cdots;$
$y_2(t) = t^{-1/2}\big(1 + \frac{1}{2}\,t - \frac{1}{24}\,t^2 - \frac{1}{80}\,t^3 + \frac{23}{2{,}688}\,t^4 + \cdots\big);\ \ -\infty < t < \infty.$

9 $r_1 = \frac{3}{2},\ \ r_2 = 1;\ \ y_1(t) = t^{3/2}\big(1 - \frac{1}{3}\,t - \frac{1}{6}\,t^2 + \frac{5}{126}\,t^3 + \frac{37}{4{,}536}\,t^4 + \cdots\big);$
$y_2(t) = t\big(1 - t - \frac{1}{6}\,t^2 + \frac{13}{90}\,t^3 + \frac{17}{2{,}520}\,t^4 + \cdots\big);\ \ -\infty < t < \infty.$

11 $r_1 = 0,\ \ r_2 = -\frac{3}{2};\ \ y_1(t) = 1 - t + \frac{1}{2}\,t^2 - \frac{7}{54}\,t^3 + \frac{29}{2{,}376}\,t^4 + \cdots;$
$y_2(t) = t^{-3/2}\big(1 - \frac{13}{4}\,t + \frac{127}{32}\,t^2 - \frac{3{,}013}{1{,}152}\,t^3 + \frac{87{,}167}{92{,}160}\,t^4 - \cdots\big);\ \ -2 < t < 2.$

13 $r_1 = 2,\ \ r_2 = \frac{1}{2};\ \ y_1(t) = t^2\big(1 - \frac{1}{5}\,t + \frac{1}{70}\,t^2 - \frac{1}{1{,}890}\,t^3 + \frac{1}{83{,}160}\,t^4 - \cdots\big);$
$y_2(t) = t^{1/2}\big(1 + t - \frac{1}{2}\,t^2 + \frac{1}{18}\,t^3 - \frac{1}{360}\,t^4 + \cdots\big);\ \ -\infty < t < \infty.$

15 $r_1 = \frac{1}{2},\ \ r_2 = 0;\ \ y_1(t) = t^{1/2}\big(1 + \frac{1}{3}\,t - \frac{1}{60}\,t^2 - \frac{13}{180}\,t^3 - \frac{131}{12{,}960}\,t^4 + \cdots\big);$
$y_2(t) = 1 + t + \frac{1}{6}\,t^2 - \frac{11}{90}\,t^3 - \frac{131}{2{,}520}\,t^4 + \cdots;\ \ -\infty < t < \infty.$

9.4.2 Equal Roots

Let $r_1 = r_2 = r_0$.

9.27 Theorem. *Suppose that $t = 0$ is a regular singular point for equation (9.3), and let $\rho > 0$ be the minimum of the radii of convergence of the power series for the analytic functions $tQ(t)/P(t)$ and $t^2R(t)/P(t)$ around $t = 0$. Then equation (9.3) has, for $t > 0$, a Frobenius solution y_1 of the form (9.8) with $r = r_0$, and a second solution y_2 of the form*

$$y_2(t) = y_1(t)\ln t + t^{r_0}\sum_{n=1}^{\infty} b_n t^n. \tag{9.9}$$

The radii of convergence of the power series involved in the construction of y_1 and y_2 are at least ρ, and their coefficients are determined, respectively, by direct substitution of (9.8) and (9.9) in equation (9.3). ■

9.28 Example. Consider the DE

$$2ty'' + (2 - t)y' + y = 0.$$

Replacing (9.8) in it, we arrive at

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n t^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1} + 2\sum_{n=0} a_n t^{n+r+1} = 0,$$

which, with the substitutions $m = n-1$ in the first and third terms and $m = n+1$ in the fourth one, changes to

$$2r^2 a_0 t^{-1+r} + \sum_{n=0}^{\infty}[2(n+r+1)^2 a_{n+1} - (n+r-1)a_n]t^{n+r} = 0;$$

hence,

$$r^2 a_0 = 0,$$
$$2(n+r+1)^2 a_{n+1} - (n+r-1)a_n = 0, \quad n = 0, 1, \dots .$$

Obviously, the roots of the indicial equation are $r_1 = r_2 = 0$. For $r = 0$, the second equality above yields the recurrence relation

$$a_{n+1} = \frac{n-1}{2(n+1)^2}\, a_n, \quad n = 0, 1, \dots ,$$

and (taking $a_0 = 1$) we obtain the terminating series solution

$$y_1(t) = 1 - \tfrac{1}{2}\, t.$$

To construct a second solution, we replace (9.9) in the DE and, after tidying up the left-hand side, find that

$$[2ty_1'' + (2-t)y_1' + y_1]\ln t + 4y_1' - y_1 + \sum_{n=0}^{\infty} 2n^2 b_n t^{n-1} - \sum_{n=0}^{\infty}(n-1)b_n t^n = 0.$$

Since y_1 is a solution of the given DE, it follows that the coefficient of $\ln t$ vanishes. Using the explicit expression of y_1 and adjusting the power of t in the first summand by writing $m = n-1$, we now bring the above equality to the form

$$-3 + \tfrac{1}{2}\, t + 2b_1 + \sum_{n=1}^{\infty}[2(n+1)^2 b_{n+1} - (n-1)b_n]t^n = 0,$$

which leads to

$$2b_1 - 3 = 0,$$
$$8b_2 + \tfrac{1}{2} = 0,$$
$$b_{n+1} = \frac{n-1}{2(n+1)^2}\, b_n, \quad n = 2, 3, \dots .$$

Then

$$b_1 = \tfrac{3}{2}, \quad b_2 = -\tfrac{1}{16}, \quad b_3 = -\tfrac{1}{288}, \quad b_4 = -\tfrac{1}{4{,}608}, \quad \dots ,$$

so

$$y_2(t) = y_1(t)\ln t + \tfrac{3}{2}\,t - \tfrac{1}{16}\,t^2 - \tfrac{1}{288}\,t^3 - \tfrac{1}{4{,}608}\,t^4 - \cdots .$$

It is easy to see that y_1 and y_2 are linearly independent; hence, the GS of our DE may be expressed as $y = c_1y_1 + c_2y_2$. The power series occurring in both y_1 and y_2 converge for all real values of t.

Verification with Mathematica®. The input

```
y1 = 1 - (1/2) * t;
y2 = y1 * Log[t] + (3/2) * t - (1/16) * t^2 - (1/288) * t^3
        - (1/4608) * t^4;
y=c1 * y1 + c2 * y2;
rem = 2 * t * D[y,t,t] + (2 - t) * D[y,t] + y;
rem // Expand
```

generates the output $c_2t^4/1{,}536$. ■

9.29 Example. Starting with (9.8) and applying the same procedure to the DE

$$t^2y'' + (t^2 + 5t)y' + (3t + 4)y = 0,$$

after the customary manipulation of the terms and powers of t we arrive at the equality

$$(r+2)^2a_0t^r + [(r+3)^2a_1 + 3a_0]t^{1+r} + \sum_{n=2}^{\infty}(n+r+2)[(n+r+2)a_n + a_{n-1}]t^{n+r} = 0,$$

from which

$$\begin{gathered}(r+2)^2a_0 = 0,\\ (r+3)^2a_1 + 3a_0 = 0,\\ (n+r+2)[(n+r+2)a_n + a_{n-1}] = 0, \quad n = 2, 3, \dots .\end{gathered}$$

