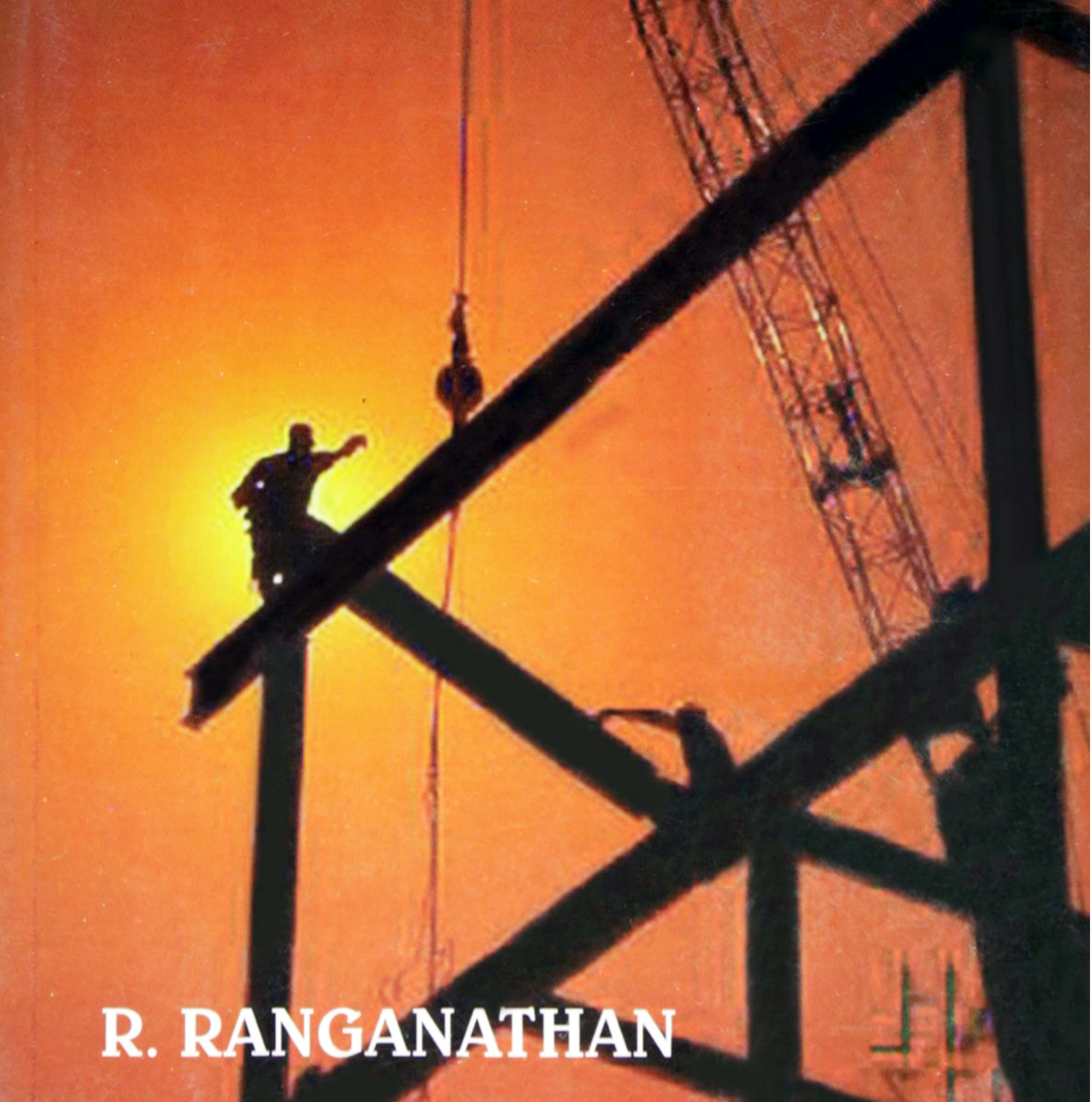
STRUCTURAL RELIABILITY

ANALYSIS AND DESIGN



R. RANGANATHAN

STRUCTURAL RELIABILITY ANALYSIS AND DESIGN

STRUCTURAL RELIABILITY ANALYSIS AND DESIGN

R. RANGANATHAN
Professor
Department of Civil Engineering
Indian Institute of Technology
Bombay

JAICO PUBLISHING HOUSE
Mumbai • Delhi • Bangalore
Calcutta • Hyderabad • Chennai

STRUCTURAL RELIABILITY ANALYSYS AND DESIGN
ISBN 81-7224-851-2

First Jaico Impression : 1999
Second Jaico Impression : 2000

Published by:
Ashwin J. Shah
Jaico Publishing House
121, M.G. Road,
Mumbai-400 023

Printed by:
Saurabh Print-O-Pack
A-16, Sector IV, Noida-201301 (U.P)

Preface

Engineering decisions must be made in the presence of uncertainties which are invariably present in practice. In the presence of uncertainties in the various parameters encountered in analysis and design, achievement of absolute safety is impossible. It is now more than twenty five years since it was proposed that the rational criterion for the safety of a structure is its reliability or probability of survival. In structural reliability, the probability of failure (which is taken as one minus reliability) is taken as a quantitative measure of structural safety. Probabilistic concepts are used in reliability analysis, and in the design of structures. Using structural reliability theory, the level of reliability of the existing structures (structures designed by existing structural standards) can be evaluated. It can also be used for developing a design criterion, that is, calibrating codes and developing partial safety factors, the use of which will result in designs with an accepted level of reliability. Structural reliability has been applied to many decision-making problems, such as development of partial safety factors, establishing inspection criteria, taking suitable decisions for improving the capability of existing structures, development of maintenance schedule etc., in the field of engineering.

Presently, only four or five books are available on this topic. These books, written by foreign authors, are very expensive and beyond the reach of Indian students and engineers. A book on this topic giving information to readers on the results of the reliability study of reinforced concrete structural elements and frames, with the field data pertaining to Indian conditions, is also presently not available. There has been an overwhelming need among students (present and past), fellow teachers in engineering institutions, scientists in research organizations, and field engineers for such a book on reliability analysis and design of structures giving fundamental concepts of structural reliability theory and illustrating its applications to practical problems. Teaching the course to postgraduate students for the last eighteen years, delivering a series of lectures given periodically to the participants of short-term courses, and research experience have motivated me to write a book which treats the topic in a simple manner so that structural reliability is easily understood and appreciated by readers.

The main aim of this book is to introduce the probabilistic bases of structural reliability, the techniques and methods of evaluating the reliability of structural components and systems, the methodology in the development of reliability-based design criteria, and the evaluation of partial safety factors (code calibration). The whole field has been developed in such a way that it is easily understood and followed by readers. Proofs and mathematical derivations are given only if they serve to explain basic ideas, otherwise, the original literature is cited for proof. Another important aim of this book is to focus the attention of academicians and engineers on the importance of an awareness of the need for a reliability-based design criterion, which is being followed in Norway, Canada, USA, UK and other countries which are in the process of modifying their standards. The book takes care of beginners as well as experienced specialists. Though the book in general deals with reinforced concrete and steel structures, the theory presented and the methods indicated could be applied to other structures also. The book will serve as a useful text and reference book for students, teachers, scientists and engineers. It is hoped that the book will provide enough foundation for further research work.

This is the only book which deals with the reliability analysis of reinforced concrete frames, adaptive sampling method (for component and system) and response surface method to estimate reliability, and fatigue reliability evaluation of bridges. The reliability analysis of structural components as well as systems is covered in a single volume. The book gives the results of analysis of field data on basic variables and the reliability study of concrete structures for Indian conditions. Nearly 110 examples are worked out. Problems with answers are given under "Exercise" at the end of each chapter.

Basic concepts of structural safety are introduced in Chapter 1. Certain inadequacies in the conventional safety checking methods are exposed and the need for a probabilistic criterion is emphasized. A history of the structural safety is also briefly traced in Chapter 1. The necessary background on statistics and probability, required for understanding the subsequent chapters and reliability analysis, is included in Chapters 2 and 3, respectively. A number of examples are given, illustrating the applications of probability theory in civil engineering.

The collection of field data on basic variables, and a statistical analysis of the same, is a very important part of reliability study. The statistical analysis of resistance variables and load variables is presented in Chapters 4 and 5, respectively. The established statistics of basic variables for Indian conditions are also given.

The computation of the structural reliability for the fundamental case of two variables, load and resistance, is treated in Chapter 6. Difficulties encountered in the probabilistic analysis of structures are brought out, and how the Monte Carlo technique can be used to tackle such problems is explained in Chapter 7. Applications of the Monte Carlo technique in structural engineering problems are outlined and illustrated.

Simple methods of computation of reliability using first-order second-moment mean-value methods, and Level 2 and advanced Level 2 methods are treated in detail in Chapter 8. Statistics of basic variables, established in Chapters 4 and 5, are used to illustrate the methods of evaluating structural components' reliability. A number of problems are solved. The results of the reliability study of the existing reinforced concrete designs as per the present IS 456-1978 code are also given. Chapter 9 deals with the computation of the partial safety factors for a specified or required level of reliability. The method of fixing optimal partial safety factors, which will ensure uniform target reliability under all design situations, is introduced. The methodology of the calibration of code is also treated in the same chapter. Results of the study of the evaluation of partial safety factors for Indian conditions are also presented.

The system performance and its reliability are of more concern and importance to engineers. The modelling of structures for the computation of reliability is demonstrated in Chapter 10. Bounds on the reliability of structural systems are introduced. Methods of generation of dominant modes, and the reliability analysis of steel and reinforced concrete frames are presented. Results of the reliability study of reinforced concrete frames, designed as per the Indian Standard Code, are also given.

Considerable research work has been done in developing methods to improve the estimate of reliability and also better sampling techniques in simulation methods. Second order reliability method is briefed in Chapter 11. Advanced simulation methods to calculate reliability, based on importance sampling method (ISM) and adaptive sampling method (ASM), are explained in detail and illustrated with examples. Response surface method is also presented which can be clubbed with ISM and ASM. Application of ISM and ASM to system reliability evaluation is included with examples.

Fatigue is one of the principal modes of failure in bridges, offshore and ship structures, pressure vessels etc. In Chapter 12, evaluation of reliability of joints / details under fatigue is explained in a simple manner using S-N curve approach. Reliability evaluation using lognormal format and Weibull format are presented. For the desired reliability level, estimation of design stree range for fatigue criterion and partial safety factors are also dealt with. Number of examples are solved to demonstrate the fatigue reliability evaluation. Application to offshore structures and bridges are explained. Fracture mechanics (FM) approach is also introduced. The method of evaluation of reliability, based on FM approach, is also presented.

While teaching the course, advanced topics discussing the generation of dominant modes in frame structures, reliability analysis of reinforced concrete frames and advanced reliability methods may be omitted. Spending only about 5 to 6 hours on Chapter 3, a course on first ten chapters can be completed in a 35-hour lecture series. In case if the students have background on basic statistics and probability, saving 6 hours in Chapters 2 and 3, Chapter 12 can also be completed.

I express my gratitude and indebtedness to Prof. P. Dayaratnam, former professor, Indian Institute of Technology, Kanpur, who initiated me into the field of structural reliability and suggested that I write a book on the said field. He encouraged me throughout with his useful comments and suggestions. I have used many of the results of the research work of students who have worked under my supervision. I sincerely thank them, viz. A. G. Deshpande, Padmini Chikkodi, Neville Kumar Shetty, David Arulraj, Bajare, C. P. Joshi, Ravi, Kulkarni, Potkar and Prabhu for their contributions to the book. Thanks are also due to the students who attended my lectures and who, through their participation and comments, led to the development of the book. My wife's encouragement throughout the hard days of writing this book is greatly appreciated.

R. RANGANATHAN

Contents

List of Symbols

a	Crack width
a_i	Initial crack width
a_f	Final crack width
b	Breadth of section or member
D	Dead load, cumulative damage
D_f	Cumulative damage at failure
D_n	Nominal value of dead load
d	Effective depth
da/dN	Crack growth rate
EA	Axial rigidity of a member
EI	Flexural rigidity of a member
E_s	Young's modulus of steel
E_{sc}	Secant modulus of concrete
E_{tc}	Tangent modulus of concrete
e_{pf}	Error in the estimate of p_f
f_r	Modulus of rupture of concrete
$F_X(\ )$	Cumulative distribution function of X
$f_X(\ )$	Probability density function of X
f_y	Yield strength of steel
$g(\ )$	Failure function
K	S-N curve parameter
L	Live load intensity or load effect
L_{apt}	Arbitrary point in time live load
L_m	Lifetime maximum live load
L_n	Nominal value of L
M	Safety margin or plastic moment
m	Slope of S-N curve
N	Number of cycles in fatigue studies
p_f	Probability of failure
p_{fi}	Probability of failure of the structure under failure mode i
p_{fs}	Probability of failure of the system
p_s	Probability of survival
p_{ss}	Probability of survival of the system

R	Resistance
R_D	Design value of R
R_n	Nominal value of R
R_o	Reliability
R_T	Theoretical resistance
S	Stress range, action
S_e	Equivalent stress range
S_{rd}	Design stress range
V	Wind speed
V_{max}	Lifetime maximum wind speed
W	Wind load
W_{apt}	Arbitrary point-in-time wind load
W_m	Lifetime maximum wind load
W_n	Nominal value of W
$\widetilde{X}$	Median of X
α_i	Directional cosine
β	Reliability index
γ	Partial safety factor
γ_D	Partial safety factor for dead load
γ_L	Partial safety factor for live load
γ_{mc}	Material reduction factor for concrete
γ_{ms}	Material reduction factor for steel
γ_R	Partial safety factor for resistance
γ_W	Partial safety factor for wind load
δ_X	Coefficient of variation of X
ϕ	Probability density function of standard normal variate or null set
Φ	Cumulative distribution function of standard normal variate
μ_X	Mean value of X
σ_X	Standard deviation of X
ρ_{XY}	Correlation between X and Y
ASM	Adaptive sampling method
BSS	British standard specifications
FORM	First order reliability method
FOSM	First-order second-moment
ISM	Importance sampling method

ISS	Indian standard specification
PWLEP	Piece-wise linear elastic-plastic
PDF	Probability distribution function
RCC	Reinforced cement concrete
PMF	Probability mass function
SDM	Standard deviation multiplier
SORM	Second order reliability method

1

Concepts of Structural Safety

1.1 GENERAL

The evaluation of the safety of structures is a task of much importance. It has been one of the subjects of interest for engineers. The safety of a structure depends on the resistance, R, of the structure and the action, S, (load or load effect) on the structure. The action is a function of loads (live load, wind load, etc.), which are random variables. Similarly, the resistance or response of the structure depends on the physical properties of materials, and the geometric properties of the structure which are also subjected to statistical variations, and are probabilisitic. Even though it was known that the above parameters were random variables, no serious attempt was made to consider their random variations, till 1960, in the analysis, and design, and evaluation of safety. It was, probably, due to the reason that engineers were not confident of applying probability theory or statistics or other mathematical tools. It was only around 1960 that engineers and research workers started realising the need for the evaluation of safety, taking into account the random variations of the design parameters.

1.2 DESIGN METHODS

General principles for checking safety define a method for calculating the behaviour and strength of structures subjected to loadings. Design methods may be classified in the following ways.

1. By the way the coefficients related to safety are introduced:

Permissible Stress Method

This is also called the working stress design (WSD) method. Here, stresses occurring under maximum service loads (working loads) are compared with fractions of the strengths of materials. These fractions of the strengths of materials are called permissible stresses. A structure is assumed to have failed if stresses developed at any point of the structure are greater than the permissible stresses. The safety is defined in terms of the factor of safety, which is given by

$$\text{Factor of safety} = \frac{\text{failure stress}}{\text{permissible stress}}$$

For ductile materials, viz. steel, the yield stress is taken as the failure stress, and for brittle materials, viz. concrete, the ultimate stress is taken as the failure stress. In this method, the elastic behaviour of the material is considered, (i.e. Hooke's law is valid) and the load deflection curve of the structure is linear.

For structural steel, the factor of safety is about 1.67. What does this mean in connection with the safety of a steel structure? It does not also convey how much load the structure will withstand. If the factor of safety is doubled, does it mean that the capacity of the structure is also doubled? Definitely it is not because the behaviour of the material and structure is inelastic near the collapse load. Just because the stress at a point is more than the permissible stress, it does not necessarily cause the collapse of the structure, especially in the case of indeterminate structures. In the case of reinforced cement concrete (RCC) structure, the use of permissible stress method by introducing two different factors of safety—one to concrete (about 3) and another to reinforcing steel bars (about 1.78) invites more criticism. What is their combined effect in defining the safety of RCC structure? The points that were raised with respect to the steel structure are more pertinent to RCC structure also where the behaviour is nonlinear and inelastic.

Whenever combinations of loads are considered, viz., dead load + live load + wind load or dead load + live load + earthquake load, an increase in the allowable stresses ($33\frac{1}{3}$ per cent) is considered since the likelihood of all the loads reaching their maximum values simultaneously, is remote. However, there is no rational basis for the selection of the value, viz. $33\frac{1}{3}$ per cent. It may be said that the safety defined in the permissible stress does not reflect the true safety, or the actual safety that is available. The structure designed by the permissible stress method is safe under service load and is assumed or expected to carry the ultimate load.

Merits of WSD are:

(i) simplicity and
(ii) familiarity.

Demerits of WSD are:

(i) A given set of permissible stresses will not guarantee a constant level of safety for all structures. For example, if two roof structures-(a) RCC shell type and (b) RCC beam and slab type, designed for the same live load using the same permissible stresses, are considered, the ratio of the dead load to live load for the shell type will be considerably much lower than the ratio for the slab and beam type. Since the dead load can be estimated and predicted more accurately than the live load, which is subjected to more probabilistic variation, the shell roof structure will have a higher chance of failure than the heavier slab and beam type roof structure. That is to say, two structures designed for the same live load using the same permissible stresses will have different levels of safety.

(ii) The working stress checking format may be unsafe when one load counteracts the other load. For example, consider a column, shown in

Fig. 1.1, subjected to dead load D and wind load, W. The column has been designed by the working stress method by limiting stresses under service loads in tension and compression to fifty per cent of their respective strength values, 3 N/mm^2 in tension and 20 N/mm^2 in compression. The stress distribution under service loads 1.0 D + 1.0 W is shown in Fig. 1.1c. When the wind load is increased by twenty six per cent, it can be seen that the stress at the point B reaches its failure level. Therefore, using the WSD method can lead to designs with safety less than conceived adequate under normal conditions, when loads counteract each other.

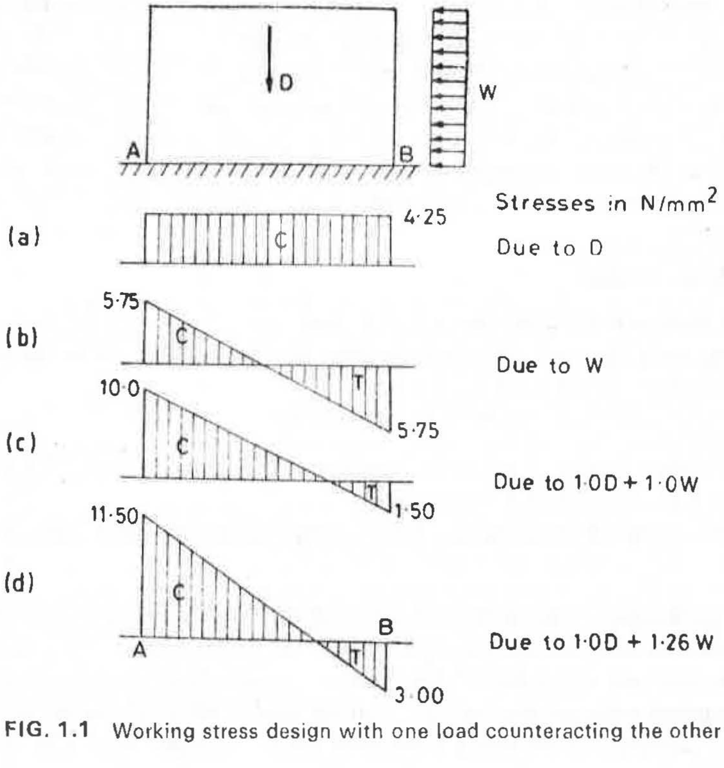


FIG. 1.1 Working stress design with one load counteracting the other

Ultimate Strength Design and Plastic Design Method

In these methods, the safety is ensured by magnifying the service loads (or load effects, such as bending moments, etc.) and checking the structure at this magnified load called collapse load. The magnification factor is called the load factor, defined as the ratio of the ultimate load to service load. In these methods, the safety is atleast related to the capacity of the structure. They take into account the inelastic behaviour of the material. In the ultimate strength design, the elastic analysis is first carried out, and only sections are designed for the factored load. Hence, moment redistribution is not taken into account in the ultimate strength design applied to RCC structures. However, the plastic design applied to steel structures takes into account the redistribution of moments, and the analysis of the structure is carried out at the collapse load. The plastic design as compared to the ultimate strength design, relatively gives a better picture of the true safety of the structure.

In these methods, separate load factors are assigned to different loads. Specifying a larger factor to the live load or wind load than the dead load, reflects that the variability in the live load or wind load is known to be larger than in the dead load. However, these factors have been selected more or less only on the subjective judgement without any rational basis.

Factored load is an imaginary load which never comes on the structure. These load factors, like factors of safety, are not related to the life of the structure. If the load factor 2, assumed to ensure a 50 year life of the structure, is increased by 50 per cent, it does not mean that the life to the structure also increases by the same amount (i.e. life of the structure need not be 75 years). The structure designed by the ultimate strength design method or the plastic design method is safe against collapse load and the same structure is assumed or expected to perform satisfactorily under service load.

Limit State Method

A limit state is a state beyond which a structure or a part of a structure, becomes unfit for use, or ceases to fulfil the function or satisfy the conditions for which it has been designed.

The limit states are placed in two categories:

(i) Ultimate limit states—these correspond to the maximum load carrying capacity (i.e. strength of the structure).

(ii) Serviceability limit states—these correspond to the criteria (durability) under normal load (service load) conditions.

The coefficients of safety are related to ultimate load conditions and service load conditions. That is, increased loads (or load effects) are compared with the relevant resistance of the structure where effects of the service load are compared with specific values. This method is definitely better than the previous methods as the safety is ensured under collapse load and service load conditions.

2. The second way of classifying the design methods is based on the safety conditions.

(i) Deterministic design methods where basic parameters (e.g. loads, strength of materials, etc.) are treated as non-random.

(ii) Probabilistic design methods where design parameters are considered as random.

In the conventional deterministic design method, it is assumed that all parameters are not subjected to probabilistic variations. However, it is well known that loads (live load on floors, wind load, ocean waves, earthquake, etc.) coming on structures are random variables. Similarly, the strengths of materials (strength of concrete, steel, etc.) and the geometric parameters (dimensions of section, effective depth, diameter of bars, etc.) are subjected to statistical variations. Hence, to be rational in the estimation of the structural safety, the random variations of the basic parameters are to be taken into account. Since load and strength are random variables, the safety of the structure is also a statistical variable.

In overcoming the uncertainties in the design parameters, the safety factor is ensured by taking the smallest value of the strength (R_s) and the largest value of the load (S_l). The safety factor, ν, is taken as R_s/S_l. This way of fixing the safety in design is very conservative and leads to uneconomical design.

The second way of fixing safety is as follows:
Let ΔR be the allowed deviation from R and ΔS the allowed deviation from S. For the safety of the structure,

$$R > S$$

$$R - \Delta R > S + \Delta S$$

$$R\left(1 - \frac{\Delta R}{R}\right) > S\left(1 + \frac{\Delta S}{S}\right)$$

$$\frac{R}{S} > \left(1 + \frac{\Delta S}{S}\right) \Big/ \left(1 - \frac{\Delta R}{R}\right)$$

Hence, the minimum, value of the safety factor is

$$\nu = \left(1 + \frac{\Delta S}{S}\right) \Big/ \left(1 - \frac{\Delta R}{R}\right)$$

If the maximum variations in R and S are 10 per cent and 20 per cent of their respective computed values, i.e.

$$\frac{\Delta S}{S} = 0.2 \qquad \text{and} \qquad \frac{\Delta R}{R} = 0.1$$

then the minimum value of ν is

$$\nu = \frac{(1 + 0.2)}{(1 - 0.1)}$$

$$= 1.33$$

The safety can also be expressed as the ratio of the mean values of R and S. This safety factor is called the central safety factor, ν_c, defined as

$$\nu_c = \frac{\text{mean value of } R}{\text{mean value of } S}$$

Definitions of safety factors vary widely and are probabilistically inaccurate. To understand the drawback in defining the safety by central safety factor consider Fig. 1.2 where probability density functions of R and S are plotted. When R and S are plotted, it will be seen that both distributions overlap. The shaded portion (overlap) in Fig. 1.2 gives an indicative measure of the probability of failure of the element or structure.

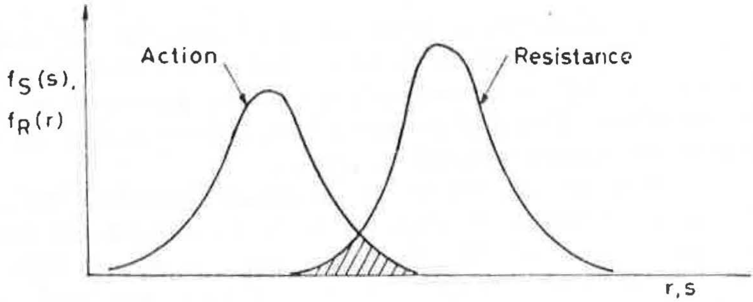


FIG. 1.2 Overlap of action and resistance distributions indicating failure probability

It will be seen now that for the same v_c, the value of p_f will be different. Consider Fig. 1.3 where mean action and mean resistance are increased in the same proportion keeping their standard deviations constant. Thus

$$v_c = \frac{k_1 R}{k_1 S} = \frac{R}{S}$$

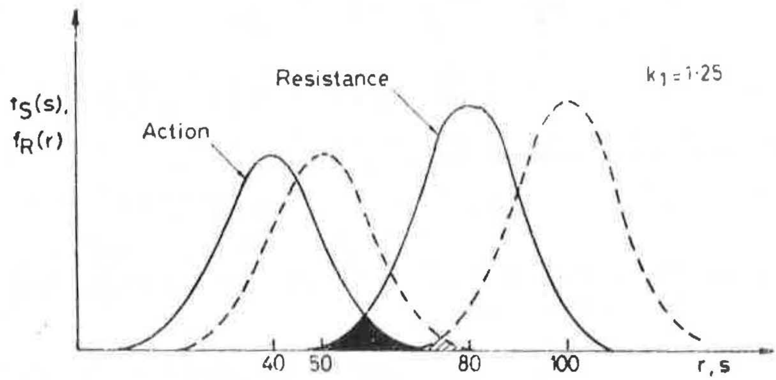


FIG. 1.3 Effect of failure probability due to proportional changes in action and resistance

It is observed from Fig. 1.3 that even though v_c remains the same, the overlap of the two curves change, meaning the change of p_f. Same things hold good when k_1 is <1.

If the mean values of R and S are kept constant and dispersions in R and S are changed (Fig. 1.4), it is seen again that the overlap of the two curves changes, indicating a change in the value of p_f. Since the mean values of R and S are not changed, v_c remains the same; but p_f is different. The probability of failure is affected by (i) the mean values R and S, (ii) the standard deviations of R and S, and (iii) the point of intersection of the two curves. This clearly shows the inadequacy of defining safety by the central safety factor. The best way to define safety is by the probability of failure or reliability. Freudenthal (1.1) said: "Because the design of a structure embodies uncertain predictions of the performance of structural

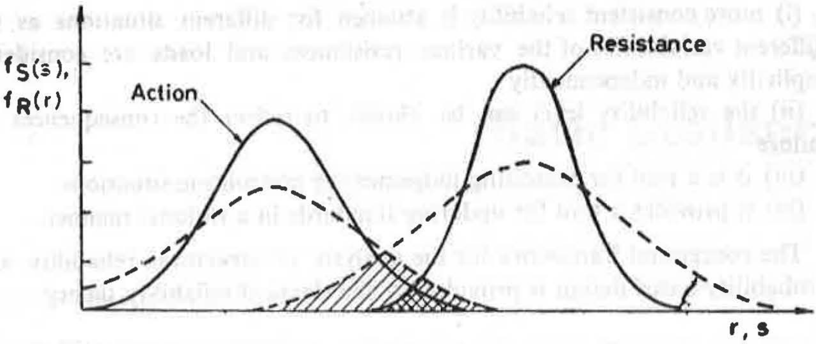


FIG. 1.4 Effect of failure probability due to changes in dispersion of action and resistance

materials as well as of the expected load patterns and intensities, the concept of probability must form an integral part of any rational analysis or design; any conceivable condition is necessarily associated with a numerical measure of the probability of its occurrence. It is by this measure alone that the structural significance of a specified condition can be evaluated". Since, the achievement of absolute safety or reliability in the uncertain world is impossible, a probabilistic approach to the evaluation of safety becomes a sensible solution. The parameters encountered in civil engineering problems are subjected to random variations. There is a need for a rational approach to the evaluation of structural safety, taking into account these random variations. The study of variability comes under the domain of statistics and probability. Using the probabilistic approach, there is a possibility of obtaining uniform reliability (uniform performance in structures under different design situations) which may probably lead to economical designs. Hence, probabilistic approach must be used. Henceforth, safety will be defined by reliability which is defined as the probability of survival of a structure under given environmental conditions. It is nothing but the ability of the structure to fulfil its assigned functions satisfactorily for some specified time. In structural analysis and design, it is the probability that a structure will not attain each specified limit during a specified reference period. For convenience, the reliability is defined in terms of the probability of failure (probability of unsatisfactory performance) which is equal to 1–the reliability of the structure. When probability theory is used in the limit state design, the method is called probability-based limit state design.

Probability-based Limit States Design

In this design method, probabilistic methods are used to guide the selection of the partial safety factors to loads and resistances of the structure or structural element or materials of the structure, and they result in the desired overall safety. The principal advantages of this design method are (1.2):

(i) more consistent reliability is attained for different situations as the different variabilities of the various resistances and loads are considered explicitly and independently

(ii) the reliability level can be chosen to reflect the consequences of failure

(iii) it is a tool for exercising judgement in nonroutine situations

(iv) it provides a tool for updating standards in a rational manner.

The conceptual framework for the analysis of structural reliability and probability-based design is provided by the classical reliability theory.

REFERENCES

1.1 Freudenthal, A.M., "Safety and the Probability of Structural Failure", *Transactions, ASCE*, Vol. 121, 1956, pp. 1337–1375.

1.2 Ellingwood, B.R., T.V. Galambos, J.G. McGregor and C.A. Cornell, "Development of a Probability Based Load Criterion for American National Standard A58", *National Bureau of Standards, Special Publication* **577**, Washington, D.C., June 1980.

EXERCISE

1.1 Is it possible to account for the uncertainties in loads in the working stress method?

1.2 Is it possible to account for the uncertainties in loads and material strengths in the ultimate load method?

1.3 What do you understand by limit state design?

1.4 What is central safety factor?

1.5 What factors affect the probability of failure of a structure?

1.6 What do you understand by uniform reliability in structure?

1.7 Do you think that the use of factor of safety is related to the life of structure?

1.8 Do you think that a design obtained using the ultimate load method with a set of load factors will ensure a particular life of the structure?

2

Basic Statistics

2.1 INTRODUCTION

In most engineering problems, experiments are generally conducted. Experiments may be carried out to study a particular property of a material, such as strength or, to study a natural phenomenon like wind velocity, earthquake intensity or, to assess the strength of a beam, etc. Decisions are to be made on the basis of these experiments. Experiments or observations are usually repeated several times under uniform or similar conditions. Even though great care is taken to keep the conditions of experiments as uniform as possible, the individual observations exhibit an intrinsic variability that cannot be eliminated.

Consider the production of concrete. If a concrete mix is prepared and a set of three or five cubes are made out of this concrete mix and they are tested for the compressive strength, it will be found that each cube will give a different strength. If another batch of concrete is prepared for the same mix ratio under the same conditions and a set of cubes are made out of this concrete and tested, it will be found again that these cubes will give another set of values for the strength of concrete. The average strength of concrete, calculated for each set, will also be different. A typical set variation of average strength of M 20 concrete, obtained from a project (2.1), is shown in Fig. 2.1. It is found that the results of the strength obtained vary and do not give the same value repeatedly for the same mix. This means that the

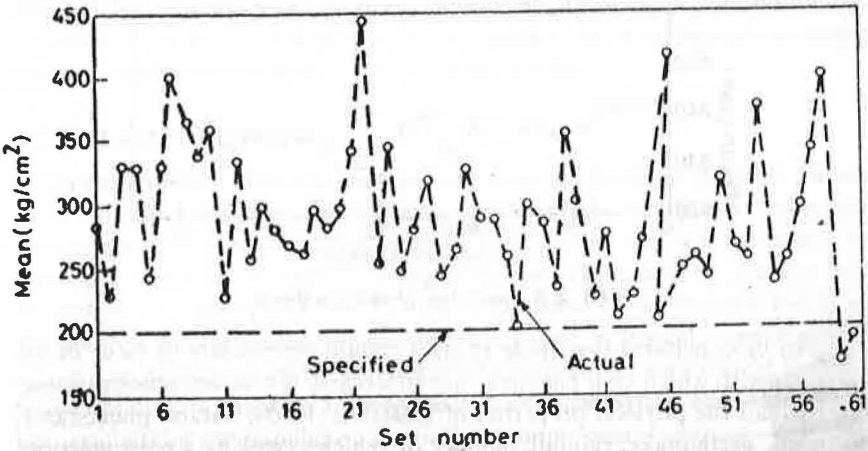


FIG. 2.1 Set variation of cube strength of concrete

strength of concrete of a mix is subject to random variations and it is not possible to predict the exact outcome of the test.

If a survey on live load on buildings is conducted, it may be observed that the intensity of live load varies from bay to bay. A typical variation of floor load intensity (FLI) of an office building (2.2, 2.3) is shown in Fig 2.2. The occurrence of live load is purely a random phenomenon. It varies with time. It has been found that there is a variation from room to room in the same floor, from floor to floor in the same building, and from building to building.

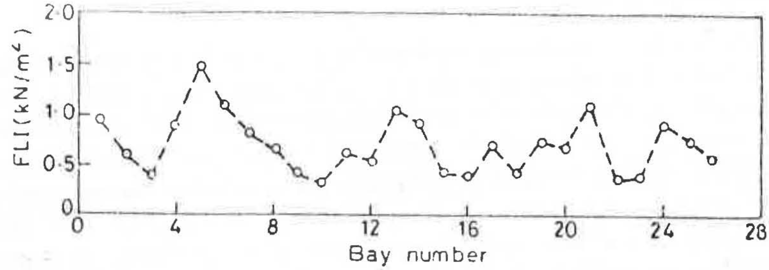


FIG. 2.2 Variation of floor load intensity

Normally there is sufficient control in the production of a particular size of a material or the raw materials in the products. However, there will be some variations. This can be observed in the production of bricks or steel bars or casting of concrete members. The variation of the mean deviation of column depth, observed in a building project (2.4), is shown in Fig. 2.3. Variations are generally small. Variations are more pronounced in natural phenomenon, e.g. wind, rainfall, stream flow, height of ocean waves, etc. Figure 2.4 gives the observed data on yearly maximum wind speed at Bangalore (2.5). It can be again observed that one cannot definitely tell what will be the maximum wind speed in the coming year. The wind speed is probabilistic in nature.

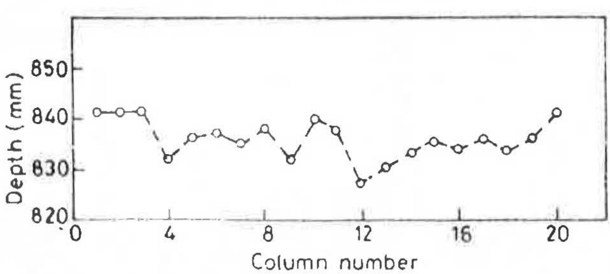


FIG. 2.3 Variation of column depth

It can be concluded that there exists a certain uncertainty in many of the variables with which civil engineers are concerned. There are inherent variations in all the physical properties of materials, loads, natural phenomena, viz. wind, earthquake, rainfall, number of vehicles crossing a road junction,

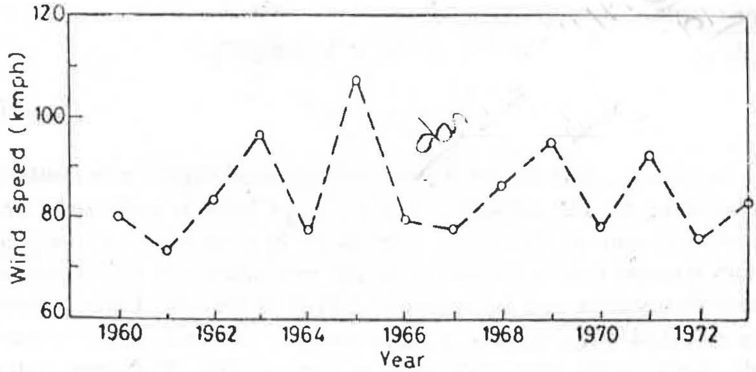


FIG. 2.4 Variation of yearly maximum wind speed at Bangalore

etc. The outcome of the event is not determinate and it is purely random or probabilistic in nature. The degree of uncertainty or variations vary from sample to sample.

Having accepted that there is bound to be some pattern of variations inherent in all observed data, it is now the problem for an engineer to take decisions based on the data. If the degree of variability is small and not of practical significance, he may decide to ignore it by simply estimating the variable with its average of all observations. If the degree of uncertainty is significant, the engineer may decide to have a conservative estimate of the variable. For example, he may choose the lowest value of the observation instead of the average. Such conservative decisions may not always result in economical designs and these estimates are at best the 'rules of thumb' for the design to circumvent the challenges posed by the uncertainty.

However, in many problems the variations are too large to be overlooked and hence call for a statistical or probabilistic approach. Data obtained from measurements or experiments needs to be manipulated so that it presents to the engineer some useful information. Statistics provides a powerful tool for this purpose. A basic notion of statistics is the notion of variation. It is the science of making decisions on incomplete informations; that is, drawing conclusions from the observed data.

2.2 DATA REDUCTION

From the collected data one can see that there is a range of possible values of a random variable. The different values of the random variable are associated with different probabilities of their occurrence. It is necessary to replace a collection of data by a single number. One wants to give the best estimate of a random variable. Probably, the mean value (average value) of the random variable is commonly used in getting a typical representation of a group of data. An infinite number of observations are required in order to determine this quantity. In practice it is impossible; one can only get a best estimate of the population mean by a sample mean.

Sample Mean

Sample mean $\bar{X}$ of a random variable X, is defined as

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{2.1}$$

where $x_1, x_2, \ldots, x_n$ is the sequence of the observed values. For illustration, a set of data on the compressive strength of brick is given and its mean value worked out in Table 2.1. The mean is a measure of the central tendency (central value). This is by far the best statistic to numerically summarize a distribution and the centre of gravity of the data. For a given data, if one is asked to give only a single number, he would probably use this sample mean as his best prediction of the variable. The mean value is highly susceptible to extreme values of the observed data. Other measures of the central tendency are the mode and the median.

TABLE 2.1 *Computation of mean, SD and CV of a set of data*

Sl. No.	Strength of brick (N/mm^2)	$\lvert x_i - \bar{X} \rvert$	$(x_i - \bar{X})^2$
1.	29.0	3.1	9.6
2.	27.7	1.8	3.24
3.	29.7	3.8	14.44
4.	21.4	4.5	20.25
5.	24.7	1.2	1.44
6.	25.2	0.7	0.49
7.	20.0	5.9	34.81
8.	27.1	1.2	1.44
9.	29.7	3.8	14.44
10.	30.8	4.9	24.01
11.	20.6	5.3	28.09
12.	20.6	5.3	28.09
13.	30.1	4.2	17.64
14.	28.0	2.1	4.41
15.	23.9	2.0	4.00

$n = 15 \quad \Sigma = 388.5$ $\qquad\qquad$ $\Sigma = 206.40$

$\bar{X} = \dfrac{388.5}{15}$ $\qquad\qquad$ Variance $= \dfrac{206.40}{n-1} = 14.74$

$= 25.9$ N/mm^2 $\qquad\qquad$ $s = \sqrt{14.74} = 3.84$ N/mm^2

$\delta = \dfrac{3.84}{25.9} = 0.148$

Mode is the most frequently observed data whereas the median is the middle value of the observation when the values are arranged in the ranked order of magnitude. If the number of observations are even, then the average of the two middle observations is taken as median. Mode is not unique.

For the given set of data, it is desirable to specify a number which gives an idea of the dispersion variability of the observations. The range, the standard deviation (SD) and the coefficient of variation (CV) are the general measures of dispersion.

Range

The range R is given by

$$R = x_l - x_s \tag{2.2}$$

where x_l and x_s are the largest and smallest values of n values of the observations respectively. It is seldom used as a descriptive parameter of population since it indicates very little about the way the distribution appears inside the interval of values. However, this measure is attractive mainly because it is computationally convenient and simple.

The amount of scatter is clearly dependent on how much the set of values deviates from the central value. The greater the scatter, the larger the total deviation. The standard deviation is a measure of dispersion.

Standard Deviation

This is defined as the positive square root of the average squared deviation from the mean, i.e.,

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{X})^2 \tag{2.3}$$

where s is the standard deviation. The above formula gives an estimate of s. An estimator whose expected value is not equal to the parameter it has estimated is said to be a biased estimator. The unbiased estimate of s is given by

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{X})^2 \tag{2.4}$$

Variance

This is defined as the square of the standard deviation. It is difficult to say, purely on the basis of standard deviation or variance whether the dispersion is large or small. This is meaningful only relative to the central value. For this reason, the coefficient of variation (CV) is often preferred and it is a convenient measure for comparing the relative dispersion of more than one kind of data.

Coefficient of Variation

This is defined as

$$\delta = \frac{s}{\bar{X}} \tag{2.5}$$

where δ is the coefficient of variation.

The calculations of variance, SD and CV, are illustrated in Table 2.1.

2.3 HISTOGRAMS

The preceding section was mainly concerned with the collection of data and the calculation of mean, SD and CV of a set of observations. Next step is the presentation of the collected data in a useful form. The observations are made and noted down as they occur and hence the collected data will be in an unorganised form. This unorganised data is arranged properly. The values are marked in increasing order. These ordered values are then divided into intervals and the number of observations (frequency of observations) in each interval is plotted as bar. The plot obtained is called a histogram. For plotting histograms, the approximate number of intervals may be selected by using the following formula (2.6):

$$a = 1 + 3.3 \log_{10} n \tag{2.6a}$$

where a = number of intervals between the minimum and maximum values and

n = sample size (number of observations)

If the proper interval for drawing a histogram is not taken, the plot may not give the correct picture of the underlying distribution of the variable.

Let the length of the brick be considered as a variable. A sample of 400 bricks are tested. Using Eq. (2.6a), the number of intervals for drawing a histogram for a sample size of 400 is

$$a = 1 + 3.3 \log_{10} 400 = 9.59 \tag{2.6b}$$

The grouped data on the length of bricks is given in Table 2.2. For this grouped data, the histogram of the length of brick is shown in Fig. 2.5a. The histogram gives the investigator an immediate impression of the range of the data, its most frequently occurring values and the degree to which it is scattered.

TABLE 2.2 *Grouped data on length of brick for drawing histogram*

Range (mm)	Frequency	Relative frequency	Cumulative frequency
221-223	1	0.0025	0.0025
223-225	3	0.0075	0.0100
225-227	25	0.0625	0.0725
227-229	71	0.1775	0.2500
229-231	92	0.2300	0.4800
231-233	88	0.2200	0.7000
233-235	75	0.1875	0.8875
235-237	33	0.0825	0.9700
237-239	10	0.0250	0.9950
239-241	2	0.0050	1.0000
	n 400	1.0000	

Relative frequency is obtained by dividing the number of observations in an interval by the total number of observations. The calculation of relative frequency is illustrated in Table 2.2. In Fig. 2.5a, the relative frequency is

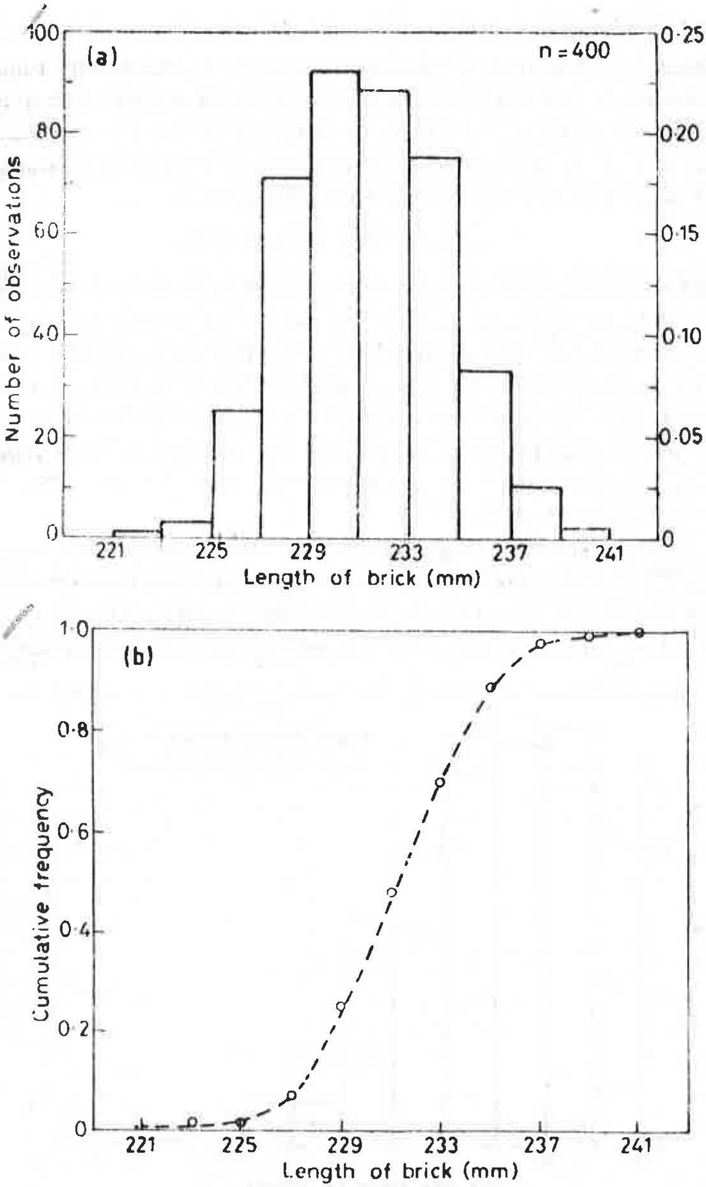


FIG. 2.5 Histogram and cumulative frequency of length of brick

also marked on the Y-axis on the right side. The relative frequency yields the investigator an immediate idea that what is the chance of the variable lying within a specified range. From Table 2.2 and Fig. 2.5a, it can be seen that the chance (probability) of a value for a length L lying between 229 and less than 231 is 0.23. That is

$$P(229 \leqslant L < 231) = 0.23$$

where $P(X)$ should be read as the probability of X.

In Table 2.2, cumulative frequency has also been computed. The cumulative frequency—"less than a particular value"—is obtained by adding the frequencies one by one starting from the top of the frequency table. Similarly, cumulative relative frequency can be computed as shown in Table 2.2. From the table it can be interpreted that the chance of getting a value for the length of brick less than 231 mm is 0.48. That is

$$P(L < 231) = 0.48$$

The cumulative frequency diagram of the length of brick is shown Fig. 2.5b. From this diagram one can quickly say what is the chance of getting a value for a length less than a particular value. For instance, this is equal to 0.7 for the specified value of a length equal to 233. Frequency distributions of the field data on the strength of M 15 concrete, floor live load in office building, yearly maximum wind speed and the strength of over-reinforced prestressed concrete beam are shown in Figs. 2.6, 2.7, 2.8 and 2.9 respectively (2.1, 2.3, 2.5, 2.7).

The histograms shown in Figs. 2.6 to 2.10 have different shapes. It can be seen in Fig. 2.5 that the histogram is symmetrical about the mean whereas other histograms are not; that is, they are skewed. Whether a histogram is symmetrical or not can be found by computing the coefficient of skewness.

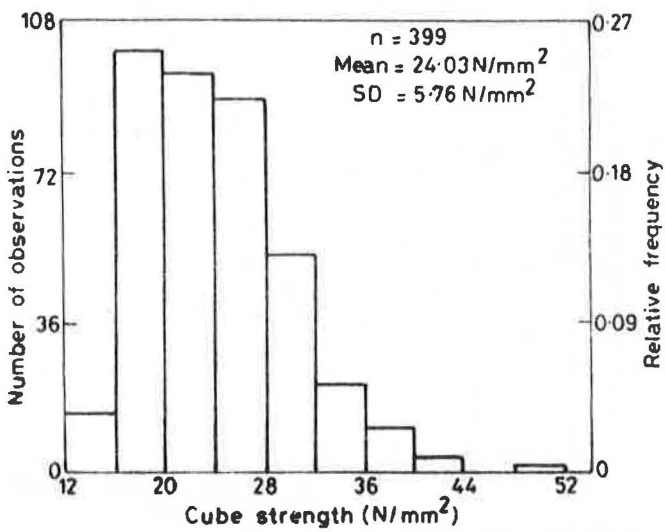


FIG. 2.6 Frequency distribution of M 15 concrete

Coefficient of Skewness

The sample coefficient of skewness is related to the third moment about the mean. The coefficient of skewness e_1 is given by

$$e_1 = \frac{1}{s^3}\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^3\right] \tag{2.7}$$

The coefficient of skewness is a measure of skewness or asymmetry about

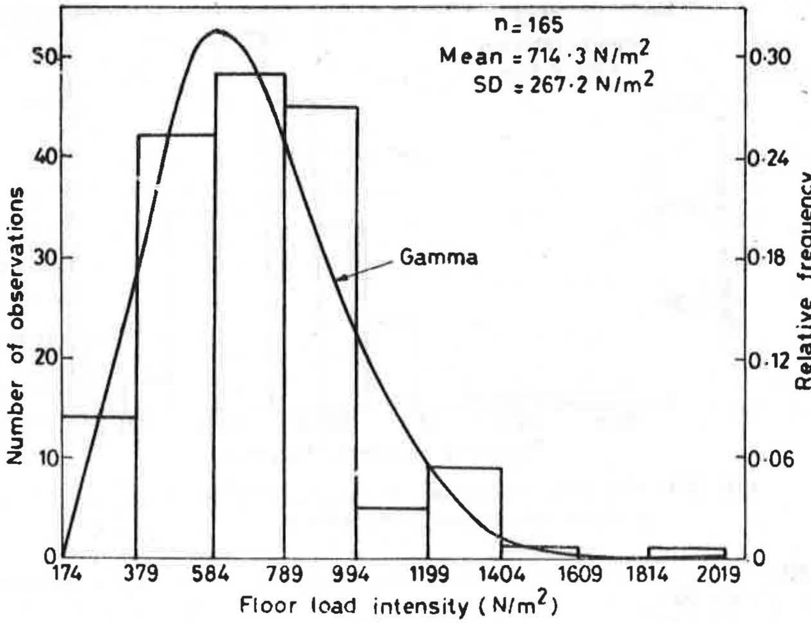


FIG. 2.7 Frequency distribution of floor live load in office building

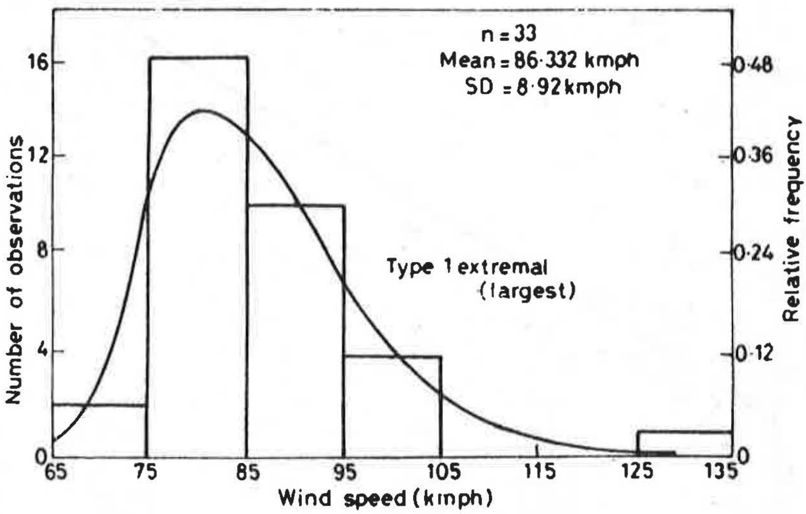


FIG. 2.8 Frequency distribution of yearly maximum wind speed at Colaba

the mean. The coefficient is positive for histograms skewed to the right (i.e. with longer tails to the right) and negative for those skewed to the left (i.e. with longer tails to the left).

2.4 SAMPLE CORRELATION

Engineers on many occasions may have to deal with two variables of

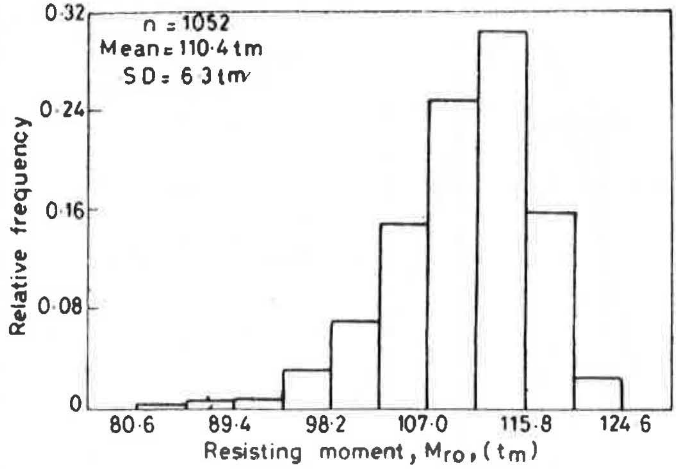


FIG. 2.9 Frequency distribution of resisting moment of an over-reinforced prestressed concrete beam

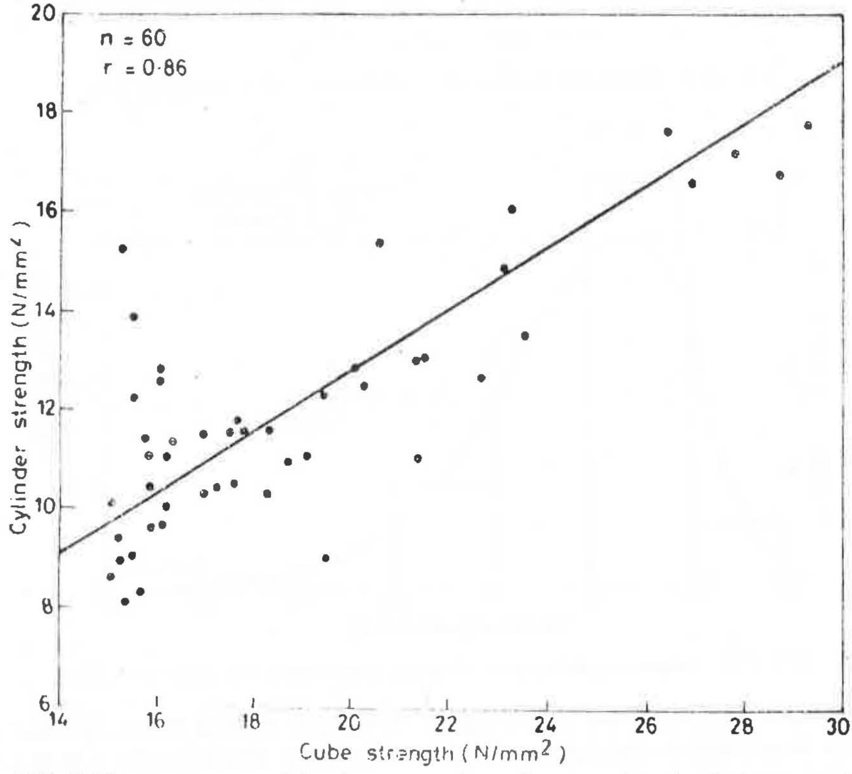


FIG. 2.10 Scattergram of the data connecting cube strength and cylinder strength of concrete (2.4)

related interest; one variable may depend on the other. When pairs of data of two variables are plotted as shown in Fig. 2.10, a plot called scattergram is obtained. A numerical summary of the tendency of the high values of one variable X pairing with the high values of the other variable Y or, the high values of X pairing with the low values of Y is given by the sample covariance s_{XY}, which is defined as

$$s_{XY} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{X})(y_i - \bar{Y}) \tag{2.8}$$

If s_{XY} is positive, it means that the high values of X pair with the high values of Y and if s_{XY} is negative, the low values of X pair with the high values of Y.

Sample Correlation Coefficient

The sample correlation coefficient is obtained by normalizing the sample covariance with standard deviations. The sample correlation coefficient, r_{XY}, is given by

$$r_{XY} = \frac{s_{XY}}{s_X s_Y} = \frac{1}{n-1} \sum_{i=1}^{n} \left(\frac{x_i - \bar{X}}{s_X}\right)\left(\frac{y_i - \bar{Y}}{s_Y}\right) \tag{2.9}$$

r_{XY} is a dimensionless quantity and its value varies from -1 to $+1$. The correlation coefficient gives a measure of the degree of the linear dependence of the two variables. If r_{XY} is equal to 1, variables are perfectly positively correlated, and if r equal to -1, variables are perfectly negatively correlated. If $r_{XY} = 0$, there is no linear dependence between the two variables. Calculations of the sample covariance are illustrated in Table 2.3.

TABLE 2.3 *Computation of sample covariance and correlation coefficient*

Sl. No.	Cube strength x_i (N/mm²)	Cylinder strength y_i (N/mm²)	$(x_i - \bar{X})$	$(y_i - \bar{Y})$	$(x_i - \bar{X})(y_i - \bar{Y})$
1.	15.17	9.86	−6.565	−3.955	+25.965
2.	17.92	11.29	−3.815	−2.525	+ 9.633
3.	20.13	12.48	−1.605	−1.335	+ 2.143
4.	22.54	14.65	+0.805	+0.835	+ 0.672
5.	24.80	15.38	+3.065	+1.565	+ 4.797
6.	18.67	11.95	−3.065	−1.865	− 5.716
7.	22.91	14.43	+1.175	+0.615	+ 0.723
8.	27.70	18.00	+5.965	+4.815	+28.721
9.	29.24	18.42	+7.505	+4.605	+34.561
10.	18.27	11.69	−3.465	−2.125	+ 7.363
	Σ 217.35	138.15			108.862

$\bar{X} = 21.735;\quad \bar{Y} = 13.815;\quad s_X = 4.533;\quad s_Y = 2.868$

$$s_{XY} = \left(\frac{1}{10-1}\right)(108.861) = 12.096$$

$$r_{XY} = \frac{12.096}{4.533 \times 2.868} = 0.93$$

$S_X = \sqrt{\frac{\Sigma d^2}{n}}$

From Table 2.3, it is noted that $r_{XY} = 0.93$, indicating that the cube strength and the cylinder strength of concrete are linearly positively correlated.

REFERENCES

2.1 Dayaratnam, P. and R. Ranganathan, "Statistical Analysis of Strength of Concrete" *Building and Environment*, Vol. II, Pergamon Press, 1976, pp. 145–152.

2.2 Prabhu, U.P., "Stochastic Analysis of Live Loads in Office Buildings", *M. Tech. Thesis*, I.I.T., Bombay, 1984.

2.3 Prabhu, U.P. and R. Ranganathan, "Stochastic Analysis of Live Loads in Office Buildings", *Proceedings of the National Conference on Quality and Reliability*, I.I.T., Bombay, Dec. 1986, pp. 275–291.

2.4 Ranganathan, R. and C.P. Joshi, "Statistical Analysis of Strengths of Concrete and Steel and Dimensional Variations", *Report No. DS and T: 4(I)/83/STP–III/2*, Civil Engineering Dept., I.I.T., Bombay, March, 1985.

2.5 Ranganathan, R., "Statistical Analysis of Wind Speed and Statistics of Wind Load for Probabilistic Criterion", *Report No. DS and T: 4(I)/83/STP–III/4*, Civil Engineering Dept., I.I.T., Bombay, March 1986.

2.6 Benjamin, J.R. and C.A. Cornell, *Probability, Statistics and Decision for Civil Enigneers*, Mc-Graw-Hill, New York, 1970.

2.7 Ranganathan, R. and P. Dayaratnam, "Reliability Analysis of Prestressed Concrete Beams", *Bridge and Structural Engineer*, Vol. 8, No. 2, June 1978, pp. 11–24.

EXERCISE

2.1 The test results of the compressive strength (N/mm²) of 50 concrete cubes obtained from a building project are given below:

17.24	16.18	16.53	15.20	18.40
19.73	17.24	20.53	19.38	23.42
17.60	18.76	20.00	20.36	20.27
19.82	20.09	21.78	19.82	19.11
21.42	22.31	21.86	21.15	20.36
13.60	14.98	15.08	18.01	14.93
13.96	15.64	15.56	16.09	13.96
13.87	15.75	12.11	17.18	16.20
15.65	16.27	14.83	13.24	15.03
13.96	15.58	17.36	16.29	16.71

Calculate the mean, the standard deviation, and the coefficient of variation of the strength of concrete for the given data. Plot a histogram. Determine the chance of getting a value less than 15 N/mm².

(Ans. $\bar{X} = 17.41$, $s = 2.76$, $P(X < 15) = 0.37$)

2.2 Samples of soil are collected from various depths below ground level and tested in the laboratory to determine their shear strength. The collected field data are given below:

Depth (m)	2	3	4	5	6	7
Shear (kN/m²) strength	14.8	20.3	32.2	39.0	42.0	56.4

Determine the sample covariance and correlation coefficient between the depth of the soil and its shear strength. What do you infer?

(Ans. $s_{XY} = 27.99$; $r_{XY} = 0.987$)

2.3 For the data given in Exercise 2.1, determine the coefficient of skewness. What do you infer? (Ans. 0.27)

2.4 What do you understand when you get a negative correlation for a given set of data? Give an example in a civil engineering field where negative correlation appears?

3

Probability Theory

3.1 INTRODUCTION

In every walk of life people make statements that are probabilistic in nature and that carry overtones of chance; example, we might talk about the probability that a bus will arrive on time, or that it may rain tomorrow, or that a child to be born will be a son, or a flood may occur in a river this year . . . , and so on. What is the characteristic feature in all the above phenomena? It is that they all lack a deterministic nature. Past informations, no matter how voluminous, will not allow us to formulate a rule, and to determine precisely what will happen when the experiment is repeated. Phenomena of the above type are called *random phenomena*. The theory of probability involves their study. Variables in engineering problems can be classified as shown in Fig. 3.1. In a deterministic study, parameters may be considered as a function of time (time variant) or in some problems they may be independent of time (time invariant). Similarly, in a probabilistic study variables may be treated as time invariant or in many cases time variant (e.g. wind load, ocean-wave height, earthquake, etc.). When a random variable assumes values as a function of time, the variable is called a stochastic variable. The probabilistic study of stochastic variables is called stochastic process or random process. In most engineering problems, random variables of interest are stochastic in nature. However, for simplicity, variables are considered as time invariant. This chapter deals with random variables which are not stochastic.

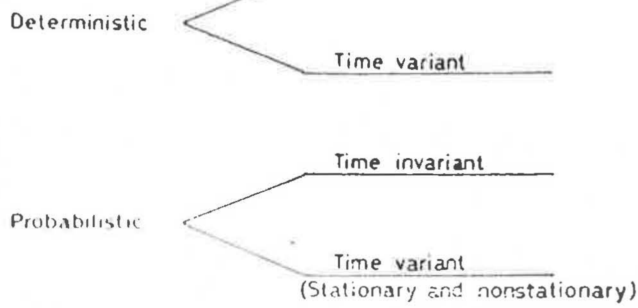


FIG. 3.1 Variables and classification

3.2 RANDOM EVENTS

Preliminary Ideas

Before discussing probability theory a few preliminary ideas that are used in subsequent discussions are introduced in this section. The first is the concept of an event.

Sample Space and Events

Consider a number of persons boarding a bus at a particular bus stop. A survey is carried out daily at the same time. The capacity of the bus is 60. Let X be the number of persons boarding the bus. It can be seen that X can assume value 0, 1, 2, 3, . . . , 60. Each one is a possible outcome of the experiment (experiment in a general sense; here experiment is the counting of the number of persons boarding the bus). Each of these outcomes is called a sample point. The collection of all these possible outcomes of the experiment is called a sample space. Hence this sample space consists of a set S of points called sample points. Each of these outcomes, a sample point, is called a simple or elementary event. Let a simple event be denoted by E_i, the subscript i, here, denoting the number of persons. Then for this example, there are sixty-one simple events denoted by E_0. $E_1, E_2, \ldots,$ $E_i, \ldots, E_{60}$, where E_i is an event representing the occurrence of the variable X taking a value i. The sample space for this experiment is

$$S = \{E_0, E_1, E_2, \ldots, E_{59}, E_{60}\}$$

One may be interested in the collection of a set of outcomes in an experiment. In this case, one may note down the number of persons boarding the bus—(i) less than 4 or (ii) greater than 5 and less than 10. Such events are called compound events or simply events. If A is the event representing the number of persons less than 4, then sample points in the event A are

$$A = \{E_0, E_1, E_2, E_3\}$$

and if the event B is defined as the number of persons boarding greater than 5 and less than 10, then sample points in the event B are

$$B = \{E_6, E_7, \ldots, E_9\}$$

If C is an event denoting the number of persons boarding the bus greater than or equal to 4, then

$$C = \{E_4, E_5, \ldots, E_{60}\}$$

when events A and C are compared, it can be seen that event C consists of all points that are not included in event A. Such two events are called complementary events. A is the complement of C.

The formal definitions of various events are given below:

(i) Simple event: An event consisting of a single sample point. A simple event cannot be decomposed into a combination of other events.

(ii) Compound event: An event made up of two or more sample points.

(iii) The complement event A^c of the event A consists of all sample points in the sample space not included in the event A. In the cited example

$$A^c = C$$

(iv) Certain event: An event constituting all sample points in the sample space.

(v) Null event is the complement of a certain event and is generally designated as ϕ.

In the experiment—number of persons boarding a bus–the sample points have individually discrete entities and are countable. Such a sample space is called a countably discrete sample space. If such a sample space has finite number of points, it is called countably finite discrete sample space. A second example of the finite sample space is the observation of the number of days in a year with temperature higher than say 30°C at a particular location. Each day of the year is a possible sample point. The sample space consists of 365 sample points. This is a discrete finite (countably) sample space. Another example is the observation of a successful bidder among the number of contractors bidding for a particular job.

Sometimes a discrete sample space may have sample point which are countably infinite. For example, the number of persons undergoing an ear operation in a year in the whole world. In this case the number of persons could be theoretically any integer from zero to infinity. Such a sample space is called a discrete (countably) infinite sample space. Another example is the observation of the number of accidents along a busy road during a year.

Many engineering problems or physical situations involve measurements. Consider the experiment, the measurement of a deflection during the load test of a reinforced concrete beam. It may be possible to get any value (noninteger) of the deflection starting from zero with the instrument (dial gauge). If the least count of the dial gauge is 0.001 mm, deflections could be obtained at an increment of 0.001 mm starting from zero. The sample points may be 0.000, 0.001, 0.002, 0.003, . . . n where n is the number of points which may be effectively large. The sample space will have a continuum of sample points. Such a sample space is called a continuous sample space. For convenience, a continuous sample space is defined from 0 to ∞, i.e. any value greater than zero is assumed as a possible outcome of an experiment. In some situations, the variable of interest may assume negative values. For example, the deflection of a simply supported prestressed concrete beam. During the initial loading stages of the beam, the beam will have an upward (negative) deflection, and after a certain level of the external load, a downward (positive) deflection. In this case, the interval $-\infty$ to $+\infty$ becomes the sample space. Another example of this case is the measurement of error.

In some physical situations, it may be known from physical conditions that a continuous variable of interest can assume a value within a finite interval only. For example, (i) wind directions can be observed from 0° to 360° and this finite interval becomes the sample space, (ii) the strength of M 15 concrete if one assumes that the strength value cannot exceed 40 N/mm² or be less than 8 N/mm². Then the interval 8 to 40 N/mm² is the sample space. Events in a continuous sample space can also be defined. A few examples are given.

EXAMPLE 3.1 Consider a traffic engineer noting down the number of vehicles on a small bridge at a particular instant. The maximum number of vehicles that can be at a time is 10. Sketch the sample space and show the events (i) observing less than four vehicles, and (ii) observing greater than 5 and less than 9 vehicles.

Solution: Let

$$E_i = \text{the event observing } i \text{ vehicles.}$$

Hence, the sample space is

$$S = \{E_0, E_1, E_2, \ldots, E_{10}\}$$

and this is shown in Fig. 3.2. This is a finite discrete sample space.

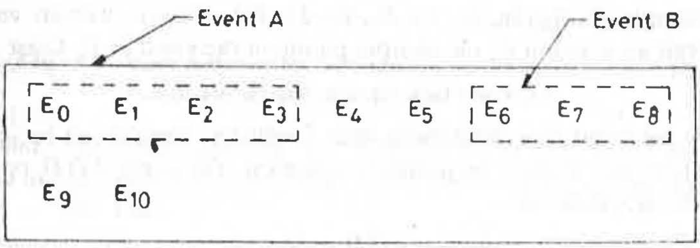


FIG. 3.2 Sample space and events—Example 3.1

Let

$$A = \text{the event observing} < 4$$

$$B = \text{the event observing} > 5 \quad \text{and} \quad < 9$$

These events are also shown in Fig. 3.2.

EXAMPLE 3.2 An engineer at an airport is measuring the wind speed at regular intervals of time. Sketch the sample space and mark the event A observing the wind speed less than 40 kmph and the event B observing the wind speed 60 kmph.

Solution The possible outcomes of the measurement of wind speed can be from 0 to ∞. Hence, this is a continuous sample space. This is sketched in Fig. 3.3. Events A and B are also marked in the same figure.

In many problems we are interested in events which are actually the combinations of two or more events. Although the reader must surely be familiar with these terms, let us review them briefly.

Let us now define relations between events. Consider counting the number

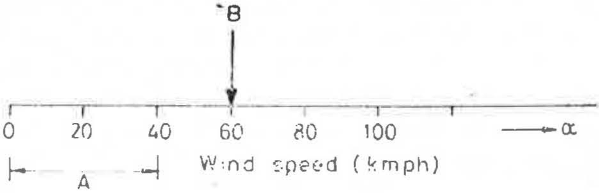


FIG. 3.3 Sample space and events—Example 3.2

of vehicles on a bridge at a time (Example 3.1). The sample space is

$$S = \{E_0, E_1, E_2, E_3, \ldots, E_{10}\}$$

Let A = the event observing < 4 vehicles

C = the event observing > 2 and < 7 vehicles

Then we have the following relations between A and C:

(i) Both A and C occur together. This situation will happen when the simple event E_3 occurs. This is written as $A \cap C$ and it is read as A intersection C.

(ii) Either A or C or both occur. In the present example, this is the event having sample points (E_0, E_1, E_2, E_3, E_4, E_5, E_6). This is written as $A \cup C$ and is read as A union C, i.e. sample points in the event $A \cup C$ are

$$A \cup C = (E_0, E_1, E_2, E_3, E_4, E_5, E_6)$$

Let D be the event observing more than 7 vehicles. Then it can be seen that the events A and D have no points in common. The event $A \cap D$ is impossible. This is written as

$$A \cap D = 0$$

and A and D are called mutually exclusive or disjoint events. Relations $A \cup C$ and $A \cap C$ are marked in Fig. 3.4.

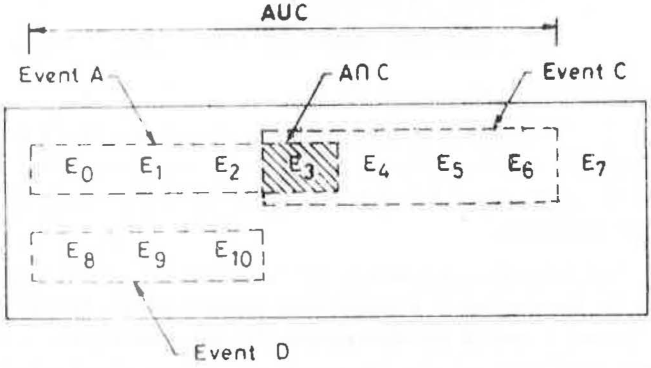


FIG. 3.4 Relationship among events

The union and the intersection of events are best understood by drawing Venn diagrams shown in Fig. 3.5. In Venn diagrams, the sample space is represented by a rectangle while events are represented by regions within the rectangle.

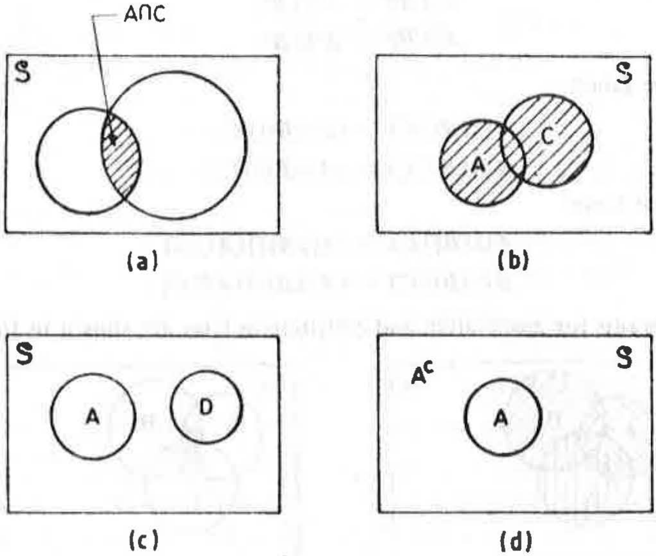


FIG. 3.5 Venn diagrams for (a) intersection, (b) union, (c) mutually exclusive and (d) complement of events

Consider the observation of the direction of wind speed at an airport. This has a continuous sample space with the variable taking any value from 0 to 360°. Let

A be the event observing wind direction $\leqslant 100°$

and B is the event observing wind direction $> 100°$

If the accuracy of the measuring instrument is 0.1°, then

$$\mathcal{S} = (0.0, 0.1, 0.2, \ldots, 100.0, 100.1, \ldots, 360)$$
$$A = (0.0, 0.1, \ldots, 100)$$
$$B = (100.1, 100.2, \ldots, 360)$$

It can be seen that the union of the two events A and B contains all sample points in the underlying sample space and these two are collectively exhaustive. In general, two or more events are collectively exhaustive if the union of all these events constitutes the underlying sample space.

A listing of a number of important laws obeyed by the combination of the events is given below without formal proofs.

Identity laws	: $A \cup \phi = A$,	$A \cap \mathcal{S} = A$
	$A \cup \mathcal{S} = \mathcal{S}$,	$A \cap \phi = \phi$
Idempotent laws	: $A \cup A = A$,	$A \cap A = A$
Complement laws	: $A \cup A^c = \mathcal{S}$,	$A \cap A^c = \phi$
Commutative laws	: $A \cup B = B \cup A$	
	$A \cap B = B \cap A$	

DeMorgan's Law The complement of the union and intersection of events is the intersection and union of their respective complements.

$$(A \cup B)^c = A^c \cap B^c \tag{3.1}$$

$$(A \cap B)^c = A^c \cup B^c \tag{3.2}$$

Associative Laws:

$$A \cup (B \cup C) = (A \cup B) \cup C \tag{3.3}$$

$$A \cap (B \cap C) = (A \cap B) \cap C \tag{3.4}$$

Distributive Laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{3.5}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{3.6}$$

Venn diagrams for associative and distributive laws are shown in Fig. 3.6.

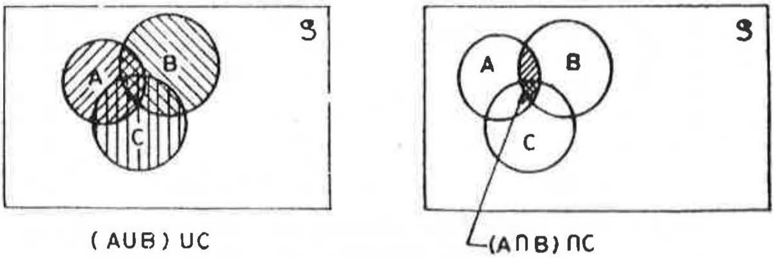


FIG. 3.6 Venn diagrams for associative and distributive laws

Probability Measure and Axioms

The empirical notion of probability is that of relative frequency; the ratio of the total number of occurrences of a situation to the total number of times the experiment is repeated. When the number of trials is large, the relative frequency provides a satisfactory measure of the probability associated with a situation of interest.

A random experiment is a repetitive process or operation that in a single trial, may result in any one of a number of possible outcomes such that a particular outcome is determined by chance, and is impossible to predict. Under a given set of conditions, a random experiment has N exhaustive, mutually exclusive and equally likely outcomes $A_1, A_2, \ldots, A_N$. If M of the outcomes are associated with the occurrence of an event A and $N-M$ outcomes with the nonoccurrence of A, the probability of the occurrence of A is (M/N), i.e.,

$$P(A) = \frac{M}{N}$$

If an experiment has a sample space and an event A is defined on S, then $P(A)$ is a real number called the probability of the event A, or the probability of A, and this $P(A)$ must satisfy the following axioms:

(i) For each event A of S

$$0 \leqslant P(A) \leqslant 1$$

(ii) $P(S) = 1$

(iii) If $A_1, A_2, \ldots$ are denumerable mutually exclusive events defined on S, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

For a finite number of mutually exclusive events, say k,

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) = \sum_{i=1}^{k} P(A_i) \tag{3.7}$$

The probability of any event is the sum of the probabilities assigned to the sample points within which it is associated.

From the axioms of the probability theory, the following formulae can be obtained (students are expected to prove them) by drawing Venn diagrams:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{3.8}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \tag{3.9}$$

This can be extended to the union of a number of events.

EXAMPLE 3.3 During the route survey of a transport mini bus, 100 observations of the total number of persons travelling by the bus on a particular length of the route yielded the following results (Table 3.1). Observations have been made at random.

TABLE 3.1 *Data for Example 3.3*

No. of persons	Number of observations	Relative frequency
0	0	0
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	1	0.01
7	2	0.02
8	1	0.01
9	3	0.03
10	4	0.04
11	4	0.04
12	2	0.02
13	1	0.01
14	5	0.05
15	6	0.06
16	8	0.08
17	16	0.16
18	14	0.14
19	17	0.17
20	16	0.16
Total number	**100**	**$\Sigma = 1.00$**

(Note: Capacity of minibus = 20)

Define

$$A = \text{more than 15 persons travelling by the bus}$$
$$B = > 12 \text{ and } < 18 \text{ persons travelling by the bus}$$

Solution Assuming the number of persons travelling are mutually exclusive events, we can use the relative frequencies given in Table 3.1 to represent the corresponding probabilities.
Hence,

$$P(A) = 0.08 + 0.16 + 0.14 + 0.17 + 0.16$$
$$= 0.71$$
$$P(B) = 0.01 + 0.05 + 0.06 + 0.08 + 0.16$$
$$= 0.36$$

The verification of Eq. 3.8 is

$$(A \cap B) = (E_{16}, E_{17})$$

Hence, $$P(A \cap B) = 0.08 + 0.16 = 0.24$$

According to Eq. 3.8,

$$P(A \cup B) = 0.71 + 0.36 - 0.24 = 0.83$$

This can also be calculated as

$$(A \cup B) = (E_{13}, E_{14}, \ldots, E_{19}, E_{20})$$
$$= 0.01 + 0.05 + 0.06 + 0.08 + 0.16 + 0.14 + 0.17 + 0.16$$
$$= 0.83$$

Hence, the theorem (Eq. 3.8) is verified.

Two Dimensional Sample Space

Consider the same experiment discussed in Sec. 3.2, namely the number of persons boarding a bus at a bus stop. Instead of counting the total number of persons, one is interested to note down how many males and females board the bus. Let

$$E_{ij} = \text{the event representing } i \text{ men and } j \text{ women boarding a bus}$$

Then the sample space for such a case can be sketched as shown in Fig. 3.7. The experiment in two-dimensional space involves an observation of 2 numbers at the same time.

Another example is that an airport engineer may be interested to note down the wind speed and the wind direction for the orientation of an airport. This is a continuous two-dimensional sample space which is shown in Fig. 3.8.

In some situations it is also possible to have a discrete-continuous sample space. For example, in Example 3.1 if the traffic engineer records not only the number of vehicles on the bridge, but also the total weight of the vehicles on the bridge at the same time, for such a case, assuming the

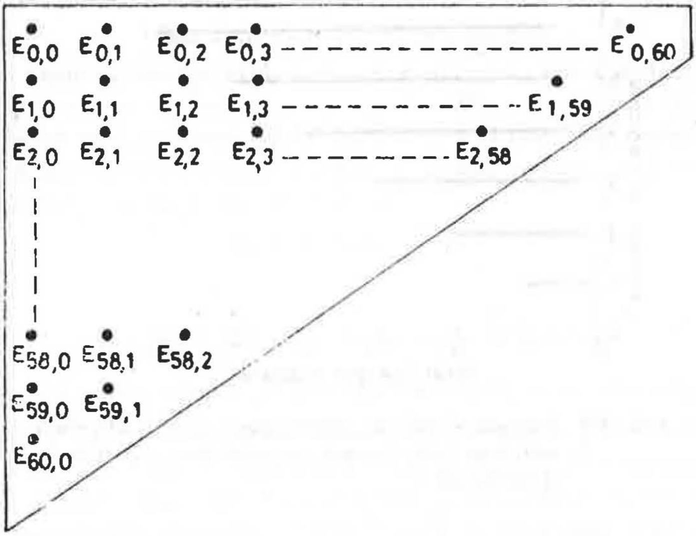


FIG. 3.7 Two-dimensional sample space-number of men and women boarding a bus (*Note* $i + j > 60$)

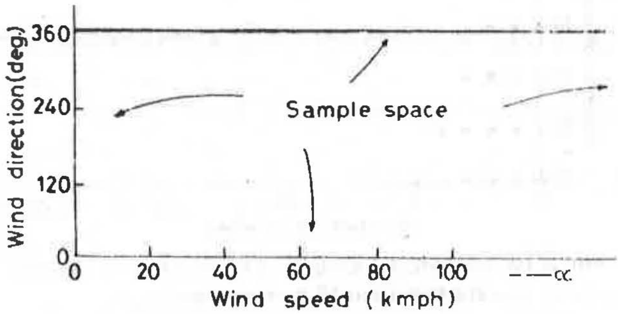


FIG. 3.8 Continuous two-dimensional sample space-wind speed and direction

minimum weight of a vehicle is 30 kN and the maximum 300 kN, and the maximum number of vehicles on the bridge at a time is 6, the sample space will be as shown in Fig. 3.9.

Conditional Sample Space

If one is interested in the possible outcomes of an experiment, given that some event A has occurred, the set of events associated with the event A can be considered a new reduced sample space. In Example 3.3, given that 15 or more men have been observed, the number of women boarding the bus will have a reduced sample space as shown in Fig. 3.10. This is a conditional sample space.

Suppose that in sampling the number of persons bording a bus at a bus stop we restrict our observations only to women. Here, there is a new sample space including only part of the elementary events in the original sample space. This new reduced sample space is also a conditional sample space.

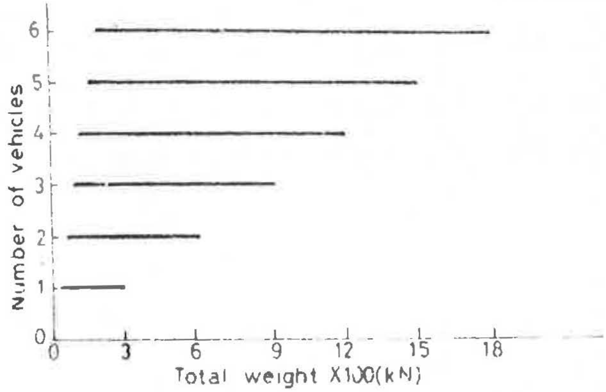


FIG. 3.9 Discrete continuous sample space: number of vehicles and their total weight observed over a small bridge at an instant

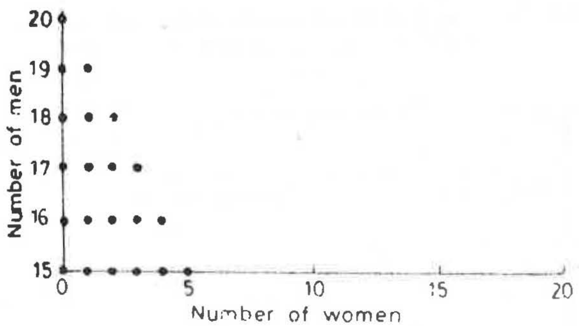


Fig 3.10 Conditional sample space-number of women in a bus given 15 or more men

Conditional Probability

As the title suggests, we are interested in the probabilities of events, given some condition. The conditional probability of an event A, given the occurrence of an event B, is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{3.10}$$

provided $P(B) \neq 0$. $P(A \mid B)$ is not defined if $P(B) = 0$.

EXAMPLE 3.4 In the previous Example 3.3, let us assume that it is given that 6 persons are travelling in the bus. Now under this condition, it is required to find out what is the chance of observing 3 or fewer women travelling in the bus.

Solution Let

T the event observing 6 persons travelling
F the event observing 3 or fewer women

The sample points in T are

$$(E_{6,0},\ E_{5,1},\ E_{4,2},\ E_{3,3},\ E_{2,4},\ E_{1,5},\ E_{0,6})$$

Let it be assumed that the probability of observing a man travelling in the bus is equal to that of observing a woman travelling in the bus. All probabilities in the reduced sample space must add up to one. The probability of observing each sample point in the reduced sample space is equal to 1/7.

Sample points in the event $(T \cap F)$ are

$$(E_{6,0},\ E_{5,1},\ E_{4,2},\ E_{3,3})$$

Hence,

$$P(F \mid T) = \frac{P(T \cap F)}{P(T)} = \frac{4}{7}$$

Note that all probabilities in the reduced sample space must add up to 1.

EXAMPLE 3.5 From a certain lot, 100 mild steel bars were selected at random and tested for their yield strength and ultimate strength. If a specimen has an yield strength less than the guaranteed yield strength and less than the guaranteed ultimate strength specified by code, we may define those cases as failures. Under this condition, it was found that 25% of the specimens had failed against yield strength, 20% against ultimate strength and 10% in both.

(i) If a specimen had failed against yield strength, what is the probability that it had also failed against ultimate strength?

(ii) If it had failed against ultimate strength, what is the probability it had also failed against yield strength?

(ii) What is the probability that a specimen failed either against yield strength or against ultimate strength?

Solution Let

$$Y = (\text{specimens which failed against yield strength})$$
$$Z = (\text{Specimens which failed against ultimate strength})$$

Given,

$$P(Y) = 0.25 \qquad P(Z) = 0.20$$
$$P(Y \cap Z) = 0.10$$

(i) The probability that a specimen also failed against ultimate strength, given that it had failed against yield strength is

$$P(Z \mid Y) = \frac{P(Y \cap Z)}{P(Y)} = \frac{0.10}{0.25}$$
$$= 0.4$$

(ii) The probability that a specimen also failed against yield strength, given that it had failed against ultimate strength is

$$P(Y \mid Z) = \frac{P(Y \cap Z)}{P(Z)} = \frac{0.10}{0.20}$$
$$= 0.5$$

(iii) The probability that a specimen failed either against yield strength or against ultimate strength or against both is

$$P(Y \cup Z) = P(Y) + P(Z) - P(Y \cap Z)$$
$$= 0.25 + 0.20 - 0.10$$
$$= 0.35$$

EXAMPLE 3.6 Two vehicles are approaching a road junction. The action of the driver of the following vehicle is dependent on the action of the leading vehicle. The probability of the leading vehicle turning right is 0.3 and the probability of the following vehicle turning right is 0.6. The probability of both the vehicles turning right is 0.1. Determine (i) the probability of the following vehicle turning right if the leading vehicle turns right.

Solution Let

L = the event that the leading vehicle is turning right

F = the event that the following vehicle is turning right

Given:

$$P(L) = 0.3, \qquad P(F) = 0.6$$
$$P(L \cap F) = 0.1$$

(i) $$P(F \mid L) = \frac{0.1}{0.3} = \frac{1}{3}$$

$$P(L \cup F) = 0.3 + 0.6 - 0.1 = 0.8$$

(ii) What is the probability of the following vehicle not turning right when the leading vehicle is not turning right? i.e. to determine $P(F^c \mid L^c)$.

$$P(F^c \mid L^c) = \frac{P(F^c \cap L^c)}{P(L^c)}$$

From DeMorgan's Law,

$$P(F^c \cap L^c) = P(F \cup L)^c$$
$$= 1 - P(F \cup L)$$
$$= 1 - 0.8 = 0.2$$

Hence

$$P(F^c \mid L^c) = \frac{0.2}{(1 - 0.3)} = \frac{2}{7}$$

(iii) What is the probability of the following vehicle not turning right when the leading vehicle turns right? i.e. determine $P(F^c \mid L)$

$$P(F^c \mid L) = \frac{P(F^c \cap L)}{P(L)}$$

But

$$P(F^c \cap L) = P(L) - P(L \cap F)$$
$$= 0.3 - 0.1 = 0.2$$

Hence

$$P(F^c \mid L) = \frac{0.2}{0.3} = \frac{2}{3}$$

Rule of Multiplication

The probability of the joint occurrence of two events is equal to the marginal probability of one of the events multiplied by the conditional probability of the other, given that the first event has occurred.

We can rewrite the formula, Eq.(3.10), to yield

$$P(A \cap B) = P(B) \cdot P(A \mid B) \tag{3.11}$$

This is called the general rule of multiplication of the probabilities and is extremely useful in many instances to find the probability that two events will occur simultaneously.

The above theorem can be extended to the joint probability of a number of random events $A_1, A_2, \ldots, A_n$:

$$P(A_1 \cap A_2, \ldots, \cap A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \ldots P(A_n \mid A_1 \cap A_2, \ldots, \cap A_{n-1})$$

EXAMPLE 3.7 Twelve concrete cubes are being cured in the laboratory. Out of them, 9 cubes were prepared from a batch of M 15 concrete mix and the other three belonged to M 42 concrete mix. During curing, the marking face of the cubes have been kept at the bottom by mistake. Now three cubes are drawn at random from the curing tank one after the other. Find the probability that all the three cubes belong to M 15 concrete.

Solution Let

A_1 = the event that the first cube is M 15 concrete

Similarly A_2 and A_3 are defined.

The probability that the first cube is M 15 concrete is 9/12 since 9 out of 12 cubes are M 15 concrete, i.e.

$$P(A_1) = \frac{9}{12}$$

If the first cube is M 15 concrete, then the probability that the next cube is M 15 concrete is 8/11 since only 8 of the remaining 11 cubes are M 15 concrete, i.e.

$$P(A_2 \mid A_1) = \frac{8}{11}$$

Similarly, it can be written that

$$P(A_3 \mid A_1 \cap A_2) = \frac{7}{10}$$

$(A_1 \cap A_2)$ implies that the first two cubes selected are M 15 concrete. Hence the probability that the first three cubes selected one after the other at random are M 15 concrete is

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)$$
$$= \left(\frac{9}{12}\right)\left(\frac{8}{11}\right)\left(\frac{7}{10}\right)$$
$$= \frac{21}{55}$$

Probability Tree Diagram

In practice, a finite sequence of experiments are conducted. Each experiment will have a finite number of outcomes with associated probabilities. A convenient way of describing such a process and computing the probability of any event is by a probability tree diagram, illustrated below. The multiplication theorem is used to compute the probability.

EXAMPLE 3.8 Consider a reinforced concrete rectangular beam. The ultimate strength of beam is a function of the cube strength of concrete, f_{cu}, the yield strength of steel, f_y, and other parameters. If the cube strength of concrete and steel and other parameters are considered as not subjected to random variations, the given beam is under-reinforced deterministically. However, if f_{cu} and f_y are subjected to random variations, then the beam may be under-reinforced or over-reinforced, depending on the values assumed by f_{cu} and f_y. When the beam is subjected to an external bending moment, the beam may fail or survive depending on whether the external moment is greater than or less than the resisting moment of the beam. It is given that under a given external moment, the probability of the beam becoming under-reinforced is 0.6 and the chance of failure of the beam is 0.1 under this given event. The probability of failure of the beam is 0.2 if the beam is over-reinforced. Assume the events under-reinforced and over-reinforced as independent. Compute the probability of failure of the beam.

Solution Let

A = the event that the beam is under-reinforced
B = the event that the beam is over-reinforced
F = the event failure
S = the event survival

the probability tree diagram is shown in Fig. 3.11. The probability of failure of the beam, p_f, is

$$p_f = P(A)P(F \mid A) + P(B)P(F \mid B)$$
$$= (0.6)(0.1) + (0.4)(0.2)$$
$$= 0.06 + 0.08 = 0.14$$

Statistical Independence

If the occurrence of an event A is not affected by the occurrence of another event B, then it is said that the two events A and B are statistically independent. Mathematically, two events are said to be independent if and only if

$$P(A \mid B) = P(A) \tag{3.12}$$

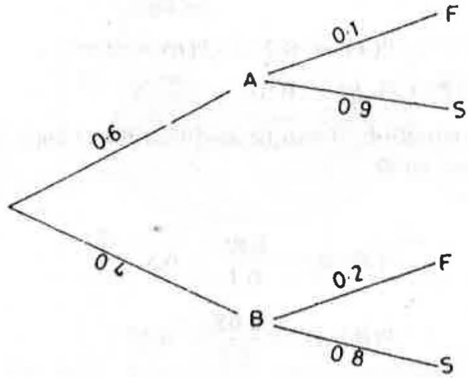


FIG. 3.11 Probability tree diagram—Example 3.8

From this definition, it can aslo be written

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$$P(A \cap B) = P(A)P(B) \tag{3.13}$$

or
$$P(B \mid A) = P(B)$$

Equation (3.12) or (3.13) is generally used to define the independence of two events. By Eq. (3.13), it is meant that if the two events are independent, the probability of their joint occurrence is equal to the product of their individual probabilities of occurrence.

Extending Eq. (3.13) to a number of N events, $A, B, \ldots, N$ are mutually statistically independent if and only if

$$P(A \cap B \cap \ldots \cap N) = P(A)P(B) \ldots P(N) \tag{3.14}$$

In practice, an engineer may postulate that two events are independent, or it may be clear from the nature of experiments, or he may be able to conclude after sampling that there is no apparent relationship between the two events.

EXAMPLE 3.9 Two lakes a and b supply water to a city. The probability of lakes a and b becoming dry in summer is 0.2 and 0.1 respectively. Lake a can supply 60% of the city's full requirement when b fails (i.e. becomes dry), and b can supply 80% of the city's full requirement if a fails. The probability that both will become dry is 0.05. Calculate the probability that the city will have its full supply of water during summer, if there is a failure of the lake.

Solution Let

A = event lake a becomes dry

B = event lake b becomes dry

Then

$$P(A) = 0.2 \qquad P(B) = 0.1$$

$$P(A \cap B) = 0.05$$

From the physical situation, it can be said that the chance of A becoming dry does not depend on B.
Hence

$$P(A \mid B) = \frac{0.05}{0.1} = 0.5$$

$$P(B \mid A) = \frac{0.05}{0.2} = 0.25$$

When there is failure, the conditional probability of the lake a not becoming dry, p_1, is given by

$$p_1 = \frac{P(A^c \cap B)}{P(A \cup B)}$$

$$= \frac{P(A^c \mid B)P(B)}{P(A) + P(B) - P(B \mid A)P(A)}$$

$$= \frac{(1 - 0.50)(0.1)}{(0.2) + (0.1) - (0.25)(0.2)} = \frac{0.05}{0.25} = 0.2$$

Similarly, when there is failure, the conditional probability of the lake b not becoming dry, p_2, is given by

$$p_2 = \frac{P(B^c \cap A)}{P(A \cup B)}$$

$$= \frac{P(B^c \mid A)P(A)}{P(A \cup B)}$$

$$= \frac{(1 - 0.25)(0.2)}{0.25} = 0.6$$

If there is a failure of the lake, the probability that the city will have its full supply of water during summer is

$$p_1 \times 0.6 + p_2 \times 0.8 = 0.6$$

Total Probability Theorem

Suppose B is an event which is accompanied by a set of events $A_1, A_2, \ldots, A_n$, which partition the sample $\mathcal{S}$ such that they are mutually exclusive and collectively exhaustive as shown in Fig. 3.12. One is interested in finding out the probability of the event B, $P(B)$, which probably is not possible to obtain directly. This is obtained as follows:
From Fig. 3.12, it is clear that

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n)$$

$$= \sum_{i=1}^{n} P(B \cap A_i) \tag{3.15}$$

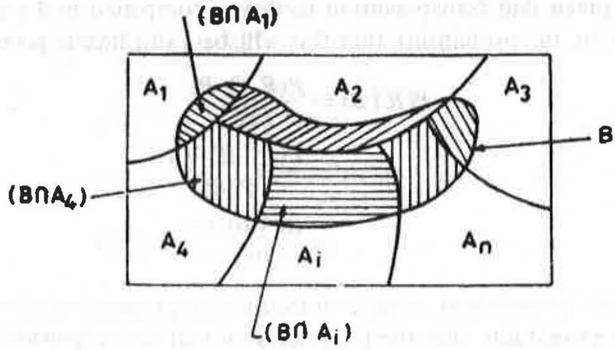


FIG. 3.12 Venn diagram for total probability

Expanding each term using the conditional probability theorem, we get,

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i) \tag{3.16}$$

This is called the total probability theorem. This is illustrated with examples.

EXAMPLE 3.10 Two cities, 500 km apart, are to be connected. Alternatives are: connecting them by rail (R), highway (H) and air (A) by constructing airports at 2 cities. The government will decide on the basis of the cost and merits of each. The chance of selecting R, H and A is 0.4, 0.5 and 0.1 respectively. However, if the government decides on constructing a railway line the probability of completing it in 3 years is 0.3; similarly, for highway and air link, the corresponding probabilities that they will be completed in 3 years are 0.7 and 0.4 respectively.

(i) What is the probability that the two cities will have the means of transportation in 3 years?

(ii) If some transportation facility between the two cities is completed in 3 years, what is the probability that it will be a rail transport?

Given:

$$P(R) = 0.4 \qquad P(H) = 0.5$$
$$P(A) = 0.1$$

Solution Let $\quad B =$ the project completed in 3 years.

Then it is given,

$$P(B \mid R) = 0.3 \qquad P(B \mid H) = 0.7 \quad \text{and} \quad P(B \mid A) = 0.4$$

(i) Using the total probability theorem, the probability that the cities will have a transport facility in 3 years is

$$\begin{aligned} P(B) &= P(B \mid R)P(R) + P(B \mid H)P(H) + P(B \mid A)P(A) \\ &= (0.3)(0.4) + (0.7)(0.5) + (0.4)(0.1) \\ &= 0.51 \end{aligned}$$

(ii) It is given that transportation facility is completed in 3 years. Under this condition, the probability that this will be a rail link is given by

$$P(R \mid B) = \frac{P(R \cap B)}{P(B)}$$

$$= \frac{P(B \mid R)P(R)}{P(B)}$$

$$= \frac{(0.3)(0.4)}{0.51} = 0.235$$

(iii) If the government rejects outright the proposal of air connection, what is the probability that the final decision will be a highway?
This is given by

$$P(H \mid R \cup H) = \frac{P[H \cap (R \cup H)]}{P(R \cup H)}$$

$$= \frac{P(H)}{P(R) + P(H)}$$

$$= \frac{0.5}{0.4 + 0.5} = \frac{5}{9}$$

EXAMPLE 3.11 A water supply system is to be designed to meet the demand during any given day during a summer. There are three demand levels, D_1, D_2 and D_3, being equal to 200,000, 300,000 and 400,000 litres/day respectively. The probabilities of meeting these demand levels are 0.7, 0.2 and 0.1 respectively. If the demand level is 200,000, the probability of the supply being adequate during any given day in the summer is 1 and the corresponding values for 300,000 and 400,000 litres/day are 0.8 and 0.6 respectively during any given day in the summer.

(i) Find the probability that the supply will be adequate during any given day in the summer.
Given:

$$P(D_1) = 0.7 \qquad P(D_2) = 0.2 \qquad P(D_3) = 0.1$$

Solution Let

A = the supply is adequate during any day in the summer

Then it is given,

$$P(A \mid D_1) = 1$$

$$P(A \mid D_2) = 0.8 \qquad \text{and} \qquad P(A \mid D_3) = 0.6$$

Using the total probability theorem, the probability of the supply being adequate on any one day in the summer is calculated.

$$P(A) = P(A \mid D_1)P(D_1) + P(A \mid D_2)P(D_2) + P(A \mid D_3)P(D_3)$$

$$= (1)(0.7) + (0.8)(0.2) + (0.6)(0.1)$$

$$= 0.7 + 0.16 + 0.06 = 0.92$$

(ii) If the adequate supply is observed, what is the probability that the demand level is 300,000 litres/day?

$$P(D_2 \mid A) = \frac{P(D_2 \cap A)}{P(A)}$$

$$= \frac{P(A \mid D_2)P(D_2)}{P(A)}$$

$$= \frac{(0.8)(0.2)}{0.92} = \frac{4}{23}$$

Bayes' Theorem

If $A_1, A_2, \ldots, A_n$ are mutually exclusive and collectively exhaustive events of the sample space $\mathcal{S}$, and B is any event in $\mathcal{S}$ as shown in Fig. 3.12, then for any event A_i,

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)} = \frac{P(B \cap A_i)}{P(B)} \tag{3.17}$$

This can be considered as a converse problem of total probability theorem. Bayes's theorem is quite useful in updating the available data.

EXAMPLE 3.12 Many government projects are executed by a contractor. The chief engineer knows from his previous experience that the chance of getting a good quality of construction from the contractor is 0.8 and a bad quality of construction 0.2. The evaluation of the quality of construction is decided by the hammer test (nondestructive testing *in situ*). If the strength of concrete *in situ* obtained from the hammer test is $>$ 20 N/mm^2, it is decided that the quality of work is good. However, it is known that the hammer test is not very reliable. The probability of a good quality work passed by the hammer test is 0.7 and that of a bad quality work 0.2.

After a project is completed by the contractor, there is a dispute between the contractor and the engineer about the quality of construction. The hammer test is then conducted. If the good quality construction has passed the test, what is the updated probability of expecting a good quality work from the contractor?

Solution Let

G = good quality of work

B = bad quality of work

C = the work passes the test

Given: $P(G) = 0.8$

$P(B) = 0.2$

$P(C \mid G) = 0.7 \quad P(C \mid B) = 0.2$

The updated probability of expecting a good quality work from the contractor is

$$P(G \mid C) = \frac{P(G)P(C \mid G)}{P(C \mid G)P(G) + P(C \mid B)P(B)}$$

$$= \frac{(0.8)(0.7)}{(0.8)(0.7) + (0.2)(0.2)}$$

$$= \frac{0.56}{0.60} = 0.93$$

In future, the engineer will use this value (0.93) as the probability of getting a good quality of work from the contractor.

If another project is executed by the same contractor and that work also passes through the hammer test, then

$$P(G \mid C) = \frac{(0.93)(0.7)}{(0.93)(0.7) + (0.07)(0.2)} = 0.9789$$

3.3 RANDOM VARIABLES

The random variable is a numerical variable whose specific value cannot be predicted with certainty before an experiment. The value assumed by a random variable associated with an experiment depends on the outcome of the experiment. This value is associated with every simple event defined on the sample space, but different simple events may have the same associated value of the random variable, e.g. the strength of concrete, the wind speed observed at a location, the number of persons waiting at a bus stop, etc. Sometimes artificial values may be assigned to a random variable associated with simple events. For example, a random variable, of the quality of a product, may assume different states: poor, satisfactory, good, very good, etc. Then each state may be artificially assigned value as 1, 2, 3, . . ., etc. A random variable X on a sample space S is a function from the sample space to a set of real numbers. The probability law of X, describing its behaviour, is characterized by the probability distribution of X.

Discrete Variables

The probability law of a discrete random variable is described by its probability mass function (PMF). For a random variable X, it is written as

$$p_X(x) = P(X = x) \tag{3.18}$$

$P(X = x)$ is read as the probability of X, taking a value x. The PMF of a random variable must satisfy the three axioms of the probability theory. Hence

(i) $0 \leqslant p_X(x) \leqslant 1$ for all x

(ii) $\sum_{\text{all } x_i} p_X(x_i) = 1$

(iii) $P[a \leqslant X \leqslant b] = \sum_{\substack{\text{all } x_i \leqslant b \\ \text{all } x_i \geqslant a}} p_X(x_i)$

If the PMF of a random variable is given or known, one can immediately tell the probability of the random variable X assuming a value x.

The probability distribution of a random variable X is also described by its cumulative distribution function (CDF), $F_X(x)$. This is defined as

$$F_X(x) = P(X \leqslant x) \quad \text{for all } x \tag{3.19}$$

For a discrete random variable,

$$F_X(x) = \sum_{\text{all } x_i \leqslant x} p_X(x_i) \tag{3.20}$$

EXAMPLE 3.13 Let X be the number of days in a week at a place having a rainfall greater than 5 cm. The following probabilities are assigned to the possible values that X can assume.

$$p_X(x) = \begin{cases} 0.05 & x = 0 \\ 0.10 & x = 1 \\ 0.15 & x = 2 \\ 0.30 & x = 3 \\ 0.20 & x = 4 \\ 0.10 & x = 5 \\ 0.08 & x = 6 \\ 0.02 & x = 7 \end{cases}$$

$$\Sigma\ 1.00$$

Note that the axioms of probability are satisfied. Plot the PMF and CDF of X. Find the probability of observing

(i) two or fewer days having a rainfall greater than 5 cm
(ii) 3 or more days having a rainfall greater than 5 cm.

Solution The plots of PMF and CDF of X are shown in Fig. 3.13 From CDF, it is easy to calculate the probabilities. Thus,

(i) $P(\text{2 or fewer days}) = F_X(2)$

$$= p_X(0) + p_X(1) + p_X(2)$$

$$= 0.3$$

(ii) The probability of observing 3 or more days having a rainfall greater than 5 cm:

$$P(\text{3 or more days}) = 1 - F_X(2)$$

$$= 1 - 0.3 = 0.7$$

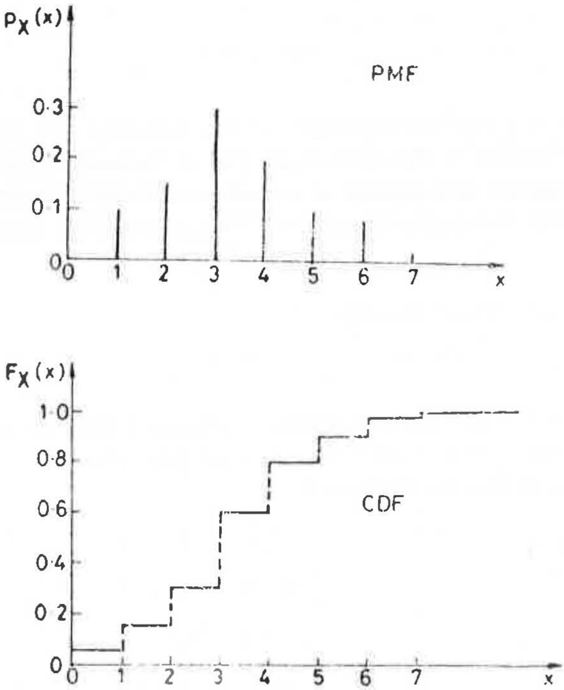


FIG. 3.13 PDF and CDF of X—Example 3.13

The PMF can be recovered from CDF. For example,

$$p_X(x = 4) = F_X(4) - F_X(3)$$
$$= 0.8 - 0.6 = 0.2$$

That is,

$$p_X(x_i) = F_X(x_i) - F_X(x_i - \epsilon) \tag{3.21}$$

where ϵ is a positive integer which is equal to 1 in this problem.

Continuous Variable

This is a function which can assume a continuum of points in a given interval. The probability of such a variable, X, assuming a particular value is zero. Its probability law is described by its probability density function, $f_X(x)$. The probability of X in the interval is given by

$$P(a \leqslant X \leqslant b) = \int_a^b f_X(x)\, dx \tag{3.22}$$

It is to be noted that $f_X(x)$ itself does not give the probability. It is only a measure of the density of probability at the point. Probabilities are given by integrals only.

The PDF of X, in fact, is defined by

$$f_X(x) = \frac{dF_X(x)}{dX} \tag{3.23}$$

where $F_X(x)$ is the CDF of X.

If X can take values right from $-\infty$ to $+\infty$, then

$$F_X(x) = P[X \leqslant x] = \int_{-\infty}^{x} f_X(x)\, dx \tag{3.24a}$$

If X can take values only from 0 to ∞, then

$$\boxed{F_X(x) = \int_{0}^{x} f_X(x)\, dx} \tag{3.24b}$$

For $F_X(x)$ to be a proper distribution function, the following conditions must be satisfied:

(i) $F_X(-\infty) = 0$
(ii) $F_X(\infty) = 1$
(iii) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
(iv) $f_X(x) \geqslant 0$
(v) $F_X(x) \geqslant 0$ and is nondecreasing with x

It is obvious that

$$\boxed{P(a < X \leqslant b)} = \int_{-\infty}^{b} f_X(x)\, dx - \int_{-\infty}^{a} f_X(x)\, dx$$
$$\boxed{= F_X(b) - F_X(a)}$$

This is displayed in Fig. 3.14.

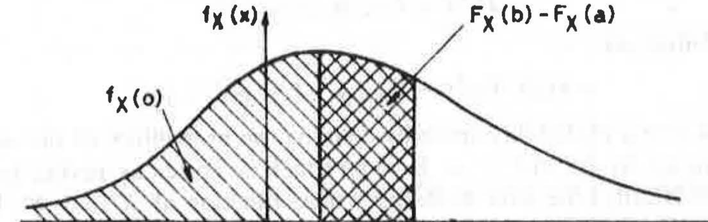


FIG. 3.14 Continuous random variable

EXAMPLE 3.14 The bearing capacity, Y, of a soil below a foundation is known to vary from 200 to 400 kN/m². Its PDF is given as

$$f_Y(y) = \begin{cases} k\left(1 - \dfrac{y}{400}\right) & 200 \leqslant y \leqslant 400 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a constant.
Determine the probability of failure of the foundation if the uniform load on the foundation is 300 kN/m².

Solution For the given function to be a proper distribution function,

$$\int_{200}^{400} f_Y(y)\, dy = 1$$

$$k\int_{200}^{400}\left(1-\frac{y}{400}\right)dy = 50k = 1$$

$$k = 1/50$$

Therefore,

$$f_Y(y) = \begin{cases} \dfrac{1}{50}\left(1-\dfrac{y}{400}\right) & 200 \leqslant y \leqslant 400 \\ 0 & \text{elsewhere} \end{cases}$$

(i) The probability of failure of the foundation is

$$P(Y < 300) = F_Y(300)$$

$$= \int_{200}^{300}\frac{1}{50}\left(1-\frac{y}{400}\right)dy = 0.75$$

Jointly Distributed Discrete Variables

Here, two or more random variables are treated simultaneously. Consider a number of persons travelling in a mini-bus having a maximum capacity of n. One is conducting a survey and finding out how many men and women are travelling in the bus for every kilometre length. Let X be the number of men and Y the number of women travelling in the bus. These random variables on a sample space with respective image sets are

$$X(S) = (x_1, x_2, \ldots, x_n)$$

$$Y(S) = (y_1, y_2, \ldots, y_n)$$

The product set

$$X(S)\cdot Y(S) = \{(x_1, y_1), (x_2, y_2) \ldots\}$$

is made into a probability space by defining the probability of the ordered pair (x_i, y_j) to be $P(X = x_i; Y = y_j)$ which is written as $p_{XY}(x_i, y_j)$. This function is called the joint probability mass function of X and Y. Hence, the joint PMF is

$$p_{XY}(x, y) = P[(X = x) \cap (Y = y)] \tag{3.25}$$

the joint CDF is defined as

$$F_{XY}(x, y) = P[(X \leqslant x) \cap (Y \leqslant y)] \tag{3.26}$$

For discrete variables X and Y,

$$F_{XY}(x, y) = \sum_{x_i \leqslant x}\ \sum_{y_j \leqslant y} p_{XY}(x_i, y_j) \tag{3.27}$$

The conditions to be satisfied are

$$p_{XY}(x_i, y_j) \geqslant 0$$

$$\sum_{\text{all } x_i}\ \sum_{\text{all } y_j} p_{XY}(x_i, y_j) = 1$$

The joint PMF and the joint CDF describe the joint probability law or the joint probabilistic behaviour of the variables.

EXAMPLE 3.15 Five RCC beams are tested in the laboratory to determine the load at the first crack, and at failure. If the load at the first crack is greater than 20 kN, it is classified as safe against cracking. Similarly, if the failure load is greater than 35 kN, it is taken that the beam is safe against collapse. Let

$$X = \text{number of beams safe against cracking}$$
$$Y = \text{number of beams safe against collapse}$$

Because of the random behaviour of the beam, if the beam is safe against cracking, it is not necessary that it should be safe against failure. The joint PDF of X and Y is given in Table 3.2 and displayed in Fig. 3.15.

TABLE 3.2 *Joint PMF—Example 3.15*

$X \backslash Y$	0	1	2	3	4	5	
0	0	0.02	0	0	0	0	$\Sigma = 0.02 = p_X(0)$
1	0.01	0.05	0	0	0	0	$\Sigma = 0.06 = p_X(1)$
2	0.02	0.01	0.1	0.01	0	0	$\Sigma = 0.14 = p_X(2)$
3	0.03	0.015	0.01	0.15	0.015	0.00	$\Sigma = 0.22 = p_X(3)$
4	0.04	0.02	0.015	0.01	0.25	0.01	$\Sigma = 0.345 = p_X(4)$
5	0.05	0.02	0.02	0.015	0.01	0.10	$\Sigma = 0.215 = p_X(5)$
Σ	0.150	0.135	0.145	0.185	0.275	0.110	1.000
	$p_Y(0)$	$p_Y(1)$	$p_Y(2)$	$p_Y(3)$	$p_Y(4)$	$p_Y(5)$	

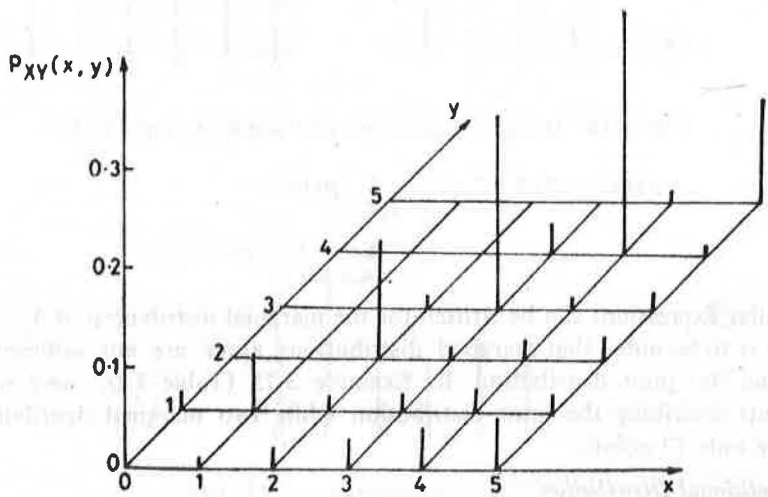


FIG. 3.15 Joint PMF—Example 3.15

Determine the probability of the event E which is defined as the count in which the number of beams safe both in cracking and in collapse are the same. (It does not imply that the same beam is safe in cracking and collapse).

Solution

$$\begin{aligned} P(E) &= P(Y = X) = \Sigma\, P(X = x_i, Y = y_i) \\ &= p_{XY}(0, 0) + p_{XY}(1, 1) + p_{XY}(2, 2) + \\ &\quad p_{XY}(3, 3) + p_{XY}(4, 4) + p_{XY}(5, 5) \\ &= 0 + 0.05 + 0.10 + 0.15 + 0.25 + 0.10 \\ &= 0.65 \end{aligned}$$

Marginal Distribution

From the joint distribution of two variables, it is possible to get the distribution of individual variables. The distributions of individual variables are called marginal distributions. The marginal distribution of X is found by summing over all the values of the other variable Y. That is,

$$p_X(x) = P[X = x] = \sum_{\text{all } y_i} p_{XY}(x, y_i) \tag{3.28}$$

The derivations of marginal distributions of X and Y are shown in Table 3.2 by adding values horizontally and vertically. Marginal distributions of X and Y are displayed in Fig. 3.16.

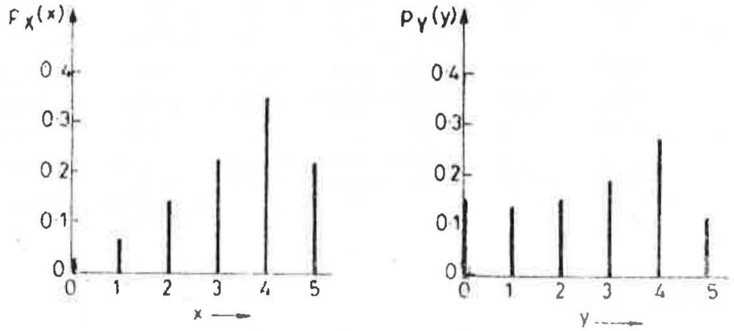


FIG. 3.16 Marginal distributions of X and Y—Example 3.15

$$F_X(x) = P[X \leqslant x] = \sum_{x_i \leqslant x} p_X(x_i)$$

$$= \sum_{x_i \leqslant x} \sum_{\text{all } y_j} p_{XY}(x_i, y_j) \tag{3.29}$$

Similar expressions can be written for the marginal distribution of Y.

It is to be noted that marginal distributions alone are not sufficient to define the joint distribution. In Example 3.15 (Table 3.2), there are 36 points describing the joint distribution while two marginal distributions have only 12 points.

Conditional Distribution

Given two discrete random variables X and Y with values x and y_l, the conditional probability mass function of X given that Y takes on the value y_l is defined as

$$p_{X|Y}(x \mid y_l) = \frac{p_{XY}(x, y_l)}{p_Y(y_l)} \tag{3.30}$$

The denominator is the marginal distribution of Y evaluated at the given value of y_l. For a proper conditional PMF of X,

$$0 \leqslant p_{X|Y}(x, y) \leqslant 1$$

and

$$\sum_{\text{all } x_i} p_{X|Y}(x_i, y) = 1$$

EXAMPLE 3.16 Consider the previous example and derive the conditional distribution of X if Y takes a value of 1.

Solution The conditional PMF of X, given $Y = 1$, is

$$p_{X|Y}(x \mid 1) = \frac{p_{X1}(x, 1)}{p_Y(1)}$$

From Table 3.2, $p_Y(1) = 0.135$

$$p_{X|Y}(0, 1) = \frac{p_{X1}(0, 1)}{0.135} = \frac{0.02}{0.135} = 0.148$$

$$p_{X|Y}(1, 1) = \frac{p_{X1}(1, 1)}{0.135} = \frac{0.05}{0.135} = 0.371$$

$$p_{X|Y}(2, 1) = \frac{p_{X1}(2, 1)}{0.135} = \frac{0.01}{0.135} = 0.074$$

$$p_{X|Y}(3, 1) = \frac{p_{X1}(3, 1)}{0.135} = \frac{0.015}{0.135} = 0.111$$

$$p_{X|Y}(4, 1) = \frac{p_{X1}(4, 1)}{0.135} = \frac{0.02}{0.135} = 0.148$$

$$p_{X|Y}(5, 1) = \frac{p_{X1}(5, 1)}{0.135} = \frac{0.02}{0.135} = \frac{0.148}{\Sigma 1.000}$$

The plot of $p_{X|Y}(x \mid 1)$ is shown in Fig. 3.17.

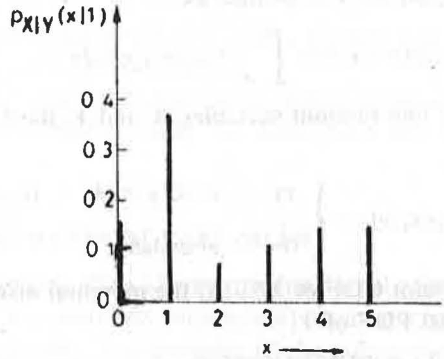


FIG. 3.17 Conditional PMF of X – Example 3.15

Jointly Distributed Continuous Variables

If X and Y are continuous random variables, their joint probability law is described by their joint probability density function, $f_{XY}(x, y)$, defined as

$$P(a_1 \leqslant X \leqslant a_2;\ \ b_1 \leqslant Y \leqslant b_2) = \int_{a_1}^{a_2}\int_{b_1}^{b_2} f_{XY}(x, y)\, dx\, dy \tag{3.31}$$

This is the volume under the function over the region.
For a proper joint PDF, the following conditions are to be satisfied:

(i) $f_{XY}(x, y) \geqslant 0$ for all values of x, y

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1$

The joint PDF is also given by

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y) \tag{3.32}$$

where $F_{XY}(x, y) = P(X \leqslant x;\ Y \leqslant y)$

The probability density function of one variable, i.e. marginal density function can be obtained by integrating out the other variable. Symbolically, it is written as

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \tag{3.33}$$

The marginal CDF of X is obtained as

$$\begin{aligned} F_X(x) = P(X \leqslant x) &= \int_{-\infty}^{x} f_X(x)\, dx \\ &= F_{XY}(x, \infty) \end{aligned} \tag{3.34}$$

The conditional PDF of X, given that Y has taken a value y_1, is defined as

$$f_{X|Y}(x \mid y_1) = \frac{f_{XY}(x, y_1)}{f_Y(y_1)} \tag{3.35}$$

The conditional CDF of X is defined as

$$F_{X|Y}(x \mid y_1) = \int_{-\infty}^{x} f_{X|Y}(x \mid y_1)\, dx \tag{3.36}$$

EXAMPLE 3.17 If two random variables, X and Y, have a joint distribution given by

$$f_{XY}(x, y) = \begin{cases} xy & 0 < x < 1 \quad 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

determine (i) the joint CDF of XY, (ii) the marginal distribution of X, and (iii) the conditional PDF of Y.

Solution (i) The joint CDF of XY is

$$\begin{aligned} F_{XY}(x, y) &= \int_0^x\int_0^y (x, y)\, dx\, dy \\ &= \frac{x^2y^2}{4} \quad 0 < x < 1;\ \ 0 < y < 2 \end{aligned}$$

(ii) The marginal density function of X is

$$f_X(x) = \int_0^2 f_{XY}(x, y)\, dy$$

$$= \int_0^2 xy\, dy$$

$$= 2x \qquad 0 < x < 1$$

(iii) The conditional PDF of Y is

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$= \frac{xy}{2x} = \frac{y}{2} \qquad 0 < y < 2$$

Independent Random Variables

Two random variables are independent if and only if

$$F_{XY}(x, y) = F_X(x) \cdot F_Y(y) \tag{3.37}$$

for all values of the random variables for which the respective functions are defined.

Two discrete variables X and Y are independent if and only if

$$p_{X|Y}(x \mid y) = p_X(x)$$
$$p_{XY}(x, y) = p_X(x) \cdot p_Y(y) \tag{3.38}$$

Two continuous variables X and Y are independent if and only if

$$f_{XY}(x, y) = f_X(x) f_Y(y) \tag{3.39}$$
$$f_{X|Y}(x \mid y) = f_X(x)$$
$$f_{Y|X}(y \mid x) = f_Y(y)$$
$$F_{X|Y}(x \mid y) = F_X(x)$$

The assumption of independence of two events permits one to get a joint distribution from marginal distributions.

In the case of jointly distributed variables, only two variables have been considered; however, whatever that has been done can be extended to multiple variables.

3.4 FUNCTIONS OF RANDOM VARIABLES

Civil engineering problems often involve the functional relationships, which predict the value of one variable (dependent) from the value of another basic (independent) variable. For example, (i) the lateral pressure on a wall is a function of the density of water and the level of water in the tank, (ii) the intensity of wind pressure is a function of the drag coefficient and the square of the wind velocity. If the basic variable (wind speed or the level of water) is random, the dependent variable (lateral pressure or the intensity of wind pressure) is also a random variable. This section deals with the determination of the probability law of one variable from the other.

Functions of Single Random Variable

Consider the case of one random variable X. It is given that

$$Z = g(X)$$

Where $g(X)$ is a monotonically increasing function and Z is a single valued function of X. For such a function, $z \geqslant z_1$ if and only if $x \geqslant x_1$, namely that the value for which $z_1 = g(x_1)$, as shown in Fig. 3.18, or when $Z = z$, $x = g^{-1}(z)$ where g^{-1} is the inverse function of g, then

$$\begin{aligned} P[Z = z] &= P[X = x] \\ &= P[X = g^{-1}(z)] \end{aligned}$$

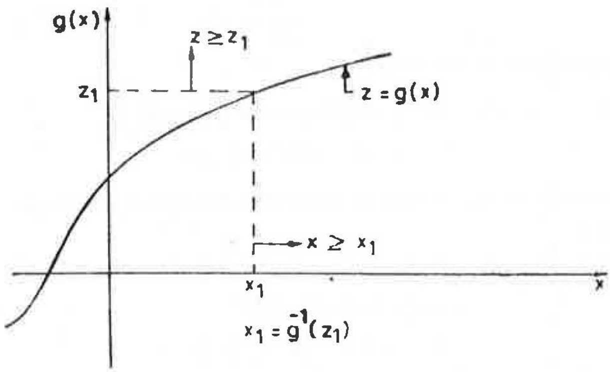


FIG. 3.18 Relation between random variable X and random variable Z

Hence if X is a discrete random variable, the PMF and CDF of Z are given by

$$p_Z(z) = p_X[g^{-1}(z)] \tag{3.40}$$

$$F_Z(z) = \sum_{\text{all } x_j \leqslant g^{-1}(z)} p_X(x_j) \tag{3.41}$$

If X is a continuous random variable, the CDF of Z is

$$\begin{aligned} F_Z(z) = P[Z \leqslant z] &= P[X \leqslant x] \\ &= F_X[g^{-1}(z)] \end{aligned}$$

Hence

$$F_Z(z) = \int_{-\infty}^{g^{-1}(z)} f_X(x)\, dx \tag{3.42}$$

Since

$$x = g^{-1}(z),$$

$$dx = \frac{d[g^{-1}(z)]}{dz}\, dz$$

Now Eq. (3.42) becomes

$$F_Z(z) = \int_{-\infty}^{x} f_X[g^{-1}(z)] \frac{d[g^{-1}(z)]}{dz}\, dz$$

Hence the PDF of Z is

$$f_Z(z) = \frac{d}{dz}[g^{-1}(z)]f_X[g^{-1}(z)]$$

$$= \frac{dx}{dz} f_X[g^{-1}(z)]$$

In general, the above equation is written as

$$\boxed{f_Z(z) = \left|\frac{dx}{dz}\right| f_X[g^{-1}(z)]} \tag{3.43}$$

The absolute value of dx/dz is necessary since for some functions $g(X)$, a positive dx corresponds to a negative dz and *vice versa* (i.e. the function may be a monotonically decreasing function).

Note: If each value of z corresponds to n values of x, i.e. the inverse function $x = g^{-1}(z)$ is multivalued, then

$$f_Z(z) = \left|\frac{dx}{dz}\right| n f_X[g^{-1}(z)] \tag{3.44}$$

Refer Example 3.21.

EXAMPLE 3.18 A column is to be designed for a load W which is equal to its self weight s and a fraction of the live load L on the beam supported by the column. That is,

$$W = s + cL$$

where c is a constant (positive). Assume that L alone is a random variable. Find the PDF of W if the PDF of L is

$$f_L(l) = \frac{1}{\sqrt{2\pi}} \exp(-l^2/2) \qquad l \geq 0$$

Solution When

$$W = w,$$

$$l = (w - s)/c = g^{-1}(z)$$

$$\frac{dl}{dw} = \frac{1}{c}$$

Using Eq. (3.43),

$$f_W(w) = \frac{1}{c} f_L\left(\frac{w - s}{c}\right)$$

$$= \frac{1}{c}\left[\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{w - s}{c}\right)^2\right\}\right] \qquad w \geq s$$

The CDF of W is

$$F_W(w) = \int_s^w f_W(w)\, dw$$

This can also be obtained if the CDF of L is given:

$$F_W(w) = P[W \leqslant w]$$
$$= P\left[L \leqslant \frac{w - s}{c}\right] = F_L\left(\frac{w - s}{c}\right)$$

In case, in a particular physical situation, the relationship is

$$W = g - cL$$

then,

$$l = \frac{g - w}{c}; \qquad \frac{dl}{dw} = -\frac{1}{c}$$

The PDF of W is

$$f_W(w) = \left|\frac{1}{c}\right| f_L\left(\frac{g - w}{c}\right)$$

and the CDF of W can be obtained as follows:

$$F_W(w) = P[W \leqslant w]$$
$$= P[g - CL \leqslant w]$$
$$= P\left[L \geqslant \frac{g - w}{c}\right]$$
$$= 1 - F_L\left(\frac{g - w}{c}\right)$$

EXAMPLE 3.19 In Example 3.18, if

$$f_L(l) = \begin{cases} \frac{1}{2} & -1 < l < +1 \\ 0 & \text{elsewhere} \end{cases}$$

what is the PDF of W?

Solution From Example 3.18,

$$f_W(w) = \frac{1}{c} f_L\left(\frac{w - s}{c}\right)$$
$$= \frac{1}{2c}, \quad s - c < w < s + c$$

Sketches of $f_L(l)$ and $f_W(w)$ are shown in Fig. 3.19.

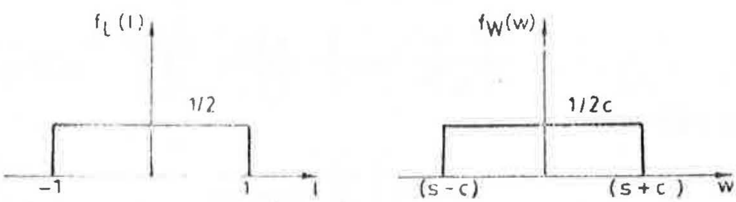


FIG. 3.19 Probability density functions of L and W—Example 3.19

EXAMPLE 3.20 The intensity of wind pressure, W, is given by the relation

$$W = \alpha V^2$$

where α is a constant (equal to 0.006 as per IS code) and V is the annual maximum wind speed. If the PDF of V, following Type II extremal (largest) distribution, is given as

$$f_V(v) = \frac{k}{u}\left(\frac{u}{v}\right)^{k+1} \exp\left[-(u/v)^k\right] \qquad v \geqslant 0$$

determine the PDF and CDF of W. k and u are parameters (constant) of V.

Solution When W assumes a value w, then

$$v = \pm\left(\frac{w}{\alpha}\right)^{1/2}$$

$$\left|\frac{dv}{dw}\right| = \frac{1}{\sqrt{\alpha w}}$$

Hence, using Eq. (3.43), the PDF of W is

$$f_W(w) = \left|\frac{1}{2\sqrt{\alpha w}}\right| \left[f_V(\sqrt{w/\alpha}) + f_V(-\sqrt{w/\alpha})\right] \tag{3.45}$$

For the given PDF of V,

$$f_V(v) = 0 \qquad \text{for} \quad v < 0$$

Hence Eq. (3.45) becomes

$$f_W(w) = \frac{1}{2\sqrt{\alpha w}} f_V(\sqrt{w/\alpha})$$

$$= \frac{1}{2\sqrt{\alpha w}}\left[\frac{k}{u}\left(\frac{u}{\sqrt{w/\alpha}}\right)^{k+1} \exp\{-(u/\sqrt{w/\alpha})^k\}\right]$$

Let

$$w_c = \alpha u^2$$

Then the PDF of W simplifies to

$$f_W(w) = \frac{k}{2w_c}\left(\frac{w_c}{w}\right)^{k/2+1} \exp\left[-(w_c/w)^{k/2}\right] \qquad w \geqslant 0 \tag{3.46}$$

(Note that W also follows the Type II extremal (largest) distribution with parameters $w_c = cu^2$ and $k/2$).

The CDF of W is derived as follows:

$$F_W(w) = \int_0^w f_W(w)\, dw$$

Substituting for $f_W(w)$ in the above equation and putting $(w_c/w) = y$, we have

$$F_W(w) = \int_\infty^y -\frac{k}{2}\,\frac{y^{k/2}}{y} \exp\left[-y^{k/2}\right] dy$$

Let
$$x = y^{k/2}$$
then
$$F_W(w) = -\int_{\infty}^{x} e^{-x}\,dx = e^{-x}$$
Hence the CDF of W is
$$F_W(w) = \exp\left[-(w_c/w)^{k/2}\right] \qquad w \geqslant 0 \tag{3.47}$$

The CDF of W can be obtained by directly using the CDF of V, which is given by

$$F_V(v) = \exp\left[-(u/v)^k\right] \qquad v \geqslant 0$$
$$\begin{aligned} F_W(w) &= P(W \leqslant w) \\ &= P(\alpha V^2 \leqslant w) \\ &= P\left(V \leqslant \sqrt{\frac{w}{\alpha}}\right) + P\left(V \leqslant -\sqrt{\frac{w}{\alpha}}\right) \end{aligned}$$

The second part is equal to zero as V cannot take a negative value. Hence, (using Eq. 3.47)

$$\begin{aligned} F_W(w) &= P(V \leqslant \sqrt{w/\alpha}) \\ &= \exp\left[-\left(\frac{u}{\sqrt{w/\alpha}}\right)^k\right] \end{aligned}$$

Let $w_c = \alpha u^2$. Then,

$$F_W(w) = \exp\left[-(w_c/w)^{k/2}\right] \qquad w \geqslant 0$$

EXAMPLE 3.21 Given

$$Z = a \sin X$$
$$f_X(x) = \begin{cases} \dfrac{1}{2\pi} & 0 < x < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of Z.

Solution Equation (3.43) has been derived on the assumption that Z is a single valued function of X. In this case X is a double valued function for each value of Z. Hence for such a function, Eq. (3.43) becomes

$$f_Z(z) = 2f_X[g^{-1}(z)]\left|\frac{dx}{dz}\right|$$

In general, if each value of z corresponds to n values of x (i.e. the inverse function of $x = g^{-1}(z)$ is multivalued), then

$$f_X(z) = nf_X[g^{-1}(z)]\left|\frac{dx}{dz}\right|$$

for the given function,

$$x = \sin^{-1}\left(\frac{z}{a}\right)$$

$$\frac{dx}{dz} = \frac{1}{\sqrt{a^2 - z^2}} \qquad -a < z < a$$

Hence,

$$fz(z) = \frac{1}{\sqrt{a^2 - z^2}}\, 2fx\left[\sin^{-1}\left(\frac{z}{a}\right)\right]$$

$$= \frac{2}{\sqrt{a^2 - z^2}}\left(\frac{1}{2\pi}\right)$$

$$fz(z) = \frac{1}{\sqrt{a^2 - z^2}}\left(\frac{1}{\pi}\right) \qquad -a < z < a$$

Functions of Two Random Variables

In many situations, an engineer may have to deal with cases where one variable depends on two or more variables. For example, (i) the total moment induced on a column may be the sum of the moments due to live load and wind load. Since live load and wind load are random variables, the total moment on the column is also a random variable. One has to derive the PDF of the total moment from the known distributions of wind load and live load. (ii) Strain in a tension member is the ratio of the force in the member to its area of cross section. If area and force are random variables, strain is also a random variable whose PDF is to be obtained from the known distributions of force and area. In general, functional relations may be of the following types:

$$Z = X + Y \qquad\qquad Z = X - Y$$

$$Z = \frac{X}{Y} \qquad\qquad Z = XY$$

Case (i): $\quad Z = X + Y \qquad\qquad (3.48)$

Let Y take a particular value y, i.e. $Y = y$. Then,

$$Z = X + y$$

The conditional PDF of X, given $Y = y$, is

$$f_{X|Y}(x, y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

Treating y temporarily as a constant,

$$x = z - y \quad \text{and} \quad \frac{dx}{dz} = |\,1\,|$$

Hence,

$$f_{Z|Y}(z, y) = |\,1\,|\, f_{X|Y}(z - y, y)$$

$$= f_{Z|Y}(z - y, y) \qquad (3.49)$$

But it is known that

$$f_{ZY}(z, y) = f_{Z\mid Y}(z, y) f_Y(y) \tag{3.50}$$

using Eq. (3.49) in Eq. (3.50), the joint distribution of ZY is obtained as

$$\begin{aligned} f_{ZY}(z, y) &= f_{X\mid Y}(z - y, y) f_Y(y) \\ &= f_{XY}(z - y, y) \end{aligned} \tag{3.51}$$

$f_{XY}(z - y, y)$ is nothing but the joint probability of X and Y evaluated at $X = z - y$ and $Y = y$.
From the joint distribution [Eq. (3.51)], the marginal distribution of Z can be obtained as

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z - y, y)\, dy \tag{3.52}$$

When X and Y are independent,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\, dy \tag{3.53}$$

Similarly for other cases, the marginal distribution of Z can be obtained.

Case (ii): $Z = X - Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z + y, y)\, dy \tag{3.54}$$

Case (iii): $Z = \dfrac{X}{Y}$

$$f_Z(z) = \int_{-\infty}^{\infty} \mid y \mid f_{XY}(zy, y)\, dy \tag{3.55}$$

Case (iv): $Z = XY$

$$f_Z(z) = \int_{-\infty}^{\infty} \left| \frac{1}{y} \right| f_{XY}\left(\frac{z}{y}, y\right) dy \tag{3.56}$$

Note: All the above equations, considering X and Y as continuous random variables, are valid for discrete variables also, keeping in mind that instead of integration, summation is to be carried out.

EXAMPLE 3.22 Let the stress in a member, X, and the area of section, Y, be independent random variables. The force Z in the member is then given by

$$Z = XY$$

It is given that

$$f_X(x) = \frac{1}{8} x \qquad 0 \leqslant x \leqslant 4$$

$$f_Y(y) = \frac{1}{a} \qquad 0 \leqslant y \leqslant a$$

Determine the PDF of Z.

Solution Since X and Y are independent, Eq. (3.56) becomes

$$f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{y}\right| f_X\left(\frac{z}{y}\right) f_Y(y)\, dy$$

For the given PDF of Y,

$$f_Z(z) = \int_0^a \frac{1}{y} f_X\left(\frac{z}{y}\right) \frac{1}{a}\, dy$$

In the above equation substitute for $f_X\left(\frac{z}{y}\right)$ using the given $f_X(x)$. For the relation $\frac{z}{y} = x$ with limits for x, $0 \leqslant x \leqslant 4$ implies that $0 \leqslant \frac{z}{y} \leqslant 4$. Corresponding limits for y are

$$\frac{z}{4} \leqslant y \leqslant \infty.$$

Hence,

$$\begin{aligned} f_Z(z) &= \int_{z/4}^{a} \frac{1}{y} \frac{1}{8}\left(\frac{z}{y}\right) \frac{1}{a}\, dy \\ &= \frac{z}{8a} \int_{z/4}^{a} \frac{1}{y^2}\, dy \\ &= \frac{1}{2a}\left[1 - \frac{z}{4a}\right] \qquad 0 \leqslant z \leqslant 4a \end{aligned}$$

EXAMPLE 3.23 A water tank is supplied with water through an inlet pipe at a constant rate for a period of time X. The water flows out through the outlet pipe from the tank at the same rate for a period of time Y. If X and Y are independent with distributions

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geqslant 0$$
$$f_Y(y) = \beta e^{-\beta y} \qquad y \geqslant 0$$

determine the PDF of $Z = X - Y$, the change of the amount of water in the tank after one cycle of inflow and outflow, assuming that the tank cannot become dry, or overflow.

Solution It is given that

$$Z = X - Y$$

Since X and Y are independent, Eq. (3.54) becomes

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z + y) f_Y(y)\, dy$$

Since $f_Y(y) = 0$ for $y < 0$,

$$\begin{aligned} f_Z(z) &= \int_0^{\infty} f_X(z + y) f_Y(y)\, dy \\ &= \int_0^{\infty} f_X(z + y)\, \beta e^{-\beta y}\, dy \end{aligned}$$

In the above equation substitute for $f_X(z+y)$ using the given $f_X(x)$.
Since $f_X(x) = 0$ for $x < 0$,

$$f_X(z+y) \text{ is zero} \qquad \text{for } z + y < 0$$

For $z < 0$, y should be $> -z$.

$$f_Z(z) = \int_{-z}^{\infty} \lambda\, e^{-\lambda(z+y)}\, \beta\, e^{-\beta y}\, dy$$

$$= \left(\frac{\lambda\beta}{\lambda + \beta}\right) e^{\beta z} \qquad z < 0$$

For $z > 0$, $y > 0$,

$$f_Z(z) = \int_{0}^{\infty} \lambda\, e^{-\lambda(z+y)}\, \beta\, e^{-\beta y}\, dy$$

$$= \left(\frac{\lambda\beta}{\lambda + \beta}\right) e^{-\lambda z} \qquad z > 0$$

Hence the PDF of Z is

$$f_Z(z) = \begin{cases} \left(\dfrac{\lambda\beta}{\lambda + \beta}\right) e^{\beta z} & z < 0 \\ \left(\dfrac{\lambda\beta}{\lambda + \beta}\right) e^{-\lambda z} & z > 0 \end{cases}$$

3.5 MOMENTS AND EXPECTATION

The complete description of a random variable requires a probability distribution in one of its various forms. However, in many applications, the form of the distribution function is not known in all details. In such situations, concise descriptors which describe the dominant features of the function may be valuable, and enough for engineering applications. These descriptors may be expectation (mean), variance, etc.

The expected value of a discrete random variable X, denoted by $E(X)$, is defined as

$$E(X) = \sum_{\text{all } x_i} x_i\, p_X(x_i) \tag{3.57}$$

If X is continuous, then

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \tag{3.58}$$

The same quantity, $E(X)$, is also called the mean of X or the first moment of the distribution of X. This should not be confused with the sample mean which is computed from the data and has statistical entity. An expectation is calculated from the probability distribution. It can be considered as the weighted average of the values of X in which each possible value is weighted by the probability of its occurrence.

A family of averages, called moments, define the probability distribution of a variable as

$$\boxed{m_i = E(X^i) = \int_{-\infty}^{\infty} x^i f_X(x)\, dx} \tag{3.59}$$

where m_i is the i^{th} moment of X about the origin. The first moment is the mean value of X and is designated by μ. That is when $i = 1$,

$$m_1 = E(X) = \mu = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Moments are generally defined, with respect to the mean and are called central moments, c_i. They are defined as

$$c_i = E[(X - \mu)^i] = \int_{-\infty}^{\infty} (X - \mu)^i f_X(x)\, dx \tag{3.60}$$

The first four moments are commonly used. The first central moment is zero. The second central moment is the variance, given by

$$\begin{aligned} \text{Var}(X) = c_2 &= E[(x - \mu)^2] \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x)\, dx \end{aligned} \tag{3.61}$$

The third central moment is related to the symmetry of the distribution and is incorporated in the dimensionless coefficient of skewness, r_1, given by

$$r_1 = \frac{c_3}{\sigma^3} \tag{3.62}$$

where σ is the standard deviation.

If the distribution is symmetrical, $r_1 = 0$. If r_1 is positive, the distribution is called positively skewed and will have a long tail (upper tail) at the right. If r_1 is negative, the distribution is called negatively skewed and will have a long tail (lower tail) on the left. The variation of the shape of the density function with r_1 is shown in Fig. 3.20.

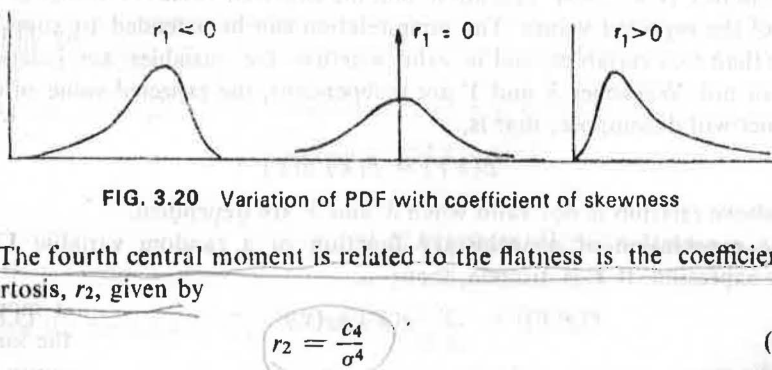


FIG. 3.20 Variation of PDF with coefficient of skewness

The fourth central moment is related to the flatness is the coefficient of kurtosis, r_2, given by

$$r_2 = \frac{c_4}{\sigma^4} \tag{3.63}$$

It is often compared to a standard value of 3 for normal distribution. If $r_2 > 3$, the distribution is said to be flatter, and if <3, the distribution is more peaked than normal.

EXAMPLE 3.24 The lateral strength, S, of a frame is subjected to random variations. The PDF of S is given as

$$f_S(s) = \begin{cases} \frac{6}{5}(2s - 1)(2 - s) & 1 \leq s \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Determine the mean and variance of S.

Solution The mean value of S is obtained using Eq. (3.58)

$$\mu = \int_1^2 s\left[\frac{6}{5}(2s - 1)(2 - s)\right] ds$$

$$= \frac{6}{5}\left[\frac{5s^3}{3} - \frac{s^4}{2} - s^2\right]_1^2$$

$$= \frac{7}{5}$$

The variance of S is given by

$$\text{Var}(S) = \int_1^2 \left[s - \frac{7}{5}\right]^2 \left[\frac{6}{5}(2s - 1)(2 - s)\right] ds$$

$$= \frac{6}{100}$$

Algebra of Expectations

The expectation has a number of convenient properties which are useful. For any random variable X, and constant a,

$$E(aX) = a\,E(X)$$

$$E(X + a) = E(X) + a$$

For any two random variables X and Y,

$$E(X + Y) = E(X) + E(Y)$$

Expectation is a linear operation and the expected values of a sum is the sum of the expected values. The same relation can be extended to sums of more than two variables, and is valid whether the variables are independent or not. Whenever X and Y are independent, the expected value of the product will decompose, that is,

$$E(XY) = E(X)\,E(Y)$$

The above relation is not valid when X and Y are dependent.

The expectation of an arbitrary function of a random variable X is easily expressed. If X is discrete, then

$$E[g(X)] = \sum_{\text{all } X_i} g(X_i)\,p_X(x_i) \tag{3.64}$$

and if X is continuous, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f_X(x)\,dx \tag{3.65}$$

In other words, $g(x)$ merely replaces x in the definition of expectation. These expressions are not a new definition, but are derived by considering a random variable $Y = g(X)$ and relating the distribution of Y to the distribution of X. If $g_1(X)$ and $g_2(X)$ are any two functions of X, then

$$E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$$

Conditional Expectation

The conditional expectation of a random variable X, given the value of a related random variable Y, is defined as

$$E(X \mid Y = y) = \sum_{\text{all } x_i} x_i \, p_{X|Y}(x_i \mid y) \tag{3.66}$$

when X and Y are discrete, and

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x \, f_{X|Y}(x \mid y) \, dx \tag{3.67}$$

when X and Y are continuous.

If X and Y are independent, then

$$E(X \mid Y = y) = E(X)$$

The expectation of marginal distribution of X is

$$E(X) = \sum_{\text{all } y} E(X \mid Y = y) \, p_Y(y) \tag{3.68}$$

when X and Y are discrete, and

$$E(X) = \int_{-\infty}^{\infty} E(X \mid Y = y) \, f_Y(y) \, dy \tag{3.69}$$

when X and Y are continuous.

A brief way to express these is

$$E(X) = E[E(X \mid Y = y)]$$

Note: $E[X \mid Y = y]$ is a constant and
$E[X \mid Y]$ is a random variable.

Properties of Variance

As given earlier [Eq. (3.61)] the variance of X is given by

$$\begin{aligned} Var(X) &= E[(X - \mu)^2] \\ &= E(X^2 + \mu^2 - 2X\mu) \\ &= E(X^2) + E(\mu^2) - 2\mu\, E(X) \\ &= E(X^2) + \mu^2 - 2\mu^2 \end{aligned}$$

Hence the variance of X is

$$\text{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2 \tag{3.70}$$

or

$$E(X^2) = \sigma^2 + \mu^2$$

The linearity property of expectation is not valid for variances.

If a and b are constants, then

(i) $\mathrm{Var}(a) = 0$
(ii) $\mathrm{Var}(aX) = a^2 \mathrm{Var}(X)$
(iii) $\mathrm{Var}(a + bX) = b^2 \mathrm{Var}(X)$

The conditional variance of X, given Y, is defined as

$$\mathrm{Var}(X \mid Y = y) = E[(X - \mu_{X|Y})^2 \mid Y = y] \tag{3.71}$$

For discrete X and Y,

$$\mathrm{Var}(X \mid Y = y) = \sum_{\text{all } x_i} (x_i - \mu_{X|Y})^2 \, p_{X|Y}(x_i \mid y)$$

For continuous variables X and Y,

$$\mathrm{Var}(X \mid Y = y) = \int_{-\infty}^{\infty} (x - \mu_{X|Y})^2 \, f_{X|Y}(x \mid y) \, dx$$

The concepts of expectation and moments can be extended to jointly distributed random variables. If Z is a function of two continuous random variables X and Y, i.e.

$$Z = g(X, Y)$$

then the expectation of Z is

$$\begin{aligned} E(Z) &= E[g(X, Y)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, f_{XY}(x, y) \, dx \, dy \end{aligned} \tag{3.72}$$

The joint moments of the order of $m + n$ of a joint distribution of X and Y are defined as

$$E[X^m Y^n] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^m y^n f_{XY}(x, y) \, dx \, dy \tag{3.73}$$

The central moments, c_{mn}, are similarly defined. Thus,

$$c_{mn} = E[(X - \mu_1)^m \, (Y - \mu_2)^n] \tag{3.74}$$

where μ_1 and μ_2 correspond to the first order moments obtained by putting ($m = 1$, $n = 0$) and ($m = 0$, $n = 1$) respectively in Eq. (3.73). That is, for example

$$\begin{aligned} \mu_1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, f_{XY}(x, y) \, dx \, dy \\ &= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{XY}(x, y) \, dy \right] dx \\ &= \int_{-\infty}^{\infty} x \, f_X(x) \, dx = E(X) \end{aligned}$$

Similarly, $\qquad \mu_2 = E(Y)$

Covariance and Correlation Coefficient

The central moment obtained by putting $m = 1$ and $n = 1$ in Eq. 3.74 is called the covariance of X and Y. That is,

$$\text{Cov}(X, Y) = \sigma_{XY} = E[(X - \mu_1)(Y - \mu_2)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_1)(y - \mu_2) f_{XY}(x, y)\, dx\, dy$$

The above expression can be shown to be equal to

$$\text{Cov}(X, Y) = E(XY) - E(X)\, E(Y) \tag{3.75}$$

Relating to mechanics, the variance corresponds to the moments of inertia about axes x and y passing through the centroid of a plate and the covariance corresponds to the product moment of inertia with respect to the axes x and y, mentioned earlier.

The correlation coefficient, a dimensionless quantity, is obtained by normalizing the covariance with standard deviations of the corresponding pair of variables. That is, the correlation coefficient between the variables X and Y is defined as

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \tag{3.76}$$

Some important points about the correlation coefficient ρ are:

(i) The value of ρ lies between -1 and $+1$, i.e.

$$-1 \leqslant \rho \leqslant +1$$

(ii) When ρ is between 0 and 1, the higher values of x will match with the higher values of y [Fig. 3.21(a)]. Variables are positively correlated.

(iii) When ρ is between -1 and 0, the higher values of x will match with the smaller values of y Fig. [3.21(b)]. Variables are negatively correlated.

(iv) ρ is a measure of the linear dependence between two variables.

(v) If $0 < \rho < 1$ or $-1 < \rho < 0$, it is said that atleast some dependence exists between X and Y.

(vi) If ρ is close to 1, it is said that a good linear relationship exists between X and Y.

(vii) If $\rho = 1$ or -1, it is said that there is a perfect linear relationship between X and Y. [Figs. 3.21(c) and 3.21(d)].

(viii) If X and Y are independent, $\rho = 0$.

(ix) If $\rho = 0$, it does not mean that X and Y are independent (unrelated). It means that the linear relationship does not exist between X and Y, but there may be a perfect nonlinear relationship (dependence) between X and Y [Fig. 3.21(e)].

(x) In engineering problems, the independence of variables is assumed (i.e. $\rho = 0$) to simplify the problem.

Mean and Variance of Functions of Variables

If Z is a linear function of variables $X_1, X_2, \ldots, X_n$, say

$$Z = \sum_{i=1}^{n} b_i X_i$$

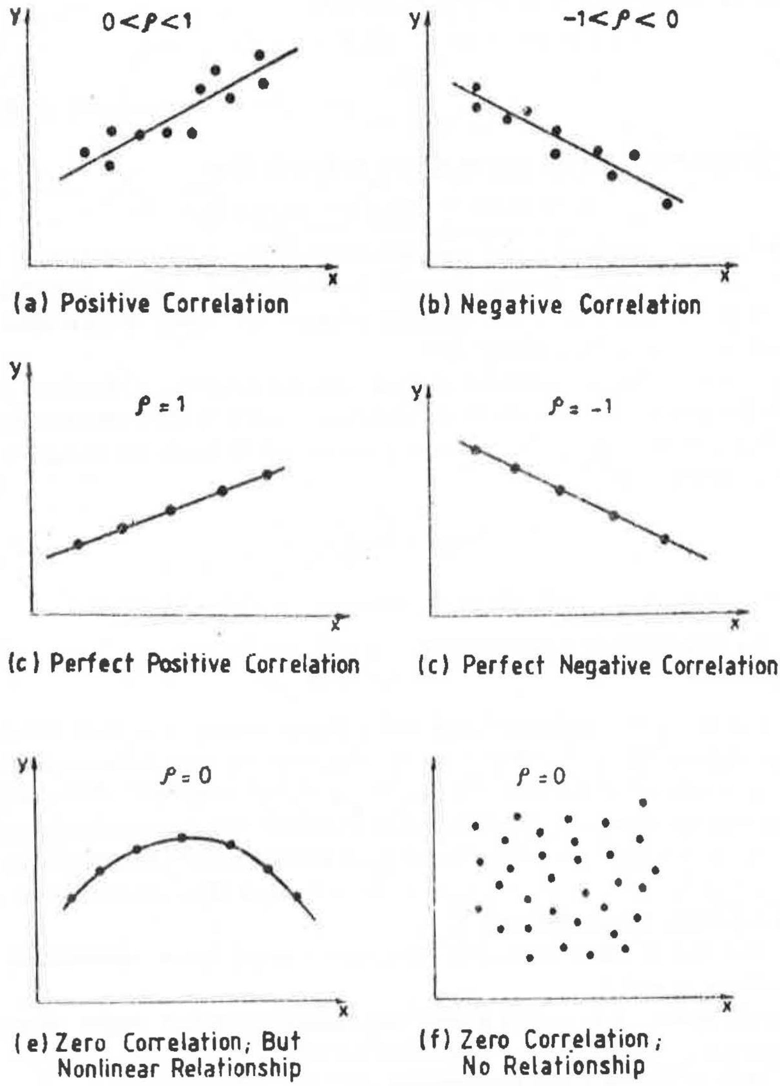


FIG. 3.21 Significance of correlation coefficient

then the expected value of Z is

$$E(Z) = \sum_{i=1}^{n} b_i \, E(X_i) \tag{3.77}$$

The above relation is valid whether variables X_i are independent or not.
If X_i are correlated, then

$$\text{Var}[Z] = \sum_{i=1}^{n} b_i^2 \, \text{Var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j>i}^{n} b_i b_j \, \text{Cov}(X_i, X_j) \tag{3.78}$$

If X_i are independent, the above equation simplifies to

$$\text{Var}(Z) = \sum_{i=1}^{n} b_i^2 \, \text{Var}(X_i) \tag{3.79}$$

When X_1 and X_2 are independent, their expectation of the product will decompose. That is, if

$$Z = X_1 X_2$$

then,

$$E(Z) = E(X_1)\, E(X_2) \tag{3.80a}$$

$$\text{Var}(Z) = \mu_1^2 \sigma_2^2 + \mu_2^2 \sigma_1^2 + \sigma_1^2 \sigma_2^2 \tag{3.80b}$$

and

$$\delta_Z^2 = \delta_1^2 + \delta_2^2 + \delta_1^2 \delta_2^2$$

where μ_i, σ_i and δ_i are the mean, the standard deviation and the coefficient of variation of X_i respectively.

If Z is a nonlinear function of several variables X_i, the approximate mean and variance of Z are obtained by using Taylor's series expansion and truncating the series to the required approximation. If

$$Z = g(X_1, X_2, \ldots, X_n)$$

the first order approximations of $E(Z)$ and Var(Z) are given by

$$E(Z) = \mu_Z \simeq g(\mu_1, \mu_2, \ldots, \mu_n) \tag{3.81}$$

$$\text{Var}(Z) = \sigma_Z^2 \simeq \sum_{i=1}^{n} \sum_{j=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_{\mu} \left.\frac{\partial g}{\partial X_j}\right|_{\mu} \text{Cov}(X_i, X_j) \tag{3.82}$$

$\left.\dfrac{\partial g}{\partial X_i}\right|_{\mu}$ means that the derivative is evaluated at the mean values of the variables.

If X_i are uncorrelated, then

$$\text{Var}(Z) \simeq \sum_{i=1}^{n} \left[\left.\frac{\partial g}{\partial X_i}\right|_{\mu}\right]^2 \text{Var}(X_i) \tag{3.83}$$

EXAMPLE 3.25 A simply supported beam is subjected to loads P_1, P_2 and P_3 as shown in Fig. 3.22.

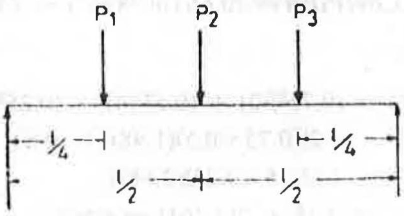


FIG. 3.22 Simply supported beam—Example 3.25

It is given that

$$E(P_1) = 20 \text{ kN} \quad \text{Var}(P_1) = 2 \text{ (kN)}^2$$
$$E(P_2) = 40 \text{ kN} \quad \text{Var}(P_2) = 4 \text{ (kN)}^2$$
$$E(P_3) = 50 \text{ kN} \quad \text{Var}(P_3) = 10 \text{ (kN)}^2$$

Determine the expected value and standard deviation of the shear force at the left end if (i) loads P_1, P_2 and P_3 are statistically independent and (ii) if loads are correlated with correlation coefficients

$$\rho_{12} = 0.7 \quad \rho_{23} = 0.8 \quad \rho_{31} = 0.6$$

Solution The shear force V at the left end of the beam is

$$V = 0.75\, P_1 + 0.5\, P_2 + 0.25\, P_3$$

The expected value of V, using Eq.(3.77), is

$$E(V) = 0.75 \times 20 + 0.5 \times 40 + 0.25 \times 50$$
$$= 47.5 \text{ kN}$$

Case (i) Loads are independent

The variance of V is calculated using Eq. (3.79):

$$\text{Var}(V) = (0.75^2)(2) + (0.5^2)(4) + (0.25^2)(10)$$
$$= 2.75$$

The standard deviation of V is equal to $= \sqrt{2.75} = 1.658$ kN.

Case (ii) Loads are correlated

The variance of V is calculated using Eq. (3.78). Before using Eq. (3.78), the covariance between the variables is to be calculated. The covariance is given by Eq. (3.76):

$$\begin{aligned}\text{Cov}(P_1 P_2) &= \rho_{12}\sigma_1\sigma_2 \\ &= (\rho_{12})\, [\text{Var}(P_1)]^{1/2}\, [\text{Var}(P_2)]^{1/2} \\ &= (0.7)(2^{1/2})(4^{1/2}) \approx 1.98\end{aligned}$$

Similarly,

$$\text{Cov}(P_2 P_3) = (0.8)(4^{1/2})(10^{1/2}) = 5.06$$
$$\text{Cov}(P_3 P_1) = (0.6)(10^{1/2})(2^{1/2}) = 2.68$$

The variance of V is

$$\begin{aligned}\text{Var}(V) &= (0.75^2)(2) + (0.5)^2(4) + (0.25^2)(10) \\ &\quad + 2[(0.75 \times 0.5)(1.98) + (0.5 \times 0.25)(5.06) \\ &\quad + (0.75 \times 0.25)(2.68)] \\ &= 2.75 + 2(2.763) = 6.505\end{aligned}$$

The standard deviation of V is equal to 2.55 kN.

3.6 COMMON PROBABILITY DISTRIBUTIONS

There are a number of discrete and continuous probability distributions which are used in engineering applications. It is always convenient to have a mathematical function (PDF or CDF) to describe a random variable. Before an engineer uses or proposes a probability distribution (probabilistic model), it is necessary and better that he knows how these models have arisen and what physical situation has given rise to the distribution. Many of the common distributions are tabulated for convenience and ready use. Out of the several probability distributions, only some of the models which are often used in reliability analysis and design of structures are dealt with. The other models which are not discussed are tabulated at the end.

Uniform Distribution

This is a continuous distribution. Here the random variable X is equally likely to have any value between the lower limit l and the upper limit u. The PDF of X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{u-l} & l \leq x \leq u \\ 0 & \text{elsewhere} \end{cases} \tag{3.84}$$

The mean and the variance are

$$\mu = \frac{l+u}{2}$$

$$\sigma^2 = \frac{(u-l)^2}{12}$$

When the uniform distribution is described between the limits 0 and 1, it is called the standard uniform distribution. In the case of the standard uniform distribution, the cumulative probability of the variable Y, taking a value y_1, is equal to the value of y_1 itself. That is,

$$F_Y(y_1) = y_1$$

This property is used in the inverse transformation technique applied for generating the random variates (Chapter 7). The sketches of the uniform probability distribution and the standard uniform distribution are shown in Fig. 3.23.

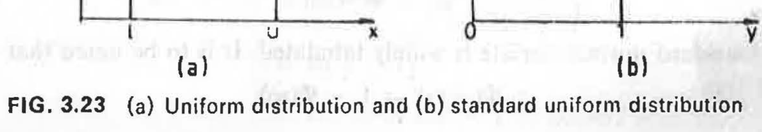


FIG. 3.23 (a) Uniform distribution and (b) standard uniform distribution

***Normal Distribution:* $N(\mu, \sigma)$**

If a phenomenon (a random variable) arises because of several factors and if the effects of these several factors act in an additive way to result the phenomenon, then the model arising out of such a situation will be a normal distribution. In short, this model arises out of an additive mechanism. This distribution is also known as the Gaussian distribution. The PDF of a normal variate is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad -\infty \leqslant x \leqslant \infty \qquad (3.85)$$

where μ and σ are the parameters, mean and standard deviation of the distribution respectively. In future, this distribution will be designated as $N(\mu, \sigma)$.

A normal distribution with parameters $\mu = 0$ and $\sigma = 1$ is called a standard normal distribution and is designated as $N(0, 1)$. The PDF of the standard normal variate U is given by

$$f_U(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) \quad -\infty \leqslant u \leqslant \infty \qquad (3.86)$$

Because it is so frequently used, the standard normal density function and its CDF are given special notations, $\phi(u)$ and $\Phi(u)$ respectively. Hence $\Phi(u)$ is the cumulative probability of a standard normal variate. That is,

$$\Phi(u) = F_U(u) = P(U \leqslant u)$$

The PDF and CDF of U are shown in Fig. 3.24. Referring to Fig. 3.24, we have

$$\Phi(u_1) = p_1$$

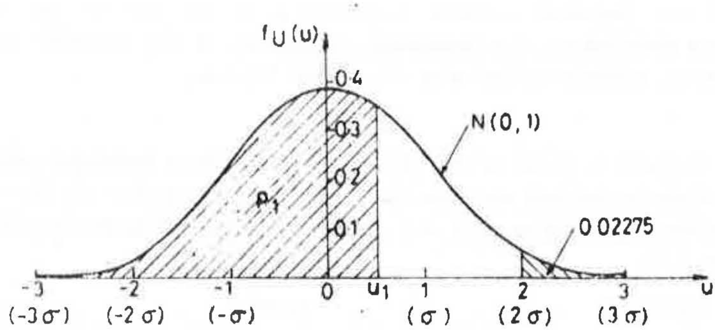


FIG. 3.24 Standard normal density function

Conversely, the value of u_1 at a cumulative probability of p_1 is given by

$$u_1 = \Phi^{-1}(p_1)$$

The standard normal variate is widely tabulated. It is to be noted that

$$\Phi(-u_2) = 1 - \Phi(u_2)$$

If $\Phi(-u_2) = p_2$, then

$$u_2 = -\Phi^{-1}(p_2)$$

because of symmetry. The CDF of X with distribution $N(\mu, \sigma)$ is written as

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]dx \tag{3.87}$$

Let
$$u = \frac{x-\mu}{\sigma}$$

Then
$$du = dx/\sigma$$

Using these in Eq.(3.87), we have

$$F_X(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x-\mu)/\sigma} \exp(-u^2/2)du$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right) \tag{3.88}$$

Hence, using normal probability tables, probabilities of any other normal distribution can be obtained. Modern computers have built-in functions to compute $\Phi(u)$. A polynomial is used to evaluate $\Phi(u)$.

EXAMPLE 3.26 The cube strength of concrete, X, follows the normal distribution with parameters, $\mu = 30$ N/mm² and $\sigma = 4.5$ N/mm² (Fig. 3.25). Calculate the probability of getting a value for a strength

(i) less than 25 N/mm² and (ii) less than 40 and greater than or equal to 30 N/mm².

X is distributed as $N(30, 4.5)$.

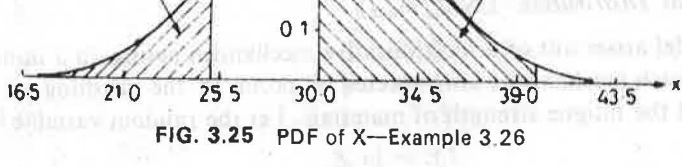


FIG. 3.25 PDF of X—Example 3.26

Solution (i) The probability of getting a value less than 25 N/mm² is

$$P(X < 25) = F_X(25) = \Phi\left(\frac{25-30}{4.5}\right)$$

$$= \Phi(-1.11)$$

$$= 1 - \Phi(1.11) = 1 - 0.8667 = 0.1333$$

(ii) The probability of getting a value less than 40 and greater than or equal to 30 is

$$P(30 < X < 40) = F_X(40) - F_X(30)$$
$$= \Phi\left(\frac{40-30}{4.5}\right) - \Phi\left(\frac{30-30}{4.5}\right)$$
$$= 0.9869 - 0.5 = 0.4869$$

Some properties of normal variables are:

(i) The distribution is symmetrical; hence the coefficient of skewness is zero.

(ii) The mean, median and mode are the same.

(iii) The coefficient of kurtosis is equal to 3.

(iv) The normal distribution is reproductive, that is the sum and the difference of two or more normally distributed random variates is itself normally distributed.

If $Z = X_1 \pm X_2 \pm \ldots \pm X_n$ and X_i are independent normal variates with parameters μ_i and σ_i, Z is also a normal variate with parameters μ_Z and σ_Z, given by

$$\mu_Z = \mu_1 \pm \mu_2 \pm \ldots \pm \mu_n$$
$$\sigma_Z^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$$

If X_i are correlated, then

$$\sigma_z^2 = \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i=1}^{n} \sum_{j>1}^{n} Cov\left(X_i, X_j\right)$$

At this stage, it is very useful to know the remarkable result established by the Central Limit Theorem which says, when stated loosely, that the sum of a large number of arbitrarily distributed random variables will tend to be normally distributed. Hence, physical process which is the result of the combined effects of several factors (irrespective of their individual distributions) would tend to be normally distributed.

***Lognormal Distribution:* $LN(\tilde{Z}, \sigma_{\ln Z})$**

This model arises out of a multiplicative mechanism acting on a number of factors. Such mechanisms are expected to occur in the crushing of aggregates and the fatigue strength of materials. Let the random variable

$$X = \ln Z \qquad (3.89)$$

be normally distributed with parameters $N(\mu_X, \sigma_X)$; then the random variable Z is said to follow the lognormal distribution whose PDF is given as

$$f_Z(z) = \frac{1}{z\sigma_{\ln Z}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left\{\frac{\ln(z/\tilde{Z})}{\sigma_{\ln Z}}\right\}^2\right] \qquad z > 0 \qquad (3.90)$$

where $\tilde{Z}$, the median of Z, and $\sigma_{\ln z}$, the standard deviation of $\ln Z$ are the parameters of the distribution. This distribution is designated as LN($\tilde{Z}$, $\sigma_{\ln z}$). The parameters are calculated by the following equations:

$$\tilde{Z} = \mu_Z \exp\left(-\frac{1}{2}\sigma^2_{\ln z}\right) \tag{3.91}$$

and
$$\sigma^2_{\ln z} = \ln(\delta_Z^2 + 1) \tag{3.92}$$

where δ_Z is the coefficient of variation of Z. The cumulative probability of a lognormal variate can be calculated using standard normal tables $\Phi(u)$. Let

$$u = \frac{1}{\sigma_{\ln z}} \ln(z/\tilde{Z})$$

$$du = \frac{1}{z\sigma_{\ln z}} dz$$

Substitution of the above values in Eq. (3.90) yields

$$Fz(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp(-u^2/2)\, du$$

$$= \Phi(u) = \Phi\left[\frac{\ln(z/\tilde{Z})}{\sigma_{\ln z}}\right] \tag{3.93}$$

Similarly the PDF of Z can be connected to the PDF of the standard normal:

$$fz(z) = \frac{1}{z\sigma_{\ln z}} \phi\left[\frac{\ln(z/\tilde{Z})}{\sigma_{\ln z}}\right] \tag{3.94}$$

the lognormal distribution for various values of $\sigma_{\ln z}$ is plotted and shown in Fig. 3.26. It can be observed that as the coefficient of variation decreases, the curve approaches the normal distribution.

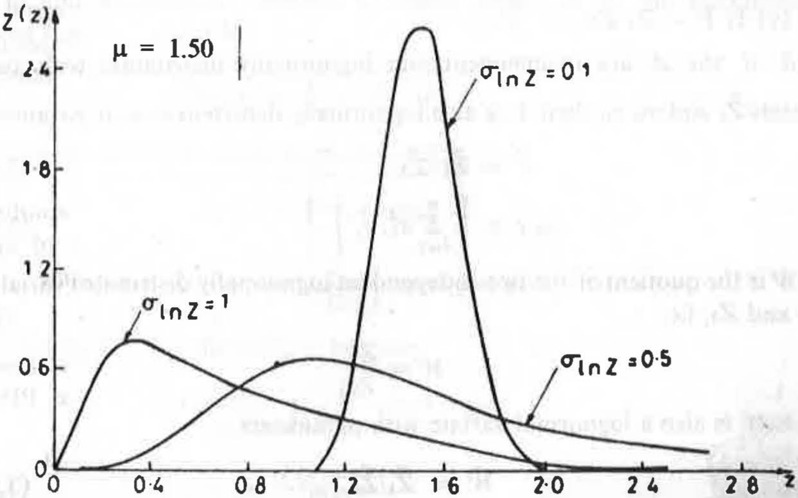


FIG. 3.26 Lognormal density functions

EXAMPLE 3.27 The compressive strength, Z, of M 15 concrete follows the lognormal distribution. It is given that

$$\mu_Z = 24.04 \text{ N/mm}^2 \qquad \sigma_Z = 5.76 \text{ N/mm}^2$$

Determine the probability of getting a strength less than the specified value, 15 N/mm².

Solution The coefficient of variation of Z is equal to

$$\delta_Z = 5.76/24.04 = 0.24$$

Using Eqs. (3.91) and (3.92), we have

$$\sigma^2_{\ln Z} = \ln(0.24^2 + 1) = 0.056$$

$$\sigma_{\ln Z} = 0.236$$

(*Note*: For $\delta_Z \leqslant 0.25$, $\quad \sigma_{\ln Z} \simeq \delta_Z$)

$$\tilde{Z} = 24.04 \exp(-0.056/2)$$

$$= 23.37 \text{ N/mm}^2$$

Hence Z is distributed as LN (23.37, 0.236). The probability of getting a value less than 15 N/mm² is

$$P(Z < 15) = F_Z(15)$$

$$= \Phi\left[\frac{\ln(15/23.37)}{0.236}\right] = 0.03$$

Some properties of the lognormal variate Z are:

(i) If $X = \ln Z$, then $\mu_X = \ln \tilde{Z}$

(ii) $\tilde{Z}$ is always less than μ_Z

(iii) Z is positively skewed

(iv) $\sigma_{\ln Z} \simeq \delta_Z \quad$ for $\quad \delta_Z \leqslant 0.25$

(v) If $Y = Z_1 \cdot Z_2 \cdot \ldots \cdot Z_n$

and if the Z_i are independent and lognormally distributed with parameters $\tilde{Z}_i$ and $\sigma_{\ln Z_i}$, then Y is also lognormally distributed with parameters

$$\tilde{Y} = \tilde{Z}_1 \cdot \tilde{Z}_2 \ldots \tilde{Z}_n \tag{3.95}$$

$$\sigma_{\ln Y} = \left[\sum_{i=1}^{n} \sigma^2_{\ln Z_i}\right]^{1/2} \tag{3.96}$$

If W is the quotient of the two independent lognormally distributed variables Z_1 and Z_2, i.e.

$$W = \frac{Z_1}{Z_2}$$

then W is also a lognormal variate with parameters

$$\tilde{W} = \tilde{Z}_1/\tilde{Z}_2 \tag{3.97}$$

$$\sigma_{\ln W} = [\sigma^2_{\ln Z_1} + \sigma^2_{\ln Z_2}]^{1/2} \tag{3.98}$$

Gamma Distribution G(k, λ)

The sum of independently and identically distributed exponential random variables results in the gamma distribution. If the occurrence of an event constitutes a poisson process, (Ref. 3.1), then the time until the kth occurrence of the event, is described by the gamma distribution. Let X_k denote the time till the kth event. Then the probability density function of the gamma variable X_k with parameters k and λ is given by

$$f_{X_k}(x) = \frac{\lambda(\lambda x)^{k-1}e^{-\lambda x}}{(k-1)!} \qquad x \geqslant 0 \tag{3.99}$$

Hereafter, the suffix k for X is removed.
Parameters k and λ are connected to the mean and variance by the following equations:

$$\mu_X = \frac{k}{\lambda} \tag{3.100}$$

$$\sigma_X^2 = \frac{k}{\lambda^2} \tag{3.101}$$

The gamma distributed variable X with parameters k and λ is designated as $G(k, \lambda)$.

The parameter k need not be integer valued. For a noninteger valued k, the PDF of X is written as

$$f_X(x) = \frac{\lambda(\lambda x)^{k-1}e^{-\lambda x}}{\Gamma(k)} \qquad x \geqslant 0 \tag{3.102}$$

$$\lambda \geqslant 0,\ k \geqslant 0$$

where
$$\Gamma(k) = \int_0^\infty e^{-t}t^{k-1}\,dt \tag{3.103}$$

The gamma distribution function is widely tabulated as the incomplete gamma function, given by

$$\Gamma(k, x) = \int_0^x e^{-t}t^{k-1}\,dt$$

This can be used to evaluate the CDF of X:

$$F_X(x) = \int_0^x f_X(x)\,dx$$

$$= \frac{\lambda^k}{\Gamma(k)}\int_0^x e^{-\lambda x}x^{k-1}\,dx$$

Substituting $y = \lambda x$, the integral becomes

$$F_X(x) = \frac{1}{\Gamma(k)}\int_0^{\lambda x} e^{-y}y^{k-1}\,dy$$

$$= \frac{\Gamma(k, \lambda x)}{\Gamma(k)} \tag{3.104}$$

Equations (3.100) and (3.101) are valid for noninteger values of k also. The shape of gamma distribution is shown in Fig. 3.27. This distribution is widely used because, like observed data from many phenomena, the variable is limited to positive values and is skewed to the right. The gamma distribution is used to describe the maximum river flows, the yield strength of the reinforced concrete members (3.2), the sustained floor load in buildings, etc. For an integer valued k, the gamma distribution [Eq. (3.99)] is also known as the Erlang distribution. The gamma distribution [(Eq. 3.102)] is also called the Pearson Type III distribution.

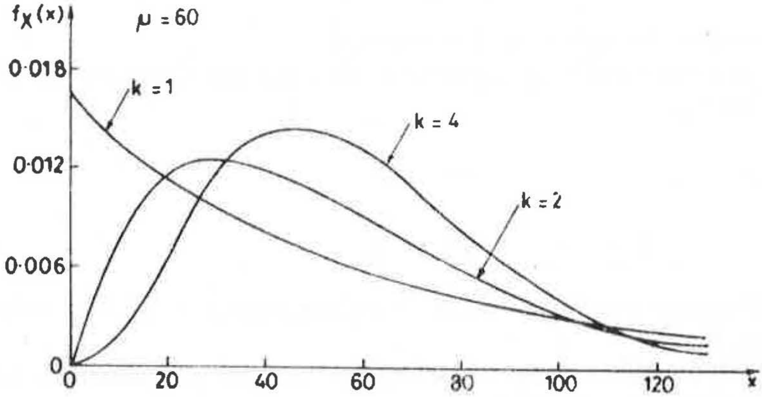


FIG. 3.27 Shapes of gamma distribution

The tables for an incomplete gamma function have been tabulated by Karl Pearson (3.3). This table directly gives the cumulative probability of an incomplete gamma variate. The algorithm AS32 given by G.P. Bhattacharjee (3.4) can be used to compute the incomplete gamma function. However, modern computers have built-in functions to compute the cumulative probability of an incomplete gamma variate.

Pearson tables give values of $I(u, p)$, where $I(u, p)$ is the cumulative probability of the variate. One enters Pearson tables with $p = k - 1$ and $u = \lambda x/\sqrt{k}$ and finds the value of $I(u, p)$.

EXAMPLE 3.28 The floor live load, X, on an office building is found to follow the gamma distribution with parameters k and λ being 3.86 and 7.55×10^{-3} respectively. Calculate the probability of the floor load exceeding the value 1500 N/m².

Solution The mean and standard deviation of X are calculated using Eqs. (3.100) and (3.101). Thus

$$\mu_X = \frac{k}{\lambda} = \frac{3.86}{7.55 \times 10^{-3}} = 511.6 \text{ N/m}^2$$

$$\sigma_X = \frac{\sqrt{k}}{\lambda} = \frac{\sqrt{3.86}}{7.55 \times 10^{-3}} = 260.35 \text{ N/m}^2$$

The probability of the floor load exceeding the value 1500 N/m² is

$$
\begin{aligned}
P(X > 1500) &= 1 - P(X \leqslant 1500) \\
&= 1 - F_X(1500) \\
&= 1 - \frac{\Gamma(k, \lambda x)}{\Gamma(k)} \\
&= 1 - \frac{\Gamma(3.86, 11.325)}{\Gamma(3.86)} \\
&= 1 - 0.9968 = 3.2 \times 10^{-3}
\end{aligned}
$$

Some properties of the gamma distribution are:

(i) It can take only positive values.

(ii) It is positively skewed.

(iii) If X_1 is $G(k_1, \lambda)$ and X_2 is $G(k_2, \lambda)$, and if $Y = X_1 + X_2$, then Y is also gamma distributed with parameters $k_1 + k_2$ and λ.

***Beta Distribution*: $BT(a, b, p, q)$**

Many of the random variables in practice, say the strength of steel or concrete, take values within certain limits. Under these conditions, the appropriate probability distribution for a random variable whose possible values lie in a restricted interval, say between limits a and b, is the beta distribution.

A standard beta distributed random variable, X, is defined over the range $0 \leqslant x \leqslant 1$. Its PDF is

$$
f_X(x) = \frac{x^{p-1}(1 - x)^{q-1}}{B(p, q)} \qquad 0 \leqslant x \leqslant 1 \tag{3.105}
$$

where $B(p, q)$ is the beta function which is tabulated directly or may be obtained from tables of the gamma function from the relation

$$
B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)} \tag{3.106}
$$

The mean and variance of X are

$$
\mu_X = \frac{p}{p + q} \tag{3.107}
$$

$$
\sigma_X^2 = \frac{pq}{(p + q)^2(p + q + 1)} \tag{3.108}
$$

The standard beta distribution is designated as $BT_X(p, q)$. When a beta distributed random variable, say Y, has a range $a \leqslant y \leqslant b$, the simplest approach is to transform Y according to

$$
X = \frac{Y - a}{b - a}
$$

Then the PDF of Y is

$$f_Y(y) = \frac{1}{b-a} f_X\left(\frac{y-a}{b-a}\right)$$

$$= \frac{(y-a)^{p-1}(b-y)^{q-1}}{B(p,q)(b-a)^{p+q-1}} \quad a \leqslant y \leqslant b \tag{3.109}$$

$$F_Y(y) = F_X\left(\frac{y-a}{b-a}\right)$$

The mean and variance of Y are given by

$$\mu_Y = a + \frac{p}{p+q}(b-a) \tag{3.110}$$

$$\sigma_Y^2 = (b-a)^2\left[\frac{pq}{(p+q)^2(p+q+1)}\right] \tag{3.111}$$

Depending on the parameters of p and q, the density function of the beta distribution will have different shapes as shown in Fig. 3.28. Whenever p and q take noninteger values, the beta function is called the incomplete beta function. The cumulative probability of the incomplete beta function is tabulated by Pearson (3.5) as $B_x(p, q)$. Hence, Pearson's tables can be used to calculate the cumulative probability of a beta variate Y. It must be noted that the tables are given for $p \geqslant q$. For $p < q$,

$$B_x(p, q) = 1 - B_{(1-x)}(q, p)$$

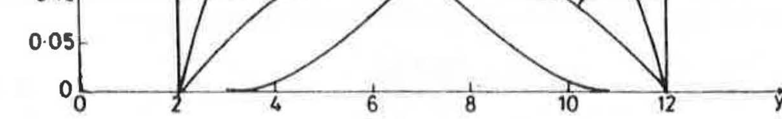


FIG. 3.28 Shapes of beta distribution

For example, if $p = 2$ and $q = 4$,

$$B_{0.3}(2, 4) = 1 - B_{0.7}(4,2)$$

Modern computers have built-in functions to compute the cumulative probability of an incomplete beta variate. The algorithm AS 63 given by Majumder and Bhattacharjee (3.6), and modified by Cran, Martin and Thomas (3.7), can be used to evaluate the incomplete beta variate.

EXAMPLE 3.29 It is given that the strength Y of M 35 concrete follows the beta distribution. The mean and standard deviation of Y are 42.50 N/mm² and 6.25 N/mm² respectively. It is found from data that the minimum and maximum values of Y are 30 N/mm^2 and 55 N/mm^2 respectively. Calculate the probability of the strength of concrete being less than 35 N/mm^2.

Solution It is given:

$$\mu_Y = 42.5 \text{ N/mm}^2 \qquad \sigma_Y = 6.25 \text{ N/mm}^2$$

$$a = 30 \text{ N/mm}^2 \qquad b = 55 \text{ N/mm}^2$$

Using Eqs. (3.110) and (3.111), we have

$$42.5 = 30 + \frac{p}{p+q}(55 - 30)$$

$$6.25^2 = (55 - 30)^2 \left[\frac{pq}{(p+q)^2(p+q+1)}\right]$$

Solving the above two equations, p and q are

$$p = 1.5 \qquad q = 1.5$$

Hence the strength of concrete is distributed as BT (30, 50, 1.5, 1.5). The probability of the strength of concrete being less than 35 is

$$P(Y < 35) = F_Y(35)$$

In terms of standard beta variate X,

$$F_Y(35) = F_X\left(\frac{y-a}{b-a}\right)$$

$$= F_X\left(\frac{35-30}{55-30}\right) = F_X(0.2)$$

As per Pearson's tables,

$$F_X(0.2) = B_{0.2}(1.5, 1.5)$$

$$= 0.1423 \text{ (from tables)}$$

3.7 EXTREMAL DISTRIBUTIONS

Civil engineers are more concerned with the occurrence of the largest or the smallest of a number of random variables in the analysis and design of structures. The structural safety of a determinate truss (system) may depend only on the extremes, for example, on the strength of the weakest of many elementary members (components). A civil engineer may be interested to know the value and the distribution of the likely maximum wind speed, or the floor load acting on a building during its lifetime.

Let X be the largest of the n random variables $Y_1, Y_2, \ldots, Y_n$. The probability that all the values in n variables will be less than a specified value x_s is

$$F_X(x_s) = P(Y \leqslant x_s)$$

$$= P(\text{all } n \text{ of the } Y_i \leqslant x_s)$$

If the Y_i are independent,

$$F_X(x_s) = P(Y_1 \leqslant x_s)P(Y_2 \leqslant x_s)\ldots P(Y_n \leqslant x_s)$$
$$= F_{Y_1}(x_s)F_{Y_2}(x_s)\ldots F_{Y_n}(x_s)$$

If all the Y_i are identically distributed with a common distribution $F_Y(y)$, then

$$F_X(x_s) = [F_Y(x_s)]^n \tag{3.112}$$

If the X_i are continuous random variables with a common PDF, $f_X(x)$, then

$$f_X(x_s) = \frac{d}{dX}F_X(x_s) = n[F_Y(x_s)]^{n-1}f_Y(x_s)$$

From past experience if an engineer knows the distribution of the maximum wind speed Y_i observed in any one year, he may be able to determine the distribution of the largest wind speed in a particular lifetime of the structure, say 50 years.

It has been found that for some parent distributions of specific general types, the extreme value distribution can be approximated by certain theoretical distributions, called asymptotic distributions, for large n. As n increases, it is more and more accurate. It is not necessary to know the underlying distribution of Y_i precisely. It is enough if the general trend of the tail portion of the Y_i is known. There are three asymptotic distributions proposed by Gumbel. They are described below:

Type 1 Extremal (Largest) Distribution: $EX_{1,L}$ (u, α)

This distribution requires that the upper tail of the parent distribution that contains the extreme value be exponential in nature (normal, Weibull, exponential, gamma, and other similarly shaped density functions). The distribution of X, the largest of many independent random variables with a common exponential type of upper tail distribution ($F_Y(y) = 1 - \exp(-h(y))$), has the form of Type 1 extremal (largest) distribution, given by

$$f_X(x) = \alpha \exp[-\alpha(x - u) - \exp\{-\alpha(x - u)\}] \quad -\infty \leqslant x \leqslant \infty \tag{3.113}$$

$$F_X(x) = \exp[-\exp\{-\alpha(x - u)\}] \quad -\infty \leqslant x \leqslant \infty \tag{3.114}$$

The parameters u (location, here it is median) and α (dispersion) are given by

$$\mu_X = u + \frac{0.5772}{\alpha} \tag{3.115}$$

$$\sigma_X^2 = \frac{\pi^2}{6\alpha^2} \tag{3.116}$$

This distribution is also called the Gumbel distribution and is positively skewed. The coefficient of skewness is 1.1396. The distribution is designated as $EX_{1,L}(u, \alpha)$. The shape of the distribution for $u = 0.275$ and $\alpha = 2.566$ is shown in Fig. 3.29. This model is used for describing the maximum annual flow in a river, the maximum annual wind speed at a location, etc.

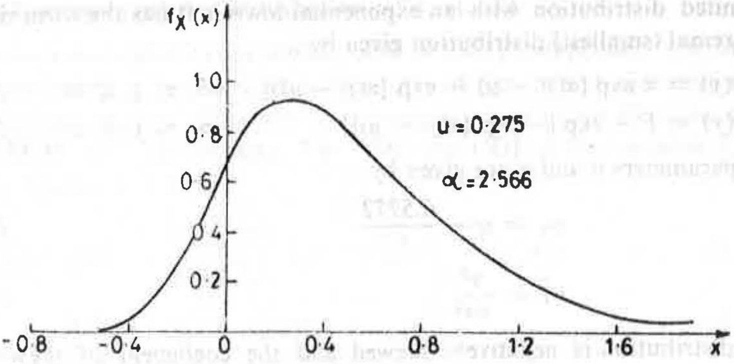


FIG. 3.29 Shape of Type 1 extremal (largest) distribution

EXAMPLE 3.30 The yearly maximum wind speed, X, observed at Pune follows the Type 1 extremal (largest) distribution. It is given:

$$\mu_X = 83.67 \text{ kmph} \qquad \sigma_X = 15.97 \text{ kmph}$$

Calculate the parameters of the distribution and determine the probability of the wind speed exceeding 117 kmph.

Solution Parameters of the distribution are calculated using Eqs. (3.115) and (3.116).

$$15.97^2 = \frac{\pi^2}{6\alpha^2}$$

$$\alpha = 0.0803$$

$$u = 83.67 - \frac{0.5772}{0.0803}$$

$$= 76.48 \text{ kmph}$$

Hence the CDF of X [Eq. (3.114)] is

$$F_X(x) = \exp[-\exp\{-0.0803(x - 76.48)\}] \qquad -\infty \leqslant x \leqslant \infty$$

The probability of the maximum wind speed exceeding 117 kmph in any one year is

$$P(X > 117) = 1 - F_X(117)$$

$$= 1 - \exp[-\exp\{-0.0803(117 - 76.48)\}]$$

$$= 1 - 0.962 = 0.038$$

Type 1 Extremal (smallest) Distribution: $EX_{1,S}(u, \alpha)$

This distribution is similar to Type 1 extremal (largest) except that the lower tail of the parent distribution has an exponential form. The distribution of Z of the smallest of many independent variables with a common unlimited distribution with an exponential lower tail has the form of Type 1 extremal (smallest) distribution given by

$$f_Y(y) = \alpha \exp[\alpha(y - u) - \exp\{\alpha(y - u)\}] \quad -\infty \leqslant y \leqslant \infty \tag{3.117}$$

$$F_Y(y) = 1 - \exp[-\exp\{\alpha(y - u)\}] \quad -\infty \leqslant y \leqslant \infty \tag{3.118}$$

The parameters u and α are given by

$$\mu_Y = u - \frac{0.5772}{\alpha} \tag{3.119}$$

$$\sigma_Y^2 = \frac{\pi^2}{6\alpha^2} \tag{3.120}$$

This distribution is negatively skewed and the coefficient of skewness is -1.1396. A typical shape of the Type 1 extremal (smallest) distribution is shown in Fig. 3.30.

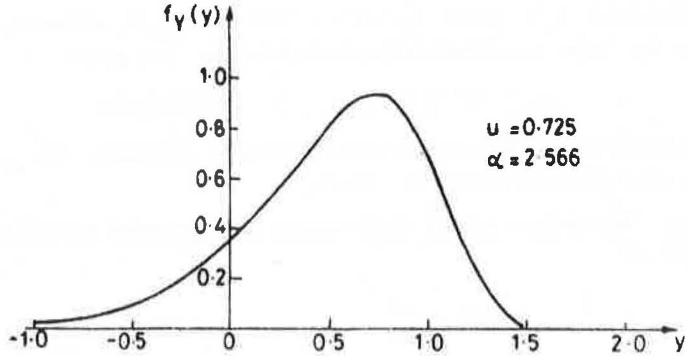


FIG. 3.30 Shape of Type 1 extremal (smallest) distribution

EXAMPLE 3.31 The minimum annual flow Y in a river is assumed to follow the Type 1 extremal (smallest) distribution. The mean and standard deviation of Y are

$$\mu_X = 5 \text{ m}^3/\text{s} \qquad \sigma_X = 2 \text{ m}^3/\text{s}$$

Calculate the probability of the minimum annual flow in a year being less than 2 m³/s.

Solution The parameters of the distribution are [Eqs. (3.119) and (3.120)]

$$\alpha = \frac{\pi}{\sqrt{6} \times 2} = 0.641$$

$$u = 5 + \frac{0.5772}{0.641} = 5.9$$

Hence the CDF of minimum annual flow is

$$F_Y(y) = 1 - \exp[-\exp\{0.641(y - 5.9)\}]$$

The probability of the minimum annual flow in a year less than 2 m³/s is

$$P(Y < 2) = F_Y(2)$$

$$= 1 - \exp[-\exp\{0.641(2 - 5.9)\}] = 0.079$$

Type 2 Extremal (largest) Distribution: $EX_{2,L}(u, k)$

This is another model for the largest value of many independent identically distributed random variables. Here, the form of the parent distribution is not generally defined. This model is generally selected on the basis of an empirical fit to a set of data. The PDF and CDF of the variable X, the largest of many Y_i are

$$f_X(x) = \frac{k}{u}\left[\frac{u}{x}\right]^{k+1} \exp\left[-(u/x)^k\right] \qquad x \geqslant 0 \tag{3.121}$$

$$F_X(x) = \exp\left[-(u/x)^k\right] \qquad x \geqslant 0 \tag{3.122}$$

where u and k are parameters of the distribution. They are connected to the mean, variance and coefficient of variation of X as follows:

$$\mu_X = u\Gamma\left(1 - \frac{1}{k}\right) \qquad k > 1 \tag{3.123}$$

$$\sigma_X^2 = u^2\left[\Gamma\left(1 - \frac{2}{k}\right) - \Gamma^2\left(1 - \frac{1}{k}\right)\right] \qquad k > 2 \tag{3.124}$$

$$\delta_X^2 = \frac{\Gamma\left(1 - \frac{2}{k}\right)}{\Gamma^2\left(1 - \frac{1}{k}\right)} - 1 \qquad k > 2 \tag{3.125}$$

The above equation has been solved and the values of k for the corresponding values of δ are given in Table 3.3. Type 2 extremal largest distribution, designated as $EX_{2,L}(u, k)$, is used to model the annual maximum wind speed, the maximum annual flood, the maximum atmospheric temperature, etc. A typical shape of Type 2 extremal (largest) distribution is shown in Fig. 3.31.

TABLE 3.3 *Values of k for corresponding values of δ for Type 2 extremal (largest) distribution*

δ	k	δ	k	δ	k
0.30	5.15	0.21	6.95	0.12	11.62
0.29	5.29	0.20	7.255	0.11	12.65
0.28	5.45	0.19	7.59	0.10	13.88
0.27	5.61	0.18	7.97	0.09	15.425
0.26	5.79	0.17	8.395	0.085	16.35
0.25	5.98	0.16	8.87	0.08	17.4
0.24	6.195	0.15	9.415	0.075	18.62
0.23	6.42	0.14	10.04	0.07	20.03
0.22	6.675	0.13	10.77	0.06	23.72

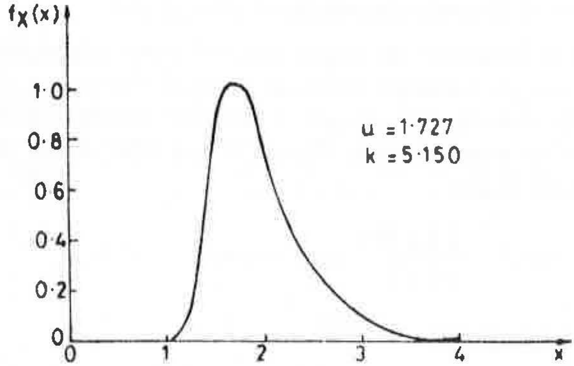


FIG. 3.31 Shape of Type 2 extremal (largest) distribution

EXAMPLE 3.32 The yearly maximum wind speed X, observed at New Delhi follows the Type 2 extremal largest distribution. It is found (from data) that

$$\mu_X = 100 \text{ kmph} \qquad \sigma_X = 23 \text{ kmph}$$

Calculate the probability of the annual maximum wind speed exceeding 120 kmph.

The parameters of the distribution are first calculated. Thus

$$\delta_X = \frac{23}{100} = 0.23$$

From Table 3.3,

$$k = 6.42 \qquad \text{for } \delta_X = 0.23$$

Using Eq. (3.123),

$$u = \frac{100}{\Gamma\left(1 - \dfrac{1}{6.42}\right)} = \frac{100}{1.12} = 89.29 \text{ kmph}$$

The probability of the wind speed exceeding 120 kmph in any one year is

$$P(X > 120) = 1 - P(X \leqslant 120) = 1 - F_x(120)$$

Using Eq. (3.122),

$$\begin{aligned} F_X(120) &= \exp\left[-(89.29/120)^{6.5}\right] \\ &= 0.864 \end{aligned}$$

Hence,

$$P(X > 120) = 1 - 0.864 = 0.136$$

Type 3 Extremal (smallest) Distribution: $EX_{3,S}(u, k)$

This model is for the smallest of the many random variables. This distribution is also called the Weibull distribution which is extensively used in reliability studies. The PDF and CDF of a variable X following Type 3 extremal (smallest) distribution are

$$f_X(x) = \frac{k}{u-l}\left(\frac{x-l}{u-l}\right)^{k-1} \exp\left[-\left(\frac{x-l}{u-l}\right)^k\right] \quad x \geqslant l \tag{3.126}$$

$$F_X(x) = 1 - \exp\left[-\left(\frac{x-l}{u-l}\right)^k\right] \quad x \geqslant l \tag{3.127}$$

The parameters u and k are given by

$$\mu_X = l + (u-l)\Gamma\left(1+\frac{1}{k}\right) \tag{3.128}$$

$$\sigma_X^2 = (u-l)^2\left[\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)\right] \tag{3.129}$$

This model has been used to represent the material strength in tension and fatigue.

For many practicable problems, it may be reasonable to assume $l = 0$. If $l = 0$, Eqs. (3.126) and (3.127) simplify greatly. The CDF of X for $l = 0$ is,

$$F_X(x) = 1 - \exp\left[-(x/u)^k\right] \quad x \geqslant 0 \tag{3.130}$$

with

$$\mu_X = u\Gamma\left(1+\frac{1}{k}\right) \tag{3.131}$$

$$\sigma_X^2 = u^2\left[\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)\right] \tag{3.132}$$

$$\delta_X^2 = \frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)} - 1 \tag{3.133}$$

The values of k corresponding to the values of δ_X are given in Table 3.4. Gumbel has studied droughts using this model with $l = 0$. A typical shape of Type 3 extremal (smallest) distribution is shown in Fig. 3.32.

TABLE 3.4 *Values of k for corresponding δ for Type 3 extremal (smallest) distribution*

δ	k	δ	k	δ	k
0.300	3.72	0.180	6.54	0.10	12.45
0.250	4.56	0.170	6.97	0.095	13.18
0.240	4.77	0.160	7.45	0.090	14.00
0.230	5.00	0.150	7.99	0.085	14.92
0.220	5.25	0.140	8.61	0.080	15.97
0.210	5.52	0.130	9.34	0.07	18.59
0.200	5.83	0.120	10.19	0.065	20.25
0.190	6.17	0.110	11.22	0.06	22.27

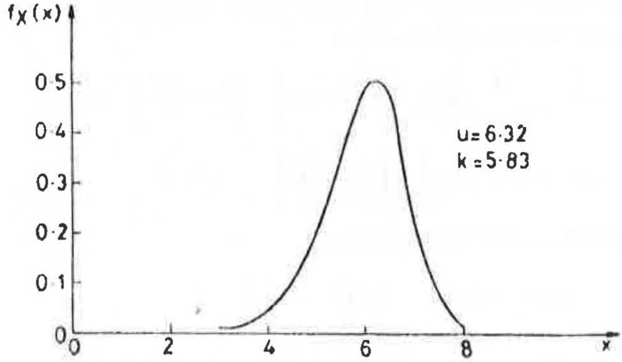


FIG. 3.32 PDF of Type 3 extremal (smallest) distribution

EXAMPLE 3.33 The flexural strength X of an over-reinforced prestressed concrete section has been found to follow the Type 3 extremal (smallest distribution with the lower bound $l = 0$. It is given:

$$\mu_X = 825.8 \text{ kN m} \qquad \sigma_X = 48.5 \text{ kN m}$$

Calculate the probability of failure of the beam when the external momen on the section is 437.5 kN m.

Solution The parameters of the distribution are first calculated as follows:

$$\delta_X = 48.5/825.8 = 0.0587$$

Using Eq. (3.133),

$$(0.0587)^2 = \frac{\Gamma\left(1 + \frac{2}{k}\right)}{\Gamma^2\left(1 + \frac{1}{k}\right)} - 1$$

Solving the above equation or using Table 3.4, the value of k is found to be 22.86. Using Eq. (3.131),

$$u = \frac{825.8}{\Gamma\left(1 + \frac{1}{21.6}\right)} = 846 \text{ kN m}$$

Hence the CDF of X is

$$F_X(x) = 1 - \exp\left[-(x/846)^{21.6}\right] \qquad x \geqslant 0$$

The beam fails when the strength of the section is less than the external moment acting on the section. Hence the probability of failure, p_f, of the section is

$$\begin{aligned} p_f &= P(X < 437.5) \\ &= 1 - \exp\left[-(437.5/846)^{22.86}\right] \\ &= 0.284 \times 10^{-6} \end{aligned}$$

The PDF of all the common distributions is listed in Table 3.5 also fo

TABLE 3.5 *Common probabilistic models and their parameters*

Distribution and designation	PDF	Relation between parameters and mean and variance
Uniform	$\frac{1}{u-l} \quad l \leqslant x \leqslant u$	$\mu_X = \frac{l+u}{12}$ $\sigma_X^2 = \frac{(u-l)^2}{12}$
Normal (Gaussian) $N(\mu_X, \sigma_X)$	$\frac{1}{\sigma_X\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right] \quad -\infty \leqslant x \leqslant \infty$	μ_X and σ_X^2
Lognormal LN $(\tilde{X}, \sigma_{\ln X})$	$\frac{1}{x\sigma_{\ln X}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left\{\frac{\ln(x/\tilde{X})}{\sigma_{\ln X}}\right\}^2\right] \quad x \geqslant 0$	$\tilde{X} = \mu_X \exp\left(-\frac{1}{2}\sigma_{\ln X}^2\right)$ $\sigma_{\ln X}^2 = \ln(\delta_X^2 + 1)$
Gamma $G(k, \lambda)$	$\frac{\lambda(\lambda x)^{k-1}e^{-\lambda x}}{\Gamma(k)} \quad x \geqslant 0;\ k, \lambda \geqslant 0$	$\mu_X = \frac{k}{\lambda};\ \sigma_X^2 = \frac{k}{\lambda^2}$
Beta BT (a, b, p, q)	$\frac{(x-a)^{p-1}(b-x)^{q-1}}{B(p, q)(b-a)^{p+q-1}} \quad a \leqslant x \leqslant b$	$\mu_X = a + \frac{p}{p+q}(b-a)$ $\sigma_X^2 = (b-a)^2\left[\frac{pq}{(p+q)^2(p+q+1)}\right]$
Type 1 extremal (largest) $EX_{1,L}(u, \alpha)$	$\alpha \exp[-\alpha(x-u) - \exp\{-\alpha(x-u)\}] \quad -\infty \leqslant x \leqslant \infty$	$\mu_X = u + \frac{0.5772}{\alpha};\ \sigma_X^2 = \frac{\pi^2}{6\alpha^2}$
Type 1 extremal (smallest) $EX_{1,S}(u, \alpha)$	$\alpha \exp[\alpha(x-u) - \exp\{\alpha(x-u)\}] \quad -\infty \leqslant x \leqslant \infty$	$\mu_X = u - \frac{0.5772}{\alpha};\ \sigma_X^2 = \frac{\pi^2}{6\alpha^2}$
Type 2 extremal (largest) $EX_{2,L}(u, k)$	$\frac{k}{u}\left(\frac{u}{x}\right)^{k+1} \exp[-(u/x)^k] \quad x \geqslant 0$	$\mu_X = u\Gamma\left(1-\frac{1}{k}\right) \quad k > 1$ $\sigma_X^2 = u^2\left[\Gamma\left(1-\frac{2}{k}\right) - \Gamma^2\left(1-\frac{1}{k}\right)\right] \quad k > 2$

(Contd.)

TABLE 3.5 *(Contd.)*

Distribution and Designation	PDF	Relation between parameters and mean and variance
Type 3 extremal (smallest) $EX_{3,S}(u, k)$	$\frac{k}{u-l}\left(\frac{x-l}{u-l}\right)^{k-1} \exp\left[-\left(\frac{x-l}{u-l}\right)^{k}\right] \quad x \geqslant l$	$\mu_X = l + (u-l)\Gamma\left(1+\frac{1}{k}\right)$
		$\sigma_X^2 = (u-l)^2\left[\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)\right]$
Poisson $P(\lambda t)$	$\frac{(\lambda t)^x e^{-\lambda t}}{x!} \quad x = 0, 1, 2, \ldots$	$\mu_X = \lambda t$
		$\sigma_X^2 = \lambda t$
Exponential EX (λ)	$\lambda e^{-\lambda x} \quad x \geqslant 0$	$\mu_X = \frac{1}{\lambda}$
		$\sigma_X^2 = \frac{1}{\lambda^2}$
Rayleigh	$\frac{x}{\alpha^2} \exp\left[-\frac{1}{2}\left(\frac{x}{\alpha}\right)^2\right] \quad x \geqslant 0$	$\mu_X = \alpha\sqrt{\pi/2}$
		$\sigma_X^2 = \alpha^2\left(2 - \frac{\pi}{2}\right)$

ready reference. The distributions, (i) t-distribution (ii) chi-square distribution and (iii) F-distribution, which are generally used for statistical tests (hypothesis testing), are not given.

Throughout, it has been assumed that the parameters of the distribution are known. They are to be estimated from the data using (i) the method of moments or (ii) the method of maximum likelihood. Readers are suggested to refer any book on probability and statistics for parameter estimation.

For a given data, the suitability of a probabilistic model is checked using chi-square test or Kolmogorov-Smirnov test (Refer 3.1).

REFERENCES

3.1 Benjamin, J.R. and C.A. Cornell, *Probability, Statistics and Decision for Civil Engineers*, McGraw-Hill, New York, 1970.

3.2 Ticky, M. and M. Vorlicek, "Safety of Reinforced Concrete Framed Structures", *Proceedings, International Symp. on Flexural Mechanics of Reinforced Concrete*, Miami, 1964, ASCE-ACI, pp 53-84.

3.3 Pearson, K., *Tables of the Incomplete Gamma Function*, Cambridge University Press, London, 1922.

3.4 Bhattacharjee, G.P., "Algorithm AS 32—Incomplete Gamma Integral", *Journal of Royal Statistical Society*, Vol. 19, No. 3, 1970.

3.5 Pearson, K., *Tables of the Incomplete Beta Function*, Cambridge University Press, Cambridge, England, 1968.

3.6 Majumder, K.L. and G.P. Bhattacharjee, "Algorithm AS 63—The Incomplete Beta Integral", *Applied Statistics*, Vol. 22, 1973, pp. 409-411.

3.7 Cran, G.W., K.J. Martin and G.E. Thomas, "Algorithm AS 109—A Remark on Algorithms AS 63: The Incomplete Beta Integral", *Applied Statistics*, Vol. 26, 1973, pp 111-114.

EXERCISE

3.1 During the monsoon season in Bombay, a strong wind may come from any direction between $\theta = 0$ (south) and $\theta = 90$ (west). The maximum wind speed cannot be greater than 200 kmph. Sketch the sample space for the wind direction and the wind speed. Show the event, the wind speed greater than 30 kmph, and the wind direction, $20 < \theta < 60$, in the sketch.

3.2 A simply supported beam of span l is to be designed for shear. There are two loads $Q_1 = 20$ kN and $Q_2 = 50$ kN which can come on the beam; but they can act only at discrete points, $0.25l$, $0.5l$ and $0.75l$ on the beam. It is not necessary that both loads should act at the same time. Sketch the sample space for the shear at the left end of the beam.

3.3 The completion of a water tank involves the successive completion of four stages. Let

A = excavation completed on time; $P(A) = 0.9$
B = foundation completed on time; $P(B) = 0.8$
C = columns and bracings completed on time; $P(C) = 0.7$
D = tank completed on time; $P(D) = 0.7$

If the events are statistically independent,

(i) what is the probability of the whole structure completed on time? (*Ans.* 0.3528)

(ii) what is the probability of the tank portion completed on time and atleast one of the other three works is not completed on time? (*Ans.* 0.3472)

3.4 There are three members in a determinate truss subjected to a given system loading. If p_i is the probability of failure of the member i, it is given as: $p_1 = 0.1$, $p_2 = 0.2$ and $p_3 = 0.3$. The performance of a member depends on other members. It is given:

$$P(F_1 \mid F_2 \cap F_3) = 0.8 \qquad P(F_2 \mid F_3) = 0.9$$

Determine the reliability of the truss.

(*Ans.* 0.784)

3.5 A policy decision, like limiting the maximum salary of an Indian to Rs. 1,500, is to be taken by the Government. This depends on the election results. Suppose if the party A wins, the probability of implementing the decision is 80%, while it is 20% for the party B and 40% for the party C. Assume there are only three parties. Without knowing which party will win in the election, one cannot say the chance of introducing the decision. If the chance of A winning the election is 0.6, of B 0.1, and of C 0.3, determine the chance of introducing the decision.

(*Ans.* 0.62)

3.6 The probability density function of rainfall in a day during the monsoon season is given by

$$f_X(x) = 32e^{-4x} \qquad x \geqslant 0$$

Calculate the mean and the variance.

(*Ans.* $\mu = 2$; $\sigma^2 = 25$)

3.7 Two variables, X and Y, follow the lognormal distribution. If $Z = XY$ and variables X and Y are statistically independent, prove that Z follows the lognormal distribution.

3.8 The cube strength of M 35 concrete, X, follows the normal distribution with parameters $\mu = 42.28$ N/mm² and $\sigma = 5.6$ N/mm².

(i) What is the probability of $X < 35$?

(*Ans.* 0.0983)

(ii) What is the probability of $30 \leqslant X \leqslant 50$?

(*Ans.* 0.9018)

3.9 The yield strength of steel, X, follows the lognormal distribution with mean $= 1568$ N/mm² and $\sigma = 48.8$ N/mm². What is the probability of getting a yield strength value less than 1500 N/mm²?

(*Ans.* 0.0793)

3.10 The fatigue life of a structural component, measured in terms of the number of cycles of a particular load, is modelled having Weibull distribution which is given by

$$f_X(x) = \alpha\beta x^{\beta-1} \exp(-\alpha x)^{\beta} \qquad x, \alpha, \beta > 0$$

α and β are parameters of the distribution given by 0.001 and 0.5 respectively. The mean value of X and the parameters are related by the equation

$$\mu_X = \alpha^{-1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right)$$

(i) How long can such structural component be expected to last.

(*Ans.* 2×10^6 cycles)

(ii) What is the probability that such a component will last more than 3×10^6 cycles?

(*Ans.* 0.1769)

4

Resistance Distributions and Parameters

4.1 INTRODUCTION

The first step in the reliability analysis and design of structures, is to study the variability of the strength of the structural (RCC, steel, prestressed concrete, masonry, etc.) members in flexure, shear, compression, bond, torsion, etc. The strength of a structural member may vary from the calculated or 'nominal strength' due to variations in the material strengths and in the dimensions of the members, as well as variabilities inherent in the equations used to calculate the strengths of members. One has to identify the sources of variability and quantify (statistics) the same. The fundamental requirement in the reliability study is the collection of data on strength and other physical properties of the materials of the structures, and the geometric parameters of the sections and the statistical analysis of the same.

The structural designer specifies the characteristic strengths of materials and the builder tries to procure the materials satisfying the specifications, and thereby, tries to achieve the same strength as assumed by the designer. However, if the quality control is poor, then the strength of the structural member will be less than that assumed. This may endanger the safety of the structure. Hence for providing a design with an assured level of reliability the systematic identification of the uncertainties in the strength of materials and the dimensional parameters and statistical analysis of the collected data becomes an important task.

In this chapter, information on statistics of basic variables, viz. physical properties of concrete, reinforcing steel bars and bricks, and dimensional variations of RCC members, based on actual field data, are furnished. Methods are also indicated to account for other uncertainties and thus to determine the allowable stresses of materials for a given reliability or probability of failure.

4.2 STATISTICS OF PROPERTIES OF CONCRETE

The cube strength (compressive strength), the modulus of rupture (flexural strength), and Young's modulus (initial tangent modulus and secant modulus) are the properties of concrete that are generally required in the design of concrete structures.

The strength of concrete in a structure may be different from the specified strength, and also, the strength may not be uniform throughout the structure. There are several sources of uncertainty which contribute to the total variation in the strength of concrete. Sources are (4.1):

(i) variations in the quality of materials
(ii) variations in the placing of concrete
(iii) variation in the supervision
(iv) variations in weighing
(v) variations in the mixing procedures
(vi) variations in the transporting methods
(vii) variations in the testing procedures
(viii) variations due to the actual strength of concrete in a structure being different from the control specimens (cube or cylinder) and
(ix) variations in the methods of curing

In construction projects, samples of concrete cubes of 150× 150 × 150 mm size are generally cast during every batching of concrete. These cubes are tested in a laboratory at the end of the 28th day of curing. The mean value and standard deviation of each set of a concrete mix can be obtained. The computed mean value of the strength of each set can be plotted as shown in Fig. 2.1. It is generally found that the in-batch variation, which may be considered as a variation in the testing procedures, mixer inefficiencies and the actual concrete strength, varies from 3 to 10 per cent. All the test results of a particular concrete grade belonging to a project can be clubbed and a histogram can be drawn. A typical plot of a histogram of the cube strength of M 15 concrete belonging to a project is shown in Fig. 4.1, (4.2). Concretes of the same strength and with the same quality control may be prepared in different projects. All these samples are combined to form a class of concrete. Figure 4.2 gives the histogram of a typical class of M15 concrete. It can be observed, as expected, that the variation and coefficient of skewness for the class is more than for a group shown in Fig. 4.1. As more and more groups are combined, the distribution may become more and more skewed.

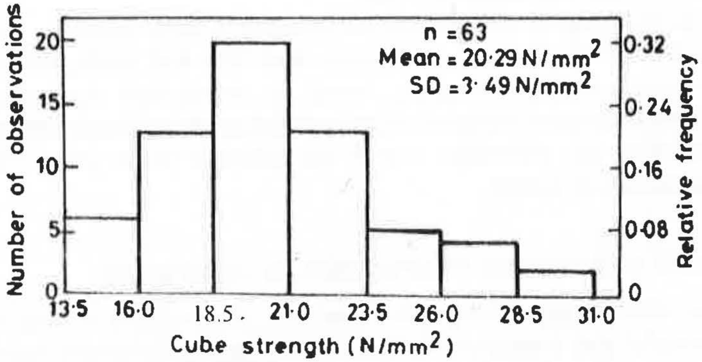


FIG. 4.1 Histogram of M15 concrete for a typical project group

A national building code must specify the coefficient of variation for a particular class of concrete, irrespective of source. It is also felt that this specified coefficient of variation must be related to the quality control. In the present code IS : 456-1978, the code specifies the values of standard deviation for various grades of conerete; but the degree of quality control is not attached to these values.

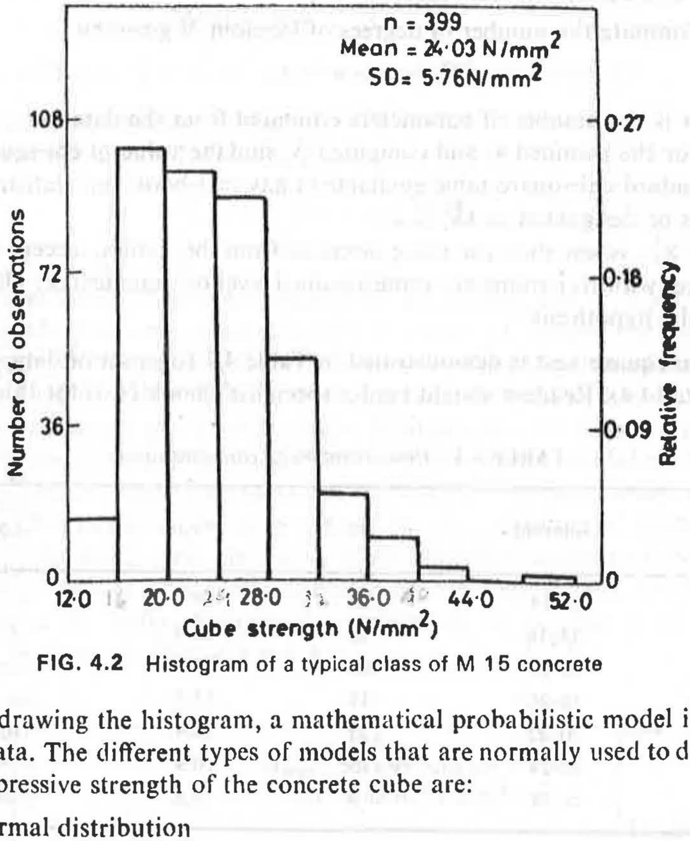


FIG. 4.2 Histogram of a typical class of M 15 concrete

After drawing the histogram, a mathematical probabilistic model is fitted to the data. The different types of models that are normally used to describe the compressive strength of the concrete cube are:

(i) normal distribution
(ii) lognormal distribution and
(iii) beta distribution

The suitability of a probabilistic model to fit the data is arrived at after applying the chi-square or the Kolmogorov–Smirnov goodness-of-fit tests (4.3). The chi-square test is briefly explained below:

Chi-square Test

1. Draw the histogram for the observed data.
2. Assume the model with its parameters calculated from the data.
3. Select the level of significance α. Generally α is taken as 5 or 1 per cent.
4. Calculate the value of chi-square as

$$\chi^2_{cal} = \sum_{i=1}^{a} \frac{(o_i - e_i)^2}{e_i} \tag{4.1}$$

where

χ^2_{cal} = the calculated value of chi-square

o_i = the observed frequency in the ith interval

e_i = the expected frequency corresponding to the assumed distribution in the ith interval

a = number of intervals considered

5. Compute the number of degrees of freedom N given by

$$N = a - r - 1 \tag{4.2}$$

where r is the number of parameters estimated from the data.

6. For the assumed α, and computed N, find the value of chi-square from the standard chi-square table available in any text-book on statistics (4.3). Let this be designated as $\chi^2_{N,\,1-\alpha}$

7 If χ^2_{cal} is less than the value obtained from the tables, accept the distribution with its parameters at the assumed level of significance. Otherwise, reject the hypothesis.

The chi-square test is demonstrated in Table 4.1 for a set of data of M 15 concrete (4.4). Readers should read a specialist's book (4.3) for this topic.

TABLE 4.1 *Demonstration of chi-square test*

Sl. No.	Interval	o_i	e_i	$(o_i - e_i)^2/e_i$
1	< 14	16	19.6	0.66
2	14–16	53	53.9	0.02
3	16–18	88	75.1	2.22
4	18–20	45	57.3	2.64
5	20–22	30	28.9	0.04
6	22–24	15	10.9	1.54
7	> 24	3	4.2	0.34
		250	250	7.46

(i) Model assumed—lognormal with parameters

$$\tilde{X} = 17.36 \text{ N/mm}^2 \text{ and } \sigma_{\ln X} = 0.152$$

(ii) α assumed = 5%

(iii) Calculation of e_i:

$$e_i = (n)p_i \qquad \text{where } n = \text{sample size}$$

$$p_i = P(X < 14) = \Phi\left\{\frac{\ln(14/\tilde{X})}{\sigma_{\ln X}}\right\} = 0.0785$$

$$e_1 = (250)(0.0785) = 19.6$$

Similarly, $e_2 = (n)p_2$,

$$p_2 = P(X < 16) - P(X \geqslant 14)$$

$$= \Phi\left\{\frac{\ln(16/\tilde{X})}{\sigma_{\ln X}}\right\} - 0.0785 = 0.2156$$

$$e_2 = (250)(0.2156) = 53.9$$

Similarly, other values of e_i are calculated and given in Table 4.1.

(iv) Degrees of freedom $N = a - r - 1 = 7 - 2 - 1 = 4$

(v) From chi-square Tables (4.3),

$$\chi^2_{N,(1-\alpha)} = \chi^2_{4,\,0.95} = 9.49$$

(vi) $\chi^2_{cal} = 7.46 < 9.49$

(vii) Hence accept lognormal model with parameters at $\alpha = 5$ per cent.

The procedure that is explained for the collection of samples for the compressive strength of concrete, can be followed for the collection of samples of cylinders (15 cm dia. $\times$ 30 cm height) and beams (10 cm $\times$ 10 cm $\times$ 50 cm) belonging to different grades of concrete. Cylinders can be tested to get the data on the initial tangent modulus, E_{tc}, and the secant modulus, E_{sc}, of concrete. Collected beam specimens can be tested in the laboratory at the end of the 28th day of curing to get data on the modulus of rupture of concrete, f_r. Table 4.2 gives the results of the statistical analysis of the data on various properties of concrete collected by the author (4.4, 4.5, 4.6) at various places in India. Frequency distributions of E_{tc} and f_r of M 15 concrete are shown in Figs. 4.3 and 4.4.

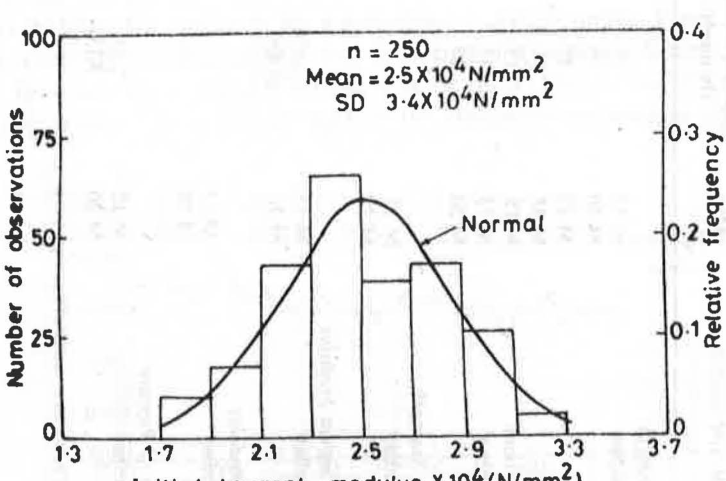


FIG. 4.3 Frequency distribution of Initial tangent modulus of M 15 concrete

TABLE 4.2 *Results of statistical analysis of various properties of concrete*

Variable and source	Mix	Specified strength (N/mm²)	μ (N/mm²)	σ (N/mm²)	δ (%)	Probability distribution	Quality control
Cube Strength							
IIT, Kanpur	M 15	15	24.03	5.76	23.96	LN	Nominal mix
,,	M 20	20	29.16	5.49	18.83	N	,,
,,	M 25	25	30.28	3.77	12.45	N	Design mix
,,	M 35	35	42.28	5.60	13.24	N	,,
REC, Calicut	M 15	15	22.67	5.01	22.10	LN	Nominal mix
IIT, Bombay	M 15	15	17.56	2.69	15.33	LN	Design mix
,,	M 20	20	26.80	4.04	15.07	N, LN	,,
Cylinder Strength							
IIT, Bombay	M 15		11.10	1.92	17.28	N, LN	,,
,,	M 20		17.21	3.34	19.40	N, LN	,,
Initial Tangent Modulus							
IIT, Bombay	M 15	22,076	25,147	3,398	13.51	N, LN	,,
,,	M 20	25,491	34,100	5.009	14.65	N, LN	,,
Secant Modulus							
IIT, Bombay	M 15	—	19,606	3,397	17.07	N	,,
,,	M 20	—	28,031	4,951	17.66	N, LN	,,
Modulus of Rupture							
IIT, Bombay	M 15	2.71	3.682	0.871	23.64	N	,,
,,	M 20	3.13	5.893	0.603	10.26	N, LN	,,

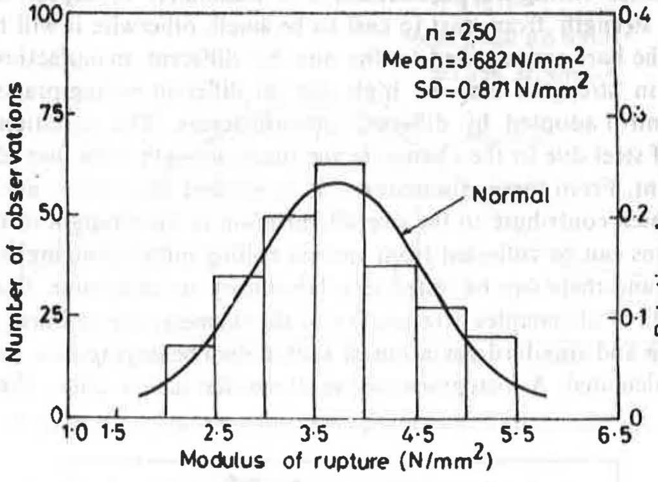


FIG. 4.4 Frequency distribution of modulus of rupture of M 15 concrete

4.3 STATISTICS OF PROPERTIES OF STEEL

The yield strength, f_y and the modulus of elasticity of steel, E_s, are the two main physical properties of steel that are used in the design of RCC and steel structures. In the case of prestressed concrete structures, the ultimate strength of high tensile steel wires is used in the design. The variation in yield strength is due to the variation in (i) material strength, (ii) cross-sectional area, (iii) rate of loading during testing, and (iv) the effect of strain at which the yield is defined (4.7). The amount of variation in strength within a single bar continuously cast for a particular length in a single cast is very small. (less than one per cent) and may be negligible as shown in Fig. 4.5. However, the in-batch variation for a given heat is slightly larger. For a construction work, the reinforcing bars may be supplied by a particular manufacturing firm having a number of steel rolling mills. Hence, the supplied bars may be from different rolling mills. If the chemical composition of steel is

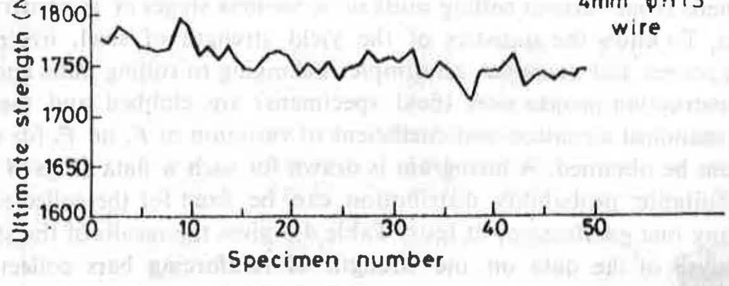


FIG. 4.5 Variation of ultimate strength in a single cast length

well controlled during the production, it is reasonable to expect the variations in strength from cast to cast to be small; otherwise it will be significant. If the bars are supplied to the site by different manufacturers, the variation in strengths may be high due to different rolling practices and quality control adopted by different manufacturers. The variation in the strength of steel due to the change in the mean strength with bar diameter is significant. From these discussions, it is evident that there are several sources which contribute to the overall variation in the strength of bars.

Specimens can be collected from various rolling mills belonging to a particular firm and these can be tested in a laboratory to determine f_y and E_s. Test results of all samples irrespective of the diameter are clubbed, and the mean value and standard deviation of such a data belonging to a particular mill are calculated. A histogram can be drawn for such a data. Figure 4.6

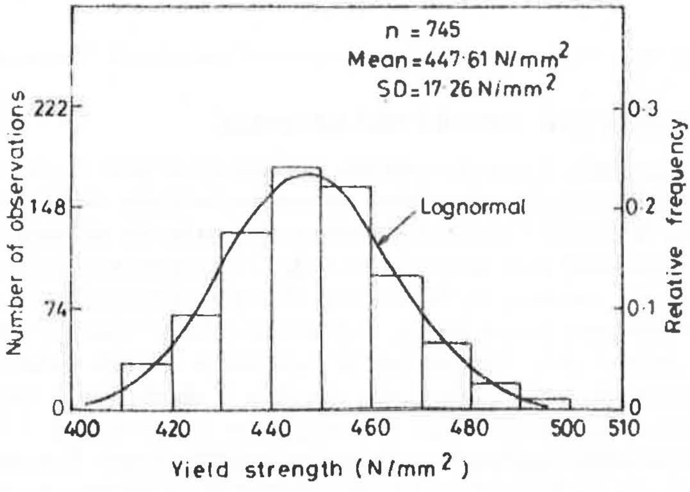


FIG. 4.6 Frequency distribution of yield strength of Fe 415 grade steel from a rolling mill

shows a typical histogram of yield strength of high yeild strength deformed bars (HYSD) belonging to a mill. The procedure is repeated by collecting specimens from various rolling mills or at various stages of a construction project. To know the statistics of the yield strength of steel, irrespective of the source and diameter, all samples belonging to rolling mills and from the construction project sites (field specimens) are clubbed and the mean value, standard deviation and coefficient of variation of F_y or E_s for such a data can be obtained. A histogram is drawn for such a data (Figs. 4.7 and 4.8). Suitable probability distribution can be fixed for the collected data using any one goodness-of-fit tests. Table 4.3 gives the results of the statistical analysis of the data on the strength of reinforcing bars collected by the author at various places in India (4.2, 4.4, 4.8).

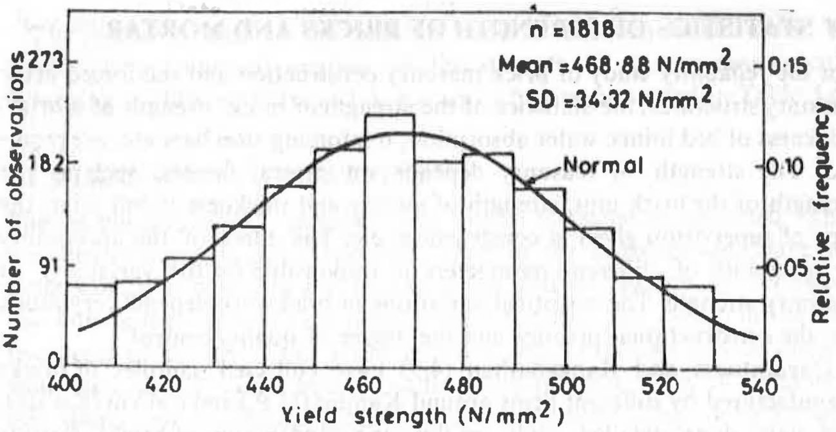


FIG. 4.7 Frequency distribution of yield strength of Fe 415 grade steel-combined data

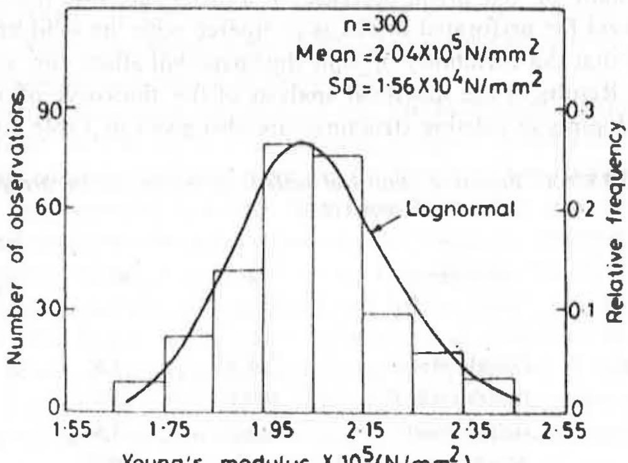


FIG. 4.8 Frequency distribution of Young's modulus of steel

TABLE 4.3 *Results of statistical analysis of yield strength and Young's modulus of steel*

Variable and source	Grade.	Diameter (mm)	μ (N/mm²)	σ (N/mm²)	δ (%)	Probability distribution
Yield Strength						
IIT, Kanpur	Fe 495	25,28,32	537.1	22.88	4.26	N
IIT, Kanpur	Fe 425	25,28,32	441.7	24.29	5.50	N
IIT, Kanpur	f_p 15,000	7	1568	48.76	3.11	N
REC, Calicut	Fe 235	6 to 20	295.3	16.24	5.50	LN
IIT, Bombay	Fe 250	8 to 20	320.0	27.50	8.61	N
IIT, Bombay	Fe 415	8 to 32	468.9	34.20	7.32	N
Modulus of Elasticity						
IIT, Bombay	Fe 250 and Fe 415	8 to 32	204100	15600	7.62	LN

Note: N—Normal; LN—Lognormal

4.4 STATISTICS OF STRENGTH OF BRICKS AND MORTAR

For the reliability study of brick masonry construction and reinforced brick masonry structures, the statistics of the strength of brick, strength of mortar, thickness of bed joints, water absorption, reinforcing steel bars etc. are required. The strength of masonry depends on several factors, such as the strength of the brick unit, strength of mortar and thickness of bed joint, the type of supervision given in construction, etc. The effect of the uncertainty or variability of different parameters is responsible for the variability of masonry strength. The statistical variations in brickwork depend very much on the constructional practice and the degree of quality control.

Dayaratnam and Ranganathan (4.9) have collected samples of bricks manufactured by different firms around Kanpur (U.P.) and Calicut (Kerala), and have done detailed study on the various properties of bricks. Results of the statistical analysis of the strength of bricks are presented in Table 4.4.

The strength of the prism decreases at a faster rate with increase in the joint thickness for perforated bricks as compared with the solid bricks. So it is expected that the variability of joint thickness will affect the strength of brickwork. Results of the statistical analysis of the thickness of horizontal and vertical joints in existing structures are also given in Table 4.4.

TABLE 4.4 *Results of statistical analysis of bricks, mortar strength and joint thickness (4.9)*

Source	Parameter	μ	$\delta(\%)$	Probability distribution
Bricks				
Kanpur Zone	Length (mm)	228.5	1.6	N
	Breadth (mm)	109.4	2.3	N
	Height (mm)	63.3	3.3	N
	Water absorption (%)	15.1	22.3	—
	Compressive strength (N/mm^2)	19.9	31.0	N
Calicut Zone	Length (mm)	232.7	1.0	N
	Breadth (mm)	116.0	2.3	N
	Height (mm)	75.7	4.8	N
	Water absorption (%)	16.4	12.9	—
	Compressive strength (N/mm^2)	9.8	29.8	N
Mortar				
Horizontal Joint	Thickness (mm)	12.9	13.5	N
Vertical Joint	Thickness (mm)	12.0	16.4	N
Mix 1:3*	Strength (N/mm^2)	21.5	13.3	N, LN
Mix 1:4*	Strength (N/mm^2)	14.3	10.0	N, LN
Mix 1:5*	Strength (N/mm^2)	10.4	18.1	N, LN

*Laboratory made specimens

Note: N—Normal; LN—Lognormal

The mortar is used as the binding material between brick units. The results of the statistical analysis of the strength of cement mortar cubes belonging to different mixes – 1.3, 1.4 and 1.5 – are presented in Table 4.4.

4.5 DIMENSIONAL VARIATIONS

The dimensions of RCC members may not be the same as specified. There may be deviations from the specified values of the cross-section shape and dimension, which may be due to size, shape and the quality of formwork, and concreting and vibrating operations. Variations also occur in the effective depth of members. The actual effective depth available may be different from the specified values because of the improper placement of reinforcing steel bars, not providing proper cover blocks and change in values when needle vibrators are used during casting of members. The amount of variation in dimensions vary from place to place and structure to structure depending on the quality of construction techniques and the training of the site personnel. Mirza and MacGregor (4.10) studied the variations in dimensions of RCC members for American conditions.

The difference between the nominal and the built-in dimensions are best characterized by the mean and standard deviation of the error. The coefficient of variation of the error increases as the size of the member decreases. Nineteen multistoreyed buildings have been visited during construction, and actual field data have been collected on the various geometric parameters of RCC members (4.11). The data are to be collected during the construction, and the measurements of members are to be taken in an unplastered condition. The results of the statistical analysis of variations in dimensions of slabs, beams, columns and foundations, carried out by Ranganathan and Joshi (4.4, 4.11), are presented in Table 4.5. The relationship connecting the coefficient of variation and nominal size of the member has been found to be

$$\delta_h = 4.9/h_n \tag{4.3}$$

where h_n is the nominal size of the member in mm. The frequency distribution of deviation in beam rib depth is shown in Fig. 4.9. All variables follow the normal distribution.

4.6 CHARACTERIZATION OF VARIABLES

The basic information required to describe behaviour of a random variable is the probability distribution with its parameters. However, in the case of first-order-second moment method of reliability, variables are characterized by their means and coefficients of variation. The concept of uncertainty is conveyed through the coefficient of variation. In reliability study, all uncertainties which affect the design reliability must be accounted for. These uncertainties must include the inherent statistical variability in the basic variables and, the additional sources of uncertainties arising due to modelling. Modelling uncertainties would include errors in the estimation of

TABLE 4.5 *Results of statistical analysis of variations in dimensions of RCC members*

Type	Mean deviation (mm)	Standard deviation (mm)	Size range (mm)
Slab (13 slabs)			100 to 110
Overall depth	+ 7.89	5.43	
Top cover	−19.75	6.89	
Bottom cover	+ 3.27	7.8	
Effective depth	+ 1.87	6.8	
Beam (252 beams)			
Breadth	+10.29	9.47	200 to 350
Overall depth	+14.37	9.38	250 to 700
Effective depth	+ 6.25	3.79	270 to 370
Top cover	− 0.56	8.41	30
Column (364 columns)			
Breadth	− 0.25	5.69	250 to 300
Depth	+ 0.11	7.89	250 to 1000
Cover (for 62 columns)	−19.09	12.13	40
Distance d_1*	6.24	11.89	360 to 710
Footing (6 footings)			
Length	−40.25	46.50	1500
Breadth	+37.73	32.28	1300

Note: *d_1 is the distance from one end of the column to the centre of bars on the other side.

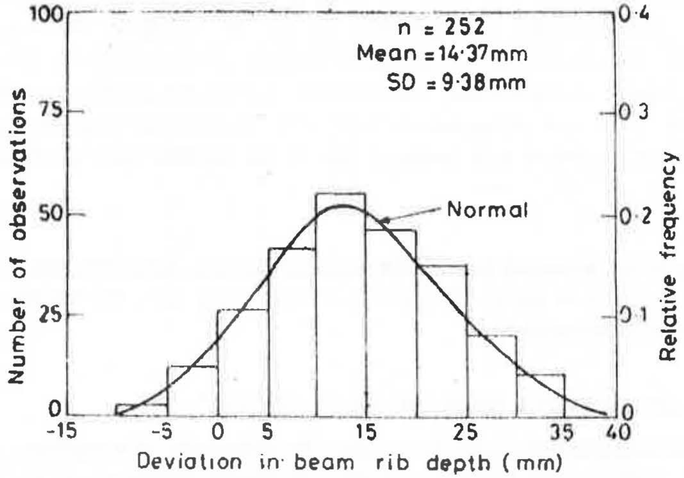


FIG. 4.9 Frequency distribution of deviation in beam rib depth

parameters, probability distribution, idealizations, testing procedures, human errors in calculation, etc.

Let

X be a basic variable

μ_X be the true mean of X

δ_X be the true coefficient of variation of X
$\bar{X}$ be the sample mean
$\bar{\delta}_X$ be the sample coefficient of variation

$\bar{X}$ and $\bar{\delta}_X$ are calculated from the data collected under carefully controlled conditions. Hence, $\bar{\delta}_X$ describes the inherent statistical variability. If the bias and coefficient of variation of uncertainties, attributed by other factors, are considered as $\bar{M}$ and δ_M, then μ_X and δ_X are estimated as (4.12)

$$\mu_X = \bar{M}\bar{X} \tag{4.4}$$

$$\delta_X = (\bar{\delta}_X^2 + \delta_M^2)^{1/2} \tag{4.5}$$

If δ_M is due to n factors, δ_M can be broken and written as

$$\delta_M = (\delta_1^2 + \delta_2^2 + \ldots \delta_n^2)^{1/2} \tag{4.6}$$

If the model is unbiased, $\bar{M}$ is taken as 1. In this approach, what is done is, μ_X is predicted by $\bar{X}$. Using this approach, the overall variation in the basic random variables can be fixed. This is illustrated below.

4.6.1 Compressive Strength of Concrete in Structure

Let

Y be the cube strength of concrete
X be the strength of concrete in structure

The mean value of the strength of concrete in structure is taken as 0.67 times the mean value of the cube strength of concrete. That is,

$$\mu_X = 0.67\mu_Y$$

In section 4.2, the coefficient of variation of Y, representating inherent variability, was obtained. Taking into account the uncertainties involved in the testing procedure (δ_{intest}) and *in situ* variation of the strength of concrete ($\delta_{in\ situ}$), the estimate of the coefficient of variation of the strength of concrete in structure can be written as (4.1).

$$\delta_X^2 = \delta_Y^2 + \delta_{\text{intest}}^2 + \delta_{in\ situ}^2$$

If

$$\delta_{\text{intest}} = 0.05 \qquad \delta_{in\ situ} = 0.1$$

then,

$$\delta_X^2 = \delta_Y^2 + 0.0125$$

From Table 4.2, for M 15 concrete (design mix),

$$\delta_Y = 0.1533$$

Hence the total variation in the strength of concrete is

$$\delta_X = [(0.1533)^2 + (0.05)^2 + (0.1)^2]^{1/2}$$
$$= 0.18$$

Similarly, the mean value and coefficient of variation of the compressive strength of concrete in structure for various grades are calculated and given in Table 4.6.

TABLE 4.6 *Statistics of strengths of concrete and steel*

Variable		Mean (N/mm^2)	Coefficient of variation	Probability distribution
(i) Compressive strength of concrete in structure				
(a) Nominal Mix	M 15	15.19	0.24	LN
	M 20	19.54	0.21	N
(b) Design Mix	M 15	11.78	0.18	LN
	M 20	17.96	0.15	N
	M 25	20.29	0.15	N
(ii) Initial tangent modulus of concrete				
Design Mix	M 15	25147	0.187	N, LN
	M 20	34100	0.206	N, LN
(iii) Modulus of rupture of concrete				
Design Mix	M 15	3.682	0.246	N
	M 20	5.893	0.125	N, LN
(iv) Yield strength of steel				
	Fe 250	320	0.10	N
	Fe 415	469	0.10	N
(v) Modulus of elasticity of steel				
		2.04×10^5	0.091	

4.6.2 Yield Strength of Steel

In the case of steel, we must include the variation that may occur due to the testing procedures and the method of specifying yield point. If the coefficient of variation in the testing procedures (δ_{intest}) is taken as 5 per cent and the coefficient of variation in the method of specifying yield ($\delta_{\text{sp. level}}$) is also taken as 5 per cent, the total variation in the yield strength of Fe 415 grade steel is

$$\delta^2_{\text{total}} = \delta^2_{\text{actual}} + \delta^2_{\text{intest}} + \delta^2_{\text{sp. level}}$$

From Table 4.3, $\delta_{\text{actual}} = 0.073$. Hence,

$$\delta_{\text{total}} = [(0.073)^2 + (0.05)^2 + (0.05)^2]^{1/2}$$

$$= 0.102$$

For mild steel (Fe 250), it is expected that

$$\delta_{\text{sp. level}} \simeq 0$$

For Fe 250 grade steel, δ_{total} is

$$\delta_{total} = [(0.0861)^2 + (0.05)^2]^{1/2} = 0.1$$

For Indian conditions, the statistics of the strength of concrete and steel, given in Table 4.6, may be used for code calibration.