The roots of the indicial equation are $r_1 = r_2 = -2$, and for $r = -2$ we use the second and third equalities above with $a_0 = 1$ to produce the Frobenius solution

$$y_1(t) = t^{-2}\big(1 - t + \tfrac{1}{2}\,t^2 - \tfrac{1}{6}\,t^3 + \tfrac{1}{24}\,t^4 + \cdots\big).$$

Continuing the procedure with (9.9) and performing the necessary computation, we find that

$$2ty_1' + ty_1 + 4y_1 + b_1t^{-1} + \sum_{n=2}^{\infty} n(nb_n + b_{n-1})t^{n-2} = 0.$$

After y_1 is replaced by its expression, this leads to

$$(b_1 - 1)t^{-1} + 4b_2 + 2b_1 + 1 + \big(9b_3 + 3b_2 - \tfrac{1}{2}\big)t + \big(16b_4 + 4b_3 + \tfrac{1}{6}\big)t^2 + \cdots = 0$$

and, hence, to the coefficients

$$b_1 = 1, \quad b_2 = -\tfrac{3}{4}, \quad b_3 = \tfrac{11}{36}, \quad b_4 = -\tfrac{25}{288}, \quad \dots,$$

which give the solution

$$y_2(t) = y_1(t)\ln t + t^{-2}\big(t - \tfrac{3}{4}\,t^2 + \tfrac{11}{36}\,t^3 - \tfrac{25}{288}\,t^4 + \cdots\big).$$

In view of the obvious linear independence of y_1 and y_2, the GS of the equation is $y = c_1 y_1 + c_2 y_2$, with the power series occurring in both y_1 and y_2 convergent for all real values of t.

VERIFICATION WITH MATHEMATICA®. The input

```
y1=t^(-2)*(1-t+(1/2)*t^2-(1/6)*t^3+(1/24)*t^4);
y2=y1*Log[t]+t^(-1)*(1-(3/4)*t+(11/36)*t^2
      -(25/288)*t^3);
y=c1*y1+c2*y2;
rem=t^2*D[y,t,t]+(t^2+5*t)*D[y,t]+(3*t+4)*y;
rem//Expand
```

generates the output $5c_1 t^3/24 + c_2[5(t^3 \ln t)/24 - 113t^3/288]$. ■

9.30 Example. The usual treatment based on (9.8) and applied to the DE

$$t^2 y'' + ty' + t^2 y = 0 \tag{9.10}$$

produces the equality

$$r^2 a_0 t^r + (r+1)^2 a_1 t^{1+r} + \sum_{n=2}^{\infty} [(n+r)^2 a_n + a_{n-2}] t^{n+r} = 0,$$

so

$$\begin{aligned} r^2 a_0 &= 0, \\ (r+1)^2 a_1 &= 0, \\ (n+r)^2 a_n + a_{n-2} &= 0, \quad n = 2, 3, \dots . \end{aligned}$$

This means that the roots of the indicial equation are $r_1 = r_2 = 0$. For $r = 0$, the above coefficient relationships (with $a_0 = 1$) yield the Frobenius solution

$$y_1(t) = 1 - \tfrac{1}{4} t^2 + \tfrac{1}{64} t^4 - \tfrac{1}{2{,}304} t^6 + \cdots .$$

A second solution y_2 is now constructed from (9.9). Carrying out the necessary differentiation and algebra, we get

$$2ty_1' + \sum_{n=1}^{\infty} n^2 b_n t^n + \sum_{n=3}^{\infty} b_{n-2} t^n = 0,$$

or, with y_1 replaced by its computed expansion,

$$\begin{aligned} b_1 t + (4b_2 - 1)t^2 + (9b_3 + b_1)t^3 + \big(16b_4 + b_2 + \tfrac{1}{8}\big)t^4 & \\ + (25b_5 + b_3)t^5 + \big(36b_6 + b_4 - \tfrac{1}{192}\big)t^6 + \cdots &= 0. \end{aligned}$$

We then have

$$b_1 = 0, \quad b_2 = \tfrac{1}{4}, \quad b_3 = 0, \quad b_4 = -\tfrac{3}{128}, \quad b_5 = 0, \quad b_6 = \tfrac{11}{13{,}824}, \quad \dots,$$

so

$$y_2(t) = y_1(t) \ln t + \tfrac{1}{4} t^2 - \tfrac{3}{128} t^4 + \tfrac{11}{13{,}824} t^6 - \cdots .$$

Since y_1 and y_2 are linearly independent, we may write the GS of the equation as $y = c_1 y_1 + c_2 y_2$. The power series occurring in y_1 and y_2 converge for all real t.

VERIFICATION WITH MATHEMATICA®. The input

```
y1 = 1 - (1/4) * t^2 + (1/64) * t^4 - (1/2304) * t^6;
y2 = y1 * Log[t] + (1/4) * t^2 - (3/128) * t^4 + (11/13824) * t^6;
y = c1 * y1 + c2 * y2;
rem = t^2 * D[y,t,t] + t * D[y,t] + t^2 * y;
rem // Expand
```

generates the output $-c_1 t^8/2{,}304 + c_2[11t^8/13{,}824 - (t^8 \ln t)/2{,}304]$. ■

9.31 Remarks. (i) The DE in the preceding example is known as *Bessel's equation of order zero*, and its solution y_1 is called *Bessel's function of the first kind and order zero*. If the coefficients a_n computed from the recurrence relation with $r = 0$ are expressed in terms of their numerical factors and we use x as the independent variable instead of t, then this function, denoted in the literature by the symbol J_0, can be written in the form

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2}\, x^{2n}.$$

There is also a *Bessel function of the second kind and order zero*, denoted by Y_0, which is defined in terms of J_0 and y_2 by

$$Y_0(x) = \frac{2}{\pi}\,[(\gamma - \ln 2)J_0(x) + y_2(x)],$$

where

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \ln n \approx 0.5772 \tag{9.11}$$

is Euler's constant. Since J_0 and Y_0 are linearly independent solutions of Bessel's equation of order zero, the GS of the latter may also be written as $y = c_1 J_0 + c_2 Y_0$.

(ii) If we divide Bessel's equation of order zero (with x as its variable) by x^2, we bring it to the alternative form

$$y'' + \frac{1}{x}\, y' + y = 0.$$

This shows that, as $x \to \infty$, the influence of the middle term diminishes, so the solutions of the given DE become asymptotically closer to those of the equation $y'' + y = 0$, therefore exhibiting an oscillatory behavior. The graphs of J_0 (lighter curve) and Y_0 (heavier curve) are shown in Fig. 9.1. ■

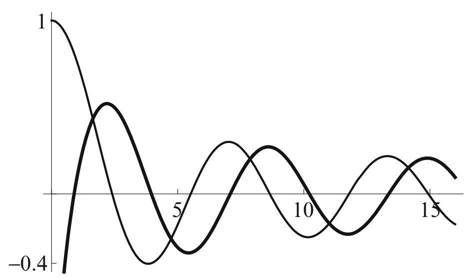

Fig. 9.1

Exercises

Find the exponents at the singularity for the given DE at the regular singular point $t = 0$ and, in each case, use (9.8) and (9.9) to construct two (linearly independent) series solutions for $t > 0$. Compute the first five nonzero terms in each of the power series occurring in these solutions and determine the minimal interval of convergence for these power series.