4.7 ALLOWABLE STRESSES BASED ON SPECIFIED RELIABILITY (4.13)

The working stress design (WSD) analyses a structure for working loads, and designs the members such that the actual stresses in the members are limited to a portion of the yield stress or critical or ultimate stress that can be carried by the material.

Design criteria in WSD can be specified as

$$f_i(\text{M, GE, GS, DL, LL, WL}) \leq f_{ai}\,(\text{M, FA, GE, GS, DL, LL, WL}) \quad (4.7)$$

where f_i is the stress developed in the structure and f_{ai} is the allowable stress. The subscript i refers to tension or compression of flexure or shear or bond stress, etc.

The stress developed in the structure can be axial or bending or shear. It is a function of the material properties (M), geometry of elements (GE), geometry of structures (GS), dead load (DL), live load (LL), wind load (WL), etc. The allowable stress is a function of the material. It also depends on the specifications for testing the materials. A liberal specification on material standards has to be compensated by lower allowable stress. f_{ai} depends on functional aspect (FA) of the structure. A high pressure vessel for liquid is to be treated differently from a high pressure vessel for the containment shell of a nuclear reactor.

Allowable stresses in compression are governed by the buckling criterion, which depends on the geometry of the element and structure. Allowable stresses in a single load condition are different from those in a combined load condition and hence it is a function of load combination. Expressions similar to Eq. (4.7) can be written for other design criteria based on allowable deflection and cracking. The code specification for permissible stresses has to take care of the many complicated situations.

Probability of Failure of Material in WSD

When the stress developed in the material is greater than the allowable stress, it is defined as a failure. Hence the probability of failure of material, p_f, can be written as

$$p_f = P(X < f_a) \quad (4.8)$$

where X is the random variable, namely the srength of the material.
If X is normally distributed.

$$p_f = \Phi\left(\frac{f_a - \mu_X}{\sigma_X}\right) \quad (4.9)$$

EXAMPLE 4.1 The cube strength of M 20 concrete, Y, follows normal distribution. Given:

$$\mu_Y = 26.8 \text{ N/mm}^2 \qquad \delta_Y = 0.18$$

$$f_a = 0.34 \times (\text{cube strength of concrete})$$
$$= 0.34(20) = 6.8 \text{ N/mm}^2$$

Determine the probability of failure of concrete in the structure.

Solution The strength of concrete in structure, X, is equal to 0.67 times the cube strength. Hence

$$\mu_X = 0.67\ \mu_Y = 17.96 \text{ N/mm}^2$$

$$\sigma_X = 0.67\ \sigma_Y$$
$$= 0.67(0.18)(26.8)$$
$$= 3.23 \text{ N/mm}^2$$

Using Eq. (4.9), the probability of failure of concrete in the structure is,

$$p_f = \Phi\left(\frac{6.80 - 17.96}{3.23}\right)$$
$$= \Phi(-3.455) = 2.75 \times 10^{-4}$$

EXAMPLE 4.2 The yield strength of HYSD bars (Grade Fe 415), X, follows the normal distribution. Given:

$$\mu_X = 468.9 \text{ N/mm}^2 \qquad \delta_X = 0.1$$

$$f_a = 190 \text{ N/mm}^2$$

Determine the probability of failure of steel.

Solution

$$\sigma_X = 0.1 \times 468.9 = 46.89 \text{ N/mm}^2$$

Using Eq. (4.9),

$$p_f = \Phi\left(\frac{190 - 468.9}{46.89}\right)$$
$$= \Phi(-5.948) = 1.4 \times 10^{-9}$$

Determination of Allowable Stress

The allowable stress can be fixed for a given reliability or probability of failure of the material. If the strength of the material follows the normal distribution, then Eq. (4.9) can be rewritten as

$$\frac{f_a - \mu_X}{\sigma_X} = \Phi^{-1}(p_f) = k \tag{4.10}$$

Substituting $\sigma_X = \delta_X \mu_X$, the above equation becomes

$$\frac{\mu_X}{f_a} = \frac{1}{1 + k\,\delta_X}$$

μ_X and σ_X are obtained from the field data and hence they are known. Knowing μ_X and δ_X and k for a given p_f, the allowable stress can be fixed.

Factor of safety, ν, is defined by the convention of WSD as the ratio of the ultimate stress or yield stress to the working stress of the material. Hence

$$\nu = \frac{\mu_X}{f_a} = \frac{1}{(1 + k\delta_X)} \tag{4.11}$$

The fixing of allowable stresses for a given reliability is illustrated with the following examples.

EXAMPLE 4.3 It is given that the ratio of the mean value of the cube strength of M 15 concrete (design mix) to its characteristic strength is 1.4 and the coefficient of variation of the strength of concrete is 0.18. Determine the allowable stress for the probability of failure of concrete equal to 10^{-3}.

Solution In the case of concrete, the allowable stress is fixed as a fraction of the characteristic cube strength of concrete. For $p_f = 10^{-3}$, $k = -3.091$ (from tables).

Let

f_{cu} = the characteristic cube strength of concrete
X = the strength of concrete in the structure

The mean value of the cube strength of concrete is given as $1.4 f_{cu}$. Hence Eq. (4.11) becomes

$$\frac{f_u}{f_{cu}} = 1.4(1 + k\delta_X)$$

As the allowable stress in the element of the structure is to be fixed, the prism strength (that is the strength of concrete in the structure) is to be used and the above equation can be written as

$$\frac{f_a}{f_{cu}} = (0.67)(1.4)[1 - (3.091)(0.18)]$$

$$= 0.416$$

If the specified cube strength of concrete is 15 N/mm², the allowable stress for $p_f = 10^{-3}$, is

$$f_a = 0.416 \times 15 = 6.24 \text{ N/mm}^2$$

Similarly, for various values of p_f, the allowable stresses can be calculated for a particular characteristic strength. They are given in Table 4.7.

TABLE 4.7 *Factor of a safety and allowable stress for M 15 concrete (design mix) for different values of probability of failure*

p_f	10^{-3}	10^{-4}	10^{-5}	10^{-6}
k	-3.091	-3.719	-4.265	-4.754
ν	2.40	3.22	4.59	7.39
f_a (N/mm²)	6.24	4.65	3.27	2.03

It can be seen from the above table that as p_f decreases, the allowable stress also decreases as expected.

EXAMPLE 4.4 In the case of steel, the allowable stress is fixed as a fraction of the yield stress. From the data it is found that the ratio of the mean value of the yield strength of steel to its characteristic strength is 1.13 (for Fe 415 - Table 4.3). The overall variation in the strength of steel has been found as 0.1 (Sec. 4.6). Determine the allowable stress for steel for $p_f = 10^{-3}$.

Solution The factor of safety for HYSD bars, using Eq. (4.11), can be written as follows:
Let

$$X = \text{yield stress of the material}$$

$$f_y = \text{the characteristic yield strength of steel}$$

It is given that

$$\mu_X = 1.13 f_y$$

Using Eq. (4.11)

$$\nu = \frac{f_y}{f_a} = \frac{1}{1.13(1 + k\,\delta_X)}$$

For $\quad p_f = 10^{-3}, \quad k = -3.091$

and $\quad \delta_X = 0.1$. Thus,

$$\nu = \frac{f_y}{f_a} = \frac{1}{1.13[1 - (3.091)(0.1)]}$$

$$\frac{f_y}{f_a} = 1.281$$

For Fe 415-grade steel, $f_y = 415\ \text{N/mm}^2$. Hence the allowable stress for $p_f = 10^{-3}$ is

$$f_a = \frac{415}{1.281} = 324\ \text{N/mm}^2$$

If $p_f = 10^{-4}$, the value of allowable stress can be similarly calculated and it is equal to 294.5 N/mm^2.

If one takes the guaranteed yield strength itself as its mean strength, then the value of allowable stress for $p_f = 10^{-3}$ is

$$f_a = f_y[1 - (3.091)(0.1)]$$

$$= (415)(1 - 0.3091) = 286.7\ \text{N/mm}^2$$

It must also be noted that the safety has been calculated based on the yielding of steel (i.e. if steel yields, it is considered as a failure). However, the actual failure (that is by breaking of steel) occurs at a value of 1.2 times the yield stress of material. Hence, the actual safety available is more.

The collection of different data on the strengths of different materials and on geometric parameters have been discussed, and the statistics of variables based on the actual field data and the published works for Indian conditions have been presented. By and large, the statistical descriptions suggested are based on published works. It is recognized that the knowledge of the behaviour of materials is continually evolving and the means, variances, and distributions of the variables may be changing as more and more data is collected, or when data is updated. Research workers also have not used the same model and parameters for reliability studies during the last two decades. For the same data, there may be a number of distributions which appear to fit the data equally well. Extreme caution should be exercised if the type of distribution is chosen on the basis of sample data. A better or preferable approach is to make use of physical reasoning about the nature of each variable to guide the choice of the distribution. In engineering problems, most of the time we may have to resort to empirically fitted distributions. It is to be noted that the variables that have been discussed are the basic variables of a resistance variable of the structure. Hence it is important, and to be recognized, that the selected models must be simple, convenient, and reasonably good for these basic variables.

Modelling of the resistance variable of a structural element and a structure is a difficult task. The resistance is a function of these basic variables, viz. strengths of materials, geometric parameters, etc. Getting field data for the resistance of an RCC column, beam or frame or steel elements and structure in civil engineering is quite expensive and impossible. One may have to resort to the simulation technique (to be discussed in a later chapter) or physical reasoning to choose appropriate models. We have already discussed different models in Chapter 3 and also the conditions under which they arise. They may be helpful in choosing a model. Normal, lognormal, Weibull, beta, and sometimes gamma distributions are generally used to characterize the resistance of a structure. Again, it is important that the selected models must be simple and convenient, otherwise it will lead to difficulties in evaluating the reliability of a structure.

The estimation of parameters is important as the accuracy of prediction depends on the parameters estimated from the data. The methods that are generally used are:

(i) method of moments
(ii) method of maximum likelihood
(iii) mean rank plot-graphical procedure

The method of moments is the simplest. The graphical procedure is easy to apply for simple probability distributions. The method of maximum liklihood is difficult to apply as it often involves iterative calculations. However, it is supposed to be the best method as the estimators have all the desirable properties, viz. unbiasedness, efficiency, and consistency. The description of the methods are beyond the scope of this book. Readers should study and refer to any of the standard books (4.3, 4.14).

REFERENCES

4.1 Mirza, S.A., M. Hatzinikolas and J.G. MacGregor, "Statistical Descriptions on Strength of Concrete", *Journal of Struct. Div., ASCE*, Vol. 105, ST6, June 1979. pp. 1021–1037.

4.2 Ranganathan, R., "Reliability Analysis and Design of Prestressed Concrete Beams at Different Limit States", *Ph.D. Thesis*, I.I.T., Kanpur, May, 1976.

4.3 Benjamin, J.R. and C.A. Cornell, *Probability, Statistics and Decision for Civil Engineers*, McGraw-Hill, New York, 1970.

4.4 Ranganathan, R. and C.P. Joshi, "Statistical Analysis of Strengths of Concrete and Steel and Dimensional Variations", *Report No. D.S. and T : 4(1)/83/STP-III/2*, Civil Engg. Dept., I.I.T., Bombay, March 1985.

4.5 Dayaratnam, P. and R. Ranganathan, *Statistical Analysis of Strength of Concrete, Building and Environment*, Vol. 11, Pergamon Press, 1976, pp. 145–152.

4.6 Alex Mathew, "Probabilistic Analysis of Concrete Poles", *M.Sc. (Engg.) Thesis*, Calicut Regional Engg. College, 1980.

4.7 Mirza, S.A. and J.G. MacGregor, "Variability of Mechanical Properties of Reinforcing Bars', *Journal of Struct. Div.*, ASCE, Vol. 105, ST5, May 1979, pp. 921–937.

4.8 Ranganathan, R. and C.P. Joshi "Variations in Strength of Reinforcing Steel Bars", *Journal of the Institution of Engineers (India)*, Civil Engg. Div., Vol. 68, May 1988, pp. 309–312.

4.9 Dayaratnam, P., R. Ranganathan, et al., "Report on Brick and Reinforced Brick Masonry", *Project Report No. DST/427/4*, Nov. 1982, Civil Engg. Dept., IIT, Kanpur.

4.10 Mirza, S.A. and J.G. MacGregor, "Variations in Dimensions of Reinforced Concrete Members", *Journal of Struct. Div., ASCE*, Vol. 105, ST4, April 1979, pp. 751–766.

4.11 Ranganathan, R. and C.P. Joshi, "Variations in Dimensions of RCC Members", *Journal of the Bridge and Structural Engineer*, Vol. 16, Sept. 1986, pp. 1–10.

4.12 Ang., A.H. and C.A. Cornell, "Reliability Basis of Structural Safety and Design", *Journal of Struct. Div., ASCE*, Vol. 100, ST9, Sept. 1974, pp. 1755–1769.

4.13 Dayaratnam, P. and R. Ranganathan, "Allowable stresses and Load Factors Based on Probability Theory", *Journal of the Institution of Engineers (India)*, Civil Engg. Div., Vol. 58, July 1977, pp. 20–25.

4.14 Siddall, J.N., *Probabilistic Engineering Design*, Marcel Dekker, New York, 1983.

EXERCISE

4.1 The cube strength of M 20 concrete follows the normal distribution with parameters $\mu = 29.16$ N/mm² and $\sigma = 5.49$ What is the characteristic strength of concrete? (*Ans.* 20.16 N/mm²)

4.2 The yield strength of steel follows the lognormal distribution with $\mu = 295.3$ N/mm² and $\sigma = 16.24$ N/mm². If the specified strength of steel is 235 N/mm², determine the characteristic strength of steel. (*Ans.* 269.4 N/mm²)

4.3 If the ratio of the mean value of the cube strength of M15 concrete to its characteristic strength is 1.51, and the coefficient of variation of the strength of concrete is 0.24, determine the allowable stress for a reliability of 0.9999? (*Ans.* 1.61 N/mm²)

4.4 If the yield strength of steel follows the normal distribution with $\mu = 468.9$ N/mm² and $\sigma = 46.89$ N/mm², determine the allowable stress for a reliability of 0.9999. (*Ans.* 294.5 N/mm²)

4.5 The flexural strength (ultimate) of a prestressed concrete beam follows the normal distribution with the coefficient of variation being 0.05. The beam is subjected to dead load and live load. Assume the loads are deterministic. If the combined load factor, F_c, is defined as the ratio of the mean value of the strength of beam to the moment due to working loads, what is the value of F_c for a desired reliability of 0.9999? (*Ans.* 1.228)

4.6 If the ratio of dead load to live load is 0.5, and load factor for dead load is 1.2, what is the load factor for live load for a desired reliability of 0.9999? (*Ans.* 1.193)

5

Probabilistic Analysis of Loads

5.1 GRAVITY LOADS

5.1.1 Introduction

The accurate evaluation of gravity loads and the proper assessment of the maximum loads that a structure will have to carry during its lifetime are very important for a safe and economical design. After the advent of high speed digital computers, accurate techniques are available to analyse and design any complex structure under given loads. However, the state of knowledge about the analysis of loads is not comparable. The loads remain an estimate based on experience, judgement, tradition, trial, and error. Recently, during the past 15 years, considerable attention has been drawn to the measurement, analysis, and modelling of loads because of the increased familiarity of the engineers with the probabilistic and statistical methodology necessary to treat the load phenomenon in the quantitative manner, which engineers expect.

Loads on structure are stochastic in nature. They vary with space and time. This spatial and temporal variability is to be taken care of in the design. In recent years, a significant amount of live load survey has been conducted in many countries (5.1–5.9). At the same time, the trend has been set up to develop probabilistic limit state design and reliability based codes. The characteristics of the loading is probably the most important parameter to a reliability based analysis and design. In the formulation of reliability based codes, considerable attention will have to be focussed on the acquisition of reliable load data of a form suitable for the estimation of key statistical parameters. Concurrent to this, there is a growing awareness to develop probabilistic models and estimate the statistical parameters. The study of floor loads in buildings with respect to how live loads are measured, analysed and modelled, is presented.

5.1.2 Load as a Stochastic Process

Loads or actions in general are the forces acting on the structures due to external influences (self weight, superimposed loads, snow, wind and wave loads) and imposed deformations (differential settlements and temperature variations). Loads are subjected to random variations in magnitude and position with time. Loads are, therefore, described as time varying, free positioning, and dynamic effect producing and hence loads are to be modelled as a stochastic process.

A single time history representing a random phenomenon is called a sample function. When this evolves in time, it leads to a process. A stochastic process is the collection of all possible sample functions, which the random phenomenon might have produced.

A sample function of a continuous time varying stochastic process of load $X(t)$ is shown in Fig. 5.1, in which $x(t_1)$ is the magnitude of a time varying load $X(t)$ at time t_1. This $x(t_1)$ is called the arbitrary point-in-time load. It is simply the load that would be measured if the load process were to be sampled at some time instant, e.g. in a load survey. This load is a random variable. If this is designated as X, the PDF of X is shown in Fig. 5.1. In the same figure, if x_{max} is represented by the random variable Z, then the PDF of Z, $f_Z(z)$, will be as shown in Fig. 5.1.

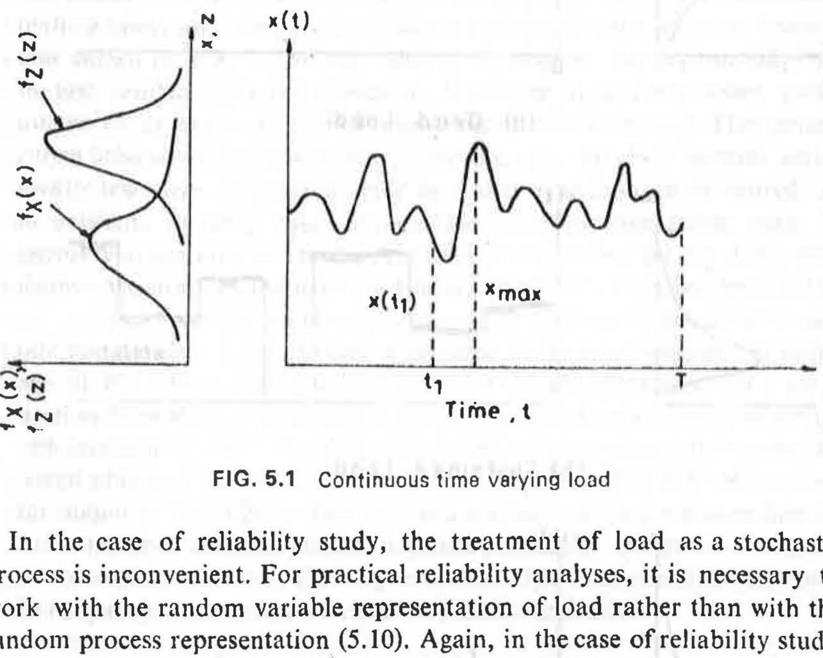


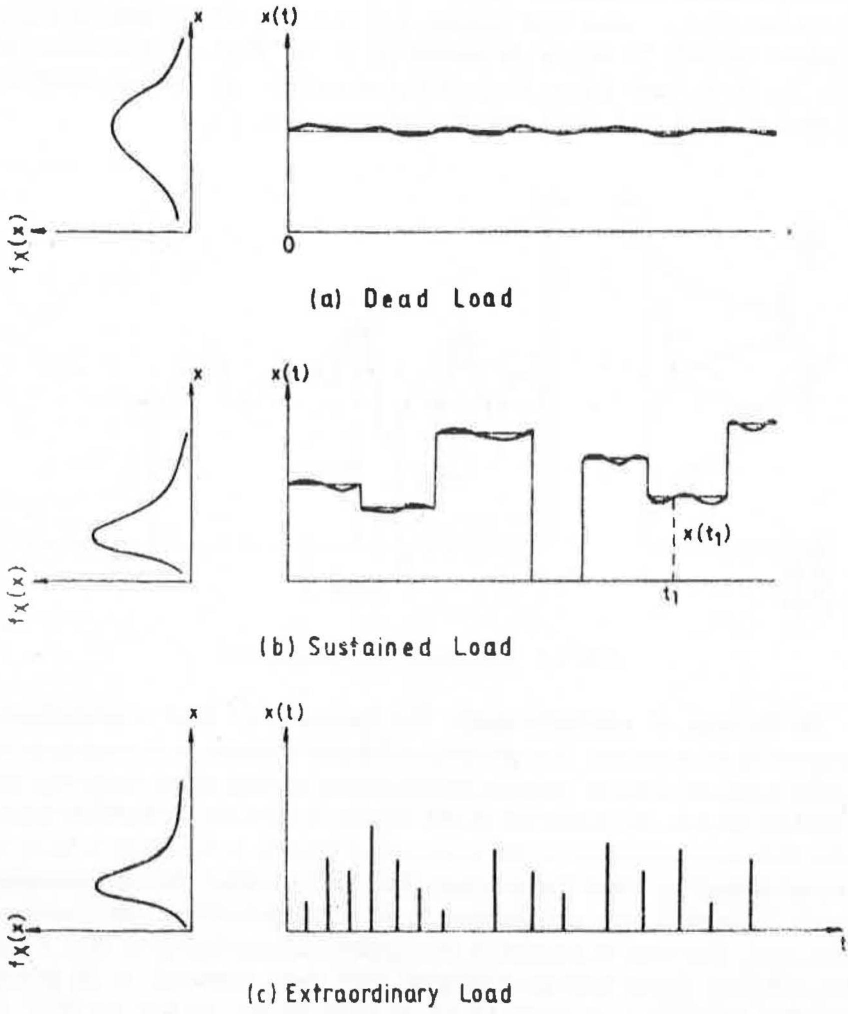
FIG. 5.1 Continuous time varying load

In the case of reliability study, the treatment of load as a stochastic process is inconvenient. For practical reliability analyses, it is necessary to work with the random variable representation of load rather than with the random process representation (5.10). Again, in the case of reliability study, the designer is interested in the value of the maximum load that is likely to occur during the life of the structure. This load is called lifetime maximum load. Ultimately one is interested to know the probability distribution of this load. This may be physically interpreted as the distribution that would be obtained if the lifetime maximum load were measured in an infinite number of identical structures (5.11). In later sections, we will see how we achieve this.

Gravity loads are divided into dead loads and live loads. Live load is again divided into (i) sustained load and (ii) transient load or extraordinary load.

5.1.3 Dead Load

Permanent loads are considered as dead load. This is mainly the weight of the structural system. This may undergo a little reduction because of wear and tear during its lifetime. This is negligible and can be ignored. Dead load may undergo increase because of the addition of some partition wall or covering during the life of the building. These may be rare events. This also induces a modest change only. Hence the dead load can be assumed to remain constant in time throughout the life of the structure. This is depicted in Fig. 5.2(a).



(c) Extraordinary Load

FIG. 5.2 Types of loads

The total dead load to be supported by a structure is generally the sum of self-weights of many parts. Hence the dead load is modelled with a normal probability distribution. The variability in dead load is strongly affected by the weights of nonstructural items, such as roofing, partitions, etc. As there is a tendency to underestimate the total dead load, it is assumed (5.10) that the ratio of the mean load to nominal load is 1.05, and the coefficient of variation is 0.10 for code calibration.

5.1.4 Live Loads

Live loads may in general, be defined as any load produced by the occupancy of the building. Nonpermanent gravity floor loads arising during the service life of the buildings are considered as live loads. That means, live loads include the weight of people and their possessions; furniture, movable partitions and other portable fixtures and equipment. The total live load on a floor is considered under two components, viz. (i) a sustained load component (long term), (ii) extraordinary load (transient load) component.

Sustained Load

A sustained load is the load of furniture, equipment and other loads needed for the activity and the normal personnel involved in the activity. Sustained loads shown in Fig. 5.2(b) may change at discrete times, but inbetween changes, remain relatively constant. A change at discrete times may be thought of as change due to change of occupancy (tenancy). The variation of load inbetween changes is due to the changes which a normal activity brings. New pieces of furniture may be added or exchanged or shifted, and the contents in desks and cabinets and other storage places vary. The persons who are involved in the activity are not present all the time which brings a variation of the load. As stated earlier, this variation between two load changes is limited and small compared to the total load. Hence a constant load between load changes is assumed in the load analysis. It may be noted in Fig. 5.2(b) that sustained loads may be entirely absent for a certain duration. This may be considered as the time gap during change of tenants.

The sustained load is the load usually measured in live load surveys. This is referred to as the arbitrary point-in-time load, L_{apt}. The PDF of this load is also shown in Fig. 5.2(b). This load is a spatially varying random function. This is assumed constant in time within a particular change of occupancy. It is therefore known as the long term load. The load changes with change of occupancy are assumed to occur as poisson arrivals.

Extraordinary Load

This arises from infrequent clustering of people above and beyond normal personnel load. That is, the extra personnel load. This extraordinary load (EL) is also due to the event when many pieces of furniture or equipment have been gathered together in one place at some instant of time, for example, at remodelling events. The EL is very unpredictable and it occurs with relatively high intensities and in short durations (in most cases a couple of hours). The term short duration is used in the sense that their durations are very small relative to permanent and sustained load. Hence they create a spike on the lifetime history of the load as shown in Fig. 5.2(c). It is very difficult and almost impossible to get data on EL. It cannot be measured in the regular live load survey.

The total load is therefore split up into three parts. This is a simplified model. The division is mainly on the difference in the time history between the loads. As the dead load, already discussed, has been considered as constant in time and can be modelled with a probabilily distribution, it can be combined with other loads. In future, live load only will be discussed.

Live Load Survey

The development of new codes, based on the reliability theory or probabilistic limit state design, needs more and more information about loads based on the actual field data. This has initiated the conduct of load survey. During the past decade, numerous load surveys have been conducted in the U.S.A., Europe, Canada, Sweden, Australia, India, etc.

J. Bryson and Gross (5.2) have developed the methodology of load surveys. The live load survey is the process of measuring the actual floor loads, L_{apt}, and collecting the extensive scientific and systematic data, and information, such as (i) building data which includes geographic region, location, height and number of storeys, age, type of occupancy, floor plans of building, layout of framing systems, number of rooms/bays, floor area of building, etc., (ii) occupancy data giving information about the types of firm, spatial orientation and duration, (iii) room/bay data, which incorporates details about the floor level, room number, location of the room, room use, room size, floor area, openings, surface finishings, floor coverings, occupants including number and weight, item description including location, contents and weights, etc., (iv) extraordinary load information about occasions of persons gathered, frequency, furniture stacking occasions, painting and remodelling, etc.

The scientific live load survey provides a sound statistical basis for (i) the adoption of an appropriate probability model for live loads, (ii) the proper assignment of parameters to the probabilistic model, (iii) the refinement of probabilistic load models, (iv) better understanding of the randomness of live loads, and (v) the modification of the existing loading standards.

Simple Statistical Analysis of Live Loads

Before we consider the rigorous statistical analysis of floor loads as an area dependent random process, let us first understand the simple treatment of the load analysis.

Assume that live load survey has been conducted in a building and the position and magnitude of loads are known on each bay (or room) of the building. Assume constant area. The floor load intensity (FLI), Q, is the total load acting on a bay (or room) in a floor divided by the floor area of the bay (or room). The actual live loads (measured in load surveys) may have any random positions and distributions. From the design point of view, the effects of actual live loads (i.e. stress resultants) developed in the floor slab or supporting beams and columns, are important. Therefore, it is necessary to convert the survey loads into uniformly distributed loads. This uniformly distributed load intensity, which would produce the same load effect as the actual set of loads, is called the equivalent uniformly distributed load (EUDL). Let EUDL be designated as L. Hence the set of point loads on each bay, with actual magnitudes and positions measured in load surveys, must be transformed to EUDL by using influence surface methods or energy methods or finite element methods, taking into account the boundary conditions and the configuration of the supporting systems. Once a probability model is selected and the parameters established for L, the characteristic load, L_k, can be fixed. If p_k is the probability of a load greater than L_k, then

$$p_k = P[L > L_k] = 1 - P(L \leqslant L_k)$$

$$= 1 - F_L(L_k) \tag{5.1}$$

or
$$L_k = F_L^{-1}(1 - p_k) \tag{5.2}$$

If the occupancy does not change during the lifetime of the building, the above calculated load gives the lifetime maximum live load with a probability of its exceedence equal to p_k.

For live load on buildings, it is usually assumed that the occupancy varies a few times during the lifetime of a building, T, in a completely independent way. Assuming that the whole building is occupied by only one tenant (i.e. single tenant model) at a time, let the building be occupied by N tenants during the lifetime of the building. The live load during each occupancy is a random variable. Let $L_1, L_2, \ldots, L_i \ldots, L_N$ be the random variables representing the maximum live load intensity (EUDL) during each occupancy. It is assumed that the live load does not change with respect to time during each occupancy. If $F_{L_i}(\ )$ is the CDF of L_i and $F_{L_m}(\ )$, the CDF of the lifetime maximum live load, L_m, then the probability of L_m less than or equal to a particular load, say characteristic load L_k, during the lifetime of the building is given by

$$P(L_m \leqslant L_k) = P[(L_1 \leqslant L_k) \cap (L_2 \leqslant L_k) \cap \ldots \cap P(L_N \leqslant L_k)]$$

If L_i are assumed as statistically independent, the above equation becomes

$$P(L_m \leqslant L_k) = P(L_1 \leqslant L_k) P(L_2 \leqslant L_k) \ldots P(L_N \leqslant L_k)$$

$$F_{L_m}(L_k) = F_{L_1}(L_k) F_{L_2}(L_k) \ldots F_{L_N}(L_k)$$

If L_i are identically distributed, the above equation simplifies to

$$F_{L_m}(L_k) = [F_{L_1}(L_k)]^N \tag{5.3}$$

where $F_{L_1}(L_k)$ is the probability distribution reflected in a histogram of live load data measured during a short period of time (initial fitted distribution for L_{apt}). If $F_{L_1}(\ )$ has an inverse at $L_k = (1 - p_k)^{1/N}$, then

$$L_k = F_{L_1}^{-1}[(1 - p_k)^{1/N}] \tag{5.4}$$

The above L_k is the lifetime maximum live load for N tenancies and p_k is the probability of live load exceeding L_k. Hence for a given number of tenancies and a specified value of p_k, the value of L_k can be calculated from the initial fitted distribution for the live load. This is illustrated with the following example.

EXAMPLE 5.1 From the statistical analysis of live load survey, it is found that live load follows the lognormal distribution with parameters

$$\tilde{L} = 1217 \text{ N/m}^2 \qquad \sigma_{\ln L} = 0.368$$

Determine the characteristic load for $p_k = 0.05$ if (i) there is no change in tenancy and (ii) the building is going to be occupied by 5 tenants during the lifetime of the building.

Solution

Case (i):
Using Eq. (5.2), the characteristic load is given by

$$L_k = F_L^{-1}(1 - p_k) = F_L^{-1}(1 - 0.05)$$

$$F_L(L_k) = 0.95$$

Since L follows the lognormal distribution, using Eq. (3.94),

$$\Phi\left[\frac{\ln (L_k/\tilde{L})}{\sigma_{\ln L}}\right] = 0.95$$

$$\begin{aligned} L &= \tilde{L} \exp [\sigma_{\ln L}\Phi^{-1}(0.95)] \\ &= 1217 \exp [0.368\Phi^{-1}(0.95)] \\ &= 2220 \text{ N/m}^2 \end{aligned}$$

Case (ii):
It is given that $N = 5$ and $p_k = 0.05$. Using Eq. (5.4), the value of L_k during the lifetime of the building is

$$\begin{aligned} L_k &= F_L^{-1}[(1 - 0.05)^{1/5}] \\ &= F_L^{-1}(0.9898) \\ &= \tilde{L} \exp [\sigma_{\ln L}\Phi^{-1}(0.9898)] \\ &= 1217 \exp [0.368\Phi^{-1}(0.9898)] \\ &= 2860 \text{ N/m}^2 \end{aligned}$$

Similarly, the values of L_k for different numbers of tenancies are calculated and given in Table 5.1. It is seen from the table that L_k increases for a given value of p_k, and L_k decreases as p_k increases for a given value of N.

Area Dependent Sustained Load Intensity Model

In the last section, it has been assumed that the bay or room area is constant and the floor load does not depend on the area, i.e. not as a function of the area. However, it is well established that the floor load depends on

TABLE 5.1 *Lifetime maximum live load for different number of tenancies—Example 5.1*

Period of tenancy	Lifetime of building (years)	N	Lifetime maximum load for $p_k = 0.05$ (kN/m²)	$p_k = 0.10$ (kN/m²)
5	25	5	2.86	2.57
5	30	6	2.92	2.65
5	40	8	3.04	2.76
5	50	10	3.17	2.76
5	100	20	3.43	3.13
10	50	5	2.86	2.57

the area. Live loads vary from building to building, floor to floor, bay to bay, point to point, and also time to time. To quantify these variations and uncertainties, to some extent rationally, the instantaneous live load survey data of arbitrary point-in-time loads on floors of selected bays of selected buildings have to be analysed to model live loads with certain assumptions and simplifications.

Statistical Assumption

The load intensity on a floor can be characterized as a stochastic process which is assumed stationary both in space and time.

The assumption of stationarity in space implies that the load in buildings, used for the same type of occupancy, can be represented with the same statistical distribution. This assumption is generally used, and is necessary so that with a proper selection of the buildings out of the whole population, good estimations of the statistical properties can be achieved.

The assumption of stationarity in time implies that the statistical distribution of the load from one point in time to another is the same. This assumption is needed. It is not possible to conduct a continuous load survey.

The procedure of analysis of live load is to start with the preposition of a probability model for the load intensity. From this, a probability model for the load effect or the equivalent uniformly distributed load (EUDL) is derived.

Load Intensity

The sustained load intensity at any location on a floor of a building is modelled as the superposition of

(i) the main trend,
(ii) the periodic components, and
(iii) the random fluctuations

According to the assumption about stationarity in space, a constant mean load intensity is chosen. Hence the main trend is the mean load intensity, which is assumed to be constant for a type of occupancy. It is to be noted that the mean load intensity will be different for different types of occupancy. That is, say between hospital buildings and office buildings.

The periodic components are the variations in the load intensity around the mean due to different buildings, different floors, and different bays.

Random fluctuations take into account unknown uncertain deviations from the mean load intensity.

The load intensity model is assumed to be noncorrelative. That is, the correlations between load intensities from floor to floor, and bay to bay, and point to point which have a very little effect on the total load (or load effect) are not considered. Therefore, the correlations are neglected for simplicity and hence the load intensity model is assumed to be noncorrelative. With the above assumptions, Pier and Cornell (5.12) proposed a model for the load intensity as

$$w(x, y) = m + r + D(x, y) \tag{5.5}$$

where

$w(x, y)$ = the load intensity at any location on a bay of a floor of a building

m = overall mean load

r = a zero-mean random variable which can be split up to represent different variations

$D(x, y)$ = a zero-mean random process which represents unknown spatial variations.

The above model has been applied by various research workers (5.9, 5.13–5.17) and it is expected that this will be the general method for the analysis of sustained load. The r term may be split up into

r_{bldg} – representing building variations
r_f – representing floor variations
r_{bay} – representing bay variations

The split is justified if every building is occupied by one organization. That is, a single tenant model is assumed. This is the case in most of the office buildings. In case if a building is occupied by many organizations (this will be in the case of tall buildings), then r_{bldg} can be considered as r_{org} representing variations between organizations.

The smallest structural unit used in the load intensity model is a structural bay. Hence the load intensity is integrated over the bay area to get the total load. It is recalled that the spatial load intensity has been assumed as a noncorrelative random process. However, the total spatial load over an area is assumed to be dependent on the area. Hence the model is an area dependent random process. Since $D(x, y)$ is a zero-mean random process,

$$E\left[\left(\iint_A D(x, y)\, dx\, dy\right)\Big/A\right] = 0 \tag{5.6}$$

The variance of $D(x, y)$ is given by

$$\mathrm{Var}\,(D) = \frac{1}{A} \iiiint_{A\ A} \mathrm{Cov}\,[D(x, y), D(u, v)]\, dx\, dy\, du\, dv$$

Since the spatial load intensity has been assumed as a noncorrelative random process, we have

$$\text{Var}\,(D) = \frac{1}{A}\ \text{Var}\left[\iint_A D(x, y)\,dx\,dy\right]$$

$$= \frac{\sigma_D^2}{A}\ \forall\ A \tag{5.7}$$

where σ_D^2 is the spatial variance.

Let L_D be the total spatial load over the area. This is dependent on the area. Hence

$$E[L_D(A)] = E\left[\iint_A D(x, y)\,dx\,dy\right]$$

$$= 0 \tag{5.8}$$

$$\text{Var}\,[L_D(A)] = \text{Var}\left[\iint_A D(x, y)\,dx\,dy\right]$$

$$= \sigma_D^2 A \tag{5.9}$$

$$\text{Cov}\,[L_D(A_1), L_D(A_2)] = \text{Cov}\left[\iint_{A_1} D(x, y)\,dx, dy, \iint_{A_2} D(x, y)\,dx\,dy\right]$$

$$= 0 \quad \text{if} \quad A_1 \cap A_2 = \phi \tag{5.10}$$

ϕ, here, means null set.

Statistical properties of $w(x, y)$ over an area can be written by using the above derived results for the spatial load. Let L be the lotal load, i.e. the sum of the load intensity over any finite area. Then the mean value of L is

$$E[L(A)] = E\left[\iint_A w(x, y)\,dx\,dy\right]$$

$$= E[mA] + E[rA] + E[D(x, y)A] \tag{5.11}$$

As r is a zero-mean-random variable,

$$E(r) = 0 \tag{5.12}$$

Using Eqs. (5.8) and (5.12) in Eq. (5.11),

$$E[L(A)] = mA \tag{5.13}$$

The variance of L is

$$\text{Var}\,[L(A)] = \text{Var}\left[\iint_A w(x, y)\,dx\,dy\right].$$

$$= \text{Var}\,[mA] + \text{Var}\,[rA] + \text{Var}\,[D(x, y)A]$$

Using Eq. (5.9),

$$\begin{aligned}\mathrm{Var}\,[L(A)] &= 0 + \sigma_r^2 A^2 + \sigma_D^2 A^2/A \\ &= \sigma_r^2 A^2 + \sigma_D^2 A \qquad (5.14)\end{aligned}$$

where σ_r^2 is the variance of r.

The covariance of loads between two different influence areas A_1 and A_2 is

$$\begin{aligned}\mathrm{Cov}\,[L(A_1), L(A_2)] &= \mathrm{Cov}\left[\iint_{A_1} w(x, y)\,dx\,dy, \iint_{A_2} w(x, y)\,dx\,dy\right] \\ &= \sigma_r^2 A_1 A_2 \quad \text{if} \quad A_1 \cap A_2 = \phi \qquad (5.15)\end{aligned}$$

To obtain the unit load, U_L, the total load over the area is divided by the area. Hence

$$U_L(A) = \frac{L(A)}{A}$$

Moments of unit load are

$$E[U_L(A)] = E\left[\frac{L(A)}{A}\right] = m \qquad (5.16)$$

$$\mathrm{Var}\,[U_L(A)] = \mathrm{Var}\left[\frac{L(A)}{A}\right] = \frac{1}{A^2}\,\mathrm{Var}\,[L(A)]$$

Using Eq. (5.14), the above equation becomes

$$\mathrm{Var}\,[U_L(A)] = \sigma_A^2 + \frac{\sigma_D^2}{A} \qquad (5.17)$$

$$\mathrm{Cov}\,[U_L(A_1), U_L(A_2)] = \frac{\mathrm{Cov}\,[L(A_1), L(A_2)]}{A_1 A_2}$$

Using Eq. (5.15), we have

$$\mathrm{Cov}[U_L(A_1), U_L(A_2)] = \sigma_r^2 \qquad (5.18)$$

So far we have not considered the load effect. This can be taken care of by determining the coefficients with which the load should be multiplied to get the load effect. The load effect is to be obtained by the influence surfaces. Instead of integrating over the influence area, the influence surface is used. As every load effect has its own influence surface, the theoretical load effect can be obtained for any case.

The correct solution for the influence surface is very complicated. To simplify the solution, two dimensional extension of influence lines is used (5.5).

Let

$$H = \text{the load effect}$$

Then the load effect over the influence area is

$$H(A) = \iint w(x, y)\, I(x, y)\, dx\, dy \qquad (5.19)$$

The equivalent uniformly distributed load, L, that produces the same load effect, is obtained by dividing load effect by the integral under the member's influence surface. (This load L is also a function of A. However, for convenience A is removed in the notation).

$$L = \frac{\iint_A w(x, y), I(x, y)\, dx\, dy}{\iint_A I(x, y)\, dx\, dy} \tag{5.20}$$

where $I(x, y)$ is the influence surface function for the particular load effect sought and A is the influence area over which $I(x, y)$ assumes nonzero values. The statistical properties of L are

$$E[L] = E\left\{\frac{\iint_A I(x, y) w(x, y)\, dx\, dy}{\iint_A I(x, y)\, dx\, dy}\right\}$$

$$= m\left\{\frac{\iint_A I(x, y)\, dx\, dy}{\iint_A I(x, y)\, dx\, dy}\right\}$$

$$= m \tag{5.21}$$

The variance of L which is a function of A, is

$$\text{Var}\,[L] = \text{Var}\left\{\frac{\iint_A w(x, y) I(x, y)\, dx\, dy}{\iint_A I(x, y)\, dx\, dy}\right\}$$

$$= \frac{\text{Var}(H(A))}{\left[\iint_A I(x, y)\, dx\, dy\right]^2} \tag{5.22}$$

The variance of $H(A)$ is

$$\text{Var}\,(H(A)) = \text{Var}\left[\iint_A w(x, y)\; I(x, y)\, dx\, dy\right]$$

$$= \text{Var}\left[\iint_A I(x, y)\{m + r + D(x, y)\}\, dx\, dy\right]$$

$$= 0 + \sigma_r^2\left(\iint_A I(x, y)\, dx\, dy\right)^2$$

$$+ \text{Var}\left[\iint_A I(x, y) D(x, y)\, dx\, dy\right] \tag{5.23}$$

It can be derived (5.5) that

$$\text{Var}\left[\iint_A I(x, y)D(x, y)\, dx\, dy\right] = \sigma_D^2 \iint_A I^2(x, y)\, dx\, dy \tag{5.24}$$

Hence the variance of L is

$$\text{Var}\,[L] = \sigma_r^2 + \frac{\sigma_D^2}{A} \frac{\iint_A I^2(x, y)\, dx\, dy}{\left[\iint_A I(x, y)\, dx\, dy\right]^2} \tag{5.25}$$

Using Eq. (5.19)

$$\text{Cov}\,[L(A_1), L(A_2)] = \sigma_r^2 \tag{5.26}$$

Let

$$k = \frac{\iint_A I^2(x, y)\, dx\, dy}{\left(\iint_A I(x, y)\, dx\, dy\right)^2} \tag{5.27}$$

The coefficient k is the mean squared influence divided by the square of the mean influence; k is always greater than or equal to 1. It depends on the type of member, its structural configuration and boundary conditions, and the type of response sought. k can be obtained for any load effect and it is relatively insensitive to load effect type (5.5). It has been found by McGuire and Cornell (5.13) and Sentler (5.5) that the values of k are

$k = 2.04$ for end moments in beams (interior bay)
$k = 2.2$ for column axial loads
$k = 2.76$ for mid-span beam moments
$k = 1.98$ for mid-span beam moments if the beam is simply supported
$k = 1.34$ to 1.5 for mid-span moment of a slab

Ellingwood and Culver (5.15) have taken an average value of 2.2 for their analysis of loads. The analysis carried out by Rao and Krishnamoorthy (5.7) shows that considering all load effects, k varies from 1.92 to 2.46. Hence we can write

$$E(L) = m \tag{5.28}$$

$$\text{Var}\,(L) = \sigma_H^2 = \sigma_r^2 + \frac{\sigma_D^2}{A} k \tag{5.29}$$

If we are considering the load effect for beams, the statistical properties of L of a beam are

$$E(L) = m$$

$$\sigma_L^2 = \sigma_r^2 + \frac{\sigma_D^2}{2A} k$$

as the influence area for a beam is twice the area of the structural bay; the value of k corresponds to the corresponding beam effect (mid-span beam moment, end moment, mid-span shear, etc.). Similarly, the statistical properties of EUDL of one storey interior column loading is

$$E(L) = m$$

$$\sigma_L^2 = \sigma_r^2 + \frac{\sigma_D^2}{4A}$$

as the influence area for one storey interior column is four times the structural bay area A. The value of k for column loading is 2.21 (5.5, 5.13). If the interior column supports n floors, then

$$E(L) = m$$

$$\sigma_L^2 = \sigma_r^2 + \frac{\sigma_D^2}{4nA}k$$

It is generally found that lognormal and gamma distributions closely fit the data (L_{apt}) from load surveys (5.5, 5.8, 5.9, 5.18). However, since a constant mean load intensity model has been assumed, the probability distribution characterizing the sustained load should have a reproductive property. The gamma distribution has this property but not the lognormal distribution.

Maximum Sustained Load Intensity Model

The maximum sustained load, L_m, is the maximum of the various sustained loads supported by a given area during the lifetime of the building. That is, this is the maximum load which will occur during the lifetime of the building. This is also called the lifetime maximum sustained load.

The following assumptions are used in the stochastic analysis of L_m (5.12):

(i) The sustained load (SL) during each occupancy is constant, but this value is random.

(ii) The stochastic load process of SL is homogeneous in time and space.

(iii) One tenant and one floor model is adequate.

(iv) The successive sustained loads on any area are independent and represented by a probability distribution over the ensemble.

(v) The probability distribution of occupancy durations are independent of each other and do not change with time.

(vi) When an occupancy change occurs, it occurs simultaneously everywhere over the area A.

(vii) The successive sustained loads follow the gamma distribution.

(viii) The load changes occur according to the Poisson process.

(ix) The duration of occupancy is exponentially distributed.

(x) A fixed number of changes occur during the lifetime of the building.

Let

L_m = maximum sustained load during the lifetime T of the building

T = duration of sustained load, i.e. lifetime of the building
$L(t)$ = sustained load on the floor at time t, i.e. instantaneous SL

Then lifetime maximum load is

$$L_m = \max [L(t)] \qquad 0 \leqslant t \leqslant T \tag{5.30}$$

L_m is also a random variable. The cumulative distribution of L_m is

$$F_{L_m}(\alpha) = P[L_m \leqslant \alpha] \tag{5.31}$$

If the number of occupancy changes is N, then $(N + 1)$ is the number of occupants (tenants) who have occupied the building during T years. Hence $(N + 1)$ load values occur during the load history. It has been assumed that the SL is constant during each occupancy of the building and has a distribution $F_L(\alpha)$. Hence if the building is subjected to N occupancy changes during the time T, then $(N + 1)$ values of L will be observed during T. This set of $(N + 1)$ values can be considered as a random sample. If it is assumed that the number of occupancy changes, N, is known, i.e. a constant or fixed, then

$$\begin{aligned} F_{L_m}(\alpha) &= P[\text{max. load} \leqslant \alpha] \\ &= P[\text{all } (N + 1) \text{ loads} \leqslant \alpha] \end{aligned} \tag{5.32}$$

Since successive sustained loads are assumed to be independent, and identically distributed, the above equation becomes

$$F_{L_m}(\alpha) = [F_L(\alpha)]^{N+1} \tag{5.33}$$

However, the duration of an occupancy of the building is not deterministic, i.e. varies randomly. In such a case, N is a random variable and the CDF of L_m is

$$F_{L_m}(\alpha) = \sum_{n=0}^{\infty} P[(N + 1) \text{ loads} \leqslant \alpha \mid N = n]P(N = n)$$

As successive sustained loads have been assumed to be independent,

$$F_{L_m}(\alpha) = \sum_{n=0}^{\infty} [F_L(\alpha)]^{N+1} P(N = n) \tag{5.34}$$

It has been assumed that the number of load changes in a period of time $(0, t)$ occur according to the Poisson process with mean rate of arrival, ν. Hence

$$P(N = n) = \frac{e^{-\nu t}(\nu t)^n}{n!} \tag{5.35}$$

using this in Eq. (5.34), the CDF of L_m during lifetime T is

$$\begin{aligned} F_{L_m}(\alpha) &= \sum_{n=0}^{\infty} [F_L(\alpha)]^{n+1} \frac{e^{-\nu t}(\nu T)^n}{n!} \\ &= F_L(\alpha) \sum_{n=0}^{\infty} e^{-\nu T} \frac{[F_L(\alpha)(\nu T)]^n}{n!} \\ &= F_L(\alpha) e^{-\nu T} \exp [\nu T F_L(\alpha)] \\ &= F_L(\alpha) \exp [-\nu T\{1 - F_L(\alpha)\}] \end{aligned}$$

For high cumulative levels, the above equation can be written as

$$F_{L_m}(\alpha) = \exp\left[-\nu T\{1 - F_L(\alpha)\}\right] \quad (5.36)$$

$$\simeq -\nu T[1 - F_L(\alpha)] \quad (5.37)$$

Hence if the probability distribution of the sustained load at any arbitrary point-in-time (obtained from load survey) is known with its parameters, the cumulative probability distribution of maximum SL can be obtained.

EXAMPLE 5.2 From the analysis of the live load survey data, it is known that (5.9)

$$E(L) = m = 717.3 \text{ N/m}^2$$

$$\sigma_L^2 = 2663 + \frac{1690000}{A} k$$

Calculate the maximum sustained load at 0.932 fractile (i.e. $F_{L_m}(\alpha) = 0.932$) for the following given conditions:

(i) $F_L(\alpha)$ follows the lognormal distribution
(ii) $\nu = 1/8$, $T = 64$ yrs
(iii) $A = 27m^2$
(iv) $k = 2.2$

Solution It is given that

$$F_{L_m}(\alpha) = 0.932$$

Using Eq. (5.36)

$$\exp\left[-\nu T\{1 - F_L(\alpha)\}\right] = 0.932$$

Substituting the values of ν and T,

$$\exp\left[-8\{1 - F_L(\alpha)\}\right] = 0.932$$

$$F_L(\alpha) = 0.991$$

Using the given probability distribution and parameters of L,

$$F_L(\alpha) = \left[\frac{\ln(\alpha/\tilde{L})}{\sigma_{\ln L}}\right] = 0.991$$

The parameters $\tilde{L}$ and $\sigma_{\ln L}$ are estimated as follows:
For $A = 27 \text{ m}^2$, $k = 2.2$

$$\sigma_L^2 = 2663 + \left(\frac{1690000}{27}\right)2.2$$

$$= 140366.7 \text{ (N/m}^2)^2$$

$$\sigma_L = 374.6 \text{ N/m}^2$$

$$\delta_L = \frac{374.6}{717.3} = 0.522$$

Using Eqs. (3.91) and (3.92), the parameters of L are

$$\sigma_{\ln L} = [\ln (0.522^2 + 1)]^{1/2}$$
$$= 0.491$$
$$\tilde{L} = 717.3 \exp\left(-\frac{(0.491)^2}{2}\right)$$
$$= 635.8 \text{ N/m}^2$$

Using the calculated values of $\tilde{L}$ and $\sigma_{\ln L}$,

$$\Phi\left[\frac{\ln (\alpha/635.8)}{0.491}\right] = 0.991$$
$$\alpha = 2048.9 \text{ N/m}^2$$

This is the maximum value of the lifetime sustained load with the probability of its exceedence during the lifetime of the building being

$$(1 - 0.932) = 0.068$$

EXAMPLE 5.3 For the same example, calculate the maximum sustained load if L follows the gamma distribution.

Solution The parameters of L following the gamma distribution are (Eqs. 3.100 and 3.101)

$$\lambda = \frac{m}{(\sigma_L)^2} = \frac{717.3}{(374.6)^2} = 0.0051$$
$$k = \lambda m = (0.0051)(717.3)$$
$$= 3.658 \text{ N/m}^2$$

From the previous example,

$$F_L(\alpha) = \frac{\Gamma(k, \lambda\alpha)}{\Gamma(k)} = 0.991$$

Using Pearson's table, it is found that the value of $\alpha = 1870 \text{ N/m}^2$.

For the code calibration or the reliability analysis of structures, it may be necessary to know the probability model of L_m with its parameters. It is also of interest to know the expected value and variance of L_m for the purpose of structural design. Approximate formulae for the mean and variance of L_m may be derived (5.15) by fitting a Type 1 extremal (largest) distribution to the upper load fractiles and calculating the mean and variance of the fitted Type 1 distribution. This involves

(i) the calculation of the values of L_m at two fractile levels in the upper tail, say $L_m = 0.932$ and $L_m = 0.992$ for various values of A

(ii) the calculation of the parameters u and α of the assumed Type 1 distribution for each area

(iii) calculation of the mean and variance from the calculated values of u and α for each value of A

(iv) the plotting of the values of $E(L_m)$ and $\mathrm{Var}(L_m)$ with the corresponding values of A.

(v) the fitting of a suitable curve to these points (may be the least square fit) connecting (a) $E(L_m)$ and A and (b) $\mathrm{Var}(L_m)$ and A.

Transient Load

This load includes (i) the weight of the probable assembly of persons during the office party or get together functions or some other activity, (ii) the weight of the probable accumulation of equipment and furnishing during remodelling of the premises, and (iii) the weight of the probable storage of the materials. Normally, the concentration of people in combination with the sustained load causes the highest load. Because of this only, the activity of persons is generally considered. Again, the clustering of people above the normal personnel load only is considered as the normal personnel load, which is the load of persons normally present in the activity already considered as one part of the sustained load.

The knowledge of transient load is very limited. Very few transient live load surveys have been carried out because of the difficulties involved in this type of survey. Transient loads are to be obtained by conducting surveys continuously in time. This would give necessary data about the magnitude and the time aspect of transient loads. This procedure is, however, difficult to employ. The other way of collecting the data is through questioning about the transient load events in the past. This method may be easier but less accurate and may bring many uncertainties. The transient load occurs for a short time and is commonly modelled as a Dirac delta-function with magnitude equal to the intensity of the maximum load applied during the event. The transient load occurs instantly and is assumed to arrive as a Poisson event. Each event is modelled by a random number of randomly positioned and sized load cells, occurring randomly in space. The EUDL associated with an extraordinary load, B, is assessed by modelling the load event as a series of randomly distributed load cells, each of which contains a cluster of loads. The model is based on Poisson occurring independent events, each of negligible duration. Basic component loads Q (weight of single concentrated load in the cell, i.e. weight of single person) are assumed with specified mean value μ_Q and variance σ_Q^2. Each load cell contains a random number R of component loads (i.e. R is the number of loads per cell, i.e. the number of persons in one load cell) with mean μ_R and variance σ_R^2. The number of load cells in a given area A is assumed to be Poisson distributed with parameter λ, which is the mean rate of load cells in A. Q is generally assumed to be independent of A. If it is assumed that Q and R are independent, the mean and variance of B are given by (5.13)

$$E[B] = \frac{E[QR]\lambda}{A} \tag{5.38}$$

If it is assumed that Q and R are independent [using Eqs. 3.80(a) and 3.80(b)], then

$$E[B] = \frac{\lambda \mu_Q \mu_R}{A} \tag{5.39}$$

$$\text{Var}[B] = \sigma_B^2 = \frac{k\lambda(\mu_Q^2 \sigma_R^2 + \mu_R^2 \sigma_Q^2 + \sigma_Q^2 \sigma_R^2)}{A^2} \tag{5.40}$$

It may be noted that even though the transient load events are probably to a certain degree area dependent, a constant mean load intensity model is assumed and the random process is made dependent on the area to reflect the fact that a high concentration of people is more likely to occur in small areas than in larger ones. The probability distribution of B is generally assumed to be gamma (5.5, 5.15) as the gamma distribution has a reproductive property. An exponential distribution has also been suggested by Sentler (5.15).

Life Time Maximum Transient Load

The distribution of the lifetime maximum transient load, B_m, is obtained in the similar way used for the sustained load. The occurrence of B is assumed to be Poisson with mean occurrence rate of ν. Hence the CDF of B_m during the lifetime T is given by

$$F_{Bm}(\alpha) = F_B(\alpha) \exp [-\nu T\{1 - F_B(\alpha)\}] \tag{5.41}$$

where $F_{Bm}(\alpha)$ = CDF of B_m

$F_B(\alpha)$ = CDF of B

Maximum Total Load Model

Two types of live load, namely sustained load and transient load have been discussed. The total live load, which is some combination of the above-mentioned live loads at any instant, is of interest. Based on certain assumptions, the total live load is derived.

It is assumed that the sustained and the transient loads are independent of each other in time and space.

As the live load has been considered in two parts, of which one is continuous in time, the total load is a two dimensional stochastic variable. The assumption of independence simplifies the problem as the joint density function is the product of the individual density functions.

Chalk and Corotis (5.14, 5.16) have suggested a load model combining all possible load cases, each weighted by its respective likelihood of occurrence. The maximum total load during the lifetime of a building may arise from one of the following situations:

$$\left.\begin{array}{ll} \text{Case I} & : L_t = L_m + B_l \\ \text{Case II} & : L_t = B_m + L \\ \text{Case III} & : L_t = L_m + B_m \\ \text{Case IV} & : L_t = L + B \end{array}\right\} \tag{5.42}$$

where L_m = the maximum sustained load, B_m = the maximum extraordinary load, B_l = largest extraordinary load occurring during the duration of L_m, L = instantaneous sustained load, and B = instantaneous extraordinary load. The case IV is not considered as the probability of its occurrence is small.

If $E(\lambda)$ is the average duration of the sustained load and T is the lifetime of the building, then the probability that Case I or Case II occurs is $[T - E(\lambda)]/T$, and the probability that Case III occurs is $E(\lambda)/T$. The probability of the maximum total load can be written as

$$P[L_t < l] = P[(L_m + B_l) < l] \cdot P[(B_m + L) < l] \frac{[T - E(\lambda)]}{T} + P[(L_m + B_m) < l] \frac{E(\lambda)}{T}$$

If it is assumed that L_m, B_m, B_t, $(L_m + B_l)$ and $(L_m + B_m)$ follow the Type 1 extremal distribution, the CDF of L_t is

$$F_{L_t}(l) = \exp[-\exp(-w_1)] \exp[-\exp(-w_2)] \left[\frac{T - E(\lambda)}{T}\right] + [-\exp(-w_3)] \frac{E(\lambda)}{T} \tag{5.43}$$

where w_1, w_2 and w_3 are reduced variates corresponding to $(L_m + B_l)$, $(B_m + L)$ and $(L_m + B_m)$ respectively.

In conclusion, the analysis of live load is complicated. The probabilistic analysis of live loads to predict the mean of lifetime maximum total load at desired reliability level is based on the live load survey data collection, data reduction, and the probability models of sustained, extraordinary, and total loads. The procedure of the analysis is summarized as (i) the estimation of parameters m, σ_r^2, σ_D^2 and ν from the survey results, (ii) establishing the statistics of sustained load and extraordinary load that are obtained from the respective load models, and (iii) the estimation of mean and variance of the maximum sustained load, the maximum extraordinary load, and the maximum total load by fitting Type 1 extremal (largest) distribution to the respective cumulative distributions.

Live load survey has been carried out on three office buildings in Bombay (5.9, 5.18). These buildings are modern office buildings occupied for a sufficient length of time for normal occupancy consolidation, and the age of the buildings varies from 20 to 40 years. All the three buildings are multistoreyed. The total area and the number of bays covered in the survey are 1800 m^2 and 386 respectively. The bay areas in the buildings vary from 27 to 67 m^2. It has been found that the floor load intensity varies from 0.1 to 4 kN/m^2. The results of the suitability of the mathematical model for FLI are given in Table 5.2. The collected data for all the buildings has been combined, and for the combined data, the mean and the coefficient of variation of FLI are 0.717 kN/m^2 and 0.52 respectively. Using the model

TABLE 5.2 *Suitability of mathematical model for bay FLI of office buildings (5.18)*

Sl. No.	Description	Distribution and parameters	Remarks of chi-square test
1.	Administrative Building, I.I.T., Bombay		
	$\mu = 0.596$ kN/m²	LN(0.54, 0.42)	Accepted $\alpha = 5\%$
	$\delta = 0.446$	G(5.027, 8.435)	Accepted $\alpha = 5\%$
2.	Head Office Walchand Building, Bombay		
	$\mu = 0.728$ kN/m²	LN(0.713, 0.202)	Accepted $\alpha = 5\%$
	$\delta = 0.207$	G(23.34, 32.06)	Accepted $\alpha = 5\%$
3.	Central Railway Administrative Building, Bombay		
	$\mu = 0.745$ kN/m²	LN(0.620, 0.618)	Accepted $\alpha = 5\%$
	$\delta = 0.67$		
4.	All Buildings combined together		
	$\mu = 0.717$ kN/m²	LN(0.636, 0.488)	Accepted $\alpha = 5\%$
	$\delta = 0.52$		

proposed by Pier and Cornell (5.12), and the method of analysis explained in the text, and the approach used by Ellingwood and Culver (5.15), the collected data have been analysed and the following values for the mean and coefficient of variation of L_{max} have been suggested by Ranganathan (5.18) for buildings. The value of ν has been taken as 8. For lifetime maximum total live load, L_{max}.

Model : Type 1 (extremal largest)

Mean : 2.48 kN/m²

Coefficient of variation : 0.283

$$\frac{\text{Mean}}{\text{Nominal}} = \frac{2.48}{4.0} = 0.62$$

For arbitrary point-in-time varying live load, L_{apt},

Model : lognormal

Mean : 0.717 kN/m²

Coefficient of variation : 0.52

$$\frac{\text{Mean}}{\text{Nominal}} : \frac{0.717}{4} = 0.179$$

5.2 WIND LOAD

5.2.1 Introduction

The wind load, W, acting on a structure can be written in the form

$$W = BV^2 \tag{5.44}$$

where B is a parameter covering all components of the wind load (except the basic wind speed), i.e. pressure coefficients, area reduction factors, velocity multipliers for height and exposure, etc. V is the wind speed, generally referred to a height of 10 m. Wind loads are random in nature due to random variations of wind speed and uncertainties in the estimation of the pressure coefficients, the exposure factor, and the gust factor. The modelling of wind load is much more complex and difficult than the modelling of speed. Because the velocity appears in the equation as a squared value, its statistics is very important. However, the uncertainties in the various factors contained in B contribute to the overall variability in the wind load.

5.2.2 Wind Speed

The wind velocity is stochastic in nature. It has spatial and temporal variation during a storm. Wind speed, $V(X, t)$, in a given direction in a point of position vector X, at time t during a storm is generally considered as the sum of two terms (5.19), viz.

$$V(X, t) = V_0(X) + V_1(X, t)$$

in which $V_0(X)$ is the steady component equal to the average velocity during the storm and $V_1(X, t)$ is a zero-mean process describing the gusts. The above model is useful when the structure under investigation behaves dynamically under wind excitations. However, many structural engineering problems are concerned with structures in the static field. If only the static behaviour of the structure is involved, the velocity is expressed in the form

$$V = V^* \alpha(z) G \tag{5.45}$$

where V^* is the steady (average) velocity at a reference height (10 m), $\alpha(z)$ the multiplication factor for height, and G the gust factor. The maximum value of V over an appropriate time interval T is of interest in structural reliability analysis. For this purpose, the mean arrival rate (or the mean occurrence interval $T = 1/\lambda$) of V must be specified. Hence, it is necessary to associate return periods T with the values of wind speed. This can be done on the basis of cumulative distribution of yearly maximum wind speed.

Wind velocities are measured in a horizontal plane with the aid of anemometers or anemographs, which are installed at the meteorological observatories at heights generally varying from 10 to 30 m. The different types of anemometers are (i) pressure anemometer (ii) rotation anemometers, and (iii) gust measuring anemometers. The one which is usually used in India is the cup anemometer which falls in the category of rotation anemometer. Very strong winds (greater than 80 kmph) are generally associated with cyclonic storms, dust storms, or vigorous monsoons. A cyclone is one in which the wind speed exceeds 80 kmph. The wind velocity recorded at any locality is extremely variable and in addition to steady wind at any time, there are effects of gusts which may last for a few seconds. Wind forces acting on structures are significantly large only during strong winds and these occur only during storms. Hence only these extreme wind forces are of interest to the structural engineer. Attempts are, therefore, always made to collect data on extreme wind speeds and suggest a suitable probabilistic model for the same.

The continuous recording of wind velocities is generally carried out in meteorological stations. Out of these values, one is interested in the extreme or the maximum. From the continuous recording, it is possible to obtain daily, monthly, and yearly maximum wind speeds. Figure 5.3 shows the

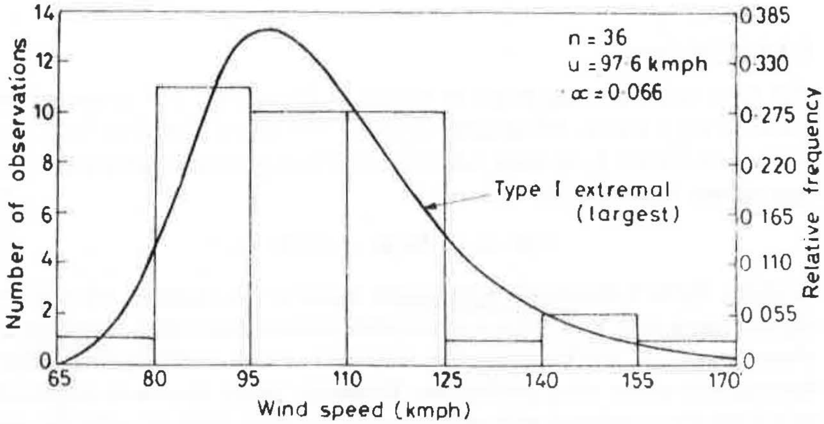


FIG. 5.3 Frequency distribution of annual maximum wind speed at New Delhi (Safdarjung)

variation of annual maximum wind speed observed at New Delhi (Safdarjung). Since the yearly maximum wind speed can be interpretated as the largest of daily values or perhaps the largest of many gusts, velocities, the statistical behaviour of the yearly maximum wind speed is represented by two types of extremal distributions with unlimited upper tail. One is the Type 1 extremal (largest) distribution, so called Gumbel distribution and the other one is the Type 2 extremal (largest) distribution, also called Frechet distribution. The choice of the underlying distribution can be made after the analysis of fitting closeness to the data. It was suggested that Type 2 distribution is an appropriate model to employ in most of North American Region (5.20), although recent studies (5.21, 5.22) have indicated that Type 1 is more appropriate. In Japan (5.23) and Australia (5.24), Type 1 is found to be more suitable on the basis of statistical analysis. In Germany, Schueller and Panggabean (5.25) have fitted Type 1 and Type 2 distributions to maximum yearly gust and average velocities. The Type 1 distribution has been used to describe the statistical behaviour of the yearly maximum wind speed in India (5.26, 5.27). The mean rank plotting (Fig. 5.4) of the data on the yearly maximum wind speed observed at Delhi shows a good straight line fit, encouraging the use of Type 1 extremal (largest) distribution. The parameters of the selected distribution are to be estimated using any-one of the methods (5.28). However, Simiu, Bietry, Filliben and Grigoriu (5.22, 5.29, 5.30) have proposed an improved technique for the analysis of wind speed data.

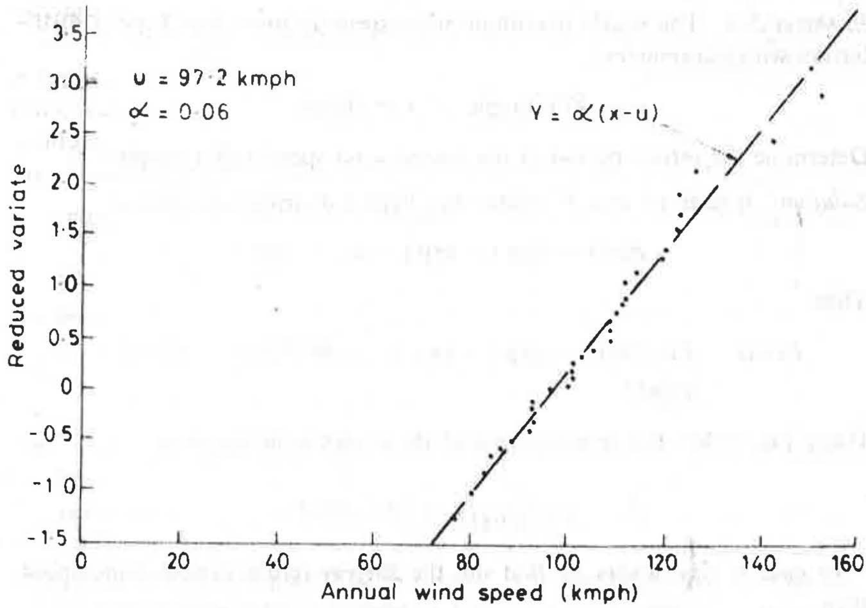


FIG. 5.4 Mean rank plot for Type 1 extremal (largest) distribution for V_{ann} observed at New Delhi (Safdarjung)

5.2.3 Return Period

A very common problem in wind analysis is to assume the return of an observed extreme wind speed or cyclone. For design purposes, one often attempts to estimate the magnitude of an extreme wind of a particular return period. The return period, R, which is called the mean recurrence interval, is defined as

$$R = \frac{1}{p} = \frac{1}{1 - F_V(v_s)} \tag{5.46}$$

where v_s is the specified design wind speed, $F_V(v)$ is the CDF of yearly maximum wind speed, V, and p is the probability of wind speed V exceeding v_s in any year.

The return period is to be understood as the interval between events. Hence a 10-year return period wind (i.e. $p = 0.1$) is the wind which could be expected to occur in the long term, about once in every 10 years. It does not mean that there will be a period of 10 years between winds of a particular size. The occurrence of wind in time is a random process and so it is quite possible that 1 in 10-year wind could be exceeded more than once in one year, or in successive years, or that there may be a period of more than 30 years in which no heavy wind as large as in the 1 in 10 year wind occurs. The one 1 in 10 year event is the event that could be expected to be equalled or exceeded about 10 times in a 100-year period. Hence the 1 in 10-year wind has a frequency of 0.1, that is, there is a 10 percent chance that it will be equalled or exceeded in any year.

EXAMPLE 5.4 The yearly maximum wind speed follows the Type I distribution with parameters

$$u = 97.6 \text{ kmph} \qquad \alpha = 0.066$$

Determine the return period of the design wind speed 158.1 kmph.

Solution It is given that V follows the Type I distribution. Hence

$$F_V(v) = \exp[-\exp\{-\alpha(v-u)\}]$$

Then

$$\begin{aligned} F_V(v_s) &= F_V(158.1) = \exp[-\exp\{-0.066(158.1-97.6)\}] \\ &= 0.9817 \end{aligned}$$

Using Eq. (5.46), the return period of the design wind speed is

$$R = \frac{1}{1-0.9817} = 54.7 \text{ years}$$

In case if one wants to find out the 20-year return period wind speed, then

$$p = \frac{1}{20} = 0.05$$

$$\frac{1}{1-F_V(v)} = \frac{1}{p} = \frac{1}{0.05}$$

$$F_V(v) = 0.95$$

Then the wind speed corresponding to this probability is given by

$$\begin{aligned} \exp[-\exp\{-0.066(v-97.6)\}] &= 0.95 \\ v &= 142.6 \text{ kmph} \end{aligned}$$

Hence the 20-year return period wind speed is 142.6 kmph.

In the current design procedures, wind loads are treated semi-probabilistically. The annual maximum wind speeds are recorded and an appropriate probability distribution is fitted to the data. A wind with some specified probability of exceedence in any one year is then selected for design purposes. Usually, a 0.02 exceedence probability for 50-year return period is used. Although 50-year return period has attained a somewhat mystical status in civil engineering, its use does not hold up well under closer examination. In fact, the 0.02 exceedence level for a Type I extreme value distribution, normally used for wind speeds, corresponds to an exceedence level of 0.63 in a lifetime of 50 years (5.31).

5.2.4 Estimation of Lifetime Design Wind Speed

Occasionally, it is necessary to design a structure against wind load for a fixed period from the period of construction. For example, if a structure is built which is only to be used for 3 years following construction, or is to be removed at the end of 3 years, the risk of damage exists only for this period. Thus what is required for design is wind speed associated with a probability of being exceeded in the fixed period starting with the building of the structure. This design wind speed is designated as lifetime design wind speed.

If v_d is the lifetime design wind speed, $1 - F_V(v_d)$ is the probability of the annual extreme wind speed exceeding the design value v_d. Hence, the probability of no extreme wind exceeding v_d in the first m years is $[F_V(v_d)]^m$. (This derivation is similar to the one that is derived for the lifetime design live load). The probability of atleast one extreme speed exceeding v_d is

$$p_m = 1 - [F_V(v_d)]^m$$

$$F_V(v_d) = [1 - p_m]^{1/m} \tag{5.47}$$

Here p_m and m are chosen by the designer. For example, if $m = 50$ years and the designer has chosen a chance of the design wind speed being exceeded to be $p_{50} = 0.05$ or one in twenty, then the value F computed by Eq. (5.47) becomes equal to 0.9989746. (It is to be noted that this corresponds to a return period of 975 years). The characteristic wind speed for the ultimate limit state is defined (5.31) as the wind gust speed with an estimated probability of exceedence of five per cent in a lifetime period of fifty years of the structure. Based on this definition, substituting $m = 50$ and $p_{50} = 0.05$ in Eq. (5.47), the computed design speed v_d becomes the characteristic wind speed for the ultimate limit state.

EXAMPLE 5.5 For the same data given in Example 5.4, calculate the lifetime design wind speed for $m = 50$ years and $p_m = 0.05$.

Solution Using Eq. (5.47),

$$F_V(v_d) = [1 - p_m]^{1/m}$$

$$= [1 - 0.05]^{1/50} = 0.9989746$$

Since V follows the type 1 extremal distribution,

$$F_V(v_d) = \exp[-\exp\{-\alpha(v - u)\}]$$

That is,

$$\exp[-\exp\{-0.066(v_d - 97.6)\}] = 0.9989746$$

$$v_d = 201.88 \text{ kmph}$$

This is the characteristic wind speed to be used for the design under ultimate limit state.

Similarly, the design wind speed can be calculated for different values of m and p_m. The variation of lifetime design wind speed with the service period of structures for different probabilities p_m, is shown in Fig. 5.5 for New Delhi (Safdarjung) station. As expected, for a given value of p_m, the design speed increases with the lifetime of the structure, and for a given lifetime of the structure, it increases with decrease in the values of risk (i.e. p_m).

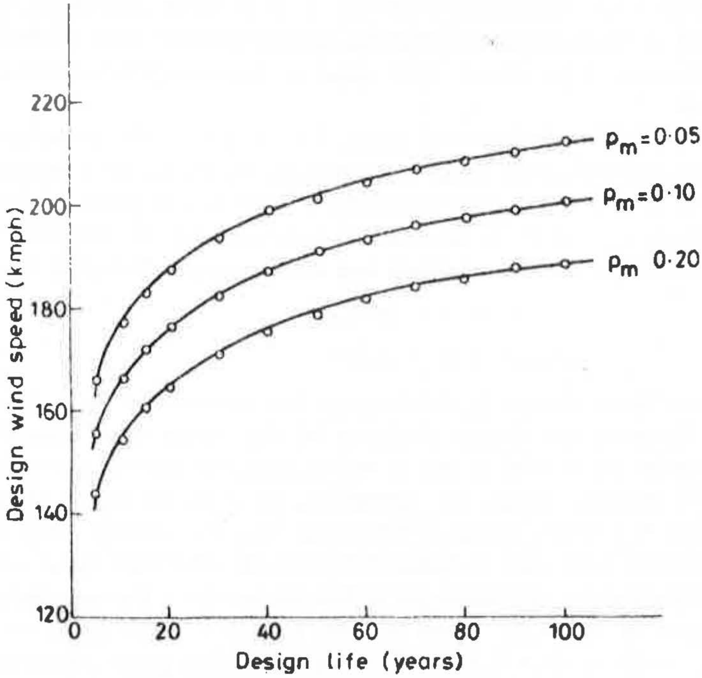


FIG. 5.5 Variation of design wind speed with fixed life period of the structure

5.2.5 Probability Model for Wind Load

Recalling Eq. (5.44), the wind load on structure can be written in the form

$$W = BV^2$$

where B is a parameter covering all components of the wind load except the basic wind speed. The parameter may be assumed to be made up of the product of the number of variables as follows

$$B = K\,C\,E\,G\,D \tag{5.48}$$

where K is the analysis factor, C is the pressure coefficient depending on the geometry of the structure, E is the exposure coefficient depending on the location (e.g. urban area or open country), G is a gust factor depending on the turbulence of wind and the dynamic interaction between the structure and wind, and D is a directionality factor to take into account the effects of the wind direction. Hence the wind load may be written as

$$W = K\,C\,E\,G\,D\,V^2 \tag{5.49}$$

If one wants to find out the probability model for W, the Monte Carlo simulation (dealt in Chapter 7) technique can be used, which requires the probability distribution and the parameters of individual variates. To determine the lifetime maximum wind load model, the probability distribution and parameters of lifetime maximum wind speed must be known. If V follows the Type 1 distribution, the lifetime design speed for m years, V_m, also follows the Type 1 distribution. The mean and coefficient of variation of V_m are given by (5.32),

$$\bar{V}_m = \bar{V}\left(1 + \frac{\sqrt{6}}{\pi}\,\delta_V \ln(m)\right) \tag{5.50}$$

$$\delta_{V_m} = \delta_V \frac{\bar{V}}{\bar{V}_m} \qquad (\text{or } \sigma_{V_m} = \sigma_V) \tag{5.51}$$

$\bar{V}_m$ and δ_{V_m} are the mean and coefficient of variation of V_m.

The approximate mean and coefficient of variation of W can be found out by the following expression assuming all variables in Eq. (5.49) as independent:

$$\bar{W} = \bar{K}\,\bar{C}\,\bar{E}\,\bar{G}\,\bar{D}\,\bar{V}^2 \tag{5.52}$$

$$(\delta_W)^2 = (\delta_K)^2 + (\delta_C)^2 + (\delta_E)^2 + (\delta_G)^2 + (\delta_D)^2 + (2\delta_V)^2 \tag{5.53}$$

Since W is the product of the number of random variables, the probabilistic model for W may tend towards the lognormal distribution. However, Ellingwood (5.32) has proposed Type 1 extremal (largest) distribution (based on Monte Carlo technique) for W for the assumed mean and coefficient of variation of the different variables in Eq. (5.49).

The author has collected data on the annual maximum wind speed observed at 48 stations, and the daily maximum wind speed observed at 4 stations in India, and has statistically analysed the collected data. The Type 1 extremal (largest) distribution, in general, is found to fit the data on annual and daily maximum wind speed. Using the results of the analysis of wind speed, the analysis of wind load has been carried out taking into account the uncertainties in various parameters affecting the wind load, and statistics of wind loads have been fixed for a probabilistic criterion. The analysis of wind load is carried out for the maximum wind load, W_{max}, corresponding to the lifetime maximum wind speed, annual maximum wind

TABLE 5.3 *Statistics of wind load*

Variable	Mean	δ	u	α	Probability distribution
W_{max}/W_n	0.804	0.334	0.683	4.75	Type 1 extremal (largest)
W_a/W_n	0.349	0.392	0.287	9.31	"
W_{apt}/W_n	0.0452	0.743	0.030	38.15	"

load, W_a, corresponding to the annual maximum wind speed, and the daily maximum wind load (which is considered as an arbitrary point-in-time varying wind load), W_{apt}, corresponding to the daily maximum wind load. The final statistics of wind load established for Indian conditions are given in Table 5.3 (5.27).

REFERENCES

5.1 Dunham, J.W., "Design Live Loads in Buildings", *Transactions, ASCE*, Vol. 112, 1947, pp. 725-744.

5.2 Bryson, J.O. and D. Gross "Techniques for the Survey and Evaluation of Live Floor Loads and Fire Loads in Modern Office Buildings", *NBS Building Science Series*, 16, U.S. Dept. of Commerce, Washington, D.C., 1968.

5.3 Mitchell, G.R. and R.W. Woodgate, "A Survey of Floor Loadings in Office Buildings", *CIRIA Report 25*, Building Research Station, London, England, Aug. 1970.

5.4 Culver, C.G., "Live Load Survey Results for Office Buildings", *Journal of the Structural Division*, ASCE, Vol. 102, No. ST112, *Proc. Paper 12615*, Dec. 1976, pp. 2269-2284.

5.5 Sentler, Lars., "A Stochastic Model for Live Loads on Floors in Buildings", *Report 60*, Division of Building Technology, Lund Institute of Technology, 1975.

5.6 Ranganathan, R. and P. Dayaratnam, "Statistical Analysis of Floor Loads and Reliability Analysis", *The Bridge and Structural Engineer*, Vol. 7, No. 1, March 1977.

5.7 Rao, P.S. and G.S. Krishnamurthy, 'Imposed Live Loads—Their Evaluation', Dept. of Civil Engineering, Indian Institute of Technology, Madras, India, 1982.

5.8 Prabhu, U.P. and R. Ranganathan, "Stochastic Analysis of Live Loads in Office Buildings", *Proc. of the National Conference on Quality and Reliability*, held at I.I.T., Bombay, Dec. 1984, pp. 275-291.

5.9 Ranganathan, R., "Statistical Analysis of Live Loads in Office Buildings", *D.S. and T. Report No. 4/1/83-STP-III(3)*, Civil Engineering Dept., I.I.T., Bombay, Oct. 1985.

5.10 Ellingwood, B., T.V. Galambos, J.G. McGregor and C.A. Cornell, 'Development of a Probability Based Load Criterion for American National Standards A58', NBS 577, U.S. Dept. of Commerce, Washington, D.C., June 1980.

5.11 Christensen, P.T. and M.J. Baker *Structural Reliability Theory and its Applications*, Springer-Verlag, Berlin, New York, 1982.

5.12 Peir, J.C. and C.A. Cornell, "Spatial and Temporal Variability of Live Loads", *Journal of the Structural Division, ASCE*, Vol. 99, No. ST5, *Proc. Paper 9749*, May 1973, pp. 903-922.

5.13 McGuire, R.K. and C.A. Cornell, "Live Load Effects in Office Buildings", *Journal of the Structural Division, ASCE*, Vol. 100, No. ST7, *Proc. Paper 10660*, July 1974, pp. 1351-66.

5.14 Corotis, R.B. *et al.*, "Area Dependent Processes for Structural Live Loads", *Journal of the Structural Division, ASCE*, Vol. 107, ST5, *Proc. Paper 16266*, May 1981, pp. 857-872.

5.15 Ellingwood, B.R. and C.G. Culver, "Analysis of Live Loads in Office Buildings", *Journal of the Structural Division, ASCE*, Vol. 103, No. ST8, *Proc. Paper 13109*, Aug. 1977, pp. 1551-1560.

5.16 Chalk, P.L. and R.B. Corotis, "Probability Models for Design Live Loads", *Journal of the Structural Division, ASCE*, Vol. 106, No. ST10, Oct. 1980, pp. 2017-2033.

5.17 Hasofer, A.M., "Statistical Model for Live Floor Loads", *Journal of the Structural Division*, ASCE, Vol. 94, No. ST10, *Proc. Paper 6146*, Oct. 1968, pp. 2183-2196.

5.18 Ranganathan, R., "Reliability Analysis and Design of RCC Slabs, Beams and Columns and Frames—Code Calibration", *D.S. and T. Report No. 4/1/83-STP* III (5), Civil Engg. Dept., I.I.T., Bombay, Sept. 1987.

5.19 Augusti, G., A. Baratta, and F. Casciati, *Probabilistic Methods in Structural Engineering*, Chapman and Hall, New York, 1984.

5.20 Thom, H.C.S., "New Distributions of Extreme Winds in the U.S.", *Journal of Structural Division, Proc. ASCE*, Vol. 94, ST7, July 1968, pp. 1787-1801.

5.21 Simiu, E. and J.J. Filliben, "Probability Distributions of Extreme Wind Speeds", *Journal of Structural Division, Proc. ASCE*, Vol. 102, ST9, Sept. 1976, pp. 1861-1878.

5.22 Simiu, E., J. Bietry and J.J. Filliben, "Sampling Error in Estimation of Extreme Winds", *Journal of Structural Division, Proc. ASCE*, Vol. 104, ST3, March 1978, pp. 491-502.

5.23 Ito, M. and Y. Fujino, "Design Wind Speed and Wind Load Factor Based on Probabilistic Rationale", *Proceedings of the Fifth International Conference on Wind Engineering*, Fort Collins, Colorado, U.S.A., July 1979, pp. 1271-1280.

5.24 Dorman, C.M.L., "Extreme Wind Gust Speeds in Australia, excluding Tropical Cyclones", *Civil Engineering Transactions*, Institution of Engineers, Australia, Vol. CE18, No. 2, 1983, pp. 96-106.

5.25 Schueller, and Panggabean, "Probabilistic Determination of Design Wind Velocity in Germany", *Proc. of Institution of Civil Engineers*, London (U.K.), Part 2, Vol. 61, Dec. 1976, pp. 673-683.

5.26 Ranganathan, R., "Statistical Analysis of Wind Speed and Wind Load for Probabilistic Criterion", *D.S. and T. Report No. 5/1/83-STP-III(4)*, Civil Engg. Dept. I.I.T., Bombay, March 1986.

5.27 Ranganathan, R., "Wind Speed and Wind Load Statistics for Probabilistic Design", *Journal of the Institution of Engineers (India)*, Civil Engg. Div., Vol, 68, May 1988, pp. 303-308.

5.28 Ang, A.H.S. and W.H. Tang, *Probability Concepts in Engineering Planning and Design*, Vol. I, John Wiley and Sons, Inc., New York, 1975.

5.29 Grigoriu, M., "Estimates of Design Wind from Short Records", *Journal of Structural Division, Proc. ASCE*, Vol. 108, ST5, May 1982, pp. 1034-1048.

5.30 Grigoriu, M., "Estimates of Extreme Winds from Short Records", *Journal of Structural Division, Proc. ASCE*, Vol. 110, ST7, July 1984, pp. 1467-1484.

5.31 Holmes, J.D. "Wind Loads and Limit States Design", *The Civil Engineering Transactions, The Institution of Engineers*, Australia, Vol. CE 27, No. 1, Feb. 1985, pp. 21-25.

5.32 Ellingwood, B., "Wind and Snow Load Statistics for Probabilistic Design", *Journal of Structural Division, Proceedings ASCE*, Vol. 107, ST7, July 1981, pp. 1345-1350.

EXERCISE

5.1 The live load on a building follows the lognormal distribution with mean $= 1.3$ kN/m^2 and $\delta = 0.381$. If the specified design load is 2.5 kN/m^2, what is the probability of exceeding the specified design load? (*Ans.* 0.0256)

What is the value of live load with a probability of exceedence of five per cent? (*Ans.* 2.22 kN/m^2)

5.2 The live load on a building follows the lognormal distribution with mean $= 1.3$ kN/m^2 and $\delta = 0.381$. The lifetime of the building is 50 years and the period of tenancy is 5 years. What is the lifetime maximum design live load for the building with a probability of exceedence of five per cent during the lifetime? (*Ans.* 3.17 kN/m^2)

5.3 The annual maximum wind speed observed at a station follows the type 2 extremal (largest) distribution with parameters $u = 81.00$ kmph and $k = 7.05$. What is the return period of the design wind speed $= 182.5$ kmph? (*Ans.* 309.6 yr)

At the same station a temporary structure is to be designed to serve for a period of 3 years only. If the engineer takes a risk of five per cent, what value of design speed will he choose for the design of the structure?

(*Ans.* 144.2 kmph)

5.4 If the annual maximum wind speed at Bombay follows the Type I extremal (largest) distribution with parameters $u = 81.4$ kmph and $\alpha = 0.126$, determine the characteristic wind speed for the ultimate limit state.

(*Ans.* 136 kmph)

What are the mean value and coefficient of variation of the 50-year lifetime maximum wind speed? (*Ans.* 116.9 kmph, 0.086)

5.5 The model for wind load is given by Eq. (5.50):

$$W = K\,C\,E\,G\,D\,V^2$$

If the variations in K and D are neglected, and if $\delta_C = 0.12$, $\delta_E = 0.16$, $\delta_G = 0.11$ and $\delta_V = 0.114$, determine δ_W. (*Ans.* 0.322)

6

Basic Structural Reliability

6.1 INTRODUCTION

The performance of a structure is assessed by its safety, serviceability, and economy. The information about input variables is never certain, precise, and complete. The sources of uncertainties may be (i) inherent randomness, i.e. physical uncertainty, (ii) limited information, i.e. statistical uncertainty, (iii) imperfect knowledge, i.e., model uncertainty, and (iv) gross errors. In the presence of uncertainties, the absolute safety of a structure is impossible due to (i) the unpredictability of (a) loads on a structure during its life, (b) in-place material strengths, and (c) human errors, (ii) structural idealizations in forming the mathematical model of the structure to predict its response or behaviour, and (iii) the limitations in numerical methods. Therefore, some risk of unacceptable performance must be tolerated. With respect to risk of life, the structural safety is important. In the conventional deterministic analysis and design methods, it is assumed that all parameters (loads, strengths of materials, etc.) are not subjected to probabilistic variations. The safety factors provided in the existing codes and standards, primarily based on practice, judgement, and experience, may not be adequate and economical.

The concept of reliability has been applied to many fields and has been interpreted in many ways. The most common definition, and accepted by all, of reliability is that reliability is the probability of an item performing its intended function over a given period of time under the operating conditions encountered. It is important to note that the above definition stresses four significant elements, viz. (i) probability, (ii) intended function, (iii) time, and (iv) operating conditions. Because of the uncertainties, the reliability is a probability which is the first element in the definition. The second point, intended function, signifies that the reliability is a performance characteristic. For a structure to be reliable, it must perform a certain function or functions satisfactorily for which it has been designed, i.e. safety against shear or flexure or torsion, etc. The reliability is always related to time. In the case of structure, it is related to the lifetime of the structure. During this specified life of the structure, it must perform the assigned function satisfactorily. The last point is the operating conditions. This establishes the actions or stresses that will be imposed on the structure. These may be loads, temperature, shock, vibrations, corrosive atmosphere, etc. Reliability also changes with respect to quality control, workmanship, production procedure, inspection, etc.

As stated in Chapter 1, in structural analysis and design, reliability is defined as the probability that a structure will not attain each specified limit (flexure or shear or torsion or deflection criteria) during a specified reference period (life of the structure). For convenience, the reliability, R_0, is defined in terms of the probability of failure, p_f, which is taken as

$$R_0 = 1 - p_f \tag{6.1}$$

In the case of the classical reliability theory, for reliability prediction informations on life characteristics of the system, operating conditions and the failure distribution are needed. Life characteristics are measured by the failure rate or the mean time between the failures or the mean time to failure. Assuming the failure rate is constant over time, the failure rate λ is defined as

$$\lambda = \frac{f}{T} \tag{6.2}$$

where f is the number of failures during a specified test interval and T is the total test time. That is, λ is a ratio of the number of failures during a specified test interval to the total test time of the components or items. The smaller the value of λ, the higher is the reliability.

If the failure rate is constant during the operating period, the mean time between the failures is the reciprocal of the constant failure rate.

If there are n components with failure times $t_1, t_2, \ldots, t_n$, then the mean time to failure is defined as

$$MF = \frac{1}{n} \sum_{i=1}^{n} t_i \tag{6.3}$$

Let a set of N items (structures) be repeatedly tested. After a time t (this may be considered as the time elapsed since the structure is put into service. i.e. the age of the structure), let n components fail (n structures in a failed condition). Then the probability of failure at time t can be expressed as

$$F(t) = \frac{n}{N} \tag{6.4}$$

This $F(t)$ is called the failure function or the lifetime failure distribution function for the set, and the reliability function or survival function, $R(t)$, is given by

$$R(t) = 1 - F(t) \tag{6.5}$$

The failure rate function is given by the derivative of the failure function. That is,

$$f(t) = \frac{dF(t)}{dt} \tag{6.6}$$

The hazard rate or hazard function is the instantaneous failure rate as the interval length tends to zero. It is defined as the probability of failure per unit of time given that the failures have not occurred prior to time t. That is,

$$\boxed{H(t) = \frac{f(t)}{R(t)}} \tag{6.7}$$

where $H(t)$ is the hazard function. If $f(t)$ is exponential and the failure rate is constant, the hazard rate is also constant and becomes equal to the constant failure rate. If a structure has a constant failure or hazard rate (say λ), and $f(t)$ is exponential, the various functions can be shown in Fig. 6.1.

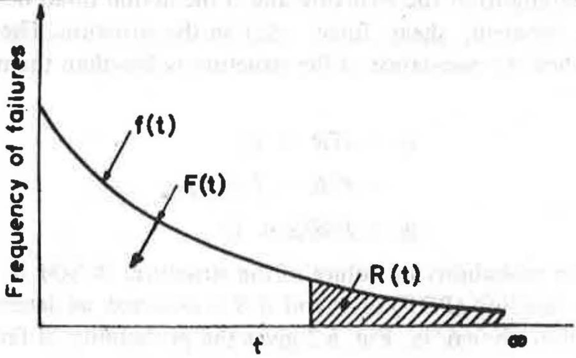


FIG. 6.1 Various reliability functions

Hence, if the informations on failure rate, time between failures or breakdowns, reliability function, and hazard function are available, based on actual data, many predictions can be made about the system performance and decisions can be taken based on that.

For structural systems it is difficult to predict the expected life or the expected failure rate or the expected time between breakdowns. In the reliability format, it is assumed that structural failures are not due to deterioration. The structures cannot be assumed to be nominally identical. The structural failures cannot be expressed in terms of the relative frequency. Thus, the structural reliability theory differs from the classical reliability theory in many such aspects except in the probabilistic nature because of the uncertainties. The probability of failure of a structure is a subjective probability. The reliability of a structure is not a unique property. It changes as the state of knowledge about the structure changes.

The acceptable probabilities for structural failures are very low, e.g. (i) of the order of 10^{-3} for serviceability limit states, meaning thereby that on an average, out of 1000 nominally identical structures, one may deform excessively or (ii) of the order of 10^{-6} for ultimate limit states, which means that out of one million identical structures, one may collapse. In practice, structures are never identical in a large number. Moreover, these low probabilities are to be estimated from the statistical properties extrapolated from the available statistical data around the central values of the random variables. Therefore, it will be proper to consider these probabilities as conventional, comparative values without having much statistical significance. In the light of this, probabilistic methods play an important role in making rational comparisons between alternative structural designs. The currently developed reliability analysis of structures aims at evaluating the probability of failure (or reliability) of a structure.

6.2 COMPUTATION OF STRUCTURAL RELIABILITY

Consider a simple structure with one element only. Let R be the resistance (capacity or strength) of the structure and S the action (load or load effect, viz. bending moment, shear force, etc.) on the structure. The structure is said to fail when the resistance of the structure is less than the action. That is,

$$p_f = P(R < S)$$

$$= P(R - S < 0) \tag{6.8}$$

Or

$$p_f = P(R/S < 1) \tag{6.9}$$

where p_f is the probability of failure of the structure. If $f_R(r)$ is the probability density function (PDF) of R and if S is assumed as deterministic, the hatched portion shown in Fig. 6.2 gives the probability of failure. This is expressed as

$$p_f = \int_{-\infty}^{S} f_R(r)\, dr \qquad -\infty \leqslant r \leqslant \infty \tag{6.10}$$

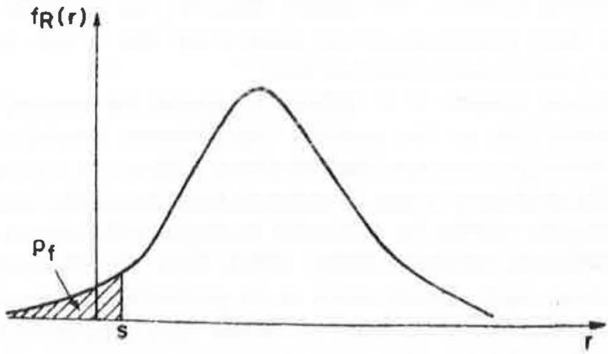


FIG. 6.2 Determination of probability of failure for deterministic action

Fundamental Case

In real situations, both R and S are random variables. The plots of the density functions of R and S are shown in Fig. 6.3. The hatched portion shown in Fig. 6.3 is an indicative measure of the probability of failure. The probability of failure is computed as follows (6.1):

The probability of S assuming a value s, is equal to the area A_1 marked in Fig. 6.4.

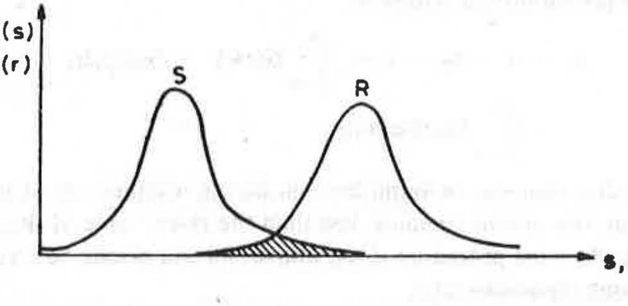


FIG. 6.3 **Probability of failure for random variations of S and R**

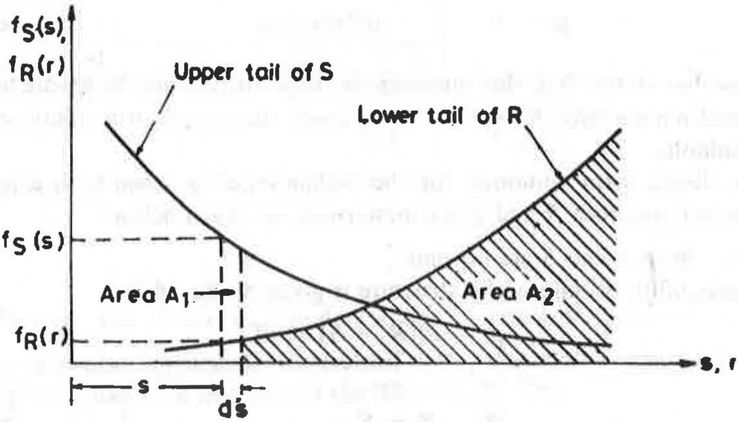


FIG. 6.4 **Determination of reliability**

$$P\left(s - \frac{ds}{2} < S < s + \frac{ds}{2}\right) = f_S(s)\, ds = A_1$$

The probability that $R > s$ is equal to the shaded area A_2 under the resistance density curve:

$$P(R > s) = \int_s^\infty f_R(r)\, dr = A_2$$

When S takes the value s, the reliability is the product of these two probabilities, i.e.

$$\boxed{dR_0 = f_S(s)\, ds \int_s^\infty f_R(r)\, dr}$$

the reliability of the structure, R_0, is the probability of R being greater than all the possible values of S:

$$\boxed{R_0 = \int dR_0 = \int_{-\infty}^{\infty} f_S(s)\left[\int_s^\infty f_R(r)\, dr\right] ds} \qquad -\infty \leqslant s \leqslant \infty \qquad (6.11)$$

Hence the probability of failure is

$$p_f = 1 - R_0 = 1 - \int_{-\infty}^{\infty} f_S(s)[1 - F_R(s)]\, ds$$

$$= \int_{-\infty}^{\infty} f_S(s) F_R(s)\, ds \tag{6.12}$$

The reliability can also be found by considering whether the structure survives when the action remains less than the given value of the resistance. Following the same procedure given above, R_0 and p_f can be expressed by the following equations also

$$R_o = \int_{-\infty}^{\infty} f_R(r) \left[\int_{-\alpha}^{r} f_S(s)\, ds \right] dr \tag{6.13}$$

$$p_f = 1 - \int_{-\infty}^{\infty} f_R(r) F_S(r)\, dr \tag{6.14}$$

It must be noted that the integrals in Eqs. (6.12) and (6.13) are to be evaluated numerically. Except for a few cases, the closed form solutions are not available.

The closed form solutions for the evaluation of p_f, when both R and S are normal and both R and S are lognormal, are given below:

Case 1: Both R and S are normal

The probability of failure of a structure is given by Eq. (6.8):

$$p_f = P[(R - S) < 0]$$

Let

$$M = R - S \tag{6.15}$$

where M is defined as the margin of safety. When R and S are independent and normally distributed, M is also normally distributed. The mean value of M, μ_M, and the standard deviation of M, σ_M, are given by

$$\mu_M = \mu_R - \mu_S \qquad \text{and} \qquad \sigma_M = (\sigma_R^2 + \sigma_S^2)^{1/2}$$

Hence the probability of failure is given by

$$p_f = P(M < 0)$$

$$= F_M(0) = \Phi\left(\frac{0 - \mu_M}{\sigma_M}\right)$$

$$p_f = \Phi\left[\frac{\mu_S - \mu_R}{(\sigma_R^2 + \sigma_S^2)^{1/2}}\right] \tag{6.16}$$

If R and S are correlated with correlation coefficient, ρ and if the joint distribution of R and S is normally distributed, the value of p_f is given by

$$p_f = \Phi\left(-\frac{\mu_M}{\sigma_M}\right)$$

$$= \Phi\left[\frac{\mu_S - \mu_R}{(\sigma_R^2 + \sigma_S^2 + 2\rho\sigma_R\sigma_S)^{1/2}}\right] \tag{6.17}$$

Let

$$\beta = \frac{\mu_M}{\sigma_M} \tag{6.18}$$

Then the value of p_f corresponding to β is given by

$$p_f = \Phi(-\beta)$$

and the value of β corresponding to a given p_f is

$$\beta = -\Phi^{-1}(p_f) \tag{6.19}$$

Hence β is related to the probability of failure and is called the 'reliability index'. The value of β is affected by the mean values and standard deviations of R and S, and also by the level at which the distributions of R and S intersect with each other.

Case 2: Both R and S are lognormal
The probability of failure of a structure is given by Eq. (6.9):

$$p_f = P\left[\left(\frac{R}{S}\right) < 1\right]$$

Let

$$\boxed{Z = \frac{R}{S}} \tag{6.20}$$

When R and S are independent and lognormally distributed, it is known that Z is also lognormally distributed with parameters $\tilde{Z}$ and $\sigma_{\ln Z}$, where $\tilde{Z}$ is the median of Z and $\sigma_{\ln Z}$ is the SD of ln Z. Thus

$$p_f = P(Z < 1)$$

$$p_f = \Phi\left[\frac{\ln(1/\tilde{Z})}{\sigma_{\ln Z}}\right] \tag{6.21}$$

When R and S are distributed as LN($\tilde{R}$, $\sigma_{\ln R}$) and LN($\tilde{S}$, $\sigma_{\ln S}$) respectively, the parameters $\tilde{Z}$ and $\sigma_{\ln Z}$ of the lognormally distributed Z are given by

$$\boxed{\tilde{Z} = \frac{\tilde{R}}{\tilde{S}}} \tag{6.22}$$

and

$$\sigma^2_{\ln Z} = \sigma^2_{\ln R} + \sigma^2_{\ln S} \tag{6.23}$$

Substituting the above equations in Eq. (6.21), we get

$$p_f = \Phi\left[\frac{\ln(\tilde{S}/\tilde{R})}{(\sigma^2_{\ln R} + \sigma^2_{\ln S})^{1/2}}\right] \tag{6.24a}$$

$$= \Phi\left[\frac{\ln(\tilde{S}/\tilde{R})}{[\ln\{(\delta_R^2 + 1)(\delta_S^2 + 1)\}]^{1/2}}\right] \tag{6.24b}$$

But if X is lognormally distributed, then

$$\ln(\tilde{X}) = \ln \mu_X - \frac{1}{2}\sigma^2_{\ln X}$$

$$= \ln\left[\frac{\mu_X}{(\delta_X^2 + 1)^{1/2}}\right]$$

Using similar equations for $\ln \tilde{R}$ and $\ln \tilde{S}$ and substituting them in Eq. (6.24b), we have

$$p_f = \Phi\left[\frac{\ln\left\{\dfrac{\mu_S}{\mu_R}\sqrt{\dfrac{\delta_R^2+1}{\delta_S^2+1}}\right\}}{\{\ln[(\delta_R^2+1)(\delta_S^2+1)]\}^{1/2}}\right] \tag{6.25}$$

It is to be noted in the above equation that p_f has been written in terms of the mean values and the coefficients of variation of R and S only. When δ_R and δ_S are less than about 0.3, Eq. (6.25) becomes

$$p_f \simeq \Phi[\ln(\mu_S/\mu_R)/(\delta_R^2 + \delta_S^2)^{1/2}] \tag{6.26}$$

If R and S follow exponential distributions with parameters λ_R and λ_S respectively, it can be easily proved that (6.2)

$$p_f = \frac{\lambda_R}{(\lambda_R + \lambda_S)}$$

For other combinations of distributions of R and S, Eq. (6.12) or (6.14) is to be used to compute the probability of failure. The closed form solutions are generally not available.

EXAMPLE 6.1 Derive an expression for the probability of failure when S (say action due to wind) follows the Type 2 extremal (largest) distribution and R (say strength of steel) follows the lognormal distribution. Given

$$F_R(r) = \Phi\left[\frac{\ln(r/\tilde{R})}{\sigma_{\ln R}}\right] \qquad r \geqslant 0 \tag{6.27}$$

and

$$f_S(s) = \frac{k}{u}\left(\frac{u}{s}\right)^{k+1} \exp[-(u/s)^k] \qquad s \geqslant 0 \tag{6.28}$$

Solution As random variables can assume only positive values, Eq. (6.12) for the probability of failure becomes

$$p_f = \int_0^\infty f_S(s)F_R(s)\,ds \tag{6.29}$$

Equation (6.28) is rewritten as

$$f_S(s) = \frac{k}{u}\left(\frac{s}{u}\right)^{-(k+1)} \exp[-(s/u)^{-k}]$$

Substituting the above equation and Eq. (6.27) in Eq. (6.29), and putting $s/u = v$, we get

$$p_f = k\int_0^{\infty} \Phi\left[\frac{\ln(uv/\tilde{R})}{\sigma_{\ln R}}\right] (v)^{-(k+1)} \exp[-v^{-k}]\, dv$$

Let

$$\frac{\tilde{R}}{u} = \beta \qquad \text{and} \qquad v^{-k} = t$$

Then

$$p_f = \int_{\infty}^{0} \Phi\left[\frac{\ln(v/\beta)}{\sigma_{\ln R}}\right] e^{-t}(-dt)$$

$$= \int_0^{\infty} \Phi\left[\frac{-\frac{1}{k}\ln t - \ln\beta}{\sigma_{\ln R}}\right] e^{-t}\, dt \tag{6.30}$$

This can be evaluated using the Laguerre-Gauss quadrature formula.

Similarly, for other combinations of probability distributions of R and S, expressions (integral form) for the reliability or probability of failure can be developed.

EXAMPLE 6.2 The axial load carrying capacity of a column, R, is normally distributed with $\mu_R = 1000$ kN and $\sigma_R = 200$ kN. The column is subjected to an axial load, S, which is normally distributed with $\mu_S = 700$ kN and $\sigma_S = 300$ kN. Calculate the reliability of the column assuming R and S are independent.

Solution The margin of safety is given by

$$M = R - S$$

Since R and S are normally distributed, M is also normally distributed. Using Eq. (6.16),

$$p_f = \Phi\left[\frac{\mu_S - \mu_R}{(\sigma_R^2 + \sigma_S^2)^{1/2}}\right]$$

$$= \Phi\left[\frac{700 - 1000}{(300^2 + 200^2)^{1/2}}\right]$$

$$= \Phi(-0.832) = 0.2027$$

$$R_0 = 1 - 0.2027$$

$$= 0.7973$$

EXAMPLE 6.3 A prestressed concrete pole is subjected to wind load, which is as shown in Fig. 6.5, lognormally distributed as LN (1000 N/m², 0.2). Determine the mean depth of the pole at the limit state of deflection for a reliability of 0.999. It is given that

(i) allowable deflection: span/325

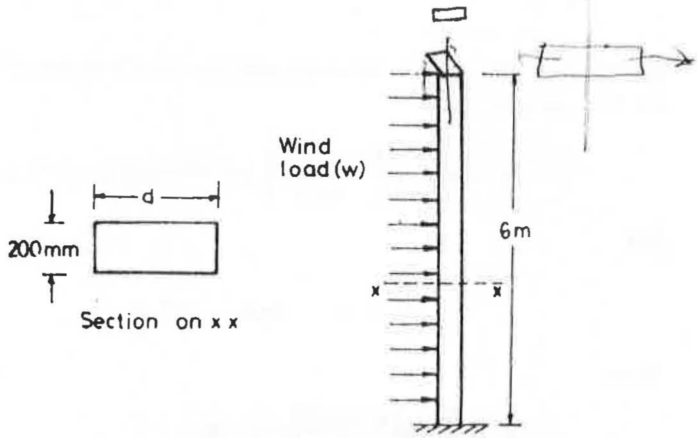


FIG. 6.5 Concrete pole—Example 6.3

(ii) Young's Modulus of concrete (E):

$$\text{LN}\ (2.6 \times 10^4\ \text{N/mm}^2,\ 0.2)$$

(iii) breadth of pole: 200 mm

(iv) variations in depth (d) and breadth of the pole are negligible.

Solution The maximum deflection, i.e. at the top of the pole, is given by

$$y_{\max} = \frac{(w \times 200)(6000)^4}{8EI}$$

where w is the wind pressure in N/mm^2 and I is the moment of inertia in mm^4.

Since,

$$I = \frac{(200d^3)}{12}$$

$$y_{\max} = \frac{w}{Ed^3} \times 1.94 \times 10^{15}$$

At the limit state of deflection, the failure will occur when the allowable deflection, y_{all}, is less than $y_{\max}$, i.e.

$$y_{all} < y_{\max}$$

or the probability of failure of the pole is given by

$$p_f = P\left[\left(\frac{y_{all}}{y_{\max}}\right) < 1\right]$$

Let

$$Z = \frac{y_{all}}{y_{\max}} = \frac{6000}{325 \times 1.94 \times 10^{15}}\left(\frac{Ed^3}{w}\right)$$

As E and w are lognormally distributed and d is deterministic, Z is also lognormally distributed. The parameters of Z are given by

$$\tilde{Z} = \frac{6000}{325 \times 1.94 \times 10^{15}}\left(\frac{\tilde{E}d^3}{\tilde{w}}\right)$$

$$= 2.48 \times 10^{-7}\, d^3$$

$$\sigma_{\ln Z} = [\sigma^2_{\ln E} + \sigma^2_{\ln W}]^{1/2}$$
$$= (0.2^2 + 0.2^2)^{1/2} = 0.28$$

Since Z is lognormally distributed, we have

$$p_f = \Phi\left[\frac{\ln(1/\tilde{Z})}{\sigma_{\ln Z}}\right]$$
$$= 1 - 0.999 = 10^{-3}$$
$$\ln(1/2.48\times 10^{-7}\, d^3) = \sigma_{\ln Z}\Phi^{-1}(10^{-3})$$
$$= (0.28)(-3.1)$$

Solving the above we get, $d = 212.56$ mm

EXAMPLE 6.4 A reinforced concrete beam of an effective span, 8 m, is subjected to live load. The cross section has been designed with M 25 concrete and steel grade Fe 250. The area of steel (A_{st}) is 1400 mm² and the self-weight of the beam 3 kN/m. It is given that the random variables, the cube strength of concrete (f_{cu}) and the yield strength of steel (f_y) are normally distributed.

Breadth of the beam (b)	= 240 mm
Effective depth of the beam (d)	= 480 mm
Mean value of f_{cu}	= 30.28 N/mm²
Mean value of f_y	= 320 N/mm²
SD of $f_{cu} = \sigma_c$	= 4.54 N/mm²
SD of $f_y = \sigma_s$	= 32.0 N/mm²

Calculate the probability of failure of the beam if the live load (L) is normally distributed with mean, 6 kN/m and standard deviation, 3 kN/m.

Solution The action, here, is the bending moment at mid-span due to dead load (D) and live load on the beam. Assuming the dead load and span length as deterministic, μ_S and σ_S are calculated as follows:
The mean value of S is

$$\mu_S = \frac{3\times 8^2}{8} + \frac{\mu_L(8^2)}{8}$$
$$= 24 + 6\times 8 = 72 \text{ kN m}$$
$$\sigma_S = \sigma_L\left(\frac{8^2}{8}\right)$$
$$= 3\times 8 = 24 \text{ kN m}$$

The resistance, here, is the ultimate resisting moment of the beam. This is given by (as per Indian Standard Code),

$$R = f_y A_{st}\, d\left[1 - \frac{0.77\, f_y A_{st}}{b\, d\, f_{cu}}\right]$$

In the above equation, only f_{cu} and f_y are considered as random variables. The approximate values of mean and standard deviation of R are calculated using Eqs. (3.81) and (3.83). It is assumed that f_y and f_{cu} are independent.

$$\mu_R = 320 \times 1400 \times 480 \left[1 - \frac{0.77 \times 1400 \times 320}{240 \times 480 \times 30.28}\right]$$

$$= 193.774 \text{ kN m}$$

Using Eq. (3.84).

$$\sigma_R^2 = \left(\frac{\partial R}{\partial f_y}\bigg|_\mu\right)^2 \sigma_s^2 + \left(\frac{\partial R}{\partial f_{cu}}\bigg|_\mu\right)^2 \sigma_c^2 \tag{6.31}$$

$$\frac{\partial R}{\partial f_y}\bigg|_\mu = A_{st}\, d \left[1 - \frac{1.54\, A_{st}\mu_{fy}}{b\, d\, \mu_{fcu}}\right]$$

$$= 1400 \times 480 \left[1 - \frac{1.54 \times 1400 \times 320}{240 \times 480 \times 30.28}\right]$$

$$= 0.54 \times 10^6$$

$$\frac{\partial R}{\partial f_{cu}}\bigg|_\mu = \left(\frac{0.77\, A_{st}^2\, \mu_{fy}^2}{b\mu_{fcu}^2}\right)$$

$$= \left(\frac{0.77 \times 1400^2 \times 320^2}{240 \times 30.28^2}\right)$$

$$= 0.70 \times 10^6$$

Using the above values in Eq. (6.31)

$$\sigma_R^2 = (0.54 \times 10^6)^2(32)^2 + (0.7 \times 10^6)^2(4.54)^2$$

$$\sigma_R = 17.56 \text{ kN m}$$

Since live load is normally distributed, S is also normally distributed in this case. Assuming R is normally distributed, the value of p_f is obtained using Eq. (6.16), i.e.

$$p_f = \Phi\left[\frac{72 - 193.774}{(24^2 + 17.56^2)^{1/2}}\right]$$

$$= \Phi(-4.0965) = 2.4 \times 10^{-5}$$

REFERENCES

6.1 Haugen, E. B. *Probabilistic Approach to Design*, John Wiley, New York, 1968.

6.2 Kapur, K. C. and L. R. Lamberson, *Reliability in Engineering Design*, John Wiley, New York, 1977.

EXERCISE

6.1 If the probability density functions of resistance R and action S are

$$f_R = \lambda_R \exp(-\lambda_R r)$$

and

$$f_S = \lambda_S \exp(-\lambda_S s)$$

derive an expression for the reliability, R_0, and prove that it is given by

$$R_0 = \frac{\lambda_S}{(\lambda_R + \lambda_S)}$$

6.2 If R and S follow gamma distributions, given by

$$f_R(r) = \frac{\lambda^n r^{n-1} e^{-\lambda r}}{\Gamma(n)} \qquad n, \lambda, r \geqslant 0$$

$$f_S(s) = \frac{\beta^m s^{m-1} e^{-\beta s}}{\Gamma(m)} \qquad m, \beta, s \geqslant 0$$

derive an expression for R_0.

$$\left(Ans. \quad R_0 = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^{t/(1+t)} (1-u)^{m-1} u^{n-1}\, du \right.$$

$$\left. = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} B_t(m, n) \text{ where } t = \beta/\lambda \right)$$

6.3 If the resistance of a structure follows the lognormal distribution, and the action on the structure due to wind follows the Type 1 extremal (largest) distribution, derive an expression for R_0 and show how will you solve it numerically.

6.4 It is assumed that the strength of a RCC column is given by the sum of the strengths of concrete, C, and reinforcing bars B_i. C and B_i follow normal distributions with parameters given by

$$\mu_R \left\langle \begin{array}{ll} \mu_C = 25 \text{ N/mm}^2 & \sigma_C = 5 \text{ N/mm}^2 \\ \mu_{Bi} = 460 \text{ N/mm}^2 & \sigma_{Bi} = 46 \text{ N/mm}^2 \end{array} \right.$$

$\mu_R = \mu_c + \mu_c$ $\quad \sigma_R^2 \neq \sigma_c^2 + \sigma_{Bi}^2$ $\quad 250 \times 400$

If the size of the column is 250×400 mm and if it is provided with four 20 mm diameter bars, determine the mean value and standard deviation of the strength of the column. The column is subjected to a dead load, D, and live load, L, with distributions $N(1500, 200)$ kN and $N(500, 200)$ kN respectively. Compute the reliability of the column. (*Ans.* $R_0 = 0.96638$)

6.5 The strength of a column, R, is given by

$$R = \frac{\pi^2 EI}{a^2}$$

where E is the Young's modulus, I the moment of inertia and a the length of the column. It is subjected to load Q. The mean values and coefficient of variations of all the variables are given below:

$$\begin{array}{ll} \mu_E = 2.03 \times 10^5 \text{ N/mm}^2 & \delta_E = 0.1 \\ \mu_I = 12.5 \times 10^6 \text{ mm}^4 & \delta_I = 0.05 \\ \mu_a = 5000 \text{ mm} & \delta_a = 0.05 \\ \mu_Q = 700 \text{ kN} & \delta_Q = 0.3 \end{array}$$

If all the variables are lognormally distributed, determine the probability of failure of the column. (*Ans.* 0.11365)

6.6 A tension member of a steel truss is subjected to an axial load, Q. The strength of the member is given by $f_y A$, where f_y is the yield strength of steel and A is the area of cross section of the member. Given:

$$\begin{array}{ll} \mu_Q = 20 \text{ kN} & \delta_Q = 0.4 \\ \mu f_y = 286 \text{ N/mm}^2 & \delta f_y = 0.1 \end{array}$$

Find the area of the member for the specified reliability of 0.99865. That is, $p_f = 1.35 \times 10^{-3}$. Assume variation in area is negligible. (*Ans.* 167.8 mm²)

7

Monte Carlo Study of Structural Safety

7.1 GENERAL

In the process of giving predictions about some physical system, the following four steps are involved: (i) observation of a physical system, (ii) formulation of a hypothesis, (iii) prediction of the behaviour of the system on the basis of the hypothesis, and (iv) performance of experiments to test the validity of the hypothesis. Sometimes it may be either impossible or extremely costly to observe certain processes in the real world. It is evident that there are many situations which cannot be represented mathematically due to the stochastic nature of the problem, complexity of the problem formulation, or the interactions needed to adequately describe the problem under study. For such situations defying mathematical formulation, simulation is the only tool that might be used to obtain relevant answers. Even if a mathematical model can be formulated to describe some system of interest from the limited data available, it may not be possible to obtain a solution to the model by straightforward analytical techniques and in turn make predictions about the behaviour of the system. For example, let us consider the probabilistic behaviour of a prestressed or reinforced concrete flanged beam. We want to determine the reliability of the beam.