1 $ty'' + (t+1)y' + 2ty = 0.$ **2** $t^2y'' + 3ty' + (1-t)y = 0.$

3 $t^2y'' - ty' + (t^2 - t + 1)y = 0.$ **4** $ty'' + (1-2t)y' + (1-t)y = 0.$

5 $ty'' + (1-t^2)y' - y = 0.$ **6** $4t^2y'' + 2t^2y' + (3t+1)y = 0.$

7 $t^2y'' + 3(t^2+t)y' + (2t+1)y = 0.$ **8** $(t^2+t)y'' + y' + 2y = 0.$

9 $(t^2-t)y'' + (t^2-1)y' - (t+2)y = 0.$ **10** $4t^2y'' + 4(t^2+2t)y' + (1-2t)y = 0.$

Answers to Odd-Numbered Exercises

1 $r_1 = r_2 = 0;\quad y_1(t) = 1 - \frac{1}{2}\,t^2 + \frac{1}{9}\,t^3 + \frac{1}{24}\,t^4 - \frac{7}{450}\,t^5 + \cdots;$
$y_2(t) = y_1(t)\ln t + \big(-t + \frac{3}{4}\,t^2 + \frac{1}{27}\,t^3 - \frac{37}{288}\,t^4 + \frac{299}{13{,}500}\,t^5 + \cdots\big);\quad -\infty < t < \infty.$

3 $r_1 = r_2 = 1;\quad y_1(t) = t\big(1 + t - \frac{1}{9}\,t^3 - \frac{1}{144}\,t^4 + \frac{1}{240}\,t^5 + \cdots\big);$
$y_2(t) = y_1(t)\ln t + t\big(-2t - \frac{1}{2}\,t^2 + \frac{13}{54}\,t^3 + \frac{43}{864}\,t^4 - \frac{67}{7{,}200}\,t^5 + \cdots\big);\quad -\infty < t < \infty.$

5 $r_1 = r_2 = 0;\quad y_1(t) = 1 + t + \frac{1}{4}\,t^2 + \frac{5}{36}\,t^3 + \frac{23}{576}\,t^4 + \cdots;$
$y_2(t) = y_1(t)\ln t + \big(-2t - \frac{1}{2}\,t^2 - \frac{7}{27}\,t^3 - \frac{287}{3{,}456}\,t^4 - \frac{15{,}631}{432{,}000}\,t^5 + \cdots\big);\quad -\infty < t < \infty.$

7 $r_1 = r_2 = -1;\quad y_1(t) = t^{-1}\big(1 + t - \frac{1}{2}\,t^2 + \frac{5}{18}\,t^3 - \frac{5}{36}\,t^4 + \cdots\big);$
$y_2(t) = y_1(t)\ln t + t^{-1}\big(-5t + \frac{9}{4}\,t^2 - \frac{137}{108}\,t^3 + \frac{563}{864}\,t^4 - \frac{6{,}361}{21{,}600}\,t^5 + \cdots\big);\quad -\infty < t < \infty.$

9 $r_1 = r_2 = 0;\quad y_1(t) = 1 - 2t + \frac{3}{4}\,t^2 + \frac{3}{64}\,t^4 + \frac{3}{160}\,t^5 + \cdots;$
$y_2(t) = y_1(t)\ln t + \big(3t - \frac{5}{2}\,t^2 + \frac{1}{36}\,t^3 - \frac{145}{1{,}152}\,t^4 - \frac{17}{400}\,t^5 + \cdots\big);\quad -1 < t < 1.$

9.4.3 Distinct Roots Differing by an Integer

Let $r_1 = r_2 + n_0$, where n_0 is a positive integer.

9.32 Theorem. *Suppose that $t = 0$ is a regular singular point for equation (9.3), and let $\rho > 0$ be the minimum of the radii of convergence of the power series for the analytic functions $tQ(t)/P(t)$ and $t^2R(t)/P(t)$ around $t = 0$. Then equation (9.3) has, for $t > 0$, a Frobenius solution y_1 of the form (9.8) with $r = r_1$, and a second solution y_2 of the form*

$$y_2(t) = cy_1(t)\ln t + t^{r_2}\left(1 + \sum_{n=1}^{\infty} b_n t^n\right), \qquad c = \text{const.} \tag{9.12}$$

The radii of convergence of the power series involved in the construction of y_1 and y_2 are at least ρ, and their coefficients are determined, respectively, by direct substitution of (9.8) and (9.12) in equation (9.3). ∎

9.33 Remark. The constant c in (9.12) may turn out to be zero. In this case, equation (9.3) has two linearly independent Frobenius solutions. ■

9.34 Example. Replacing series (9.8) in the DE

$$t^2y'' + t^2y' - (t+2)y = 0,$$

we arrive at the equality

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r} + \sum_{n=0}^{\infty}(n+r)a_nt^{n+r+1} - \sum_{n=0}^{\infty}a_nt^{n+r+1} - 2\sum_{n=0}^{\infty}a_nt^{n+r} = 0.$$

The substitution $m = n+1$ in the second and third terms (followed by a change from m back to n) leads to

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1}t^{n+r} - \sum_{n=1}^{\infty}a_{n-1}t^{n+r} - 2\sum_{n=0}^{\infty}a_nt^{n+r} = 0,$$

or, equivalently,

$$(r^2-r-2)a_0t^r + \sum_{n=1}^{\infty}\{[(n+r)(n+r-1)-2]a_n + (n+r-2)a_{n-1}\}t^{n+r} = 0.$$

After simplifying the coefficients, we equate them to zero and find that

$$(r^2-r-2)a_0 = 0, \tag{9.13}$$

$$(n+r-2)[(n+r+1)a_n + a_{n-1}] = 0, \quad n = 1, 2, \dots. \tag{9.14}$$

As expected, (9.13) yields the indicial equation $r^2 - r - 2 = 0$, with roots $r_1 = 2$ and $r_2 = -1$. For $r = 2$, we have $n + r - 2 = n \neq 0$ for all $n \geq 1$, so, from (9.14),

$$a_n = -\frac{1}{n+3}\,a_{n-1}, \quad n = 1, 2, \dots.$$

Taking $a_0 = 1$, we use this formula to compute the coefficients a_n, which, replaced in (9.8), give rise to the Frobenius solution

$$y_1(t) = t^2\bigl(1 - \tfrac{1}{4}\,t + \tfrac{1}{20}\,t^2 - \tfrac{1}{120}\,t^3 + \tfrac{1}{840}\,t^4 + \cdots\bigr).$$

If we now set $r = -1$ in (9.14), we get

$$(n-3)(na_n + a_{n-1}) = 0, \quad n = 1, 2, \dots.$$

This yields, in turn,

$$a_1 = -a_0, \quad a_2 = -\tfrac{1}{2}\,a_1 = \tfrac{1}{2}\,a_0, \quad 0 = 0,$$

$$a_4 = -\tfrac{1}{4}\,a_3, \quad a_5 = -\tfrac{1}{5}\,a_4 = \tfrac{1}{20}\,a_3, \quad a_6 = -\tfrac{1}{6}\,a_5 = -\tfrac{1}{120}\,a3, \quad \dots,$$

with a_3 undetermined and, therefore, arbitrary. The above coefficients produce the series

$$t^{-1}\bigl[a_0\bigl(1 - t + \tfrac{1}{2}\,t^2\bigr) + a_3\bigl(t^3 - \tfrac{1}{4}\,t^4 + \tfrac{1}{20}\,t^5 - \tfrac{1}{120}\,t^6 + \cdots\bigr)\bigr] = a_0\bigl(t^{-1} - 1 + \tfrac{1}{2}\,t\bigr) + a_3y_1.$$

Consequently, without loss of generality, we may take $a_3 = 0$ (and, as usual, $a_0 = 1$) to end up with the second Frobenius solution

$$y_2(t) = t^{-1} - 1 + \tfrac{1}{2}\,t. \tag{9.15}$$

Given that, as is obvious, y_1 and y_2 are linearly independent, the GS of the equation is $y = c_1y_1 + c_2y_2$, with the power series in y_1 and y_2 convergent for all real values of t.

The same result is achieved if we substitute (9.12) with $r = -1$ in the equation. Proceeding as in Example 9.28, we then arrive at the equality

$$2cty_1' - cy_1 + cty_1 - 2 + \sum_{n=1}^{\infty}(n-1)(n-2)b_nt^{n-1} + \sum_{n=1}^{\infty}(n-1)b_nt^n - \sum_{n=1}^{\infty}b_nt^n - 2\sum_{n=1}^{\infty}b_nt^{n-1} = 0,$$

from which, if we replace y_1 by its expression and perform the necessary operations and simplifications, we obtain

$$-2(b_1+1) - (2b_2+b_1)t + 3ct^2 + \big(-\tfrac{1}{4}\,c + 4b_4 + b_3\big)t^3 + \big(\tfrac{1}{10}\,c + 10b_5 + 2b_4\big)t^4 + \big(-\tfrac{1}{40}\,c + 18b_6 + 3b_5\big)t^5 + \big(\tfrac{1}{336}\,c + 28b_7 + 4b_6\big)t^6 + \cdots = 0,$$

so

$$b_1 = -1, \quad b_2 = \tfrac{1}{2}, \quad c = 0, \quad b_4 = -\tfrac{1}{4}\,b_3, \quad b_5 = \tfrac{1}{20}\,b_3, \quad b_6 = -\tfrac{1}{120}\,b_3, \quad b_7 = \tfrac{1}{840}\,b_3, \quad \dots .$$

With these coefficients we now construct the series

$$t^{-1} - 1 + \tfrac{1}{2}\,t + b_3\big(t^2 - \tfrac{1}{4}\,t^3 + \tfrac{1}{20}\,t^4 - \tfrac{1}{120}\,t^5 + \tfrac{1}{840}\,t^6 + \cdots\big),$$

which for $b_3 = 0$ reduces to (9.15).

Verification with Mathematica®. The input

```
y1 = t^2(1 - (1/4)*t + (1/20)*t^2 - (1/120)*t^3 + (1/840)*t^4);
y2 = t^(-1) - 1 + (1/2)*t;
y = c1*y1 + c2*y2;
rem = t^2*D[y,t,t] + t^2*D[y,t] - (t + 2)*y;
rem//Expand
```

generates the output $c_1t^7/168$. ■

9.35 Example. The same treatment applied to the DE

$$(t^2+t)y'' + (t+3)y' + (2t+1)y = 0$$

leads in the first instance to the equality

$$(r^2+2r)a_0t^{-1+r} + [(r+1)(r+3)a_1 + (r^2+1)a_0]t^r + \sum_{n=1}^{\infty}\{(n+r+1)(n+r+3)a_{n+1} + [(n+r)^2+1]a_n + 2a_{n-1}\}t^{n+r} = 0$$

and, hence, to the indicial equation and recurrence formulas

$$r^2 + 2r = 0,$$
$$(r+1)(r+3)a_1 + (r^2+1)a_0 = 0,$$
$$(n+r+1)(n+r+3)a_{n+1} + [(n+r)^2+1]a_n + 2a_{n-1} = 0, \quad n = 1, 2, \ldots.$$

The roots of the top equation are $r_1 = 0$ and $r_2 = -2$. For $r = 0$, the other two give

$$a_1 = -\tfrac{1}{3}\, a_0,$$
$$a_{n+1} = -\frac{1}{(n+1)(n+3)}\,[(n^2+1)a_n + 2a_{n-1}], \quad n = 1, 2, \ldots,$$

which, with $a_0 = 1$, yield the Frobenius solution

$$y_1(t) = 1 - \tfrac{1}{3}\,t - \tfrac{1}{6}\,t^2 + \tfrac{1}{10}\,t^3 - \tfrac{1}{36}\,t^4 + \cdots.$$

If we now try $r = -2$ in the above recurrence relations, we get

$$-a_1 + 5a_0 = 0,$$
$$(n^2-1)a_{n+1} + (n^2 - 4n + 5)a_n + 2a_{n-1} = 0, \quad n = 1, 2, \ldots,$$

and, since $a_0 \neq 0$, we arrive at two contradictory results: $a_1 = 5a_0$ from the former and, for $n = 1$, $a_1 = -a_0$ from the latter. This means that the method fails and we must use (9.12) with $r = -2$ to construct a second solution.

Doing so, after simplification we obtain

$$\begin{aligned} &2cty_1' + 2cy_1' + 2ct^{-1}y_1 + 5t^{-2} + 2t^{-1} \\ &\quad + \sum_{n=1}^{\infty}(n-2)(n-3)b_nt^{n-2} + \sum_{n=1}^{\infty}(n-2)(n-3)b_nt^{n-3} + \sum_{n=1}^{\infty}(n-2)b_nt^{n-2} \\ &\qquad + 3\sum_{n=1}^{\infty}(n-2)b_nt^{n-3} + 2\sum_{n=1}^{\infty}b_nt^{n-1} + \sum_{n=1}^{\infty}b_nt^{n-2} = 0. \end{aligned}$$

Substituting the expression of y_1 in this equality and computing the first few terms yields

$$\begin{aligned} &(5 - b_1)t^{-2} + (2c + 2 + 2b_1)t^{-1} + \big(-\tfrac{4}{3}\,c + 3b_3 + b_2 + 2b_1\big) \\ &\quad + \big(-\tfrac{5}{3}\,c + 8b_4 + 2b_3 + 2b_2\big)t + \big(\tfrac{2}{15}\,c + 15b5 + 5b_4 + 2b_3\big)t^2 \\ &\qquad + \big(\tfrac{29}{90}\,c + 24b_6 + 10b_5 + 2b_4\big)t^3 + \cdots = 0, \end{aligned}$$

from which

$$b_1 = 5, \quad c = -6, \quad b_3 = -6 - \tfrac{1}{3}\,b_2, \quad b_4 = \tfrac{1}{4} - \tfrac{1}{6}\,b_2,$$
$$b_5 = \tfrac{77}{100} + \tfrac{1}{10}\,b_2, \quad b_6 = -\tfrac{47}{180} - \tfrac{1}{36}\,b_2, \quad \ldots,$$

where b_2 remains arbitrary. When these coefficients are replaced in (9.12), we see that the series multiplied by b_2 is, in fact, y_1, so, as in the preceding example, we may take $b_2 = 0$ and find the second solution

$$y_2(t) = -6y_1(t)\ln t + t^{-2}\big(1 + 5t - 6t^3 + \tfrac{1}{4}\,t^4 + \tfrac{77}{100}\,t^5 + \cdots\big).$$

Since y_1 and y_2 are linearly independent, it follows that $y = c_1y_1 + c_2y_2$ is the GS of the equation. The power series in y_1 and y_2 converge for all real t.