7.1.1 Failure of a Flanged Section (7.1, 7.2)

Prestressed concrete members usually have symmetrical or unsymmetrical I sections. Because of the random variations of the parameters of the resistance of a section, the failure of a flanged section can take place with the occurrence of any one of the following events:

Y_1 – the section is under-reinforced with the neutral axis in the flange

Y_2 – the section is under-reinforced with the neutral axis in the web

Y_3 – the section is over-reinforced with the neutral axis in the flange

Y_4 – the section is over-reinforced with the neutral axis in the web

The occurrence of each event has a certain probability. The probability tree for the failure of a section at the limit state of strength is given in Fig. 7.1. If the above events are assumed mutually exclusive, it can be seen that the probability of failure, p_f, of a flanged section is the sum of the conditional probabilities of failures of the section under each given event, and the same can be written as

$$p_f = \sum_{i=1}^{4} P(F \mid Y_i)\, P(Y_i) \tag{7.1}$$

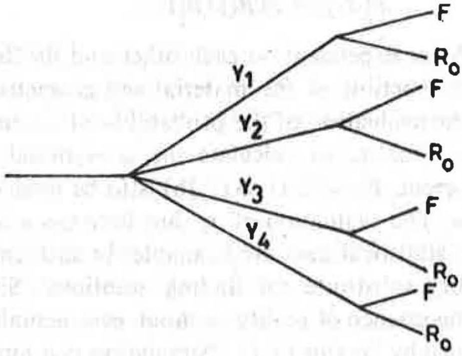


FIG. 7.1 Probability tree diagram

where $P(F \mid Y_i)$ denotes the conditional probability of F for a given event Y_i. F denotes the event 'failure'. In Fig. 7.1, R_0 represents the event 'reliable' (i.e. safe). The conditional probability of failure of a section for any given event (say Y_i) is given by

$$P(F \mid Y_i) = P[(R-S) < 0 \mid Y_i] \tag{7.2a}$$

or

$$P(F \mid Y_i) = P[(R/S) < 1 \mid Y_i] \tag{7.2b}$$

where S is the action (load or bending moment) on the section and R is the resistance of the section. The resistance of a section is a function of the various material and geometric properties of the section:

$$R = g(X_1, X_2, \ldots, X_n) \tag{7.3}$$

Because parameters X_j are usually random variables, the resistance is also a random variable with density function f_R and cumulative distribution F_R. If X_j are correlated, their joint distribution must be known. Assuming the X_j in Eq. (7.3) are statistically independent, their joint density function is

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \boxed{\prod_{j=1}^{n} f_{X_j}(x_j)} \tag{7.4}$$

and its cumulative probability is

$$F_R(r) = P(R \leqslant r) = \int_G \ldots \int \prod_{j=1}^{n} f_{X_j}(x_j)\, dx_j \tag{7.5}$$

The restriction $R \leqslant r$ defines the region of integration G in Eq. (7.5). The integral contained in the equation cannot be evaluated in a closed form.

Added to this, the evaluation of p_f requires the evaluation of the probability of occurrence of each given event Y_i. Defining:

B_1 – the event that the section is under-reinforced
B_2 – the event that the section is over-reinforced
B_3 – the event that the neutral axis lies in the flange
B_4 – the event that the neutral axis lies in the web

the probability of occurrence of the event Y_1 is

$$P(Y_1) = P(B_1 \cap B_3) \tag{7.6}$$

The events B_1 and B_3 are dependent on each other and the density function of each is again the function of the material and geometric properties of the section. Hence, the evaluation of the probability of occurrence of each event Y_i is difficult. Finally, to calculate the conditional probability of failure for the given event, Eq. (7.2a) or (7.2b) is to be used which involves numerical integration. The evaluation of p_f thus becomes a formidable task even when adequate statistical data are available. In such cases, simulation becomes a satisfactory substitute for finding solutions. Simulation is a process of creating the essence of reality without ever actually attaining the reality itself. As defined by Naylor (7.3): "Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real world system over extended periods of time".

7.2 MONTE CARLO METHOD

7.2.1 Introduction

The Monte Carlo method is a simulation technique. One of the usual objectives in using the Monte Carlo technique is to estimate certain parameters and probability distributions of random variables whose values depend on the interactions with random variables whose probability distributions are specified. As it is known that the ultimate resisting moment, M_r, of a section is a function of several random variables, the probability distribution of M_r depends on the equation connecting these random variables. As explained in the previous section, as closed form solution for the calculation of the cumulative probability of M_r is not possible, the Monte Carlo method can be used to study the statistical properties of M_r. Secondly, as explained in Sec. 7.1.1, the failure of a flanged section can take place under different events. Hence to study and simulate the complete random behaviour of the section at the limit state of strength, the Monte Carlo technique is the best suited method.

7.2.2 Monte Carlo Method (7.4)

Provided high speed digital computing facilities are available, a simple

Monte Carlo technique can often be useful in obtaining the distribution $F_R(r)$. Let R be a function of n independent random variables Y_i:

$$R = g(Y_1, Y_2, \ldots, Y_n)$$

The technique consists of three steps:

1. Generating a set of values y_{ik} for the material properties and geometric parameters Y_i in accordance with the empirically determined or assumed density functions f_{Y_i}. The suffix i is used to denote the ith variable and suffix k is used to represent the kth set of values $(y_{1k}, y_{2k}, \ldots, y_{ik}, \ldots, y_{nk})$ of the corresponding variables $(Y_1, Y_2, \ldots, Y_i, \ldots, Y_n)$.
2. Calculating the value r_k corresponding to the set of values y_{ik} obtained in step 1, by means of the appropriate response equation for resistance of the section. That is

$$r_k = g(y_{1k}, y_{2k}, \ldots, y_{ik}, \ldots, y_{nk})$$

3. Repeating steps 1 and 2 to obtain a large sample of the values of R and therefore, estimating $f_R(r)$.

This method can also be used to obtain distributions for M and Z where

$$M = R - S \tag{7.7}$$

$$Z = \frac{R}{S} \tag{7.8}$$

Here, R is the resistance and S the action. It is then only necessary to obtain additional sample values for S in accordance with the density function f_S and to combine the equation for resistance with Eq. (7.7) or (7.8) to provide the direct means of calculating the values of M or Z.

The procedure for generating a random deviate from a specified distribution generally follows this pattern:

1. Generate a random number from the standard uniform distribution.
2. Perform a mathematical transformation of the standard uniform random number (or numbers) which produces a random deviate from the desired distribution.
3. Use the transformed deviate in the experiment as required.

Various methods have been developed for the generation of uniform pseudo-random numbers. Subroutines for this purpose are readily available (7.3, 7.5). Built-in programmes are generally available in all the computer centres to generate uniform random numbers. The transformation of the uniform random number to the random variate of the desired distribution is obtained by the inverse transformation method, if possible.

Inverse Transformation Technique

Consider the cumulative distribution function, $F_Y(y)$, of the distribution to be simulated. $F_Y(y)$ is defined over the interval (0, 1). Consider the standard uniform variate V, which is also defined over the interval (0, 1).

Generate a value v for the standard uniform random deviate. For a standard uniform variate V, the cumulative probability of $V \leqslant v$ is equal to v. That is, $F_V(v) = v$. Hence if we set

$$F_Y(y) = v$$

Then y is uniquely determined by the relation

$$F_Y(y) = v \Leftrightarrow y = F_Y^{-1}(v)$$

This is graphically shown in Fig. 7.2. The PDF of V is shown on the left side. The generated uniform random number v is projected on the curve of the CDF of Y. The point C on the curve is projected down on the horizontal axis to get the corresponding value y. Hence, $y = F_Y^{-1}(v)$ is the variate desired from the given distribution of Y.

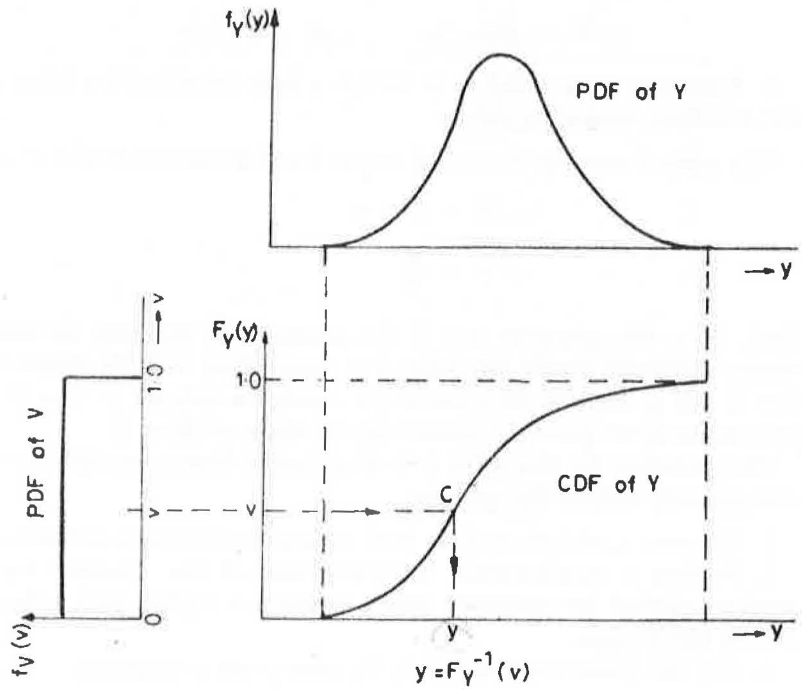


FIG. 7.2 Inverse transformation technique

When the inverse of $F_Y(y)$ i.e. $F_Y^{-1}(v)$, does not exist or, it is so complicated as to be impracticable, other techniques such as rejection technique, composition method, and approximation methods (7.5) are to be used. Hence, the suggested procedure for drawing the kth set of input values y_{ik} from the corresponding distributions of F_{Yi} is to generate first a set of n random numbers, v_{ik}, with uniform density in the range $0 \leqslant v \leqslant 1.0$. The values of y_{ik} are then obtained from

$$y_{ik} = F_{Yi}^{-1}(v_{ik}) \tag{7.9}$$

The method of obtaining a random deviate of $F_{Y_i}(\ )$ using the inverse transformation technique is illustrated in the following example.

EXAMPLE 7.1 Using inverse transformation technique, develop expressions for generating random deviates of Y having the following distributions: (i) Uniform distribution, (ii) Exponential distribution, (iii) Weibull distribution, (iv) Type 1 extremal distribution, (v) Type 2 extremal distribution, and (vi) Type 3 extremal distribution.

Solution Uniform distribution:

Given
$$f_Y(y) = \begin{cases} \dfrac{1}{b-a} & a \leqslant y \leqslant b \\ 0 & \text{elsewhere} \end{cases}$$

Then
$$F_Y(y) = \int_a^y \left(\frac{1}{b-a}\right) dt$$
$$= \left(\frac{y-a}{b-a}\right)$$

Set
$$v = F_Y(y) = \left(\frac{y-a}{b-a}\right)$$

The inverse transformation is
$$y = F_Y^{-1}(v) = a + (b-a)\,v \tag{7.10}$$

where v is a uniform random number with uniform density in the range 0 and 1.

(ii) Exponential distribution:

Given
$$f_Y(y) = \lambda e^{-\lambda y} \qquad y \geqslant 0$$
$$F_Y(y) = 1 - e^{-\lambda y}$$

Set
$$v = F_Y(y) = 1 - e^{-\lambda y}$$
$$y = \frac{-\ln(1-v)}{\lambda} \tag{7.11}$$

However, one can straightaway use the following equation:
$$y = \frac{-\ln(v)}{\lambda} \tag{7.12}$$

rather than Eq. (7.11), since $(1-v)$ is also from the uniform distribution.

(iii) Weibull distribution:

Given
$$f_Y(y) = \alpha\beta y^{\beta-1} e^{-\alpha y^\beta} \qquad y \geqslant 0$$
$$F_Y(y) = 1 - e^{-\alpha y^\beta}$$

Set
$$v = 1 - e^{-\alpha y^\beta}$$

Hence
$$y = \left[-\frac{1}{\alpha}\ln v\right]^{1/\beta} \tag{7.13}$$

(iv) Type 1 Extremal (largest) distribution:

Given $\quad F_Y(y) = \exp[-\exp\{-\alpha(y-u)\}] \quad -\infty \leqslant y \leqslant \infty$

Set $\quad v = \exp[-\exp\{-\alpha(y-u)\}]$

Then
$$y = u - \frac{\ln[-\ln(v)]}{\alpha} \tag{7.14}$$

(v) Type 2 Extremal (largest) distribution:

Given $\quad F_Y(y) = \exp\left[-\left(\frac{u}{y}\right)^k\right] \quad y \geqslant 0$

Set $\quad v = \exp\left[-\left(\frac{u}{y}\right)^k\right]$

Then
$$y = u/(-\ln v)^{1/k} \tag{7.15}$$

(vi) Type 3 Extremal (smallest) distribution:

Given $\quad F_Y(y) = 1 - \exp\left[-\left(\frac{y}{u}\right)^k\right] \quad y \geqslant 0$

Set $\quad v = 1 - \exp\left[-\left(\frac{y}{u}\right)^k\right]$

Then
$$y = u[-\ln(1-v)]^{1/k} \tag{7.16}$$

One can straightaway use the expression
$$y = u[-\ln(v)]^{1/k} \tag{7.17}$$
since $(1-v)$ is also from the uniform distribution.

For normal distribution, the Box and Muller technique is used to generate normal variates. Here, standard normal deviates are obtained by generating two uniform random numbers v_1 and v_2 (with a uniform density range between 0 and 1) at a time. Then the desired standard normal variates are given by (7.5)
$$u_1 = [2 \ln 1/v_1]^{1/2} \cos(2\pi v_2) \tag{7.18}$$
$$u_2 = [2 \ln 1/v_1]^{1/2} \sin(2\pi v_2) \tag{7.19}$$

EXAMPLE 7.2 (Normal distribution) Generate normal variates from the distribution of Y following the normal distribution with mean μ and variance σ^2.

Solution First generate two uniform random numbers v_1 and v_2 in the range 0 and 1. Then, the standard normal variates are given by Eqs. (7.18) and (7.19). We know that the standard normal variate is connected to the normal variate Y as follows:
$$\frac{Y-\mu}{\sigma} = U \tag{7.20}$$
where U is the standard normal variate. Hence we can get two normal variates y_1 and y_2, using Eqs. (7.18) – (7.20). Thus.
$$y_1 = \sigma u_1 + \mu$$
$$y_2 = \sigma u_2 + \mu$$

That is

$$y_1 = \mu + \sigma[2 \ln 1/v_1]^{1/2} \cos (2\pi v_2) \tag{7.21}$$

$$y_2 = \mu + \sigma[2 \ln 1/v_1]^{1/2} \sin (2\pi v_2) \tag{7.22}$$

EXAMPLE 7.3 (Lognormal distribution) Generate the lognormal variates from the distribution of Y following the lognormal distribution with parameters $\tilde{Y}$ and $\sigma_{\ln Y}$.

Solution As in the case of normal distribution, here also, we first generate two uniform random numbers v_1 and v_2 and get two standard normal variates using Eqs. (7.18) and (7.19). Using the following transformation

$$\frac{\ln (y/\tilde{Y})}{\sigma_{\ln Y}} = u \tag{7.23}$$

for transforming the lognormal variate to the standard normal variate, we get two values of the lognormal variate Y:

$$y_1 = \tilde{Y} \exp (u_1 \sigma_{\ln Y})$$

$$y_2 = \tilde{Y} \exp (u_2 \sigma_{\ln Y})$$

Using Eqs. (7.18) and (7.19),

$$y_1 = \tilde{Y} \exp [\sigma_{\ln Y}(2 \ln 1/v_1)^{1/2} \cos (2\pi v_2)] \tag{7.24}$$

$$y_2 = \tilde{Y} \exp [\sigma_{\ln Y}(2 \ln 1/v_1)^{1/2} \sin (2\pi v_2)] \tag{7.25}$$

EXAMPLE 7.4 (Beta distribution) The PDF of the standard beta distribution is given by Eq. (3.105) with parameters p and q, i.e.,

$$f_X(x) = \frac{1}{B(p, q)} x^{p-1}(1 - x)^{q-1} \qquad 0 \leqslant x \leqslant 1$$

The procedure to generate beta deviates is as follows (7.5):

Generate two standard uniform random numbers v_1 and v_2.

Set $\qquad g = (v_1)^{1/p} \qquad$ and $\qquad h = (v_2)^{1/q}$

Check whether $\qquad g + h \leqslant 1$

If $g + h \leqslant 1$, the standard beta deviate is given by

$$x = \frac{g}{(g + h)} \tag{7.26}$$

If we want to generate a random deviate from the beta distribution of Y, given by [Eq. (3.109)]

$$f_Y(y) = \frac{(y - a)^{p-1}(b - y)^{q-1}}{B(p, q)(b - a)^{p+q-1}} \qquad a \leqslant y \leqslant b$$

then use the transformation to transform the beta variate to the standard beta variate, i.e.

$$x = \frac{(y - a)}{(b - a)} \cdot$$

Hence, the required beta random deviate is given by

$$y = x(b - a) + a \tag{7.27}$$

EXAMPLE 7.5 (Gamma distribution) We are interested in generating gamma distributed random deviates. The PDF of the gamma distribution is given by Eq. (3.102), i.e.

$$f_X(x) = \frac{\lambda(\lambda x)^{k-1}e^{-\lambda x}}{\Gamma(k)} \qquad x \geqslant 0$$

$$\lambda, k \geqslant 0$$

where λ and k are parameters of the distribution. The procedure to generate gamma deviates is as follows (7.6):

(i) Let $k' \geqslant 1$ be the integer part of k.

(ii) Generate $k' + 3$ standard uniform random numbers, i.e. $v_1, v_2, \ldots, V_{k'+3}$, satisfying the condition

$$v_1^{1/k} + v_2^{1/(1-k)} \leqslant 1 \tag{7.28}$$

(iii) The gamma distributed deviate is given by

$$x = -\frac{1}{\lambda}\sum_{i=4}^{k'+3} \ln v_i + \frac{1}{\lambda}(-\ln v_3)\frac{v_1^{1/k}}{v_1^{1/k} + v_2^{1/(1-k)}} \tag{7.29}$$

7.3 APPLICATIONS

The Monte Carlo method has a variety of applications. It can be used to study the distribution of a variable, which is a function of several random variables, to simulate the performance or behaviour of a system, and to determine the reliability or probability of failure of a system or a component. The simulation technique has been used in the reliability study of structures by several research workers. Some of the applications are illustrated through the following examples.

EXAMPLE 7.6 The strength of an axially loaded short column is given by

$$R = 0.67\ CA_c + A_s$$

where C is the cube strength of concrete, F the yield strength of the reinforcing bars, A_c the area of concrete and A_s the area of steel. Given:

Size of the column = 250 mm × 500 mm

$\mu_C = 19.54$ N/mm² $\qquad \sigma_C = 4.1$ N/mm²

$\mu_F = 469$ N/mm² $\qquad \sigma_F = 46.9$ N/mm²

$A_s = 1250$ mm²

C and F are normally distributed. The problem is to determine the distribution of R using the Monte Carlo method.

Solution Area of concrete (A_c) = 250 × 500 − 1250

= 123750 mm²

$$R = 0.67 \times 123750\ C + 1250\ F$$

$$= 82912.5\ C + 1250\ F \tag{7.30}$$

Using Eqs. (7.21) and (7.22), the random deviates of the normal variates of C and F are first generated. Using these values in the prediction equation for R, random deviates for R are generated using the Monte Carlo method.

The mean value and standard deviation of R, calculated after the generation of 500 and 1000 values, are given below:

$$\mu_R = 2.216\times 10^6 \qquad \sigma_R = 3.466\times 10^5 \text{ (after 500 values)}$$

$$\mu_R = 2.207\times 10^6 \qquad \sigma_R = 3.460\times 10^5 \text{ (after 1000 values)}$$

These values, when verified with the theoretical exact values

$$\mu_R = 82912.5\ \mu_C + 1250\ \mu_F$$

$$= 2.206\times 10^6 \text{ N}$$

and

$$\sigma_R = [(82912.5\ \sigma_C)^2 + (1250\ \sigma_F)^2]^{1/2}$$

$$= 3.4496\times 10^5 \text{ N}$$

agree very well. The error on the estimates of mean is almost nil and on standard deviation about 0.3 per cent.

The frequency distribution of generated R is shown in Fig. 7.3. The coefficients of skewness are -0.01 and $+\ 0.016$ at the end of 500 and 1000 generated samples respectively. Coefficients of Kurtosis are 2.637 and 2.989 at the end of 500 and 1000 simulations respectively. R, being normal, the theoretical values of the coefficient of skewness and Kurtosis are zero and 3 respectively. The normal distribution fits very well for the generated data. Theoretically also, R should follow the normal distribution.

7.3.1 Sample Size

We have seen in Example 7.6 that the generated data is used for estimating the mean and standard deviation of the resistance of the column. As larger and larger samples are used, the estimates are closer to the population values. The minimum size of the sample depends on the desired accuracy of the estimates.

For the estimate of the population mean of a random variable X, the minimum sample size is specified (7.3) such that the probability of the true mean falling within the confidence interval

$$\bar{X}_m \pm \Phi_{\alpha/2}\left(\frac{\bar{s}_x}{\sqrt{n}}\right) \tag{7.31}$$

is $(1 - \alpha)$ per cent where $\bar{X}_m$ and $\bar{s}_x$ are the sample mean and standard deviation of X, and α is the level of significance. $\Phi_{\alpha/2}$ is the value of the standard normal variate at a cumulative probability of $\alpha/2$. If e_m is the specified acceptable error in the estimate of the mean value of X, then

$$e_m = \Phi_{\alpha/2}\left(\frac{\bar{s}_x}{\sqrt{n}}\right) \tag{7.32}$$

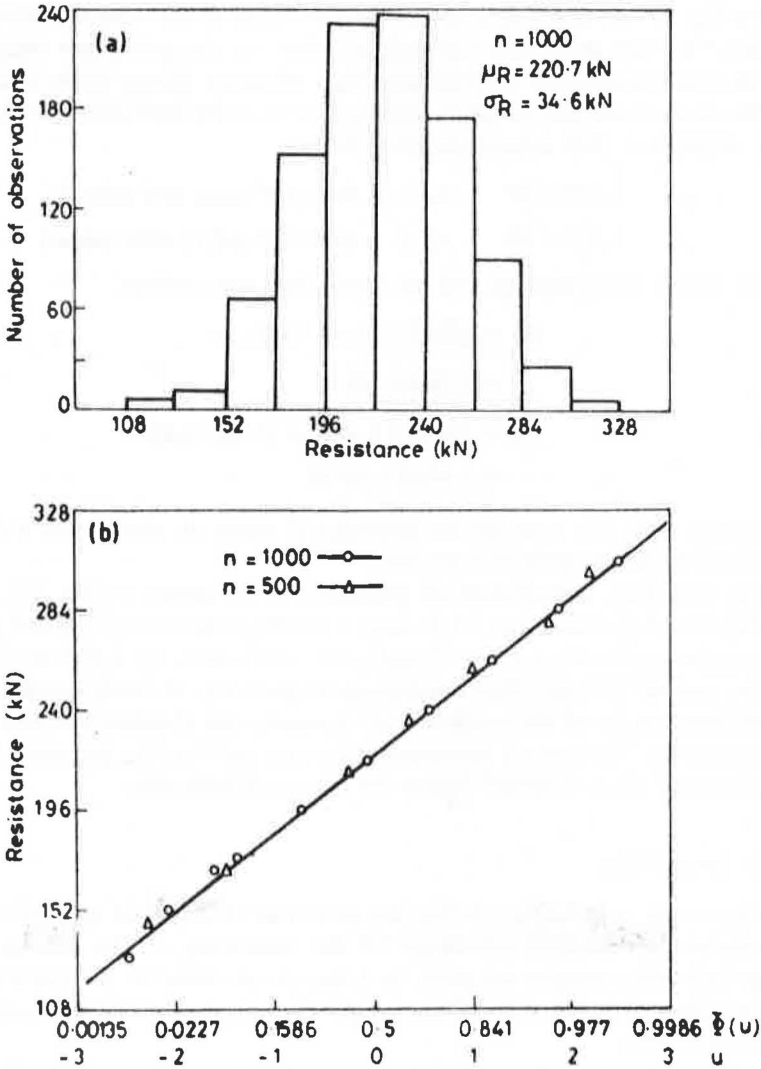


FIG. 7.3 Generated distribution of R: (a) frequency distribution of R and (b) CDF on normal probability paper—Example 7.6

then the minimum size of the sample for the estimate of the population of X is given by

$$n = \Phi^2_{\alpha/2}\left[\frac{\bar{s}_x}{e_m}\right]^2 \tag{7.33}$$

For a large sample size (say $n > 120$), the standard deviation of $\bar{s}_x$ is equal to $\bar{s}_x/\sqrt{2n}$. Hence for the estimate of the standard deviation of X, the minimum size is specified such that the probability of the true standard deviation falling within the confidence interval (7.3)

$$\bar{s}_x \pm \Phi_{\alpha/2}\left(\frac{\bar{s}_x}{\sqrt{2n}}\right) \tag{7.34}$$

is $(1-\alpha)$ per cent. Specifying

$$e_s = \Phi_{\alpha/2}\left(\frac{\bar{s}_x}{\sqrt{2n}}\right) \tag{7.35}$$

the minimum sample size for the estimate of the population standard deviation of X is given by

$$n = \frac{1}{2}\,\Phi^2_{\alpha/2}\left[\frac{\bar{s}_x}{e_s}\right]^2 \tag{7.36}$$

where e_s is the acceptable error in the estimate of the standard deviation of X. Generally, the acceptable error $= 5\%$ and $\alpha = 5\%$ are taken.

If the Monte Carlo technique is used to generate straightaway the samples for the margin of safety and determine the probability of failure, Shooman (7.7) has proposed the following expression for the percentage of error on the estimated probability of failure:

$$\%\ \text{Error} = 200\left[\frac{1-p_f}{np_f}\right]^{1/2} \tag{7.37}$$

Using this equation, the sample size can be calculated for the required accuracy.

EXAMPLE 7.7 Calculate the sample size, required for the case study in Example 7.6, to estimate the mean and standard deviation for an acceptable error of five per cent on the estimates of the mean and standard deviation, and a level of significance equal to five per cent.

Solution If the mean value and standard deviation of the generated samples for R (say after 500 samples) are

$$\bar{R} = 2.216\times 10^6 \qquad \bar{s}_R = 3.466\times 10^5$$

then the sample size required to terminate the simulation process, using Eq. (7.33), is

$$n = \Phi^2_{\alpha/2}\left[\frac{\bar{s}_x}{e_m}\right]^2$$

For $\alpha = 0.05$, confidence level $= 1-\alpha = 0.95$. Thus

$$\Phi_{\alpha/2} = \Phi_{0.025} = \Phi^{-1}(0.975) = 1.96$$

The allowable error on the mean $= \dfrac{5}{100}\,\bar{R}$

Hence the sample size required to estimate the mean with $\alpha = 5\%$ and $e_m = 5\%$ is

$$n = (1.96)^2\left[\frac{3.466\times 10^5}{0.05\times 2.21\times 10^6}\right]^2$$

$$= 38$$

The sample size required to estimate the standard deviation with $\alpha = 5\%$ and $e_s = 5\%$, using Eq. (7.36), is

$$n = \frac{1}{2}\,\Phi^2_{\alpha/2}\left[\frac{\bar{s}_R}{e_s}\right]^2$$

$$= \frac{1}{2} \times 1.96^2 \left(\frac{3.466 \times 10^5}{0.05 \times 3.466 \times 10^5}\right)$$

$$= 768$$

Considering both, the minimum sample size required is 768. In Example 7.6, it can be seen that at the end of 1000 simulations ($n = 1000$), the error on the estimate of the standard deviation of R is less than five per cent.

EXAMPLE 7.8 Consider the column in Example 7.6, the strength of which is given by Eq. (7.30). The column is subjected to an axial load Q. Given:

$$\mu_Q = 1.2 \times 10^6 \text{ N} \qquad \sigma_Q = 0.35 \times 10^6 \text{ N}$$

$$\mu_C = 19.54 \text{ N/mm}^2 \qquad \sigma_C = 4.1 \text{ N/mm}^2$$

$$\mu_F = 469 \text{ N/mm}^2 \qquad \sigma_F = 46.9 \text{ N/mm}^2$$

Variables Q, C and F are normally distributed. Determine the probability of failure of the column using the Monte Carlo method.

Solution The resistance of the column is given by [Eq. (7.30)]

$$R = 82912.5\, C + 1250\, F$$

The safety margin equation is

$$M = 82912.5\, C + 1250\, F - Q \tag{7.38}$$

Using the given distributions and the corresponding parameters of C, F and Q, the simulation is carried out and 20,000 samples are generated for M. During the process of generation, the number of values of M falling below zero are counted. At the end of 20,000 simulations, the number of sample values of M falling below zero is obtained as 417. Hence, the probability of failure of the column is

$$p_f = \frac{417}{20000} = 0.02085$$

Using Eq. (7.37), if we want to have an estimate of p_f (say 0.02) with an error $\pm$ 10 per cent, the sample size required is

$$n = \frac{200^2\,(1 - 0.02)}{0.02 \times 10^2}$$

$$= 19600$$

We have generated 20000 samples. Hence there is a 95 per cent chance that the percentage error in the estimated p_f is less than 10 per cent.

The theoretical values of μ_M and σ_M, using Eqs. (3.77) and (3.79), are

$$\mu_M = 1.006 \times 10^6 \qquad \sigma_M = 0.489 \times 10^6$$

Since C, F and Q are normal, M also follows the normal distribution. Hence the theoretical value of p_f is

$$p_f = P(M < 0)$$
$$= \Phi\left[\frac{0 - \mu_M}{\sigma_M}\right]$$
$$= \Phi(-2.0488) = 0.0202$$

From the Monte Carlo method, the value of p_f has been obtained as 0.02085 at the end of 20000 samples values of M.

During the process of code calibration, reliability analyses of existing designs as per the current codal provisions are carried out for various limit states criteria. For this, the probability distribution and statistics of the strengths of members (say, in flexure, tension, shear, torsion, etc.) for various failure criteria are to be known. Statistics of the strengths of members are established using the Monte Carlo method (7.8, 7.9). The determination of the statistics of the flexural strength of RCC beam is illustrated below.

EXAMPLE 7.9 A simply supported reinforced concrete beam of span l is subjected to a uniformly distributed live load L and a dead load D. The breadth, effective depth, and area of steel on the tension side are b, d and A_s respectively. It is given:

$b = 300$ mm $\quad d = 550$ mm $\quad A_s = 1039.5\ \text{mm}^2$

$\mu_C = 17.58\ \text{N/mm}^2 \quad \sigma_C = 3.164\ \text{N/mm}^2$

$\mu_F = 469\ \text{N/mm}^2 \quad \sigma_F = 45.9\ \text{N/mm}^2$

b: mean deviation $= +10.29$ mm $\quad \sigma = 9.47$ mm

d: mean deviation $= 6.25$ mm $\quad \sigma = 3.79$ mm

C and F are the cube strength of concrete and the yield strength of reinforcing bars respectively. Their nominal values are 15 N/mm^2 and 415 N/mm^2 respectively. The above data are based on the actual field data, given in Chapter 4, for Indian conditions (7.9). C follows the lognormal distribution and all other variables are normally distributed.

Study of Distribution of Strength in Flexure

The theoretical model for the ultimate resisting moment of a RCC beam is

$$R = FA_s\, d\left[1 - \frac{0.77\ FA_s}{bdC}\right] \tag{7.39}$$

This equation is obtained when the material reduction factors attached to the strengths of concrete and steel are removed in the equation given by IS: 456-1978 for computing the design strength of a singly reinforced beam. There will be, in general, a certain model error associated with every prediction equation for the strength of a member. If B is the model parameter, let μ_B and σ_B be the mean and standard deviation of B. For flexural strength, $\mu_B = 1.01$ and $\sigma_B = 0.0465$ (7.8). Attaching this model parameter B to the prediction equation, Eq. (7.39) becomes

$$R = BFA_s d\left[1 - \frac{0.77 FA_s}{bdC}\right] \tag{7.40}$$

From the given data, $\mu_b = 300 + 10.29 = 310.29$ mm and $\mu_d = 550 + 6.25 = 556.25$ mm.

Using the Monte Carlo technique, random deviates of various variables are generated (B is assumed to follow normal) and then, using the same in the prediction equation, sample values of R are generated.

Generally, the values of R are normalized with its corresponding nominal value R_n, so that the statistics of R of different designs could be compared. R_n is obtained by substituting the nominal values of the variables in the prediction equation. For this problem,

$$R_n = (1.0)(415)(1039.5)(550)\left[1 - \frac{0.77 \times 415 \times 1039.5}{300 \times 550 \times 15}\right]$$

$$= 2.055 \times 10^8 \text{ N mm}$$

Hence, instead of studying the distribution of R, the distribution of R/R_n is studied. It is to be noted that R_n is deterministic and is constant for a particular design. The frequency distribution of the generated samples of R/R_n and the statistics of R/R_n are given in Fig. 7.4. It is found that the normal distribution fits the generated data well (based on the chi-square test at five per cent level of significance).

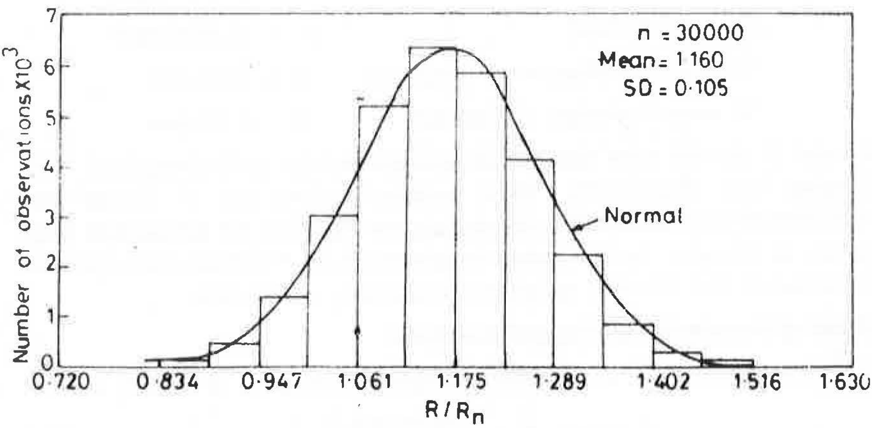


FIG. 7.4 Frequency distribution of the resistance of RCC beam—Example 7.9

During the reliability analysis of the present designs, the statistics of the strengths of members for various combinations of basic variables for each failure criteria (shear, flexure, torsion, etc.) are studied in detail using the Monte Carlo technique, and then fixed. To be consistent, Ellingwood, et al., (7.8) have fitted a normal distribution to the lower tail below five per cent fractile of the generated strength distribution, and the statistics (mean and standard deviation) of R/R_n are established. Typical values of resistance statistics of RCC members, established for Indian conditions, are given in Table 7.1 (7.9). In Table 7.1, γ_R is the ratio of the design value of R to its nominal value.

TABLE 7.1 *Typical resistance statistics of RCC members*

Member	Steel grade	Concrete grade	μ_{R/R_n}	δ	γ_R
Slabs					
One way (SS)	Fe 250	M 15	1.433	0.124	
	Fe 415	M 15	1.275	0.124	Range
Two way (SS)	Fe 415	M 15	1.281	0.124	0.835–0.865
One way (C)	Fe 415	M 15	1.263	0.136	Average
Two way (C)	Fe 415	M 15	1.286	0.129	0.85
Beams (flexure)					
Singly reinforced	Fe 250	M 15	1.288	0.104	
	Fe 415	M 20	1.179	0.103	Range
	Fe 415	M 25	1.169	0.101	0.835–0.845
	Fe 415	M 15*	1.197	0.105	Average
Doubly reinforced	Fe 415	M 15	1.151	0.103	0.84
Beams (shear)					
	Fe 250	M 15	1.355	0.166	Range
	Fe 415	M 15	1.277	0.165	0.855–0.865
					Average
					0.86
Columns					
Compression failure	Fe 415	M 20	1.29	0.152	Range
	Fe 415	M 20*	1.38	0.224	0.68–0.79
					Average
					0.725
Tension failure	Fe 415	M 20	1.19	0.13	Range
	Fe 415	M 20*	1.22	0.15	0.68–0.89
					Average
					0.8

Note: SS = Simply supported
C = Continuous
* = Indicates nominal mix

Sometimes in engineering problems we may have to deal with situations while studying the performance of a system under two failure criteria or two different designs when they are correlated. Under such conditions the correlated sampling technique may be used. This is illustrated in the following example.

EXAMPLE 7.10 Consider the portal frame shown in Fig. 7.5. Consider the two failure modes shown in Figs. 7.5b and 7.5c. It is given that

$$\mu_{M3} = \mu_{M4} = \mu_{M5} = 300 \text{ kN m}$$
$$\sigma_{M3} = \sigma_{M4} = \sigma_{M5} = 30 \text{ kN m}$$
$$\mu_{M1} = \mu_{M2} = \mu_{M6} = \mu_{M7} = 50 \text{ kN m}$$
$$\sigma_{M1} = \sigma_{M2} = \sigma_{M6} = \sigma_{M7} = 5 \text{ kN m}$$

$$\mu_Q = 180 \text{ kN} \qquad \sigma_Q = 40 \text{ kN}$$
$$\mu_H = 40 \text{ kN} \qquad \sigma_H = 10 \text{ kN}$$

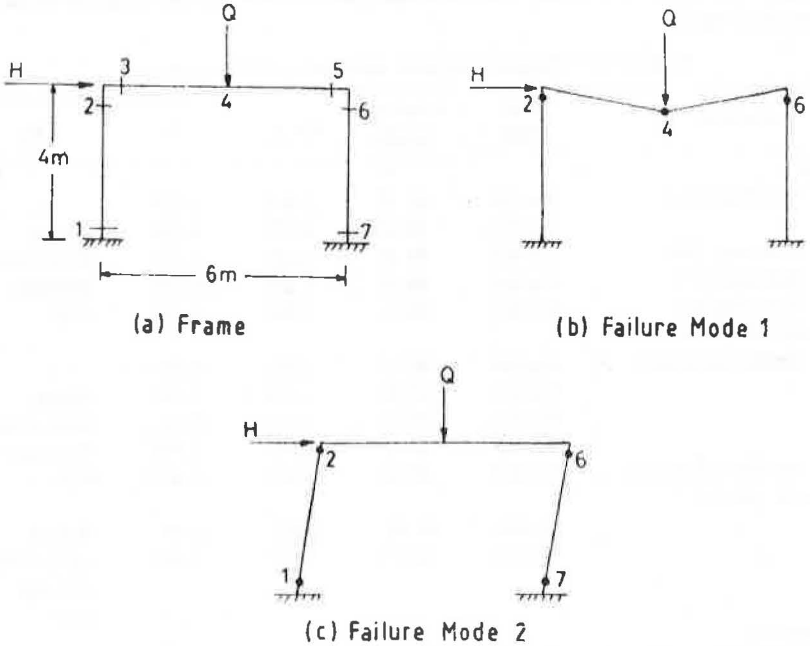


FIG. 7.5 Correlated failure modes—Example 7.10

where M_i is the plastic moment capacity of section i. All variables are normally distributed.

Using the mechanism method of analysis (7.10), sefety margin equations for the two failure modes can be written as

$$Z_1 = M_2 + 2M_4 + M_6 - 3Q \tag{7.41}$$
$$Z_2 = M_1 + M_2 + M_6 + M_7 - 4H \tag{7.42}$$

The probability of failure of the frame under failure mode i is

$$p_{fi} = P(Z_i < 0)$$

The probability of failure of the frame under failure modes Z_1 and Z_2 is

$$p_{f12} = P(Z_1 < 0 \cap Z_2 < 0) \tag{7.43}$$

The problem is to generate the joint distribution of Z_1 and Z_2 and then calculate p_{f12}.

It can be seen that Z_1 and Z_2 are correlated as they depend on the same basic variables M_2 and M_6. The correlation sampling technique can be used to generate the joint distribution of Z_1 and Z_2, and to calculate p_{f12}. The procedure is to generate normal deviates of M_1, M_2, M_4, M_6, M_7, Q and H using their respective parameters. Substituting the generated deviate of each variable in the equations for Z_1 and Z_2, the random deviates of Z_1 and Z_2 are generated. While generating random values z_1 and z_2 for Z_1 and Z_2, a count is made when $z_1 < 0$ and $z_2 < 0$ are simultaneously observed. The process is repeated for generating a number of samples. The procedure is outlined in the flow chart given in Fig. 7.6, where the number of simulations has been fixed at 20000.

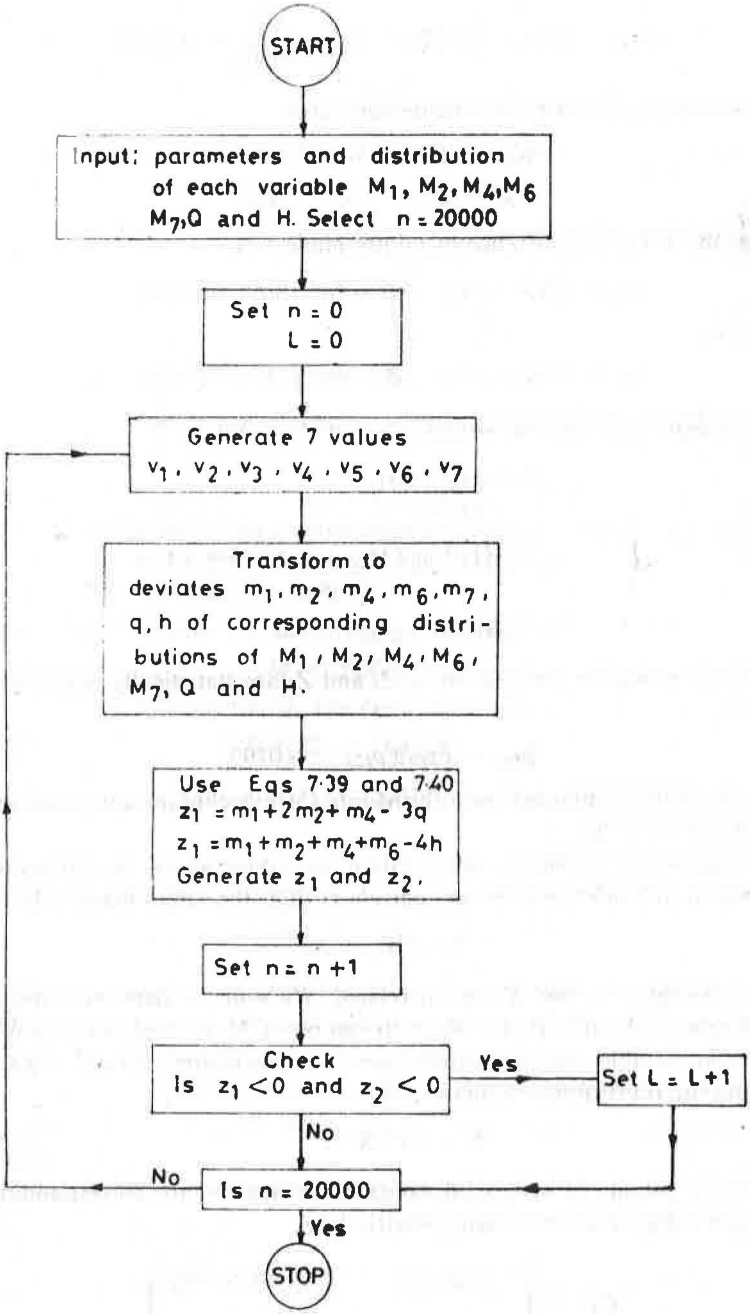


FIG. 7.6 Flow chart—Example 7.10

Result:
Number of samples for the condition
($z_1 < 0$ and $z_2 < 0$) = 374
Hence

$$p_{f12} = P(Z_1 < 0 \cap Z_2 < 0) = \frac{374}{20000} = 0.0187$$

Let us compare this with the theoretical value:

$$\mu_{Z1} = 160 \qquad \sigma_{Z1} = 134$$
$$\mu_{Z2} = 40 \qquad \sigma_{Z2} = 41.2$$

The probability of occurrence of failure mode 1 is

$$p_{f1} = P(Z_1 < 0) = \Phi(-160/134) = 0.1162$$

Similarly,

$$p_{f2} = P(Z_2 < 0) = \Phi(-40/41.2) = 0.1658$$

The correlation coefficient between Z_1 and Z_2 is (Eq. 3.77)

$$\begin{aligned}\rho &= \frac{\text{Cov}\,(Z_1, Z_2)}{\sigma_{Z1}\sigma_{Z2}} \\ &= \frac{(1)(1)\,\text{Var}\,(M_2) + (1)(1)\,\text{Var}\,(M_6)}{\sigma_{Z1}\sigma_{Z2}} \\ &= 0.009\end{aligned}$$

This being negligible, and assuming Z_1 and Z_2 are statistically independent, we have

$$p_{f12} = (p_{f1})(p_{f2}) = 0.0193$$

The value 0.0187 obtained from the Monte Carlo technique agrees well with the theoretical value.

In engineering problems, quite often we come across situations when variables in the safety margin are correlated. Let the safety margin M be

$$M = X_1 - X_2$$

The variables X_1 and X_2 are correlated. We want to determine the joint distribution of X_1 and X_2, i.e. the distribution of M. In such situations, the correlated variables are first transformed to uncorrelated variables Y_1 and Y_2 using the transformation matrix $[T]$.

$$\mathbf{Y} = [T]^t\mathbf{X} \tag{7.44}$$

where each column of matrix $[T]$ contains an eigen vector corresponding to the eigen value of the covariance matrix $[C_X]$.

$$[C_X] = \begin{bmatrix} \text{Var}\,(X_1) & \text{Cov}\,(X_1, X_2) \\ \text{Cov}\,(X_1, X_2) & \text{Var}\,(X_2) \end{bmatrix}$$

If λ_1 and λ_2 are the eigen values of the matrix $[C_X]$ and $\mathbf{e}_1$ and $\mathbf{e}_2$ are the corresponding eigen vectors,

$$[T] = [\mathbf{e}_1, \mathbf{e}_2] = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

The expected value and covariance of variables Y_1 and Y_2 are

$$E(\mathbf{Y}) = [T]'E(\mathbf{X}) \tag{7.45}$$

$$[C_Y] = [T]'[C_X][T] \tag{7.46}$$

where $[C_Y]$ is the covariance matrix of the variables Y_1 and Y_2. The diagonal elements of $[C_Y]$ are Var $(\mathbf{Y})$ which are equal to the eigen values of $[C_X]$. $[T]'$ is read as the transpose of $[T]$.

Var $(\mathbf{Y})$ is nothing but a matrix having diagonal elements equal to the eigen values and other terms zero. That is

$$\text{Var}(\mathbf{Y}) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{7.47}$$

Since $[T]$ is an orthogonal matrix,

$$\mathbf{X} = [T]\mathbf{Y} \tag{7.48}$$

Hence, the given equation for M can be written in terms of the uncorrelated variables $\mathbf{Y}$. Knowing the mean and standard deviation of $\mathbf{Y}$, the sample values for M can be generated using the Monte Carlo method. This is illustrated in the following example:

EXAMPLE 7.11 Consider the safety margin equation

$$M = X_1X_2 - X_3$$

where X_1 and X_2 are correlated. The covariance matrix is given as

$$[C_X] = \begin{bmatrix} 0.0222 & 0.0111 & 0 \\ 0.0111 & 0.011 & 0 \\ 0 & 0 & 0.0308 \end{bmatrix}$$

$$\boldsymbol{\mu}_X = \begin{bmatrix} 1.222 \\ 1.050 \\ 0.620 \end{bmatrix} \qquad \boldsymbol{\sigma}_X = \begin{bmatrix} 0.149 \\ 0.105 \\ 0.1755 \end{bmatrix}$$

It is given that all X_i are normally distributed. The problem is to determine the distribution of M.

Eigen values of the matrix $[C_X]$ are $\lambda_1 = 0.02903$; $\lambda_2 = 0.004167$; $\lambda_3 = 0.0308$. (*Note*: The computation of eigen values is illustrated in Example 8.11).

The corresponding normalized eigen vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 0.8516 \\ 0.5242 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} -0.5242 \\ 0.8516 \\ 0 \end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the transformation matrix is

$$[T] = \begin{bmatrix} 0.8516 & -0.5242 & 0 \\ 0.5242 & 0.8516 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using Eqs. (7.43) and (7.44),

$$\mu_Y = \begin{bmatrix} 1.591 \\ 0.2536 \\ 0.62 \end{bmatrix} \qquad \sigma_Y = \begin{bmatrix} 0.1704 \\ 0.06456 \\ 0.1755 \end{bmatrix}$$

Using Eq. (7.46),

$$X_1 = 0.8516\, Y_1 - 0.5242 Y_2$$
$$X_2 = 0.5242\, Y_1 + 0.8516 Y_2$$
$$X_3 = Y_3$$

Hence

$$M = (0.8516 Y_1 - 0.5242 Y_2)(0.5242 Y_1 + 0.8516 Y_2) - Y_3$$

Now Y_1, Y_2 and Y_3 are independent variables. Since X_1, X_2 and X_3 are normal, Y_1, Y_2 and Y_3 are also normal. Knowing the mean and standard deviation of Y_i, the normal deviates of Y_i can be generated. Using the usual Monte Carlo technique, the required samples for M can now be generated to study the distribution and establish the mean and standard deviation of M. Figure 7.7 shows the generated cumulative distribution of M.

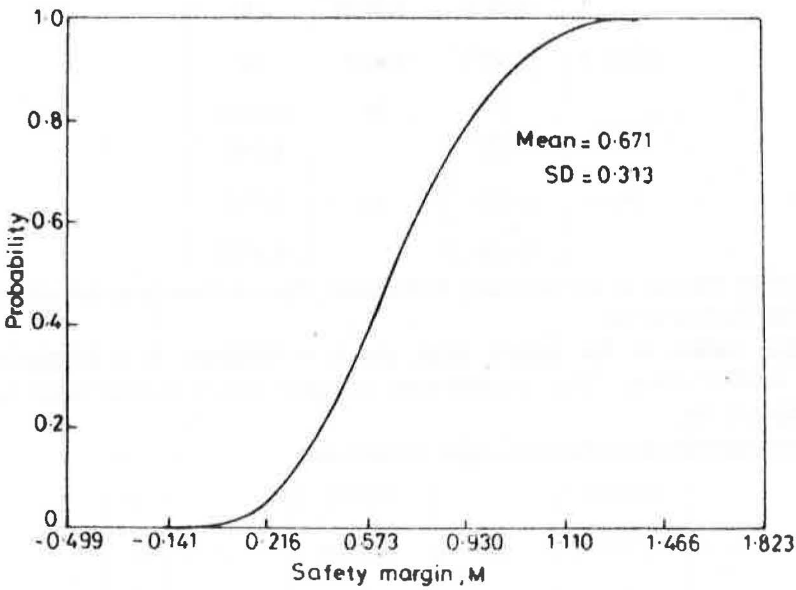


FIG. 7.7 CDF of safety margin with correlated variables—Example 7.11

REFERENCES

7.1 Ranganathan, R., "Reliability Analysis and Design of Prestressed Concrete Beams at Different Limit States", *A Ph.D. Thesis,* Civil Engineering Dept., IIT, Kanpur, May 1976.

7.2 Ranganathan, R. and P. Dayaratnam, "Reliability Analysis of Prestressed Concrete Beams", ***Journal of Bridge and Structural Engineer,*** Vol. 8, No. 2, June 1978, pp. 11–24.

7.3 Naylor, T.E., ***Computer Simulation Experiments with Models of Economical Systems,*** John Wiley, New York, 1971.

7.4 Warner, R.F. and A.P. Kabaila, "Monte Carlo Study of Structural Safety", ***Journal of Struct. Div., Proc. ASCE,*** Vol. 94, ST-12, Dec. 1968, pp. 2847–2860.

7.5 Philips Don T., A. Ravindran, and J.J. Solberg, ***Operations Research: Principles and Practice,*** John Wiley, New York, 1976.

7.6 Ang, A.H.S. and W.H. Tang, ***Probability Concepts in Engineering Planning and Design – Vol. II,*** John Wiley, New York, 1984.

7.7 Shooman, M.L., ***Probabilistic Reliability: An Engineering Approach,*** McGraw-Hill, New York, 1968.

7.8 Ellingwood, B.R., T.V. Galambos, J.G. McGregor and C.A. Cornell, "Development of a Probability Based Load Criterion for American National Standard A58", NBS special publication 577, U.S. Deptt. of Commerce, Washington, D.C., June 1980.

7.9 Padmini Chikkodi and R. Ranganathan, "Partial Safety Factors for RCC Design", *International Journal of Structures,* Vol. 8, July-Dec. 1988, pp. 127–149.

7.10 Neal, B.G., *The Plastic Methods of Structural Analysis,* Halsted, 3rd edition, 1977.

EXERCISE

7.1 It is given that

$$Y = X_1 X_2$$

where X_1 and X_2 are statistically independent lognormal variates. Given the parameters

$$\tilde{X}_1 = 10 \qquad \tilde{X}_2 = 5$$

$$\sigma_{\ln X_1} = 0.3 \qquad \sigma_{\ln X_2} = 0.05$$

determine the distribution of Y using the Monte Carlo method and check whether it is lognormal with parameters

$$\tilde{Y} = 50 \quad \text{and} \quad \sigma_{\ln Y} = 0.304$$

7.2 If the variable Y is

$$Y = X_1 + X_2$$

where X_1 and X_2 are exponentially distributed independent variates with respective parametres λ_1 and λ_2, where being 6 and 12 respectively, determine the distribution of Y using the Monte Carlo method and check whether it follows the exponential distribution with parameters $\lambda = 4$ and $\mu_Y = 1/4$

7.3 The annual maximum wind speed observed at a station follows the Type 1 extremal (largest) distribution with parameters

$$u = 81.4 \text{ kmph} \qquad \alpha = 0.126$$

Determine the distribution of a 20 year maximum wind and the probability of the lifetime maximum wind speed exceeding the specified design speed = 120 kmph. Use the Monte Carlo method.

7.4 The total load Y on a structure is given by

$$Y = D + L$$

where D and L are correlated with the correlation coefficient 0.5. It is given that

$$\mu_D = 50 \qquad \mu_L = 100$$

$$\sigma_D = 5 \qquad \sigma_L = 40$$

Generate the distribution of Y if D and L are normally distributed. Check whether it is normal.

7.5 The distribution of L_{apt} follows the gamma distribution with parameters

$$\lambda = 23.87 \qquad k = 0.328$$

Determine the distribution of the lifetime maximum live load for 10 occupancy changes during the life of the building using the Monte Carlo technique.

7.6 The ultimate strength of an axially loaded short RCC column is given by

$$R = kCA + Ys$$

where k is a constant, C is the cube strength of concrete, A is the area of concrete, Y is the yield strength of steel, and s is the area of steel. It is given that $k = 0.67$ and $s = 1250$ mm². Variables C, Y and A follow uniform distributions as given below:

$$f_C(c) = -\frac{1}{c_1 - c_2} \qquad c_1 \leqslant C \leqslant c_2$$

$$f_Y(y) = -\frac{1}{y_1 - y_2} \qquad y_1 \leqslant Y \leqslant y_2$$

$$f_A(a) = -\frac{1}{a_1 - a_2} \qquad a_1 \leqslant A \leqslant a_2$$

where $c_1 = 18$ N/mm² $\qquad c_2 = 25$ N/mm²

$y_1 = 420$ N/mm² $\qquad y_2 = 460$ N/mm²

$a_1 = 1000$ cm² $\qquad a_2 = 1100$ mm²

Determine the distribution of R using the Monte Carlo technique.

8

Level 2 Reliability Methods

8.1 INTRODUCTION

The Joint Committee on Structural Safety (8.1) classified the structural reliability analysis and the safety checking into three groups. They are termed as Level 1, Level 2, Level 3 methods. The levels are defined as follows (8.1, 8.2).

Level 1

A design method in which appropriate levels of structural reliability are provided on a structural element basis (exceptionally on a structural basis) by the specification of a number of partial safety factors, related to some predefined characteristic values of the basic variables.

Level 2

A design method incorporating safety checks only at a selected point (or points) on the failure boundary (as defined by the appropriate limit state equation in the space of the basic variables) – rather than as a continuous process, as in Level 3.

Level 3

Safety checking based on 'exact' probabilistic analysis for whole structural systems or structural elements, using a full distributional approach based on failure probabilities, possibly being derived from optimisation studies or assessed by other approach criteria.

The present structural design (8.3) with explicit consideration of the number of limit states (being called as limit state design) is nothing but Level 1 design. It is advocated that the present design be called as Level 1 design. The limit state is a criterion to define a particular failure or performance condition. In Level 2 methods, certain idealisations and assumptions are used. Mean values and variances of the random variables only are required. In advanced Level 2 methods, distributions also can be taken care of in-an-approximate way. Reliability levels are defined by safety indices or equivalent "operational" or "notional" probabilities. Level 2 methods are approximate compared to Level 3 methods where full joint probabilistic description of the random variables are used, and they are purely probabilistic methods and are exact in estimating the reliability. It is recognised that Level 3 methods will be used rarely – for checking special structures or at research level. Level 2 methods are more practical-oriented and are quite suitable for design. They are suitable for calibrating codes on reliability basis. Level 2 methods will be used by committees engaged in calibrating codes for the evaluation of partial safety factors in a rational manner. It is realised that structural designers will be working with Level 1 methods of checking. It is also to be understood that Level 1 method is not a reliability method.

This chapter deals with Level 2 methods (including advanced Level 2 methods) of reliability analysis.

8.2 BASIC VARIABLES AND FAILURE SURFACE

In any engineering problem, several random variables are involved. In structural engineering problems, geometric parameters of the section (i.e. dimensional variations), physical properties of the materials (cube strength of concrete, yield strength of steel, Young's modulus of steel and concrete, etc.) and loads (live load on floors, wind load, etc.) coming on structures are subjected to random variations. If the coefficient of variation of a random variable is very small (e.g. dimensional variations in many cases), probably this may be ignored and the variable may be considered as deterministic. Hence in any engineering problem, the parameters which are to be considered as random variables are initially fixed and those random variables are called as basic variables. Let these basic variables be $X_1, X_2, \ldots, X_n$. Any equation that is developed for a particular limit state condition (failure condition) of the structure will be interconnecting these basic variables and hence it is a function of these variables.

Let this function be

$$g(X_1, X_2, \ldots, X_n) \tag{8.1}$$

This function is called a failure function. This is nothing but representing the margin of safety, M, which can be written as

$$M = R - S \tag{8.2}$$

where the resistance R and the action S will be in terms of the basic variables $X_1, X_2, \ldots, X_n$. Hence,

$$M = g(X_1, X_2, \ldots, X_n) \tag{8.3}$$

When this failure function is made equal to zero, i.e.

$$g(X_1, X_2, \ldots, X_n) = 0$$

it is called a failure surface (or limit state surface). The safety is ensured by specifying a small value for the probability of reaching a particular limit state. The magnitude assigned depends on the serviceability of the consequences of reaching the particular limit state. If $f_{\mathbf{X}}(\mathbf{x})$ is the probability density function of the jointly distributed variables $X_1, X_2, \ldots, X_n$, then the probability of failure (or probability of reaching the limit state) is

$$p_f = \int \int \int_{g<0} \cdots \int f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{8.4}$$

where
$$\mathbf{X} = (X_1, X_2, X_3, \ldots, X_n)$$
$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$$
$$d\mathbf{x} = (dx_1, dx_2, \ldots, dx_n)$$

The multiple integral is to be evaluated over the region $g < 0$.

The failure surface equation divides the design space into two regions, viz. (i) safe and (ii) unsafe failure regions. For the two variable case, i.e. if the failure function is $g(X_1, X_2)$, this is shown in Fig. 8.1. It may be noted that the same failure surface may be represented by different equivalent failure functions.

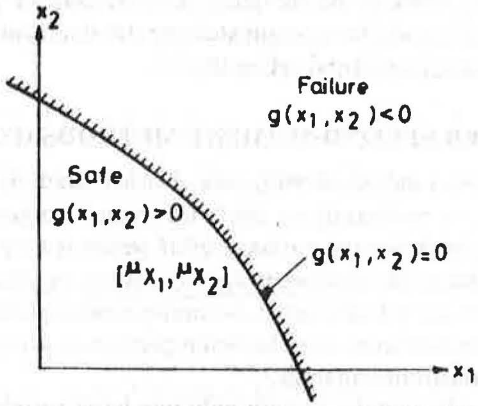


FIG. 8.1 Concept of design space, failure surface and failure and safe regions

Recall the fundamental case–A structure with resistance R subjected to an action S – discussed in Chapter 6:
$$M = g(R, S) = R - S$$
the failure surface equation is
$$g(R, S) = R - S = 0$$
It has already been derived [Eq. (6.12)], assuming R and S independent, i.e.
$$\boxed{p_f = \int_{-\infty}^{\infty} f_S(x) F_R(x)\, dx} \tag{8.5}$$

Equation (8.5) is a particular case of Eq. (8.4) and differs in two main respects. Equation (8.5) is not expressed in terms of the basic variables X_i; but in terms of state variables R and S. Equation (8.5) is concerned with a specific failure mode related to the form of R and S. In general, R and S will be in terms of the basic variables X_i. The PDF of R and S will depend on the PDF of individual basic variables and the nature of functions relating them to particular state variables R and S. In many practical cases, R and S will be related to some of the same basic variables and hence will be correlated. Initially, the structural safety was assessed using Eq. (8.5), ignoring correlation between R and S, if it exists. Use of Eq. (8.5) is not satisfactory because of the lack of statistical data for the variables R and S. If the distributions of R and S are directly known and if they are uncorrelated, Eq. (8.5) will give the exact value of the probability of failure.

The probability of failure provides a basis for quantifying structural reliability. All uncertainties in the joint probability law of all basic variables (in the fundamental case, R and S) must be known. However, in practice, these probability laws are seldom precisely known due to general scarcity of data. In many cases, the probability laws of individual basic variables will not be known and it may be difficult to obtain. The joint distribution of all the basic variables, in general, is impossible to get in the field. If the failure function is highly nonlinear, it may be difficult to numerically evaluate the integral [Eq. (8.4)] even if the marginal distributions of the variables are known. These difficulties have motivated the development of approximate methods of evaluating structural reliability.

8.3 FIRST-ORDER SECOND-MOMENT METHODS (FOSM)

In these metods, the random variables are characterized by their first and second moments. In evaluating the first and second moments of the failure function (i.e. say, the mean and variance of M which is a nonlinear function of the basic variables), the first order approximation is used. That is why these methods are called first-order second-moment methods. In the case of nonlinear failure functions, linearisation is performed using Taylor's series expansion in the reliability analysis.

Consider the fundamental case with only two basic variables R and S:

$$p_f = P(R < S)$$

$$M = g(R, S) = R - S \tag{8.6}$$

The failure surface equation is

$$R - S = 0 \tag{8.7}$$

Cornell (8.3) first defined the reliability index β as

$$\boxed{\beta = \frac{\mu_M}{\sigma_M}} \tag{8.8}$$

where μ_M and σ_M are the mean value and standard deviation of M. That is, β is the reciprocal of the coefficient of variation in M. The concept of β is illustrated in Fig. 8.2a which shows the PDF of M for the fundamental case – two variable problem. The safety is defined by the condition $M > 0$ and therefore, failure by $M < 0$. The reliability index may be thought of as the distance from the origin ($M = 0$) to the mean μ_M measured in standard deviation units. As such, β is a measure of the probability that M will be less than zero. If

$$\mu_M - \beta\sigma_M \geqslant 0 \tag{8.9}$$

then the reliability in terms of the safety index is atleast β.

When both R and S are normal and independent,

$$\mu_M = \mu_R - \mu_S \qquad \sigma_M = (\sigma_R^2 + \sigma_S^2)^{1/2}$$

$$\beta = \frac{\mu_R - \mu_S}{(\sigma_R^2 + \sigma_S^2)^{1/2}} \tag{8.10}$$

When both R and S are lognormal and independent, the alternative formulation for failure [Refer Eq. (6.9)] is taken. That is, for failure

$$\left(\frac{R}{S}\right) < 1$$

$$\ln\left(\frac{R}{S}\right) < 0$$

The failure surface equation is

$$M = \ln\left(\frac{R}{S}\right) = 0$$

Using the small variance approximations,

$$\mu_M = E\left[\ln\left(\frac{R}{S}\right)\right] \simeq \ln\left(\frac{\mu_R}{\mu_S}\right)$$

$$\sigma_M^2 = \mathrm{Var}\left[\ln\left(\frac{R}{S}\right)\right] \simeq (\delta_R^2 + \delta_S^2)$$

$$\beta = \frac{\ln(\mu_R/\mu_S)}{(\delta_R^2 + \delta_S^2)^{1/2}} \tag{8.11}$$

The above format of Eq. (8.11) (the corresponding reliability concept depicted in Fig. 8.2b) has been used for the development of probability based load and resistance factors for the design of steel structures (8.5).

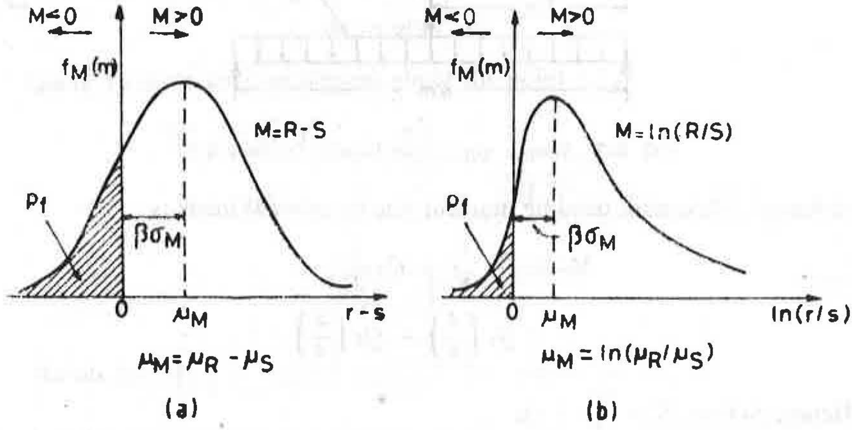


FIG. 8.2 Concept of reliability index (a) $M = R - S$; (b) $M = \ln(R/S)$

If the safety margin is a linear function of basic variables and if basic variables are normally distributed, the safety margin M is also normally distributed.

Let

$$M = b_0 + b_1X_1 + b_2X_2 + \ldots + b_nX_n \tag{8.12}$$

Using Eqs. (3.77) and (3.78),

$$\mu_M = b_0 + \sum_{i=1}^{n} b_i \mu_i \tag{8.13}$$

$$\sigma_M^2 = \sum_{i=1}^{n} b_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \rho_{ij} b_i b_j \sigma_i \sigma_j \tag{8.14}$$

where b_0 and b_i are constants and ρ_{ij} is the correlation coefficient between X_i and X_j, and $\mu_i = \mu_{X_i}$ and $\sigma_i = \sigma_{X_i}$. The probability of failure is related to the reliability index as follows:

$$p_f = \Phi(-\beta) \tag{8.15}$$

or

$$\beta = -\Phi^{-1}(p_f) \tag{8.16}$$

For a linear combination of the normally distributed variables, using β the true value of reliability can be obtained.

EXAMPLE 8.1 Calculate the reliability index of the beam (against the limit state of collapse in flexure), shown in Fig. 8.3, subjected to a self-weight Q_1 and a live load Q_2. The flexural resisting moment capacity of the beam is R. It is given that

$$\mu_{Q_1} = 400 \text{ N} \qquad \sigma_{Q_1} = 10 \text{ N}$$

$$\mu_{Q_2} = 5000 \text{ N} \qquad \sigma_{Q_2} = 2000 \text{ N}$$

$$\mu_R = 10000 \text{ Nm} \qquad \sigma_R = 1000 \text{ Nm}$$

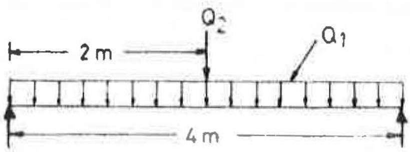


FIG. 8.3 Simply supported beam—Example 8.1

Solution Maximum bending moment due to external loads is

$$M_c = Q_1 \frac{l}{8} + Q_2 \frac{l}{4}$$

$$= Q_1 \left(\frac{4}{8}\right) + Q_2 \left(\frac{4}{4}\right)$$

Hence, Action $S = \frac{Q_1}{2} + Q_2$

The failure function $(R - S)$ is

$$g(Q_1, Q_2, R) = R - \frac{Q_1}{2} - Q_2$$

This is a linear function of variables R, Q_1 and Q_2.

$$M = R - \frac{Q_1}{2} - Q_2$$

Using Eqs. (8.13) and (8.14) and assuming R, Q_1 and Q_2 are mutually independent,

$$\mu_M = \mu_R - \frac{1}{2}\mu_{Q_1} - \mu_{Q_2}$$

$$\sigma_M^2 = \sigma_R^2 + \left(\frac{1}{2}\right)^2 \sigma_{Q_1}^2 + \sigma_{Q_2}^2$$

Substituting the given data, we have

$$\mu_M = 10 - \frac{1}{2}(0.4) - 5 = 4.8 \text{ kN}$$

$$\sigma_M^2 = (1)^2 + \left(\frac{1}{2}\right)^2 (0.01)^2 + (2)^2$$

$$\sigma_M = 2.236 \text{ kN}$$

Hence the reliability index is

$$\beta = \left(\frac{4.8}{2.236}\right) = 2.147$$

It has so far been assumed that the failure function is a linear combination of the basic variables. However, this may not be true most of the times in practical cases. If the function for M is nonlinear, the approximate values of μ_M and σ_M are obtained using Taylor's series expansion of linearised safety margin M. Let

$$M = g(X_1, X_2, \ldots, X_n)$$

Using Taylor's series expansion about the point

$$\mathbf{X}^* = (X_1^*, X_2^*, \ldots, X_n^*)$$

$$M = g(X_1^*, X_2^*, \ldots, X_n^*) + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i}\bigg|_{X^*}\right)(X_i - X_i^*)$$

$$+ \sum_{i=1}^{n} \left(\frac{\partial^2 g}{\partial X_i^2}\bigg|_{X^*}\right)\frac{(X_i - X_i^*)^2}{2} + \ldots \quad (8.17)$$

Recall that $\left(\frac{\partial g}{\partial X_i}\right)_{X^*}$ means that $\frac{\partial g}{\partial X_i}$ is evaluated at $\mathbf{X}^*$.

Retaining only the linear terms, we get

$$M \simeq g(X_1^*, X_2^*, \ldots, X_n^*) + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)_{X^*}(X_i - X_i^*) \quad (8.18)$$

In the case of mean value methods, the point $X_i^* = \mu_{X_i} = \mu_i$. That is, the expansion is about the mean point. In such a case, for Eq. (8.18),

$$\mu_M = E[g(\mathbf{X})] \simeq g(\mu_1, \mu_2, \ldots, \mu_n) + 0 \quad (8.19)$$

as
$$E(X_i - \mu_i) = 0$$

$$\sigma_M^2 = \text{Var}\,[g(\mathbf{X})] \simeq \text{Var}\,[g(\mu_1, \ldots \mu_n)]$$
$$+ \text{Var}\left[\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\bigg|_{\mu}\right)(X_i - \mu_i)\right]$$

$\partial g/\partial X_i|_\mu$ means that $\partial g/\partial X_i$ is evaluated at $\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}$.

Since $\text{Var}\,[g(\mu_1, \mu_2, \ldots, \mu_n)] = 0$, and assuming X_i are uncorrelated,

$$\sigma_M^2 = \text{Var}\,[g(\mathbf{X})] \simeq \sum_{i=1}^{n}\left[\frac{\partial g}{\partial X_i}\bigg|_{\mu}\right]^2(\sigma_i)^2 \tag{8.20}$$

where $\sigma_i = \sigma_{X_i}$. It is to be noted that both μ_M and σ_M are only first order approximations.

If the second order terms in Eq. (8.17) are taken into account, the second order approximation of μ_M is obtained as

$$\mu_M \simeq g(\mu_1, \mu_2, \ldots, \mu_n) + \sum_{i=1}^{n}\frac{\partial^2 g}{\partial X_i^2}\bigg|_{\mu}(\sigma_i^2/2) \tag{8.21}$$

Even in the second order approximation of μ_M, only the mean value and variance are required. Hence in practice, the second order approximation for μ_M and first order approximation for σ_M are used. However, in Level 2 methods the nonlinear function is linearized retaining only linear terms in the Taylor's series expansion, and hence the first order approximate values of μ_M and σ_M are used. The extent to which the values for μ_M and σ_M obtained by using Eqs. (8.19) and (8.20) are accurate, depends on the effect of neglecting higher order terms in the Taylor's series expansion and the magnitudes of the coefficient of variation of X_i. If $g(\ \ )$ is linear and the basic variables are uncorrelated, Eqs. (8.19) and (8.20) are exact. If X_i are correlated, the first order approximation of σ_M is obtained as

$$\sigma_M^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\bigg|_{\mu}\right)\left(\frac{\partial g}{\partial X_j}\bigg|_{\mu}\right)\text{Cov}\,(X_i, X_j) \tag{8.22}$$

EXAMPLE 8.2 Determine the reliability index for a steel tension member, having tensile strength R, subjected to a tensile load Q. Given:

$$\mu_R = 280\ \text{N/mm}^2 \qquad \sigma_R = 28\ \text{N/mm}^2$$
$$\mu_Q = 5000\ \text{N} \qquad \sigma_Q = 2000\ \text{N}$$
$$\mu_D = 6\ \text{mm} \qquad \sigma_D = 0.6\ \text{mm}$$

The member is circular in cross-section of diameter D.

Solution The induced stress in the member is $4Q/\pi D^2$ and this is the action (i.e. load effect). Hence the safety margin is

$$M = R - \frac{4Q}{\pi D^2}$$

Using Eqs. (8.19) and (8.20)

$$\mu_M \simeq \mu_R - \frac{4}{\pi}\left(\frac{\mu_Q}{\mu_D^2}\right)$$

$$= 280 - \frac{4}{\pi}\left(\frac{5000}{36}\right) = 103.16 \text{ N/mm}^2$$

$$\sigma_M^2 \simeq \left(\frac{\partial M}{\partial R}\bigg|_\mu\right)^2 (\sigma_R^2) + \left(\frac{\partial M}{\partial Q}\bigg|_\mu\right)^2 (\sigma_Q^2) + \left(\frac{\partial M}{\partial D}\bigg|_\mu\right)^2 (\sigma_D^2)$$

$$= (1)^2 \times (28^2) + \left(\frac{4}{\pi D^2}\right)_\mu^2 (\sigma_Q^2) + \left(\frac{8Q}{\pi D^3}\right)_\mu^2 (\sigma_D^2)$$

$$= 28^2 + \left(\frac{4}{\pi \mu_D^2}\right)^2 (\sigma_Q^2) + \left(\frac{8\mu_Q}{\pi \mu_D^3}\right)^2 (\sigma_D^2)$$

$$= (28^2) + (0.00125)(2000)^2 + (3474.7)(0.6)^2$$

$$\sigma_M^2 = 784 + 5003.5 + 1250.9$$

$$\sigma_M = 83.9$$

$$\beta = (103.16/83.9) = 1.23$$

EXAMPLE 8.3 The reliability index for the beam given in Example 8.2 is calculated using a different failure function.

Solution Let us consider Q as the action and the capacity of the section as $(R)(\pi D^2/4)$. Then the margin of safety M is

$$M = (R)\left(\frac{\pi D^2}{4}\right) - Q$$

The failure occurs when M is less than zero. The mean value and variance of M are calculated using Eqs. (8.19) and (8.20):

$$\mu_M = (\mu_R)\left(\frac{\pi \mu_D^2}{4}\right) - \mu_Q$$

$$= (280)\left(\frac{\pi 6^2}{4}\right) - 5000 = 2916.8 \text{ N}$$

$$\sigma_M^2 = \left(\frac{\partial M}{\partial R}\right)_\mu^2 (\sigma_R^2) + \left(\frac{\partial M}{\partial D}\right)_\mu^2 (\sigma_D^2) + \left(\frac{\partial M}{\partial Q}\right)_\mu^2 (\sigma_Q^2)$$

$$= \left(\frac{\pi \mu_D^2}{4}\right)^2 (\sigma_R^2) + \left(\frac{\pi \mu_R \mu_D}{2}\right)^2 (\sigma_D^2) + \sigma_Q^2$$

$$= \left(\frac{\pi 6^2}{4}\right)^2 (28^2) + (\pi\, 280 \times 6/2)^2 (0.6^2) + (2000^2)$$

$$\sigma_M = 2670.9 \text{ N}$$

Hence the reliability index β is

$$\beta = (2916.8/2670.9) = 1.0918$$

In the last two examples, the safety margin M is a nonlinear function of the basic variables. The mean value method has been used and linearization of M is about the mean values. It can be observed that different values of β have been obtained for the same problem. That is, β changes when different but equivalent nonlinear failure functions are used. This can also be demonstrated again.

For the fundamental case when R and S are uncorrelated,

$$M = R - S$$

$$\beta_1 = \frac{\mu_M}{\sigma_M} = \left[\frac{\mu_R - \mu_S}{(\sigma_R^2 + \sigma_S^2)^{1/2}}\right] \tag{8.23}$$

If the equivalent failure function, given below,

$$M = \ln\left(\frac{R}{S}\right) = \ln R - \ln S \tag{8.24}$$

is selected, we get

$$\beta_2 = \frac{\mu_{\ln(R/S)}}{\sigma_{\ln(R/S)}}$$

If linearization of the safety margin

$$M = \ln R - \ln S$$

is done about μ_R and μ_S, then

$$\beta_2 = \frac{\ln(\mu_R/\mu_S)}{(\delta_R^2 + \delta_S^2)^{1/2}} \tag{8.25}$$

It is clear that β_1 and β_2 are not equal. Hence the reliability index β, defined by the equation $\beta = \mu_M/\sigma_M$, is thus not invariant with regard to the choice of the failure function. If the linearization is done about the mean value, the method can give different values of β, that is different values of p_f, for the same problem. When the failure functions are linear functions of the basic variables, they will yield same values of β, and hence the same p_f. In general, an expansion of M about the mean point should not be used. Mean value FOSM methods have two basic shortcomings:

(i) $g(\ )$ is linearized at the mean value of basic variables. When g is nonlinear, significant errors may be introduced at increasing distances from the linearizing point by neglecting higher order terms in the Taylor's series expansion. In most structural engineering problems, the mean point is, in fact, at some distance from the failure surface $g(\ ) = 0$ and thus there are likely to be unacceptable errors in approximating the equation

$$M = g(X_1, X_2, \ldots X_n) = 0$$

by the equation

$$M \simeq g(X_1^*, X_2^*, \ldots, X_n^*) + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)_{\mathbf{X}^*} (X_i - X_i^*)$$

(ii) Mean value methods fail to be invariant to different mechanically equivalent formulations of the same problem. The lack of invariance arises because linear expansions are taken about the mean value point. The problem is avoided by linearizing $g(\ )$ at some point on the failure surface. This is because $g(\ )$ and its partial derivatives are independent of how the problem is formulated only on the failure surface $g(\ ) = 0$.

Consider again the fundamental case of the two variable problem, i.e.

$$M = R - S$$

The failure surface equation for a set of the realization of values of R and S is

$$r - s = 0 \tag{8.26}$$

The above Eq. (8.26) is shown in Fig. 8.4(a).

Let

$$Z_1 = \frac{(R - \mu_R)}{\sigma_R} \qquad Z_2 = \frac{(S - \mu_S)}{\sigma_S}$$

For a set of realization of R and S,

$$z_1 = \frac{(r - \mu_R)}{\sigma_R} \qquad z_2 = \frac{(s - \mu_S)}{\sigma_S} \tag{8.27}$$

Hence, the safety margin Eq. (8.26) becomes

z_1 $\frac{(\mu_R * \mu_s)}{\sigma_R}$

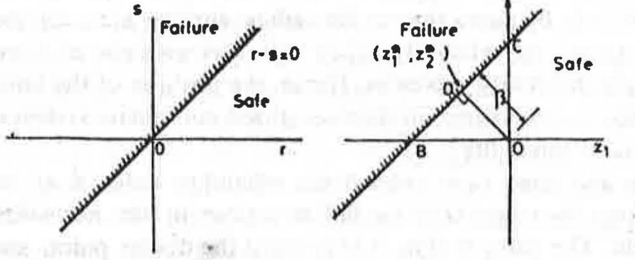
$$z_1\sigma_R + \mu_R - z_2\sigma_S - \mu_S = 0 \tag{8.28}$$

$$z_1\sigma_R - z_2\sigma_S + \mu_R - \mu_S = 0$$

The above equation is represented in Fig. 8.4b. This is in the normalized coordinate system since R and S have been normalized with respect to their corresponding mean values. The mean values Z_1 and Z_2 are equal to zero and their variances are equal to one. In Fig. 8.4b, OD is drawn perpendicular to the failure surface and it can be proved easily that $OD = \beta$.

Proof : In Fig. 8.4b,

$$OB = \frac{(\mu_R - \mu_S)}{\sigma_R}$$



(a) Original Coordinate System (b)Normalized Coordinate System

FIG. 8.4 Linear failure surface

$$OC = \frac{(\mu_R - \mu_S)}{\sigma_S}$$

$$\tan \angle OBC = \frac{\sigma_R}{\sigma_S}$$

$$\sin \angle OBC = \frac{\sigma_R}{(\sigma_S^2 + \sigma_R^2)^{1/2}}$$

$$OD = OB \sin \angle OBC$$

$$= \frac{\sigma_R}{(\sigma_S^2 + \sigma_R^2)^{1/2}}\left[\left(\frac{\mu_R - \mu_S}{\sigma_R}\right)\right]$$

$$= \frac{\mu_R - \mu_S}{(\sigma_S^2 + \sigma_R^2)^{1/2}} = \beta$$

Hence it is proved that β is the shortest distance to the linear failure surface from the origin O in the normalized coordinate system. This is used in the definition of the reliability index defined by Hasofer and Lind.

8.3.1 Hasofer and Lind's Method (8.6)

Let the failure function g be a function of independent basic variables $X_1, X_2, \ldots, X_n$, i.e. $g(X_1, X_2, \ldots, X_n)$. The basic variables are then normalized using the relationship

$$Z_i = \frac{X_i - \mu_i}{\sigma_i} \qquad i = 1, 2, \ldots, n \tag{8.29}$$

where $\mu_i = \mu_{X_i}$ and $\sigma_i = \sigma_{X_i}$. In the z coordinate system, the failure surface is a function of z_i. Using Eq. (8.29) in the failure function and equating it to zero, the failure surface equation is written in the normalized coordinate system, i.e. the z coordinate system. This failure surface also divides the design sample space into two regions, safe and failure. Because of the normalization of the basic variables,

$$\mu_{Z_i} = 0 \qquad \text{and} \qquad \sigma_{Z_i} = 1 \tag{8.30}$$

It is also to be noted that the z coordinate system has a rotational symmetry with respect to the standard deviation and the origin O will usually lie in the safe region. A two dimensional example is shown in Fig. 8.5. It is to be noted that as the failure surface $g(z_1, z_2)$ moves away from the origin, the reliability, $g(Z) > 0$, increases and as it moves closer to the origin, reliability decreases. Hence, the position of the failure surface with respect to the origin in the normalized coordinate system determines the measure of reliability.

Hasofer and Lind (8.6) defined the reliability index β as the shortest distance from the origin O to the failure surface in the normalized coordinate system. The point D (Fig. 8.5) is called the design point, and it is on the failure surface. This point is also called the check point for the safety of the structure. Now β is related to the failure surface (and not to the

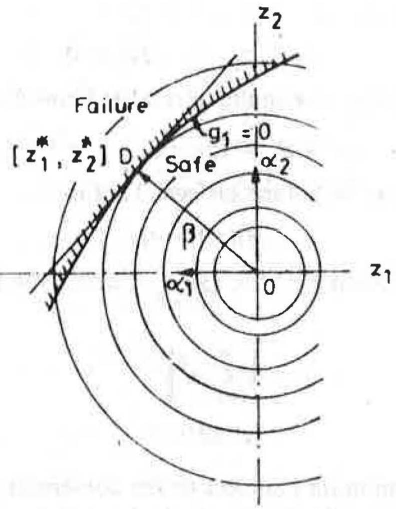


FIG. 8.5 Formulation of safety analysis in normalized coordinates

failure functions). The safety measure obtained is invariant to the failure function, since equivalent failure functions will result the same failure surface. The reliability index, $\beta = \mu_M/\sigma_M$, defined by Cornell, will coincide with the value obtained by Hasofer and Lind when the failure function is a linear function of basic variables. Hence in this method also (Hasofer and Lind), the important relation,

$$\beta = -\Phi^{-1}(p_f) \Leftrightarrow p_f = \Phi(-\beta) \tag{8.31}$$

can be used, provided the failure function is a linear function of the normally distributed basic variables.

From the above discussions, it is obvious that β defined by μ_M/σ_M can be obtained for a nonlinear function by expanding the function about the design point D. This corresponds to approximating the nonlinear failure surface by its tangent plane at the design point D as shown in Fig. 8.5. For a nonlinear failure surface, the shortest distance of the origin (normalized coordinate system) to the failure surface is not unique as in the case of a linear failure surface. The computation of the probability of failure involves numerical integration. For practical purposes, an approximation to the exact value is required. Shinozuka (8.7) has proved that the point D on the failure surface with minimum distance to the origin (normalized coordinate system) is the most probable failure point. The tangent plane to the design point D may then be used to approximate the value of β. If the failure surface is concave towards the origin, the approximation will be on the safer side, while for the surface convex towards the origin it will be on the unsafe side.

The problem therefore reduces to finding out the minimum value of the distance OD (Fig. 8.5).

Let

$$g_1(z_1, z_2, \ldots, z_n) = 0 \tag{8.32}$$

be a nonlinear failure surface in the normalized coordinate system and

$$D = \mathbf{z}^* = (z_1^*, z_2^*, \ldots, z_n^*) \tag{8.33}$$

be the design point on the failure surface. That is

$$g_1(\mathbf{z}^*) = 0$$

The distance from a point $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ on the failure surface to the origin is

$$r = \left[\sum_{i=1}^{n} z_i^2\right]^{1/2} \tag{8.34a}$$

$$= (\mathbf{z}^t\, \mathbf{z})^{1/2} \tag{8.34b}$$

The problem is to minimize r subject to the constraint $g_1(\mathbf{z}) = 0$

Using the Lagrange multiplier method, the problem can be solved. The Lagrange function L is

$$L = r + \lambda g_1(\mathbf{z})$$

$$= (\mathbf{z}^t\, \mathbf{z})^{1/2} + \lambda g_1(\mathbf{z}) \tag{8.35}$$

For minimum

$$\frac{\partial L}{\partial z_i} = \frac{z_i}{(\mathbf{z}^t\, \mathbf{z})^{1/2}} + \lambda \frac{\partial g_1}{\partial z_i} = 0 \qquad i = 1, 2, \ldots n \tag{8.36}$$

$$\frac{\partial L}{\partial \lambda} = g_1(z_1, z_2, \ldots, z_n) = 0 \tag{8.37}$$

There are $n + 1$ equations. In matrix notation, the n equations [Eq. (8.36)] can be written as

$$\frac{\mathbf{z}}{(\mathbf{z}^t\, \mathbf{z})^{1/2}} + \lambda \mathbf{G} = 0 \tag{8.38}$$

where
$$\mathbf{G}^t = \left(\frac{\partial g_1}{\partial z_1}, \frac{\partial g_1}{\partial z_2}, \ldots, \frac{\partial g_1}{\partial z_n}\right) \tag{8.39}$$

The solutions for $\mathbf{z}_*$ and λ^* are obtained as

$$\mathbf{z}_* = -\lambda^* r \mathbf{G}_* \tag{8.40}$$

$$\lambda_* = (\mathbf{G}_*^t\, \mathbf{G}_*)^{-1/2} \tag{8.41}$$

Premultiplying both sides of Eq. (8.40) by $\mathbf{G}_*^t$ and using Eq. (8.41), it is obtained as

$$r = -\frac{\mathbf{z}_*^t\, \mathbf{G}_*}{(\mathbf{G}_*^t\, \mathbf{G}_*)^{1/2}} \tag{8.42}$$

This r is the minimum distance and is equal to β. $\mathbf{G}_*$ is the gradient vector at the design point $(z_1^*, z_2^*, \ldots, z_n^*)$. In scalar form, the above equation is written as

$$\beta = -\frac{\sum_{i=1}^{n} z_i^* \left(\frac{\partial g_1}{\partial z_i}\right)_*}{\left[\sum_{i=1}^{n} \left(\frac{\partial g_1}{\partial z_i}\right)_*^2\right]^{1/2}} \tag{8.43}$$

$(\partial g/\partial z_i)_*$ means that the derivative is evaluated at $(z_1^*, z_2^*, \ldots, z_n)$. Using the minimum value of r, equal to β, in Eq. (8.40) and using Eq. (8.41), the design point on the failure surface becomes

$$\mathbf{z}_* = \frac{-\beta \mathbf{G}_*}{(\mathbf{G}_*^t \mathbf{G}_*)^{1/2}} \tag{8.44}$$

In scalar form, the components of $\mathbf{z}_*$ are

$$z_i^* = \alpha_i^* \beta \qquad i = 1, 2, \ldots, n \tag{8.45}$$

where

$$\alpha_i^* = \frac{-(\partial g_1/\partial z_i)_*}{\left[\sum_{i=1}^{n} (\partial g_1/\partial z_i)_*^2\right]^{1/2}} \tag{8.46}$$

are the direction cosines along the axes z_i.

Let the nonlinear failure surface function $g_1(z)$ be expanded around the point D using Taylor's series expansion, i.e.

$$g_1(\mathbf{z}) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\sum_{i=1}^{n} \left(\frac{\partial g_1}{\partial z_i}\right)_*^k (z_i - z_i^*)^k\right] \tag{8.47}$$

Using the linear approximation, i.e. deleting the terms with $k \geqslant 2$ in the above equation, we get

$$\boxed{g_1(\mathbf{z}) = \sum_{i=1}^{n} \left(\frac{\partial g_1}{\partial z_i}\right)_* (z_i - z_i^*)} \tag{8.48}$$

The expected value and standard deviation of the above function $g_1(\mathbf{z})$, assuming statistical independence of the variables, are

$$E[g_1(\mathbf{z})] = -\sum_{i=1}^{n} z_i^* \left(\frac{\partial g_1}{\partial z_i}\right)_* \tag{8.49}$$

$$\sigma_{g_1(z)} = \left[\sum_{i=1}^{n} \left(\frac{\partial g_1}{\partial z_i}\right)_*^2\right]^{1/2} \tag{8.50}$$

If β is taken as

$$\beta = \frac{E[g_1(\mathbf{z})]}{\sigma_{g_1(z)}} \tag{8.51}$$

$$= -\frac{\sum_{i=1}^{n} z_i^* \left(\frac{\partial g_1}{\partial z_i}\right)_*}{\left[\sum_{i=1}^{n} \left(\frac{\partial g_1}{\partial z_i}\right)_*^2\right]^{1/2}} \tag{8.52}$$

the comparison of Eqs. (8.43) and (8.52) indicates that both are same. The ratio, defined by Eq. (8.52), is also the distance from the tangent plane of the failure surface at the design point $D = \mathbf{z}^*$ to the origin in the normalized variate coordinates.

The problem of finding the minimum value of β for a nonlinear failure surface is solved iteratively. The problem can be solved in many ways. One simple method is solving the following n equations [Eq. (8.46)]

$$\alpha_i = -\frac{(\partial g_1/\partial z_i)_*}{K} \qquad i = 1, 2, \ldots, n$$

with $(n + 1)$th equation

$$g_1(\overset{\bullet}{z}_1, \overset{\bullet}{z}_2, \ldots, \overset{\bullet}{z}_n) = 0$$

where
$$K = \left[\sum_{i=1}^{n} \left(\frac{\partial g_1}{\partial z_i}\right)_*^2\right]^{1/2} \tag{8.53}$$

$$\overset{\bullet}{z}_i = \alpha_i \beta$$

and searching for directional cosines which minimize β.

The following steps are involved in the method:

1. Write the limit state equation, $g(x_1, x_2, \ldots, x_n) = 0$, in terms of the basic variables.
2. Normalize the basic variables using Eq. (8.29). $Z_i = \frac{X_i - \mu_i}{\sigma_i}$
3. Write the (limit state) failure surface equation in terms of the normalized coordinate system. i.e.

$$g_1(z_1, z_2, \ldots, z_n) = 0$$

Write expressions for $\partial g_1/\partial z_i$, $i = 1, 2, \ldots, n$

At the design point $z_i = \alpha_i \beta$. Using this, write $g_1(z)$ in terms of β and α_i. Write the equation such that

$$\beta = g(\beta, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

for computation purposes.

4. Select a value for β and values for $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfying $\Sigma \alpha_i^2 = 1$.

While choosing values for α_i, select positive values for load variables and negative values for resistance variables.

5. Start the iteration. Calculate the new value of β using the equation

$$\beta = g(\beta, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

6. Calculate

$$K = \left[\sum_{i=1}^{n} \left(\frac{\partial g_1}{\partial z_i}\right)_*^2\right]^{1/2}$$

7. Determine new values of α_i

$$\alpha_i = -\frac{1}{K}\left(\frac{\partial g_1}{\partial z_i}\right)_* \qquad i = 1, 2, \ldots, n$$

8. With these new values of β and α_i, start the next iteration. Go to step 5.

9. Stop the procedure when the values of β obtained from two successive iterations is within the acceptable error.

(*Note*: If the equation $g_1(z)$ is linear or quadratic, it may not be necessary to start the procedure assuming a value of β.)

The procedure is explained with examples.

EXAMPLE 8.4 Determine the reliability index of a simply supported I beam at the limit state of shear. The beam is subjected to a point load Q at mid-span. It is given that

$$\mu_Q = 4000 \text{ N} \qquad \sigma_Q = 1000 \text{ N}$$

$$\mu_{f_s} = 95 \text{ N/mm}^2 \qquad \sigma_{f_s} = 10 \text{ N/mm}^2$$

$$\sigma_d = 2.5 \text{ mm} \qquad \frac{d}{t_w} = 40 \qquad \mu_d = 50 \text{ mm}$$

where d is the depth of the beam, t_w is the thickness of the web, f_s is the shear strength of the material. The coefficient of variation of t_w is negligible.

Solution

$$\text{Maximum shear force} = \frac{Q}{2}$$

It is assumed that the web resists the whole shear. The beam fails in shear if

$$f_s t_w d - \frac{Q}{2} \leqslant 0$$

Hence the failure surface equation is

$$g(X) = f_s t_w d - \frac{Q}{2} = 0$$

As variation in t_w is negligible, t_w is considered as deterministic.

Let

$$z_1 = \frac{(f_s - \mu_{f_s})}{\sigma_{f_s}}$$

$$z_2 = \frac{(d - \mu_d)}{\sigma_d}$$

$$z_3 = \frac{(Q - \mu_Q)}{\sigma_Q}$$

Substituting them in the equation for $g(X)$, we get

$$g_1(z) = t_w(\sigma_{f_s} z_1 + \mu_{f_s})(\sigma_d z_2 + \mu_d) - \frac{1}{2}\sigma_Q z_3 - \frac{\mu_Q}{2} = 0$$

$\frac{4d}{40} = t_w.$

$$= \frac{50}{40}[\sigma_{f_s} z_1 \sigma_d z_2 + \sigma_{f_s} z_1 \mu_d + \mu_{f_s}\sigma_d z_2 + \mu_{f_s}\mu_d]$$

$$- \frac{\sigma_Q z_3}{2} - \frac{\mu_Q}{2} = 0$$

Substituting the given data, we have

$$g_1(\mathbf{z}) = 625z_1 + 296.88z_2 + 31.25z_1z_2 - 500z_3 + 3937.5 = 0$$

At the design point, $z_i = \beta\alpha_i$ [Eq. (8.45)].

$$g_1(\mathbf{z}) = 625\beta\alpha_1 + 296.88\beta\alpha_2 + 31.25\beta^2\alpha_1\alpha_2$$
$$-500\beta\alpha_3 + 3937.5 = 0$$

$$\beta = \frac{-3937.5}{625\alpha_1 + 296.88\alpha_2 + 31.25\beta\alpha_1\alpha_2 - 500\alpha_3} \tag{8.54}$$

Taking partial deviatives of $g_1(\mathbf{z})$,

$$\left(\frac{\partial g_1}{\partial z_1}\right)_* = (625 + 31.25z_2)_*$$
$$= 625 + 31.25\beta\alpha_2 \tag{8.55}$$

$$\left(\frac{\partial g_1}{\partial z_2}\right)_* = (296.88 + 31.25z_1)_*$$
$$= 296.88 + 31.25\beta\alpha_1 \tag{8.56}$$

$$\left(\frac{\partial g_1}{\partial z_3}\right)_* = -500 \tag{8.57}$$

Start with

$$\beta = 6 \qquad \alpha_1 = -0.58 \qquad \alpha_2 = -0.58 \qquad \alpha_3 = +0.58$$

Using these in Eqs. (8.54) to (8.57), we have

$$\beta = \frac{-3937.5}{625(-0.58) + 296.88(-0.58) + 31.25(6)(-0.58)(-0.58) - 500(0.58)}$$
$$= 3937.5/761.62 = 5.17$$

Using Eqs. (8.46) and (8.53)

$$\alpha_i = -\frac{1}{K}\left(\frac{\partial g_1}{\partial z_i}\right)_*$$

$$\alpha_1 = -\frac{1}{K}[625 + 31.25(5.17)(-0.58)] = -\frac{531.29}{K}$$

$$\alpha_2 = -\frac{1}{K}[296.88 + 31.25(5.17)(-0.58] = -\frac{203.18}{K}$$

$$\alpha_3 = -\frac{1}{K}[-500] = \frac{500}{K}$$

$$K^2 = (-531.29)^2 + (-203.18)^2 + (500)^2$$
$$= 573551.17$$
$$K = 757.33$$

Hence

$$\alpha_1 = -\frac{531.29}{757.33} = -0.702$$

$$\alpha_2 = -\frac{203.18}{757.33} = -0.263$$

$$\alpha_3 = \frac{500}{757.33} = +0.66$$

With these new values of β, α_1, α_2 and α_3, the cycle is repeated till β converges to the minimum. Summarized results are given in Table 8.1.

TABLE 8.1 *Computation of β—Example 8.4*

Variable	Iteration			
	Start	1	2	3
β	6	5.17	4.82	4.796
α_1	−0.58	−0.702	−0.738	−0.741
α_2	−0.58	−0.263	−0.241	−0.234
α_3	+0.58	+0.660	+0.63	+0.629

The solution is: $\beta = 4.796$ $\quad p_f = \Phi^{-1}(-4.796) = 6\times10^{-7}$

$\alpha_1 = -0.741 \quad \alpha_2 = -0.234 \quad \alpha_3 = 0.629$

The design point is: $z^* = (\beta\alpha_1, \beta\alpha_2, \beta\alpha_3)$

EXAMPLE 8.5 For the same failure case, in Example 8.4, determine the mean depth of the beam for a reliability index of 5. The beam is subjected to a point load Q at mid-span. It is given that

$$\mu_Q = 300 \text{ kN} \qquad \sigma_Q = 80 \text{ kN}$$

$$\mu_{f_s} = 95 \text{ N/mm}^2 \qquad \sigma_{f_s} = 10 \text{ N/mm}^2$$

$$\sigma_d = 5 \text{ mm} \qquad \frac{d}{t_w} = 40$$

Coefficient of variation of t_w is negligible.

Solution As the coefficient of variation of t_w is negligible, it is considered as deterministic.

The failure surface is

$$f_s t_w d - \frac{Q}{2} = 0 \tag{8.58}$$

Let

$$z_1 = \frac{f_s - \mu_{f_s}}{\sigma_{f_s}}$$

$$z_2 = \frac{d - \mu_d}{\sigma_d}$$

$$z_3 = \frac{Q - \mu_Q}{\sigma_Q}$$

Substituting the above equations in Eq. (8.58), we get

$$t_w[(\sigma_{f_s} z_1 + \mu_{f_s})(\sigma_d z_2 + \mu_d)] - \frac{\sigma_Q z_3}{2} - \frac{\mu_Q}{2} = 0 \qquad t_w = \frac{\mu_d}{40}$$

Using the given data, we have

$$\mu_d[50 z_1 z_2 + 10\mu_d z_1 + 475 z_2 + 95\mu_d] - 1600\times 10^3 z_3 - 6000\times 10^3 = 0 \tag{8.59}$$

At the design point, using $z_i = \alpha_i \beta$, the above equation becomes $\beta = 25$

$$\mu_d^2(50\alpha_1 + 95) + \mu_d(1250\alpha_1\alpha_2 + 2375\alpha_2) - 8000\times 10^3\alpha_3 - 6000\times 10^3 = 0 \tag{8.60}$$

Using Eq. (8.46),

$$\alpha_1 = -\frac{1}{K}\left[\frac{\mu_d}{40}(250\alpha_2 + 10\mu_d)\right] \tag{8.61}$$

$$\alpha_2 = -\frac{1}{K}\left[\frac{\mu_d}{40}(250\alpha_1 + 475)\right] \tag{8.62}$$

$$\alpha_3 = \frac{40\times 10^3}{K} \tag{8.63}$$

Start with $\alpha_1 = -0.58 \qquad \alpha_2 = -0.58 \qquad \alpha_3 = 0.58$

Substituting the values of α_1, α_2 and α_3 in Eq. (8.60),

$$(\mu_d)^2(66) + \mu_d(2795.5) - 10640\times 10^3 = 0$$

Solving the above equation,

$$\mu_d = 380.9 \text{ mm}$$

Using this value of $\mu_d = 380.9$ mm, new values of α_1, α_2 and α_3 are obtained.

Using Eqs. (8.61) to (8.63),

$$\alpha_1 = -\frac{1}{K}\left[\frac{380.9}{40}(250\times -0.58 + 10\times 380.9)\right]$$

$$= -\left(\frac{34890}{K}\right)$$

$$\alpha_2 = -\frac{1}{K}\left[\frac{380.9}{40}(250\times (0.58) + 475)\right]$$

$$= -\left(\frac{3142}{K}\right)$$

$$\alpha_3 = \frac{40\times 10^3}{K}$$

Using the relation $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$,

$$K = 51371$$

Hence $\qquad \alpha_1 = -0.656 \qquad \alpha_2 = -0.059 \qquad \alpha_3 = 0.752$

Now the whole process is repeated till the maximum value of μ_d is obtained. Summarized results are given in Table 8.2.

TABLE 8.2 *Summarized results—Example 8.5*

Variable	Start	Iteration		
		1	2	3
α_1	−0.58	−0.656	−0.729	−0.733
α_2	−0.58	−0.059	−0.083	−0.082
α_3	+0.58	+0.752	+0.688	+0.685
μ_d (mm)	380.9	420.0	422.8	423.0

The solution is:

$$\mu_d = 423 \text{ mm}$$

8.3.2 Non-normal Distributions

So far, the mean values and standard deviations of basic variables X_i only have been used in evaluating the reliability index. Probability distributions of the variables X_i have not been considered. If the safety margin equation is linear and X_i are normally distributed, the evaluated reliability index can be connected to the true value of the probability of failure of the structure (Eq. 8.15), as M is normally distributed. However, in practical situations, many of the basic variables are non-normal, e.g. wind speed, live load, strength of low strength concretes, etc. In such cases, the value of β (or p_f) can be obtained using equivalent normal distributions (8.8, 8.9) at the design point. The transformation of a non-normal variable to a normal variable at the design point is done as follows:

At the failure point (i.e. the design point D)x_i^*,

(i) the probability density ordinate of the original non-normal variable X_i is made equal to the probability density ordinate of the equivalent normal variable X_i'. That is

$$f_{X_i'}(x_i^*) = f_{X_i}(x_i^*) \tag{8.64}$$

(ii) the cumulative probability of the original non-normal variable X_i is made equal to the cumulative probability of the equivalent normal variable X_i'. That is

$$F_{X_i'}(x_i^*) = F_{X_i}(x_i^*) \tag{8.65}$$

If μ'_{X_i} and σ'_{X_i} are the unknown mean and standard deviation of X_i', then Eq. (8.65) becomes

$$F_{X_i}(x_i^*) = \Phi\left(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}\right) \tag{8.66}$$

The above equation leads to

$$\mu'_{X_i} = -\sigma'_{X_i}\Phi^{-1}[F_{X_i}(x_i^*)] + x_i^* \tag{8.67}$$

Considering Eq. (8.64),

$$f_{X_i}(x_i^*) = \frac{1}{\sigma'_{X_i}}\phi\left[\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}\right] \tag{8.68}$$

since X_i' is a normal variable.

Substituting Eq. (8.67) in Eq. (8.68), we get

$$\sigma'_{X_i} = \frac{\phi\{\Phi^{-1}[F_{X_i}(x_i^*)]\}}{f_{X_i}(x_i^*)} \tag{8.69}$$

Since F_{X_i} and f_{X_i} are given or known, the values of μ'_{X_i} and σ'_{X_i} of equivalent non-normal can be obtained using Eqs. (8.67) and (8.69). The procedure of determining β for the failure surface having non-normal basic variables involves the following steps:

(i) Write the limit state equation in terms of the basic variables, i.e.

$$g(X_1, X_2, \ldots, X_n) = 0$$

(ii) Normalize the basic variables using Eq. (8.29).

For normal variable X_i, $\quad Z_i = \dfrac{X_i - \mu_{X_i}}{\sigma_{X_i}}$

For non-normal variable X_j, $\quad Z_j = \dfrac{X_j - \mu'_{X_j}}{\sigma'_{X_j}}$

where μ'_{X_j} and σ'_{X_j} are the unknown mean and standard deviation of equivalent normal X_j' of non-normal X_j at the failure point.

(iii) Write the limit state equation in terms of the normalized variables and unknown values of

$$\mu'_{X_j} \quad \text{and} \quad \sigma'_{X_j}$$

(iv) Select values for β and $\alpha_1, \alpha_2, \ldots, \alpha_n$ as explained in the previous section and values for μ'_{X_j} and σ'_{X_j}.

(v) Start the iteration. Calculate new values of β, $\alpha_1, \alpha_2, \ldots, \alpha_n$ as explained in the previous section.

(vi) For non-normal variables (say X_j), the design point is

$$x_j^* = \mu'_{X_j} + \alpha_j\beta\sigma'_{X_j}$$

(vii) At this design point x_j^*, find new values of μ'_{X_j} and σ'_{X_j} using Eqs. (8.67) and (8.69).

Go to step 5 and repeat the procedure till β converges to the minimum.

The procedure is illustrated with the following examples.

Example 8.6 A cantilever steel beam (ISLB 450) of span l is subjected to a load P at the free end. The resisting moment capacity of a section is taken as $F_y Z$, where F_y is the yield stress and Z is the section modulus. Hence at the limit state of collapse in flexure, the safety margin can be written as

$$M = F_y Z - Pl$$

Given:

For	F_y,	$\mu_1 = 0.32$ kN/mm^2	$\sigma_1 = 0.032$ kN/mm^2
	Z,	$\mu_2 = 1400 \times 10^3$ mm^3	$\sigma_2 = 70 \times 10^3$ mm^3
	P,	$\mu_3 = 100$ kN	$\sigma_3 = 40$ kN

F_y and Z are normally distributed and P is lognormally distributed. Calculate β if $l = 2$ m.

Solution Denote

$$X_1 = F_y \qquad X_2 = Z \qquad X_3 = P$$

$$\mu_i = \mu_{X_i} \qquad \sigma_i = \sigma_{X_i}$$

Then the failure surface equation is rewritten as

$$g(\mathbf{X}) = X_1 X_2 - 2000 X_3 = 0$$

Let μ_3' and σ_3' be the mean value and standard deviation of the equivalent normal X_3' of the non-normal X_3 at the design point. Normalizing the variables

$$z_1 = \frac{X_1 - \mu_1}{\sigma_1} \qquad z_2 = \frac{X_2 - \mu_2}{\sigma_2}$$

$$z_3 = \frac{X_3 - \mu_3'}{\sigma_3'}$$

Substituting these in the failure surface equation and using the given data,

$$\begin{aligned} g_1(\mathbf{z}) &= \sigma_1\sigma_2 z_1 z_2 + \sigma_1\mu_2 z_1 + \sigma_2\mu_1 z_2 + \mu_1\mu_2 \\ &\quad - 2000(\sigma_3' z_3 + \mu_3') \\ &= 2240 z_1 z_2 + 44800 z_1 + 22400 z_2 - 2000\sigma_3' z_3 \\ &\quad + 448000 - 2000\mu_3' \end{aligned} \tag{8.70}$$

At the design point, $z_i = \alpha_i \beta$ and $g_1(\mathbf{z}) = 0$. Using these in Eq. (8.70),

$$\beta = \frac{2000\mu_3' - 448000}{2240\alpha_1\alpha_2\beta + 44800\alpha_1 + 22400\alpha_2 - 2000\sigma_3'\alpha_3} \tag{8.71}$$

Taking partial derivatives of $g_1(\mathbf{z})$,

$$\alpha_1 = -\frac{1}{K}\left(\frac{\partial g_1}{\partial z_1}\right) = -\frac{1}{K}(2240\alpha_2\beta + 44800) \tag{8.72}$$

$$\alpha_2 = -\frac{1}{K}\left(\frac{\partial g_1}{\partial z_2}\right) = -\frac{1}{K}(2240\alpha_1\beta + 22400) \tag{8.73}$$

$$\alpha_3 = -\frac{1}{K}\left(\frac{\partial g_1}{\partial z_3}\right) = -\frac{1}{K}(-2000\sigma_3') \tag{8.74}$$

As X_3 is lognormal, the parameters of X_3 are first calculated.

$$\sigma_{\ln X_3} = [\ln (\delta_{X_3}^2 + 1)]^{1/2} = 0.385$$

$$\tilde{X}_3 = \mu_{X_3} \exp (-\tfrac{1}{2}\sigma_{\ln X_3}^2) = 92.85 \text{ kN}$$

Using Eq. (8.69) and CDF of X_3 [Eq. (3.93)],

$$\sigma'_{X_3} = \frac{\phi\{\Phi^{-1}[\Phi(\ln (x_3^*/\tilde{X}_3)/\sigma_{\ln X_3})]\}}{f_{X_3}(x_3^*)}$$

$$= \frac{\phi[\ln (x_3^*/\tilde{X}_3)/\sigma_{\ln X_3}]}{f_{X_3}(x_3^*)}$$

But the PDF of lognormal X_3 is given by

$$f_{X_3}(x_3^*) = \frac{1}{x_3^*\sigma_{\ln X_3}} \phi[\ln(x_3^*/\tilde{X}_3)/\sigma_{\ln X_3}] \tag{8.75}$$

Using Eq. (8.75) in the above equation for σ'_{X_3}, we have

$$\sigma'_{X_3} = x_3^*\sigma_{\ln X_3} \tag{8.76}$$

The mean value μ'_{X_3} is calculated using Eq. (8.76) in Eq. (8.67).

$$\mu'_{X_3} = -x_3^*\,\sigma_{\ln X_3})\Phi^{-1}[\Phi\{\ln(x_3^*/\tilde{X}_3)/\sigma_{\ln X_3}\}] + x_3^*$$

$$= x_3^*(1 - \ln x_3^* + \ln \tilde{X}_3) \tag{8.77}$$

Assume

$$\beta = 5,$$

$$\alpha_1 = -0.5 \qquad \alpha_2 = -0.5 \qquad \alpha_3 = +0.707$$

$$\mu'_3 = \mu_3 = 100 \qquad \sigma'_3 = \sigma_3 = 40 \text{ kN}$$

and start the procedure. Using Eq. (8.71),

$$\beta = \frac{(2000)(100) - 448000}{(2240)(0.5)(5) - (44800)(0.5) - 22400(0.5) - 2000(40)(0.70)}$$

$$= 2.839$$

Using Eqs. (8.72), (8.73) and (8.74),

$$\alpha_1 = -\frac{1}{K}[2240(-0.5)(2.839) + 44800]$$

$$= \frac{41620}{K}$$

$$\alpha_2 = -\frac{1}{K}[2240(-0.5)(2.839) + 22400]$$

$$= -\frac{19220}{K}$$

$$\alpha_3 = -\frac{1}{K}[-2000(40)]$$

$$= \frac{8000}{K}$$

Using $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, $K = 92200$. Hence

$$\alpha_1 = -0.451 \qquad \alpha_2 = -0.209 \qquad \alpha_3 = 0.868$$

The design point, $x_3^* = \mu'_{x_3} + \alpha_3\beta\sigma'_{x_3}$

$$= 100 + 0.868(2.839)(40)$$

$$= 198.5$$

As X_3 follows the lognormal distribution, using Eqs. (8.76) and (8.77), new values of μ_3' and σ_3' are calculated:

$$\sigma_3' = (198.5)(0.385) = 76.42$$

$$\mu_3' = (198.5)(1 - \ln 198.5 + \ln 92.85)$$

$$= 47.75$$

Carry out the second iteration with the new values of β, α_1, α_2, α_3, σ_3' and μ_3'. The whole process is repeated till the convergence is achieved. The results of each iteration are given in Table 8.3.

TABLE 8.3 *Computation of β—Example 8.6*

Variable	Start	Iteration			
		1	2	3	4
β	5	2.839	2.247	2.192	2.192
α_1	−0.500	−0.451	−0.273	−0.260	−0.264
α_2	−0.500	−0.209	−0.126	−0.124	−0.126
α_3	+0.707	+0.868	+0.954	+0.958	+0.956
x_3^*	—	198.5	211.5	208.4	208.2
σ_3'	40	76.42	81.40	80.21	80.12
μ_3'	100	47.75	37.51	40.04	40.23

Results are:

$$\beta = 2.192 \qquad p_f = \Phi^{-1}(-2.192) = 0.0142$$

Design point: $(z_1^*, z_2^*, z_3^*) = \beta\,(-0.264,\ -0.126,\ 0.956)$

$$= (-0.579,\ -0.276,\ 2.095)$$

$$(x_1^*, x_2^*, x_3^*) = (0.3149,\ 1381000,\ 208.2)$$

(*Note*: $x_3^* = 40.23 + (0.956)(40.23)(2.192) = (208.2)$

The same problem has been solved for various values of the coefficient of variation of P, and corresponding values of β have been computed. The variation of β with δ_P is shown in Fig. 8.6. As expected, β decreases (i.e. reliability decreases) as δ_P increases.

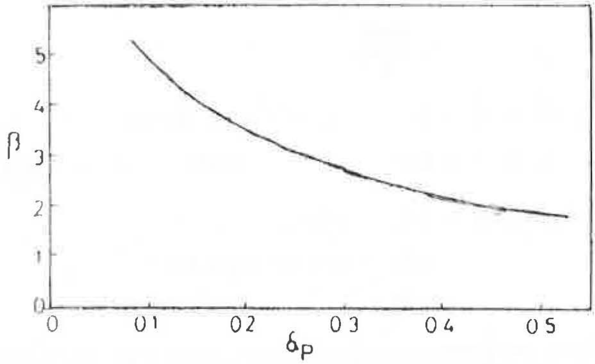


FIG. 8.6 Variation of β with δ_p --- Example 8.6

EXAMPLE 8.7 An RSJ section is used as a column. The height H of the column above ground level is 10 m. It is subjected to a wind load W which follows the Type 2 extremal largest distribution. The allowable deflection at the top of the column is $H/250$.
Given:

For Young's Modulus (E): (Normal)
$\mu_1 = 2.041 \times 10^2$ kN/mm² $\quad \sigma_1 = 0.156 \times 10^2$ kN/mm² $\quad (\delta = 7.62\%)$
For Moment of Inertia (I): (Normal)

$$\mu_2 = 315 \times 10^6 \text{ mm}^4 \qquad \sigma_2 = 15.75 \times 10^6 \text{ mm}^4 \qquad (\delta = 5\%)$$

For wind load (W): [Type 2 extremal (largest)]

$$\mu_3 = 6 \text{ kN} \qquad \sigma_3 = 1.38 \text{ kN} \qquad (\delta = 23\%)$$

Parameters: $\quad u = 5.358 \qquad k = 6.42$

Compute the reliability of the column at the limit state of deflection.
Solution For a uniformly distributed wind load,

$$\text{Maximum deflection} = \frac{WH^3}{8EI}$$

The failure surface equation is

$$\frac{H}{250} - \frac{WH^3}{8EI} = 0$$

H is considered as a deterministic variable. Substituting the given value of H, the above equation becomes

$$40 - \frac{(10000)^3}{8EI} W = 0$$

$$EI - 0.3125 \times 10^{10} W = 0$$

Let

$$X_1 = E \qquad X_2 = I \text{ and } X_3 = W$$

Hence the failure surface equation is

$$X_1X_2 - (0.3125\times 10^{10})\, X_3 = 0 \tag{8.78}$$

Let μ_3' and σ_3' be the mean and standard deviation of equivalent normal X_3' at the design point. After normalizing the variables, the above equation becomes

$$g_1(\mathbf{z}) = \sigma_1\sigma_2 z_1 z_2 + \sigma_1\mu_2 z_1 + \sigma_2\mu_1 z_2 + \mu_1\mu_2 - 0.3125\times 10^{10}(\sigma_3' z_3 + \mu_3')$$

Using the given data,

$$g_1(\mathbf{z}) = az_1z_2 + bz_1 + cz_2 + 6429\times 10^7 - d\sigma_3' z_3 \times d\mu_3'$$

where $a = 245.7\times 10^6$ $\quad b = 4914\times 10^6$ $\quad c = 3215\times 10^6$
$d = 3125\times 10^6$

At the design point, $z_i = \alpha_i\beta$ and $g_1(\mathbf{z}) = 0$
Using these, it can be written that

$$\beta = \frac{d\mu_3' - 6429\times 10^7}{a\alpha_1\alpha_2\beta + b\alpha_1 + c\alpha_2 - d\sigma_3'\alpha_3} \tag{8.79}$$

The directional cosines are given by

$$\alpha_1 = -\frac{1}{K}(a\alpha_2\beta + b) \tag{8.80}$$

$$\alpha_2 = -\frac{1}{K}(a\alpha_1\beta + c) \tag{8.81}$$

$$\alpha_3 = -\frac{1}{K}(-d\sigma_3') \tag{8.82}$$

where $\quad K = \left[\sum_{i=1}^{3}\left(\frac{\partial g_1}{\partial z_i}\right)_*^2\right]^{1/2}$

Start with

$$\beta = 5 \qquad \alpha_1 = -0.5 \qquad \alpha_2 = -0.5 \qquad \alpha_3 = 0.707$$

$$\mu_3' = \mu_3 = 6 \qquad \sigma_3' = \sigma_3 = 1.38$$

Substituting these values in Eq. (8.79), we get

$$\beta = 6.691$$

Using Eqs. (8.80), (8.81) and (8.82),

$$\alpha_1 = -\frac{1}{K}(4092\times 10^6)$$

$$\alpha_2 = -\frac{1}{K}(2393\times 10^6)$$

$$\alpha_3 = \frac{1}{K}(4313\times 10^6)$$

Using $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, $K = 6408\times 10^6$. Hence

$$\alpha_1 = -0.6385 \qquad \alpha_2 = -0.3734 \qquad \alpha_3 = 0.6730$$

The design point x_3^* is given by

$$x_3^* = \mu_3' + \alpha_3 \beta \sigma_3' = 12.21$$

X_3 follows the Type 2 extremal largest distribution. The PDF and CDF of X_3 are [Eqs. (3.121) and (3.122)]

$$F_{X_3}(x_3^*) = \exp\left[-\left(\frac{u}{x_3^*}\right)^k\right]$$

$$f_{X_3}(x_3^*) = \frac{k}{u}\left[\frac{u}{x_3^*}\right]^{k+1} \exp\left[-\left(\frac{u}{x_3^*}\right)^k\right]$$

Using $x_3^* = 12.21$ $\quad u = 5.358,$ and $k = 6.42$, we get

$$F_{X_3}(x_3^*) = 0.995 \qquad f_{X_3}(x_3^*) = 0\ 002637$$

Using Eqs. (8.67) and (8.69), the new values of σ_3' and μ_3' are calculated:

$$\sigma_3' = \frac{\phi\{\Phi^{-1}(0.995)\}}{0.002637} = 5.507$$

$$\begin{aligned} \mu_3' &= x_3^* - \Phi^{-1}[F_{X_3}(x_3^*)]\sigma_3' \\ &= 12.21 - \Phi^{-1}(0.995)(5\ 507) \\ &= -1.961 \end{aligned}$$

Using these new values of β, α_1, α_2, μ_3' and σ_3', successive iterations are carried out. Results are summarized in Table 8.4.

TABLE 8.4 *Computation of β—Example 8.7*

Variable	Start	Iteration				
		1	2	3	4	5
β	5.000	6.691	4.536	3.587	3.531	3.528
α_1	−0.500	−0.639	−0.25	−0.113	−0.128	−0.1342
α_2	−0.500	−0.373	−0.139	−0.071	−0.082	−0.086
α_3	+0.707	+0.673	+0.958	+0.991	+0.988	+0.987
x_3^*	—	12.21	21.97	20.28	19.58	19.53
σ_3'	1.380	5.507	13.41	11.98	11.40	11.36
μ_3'	6.000	−1.961	−27.40	−22.24	−20.19	−20.03

Result: Reliability index = 3.528
Probability of failure = $\Phi(-3.528) = 2.093 \times 10^{-4}$

EXAMPLE 8.8 An under-reinforced concrete beam of breadth (b) 240 mm and effective depth (d) 480 mm is reinforced with steel bars (grade Fe 250) of area (A_s) 1400 mm². The grade of concrete used is M 15 (nominal mix). The beam is subjected to a moment M. Given:

Variable F_y: Normal

$$\mu = 320\ \text{N/mm}^2 \qquad \sigma = 32\ \text{N/mm}^2$$

f_{cu}: Lognormal

$$\mu = 22.67 \text{ N/mm}^2 \qquad \sigma = 5.44 \text{ N/mm}^2$$

Parameters: $\tilde{f}_{cu} = 22.04 \qquad \sigma_{\ln f_{cu}} = 0.237$

M: Type I extremal (largest):

$$\mu = 72\times 10^6 \text{ N mm} \qquad \sigma = 24\times 10^6 \text{ N mm}$$

Parameters: $u = 61.2\times 10^5 \qquad \alpha = 0.534\times 10^{-8}$

Determine the reliability of the beam at the limit state of collapse in flexure.

Solution The ultimate strength of the beam is given by

$$R = A_s F_y\, d\left[1 - \frac{0.77\, F_y A_s}{f_{cu} b\, d}\right]$$

The failure surface equation is given by

$$g(\;) = R - M$$

$$= A_s F_y\, d\left[1 - \frac{0.77\, F_y A_s}{f_{cu}\, b\, d}\right] - M \tag{8.83}$$

Let

$$X_1 = F_y \qquad X_2 = f_{cu} \qquad X_3 = M$$

Using the given values of A_s, b and d in Eq. (8.83), the failure surface equation becomes

$$g(\mathbf{X}) = a_1 X_1 X_2 - a_2 X_1^2 - a_3 X_3 X_2 = 0 \tag{8.84}$$

where $a_1 = A_s\, d = 672000$

$$a_2 = (a_1)\left(\frac{0.77\, A_s}{b\, d}\right) = 6288 \qquad a_3 = 1$$

Let μ_2' and σ_2', and μ_3' and σ_3' be the values of the mean and standard deviation of the equivalent normals X_2' and X_3' at the design point.

Normalizing the variables X_i, Eq. (8.84) becomes

$$g_1(\mathbf{z}) = a_1(\sigma_1\sigma_2' z_1 z_2 + \sigma_1\mu_2' z_1 + \sigma_2'\mu_1 z_2) - a_2(\sigma_1^2 z_1^2 + 2\sigma_1 z_1\mu_1)$$
$$- a_3(\sigma_2'\sigma_3' z_2 z_3 + \sigma_2'\mu_3' z_2 + \sigma_3'\mu_2' z_3)$$
$$+ a_1\mu_1\mu_2' - a_2\mu_1^2 - a_3\mu_2'\mu_3'$$

At the design point, $z_i = \alpha_i\beta$ and $g_1(\mathbf{z}) = 0$.

Using these, the above equation can be rewritten as

$$\beta = \frac{-\left(a_1\mu_1\mu_2' - a_2\mu_1^2 - a_3\mu_2'\mu_3'\right)}{b_1 - b_2 - b_3} \tag{8.85}$$

where $b_1 = a_1(\sigma_1\sigma_2'\alpha_1\alpha_2\beta + \sigma_1\mu_2'\alpha_1 + \sigma_2'\mu_1\alpha_2)$

$$b_2 = a_2(\sigma_1^2\alpha_1^2\beta + 2\sigma_1\alpha_1\mu_1)$$

$$b_3 = a_3(\sigma_2'\sigma_3'\alpha_2\alpha_3\beta + \sigma_2'\mu_3'\alpha_2 + \sigma_3'\mu_2'\alpha_3)$$

The directional cosines are given by

$$\alpha_1 = -\frac{1}{K}\left[a_1(\sigma_1\sigma_2'\alpha_2\beta + \sigma_1\mu_2') - 2a_2(\sigma_1^2\alpha_1\beta + \sigma_1\mu_1)\right] \tag{8.86}$$

$$\alpha_2 = -\frac{1}{K}\left[a_1(\sigma_1\sigma_2'\alpha_1\beta + \sigma_2'\mu_1) - a_3(\sigma_2'\sigma_3'\alpha_3\beta + \sigma_2'\mu_3')\right] \tag{8.87}$$

$$\alpha_3 = -\frac{1}{K}\left[-a_3(\sigma_2'\sigma_3'\alpha_2\beta + \sigma_3'\mu_2')\right] \tag{8.88}$$

where $$K = \left[\sum_{i=1}^{3}\left(\frac{\partial g_1}{\partial z_i}\right)_*^2\right]^{1/2}$$

Start with

$$\beta = 5 \qquad \alpha_1 = -0.5 \qquad \alpha_2 = -0.5 \qquad \alpha_3 = 0.707$$

$$\mu_2' = \mu_2 = 22.67 \qquad \sigma_2' = \sigma_2 = 5.44$$

$$\mu_3' = \mu_3 = 72\times 10^6 \qquad \sigma_3' = \sigma_3 = 24\times 10^6$$

Substituting these values in Eq. (8.85), we have

$$\beta = 4.449$$

Using Eqs. (8.86), (8.87) and (8.88), we get

$$\alpha_1 = -\frac{1}{K}(127.1\times 10^6)$$

$$\alpha_2 = -\frac{1}{K}(107.3\times 10^6)$$

$$\alpha_3 = -\frac{1}{K}(-253.6\times 10^6)$$

Using $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, $\quad K = 303.3\times 10^6$
the directional cosines are

$$\alpha_1 = -0.4191 \qquad \alpha_2 = -0.3537 \qquad \alpha_3 = 0.8362$$

The design point x_2^* and x_3^* are given by

$$x_2^* = \mu_2' + \alpha_2\beta\sigma_2'$$

$$= 22.67 + (-0.3537)(4.449)(5.44) \qquad = 14.11$$

$$x_3^* = \mu_3' + \alpha_3\beta\sigma_3'$$

$$= 72\times 10^6 + 0.8362(4.449)(24\times 10^6) = 161.3\times 10^6$$

It is given that X_2 follows the lognormal distribution. Using Eqs. (8.76) and (8.77), new values of μ_2' and σ_2' are calculated:

$$\sigma_2' = 3.336 \qquad \mu_2' = 20.4$$

X_3 follows Type 1 extremal (largest) distribution. The PDF and CDF of X_3 are [Eqs. (3.114) and (3.115)]

$$f_{X_3}(x_3^*) = \alpha \exp[-\alpha(x_3^* - u) - \exp\{-\alpha(x_3^* - u)\}] \tag{8.89}$$

$$F_{X_3}(x_3^*) = \exp[-\exp\{-\alpha(x_3^* - u)\}] \tag{8.90}$$

Using the above equations, at the design point x_3^*,

$$F_{X_3}(x_3^*) = 0.9953 \qquad f_{X_3}(x_3^*) = 0.253 \times 10^{-9}$$

Using Eqs. (8.67) and (8.69), the new values σ_3' and μ_3' are

$$\sigma_3' = \frac{\phi[\Phi^{-1}(F_{X_3}(x_3^*))]}{f_{X_3}(x_3^*)} = 54.50 \times 10^6$$

$$\begin{aligned} \mu_3' &= x_3^* - \Phi^{-1}[F_{X_3}(x_3^*)]\sigma'_{X_3} \\ &= 161.3 \times 10^6 - \Phi^{-1}(0.9953)(54.50 \times 10^6) \\ &= 19.89 \times 10^6 \end{aligned}$$

Using these new values of β, α_1, α_2, α_3, μ_2', σ_2', μ_3' and σ_3', the whole process is repeated and successive iterations carried out till the required convergence is achieved. Results are summarized in Table 8.5.

TABLE 8.5 *Computation of reliability—Example 8.8*

Variable	Start	Iteration			
		1	2	3	4
β	5.000	4.449	3.310	2.932	2.904
α_1	−0.500	−0.419	−0.261	−0.241	−0.259
α_2	−0.500	−0.358	−0.0518	−0.084	−0.104
α_3	+0.707	+0.836	+0.964	+0.967	+0.960
x_2^*	—	14.11	19.83	20.77	20.52
σ_2'	5.44	3.34	4.69	4.917	4.858
μ_2'	22.67	20.40	21.93	22.00	21.99
x_3^*	—	161.3×10^6	193.7×10^6	174.1×10^6	172.2×10^6
σ_3'	24×10^6	54.50×10^6	63.88×10^6	58.34×10^6	57.78×10^6
μ_3	72×10^6	19.89×10^6	-7.0×10^6	-9.5×10^6	-11.70×10^6

Result: Reliability index β = 2.904
Design point : $\beta(-0.259, -0.104, 0.96)$

8.3.3 Determination of β for Present Designs

During the process of code calibration, the reliability analysis of structural components designed as per the present code are first carried out, and then the reliability levels of the present designs under different design situations are established. Different design situations may mean different load combinations, viz. $D + L$ or $D + W$ or $D + L + W$. Again, different ratios of loads under each load combination may also be considered. The process of establishing the reliability level is explained in the following section.

Consider the case: the reliability study of RCC beams at the limit state of collapse in flexure. Let R be the ultimate strength of a RCC section in flexure. This is the true or actual strength based on the theoretical model to compute the ultimate strength. After identifying the basic variables in the theoretical model, the statistics of R are established using the Monte Carlo method, (explained in Chapter 7) taking into account the model error also. Let R_n be the nominal strength of the member.

When the nominal values of variables are used in the theoretical model, the strength obtained is called the nominal strength. When strength is calculated substituting nominal values of variables in the equation given by the code, it is called the design strength R_D. Let γ_R be the strength reduction factor given by R_D/R_n. If the load combination $D + L$ is considered, the code (8.3) specifies that member is to be designed for $\gamma_D D_n + \gamma_L L_n$, where γ_D and γ_L are the partial safety factors for dead load and live load respectively, and D_n and L_n are nominal values of D and L respectively. As per IS Code (8.3), $\gamma_D = \gamma_L = 1.5$. Hence the design strength of the member is

$$R_D = 1.5(D_n + L_n) \tag{8.91}$$

Since $R_D = R_n\gamma_R$, the above equation becomes

$$\frac{R_n}{D_n} = \frac{1.5\left(1 + \frac{L_n}{D_n}\right)}{\gamma_R} \tag{8.92}$$

Let the failure surface equation under the $D + L$ load case be

$$R - D - L = 0$$

Dividing by D_n, the above equation can be written as

$$\left(\frac{R}{R_n}\right)\left(\frac{R_n}{D_n}\right) - \left(\frac{D}{D_n}\right)\left(\frac{D_n}{D_n}\right) - \left(\frac{L}{L_n}\right)\left(\frac{L_n}{D_n}\right) = 0$$

Using Eq. (8.92),

$$\left(\frac{R}{R_n}\right)\left[\frac{1.5(1 + L_n/D_n)}{\gamma_R}\right] - \left(\frac{D}{D_n}\right) - \left(\frac{L}{L_n}\right)\left(\frac{L_n}{D_n}\right) = 0 \tag{8.93}$$

For a known value of γ_R, the reliability analysis can be carried out and β computed for various ratios L_n/D_n. When the reliability is estimated at the limit state of collapse (ultimate limit states), the statistics of the lifetime maximum live load is to be used. This is illustrated in the following example.

EXAMPLE 8.9 From the statistical study of the flexural strength of doubly reinforced sections, with M 20 grade of concrete (nominal mix) and Fe 415 grade of steel, it has been found that the mean value and standard deviation of R/R_n are 1.222 and 0.149 respectively. $\gamma_R = 0.844$. If $L_n/D_n = 0.5$,

determine the reliability of the beam under the load combination $D + L$.
Given:
For

$$\left(\frac{D}{D_n}\right): \mu = 1.05 \qquad \sigma = 0.105$$

$$\left(\frac{L}{L_n}\right): \mu = 0.62 \qquad \sigma = 0.1755$$

R and D are normal and L is Type 1 extremal (largest).

Let

$$X_1 = \frac{R}{R_n} \qquad X_2 = \frac{D}{D_n} \qquad X_3 = \frac{L}{L_n}$$

$$a_1 = 1.5\left(1 + \frac{L_n/D_n}{\gamma_R}\right) \quad ; \quad a_3 = \frac{L_n}{D_n}$$

For $\gamma_R = 0.844$ and $L_n/D_n = 0.5$, $a_1 = 2.666$. The safety margin given by Eq. (8.93) becomes

$$M = a_1X_1 - X_2 - a_3X_3 = 0 \tag{8.94}$$

In the present case, $a_1 = 2.666$ and $a_3 = 0.5$.
Let μ_3' and σ_3' be the mean and standard deviation of equivalent normal X_3' at the design point. The failure surface equation, being a linear equation,

$$\mu_M = a_1\mu_1 - \mu_2 - a_3\mu_3'$$

$$\sigma_M = [a_1^2\sigma_1^2 + \sigma_2^2 + (a_3\sigma_3')^2]^{1/2}$$

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{a_1\mu_1 - \mu_2 - a_3\mu_3'}{[a_1^2\sigma_1^2 + \sigma_2^2 + (a_3\sigma_3')^2]^{1/2}} \tag{8.95}$$

The above equation can be verified by normalizing the variables and following the usual procedure in the previous examples. Start the procedure assuming values for the unknown σ_3' and μ_3'.

Start with $\qquad \sigma_3' = \sigma_3 = 0.1755$

$$\mu_3' = \mu_3 = 0.62$$

Using Eq. (8.95),

$$\beta = \frac{2.666(1.222) - (1.05) - 0.5(0.62)}{[2.666^2(0.149)^2 + 0.105^2 + 0.5^2(0.1755)^2]^{1/2}}$$

$$= \frac{1.898}{0.4201} = 4.517$$

$$\alpha_3 = -\left(-\frac{a_3\sigma_3'}{\sigma_M}\right)$$

$$= \frac{(0.5)(0.1755)}{0.4201} = 0.2089$$

The design point x_3^* is

$$x_3^* = \mu_3' + \alpha_3\beta\sigma_3'$$
$$= 0.62 + (0.2089)(4.517)(0.1755)$$
$$= 0.7856$$

X_3 follows the Type 1 extremal largest distribution.
Following the procedure in Example 8.8,

$$F_{X_3}(x_3^*) = 0.8458 \qquad f_{X_3}(x_3^*) = 1.035$$

Using Eqs. (8.67) and (8.69), new values of σ_3' and μ_3' are

$$\sigma_3' = 0.2466 \qquad \mu_3' = 0.5327$$

The whole process is repeated till β converges. Summarized results are given in Table 8.6.

TABLE 8.6 *Computation of reliability—Example 8.9*

Variable	Start	Iteration			
		1	2	3	4
β	—	4.517	4.529	4.526	4.525
σ_3'	0.1755	0.229	0.247	0.255	0.259
μ_3'	0.62	0.552	0.533	0.522	0.517

Result : Reliability index $\beta = 4.525$
Probability of failure $p_f = \Phi(-4.525)$
$= 2.465 \times 10^{-6}$

The same problem has been solved for various values of L_n/D_n and the variation of β with L_n/D_n is shown in Fig. 8.7. It may be observed that the values of β range from 4.33 to 4.66, which are high. Normally, for compo-

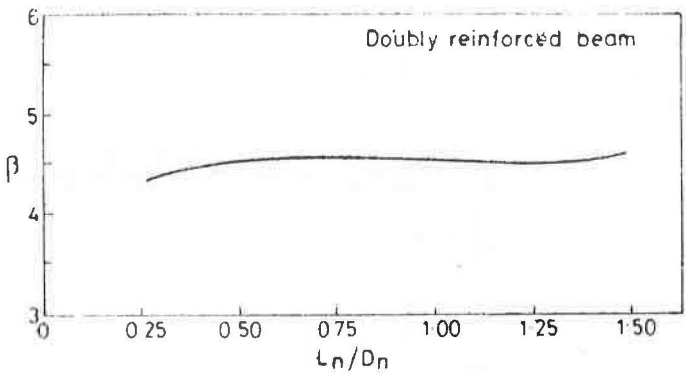


FIG. 8.7 Variation of β for doubly reinforced beam with L_n/D_n —Example 8.9

ment failure, a value of β about 3(8.10) is considered for code calibration. The high value obtained is due to the fact that for office buildings the value of nominal live load specified by IS Code (8.11) is quite high.

The formulation for determining β for the load combination $D + W$ is same as for the $L + D$ case. L is to be replaced with W in all Equations from (8.91) to (8.93). The failure surface equation is

$$R - D - W = 0$$

or

$$b_1\left(\frac{R}{R_n}\right) - \frac{D}{D_n} - b_3\left(\frac{W}{W_n}\right) = 0$$

where

$$b_3 = \frac{W_n}{D_n} \qquad b_1 = 1.5\left(\frac{1 + W_n/D_n}{\gamma_R}\right) \tag{8.96}$$

The value 1.5 in the above equation is as per the present IS Code (8.3). If we consider the shear strength of the beam (limit state of collapse in shear), it has been found that the statistics of R/R_n for a RCC beam (with M 15 design mix and Fe 250 steel grade) are $\mu = 1.355$, $\sigma = 0.225$, and $\gamma_R = 0.85$ (8.12). The normal distribution has been fitted to the tail region. Using the statistics of R/R_n, D/D_n and W_m/W_n, given in Table 5.3, the reliability analysis can be carried out for various ratios of W_n/D_n. The variation of β with W_n/D_n for the case of a beam in shear under the load combination $D + W$ case is shown in Fig. 8.8 (Ref. 8.12). Values of β vary from 3 to 3.5. Instead of the steel grade Fe 250, if Fe 415 is used, the statistics of R/R_n for the same case change to $\mu = 1.277$ and $\sigma = 0.2105$. The results of the reliability analysis for beams in shear using Fe 415 grade is also shown in Fig. 8.8. It can be observed that beams with stirrups of Fe 250 grade have higher reliability than those with stirrups of Fe 415 grade. It is mainly because the ratio of the mean value of the yield strength to its specified strength for the Fe 250 grade steel is much higher than the ratio for the Fe 415 grade steel (Table 4.3).

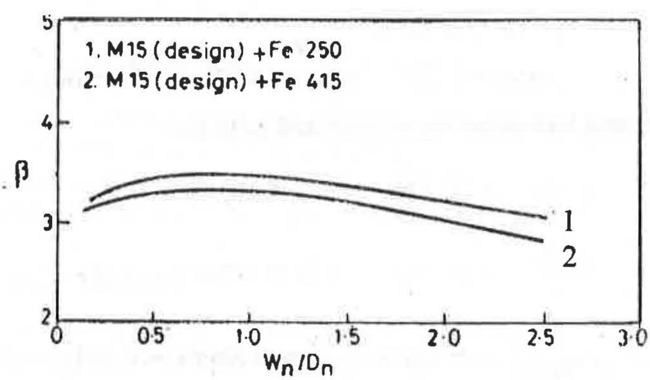


FIG. 8.8 Variation of β with W_n/D_n for RCC beam in shear under load: $D + W_m$

The reliability analysis of members under load combination $D + L + W$ can be formulated as follows. The failure surface equation is

$$R - D - L - W = 0 \tag{8.97}$$

Dividing each term by D_n, the equation can be rewritten as

$$\left(\frac{R}{R_n}\right)\left(\frac{R_n}{D_n}\right) - \left(\frac{D}{D_n}\right)\left(\frac{D_n}{D_n}\right) - \left(\frac{L}{L_n}\right)\left(\frac{L_n}{D_n}\right) - \left(\frac{W}{W_n}\right)\left(\frac{W_n}{D_n}\right) = 0 \tag{8.98}$$

As per the IS code (8.3),

$$R_D = 1.2(D_n + L_n + W_n)$$

Dividing by D_n, and using the relation $R_n = R_D/\gamma_R$,

$$\frac{R_n}{D_n} = 1.2\left(\frac{1 + L_n/D_n + W_n/D_n}{\gamma_R}\right) \tag{8.99}$$

Hence the failure surface equation becomes,

$$C_1\left(\frac{R}{R_n}\right) - \frac{D}{D_n} - C_3\left(\frac{L}{D_n}\right) - C_4\left(\frac{W}{W_n}\right) \tag{8.100}$$

where
$$C_1 = 1.2\left(\frac{1 + L_n/D_n + W_n/D_n}{\gamma_R}\right)$$

$$C_3 = \frac{L_n}{D_n} \qquad C_4 = \frac{W_n}{D_n}$$

Now for different combinations of L_n/D_n and W_n/D_n ratios, the reliability analysis can be carried out.

EXAMPLE 8.10 It is desired to determine the reliability of a column under a load combination of gravity plus wind loads, viz. $D + L + W$. From the statistical study of rectangular RCC columns subjected to axial load and uniaxial bending, it has been found that (8.12) for compression failure,

$$\gamma_R = 0.725$$

$$\mu_{R/R_n} = 1.22 \qquad \sigma_{R/R_n} = 0.171 \qquad (\delta = 14\%)$$

Mean and standard deviation of D/D_n and L/L_n are

For $\dfrac{D}{D_n}$: $\mu = 1.05$ $\quad\sigma = 0.105$ ($\delta = 0.1$)

For $\dfrac{L}{L_n}$: $\mu = 0.62$ $\quad\sigma = 0.1755$ ($\delta = 0.28$)

For $\dfrac{W}{W_n}$: $\mu = 0.804$ $\quad\sigma = 0.268$ ($\delta = 0.334$)

R and D follow normal. L and W follow Type I extremal (largest). Determine β if $L_n/D_n = 0.5$ and $W_n/D_n = 1.0$.

Solution The failure surface is given by Eq. (8.100) Let

$$X_1 = \frac{R}{R_n} \qquad X_2 = \frac{D}{D_n}$$

$$X_3 = \frac{L}{L_n} \qquad X_4 = \frac{W}{W_n}$$

The failure surface Eq. (8.100) becomes

$$M = C_1X_1 - X_2 - C_3X_3 - C_4X_4 = 0 \tag{8.101}$$

where
$$C_1 = 1.2\frac{(1 + L_n/D_n + W_n/D_n)}{\gamma_R}$$

$$C_3 = \frac{L_n}{D_n} \qquad C_4 = \frac{W_n}{D_n}$$

For $\frac{L_n}{D_n} = 0.5, \qquad \frac{W_n}{D_n} = 1.0, \qquad \gamma_R = 0.725$

$$C_1 = 4.138, \qquad C_3 = 0.5, \qquad C_4 = 1.0$$

It is given that the live load and wind load follow the Type 1 extremal distribution. Let μ_3' and σ_3' be the mean value and standard deviation of the equivalent normal X_3' at the design point. Similarly, μ_4' and σ_4' are for X_4'. The failure surface equation being linear

$$\mu_M = C_1\mu_1 - \mu_2 - C_3\mu_3' - C_4\mu_4'$$

$$\sigma_M = [(C_1\sigma_1)^2 + \sigma_2^2 + (C_3\sigma_3')^2 + (C_4\sigma_4')^2]^{1/2}$$

$$\beta = \frac{\mu_M}{\sigma_M}$$

$$\alpha_3 = -\frac{(-C_3\sigma_3')}{\sigma_M} \qquad \alpha_4 = -\frac{(-C_4\sigma_4')}{\sigma_M}$$

$$x_3^* = \mu_3' + \alpha_3\beta\sigma_3' \qquad x_4^* = \mu_4' + \alpha_4\beta\sigma_4'$$

The procedure of computation is similar to the one explained in the previous example. Summarized results are given in Table 8.7.

The variation of β with W_n/D_n and L_n/D_n for RCC columns under the load combination $D + L_m + W_m$ is shown in Fig. 8.9.

8.3.4 Correlated Variables

In all the previous problems, we have assumed that all variables in the failure surface equation are statistically independent, and have computed reliability index based on this assumption. In practice, we may have to deal with cases when all variables or some of the variables are correlated. The procedure of solving such cases is explained below.

TABLE 8.7 *Computation of reliability—Example 8.10*

Variable	Start	Iteration				
		1	2	3	4	5
β	—	3.755	3.728	3.698	3.689	3.687
σ_3'	0.1755	0.1955	0.185	0.182	0.181	0.181
μ_3'	0.6200	0.5795	0.585	0.586	0.587	0.587
σ_4'	0.268	0.387	0.455	0.495	0.516	0.527
μ_4'	0.804	0.657	0.555	0.481	0.438	0.416

Result : Reliability index $\beta = 3.687$

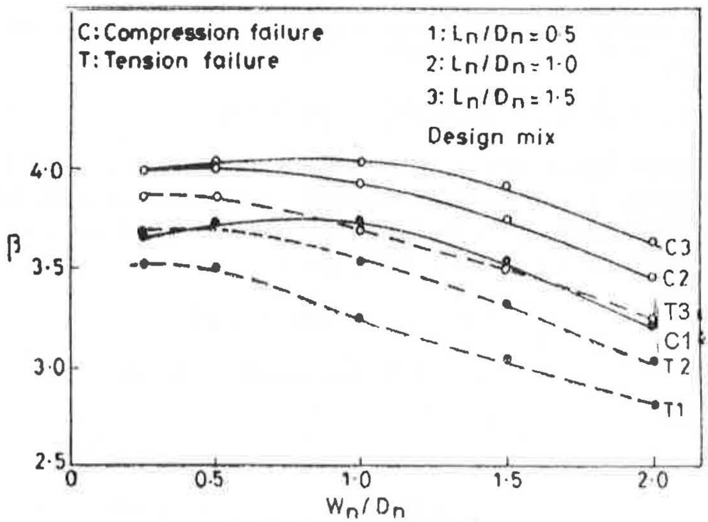


FIG. 8.9 Variation of β with W_n/D_n and L_n/D_n for RCC columns under load; D ; L_m W_m

Let $X_1, X_2, \ldots, X_n$ be the set of correlated variables appearing in the failure surface equation. Let $[Cx]$ be the covariance matrix of the correlated variables. That is

$$[Cx] = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \text{Cov}(X_1, X_3) & \ldots & \text{Cov}(X_1, X_n) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) & \text{Cov}(X_2, X_3) & \ldots & \text{Cov}(X_2, X_n) \\ \vdots & \vdots & \vdots & & \vdots \\ \text{Cov}(X_n, X_1) & \text{Cov}(X_n, X_2) & \ldots & \ldots & \text{Var}(X_n) \end{bmatrix} \tag{8.102}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values and $[V]$ be the matrix having each column represented by an eigen vector corresponding to each eigen value. That is, if e_{ij} is the eigen vector for λ_j, then the elements of matrix $[V]$ are

$$[V] = \begin{bmatrix} e_{11} & e_{12} & \ldots, & e_{1j} & \ldots & e_{1n} \\ e_{21} & e_{22} & \ldots & e_{2j} & \ldots & e_{2n} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ e_{n1} & e_{n2} & \ldots & e_{nj} & \ldots & e_{nn} \end{bmatrix} \tag{8.103}$$

This matrix $[V]$ is an orthogonal transformation matrix. Then the required set of uncorrelated variables Y is given by (8.2)

$$\mathbf{Y} = [V]^t\mathbf{X} \tag{8.104}$$

where $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_n\}^t$ and $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}^t$. The superscript t denotes the transpose. Since $[V]$ is orthogonal, $[V]^{-1} = [V]^t$. Hence,

$$\mathbf{X} = [V]\mathbf{Y} \tag{8.105}$$

The expected values of Y are given by [Eq. (8.104)]

$$E(\mathbf{Y}) = [V]^t E(\mathbf{X}) \tag{8.106}$$

The variance matrix of Y, $[C_Y]$, is given by

$$[C_Y] = [V]^t[C_X][V]$$

$$= [\lambda] = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n \end{bmatrix} \tag{8.107}$$

That is, the eigen values of $[V]$ are also the variances of respective variables $Y_1, Y_2, \ldots, Y_n$. Knowing the mean values and standard deviations of $\mathbf{Y}$, the variates Y_i can be normalized as usual, i.e.

$$Z_i = \frac{(Y_i - \mu_{Y_i})}{\sigma_{Y_i}} \tag{8.108}$$

and β can be determined following the procedure given in the previous sections. Hence the following steps are involved for correlated variables:

(i) Determine the eigen values and the corresponding eigen vectors of the covariance matrix. That is, determine the matrix $[V]$.

(ii) Write the safety margin equation in terms of the uncorrelated variables using Eq. (8.105).

(iii) Determine the mean values of $\mathbf{Y}$ using Eq. (8.106) and the variances of $\mathbf{Y}$ using Eq. (8.107).

(iv) Normalize the variables Y_i using Eq. (8.108) and write the safety margin equation in terms of the normalized variables Z_i. Note that using Eqs. (8.105) and (8.108), we have

$$\begin{aligned} \mathbf{X} &= [V]\Big[[\sigma_Y]\mathbf{Z} + \boldsymbol{\mu}_\mathbf{Y} \Big] \\ &= [V][\sigma_Y]\mathbf{Z} + [V]\boldsymbol{\mu}_\mathbf{Y} \end{aligned} \tag{8.109}$$

where

$$[\sigma_Y] = \begin{bmatrix} \sigma_{Y_1} & 0 & 0 \ldots 0 \\ 0 & \sigma_{Y_2} & 0 \ldots 0 \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ 0 & \ldots\ldots\ldots & \sigma_{Y_n} \end{bmatrix} \tag{8.110}$$

$$\boldsymbol{\mu}_\mathbf{Y} = (\mu_{Y_1}, \mu_{Y_2}, \ldots, \mu_{Y_n})^t$$

$$\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)^t$$

(v) Determine β.
With the orthogonal transformation of Eq. (8.104), it can be shown (8.7) that the reliability index of Eq. (8.44) becomes

$$\beta = \frac{-\mathbf{z}_*'\mathbf{G}_*}{(\mathbf{G}_*'[C]\mathbf{G}_*)^{1/2}} \tag{8.111}$$

EXAMPLE 8.11 For the same problem in Example 8.9, assume that R and D are correlated. The correlation arises because both depend on the dimensions of the beam. Assuming Cov $(R, D) = 0.0111$, determine the reliability index.

Solution The failure surface equation is [Eq. (8.94)]

$$a_1X_1 - X_2 - a_3X_3 = 0$$

where $\quad X_1 = R, \quad X_2 = D \quad \text{and} \quad X_3 = L$

$$a_1 = 2.666 \qquad a_3 = 0.5$$

The covariance matrix is

$$[C_X] = \begin{array}{c} \\ R \\ D \\ L \end{array} \!\! \begin{array}{c} \begin{array}{ccc} R \quad\quad & D \quad\quad & L \end{array} \\ \begin{bmatrix} 0.0222 & 0.0111 & 0 \\ 0.0111 & 0.0110 & 0 \\ 0 & 0 & 0.0308 \end{bmatrix} \end{array}$$

The corresponding determinant equation is

$$\text{Det} \begin{bmatrix} (0.0222 - \lambda) & 0.0111 & 0 \\ 0.0111 & (0.011 - \lambda) & 0 \\ 0 & 0 & (0.0308 - \lambda) \end{bmatrix} = 0$$

The characteristic equation of $[C_X]$ is

$$(0.0222 - \lambda)[(0.011 - \lambda)(0.0308 - \lambda)] - 0.0111[0.0111(0.0308 - \lambda)] = 0$$

$$\lambda^3 - 0.064\lambda^2 + 11.436\times 10^{-4}\lambda - 3.725\times 10^{-6} = 0$$

The eigen values, given by the roots of the equation are

$$\lambda_1 = 0.02903 \qquad \lambda_2 = 0.004167 \qquad \lambda_3 = 0.0308$$

The corresponding normalized eigen vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 0.8516 \\ 0.5242 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} -0.5242 \\ 0.8516 \\ 0 \end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the orthogonal transformation matrix is

$$[V] = \begin{bmatrix} 0.8516 & 0.5242 & 0 \\ 0.5242 & 0.8516 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, using Eq. (8.104),

$$\mathbf{Y} = [V]^t \mathbf{X}$$

The expected values of variates $\mathbf{Y}$ are [Eq. (8.106)]

$$\begin{bmatrix} \mu_{Y_1} \\ \mu_{Y_2} \\ \mu_{Y_3} \end{bmatrix} = \begin{bmatrix} 0.8516 & 0.5242 & 0 \\ -0.5242 & 0.8516 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.222 \\ 1.050 \\ 0.620 \end{bmatrix}$$

This yields

$$\mu_{Y_1} = 1.591 \qquad \mu_{Y_2} = 0.2536 \qquad \mu_{Y_3} = 0.62$$

The variances of $\mathbf{Y}$, given by eigen values, are,

$$\sigma^2_{Y_1} = 0.02903 \qquad \sigma^2_{Y_2} = 0.004167 \qquad \sigma^2_{Y_3} = 0.0308$$

that is

$$\sigma_{Y_1} = 0.1704 \qquad \sigma_{Y_2} = 0.06456 \qquad \sigma_{Y_3} = 0.1755$$

Using Eq. (8.109), it is obtained as

$$X_1 = 0.1451Z_1 - 0.0338Z_2 + 1.222 \tag{8.112}$$

$$X_2 = 0.0893Z_1 + 0.055Z_2 + 1.050 \tag{8.113}$$

$$X_3 = 0.1755Z_3 + 0.62 \tag{8.114}$$

Substituting these in the failure surface equation, the equation in terms of the uncorrelated normalized variates becomes

$$a_1(0.1451Z_1 - 0.0338Z_2 + 1.222) - (0.0893Z_1 + 0.055Z_2 + 1.05)$$
$$- a_3(0.1755Z_3 + 0.62) = 0$$

Substituting the values of $a_1 = 2.666$ and $a_3 = 0.5$ in the above equation, we get

$$0.2975z_1 - 0.1452z_2 - 0.0878z_3 + 1.898 = 0$$

The above equation being linear, and since $\mu_{Z_i} = 0$ (normalized variates),

$$\beta = \frac{1.898}{[(0.2975)^2 + (0.1452)^2 + (0.0878)^2]^{1/2}}$$
$$= 5.54$$

The same problem can be solved straightway without using the transformation matrix, since the failure surface equation is linear. The given equation is

$$M = a_1X_1 - X_2 - a_3X_3 = 0$$

The reliability index is

$$\beta = \frac{\mu_M}{\sigma_M}$$

$$\mu_M = a_1\mu_{X_1} - \mu_{X_2} - a_3\mu_{X_3} = 1.8978$$

$$\sigma_M = [(a_1\sigma_{X_1})^2 + \sigma_{X_2}^2 + (a_3\sigma_{X_3})^2 - 2a_1 \text{ Cov } (X_1, X_2)]^{1/2}$$
$$= [(2.666 \times 0.149)^2 + (0.105)^2 + (0.5 \times 0.1755)^2 - 2 \times 2.666 \times 0.0111]^{\frac{1}{2}}$$
$$= 0.342$$

$$\beta = \frac{1.8978}{0.342} = 5.54$$

EXAMPLE 8.12 Consider the same problem of the cantilever beam, given in Example 8.6. The limit state failure equation is of the form

$$X_1X_2 - X_3 = 0$$

If

$$\mu_{X_1} = 14 \times 10^2 \qquad \sigma_{X_1} = 0.7 \times 10^2$$
$$\mu_{X_2} = 32 \qquad \sigma_{X_2} = 3.2$$
$$\mu_{X_3} = 20 \times 10^3 \qquad \sigma_{X_3} = 8 \times 10^3$$
$$\text{Cov } (X_1, X_2) = 1.12 \times 10^2 \qquad \text{Cov } (X_2, X_3) = 8 \times 10^3$$
$$\text{Cov } (X_2, X_3) = 45 \times 10^4$$

determine the reliability index if all, X_1, X_2 and X_3 follow the normal distribution.

Solution The variance matrix is

$$[C_X] = \begin{bmatrix} 4900 & 112 & 45 \times 10^4 \\ 112 & 10.24 & 8000 \\ 45 \times 10^4 & 8000 & 64 \times 10^6 \end{bmatrix}$$

The eigen values are

$$\lambda_1 = 64\times10^6 \quad \lambda_2 = 1738 \quad \lambda_3 = 7.442$$

The corresponding $[V]$ matrix given by Eq. (8.103) is

$$[V] = \begin{bmatrix} 0.7031\times10^{-2} & 0.9995 & -0.03224 \\ 0.125\times10^{-3} & 0.03224 & 0.9995 \\ 1.0 & -0.7032\times10^{-2} & 0.1017\times10^{-3} \end{bmatrix}$$

The uncorrelated variables $\mathbf{Y}$ are given by [Eq. (8.104)]

$$\mathbf{Y} = [V]^t\mathbf{X}$$

The expected values and standard deviations of Y_i are [use Eqs. (8.106) and (8.107)]

$$\mu_{Y_1} = 20010 \quad \mu_{Y_2} = 1260 \quad \mu_{Y_3} = -11.11$$

$$\sigma_{Y_1} = 8000 \quad \sigma_{Y_2} = 41.69 \quad \sigma_{Y_3} = 2.728$$

Since all X_i follow normal distributions and Y_i is a linear combination of X_i, Y_i also follows the normal distribution with the corresponding parameters μ_{Y_i} and σ_{Y_i}. The original failure equation is written in terms of the uncorrelated variables using

$$\mathbf{X} = [V]\mathbf{Y} \tag{8.105}$$

as

$$X_1 = 0.7032\times10^{-2}Y_1 + 0.9995Y_2 - 0.03224Y_3$$

$$X_2 = 0.125\times10^{-3}Y_1 + 0.03224Y_2 + 0.9995Y_3$$

$$X_3 = Y_1 - 0.7032\times10^{-2}Y_2 + 0.1017\times10^{-3}Y_3$$

Normalizing the variables Y_i,

$$Z_i = \frac{(Y_i - \mu_{Y_i})}{\sigma_{Y_i}} \qquad Y_i = Z_i\sigma_{Y_i} + \mu_{Y_i}$$

X_i can be written in terms of Z_i.

$$\mathbf{X} = [V][\sigma_Y]\mathbf{Z} + [V]\mu_\mathbf{Y} \tag{8.115}$$

where

$$[\sigma_Y] = \begin{bmatrix} 8000 & 0 & 0 \\ 0 & 41.69 & 0 \\ 0 & 0 & 2.728 \end{bmatrix}$$

$$\mu_\mathbf{Y} = \begin{bmatrix} 20010 \\ 1260 \\ -11.11 \end{bmatrix}$$

Substitution of $[V]$, $[\sigma_Y]$ and μ_Y in the above equation yields

$$X_1 = 58.48Z_1 + 41.67Z_2 - 0.08795Z_3 \tag{8.116}$$

$$X_2 = Z_1 + 1.344Z_2 + 2.727Z_3 \tag{8.117}$$

$$X_3 = 8000Z_1 - 0.293Z_2 + 2.782\times10^{-4}Z_3 \tag{8.118}$$

The given failure surface equation is

$$X_1X_2 - X_3 = 0$$

Using Eqs. (8.107) to (8.109), the failure surface equation in terms of the uncorrelated normalized variates becomes

$$\begin{aligned} g_1(\mathbf{z}) = {} & 56.25z_1^2 + 56.0z_2^2 - 0.24z_3^2 + 117.26z_1z_2 + 153.3z_1z_3 \\ & + 113.5z_2z_3 - 4799z_1 + 3217z_2 + 3816z_3 + 24850 \end{aligned}$$

The procedure of determining β is the same as explained in Example 8.6. Results of iterations are summarized in Table 8.8.

TABLE 8.8 *Computation of β—Example 8.12*

Variable	Start	Iteration			
		1	2	3	4
β	5.000	−7.519	3.721	3.533	3.533
α_1	−0.5	−0.624	−0.714	−0.699	
α_2	−0.5	−0.457	−0.424	−0.437	
α_3	0.707	+0.634	0.557	0.565	

From Table 8.8

$$\beta = 3.533 \qquad p_f = \Phi(-3.533) = 2.023\times10^{-4}$$

the design point: $z_i^* = \alpha_i^*\beta$

$$(z_1^*, z_2^*, z_3^*) = (-2.47, -1.544, 1.994)$$

$$\begin{aligned} x_1^* &= 58.48z_1^* + 41.67z_2^* - 0.08795z_3^* \\ &= 208.78 \end{aligned}$$

$$\begin{aligned} x_2^* &= z_1^* + 1.344z_2^* + 2.727z_3^* \\ &= -4.544 \end{aligned}$$

$$\begin{aligned} x_3^* &= 8000z_1^* - 0.293z_2^* + 2.782\times10^{-4}z_3^* \\ &= 19759.5 \end{aligned}$$

REFERENCES

8.1 *Joint Committee on Structural Safety, CEB–CECM–CIB–FIP–IABSE*, "First Order Reliability Concepts for Design Codes", *CEB Bulletin No. 112*, July 1976.

8.2 CIRIA, "Rationalisation of Safety and Serviceability Factors in Structural Codes", *Construction Industry Research and Information Association, Report No. 63*, London, 1977.

8.3 IS: 456-1978, "Code of Practice for Plain and Reinforced Concrete", *Indian Standards Institution*, New Delhi, 1980.
8.4 Cornell, C. A., "A Probability Based Structural Code", *Journal of ACI*, Vol. 66, Dec. 1969, pp. 975–985.
8.5 Galambos, T.V. and M.K. Ravindra, "Load and Resistance Factor Design", *Journal of Struct. Div., ASCE*, Vol. 104, ST-9, Sept. 1978, pp. 1325–1336.
8.6 Hasofer, A.M. and N.C. Lind, "An Exact and Invariant First Order Reliability Format", *Journal of Engg. Mech., ASCE*, Vol. 100, EM-1, Feb. 1974, pp. 111–121.
8.7 Shinozuka, M., "Basic Analysis of Structural Safety", *Journal of Struct. Div., ASCE*, Vol. 109, ST-3, March 1983, pp. 721–740.
8.8 Paloheimo, E. and H. Hannus, "Structural Design Based on Weighted Fractiles", *Journal of Struct. Div., ASCE*, Vol. 100, ST-7, July 1974, pp. 1367–1378.
8.9 Rackwitz, N. and B. Fiessler, "Structural Reliability Under Combined Random Load Sequences", *Computers and Structures*, Vol. 9, 1978, pp. 489–494.
8.10 Ellingwood, B.R., T.V. Galambos, J.G. MacGregor, and C.A. Cornell, "Development of a Probability Based Load Criterion for American National Standards A58", *National Bureau of Standards, Special Publication 577*, Washington, D.C., June 1980.
8.11 IS: 875-1982, "Code of Practice for Structural Safety of Buildings, Loading Standards Part II—Imposed (Live) Loads", *Indian Standards Institution*, New Delhi, 1986.
8.12 Ranganathan, R., "Reliability Analysis and Design of RCC Slabs, Beams, Columns and Frames—Code Calibration", *Report No. DS and T: 4/1/83-STP-III/5*, Civil Engg. Dept., IIT, Bombay, Sept. 1987.

EXERCISE

8.1 For the problem in Example 8.6, what is the reliability of the beam if the coefficient of variation of the load is 20%. All other data remain the same.

(*Ans.* $\beta = 3.558$)

8.2 For the problem in Example 8.8, what is the reliability of the beam if the mean value and standard deviation of the strength of concrete are 30.28 N/mm^2 and 4.54 N/mm^2 respectively. All other data are the same.

(*Ans.* $\beta = 3.293$)

8.3 For the same problem in Example 8.6, what is the reliability of the beam if P follows the Type 1 extremal largest distribution with mean, 100 kN and standard deviation, 30 kN.

(*Ans.* $\beta = 2.608$)

8.4 For the same problem in Example 8.6, what is the reliability of the beam if P follows the Type 2 extremal largest distribution with parameters $u = 89.3$ kmph and $k = 6.42$. The corresponding mean = 100 and standard deviation = 23 kmph.

(*Ans.* $\beta = 2.7$)

8.5 For the same problem in Example 8.11, determine the reliability index if

(i) the correlation coefficient between the variables X_1 and X_2 is 0.5

(*Ans.* $\beta = 5.169$)

(ii) ρ between X_1 and X_2 is 0.5 and ρ between X_2 and X_3 is 0.8.

(*Ans.* $\beta = 4.901$)

8.6 (a) The shear strength, R, of a RCC beam is given by the following model equation

$$R = B\left[1.1A_{sv}f_y\frac{d}{s} + 1.8566bd\left(f_{cu}\frac{A_s}{bd}\frac{d}{a}\right)^{1/3}\right]$$

where B is the model error, A_{sv} is the area of stirrups, b is the breadth, d is the effective depth, s is the spacing of stirrups, A_s is the area of tension steel and a is the shear span.

It is given:

$$
\begin{aligned}
f_y &: \mu = 469\ \text{N/mm}^2 & \sigma &= 46.9\ \text{N/mm}^2\\
f_{cu} &: \mu = 17.58\ \text{N/mm}^2 & \sigma &= 3.16\ \text{N/mm}^2\\
b &: \mu = 310.3\ \text{mm} & \sigma &= 9.47\ \text{mm}\\
d &: \mu = 556.3\ \text{mm} & \sigma &= 3.79\ \text{mm}\\
s &: \mu = 150\ \text{mm} & \sigma &= 13.5\ \text{mm}\\
B &: \mu = 1.2 & \sigma &= 0.13\\
V_D &: \mu = 73430\ \text{N} & \sigma &= 7343\ \text{N}\\
V_L &: \mu = 57830\ \text{N} & \sigma &= 16400\ \text{N}
\end{aligned}
$$

where V_D and V_L are the shear forces due to dead load and live load, respectively. f_{cu} and V_L follow the lognormal and Type 1 extremal (largest) distributions, respectively. All other variables are normally distributed. Determine the reliability index of the beam at the limit state of collapse in shear if $A_{sv} = 100.5\ \text{mm}^2$, $A_s/bd=0.008$ and $a/d=4$

(*Ans.* $\beta = 7.68$)

(ii) If the shear strength of the beam is predicted by the following model

$$R = B\left[f_y A_{sv}\frac{d}{s} + \frac{bd}{6}\sqrt{0.8 f_{cu}}\left\{\frac{\sqrt{1+5\theta}-1}{\theta}\right\}\right]$$

what is the reliability of the beam if θ is 2.175

(*Ans.* $\beta = 4.45$)

8.7 (i) The safety checking format of a steel column subjected to axial load P and bending moment M is as follows.

$$\left(\frac{M}{M_p}\right) + \left(\frac{P}{P_u}\right) \leqslant 1$$

where M_p is the plastic moment capacity of the column when there is no axial load and P_u is the ultimate axial load carrying capacity of the column under pure axial load case.

Area of the cross-section is 6496 mm² and plastic section modulus of the section is 678700 mm³. It is given:

For

$$
\begin{aligned}
f_y &: \mu = 262.5\ \text{N/mm}^2 & \sigma &= 26.25\ \text{N/mm}^2\\
P_D &: \mu = 0.398\times 10^6\ \text{N} & \sigma &= 0.398\times 10^5\ \text{N}\\
P_L &: \mu = 0.3108\times 10^6\ \text{N} & \sigma &= 0.870\times 10^5\ \text{N}\\
M_D &: \mu = 0.1785\times 10^8\ \text{N mm} & \sigma &= 0.1785\times 10^7\ \text{N mm}\\
M_L &: \mu = 0.1394\times 10^8\ \text{N mm} & \sigma &= 0.3945\times 10^7\ \text{N mm}
\end{aligned}
$$

where P_D and P_L are axial loads due dead load and live load respectively. M_D and M_L are moments due to dead load and live load respectively. Determine the reliability of the column.

(*Ans.* $\beta = 3.463$)

(ii) If the safety checking format uses a nonlinear model given by the following equation

$$\left(\frac{M}{M_p}\right) + \left(\frac{P}{P_u}\right)^2 \leqslant 1$$

what is the reliability of the column?

(*Ans.* $\beta = 4.22$)

9

Reliability Based Design

9.1 INTRODUCTION

In the last chapter, we studied the Level 2 (including advanced Level 2) methods in detail. Using the same methods, the evaluation of the reliability of structural elements was illustrated. Now the problem is reverse. One wants to produce a structural design which will ensure a certain level of reliability. That is to say, to provide a design for a specified level of risk/reliability. This was demonstrated in Example 8.5 also, where the depth of the girder was calculated to be safe against the limit state of collapse in shear, ensuring the required reliability level.

Consider the fundamental case: a structural element/system with a resistance R subjected to an action S. If R and S are independent normal variates,

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{9.1}$$

Therefore, the mean resistance (representing the design) required to ensure the specified reliability or target reliability, β_0, is

$$\mu_R = \mu_S + \beta_0\sqrt{\sigma_R^2 + \sigma_S^2} \tag{9.2}$$

If one uses the other safety format, assuming R and S are independent lognormal variates, the median value of the required resistance of the design is

$$\widetilde{R} = \widetilde{S}\exp[\beta_0(\delta_R^2 + \delta_S^2)^{1/2}] \tag{9.3}$$

But in practice, R is represented in terms of several resistance variables and design constants, and S in terms of load variables and design constants. For safety,

$$g_R(X_1, X_2, \ldots, X_m, C_1, C_2 \ldots) \geqslant g_S(X_{m+1}, X_{m+2}, \ldots, X_n, C_j, C_{j+1} \ldots) \tag{9.4}$$

where, $X_1, \ldots, X_m$ are the resisting variables, $X_{m+1}, \ldots, X_n$ are the loading variables, and $C_1, C_2, \ldots, C_j \ldots$ are design constants. g_R and g_S are resistance and load effect functions respectively.

If x_i^* are the design values of variables, then the design equation is

$$g_R(x_1^*, x_2^*, \ldots, x_m^*, C_1, C_2 \ldots) \geqslant g_S(x_{m+1}^*, x_{m+2}^*, \ldots, C_j, C_{j+1} \ldots) \tag{9.5}$$

The partial safety factor or the safety factor is defined with respect to a particular value of the variable. If it is defined with respect to the mean value, as given below,

$$\gamma_{ci} = \frac{x_i^*}{\mu_i} \tag{9.6}$$

It is called the central safety factor. μ_i is the mean value of X_i. If the partial safety factor is specified with respect to the specified characteristic value, x_{ki} of X_i (corresponding to five per cent fractile in the case of resistance variable and 95 per cent fractile in the case of load variable), then

$$\gamma_{ki} = \frac{x_i^*}{x_{ki}} \tag{9.7}$$

The partial safety factor, γ_i, defined with respect to the nominal value, x_{ni}, of the variable X_i is given by

$$\gamma_i = \frac{x_i^*}{x_{ni}} \tag{9.8}$$

In this text, whenever γ_i is used, it refers to the partial safety factor with respect to the nominal value. Using these γ_i, the design equation (Eq. 9.5) becomes

$$\begin{aligned} &g_R(\gamma_1 x_{n1}, \gamma_2 x_{n2}, \ldots, \gamma_k x_{nk}, C_1, C_2, \ldots) \geqslant \\ &g_S(\gamma_{k+1}\, x_{n,k+1}, \gamma_{k+2}\, x_{n,k+2}, \ldots, C_j, C_{j+1} \ldots) \end{aligned} \tag{9.9}$$

Presently, the reliability based design means arriving at these values of partial safety factors for a given target reliability for a particular failure criteria. Once safety factors are calculated, the design values are known and hence the design is proposed for the specified reliability. The computation of partial safety factors and the process of reliability based code calibration are dealt with in this chapter.

9.2 DETERMINATION OF PARTIAL SAFETY FACTORS

The reliability based design criteria is developed using the first-order second-moment approach. In the last chapter, the reliability analysis was introduced and illustrated using the Level 2 method. The probability of failure or reliability (in terms of β) was calculated for given safety factors for a given limit state. Now the process is reverse: partial safety factors are to be evaluated for the given target β. The same Level 2 reliability method can be used. In the normalized coordinate system, for a given failure surface, the shortest distance from the origin O to the failure surface defines the safety of the design. Different levels of safety (i.e. β) will yield different failure surfaces, as shown in Fig. 9.1, amounting to different designs. Hence, in the reliability based design, the problem is to determine the design values of the variables that will result in designs having failure surfaces that comply with a required safety index β. If x_i^* is the design value of the original variable X_i, the failure surface equation is

$$g(x_1^*, x_2^*, \ldots, x_n^*) = 0 \tag{9.10}$$

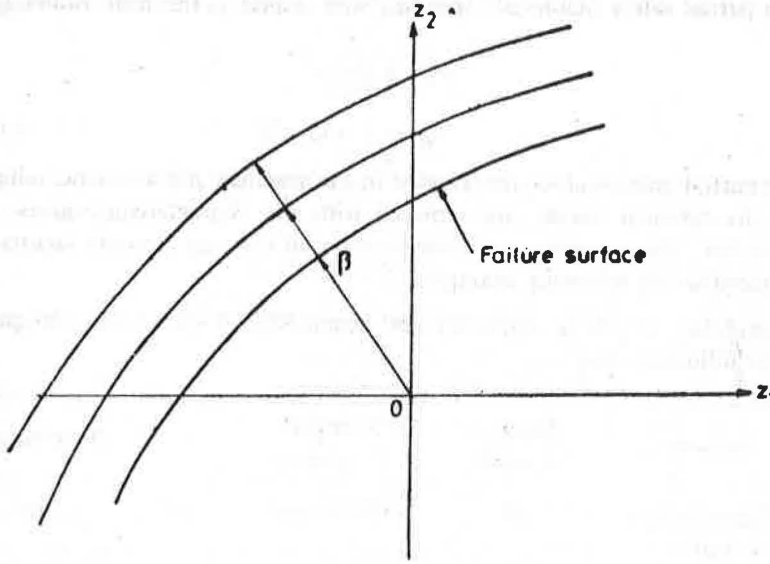


FIG. 9.1 Design corresponding to different reliability indices

If the partial safety factors are attached to the nominal values of variables, the above equation becomes

$$g(\gamma_1 x_{n1}, \gamma_2 x_{n2}, \ldots, \gamma_n x_{nn}) = 0 \tag{9.11}$$

The design point should be the most probable failure point. Now the problem is to determine the most probable failure point. In the normalized coordinate system, the most probable failure point is given by [Sec. 8.3.1: Eqs. (8.45) and (8.46)]

$$z_i^* = \alpha_i^* \beta \tag{9.12}$$

where
$$\alpha_i^* = \frac{-(\partial g_1/\partial z_i)^*}{[\Sigma(\partial g_1/\partial z_i)_*^2]^{1/2}} \tag{9.13}$$

The original variates are given by

$$x_i^* = \mu_i + \sigma_i z_i^*$$

$$= \mu_i + \sigma_i \alpha_i^* \beta \tag{9.14}$$

This equation can also be written as

$$x_i^* = \mu_i(1 + \delta_i \alpha_i^* \beta) \tag{9.15}$$

where δ_i is the coefficient of variation of X_i.
Hence the partial safety factor required for the given β is

$$\gamma_i = \frac{x_i^*}{x_{ni}} = \mu_i \frac{(1 + \delta_i \alpha_i^* \beta)}{x_{ni}} \tag{9.16}$$

If the partial safety factors are specified with respect to the mean values, i.e.

$$\gamma_{ci} = \frac{x_i^*}{\mu_i}$$

then
$$\gamma_{ci} = 1 + \delta_i \alpha_i^* \beta \tag{9.17}$$

If the partial safety factors are referred to the specified characteristic values, then the nominal values are replaced with the characteristic values in Eq. (9.16). The procedure of computation of the partial safety factors is illustrated in the following examples.

EXAMPLE 9.1 A simply supported steel beam (RSJ) of span 8 m is designed for the following data:

Variable	$\frac{\text{Mean}}{\text{Nominal}}$	Nominal value	δ	Distribution
X_1: Yield strength of steel	1.10	250 N/mm²	0.10	normal
X_2: Dead load	1.05	11.0 N/mm	0.10	normal
X_3: Live load	0.70	12.0 N/mm	0.40	normal

Determine the partial safety factors for the design variables X_i if the target reliability is 4.0.

Solution The limit state equation in the original space, $g(\mathbf{X}) = 0$, is

$$X_1 Z_p - X_2\left(\frac{l^2}{8}\right) - X_3\left(\frac{l^2}{8}\right) = 0 \tag{9.18}$$

where l is the span and Z_p is the plastic section modulus of the section. Normalizing the variables by using the equation

$$Z_i = \frac{X_i - \mu_i}{\sigma_i}$$

the limit state equation in the z space is

$$g_1(\mathbf{z}) = Z_p(\sigma_1 z_1 + \mu_1) - \frac{l^2}{8}(\sigma_2 z_2 + \mu_2) - (\sigma_3 z_3 + \mu_3)\frac{l^2}{8} = 0$$

Let

$$A = 8\frac{Z_p}{l^2}$$

Then

$$g_1(\mathbf{z}) = A(\sigma_1 z_1 + \mu_1) - (\sigma_2 z_2 + \mu_2) - (\sigma_3 z_3 + \mu_3) = 0 \tag{9.19}$$

Using Eq. (8.46),

$$\alpha_i = -\frac{1}{K}\left(\frac{\partial g_1}{\partial z_i}\right)_*$$

where

$$K = \left[\sum_{i=1}^{n}\left(\frac{\partial g_1}{\partial z_i}\right)_*^2\right]^{1/2}$$

$$\alpha_1 = -\frac{1}{K}(\sigma_1 A)$$

$$\alpha_2 = -\frac{1}{K}(-\sigma_2)$$

$$\alpha_3 = -\frac{1}{K}(-\sigma_3)$$

Since the limit state equation is linear, and all variables follow the normal distribution, the reliability index is given by

$$\beta = \frac{A\mu_1 - \mu_2 - \mu_3}{[(A\sigma_1)^2 + \sigma_2^2 + \sigma_3^2]^{1/2}} \tag{9.20}$$

In this design problem, the value of β is already given as 4. Hence

$$\frac{A\mu_1 - \mu_2 - \mu_3}{(A^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2)^{1/2}} = \beta$$

$$(A\mu_1 - \mu_2 - \mu_3)^2 = \beta^2(A^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

The quadratic equation in A becomes

$$b_1A^2 - b_2A + b_3 = 0$$

where

$$b_1 = \mu_1^2 - \beta^2\sigma_1^2$$

$$b_2 = 2\mu_1(\mu_2 + \mu_3)$$

$$b_3 = \mu_2^2 + \mu_3^2 + 2\mu_2\mu_3 - \beta^2(\sigma_2^2 + \sigma_3^2)$$

Substituting the given values of μ_i, σ_i and β, and solving the quadratic equation, we have

$$b_1 = 63525, \qquad b_2 = 10972.5, \qquad b_3 = 196$$

$$A = 0.153$$

Using the computed value of A, the directional cosines α_i can be calculated.

$$\alpha_1 = -\frac{1}{K}(27.53 \times 0.153)$$

$$\alpha_2 = -\frac{1.15}{K} \qquad \alpha_3 = \frac{3.36}{K}$$

Using $\quad \Sigma\,\alpha_i^2 = 1$ and $K = 5.507$, we have

$$\alpha_1 = -0.764, \qquad \alpha_2 = 0.21, \qquad \alpha_3 = 0.61$$

Design points in the original space are

$$x_1^* = \mu_1 + \alpha_1 \beta \sigma_1$$

$$= 275 - 0.764 \times 4 \times 27.5 = 190.96$$

$$x_2^* = 11.55 + 0.21 \times 4 \times 1.155 = 12.52$$

$$x_3^* = 8.4 + 0.61 \times 4 \times 3.36 = 16.6$$

Hence, the partial safety factors with respect to the nominal values are

$$\gamma_1 = \frac{190.96}{250} = 0.764$$

$$\gamma_2 = \frac{12.52}{11} = 1.138$$

$$\gamma_3 = \frac{16.6}{12} = 1.383$$

Here γ_1 is the partial safety factor (multiplying factor) for the yield strength of steel. (Note: In IS and British codes, $1/\gamma_1$ is taken as the partial safety factor for materials. That is, $1/0.764 = 1.309$). Hence the design equation

$$0.764 \, f_{yn} Z_p \geqslant \frac{l^2}{8} (1.138 \, D_n + 1.383 \, L_n)$$

will ensure a reliability level of β equal to atleast 4 for the given data. f_{yn} is the nominal value of f_y.

For example, if a beam is to be designed for a span of 6 m and for the same nominal loads, the section modulus required is given by the condition

$$0.764 \times 250 \, Z_p \geqslant \frac{36 \times 10^6}{8} (1.138 \times 11 + 1.383 \times 12)$$

Z_p required is 685930 mm^3. If this Z_p is provided, the reliability analysis can be performed and it will be found that $\beta = 4$ for the same mean values and standard deviations of f_y, D and L.

Example 9.2 For the same problem in Example 9.1, what are the values of the partial safety factors with respect to (i) the mean values and (ii) the characteristic values.

Solution Case (i)
From Example 9.1, the design points in the original space are

$$x_1^* = 190.96, \qquad x_2^* = 12.52, \qquad X_3^* = 16.6$$

Hence, the partial safety factors with respect to the mean values are

$$\gamma_i = \frac{x_i^*}{\mu_i}$$

$$\gamma_1 = \frac{190.96}{275} = 0.694$$

$$\gamma_2 = \frac{12.52}{11.55} = 1.084$$

$$\gamma_3 = \frac{16.60}{8.4} = 1.976$$

Hence the design equation is

$$0.694\, \mu_{fy} Z_p \geqslant \frac{l^2}{8}(1.084\, \mu_D + 1.976\, \mu_L)$$

to ensure a reliability level of $\beta = 4$.

Case (ii)

The partial safety factors with respect to the characteristic values are

$$\gamma_i = \frac{x_i^*}{x_{ki}}$$

where x_{ki} is the characteristic value of x_i. For the yield strength of steel, (5% fractile)

$$\begin{aligned} x_{k1} &= \mu_1 - 1.64\, \sigma_1 \\ &= 275 - 1.64 \times 27.5 = 229.9 \end{aligned}$$

For dead load (95% fractile)

$$\begin{aligned} x_{k2} &= \mu_2 + 1.64\, \sigma_2 \\ &= 11.55 + 1.64 \times 1.155 = 13.44 \end{aligned}$$

For live load (95% fractile)

$$\begin{aligned} x_{k3} &= \mu_3 + 1.64\, \sigma_3 \\ &= 8.4 + 1.64 \times 3.36 = 13.9 \end{aligned}$$

Hence, the partial safety factors with respect to the characteristic values are

$$\gamma_{k1} = \frac{190.96}{229.9} = 0.831$$

$$\gamma_{k2} = \frac{12.52}{13.44} = 0.932$$

$$\gamma_{k3} = \frac{16.60}{13.90} = 1.194$$

Hence the design equation for $\beta = 4$ is

$$0.831\, f_{yk} Z_p \geqslant \frac{l^2}{8}(0.932\, D_k + 1.194\, L_k)$$

where f_{yk}, D_k and L_k are the characteristic values of f_y, D and L respectively.

EXAMPLE 9.3 For the same problem in Example 9.1, what is the value of the combined load factor ?

Solution From Example 9.1, the design equation is

$$0.764\, f_{yn} Z_p \geqslant \frac{l^2}{8}\,(1.138\, D_n + 1.383\, L_n)$$

We want to propose the design equation

$$\left|\ 0.764\ f_{yn}\ Z_p \geqslant \frac{l^2}{8}\,[\gamma_T(D_n + L_n)]\right.$$

such that it will ensure β equal to alteast 4. γ_T is the combined load factor on total load. This is computed as follows:

$$1.138\, D_n + 1.383\, L_n = \gamma_T(D_n + L_n)$$

$$\gamma_T = \frac{1.138\, D_n + 1.383\, L_n}{D_n + L_n}$$

$$= \frac{1.138 \times 11 + 1.383 \times 12}{11 + 12}$$

$$= 1.266$$

Hence the design equation becomes

$$0.764\, f_{yn} Z_p \geqslant \frac{l^2}{8}\,[1.266\,(D_n + L_n)]$$

In Example 9.1, no iteration is involved as the failure surface equation is a linear function of the normal variables. If the failure function is nonlinear and/or the variables are nonnormal, the problem is to be solved iteratively. This is illustrated in the following example.

EXAMPLE 9.4 Consider the same problem in Example 9.1. Determine the partial safety factors for $\beta = 4$, if the yield strength of steel (X_1) and live load (X_3) follow the lognormal and Type 1 extremal (largest) distribution respectively.

Solution The failure surface equation is

$$Z_p X_1 - \frac{l^2}{8} X_2 - \frac{l^2}{8} X_3 = 0$$

Let the design constant A be

$$A = 8\,\frac{Z_p}{l^2} \tag{9.21}$$

If x_1^*, x_2^* and x_3^* are the design points, then

$$A x_1^* - x_2^* - x_3^* = 0 \tag{9.22}$$

Since X_1 and X_3 are nonnormal, the equivalent means and standard deviations of nonnormal variables (X_1, X_3) are to be used.

Hence the failure surface equation in normalized variables becomes

$$A(\sigma_1' z_1 + \mu_1') - (\sigma_2 z_2 + \mu_2) - (\sigma_3' z_3 + \mu_3') = 0$$

Using Eq. (8.46), the directional cosines are computed:

$$\alpha_1 = \frac{-(A\,\sigma_1')}{K} \tag{9.23}$$

$$\alpha_2 = \frac{\sigma_2}{K} \tag{9.24}$$

$$\alpha_3 = \frac{\sigma_3'}{K} \tag{9.25}$$

The procedure of computation of the partial safety factors is as follows:

(i) Start with any x_1^*, x_2^* and x_3^*.

Using Eqs. (8.67) and (8.69), compute σ_1' and μ_1' and σ_3' and μ_3' of the non-normal variables X_1 and X_3 at the design point $\mathbf{x}*$.

(iii) Compute A using Eq. (9.22).

(iv) Determine the directional cosines α_i using Eqs. (9.23) to (9.25.)

(v) Determine the new design point

$$x_i^* = \mu_i' + \sigma_i' \alpha_i \beta$$

(vi) Go to step (ii) and repeat the procedure till the required convergence is achieved.

For example, stop if

$$\left[\frac{A_j - A_{j-1}}{A_j}\right] \leqslant 0.005 \tag{9.26}$$

and/or

$$\left[\frac{x_{ji} - x_{j-1,i}}{x_{ji}}\right] \leqslant 0.005 \qquad i = 1, 2, 3 \tag{9.27}$$

where j stands for the jth iteration.

For calculating the equivalent σ_i' and μ_i' of X_i, parameters $\sigma_{\ln X1}$ an $\tilde{X}_1$ are computed using Eqs. (3.91) and (3.92):

$$\sigma_{\ln X1} = [\ln(\delta_X^2 + 1)]^{1/2} = 0.1$$

$$\tilde{X}_1 = \mu_1 \exp\left(-\frac{1}{2}\,\sigma_{\ln X1}^2\right) = 273.6$$

Using Eqs. (3.115) and (3.116), the parameters of X_3 following the Type extremal distribution are calculated:

$$\alpha = \frac{\pi}{\sqrt{6}\,\sigma_3} = 0.382$$

$$u = 8.4 - \frac{0.57722}{0.382} = 6.839$$

Start with

$$x_1^* = \mu_1 = 275$$

$$x_2^* = \mu_2 = 11.55 \qquad x_3^* = \mu_3 \doteq 8.4$$

At $x_1^* = 275$, using Eqs. (8.76) and (8.77),

$$\sigma_1' = x_1^* \sigma_{\ln X1} = 27.43$$

$$\mu_1' = x_1^*(1 - \ln x_1^* + \ln \tilde{X}_1) = 273.6$$

At $x_3^* = 8.4$ using Eqs. (3.113) and (3.114),

$$F(x_3^*) = \exp[-\exp\{-0.382(8.4 - 6.839)\}]$$
$$= 0.5704$$

$$f(x_3^*) = 0.382 \exp[-0.382(8.4 - 6.389)$$
$$-\exp\{-0.382(8.4 - 6.839)\}]$$
$$= 0.1222$$

Using Eqs. (8.67) and (8.69),

$$\sigma_3' = \frac{\phi[\Phi^{-1}(0.5704)]}{0.1222}$$
$$= 3.213$$

$$\mu_3' = 8.4 - \sigma_3'\{\Phi^{-1}(0.5704)\}$$
$$= 7.831$$

Using Eq. (9.22) compute A:

$$A = (11.55 + 8.4)/275$$
$$= 0.0725$$

The directional cosines are

$$\alpha_1 = -0.504 \qquad \alpha_2 = 0.292 \qquad \alpha_3 = 0.813$$

New design points in the original space are

$$x_1^* = 273.6 - 0.504 \times 27.43 \times 4 = 218.4$$

$$x_2^* = 11.55 + 0.292 \times 4 \times 1.155 = 12.9$$

$$x_3^* = 7.831 + 0.813 \times 4 \times 3.213 = 18.28$$

With these new values of x_i^*, the whole process is repeated till the required convergence is achieved. The results are summarized in Table 9.1.

At the end of the fourth iteration,

$$A = 0.1812$$

$$x_1^* = 235.1 \qquad x_2^* = 12.03 \qquad x_3^* = 30.57$$

The partial safety factors with respect to the nominal values are

$$\gamma_1 = \gamma_{fy} = \frac{235.1}{250} = 0.94$$

TABLE 9.1 *Computation of partial safety factors—Example 9.4*

Variable	Start	Iteration 1	Iteration 2	Iteration 3	Iteration 4
x_1^*	275.0	218.4	231.8	234.8	235.1
x_2^*	11.55	12.9	12.26	12.07	12.03
x_3^*	8.40	18.28	27.54	30.38	30.57
A	0.0725	0.1428	0.1717	0.1807	0.1812
σ_1'	27.43	21.76	23.11	23.42	
μ_1'	273.6	267.6	270.2	270.7	
σ_3'	3.21	6.79	9.50	10.21	
μ_3'	7.83	3.12	−4.47	−6.97	
α_1	−0.504	−0.411	−0.383	−0.381	
α_2	0.292	0.153	0.112	0.104	
α_3	0.813	0.899	0.917	0.919	

$$\gamma_2 = \gamma_D = \frac{12.03}{11.0} = 1.093$$

$$\gamma_3 = \gamma_L = \frac{30.57}{12} = 2.548$$

Hence the design equation is

$$0.94\, Z_p f_y \geqslant \frac{l^2}{8}\,[1.093\, D + 2.548\, L]$$

to ensure a reliability level of $\beta = 4$.

EXAMPLE 9.5 The ultimate strength of a RCC beam is given by

$$R = f_y A_s d\left[1 - \frac{0.77\, f_y\, A_s}{b\, d\, f_{cu}}\right] \qquad (9.28)$$

$f_y A_s d - \frac{0.77 f_y A_s^2}{b f_{cu}}$

Let the beam be subjected to a bending moment M due to the dead load and live load. Then the failure surface equation is

$$R - M = 0 \qquad (9.29)$$

The main basic variables in this case are f_y, f_{cu} and M. However, if we compute the partial safety factors for f_y, f_{cu} and M, we may end up with a value of γ for concrete > 1.0 and sometimes with high values more than 1.5. This can be quite misleading. This happens because the compressive strength of concrete does not play a significant role in determining the flexural strength of the RCC beam. Hence what is done is, the partial safety factor for concrete is prefixed or selected to account for the various uncertainties. The concrete strength may play a significant role in columns.

Let the partial safety factor for concrete strength be 0.667 (given in the present code as $1/\gamma_{mc} = 1/1.5$). Therefore, the design strength of M 15 concrete is 10 N/mm².

It is given:
Variable f_y: (normal)

$$\text{Mean} = 320 \text{ N/mm}^2 \qquad \sigma = 32 \text{ N/mm}^2$$

$$\text{Nominal value} = 250 \text{ N/mm}^2$$

Variable M: (Normal)

$$\text{Mean} = 0.82 \times 10^8 \text{ N mm} \qquad \sigma = 0.12 \times 10^8 \text{ N mm}$$

$$\text{Nominal value} = 0.8 \times 10^8 \text{ N mm}$$

Compute the partial safety factors for steel strength and bending moment for a reliability index $\beta = 4$, $b = 240$ mm and $d = 480$ mm.

Solution Let

$$X_1 = f_y \qquad X_2 = M$$

Using Eqs. (9.28) and (9.29), the failure surface equation becomes

$$10\, A_s\, d\, X_1 - \left(\frac{0.77}{b}\right) A_s^2 X_1^2 - 10\, X_2 = 0 \tag{9.30}$$

Start with

$$x_1^* = 320 \qquad x_2^* = 0.82 \times 10^8$$

Substituting the above values, and given values of b and d in Eq. (9.30) and solving the same, we have

$$A_s = 614.7 \text{ mm}^2$$

The directional cosines are

$$\alpha_1 = -\frac{1}{K}\left[\left(10\, A_s\, d - \frac{1.44}{b}\, A_s^2\, x_1^*\right) \sigma_1\right] \tag{9.31}$$

$$\alpha_2 = \frac{1}{K}\, (10\, \sigma_2) \tag{9.32}$$

Using the calculated value of $A_s = 614.7$ and other data, the directional cosines can be evaluated. They are

$$\alpha_1 = -0.502 \qquad \alpha_2 = 0.865$$

The new values of design points, using

$$x_i^* = \mu_i + \alpha_i\, \beta \sigma_i$$

are given by

$$x_1^* = 255.8 \qquad x_2^* = 0.124 \times 10^9$$

With these new values of x_i^*, the whole process is repeated till the required convergence is achieved. Results of iterations are given in Table 9.2.
The design points are

$$x_1^* = 229.3 \qquad x_2^* = 11.6 \times 10^7$$

TABLE 9.2 *Summary of calculations—Example 9.5*

Variable	Start	Iteration 1	2	3	4
x_1^*	320.0	255.8	233.2	229.8	229.3
x_2^*	8.2×10^7	12.4×10^7	11.7×10^7	11.6×10^7	11.6×10^9
A_s	614.7	1291	1318.8	1319.6	1319.7
α_1	−0.502	−0.678	−0.705	−0.709	
α_2	0.865	0.735	0.709	0.706	

The partial safety factors are

$$\gamma_1 = \gamma_{fy} = \frac{229.3}{250} = 0.917$$

$$\gamma_2 = \gamma_M = \frac{11.6\times10^7}{0.80\times10^8}$$

$$= 1.45$$

EXAMPLE 9.6 The shear strength of a RCC beam is given by (9.1)

$$R = 1.1\, A_{sv} f_y \frac{d}{s} + 1.8566\, bd\left(f_{cu}\frac{A_s}{bd}\frac{d}{a}\right)^{1/3} \tag{9.33}$$

where A_{sv} is the area of the stirrups, s is the spacing of the stirrups, A_s is the area of the tension steel and (a/d) is the shear span ratio. For the limit state of collapse in shear, the failure surface equation is

$$R - V_D - V_L = 0$$

where V_D and V_L are the shear forces due to dead load and live load respectively. It is given:

$$b = 300 \text{ mm} \qquad d = 580 \text{ mm} \qquad s = 100 \text{ mm}$$

$$\frac{a}{d} = 4 \qquad \frac{A_s}{bd} = 0.008$$

Variable f_y: (Nominal value = 250 N/mm²)

$$\mu = 320 \text{ N/mm}^2 \qquad \sigma = 32 \text{ N/mm}^2$$

Variable f_{cu}: (Nominal value = 20 N/mm²)

$$\mu = 26.8 \text{ N/mm}^2 \qquad \sigma = 4.02 \text{ N/mm}^2$$

Variable V_D (Nominal value = 70.0 kN)

$$\mu = 73.5 \text{ kN} \qquad \sigma = 7.35 \text{ kN}$$

Variable V_L: (Nominal value = 50 kN)

$$\mu = 41.35 \text{ kN} \qquad \sigma = 11.70$$

Determine the partial safety factors for f_y, f_{cu}, V_D and V_L for $\beta_0 = 5$, assuming all variables are normally distributed.

Solution Let

$$X_1 = f_y \qquad X_2 = f_{cu}$$

$$X_3 = V_D \qquad X_4 = V_L$$

$$a_1 = 1.1\frac{d}{s} = 1.1 \times \frac{580}{100} = 6.38$$

$$a_2 = 1.8566\, bd\left(\frac{A_s}{bd}\frac{d}{a}\right)^{1/3}$$

$$= 1.8566 \times 300 \times 580\left(\frac{0.008}{4}\right)^{1/3} = 40701$$

Then the failure surface equation can be written as

$$a_1 A_{sv} X_1 + a_2 X_2^{1/3} - X_3 - X_4 = 0 \tag{9.34}$$

Start with

$$x_1^* = 220 \qquad x_2^* = 18$$

$$x_3^* = 80{,}000 \qquad x_4^* = 70{,}000$$

Using these values, and a_1 and a_2 in Eq. (9.34),

$$A_{sv} = 30.87$$

The directional cosines are

$$\alpha_1 = -\frac{1}{K}(\sigma_1 A_{sv} a_1)$$

$$\alpha_2 = -\frac{1}{K}\left(\frac{\sigma_2 a_2}{3(x_2^*)^{2/3}}\right)$$

$$\alpha_3 = \frac{\sigma_3}{K} \qquad \alpha_4 = \frac{\sigma_4}{K}$$

Substituting the computed value of A_{sv} and other given data in the above equations, the computed directional cosines are

$$\alpha_1 = -0.368 \qquad \alpha_2 = -0.463$$

$$\alpha_3 = 0.429 \qquad \alpha_4 = 0.683$$

The new values of x_i^* using

$$x_i^* = \mu_i + \alpha_i \beta \sigma_i$$

are given by

$$x_1^* = 261 \qquad x_2^* = 17.5$$

$$x_3^* = 89260 \qquad x_4^* = 81300$$

With these new values, the whole process is repeated till the required convergence is achieved. At the end of the second iteration, the final values of x_i^* are (Table 9.3):

$$x_1^* = 248.8 \qquad x_2^* = 17.69$$

$$x_3^* = 88600 \qquad x_4^* = 79610$$

TABLE 9.3 *Results of iterations—Example 9.6*

Variable	Start	Iteration 1	Iteration 2
x_1^*	220.0	261.0	248.8
x_2^*	18.00	17.50	17.69
x_3^*	80000	89260	88600
x_4^*	70000	81300	79610
A_{sv}	30.87	38.96	39.16
α_1	−0.368	−0.445	−0.449
α_2	−0.463	−0.453	−0.451
α_3	0.429	0.411	0.413
α_4	0.683	0.654	0.657

The partial factors are

$$\gamma_1 = \gamma_{f_y} = \frac{248.8}{250} = 0.995$$

$$\gamma_2 = \gamma_{f_c} = \frac{17.69}{20} = 0.885$$

$$\gamma_3 = \gamma_{V_D} = \frac{88600}{70000} = 1.266$$

$$\gamma_4 = \gamma_{V_L} = \frac{79760}{50000} = 1.595$$

Note: Readers' attention is drawn to the point that in the text all γ_i are multiplying factors.

As stated in Example 9.1, in the IS and British codes, $1/\gamma_i$ is taken as the partial safety factor for materials, and are collectively called as the material reduction factors. As per this, the partial material reduction factors are

$$\gamma_{ms} = \frac{1}{0.995} = 1.005$$

$$\gamma_{mc} = \frac{1}{0.885} = 1.131$$

9.3 SAFETY CHECKING FORMATS

The safety checking format for a code is defined as the number of partial safety factors and the way in which they are introduced into the design equations. For the safety of the structure,

Factored resistance $\geqslant$ effect of factored loads

In the development of probability based limit state design criteria, different national codes use different formats.

NBC-(Canada) Format

The National Building Code of Canada (9.2) uses the following probability factor format

$$\gamma_R R_n \geqslant g_S \{\gamma_D D_n + \Psi(\gamma_L L_n + \gamma_W W_n + \ldots + \gamma_E E_n)\} \tag{9.35}$$

where g_S refers to the function that converts the loads to load effects in brackets, and γ_D, γ_L, ... are the corresponding partial safety factors or load factors for the loads. Ψ is a load combination probability factor depending on one, two, or three loads included in the brackets. The value of Ψ is less than or equal to 1. This factor takes care of the reduced probability of the simultaneous occurrence of loads. The values given are 1.0, 0.7, and 0.6, respectively, for one or two or three loadings acting simultaneously. The terms γ_D, γ_L, ... take care of variations in the load itself plus variations in the load effects due to uncertainties in the load model and the structural analysis.

The factor γ_R represents the overall resistance factor, based on characteristic strengths, material properties, dimension, etc. This factor is intended to reflect the probability that the member as a whole is understrength.

CEB Format

CEB committee (9.3) recommends the following format

$$g_R\left(\frac{f_k}{\gamma_{m1}\gamma_{m2}\gamma_{m3}}\right) \geqslant g_S(\gamma_{f1}\gamma_{f2}\gamma_{f3}Q_k) \tag{9.36}$$

where g_R and g_S are the resistance and load effect functions which convert the terms in the brackets to resistance and load effects respectively. f_k and Q_k are the characteristic strengths and loads respectively. γ_{mi} is the material reduction factor. It is to be noted that $\gamma_{mi} \geqslant 1$. γ_{fi} is the multiplicative factor on the load.

The material reduction factor γ_{mi} is intended to take into account (9.3)

(i) the material strengths occasionally falling below the specified characteristic value

(ii) the possible difference between the strength of the material in the structure obtained from control test specimens

(iii) the possible weakness in the structural material or element structure resulting from the construction process

(iv) the possible inaccurate assessment of the resistance of a structural element resulting from modelling errors (say, models derived from the elementary strength of materials)

(v) the effects of poor dimensional accuracy in the finished structure on the resistance of a section

The partial factors for loads, γ_{fi}, are introduced to account for the following factors:

(i) γ_{f1}: for the possibility of loads occasionally exceeding their characteristic values

(ii) γ_{f2}: multiplicative load combination factor for the reduced probability of all loads exceeding their characteristic values simultaneously

(iii) γ_{f3}: multiplicative factor on load effects for possible errors in predicting load effects as a result of inaccurate structural analysis and as a result of neglecting dimensional inaccuracies.

In addition, either γ_m or γ_f may be modified to take care of the nature of the structure and the seriousness of attaining the limit state.

The European Concrete Committee Model Code (9.4) recommends the following equation:

$$g_R\left(\frac{f_k}{\gamma_{m1}\gamma_{m2}\gamma_{m3}}\right) \geqslant g_S\left\{\gamma_D\mu_D + \gamma_Q\left[Q_{1k} + \sum_{j>1}^{n} (\Psi_{0j}Q_{jk})\right]\right\} \quad (9.37)$$

where Q_{1k} represents the characteristic value of the main time varying load Q_1, and $Q_{2k} \ldots, Q_{nk}$ are the characteristic values of other less dominant time varying loads $Q_2, \ldots, Q_n$. Ψ_{0j} is considered as the ratio of the arbitrary point-in-time value of the jth load to the characteristic value of that load. γ_Q is the load factor on the combination of time varying loads. It consists of $\gamma_{f1}\gamma_{f3}$. While determining the maximum factored load effect for a case involving several time varying loads, it may be necessary to consider several combinations with each of the loads considered as the most dominant load (i.e. Q_{1k}) in turn. Hence, in the above format, when a structure has to resist a number of stochastically independent time varying loads, a number of load combinations are to be considered. For a situation with dead, live, wind and snow loading, the CEB format requires a checking of 32 load combinations. If the NBC format [Eq. (9.35)] is selected for loads, viz. dead load, live load, wind load and snow load, a total of 14 load combinations are to be considered.

However, the Load Resistance Factor Design (LRFD) checking format, discussed below, requires only four load combinations to be considered.

LRFD Format

The load and resistant factor design checking format, proposed by Ravindra Galambos, Ellingwood, et al. (9.5, 9.6) recommends only four load combinations to be considered. They are

$$\gamma_R R_n \geqslant \gamma_D\mu_D + \gamma_L\mu_{L_m} \quad (9.38)$$

$$\gamma_R R_n \geqslant \gamma_D\mu_D + \gamma_{apt}\mu_{L_{apt}} + \gamma_W\mu_{W_m} \quad (9.39)$$

$$\gamma_R R_n \geqslant \gamma_D\mu_D + \gamma_{apt}\mu_{L_{apt}} + \gamma_S\mu_{S_m} \quad (9.40)$$

$$\gamma_R R_n \geqslant \gamma_W\mu_{W_m} - \gamma_D\mu_D \quad (9.41)$$

where μ_D is the load effect due to the mean dead load, μ_{L_m}, μ_{W_m} and μ_{S_m} are the load effects due to means of the maximum lifetime live load, maximum lifetime wind load, and maximum lifetime snow load respectively. Here the term $\gamma_{apt}\,\mu_{L_{apt}}$ in Eq. (9.39) is equivalent to

$g_S(\gamma_{f1}\gamma_{f3}\Psi_{Oj}Q_{jk} = \gamma_Q\Psi_{Oj}Q_{jk})$ in Eq. (9.37), the major difference being that the load is given as a multiple of the maximum load $(\Psi_{Oj}Q_{jk})$ in Eq. (9.37) but as a separate loading case with its own load factors in Eq. (9.39). The load factors, in general, should be applied to the loads before performing the analysis which transforms loads to load effects. If the relation between load and load effect is linear, load factors can be applied directly to load effects.

9.4 DEVELOPMENT OF RELIABILITY BASED DESIGN CRITERIA

Before starting the procedure for the development of design criteria (evaluation of partial safety factors), the precise scope of the work should be defined. That is, the types of structures for which it is applicable, the types of materials that will be used, and the range of parameters that will be covered. The proposed work should be compatible with the present code. It should also specify the range of application of the code and the different limit states (ultimate and serviceability) considered in the work.

One must specify the safety checking format selected. By format is meant the number of partial factors and the way in which they are introduced in the design equations (i.e. on loads, load effects, material strengths, resistances, etc.).

It must also specify the basis on which the loads have been developed. That is to say, whether the loads have been developed for a 50-year design period or a 25-year design period. It means specifying the selection of the period for a risk assessment for the class of structures being considered.

The development of a reliability based design criteria involves the following steps:

(i) collection and statistical analysis of the data on basic variables. Defining of the probability distribution of each variable—at least in terms of mean values, standard deviations and probability distribution type

(ii) statistical study of the strengths (resistances) of members and establishing their statistics

(iii) reliability analysis and determination of the reliability index β for the members designed as per the present code for each load combination

(iv) selection of the target reliability index, β_0, (i.e. accepted or specified level of reliability)

(v) determination of the partial safety factors for the desired uniform reliability β_0 under all design situations within the scope of the work.

For illustration, let us assume that the scope of our work is to determine the partial safety factors for RCC members (slabs, beams, columns) for Indian conditions. The limit states considered are the limit states of collapse in flexure, shear, combined axial load and bending moment in columns. The lifetime of structures is selected as 50 years. The safety checking format used is as per the LRFD method.

An extensive data on the basic variables, viz. the mechanical properties of different grades of steel and concrete, structural dimensions of RCC members slabs, beams, and columns, including the position of steel for Indian conditions, has been collected and statistically analysed, and the results of the same (9.7–9.10) are given in Chapter 4. The consolidated list of statistics of the basic variables is given in Table 9.4. The statistical analysis of the data on live loads on office buildings, and wind loads based on wind speeds observed at various stations in India is given in Table 9.4 (9.11, 9.12, 9.13). The statistics of the lifetime maximum live load L_m and the lifetime maximum wind load W_m, given in Table 9.4, are based on the selected design period of 50 years.

TABLE 9.4 *Statistics of basic variables*

Variable X		μ_X/X_n	δ	Probability distribution
f_{cu}				
Nominal Mix	M 15	1.51	0.24	Lognormal
	M 20	1.46	0.21	Normal
Design Mix	M 15	1.17	0.18	Lognormal
	M 20	1.34	0.15	Normal
	M 25	1.21	0.15	Normal
f_y				
Fe 250		1.28	0.10	Normal
Fe 415		1.13	0.10	Normal
Slabs				
d (mm)		1.87*	4.17**	Normal
Beams				
b (mm)		10.29*	9.47**	Normal
d (mm)		6.25*	3.79**	Normal
s (mm)		0.00	13.50**	Normal
Columns				
b (mm)		−0.25*	5.69**	Normal
D (mm)		0.113*	9.89**	Normal
Bar placement (mm)		0.640*	12.00**	Normal
Loads				
D		1.05	0.10	Normal
L_m		0.620	0.28	$EX_{I,l}$
L_{apt}		0.179	0.55	Lognormal
W_m		0.804	0.334	$EX_{I,l}$ +
W_{apt}		0.045	0.743	$EX_{I,l}$

*Deviation from mean (mm); **Standard deviation (mm)
+ $EX_{I,l}$ denotes Type I extremal (largest).

Statistical Study of Strength of Members

The concrete members, considered here, are slabs, beams, and columns. The limit states considered are the limit states of (i) collapse due to flexure and shear in beams, (ii) collapse due to flexure in slabs, and (iii) collapse of columns subjected to axial load and uniaxial bending. Hence, the partial factors presented apply to only these cases.

The strength of RCC members vary from the calculated nominal strength due to variations in the material strengths and dimensions of members, as well as due to uncertainties inherent in the theoretical model chosen to compute the member strength. The Monte Carlo technique, dealt with in detail in Chapter 7, is used to establish the statistics of the strengths of members in flexure, shear, etc. The procedure involves the following steps: (i) Selection of a theoretical model to calculate the member strength for a particular limit state and the model error associated with the same. The model error, say for the flexural strength of beams, is to be obtained by collecting data on the experimental results of beams tested for the ultimate strength in flexure and comparing these values with values obtained by using the theoretical model equation for predicting the ultimate strength of beam. The collected data can be statistically analysed and the mean and standard deviation of the model error can be fixed. (ii) Choosing a series of representative cross sections or members (different sizes, different boundary conditions, different spans, different percentages of steel, different grades of concrete and steel, etc.), each defined by a set of nominal strengths and dimensions. (iii) Establishing the statistics of the resistance of each selected member is carried out as follows: For the selected member nominal resistance, R_n, is computed based on the nominal material strengths and dimensions substituted in the theoretical model with the resistance factor as unity. This value of R_n corresponds to the failure mode expected when nominal strengths exist in the members. The design resistance, R_D, is computed from the model equation given by the present code using nominal values with partial factors or material reduction factors (for concrete $\gamma_{mc} = 1.5$, for steel $\gamma_{ms} = 1.15$). The resistance reduction factor γ_R is evaluated using $\gamma_R = R_D/R_n$. A set of material strengths and dimensions is generated randomly from the statistical distributions of each variable and are used to calculate the theoretical resistance, R, along with the randomly generated value for the model error. Then strength ratio R/R_n is determined. This procedure is repeated and a large number of samples of R/R_n is generated. A probability model is fitted to the generated data. A normal distribution is fitted to the lower tail of the data and the statistics of R/R_n are established. By repeating steps (ii) and (iii), the statistics of the strength ratio of different members are established. The procedure of the Monte Carlo method was dealt with in detail in Chapter 7. A few typical values of the established resistance statistics and the range of γ_R values observed for RCC members are given in Table 9.5.

Using the established statistics of resistance ratios and loads for Indian conditions, the reliability analysis of RCC members designed according to the present code (9.14) procedures is carried out using Level 2 methods described in Chapter 8. The reliability levels of the present designs are found out using the Level 2 method for various load combinations

TABLE 9.5 *Typical resistance statistics of RCC members*

Member	Steel grade	Concrete grade	$\mu_{R/Rn}$	δ	γ_R
Slabs					
One way (SS)	Fe 250	M 15	1.433	0.124	Range
	Fe 415	M 15	1.275	0.124	0.835–0.865
Two way (SS)	Fe 415	M 15	1.281	0.124	Average
One way (C)	Fe 415	M 15	1.263	0.136	0.85
Two way (C)	Fe 415	M 15	1.286	0.129	
Beams (flexure)					
Singly reinforced	Fe 250	M 15	1.288	0.104	
	Fe 415	M 15	1.170	0.104	Range
	Fe 415	M 20	1.179	0.103	0.835–0-845
	Fe 415	M 25	1.169	0.101	Average
	Fe 415	M 15*	1.197	0.105	0.84
Doubly reinforced	Fe 415	M 15	1.151	0.103	
Beams (shear)					
	Fe 250	M 15	1.355	0.166	Range 0.855–0.865
	Fe 415	M 15	1.277	0.165	Average 0.86
Columns					
Compression	Fe 415	M 20	1.29	0.152	Range 0.68–0.79
	Fe 415	M 20*	1.38	0.224	Average 0.725
Tension	Fe 415	M 20	1.19	0.13	Range 0.68–0.89
	Fe 415	M 20*	1.22	0.15	Average 0.8

Note: SS = simply supported; C = continuous
* = indicates nominal mix.

(i) $D + L_m$ (ii) $D + W_m$, and (iii) $D + L_m + W_m$. A summary of the results of the same is given in Table 9.6. Based on the above study, a proper target reliability is selected. For the selected target reliability, partial safety factors are evaluated for different load combinations for each member/limit state. The evaluation of the partial safety factors is illustrated below.

EXAMPLE 9.7 (Load Combination: $D + L_m$) Consider a RCC beam. After considering all the possible combinations of the grades of concrete and steel statistics of the flexural strength of RCC beams have been taken, as given below, for the study of the partial safety factors at the limit state of collapse in flexure (9.12).

Grade of steel	$\mu_{R/Rn}$	σ	Distribution
Fe 415	1.17	0.122	Normal
Fe 250	1.289	0.1289	Normal

TABLE 9.6 *Range of reliability indexes for RCC members*

Load combination	Member	Range of β	Average β	Remark
$D + L_m$				
	Slabs	4.2 to 4.8	4.5	
	Beams (flexure)	4.3 to 5.5	4.9	Range of L_n/D_n
	Beams (shear)	3.3 to 3.8	3.6	0.25 to 2.0
	Columns	3.3 to 4.6	3.9	
$D + W_m$				
	Beams (flexure)	3.5 to 5.1	4.3	Range of W_n/D_n
	Beams (shear)	3.2 to 3.5	3.33	0.25 to 2.0
	Columns	3.2 to 4.2	3.50	
$D + L_m + W_m$				
	Beams (flexure)	2.9 to 4.6	3.75	Range of L_n/D_n
	Beams (shear)	2.9 to 3.4	3.15	0.5, 1.0, 1.5
	Columns	2.8 to 4.1	3.5	Range of W_n/D_n 0.25 to 2.0

The load statistics (from Table 9.4) and resistance statistics of the beam at the limit state of collapse in flexure (for the steel grade Fe 415) are

Variable R/R_n: (Normal)

$$\mu = 1.17 \qquad \sigma = 0.122$$

Variable D/D_n: (Normal)

$$\mu = 1.05 \qquad \sigma = 0.105$$

Variable L_m/L_n: Type 1 extremal (largest)

$$\mu = 0.62 \qquad \sigma = 0.1755$$

Parameters $\qquad u = 0.315 \qquad \alpha = 1.895$

If Fe 415 steel grade is used for reinforcing bars, determine the partial safety factors for the limit state of collapse in flexure under the load combination $D + L$ if $L_n/D_n = 1.0$ and $\beta_0 = 4.5$.

Solution The safety checking format (LRFD) under dead load D and live load L is

$$\gamma_R R_n \geqslant \gamma_D D_n + \gamma_L L_n$$

The limit state equation is

$$R - D - L = 0$$

The equation can be rewritten as

$$\left(\frac{R}{R_n}\right) R_n - \left(\frac{D}{D_n}\right) D_n - \left(\frac{L}{L_n}\right) L_n = 0 \tag{9.42}$$

Let

$$\frac{L_n}{D_n} = a_3 \qquad X_1 = \frac{R}{R_n}$$

$$X_2 = \frac{D}{D_n} \qquad X_3 = \frac{L}{L_n}$$

Then Eq. (9.42) becomes

$$R_n X_1 - X_2 D_n - a_3 X_3 D_n = 0$$

It is to be remembered that for L, the statistics of L_m must be used for the ultimate limit state. The reliability index is given by

$$\beta = \frac{\mu_M}{\sigma_M}$$

$$= \frac{R_n \mu_1 - \mu_2 D_n - a_3 \mu_3' D_n}{[(R_n \sigma_1)^2 + (\sigma_2 D_n)^2 + (a_3 \sigma_3' D_n)^2]^{1/2}} \tag{9.43}$$

where μ_3' and σ_3' are the mean and standard deviation of the equivalent normal X_3' of the nonnormal variable X_3 at the design point. Let the starting design point be

$$x_1^* = \mu_1 \qquad x_2^* = \mu_2 \qquad x_3^* = \mu_3$$

At $x_3^* = \mu_3 = 0.62$, the parameters σ_3' and μ_3' for the Type 1 extremal (largest) distribution are calculated as illustrated in Example 9.4. They are

$$\sigma_3' = 0.1678 \qquad \mu_3' = 0.5903$$

Substituting the values of $\beta = 4.5$, $a_3 = 1$, σ_3', μ_3', and other σ_i and μ_i values in Eq. (9.43), we have

$$4.5 = \frac{1.17 R_n - 1.05 D_n - 0.59 D_n}{[(0.122 R_n)^2 + (0.105 D_n)^2 + (0.168 D_n)^2]^{1/2}}$$

Solving the above quadratic equation in R_n, we get

$$R_n = 3.004 D_n$$

The directional cosines are

$$\alpha_1 = -\frac{1}{K}(R_n \sigma_1)$$

$$= -\frac{1}{K}(3.004 \times 0.122) D_n = -\frac{1}{K}(0.366 D_n)$$

$$\alpha_2 = \frac{1}{K}(\sigma_2 D_n) = \frac{1}{K}(0.105 D_n)$$

$$\alpha_3 = \frac{1}{K}(a_3 \sigma_3' D_n) = \frac{1}{K}(0.168 D_n)$$

Using $\Sigma\, \alpha_i^2 = 1$ and $K = 0.417 D_n$, we have

$$\alpha_1 = -0.878 \qquad \alpha_2 = 0.252 \qquad \alpha_3 = 0.403$$

The new design point x_3^* is given by

$$x_3^* = \mu_3' + \alpha_3 \beta \sigma_3'$$

$$= 0.5903 + 0.403 \times 4.5 \times 0.168 = 0.894$$

At this new design point, new values of σ_3' and μ_3' are calculated and the whole process is repeated till the required convergence is achieved. Results of subsequent iterations are given in Table 9.7.

TABLE 9.7 *Summary of computations—Example 9.7*

Variable	Start	Iteration			
		1	2	3	4
x_1^*	1.17	0.687	0.730	0.772	0.802
x_2^*	1.05	1.169	1.152	1.139	1.131
x_3^*	0.62	0.894	1.177	1.141	1.553
σ_3'	0.168	0.270	0.368	0.440	0.479
μ_3'	0.590	0.501	0.313	0.126	0.0056
R_n/D_n	3.004	3.193	3.304	3.348	3.3591
α_1	−0.878	−0.802	−0.725	−0.671	
α_2	0.252	0.216	0.189	0.172	
α_3	0.403	0.556	0.662	0.722	

At the end of the sixth iteration

$$\frac{R_n}{D_n} = 3.361$$

$$x_1^* = 0.82 \qquad x_2^* = 1.13 \qquad x_3^* = 1.62$$

The partial safety factors with respect to the nominal values are

$$\gamma_i = \frac{x_i^*}{x_{in}} = x_i^*$$

since the variables X_i have been initially normalised with respect to their corresponding nominal values. Hence

$$\gamma_1 = \gamma_R = 0.82 \qquad \gamma_2 = \gamma_D = 1.13$$
$$\gamma_3 = \gamma_L = 1.62$$

The design equation is

$$\gamma_R R_n \geqslant \gamma_D D_n + \gamma_L L_n$$

$$0.82 R_n \geqslant 1.13 D_n + 1.62 L_n$$

The same problem has been solved for various values of L_n/D_n equal to 0.25, 0.5, 1.0, 1.5 and 2.0, and the variation of the partial safety factors with L_n/D_n is shown in Fig. 9.2. If the steel grade Fe 250 is used, $\mu_{R/Rn}$ = 1.289, $\sigma_{R/R_n} = 0.1289$. For this case also, the variations of γ_R, γ_D and γ_L with L_n/D_n are shown in the same Fig. 9.2. It is observed that γ_R increases slightly with an increase in the L_n/D_n ratios. This is due to the use of the higher values of γ_L at higher L_n/D_n ratios. The dead load factor γ_D shows a slight fall with an increase in the L_n/D_n ratio; but can be treated to be a fairly constant value. The variation in γ_D is very small because the variation in dead load is small compared to other load variables. γ_L increases with increase in the L_n/D_n ratio as its higher variability becomes increasingly more dominant in determining the total load effect.

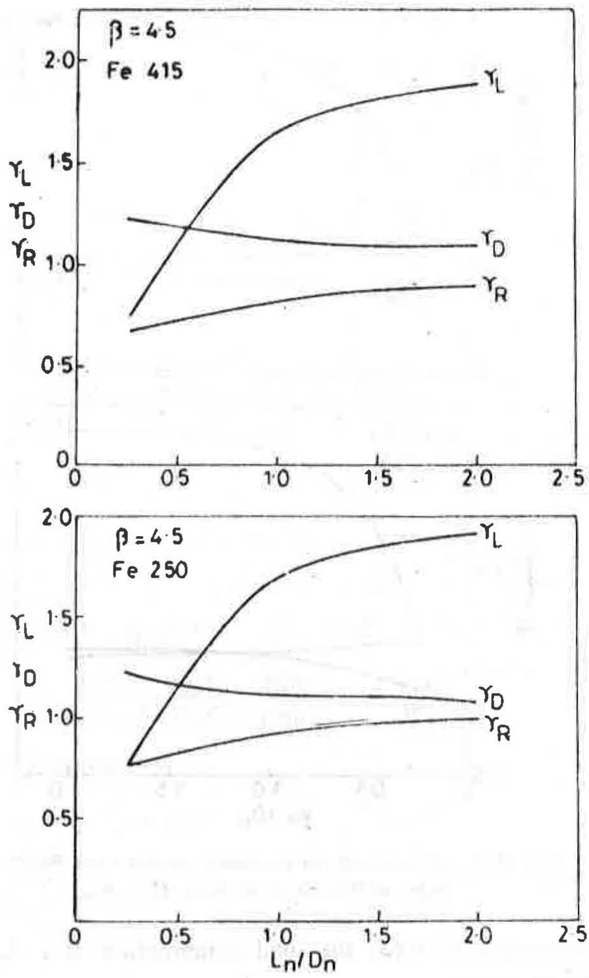


FIG. 9.2 Variation of partial safety factors for RCC beam in flexure under load: $D + L_m$

Load Combination: $D + W_m$

The procedure of computation of the safety factors for the load combination $D + W_m$ is same as used for the load case $D + L_m$, explained and illustrated in Example 9.7. The only difference is that the corresponding statistics of W_m are to be used instead of those of L_m. Typical curves showing variation of the wind load factors γ_W, γ_D, and γ_R with respect to the W_n/D_n ratio are shown in Fig. 9.3. Here also, similar observations are made about γ_R, γ_D, and γ_W as in the previous case $D + L_m$, i.e. γ_R increases slightly with increase in W_n/D_n, γ_D remains fairly constant, and γ_W increases as W_n/D_n increases.

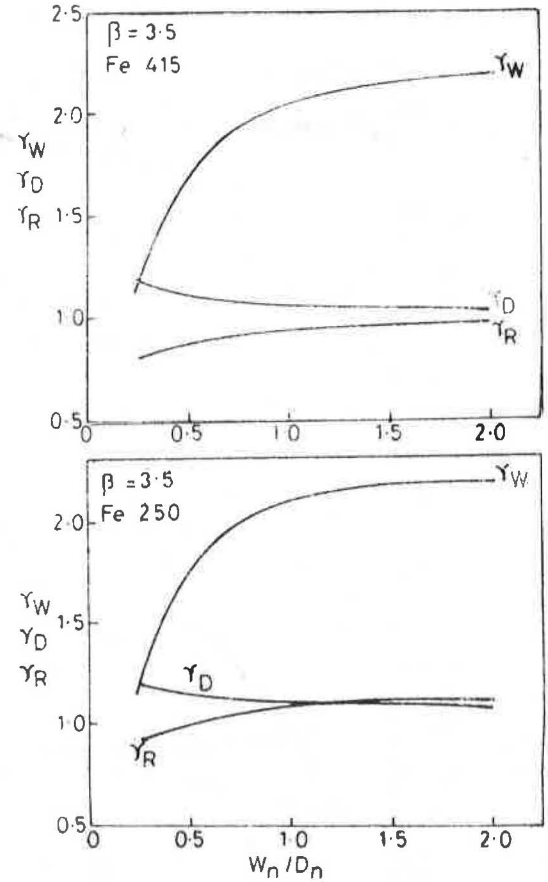


FIG 9.3 Variation of partial safety factors for RCC beam in flexure under load: $D + W_m$

The determination of β for the load combination $D + L_{apt} + W_m$ is illustrated below.

EXAMPLE 9.8 (Load combination: $D + L_{apt} + W_m$) Consider the same problem in Example 9.7. The beam is subjected to wind load along with the gravity loads. From Table 9.4, the following load statistics are taken.

Variable: D/D_n: (normal)

$$\mu = 1.05 \qquad \sigma = 0.105 \qquad \delta = 0.10$$

Variable: L_{apt}/L_n: (lognormal)

$$\mu = 0.179 \qquad \sigma = 0.098 \qquad \delta = 0.55$$

Variable W_m/W_n: (Type 1 extremal (largest))

$$\mu = 0.804 \qquad \sigma = 0.269 \qquad \delta = 0.334$$

The resistance statistics are the same as given in Example 9.7. That is, the mean and standard deviation of R/R_n are 1.17 and 0.122 respectively. Determine the partial safety factors for $\beta_0 = 4$. It is also given that

$$\frac{L_n}{D_n} = 1.0 \qquad \frac{W_n}{D_n} = 1.0$$

Solution The failure surface equation is

$$R - D - L_{apt} - W_m = 0 \tag{9.44}$$

Rewriting the equation, we have

$$\left(\frac{R}{R_n}\right)R_n - \left(\frac{D}{D_n}\right)D_n - \left(\frac{L_{apt}}{L_n}\right)L_n - \left(\frac{W_m}{W_n}\right)W_n = 0 \tag{9.45}$$

Let

$$X_1 = \frac{R}{R_n} \qquad X_2 = \frac{D}{D_n}$$

$$X_3 = L_{apt}/L_n \qquad X_4 = \frac{W_m}{W_n}$$

$$\frac{L_n}{D_n} = a_3 \qquad \frac{W_n}{D_n} = a_4$$

Then the failure surface equation becomes

$$R_nX_1 - X_2D_n - X_3a_3D_n - X_4a_4D_n = 0 \tag{9.46}$$

The reliability index is given by

$$\beta = \frac{\mu_1R_n - \mu_2D_n - \mu_3'a_3D_n - \mu_4'a_4D_n}{[(\sigma_1R_n)^2 + (\sigma_2D_n)^2 + (\sigma_3'a_3D_n)^2 + (\sigma_4'a_4D_n)^2]^{1/2}} \tag{9.47}$$

The directional cosines are

$$\alpha_1 = -\frac{1}{K}(\sigma_1R_n) \qquad \alpha_2 = \frac{1}{K}(\sigma_2D_n)$$

$$\alpha_3 = \frac{1}{K}(\sigma_3'a_3D_n) \qquad \alpha_4 = \frac{1}{K}(\sigma_4'a_4D_n)$$

Start with

$$x_1^* = \mu_1 = 1.17 \qquad x_2^* = \mu_2 = 1.05$$

$$x_3^* = \mu_3 = 0.179 \qquad x_4^* = \mu_4 = 0.804$$

The procedure of computation is the same as explained in Example 9.5. Summary of the results is given in Table 9.8. After the fifth iteration,

$$x_1^* = 0.908 \qquad x_2^* = 1.1$$

$$x_3^* = 0.199 \qquad x_4^* = 2.29$$

The partial safety factors are

$$\gamma_1 = \gamma_R = 0.908 \qquad \gamma_2 = \gamma_D = 1.1$$

$$\gamma_3 = \gamma_L = 0.199 \qquad \gamma_4 = \gamma_W = 2.29$$

TABLE 9.8 *Summary of results—Example 9.8*

Variable	Start	Iteration 1	2	3
x_1^*	1.170	0.883	0.877	0.896
x_2^*	1.050	1.172	1.116	1.103
x_3^*	0.179	0.249	0.231	0.209
x_4^*	0.804	1.489	2.000	2.213
σ_3'	0.092	0.127	0.119	0.107
μ_3'	0.156	0.134	0.142	0.149
σ_4'	0.257	0.508	0.669	0.728
μ_4'	0.759	0.457	0.060	−0.120
R_n/D_n	1.738	3.294	3.819	3.935
α_1	−0.587	−0.601	−0.561	−0.543
α_2	0.291	0.157	0.127	0.119
α_3	0.254	0.191	0.143	0.121
α_4	0.711	0.760	0.805	0.823

Hence the design equation is

$$0.908R_n \geqslant 1.1D_n + 0.199L_n + 2.29W_n$$

Similarly, for various values of W_n/D_n, the values of γ_R, γ_D, and γ_W can be determined. The variation of the partial safety factors for various values of W_n/D_n is shown in Fig. 9.4. It can be observed that γ_D remains fairly constant. γ_L decreases with increase in W_n/D_n up to $W_n/D_n = 1$, and for $W_n/D_n \geqslant 1$, γ_L remains fairly constant. γ_W increases with increase in W_n/D_n, increase is more up to $W_n/D_n = 1.0$. The region of interest in design is up to $L_n/D_n \leqslant 1$ and $W_n/D_n \leqslant 1$. In this region, the variation of load factors is observed to be high (Figs. 9.2, 9.3 and 9.4). The process of code calibration involves proposing one set of partial factors for Level 1 code, irrespective of the load ratios (e.g. L_n/D_n or W_n/D_n), and probably other design situations-different limit states, ensuring uniform reliability. For this, the simple optimisation technique proposed by Ellingwood, et al. (9.6) or the method used by Baker (9.15), can be used after assigning weighting factors to load occurrence. They are explained in the following section.

9.5 OPTIMAL SAFETY FACTORS

As seen in the previous section, the partial safety factors are not constant for a given safety checking format and a given target β. For convenience, the partial safety factors in the code checking format are to be constant atleast over a large group of design situations.

As said earlier, the aim of code calibration is to determine a set of safety factors which will ensure the best approximate uniform reliability over

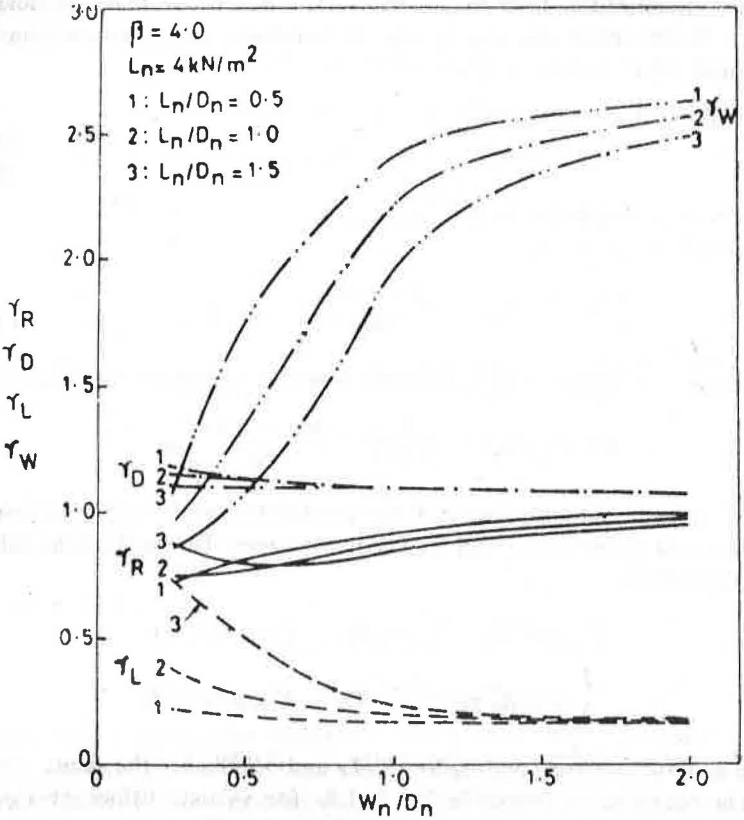


FIG. 9.4 Variation of partial safety factors for RCC beam in flexure under load: $D + L_{apt} + W_m$

different design situations. If a constant set of factors are prescribed, the associated reliabilities will deviate from the target reliability β_0. To select one set (optimal set) of load factors, a function, $S(\gamma_i)$, which measures the "closeness" between the target reliability and reliability associated with the proposed partial factors set. is defined and this function is minimised to get the optimal safety factors.

For a given set of partial factors with an associated β_0, there is some corresponding nominal resistance. Let it be called R_n^{II}, obtained using the Level 2 method. This is a function of the load ratio and load combination. Let the nominal resistance corresponding to a design equation, which prescribes a set of partial factors that are constant for all load ratios be R_n^I which may differ from $R_n^{II} \cdot R_n^I$ corresponds to Level 1 code. The problem is therefore, to find γ_i, minimizing the function, S, defined by (9.6)

$$S(\gamma_i) = \sum_i (R_n^{II} - R_n^I)^2 w_i \qquad (9.48)$$

over a predefined set of combinations of dead, live and wind loads wherein w_i is the relative weight assigned to the ith load ratio. The function selected is the square of the difference between R_n^{II} and R_n^{I} so that the deviations from β_0 on either side can be equally penalized. The determination of the optimal safety factors is illustrated below.

EXAMPLE 9.9 Let the safety checking format be

$$\gamma_R R_n = \gamma_D D_n + \gamma_L L_n \tag{9.49}$$

for the load combination $D + L_m$.
Taking $D_n = 1$,

$$R_n^{I} = \frac{(\gamma_D + \gamma_L a_i)}{\gamma_R} \tag{9.50}$$

where $a_i = (L_n/D_n)_i$. The S function, given by Eq. (9.48), becomes

$$S(\gamma_R, \gamma_D, \gamma_L) = \sum_i \left[R_n^{II} - \frac{\gamma_D + a_i\gamma_L}{\gamma_R} \right]^2 w_i \tag{9.51}$$

To find the minimum value of S, the partial derivatives of S with respect to γ_R, γ_D, and γ_L are taken and made equal to zero. This leads to the following two equations:

$$\sum_i w_i a_i R_n^{II} \gamma_R - \sum_i w_i a_i \gamma_D - \sum_i w_i a_i^2 \gamma_L = 0 \tag{9.52}$$

$$\sum_i w_i R_n^{II} \gamma_R - \sum_i w_i \gamma_D - \sum_i w_i a_i \gamma_L = 0 \tag{9.53}$$

The equations corresponding to $\partial S/\partial \gamma_R$ and $\partial S/\partial \gamma_D$ are the same.

The computed values of γ_R, γ_D, and γ_L for various values of L_n/D_n for RCC beam in flexure are given in Table 9.9. The weights to be assigned should be based on the likelihood of different load situations in practice. The assumed weighting factors w_i in percentage (9.6) are also given in the same table.

TABLE 9.9 *Values of partial safety factors for beam in flexure—Load combination $D + L_m$*

Partial factor	L_n/D_n: 0.25	0.5	1.0	1.5	2.0	Remark
γ_R	0.802	0.860	0.937	0.964	0.977	$\beta_0 = 3.5$
γ_D	1.200	1.154	1.105	1.087	1.078	$\mu_{Lm/Ln} = 0.827$
γ_L	1.057	1.505	1.859	1.954	1.995	$\delta L_m = 0.283$ $L_n = 3$ kN/m²
Weights w_i (per cent)	10	45	30	10	5	For RCC beams

The values of R_n^{11} for $i = 1$ to 5 are calculated using the expression

$$R_n^{11} = \frac{(\gamma_{D_i} + \gamma_{L_i} a_i)}{\gamma_{R_i}} \tag{9.54}$$

and the values of γ_i given in Table 9.9. For example, for $i = 1$, $L_n/D_n = 0.25$.

$$R_n^{11} = \frac{(1.20 + 1.057 \times 0.25)}{0.802} = 1.82$$

Using weighting factors given in Table 9.9, we get

$$\sum_{i=1}^{5} w_i a_i R_{ni}^{11} = 2.6369$$

$$\sum_{i=1}^{5} w_i R_{ni}^{11} = 2.8049$$

$$\sum_{i=1}^{5} w_i a_i = 0.8 \qquad \sum_{i=1}^{5} w_i = 1.0$$

Using these values, Eqs. (9.52) and (9.53) become

$$2.6369\gamma_R - 0.8\gamma_D - 0.844\gamma_L = 0 \tag{9.55}$$

$$2.8049\gamma_R - \gamma_D - 0.800\gamma_L = 0 \tag{9.56}$$

From the study of the results (Table 9.9), it is observed that γ_D remains fairly constant around 1.1. This has been observed for various load combinations and failure states (9.12). Since the value 1.1 is low and may not be acceptable by the profession, the value of γ_D is fixed as 1.2. Using this value, Eqs. (9.55) and (9.56) become

$$2.6369\gamma_R - 0.844\gamma_L = 0.96 \tag{9.57}$$

$$2.8049\gamma_R - 0.800\gamma_L = 1.2 \tag{9.58}$$

Solving the above equations

$$\gamma_R = 0.9495 \qquad \gamma_L = 1.829$$

If it is desired that γ_R must be around 0.85, as existing in the present designs corresponding to the material reduction factors $\gamma_{mc} = 1.5$, $\gamma_{ms} = 1.15$ and other material specifications, so that the partial safety factors for material strengths and other material specifications on the resistance side are not changed, then keeping the present value of $\gamma_R = 0.85$, and using the same in Eqs. (9.57) and (9.58), two values of γ_L are obtained. They are 1.52 and 1.48. Taking the average, γ_L is fixed as 1.50. Hence, the optimal safety factors for this case in this example are

$$\gamma_R = 0.85 \qquad \gamma_D = 1.2 \qquad \gamma_L = 1.5$$

Similar studies can be done for other combinations of variables and other limit states.

The approach used was suggested by Ellingwood, et al. (9.6). The approach used by Baker (9.15) is given below.

The function used for S is

$$S = \sum_i (\log_{10} p_f - \log_{10} p_{ft})_i^2 w_i \tag{9.59}$$

where

$(p_f)_i$ = is the failure probability for the case (say ith load ratio L_n/D_n)
$(p_{ft})_i$ = corresponding target failure probability

To determine the partial factors for the new code checking format, trial values of partial factors are used in the new code format and β_i values and corresponding $(p_f)_i$ are computed. These values are substituted in Eq. (9.59) and the value of S is calculated. The process is repeated for different trial values of γ_i. Finally, the set of partial factors corresponding to the minimum value of S is taken for the new code checking format. This method is illustrated below.

EXAMPLE 9.10 For the same problem in Example 9.9, determine the optimal partial safety factors using Baker's approach (9.15). $\beta_0 = 3.5$.

Solution As in the previous case, let us fix

$$\gamma_R = 0.85 \qquad \gamma_D = 1.2$$

The problem is to find the optimal value of γ_L.
First select a trial value for γ_L. say 1.3.
Using $\gamma_R = 0.85$, $\gamma_D = 1.2$, and $\gamma_L = 1.3$, determine β (as explained in Sec. 8.3.3 and Example 8.9) for each value of a_i. Find the corresponding value of $(p_f)_i = \Phi(-\beta)_i$. Using Eq. (9.59), calculate S. A summary of the calculations for $\gamma_L = 1.3$ are shown in Table 9.10. Repeat the process for different trial values of γ_L and calculate the corresponding values of S. γ_L corresponding to the lowest value of S is the optimal value of γ_L. The optimum value can be obtained by plotting γ_L versus S. The optimum value of S is 1.45.

TABLE 9.10 *Summary of calculations—Example 9.10*

L_n/D_n	β	$p_f \times 10^{-4}$	w_i	$(\log p_f - \log p_{ft})^2 w_i$	Remark
$\gamma_L = 1.3$					
0.25	3.41	3.305	0.10	0.00233	$p_{ft} = \Phi(-3.5)$
0.50	3.42	3.133	0.45	0.00753	$= 2.326 \times 10^{-4}$
1.00	3.19	7.094	0.30	0.07036	
1.50	3.04	11.665	0.10	0.04904	
2.00	2.95	15.655	0.05	0.03429	
			$\Sigma = 1.00$	$S = \Sigma = 0.1635$	
For	$\gamma_L = 1.4$		$S = 0.0531$		
	$\gamma_L = 1.5$		$S = 0.0547$		
	$\gamma_L = 1.6$		$S = 0.16108$		

The procedure of calculation of optimal safety factors for the load combination $D + L_{apt} + W_m$ is same as explained in the previous load case: $D + L_m$. This is illustrated in the following example for the same member RCC beam in flexure.

EXAMPLE 9.11 Consider RCC beams in the limit state of collapse in flexure under load combination $D + L_{apt} + W_m$, as considered in Example 9.8.

The safety checking format is

$$\gamma_R R_n = \gamma_D D_n + \gamma_L L_m + \gamma_W W_n \tag{9.60}$$

Taking $D_n = 1$,

$$R_n^I = \frac{(\gamma_D + a_i \gamma_L + a_j \gamma_W)}{\gamma_R} \tag{9.61}$$

where $a_i = \dfrac{L_n}{D_n} \quad a_j = \dfrac{W_n}{D_n}$

The function S defined by Eq. (9.48) becomes

$$S(\gamma_R, \gamma_D, \gamma_L, \gamma_W) = \sum_i \sum_j \left\{ \left[R_n^{II} - \frac{\gamma_D + a_i \gamma_L + a_j \gamma_W}{\gamma_R} \right]^2 w_i \right\} w_j \tag{9.62}$$

The partial derivatives of S with respect to γ_R, γ_D, γ_L, and γ_W result in the following equations:

$$\sum_i \sum_j R_n^{II} w_i w_j \gamma_R - \sum_i \sum_j w_i w_j \gamma_D - \sum_i \sum_j a_i w_i w_j \gamma_L - \sum_i \sum_j w_i w_j a_j \gamma_W = 0 \tag{9.63}$$

$$\sum_i \sum_j R_n^{II} w_i w_j a_i \gamma_R - \sum_i \sum_j w_i w_j a_i \gamma_D - \sum_i \sum_j a_i^2 w_i w_j \gamma_L - \sum_i \sum_j w_i w_j a_j \gamma_W = 0 \tag{9.64}$$

$$\sum_i \sum_j R_n^{II} w_i w_j a_j \gamma_R - \sum_i \sum_j w_i w_j a_j \gamma_D - \sum_i \sum_j a_i a_j w_i w_j \gamma_L - \sum_i \sum_j w_i w_j a_j^2 \gamma_W = 0 \tag{9.65}$$

It is to be noted that $\partial S/\partial \gamma_R$ and $\partial S/\partial \gamma_D$ will yield the same equation. Here four variables are to be determined with three equations. Hence, the value of one of the partial factor, generally γ_D, is assigned or selected and the other three factors are evaluated. The procedure is similar to that of the gravity load case, $D + L$.