VERIFICATION WITH MATHEMATICA®. The input

```
y1 = 1 - (1/3) * t - (1/6) * t^2 + (1/10) * t^3 - (1/36) * t^4;
y2 = -6 * y1 * Log[t] + t^(-2) * (1 + 5 * t - 6 * t^3 + (1/4) * t^4
        + (77/100) * t^5);
y = c1 * y1 + c2 * y2;
rem = (t^2 + t) * D[y,t,t] + (t + 3) * D[y,t] + (2 * t + 1) * y;
rem // Expand
```

generates the output $c_1\big(-\frac{49}{180}\,t^4 - \frac{1}{18}\,y^5\big) + c_2\big[\frac{94}{15}\,t^3 + \frac{431}{150}\,t^4 + \big(\frac{49}{30}\,t^4 + \frac{1}{3}\,t^5\big)\ln t\big]$. ■

9.36 Example. Consider the DE

$$t^2y'' + ty' + (t^2 - 1)y = 0.$$

Starting with expansion (9.8) and following the usual procedure, we derive the equality

$$(r^2-1)a_0t^r + r(r+2)a_1t^{1+r} + \sum_{n=2}^{\infty}[(n+r-1)(n+r+1)a_n + a_{n-2}]t^{n+r} = 0,$$

which leads to

$$\begin{aligned} (r^2-1)a_0 &= 0,\\ r(r+2)a_1 &= 0,\\ (n+r-1)(n+r+1)a_n + a_{n-2} &= 0, \quad n = 2, 3, \dots. \end{aligned}$$

Hence, the indicial equation is $r^2 - 1 = 0$, with roots (exponents at the singularity) $r_1 = 1$ and $r_2 = -1$.

For $r = 1$, we see that $a_1 = 0$ and

$$a_n = -\frac{1}{n(n+2)}\,a_{n-2}, \quad n = 2, 3, \dots,$$

so, computing the first few coefficients and taking $a_0 = 1/2$ (see Remark 9.37 below), we obtain the Frobenius solution

$$y_1(t) = \tfrac{1}{2}\,t\big(1 - \tfrac{1}{8}\,t^2 + \tfrac{1}{192}\,t^4 - \tfrac{1}{9{,}216}\,t^6 + \cdots\big).$$

A second solution can be constructed by means of (9.12). We replace this expression, with $r_2 = -1$, in the DE and, making use of the fact that y_1 is a solution, after some straightforward manipulation we establish that

$$2cty_1' + \sum_{n=1}^{\infty} n(n-2)b_nt^{n-1} + \sum_{n=3}^{\infty} b_{n-2}t^{n-1} = 0.$$

If we plug the series for y_1 in this equality and write the first few terms explicitly, we get

$$\begin{aligned} &-b_1 + (c+1)t + (3b_3 + b_1)t^2 + \big(-\tfrac{3}{8}\,c + 8b_4 + b_2\big)t^3 + (15b_5 + b_3)t^4\\ &\quad + \big(\tfrac{5}{192}\,c + 24b_6 + b_4\big)t^5 + (35b_7 + b_5)t^6 + \big(-\tfrac{7}{9{,}216}\,c + 48b_8 + b_6\big)t^t + \cdots = 0; \end{aligned}$$

therefore,

$$\begin{gathered} b_1 = 0, \quad c = -1, \quad b_3 = 0, \quad b_4 = -\tfrac{3}{64} - \tfrac{1}{8}\,b_2, \quad b_5 = 0,\\ b_6 = \tfrac{7}{2{,}304} + \tfrac{1}{192}\,b_2, \quad b_7 = 0, \quad b_8 = -\tfrac{35}{442{,}368} - \tfrac{1}{9{,}216}\,b_2, \quad \dots, \end{gathered}$$

where b_2 remains undetermined. The series multiplying b_2 when the above coefficients are inserted in (9.12) is twice the series for y_1. However, instead of setting the arbitrary coefficient equal to zero, as we did in Examples 9.34 and 9.35, this time we take $b_2 = 1/4$ and so,

$$y_2(t) = -y_1(t)\ln t + t^{-1}\big(1 + \tfrac{1}{4}\,t^2 - \tfrac{5}{64}\,t^4 + \tfrac{5}{1{,}152}\,t^6 - \tfrac{47}{442{,}368}\,t^8 + \cdots\big).$$

The linear independence of y_1 and y_2 now shows that we can write the GS of the given equation as $y = c_1 y_1 + c_2 y_2$. The power series involved in this solution converge for all real values of t.

Verification with Mathematica®. The input

```
y1 = (1/2) * t * (1 - (1/8) * t^2 + (1/192) * t^4 - (1/9216) * t^6);
y2 = -y1 * Log[t] + t^(-1) * (1 + (1/4) * t^2 - (5/64) * t^4
        + (5/1152) * t^6 - (47/442368) * t^8);
y = c1 * y1 + c2 * y2;
rem = t^2 * D[y,t,t] + t * D[y,t] - (t^2 - 1) * y;
rem // Expand
```

generates the output $-c_1 t^9/18{,}432 + c_2 t^9[-47/442{,}368 + (\ln t)/18{,}432]$. ■

9.37 Remark. The DE in Example 9.36 is *Bessel's equation of order* 1. Its solution y_1 (with the choice $a_0 = 1/2$ intentionally made by us for the leading coefficient) is called *Bessel's function of the first kind and order* 1 and is denoted by J_1. Switching the independent variable from t to x and expressing the coefficients a_n from the recurrence relation with $r = 1$ in factor form, we have

$$J_1(x) = \frac{1}{2}\,x \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(n+1)!}\,x^{2n}.$$

The *Bessel function of the second kind and order* 1, denoted by Y_1, is defined as

$$Y_1(x) = \frac{2}{\pi}\,[(\gamma - \ln 2)J_1(x) - y_2(x)],$$

where y_2 is computed with $b_2 = 1/4$ (our choice in Example 9.36) and γ is Euler's constant given by (9.11). Since $\{J_1, Y_1\}$ is an FSS for Bessel's equation of order 1, the GS of this equation may also be written as $y = c_1 J_1 + c_2 Y_1$.

The graphs of J_1 (lighter curve) and Y_1 (heavier curve) are shown in Fig. 9.2. ■

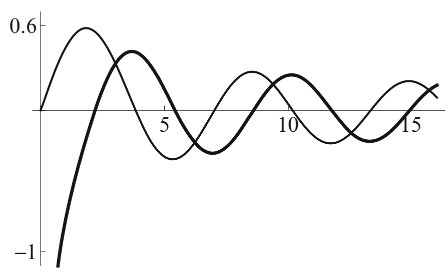

Fig. 9.2

Exercises

Find the exponents at the singularity for the given DE at the regular singular point $t = 0$ and, in each case, use (9.8) and (9.12) to construct two (linearly independent) series solutions for $t > 0$. (In 1–12, expect two Frobenius solutions.) Compute the first five nonzero terms in each of the power series occurring in these solutions and determine the minimal interval of convergence for these power series.