The computed values of γ_i for the various values of W_n/D_n and L_n/D_n are given in Table 9.11 for $\beta_0 = 3.5$. The assumed weighting factors w_i and w_j are given in Table 9.12.

Since γ_D is fairly constant as can be seen in Table 9.11, γ_D can be fixed. The value of γ_R is also fixed. Selecting $\gamma_D = 1.2$ and $\gamma_R = 0.85$, and using values given in Tables 9.11 and 9.12 in Eqs. (9.63)–(9.65), the following three equations are obtained:

$$0.725 \ \gamma_L + 0.713 \ \gamma_W = 1.3748$$
$$0.7125 \ \gamma_L + 0.5218 \ \gamma_W = 1.1356$$
$$0.5218 \ \gamma_L + 0.7127 \ \gamma_W = 1.32$$

TABLE 9.11 *Values of partial safety factors for RCC beams in flexure—Load combination: $D + L_{apt} + W_m$—Example 9.11*

$\frac{L_n}{D_n}$*	Partical factor	$\frac{W_n}{D_n}$ 0.25	0.50	1.0	1.5	2.0	Remark
0.5	γ_R	0.791	0.836	0.94	0.984	1.006	$\beta_0 = 3.5$
	γ_D	1.163	1.137	1.098	1.082	1.074	$\mu_{Lapt}/L_n = 0.239$
	γ_L	0.324	0.281	0.240	0.229	0.224	$L_n = 3$ kN/m²
	γ_W	0.996	1.436	1.998	2.157	2.224	$\delta_{Lapt}/L_n = 0.55$
1.0	γ_R	0.854	0.827	0.900	0.954	0.984	
	γ_D	1.123	1.121	1.077	1.082	1.074	
	γ_L	0.647	0.456	0.290	0.255	0.241	
	γ_W	0.884	1.137	1.802	2.052	2.157	
1.5	γ_R	0.932	0.900	0.87	0.927	0.962	
	γ_D	1.095	1.097	1.095	1.082	1.074	
	γ_L	0.911	0.772	0.397	0.292	0.263	
	γ_W	0.828	0.937	1.527	1.921	2.076	

TABLE 9.12 *Weighting factors in percentage for load combination $D + L_{apt} + W_m$—Example 9.11*

$\frac{L_n}{D_n}$	Weighting factor for $\frac{L_n}{D_n}$ (w_j)	$\frac{W_n}{D_n}$ (w_i) 0.25	0.5	1.0	1.5	2.0
0.5	0.55	10	45	30	10	5
1.0	0.35	30	45	15	7	3
1.5	0.10	45	30	15	7	3

Using any two equations, three sets of γ_L and γ_W can be obtained:

$$\text{(i) } \gamma_L = 0.712 \qquad \gamma_W = 1.2$$
$$\text{(ii) } \gamma_L = 0.512 \qquad \gamma_W = 1.477$$
$$\text{(iii) } \gamma_L = 0.267 \qquad \gamma_W = 1.656$$

Any one set or taking the average of the three values, $\gamma_L = 0.496$ and $\gamma_W = 1.44$ may be selected with $\gamma_R = 0.85$ and $\gamma_D = 1.2$.

This exercise of establishing the optimal safety factors for the various values of target β_0 can be done for various cases. A typical variation of the optimal values of partial safety factors for RCC beams in flexure for various load combinations are shown in Figs. 9.5, 9.6 and 9.7 (9.16). These are applicable to Indian conditions. There are three values of L_n given in the figures. The data used for live load is the one based on the load survey of office buildings. The Indian Standard Code (9.17) suggests nominal live

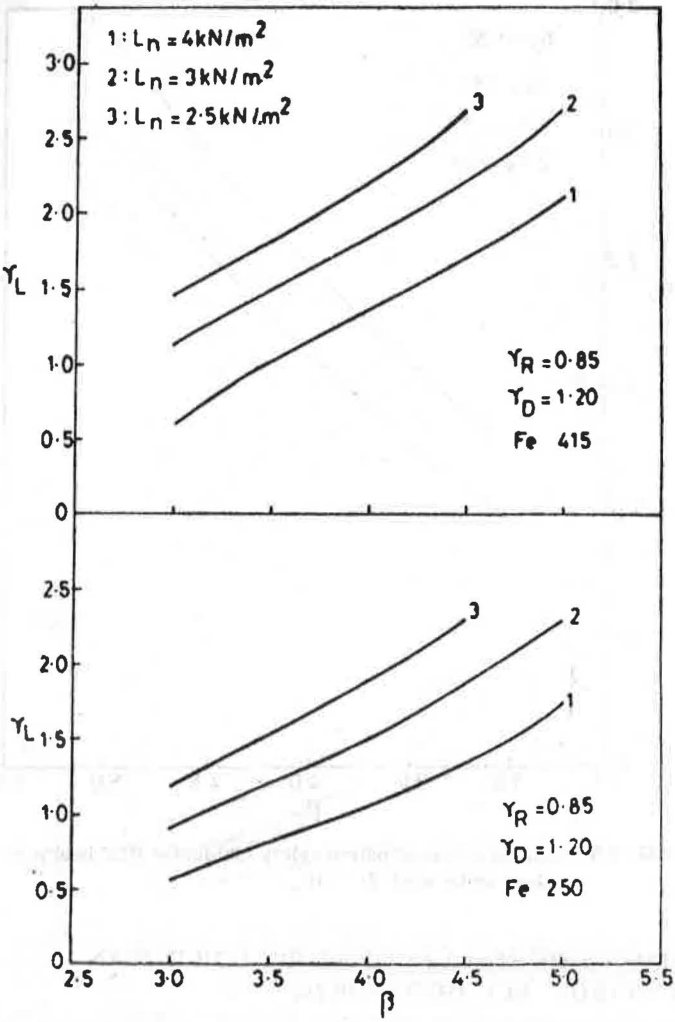


FIG. 9.5 Optimal values of partial safety factors for RCC beams in flexure under load: $D + L_m$

load of 2.5 to 4 kN/m² for office buildings depending on the separate storage facilities available. Office buildings are generally designed for a nominal live load of 4 kN/m² assuming no separate storage facilities. The analysis of live load on office buildings indicates the mean value of L_m as 2.48 kN/m². However, the whole study has been carried out assuming other values of L_n equal to 3.0 and 2.5 kN/m² (i.e. $\mu_{Lm/Ln}$ = 0.827 and 1.00), with a view whether it is possible to reduce the design loads for office buildings in India.

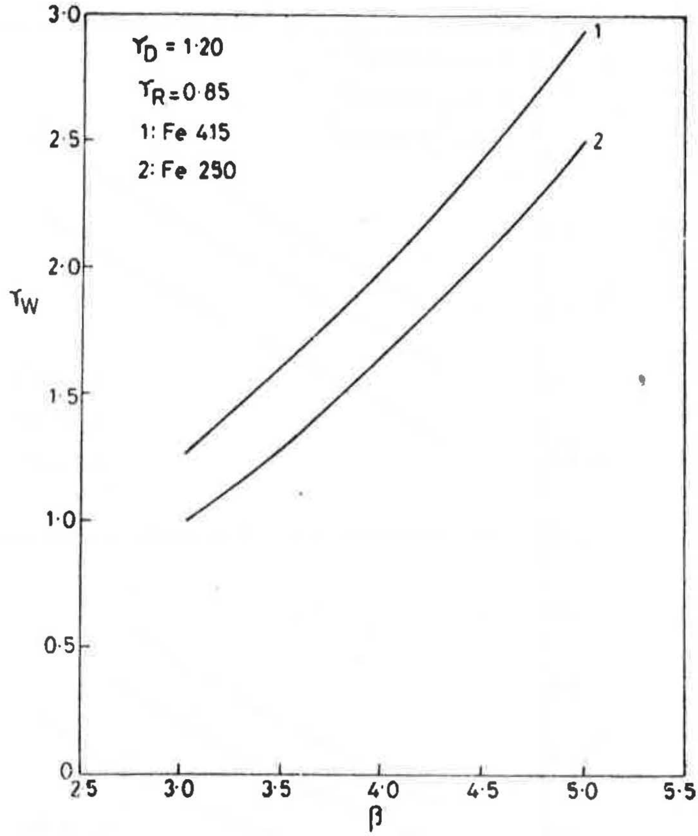


FIG. 9.6 Optimal values of partial safety factors for RCC beams in flexure under load: $D + W_m$

9.6 SUMMARY OF RESULTS OF STUDY FOR INDIAN STANDARDS – RCC DESIGN (9.16)

The development of reliability based design was illustrated for RCC beams for limit state of collapse in flexure. A similar study (9.12) for Indian conditions has been made for RCC slabs, RCC beams for limit state of collapse in shear, and RCC columns for limit state of collapse under combined axial load and uniaxial bending moment. A summary of typical resistance statistics and results of reliability analyses of the RCC members mentioned above have been given in Tables 9.5 and 9.6. While proposing a set of partial safety factors for Indian conditions, the same material factors given in the present code (9.14) have been retained, and hence the corresponding resistance factor (γ_R), evaluated by using nominal values during the statistical study of RCC members, has been kept constant. The variation in γ_D has been found to be small in all cases and can be considered almost constant around 1.1. However, this value being very small and that the profession may not accept this, a value of 1.2 has been selected for γ_D. The

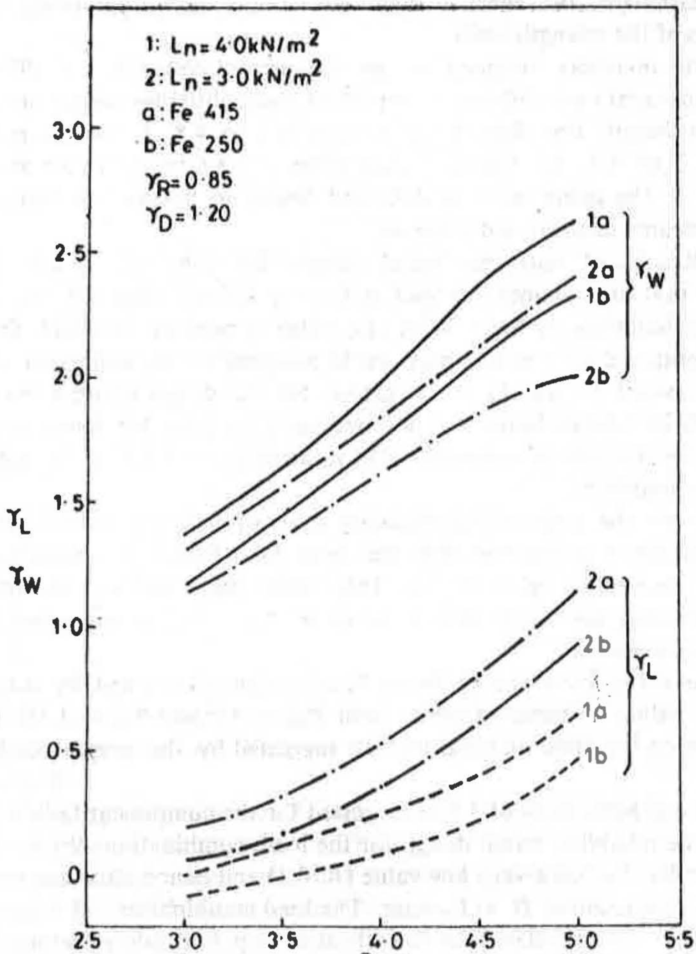


FIG. 9.7 Optimal values of partial safety factors for RCC beams in flexure under load: $D + L_{\text{apt}} + W_{\text{m}}$

optimum values of γ_L and γ_W have been fixed based on the above conditions for all the cases for the target reliability β_0. A set of curves for slabs and beams in shear are given in Figs. B1–B5 (given in Appendix), connecting optimal values of γ_L and γ_W with β_0. Table B1 is also given in the Appendix for the optimal values of γ_L and γ_W for columns. Some of the observations and conclusions on safety factors for concrete design in Indian conditions are given below:

(i) The yield strength of steel has a significant effect on the statistics of the strength ratio R/R_n for all RCC members.

In the case of columns, the concrete grade also has a significant effect on the statistics of the strength ratio in the region of compression failure. In the case of slabs, the effective depth also has a significant effect on the statistics of the strength ratio.

(ii) The members designed as per the present code (IS: 456–1978 limit state approach) have different safety levels under different design situations and vary widely. For slabs, β varies from 4.2 to 4.8, for beams in flexure from 3.2 to 4.7, for beams in shear from 3 to 3.8, and for columns from 2.9 to 4.6. The safety levels of slabs and beams in flexure are higher than that of beams in shear and columns.

(iii) Results of reliability based designs for slabs and beams clearly indicate that the nominal live load of $L_n = 4$ kN/m^2, used for the design of office buildings is high. With this value of nominal live load, the load factors obtained are low and may not be accepted by the profession. Hence, it is proposed to use $L_n = 3$ kN/m^2 for the design of office buildings. Although in column design it is not necessary to take the lower value of L_n but, for the sake of uniformity, the value of $L_n = 3$ kN/m^2 is suggested for office buildings.

(iv) In all the cases of the reliability study of members and for all load combinations, it is observed that the dead load factor, γ_D, remains fairly constant around a value of 1.1. This value being very low and that the profession may not accept this, a value of $\gamma_D = 1.2$ is suggested for all load combinations.

(v) The values of resistance factor γ_R are taken as obtained by using the nominal values of basic variables with $\gamma_{mc} = 1.5$ and $\gamma_{ms} = 1.15$. This is done so that the same material factors, suggested by the present code, can be used.

(vi) A reliability level of 3.5 is suggested for the component failure.

(vii) The reliability based design for the load combination $D + L_m + W_{apt}$ indicates that γ_W has a very low value (< 0.1) and hence, this case tends to the load combination, $D + L_m$ case. The load combination, $D + L_m + W_{apt}$, is therefore not considered for the selection of partial safety factors for the gravity load plus wind load combination.

(viii) For $\gamma_D = 1.2$, the target reliability $\beta_0 = 3.5$, and for the resistance factor γ_R corresponding to the material safety factors, $\gamma_{mc} = 1.5$ and $\gamma_{ms} = 1.15$ of the present code, the values of the live load factor and the wind load factor to be used for different load combinations are given in Table 9.13.

(ix) For slabs, beams in shear, and beams in flexure and columns, curves or tables are also presented in Appendix B to choose the load factors γ_L and γ_W corresponding to the different reliability levels as desired by the designer.

(x) In the case of columns, the quality of concrete (design mix or nominal mix) significantly affects the partial safety factors for live and wind loads.

(xi) For columns, now-a-days at least M 20 concrete is used and the concrete is prepared based on the design mix proportions in major constructions. Hence a partial safety factor for loads ranging from 1.4 to 1.8 for different load combinations, as given in Table 9.13, is suggested. In the

TABLE 9.13 *Partial safety factors for different components and load combinations at ultimate limit states $\beta_o = 3.5, \gamma_D = 1.2$ and $L_n = 3\ kN/m^2$*

S. No.	Load combination	Component	γ_R	γ_L	γ_W
1.	$D + L_m$	Slab	0.85	1.4	
		Beam in flexure	0.85	1.5	
		Beam in shear	0.85	2.0	
		Column			
		Comp* failure/Design mix	0.725	1.4	
		Tens+ failure/Design mix	0.80	1.8	
2.	$D + W_m$	Beam in flexure	0.85		1.6
		Beam in shear	0.85		2.0
		Column			
		Comp failure/Design mix	0.725		1.5
		Tens failure/Design mix	0.80		2.0
3.	$D + L_{apt} + W_m$	Beam in flexure	0.85	0.45	1.4
		Beam in shear	0.85	0.90	1.5
		Column			
		Comp failure/Design mix	0.725	0.27	1.5
		Tens failure/Design mix	0.8	0.24	1.8

Note: *Comp — Compression
+Tens — Tension

case of minor works where nominal mix is used for columns, higher safety factors for live and wind load are to be used as suggested in Table B1 in Appendix B.

(xii) The suggested values of γ_L and γ_W are for the case when steel grade Fe 415 is used. If steel grade Fe 250 is used, these values of load factors will ensure a slightly higher reliability than that conceived with the use of steel grade Fe 415. If the same reliability is to be achieved, irrespective of steel grade, then slightly lower values of γ_L and γ_D may be used when steel grade Fe 250 is used. However, the difference is very marginal. Hence, the safety factors based on steel grade Fe 415 are finally suggested to be on the safer side.

(xiii) The proposed partial safety factors for loads, given in Table 9.13, will lead to more economical designs compared to the present values given in the code (9.14).

(xiv) Even though the live load data on office buildings has been used in the study, the curves or tables are presented for various ratios of $\mu_{Lm/Ln}$ so that they could be used for any case of known $\mu_{Lm/Ln}$ assuming the coefficient of variation of L_m does not change significantly.

(xv) The Indian standard code for RCC design has not been yet calibrated by the Indian Standard Institution. It is expected that they may use the CEB checking format as followed by the British Standards. The LRFD format has been used in arriving at the results given in Table 9.13. However it is felt that the results will be only marginally affected, and negligible, since, while arriving at these values of γ_D, γ_L, and γ_W, the value of γ_R for each case corresponding to the same material reduction factors and material specifications as per present IS: 456–1978 (9.14) is used. Hence, the present code format with new optimal values of γ_D, γ_L and γ_W may be used.

The study has revealed many things for Indian conditions. It has given insight into (i) the present level of reliability available in RCC members, (ii) how the safety factors vary with target β, (iii) what is the reasonable value of β_0, and (iv) how optimum safety factors could be fixed and how these change for different failure criteria.

REFERENCES

9.1 CP 110: Part I-1972, "Code of Practice for the Structural Use of Concrete, Part I; Design, Materials and Workmanship", *British Standards Institution*, London, 1972.

9.2 NBCC, "National Building Code of Canada", *National Research Council of Canada*, Ottawa, 1977.

9.3 CEB, "Common Unified Rules for Different Types of Construction and Material" (3rd draft), *Bulletin D'Information, No. 116-E*, Comite European Du Beton, Paris, 1976.

9.4 "Common Unified Rules for Different Types of Construction and Material", *Bulletin D'Information No. 124 E*, Comite Euro-International Du Beton (CEB), Paris, April 1978.

9.5 Galambos, T.V. and M.K. Ravindra, "Load and Resistance Factor Design for Steel", *Journal of Structural Div., ASCE*, Vol. 104, ST9, Sept. 1978, pp. 1337-1353.

9.6 Ellingwood, B., J.G. MacGregor, T.V. Galambos, and C.A. Cornell, "Probability Based Load Criteria: Load Factors and Load Combinations", *Journal of Structural Div., ASCE*, Vol. 108, ST5, May 1982, pp. 978-997.

9.7 Dayaratnam, P. and R. Ranganathan, *Statistical Analysis of Strength of Concrete*, Building and Environment, Vol. 11, Pergamon Press, 1976, pp. 145-152.

9.8 Ranganathan, R. and C.P. Joshi, "Variations in Dimensions of RCC Members", *Journal of Bridge and Structural Engineer*, Vol. 16, Sept. 1986, pp. 1-10.

9.9 David Arulraj and R. Ranganathan, "Reliability Design Criteria for Slabs", *International Journal of Structures*, Vol. 7, July-Dec. 1987, pp. 155-174.

9.10 Joshi, C.P. and R. Ranganathan, "Variations in Strength of Reinforcing Steel Bars" *Journal of the Institution of Engineers (India)*, Civil Engg. Div., Vol. 68, May 1988, pp. 309-312.

9.11 Ranganathan, R., "Statistical Analysis of Floor Loads in Office Buildings", *DS and T Report No. 4/1/83/STP-III/3*, Civil Engineering Dept., IIT, Bombay, Oct. 1985.

9.12 Ranganathan, R., "Reliability Analysis and Design of RCC Slabs, Beams and Columns and Frames – Code Calibration", *DS and T Report No. 4/1/83/STP-III/5*, Civil Engg. Dept., IIT Bombay, Sept. 1987.

9.13 Ranganathan, R., "Wind Speed and Wind Load Statistics for Probabilistic Design", *Journal of the Institution of Engineers (India)*, Civil Engg. Div., Vol. 68, May 1988, pp. 303-308.

9.14 IS: 456-1978, "Code of Practice for Plain and Reinforced Concrete", *Indian Standards Institution*, New Delhi, 1980.

9.15 CIRIA, Rationalisation of Safety and Serviceability Factors in Structural Codes" *Construction Industry Research and Information Association, Report No. 63*, London, 1977.

9.16 Padmini Chikkodi and R. Ranganathan, "Partial Safety Factors for RCC Design" *International Journal of Structures*, Vol. 8, July-Dec. 1988, pp. 127-149.
9.17 IS: 875–1982, "Code of Practice for Structural Safety of Buildings, Loadings, Standards, Part II – Imposed (live) Loads", Jan. 1982.

EXERCISE

9.1 Determine the partial safety factors for the variables, the yield strength of steel, dead load, and live load for the given limit state equation

$$f_y Z_p - D\frac{l^2}{8} - L\frac{l^2}{8} = 0$$

where l is the span, f_y is the yield strength of steel and Z_p is the plastic section modulus.
It is given:

Variable f_y: $\mu = 275$ N/mm² $\qquad \sigma = 27.5$ N/mm²
Variable D: $\mu = 11.55$ N/mm² $\qquad \sigma = 1.155$ N/mm²
Variable L: $\mu = 8.4$ N/mm² $\qquad \sigma = 3.36$ N/mm²

The nominal values of f_y, D, and L are 250 N/mm², 11 N/mm², and 12 N/mm² respectively.

(i) If the span is 8 m, determine the partial safety factors for $\beta_0 = 3$ assuming f_y and D are normal and L is Type 1 extremal (largest).
(*Ans*: 0.955, 1.091, 1.787)
(ii) Determine the combined load factor. (*Ans*: 1.454)
(iii) Determine the partial safety factors with respect to mean values.
(*Ans*. 0.868, 1.039, 2.553)

9.2 For the problem in Exercise 9.1, if the standard deviation of Z_p is 60000 mm³ and the mean deviation zero, determine the mean and partial safety factor for Z_p and the combined resistance factor for $\beta_0 = 3$. (*Ans*. $\gamma_{Z_p} = 0.968$
$\gamma_{comb} = 0.928$, mean of $Z_p = 1.139 \times 10^6$ mm³,
$\gamma_{f_y} = 0.956$, $\gamma_D = 1.091$, $\gamma_L = 1.75$)

9.3 The limit state equation for the shear strength of steel beam is taken as

$$f_y t_w d - V_D - V_L = 0$$

It is given:

Variable f_y: $\mu = 275$ N/mm² $\qquad \sigma = 27.5$ N/mm²
Nominal = 250 N/mm²
Variable V_D: $\mu = 270.9$ kN $\qquad \sigma = 27.09$ kN
Nominal = 258 kN
Variable V_L: $\mu = 224$ kN $\qquad \sigma = 63.4$ kN
Nominal = 361 kN

$t_w = 8.9$ mm. If the standard deviation of d is 20 mm and the mean deviation 0, determine the partial safety factors of f_y, V_D, V_L, and d for $\beta_0 = 5$, assuming f_y, V_D, V_L and d follow lognormal, normal, Type 1 extremal (largest) and normal respectively. What is the combined resistance factor? (*Ans*. $\gamma_d = 0.973$
$\gamma_{f_y} = 0.878$, $\gamma_D = 1.105$, $\gamma_L = 2.229$, $\gamma_{comb} = 0.855$)

9.4 The shear strength of RCC beam is given by

$$R = f_y A_{sv}\frac{d}{s} + \frac{bd}{6}\sqrt{0.8 f_{cu}}\left\{\sqrt{\frac{1 + 5\beta' - 1}{\beta'}}\right\}$$

The limit state equation is

$$R - V_D - V_L = 0$$

where V_D and V_L are shears due to dead and live load. It is given: $\beta' = 2.9$, $b = 300$ mm, $d = 580$ mm, $s = 100$ mm.

Variable f_y: $\mu = 320$ N/mm² $\quad \sigma = 32$ N/mm²
Variable f_{cu}: $\mu = 26.7$ N/mm² $\quad \sigma = 4.02$ N/mm²
Variable V_D: $\mu = 94.5$ kN $\quad \sigma = 9.45$ kN
Variable V_L: $\mu = 75.0$ kN $\quad \sigma = 21.22$ kN

The nominal values of f_y, f_{cu}, V_D and V_L are 250 N mm², 20 N mm², 90 kN and 90.69 kN respectively.
Determine the partial safety factors of f_y, f_{cu}, V_D and V_L for $\beta_0 = 5$ if all variables are normally distributed.

(*Ans.* $\gamma_{f_y} = 0.898$, $\gamma_{f_{cu}} = 0.97$, $\gamma_{V_D} = 1.202$ and $\gamma_{V_L} = 1.588$)

9.5 The ultimate strength of a RCC beam in shear is given by Eq. 9.32. Consider the problem in Example 9.6.

(i) The statistics of shear force due to dead and live load are as follows:

Variable V_D: (Nominal value $= 79.5$ kN)
$\mu = 83.5$ kN $\quad \sigma = 8.35$ kN
Variable V_L: (Nominal value $= 54.41$ kN)
$\mu = 45.0$ kN $\quad \sigma = 12.73$ kN

The statistics of f_{cu} and f_y, and all other data are the same as given in Example 9.6. Determine the partial safety factors for f_{cu}, f_y, V_D, and V_L for $\beta_0 = 5$.

(*Ans.* $\gamma_{fy} = 0.9438$; $\gamma_{fcu} = 0.9566$; $\gamma_D = 1.269$ and $\gamma_L = 1.571$)

(ii) If the code committee fixes the material reduction factor for concrete as 1.5, determine the partial safety factors for f_y, V_D, and V_L for $\beta_0 = 5$.

(*Ans.* $\gamma_{fy} = 0.859$; $\gamma_D = 1.267$; $\gamma_L = 1.563$)

(iii) If the code committee fixes the material reducation factor for steel as 1.15, determine the partial safety factors for f_{cu}, V_D, and V_L.

(*Ans.* $\gamma_{fcu} = 0.817$; $\gamma_D = 1.288$ and $\gamma_L = 1.673$)

9.6 A column is subjected to combined axial load and bending moment. Under this combined action, let the equivalent strength of column be R. The column is subjected to dead, live and wind load. It is given:

Variable R/R_n: (normal)
$\mu = 1.22 \quad \delta = 0.14$
Variable D/D_n: (normal)
$\mu = 1.05 \quad \delta = 0.1$
Variable L_{apt}/L_n: (lognormal)
$\mu = 0.179 \quad \delta = 0.55$
Variable W_m/W_n: [Type 1 extremal (largest)]
$\mu = 0.804 \quad \delta = 0.334$

Study the variation of the partial safety factors for various values of $W_n/D_n = 0.5$ 0.75, 1.0, 1.5 and 2 for $\beta_0 = 3$ and plot the same. Assume $L_n/D_n = 1.0$.

9.7 The limit state equation of a structural component subjected to dead and wind load is given as

$$R - D - W = 0$$

The statistics of the variables are given below.

Variable R/R_n: $\mu = 1.355$ $\sigma = 0.225$ (normal)
Variable D/D_n: $\mu = 1.05$ $\sigma = 0.105$ (normal)
Variable W/W_n: $\mu = 0.804$ $\sigma = 0.269$ [Type 1 extremal (largest)]

Plot the variation of the partial safety factors γ_R, γ_D, and γ_W with W_n/D_n ranging from 0.5, 0.75, 1.0, 1.5 and 2 for $\beta_0 = 3.5$. Determine the optimal values of γ_R, γ_D and γ_W using the method adopted by Baker (9.15) assuming suitable weighting factors for the occurrence of each W_n/D_n ratio.

9.8 The limit state equation for an axially loaded short column is assumed as:

$$0.67 f_{cu} A_c + f_y A_s - D - L = 0$$

The area of concrete, A_c, is 113000 mm². It is given:

Variable f_{cu} : (normal)

$\mu = 26.8$ N/mm² $\sigma = 4.02$ N/mm²

Variable f_y: (normal)

$\mu = 469$ N/mm² $\sigma = 46.9$ N/mm²

Dead load D: (Normal)

$\mu = 420$ kN $\sigma = 42$ kN

Live load L: [Type 1 extremal (largest)]

$\mu = 166.8$ kN $\sigma = 47.2$ kN

If the nominal values of f_{cu}, f_y, D, and L are 20 N/mm², 415 N/mm², 400 kN and 269 kN respectively, determine the partial safety factors for variables for $\beta_0 = 5$.

(*Ans.* $\gamma_{fcu} = 0.458$; $\gamma_{fy} = 1.044$; $\gamma_D = 1.15$; $\gamma_L = 1.075$)

10

Reliability of Structural Systems

10.1 GENERAL

We have so far studied the reliability analysis and design of structural components. The code calibration based on component reliability was also introduced and illustrated in Chapter 9. But a structure or a structural system, viz. building, bridge, offshore platform, water tank, etc. is built up of many components (elements). The capacity of a structural system will depend on the capacities of its components. The behaviour of the system is probabilistic as it depends on the performance of its components whose behaviour is random. Civil engineering structures are invariably a kind of system. Information is available only on the statistical performance of components. With this information, the reliability of the structural system must be determined. A structural system may have several failure modes. These failure modes are to be identified, modelled, and combined to determine the system reliability. Hence, the reliability of structures/structural systems of multiple components and with multiple failure modes is to be considered from the system point of view.

10.2 SYSTEM RELIABILITY

One of the important applications of probability theory is the evaluation of the reliability of a system which is made up of components with known reliabilities. The reliability of a component is the probability of its satisfactory performance against the purpose for which it has been designed. Block diagrams are used to demonstrate the computation of the reliability of a system. Systems are classified basically into three groups as given below:

(i) series system
(ii) parallel redundant system
(iii) mixed system

10.2.1 Series System

The term, commonly used in the field of electrical engineering, is easily understood by everyone. In this system, even if one component fails to function satisfactorily, the whole system will fail. Therefore, a series system performs satisfactorily only when every component works satisfactorily.

The block diagram for this system is as shown in Fig. 10.1 and the reliability of the system is calculated as explained below:
Let

A_i = the event that component i works satisfactorily
p_{ss} = probability of survival of the system
p_{fs} = probability of failure of the system
$p_{ss} = 1 - p_{fs}$

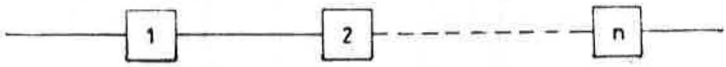


FIG. 10.1 Series system

As every component should function satisfactorily for the system to be reliable,

$$p_{ss} = P(A_1 \cap A_2 \cap \ldots \cap A_n) \tag{10.1}$$

If the events A_i are independent, the above equation simplifies to

$$p_{ss} = P(A_1)P(A_2) \ldots P(A_n)$$

$$= \prod_{i=1}^{n} (1 - p_{fi}) \tag{10.2}$$

where p_{fi} = the probability of failure of the component i, and n = the number of components.

The model is also called the "weakest link model".

In the case of structural systems in civil engineering, the values of p_{fi} are very small. If $p_{fi} \ll 1$, Eq. (10.2) can be rewritten as

$$p_{ss} \simeq 1 - \sum_{i=1}^{n} p_{fi} \tag{10.3}$$

and

$$p_{fs} \simeq \sum_{i=1}^{n} p_{fi} \tag{10.4}$$

10.2.2 Parallel Redundant System

In this case, the system survives even if one component has failed. The system fails to function satisfactorily only when every component of the system has failed to function satisfactorily. The block model diagram for the computation of reliability is shown in Fig. 10.2. The reliability of the system is given by

$$p_{ss} = 1 - p_{fs}$$

$$= 1 - P(A_1^c \cap A_2^c \cap \ldots \cap A_n^c) \tag{10.5}$$

where A_i^c = the event that component i does not function satisfactorily. If events A_i^c are independent, Eq. (10.5) simplifies to

$$p_{ss} = 1 - [P(A_1^c)P(A_2^c) \ldots P(A_n^c)]$$

$$= 1 - \prod_{i=1}^{n} p_{fi} \tag{10.6}$$

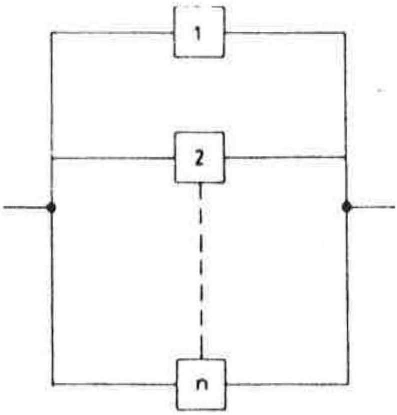


FIG. 10.2 Parallel redundant system

In structural engineering, this system may be referred to as a parallel system with n perfectly ductile elements.

10.2.3 Mixed System

This is a combination of series and parallel redundant systems. The block model diagram for the computation of the reliability of a mixed system is shown in Fig. 10.3. This is visualised to consist of subsystems S_1 and S_2 as shown in Fig. 10.3. S_1 is a series system and S_2 a parallel redundant system, and subsystems S_1 and S_2 are connected in series. For this mixed system to survive, each subsystem should survive under the given conditions. Hence the reliability of the system is given by

$$p_{ss} = P(E_1 \cap E_2)$$

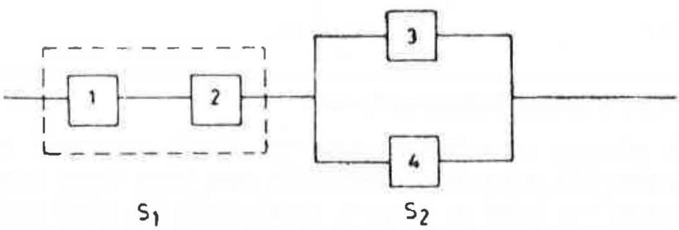


FIG. 10.3 Mixed system

where E_1 = the event that subsystem 1 functions satisfactorily and E_2 = the event that subsystem 2 functions satisfactorily. Knowing how to compute the system reliability of the series and parallel redundant systems, the probability of the survival of this mixed system, shown in Fig. 10.3, is given by

$$\begin{aligned} p_{ss} &= P(E_1)P(E_2) \\ &= (1 - p_{fs1})(1 - p_{fs2}) \end{aligned} \tag{10.7}$$

where p_{fsi} is the probability of failure of the subsystem i. It has been assumed that events A_i are statistically independent.

EXAMPLE 10.1 Calculate the reliabilities of the systems shown in Figs. 10.3 and 10.4 assuming the performance of components is statistically independent. Compare the results. Given:

$$p_{f1} = 0.1 \qquad p_{f2} = p_{f4} = 0.2 \qquad p_{f3} = 0.3$$

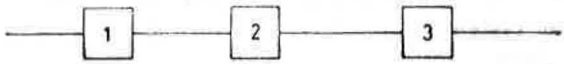


FIG. 10.4 Block model—Example 10.1

(i) The reliability of the mixed system (Fig. 10.3):
The reliability of the subsystem 1, using Eq. (10.2), is

$$P(E_1) = (1 - 0.1)(1 - 0.2)$$
$$= 0.72$$

Using Eq. (10.6), the reliability of the subsystem 2 is

$$P(E_2) = 1 - p_{f3}p_{f4}$$
$$= [1 - (0.3)(0.2)]$$
$$= 0.94$$

Hence the reliability of the mixed system is (Eq. 10.7)

$$p_{ss} = P(E_1)P(E_2)$$
$$= (0.72)(0.94) = 0.6768$$

(ii) The reliability of the series system, shown in Fig. 10.4, is

$$p_{ss} = (1 - 0.1)(1 - 0.2)(1 - 0.3)$$
$$= 0.504$$

When the reliabilities of the two systems are compared, it can be seen that at the cost of an additional redundant component 4, the mixed system is more reliable than the one shown in Fig. 10.4.

EXAMPLE 10.2 Consider a nuclear power plant designed for a level of earthquake intensity. At this particular level of the earthquake intensity, the controlled shutdown of the reactor depends on the functioning of the control systems; the cooling systems and the primary containment vessel. There are two redundant control systems, two redundant cooling systems, and a single primary containment vessel with two components A and B in series.

(a) Draw the block model for the computation of the reliability of the plant with respect to shutdown at the given earthquake level.

(b) If it is assumed that there will be no major accident if either the shutdown is controlled or the reinforced concrete secondary containment vessel C performs properly, model the total system with respect to the major accident reliability.

Solution Let

$$p_{fi} = \text{probability of failure of the component } i$$

(a) The block model diagram for the computation of reliability is shown in Fig. 10.5(a). The reliability of the plant with respect to shutdown at the given earthquake level is

$$p_{ss} = (1 - p_{f1}p_{f2})(1 - p_{f3}p_{f4})[(1 - p_{fA})(1 - p_{fB})]$$

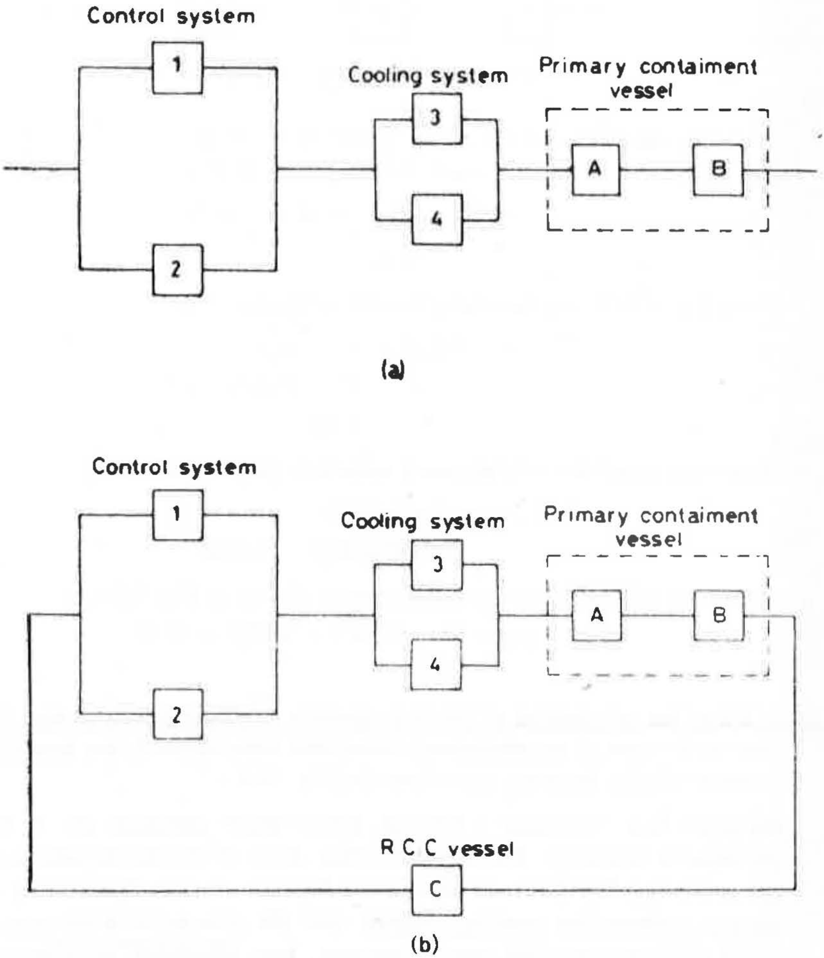


FIG. 10.5 (a) Block model for case a and (b) block model for case b—Example 10.2

(b) In this case, there will be no major accident if either shutdown is controlled or the secondary concrete containment vessel C performs satisfactorily. The block diagram is shown in Fig. 10.5(b). The reliability of the system with respect to no major accident is

$$P_{ss} = 1 - [1 - \{(1 - p_{f1}p_{f2})(1 - p_{f3}p_{f4})(1 - p_{fA})(1 - p_{fB})\}](p_{fC})$$

10.3 MODELLING OF STRUCTURAL SYSTEMS

10.3.1 General

Structures can be considered as a system and can be modelled into any-one of the three basic systems, depending on the physical behaviour for computing its reliability. The modelling of a few structural systems for the computation of reliability is explained in the following sections.

10.3.2 Simple Beams

If a simply supported beam is subjected to a load as shown in Fig. 10.6, the failure of the beam occurs when the strength of the critical section (the section subjected to the maximum moment) is less than the external load. The beam can be considered as a system with one component (critical section) only. It is similar to a case of a tension member subjected to a load shown in Fig. 10.6(c).

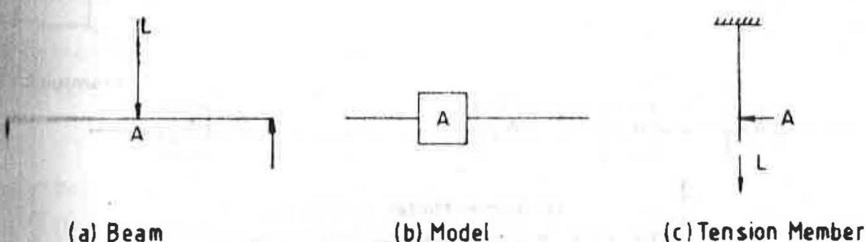


FIG. 10.6 Single member single load condition

If the beam shown in Fig. 10.7(a) or the tension member, shown in Fig. 10.7(b) is subjected to several independent load conditions, $L_1, L_2, \cdots, L_m$ or independent repetitions of a single load, the reliability model is a series system, since for the beam to be reliable, the critical section should survive or be reliable under each load. The block model is shown in Fig. 10.7(c), where the Section A under each load is imagined as a component and is denoted as A_i.

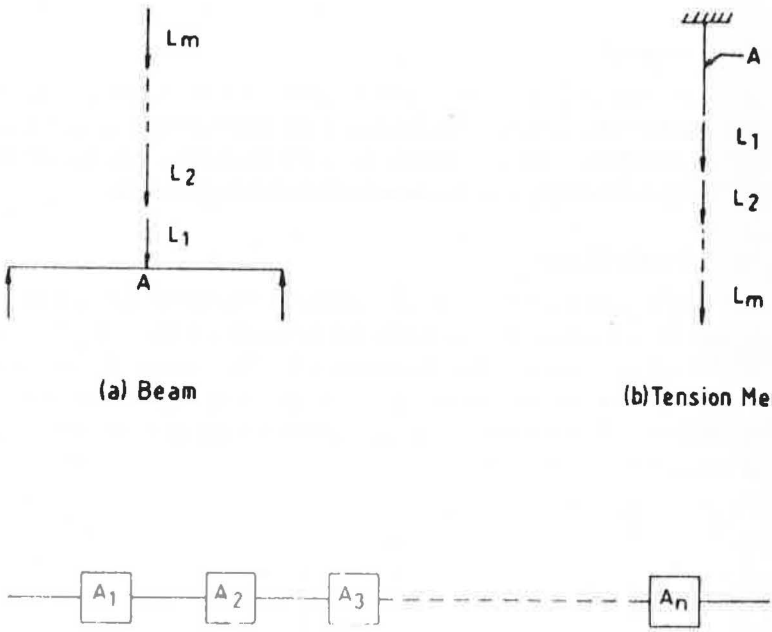
EXAMPLE 10.3 Consider a steel tension member, shown in Fig. 10.7(b), subjected to m independent repetitions of load L. It is given that the means and standard deviations of the resistance of the member R and L are

$$\mu_R = 50 \text{ kN} \qquad \sigma_R = 5 \text{ kN}$$
$$\mu_L = 25 \text{ kN} \qquad \sigma_L = 12 \text{ kN}$$
$$m = 5$$

Compute the reliability of the member if R and L are normally distributed.

Solution Consider the member as a system subjected to m independent repetitions of L. The block model for the system to compute the reliability will be a series system. The reliability of the system is

$$p_{ss} = [(1 - p_f)]^m$$



(a) Beam (b) Tension Member

(c) Block Model

FIG. 10.7 Single member m load conditions

where p_f is the probability of failure of the member under L. The value of p_f is computed as given below:

$$p_f = P[R < L]$$
$$= P[(R - L) < 0]$$

Since R and L are normal, using Eq. (6.16), we get

$$p_f = \Phi(-\beta)$$
$$= \Phi\left[\frac{\mu_L - \mu_R}{(\sigma_R^2 + \sigma_L^2)^{1/2}}\right]$$

Substituting the given data, we have

$$p_f = \Phi\left[\frac{25 - 50}{(5^2 + 12^2)^{1/2}}\right]$$
$$= \Phi(-1.92) = 0.02743$$

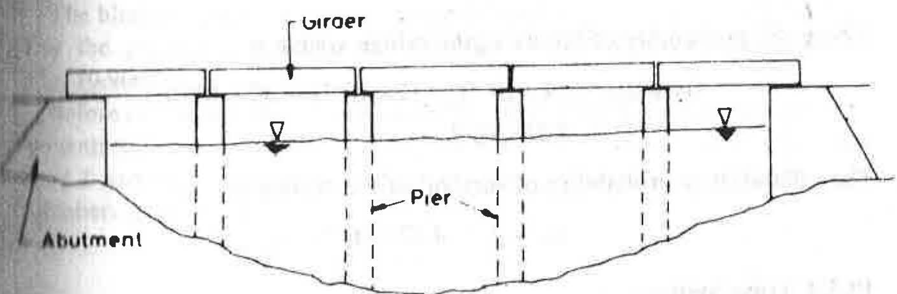
The probability of survival of the system under repetition of L for five times is

$$p_{ss} = (1 - 0.02743)^5$$
$$= 0.87$$

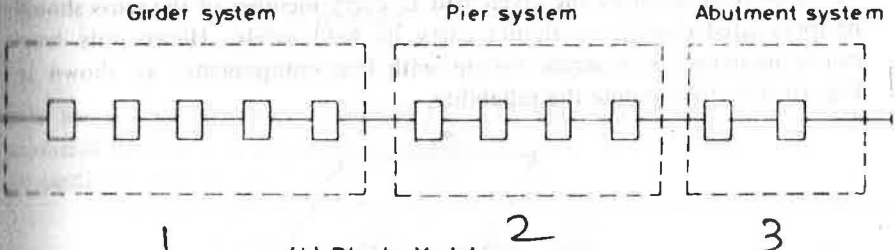
It is to be noted that the reliability of the system decreases as m increases.

10.3.3 Bridge System

A bridge structural system consists of girders, piers, and abutments, as shown in Fig. 10.8(a). For the reliability of the system of piers, each pier should function satisfactorily. Hence, the reliability model for the system of piers will be a series system. Similarly, for the reliability of the system of abutments, each abutment should be reliable under the given loading condition. Hence, the reliability model for the satisfactory performance of the system of abutments is a series system. Likewise, for the reliability of the system of girders, each girder should be reliable under the given conditions. Hence the reliability model for the system of girders is a series system.



(a) Bridge System



(b) Block Model

FIG. 10.8 Modelling of a bridge system

For the whole bridge system to survive under the given loading condition, each subsystem, i.e. system of girders, system of piers, and system of abutments, should survive. Hence, all the three subsystems are to be connected in series to compute the reliability of the system. The block model diagram is shown in Fig. 10.8(b).

EXAMPLE 10.4 Compute the reliability of the bridge system, shown in Fig. 10.8(a), having four piers, five girders and two abutments. The probability of failure of each pier, girder and abutment is 10^{-4}, 10^{-5} and 10^{-6} respectively. Compute the reliability of the bridge system.

Solution The probability of failure of the pier system is [Eq. (10.4)]

$$(p_f)_{\text{pier}} = \sum_{i=1}^{4} p_{fi} = 4\times 10^{-4}$$

The probability of failure of the girder system is [Eq. (10.4)]

$$(p_f)_{\text{Girder}} = \sum_{i=1}^{5} p_{fi} = 5\times 10^{-5}$$

The probability of failure of the abutment system is [Eq. (10.4)]

$$(p_f)_{\text{Abut}} = \sum_{i=1}^{2} p_{fi} = 2\times 10^{-6}$$

Hence the probability of failure of the bridge system is

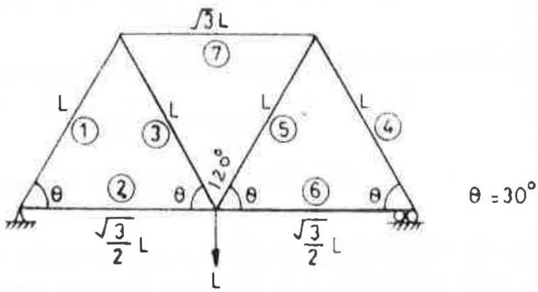
$$(p_f)_{\text{Bridge}} = (4\times 10^{-4}) + (5\times 10^{-5}) + (2\times 10^{-6})$$
$$= 4.52\times 10^{-4}$$

The reliability or probability of survival of the system is

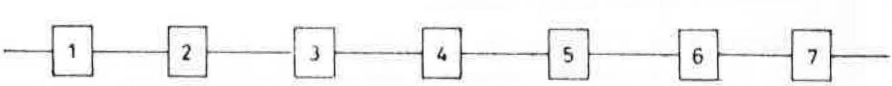
$$p_{ss} = 1 - 4.52\times 10^{-4}$$

10.3.4 Truss System

Consider the truss shown in Fig. 10.9(a). It is subjected to a load L as shown in the figure. This is a determinate structure which is considered as a system having six members (components). For the truss to perform satisfactorily, i.e. to be reliable under the given load L, every member of the truss should perform satisfactorily, i.e. should carry its load safely. Hence, this truss can be modelled as a series system with five components, as shown in Fig. 10.9(b), to compute the reliability.



(a) Truss System



(b) Block Model

FIG. 10.9 Modelling of a truss system

EXAMPLE 10.5 The truss shown in Fig. 10.9 is subjected to a random load L with parameters

$$\mu_L = 25 \text{ kN} \qquad \sigma_L = 12 \text{ kN}$$

The parameters of resistances of members are given as

$$\mu_{Ri} = 50 \text{ kN} \qquad \sigma_{Ri} = 5 \text{ kN} \qquad i = 1, 2, \ldots, 6$$

$$\mu_{R7} = 60 \text{ kN} \qquad \sigma_{R7} = 6 \text{ kN}$$

Compute the reliability of the truss system assuming resistances of the members are independent, and also that resistance and load are independent and R_i and L are normal.

The block model for the computation of reliability is shown in Fig. 10.9(b). For the given load, the forces developed in the members are given in Fig. 10.9(a).

Before calculating the system reliability, the reliability of individual components must be computed. Since members 1, 3, 4 and 5 carry the same load L and their resistances are the same, the probability of failure of these members is the same:

$$p_{f1} = \Phi\left[\frac{\mu_L - \mu_{R1}}{(\sigma_L^2 + \sigma_{R1}^2)^{1/2}}\right]$$

$$= \Phi\left[\frac{25 - 50}{(5^2 + 12^2)^{1/2}}\right]$$

$$= \Phi(-1.92)$$

$$= 0.02743$$

$$p_{f3} = p_{f4} = p_{f5} = p_{f1} = 0.02743$$

The force in the members 2 and 6 is $L\sqrt{3}/2$. Hence, the mean value and standard deviation of the force in member 2 are $(\sqrt{3}/2)\mu_L$ and $(\sqrt{3}/2)\sigma_L$ respectively. Hence,

$$p_{f2} = p_{f6} = \Phi\left[\frac{\frac{\sqrt{3}}{2}\mu_L - \mu_{R2}}{\left\{\left(\frac{\sqrt{3}}{2}\sigma_L\right)^2 + \sigma_{R2}^2\right\}^{1/2}}\right]$$

$$= \Phi\left[\frac{21.65 - 50}{\{(10.392)^2 + 5^2\}^{1/2}}\right]$$

$$= \Phi(-2.458) = 0.006986$$

Similarly,

$$p_{f7} = \Phi\left[\frac{\sqrt{3}\,\mu_L - \mu_{R7}}{\{(\sqrt{3}\;\sigma_L)^2 + \sigma_{R7}^2\}^{1/2}}\right]$$

$$= \Phi\left[\frac{43.3 - 60}{(20.78^2 + 6^2)^{1/2}}\right]$$

$$= \Phi\left(\frac{16.7}{21.63}\right) = \Phi(-0.772)$$

$$= 0.224338$$

Using Eq. (10.2), the probability of survival of the system is

$$p_{ss} = \prod_{i=1}^{7} (1 - p_{fi})$$

$$= (1 - 0.02743)(1 - 0.006986)(1 - 0.02743) \times (1 - 0.02743)(1 - 0.02743)(1 - 0.006986) \times (1 - 0.22438)$$

$$= 0.68625$$

Consider the trusses shown in Figs. 10.10(a) and 10.10(b) in which the number of members are 3 and 11 respectively. Reliability of these trusses can be computed as above and are 0.948 and 0.148 respectively. Including the result for the same type of truss with seven members, it can be observed that as the number of members increases, the reliability of the system decreases when the performance of the members are statistically independent.

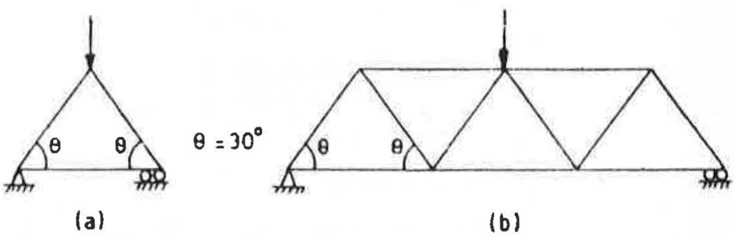


FIG. 10.10 Trusses

In the case of truss systems, for a given number of members, if the resistances of members are correlated, the reliability of the system increases with the increase in the correlation coefficient between member performances (resistances of members).

10.3.5 Indeterminate Beam

Consider a fixed steel beam shown in Fig. 10.11. This is a redundant, perfectly ductile structure. In this case, the failure of the structure does not occur if one section yields; failure occurs only when a sufficient number of sections have yielded to form a collapse mechanism. In the case of the fixed beam shown in Fig. 10.11, the beam fails only when the critical sections 1, 2 and 3 (positions of maximum moments) have yielded. Hence, the given beam can be considered as a parallel redundant system, the block model diagram of which is shown in Fig. 10.11(b).

In the case of redundant structures, the reliability of the system increases as the number of redundant components increases if R_i are statistically

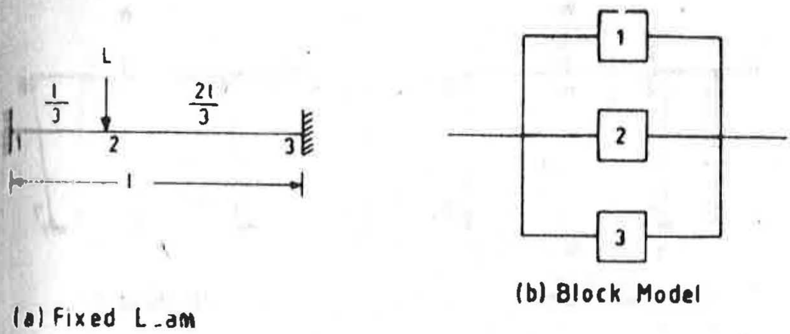


FIG. 10.11 Modelling of a fixed beam

independent. It can be proved also that in the case of a redundant parallel system, if the resistances of members are correlated, the reliability of the system decreases with the increase in correlation.

10.3.6 Frame Structural Systems

Frame structures are highly redundant structures. In this case, the failure of a single section (component) does not result in the failure of the frame (system). Assuming a perfect ductile structure, the frame fails only when a sufficient number of plastic hinges are developed to cause a collapse mechanism. Again, there may be a number of possible collapse mechanisms in a frame structure. These possible collapse mechanisms are to be synthesized and the system failure probability is to be computed.

A failure mode, i.e. a collapse mechanism is composed of component (section) failure events that are in parallel. For a failure mode to be formed, every critical section in that mode must have failed. Hence, to compute the reliability of the frame under a particular failure mode, the critical sections in that mode are to be connected in parallel. For the frame to be reliable, it has to survive under all the possible failure modes. Hence, to compute the system reliability, these parallel subsystems are to be combined as a series system. The block model for the computation of the reliability of a frame structure is shown in Fig. 10.12(e). It is clear that this is a mixed system. The individual failure modes may be correlated because of common load and resistance variables. There may be correlations between single elements in the same failure mode. The system reliability depends on (i) topology, (ii) post failure behaviour of components, and (iii) correlation characteristics of different variables and different failure modes.

In the case of ideal plastic structures, each collapse mode, called collapse mechanism, (i.e. limit state) can be represented directly by an equation in terms of the plastic moments of hinged sections in the mechanism and the length factors multiplied by loads (10.1), using the mechanism method of analysis (10.2). Hence, the safety margin equations for failure modes can be directly written and the reliability of a frame under each failure mode can

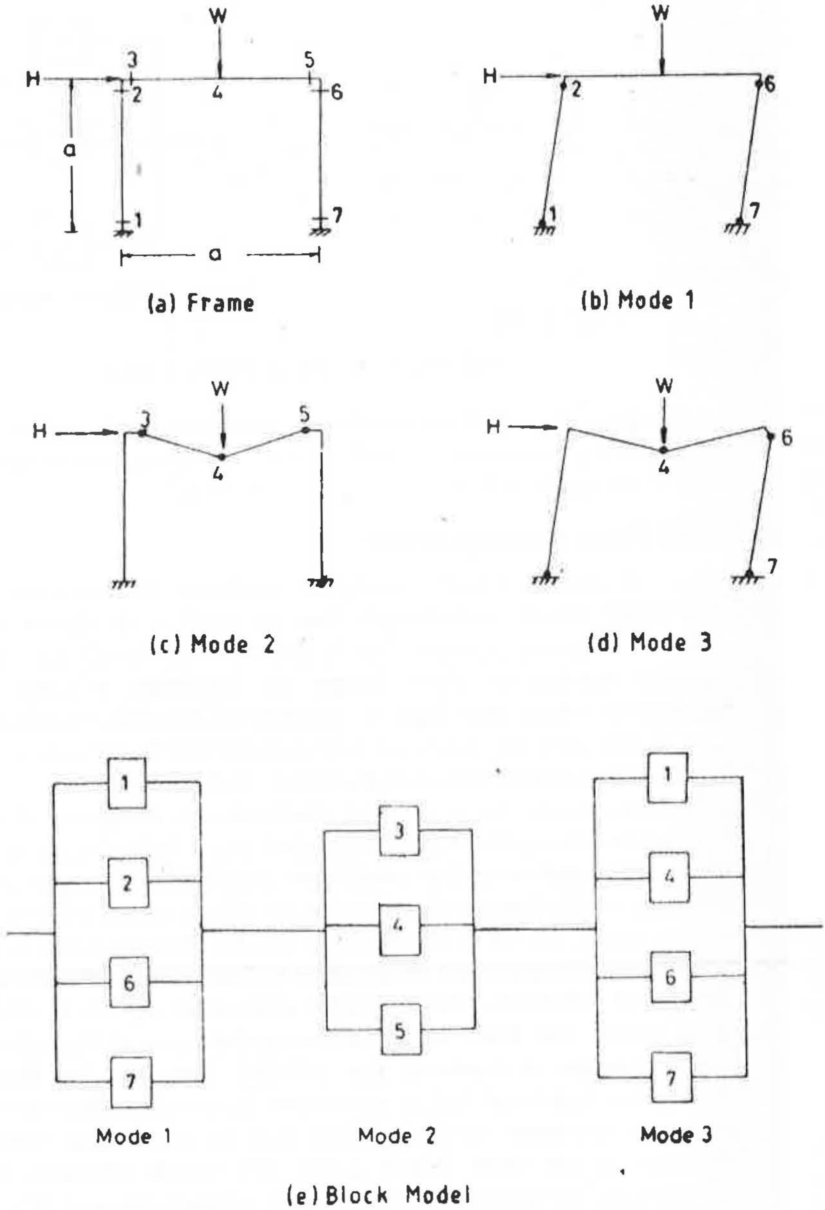
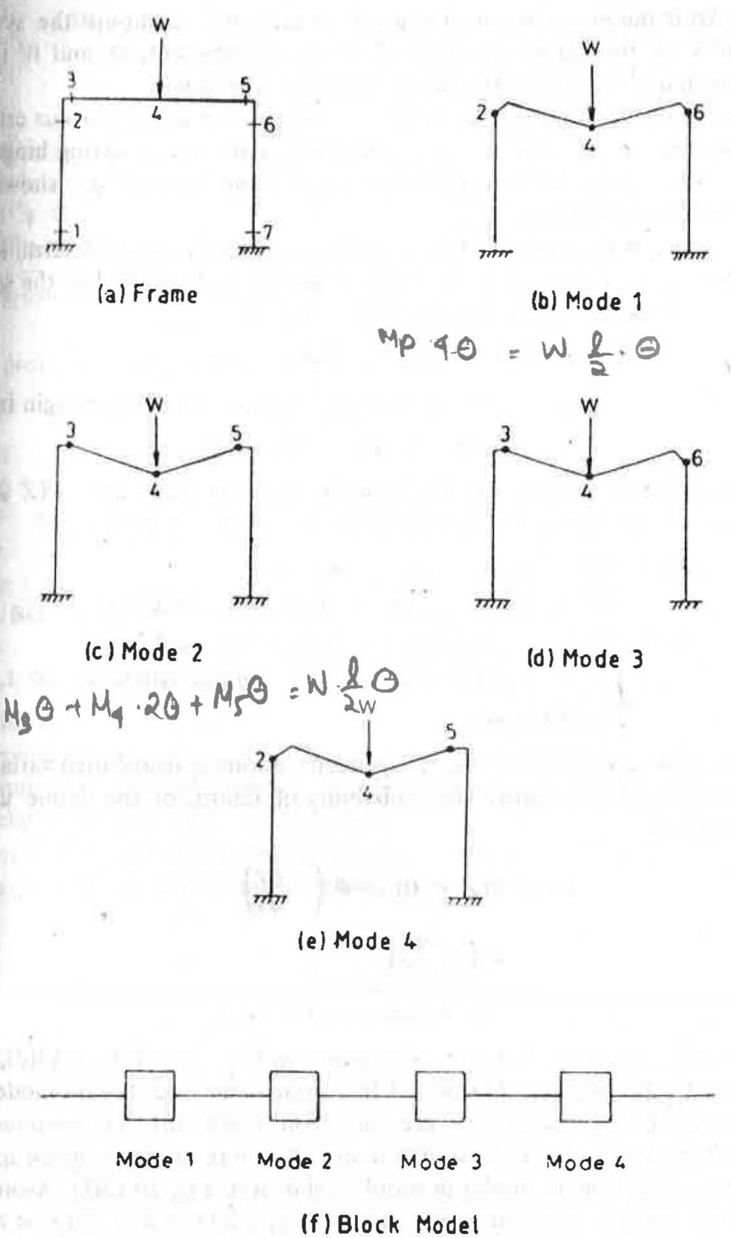


FIG. 10.12 Modelling of a frame system

be calculated. Then the structure is modelled as a series system with the failure modes as components, and the system reliability is determined. This is illustrated in the following example.

EXAMPLE 10.6 Consider the rigid steel frame shown in Fig. 10.13(a). It is given that



Mp 4θ = W l/2 · θ

M3 θ + M4 · 2θ + M5 θ = W l/2 θ

FIG. 10.13 Frame, failure modes and block model—Example 10.6

$$\mu_{M1} = \mu_{M2} = \mu_{M6} = \mu_{M7} = 490 \text{ kN m}$$
$$\sigma_{M1} = \sigma_{M2} = \sigma_{M6} = \sigma_{M7} = 73.5 \text{ kN m}$$
$$\mu_{M3} = \mu_{M4} = \mu_{M5} = 653 \text{ kN m}$$
$$\sigma_{M3} = \sigma_{M4} = \sigma_{M5} = 97.95 \text{ kN m}$$
$$\mu_W = 446 \text{ kN} \qquad \sigma_W = 69.9 \text{ kN}$$

where M_i is the plastic moment capacity of section i. Compute the system reliability of the frame assuming all M_i are independent, M_i and W independent and all variables M_i and W normally distributed.

Because of the random behaviour of the resistances of various critical sections and load, the frame may fall under failure modes having hinges at 2, 4 and 6, or 3, 4 and 5, or 3, 4 and 6, or 2, 4 and 5 which are shown in Figs. 10.13(b)–10.13(e).

The virtual work method of plastic analysis (10.2) is used to determine the resistance of the frame, and action at collapse for each mode. For the safety of the frame under failure mode 1 [Fig. 10.13(b)]:

$$M_2\theta + 2M_4\theta + M_6\theta > 3W\theta$$

where θ is the virtual rotation at section 2. Hence, the safety margin is

$$Z = M_2 + 2M_4 + M_6 - 3W$$

The probability of survival of the frame under a mode is $p_s \doteq P(Z \geqslant 0)$. As Z is a linear function of the variables M_i and W, we have

$$\begin{aligned}
\mu_Z &= \mu_{M2} + 2\mu_{M4} + \mu_{M6} - 3\mu_W \\
&= 490 + 2(653) + 490 - (3)(446) = 948 \text{ kN m} \\
\sigma_Z &= [(1)^2\sigma_{M2}^2 + (2)^2\sigma_{M4}^2 + (1)^2\sigma_{M6}^2 + (3)^2\sigma_W^2]^{1/2} \\
&= [73.5^2 + (4)(97.95)^2 + 73.5^2 + (9)(69.9)^2]^{1/2} \\
&= 305.2 \text{ kN m}
\end{aligned}$$

As Z is a linear function of the independent, normally distributed variables, Z is also a normal variable. The probability of failure of the frame under the mode 1 is

$$\begin{aligned}
p_{f1} &= P(Z < 0) = \Phi\left(-\frac{\mu_Z}{\sigma_Z}\right) \\
&= \Phi\left(\frac{-948}{305.2}\right) \\
&= \Phi(-3.106) = 9.35 \times 10^{-4}
\end{aligned}$$

Similarly, for other failure modes shown in Figs. 10.13(c), 10.13(d), and 10.13(e), the probability of failure of the frame under each failure mode can be calculated; the calculations are shown in Table 10.1. To compute the probability of survival of the system under all failure modes, a block model is drawn connecting all modes in series as shown in Fig. 10.13(f). Assuming all failure modes are statistically independent (Note: this is not true as all Z_i are correlated as seen in Table 10.1) and using Eq. (10.2), we have

$$\begin{aligned}
p_{ss} &= \prod_{i=1}^{4} (1 - p_{fi}) \\
&= (1 - 9.35 \times 10^{-4})(1 - 3.97 \times 10^{-5}) \\
&\quad \times (1 - 1.85 \times 10^{-4})(1 - 1.85 \times 10^{-4}) \\
&= 0.9986553
\end{aligned}$$

TABLE 10.1 *Reliability analysis of the portal frame—Example 10.6*

Mode No.		Z_i	μ_{Zi} (kN m)	σ_{Zi} (kN m)	p_{fi}	β
1	[Fig. 10.13(b)]	$M_2 + 2M_4 + M_6 - 3W$	948	305.2	9.35×10^{-4}	3.106
2	[Fig. 10.13(c)]	$M_3 + 2M_4 + M_5 - 3W$	1274	318.7	3.17×10^{-5}	3.997
3	[Fig. 10.13(d)]	$M_3 + 2M_4 + M_6 - 3W$	1111	312.0	1.85×10^{-4}	3.561
4	[Fig. 10.13(e)]	$M_2 + 2M_4 + M_5 - 3W$	1111	312.0	1.85×10^{-4}	3.561

Simple bounds: $9.35\times 10^{-4} \leqslant p_{fs} \leqslant 13.37\times 10^{-4}$

The probability of failure of the structural system is

$$p_{fs} = 1 - 0.9986553$$
$$= 1.3447\times 10^{-3}$$

(Note: $p_{fs} \simeq \sum_{i=1}^{4} p_{fi}$)

10.4 BOUNDS ON SYSTEM RELIABILITY

10.4.1 Introduction

In the previous problem, it has been assumed that Z_i are statistically independent during the computation of the reliability of the system. It is obvious when the equations of Z_i are examined (Table 10.1), that Z_i are correlated, as the same random variables appear in the equations. For example, if Z_1 and Z_2 are considered, M_4 and W appear in both the equations. Hence, Z_1 and Z_2 are correlated.

If

$$Z_i = \sum_{i=1}^{n} a_i X_i \tag{10.8}$$

and

$$Z_k = \sum_{i=1}^{n} b_i X_i \tag{10.9}$$

(a_i and b_k are constants)
the covariance between Z_i and Z_k is given by

$$\mathrm{Cov}(Z_i, Z_k) = \sum_{i=1}^{n} a_i b_i \sigma_{Xi}^2 \tag{10.10}$$

The correlation coefficient between Z_i and Z_k is given by

$$\rho_{Zi,Zk} = \frac{\mathrm{Cov}(Z_i, Z_k)}{\sigma_{Z_i}\,\sigma_{Z_k}} \tag{10.11a}$$

Correlation coefficients can also be calculated using directional cosines α_i

directly obtained during the computation of reliability index using Level 2 method. The correlation coefficient between Z_i and Z_k is given by

$$\rho_{Z_i, Z_k} = \sum_{l=1}^{n} \alpha_{il}\alpha_{kl}$$

The above equation and Eq. (10.11a) are same. This can be easily verified for linear Eqs. (10.8) and (10.9) for Z_i and Z_k. For nonlinear equations, directional cosines evaluated at the design point, $\alpha_i^{\bullet}$, on the failure surface are used. That is, in general,

$$\rho_{Z_i, Z_k} = \sum_{l=1}^{n} \alpha_{il}^{\bullet}\alpha_{kl}^{\bullet} \tag{10.11b}$$

The probability of survival of the system is given by Eq. (10.1). That is,

$$p_{ss} = P[\text{All } Z_i > 0]$$

$$= \underbrace{\int_0^{\infty} \cdots \int_0^{\infty}}_{n=\text{fold}} f_{Z1, Z2, \ldots, Z_n}(z_1, z_2, \ldots, z_n)\, dz_1, dz_2, \ldots, dz_n$$

where $f_{Z_1, Z_2, \ldots, Z_n}(z_1, z_2, \ldots, z_n)$ is the n-dimensional joint probability density function of $Z_1, Z_2, \ldots, Z_n$. The joint probabilities are generally not known and the computation of n-fold integration is very difficult, and may not be possible. Therefore, the above equation is simplified by certain assumptions to derive bounds on the probability of failure. It is generally not possible to compute the unique value of the reliability of the system and therefore, the reliability of the system is specified by its bounds.

10.4.2 Simple Bounds

Cornell (10.3) has established simple bounds on the reliability of structural systems subjected to n failure modes and m load conditions. The assumption that all failure modes, i.e. Z_i, are perfectly correlated yields the upper bound as

$$p_{ss} = 1 - \max_i P(Z_i \leqslant 0) = 1 - \max_i p_{fi}$$

The assumption of all failure modes to be statistically independent yields the lower bound as

$$p_{ss} = \prod_{i=1}^{n} (1 - p_{fi})$$

Hence, the reliability of the system is bounded by

$$\prod_{i=1}^{n} (1 - p_{fi}) \leqslant p_{ss} \leqslant 1 - \max_i p_{fi} \tag{10.12}$$

If $p_{fi} \ll 1$, the bounds on p_{ss} become

$$1 - \sum_{i=1}^{n} p_{fi} \leqslant p_{ss} \leqslant 1 - \max_i p_{fi} \tag{10.13}$$

The bounds on the probability of failure of the system can be written as

$$\max_i p_{fi} \leqslant p_{fs} \leqslant \sum_{i=1}^{n} p_{fi} \tag{10.14}$$

If the system is subjected to several m load conditions, the bounds on p_{ss} and p_{fs} are given by (10.3)

$$1 - \sum_{i=1}^{n} \sum_{j=1}^{n} p_{fij} \leqslant p_{ss} \leqslant 1 - \max_{ij} p_{fij} \tag{10.15}$$

and

$$\max_{ij}^{nm} p_{fij} \leqslant p_{fs} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{m} p_{fij} \tag{10.16}$$

where p_{fij} is the probability of failure of the frame under mode i and load condition j. These bounds are very wide for practical purposes.

10.4.3 Narrow Bounds

The assumption of perfect correlation or, no correlation between failure modes, is not proper. The modes are usually positively correlated. The correlation coefficients between modes can be calculated using Eqs. (10.10) and (10.11a). Ditlevsen (10.4) has developed narrow bounds for the structural system failure probability through indicator function algebra. The lower bound on p_{fs} is

$$p_{fs} \geqslant P(Z_1 < 0) + \sum_{i=2}^{n} \max \{P(Z_i < 0) - \sum_{j=1}^{i-1} P[(Z_i < 0) \cap (Z_j < 0)], 0\} \tag{10.17}$$

and the upper bound is

$$p_{fs} \leqslant \sum_{i=1}^{n} P(Z_i < 0) - \sum_{i=2,\, j<i}^{n} \max P[(Z_j < 0) \cap (Z_j < 0)] \tag{10.18}$$

Let

$$E_i = (Z_i < 0)$$
$$E_j = (Z_j < 0)$$

Then the above Eqs. (10.17) and (10.18) become

$$p_{fs} \geqslant p_{f1} + \sum_{i=2}^{n} \max [p_{fi} - \sum_{j=1}^{i-1} P(E_i \cap E_j), 0] \tag{10.19}$$

and

$$p_{fs} \leqslant \sum_{i=1}^{n} p_{fi} - \sum_{i=2,\, j<i}^{n} \max P(E_i \cap E_j) \tag{10.20}$$

The joint probability, $P(E_i \cap E_j)$, may be approximated as follows. For lower bound (Eq. 10.19)

$$P(E_i \cap E_j) = P(A) + P(B) \tag{10.21}$$

upper bound (Eq. 10.20)

$$P(E_i \cap E_j) = \max [P(A), P(B)] \tag{10.22}$$

where $$P(A) = \Phi(-\beta_i)\,\Phi\left[-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}}\right] \tag{10.23}$$

$$P(B) = \Phi(-\beta_j)\,\Phi\left[-\frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}}\right] \tag{10.24}$$

where $\beta_i = \frac{\mu_{Z_i}}{\sigma_{Z_i}}$ $\quad \beta_j = \frac{\mu_{Z_j}}{\sigma_{Z_j}}$

EXAMPLE 10.7 For the same portal frame in Example 10.6, compute the simple and narrow bounds on the probability of failure of the frame.

Simple bounds:
For the possible four failure modes, the probability of failure of each mode has already been calculated and given in Table 10.1. The bounds on the probability of failure of the system, using Eq. (10.14), are

$$\begin{aligned}
\text{Upper bound} &= \sum_{i=1}^{4} p_{fi} \\
&= (9.35\times10^{-4}) + (3.17\times10^{-5}) + (1.85\times10^{-4}) \\
&\quad + (1.85\times10^{-4}) \\
&= 13.37\times10^{-4}
\end{aligned}$$

$$\text{Lower bound} = \max_i p_{fi} = p_{f1}$$

$$= 9.35\times10^{-4}$$

Hence, the bounds on p_{fs} are

$$9.35\times10^{-4} \leqslant p_{fs} \leqslant 13.37\times10^{-4}$$

Narrow bounds:
The failure modes are first renumbered, or ordered, in the descending order of p_{fi} values. Hence, from Table 10.1,

$$\begin{aligned}
&\text{Mode 1}: Z_1 = M_2 + 2M_4 + M_6 - 3W \qquad \beta = 3.106 \\
&\text{Mode 2}: Z_2 = M_3 + 2M_4 + M_6 - 3W \qquad \beta = 3.561 \\
&\text{Mode 3}: Z_3 = M_2 + 2M_4 + M_5 - 3W \qquad \beta = 3.561 \\
&\text{Mode 4}: Z_4 = M_3 + 2M_4 + M_5 - 3W \qquad \beta = 3.997
\end{aligned}$$

The correlations among failure modes (that is safety margins Z_i and Z_j) are next computed.

Using Eq. (10.10),

$$\begin{aligned}
\text{Cov}(Z_1, Z_2) &= (2)(2)\sigma_{M4}^2 + (1)(1)\sigma_{M6}^2 + (-3)(-3)\sigma_W^2 \\
&= 4\times97.95^2 + 73.5^2 + 9\times69.9^2 \\
&= 87753
\end{aligned}$$

Using Eq. (10.11a),

$$\rho_{Z_1, Z_2} = \frac{87753}{\sigma_{Z_1}\sigma_{Z_2}}$$

$$= \frac{87753}{305.2\times312} = 0.922$$

the correlation between Z_1 and Z_3 is

$$\rho_{Z_1, Z_3} = \frac{(1)(1)\sigma_{M2}^2 + (2)(2)\sigma_{M4}^2 + (-3)(-3)\sigma_W^2}{\sigma_{Z_1}\,\sigma_{Z_3}}$$

$$= \frac{73.5^2 + 4\times 97.95^2 + 9\times 69.9^2}{305.2\times 312} = 0.922$$

Similarly the correlation between other pairs of Z_iZ_j can be computed. They are

$$\rho_{Z_1, Z_4} = 0.847 \qquad \rho_{Z_2, Z_3} = 0.846$$

$$\rho_{Z_2, Z_4} = 0.925 \qquad \rho_{Z_3, Z_4} = 0.925$$

For the calculation of bounds, bounds on joint probabilities, $P(E_iE_j)$, are to be computed first.

Bounds on $P(E_1E_2)$:

Using Eq. (10.23),

$$P(A) = \Phi(-\beta_1)\,\Phi\left[-\frac{\beta_2 - \rho_{12}\beta_1}{(1-\rho_{12}^2)^{1/2}}\right]$$

$$= \Phi(-3.106)\,\Phi\left[-\frac{3.561 - 0.922\times 3.106}{(1-0.922^2)^{1/2}}\right]$$

$$= \Phi(-3.106)\,\Phi(-1.802)$$

$$= 0.334\times 10^{-4}$$

$$P(B) = \Phi(-\beta_2)\,\Phi\left[-\frac{\beta_1 - \rho_{12}\beta_2}{(1-\rho_{12}^2)^{1/2}}\right]$$

$$= \Phi(-3.561)\,\Phi\left[-\frac{3.106 - 0.922\times 3.561}{(1-0.922^2)^{1/2}}\right]$$

$$= \Phi(-3.561)\,\Phi(0.458)$$

$$= 1.25\times 10^{-4}$$

Lower bound on $P(E_1 E_2) = P(A) + P(B)$

$$= (0.334 + 1.25)\times 10^{-4}$$

$$= 1.584\times 10^{-4}$$

Upper bound on $P(E_1E_2) = \max\,[P(A); P(B)]$

$$= 1.25\times 10^{-4}$$

Since $\rho_{12} = \rho_{13}$ and $\beta_3 = \beta_2$,

$$P(E_1E_3) = P(E_1E_2)$$

Bounds on $P(E_1E_4)$:

$$P(A) = \Phi(-3.106)\,\Phi\left[-\frac{3.997 - 0.847 \times 3.106}{(1 - 0.847^2)^{1/2}}\right]$$

$$= \Phi(-3.106)\,\Phi(-2.569) = 0.048 \times 10^{-4}$$

$$P(B) = \Phi(-3.997)\,\Phi\left[-\frac{3.106 - 0.847 \times 3.997)}{(1 - 0.847^2)^{1/2}}\right]$$

$$= \Phi(-3.997)\,\Phi(0.5258)$$

$$= 0.222 \times 10^{-4}$$

$$P(A) + P(B) = 0.27 \times 10^{-4}$$

$$\max\,[P(A);\,P(B)] = 0.222 \times 10^{-4}$$

Joint probability: $P(E_2E_3)$

$$P(A) = \Phi(-3.561)\,\Phi\left[-\frac{3.561 - 0.846 \times 3.561}{(1 - 0.846^2)^{1/2}}\right]$$

$$= \Phi(-3.561)\,\Phi(-1.029) = 0.281 \times 10^{-4}$$

$$P(B) = P(A) \qquad \because \quad \beta_3 = \beta_2$$

$$P(A) + P(B) = 0.562 \times 10^{-4}$$

$$\max\,[P(A);\,P(B)] = 0.281 \times 10^{-4}$$

Joint probability: $P(E_2E_4)$

$$P(A) = \Phi(-3.561)\,\Phi\left[-\frac{3.997 - 0.925 \times 3.561}{(1 - 0.925^2)^{1/2}}\right]$$

$$= \Phi(-3.561)\,\Phi(-1.85) = 0.059 \times 10^{-4}$$

$$P(B) = \Phi(-3.997)\,\Phi(0.358) = 0.203 \times 10^{-4}$$

$$P(A) + P(B) = 0.262 \times 10^{-4}$$

$$\max\,[P(A);\,P(B)] = 0.203 \times 10^{-4}$$

Joint probability: $P(E_3\,E_4)$

$$P(A) = \Phi(-3.561)\,\Phi\left[-\frac{3.997 - 0.925 \times 3.561}{(1 - 0.925^2)^{1/2}}\right]$$

$$= 0.059 \times 10^{-4}$$

$$P(B) = 0.203 \times 10^{-4}$$

$$P(A) + P(B) = 0.262 \times 10^{-4}$$

$$\max\,[P(A);\,P(B)] = 0.203 \times 10^{-4}$$

Bounds on the probability of failure of the system are calculated using Eqs. (10.19) and (10.20).

Lower bound:

$$p_{fs} \geqslant p_{f1} + \sum_{i=2}^{n} \max\,[\{p_{fi} - \sum_{j=1}^{i-1} P(E_iE_j)\};\,0]$$

$$\geqslant p_{f1} + \max [\{p_{f2} - P(E_2E_1)\}; 0]$$
$$+ \max [\{p_{f3} - P(E_3E_1) - P(E_3E_2)\}; 0]$$
$$+ \max [\{p_{f4} - P(E_4E_1) - P(E_4E_2) - P(E_4E_3)\}; 0]$$
$$\geqslant \{9.35 + \max [(1.85 - 1.584), 0]$$
$$+ \max [(1.85 - 1.584 - 0.562); 0]$$
$$+ \max [(0.317 - 0.27 - 0.262 - 0.262); 0\} \times 10^{-4}$$
$$\geqslant 9.616 \times 10^{-4}$$

Upper bound:

$$p_{fs} \leqslant \sum_{i=1}^{n} p_{fi} - \sum_{i=2,\, j<i}^{n} \max [P(E_iE_j)]$$
$$\leqslant \sum_{i=1}^{n} p_{fi} - \max[P(E_2E_1)] - \max[P(E_3E_1); P(E_3E_2)]$$
$$- \max[P(E_4E_1); P(E_4E_2); P(E_4E_3)]$$
$$\leqslant [(9.35 + 1.85 + 1.85 + 0.317) - 1.25 - \max (1.25; 0.281)$$
$$- \max (0.22; 0.203; 0.203)] \times 10^{-4}$$
$$\leqslant 10.648 \times 10^{-4}$$

Hence, bounds on p_{fs} of the system are

$$9.616 \times 10^{-4} \leqslant p_{fs} \leqslant 10.648 \times 10^{-4}$$

EXAMPLE 10.8 **An under-reinforced concrete beam of breadth (b) 240 mm and effective depth (d) 480 mm is reinforced with steel bars of area (A_s) 1400 mm². The span of the beam (l) is 6 m. The beam is subjected to a total uniformly distributed load Q over the entire span and a torsional moment T at a distance of 1 m from one end. It is given:**

Variable f_y : (Fe 250)	$\mu = 320$ N/mm²;	$\sigma = 32$ N/mm²
Variable f_{cu} : (Mix M 15)	$\mu = 22.67$ N/mm²;	$\sigma = 5.44$ N/mm²
Variable Q :	$\mu = 16$ N/mm;	$\sigma = 5$ N/mm
Variable T :	$\mu = 5 \times 10^6$ N mm;	$\sigma = 1.5 \times 10^6$ N mm

The beam is reinforced with shear stirrups of area, $A_{sv} = 56.57$ mm². Spacing of stirrups, $s = 300$ mm. Three limit states of collapse (i) in flexure, (ii) in shear and (iii) in combined bending and shear are considered. Determine the probability of failure of the beam considering all the three failure modes. Assume all variables are normally distributed.

Solution *Collapse in flexure* The ultimate resisting moment of the beam is

$$R = f_y A_s d \left[1 - \frac{0.77\, f_y A_s}{bd\, f_{cu}} \right]$$

The failure surface equation is

$$Z = R - Q\frac{l^2}{8} = 0$$

Substituting the given data, and designating

$$X_1 = f_{cu}\,; \quad X_2 = f_y\,; \quad X_3 = Q$$

the failure surface equation becomes

$$Z = 672000\, X_1X_2 - 6288\, X_2^2 - 45 \times 10^5\, X_1X_3 = 0$$

Using Level 2 method (Sec. 8.3.1), the problem is solved and the following results are obtained.

$$\beta = 3.305 \qquad p_f = 47.42 \times 10^{-4}$$

$$\alpha_1^\bullet = -0.9878; \quad \alpha_2^\bullet = 0.0325; \quad \alpha_3^\bullet = 0.152$$

Collapse in shear The shear strength of the beam is given by

$$R = f_y A_{sv}\frac{d}{s} + \frac{bd}{6}\sqrt{0.8\, f_{cu}}\left[\frac{\sqrt{1+5\theta}-1}{\theta}\right]$$

where $$\theta = \frac{0.8\, f_{cu}}{6.89\, p_t} \nless 1$$

$$p_t = \frac{100\, A_s}{bd}$$

θ is an empirical coefficient depending on f_{cu} and p_t. In this problem, θ is assumed deterministic constant. For $f_{cu} = 15$, $A_s = 1400$, $b = 240$, $d = 480$, $\theta = 1.439$. The failure surface is given by

$$Z = R - Q\frac{l}{2} = 0$$

Using the given data, the above equation becomes,

$$Z = 22222\, X_1^{0.5} + 90.5\, X_2 - 3000\, Q = 0$$

Using Level 2 method explained in Ch. 8, following results are obtained.

$$\beta = 3.814 \qquad p_f = 6.847 \times 10^{-4}$$

$$\alpha_1^\bullet = -0.8753\,; \quad \alpha_2^\bullet = -0.0917\,; \quad \alpha_3^\bullet = 0.4749$$

Collapse in combined shear and torsion For checking under combined shear and torsion, IS: 456–1978 gives the following equation to calculate the equivalent shear (V_e).

$$V_e = V + 1.6\,\frac{T}{b}$$

where V is the shear at the section due to load Q. At the section (1 m from end) where torsional moment is acting,

$$V_e = Q\left(\frac{l}{2} - 1000\right) + 1.6\,\frac{T}{240}$$

$$= 2000\, Q + 0.00667\, T$$

Using the resistance part in the failure surface equation derived in the previous failure case, the failure surface equation under combined shear and torsion becomes

$$Z = 22222\, X_1^{0.5} + 90.5\, X_2 - 2000\, X_3 - 0.00667\, X_4$$

where $X_4 = T$. Using Level 2 method, following results are obtained for this failure case.

$$\beta = 3.262; \qquad p_f = 55.3\times 10^{-4}$$

$$\alpha_1^* = -0.8223; \qquad \alpha_2^* = -0.1142$$

$$\alpha_3^* = 0.3942; \qquad \alpha_4^* = 0.3942$$

Considering the beam as a system under the three failure modes, simple bounds on p_{fs} of the beam [Eq. (10.14)], are

$$55.3\times 10^{-4} \leqslant p_{fs} \leqslant (47.42 + 6.847 + 55.3)\times 10^{-4}$$

$$5.53\times 10^{-3} \leqslant p_{fs} \leqslant 10.96\times 10^{-3}$$

Narrow bounds The failure modes are numbered in the descending order of their p_f values.

Mode 1 : $Z_1 = 22222\, X_1^{0.5} + 90.5\, X_2 - 2000\, X_3 - 0.00667\, X_4$

Mode 2 : $Z_2 = 672000\, X_1X_2 - 6288\, X_1^2 - 45\times 10^5\, X_1X_3$

Mode 3 : $Z_3 = 22222\, X_1^{0.5} + 90.5\, X_2 - 3000\, X_3$

Correlation between mode 1 and mode 2: Using Eq. (10.11b),

$$\begin{aligned}\rho_{Z_1, Z_2} &= \sum_{i=1}^{4} \alpha_{1i}^*\alpha_{2i}^* \\ &= (-0.8223)(-0.9878) + (-0.1142)(0.0325) \\ &\quad + (0.3942)(0.152) \\ &= 0.8685\end{aligned}$$

Joint probability : $P(E_1\, E_2)$

Using Eqs. (10.23) and (10.24),

$$P(A) = \Phi(-3.262)\, \Phi\left[-\frac{3.305 - 0.8685\times 3.262}{\sqrt{1 - 0.8685^2}}\right]$$

$$= 9.346\times 10^{-4}$$

$$P(B) = \Phi(-3.305)\, \Phi\left[-\frac{3.262 - 0.8685\times 3.305}{\sqrt{1 - 0.8685^2}}\right]$$

$$= 10.184\times 10^{-4}$$

$$P(A) + P(B) = 19.53\times 10^{-4}$$

$$\max[P(A); P(B)] = 10.184\times 10^{-4}$$

Correlation between mode 2 and mode 3: Using Eq. (10.11b),

$$\rho_{Z_2, Z_3} = (-0.9878)(-0.8753) + (0.0325)(-0.09168) + (0.152)(0.4749) = 0.9338$$

Joint probability: $P(E_2 E_3)$
Using Eqs. (10.23) and (10.24)

$$P(A) = \Phi(-3.305)\,\Phi\left[-\frac{3.814 - 0.9338\times 3.305}{\sqrt{1-0.9338^2}}\right] = 1.004\times 10^{-4}$$

$$P(B) = \Phi(-3.814)\,\Phi\left[-\frac{3.305 - 0.9338\times 3.814}{\sqrt{1-0.9338^2}}\right] = 5.219\times 10^{-4}$$

$$P(A) + P(B) = 6.223\times 10^{-4}$$

$$\max\,[P(A); P(B)] = 5.219\times 10^{-4}$$

Correlation between modes 3 and 1:

$$\rho_{Z_3, Z_1} = (-0.8753)(-0.8223) + (-0.09168)(-0.1142) + (0.4749)(0.3942) + (0)(0.3942) = 0.9174$$

For failure modes 3 and 1, we have

$$P(A) = \Phi(-3.814)\,\Phi\left[-\frac{3.262 - 0.9174\times 3.814}{\sqrt{1-0.9174^2}}\right] = 4.958\times 10^{-4}$$

$$P(B) = \Phi(-3.262)\,\Phi\left[-\frac{3.814 - 0.9174\times 3.262}{\sqrt{1-0.9174^2}}\right] = 1.073\times 10^{-4}$$

$$P(A) + P(B) = 6.031\times 10^{-4}$$

$$\max[P(A); P(B)] = 4.958\times 10^{-4}$$

Lower bound on p_{fs} is (Eq. 10.19)

$$\begin{aligned} p_f &\geq 55.3\times 10^{-4} + \max\,[\{p_{f2} - P(E_1E_2)\}; 0] \\ &\quad + \max[\{p_{f3} - P(E_3E_1) - P(E_3E_2)\}; 0] \\ &\geq 55.3\times 10^{-4} + \max\,[(47.42 - 19.53)\times 10^{-4}; 0] \\ &\quad + \max[(6.847 - 6.031 - 6.223)\times 10^{-4}; 0] \\ &\geq (55.3 + 27.89 + 0)\times 10^{-4} = 83.19\times 10^{-4} \end{aligned}$$

Upper bound on p_{fs} is [Eq. (10.20)]

$$
\begin{aligned}
p_f &\leqslant \sum_{i=1}^{3} p_{fi} - \max\,[P(E_2E_1)] \\
&\quad - \max\,[P(E_3E_1);\ P(E_3E_2)] \\
&\leqslant [(55.3 + 47.42 + 6.847) \\
&\quad -10.184 - \max\,(4.958;\ 5.219)] \times 10^{-4} \\
&\leqslant 94.164 \times 10^{-4}
\end{aligned}
$$

Hence bounds on p_{fs} of the beam are

$$8.289 \times 10^{-3} \leqslant p_{fs} \leqslant 9.416 \times 10^{-3}$$

10.5 AUTOMATIC GENERATION OF A MECHANISM

The problem of the reliability analysis of a frame structure becomes formidable if one uses sophisticated probability models for the basic random variables and safety margins, as well as nonlinear analysis of structures. In order to obtain tractable analytical models, the methodology of the reliability analysis of plane frame structures is developed using the stiffness matrix method, the linear elastic and piecewise linear elastic-plastic (PWLEP) structural analysis, and the first-order second-moment method of reliability. Along with the usual assumptions in the conventional plastic analysis of structures, it is also assumed that (i) applied loads are concentrated forces, (ii) the PWLEP analysis is based on the mean values of basic random variables, (iii) plastic moment capacities of sections, M_i, and applied loads, Q_i, are the only random variables, and (iv) plastic moment capacities of sections are statistically independent of applied loads.

10.5.1 Failure Models

At any stage of the structural analysis, the failure of a section (member end) is assumed to take place when the plastic moment capacity of the section is reached. This failure is called the formation of the plastic hinge at the section.

In a redundant structure, a collapse mode forms only when a sufficient number of hinges have developed. The failure mode of a structure is defined as the formation of a collapse mechanism. When the PWLEP analysis is carried out by moving from one hinge to another, the criterion of determining the formation of a mechanism is given by the singularity of the stiffness matrix, $[K]$, i.e. $|[K]| = 0$. $|[K]|$ is read as the determinant of matrix $[K]$. The finally formed hinge converts the structure into a collapse mechanism and the failure model of the finally hinged section corresponds to the collapse mechanism.

10.5.2 Safety Margin Equation

The results of PWLEP analysis enable one to write the safety margin equations of a potential hinge section (the section selected for forming a hinge) at any stage of a failure path in terms of M_i and Q_j. The safety margin is the difference between the plastic moment capacity of the section and the bending moment at the section, just before forming the hinge due to applied loads and plastic moment capacity of earlier sections. For example, if the frame shown in Fig. 10.14 is considered and, if the sequence of hinges formed in a failure path are at member ends 2, and 4,. and at 6 (the potential hinge), the hinge is going to be formed, the safety margin, Z_6, of the section 6 at this stage can be written as

$$Z_6 = a_{62}M_2 + a_{64}M_4 + a_{66}M_6 - \sum_{j=1}^{n} b_{6j}Q_j \tag{10.25}$$

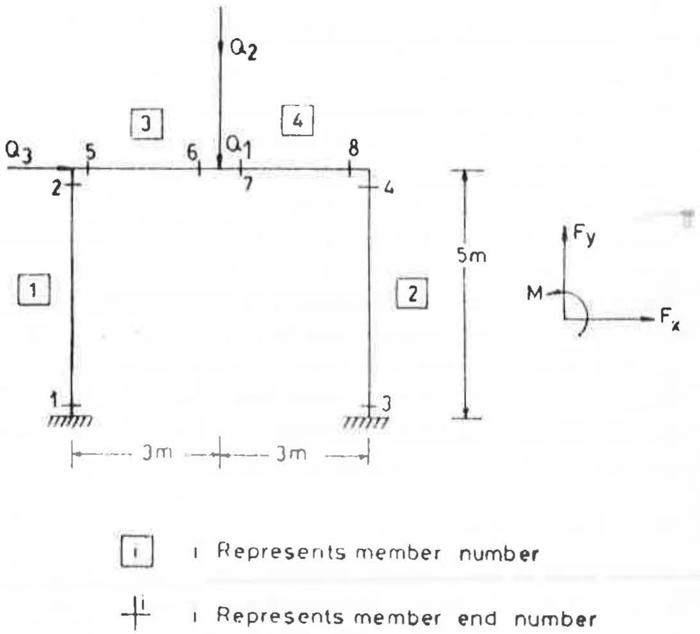


FIG. 10.14 One-bay one storey frame—Example 10.9

where M_i is the plastic moment capacity of section i, a_{6i} is the moment at section 6 due to unit M_i, b_{6j} is the bending moment at section 6 due to unit load Q_j, and n is the number of loads. It is to be noted that a_{66} is unity. $P(Z_6 < 0)$ gives the probability of failure of the section 6, given that sections 2 and 4 have already failed. As the analysis progresses, at every stage of the progressive failure tree the safety margin equation for the hinge to be formed can be written. As PWLEP analysis is carried out by moving from one hinge to another, when the stiffness matrix of the structure becomes singular, the finally formed hinge converts the structure into a mechanism and the failure model of the finally hinged section corresponds to the collapse mechanism. The safety margin of the finally formed hinge becomes the safety margin equation of the mechanism. This safety margin equation coincides with the safety margin equation obtained from the conventional mechanism method of plastic analysis. When a mechanism is formed, $P(Z_i < 0)$ gives the probability of occurrence of the failure mode i.

The safety margin, in general, for a potential hinge section i or, the safety margin of a mechanism having the last hinge at section i, is expressed as

$$Z_i = \sum_{j=1}^{m} a_{ij}M_j - \sum_{k=1}^{n} b_{ik}Q_k \tag{10.26}$$

where m is the number of critical sections (member ends) in the given frame. In a particular failure path, if there is no hinge at the member end j, the corresponding coefficient $a_{ij} = 0$. For $i = j$, $a_{ij} = 1$.

If random variables M and Q are grouped in X, Eq. (10.26) can be written in the generalized matrix form as

$$Z_i = [A]\{X\} \tag{10.27}$$

The mean value and standard deviation of Z_i are

$$\mu_{Zi} = [A]\{\mu_X\} \tag{10.28}$$

$$\sigma_{Zi}^2 = [A][C_X][A]' \tag{10.29}$$

where $[A]$ is a row matrix of coefficients a_{ij} and b_{ik} for all variables X_j, $[A]'$ is the transpose of matrix $[A]$, $\{\mu_X\}$ is a column matrix of the means of all random variable X_j, and $[C_X]$ is a covariance matrix of all random variables X_j. The reliability index β_i for the safety margin Z_i is given by

$$\beta_i = \frac{\mu_{Zi}}{\sigma_{Zi}}$$

As Z_i is a linear function of the number of variables X_j, the distribution of Z_i tends to normal, [based on the central limit theorem (10.5)] irrespective of the individual distributions of the variables. Hence, assuming normal distribution for Z_i, the probability of a structure under a collapse mechanism i can be computed.

The methodology of the reliability analysis of ductile structural systems involves the following steps:

(i) Data

(a) structural data

(b) probability description of the random variables

X_j in terms of μ_{Xj}, σ_{Xj} and $\rho_{Xj,Xk}$

(ii) Linear elastic analysis

(a) determination of coefficients a_{ij}, b_{ik}

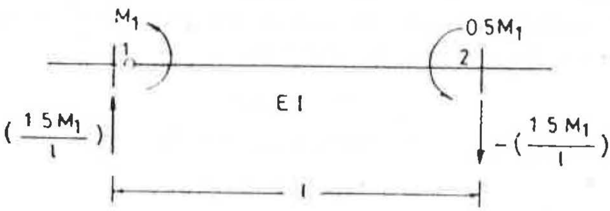
(iii) For any potential hinge location

(a) writing the safety margin Z_i from Eq. (10.27)

(b) computation of μ_{Zi} and σ_{Zi} using Eqs. (10.28) and (10.29)
(c) computation of β_i

(iv) Selection of the next hinge location
(v) Formation of the plastic hinge at the selected member end
(vi) Modification of the member stiffness matrix having plastic hinges at the ends, as shown in Figs. 10.15, 10.16 and 10.17
(vii) Application of a plastic moment at the hinge in the form of equivalent forces, as shown in Figs. 10.15, 10.16 and 10.17.
(viii) Determination of the structure stiffness matrix $[K]$
(ix) Linear elastic-plastic analysis and determination of coefficients a_{ij}.
(x) Repetition of steps (iii) to (ix) until the formation of a mechanism.

The above procedure is illustrated with an example.

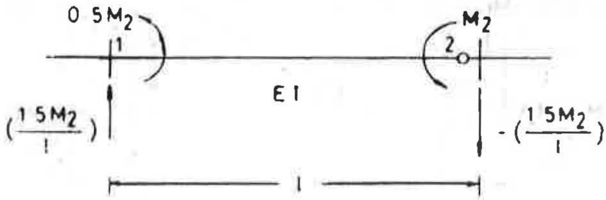


(a) Member Loading Induced by M_1

$$
\begin{bmatrix}
\frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\
 & \frac{3EI}{l^3} & 0 & 0 & -\frac{3EI}{l^3} & \frac{3EI}{l^2} \\
 & & 0 & 0 & 0 & 0 \\
 & \text{SYM} & & \frac{EA}{l} & 0 & 0 \\
 & & & & \frac{3EI}{l^3} & -\frac{3EI}{l^2} \\
 & & & & & \frac{3EI}{l}
\end{bmatrix}
$$

(b) Member Stiffness Matrix

FIG. 10.15 Effect of hinge: left end of member hinged



(a) Member Loading Induced by M_2

$$\begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ & \frac{3EI}{l^3} & \frac{3EI}{l^2} & 0 & -\frac{3EI}{l^3} & 0 \\ & & \frac{3EI}{l} & 0 & -\frac{3EI}{l^2} & 0 \\ \text{SYM} & & & \frac{EA}{l} & 0 & 0 \\ & & & & \frac{3EI}{l^3} & 0 \\ & & & & & 0 \end{bmatrix}$$

(b) Member Stiffness Matrix

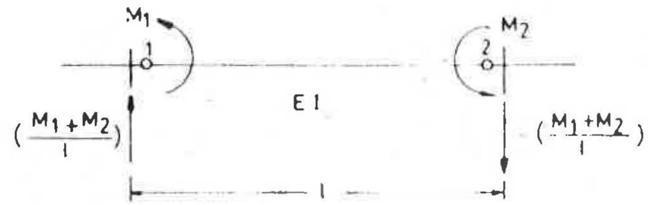
FIG. 10.16 Effect of hinge : right end of member hinged

EXAMPLE 10.9 A simple one-storey, one-bay portal frame is subjected to vertical and horizontal loads, is shown in Fig. 10.14. The data for the problem is given in Table 10.2.

For this frame the degree of redundancy is three, and the maximum number of hinges required for a mechanism is four. The stepwise procedure of generating a mechanism is illustrated below.

(i) The linear elastic analysis of the structure is performed to compute the bending moments at the member ends, expressed in terms of the coefficients a_{ij} and b_{ik}. The structure at this stage is considered intact and this stage is called the first stage. The number of critical sections (potential hinge sections), m, is equal to eight. They are marked in Fig. 10.14. The number of loads, n, is equal to three.

(ii) At this stage, for all critical sections the safety margin equations in terms of a_{ij}, M_j, b_{ik} and Q_{ik} are generated and reliability indices, β_i, are computed.



(a) Member Loading Induced by M_1 and M_2

$$\begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ \text{SYM} & & & \frac{EA}{l} & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

(b) Member Stiffness Matrix

FIG. 10.17 Effect of hinge : both ends of member hinged

TABLE 10.2 *Data for frame in Fig. 10.14—Example 10.9*

Section or variable	EA (kN)	EI (kN m²)	μ	σ
Section				
1, 2, 3, 4	0.367×10^7	0.92×10^4		
5, 6, 7, 8	0.965×10^7	0.406×10^4		
Variable				
M_1, M_2, M_3, M_4			121.57 kN m	9.969 kN m
M_5, M_8			135.04	12.424
M_6, M_7			344.65	32.74
Q_1			105.0 kN	10.5 kN
Q_2			36.0	14.4
Q_3			2.4	1.032

Note: All variables are statistically independent.

When the external acting moment, S, at the section due to loads is positive, the safety margin is written as

$$\text{Safety margin} = \text{resistance} - \text{action}$$
$$= R - S = R - (+S)$$

If S is negative,

$$\text{Safety margin} = R + S = R + (-S)$$

For example, if the potential hinge section 7 is considered, the safety margin for section 7 at this stage 1, using Eq. (10.26), is

$$Z_7 = \sum_{j=1}^{8} a_{7j}M_j - \sum_{k=1}^{3} b_{7k}Q_k$$

From the elastic analysis of the frame,

$$b_{71} = -1.24 \qquad b_{72} = -1.24 \qquad b_{73} \simeq 0$$

The negative sign shows that the direction of the bending moment is clockwise. The sign conventions for forces are shown in Fig. 10.14. Since this is stage I, there is no hinge at any member end j. All a_{ij} coefficients are zero except $a_{77} = 1$. The capacity of the section is M_7. The action is $-(1.24\,Q_1 + 1.24\,Q_2)$. Hence, the safety margin equation for section 7 is

$$Z_7 = 1.0\,M_7 - 1.24\,Q_1 - 1.24\,Q_2$$

The mean value and standard deviation of Z_7 are

$$\mu_{Z7} = 344.65 - 1.24\times 105 - 1.24\times 36$$
$$= 169.81 \text{ kN m}$$
$$\sigma_{Z7} = [(32.74)^2 + (1.24\times 10.5)^2 + (1.24\times 14.4)^2]^{1/2}$$
$$= 39.4 \text{ kN m}$$
$$\beta_7 = \frac{169.81}{39.4} = 4.31$$

Similarly, for all other potential hinge sections, safety margin equations can be written and β values found out at this stage.

(iii) Let us now assume that the first hinge is formed at section 7. Now the stiffness matrix of the member 4 having plastic hinge at the left end is modified as given in Fig. 10.15.

(iv) A clockwise moment of 344.65 kN m, (i.e. negative moment), equal to the plastic moment capacity of section 7 is applied at the left end. Corresponding to this moment, a self-equilibriating force system, shown in Fig. 10.15, is considered as an additional load case for further analysis.

(v) The structure stiffness matrix is assembled and the determinant $|[K]|$ is computed and found to be greater than zero.

(vi) The linear elastic-plastic analysis at this stage 2 is carried out to determine a_{ij} and b_{ik} for all the potential hinge sections. Knowing a_{ij} and

b_{ik}, safety margin equations for all potential hinge sections can be written. For example, if section 4 is considered,

$$a_{47} = -1.0 \qquad b_{41} = 1.5 \qquad b_{42} = 1.5$$

$$b_{43} = 1.2 \qquad a_{44} = 1.0$$

The capacity of the section is M_4. $a_{44} = 1.0$. Hence, the safety margin equation for section 4, given the hinge at section 7 has formed, is

$$\begin{aligned} Z_4 &= 1.0\, M_4 - (-1.0\, M_7 + 1.5\, Q_1 + 1.5\, Q_2 + 1.2\, Q_3) \\ &= M_4 + M_7 - 1.5\, Q_1 - 1.5\, Q_2 - 1.2\, Q_3 \end{aligned}$$

Using the given data in Table 10.2,

$$\begin{aligned} \mu_{Z4} &= 121.57 + 344.65 - 1.5\times 105 - 1.5\times 36 - 1.2\times 2.4 \\ &= 251.84 \end{aligned}$$

$$\begin{aligned} \sigma_{Z4} &= [(9.969)^2 + (32.74)^2 + (1.5\times 10.5)^2 + (1.5\times 14.4)^2 \\ &\quad + (1.2\times 1.032)^2]^{1/2} \\ &= 43.44 \end{aligned}$$

$$\beta_4 = \frac{251.84}{43.44} = 5.79$$

Similarly, for all potential hinge sections (excluding section 7 where the hinge is already formed) at this stage, the safety margin equations can be written and β found out given that the hinge at section 7 is already formed.

(vii) Now another hinge (second hinge) location, say section 4, is selected and the hinge is formed at that section, and the whole process is repeated from steps (iii) to (v). Now the stiffness matrix of the member 2 having plastic hinge at the right hand side member end 4 is modified as given in Fig. 10.16. At section 4 an anti-clockwise moment of 121.57 kN m, equal to the plastic moment capacity of section 4, is applied. Corresponding to this moment, a self-equilibriating force system, shown in Fig. 10.16, is considered as an additional load case for further analysis.

The stiffness matrix of the structure is assembled and $|[K]|$ is found to be greater than zero. The linear elastic–plastic analysis is carried out at this stage 3 to determine the coefficients a_{ij}. If section 2 is considered, from analysis

$$a_{27} = 2.0 \qquad a_{24} = 1.0$$

$$b_{21} = -3 \qquad b_{22} = -3.0 \qquad b_{23} \simeq 0$$

$$a_{22} = 1$$

Hence, the safety margin equation for section 2, given hinges at sections 7 and 4 have formed, is

$$Z_2 = 1.0\, M_2 + 2\, M_7 + M_4 - 3Q_1 - 3Q_2 \tag{10.30}$$

Using mean values and standard deviations of M_2, M_7, M_4, Q_1, and Q_2,

$$\mu_{Z2} = 509.4 \qquad \sigma_{Z2} = 85.7$$

$$\beta_2 = 5.94$$

If section 2 is selected for the next hinge location, a hinge is formed at the section. A clockwise moment of 121.57 kN m, equal to the moment capacity of section 2 is applied at the member end 2. The stiffness matrix of member 1 with a hinge at the right member end 2 is modified as shown in Fig. 10.15. The stiffness matrix of the structure is now assembled and $|[K]|$ is found to be $\leqslant$ zero. This shows that when hinges are formed at sections 7, 4 and 2, a mechanism is formed. This is a beam mechanism. Using the mechanism method of plastic analysis (10.2), one can directly write the safety margin equation for this failure mode:

$$Z = M_2 + 2M_7 + M_4 - Q_1 \times 3 - Q_2 \times 3$$

It can be observed that this equation, i.e. the safety margin equation for the mechanism, coincides with the safety margin equation (Eq. 10.30) for the potential hinge section 2, written just before the hinge is formed there.

It has been shown in the example how a mechanism can be generated, and how the safety margin equations are written at every stage of analysis and β_i values for potential hinge sections computed, and how the failure surface equation or the failure model of the finally hinged section corresponds to the collapse mechanism. β_i of the last hinged section becomes β of the mechanism.

10.6 GENERATION OF DOMINANT MECHANISMS

In the last example, only one mechanism was generated out of 15 possible failure mechanisms. A plane frame structure may fail in different collapse mechanisms, called failure modes. The reliability analysis of frames mainly involves identification, modelling and synthesis of all possible failure modes to estimate the system reliability. In the case of a frame structure of a high degree of indeterminancy, the number of possible collapse mechanisms is quite large. To illustrate, for a one-bay two-storey rectangular frame with fixed bases, the number of elementary mechanisms, N_e, is equal to 8. The number of possible collapse mechanisms is given by $2^{N_e} - 1 = 255$. Due to uncertainties of load and resistance variables, it is likely that the structure may fail under any of the possible collapse mechanisms. Hence, the reliability of frames of multiple components and with multiple failure modes is considered from the system point of view. Out of the innumerable possible collapse mechanisms, generally only a few mechanisms, having comparatively large failure probabilities, contribute significantly to the system failure probability, p_{fs}. These collapse mechanisms are called stochastically dominant failure modes. The identification and combination of these dominant collapse mechanisms are necessary in the reliability analysis of a frame structure to estimate its system reliability. It is practically difficult and rather impossible to identify these dominant failure modes. There is no method which assures, and mathematically proves, that all stochastically dominant modes are generated. However, the methods, namely (i) exhaustive enumeration, (ii) simulation, and (iii) heuristic search, are generally used for this purpose. The earlier studies (10.6, 10.7) concentrated on the reliability analysis of known failure modes. The foremost essential step of the identification of dominant failure modes in a frame structure has been the subject of research during the past eight years. Ma and Ang (10.8) have suggested a method of determining the most probable modes by using a mathematical programming technique, based on independent failure modes, obtained deterministically by Watwood's (10.9) method. Murotsu (10.10, 10.11) has proposed a complex method, based on the joint probabilities of hinged sections, for the automatic generation of stochastically dominant failure modes. Moses (10.12) has proposed a strategy, using the incremental load approach, to identify and enumerate the significant failure modes of trusses. Tang and Melchers (10.13) have proposed a truncated enumeration method to search for stochastically dominant failure modes. Ranganathan and Deshpande (10.14) have proposed a heuristic search technique to generate dominant modes in frames. This is explained below.

10.6.1 Heuristic Technique (10.14)

The strategy developed for a sequential search of plastic hinge locations leading to dominant mechanisms is explained below.

Search for Plastic Hinge Locations

It is quite logical to select the potential hinge section with the lowest reliability index, for the plastic hinge at any stage of the analysis, to get stochastically dominant mechanisms. However, it has been observed that this logic fails in certain situations. Dominant failure paths of equal likelihood of occurrence may intermix, resulting in a nondominant mechanism. Therefore, the following strategy for the selection of plastic hinge locations is suggested.

Selection of First Hinge and First Dominant Mechanism After performing the intact analysis and computing β for all potential hinge sections, the potential hinge section having the lowest value of β is selected as the location for the first plastic hinge. This β is called the first damage reliability index and denoted as β_0. The arithmetic mean of the reliability indices of all potential hinge sections at the initial stage is termed as the average reliability index β_{av}. The reliability of a mechanism is always higher than the reliability of its hinges at each stage, and the reliability index of a mechanism corresponds to β for the member end hinged at the stage of mechanism. Hence, after selecting the first hinge, it is logical to select the subsequent plastic hinge locations such that the reliability index of the section is the lowest at that stage and is also greater than β_0. Following the above strategy, hinges are selected and the first probabilistically dominant mechanism is generated.

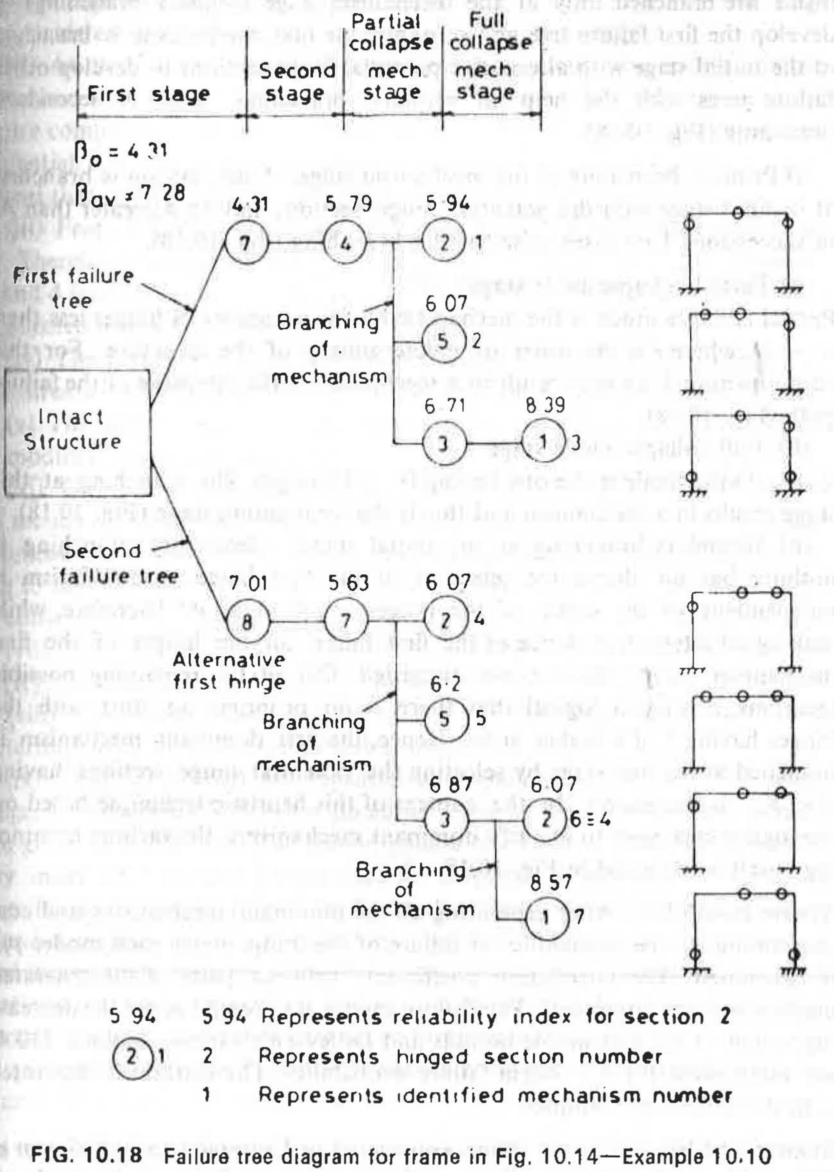


FIG. 10.18 Failure tree diagram for frame in Fig. 10.14—Example 10.10

Branching Strategy After generating the first dominant mechanism, it is obvious that if this mechanism is branched at all stages with alternative potential hinge sections in succession, it may be possible to identify all the possible mechanisms. This procedure is computationally prohibitive as there can be a very large number of reanalyses to perform. Moreover, during this process the same mechanisms may be repeatedly generated and insignificant mechanisms identified, which are of no interest from the viewpoint of the system failure probability. In the light of this, primarily, dominant mechanisms are branched only at the mechanism stage (primary branching) to develop the first failure tree and secondly, the first mechanism is branched at the initial stage with alternative potential hinge sections to develop other failure trees with the help of primary branching. This is secondary branching (Fig. 10.18).

(i) Primary branching at the mechanism stage: A mechanism is branched at its final stage with the potential hinge sections having β greater than β_0 in succession. Two cases arise for this branching (Fig. 10.18).

(a) Partial collapse mode stage
Partial collapse mode is the mechanism having a number of hinges less than $(r + 1)$, where r is the order of indeterminacy of the structure. For this case, the branching may result in a mechanism or the extension of the failure path (Fig. 10.18).

(b) Full collapse mode stage
Full collapse mode is the one having $(r + 1)$ hinges. The branching at this stage results in a mechanism and this is the terminating stage (Fig. 10.18).

(ii) Secondary branching at the initial stage: Secondary branching is nothing but an alternative selection of the first hinge. A mechanism is independent of the order of the hinges involved in it. Therefore, while making an alternative choice of the first hinge, all the hinges of the first mechanism except the last are discarded. Out of the remaining possible locations, it is again logical that there is no propriety to start with the hinges having β of a higher order. Hence, the first dominant mechanism is branched at the first stage by selecting the potential hinge sections having $\beta < \beta_{av}$ in succession. In the context of this heuristic technique based on the logical strategies to identify dominant mechanisms, the various terminology used is indicated in Fig. 10.18.

System Reliability After generating all the dominant mechanisms and corresponding Z_i, the probability of failure of the frame under each mode, p_{fi}, is calculated. The correlation coefficients between pairs of the generated mechanisms are computed. The failure modes are ordered as per the decreasing values of p_{fi}, and simple bounds and Ditlevsen's narrow bounds (10.4) are established for the system failure probability. The method is illustrated with the following examples:

EXAMPLE 10.10 The same frame, considered in Example 10.9 and shown in Fig. 10.14, is taken here to illustrate the generation of dominant mechanisms. The data required for the reliability analysis of the frame is given in Table 10.2. For this frame, the degree of redundancy is 3, the maximum number of hinges required for a mechanism is 4, and the number of elementary mechanisms is 4, whereas the number of possible mechanisms is 15. The stepwise procedure of generating dominant mechanisms, and reliability analysis, is explained below:

(i) The linear elastic analysis of the structure is performed to compute the bending moments at the member ends, expressed in terms of the coefficients

(i) The signs of the bending moments are noted. The structure at this stage is considered as intact and this stage is called the first stage.

(ii) At this stage, for all potential hinge sections, the safety margin equations in terms of a_{ij}, M_j, b_{ik}, and Q_k are generated and reliability indices β_i are computed (explained in Example 10.9) as given in Table 10.3. The potential hinge sections are ordered with increasing reliability index. It is found that $\beta_0 = 4.31$ and $\beta_{av} = 7.28$.

(iii) From Table 10.3, it is noted that the sections 6 and 7 have the same β_0. Therefore, comparing the reliability indices at both ends of the members 3 and 4 (sections 5 and 8), section 7 with the lowest reliability index of 4.31 is selected as the first hinge.

(iv) The first plastic hinge is formed at section 7. The safety margin of the structure at this stage, with the hinge at section 7, is given in Table 10.3.

(v) The stiffness matrix of member 4 having a plastic hinge at the left end is modified as given in Fig. 10.15.

(vi) The moment of 344.65 kN m, equal to the plastic moment capacity of section 7 is applied at the member end, in the direction of the bending moment developed at the member end in the elastic analysis. Corresponding to this moment, a self-equilibriated force system, as shown in Fig. 10.15, is considered as an additional load case for further analysis.

(vii) The structure stiffness matrix $[K]$ is assembled and the determinant $|[K]|$ is computed and found to be greater than zero.

(viii) The linear elastic-plastic analysis at this stage is carried out to determine a_{ij} and b_{ik} for the potential hinge sections.

(ix) Steps (ii) to (viii) are repeated as explained below. At the second stage, the reliability indices for potential hinge sections are computed, which are given in Table 10.3. The second plastic hinge, having the lowest reliability index of 5.79 and greater than β_0, is formed at section 4, as shown in Fig. 10.18. Corresponding to the first and second plastic hinges at sections 7 and 4 respectively, the modified member stiffness matrices for members 4 and 2, and additional load cases equivalent to plastic moment capacities of sections 7 and 4, as shown in Figs. 10.15 and 10.16, are considered for further analysis. The safety margin of the structure having the second hinge at section 4 is given in Table 10.3. The determinant of the structure stiffness matrix is found to be greater than zero.

At the third stage, according to the selection strategy explained earlier, the plastic hinge is formed at section 2, as shown in Fig. 10.18. As $|[K]| \leqslant 0$, the first dominant mechanism is generated.

(x) This mechanism consists of three hinges at 7, 4, and 2. It is therefore a partial collapse mechanism. The safety margin equation of this mechanism is same as the safety margin equation of the hinge at section 2 (Table 10.3). This mechanism is the most dominant mechanism, having a reliability index 5.94 and a probability of failure 0.145×10^{-8}.

(xi) As per the branching strategy, this mechanism is branched as shown in Fig. 10.18. To do this, the last hinge of this mechanism at section 2 is

TABLE 10.3 *Details of development of first failure tree in Fig 10.14—Example 10.10*

Stages of analysis	Reliability index of sections								Safety margin equation	Selected	
	1	2	3	4	5	6	7	8		Hinge section	β
First	10.3	7.87	9.67	7.35	7.43	4.31	4.31	7.01	$Z_7 = 1.0M_7 - 1.24Q_1 - 1.24Q_3$	7	4.31
Second	8.17	5.92	7.90	5.79	6.13	0.0	0.0	6.01	$Z_4 = 1.0M_4 + 1.0M_7 - 1.5Q_1 - 1.5Q_2 - 1.20Q_3$	4	5.79
Third (Mechanism 1)	11.9	5.94	6.71	0.0	6.07	0.0	0.0	0.0	$Z_2 = 1.0M_2 + 1.0M_4 + 2.0M_7 - 3.0Q_1 - 3.0Q_2$	2	5.94
Third (Mechanism 2)	11.9	5.94	6.71	0.0	6.07	0.0	0.0	0.0	$Z_5 = 1.0M_4 + 1.0M_5 + 2.0M_7 - 3.0Q_1 - 3.0Q_2$	5	6.07
Third	11.9	5.94	6.71	0.0	6.07	0.0	0.0	0.0	$Z_3 = 1.0M_3 + 1.0M_4 + 1.5M_7 - 2.25Q_1 - 2.25Q_2 - 2.50Q_3$	3	6.71
Fourth (Mechanism 3)	8.39	5.94	0.0	0.0	6.07	0.0	0.0	0.0	$Z_1 = 1.0M_1 + 1.0M_3 + 2.0M_4 + 2.0M_7 - 3.0Q_1 - 3.0Q_2 - 5.0Q_3$	1	8.39

suppressed and replaced by sections, in succession, except 7, 4 and 2 and those having a reliability index at this stage greater than β_0.

(xii) As shown in Fig. 10.18, the selection of section 5 with β as 6.07 results in a dominant mechanism. The failure path continues with the selection of the section 3 with β as 6.71, as shown in Fig. 10.18. Again, according to the selection strategy, the next hinge, i.e. the fourth plastic hinge is formed at section 1 having β as 8.39. Since $|\,[K]\,| \leqslant 0$, the full collapse mechanism is formed at this stage. The branching of this mechanism does not result in any new mechanism. As this is the terminating stage, the development of the first failure tree is completed.

(xiii) To initiate other failure trees, according to the strategy of the selection of alternative first hinges, only one alternative is possible in this case, as shown in Table 10.4.

TABLE 10.4 *Selection of first hinge for failure trees other than the first in Fig. 10.14—Example 10.10*

Section	β at first stage	Possibility of selection of first hinge	Remarks
1	10.3	Not possible	$\beta > \beta_{av}$
2	7.87	Not possible	$\beta > \beta_{av}$
3	9.67	Not possible	$\beta > \beta_{av}$
4	7.35	Not possible	Involved in the first mechanism
5	7.43	Not possible	$\beta > \beta_{av}$
6	4.31	Not possible	Equinodal to 7
7	4.31	Not possible	Involved in the first mechanism
8	7.01	Possible	$\beta < \beta_{av}$

(xiv) The first hinge of the second failure tree is selected at section 8, and this failure tree is developed using the procedure similar to the first tree. The failure tree diagram for this example, including the dominant mechanisms generated, is shown in Fig. 10.18.

(xv) The identified mechanisms are arranged in the increasing order of the reliability index, as given in Table 10.5.

(xvi) The correlation coefficients of mechanisms, calculated using Eqs. (10.10) and (10.11a), are presented in Table 10.6.

(xvii) Simple bounds are computed using Eq. (10.14), and narrow bounds using Eqs. (10.19) and (10.20). They are also shown in Table 10.5.

From the rusults it can be seen that for this example, the first dominant mechanism, having a probability of failure of 0.145×10^{-8}, is the most significant mechanism. Therefore, the lower bound on the system collapse probability of 0.159×10^{-8}, is close to the failure probability of the first dominant mechanism.

TABLE 10.5 *Details of generated dominant mechanisms and results of reliability analysis of frame in Fig. 10.14—Example 10.10*

Sl. No.	Hinged sections	Z	β	p_f	Failure tree
1.	2, 4, 7	$1.0M_2 + 1.0M_4 + 2.0M_7 - 3.0Q_1 - 3.0Q_2$	5.94	0.145×10^{-8}	1
2.	4, 5, 7	$1.0M_4 + 1.0M_5 + 2.0M_7 - 3.0Q_1 - 3.0Q_2$	6.07	0.642×10^{-9}	1
3.	2, 7, 8	$1.0M_2 + 2.0M_7 + 1.0M_8 - 3.0Q_1 - 3.0Q_2$	6.07	0.642×10^{-9}	2
4.	5, 7, 8	$1.0M_5 + 2.0M_7 + 1.0M_8 - 3.0Q_1 - 3.0Q_2$	6.20	0.282×10^{-9}	2
5.	1, 3, 4, 7	$1.0M_1 + 1.0M_3 + 2.0M_4 + 2.0M_7 - 3.0Q_1 - 3.0Q_2 - 5.0Q_3$	8.39	0.239×10^{-16}	1
6.	1, 3, 7, 8	$1.0M_1 + 1.0M_3 + 2.0M_7 + 2.0M_8 - 3.0Q_1 - 3.0Q_2 - 5.0Q_3$	8.57	0.542×10^{-17}	2

System failure probability

Simple bounds $0.145\times10^{-8} \leqslant p_{fs} \leqslant 0.302\times10^{-8}$

Narrow bounds $0.159\times10^{-8} \leqslant p_{fs} \leqslant 0.214\times10^{-8}$

TABLE 10.6 *Correlations between generated mechanisms—Example 10.10*

Mechanism No.	Correlations ρ_{ij}					
	1	2	3	4	5	6
1	1.0	0.982	0.982	0.965	0.972	0.932
2		1.0	0.965	0.963	0.968	0.928
3			1.0	0.983	0.942	0.969
4				1.0	0.938	0.965
5					1.0	0.935
6	Symmetrical					1.0

EXAMPLE 10.11 An unsymmetrical two-storey two-bay frame, carrying vertical and horizontal loads, is shown in Fig. 10.19. The data for the

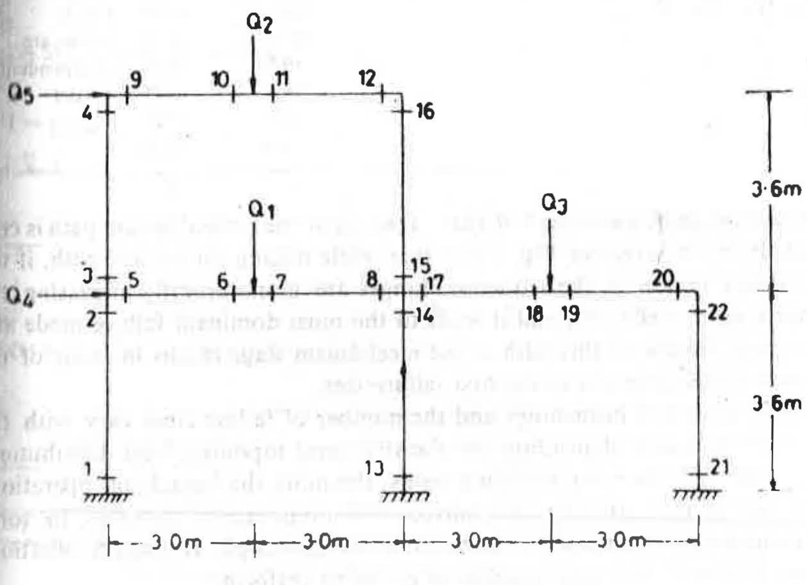


FIG. 10.19 Two-storey two-bay unsymmetrical frame—Example 10.11

example is given in Table 10.7. For this structure, the number of elementary mechanisms is 10 and the number of possible mechanisms is 1023. The identified dominant mechanisms for this example are indicated in Fig. 10.20. The results of the identified dominant mechanisms and of the reliability analysis are given in Table 10.8. The correlation coefficient matrix, representing the correlation between pairs of mechanisms, is shown in Table 10.9. For this example, the dominant mechanisms are very close to each other. Most of the dominant mechanisms are identified in the first tree only.

The same problem has been solved by Ma and Ang (10.8) and Murotsu (10.10), and the results of the generation of dominant mechanisms are compared with their results (Table 10.8).

TABLE 10.7 *Data for frame in Figure 10.19—Example 10.11*

Section/Variable	EA (kN)	EI (kN m²)	μ	δ	ρ
Section					
1, 2, 3, 4, 21, 22	0.105×10^{7}	0.105×10^{6}			
5, 6, 7, 8	0.132×10^{7}	0.277×10^{5}			
9, 10, 11, 12	0.101×10^{7}	0.154×10^{6}			
13, 14, 15, 16	0.101×10^{7}	0.758×10^{4}			
17, 18, 19, 20	0.116×10^{7}	0.207×10^{5}			
Variable					
$M_1, M_2, M_{13}, M_{14}, M_{21}, M_{22}$			95.0 kN m	0.15	1.0
M_3, M_4, M_{15}, M_{16}			95.0	0.15	1.0 Otherwise uncorrelated
M_5, M_6, M_7, M_8			204.0	0.15	1.0
$M_9, M_{10}, M_{11}, M_{12}$			122.0	0.15	1.0
$M_{17}, M_{18}, M_{19}, M_{20}$			163.0	0.15	1.0
Q_1			169.0	0.15	Loads are
Q_2			89.0	0.25	independent
Q_3			116.0	0.25	except
Q_4			62.0	0.25	$\rho_{Q4,Q5} = 1$
Q_5			31.0	0.25	

Discussion and Conclusion (10.14) Tracing of the critical failure path is crucial. It is observed in Fig. 10.18 that while tracing the critical path, if the reliability indices of the sequential hinges are monotonically increasing, the failure path is efficient, and it leads to the most dominant failure mode and also that branching this path at the mechanism stage results in many of the dominant mechanisms in the first failure tree.

The number of branchings and the number of failure trees vary with the type of problem, depending on the structural topology, load distribution, etc. The more the parallel failure paths, the more the branching operations will be. If parallel paths get mixed, the randomness increases. In some circumstances, inadmissible mechanisms are generated. In certain situations there can be a very large number of cycles to perform.

It is observed that in Example 10.11, all the dominant modes generated by other research workers have also been obtained using the proposed method. However, more modes, including a few insignificant modes, are generated in the process. It is found from Tables 10.5 and 10.8 that the most dominant failure mode is obtained in the first tree for both the problems. In Example 10.11 (Table 10.8), all the modes identified by Ma and Ang (10.8) and Murotsu (10.10), except one have been generated in the first tree itself.

It is observed that the accuracy of estimating p_{fs} may be improved marginally by generating more failure trees, but is quite expensive. For all practical purposes, the generation of the first failure tree and the system failure probability calculated based on that appears to be adequate.

It is concluded that the proposed method used simple logical strategies

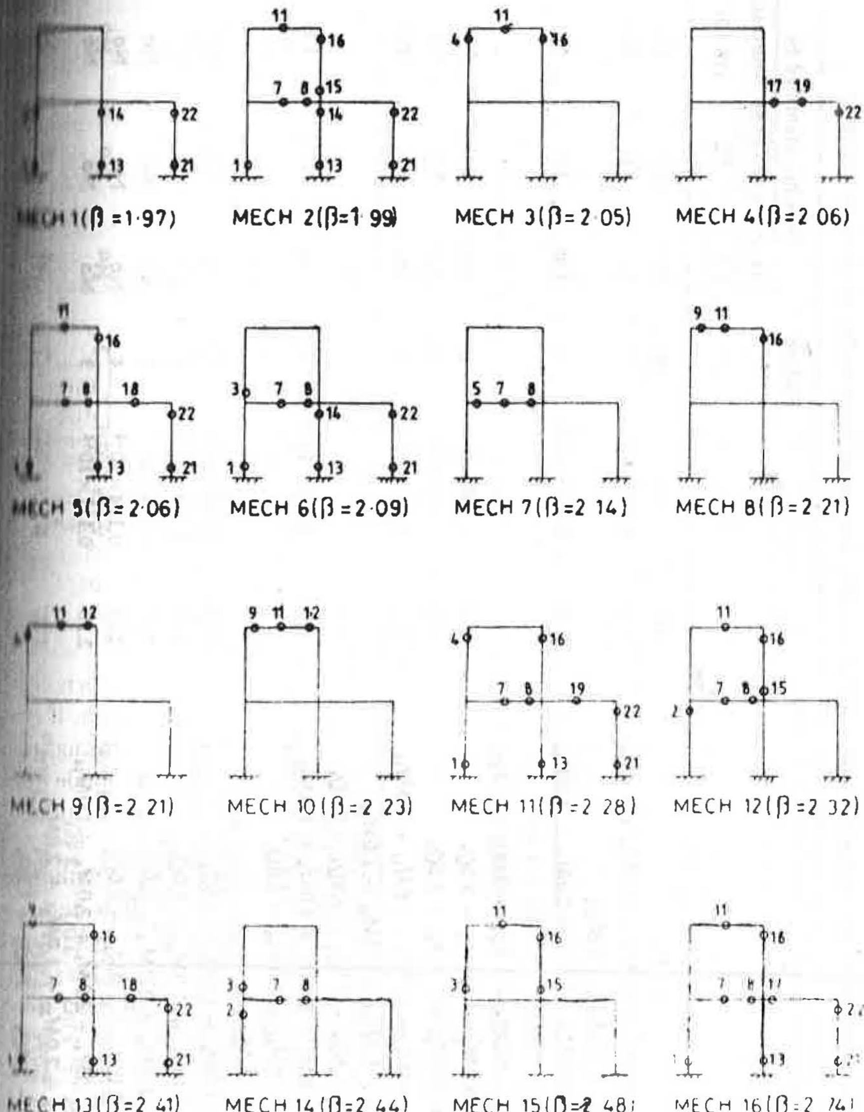


FIG. 10.20 Location of hinges in identified dominant mechanisms—Example 10.11

for the selection of hinges and the branching of failure paths to identify the probabilistically dominant mechanisms. It is simple, fairly efficient, and is capable of generating the dominant mechanisms for a practical complex structure. However, being a heuristic technique, it is not possible to prove theoretically whether all dominant mechanisms can be generated by using the proposed technique. For practical problems, it is felt that it is enough if the first failure tree is generated and the system failure probability calculated based on the generated mechanisms in the first failure tree.

TABLE 10.8 *Details of generated dominant mechanisms and results of reliablity analysis of frame in Fig. 10.19—Example 10.11*

Sl. No.	Hinged sections	Z	β	p_f	Failure tree	Whether identified by		
						Ma and Ang (10.8)		Murotsu
						F1	F2	(10.10)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1.	1, 2, 13, 14 21, 22	$1.0M_1 + 1.0M_2 + 1.0M_{13} + 1.0M_{14} + 1.0M_{21} + 1.0M_{22} - 3.60Q_4 - 3.60Q_5$	1.97	0.247×10^{-1}	1	Yes	Yes	Yes
2.	1, 7, 8, 11, 13, 14, 15, 16, 21, 22	$1.0M_1 + 2.0M_7 + 1.0M_8 + 2.0M_{11} + 1.0M_{13} + 1.0M_{14} + 1.0M_{15} + 2.0M_{16} + 1.0M_{21} + 1.0M_{22} - 3.0Q_1 - 3.0Q_2 - 3.6Q_4 - 7.2Q_5$	1.99	0.231×10^{-1}	1	Yes	Yes	Yes
3.	4, 11, 16	$1.0M_4 + 2.0M_{11} + 1.0M_{16} - 3.0Q_2$	2.05	0.200×10^{-1}	1	Yes	Yes	Yes
4.	17, 19, 22	$1.0M_{17} + 2.0M_{19} + 1.0M_{22} - 3.0Q_3$	2.06	0.198×10^{-1}	3	Yes	Yes	Yes
5.	1, 7, 8, 11, 13, 16, 18, 21, 22	$0.5M_1 + 1.0M_7 + 1.0M_8 + 1.0M_{11} + 0.5M_{13} + 1.0M_{16} + 1.0M_{18} + 0.5M_{21} + 1.0M_{22} - 1.5Q_1 - 1.5Q_2 - 1.5Q_3 - 1.8Q_4 - 3.6Q_5$	2.06	0.197×10^{-1}	1	Yes	No	Yes
6.	1, 3, 7, 8 13, 14, 21 22	$1.0M_1 + 1.0M_3 + 2.0M_7 + 1.0M_8 + 1.0M_{13}$ $1.0M_{14} + 1.0M_{21} + 1.0M_{22} - 3.0Q_1 - 3.6Q_4 - 3.6Q_5$	2.09	0.183×10^{-1}	1	Yes	No	Yes
7.	5, 7, 8	$1.0M_5 + 2.0M_7 + 1.0M_8 - 3.0Q_1$	2.14	0.160×10^{-1}	1	Yes	Yes	No
8.	9, 11, 16	$1.0M_9 + 2.0M_{11} + 1.0M_{16} - 3.0Q_2$	2.21	0.134×10^{-1}	1	Yes	No	No
9.	4, 11, 12	$1.0M_4 + 2.0M_{11} + 1.0M_{12} - 3.0Q_2$	2.21	0.134×10^{-1}	4	No	No	No
10.	9, 11, 12	$1.0M_9 + 2.0M_{11} + 1.0M_{12} - 3.0Q_2$	2.23	0.124×10^{-1}	4	No	No	No
11.	1, 4, 7, 8, 13, 16, 19, 21, 22	$1.0M_1 + 1.0M_4 + 2.0M_7 + 2.0M_8 + 1.0M_{13} + 1.0M_{16} + 2.0M_{19} + 1.0M_{21} + 2.0M_{22} - 3.0Q_1 - 3.0Q_3 - 3.6Q_4 - 7.2Q_5$	2.28	0.112×10^{-1}	3	No	No	No

	13, 16, 19, 21, 22	$+ 1.0M_{16} + 2.0M_{19} + 1.0M_{21} + 2.0M_{22} - 3.0Q_1 - 3.0Q_2 - 3.6Q_4 - 7.2Q_5$						
12.	2, 7, 8, 11, 15, 16	$1.0M_2 + 2.0M_7 + 1.0M_8 + 2.0M_{11} + 1.0M_{15} + 2.0M_{16} - 3.0Q_1 - 3.0Q_2 - 3.6Q_5$	2.32	0.102×10^{-1}	2	No	No	No
13.	1, 7, 8, 9, 13, 16, 18, 21, 22	$1.0M_1 + 2.0M_7 + 2.0M_8 + 1.0M_9 + 1.0M_{13} + 1.0M_{16} + 2.0M_{18} + 1.0M_{21} + 2.0M_{22} -3.0Q_1 - 3.0Q_2 - 3.6Q_4 - 7.2Q_5$	2.41	0.795×10^{-2}	3	No	No	No
14.	2, 3, 7, 8	$1.0M_2 + 1.0M_3 + 2.0M_7 + 1.0M_8 - 3.0Q_1$	2.44	0.734×10^{-2}	4	No	No	No
15.	3, 11, 15, 16	$1.0M_3 + 2.0M_{11} + 1.0M_{15} + 2.0M_{16} - 3.0Q_1 - 3.6Q_5$	2.48	0.664×10^{-2}	2	No	No	No
16.	1, 7, 8, 11, 13, 16, 17 21, 22	$1.0M_1 + 2.0M_7 + 2.0M_8 + 2.0M_{11} + 1.0M_{13} +2.0M_{16} + 1.0M_{17} + 1.0M_{21} + 1.0M_{22} - 3.0Q_1 - 3.0Q_2 - 3.6Q_4 - 7.2Q_5$	2.74	0.307×10^{-2}	1	No	No	No

System failure probability

Simple bounds $0.247\times10^{-1} \leqslant p_{f_s} \leqslant 0.205$

Narrow bounds $0.702\times10^{-1} \leqslant p_{f_s} \leqslant 0.147$

[($p_{f_s} = 0.116$ given by Ma and Ang (10.8) using Monte Carlo simulation with sample size 5000 and $0.745\times10^{-1} \leqslant p_{f_s} \leqslant 0.907\times10^{-1}$ given by Murotsu (10.10)]

TABLE 10.9 *Correlations between generated mechanisms—Example 10.11*

Mechanism No.	Correlations ρ_{ij}															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.0	0.65	0.0	0.09	0.551	0.672	0.0	0.0	0.0	0.0	0.582	0.197	0.584	0.084	0.197	0.568
2		1.0	0.435	0.045	0.894	0.905	0.595	0.407	0.407	0.363	0.814	0.866	0.818	0.660	0.577	0.978
3			1.0	0.0	0.347	0.031	0.0	0.965	0.965	0.866	0.045	0.571	0.06	0.041	0.921	0.389
4				1.0	0.453	0.054	0.0	0.0	0.0	0.0	0.479	0.012	0.48	0.015	0.0	0.109
5					1.0	0.835	0.615	0.335	0.335	0.307	0.946	0.796	0.953	0.651	0.454	0.921
6						1.0	0.726	0.014	0.014	0.0	0.883	0.739	0.88	0.784	0.195	0.906
7							1.0	0.0	0.0	0.0	0.651	0.782	0.652	0.977	0.00	0.690
8								1.0	1.0	0.977	0.021	0.535	0.062	0.019	0.838	0.375
9									1.0	0.977	0.021	0.535	0.062	0.019	0.838	0.375
10				Symmetrical						1.0	0.0	0.477	0.061	0.0	0.727	0.345
11											1.0	0.668	0.095	0.689	0.216	0.849
12												1.0	0.672	0.823	0.602	0.886
13													1.0	0.683	0.210	0.857
14														1.0	0.068	0.722
15															1.0	0.509
16																1.0

If Z_i is the safety margin for the potential hinge section i, let the event E_i be

$$E_i = P(Z_i < 0)$$

In the illustrated method, when the hinges were selected at any stage along a failure path before a mechanism is formed, the hinge location E_j (over the set of all eligible locations) with the greatest probability of failure (i.e. min. β) was selected. A better, or more rational logic may be to select the next hinge location such that the joint probability of occurrence of the hinge sequence, including the selected hinge location, is maximum. That is, at the jth stage, select jth hinge location such that

$$P(E_1 \cap E_2 \cap \ldots \cap E_j) = \max_i [P_i(E_1 \cap E_2 \cap \ldots \cap E_j)] \tag{10.31}$$

where the maximum is over the set of all possible hinge locations at that selection stage, that is, all hinge locations other than the sections where hinges have already formed. All events E_i are correlated because of the common load variables in all equations for Z_i. The evaluation of the joint probability is complicated and time consuming. In the proposed method, this joint probability was never used. Murotsu (10.10) used these approximations, given below, which might be the upper bounds for Eq. (10.31).

$$P(E_1 \cap E_2 \ldots \cap E_j) \leq \max_i [P_i(E_j)] \qquad j \geq 1 \tag{10.32}$$

or

$$P(E_1 \cap E_2 \cap \ldots \cap E_j) \leq \max_i [P_i(E_i \cap E_j)] \qquad j \geq 2 \tag{10.33}$$

Using these, he proposed a branching and bounding algorithm for the generation of dominant modes (10.10, 10.11).

10.7 RELIABILITY ANALYSIS OF RCC FRAMES

10.7.1 Introduction

The failure of a frame structure by the formation of a collapse mechanism requires a large rotational capacity of plastic hinges. Steel structures satisfy these requirements. Nonlinear and inelastic deformation characteristics of RCC structures do not allow to use the available resistance of sections to maximize the structural reliability. Moment-rotation relationships and the limited rotation capacity of RCC sections pose difficulties and limitations in the reliability analysis of plane frame structures.

The reliability analysis of RCC frame structures was initiated by Ticky and Vorlicek (10.15). They formulated the reliability of RCC structures subjected to loads from one or several sources based on the ultimate load, and it was shown how the deformability (ductility) of critical sections could be taken into account in studying RCC frames. Webster (10.16) presented a probabilistic procedure to forecast the performance of RCC frames subjected to an arbitrary number of sequential loads. Chou, McIntosh and Corotis (10.17) had investigated the correlation between resistance and reliability for a simple RCC frame with known collapse mechanisms. Ranganathan and Deshpande (10.18) presented a method of the reliability analysis of RCC frames, considering the limited rotation capacity of concrete sections. A reliability model compatible with the collapse mechanism was proposed for the rotation failure mode and the method was illustrated with examples. The same thing is presented in this section.

In the case of RCC frames, any critical section, hinged earlier, may fail due to an insufficient plastic rotation capacity before a collapse mechanism is formed. This mode of failure will be called a rotation failure mode. Considering the limited plastic rotation capacity in RCC frames, a method is suggested to verify and analyse the identified dominant mechanisms and to generate rotation failure modes, if necessary. A reliability model compatible with the collapse mechanism is proposed for a rotation failure mode on the basis of partial utilization of the plastic moment capacity of an incipient hinge section at the failure stage. The rotation failure modes are then combined with other possible mechanisms to assess the system reliability of a RCC frame. The method is illustrated with examples.

10.7.2 Strength and Stiffness of RCC Sections in Flexure

Idealisations

The nonlinear stress–strain and moment-rotation relationships of RCC sections pose some difficulty in the assessment of their strength–stiffness properties, required to carry out a reliability analysis for RCC frames. Using the idealized stress–strain curves for concrete and steel, shown in Fig. 10.21, and the bilinear moment rotation diagram of RCC plastic hinges, shown in Fig. 10.22(a), computational methods are developed to determine the moment capacity, flexural rigidity, and rotational capacity of RCC plastic hinges. Furthermore, nonlinear behaviour is approximated as linear elastic and piecewise linear elastic–plastic (PWLEP) to simplify the structural analysis. Also, first-order second-moment (FOSM) method is used to formulate the reliability analysis.

The limits l_1 and l_2, shown in Fig. 10.22(a), are considered as idealised elastic and plastic limits respectively (10.19). The elastic limit l_1 corresponds to either a maximum compressive strain in the concrete, e_{c1}, equal to 0.002 [Fig. 10.21(a),] or the yielding of steel by attaining the yield strain e_{sy} in mild steel bars as shown in Fig. 10.21(b), or offset strain of 0.001 in high yield strength deformed bars as shown in Fig. 10.21(c), according to whichever condition is attained first. The resisting moment of the section corresponding to l_1 in Fig. 10.22(a) is termed as the yield moment, M_y. The plastic limit l_2 is attained when either the concrete or steel fails corresponding to the maximum strain e_{c2} (taken as 0.0035) for concrete as shown in Fig. 10.21(a), or strain of 0.01 in steel as shown in Fig. 10.21(b) or 10.21(c). The resisting moment of the section at l_2 is taken as the plastic moment, M.

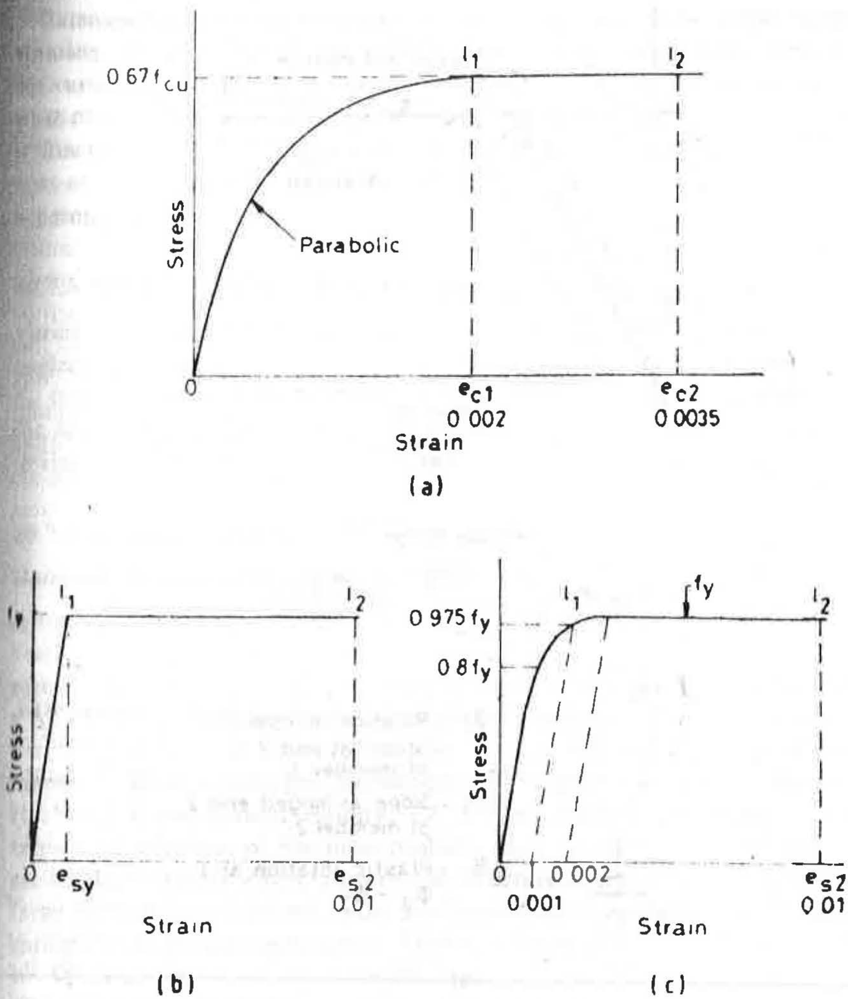


FIG. 10.21 Idealized stress-strain curves for (a) concrete (b) mild steel and (c) cold-worked steel

Resisting Moment and Flexural Rigidity

The resisting moment of a rectangular or tee section at limits l_1 or l_2 can be computed by satisfying force equilibrium and strain compatibility.

Referring to the bilinear moment-rotation diagram, shown in Fig. 10.22(a), the flexural rigidity, EI, of a RCC member is assumed to be constant in the range 0 to l_1. Also, EI is assumed constant between critical sections of the member. The value of a uniform EI obtained from stress and strain conditions at limit l_1 is given by (10.19)

$$EI = \frac{M_y c_1}{e_{c1}} = \frac{M_y(d - c_1)}{e_{sy}} \tag{10.34}$$

where c_1 is the depth of neutral axis at limit l_1 and d is the effective depth.

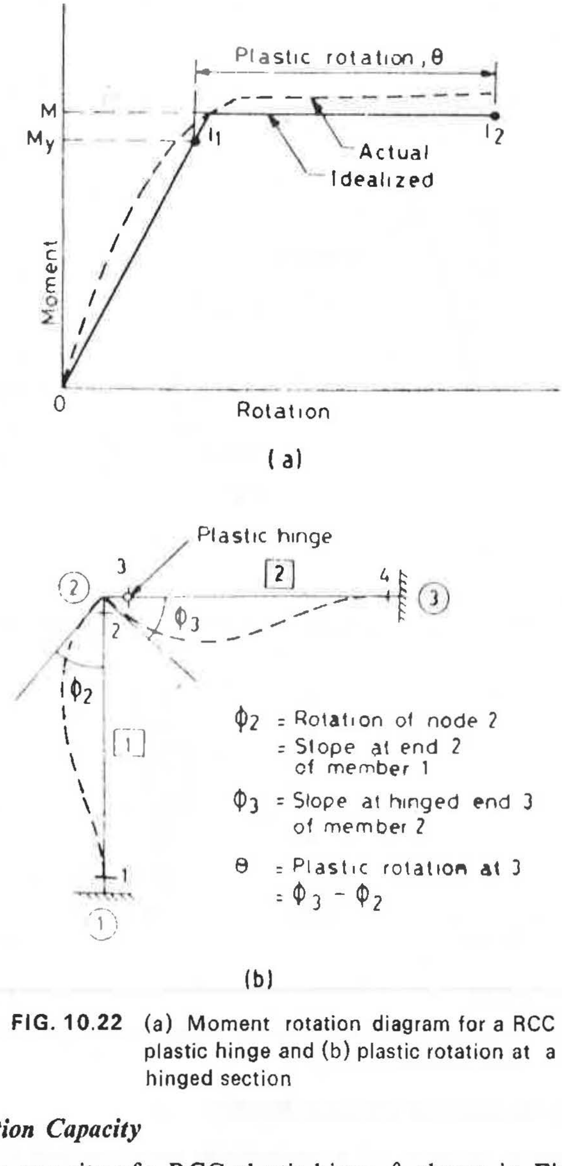


FIG. 10.22 (a) Moment rotation diagram for a RCC plastic hinge and (b) plastic rotation at a hinged section

Plastic Rotation Capacity

The rotation capacity of a RCC plastic hinge, θ, shown in Fig. 10.22(a), is the angular rotation which the section can sustain under the constant plastic moment without the local failure of the section due to limiting strain conditions at the plastic limit l_2 defined earlier. The plastic rotation capacity of a section depends on the (i) material properties, (ii) amount of reinforcement, (iii) confinement percentage, and (iv) axial load. The plastic rotation capacity can be derived (10.19) as a function of the (i) ultimate strain of concrete, (ii) strain variation in concrete from l_1 to l_2, (iii) spread of the plastic zone, i.e., the length of the plastic hinge, and (iv) position of the neutral axis.

Extensive experimental investigations show large variations in the plastic rotation capacity of RCC regions. Empirical formulae have been reported by various research workers. Baker and Amarakone (10.20) have proposed a set of curves representing the permissible plastic rotation capacity, θ_p, as a function of the depth of the neutral axis at limit l_2 for different percentages of confinement in the member. Hence, one can use these curves to get a permissible rotation.

10.7.3 Statistics of Plastic Moment Capacity

Variations in geometric parameters of a section, generally being small, are neglected. Hence, for establishing statistics of M, the random variations of f_{cu} and f_y only are considered. Using the developed prediction equations for M, and using a first order approximation, the mean value and standard deviation of M are calculated from the known statistics of f_{cu} and f_y.

10.7.4 Reliability Analysis of RCC Frames (10.18)

Automatic Generation of Dominant Mechanisms

After establishing the strength and stiffness properties of RCC members and the statistics of M of various critical sections, stochastically dominant mechanisms are generated using the stiffness method of linear elastic and PWLEP analysis of the structure, and FOSM method of reliability analysis, assuming, initially an unlimited rotational capacity available for all plastic hinges to form a collapse mechanism. To simplify the analysis, the axial rigidity, EA, and flexural rigidity, EI, are assumed as deterministic. The sequential selection of the most probable hinge locations, to determine the set of plastic hinges which converts the structure to a mechanism having a large probability of failure, is the key consideration in the process of generating the dominant mechanisms. Methods suggested by Murotsu (10.10) or Ma and Ang (10.8) or Tang and Melchers (10.13) can be used to generate stochastically dominant mechanisms; however, using the technique (10.14) explained and illustrated in the previous section, stochastically dominant mechanisms are generated.

Checking of Plastic Rotations

The technique for generating stochastically dominant mechanisms selects the plastic hinges on the basis of β and determines the set of hinges which converts the structure into a mechanism without verifying the plastic rotations of the hinged regions. It is observed that this set of hinges may consist of inactive hinges; moreover, the sequence of hinges may be random with respect to load factors. When the actual plastic rotation of plastic hinges is to be checked against the permissible plastic rotation, the physical process of the sequential occurrence of plastic hinges due to load increments has to be considered. Therefore, the sequential analysis of dominant mechanisms based on the load factor is employed to check the plastic rotation of hinges at each stage of the sequential analysis, and to formulate the failure probability if the rotation check fails.

The plastic rotation is assumed to be concentrated at the critical section. Therefore, the relative slope at the node of the plastic hinge of the section is considered as the plastic rotation of the section, i.e. the angle of discontinuity as represented in Fig. 10.22(b). The rotations of nodes, obtained from the analysis, correspond to the slopes at the intact ends of the members meeting at the node, and not to the hinged ends. The slope at the hinged end of a member is obtained by slope deflection equations of the corresponding member. Then the plastic hinge rotation, as shown in Fig. 10.22(b), is given by the difference between the slope at the hinged end and the rotation of the corresponding node.

An identified mechanism with known active hinge sections is regenerated sequentially for checking the rotations of hinge sections on the basis of load factors. The load factor, η_i for a potential hinge section i at any stage is given by

$$\eta_i = \frac{\sum_{j=1}^{m} a_{ij} M_j}{\sum_{k=1}^{n} b_{ik} Q_k} \tag{10.35}$$

At any stage, let the selected potential hinge section be i having the lowest load factor η_i, and the earlier hinged sections be j and k. The actual plastic rotation at j (or k), θ_j is

$$\theta_j = \sum_{s=1}^{m} \theta_{js} M_s + \eta_i \left[\sum_{t=1}^{n} \theta_{jt} Q_t \right] \tag{10.36}$$

where M_s is the plastic moment capacity of the critical section s, Q_t is the tth applied load on the structure, θ_{js} and θ_{jt} are the plastic rotations at the hinged section j due to the unit plastic moment M_s and unit load Q_t respectively. θ_{js} corresponding to M_s of the nonhinged section is zero. It is possible that θ_j and/or θ_k may exceed permissible plastic rotation capacities θ_{pj} and θ_{pk} respectively. In such a case it is not possible for a hinge to be formed at section i as indicated in Fig. 10.23. If $\theta_j > \theta_{pj}$, then, considering

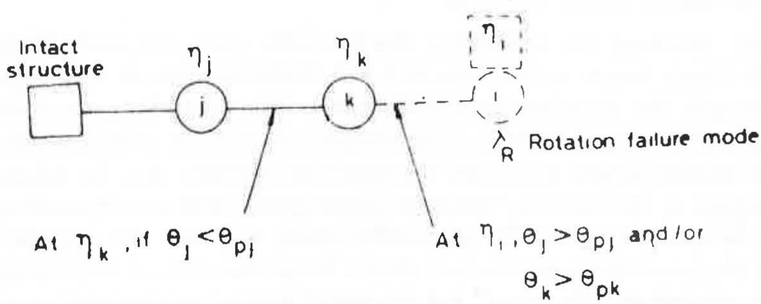


FIG. 10.23 **Checking of plastic rotation of hinged sections during regeneration of a mechanism**

the rotation capacity, the load factor λ_j at which $\theta_j = \theta_{pj}$ is given by (10.18)

$$\lambda_j = \frac{[\theta_{pj} - \sum_{s=1}^{m} \theta_{is} M_s]}{[\sum_{l=1}^{n} \theta_{sl} Q_l]} \tag{10.37}$$

Similarly, if $\theta_k > \theta_{pk}$, the expression for the load factor λ_k, at which $\theta_k = \theta_{pk}$, can also be obtained. λ_j and λ_k are now compared and the lowest selected and denoted by λ_R. At this value of λ_R, the RCC frame is assumed to fail under the rotation failure mode, prior to the formation of the mechanism. As the rotation check fails, a full strength of section i is not utilized. Whereas a plastic hinge at section i forms at the load factor λ_R, the coefficient a_{ii}, instead of being unity, is modified as (10.18)

$$a_{ii} = [\sum_{\substack{j=1 \\ j \neq i}}^{m} a_{ij} M_j] - \lambda_R [\sum_{k=1}^{n} b_{ik} Q_k] \left(\frac{1}{M_i}\right) \tag{10.38}$$

Substituting this value of a_{ii} in Eq. (10.26), the safety margin Z_i of the rotation failure mode is formulated and the reliability index β_i and probability of failure p_{fi} are calculated as usual. The process of regeneration of the mechanism is terminated at this stage.

Likewise, all dominant mechanisms identified earlier, assuming full redistribution, are regenerated and analysed, in addition, a plastic rotation check is performed for hinged sections at every stage. All failure modes are combined for establishing bounds on the system failure probability. Hence, the proposed formulation of the reliability analysis of RCC frames involves the following steps: (i) analysis of RCC cross sections of beams and columns and establishing statistics of M_i, (ii) generation of dominant mechanisms assuming unlimited rotation capacity of sections, and a reliability analysis, (iii) regeneration of failure modes with checking of plastic rotations of hinged sections, and (iv) synthesis of all failure modes and assessment of p_{fs}. A flowchart for the reliability analysis of RCC frames is given in Fig. 10.24. The proposed method is illustrated in the following examples (10.18).

EXAMPLE 10.12 The simple one-bay one-storey RCC frame, shown in Fig. 10.25, has been designed as per ISS (10.21) with the following data:

(i) Characteristic loads:

Live load : 4 kN/m^2
Wind load : 1.5 kN/m^2

(ii) Load combinations with partial safety factors:

(a) $1.5(D + L)$
(b) $1.5(D + W)$
(c) $1.2(D + L + W)$

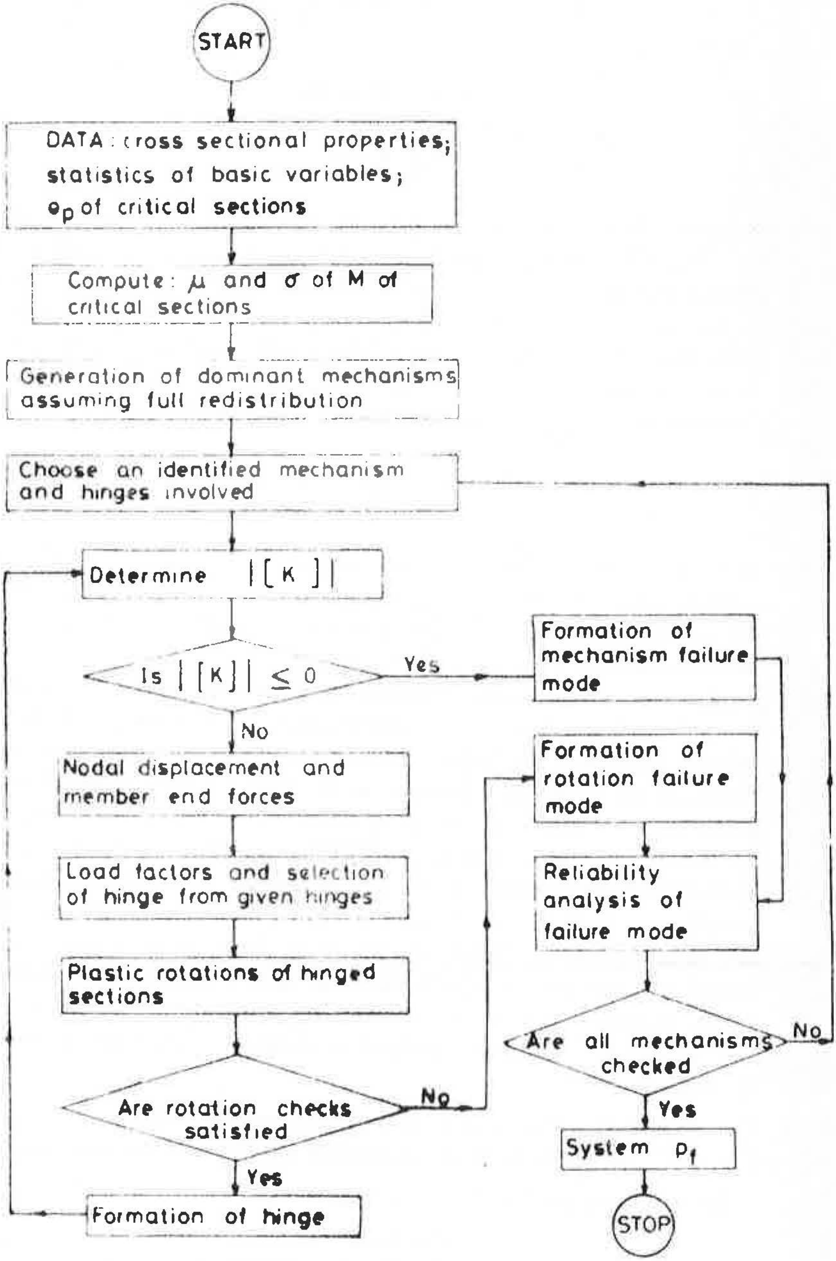


FIG. 10.24 Flowchart for reliability analysis of RCC frames

(iii) Characteristic strength of materials:

Concrete (M 20) : 20 N/mm^2
Steel (Fe 415) : 415 N/mm^2

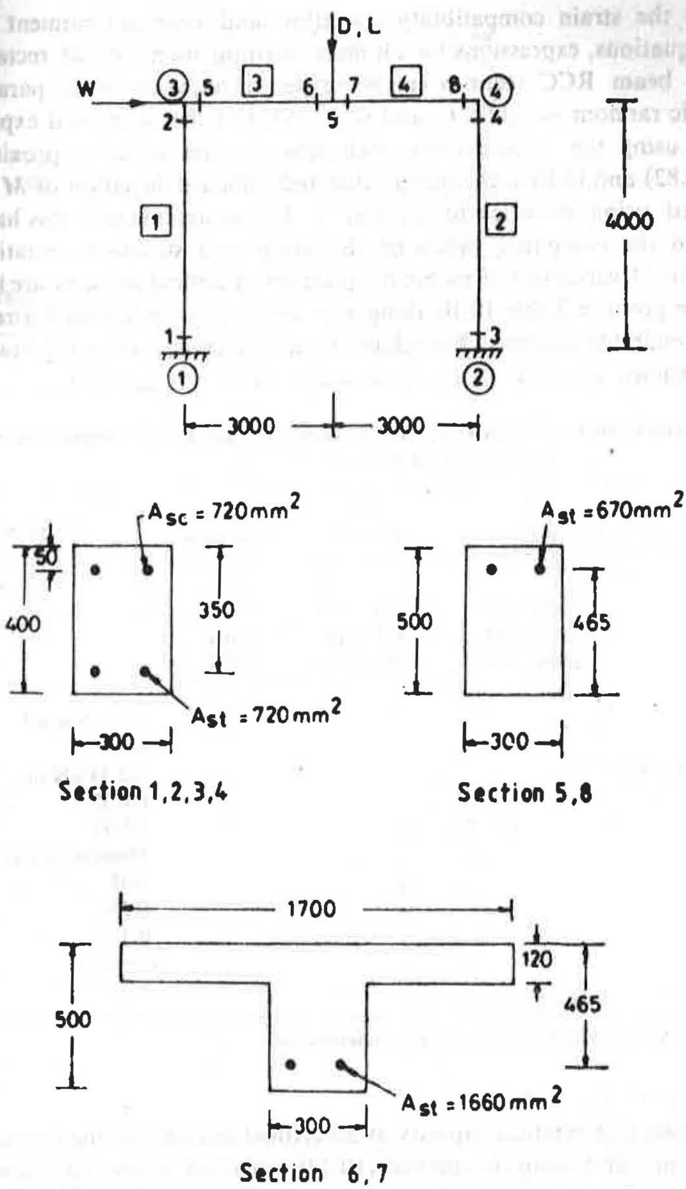


FIG. 10.25 One-bay one-storey RCC frame and details of cross-sections—Example 10.12

(iv) Partial safety factors for material strengths:

Concrete : $\gamma_{mc} = 1.5$

Steel : $\gamma_{ms} = 1.15$

(v) Young's modulus of elasticity:

Concrete (M 20) : 25.5 kN/mm^2

Steel : 200 kN/mm^2

Using the strain compatibility condition, and force and moment equilibrium equations, expressions for ultimate resisting moments of rectangular and tee beam RCC sections can be written in terms of design parameters and basic random variables f_{cu} and f_y (10.19). For the developed expression for M, using the Monte Carlo technique or first order approximation [Eqs. (3.82) and (3.84)], the mean value and standard deviation of M can be computed using statistics of f_{cu} and f_y. Let us assume that this has been done and the computed values of the mean and standard deviation (or coefficient of variation) of moment capacities of critical sections are known. They are given in Table 10.10 along with other data (including θ_p) required for the reliability analysis. The reliability analysis is carried out for two load combinations, viz. (i) $D + L_m + W_{apt}$ and (ii) $D + L_{apt} + W_m$.

TABLE 10.10 *Properties of cross sections and statistics of variables for RCC frame-Example 10.12*

Section or variable	EA (kN)	EI (kN m²)	θ_p (radian)	μ	δ
Section					
1, 2, 3, 4	0.356×10^7	0.915×10^4	0.017		
5, 8	0.949×10^7	0.392×10^5	0.018		
6, 7	0.949×10^7	0.392×10^5	0.019		
Variable					
f_{cu}				26.81 N/mm²	0.150
f_y				469	0.100
M_1, M_2, M_3, M_4				122.34 kN m	0.082
M_5, M_8				136.37	0.093
M_6, M_7				332.97	0.099
				Mean/nominal	
D				1.05	0.100
$L(L_m)$				0.558	0.334
$L(L_{apt})$				0.319	0.397
$W(W_m)$				0.693	0.236
$W(W_{apt})$				0.200	0.420

Remark: All variables are statistically independent.

Case (i) $D + L_m + W_{apt}$

Assuming full rotation capacity at all critical sections for the formation of mechanisms, and using the method (10.14) explained in Sec. 10.6, stochastically dominant modes are first generated, as shown in Fig. 10.26, for the load combination $D + L_m + W_{apt}$. The identified mechanisms are ordered and the system reliability is assessed from the synthesis of theses mechanisms, which are represented by their hinges as shown in Fig. 10.27, and safety margins as given in Table 10.11. The correlations between dominant failure modes are computed (Table 10.12). Details of the identified mechanisms and results of the reliability analysis of the frame, assuming full redistribution (i.e. without limiting the plastic rotations of hinges), are presented in Table 10.11.

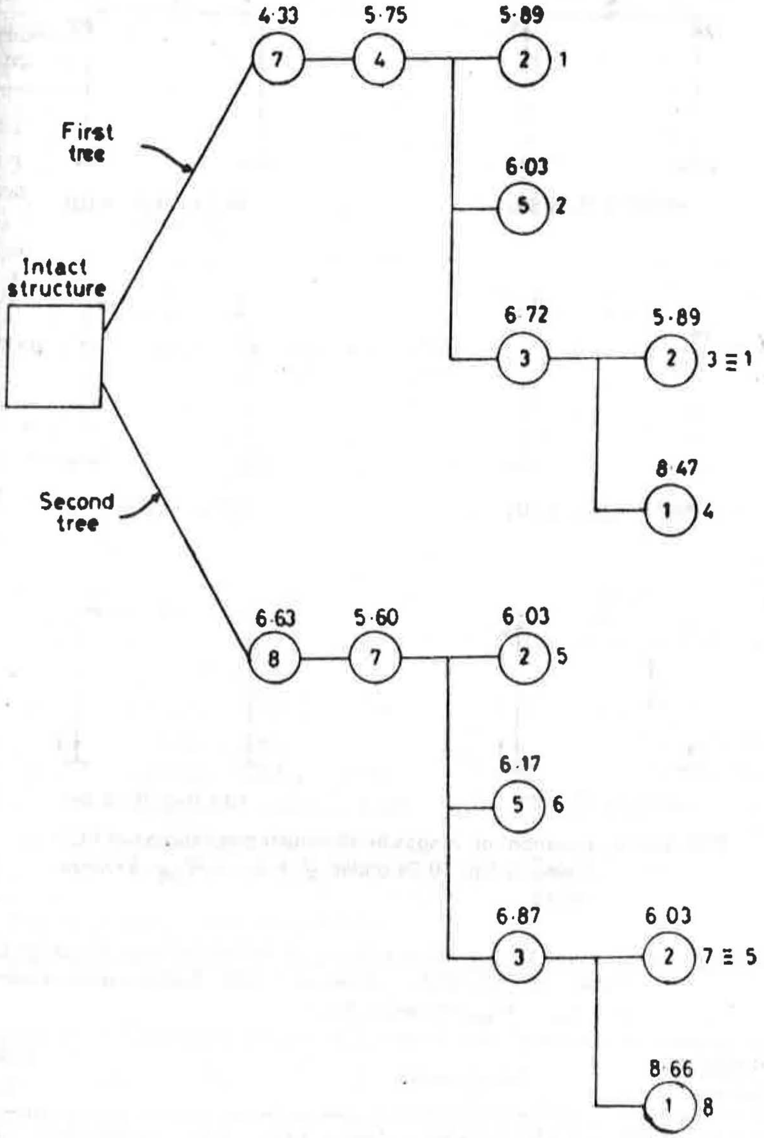


FIG. 10.26 Failure tree diagram for RCC frame in Fig. 10.25 under $D + L_{in} + W_{apl}$—Example 10.12

Each identified mechanism, indicated in Fig. 10.27, is regenerated, as shown in Fig. 10.28, according to the procedure outlined in the flowchart, given in Fig. 10.24, for checking the plastic rotations of hinged sections against their permissible plastic rotations, θ_p. The final failure modes, corresponding to the possible mechanisms or rotation failures are generated as explained in Sec. 10.7.3, and are given in Table 10.13. Correlations

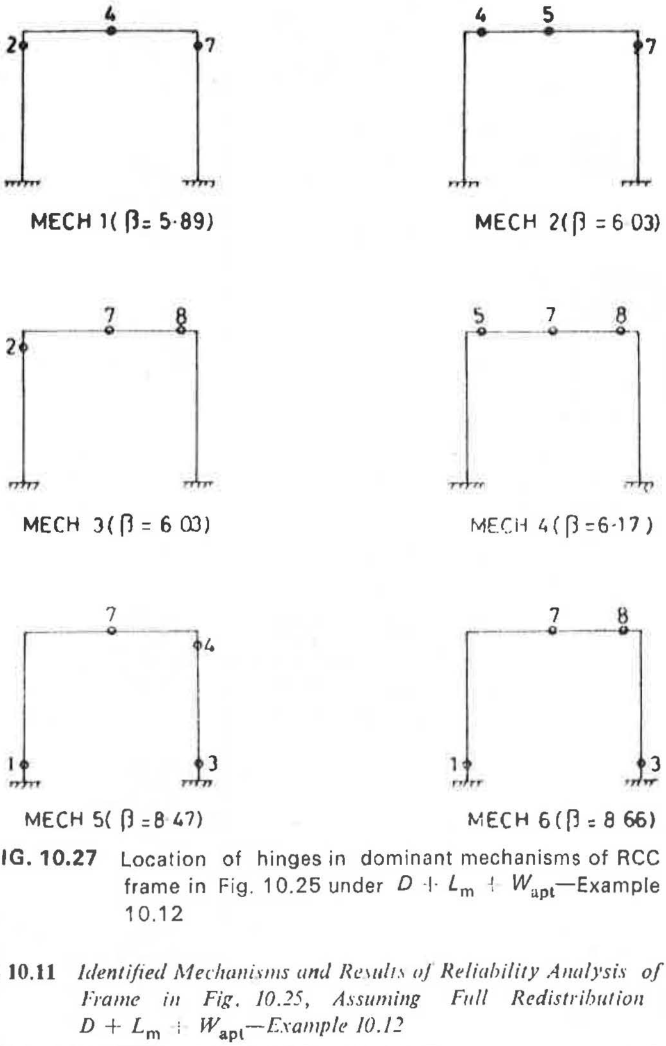


FIG. 10.27 Location of hinges in dominant mechanisms of RCC frame in Fig. 10.25 under $D + L_m + W_{apt}$—Example 10.12

TABLE 10.11 *Identified Mechanisms and Results of Reliability Analysis of RCC Frame in Fig. 10.25, Assuming Full Redistribution Under $D + L_m + W_{apt}$—Example 10.12*

Sl. No.	Hinged sections	Safety margin	β	p_f	Failure tree
1.	2, 4, 7	$1.0M_2 + 1.0M_4 + 2.0M_7 - 3.0D - 3.0L$	5.89	0.197×10^{-8}	1
2.	4, 5, 7	$1.0M_4 + 1.0M_5 + 2.0M_7 - 3.0D - 3.0L$	6.03	0.821×10^{-9}	1
3.	2, 7, 8	$1.0M_2 + 2.0M_7 + 1.0M_8 - 3.0D - 3.0L$	6.03	0.821×10^{-9}	2
4.	5, 7, 8	$1.0M_5 + 2.0M_7 + 1.0M_8 - 3.0D - 3.0L$	6.17	0.338×10^{-9}	2
5.	1, 3, 4, 7	$1.0M_1 + 1.0M_3 + 2.0M_4 + 2.0M_7 - 3.0D - 3.0L - 4.0W$	8.47	0.124×10^{-16}	1
6.	1, 3, 7, 8	$1.0M_1 + 1.0M_3 + 2.0M_7 + 2.0M_8 - 3.0D - 3.0L - 4.0W$	8.66	0.239×10^{-17}	2

Bounds on system probability of failure

Simple $0.197\times10^{-8} \leqslant p_{fs} \leqslant 0.395\times10^{-8}$

Narrow $0.215\times10^{-8} \leqslant p_{fs} \leqslant 0.284\times10^{-8}$

TABLE 10.12 *Correlations between mechanisms given in Table 10.11—Example 10.12*

Mechanism No.	ρ_{ij} 1	2	3	4	5	6
1	1.0	0.981	0.981	0.962	0.971	0.927
2		1.0	0.962	0.981	0.966	0.923
3			1.0	0.981	0.938	0.968
4				1.0	0.934	0.963
	Symmetrical					
5					1.0	0.929
6						1.0

TABLE 10.13 *Regenerated failure modes and results of reliability analysis of RCC frame in Fig. 10.25 under $D + L_m + W_{apt}$—Example 10.12*

Sl. No.	Hinged sections	Safety margin	β	p_f	Remarks
1	2, 4, 7	$1.0M_2 + 1.0M_4 + 2.0M_7 - 3.0D - 3.0L$	5.89	0.197×10^{-8}	Mechanism failure
2	4, 5, 7	$1.0M_4 + 0.939M_5 + 2.0M_7 - 3.0D - 3.0L$	5.94	0.144×10^{-8}	Rotation failure
3	2, 7, 8	$1.0M_2 + 2.0M_7 + 1.0M_8$ $3.0D - 3.0L$	6.03	0.821×10^{-9}	Mechanism failure
4	5, 7, 8	$0.991M_5 + 2.0M_7 + 1.0M_8 - 3.0D - 3.0L$	6.16	0.368×10^{-9}	Rotation failure
5	3, 4, 7	$0.620M_3 + 1.0M_4 + 1.5M_7 - 0.225D - 0.225L - 2.0W$	6.02	0.861×10^{-9}	Rotation failure
6	3, 7, 8	$0.636M_3 + 1.5M_7 + 1.0M_8 - 0.225D - 0.225L - 2.0W$	6.23	0.235×10^{-9}	Rotation failure

Bounds on system probability of failure:

Simple $0.197\times10^{-8} \leqslant p_{fs} \leqslant 0.57\times10^{-8}$

Narrow $0.245\times10^{-8} \leqslant p_{fs} \leqslant 0.371\times10^{-8}$

TABLE 10.14 *Correlations between failure modes given in Table 10.13-Example 10.12*

Failure mode No.	ρ_{ij} 1	2	3	4	5	6
1.	1.0	0.982	0.962	0.986	0.981	0.959
2.		1.0	0.981	0.983	0.983	0.956
3.			1.0	0.959	0.981	0.982
4.	Symmetrical			1.0	0.963	0.967
5.					1.0	0.986
6.						1.0

between failure modes are computed (Table 10.14) and the bounds on system failure are established. Results of the same are given in Table 10.13.

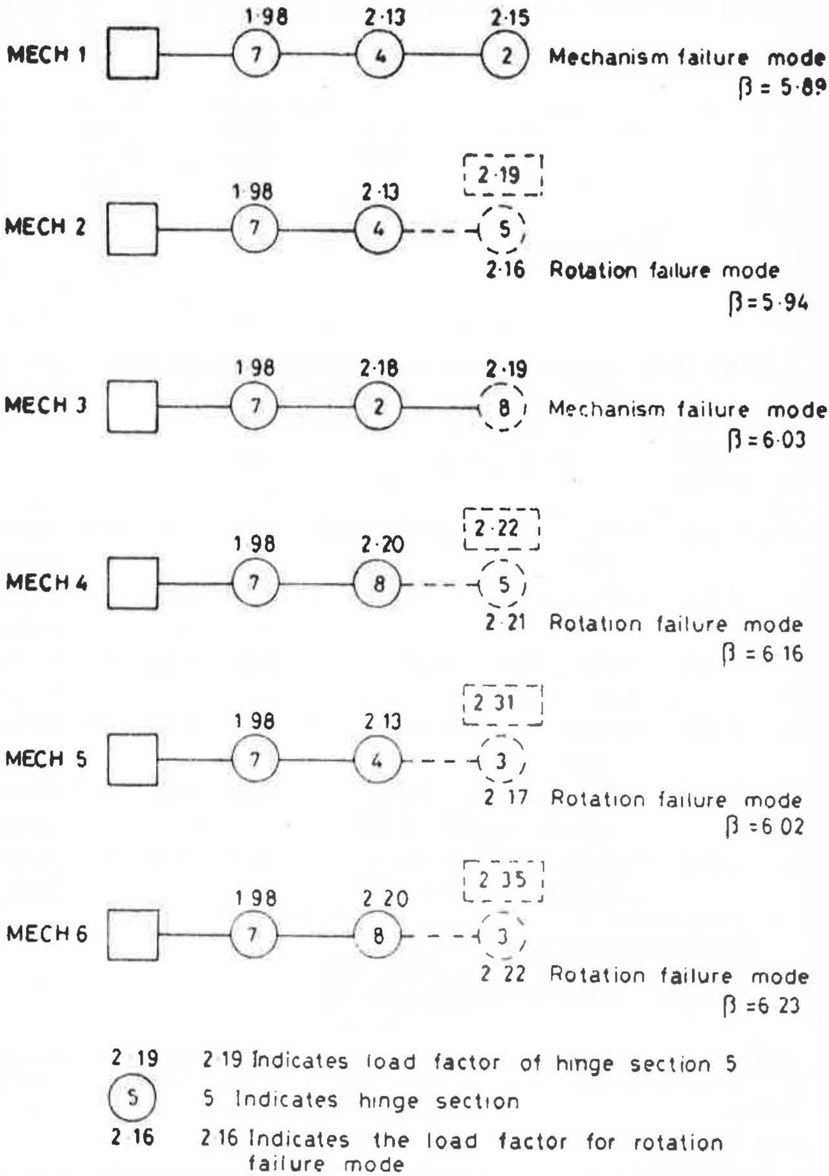


FIG. 10.28 Regeneration of individual mechanisms in Fig. 10.27 for checking plastic rotations under load case $D + L_m + W_{apt}$—Example 10.12

Case (ii) $D + L_{apt} + W_m$

The procedure is repeated for this example to assess the system reliability under the second load combination $D + L_{apt} + W_m$. The generated dominant mechanisms, assuming full redistribution, and their safety margins and results of the reliability analysis based on these are given in Table 10.15. The mechanisms are regenerated as shown in Fig. 10.29. Table 10.16 gives

TABLE 10.15 *Identified mechanisms and results of reliability analysis of RCC frame in Fig. 10.25, assuming full redistribution under $D + L_{apt} + W_m$—Example 10.12*

Sl. No.	Hinged sections	Safety margin	β	p_f	Failure tree
1	2, 4, 7	$1.0M_2 + 1.0M_4 + 2.0M_7 - 3.0D - 3.0L$	6.79	0.567×10^{-11}	1
2	4, 5, 7	$1.0M_4 + 1.0M_5 + 2.0M_7 - 3.0D - 3.0L$	6.93	0.206×10^{-11}	1
3	2, 7, 8	$1.0M_2 + 2.0M_7 + 1.0M_8 - 3.0D - 3.0L$	6.93	0.206×10^{-11}	2
4	5, 7, 8	$1.0M_5 + 2.0M_7 + 1.0M_8 - 3.0D - 3.0L$	7.08	0.736×10^{-12}	2
5	1, 3, 4, 7	$1.0M_1 + 1.0M_3 + 2.0M_4 + 2.0M_7$ $3.0D - 3.0L - 4.0W$	9.14	$< 10^{-18}$	1
6	1, 3, 7, 8	$1.0M_1 + 1.0M_3 + 2.0M_7 + 2.0M_8 - 3.0D - 3.0L - 4.0W$	9.32	$< 10^{-18}$	2

Bounds on system probability of failure:

Simple $0.567\times 10^{-11} \leqslant p_{fs} \leqslant 0.105\times 10^{-10}$

Narrow $0.626\times 10^{-11} \leqslant p_{fs} \leqslant 0.812\times 10^{-11}$

TABLE 10.16 *Regenerated failure modes and results of reliability analysis of RCC frame in Fig. 10.25 under $D + L_{apt} + W_m$-Example 10.12*

Sl. No.	Hinged sections	Safety margin	β	p_f	Remarks
1	2, 4, 7	$0.877M_2 + 1.0M_4 + 2.0M_7 -3.0D - 3.0L$	6.61	0.193×10^{-10}	Rotation failure
2	4, 5, 7	$1.0M_4 + 0.786M_5 + 2.0M_7 -3.0D - 3.0L$	6.60	0.211×10^{-10}	Rotation failure
3	2, 7, 8	$0.931M_2 + 2.0M_7 + 1.0M_8 -3.0D - 3.0L$	6.83	0.414×10^{-11}	Rotation failure
4	5, 7, 8	$0.835M_5 + 2.0M_7 + 1.0M_8 -3.0D - 3.0L$	6.82	0.457×10^{-11}	Rotation failure
5	3, 4, 7	$0.729M_3 + 1.0M_4 + 1.5M_7 -2.25D - 2.25L - 2.0W$	6.93	0.218×10^{-11}	Rotation failure
6	3, 7, 8	$0.749M_3 + 1.5M_7 + 1.0M_8 -2.25D - 2.25L - 2.0W$	7.14	0.469×10^{-12}	Rotation failure

Bounds on system probability of failure:

Simple $0.211\times 10^{-10} \leqslant p_{fs} \leqslant 0.518\times 10^{-10}$

Narrow $0.289\times 10^{-10} \leqslant p_{fs} \leqslant 0.378\times 10^{-10}$

the final failure modes obtained after checking the plastic rotations of the hinges. The correlations are computed. The estimated system reliability is given in the same Table 10.16.

EXAMPLE 10.13 The two-bay two-storey RCC frame, shown in Fig. 10.30, has been designed as per the ISS (10.21) with the same data given in

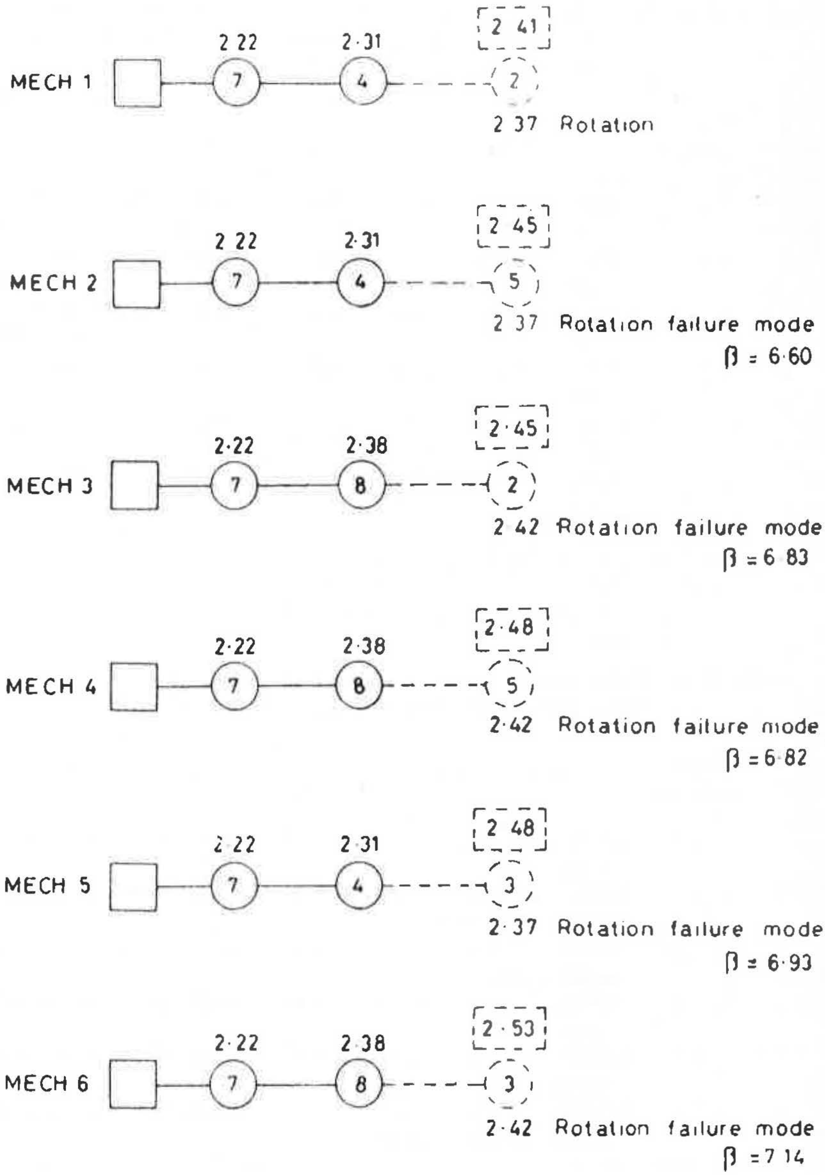


FIG. 10.29 Regeneration of individual mechanisms in Table 10.15 for checking plastic rotations under load case $D + L_{apt} + W_m$— Example 10.12

Example 10.12. Details of cross-sections of the frame are given in Fig. 10.30. Flexural rigidities, plastic moment capacities, and permissible plastic rotation capacities of sections are calculated and given in Table 10.17. The results of the reliability analysis for the two load combinations (i) $D + L_m + W_{apt}$ and (ii) $D + L_{apt} + W_m$ are given in Tables 10.18, 10.19, 10.20, and 10.21.

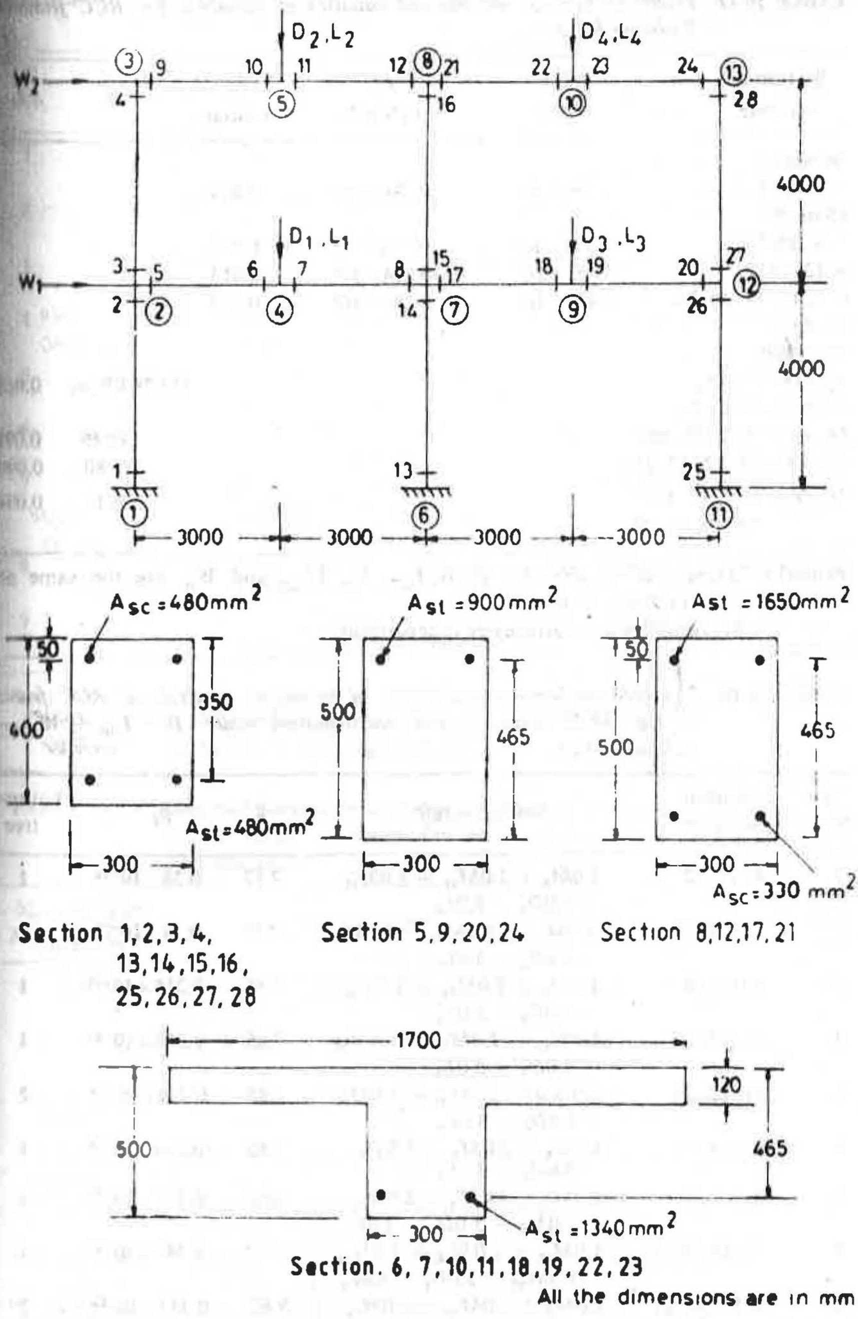


FIG. 10.30 Two-bay two-storey RCC frame and details of cross-sections—Example 10:23

TABLE 10.17 *Properties of cross sections and statistics of variables for RCC frame—Example 10.13*

Section or variable	EA (kN)	EI (kN m²)	θ_p (radian)	μ	δ
Section					
1 to 4, 13 to 16, 25 to 28	0.356×10^7	0.786×10^4	0.014		
5, 9, 20, 24	0.453×10^7	0.294×10^5	0.017		
8, 12, 17, 21	0.453×10^7	0.294×10^5	0.013		
6, 7, 10, 11, 18, 19, 22, 23	0.453×10^7	0.294×10^5	0.017		
Variable					
M_i, (i = 1 to 4, 13 to 16, 25 to 28)				114.78 kN m	0.058
M_j, (j = 5, 9, 20, 24)				178.49	0.092
M_k, (k = 8, 12, 17, 21)				315.80	0.090
M_t, (t = 6, 7, 10, 11, 18, 19, 22, 23)				274.11	0.098

Remarks: Statistics of variables f_{cu}, f_y, D, L_{apt}, L_m, W_{apt} and W_m are the same as given in Table 10.10
All variables are statistically independent.

TABLE 10.18 *Identified mechanisms and results of reliability analysis of RCC frame in Fig. 10.30 assuming full redistribution under* $D + L_m + W_{apt}$*—Example 10.13*

Sl. No.	Hinged sections	Safety margin	β	p_f	Failure tree
1	4, 11, 12	$1.0M_4 + 2.0M_{11} + 1.0M_{12} - 3.0D_2 - 3.0L_2$	7.17	0.38×10^{-12}	1
2	21, 22, 28	$1.0M_{21} + 2.0M_{22} + 1.0M_{28} - 3.0D_4 - 3.0L_4$	7.17	0.38×10^{-12}	2
3	9, 11, 12	$1.0M_9 + 2.0M_{11} + 1.0M_{12} - 3.0D_2 - 3.0L_2$	7.85	0.216×10^{-14}	1
4	17, 18, 20	$1.0M_{17} + 2.0M_{18} + 1.0M_{20} - 3.0D_3 - 3.0L_3$	7.85	0.216×10^{-14}	1
5	21, 22, 24	$1.0M_{21} + 2.0M_{22} + 1.0M_{24} - 3.0D_4 - 3.0L_4$	7.85	0.216×10^{-14}	2
6	5, 6, 8	$1.0M_5 + 2.0M_6 + 1.0M_8 - 3.0D_1 - 3.0L_1$	7.85	0.216×10^{-14}	1
7	2, 3, 6, 8	$1.0M_2 + 1.0M_3 + 2.0M_6 + 1.0M_8 - 3.0D_1 - 3.0L_1$	8.62	0.347×10^{-17}	2
8	17, 18, 26, 27	$1.0M_{17} + 2.0M_{18} + 1.0M_{26} + 1.0M_{27} - 3.0D_3 - 3.0L_3$	8.62	0.347×10^{-17}	1
9	4, 10, 16, 21	$1.0M_4 + 2.0M_{10} + 1.0M_{16} + 1.0M_{21} - 3.0D_3 - 3.0L_3$	8.62	0.347×10^{-17}	2

Bounds:

Simple $0.380\times10^{-12} \leqslant p_{fs} \leqslant 0.769\times10^{-12}$

Narrow $0.765\times10^{-12} \leqslant p_{fs} \leqslant 0.765\times10^{-12}$

TABLE 10.19 *Regenerated failure modes and results of reliability analysis of RCC frame in Fig. 10.30 under $D + L_m + W_{apt}$—Example 10.13*

Sl. No.	Hinged sections	Safety margin	β	p_f	Remarks
1	4, 11, 12	$1.0M_4 + 2.0M_{11} + 1.0M_{12} - 3.0D_2 - 3.0L_2$	7.17	0.38×10^{-12}	Mechanism failure
2	21, 22, 28	$1.0M_{21} + 2.0M_{22} + 1.0M_{28} - 3.0D_4 - 3.0L_4$	7.17	0.38×10^{-12}	Mechanism failure
3	9, 11, 12	$0.671M_9 + 2.0M_{11} + 1.0M_{12} - 3.0D_2 - 3.0L_2$	7.19	0.330×10^{-12}	Rotation failure
4	17, 18, 20	$1.0M_{17} + 2.0M_{18} + 1.0M_{20} - 3.0D_3 - 3.0L_3$	7.85	0.216×10^{-14}	Mechanism failure
5	21, 22, 24	$1.0M_{21} + 2.0M_{22} + 0.714M_{24} - 3.0D_4 - 3.0L_4$	7.28	0.172×10^{-12}	Rotation failure
6	5, 6, 8	$1.0M_5 + 2.0M_6 + 1.0M_8 - 3.0D_1 - 3.0L_1$	7.85	0.216×10^{-14}	Mechanism failure
7	3, 6, 8	$0.955M_3 + 0.967M_6 + 0.484M_8 - 1.45D_1 - 0.325W_1$	8.27	0.605×10^{-16}	Rotation failure
8	17, 18, 26, 27	$1.0M_{17} + 2.0M_{18} + 0.792M_{26} + 1.0M_{27} - 3.0D_3 - 3.0L_3$	8.32	0.438×10^{-16}	Rotation failure
9	4, 10, 16 21	$1.0M_4 + 2.0M_{10} + 0.374M_{16} + 1.0M_{21} - 3.0D_2 - 3.0L_2$	7.72	0.593×10^{-14}	Rotation failure

Bounds:
Simple $0.380 \times 10^{-12} \leqslant p_{f_s} \leqslant 0.127 \times 10^{-11}$
Narrow $0.952 \times 10^{-12} \leqslant p_{f_s} \leqslant 0.108 \times 10^{-11}$

TABLE 10.20 *Identified mechanisms and results of reliability analysis of RCC frame in Fig. 10.30 assuming full redistribution under $D + L_{apt} + W_m$—Example 10.13*

Sl. No.	Hinged sections	Safety margin	β	p_f	Failure tree
1	4, 11, 12	$1.0M_4 + 2.0M_{11} + 1.0M_{12} - 3.0D_2 - 3.0L_2$	8.25	0.809×10^{-16}	1
2	21, 23, 28	$1.0M_{21} + 2.0M_{23} + 1.0M_{28} - 3.0D_4 - 3.0L_4$	8.25	0.809×10^{-16}	1
3	17, 18, 20	$1.0M_{17} + 2.0M_{18} + 1.0M_{20} - 3.0D_3 - 3.0L_3$	8.94	0.217×10^{-18}	1
4	5, 6, 8	$0.5M_5 + 1.0M_6 + 0.5M_8 - 1.5D_1 - 1.5L_1$	8.94	0.217×10^{-18}	1
5	9, 11, 12	$1.0M_9 + 2.0M_{11} + 1.0M_{12} - 3.0D_2 - 3.0L_2$	8.94	0.217×10^{-18}	1
6	17, 18, 26, 27	$1.0M_{17} + 2.0M_{18} + 1.0M_{26} + 1.0M_{27} - 3.0D_3 - 3.0L_3$	9.78	$< 10^{-18}$	1
7	12, 16, 23, 28	$1.0M_{12} + 1.0M_{16} + 2.0M_{23} + 1.0M_{28} - 3.0D_4 - 3.0L_4$	9.78	$< 10^{-18}$	1

Bounds on system probability of failure:
Simple $0.809 \times 10^{-16} \leqslant p_{f_s} \leqslant 0.162 \times 10^{-15}$
Narrow $0.162 \times 10^{-15} \leqslant p_{f_s} \leqslant 0.162 \times 10^{-15}$

TABLE 10.21 *Regenerated failure modes and results of reliability Analysis of RCC frame in Fig. 10.30 under $D + L_{apt} + W_m$—Example 10.13*

Sl. No.	Hinged section	Safety margin	β	p_f	Remarks
1	4, 11, 12	$0.952M_4 + 2.0M_{11} + 1.0M_{12} - 3.0D_2 - 3.0L_2$	8.18	0.148×10^{-15}	Rotation failure
2	21, 23, 28	$1.0M_{21} + 2.0M_{23} + 1.0M_{28} - 3.0D_4 - 3.0L_4$	8.25	0.809×10^{-16}	Mechanism failure
3	17, 18, 20	$1.0M_{17} + 2.0M_{18} + 1.0M_{20} - 3.0D_3 - 3.0L_3$	8.94	0.217×10^{-18}	Mechanism failure
4	5, 6, 8	$0.802M_5 + 2.0M_6 + 1.0M_8 - 3.0D_1 - 3.0L_1$	8.54	0.694×10^{-17}	Rotation failure
5	9, 11, 12	$0.612M_9 + 2.0M_{11} + 1.0M_{12} - 3.0D_2 - 3.0L_2$	8.13	0.217×10^{-15}	Rotation failure
6	17, 18, 26, 27	$1.0M_{17} + 2.0M_{18} + 1.0M_{26} + 1.0M_{27} - 3.0D_3 - 3.0L_3$	9.78	$< 10^{-18}$	Mechanism failure
7	12, 16, 23, 28	$1.0M_{12} + 0.123M_{16} + 2.0M_{23} + 1.0M_{28} - 3.0D_4 - 3.0L_4$	8.44	0.158×10^{-16}	Rotation failure

Bounds on system probability of failure:

Simple $\quad 0.217 \times 10^{-15} \leqslant p_{fs} \leqslant 0.468 \times 10^{-15}$

Narrow $\quad 0.379 \times 10^{-15} \leqslant p_{fs} \leqslant 0.416 \times 10^{-15}$

10.7.5 Discussion

A simple and practical method of the reliability analysis of RCC frames, considering the limited rotation capacity of RCC sections, had been developed and illustrated. The probability of failure of a rotation failure mode, generated from the mechanism through a check for plastic hinge rotation, is found to be higher than that for mechanisms with unlimited rotational capacity, which is expected. This increase in p_f is observed to be considerable in the case of the least dominant mechanism.

A comparison of results for limited ductility and full redistribution shows that the bounds on p_{fs} are generally higher and wider for limited ductility. For the two case studies, it is noted that the probability of failure of the frame under the load combination $D + L_m + W_{apt}$ is more than that of under $D + L_{apt} + W_m$. The effect of limited rotational capacity on p_{fs} is found to be more critical under $D + L_{apt} + W_m$ than under $D + L_m + W_{apt}$ for these two case studies.

For the two case studies of RCC frames (design according to ISS), the system failure probability is found to be of the order of 10^{-9} for the one-bay one-storey frame and 10^{-12} for the two-bay two-storey frame. These values of failure probability are very small. This is due to high design loads and low material design strengths specified by the IS code.

The checking of plastic rotations of hinges and remodelling of the failure modes improves the accuracy of the system reliability of RCC frames. However, the improvement in the present case studies, where the load combination $D + L_m + W_{apt}$ is more dominant than $D + L_{apt} + W_m$, is not significant in the context of computer effort.

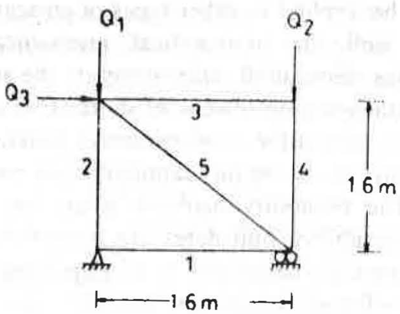


FIG. 10.31 Five member truss

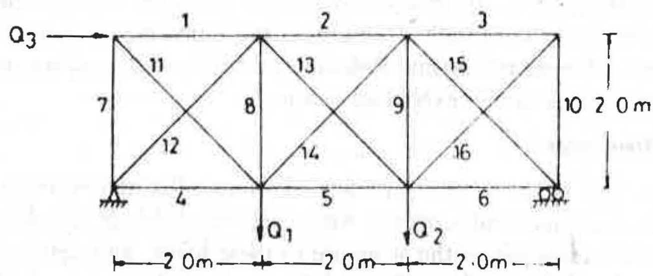


FIG. 10.32 Indeterminate truss

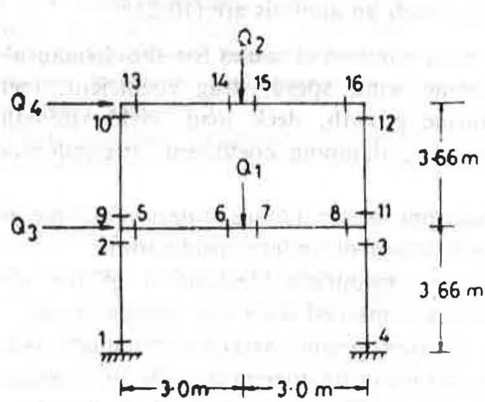


FIG. 10.33 Two-storey one-bay frame

10.8 STRUCTURAL SAFETY IN OTHER FIELDS

Reliability analysis is a tool in the design process. It can be applied to any field. The importance of making reliability assessments, especially for the purpose of making comparative design judgements, has received recognition in the last decade. Even though the reliability analysis and design of steel and RCC building structures are mainly treated in the examples of this book, methods can be applied to other types of strucures and other fields of engineering, as well, viz. aeronautical, mechanical and nuclear. The reliability theory has been used extensively in the analysis and design of bridges, buildings, transmission towers, offshore structures, ship structures, nuclear power plants, and in the development of general purpose structural design codes. In this book, mainly failure criteria based on strength have been considered. The reliability methods given can be applied to other criteria, such as serviceability limit states, viz. deflection, cracking, corrosion, etc. Fatigue and fracture behaviour is an important consideration in the design of bridges, offshore structures, aircraft structures, pressure-vessels, cranes, etc. Hence, reliability predictions against fatigue crack initiation, growth, and fracture is important. A considerable research has been done and is going on in developing analytical techniques for fatigue reliability. In the case of dynamically sensitive structures subjected to dynamic loads, the reliability analysis of such structures is more involved. This is so in the case of deep offshore platforms. Reliability analysis with respect to such a type of structure is briefly explained below.

Offshore Structures

The safety of an offshore structure depends on predicting the environmental phenomena, such as wind, current, wave, seismic loading, accurate calculation of the response of the structure to these loads, and determining the strength of the structure. Level 2 reliability methods have been used in the evaluation of component reliabilities in jacket structures. The various steps that are involved for such an analysis are (10.22):

(i) defining the basic random variables for the structural resistance and loading, viz. extreme wind speed, drag coefficient, inertial coefficient, current speed, marine growth, deck load, yield strength of steel, tube thickness, leg diameter, damping coefficient, strength model uncertainty, etc.

(ii) selecting the appropriate failure criterion and the associated model uncertainty for the component under consideration

(iii) developing an appropriate idealisation of the structure for the purposes of evaluating combined wave and current forces

(iv) developing an appropriate mathematical model relating the natural frequency of the structure in its dominant mode of vibration to the basic random variables which affect it, such as the soil and structure stiffness, superimposed deck loads, thickness of marine growth, and the coefficient of the added mass

(v) developing an efficient algorithm to determine the stochastic response of the structure under dynamic loads

(vi) obtaining the relationship between the displaced shape of the structure and the loads and moments in the individual components of the structure, by an appropriate structural analysis

(vii) combining the mathematical models given by steps (ii) to (vii) above to obtain the safety margin equation and

(viii) determining β for the failure criteria.

Baker (10.22) has done the reliability analysis of jacket platforms in the North Sea. The Level 2 reliability methods have been applied for taking decisions for the safety of offshore structures against fatigue (10.23).

REFERENCES

10.1 Stevensen, J. and F. Moses, "Reliability Analysis of Frame Structures", *Journal of Struct. Div., ASCE,* Vol. 96, ST-11, Nov. 1970, pp. 2409-2427.

10.2 Neal, B.G., *The Plastic Methods of Structural Analysis*, Halsted Press, Third Edition, 1977.

10.3 Cornell, C.A., "Bounds on the Reliability of Structural Systems", *Journal of Struct. Div. ASCE,* Vol. 93, ST-1, Feb. 1967, pp. 171-200.

10.4 Ditlevsen, O., "Narrow Reliability Bounds for Structural Systems", *Journal of Structural Mech.*, Vol. 1, No.4, 1979, pp. 453-472.

10.5 Benjamin, J.R. and C.A. Cornell, *Probability, Statistics and Decision for Civil Engineers*, McGraw-Hill, New York, 1970.

10.6 Ang, A.H.S. and M. Amin, "Reliability of Structures and Structural Systems", *Journal of Engg. Mech. Div., ASCE,* Vol. 94, EM4. April 1968, pp. 671-691.

10.7 Stevenson, J. and F. Moses, "Reliability Analysis of Frame Structures", *Journal of Struct. Div., ASCE,* Vol. 96, ST11, November 1970, pp. 2409-2427.

10.8 Ma, H.F. and A.H.S. Ang, "Reliability Analysis of Redundant Ductile Structural Systems", *Report UILEENG—81.2013*, University of Illinois, Aug. 1981.

10.9 Watwood, V.B., "Mechanism Generation for Limit Analysis of Frames", *Journal of Struct. Div., ASCE,* Vol. 105, ST1, Jan. 1979, pp. 1-15.

10.10 Murotsu, K., "Reliability Analysis of Frame Structure through Automatic Generation of Failure Modes", *Reliability Theory and its Structural and Soil Mechanics*, Ed. by P. Thoft Christensen, NATO, ASI Series, 1983, pp. 525-540.

10.11 Thoft Christensen, P. and Y. Murotsu, *Application of Structural Systems Reliability Theory*, Springer-Verlag, Berlin, 1986.

10.12 Moses, F., "System Reliability Developments in Structural Engineering", *Structural Safety*, No. 1, 1982, pp. 3-13.

10.13 Tang, L.K. and R.E. Melchers, "Reliability of Large Structural Systems", *Proc. of the Institution of Engineers*, Australia, Civil Engg. Transactions, 1985. pp. 136-143.

10.14 Ranganathan, R. and A.G. Deshpande, "Generation of Dominant Modes and Reliability Analysis of Frames", *Structural Safety*, No. 4, 1987, pp. 217-228.

10.15 Ticky, M. and M. Vorlicek, "Safety of Reinforced Concrete Framed Structures", *Proc. International Symp. on Flexural Mechanics of Reinforced Concrete*, Miami, ASCE, Nov. 1964, pp. 53-84.

10.16 Webster, F., "Probabilistic Analysis of a Simple Portal Structure", *Journal of American Concrete Institute*, Vol. 70, No. 9, Sept. 1973, pp. 649-651.

10.17 Chou, K.C., C. McIntosh and R.B. Corotis, "Observations on Structural System Reliability and the Role of Modal Correlations", *Structural Safety*, No. 1, 1983, pp. 189-198.

10.18 Ranganathan, R. and A.G. Deshpande, "Reliability Analysis of Reinforced Concrete Frames", *Journal of Struct. Div., ASCE*, Vol. 113, No. 6, June 1987, pp. 1315-1328.

10.19 Baker, A.L.L., "Limit State Design of Reinforced Concrete", *Cement and Concrete Association*, London, 1970.

10.20 Baker, A.L.L. and A.M.N. Amarakone, "Inelastic Hyperstatic Frame Analysis", *Proc. International Symp. on Flexural Mechanics of Reinforced Concrete*, Miami, ASCE, Nov. 1964, pp. 85-142.

10.21 IS: 456-1978, "Code of Practice for Plain and Reinforced Concrete", *Indian Standards Institutions*, New Delhi, 1980.

10.22 Baker, M.J. and T.A. Wyatt, "Methods of Reliability Analysis for Jacket Platforms", *Second International Conference on Behaviour of Offshore Structures*, London, August, 1979, pp. 499–520.

10.23 Manners, W. and M.J. Baker, "Reliability Analysis in Fatigue", *Second International Symposium on the Integrity of Offshore Structures*, Scotland, July 1981.

EXERCISE

10.1 Consider the structural system (5 member truss) shown in Fig. 10.31. It is given:

$$A_1 = 4.5 \text{ cm}^2 \quad A_2 = A_4 = 1.67 \text{ cm}^2$$
$$A_3 = 1.2 \text{ cm}^2 \quad A_5 = 4.5 \text{ cm}^2$$

Variable $f_y : \mu = 276 \text{ kN/m}^2 \quad \sigma = 27.6 \text{ kN/m}^2$

$Q_1\ Q_2 : \mu = 30 \text{ kN} \quad \sigma = 6 \text{ kN}$

$Q_3 : \mu = 50 \text{ kN} \quad \sigma = 15 \text{ kN}$

Assume all variables are independent and normal.

(i) Compute simple bounds on the reliability of the system (p_{f_s}).

(*Ans.* $0.0602 \leqslant p_{f_s} \leqslant 0.0896$)

(ii) Compute Ditlevsen's narrow bounds on p_{f_s}.

(*Ans.* $0.0732 \leqslant p_{f_s} \leqslant 0.0773$)

10.2 For the same problem given above, determine the narrow bounds on p_{f_s} if Q_1, Q_2, and Q_3 follow the Type 1 extremal (largest) distribution and f_y follows the lognormal distribution.

(*Ans.* $0.0223 \leqslant p_{f_s} \leqslant 0.0236$)

10.3 Consider the indeterminate truss shown in Fig. 10.32. It is given that for Variable:

R_1, R_{15}, R_{16}	$\mu = 77.6$ kN	$\sigma = 7.76$ kN
R_2	$\mu = 88.6$ kN	$\sigma = 8.86$ kN
R_3	$\mu = 50$ kN	$\sigma = 5$ kN
R_4	$\mu = 67.6$ kN	$\sigma = 6.76$ kN
R_5	$\mu = 78.6$ kN	$\sigma = 7.86$ kN
R_6	$\mu = 40$ kN	$\sigma = 4$ kN
R_7, R_8, R_9, R_{10}	$\mu = 75$ kN	$\sigma = 7.5$ kN
R_i ($i = 11$ to 14)	$\mu = 50$ kN	$\sigma = 5$ kN
Q_1, Q_2	$\mu = 50$ kN	$\sigma = 10$ kN
Q_3	$\mu = 20$ kN	$\sigma = 6$ kN

Assuming all variables are normally distributed and statistically independent, determine simple bounds on the p_{f_s} of the system.

(*Ans.* $0.00866 \leqslant p_{f_s} \leqslant 0.019$)

10.4 Consider the RCC frame, shown in Fig. 10.25, and given in Example 10.12. All data are the same as given in Table 10.10 except that for L_m, Mean/Nominal $= 1.38$ and $\delta = 0.25$. Generate dominant modes for the load combination $D + L_m + W_{apt}$ and determine

(i) The bounds on p_{f_s} assuming full redistribution.

(*Ans.* $0.726 \times 10^{-3} \leqslant p_{f_s} \leqslant 0.978 \times 10^{-3}$)

(ii) The bounds on p_{f_s} assuming limited ductility.

(*Ans.* $0.724 \times 10^{-3} \leqslant p_{f_s} \leqslant 0.818 \times 10^{-3}$)

10.5 The steel frame shown in Fig. 10.33 is taken from Reference 10.8. The data for the frame is given in Table E 10.5. Generate dominant modes and determine Ditlevsen's narrow bounds on the probability of failure of the system. Results are available in Reference 10.14.

(*Ans.* $0.227\times10^{-3} \leqslant p_{f_s} \leqslant 0.322\times10^{-3}$)

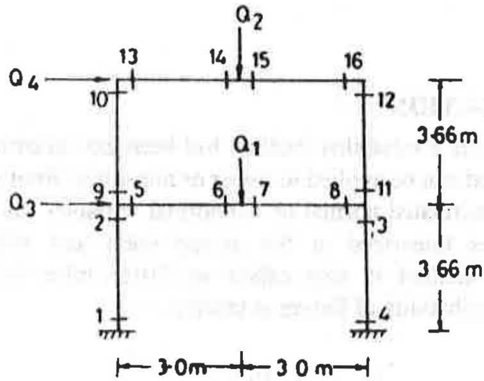


FIG. 10.5 Two-storey one bay frame

TABLE E 10.5 *Data for frame in Fig*. E *10.5*

Section/ variable	*EA* (kN)	*EI* (kN m²)	μ	δ	ρ
Section					
1, 2, 3, 4, 9, 10, 11, 12	0.105×10^7	0.84×10^4			
5, 6, 7, 8, 13, 14, 15, 16	0.168×10^7	0.336×10^5			
Variable					
M_1, M_2, M_3, M_4 $M_9, M_{10}, M_{11}, M_{12}$			110.0 kN m	0.15	1.0 } Independent
M_5, M_6, M_7 $M_8, M_{13}, M_{14}, M_{15}, M_{16}$			275.0	0.15	1.0
Q_1			180.0 kN	0.15	Loads are independent except $\rho Q_3, Q_4 = 1$
Q_2			90.0	0.25	
Q_3			32.0	0.25	
Q_4			16.0	0.25	

11

Advanced Reliability Methods

11.1 INTRODUCTION

In Chapter 7, Level 2 reliability method has been explained and illustrated in detail. The method can be applied to linear or non-linear limit state functions of correlated or uncorrelated normal or nonnormal variables. In this method, the failure surface is linearized at the design point and reliability index is calculated. The method is also called as First Order Reliability Method (FORM). Here, probability of failure is taken as

$$p_f = \Phi^{-1}(-\beta) \qquad (11.1)$$

given by Eq. 8.31. Only in the case of linear function of normal variables, the value of probability of failure estimated by the above equation gives the exact value. In other cases, it gives only approximate value called as notional value of probability of failure. In general, the probability of failure estimated by Eq. 11.1 is sufficiently accurate and holds good for the majority of complex engineering problems with number of variables as long as the probabilty of failure is not too small and the distributions of the variables do not deviate too far from the normal distribution. This estimate of probability of failure is enough and quite adequate for decision making problems in the field viz. fixing partial safety factors, calibrating codes, development of inspection strategy and maintenance schedule etc. The estimated p_f by Eq. 11.1 gives significant error when the failure surface has large curvature and highly nonlinear and the function is in terms of correlated nonnormal variables. In such cases, when one is interested in estimating more accurate value of p_f, he may have to use Second Order Reliability Methods (SORM). Basic Monte Carlo technique explained in Chapter 7 gives true value of p_f; however, it takes more time and large number of samples are to be generated to estimate p_f with a certain minimum confidence level in the estimated p_f. Better sampling methods, which are called here as advanced simulation methods, are available to estimate p_f without much statistical error. In this chapter, the principle behind second order reliability method is just introduced and advanced simulation methods are explained in detail and illustrated with examples.

11.2 SECOND ORDER RELIABILITY METHOD

The first order reliability methods are easy and simple to apply but approximation used to linearise the failure surface at design point does not always hold good. When the failure surface is very non-linear, the estimated reliability index shows an erroneous p_f value. The figure 11.1 brings out the drawback of FORM. In Fig. 11.1 two failure surfaces are shown. Surface B is more non-linear than the surface A. It can be easily seen that the probability

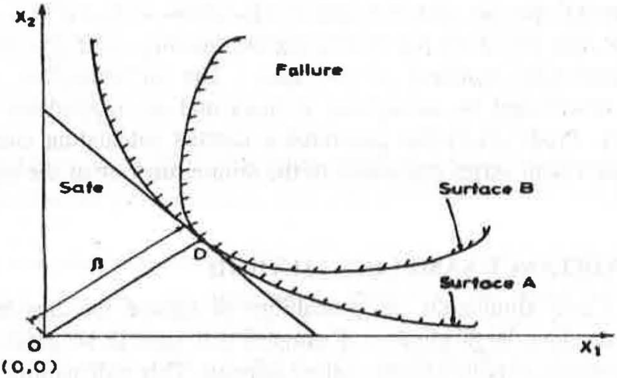


FIG. 11.1 Drawback of FORM

of failure of B is less than that of the surface A. But using the Hasofer-Lind method, the values of reliability index β evaluated for both surfaces for linearization at design point D are the same. This shows that not only the distance of a design point D from the origin in the independent standardized co-ordinate system but also the nature of the failure surface affects the failure probability. Thus it becomes essential to take into account the nature of the failure surface while evaluating the probability of failure in problems involving non-linear surfaces. It is drawn to the attention of the readers that if the original distributions of the variables significantly deviate from the normal

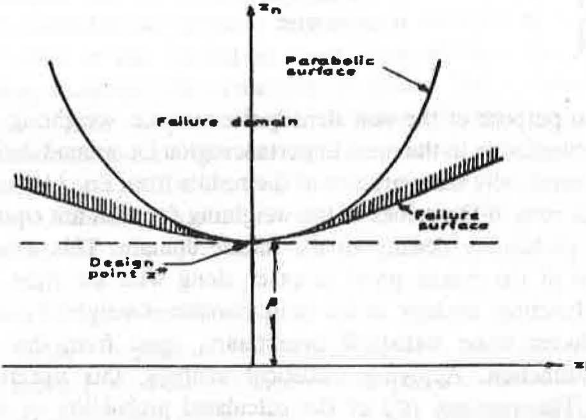


FIG. 11.2 Parabolic approximation to failure domain

distribuiton, original smooth surface (even if the original equation is linear) can become distinctively curved in the normalized space. But it is particularly difficult to find the exact nature of the surface every time. The second order approach in the space of normalized variables will yield results close to the exact value. In the second order reliability method the failure surface in the standard normal space is approximated by a parabolic surface (Fig. 11.2) at the design point, the axis of the parobola being the direction of **z*** (the design point in the independent standard normal space) The corresponding probability content is determined by asymptotic formula and by approximate formulae (11.1, 11.2). Tvedt (11.3) has presented a method calculating from the full second order Taylor series expansion of the failure function at the design point **z*** .

11.3 IMPORTANCE SAMPLING METHOD

In Monte Carlo simulation, as probability of failure for any structure is generally very low, large number of samples will have to be generated to get sufficient number of points in the failure domain. This will require evaluation of structural response for large number of times which affects the efficiency of the method. This drawback is overcome by replacing the joint density function $f_X(\mathbf{x})$ by new sampling function $h_X(\mathbf{x})$ which ensures the sampling in the region which contribute most to the probability of failure. The probability of failure is given by,

$$p_f = \frac{\sum_{i=1}^{N} I\left\{ g(x) \le 0 \right\} \dfrac{f_X(x_i)}{h_X(x_i)}}{N_s} \tag{11.2}$$

where

N_s is the number of simulations and

I { } is an indicator function given by

$$I\{\ \} = \begin{cases} 1 & \text{for } g(x) \le 0 \\ 0 & \text{otherwise} \end{cases}$$

The main purpose of the new density function i.e. weighting function is to centre the simulation in the most important region i.e. around design point. It is possible theoretically that variance of the results from Eq. 11.2 can be reduced to as low as zero, if the values of the weighting function are equal to values of the actual probability density in the failure domain. This assumes that the information of the design point is exact along with the right choice of the weighting function. Because of the finite number of weighted simulation, there will be always some statistical uncertainty, apart from due to choice of weighting function. Applying statistical analysis, this uncertainty can be estimated. The variance (s^2) of the calculated probability of failure can be estimated as,

$$s^2 = \frac{1}{N_s(N_s-1)}\left\{\sum_{i=1}^{N_s}\left[I\{g(x)\le 0\}\frac{f_X(x_i)}{h_X(x_i)}-p_f\right]^2\right\} \tag{11.3}$$

The so called standard error (or statistical error) in the estimate of p_f is given by,

$$e_{pf} = \frac{\sqrt{s^2}}{p_f} \tag{11.4}$$

From the above equation it is clear that statistical error is not only dependent upon the number of simulations but also type of weighting function $h_X(x)$.

Choice of weighting function

It is obvious from Eq. 11.4 that, whole success of the importance sampling approach depends upon the choice of the weighting function $h_X(x)$. Several suggestions have been made for the choice of $h_X(x)$ in the importance sampling. Harbitz (11.4) suggested the weighting function as the same original joint density function but only shifted at the design point, which is calculated by Level 2 method. But the question comes, as once the design point by Level 2 is known, why to go for further analysis, unless some improvement in accuracy is needed. Also original distribution may be complex when variables are correlated, which causes difficulty in sampling process.

Another choice for weighting function $h_X(x)$ is to use independent standard multinormal density function, centred at the design point and standard deviation equal to or greater than the original standard deviation (11.5, 11.6, 11.7). Design point can be calculated based on the assumption of uncorrelated Gaussian variables or uncorrelated with original distributions. Generally $h_X(x)$ is taken as independent n-dimensional multinormal density function, centred at the design point calculated on the assumption of uncorrelated Gaussian variables. The standard deviation is taken as, one to three times of the original standard deviation. As this choice for $h_X(x)$ will produce the sample points unbiased with respect to all variables, it will cover the wide region around the design point. Due to this advantage exact form of limit state g(x) is not necessary while evaluating the probability of failure. Due to simplicity of the $h_X(x)$ generation of the sample points can be done very efficiently. As $h_X(x)$ is the independent multinormal density function, unless the failure surface is highly nonlinear, there will be 50 % probability that sample point falls in the failure domain. Random deviates for the normal distribution are generated using Box and Muller technique explained in Chapter 7.

Correlated nonnormal variables

In Level 2 method explained in Chapter 8, the treatment of correlated nonnormal variables has been explained when the covariance matrix $[C_X]$ is known. If the correlation matrix $[\rho_X]$ is given, the procedure is slightly modified and is explained below. It can be easily proved that the correlation matrix of original variates becomes covariance matrix of reduced variates. Here reduced variate Z_i means,

$$Z_i = \frac{X_i - \mu_i}{\sigma_i} \tag{11.5}$$

where μ_i and σ_i are mean value and standard deviation of X_i. In the case of correlated non-normal variables, the original probability density $f_X(\mathbf{x})$ is found at the sample point in consideration by transforming them into equivalent independent Gaussian components. This is done by first transforming them (nonnormal variables) into equivalent normal at the sample point by using the procedure explained in Chapter 8. The Gaussian components obtained are then transformed into independent components by orthogonal transformation. For the correlation matrix $[\rho_X]$ the eigen values are evaluated from which eigen vectors are found out for each eigen value. Then the transformation matrix [T] will be the matrix with each column as eigen vector for respective eigen value. The independent standard normal variates $Y_1, Y_2, \ldots\ldots Y_n$ will be given by,

$$\mathbf{Y} = [T]^t\,\mathbf{Z} \tag{11.6}$$

$$E[Y] = [T]^t\,E[Z] \tag{11.7}$$

$$[C_Y] = [T]^t\,[\rho_X]\,[T] \tag{11.8}$$

That is eigen values of $[\rho_X]$ are the variances of the respective variates Y_i. Though this transformation is approximate, it can be applied very efficiently and gives results within good approximation.

Following steps are involved in the computation of p_f using ISM when statistics of all variables, the correlation matrix and the limit state function are given:

1. An eigen value analysis of the correlation matrix is carried out to find the transformation matrix [T]. Each column of the transformation matrix is an eigen vetor corresponding to the respective eigen value (Refer Chapter 7).
2. Find the design point $\mathbf{x}^*$ using Level 2 method. For simplicity, assuming all variables as uncorrelated normal variables, $\mathbf{x}^*$ can be found out and this may be used as a sampling point.
3. Two uniform random numbers v_1 and v_2 are generated between 0 and 1 for each variable.
4. A standard normal variate u for each variable is obtained as
$$u = [2\ln(1/v_1)]^{1/2}\cos(2\pi v_2)$$
5. Select a value for standard deviation multiplier, S_{dm}, from 1 to 3. A sample point $\mathbf{x}$ is obtained as
$$\mathbf{x} = \mathbf{x}^* + S_{dm}\,\mathbf{U}\,\sigma$$
6. The value of limit state function $g(\mathbf{x})$ is evaluated.
7. If $g(\mathbf{x}) < 0$ proceed ; otherwise go to Step 3.

8. The equivalent mean and standard deviation at point x are found out as explained in Chapter 8. They are given by Eqs. 8.67 and 8.69.

$$\mu'_{X_i} = -\sigma'_{X_i}\Phi^{-1}\left[F_{X_i}(x_i)\right] + X_i \tag{11.9}$$

$$\sigma'_{X_i} = \frac{\phi\left[\Phi^{-1}\left\{F_{X_i}(x_i)\right\}\right]}{f_{X_i}(x_i)} \tag{11.10}$$

9. The equivalent normal variables Z are found at the point X as

$$Z_i = \frac{X_i - \mu'_{X_i}}{\sigma'_{X_i}} \tag{11.11}$$

10. The independent variables Y are found at point X as

$$\mathbf{Y} = [\mathbf{T}]^t\,\mathbf{Z} \tag{11.12}$$

11. The probability density and the sampling density at X are found out as

$$f_X = \frac{1}{\left(\sqrt{2\pi}\right)^n}\prod_{i=1}^{n}\frac{1}{\sigma_{Y_i}}\exp\sum_{i=1}^{n}\left(-\frac{1}{2}\frac{Y_i^2}{c_{Y_i}}\right) \tag{11.13}$$

$$h_X = \frac{1}{\left(\sqrt{2\pi}\right)^n}\prod_{i=1}^{n}\frac{1}{\sigma_{X_i}}\exp\sum_{i=1}^{n}-\frac{1}{2}\left(\frac{X_i - X_i^*}{\sigma_{X_i}}\right)^2 \tag{11.14}$$

Here h_X is independent multinormal density function at X

12. Calculate f_X / h_X. Go to Step 3.
The whole process is repeated from Step 3 to Step 12 for number of required simulations N_s.

13. Compute p_f using Eq. 11.2 and e_{pf} using Eq. 11.4.

The procedure explained above is shown in the flow chart given in Fig. 11.3. The importance sampling method is illustrated with the following examples.

EXAMPLE 11.1 The limit state function is given by

$$g(X) = X_1 X_2 - X_3$$

Here the number of variables is 3. The statistics of the variables are given in Table 11.1. The correlation matrix is follows:

$$[\rho_x] = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

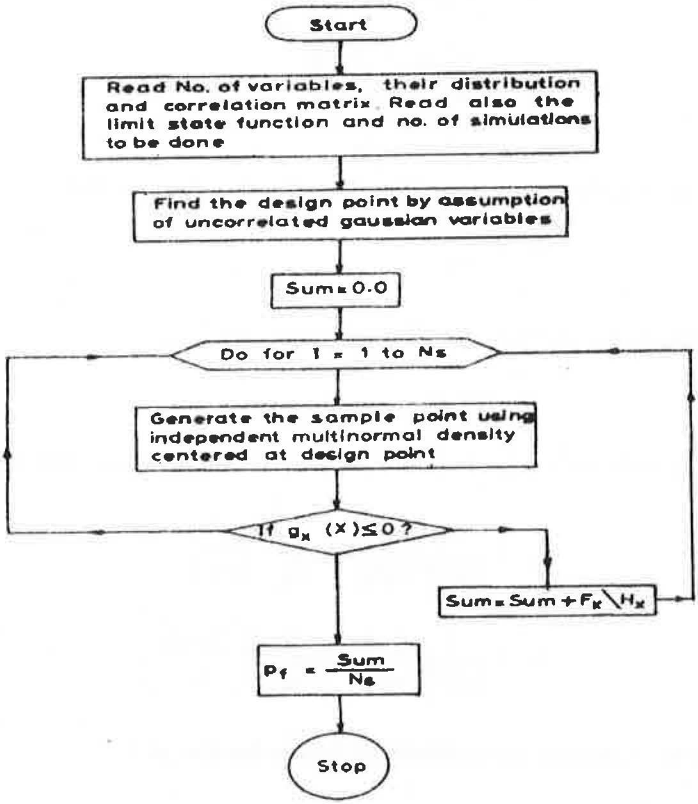


FIG. 11.3 Flow Chart for importance sampling

TABLE 11.1 *Statistics of variables - Example 11.1*

Variable	Mean	Standard deviation	Distribution
X_1	40.0	5.0	Type I extremal (largest)
X_2	50.0	2.5	Normal
X_3	1500.0	100.0	Lognormal

Compute p_f by using ISM taking SDM equal to 1.

Detailed stepwise calculation for the computation of p_f is given below.

Step 1: For the given correlation matrix $[\rho_X]$ the eigen value analysis is carried out.

The eigen values of the correlation matrix are $\lambda_1 = 1.5$; $\lambda_2 = 0.5$; $\lambda_3 = 1.0$. The transformation matrix is obtained as

$$[T] = \begin{bmatrix} 0.707 & -0.707 & 0.000 \\ 0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

Step 2: Using Level 2 method, the design point $\mathbf{x}^*$ and corresponding β are obtained.

$$\mathbf{x}^* = \begin{Bmatrix} 33.029 \\ 47.64 \\ 1623.47 \end{Bmatrix} \quad ; \quad \beta = 2.0868$$

Step 3: The two uniform random numbers v_1 and v_2 generated for first variable, are

$$v_1 = 0.8704 \; ; \qquad v_2 = 0.3995$$

Step 4: The normal variate is

$$u_1 = \left[2 \ln\left(\frac{1}{v_1}\right)\right]^{1/2} \cos(2\pi v_2)$$
$$= -0.4254$$

Step 5: The sample point x_1 is given by

$$x_1 = x_1^* + S_{dm}\, u_1\, \sigma_{X_1}$$
$$= 33.029 + (1.0)(-0.4254)\, 5$$
$$= 30.901$$

Similarly random numbers are generated for random variables X_2 and X_3 and the values for the other two variables are evaluated as

$$x_2 = 49.6598 \; ; \qquad x_3 = 1582.928$$

Step 6: The value of the limit state function is

$$g(\mathbf{x}) = x_1\, x_2 - x_3$$
$$= (30.9018)\,(49.6598) - 1582.928$$
$$= -48.350$$

Step 7: Check $g(\mathbf{x})$. Since $g(\mathbf{x})$ is negative, it is proceeded further.

Step 8: The equivalent mean and standard deviation of variables at the sample point are calculated. The parameters α and u of X_1 following Type 1 extremal (largest) distribution, are determined as follows using Eqs. 3.115 and 3.116.

$$\alpha = \frac{1}{\sqrt{6}}\frac{\pi}{\sigma_{X_1}}$$

$$= \frac{1}{\sqrt{6}}\frac{\pi}{5.0} \qquad = 0.2565$$

$$u = \mu_{X_1} - \frac{0.5772}{\alpha}$$

$$= 40.0 - \frac{0.5772}{0.2565} = 37.75$$

The probability density function of random variable X_1, following Type 1 extremal (largest) distribution is given by Eq. 3.113.

$$f_{X_1} = \alpha \exp[-\alpha\,(x_1 - u) - \exp\{-\alpha\,(x_1 - u)\}]$$

The value of f_{X_1} at $x = x_1$ is

$$f_{X_1} = 0.2565 \exp[-0.2565(30.9018\text{-}37.75)\text{-}\exp\{-0.2565(30.9018\text{-}37.75)\}]$$
$$= 0.004532$$

The cumulative distribution function of X_1 is given by Eq. 3.114.

$$F_{X_1} = \exp[-\exp\{-\alpha\,(x_1 - u)\}]$$

The value of F_{X_1} at $x = x_1$ is

$$F_{X_1} = \exp[-\exp\{-0.2565\,(30.9018\text{-}37.75)\}]$$
$$= 0.00305$$

Using Eqs. 11.9 and 11.10, mean and standard deviation of equivalent normal at x_1 are obtained. They are calculated as

$$\sigma'_{X_1} = \frac{\phi\{\Phi^{-1}(F_{X_1})\}}{f_{X_1}}$$

$$= \frac{\phi\{\Phi^{-1}(0.00305)\}}{0.004532} \quad = \quad 2.0012$$

$$\mu'_{X_1} = -\sigma_{X_1}\Phi^{-1}[F_{X_1}(\,)] + x_1$$
$$= (-2.0012)\,\Phi^{-1}\,[0.00305] + 30.9018 \quad = \quad 36.3942$$

Similar procedure is followed to evaluate the equivalent mean and standard deviation of variables X_2 and X_3. They are given as

$$\mu'_{X_2} = 50.0 \quad ; \quad \sigma'_{X_2} = 2.5$$

$$\mu'_{X_3} = 1494.239 \quad ; \quad \sigma'_{X_3} = 105.412$$

Step 9: The equivalent normal variables at the sample point in the normalised co-ordinate system are obtained using Eq. 11.11

$$Z_1 = \frac{X_1 - \mu'_{X_1}}{\sigma'_{X_1}} = \frac{30.9018 - 36.3942}{2.0012} = -2.7445$$

$$Z_2 = \frac{X_2 - \mu'_{X_2}}{\sigma'_{X_2}} = \frac{49.6598 - 50.0}{2.5} = -0.1361$$

$$Z_3 = \frac{X_3 - \mu'_{X_3}}{\sigma'_{X_3}} = \frac{1582.928 - 1494.239}{105.412} = 0.8414$$

Step 10: Using the transformation matrix [T] the variables are converted into independent variables using Eq. 11.12.

$$\mathbf{Y} = \begin{bmatrix} 0.707 & 0.707 & 0.000 \\ -0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{Bmatrix} -2.7445 \\ -0.1369 \\ 0.8414 \end{Bmatrix} = \begin{Bmatrix} -2.0369 \\ 1.8444 \\ 0.8414 \end{Bmatrix}$$

Variance of Y_i is given by eigen value λ_i. Hence

$$\sigma_{Y_1} = \sqrt{1.5} = 1.225 \quad ; \quad \sigma_{Y_2} = \sqrt{0.5} = 0.707$$

$$\sigma_{Y_3} = \sqrt{1.0} = 1.0$$

Step 11: Using Eqs. 11.13 and 11.14 the probability density f_X and the sampling density h_X are computed as follows:

$$f_X = \frac{1}{\left(\sqrt{2\pi}\right)^3} \frac{1}{(2.001x1.225)(2.5x0.707)(105.412x1)} x$$

$$\exp\left\{-\frac{1}{2}\left[\left(\frac{-2.0369}{1.225}\right)^2 + \left(\frac{1.8444}{0.707}\right)^2 + \left(\frac{0.8414}{1.0}\right)^2\right]\right\}$$

$$= 8.14 \times 10^{-7}$$

$$h_X = \frac{1}{\left(\sqrt{2\pi}\right)^3} \frac{1}{(5.0)(2.5)(100.0)} x$$

$$\exp\left\{-\frac{1}{2}\left[\left(\frac{30.901 - 33.029}{5.0}\right)^2 + \left(\frac{49.6958 - 47.64}{2.5}\right)^2 + \left(\frac{1582.928 - 1623.47}{100.0}\right)^2\right]\right\}$$

$$= 3.084 \times 10^{-5}$$

The ratio of f_X and h_X is calculated and stored. The process is repeated for the specified number of simulations. The value of p_f is computed using Eq. 11.2 and the statistical error using Eq. 11.4. The results obtained for different values of specified number of simulations are given below.

Sl. No.	N_s	p_f	e_{pf} (%)
1.	500	0.0291	10.567
2.	1000	0.032	7.178
3.	1500	0.0328	5.950

The exact value of p_f, by Monte Carlo method is 0.032. It can be observed that as number of simulation increases, the accuracy of p_f also increases and percentage error decreases. It should be noted that the set of random numbers obtained for different starting points will be different. Hence for the same number of simulations, the value of p_f obtained will not be exactly same.

EXAMPLE 11.4 The limit state function is given as

$$g(\mathbf{X}) = -\frac{1}{8}\left(X_1^2+X_2^2+X_3^2\right)-X_4+4.0$$

Variables X_i are normally distributed with mean and standard deviation of each variable are 0 and 1 respectively. That is, they are standard normal variables. The variables are uncorrelated. Determine the probability of failure by using ISM.