1 $4t^2y'' + (2t^2 + 4t)y' + (t - 1)y = 0.$

2 $ty'' + (t^2 + 2)y' + 2ty = 0.$

3 $(2t^2 + t)y'' + 3y' - (t^2 + 4)y = 0.$

4 $9t^2y'' + (3t^2 - 3t)y' - (2t + 5)y = 0.$

5 $2ty'' + (t + 6)y' + (t^2 + 1)y = 0.$

6 $2ty'' + (2t^2 - t + 4)y' + (t^2 + 4t - 1)y = 0.$

7 $t^2y'' - (t^2 + 2t)y' + (3t - 4)y = 0.$

8 $t^2y'' - t^2y' + (2t - 6)y = 0.$

9 $t^2y'' + (2t^2 + 3t)y' - 3y = 0.$

10 $ty'' + (t^2 - t + 4)y' + (2t - 1)y = 0.$

11 $(t^2 - t)y'' + (t - 2)y' + (t^2 - 1)y = 0.$

12 $t^2y'' + (t^2 - 5t)y' + (5 - 4t)y = 0.$

13 $t^2y'' + (t^2 + 4t)y' + (4t^2 - t + 2)y = 0.$

14 $t^2y'' - 2ty' + (t^2 + t + 2)y = 0.$

15 $4t^2y'' + (2t^2 + 8t)y' - 3y = 0.$

16 $t^2y'' + (5t - t^2)y' + (t + 3)y = 0.$

17 $t^2y'' + (2t^2 - 3t)y' + 3y = 0.$

18 $t^2y'' + 2ty' + 2(t - 1)y = 0.$

19 $t^2y'' + t^2y' + (3t - 2)y = 0.$

20 $t^2y'' + 3ty' + (2t - 3)y = 0.$

21 $t^2y'' - (t^2 + t)y' + (2t^2 - t - 3)y = 0.$

22 $ty'' + (2 - t)y' + (1 - t)y = 0.$

23 $(t^2 + 2t)y'' + (4 - 2t)y' + (6t + 2)y = 0.$

24 $ty'' + 4(t + 1)y' + (t^2 + 4)y = 0.$

Answers to Odd-Numbered Exercises

1 $r_1 = \frac{1}{2},\ r_2 = -\frac{1}{2};\ \ y_1(t) = t^{1/2}\big(1 - \frac{1}{4}\,t + \frac{1}{24}\,t^2 - \frac{1}{192}\,t^3 + \frac{1}{1{,}920}\,t^4 - \cdots\big);$
$y_2(t) = t^{-1/2};\ \ -\infty < t < \infty.$

3 $r_1 = 0,\ r_2 = -2;\ \ y_1(t) = 1 + \frac{4}{3}\,t + \frac{2}{3}\,t^2 + \frac{1}{15}\,t^3 + \frac{1}{30}\,t^4 + \cdots;$
$y_2(t) = t^{-2}\big(1 + 8t + \frac{1}{3}\,t^3 + \frac{7}{6}\,t^4 + \frac{1}{72}\,t^6 + \cdots\big);\ \ -\infty < t < \infty.$

5 $r_1 = 0,\ r_2 = -2;\ \ y_1(t) = 1 - \frac{1}{6}\,t + \frac{1}{48}\,t^2 - \frac{17}{480}\,t^3 + \frac{37}{5{,}760}\,t^4 + \cdots;$
$y_2(t) = t^{-2}\big(1 - \frac{1}{2}\,t - \frac{1}{6}\,t^3 + \frac{5}{96}\,t^4 - \frac{1}{192}\,t^5 + \cdots\big);\ \ -\infty < t < \infty.$

7 $r_1 = 4,\ r_2 = -1;\ \ y_1(t) = t^4\big(1 + \frac{1}{6}\,t + \frac{1}{42}\,t^2 + \frac{1}{336}\,t^3 + \frac{1}{3{,}024}\,t^4 + \cdots\big);$
$y_2(t) = t^{-1}\big(1 + t + \frac{1}{2}\,t^2 + \frac{1}{6}\,t^3 + \frac{1}{24}\,t^4\big);\ \ -\infty < t < \infty.$

9 $r_1 = 1,\ r_2 = -3;\ \ y_1(t) = t\big(1 - \frac{2}{5}\,t + \frac{2}{15}\,t^2 - \frac{4}{105}\,t^3 + \frac{1}{105}\,t^4 - \cdots\big);$
$y_2(t) = t^{-3}\big(1 - 2t + 2t^2 - \frac{4}{3}\,t^3\big);\ \ -\infty < t < \infty.$

11 $r_1 = 0,\ r_2 = -1;\ \ y_1(t) = 1 - \frac{1}{2}\,t + \frac{1}{12}\,t^3 + \frac{1}{120}\,t^4 + \frac{1}{240}\,t^5 + \cdots;$
$y_2(t) = t^{-1}\big(1 + \frac{1}{6}\,t^3 + \frac{1}{24}\,t^4 + \frac{1}{60}\,t^5 + \frac{1}{72}\,t^6 + \cdots\big);\ \ -1 < t < 1.$

13 $r_1 = -1,\ r_2 = -2;\ \ y_1(t) = t^{-1}\big(1 + t - \frac{1}{2}\,t^2 - \frac{1}{3}\,t^3 + \frac{7}{60}\,t^4 + \cdots\big);$
$y_2(t) = 3y_1(t)\ln t + t^{-2}\big(1 - 8t^2 - \frac{7}{12}\,t^3 + \frac{27}{8}\,t^4 - \frac{383}{2{,}400}\,t^5 + \cdots\big);\ \ -\infty < t < \infty.$

15 $r_1 = \frac{1}{2}$, $r_2 = -\frac{3}{2}$; $y_1(t) = t^{1/2}\bigl(1 - \frac{1}{12}\,t + \frac{1}{128}\,t^2 - \frac{1}{1{,}536}\,t^3 + \frac{7}{147{,}456}\,t^4 - \cdots\bigr)$;
$y_2(t) = -\frac{3}{32}\,y_1(t)\ln t + t^{-3/2}\bigl(1 - \frac{3}{4}\,t + \frac{1}{192}\,t^3 - \frac{7}{16{,}384}\,t^4 + \frac{9}{327{,}680}\,t^5 - \cdots\bigr)$;
$-\infty < t < \infty$.

17 $r_1 = 3$, $r_2 = 1$; $y_1(t) = t^3\bigl(1 - 2t + 2t^2 - \frac{4}{3}\,t^3 + \frac{2}{3}\,t^4 + \cdots\bigr)$;
$y_2(t) = -4y_1(t)\ln t + t\bigl(1 + 2t - 8t^3 + 12t^4 - \frac{88}{9}\,t^5 + \cdots\bigr)$; $-\infty < t < \infty$.

19 $r_1 = 2$, $r_2 = -1$; $y_1(t) = t^2\bigl(1 - \frac{5}{4}\,t + \frac{3}{4}\,t^2 - \frac{7}{24}\,t^3 + \frac{1}{12}\,t^4 + \cdots\bigr)$;
$y_2(t) = -2y_1(t)\ln t + t^{-1}\bigl(1 + t + \frac{3}{2}\,t^2 - \frac{21}{8}\,t^4 + \frac{19}{8}\,t^5 + \cdots\bigr)$; $-\infty < t < \infty$.

21 $r_1 = 3$, $r_2 = -1$; $y_1(t) = t^3\bigl(1 + \frac{4}{5}\,t + \frac{1}{6}\,t^2 - \frac{1}{35}\,t^3 - \frac{1}{60}\,t^4 + \cdots\bigr)$;
$y_2(t) = t^{-1}\bigl(1 + \frac{1}{2}\,t^2 - \frac{1}{3}\,t^3 + \frac{77}{150}\,t^5 + \frac{17}{72}\,t^6 + \cdots\bigr)$; $-\infty < t < \infty$.

23 $r_1 = 0$, $r_2 = -1$; $y_1(t) = 1 - \frac{1}{2}\,t - \frac{1}{2}\,t^2 + \frac{1}{8}\,t^3 + \frac{11}{160}\,t^4 + \cdots$;
$y_2(t) = -3y_1(t)\ln t + t^{-1}\bigl(1 - 6t^2 - \frac{9}{8}\,t^3 + \frac{53}{32}\,t^4 + \frac{331}{1{,}600}\,t^5 + \cdots\bigr)$; $-2 < t < 2$.

Appendix A
Algebra Techniques

A.1 Partial Fractions

The integration of a rational function—that is, a function of the form P/Q, where P and Q are polynomials—becomes much easier if the function can be written as a sum of simpler expressions, commonly referred to as *partial fractions*. This is done by means of a procedure that consists of the following steps:

(i) Using long division, if necessary, we isolate the fractional part of the given function, for which the degree of P is strictly less than the degree of Q.

(ii) We split Q into a product of linear factors (first-degree polynomials) and irreducible quadratic factors (second-degree polynomials with complex roots). Some of these factors may be repeated.

(iii) For each single linear factor $ax+b$, we write a fraction of the form $C/(ax+b)$. For each repeated linear factor $(ax+b)^n$, we write a sum of fractions of the form

$$\frac{C_1}{ax+b}+\frac{C_2}{(ax+b)^2}+\cdots+\frac{C_n}{(ax+b)^n}.$$

(iv) For each single irreducible quadratic factor ax^2+bx+c, we write a fraction of the form $(C_1x+C_2)/(ax^2+bx+c)$. For each repeated irreducible quadratic factor $(ax^2+bx+c)^n$, we write a sum of fractions of the form

$$\frac{C_{11}x+C_{12}}{ax^2+bx+c}+\frac{C_{21}x+C_{22}}{(ax^2+bx+c)^2}+\cdots+\frac{C_{n1}x+C_{n2}}{(ax^2+bx+c)^n}.$$

(v) The fractional part of the given function is equal to the sum of all the partial fractions and sums of partial fractions constructed in steps (iii) and (iv). The unknown coefficients C, C_i, and C_{ij} are determined (uniquely) from this equality.

A.1 Example. The denominator of the fraction $(x-5)/(x^2-4x+3)$ is a quadratic polynomial, but it is not irreducible. Its roots are 1 and 3, so we write

$$\frac{x-5}{x^2-4x+3}=\frac{x-5}{(x-1)(x-3)}=\frac{A}{x-1}+\frac{B}{x-3}.$$

If we eliminate the denominators, we arrive at the equality

$$x-5=A(x-3)+B(x-1)=(A+B)x-3A-B,$$

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_10,
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which, in fact, is an identity, meaning that it must hold for all admissible values (in this case, all real values) of x. Then, matching the coefficients of x and the constant terms on both sides, we obtain the system

$$A + B = 1, \quad 3A + B = 5,$$

with solution $A = 2$ and $B = -1$. Hence,

$$\frac{x-5}{x^2-4x+3} = \frac{2}{x-1} - \frac{1}{x-3}. \quad \blacksquare$$

A.2 Example. Similarly, we have

$$\frac{x^2+7x+4}{x^3+4x^2+4x} = \frac{x^2+7x+4}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2},$$

or

$$x^2 + 7x + 4 = A(x+2)^2 + Bx(x+2) + Cx.$$

Equating the coefficients of x^2 and x and the constant terms on both sides would yield a system of three equations in three unknowns, whose solution would require a little time and effort. This can be avoided if we recall that the above equality, which—we must emphasize—has been set up correctly, holds for all real values of x. Thus, for $x = 0$ we obtain $4 = 4A$, so $A = 1$, and for $x = -2$ we get $-6 = -2C$, so $C = 3$. We chose these particular values of x because they made some of the terms on the right-hand side vanish. To find B, we can take x to be any other number, for example, 1, and replace A and C by their already determined values. Then $12 = 9A + 3B + C = 12 + 3B$, from which $B = 0$; hence,

$$\frac{x^2+7x+4}{x^3+4x^2+4x} = \frac{1}{x} + \frac{3}{(x+2)^2}. \quad \blacksquare$$

A.3 Example. Since $x^2 + 1$ is an irreducible quadratic polynomial, we have

$$\frac{2+x-x^2}{x^3+x^2} = \frac{2+x-x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1},$$

or

$$2 + x - x^2 = A(x^2+1) + (Bx+C)x.$$

Using either the method in Example A.1 or that in Example A.2, we find that $A = 2$, $B = -3$, and $C = 1$, so

$$\frac{2+x-x^2}{x^3+x^2} = \frac{2}{x} + \frac{1-3x}{x^2+1}. \quad \blacksquare$$

A.4 Example. Without giving computational details but mentioning the use of long division, we have

$$\begin{aligned}\frac{x^5+2x^2-3x+2}{x^4-x^3+x^2} &= x+1+\frac{x^2-3x+2}{x^4-x^3+x^2} = x+1+\frac{x^2-3x+2}{x^2(x^2-x+1)}\\ &= x+1+\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2-x+1}\\ &= x+1-\frac{1}{x}+\frac{2}{x^2}+\frac{x-2}{x^2-x+1}. \quad \blacksquare\end{aligned}$$

A.2 Synthetic Division

Consider a polynomial equation of the form

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0, \tag{A.1}$$

where all the coefficients $a_0, \dots, a_n$ are integers. If a root of this equation is an integer, then that root is a divisor of the constant term a_n and can be determined by means of a simple algorithm.

(i) Let r_0 be a divisor of a_n. We line up the coefficients of the equation in decreasing order of the powers of r (writing 0 if a term is missing) as the first row in Table A.1.

Table A.1

a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n	

(ii) Next, we complete the second row in Table A.1 with numbers $b_0, \dots, b_n$ computed as shown in Table A.2.

Table A.2

a_0	a_1	a_2	$\cdots$	a_n	
$b_0 = a_0$	$b_1 = b_0 r_0 + a_1$	$b_2 = b_1 r_0 + a_2$	$\cdots$	$b_n = b_{n-1} r_0 + a_n$	r_0

(iii) If $b_n \neq 0$, then r_0 is not a root of the equation, and we repeat the procedure with the next candidate—that is, the next divisor of the constant term. If $b_n = 0$, then r_0 is a root; furthermore, b_0, b_1, $\dots$, b_{n-1} are the coefficients of the polynomial equation of degree $n-1$ which yields the remaining roots.

(iv) When this new equation is not easily solvable, we may try applying the procedure again, starting with the new set of coefficients b_i.

A.5 Example. For the equation

$$r^4 - 2r^3 - 3r^2 + 4r + 4 = 0,$$

the divisors of the constant term 4 are ± 1, ± 2, and ± 4. To check whether, say, 1 is a root, we apply the algorithm described above and arrive at Table A.3.

Table A.3

1	−2	−3	4	4	
1	−1	−4	0	4	1

The number 4 in the last place before the vertical bar in the second row means that 1 is not a root, so we try the next candidate, −1. This time we get Table A.4.

Table A.4

1	−2	−3	4	4	
1	−3	0	4	0	−1

Since the last number in the second row before the bar is 0, we conclude that -1 is a root. We can try -1 again, starting the operation directly from the second row, as shown in Table A.5.

Table A.5

1	-2	-3	4	4	
1	-3	0	4	0	-1
1	-4	4	0		-1

Hence, -1 is a double root, and the coefficients 1, -4, and 4 in the third row tell us that the remaining two roots of the characteristic equation are given by the equation $r^2 - 4r + 4 = 0$, which yields another double root, namely 2. ■

We can extend the above algorithm to determine whether an equation of the form (A.1) has rational roots; that is, roots of the form a/b, where $b \neq 0$ and a and b are integers with no common factor greater than 1. If this is the case, then the numerator a is a divisor of the constant term a_n and the denominator b is a divisor of the leading coefficient a_0.

A.6 Example. For the equation

$$2r^3 - 9r^2 + 14r - 5 = 0,$$

the divisors of the constant term -5 are ± 1 and ± 5, but none of them satisfies the equation. Since the divisors of the leading coefficient 2 are ± 1 and ± 2, it follows that any possible nonintegral rational roots need to be sought amongst the numbers $\pm 1/2$ and $\pm 5/2$. Trying the first one, $1/2$, with the procedure described above, we construct Table A.6.

Table A.6

2	-9	14	-5	
2	-8	10	0	$1/2$

This shows that $r_1 = 1/2$ is a root and that the other two roots are given by the quadratic equation $r^2 - 4r + 5 = 0$ (for convenience, we have divided the coefficients 2, -8, and 10 in the second row by 2); they are $r_2 = 2 + i$ and $r_3 = 2 - i$. ■

Appendix B
Calculus Techniques

B.1 Sign of a Function

If $f = f(x)$ is a continuous function on an interval J and a and b are two points in J such that $f(a)$ and $f(b)$ have opposite signs, then, by the intermediate value theorem, there is at least one point c between a and b such that $f(c) = 0$. In other words, a continuous function cannot change sign unless it goes through the value 0. This observation offers a simple way to find where on J the function f has positive values and where it has negative values.

B.1 Example. To determine the sign of

$$f(x) = (x+3)(x-1)(5-x)$$

at all points x on the real line, we note that the roots of the equation $f(x) = 0$ are $x_1 = -3$, $x_2 = 1$, and $x_3 = 5$. Since the sign of f does not change on any of the subintervals $x < -3$, $-3 < x < 1$, $1 < x < 5$, and $x > 5$, we can test it by computing the value of f at an arbitrarily chosen point in each of these subintervals; here we picked the points -4, 0, 2, and 6. The conclusions are listed in Table B.1.

Table B.1

x		(-4)		-3		(0)		1		(2)		5		(6)	
$f(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$

This makes it unnecessary to study and combine the sign of every factor in the expression of $f(x)$. ■

B.2 Integration by Parts

If u and v are continuously differentiable functions of a variable x, then

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx,$$

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_11,

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or, in abbreviated form,

$$\int u\,dv = uv - \int v\,du.$$

B.2 Example. Let

$$I_1 = \int e^{ax}\cos(bx)\,dx, \quad I_2 = \int e^{ax}\sin(bx)\,dx.$$

Then

$$I_1 = \frac{1}{a}\,e^{ax}\cos(bx) - \int \frac{1}{a}\,e^{ax}(-\sin(bx))b\,dx = \frac{1}{a}\,e^{ax}\cos(bx) + \frac{b}{a}\,I_2,$$
$$I_2 = \frac{1}{a}\,e^{ax}\sin(bx) - \int \frac{1}{a}\,e^{ax}\cos(bx)b\,dx = \frac{1}{a}\,e^{ax}\sin(bx) - \frac{b}{a}\,I_1.$$

This is a simple algebraic system for I_1 and I_2, which yields

$$I_1 = \int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}[a\cos(bx) + b\sin(bx)] + C, \tag{B.1}$$
$$I_2 = \int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}[a\sin(bx) - b\cos(bx)] + C. \;\blacksquare \tag{B.2}$$

B.3 Integration by Substitution

Let $f = f(x)$ be a continuous function on an interval $[a, b]$, and let $x = x(t)$ be a continuously differentiable function on an interval $[c, d]$, which takes values in $[a, b]$ and is such that $x(c) = a$ and $x(d) = b$. Then

$$\int_a^b f(x)\,dx = \int_c^d f(x(t))x'(t)\,dt.$$

B.3 Example. If $g = g(t)$, then, in the absence of prescribed limits,

$$\int \frac{g'(t)}{g(t)}\,dt = \int \frac{1}{g(t)}\,g'(t)\,dt = \int \frac{dg}{g} = \ln|g| + C. \;\blacksquare$$

Appendix C
Table of Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$f^{(n)}(t)$ (nth derivative)	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
2	$f(t-a)H(t-a)$	$e^{-as}F(s)$
3	$e^{at}f(t)$	$F(s-a)$
4	$(f * g)(t)$	$F(s)G(s)$
5	1	$\dfrac{1}{s}$ $(s > 0)$
6	t^n (n positive integer)	$\dfrac{n!}{s^{n+1}}$ $(s > 0)$
7	e^{at}	$\dfrac{1}{s-a}$ $(s > a)$
8	$\sin(at)$	$\dfrac{a}{s^2+a^2}$ $(s > 0)$
9	$\cos(at)$	$\dfrac{s}{s^2+a^2}$ $(s > 0)$
10	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$ $(s > \|a\|)$
11	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$ $(s > \|a\|)$
12	$\delta(t-a)$ $(a \geq 0)$	e^{-as}
13	$\delta(c(t-a))$ $(c,\ a > 0)$	$\dfrac{1}{c}\, e^{-as}$
14	$f(t)\delta(t-a)$ $(a \geq 0)$	$f(a)e^{-as}$
15	$\int_0^t f(\tau)\, d\tau$	$\dfrac{1}{s}\, F(s)$
16	$t^n f(t)$ (n positive integer)	$(-1)^n F^{(n)}(s)$

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_12,

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Appendix D
The Greek Alphabet

Below is a table of the Greek letters most used by mathematicians. The recommended pronunciation of these letters as symbols in an academic context, listed in the third column, is that of classical, not modern, Greek.

Letter	Name	Greeks say...
α	Alpha	*Ahl*-fah
β	Beta	*Beh*-tah
γ, $\varGamma$	Gamma	*Gahm*-mah
δ, $\varDelta$	Delta	*Del*-tah
ε, ϵ	Epsilon	*Ep*-sih-lohn
ζ	Zeta	*Zeh*-tah
η	Eta	*Eh*-tah
θ, $\varTheta$	Theta	*Theh*-tah
ι	Iota	*Yoh*-tah (as in 'York')
κ	Kappa	*Kahp*-pah
λ, $\varLambda$	Lambda	*Lahmb*-dah
μ	Mu	Mu
ν	Nu	Nu
ξ, $\varXi$	Xi	Xih
π, $\varPi$	Pi	Pih
ρ	Rho	Roh
σ, $\varSigma$	Sigma	*Sig*-mah
τ	Tau	Tau (as in 'how')
υ, $\varUpsilon$	Upsilon	*Ewe*-psih-lohn
ϕ, φ, $\varPhi$	Phi	Fih
χ	Chi	Khih
ψ, $\varPsi$	Psi	Psih
ω, $\varOmega$	Omega	Oh-*meh*-gah

One allowed exception: π (and its upper case version $\varPi$) can be mispronounced 'pie', to avoid the objectionable connotation of the original sound.

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1_13,
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Further Reading

The following is a short list, by no means exhaustive, of textbooks on ordinary differential equations that contain additional material, technical details, and information on the subject going beyond the scope of this book.

1 Birkhoff, G., Rota, G.-C.: Ordinary Differential Equations, 4th edn. Wiley, New York (1989)

2 Borrelli, R.L., Coleman, C.S.: Differential Equations: A Modeling Perspective, 2nd edn. Wiley, New York (2004)

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11 Zill, D.G.: A First Course in Differential Equations with Modeling Applications, 10th edn. Brooks Cole, Pacific Grove (2012)

C. Constanda, *Differential Equations: A Primer for Scientists and Engineers*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4614-7297-1,

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