The starting point is the design point obtained by Level 2 method. That is

$$\mathbf{x}^* = \begin{Bmatrix} 0.250 \\ 0.250 \\ 0.250 \\ 3.997 \end{Bmatrix}$$

The procedure of computation of p_f is same as explained in the previous example. It is to be noted that since the given variables are uncorrelated standard normal variables,

$$\mathbf{y} = \mathbf{z} = \mathbf{x}$$

where $\mathbf{x}$ is the sampling point obtained by generating random numbers and using standard deviation multiplier. All the intermediate steps in the computation for SDM = 1 for the first simulation are given in Table 11.2. The whole process is repeated for number of simulations and the values of p_f and e_{pf} are computed using Eqs. 11.2 and 11.4 respectively.

TABLE 11.2 *Results of the analysis in the first simulation using ISM - Example 11.2*

Initial point	Random numbers	$\mathbf{x}$	$g(\mathbf{x})$	$\mathbf{y} = \mathbf{z} = \mathbf{x}$	F_X	h_X
$\mathbf{x} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 3.997 \end{Bmatrix}$	$\begin{Bmatrix} 0.5134 \\ 0.3206 \end{Bmatrix}$ $\begin{Bmatrix} 0.6311 \\ 0.2595 \end{Bmatrix}$ $\begin{Bmatrix} 0.5278 \\ 0.4401 \end{Bmatrix}$ $\begin{Bmatrix} 0.7832 \\ 0.9289 \end{Bmatrix}$	$\begin{Bmatrix} -0.2456 \\ 0.1930 \\ -0.8012 \\ 4.6072 \end{Bmatrix}$	-0.6996	$\begin{Bmatrix} -0.2456 \\ 0.1930 \\ -0.8012 \\ 4.6072 \end{Bmatrix}$	0.431×10^{-6}	0.1056×10^{-1}

The problem is solved for various values of number of simulations and SDM. The results are given in Table 11.3. It can be observed from the table, that the statistical error decreases for SDM equal to 2 at which the results are very consistent in successive runs. The corresponding value of p_f agrees with the exact value of 0.423×10^{-3}

TABLE 11.3 *Results obtained by ISM - Example 11.4*

SDM	Number of simulations					
	1000		1500		2000	
	p_f	e_{pf} (%)	p_f	e_{pf} (%)	p_f	e_{pf} (%)
1.0	0.326×10^{-3}	23.77	0.315×10^{-3}	17.54	0.303×10^{-3}	14.76
1.5	0.412×10^{-3}	10.90	0.422×10^{-3}	8.74	0.435×10^{-3}	7.68
2.0	0.423×10^{-3}	9.59	0.417×10^{-3}	7.87	0.409×10^{-3}	6.97

11.4 ADAPTIVE SAMPLING METHOD

The main limitation to the importance sampling method is the difficulty in selecting a good sampling density. To choose such a density one needs to know which part of the failure domain has a relatively high probability density. This knowledge is not usually available priori and hence it is difficult to choose a good sampling density. Adaptive sampling method (ASM) can be used to overcome this difficulty. This technique utilises the fact that even with a poor initial choice of importance sampling density, the knowledge about the failure domain increases with the sampling process. Hence after each sample the importance sampling density can be modified for this increased knowledge and finally a good sampling density can be obtained.

It is already said that the much prior knowledge about the important region or the region where probability density is relatively high is not available. This may result into the poor initial choice for the sampling density. If such a poor density is used, the sample points generated may lie in the region where probability density is relatively low. Thus the sample points are clustered around an unimportant region. However, while sampling with such a poor density some sampling points may have relatively more probability density than that of the chosen point. Thus while sampling, the knowledge about the important region increases i.e. the region of relatively more probability density is known. Adaptive sampling technique makes use of this knowledge to move towards the more useful density. For this, the sample point having more probability density is chosen as the new centre of the sampling density. Thus the sampling density is moved towards the more important region. Figures 11.4 and 11.5 show the poor and improved choice of sampling density respectively. The steps involved in the procedure of computation of p_f using ASM are almost same as given for ISM except the following changes.

In Step 2, any point $\mathbf{x}_c$ can be chosen as the starting point for mean of sampling density. At this point, the original variables are converted into independent normal variables Y_c. The probability density at point x_c is calculated as independent multinormal density and is taken as f_{max}

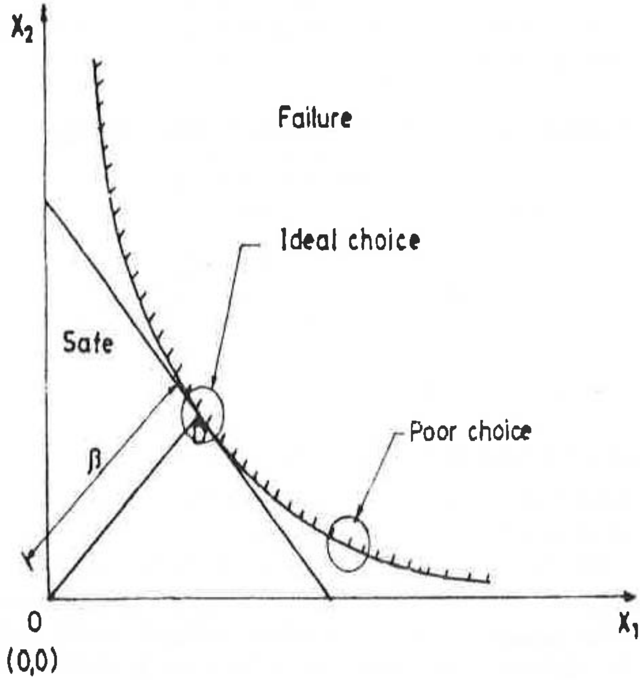


FIG. 11.4 Poor Choice of sampling density

All the steps from 3 to 11 are same.

In Step 12, after calculating f_X / h_X check whether $f_X > f_{max}$

If $f_X > f_{max}$, shift the point $\mathbf{x}_c$ to $\mathbf{x}$ and go to Step 3 and repeat the whole process. Otherwise go to Step 3 directly and repeat the process for number of simulations.

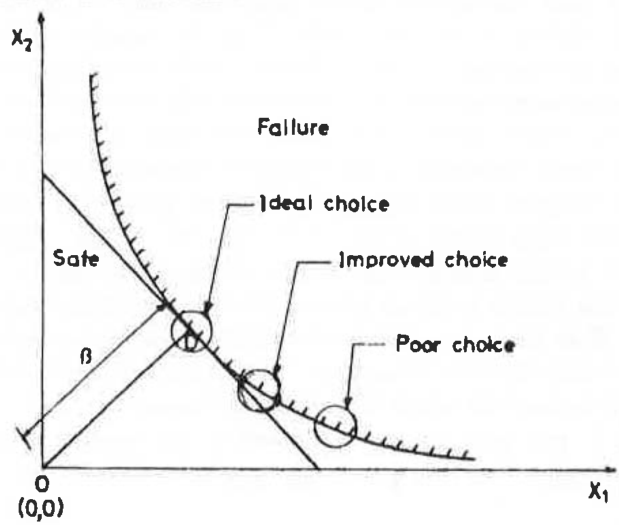


FIG. 11.5 Improved choice of sampling density

Step 13: Compute p_f using Eq. 11.2 and e_{pf} by using Eq. 11.4.
The whole procedure is shown in the flowchart given in Fig. 11.6. The procedure of computation of p_f using ASM is illustrated with the following examples.

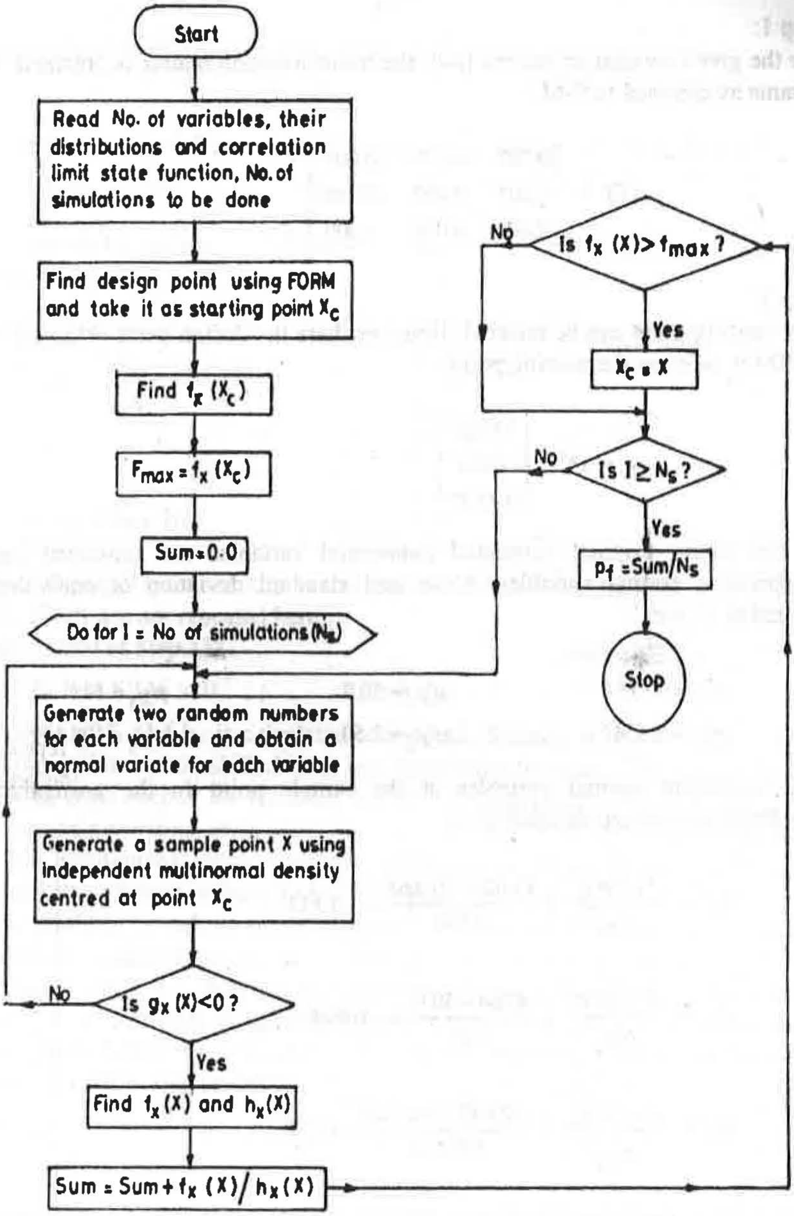


FIG. 11.6 Flow chart for ASM

EXAMPLE 11.3 The same problem given in Example 11.1 is considered here.

$$g(X) = X_1 X_2 - X_3$$

Statistics of the variables are same as given in Example 11.1. Determine p_f using ASM.

Step 1:
For the given correlation matrix $[\rho_X]$, the transformation matrix is obtained. it is same as obtained in ISM.

$$[T] = \begin{bmatrix} 0.707 & -0.707 & 0.000 \\ 0.7u7 & 0.707 & 0.0000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

Step 2:
Any starting point can be selected. However, here the design point obtained by FORM is taken as the starting point.

$$\mathbf{x}_c = \mathbf{x}^* = \begin{Bmatrix} 33.029 \\ 47.64 \\ 1623.47 \end{Bmatrix}$$

At this point, original correlated nonnormal variables are converted into independent normal variables. Mean and standard deviation of equivalent normal at x^* are

$$\mu'_{X_1} = 37.665 \quad ; \quad \mu'_{X_2} = 50.0 \quad ; \quad \mu'_{X_3} = 1491.45$$

$$\sigma'_{X_1} = 2.554 \quad ; \quad \sigma'_{X_2} = 2.50 \quad ; \quad \sigma'_{X_3} = 108.112$$

The equivalent normal variables at the sample point in the normalized coordinate system are obtained as

$$Z_1 = \frac{X_1 - \mu'_{X_1}}{\sigma'_{X_1}} = \frac{33.029 - 37.665}{2.554} = -1.815$$

$$Z_2 = \frac{X_2 - \mu'_{X_2}}{\sigma'_{X_2}} = \frac{47.64 - 50.0}{2.5} = -0.944$$

$$Z_3 = \frac{X_3 - \mu'_{X_3}}{\sigma'_{X_3}} = \frac{1623.47 - 1491.45}{108.112} = 1.221$$

Using the transformation matrix, the variables are converted into independent variables using Eq. 11.12.

$$\mathbf{Y} = \begin{bmatrix} 0.707 & 0.707 & 0.000 \\ -0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{Bmatrix} -1.815 \\ -0.944 \\ 1.221 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.951 \\ 0.616 \\ 1.221 \end{Bmatrix}$$

Standard deviation of **Y** are square roots of eigen values of $[\rho_X]$. They are

$\sigma_{Y_1} = 1.225$; $\sigma_{Y_2} = 0.707$; $\sigma_{Y_3} = 1.0$

The probability density is computed at point $\mathbf{x}^*$ as independent multinormal density and is taken as

$$f_{max} = \frac{1}{\left(\sqrt{2\pi}\right)^3} \frac{1}{(2.554x1.225)(2.5x0.707)(108.112x1.0)} x$$

$$\exp\left\{-\frac{1}{2}\left[\left(\frac{-1.951}{1.225}\right)^2 + \left(\frac{0.616}{0.707}\right)^2 + \left(\frac{1.221}{1.0}\right)^2\right]\right\}$$

$$= 0.9698 \times 10^{-5}$$

Step 3 to Step 11 are same as in the previous case solving by ISM in Example 11.1. (They are not repeated here).

At the end of Step 11,

$f_X = 8.146 \times 10^{-7}$; $h_X = 3.084 \times 10^{-5}$

The ratio of f_X and h_X is calculated and stored. Since f_X is less than f_{max}, the starting point is not shifted and one simulation is over. The procedure is repeated from Step 3 for specified number of simulations. The probability of failure and percentage error are calculated using Eqs. 11.2 and 11.4. At the end of 500 simulations using standard deviation multiplier equal to one, the probability of failure is found to be 0.039 with $e_{pf} = 11.4$ %.

The problem is solved using different standard deviation multipliers. The simulation is carried out for 500, 1000 and 2000 number of simulations with different "SEED" i.e. different starting points for generating random numbers. (Note: In all available programmes for generating random numbers, starting point, called SEED, is to be given). The results obtained by ASM are given in Table 11.4. The exact probability of

TABLE 11.4 *Results obtained by ASM - Example 11.3*

SDM	SEED	500 p_f	500 S_e (%)	1000 p_f	1000 S_e (%)	2000 p_f	2000 S_e (%)
	1	0.0390	11.40	0.0355	7.59	0.0340	5.23
1.0	2	0.0335	12.58	0.0330	7.76	0.0328	5.01
	3	0.0300	12.69	0.0320	7.88	0.0350	4.99
	1	1.0373	14.19	0.0345	9.30	0.0326	6.38
1.25	2	0.0310	15.42	0.0316	9.72	0.0320	6.39
	3	0.0280	14.43	0.0312	9.15	0.0340	6.06

Note: The exact probability of failure is 0.032 as per Monte Carlo Method

failure is found to be 0.032. From Table 11.4, it is observed that the statistical error is found to be decreasing with the increasing number of simulations. For 2000 simulations the statistical error is very low and the results are very close to the exact value.

EXAMPLE 11.4 Consider the same limit state function given below (11.4).

$$g(\mathbf{X}) = X_2X_3X_4 - \frac{X_5 X_3^2 X_4^2}{X_6 X_7} - X_1$$

All the variables are normally distributed and uncorrelated. The statistics of the variables are given in Table 11.5. Compute p_f using ASM.

TABLE 11.5 *Statistics of variables - Example 11.4*

Variable	Mean	Standard deviation	Distribution
X_1	0.01	0.003	Normal
X_2	0.30	0.015	Normal
X_3	360.0	36.0	Normal
X_4	2.26×10^{-4}	1.13×10^{-5}	Normal
X_5	0.50	0.05	Normal
X_6	0.12	0.006	Normal
X_7	40.0	6.0	Normal

The results obtained by using ASM for different starting points, different seeds, and for different number of simulations are given in Tables 11.6 and 11.7. From the tables, it can be seen that certain minimum number of simulations are required to get probability of failure close to the exact value. In general, as number of simulations increases, statistical error in estimated p_f decreases. It should be noted also, that the value of standard deviation multiplier plays a role. For this problem, it appears that percentage error is very less for SDM =

1.25. It can also be noted that the selection of starting point affects the result in this case. The choice of seed is not affecting the result significantly when large

number of simulations is used. The probability of failure is found to be very close to the exact value 0.34×10^{-3} .

TABLE 11.6 *Results by ASM with starting point 1 - Example 11.4*

SDM	SEED	Starting point 2 : $\mathbf{x}_c$= $(0.015, 0.25, 300.0, 2.26 \times 10^{-4}, 0.5, 0.12, 40.0)^t$ Number of simulations							
		500		1000		1500		2000	
		p_f $\times 10^{-4}$	e_{pf} (%)	p_f $\times 10^{-4}$	e_{pf} (%)	p_f $\times 10^{-4}$	e_{pf} (%)	p_f $\times 10^{-4}$	E_{pf} (%)
	1	3.92	41.84	4.27	25.75	3.74	21.06	3.60	17.00
1.25	2	8.00	30.38	5.21	23.81	4.58	18.35	4.10	15.57
	3	4.04	30.74	3.28	19.88	3.38	13.85	3.30	11.43
	1	2.35	38.16	4.13	27.18	3.91	20.21	3.37	16.59
1.5	2	3.33	57.08	2.50	23.53	3.35	20.65	3.31	16.72
	3	2.77	42.72	3.80	18.60	3.85	15.16	3.60	13.56
	1	1.95	78.76	1.44	34.08	2.66	31.63	2.70	26.83
2.0	2	2.20	40.46	3.12	35.00	3.21	26.48	2.65	24.20
	3	1.30	39.73	2.56	28.73	2.57	25.53	2.60	23.45

General points

- The value of SDM is generally found to vary between 1 and 2.
- While simulating using ASM or ISM, a suitable value of SDM is chosen by performing a few number of simulations with different SDM values between 1 and 2 and the best one is to be selected and it is the one with the least statistical error. The same value is to be used for calculating p_f by conducting enough number of simulations.
- Generally, while solving problems with ASM and ISM, it is suggested that the termination criterion to stop the simulation, may be where statistical error in computing p_f is less than 20 percent.
- The starting point affects the convergence of p_f value. For a good starting point, the value of p_f converges to the exact value with less number of simulations and less statistical error, as compared to the poor starting point. The design point obtained by Level 2 method is a good starting point.
- Generally ASM requires less number of simulations to evaluate p_f value by maintaining the same statistical error as that of the ISM.
- For any arbitrary starting point, ASM is preferable as it is requires less number of simulations.

TABLE 11.7 *Results by ASM with starting point 2 - Example 11.4*

SDM	SEED	Starting point 2 : $\mathbf{x}_c = (0.015, 0.25, 300.0, 2.26 \times 10^{-4}, 0.5, 0.12, 40.0)^t$ Number of simulations							
		500		1000		1500		2000	
		$p_f \times 10^{-4}$	e_{pf} (%)	$p_f \times 10^{-4}$	e_{pf} (%)	$p_f \times 10^{-4}$	e_{pf} (%)	$p_f \times 10^{-4}$	e_{pf} (%)
	1	3.86	20.53	3.91	14.28	3.82	11.06	3.75	9.11
1.25	2	3.09	15.74	3.20	13.00	2.91	10.51	3.03	10.09
	3	2.27	18.46	2.54	12.27	2.80	10.53	3.02	10.84
	1	2.42	47.61	3.23	24.13	3.55	17.44	3.47	16.95
1.5	2	2.09	32.31	2.87	24.56	2.67	18.75	2.55	15.38
	3	1.36	26.84	2.90	24.22	3.45	17.77	3.29	14.84
	1	1.11	58.96	1.85	26.23	1.80	21.77	2.25	25.46
2.0	2	8.80	45.99	7.27	31.93	6.00	28.31	5.63	24.45
	3	4.40	67.06	3.75	43.08	4.18	35.20	3.96	29.49

11.5 RESPONSE SURFACE METHOD

Advanced Monte Carlo simulation methods are exact and computationally efficient from probabilistic point of view. While evaluating the structural reliability, the maximum time is spent for evaluating structural response only. Since simulation methods are numerical experiments carried out randomly, they require the full analysis of the structural system for each generated set of load, resistance and system random variables. This may result in large computational efforts to an unacceptable level. Hence it is desirable to simplify the whole mechanical process by a new mechanical model for evaluating the structure/system response. While developing the new model, it is important that it will allow an easy and efficient computation of failure function response under loading/ system conditions but still preserves the essential features of the structure/ system. This new mechanical model representing the original limit state function is called response surface.

The representation of the limit state function by response surface should be independent of properties of the basic variables involved. However for improving efficiency and accuracy of the method including subsequent reliability analysis, some prior knowledge of the stochastic properties of the variables is to be used. In most of the cases, mean value and standard deviation of variables are known. Use of such information will produce response surface suitable for wide range of stochastic properties of basic variables.

The aim of the response surface is to replace the original failure function g(X) by an equivalent function R(X) by which computational procedure can be simplified maintaining the accuracy. The limit state surface can be represented in polynomial form (11.8).

$$R(X) = a + \sum_{i=1}^{n} b_i X_i + \sum_{i=1}^{n} c_i X_i^2 \tag{11.15}$$

where X_i , i = 1, 2, ..., n are basic variables and parameters a, b_i , c_i , i = 1, 2, ..., n are constants to be determined. Here Eq. 11.15 does not contain mixed terms X_iX_j , hence the function R(X) basically represents the original function g(X) along the coordinate axes X_i only. As number of free parameters in Eq. 11.15 are less i.e. 2n+1, only few numerical experiments are required to obtain unique R(X). However this implies that, in general, sample between the axes will not be covered sufficiently. This is improved by using information on mean and standard deviation of basic variables while updating the R(X).

Bucher and Bourgand (11.8) suggested the way of obtaining R(X) by interpolation using points along X_i. The starting central point chosen is the mean vector. (see Fig. 11.7). Around this starting point 2n points are generated as $X_i = \mu_i + t\sigma_i$, i = 1, 2, ..., n, in which t is the arbitrary factor varying from 1

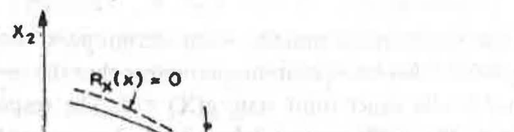


FIG. 11.7 Starting approximation for response surface

to 3. Using function values of original surface at 2n+1 points, the parameters a, b_i, c_i are obtained by solving the set of 2n+1 linear simultaneous equations. Thus first approximation to R(X) is obtained.

In the next stage, function R(X) is used along with the information on mean and standard deviations of basic variables to obtain the estimate of the design point. The estimate is based on the assumption of uncorrelated Gaussian variables. This design point obtained by Level 2 method is used for interpolation to find the new centre point on original failure surface which is in the area of interest i.e. area from which maximum contribution to the probability of failure is made (see Fig. 11.8). So the new centre point for interpolation of R(X) can be obtained as,

$$X_N = \mu + (X_D - \mu)\frac{g(\mu)}{g(\mu) - g(X_D)} \tag{11.16}$$

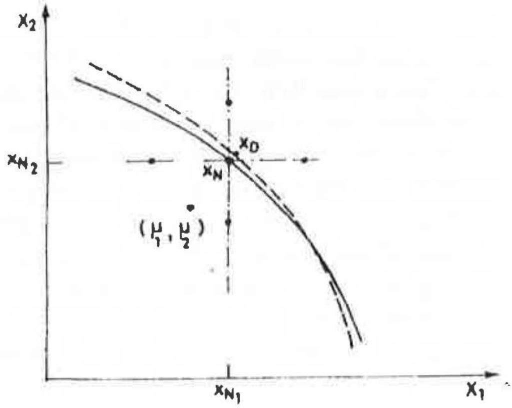


Fig.11.8 Upating the interpolation point for response surface

where X_N is the new centre point and X_D is the design point obtained for first approximation to R(**X**). This interpolation guarantees that the new centre point is sufficiently close to the exact limit state g(**X**) = 0. The response surface is updated by evaluating the coefficients a, b_i, c_i, i = 1, 2, ..., n at the new centre point X_N. So the total number of evaluation of the original limit state equation required is 4n+3.

The update of the polynomial ensures that the critical domain is sufficiently covered by the numerical experiments from the full mechanical model. Once the R(**X**) is found, the reliability analysis can be proceeded in any suitable way, preferably using advanced Monte Carlo technique - Importance sampling method or adaptive sampling method. R(**X**) need not produce the exact limit state surface in entire space but, only sign of original limit state near the design point (i.e. in the region which contributes most to the failure probability) is important (11.8). A simple computer program can be easily written combining response surface method with ISM or ASM. A flow chart for RSM is given in Fig. 11.9.

In some problems, the response surface obtained by using Eq. 11.15 may not give sufficiently accurate mechanical model. To improve the accuracy, mixed terms may be added to Eq. 11.15 as given below:

$$g(\mathbf{X}) = a + \sum_{i=1}^{n} b_i X_i + \sum_{i=1}^{n} c_i X_i^2 + \sum_{i \neq j} \sum d_{ij} X_i X_j \tag{11.17}$$

Various numerical and structural engineering problems solved using the response surface method with ISM are given below. Probability of failure is calculated using Level 2 and Importance sampling technique for both, original failure surface and response surface and results are compared.

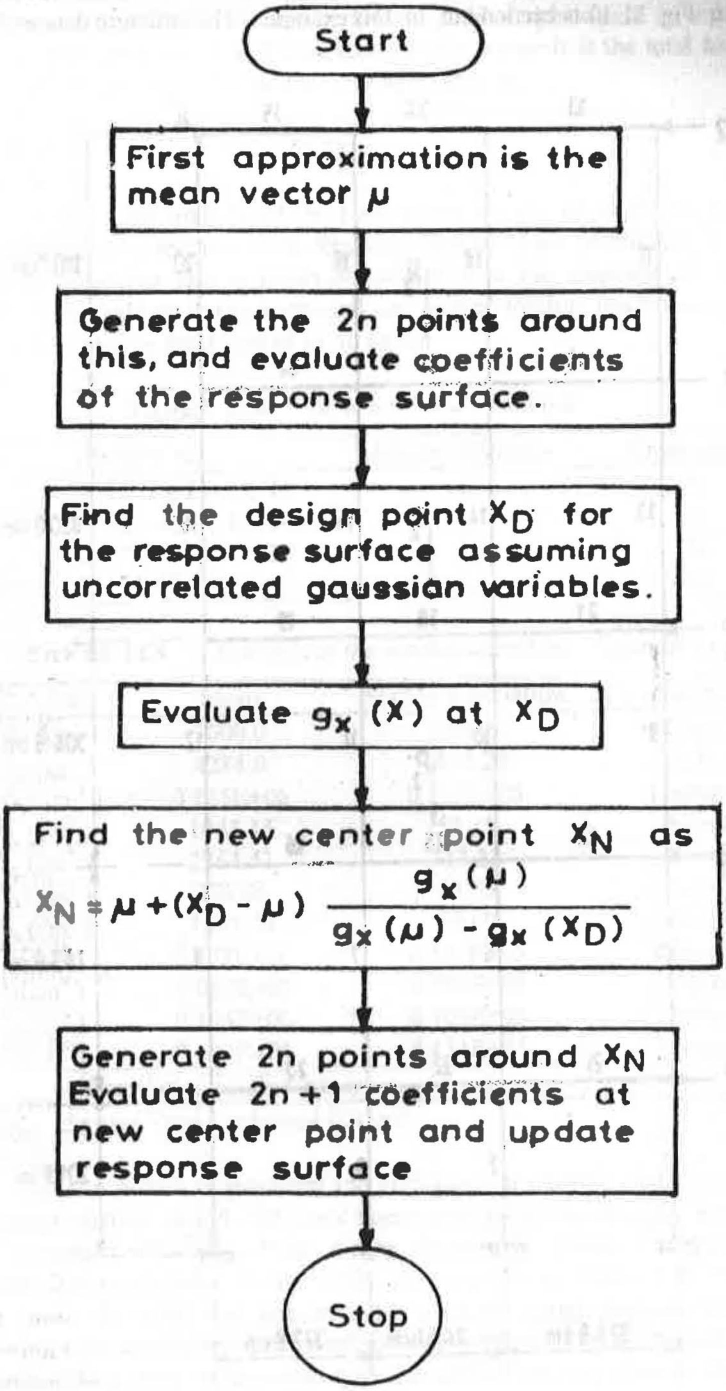


FIG. 11.9 Flow chart for response surface method

EXAMPLE 11.5 Reliability analysis of the three bay five storey R.C.C. frame shown in Fig. 11.10 is carried out in this example. The structure data and the

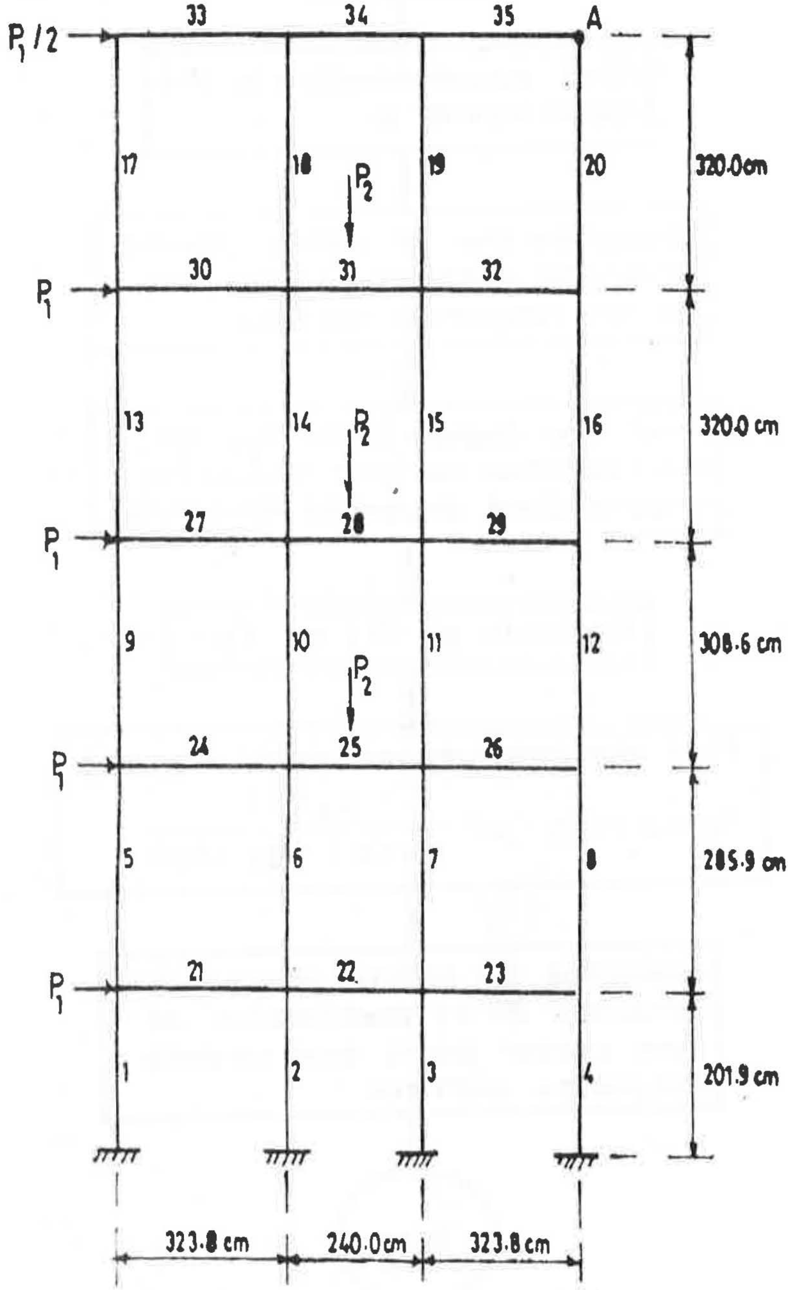


FIG. 11.10 Three bay five storeyed portal frame Example 11.5

statistical data of the random variables involved are given in Tables 11.8 and 11.9 respectively. The limit state criterion is the displacement at top of the frame (i.e. at point A) should not exceed h/350, where h is the total height of the structure. So limit state function can be written as,

$$g(X) = 4.10 - \delta_h(A) \qquad (11.18)$$

where $\delta_h(A)$ is the function of the loads acting on the structure and material and geometrical properties of the structure. Here all these parameters i.e. loads, material properties and geometrical properties of the structure are random variables. In addition to the horizontal and vertical random loads, each beam is carrying a constant dead load of 24.50 kg/cm.

TABLE 11.8 *Structure data - Example 11.5*

Element No.	Moment of inertia	Cross section
1, 4, 5, 8, 24, 25, 26, 28, 31, 34	I_1	A_1
2, 3, 6, 7, 10, 11, 14, 15, 18, 19	I_2	A_2
9, 12, 13, 16, 17, 20, 21, 22, 23	I_3	A_3
27, 29, 30, 32, 33, 35	I_4	A_4

TABLE 11.9 *Statistics of the random variables - Example 11.5*

Variable	Mean	Standard deviation	Distribution
P_1 (kg)	3000.0	1200.0	$EX_{I,L}$ *
P_2 (kg)	4284.0	1235.20	$EX_{I,L}$
E (kg/cm^2)	0.225E+06	0.225E+05	Lognormal
A_1 (cm^2)	1045.15	52.25	Normal
A_2 (cm^2)	2264.51	113.22	Normal
A_3 (cm^2)	870.96	43.55	Normal
A_4 (cm^2)	1393.54	69.67	Normal
I_1(cm^4)	0.182E+06	0.182E+05	Normal
I_2(cm^4)	0.986E+05	0.986E+04	Normal
I_3(cm^4)	0.105E+06	0.105E+05	Normal
I_4(cm^4)	0.431E+06	0.431E+05	Normal

Correlation coefficients are $\rho_{A,A} = \rho_{I,A} = \rho_{I,I} = 0.30$

* $EX_{I,L}$ denotes Type 1 extremal (largest)

The response surface is generated and probability of failure is found out for the generated surface using Level 2 and importance sampling methods. The results are compared with that of the results obtained by Level 2 analysis using original failure surface. All the results are presented in Table 11.10. It can be seen from the table that the reliability analysis using response surface is showing considerable computational advantage over that of the use of original failure surface. Also, results with response surface are very close to that using original surface.

TABLE 11.10 *Results of the reliability analysis - Example 11.5*

	Using $R_x(X)$	Using $g_x(X)$
1. Level 2 method		
Beta	3.255	3.274
P_f	0.567E-03	0.528E-03
Computer time (CP sec)	24.90 (For RS)	239.88
	0.65 (for p_f)	
2. Importance sampling		
Number of simulations	5000	
SDM	1.0	
p_f	0.520E-03	
e_{pf} (%)	11.28	
Computer time (CP sec)	25.16 (for RS)	
	14.18 (for p_f)	

EXAMPLE 11.6 Reliability analysis for the 25 bar transmission tower shown in Fig. 11.11 is carried out. The tower is considered as a space truss. The structure data and the statistical data for the random variables involved are given in Table 11.11. Failure criterion is the displacement at top (i.e. at point P in Fig. 11.11) should not exceed h/250, where h is the total height of the structure. For this, failure function is given by,

$$g(X) = 0.02 - \delta_h(P)$$

Here $\delta_h(P)$ is the function of loads acting on the truss and geometrical and material properties of the structure which are random variables. Reliability analysis is carried out for the response surface, using Level 2 and importance sampling method. Results can been compared with Level 2 analysis using original failure surface. All the results are given in Table 11.12.

TABLE 11.11 *Statistics of the random variables - Example 11.6*

Element No.	Variable	Mean	Standard deviation	Distribution
1	A_1 (m^2)	6.45E-05	9.675E-06	Normal
2 to 3	A_2 (m^2)	2.43E-04	3.635E-05	Normal
6 to 9	A_3 (m^2)	3.04E-04	4.560E-05	Normal
10 to 13	A_4 (m^2)	6.45E-05	1.290E-05	Normal
14 to 21	A_5 (m^2)	1.79E-04	2.680E-05	Normal
22 to 25	A_6 (m^2)	2.45E-04	3.678E-05	Normal
	E (KN/m^2)	2.04E+08	1.860E+07	Normal
	P_1 (KN)	10.0	3.5	$EX_{L,L}$
	P_2 (KN)	15.0	4.0	$EX_{L,L}$
Correlation coefficients are ρ Ai, Aj = 0.25 ; ρ P1,P2 = 0.5				

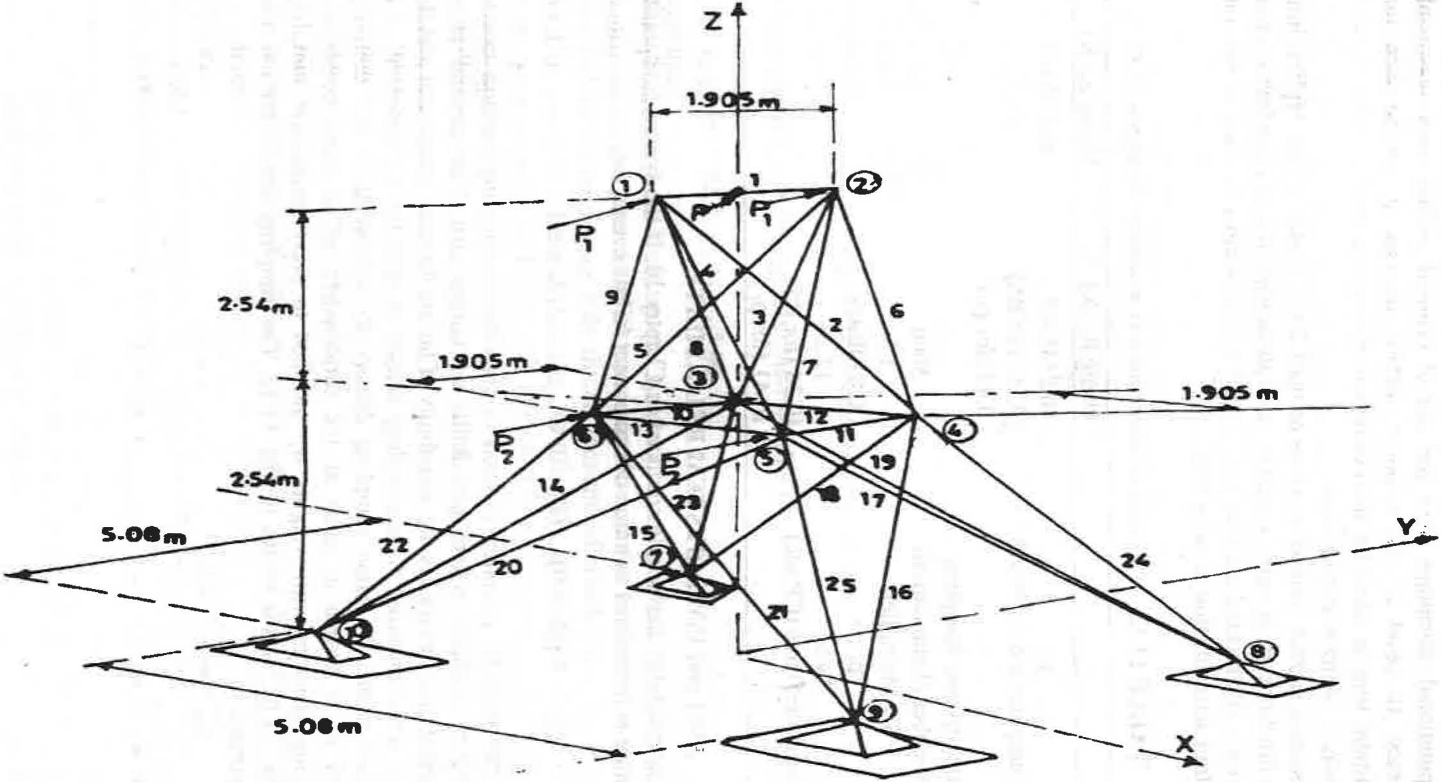


FIG. 11.11 Twenty five bar transmission tower – Example 11.6

Here also, reliability analysis using response surface is showing considerable computational advantage over the use of original surface with reasonable accuracy. In Level 2 with response surface analysis, it can be seen that maximum time is taken for the evaluation of response surface, while Level 2 analysis is taking negligible time.

Response surface method is not to be used for the cases where explicit limit state functions are directly available. It is advocated only in those cases where repetition of structural analysis is to be carried out number of times to generate the limit state function at every time.

TABLE 11.12 *Results of the reliability analysis - Example 11.6*

	Using R_x (X)	Using g_x (X)
p_f	0.934E-05	0.621E-05
Computer time (CP sec)	10.16 (For RS)	84.21
	0.42 (for p_f)	
2. Importance Sampling		
Number of simulations	5000	
SD Multiplier	1.3	
p_f	0.810E-05	
% S_e	12.20	
Computer Time (CP sec)	10.38 (for RS)	
	10.41 (for p_f)	

11.6 ASM and ISM in SYSTEM RELIABILITY

System reliability has been introduced in Chapter 10. If the system probability of failure is formulated as union of component failure events, i.e.

$$p_{fs} = P\,[(Z_1 < 0) \cup (Z_2 < 0) \cup \dots\dots\dots \cup (Z_n < 0)] \tag{11.19}$$

then the adaptive sampling method or importance sampling method can be applied to evaluate system probability of failure also. The method is so developed that it evaluates the probability of failure for each component and the system simultaneously. As a sampling density is used for a component, it is required to use the same sampling density for system also. The sampling density for a system is taken as the combination of all these component sampling densities with weights w_i attached to every component sampling density (11.6). This is shown in Fig. 11.12. The sampling density for a system is expressed as

$$h_{sys} = w_1 h_1 + w_2 h_2 + \dots\dots\dots + w_n h_n \tag{11.20}$$

where $w_1 + w_2 + \dots\dots\dots + w_n = 1$

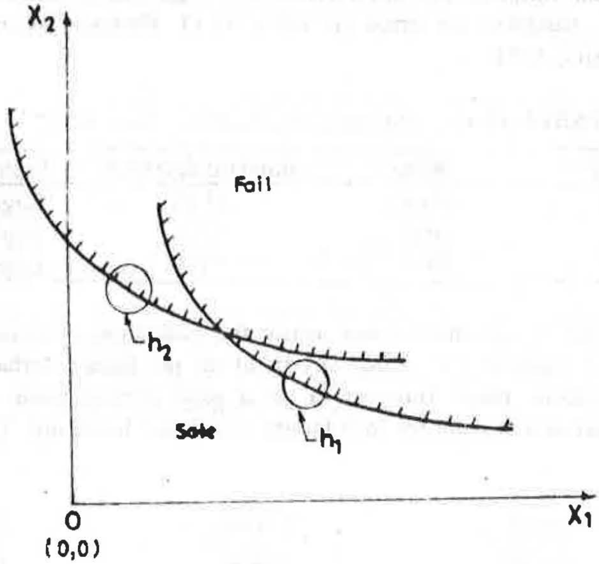


FIG. 11.12 System sampling density

Karamchandani (11.6) suggests equal weights i.e.

$$w_1 = w_2 = \ldots\ldots\ldots\ldots\ldots = w_n$$

The procedure for evaluating the system probability of failure is very much similar to the procedure used for component except that the sample generated is checked for failure for not only the component (of which sampling density is used to generate a sample) but also for all other components. If the sample point is observed to be failed with reference to one or more components, the sampling density for these components is updated if required. Also the sampling density for a system is updated. The probability of failure of the system is given by

$$p_{fs} = \frac{1}{N_s} \sum_{i=1}^{N_s} I\{g(\mathbf{X}) \le 0\} \frac{f_X(\mathbf{x})}{h_{sys}(\mathbf{x})} \tag{11.21}$$

The procedure of computation of probability of failure of a structural system using ASM is illustrated with the following examples.

EXAMPLE 11.7 A structure can collapse under any one of the three limit states whose equations are given below.

$$g_1(X) = Z_1 = X_1 + X_2 + X_4 + X_5 - 5.0X_6 \tag{11.22}$$
$$g_2(X) = Z_2 = X_1 + 2.0X_3 + 2.0X_4 + X_5 - 5.0X_6 - 5.0X_7 \tag{11.23}$$
$$g_3(X) = Z_3 = X_2 + 2.0X_3 + X_4 - 5.0X_7 \tag{11.24}$$

The individual variables are uncorrelated and lognormally distributed. The statistics of variables are given in Table 11.13. Compute system failure probability using ASM.

TABLE 11.13 *Statistics of Variables - Example 11.7*

Variable	Mean	Standard deviation	Distribution
X_1 to X_5	134.9	13.49	Lognormal
X_6	50.0	15.0	Lognormal
X_7	40.0	12.0	Lognormal

Here the variables are uncorrelated: hence the evaluation of transformation matrix is not required. For ASM, any point on the failure surface can be selected as starting point. However to get a good starting point, FORM is carried out and design point for each failure criterion is found out. They are as follows:

$$\mathbf{x}_1^* = \begin{Bmatrix} 131.281 \\ 131.281 \\ 134.90 \\ 131.281 \\ 131.281 \\ 90.484 \\ 40.0 \end{Bmatrix} \qquad \mathbf{x}_2^* = \begin{Bmatrix} 131.616 \\ 134.90 \\ 128.473 \\ 128.473 \\ 131.616 \\ 84.22 \\ 59.528 \end{Bmatrix} \qquad \mathbf{x}_3^* = \begin{Bmatrix} 134.900 \\ 129.878 \\ 125.231 \\ 129.878 \\ 134.900 \\ 50.000 \\ 81.053 \end{Bmatrix}$$

A sample point is found out from the first starting point $\mathbf{x}_1^*$ using the generated random numbers as explained in Example 11.3. The sample point (for SDM = 2) is given by

$$\mathbf{x} = \begin{Bmatrix} 130.0149 \\ 109.3791 \\ 156.8981 \\ 122.4430 \\ 114.6321 \\ 115.6861 \\ 26.9407 \end{Bmatrix}$$

The values of limit state functions are calculated by substituting the value of sample point in Eqs. 11.22 - 11.24. They are as follows:

$$g_1(\,) = -101.94 \;;\; g_2(\,) = 90.2152 \;; \qquad g_3(\,) = 410.9148$$

Since only $g_1 < 0$, the sample point fails under first failure criterion only. The equivalent mean and standard deviation of variables are calculated as explained in Example 11.1. They are as follows:

$$\mu'_X = \begin{Bmatrix} 134.1642 \\ 131.7732 \\ 132.4160 \\ 133.6971 \\ 132.7642 \\ 13.6572 \\ 36.4279 \end{Bmatrix} \qquad \sigma'_X = \begin{Bmatrix} 12.9711 \\ 10.9107 \\ 15.6508 \\ 12.2138 \\ 11.4347 \\ 33.9608 \\ 7.9087 \end{Bmatrix}$$

The independent equivalent normals are calculated as explained in Example 11.1. They are given as:

$$Y_1 = \begin{Bmatrix} -0.3183 \\ -2.0525 \\ 1.5643 \\ -0.9214 \\ -1.5822 \\ 3.0043 \\ -1.1996 \end{Bmatrix}$$

The probability density f_X and sampling density h_X which are multivariable normal densities are calculated as explained in Example 11.3. As there are three failure criteria, there are three different sampling densities namely h_{X1}, h_{X2} and h_{X3}. Their values are given below:

$$f_X = 0.4092 \times 10^{-12}$$

$$h_{X1} = 0.23767 \times 10^{-10} \quad ; \quad h_{X2} = 0.20109 \times 10^{-12} \quad ; \quad h_{X3} = 0.32618 \times 10^{-10}$$

The probability density evaluated at x_1^* is taken as f_{max}. This is given as

$$f_{max} = 0.69226 \times 10^{-9}$$

Since the sample point Y_1 fails under first failure criterion only, for evaluation of probability of failure under first failure criterion, the term f_X/h_{X_1} is used in Eq. 11.2.

Here the system is expressed as union of three failure criteria. Hence failure of any criterion causes the failure of the system. Thus here the generated sample point which fails under first failure criterion causes the failure of the system. For evaluation of system failure probability, the term $f_X/h_{X_{sys}}$ is used in Eq. 11.21 where $h_{X_{sys}}$ is given by Eq. 11.20. Attaching equal weights i.e. $w_1 = w_2 = w_3 = 1/3$,

$$h_{X_{sys}} = \frac{1}{3}h_{X_1} + \frac{1}{3}h_{X_2} + \frac{1}{3}h_{X_3}$$

$$= 0.1862 \times 10^{-10}$$

As the probability density f_X at sample point is less than f_{max}, the sampling density is not improved and is kept the same. That means, the point x_1^* is not shifted and remains same for the next simulation. Similarly using points x_2^* and x_3^*, sample points Y_2 and Y_3 are generated as explained earlier. The procedure of calculation for $h_{X_{syi}}$ for points Y_2 and Y_3 is as explained for Y_1.

All the intermediate values are given in Table 11.14. With this the three simulations are completed.

TABLE 11.14 *Intermediate values in computing system probability of failure using ASM - Example 11.7*

Initial point	Independent Normal point	f_X $\times 10^{-12}$	h_X $\times 10^{-11}$	h_{syi} $\times 10^{-10}$	Next point
$X_1^* = \begin{Bmatrix} 131.281 \\ 131.281 \\ 134.90 \\ 131.281 \\ 131.281 \\ 90.484 \\ 40.0 \end{Bmatrix}$	$Y_1 = \begin{Bmatrix} -0.3183 \\ -2.0525 \\ 1.5643 \\ -0.9214 \\ -1.5822 \\ 3.0043 \\ -1.1996 \end{Bmatrix}$	0.41	2.4 0.02 3.2	0.18	$X_1^* = \begin{Bmatrix} 131.281 \\ 131.281 \\ 134.90 \\ 131.281 \\ 131.281 \\ 90.484 \\ 40.0 \end{Bmatrix}$ Not improved
$X_2^* = \begin{Bmatrix} 131.616 \\ 134.90 \\ 128.473 \\ 128.473 \\ 131.616 \\ 84.22 \\ 59.528 \end{Bmatrix}$	$Y_2 = \begin{Bmatrix} -0.6816 \\ 1.6801 \\ -0.2101 \\ -1.4138 \\ -1.3007 \\ 3.2178 \\ -1.2698 \end{Bmatrix}$	0.75	2.0 0.018 2.7	0.16	$X_2^* = \begin{Bmatrix} 131.616 \\ 134.90 \\ 128.473 \\ 128.473 \\ 131.616 \\ 84.22 \\ 59.528 \end{Bmatrix}$ Not improved
$X_3^* = \begin{Bmatrix} 134.90 \\ 129.878 \\ 125.231 \\ 129.878 \\ 134.90 \\ 50.0 \\ 81.053 \end{Bmatrix}$	$Y_2 = \begin{Bmatrix} -0.5013 \\ -0.5956 \\ 0.3361 \\ -0.3603 \\ 0.4752 \\ 2.1243 \\ 2.2274 \end{Bmatrix}$	24	2.9 0.017 3.2	0.21	$X_3^* = \begin{Bmatrix} 134.90 \\ 129.878 \\ 125.231 \\ 129.878 \\ 134.90 \\ 50.0 \\ 81.053 \end{Bmatrix}$ Not improved

The same procedure is repeated for a number of simulations and the probability of failure under each failure criterion and the probability of failure of the system is evaluated using Eq. 11.21. After 500 simulations and using standard deviation multiplier as 2.0, the probability of failure under the three failure criteria and the probability of failure of the system are found to be,

$$p_{f_1} = 2.68 \times 10^{-3} \quad p_{f_2} = 2.98 \times 10^{-3} \qquad p_{f_3} = 2.65 \times 10^{-4}$$

and $\quad p_{f_s} = 5.05 \times 10^{-3}$

where p_{f_i} is probability of failure under i^{th} criterion.

In the above illustration, equal weight has been attached to each sampling density. One may try by attaching different weights to each sampling densities probably according to p_{f_i} values. But it is found that this technique doesn't give good results and also gives large statistical error. In general, attaching equal weight to each sampling density is found to give better results with less statistical error.

The procedure for evaluating system probability of failure using ISM is very much similar to the procedure of ASM. In ISM starting point is found out using FORM and the simulation is carried out. The difference between ISM and ASM is that the sampling density in ISM once selected is not improved during further simulations . That means once the starting point is taken it is not changed throughout the simulation process. The remaining procedure for ISM is similar to the ASM procedure.

The method of computing system reliability using the method explained in Example 11.7 has been applied to roof trusses and frames (11.10).

11.7 APPLICATION OF ASM TO STRUCTURAL SYSTEMS

Application of ASM to compute reliability of a steel truss is illustrated. Reliability analysis of a steel truss, shown in Fig. 11.13 is to be carried out and the system reliability is to be found out. The truss is located in Mumbai and the height of the building is assumed to be less than 10m. The truss has been designed as per Indian Standard specifications (11.12, 11.13). The loading cases considered for design are as follows:

1. Dead load + Live load
2. Dead load + Wind load

Formulation of safety margin equation

The safety margin equation is basically formulated as

$$M = R - S$$

where R is a resistance and S is an action. The resistance and action are further modelled as explained below.

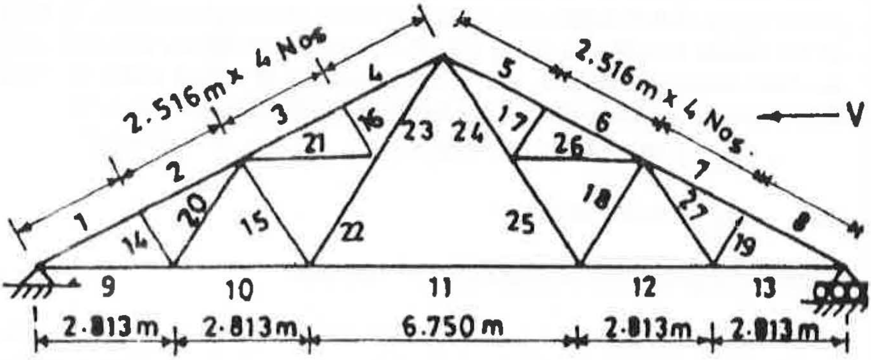


FIG. 11.13 Roof truss

The action S here is developed force in member due to dead load, live load, wind load. Thus S is expressed as

$$S = B\ f(W, D, L) \tag{11.24}$$

where

B = Uncertainty due to assumption in analysis

f(W, D, L) = Force in member due to dead load, live load, wind load

The resistance R of a member is a resistance in tension or compression. R is expressed as

$$R = A\ Y_n\ f(M, F, P) \tag{11.25}$$

where

A = Cross sectional area of a member

Y_n = Nominal value of yield strength

M = Material variability

F = Fabrication variability

P = Professional Factor

Statistics of strength variables

The length of each member is assumed to be statistically independent of each other. The yield strength of a member is expresses as

$$Y = Y_n\ M\ F\ P\ A \tag{11.26}$$

where A is assumed to be deterministic. The statistics of variables M, F, P are given in Table 11.15. For compression member buckling is considered by taking into account the effective slenderness ratio.

$$\omega = \frac{KL}{r}\frac{1}{\pi}\sqrt{\frac{Y}{E}} \tag{11.27}$$

where

$\frac{KL}{r}$ = Effective slenderness ratio

E = Modulus of elasticity

ω = Slenderness constant

The critical strength in buckling, Y_{cr} is obtained from the following equations depending upon value of ω.

$$Y_{cr} = (1 - 0.25\,\omega^2)\,Y \qquad \text{if } \omega < 1.414 \qquad (11.28)$$

$$Y_{cr} = \frac{Y}{\omega^2} \qquad \text{if } \omega \geq 1.414 \qquad (11.29)$$

Statistics of load variables

i) Basic wind velocity

The truss is located in Mumbai for which the mean, standard deviation and coefficient of variation of 50 year life time maximum wind speed are

$$\mu_V = 32.239 \text{ m/s} \quad \sigma_V = 3.417 \text{ m/s} \quad \delta_V = 0.106$$

The model for wind load can be expressed as (Refer Eq. 5.49)

$$W = A\,\rho\,K\,C\,E\,G\,V^2 \qquad (11.30)$$

where A = Projected surface area, ρ = Air density, K = Uncertainty in modelling of load, C = Pressure coefficient depending on geometry of a structure, E = Exposure coefficient and G = Gust factor. The nominal design wind load is given by

$$W_n = A_n\,\rho_n\,K_n\,C_n\,E_n\,G_n\,V_n^2 \qquad (11.31)$$

The variable is considered as ratio of probabilistic wind load to the nominal wind load and is following Type 1 extremal (largest) distribution.

$$\frac{W}{W_n} = \frac{A\rho KCEGV^2}{A_n\rho_n K_n C_n E_n G_n V_n^2} \qquad (11.32)$$

Considering A and ρ as deterministic, mean of W/W_n is given by

$$\mu(W/W_n) = \frac{\mu_K\,\mu_C\,\mu_E\,\mu_G}{K_n C_n E_n G_n}\left(\frac{\mu_V}{V_n}\right)^2 \qquad (11.33)$$

$$\delta(W/W_n) = \left[\delta_K^2 + \delta_C^2 + \delta_E^2 + \delta_G^2 + 4\delta_V^2\right]^{1/2} \qquad (11.34)$$

The combined mean of (K E C G) is taken as unity and the coefficient of variation of C, E and G are given as

$$\delta_C = 0.12 \qquad \delta_E = 0.16 \qquad \delta_G = 0.11$$

The nominal wind speed for Mumbai is 40.04 m/s. Using the same in Eqs. 11.33 and 11.34,

$$\mu(W/W_n) = \left(\frac{\mu_V}{V_n}\right)^2 = \left(\frac{32.239}{40.040}\right)^2 = 0.648$$

$$\delta(W/W_n) = \left[(0.12)^2 + (0.16)^2 + (0.11)^2 + 4(0.106)^2\right]^{1/2} = 0.312$$

ii) Live load

For 50 year life time maximum live load, the following data are taken.

$$\frac{\mu_L}{L_n} = 0.65 \qquad \delta_L = 0.3$$

The live load is following Type 1 extremal (largest) distribution

iii) Dead load

The mean and coefficient of variation for dead load D are given as follows.

$$\frac{\mu_D}{D_n} = 1.05 \qquad \delta_D = 0.1$$

iv) Uncertainty due to assumption in analysis

From analysis point of view, members of a structure are assumed to be connected by pin jointed frictionless hinges; but each joint has some rigidity which actually decreases the force in a member. The statistical data for the variable B is taken as

$$\mu_B = 0.909 \qquad \delta_B = 0.1$$

The statistical data of all the variables are given in Table 11.15. In the table, D_n , L_n, and W_n , are the nominal values of D, L and W respectively. Failure of a member in direct tension or compression is called as a failure mode in a determinate roof truss. The analysis of truss is carried out and the forces in members under the load combination of (a) Dead load + Live load (b) Dead load + Wind load are determined. The truss, being determinate, fails even if one member fails. After analysing the truss and knowing the force in each member, the safety margin equation for each member is written. Using the

TABLE 11.15 *Statistics of variables (Roof truss - Fig. 11.13)*

Variable	Mean	Standard deviation	Distribution
D/D_n	1.05	0.105	Normal
L/L_n	0.65	0.195	Gumbel
W/W_n	0.648	0.312	Gumbel
Y	305.29 N/mm^2	22.77 N/mm^2	Normal
B	0.909	0.0909	Normal
In tension			
M	1.0	0.0898	Normal
F	1.0	0.05	Normal
P	1.0	0.001	Normal
In compression			
M	1.0	0.0925	Normal
F	1.0	0.05	Normal
P	1.0	0.016	Normal

statistics of variables given in Table 11.15 and FORM, the reliability index is calculated for each member. It is found that the value of β for members 9, 12, 22 and 23 are very small compared to the values of β for the remaining members. Hence, the members 9, 12, 22, 23 will only contribute significantly to the system probability of failure. Safety margin equations are given in Table 11.16 only for these dominant members. Probability of failure of these members is evaluated using ASM. System reliability is calculated using ASM with values of standard deviation multiplier and number of simulations 1.0 and 1000 respectively as explained in Example 11.2. The same problem is solved using ISM also and the system reliability is evaluated. Using FORM, the value of β for each dominant member is determined and using these results, bounds on system probability of failure are established as explained in Chapter 10. These results are also given in Table 11.17. From the table it is seen that the probability of failure of the truss is about 0.006 and the corresponding value of β is 2.522.

TABLE 11.16 *Safety Margin Equations (Roof truss - Fig. 11.13)*

Failure Mode	Member No.	Safety Margin Equation	Failure in
1	9	497.76 M F P Y+B(68276.25 D-131245W)	Compression
2	23	347.4 M F P Y+B(29261.25 D-74524.77W)	Compression
3	12	497.76 M F P Y+B(58522.5 D-106404.7W)	Compression
4	22	347.4 M F P Y+B(19507.5 D-49683.18W)	Compression

TABLE 11.17 *Results by Adaptive Sampling Method (Roof truss - Fig. 11.13)*

Failure mode	Reliability Index (β)	Probability of failure				
		FORM	Importance Sampling		Adaptive Sampling	
		p_f	p_f	S_e (%)	p_f	S_e (%)
1	2.537	0.00562	0.00596	12.22	0.00584	11.34
2	2.722	0.00325	0.00407	17.41	0.00412	15.36
3	2.962	0.00154	0.00208	23.23	0.00188	19.96
4	3.587	0.000168	0.000629	21.16	0.000618	19.54

System failure probability bounds are $0.00578 < p_{fs} < 0.00629$
System failure probability obtained by ISM $p_{fs} = 0.00605$
System failure probability obtained by ASM $p_{fs} = 0.00594$

In this chapter, advanced reliability methods have been explained and illustrated. It must be remembered that in general, when explicit functions for limit states are available, response surface method is not to be used. For decision making problems, application of FORM is sufficient. Only in cases where more accurate values of probability of failure are to be estimated, SORM, ISM and ASM are to be used.

REFERENCES

11.1 Fiessler, B., Neuman, H. J. and Rackwitz, R., "Quadratic Limit State in Structural Reliability", *Journal of Engg. Mechanics, ASCE*, Vol. 100, 1979, pp 661-676.

11.2 Breitung, K., "Asymptotic Approximations for Multinormal Integrals", *Journal of Engg. Mechanics*, ASCE, Vol. 110, 1984, pp 357-366.

11.3 Tvedt, L., "Distribution of Quadratic Form in Normal Space Application to Structural Reliability", *Journal of Engg. Mechanics, ASCE*, Vol. 116, 1987, pp 1183-1197.

11.4 Harbiz, A., "An Efficient Method for Probability Failure Calculation", *Structural Safety*, Vol. 3, 1986, pp 109-116.

11.5 Melchers, R. E., "Importance Sampling in Structural Systems", *Structural Safety*, Vol. 6, 1989, pp 3-10.

11.6 Karamchandani, A., "New Methods in System Reliability", *Ph.D. Thesis*, Department of Civil Engg., Stanford University, 1990.

11.7 Melchers, R. E., "Search Based Importance Sampling", *Structural Safety*, Vol. 9, 1990, pp 117-128.

11.8 Bucher, C. G. and Bourgand, U., "A Fast and Efficient Response Surface Approach for Structural Reliability", *Structural Safety*, Vol. 7, 1990, pp 57-66.

11.9 Ang, A. and Tang, W. H., "*Probability Concepts in Engineering Planning and Design*", Vol. 2, John Wiley, Canada, 1984.

11.10 Kulkarni, R. R., "Structural Reliability using Response Surface with Importance Sampling", *M.Tech. Thesis*, Department of Civil Engg. , IIT, Bombay, 1993.

11.11 Himanshu, P., "Simulation Based Reliability Assessment of Structures using Adaptive Sampling", *M.Tech. Thesis*, Department of Civil Engg. , IIT, Bombay, 1999.

11.12 IS:800-1994, "Indian Standard Code of Practice for General Construction in Steel", *Indian Standards Institution*, New Delhi, 1994.

11.13 IS:875-1992(Part 3), "Indian Standard Code of Practice for Design Loads (other than Earthquake) for Buildings and Structures", *Indian Standards Institution*, New Delhi, 1994.

EXERCISE

11.1 For the problem 8.6(a) under Exercise in Chapter 8, determine the probability of failure of the RCC beam in shear using (a) ISM, (b) ASM and (c) response surface with ISM

(Ans. $p_f = 4.1 \times 10^{-5}$)

11.2 For the problem 8.7(ii) under Exercise given in Chapter 8, determine probability of failure of the steel column under combined bending and axial load using (a) ISM, (b) ASM and (c) importance surface with ASM.

(Ans. $p_f = 1.22 \times 10^{-5}$)

12

Fatigue Reliability

12.1 INTRODUCTION

The word "fatigue" refers to the behaviour of materials under the action of repeated stresses or strains as distinguished from the behaviour under monotonic or static stresses. Fatigue is defined as follows (12.1)

> * "Fatigue is defined as the process of progressive localized permanent structural change occurring in a material subjected to condition, which produce fluctuating stresses and strains at some point or points and which may culminate in cracks or complete fracture after a sufficient number of fluctuations".

This definition implies that fatigue process occurs over a period of time or usage and operates at local areas rather than throughout the entire component or structure. The ultimate cause of all fatigue failures is that a crack has grown to a point at which the remaining material can no longer resist the stresses or strains and sudden fracture (i.e. the separation of the component into two or more parts) occurs.

The fatigue life of a structure is determined by the sum of the elapsed cycles required to (i) initiate a fatigue crack and (ii) to propagate the crack from sub-critical dimensions to the critical size. The size of the crack at the transition from initiation to propagation is usually unknown and often depends on the point of view of the analyst and the size of the component being analyzed. For a research worker using microscope to measure crack size, it may be on the order of crystal imperfection or location of a 0.1 mm crack while to the engineer on the field, it may be the smallest crack that can be detected with the available equipment for nondestructive tests. Depending on the nature of the structure and the service loads applied to it, either crack initiation or crack propagation or both phases may be important in assessing structural performance.

The need to consider fatigue damage in the design of structural components arises when the service loading conditions involve cyclic or pulsating variations. Fatigue can be classified into two categories; low cycle fatigue and high cycle fatigue. For low cycle fatigue, plastic strain predominates and ductility controls performance. For high cycle fatigue, elastic strain dominates and strength controls performance. The dividing line between low and high cycle fatigue depends on the material being considered; but usually falls between 10 and 10^5 cycles. In the case of transmission towers, offshore structures and bridges, their vibration amplitudes are within the elastic range. They come under high cycle fatigue (their life span excess of 10^4 cycles). For many components in high cycle fatigue, the fatigue life is dominated by

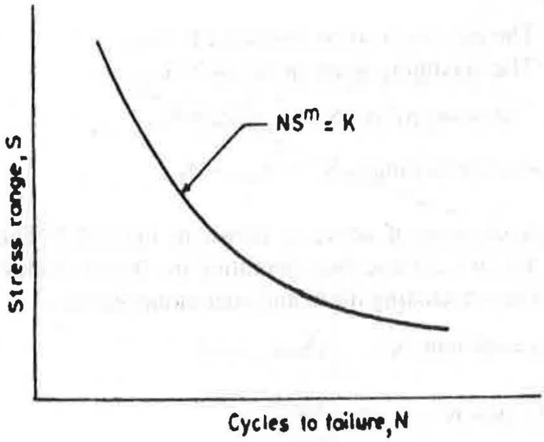


FIG. 12.1 S-N Curve obtained from constant amplitude test results

crack initiation. On the other hand, when stress fluctuations are high or cracks, notches and other stress risers are present, fatigue crack initiates quite early and a significant life portion of the service life may be spent propagating the crack to critical size.

The classical approach to fatigue has focussed on the S-N diagram (Fig.12.1) which relates fatigue life (cycles to failure, N) to cyclic stress, S, which may be specified in terms of stress amplitude or cyclic stress range. Common terms used with S-N diagram are fatigue life, fatigue strength and fatigue limit. The

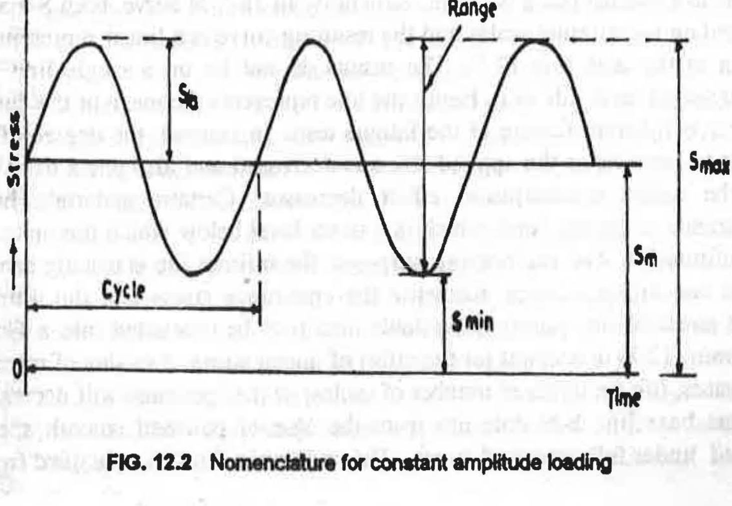


FIG. 12.2 Nomenclature for constant amplitude loading

fatigue life, N, is the number of cycles of stress or strain of a specified character that a given specimen sustains before failure occurs. There are four possible basic parameters, which can be used in the definition of stress cycle to which a fatigue test specimen is subjected. Referring to Fig. 12.2, they are

(i) The minimum stress in the cycle S_{min}

(ii) The maximum stress in the cycle S_{max}

(iii) The mean stress : $S_m = \frac{1}{2}(S_{min} + S_{max})$ (12.1)

(iv) The stress range : $S_r = S_{max} - S_{min}$ (12.2)

Graphical representation of above is shown in Fig. 12.2. The cycle is fully defined when any two of these four quantities are known. Following definitions are also used when discussing mean and alternating stresses.

Stress amplitude $S_a = \frac{1}{2}(S_{max} - S_{min})$ (12.3)

Stress ratio $R = \frac{S_{min}}{S_{max}}$ (12.4)

Amplitude ratio $A = \frac{S_a}{S_m}$ (12.5)

Most design engineers find it convenient to think in terms of minimum stress and maximum stress in the cycle, which in many cases corresponding to dead load stress and dead load plus live load stress respectively. Some times the cycle is referred to by the stress ratio R which is defined as the algebraic ratio of minimum stress to the maximum stress. Tensile stress is being taken as positive and compressive stress as negative. Baseline fatigue data usually are obtained by cycling testing specimens at constant amplitude stress (or strain) until the specimen fails. Such tests are repeated several times at different stress levels to establish the S-N curve. Generally, in an S-N curve, both S and N are plotted on logarithmic scales and the resulting curve is a linear representing the mean of the data (Fig.12.3). The results do not lie on a single line but are scattered on each side of it. Hence the line represents the mean of the data. This scatter is inherent feature of the fatigue tests. In general, the degree of scatter tends to increase as the applied stress is decreased and also tends to be greater as the stress concentration effect decreases. Certain materials have an endurance or fatigue limit which is a stress level below which the material has an infinite life. For engineering purposes, the infinite life is usually considered to be one million cycles. Knowing the endurance stress and the ultimate or yield stress of the material, available data may be converted into a Goodman diagram (12.2) to account for the effect of mean stress. As value of mean stress increases, life (in terms of number of cycles) of the specimen will decrease. The fatigue base line S-N data are from the case of polished smooth specimens loaded under fully reversed stress. The endurance limit S_{en} obtained from this test is to be modified for design taking into account effect of various factors viz. size, type of loading, surface finish, surface treatment (notches, residual

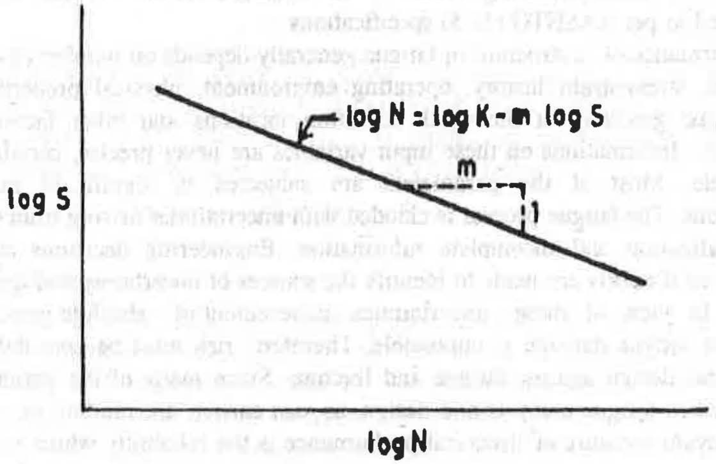


FIG. 12.3 S-N Curve on log-log plot

(notches, residual stress) temperature and environment. The effects of these factors are quantified experimentally through modification factors which are applied to S'_{en} which is obtained from the baseline S-N data.

$$S_e = (S'_{en})(K_{size})(K_{load})(K_{surf.fin.}) \tag{12.6}$$

where K_{size} is the modification factor for the size effect. The modification factors are supposed to be applied to determine endurance limit and the modification for the reminder of the S-N curve is not clearly defined. However, a conservative approach is to use these modification factors on the entire S-N curve.

The fatigue strength at any particular life is defined as the stress at which the S-N curve cuts the particular value of N. Further the curve being a straight line in log-log plot, a linear equation can be formulated to predict the value of S for any given value of N and vice versa. Fatigue is one of the principal modes of failure in bridges, offshore structures pressure vessels etc. However, most of the civil engineers in India may not know how to check the safety and evaluate a given bridge under fatigue. Presently there is no Indian standard code for fatigue design and evaluation of a bridge. Even though considerable development has taken place in reliability analysis and design, most fatigue assessment procedures, currently used, do not take advantage of such developments. Instead, typical fatigue assessment guidelines for structural elements require that engineers refer to stress range cycle life curves (S-N curves). Fatigue strength is determined from S-N curve drawn approximately setting at two standard deviations below mean curve obtained from laboratory testing. This approach does not consider the inherent variation in loading models. The evaluation of safety also does not consider the interaction between resistance and action. Hence this way of checking does not provide consistent evaluation or safety of joints. This has been observed by Albrecht and Moses (12.3,12.4) while checking reliability of structural joints of steel bridges designed as per AASHTO (12.5) specifications.

Performance of a structure in fatigue generally depends on number of cycles of load, stress-strain history, operating environment, physical properties of materials, geometry at the crack initiation locations and other factors. In practice, informations on these input variables are never precise, certain and complete. Most of the parameters are subjected to significant random variations. The fatigue process is clouded with uncertainties arising from errors in idealization and incomplete information. Engineering decisions can be improved if efforts are made to identify the sources of uncertainty and quantify them. In view of these uncertainties, achievement of absolute prevention of some fatigue damage is impossible. Therefore, risk must be considered in structural design against fatigue and fracture. Since many of the parameters involved in fatigue analysis and design, as said earlier, are random in nature, the relevant measure of structural performance is the reliability which is taken as

$$\text{Reliability} = 1 - p_f$$

where p_f is the probability of failure. The application of structural reliability theory to design has several advantages (i) The use of reliability (or probability of failure) is the most meaningful index of structural performance (ii) It provides a systematic method of treatment of uncertainties (iii) Provides a tool for making rational decisions (iv) All components can be designed to a balanced reliability level thereby producing an efficient system (v) The technique permits the sensitive studies of uncertainties with the greatest impact on the solution to be evaluated (vi) It is a tool for establishing partial safety factors to result designs with uniform reliability under different design situations (vii) It is a tool for updating standards (viii) It is a tool to develop an inspection criteria or remedial measures on existing structures.

Evaluation of fatigue reliability of joints in bridges appears to have started in 1981 (12.1, 12.3). The problem has been initially formulated on S-N curve. In 1982, the ASCE Committee on Fatigue and Fracture Reliability (12.1) presented a series of papers dealing with the state of art on fatigue reliability aspects and introducing fatigue reliability models for reliability analysis and development of criteria for assuring integrity against fatigue and fracture using principles of structural reliability. Afterwards, the attention of research workers was diverted to evaluation of fatigue reliability using system approach based on S-N curve. Fracture mechanics approach is essential for the development of inspection and maintenance strategy. Research was carried out in applying fracture mechanics approach for the evaluation of fatigue reliability of bridge structures. The formulation of fatigue reliability analysis and design based on S-N curve and fracture mechanics approaches are presented in this chapter.

12.2 S-N CURVE APPROACH

The most commonly used model for fatigue behaviour under constant amplitude loading is of the form

$$N\ S^m = K \tag{12.7}$$

In which m and K are empirical constants denoting slope of S-N line and intercept on S axis respectively. N is number of cycles to failure and S is the applied stress range. When Eq. 12.7 is plotted on log-log scale, the S-N relationship has a linear form (Fig. 12.3) as given below.

$$\log N = \log K - m \log S \tag{12.8}$$

12.2.1 Equivalent Stress Range

In practice, the loading on structures does not take the form of a cyclic constant amplitude stress. Rather the loading is a sequence of variable amplitudes and frequencies, which do not repeat themselves. For variable amplitude loading the concept of equivalent stress range based on Palmgren-Miner's (P-M) cumulative damage hypothesis is generally used. It states that failure occurs when the total strain energy due to n cycles of variable amplitude loading is equal to the total strain energy from N cycles of constant amplitude loading. That is the cumulative damage, D, is written as

$$D = \sum_{i}^{B} D_i = \sum_{i}^{B} n_i / N_i \tag{12.9}$$

where D_i is the damage incurred at stress level S_i , n_i is the number of stress cycles at stress range level S_i and N_i is the number of cycles at constant stress range level S_i (from S-N curve) to cause failure. B is the number of stress range blocks. D is generally taken as 1 at failure. Equivalent stress range is calculated as given below (12.6)

If N_T is the total number of cycles in the life of the structure, then number of cycles, n_i , in the stress range block i is given by (Fig. 12.4)

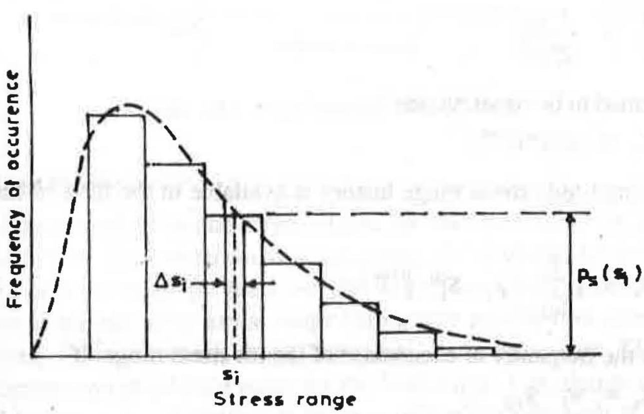


FIG. 12.4 Histogram and probability density function for induced stress range

$$n_i = N\,[p_S(s_i)\,\Delta s]\,. \tag{12.10}$$

where $p_S(s_i)$ is the probability density function for induced stress. Using Eq. 12.10 in Eq. 12.9.

$$D = \sum_{i=1}^{B} \frac{N\,[p_S(s_i)\,\Delta s]}{N_i} \tag{12.11}$$

From Eq. 12.7,

$$N_i = K/S_i^m$$

Substituting the same in Eq. 12.11

$$D = \sum_{i=1}^{B} \frac{N}{K}\; S_i^m\,[p_S(s_i)\,\Delta s]$$

$$= \frac{N}{K} \sum_{i=1}^{B} S_i^m\,[p_S(s_i)\,\Delta s]$$

$$= \frac{N}{K}\; E(S^m) \tag{12.12}$$

$E(S^m)$ is read as expected value of S^m. For continuous random variable S, as $\Delta s \to 0$, Eq. 12.11 becomes

$$D = \int_0^{\infty} \frac{N S^m\, p_S(s)}{K}\; ds$$

$$= \frac{N}{K}\; E(S^m) \tag{12.13}$$

If S_e is the equivalent constant amplitude stress range for random variable amplitude, then

$$\frac{N}{K} = \frac{1}{S_e^m} \tag{12.14}$$

If D is assumed to be equal to one,

$$S_e = [E(S^m)]^{1/m} \tag{12.15}$$

If variable amplitude stress range history is available in the form of histogram, then

$$S_e = \left[\sum_{i=1}^{B} p_i\; S_i^m\right]^{1/m} \tag{12.16}$$

Where p_i is the frequency of occurrence of the ith stress range. If

$$S_i = \Psi_i\; S_{rd} \tag{12.17}$$

where Ψ_i is the ratio of the mean value of the stress range block i to the design stress range and S_{rd} is the allowable design stress based on design load, then

$$S_e = \left[\sum_{i=1}^{B} p_i \quad \Psi_i^m \right]^{1/m} S_{rd} \tag{12.18}$$

In 1983, Albrecht (12.3) presented a lognormal format method of calculating reliability of a structural detail of a highway bridge, maintaining the concept, the resistance is given by number of cycles to failure and the load by the applied stress range history. Load spectra in the form of stress histograms is replaced by a lognormal distribution of equivalent stress ranges. The fatigue properties of a detail are represented by an S-N curve. At any point S on the mean regression line, the fatigue life considered as resistance, is found to be lognormally distributed about the point with mean $\mu_R = log$ N and standard deviation $\sigma_R = \sigma_{\log N}$. This defines the resistance. This is shown in Fig. 12.5.

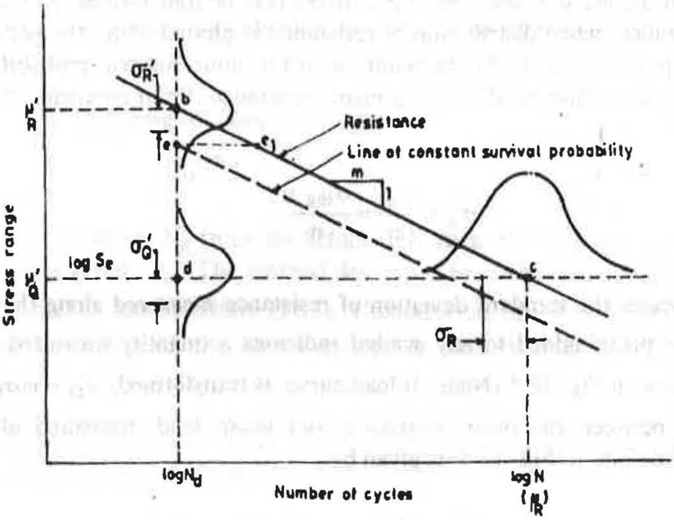


FIG. 12.5 Transformation of resistance

12.2.2 Load Curve

Field measurements of actual stress ranges by the application of live load or actual load from loadometer surveys are generally available in the form of a histogram of stress range (or truck weight) versus frequency of occurrence. For development of load curve, stress range histograms recorded on several bridges are required. For each stress range histogram, S_e is calculated. This replaces that histogram and provides a point for the load curve. Calculation of values of S_e for all histograms and plotting them on a vertical line through N_d results the load curve. N_d is the total number of cycles estimated to occur in the design life of the structure. Mean value of S_e and coefficient of variation of S_e are computed. On S-N curve (Fig.12.5) plotted on log-log scale, point d represents mean value of log S_e along a vertical line through log N_d. For lognormally distributed S_e, standard deviation of *log* S_e, $\sigma_{\log S_e}$, is given by

$$\sigma_{\log S_e} = [0.4343 \log (1 + \delta_{se}^2]^{1/2} \tag{12.19}$$

This is called as standard deviation of load, (action or load effect) σ_Q. Hence δ_{Se} represents coefficient of variation of S_e. The prime added to Q represents that it is measured along vertical line.

12.2.3 Transformation of Resistance

Calculation of reliability requires that load and resistance are expressed in terms of the same basic quantities i.e. either cycles to failure or stress range. That is both load and resistance curves are to be plotted on the same axis. Hence, in the present case, one of the curves is to be transformed. Transforming the resistance, when distribution of resistance is plotted along the vertical line through point b (Fig.12.5), the points with the same survival probability must lie on the same line parallel to the mean resistance. From geometry, it is clear that

$$\sigma_R' = \frac{\sigma_R}{m} = \frac{\sigma_{\log N}}{m} \tag{12.20}$$

σ_R' indicates the standard deviation of resistance measured along the vertical line. The prime added to any symbol indicates a quantity measured along a vertical line in Fig. 12.5 (Note : If load curve is transformed, $\sigma_Q = m\sigma_Q'$). The distance between the mean resistance and mean load, measured along the vertical line d-b, in Fig. 12.5, is given by

$$\mu_R' - \mu_Q' = \frac{1}{m} (\mu_R - \mu_Q) \tag{12.21}$$

Reliability index is given by

$$\beta = \frac{\mu_M}{\sigma_M}$$

$$\beta = \frac{\frac{1}{m}(\mu_R - \mu_Q)}{(\sigma_Q' + \sigma_R')^{1/2}}$$

$$= (\log N - \log N_d)/\sigma_t \tag{12.22}$$

where

$$\sigma_t = [(m\sigma_{\log S_e})^2 + (\sigma_{\log N})^2]^{1/2} \tag{12.23}$$

If one is interested in evaluating design life for specified reliability index β_o, it can be calculated as follows. Using Eq. 12.22,

$$\log N_d = \log N - \beta_o \ \sigma_t \tag{12.24}$$

Using Eq. 12.8, and substituting for log N,

$$\text{Log } N_d = (\log K - \beta_o \ \sigma_t) - m \log F_{re} \tag{12.25}$$

Here F_{re} is the allowable equivalent stress range. The above equation can be rewritten as

$$N_d = \frac{10^{(\log K - \beta_o \sigma_t)}}{(F_{re})^m} \tag{12.26}$$

For the known value of N_d from the actual field data, the value of F_{re} can be calculated for a given β_o. The method developed has been applied to designs meeting AASHTO specifications (12.3). Computation of β against fatigue failure criterion based on the above method is illustrated in the following examples.

EXAMPLE 12.1 For a particular joint or detail in a highway bridge, the value of K and m from test results are

$$K = 0.37 \times 10^{12} \text{ and } m = 3.0$$

The coefficient of variation of N is 0.24. From the field data, mean value and coefficient of variation of equivalent stress range S_e calculated from 100 histograms are 36.5 N/mm^2 and 0.114 respectively. The actual number of cycles for a 50 year design life is estimated to be 4.56 x 10^7 cycles. Determine the probability of failure of the joint against fatigue.

The resistance mean S-N curve plotted on log-log scale is shown in Fig. 12.6. The position of the actual design point d and load curve are also shown in the same figure. Using Eq. 12.19, standard deviation of log S_e is calculated.

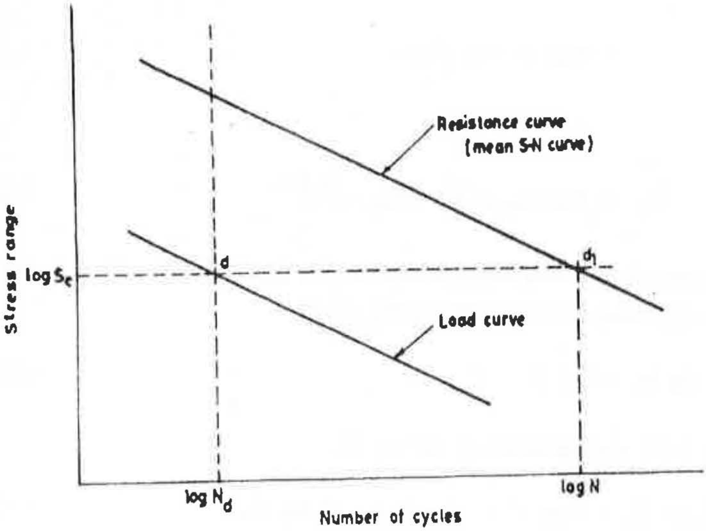


FIG. 12.6 S-N curve for example 12.1

$$\sigma_{\log\ S_e} = [0.4343\ \log(1 + 0.14^2)]^{1/2}$$

$$= 0.0494$$

Here log S_e is considered as load (action). Hence

$$\sigma_Q = \sigma_{\log\ S_e} = 0.0494$$

Similarly, using Eq. 12.19,

$$\sigma_{\log\ N} = [0.4343\ \log(1 + 0.24^2)]^{1/2}$$

$$= 0.1028$$

Here log N is considered as resistance. Hence

$$\sigma_R = \sigma_{\log\ N} = 0.1028$$

Using Eq. 12.23,

$$\sigma_t = [(m \log S_e)^2 + (\log N)^2]^{1/2}$$

$$= [(3 \times 0.0494)^2 + (0.1028)^2]^{1/2} = 0.1804$$

At this given value of $S_e = 36.5$ N/mm^2, the actual number of cycles that the joint can withstand is obtained using the resistance mean S-N curve. It is given by,

$$\log N = \log K - 3 \log S_e$$
$$= \log (0.37 \times 10^{12}) - 3 \log 36.5$$
$$= 6.881$$
$$N = 7.603 \times 10^6 \text{ cycles}$$

The reliability index β is obtained using Eq. 12.22 for given $N_d = 4.56 \times 10^7$.

$$\beta = \frac{\log N - \log N_d}{\sigma_t}$$
$$= \frac{6.881 - 6.676}{\sigma_t} = \frac{0.205}{\sigma_t}$$

The numerator of the above expression is equal to $2\,\sigma_{\log N}$. That is

$$\beta = \frac{0.205}{0.1804} = \frac{2\sigma_{\log N}}{0.1804}$$
$$= 1.136$$

The fatigue specifications for bridges give the allowable stress range as a function of type of detail and number of loading cycles N_d. They specify allowable S-N line which is set at two standard deviations, $2\,\sigma_R$, to the left of the resistance. The specifications do not make any allowance for load variability.

EXAMPLE 12.2 Consider the same problem in Example 12.1. The allowable design S-N curve is given by (12.7),

$$\log N = 0.2306 \times 10^{12} - 3 \log S_{rd}$$

The value of K for design S-N curve, K_d, can be obtained from the mean curve by using the following equation (12.7).

$$K_d = (K)_{mean}\,\Delta^d$$

where $d = 2$, when design curve is drawn at two standard deviations from the mean curve and $\Delta = 0.7893$

$$K_d = (0.37 \times 10^{12})\,(0.7893)^2 = 0.2306 \times 10^{12}$$

The mean S-N curve and allowable (design) S-N curve are shown in Fig. 12.7. If the detail is to be designed for 2×10^6 cycles, the design stress is given by the point e in Fig. 12.7.

$$\log S_{rd} = \frac{1}{3}\,[\log 0.2306 \times 10^{12} - \log 2.0 \times 10^6]$$
$$S_{rd} = 48.672 \text{ N/mm}^2$$

This is the hypothetical design point. To locate the actual design point, d, one must find equivalent stress range, S_e, and the actual number of loading cycles, N_d estimated from the data. Let us assume that the gross vehicle weight distribution based on lodometer survey yielded.

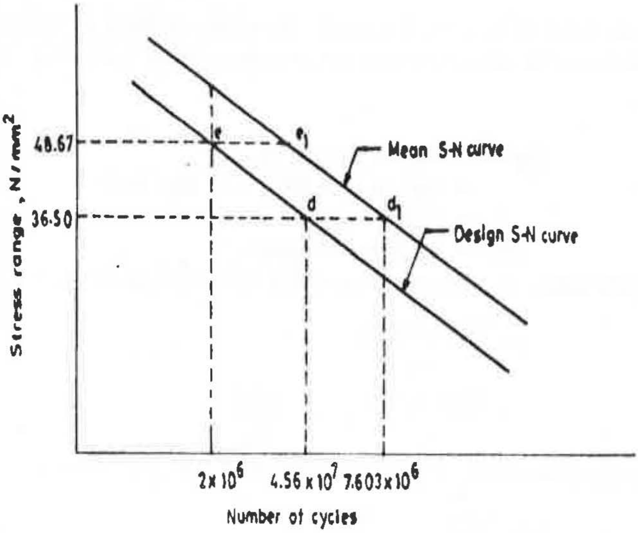


FIG. 12.7 Mean and design S-N curves on log-log-plot- Example 12.2

$$\rho = \Sigma\, p_i\, \psi_i^m = 0.35$$

Then
$$S_e = (0.35)^{1/3} S_{rd}$$
$$= 0.705\ S_{rd}$$
$$= (0.705)\ (48.672) = 36.5\ \text{N/mm}^2$$

Corresponding value of N_d (from design S-N curve) is

$$\text{Log } N_d = \log\ 0.2306 \times 10^{12} - 3 \log\ 36.5$$
$$N_d = 4.56 \times 10^6 \text{ cycles.}$$

Generally fatigue design specifications do not reflect the actual fatigue conditions that occur. High stress range is specified with low number of stress cycles to produce a reasonable design. But in actual field conditions, fatigue stresses are well below this value (equivalent stress is very much lower) ; but a much higher number of cycles. For the above value of N_d, point on the design curve is given by the point d in Fig. 12.7. This is the actual design point. The actual number of cycles that the joint can withstand at S_e = 36.5 N/mm² is 7.603×10^6 cycles (Refer Example 12.1).

The reliability index β is given by

$$\beta = \frac{\log(7.603 \times 10^6) - \log(4.744 \times 10^6)}{\sigma_t}$$

$$= \frac{6.881 - 6.676}{\sigma_t} = \frac{0.205}{\sigma_t}$$

$$= \frac{2\,\sigma_{\log N}}{\sigma_t}$$

If the coefficient of variation of load is zero, i.e. $\sigma_Q = 0$, or if this is not considered,

then $\sigma_t = \sigma_{\log N}$. Hence

$$\beta = \frac{2\sigma_{\log N}}{\sigma_{\log N}} = 2$$

The value of probability of failure corresponding to $\beta = 2$, is

$$\beta = \Phi^{-1}(-2) = 0.0227 \cong 2.3\,\%$$

In the conventional fatigue design, the uncertainty in load is not taken into account. Because of this, for different values of $\delta_Q = \delta_{\log Se}$ the values of β will differ significantly. For example,

For $\delta_{\log Se} = 0.114,$

$$\sigma_t = 0.1804\,,\ \beta = 1.136$$

Similarly for $\delta_{\log Se} = 0.25,$

$$\sigma_t = 0.337\,,\ \beta = 0.2716$$

Hence the conventional design will not give consistent level of safety in different design situations.

In the regular design, one would have selected the value of design stress range 48.67 N/mm^2 for the desired life 2 x 10^6 cycles. This is given by the point e in Fig. 12.7. The detail would have been designed for this stress. But the actual strength of joint is given by point e, for which the number of cycles that the joint can withstand is 3.207 x 10^6 cycles. The distance e-e$_1$, is equal to d-d$_1$ and is equal to $2\,\sigma_{\log N}$. Value of β is equal to 2 when uncertainty in load is not taken in to account.

EXAMPLE 12.3 The mean resistance S-N curve and the allowable resistance curve, shown in Fig.12.8, for a detail are given by

$$\log N = \log(0.37 \times 10^{12}) - 3 \log S$$

$$\log N = \log(0.2306 \times 10^{12}) - 3 \log S$$

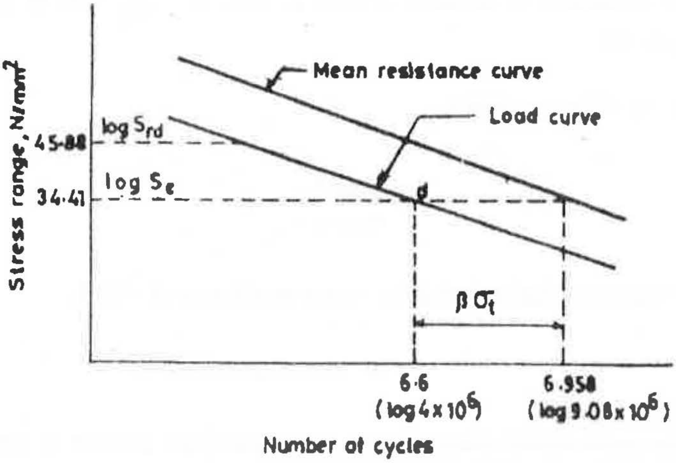


FIG. 12.8 Determination of allowable design stress range – Example 12.3

From the load history, equivalent constant amplitude stress range is equal to 0.75 S_{rd} and the number of loading cycles is 4 x 10^6 cycles. It is given :

$$\delta_N = 0.2; \quad ; \quad \delta_{Se} = 0.12$$

Determine the allowable stress range for design based on equivalent truck weight and for design based on design load for the desired level $\beta_o = 2$.

Using Eq. 12.19,

$$\sigma_{\log Se} = [0.4343 \log (1 + 0.12^2)]^{1/2} = 0.052$$

$$\sigma_{\log N} = [0.4343 \log (1 + 0.2^2)]^{1/2} = 0.086$$

Using Eq. 12.23,

$$\sigma_t = [(3 \times 0.052)^2 + (0.086)^2]^{1/2} = 0.178$$

Reliability index is given by Eq. 12.22

$$\beta = \frac{\log N - \log N_d}{\sigma_t}$$

For the desired reliability level $\beta = \beta_o = 2$,

$$\log N - \log N_d = 2 \times 0.178 = 0.356$$

The design life N_d is

$$\log N_d = \log N - \beta \sigma_t$$

Using the mean resistance S-N curve,

$$\log N_d = (\log K - m \log S_e) - \beta \sigma_t$$

$$= \log (0.37 \times 10^{12}) - 3 \log S_e - 0.356$$

$$= 11.212 - 3 \log S_e$$

Using the above equation, S_e can be calculated for given $N_d = 4 \times 10^6$. Hence,

$$S_e = \frac{1}{3} [11.212 - \log (4 \times 10^6)]$$

$$S_e = 34.408 \text{ N/mm}^2$$

This is the allowable equivalent stress range for $\beta_0 = 2$. Knowing $S_e = 0.75 \times S_{rd}$, the allowable stress based on design load

$$S_{rd} = \frac{34.408}{0.75} = 45.88 \text{ N/mm}^2$$

For $S_e = 34.408$ N/mm^2, the corresponding value of N from resistance curve is

$$\log N = \log (0.37 \times 10^{12}) - 3 \log 34.408$$
$$= 11.568 - 4.61 = 6.958$$
$$N = 9.078 \times 10^6 \text{ cycles}$$

12.3 LRFD FORMAT

In LRFD format(Refer Chapter 9) uncertainty in random loading can be taken care of explicitly. Adoption of the format makes the designer to determine partial safety factors to resistance, γ_R, and partial safety factor to load, γ_S, for the desired reliability level. Smith and Hirt (12.8) proposed a safety format similar to LRFD format for calibrating European convention for constructional steel works (ECCS) 1985 standards. For safety

$$S_R / \gamma_R \geq \gamma_S \, S_e \qquad (12.27)$$

The fatigue strength S_R is defined by the S-N curve corresponding to the detail/joint which is evaluated. The equivalent constant amplitude stress range S_e is calculated from the resulting stress histories due to the application of design load spectra and applying the reservoir or rain flow method of cycle counting. The safety factor γ_R reflects the uncertainty quantified by

- variations in effects of fabrication, workmanship, size, shape, local stress concentration and fatigue crack shapes
- size of detail, residual stresses, metallurgical effects.

Total uncertainty in fatigue strength is represented by δ_R.

The partial safety factor γ_S reflects the uncertainty

- in estimating the effects of stress analysis
- due to errors in fatigue model and use of Miner's rule
- in developing stress histories due to loads and determining stress ranges and counting number of cycles using rain flow or reservoir method
- in estimating the equivalent constant amplitude effects of the design spectrum.

Total uncertainty in load (action) is represented by δ_{Se}. In the lognormal safety format all variables are assumed to be lognormally distributed. When limit state equation is written in terms of stress ranges, Eq. 12.22 becomes

$$\beta = \frac{\log S_R - \log S_e}{[(\sigma_R^{'})^2 + (\sigma_S^{'})^2]^{1/2}}$$

If the resistance curve is defined in terms of design S-N curve drawn at $2\,\sigma_{\log\ N}$ from mean S-N curve, then the above equation becomes

$$\beta = \frac{\log S_R + 2\sigma_R - \log S_e}{(\sigma_t / m)} \tag{12.28}$$

Taking logarithm on both sides of Eq. 12.27

$$\log S_R = \log S_e + \log \gamma_S + \log \gamma_R$$

Substituting the same in Eq. 12.29, reliability index expressed in terms of partial safety factors becomes,

$$\beta = \frac{\log \gamma_S + \log \gamma_R + 2\,\sigma_R^{'}}{(\sigma_t / m)} \tag{12.29}$$

For given γ_R, γ_S and m, one can compute β if δ_R and δ_{Se} are known from the field data.

Since in fatigue design, design S-N curves are drawn at mean minus two standard deviations to take care of variation in R, γ_R is taken as one. Considering the same Smith and Hirt (12.8) have found that β varies from 2 to 3.5 at the end of service life for fatigue designs of details designed as per ECCS. With the above format it is possible to establish partial safety factors γ_R and γ_S directly for the specified reliability index β_0. This approach has been used in updating fatigue provisions of Swiss code for steel design. This method of calculating β for given γ_S *and* γ_R and (ii) calculating γ_R and γ_S for desired β_o is illustrated below.

EXAMPLE 12. For a given detail used in a bridge, it is found from the field data that the values of δ_R and δ_{Se} are 0.36 and 0.2 respectively. For the particular detail, m = 3, the code has specified γ_R = 1 and γ_S = 1.8. Determine β.

Using Eq. 12.19, standard deviation of log S_e and log S_R can be computed.

$$\sigma_{\log Se} = [0.4343 \log(1 + 0.2^2)]^{1/2} = 0.086$$

$$\sigma_R = \sigma_{\log N} = [\,0.4343 \quad \log(1 + 0.36^2)]^{1/2} = 0.15$$

$$\sigma_R = \sigma_R/m$$

$$= 0.15/3 = 0.05$$

Using Eq. 12.23,

$$\sigma_t = [(3 \times 0.086)^2 + (0.0647)^2]$$
$$= 0.298$$

The value of β is calculated using Eq. 12.29.

$$\beta = \frac{\log 1.8 + \log 1.0 + 2 \times 0.05}{(0.298/3)}$$

$$= 3.3$$

EXAMPLE 12.4 A detail is to be designed for a reliability level of $\beta_0 = 2.5$. Determine γ_S fixing $\gamma_R = 1$. It is given

$$\delta_{Se} = 0.2 \; ; \; \delta_R = 0.36 \; ; \; m = 3$$

From the previous example, for the above values of δ_{Se} and δ_R.

$$\sigma_{\log S_e} = 0.086; \; \sigma_{\log R} = 0.15; \sigma_t = 0.298$$

Using the above values and given values of β_0 and m in Eq. 12.29,

$$2.5 = \frac{\log \gamma_S + \log(1.0) + 2(0.15/3)}{(0.298/3)}$$

$$\log \gamma_S = 0.148$$

$$\gamma_S = 1.407$$

Hence the partial safety factor for stress range is 1.487. Similarly for different valus of δ_{Se} and β, corresponding values of γ_S can be calculated. Variations γ_S with δ_{Se} and β are shown in Fig. 12.9. It can be noted that as δ_S increases, γ_S increases for given β. Again for given δ_{Se}, as β increases γ_S decreases.

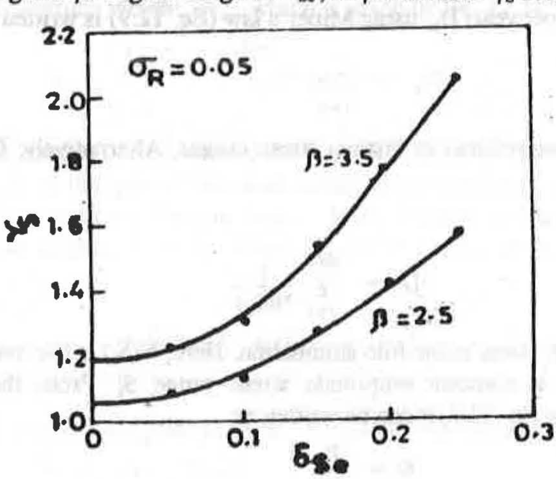


Fig. 12.9 Variation of γ_S with δ_{Se}

So far failure function has been formulated based on number of cycles or stress range. In general limit state function for evaluation of fatigue reliability can be of any one of these when S-N curve approach is used.

i) $p_f = P[\,T_f < T_s\,]$ (12.30)

ii) $p_f = P(D_f < D_s)$ (12.31)

iii) $p_f = P(N < N_T)$ (12.32)

iv) $p_f = P(S_e < S_R)$ (12.33)

Here T_f denotes actual time to fatigue failure and T_s is the service life (desired life) of the structure with which is deterministic. T is a function of several random variables. D_f is the cumulative damage at failure and D_s is the specified damage. N is the actual number of cycles that the detail/joint can with stand and N_T is the total number of cycles in time T_s (desired number of cycles). Moses et al (12.9), in 1985, have dealt with modelling of bridge loads and its application to fatigue design of bridges in accordance with AASHTO specifications using damage based failure criterion.

12.4 APPLICATIONS IN BRIDGES

Ravi and Ranganathan (12.10) started the formulation for fatigue reliability assessment from Eq. 12.30. For a particular bridge in service, general formulation of limit state equation for a bridge is explained below.

Let the limit state equation under fatigue loading is defined by

$$Z = Y_f - Y_s \tag{12.34}$$

Where Y_f is the life at failure and Y_s is the specified life. Both Y_f and Y_s are in terms of years. Limit state is reached when Z is equal to zero. The damage accumulated per year, D_y, using Miner's law (Eq. 12.9) is written as

$$D_y = \sum_{i=1}^{j} n_i / N_i$$

where j is the number of distinct stress ranges. Alternatively, D_y can also be written as

$$D_y = \sum_{i=1}^{\text{all S}} \frac{1}{N(S_i)} \tag{12.35}$$

by taking each stress range into summation. Here, $N(S_i)$ is the number of cycles to failure at a constant amplitude stress range S_i. From the S-N curve, represented by Eq. 12.7, it can be written as

$$N = \frac{K}{S^m} \tag{12.36}$$

Substituting this value of N in Eq. 12.35,

$$D_y = \frac{1}{K} \sum_{i=1}^{all\ S} S_i^m \tag{12.37}$$

The true stress range for any truck crossing of a bridge depends on several variables and may be written as

$$S_i = \frac{W_i(1+i_{fi})(i_{\ell i})(g)(h)}{z_x} \tag{12.38}$$

Where W_I = i th truck crossing gross vehicle weight,
i_{fi} = impact factor
g = lateral girder distribution (expressed as percentage of gross span moment or force carried by single member)
h = factor to account for closely spaced or multilane presence of vehicles which amplify the load effect
z_x = the actual section modulus or cross sectional area
$i_{\ell i}$ = the influence factor which converts the load to load effect.

Influence factor is defined as

$$i_f = \frac{\text{absolute maximum load effect}}{\text{total load on span}} \tag{12.39}$$

Representing the volume in total number of equivalent stress cycles in a year by V, D_y is written as

$$D_y = \frac{V}{K}\left[\frac{(1+i_{fi})(i_{\ell i})(g)(h)}{z_x}\right]^3 \sum \frac{W_i^m}{V} \tag{12.40}$$

The term within the summation in the above equation can be represented by equivalent fatigue truck weight, W_{eq}, which is given by Eq. 12.16

$$W_{eq} = \left[\sum_{i=1}^{n_L} f_i\, w_i^m\right]^{1/m} \tag{12.41}$$

where n_L is the number of load categories, f_i is the relative frequency of the load category *i* and W_i is that part of the load acting on the structure corresponding to maximum load effect for category load *i*. Here, maximum load effect can be bending moment or shear force etc. Hence Eq. 12.40 can be written as

$$D_y = \frac{V}{K}\left[\frac{W_{eq}(1+i_{fi})(i_{\ell i})(g)(h)}{z_x}\right]^3 \tag{12.42}$$

The equivalent number of cycles per year, V, can be expressed as

$$V = (N_T)(N_{eq}) \tag{12.43}$$

in which N_T is annual traffic (truck traffic in vehicles per day x 365 or train traffic) and N_{eq} is equivalent number of stress range cycles per passage of train or truck crossing. Thus Eq. 12.42 is rewritten as

$$D_y = \frac{N_T \ N_{eq}}{K}\left[\frac{W_{eq}(1+i_{fi})(i_{\ell i})(g)(h)}{z_x}\right]^3 \tag{12.44}$$

Y_f represents the life at failure when the cumulative damage of Miner's model is equal to one. However, this cumulative damage is seen to be a random variable, its value lying anywhere between 0.84 to 2.06 (12.1). Hence cumulative damage at failure, X, is treated as a random variable. Knowing the damage accumulated per year as D_y , Y_f can be written as

$$Y_f = \frac{X \ K}{N_T \ N_{eq}}\left[\frac{z_x}{W_{eq}\ (1+i_{fi})(i_{\ell i})(g)(h)}\right]^3 \tag{12.45}$$

Hence the limit state equation 12.34 becomes

$$Z = \frac{X \ K}{N_T \ N_{eq}}\left[\frac{z_x}{W_{eq}\ (1+i_{fi})(i_{\ell i})(g)(h)}\right]^3 - Y_s \tag{12.46}$$

The S – N curve intercept, K , is expressed as

$$K = N_s S^3$$

where N_s is the desired life in cycles. It is calculated as

$$N_s = \overline{N}_T \ \overline{N}_{eq} \ Y_s \tag{12.47}$$

where $\overline{N}_T$ and $\overline{N}_{eq}$ are the mean values of N_T and N_{eq}. Hence

$$Z = \frac{\overline{N}_T \ \overline{N}_{eq} \ Y_s}{N_T \ N_{eq}}\left[\frac{z_x \ S}{W_{eq}(1+i_{fi})(i_{\ell i})(g)(h)}\right]^3 - Y_s \tag{12.48}$$

Let $$A = N_T/\overline{N}_T \tag{12.49}$$

$$B = \frac{N_{eq}}{\overline{N}_{eq}} \tag{12.50}$$

and $$I_{fi} = 1+i_{fi} \tag{12.51}$$

where A and B represent the volume ratio and equivalent cycle ratio respectively. I_{fi} is the combined impact factor which takes care of live load and impact effects. Hence Eq. 12.48 becomes

$$Z = \frac{X\ Y_s}{A\ B}\left[\frac{z_x\ S}{W_{eq}\ I_{fi}\ i_{\ell i}\ g\ h}\right]^3 - Y_s \qquad (12.52)$$

The above equation represents the limit state equation in terms of actual values. This equation is normalized as follows. Defining

$$P = \frac{z_x}{z_d} \qquad (12.53)$$

in which z_d is the section modulus as per design. This is given by

$$z_d = \frac{W_d(1+i_{fd})\,i_{\ell d}\ h_d\ g_d}{S_{rd}} \qquad (12.54)$$

Using Eqs. 12.53 and 12.54, the expression for Z (Eq. 12.52) becomes

$$Z = \frac{X\ Y_s}{A\ B}\left[\frac{P\,W_d(1+i_{fd})\,i_{\ell d}\ g_d\ h_d\ S}{W_{eq}\ (1+i_{fi})\,i_{\ell i}\ S_{rd}\ g\ h}\right]^3 - Y_s \qquad (12.55)$$

Let

$$W = W_{eq}/W_d \qquad (12.56)$$

$$I_F = \frac{1+i_{fi}}{1+i_{fd}} \qquad (12.57)$$

$$I_L = \frac{i_{\ell i}}{i_{\ell d}} \qquad (12.58)$$

$$G = \frac{g}{g_d} \qquad (12.59)$$

$$S_n = S/S_{rd} \qquad (12.60)$$

$$H = h/h_d \qquad (12.61)$$

Using the same Eq. 12.55 becomes

$$Z = \frac{X\,Y_s}{A\,B}\left[\frac{P\,S_n}{W.I_F\ I_L\ G\,H}\right]^3 - Y_s \qquad (12.62)$$

The failure surface equation becomes (ie $Z = 0$),

$$Z = \frac{X}{A\,B}\left[\frac{P\,S_n}{W.I_F\ I_L\ G\,H}\right]^3 - 1 = 0 \qquad (12.63)$$

The above equation represents the failure surface in normalized format. The random variables included in the above fatigue criterion contains material terms X, P and S_n, truck variables, W, A, B, I_L and H and analysis uncertainties I_F and G. Once the probability distribution and parameters of all random variables are known, probability of failure can be evaluated using any reliability method. This is demonstrated with examples.

EXAMPLE 12.5 The fatigue reliability of a riveted railway plate girder bridge of span (L) 32 m is to be evaluated. Here reliability for a joint in tension flange at mid span is computed. The joint detail comes under category class D as per British standards (12.7). Statistics of variables are given in Table 12.1. In the case of railway bridges, the factors G and H are not considered in Eq. 12.63.

TABLE 12.1 *Statistics of variables - Example 12.5*

Sr. No	Variable	Mean	δ	Median	$\sigma_{\ell n}$
1.	*X - Model uncertainty*	1.04	0.300	0.999971	0.293560
2.	*P – Sec. Mod. ratio*	0.855	0.100	0.851055	0.099751
3.	*S – Stress range ratio*	1.380	0.142	1.365799	0.141292
4.	*A – Volume ratio*	1.000	0.100	0.995037	0.099751
5.	*B – Equivalent cycle ratio*	1.000	0.011	0.999940	0.011000
6.	*W – Weight ratio*	0.536	0.100	0.532842	0.099751
7.	*I_F – Impact factor ratio*	1.000	0.150	0.988936	0.149166
8.	*I_L – Influence factor ratio*	0.986	0.111	0.979981	0.110660

The mean value of sectional modular ratio is first computed as follows.

Assuming that the live load, given by IRS bridge rules (12.11), holds good for fatigue design also, the design value of section modulus is calculated from Eq. 12.54 deleting factors G and H.

$$z_d = \frac{W_d(1+i_{fd})(i_{\ell d})}{S_{rd}} \tag{12.64}$$

But

$$S_{rd} = (K/N)^{1/m}$$

From British standards (12.7), for design S – N curve of class D detail,

$$K = 1.52 \times 10^{12} \text{ and } m = 3.0$$

For a desired life of 2×10^6 cycles,

$$S_{rd} = \left[\frac{1.52 \times 10^{12}}{2 \times 10^6}\right]^{1/3} = 91.258 \text{ N/mm}^2$$

For plate girder of span 32 m, design values of W_d and I_{fd} obtained from IRS bridge rules (12.11) are,

$$W_d = 1.437 \times 10^6 \text{ N;}$$

For single track spans,

$$i_{fd} = 0.15 + \left\{\frac{8}{6+L}\right\}$$

For L = 32 m, i_{fd} = 0.361

For simply supported uniform distributed beam,

$$i_{\ell d} = \frac{\text{span}}{8} = \frac{32}{8} = 4 \text{ m}$$

Hence

$$z_d = \frac{1.437 \times 10^6 \, (1+0.361)\,(4 x 10^3)}{91.258}$$

$$= 8.569 \times 10^7 \text{ mm}^3$$

But the section modulus provided by Railways is

$$z_x = 7.33 \times 10^7 \text{ mm}^3$$

Mean value of z_x is taken as the provided section modulus. Hence the mean value of P is

$$\bar{P} = \frac{7.330 \times 10^7}{8.569 \times 10^7} \mp 0.855$$

Considering Eq. 12.63, let

$$R = X\ (P\,S)^3 \tag{12.65}$$

$$\text{and} \quad Q = A\,B\,(W\,I_f\,I_L)^3 \tag{12.66}$$

in the problem all variables are lognormally distributed. Their parameters are given in Table 12.1. Using them, parameters of lognormally distributed R and Q can be calculated as follows.

$$\tilde{R} = \tilde{X}\,(\tilde{P}\tilde{S})^3 \tag{12.67}$$

$$= (0.999971)\,(0.851 \times 1.366)^3 = 1.57$$

$$\tilde{Q} = \tilde{A}\,\tilde{B}(\tilde{W}\,\tilde{I}_f\,\tilde{I}_L)^3 \tag{12.68}$$

$$= 0.995 \times 0.99994\,(0.533 \times 0.989 \times 0.98)^3 = 0.137$$

Using the given values of coefficients of variations of variables, values of $\sigma_{\ell n\ R}$ and $\sigma_{\ell n\ Q}$ are calculated as follows.

$$\sigma^2_{\ell n\ R} = \ell n[(1+\delta_X^2)(1+\delta_P^2)^9\,(1+\delta_s^2)^9] \tag{12.69}$$

$$= \ell\, n[(1+0.3^2)\,(1+0.1^2)^9 + (1+0.142^2)^9]$$

$$\sigma_{\ell n\ R} = 0.596$$

$$\sigma^2_{\ell n\ Q} = \ell n[(1+\delta_A^2)(1+\delta_B^2)(1+\delta_W^2)^9(1+\delta_{I_f}^2)^9(1+\delta_{I_L}^2)^9] \tag{12.70}$$

$$= \ell\, n\,[(1+0.1^2)\,(1+0.011^2)\,(1+0.1^2)^9\,(1+0.15^2)^9\,(1+0.111^2)^9]$$

$$\sigma_{\ell n\ Q} = 0.64$$

Since R and S are lognormally distributed, (R/S) is also lognormally distributed. Hence reliability index is given by (Refer chapter 6).

$$\beta = \frac{\ell n\left[\frac{\tilde{R}}{\tilde{Q}}\right]}{\sqrt{\sigma_{\ell n R}^2 + \sigma_{\ell n Q}^2}}$$

$$= \frac{\ell n\left[\frac{1.57}{0.137}\right]}{\sqrt{0.596^2 + 0.64^2}} = 2.788$$

This is the fatigue reliability index of the joint in the tension flange at mid span of the bri=dge.

EXAMPLE 12.6 Fatigue reliability of the lower chord member L_2 L_3 of the riveted truss bridge of span 36 m, shown in Fig. 12.10, is to be evaluated. The statistics of the variables are given in Table 12.2.

The mean value of section area ratio is computed as follows. The design value of sectional area z_d is

$$z_d = \frac{W_d(1 + i_{fd})(i_{\ell d})}{S_{rd}}$$

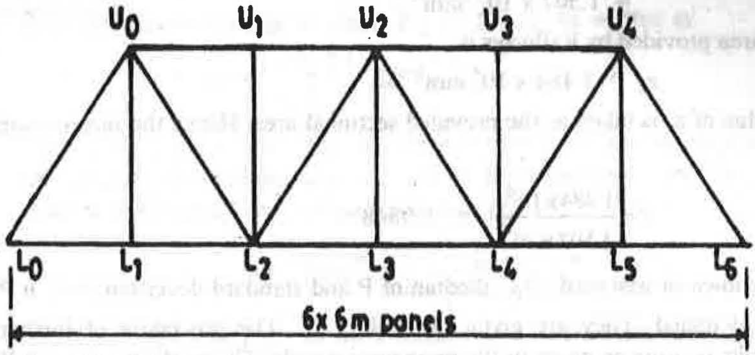
TABLE 12.2 *Statistics of variables – Example 12.6*

Sr. No.	Variable	Mean	δ	Median	$\sigma_{\ell n}$
1.	*X – Model uncertainty*	1.040	0.3000	0.999971	0.293560
2.	*P – Cross sectional area ratio*	0.985	0.1000	0.985	0.099751
3.	*S – Stress range ratio*	1.380	0.1420	1.366	0.141292
4.	*A – Volume ratio*	1.000	0.1000	0.995	0.099751
5.	*B – Equivalent cycle ratio*	1.000	0.0065	0.99994	0.00647
6.	*W – Weight ratio*	0.513	0.1000	0.510	0.10
7.	*I_F – Impact factor ratio*	1.000	0.1500	0.989	0.150
8.	*I_L – Influence factor ratio*	0.005	0.0990	0.990	0.099

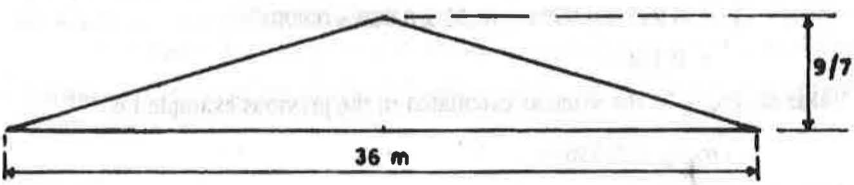
From British standards (12.7), for design S – N curve of class D detail,

$$K = 1.52 \times 10^{12} \text{ and } m = 3.0$$

For desired life of 2×10^6 cycles,



(a) Truss bridge configution – Span-36M.



(b) Influence line for force in member L_2L_3

Fig. 12.10 Truss bridge – Example 12.10

$$S_{rd} = \left[\frac{1.52 \times 10^{12}}{2 \times 10^{6}}\right]^{1/3} = 91.258 \text{ N/mm}^2$$

For truss bridge of 36 m, design values of W_d and I_{fd} are obtained using IRS bridge rules (12.11). They are

$$W_d = 1595.5 \text{ kN};$$
$$i_{fd} = 0.34$$

The influence line diagram for force in member L_2 L_3 is shown in Fig. 12.10. Using this

$$i_{\ell d} = \left(\frac{9}{7} x \frac{L\, w_d}{2}\right) \frac{1}{w_d\, L}$$
$$= 0.643$$

Hence $\quad z_d = \dfrac{1595.5 \times 10^3 \times (1.34)(0.643)}{91.258}$

$$= 1.507 \times 10^4 \ \text{mm}^2$$

But the area provided by Railways is

$$z_x = 1.484 \times 10^4 \ \text{mm}^2$$

Mean value of z_x is taken as the provided sectional area. Hence the mean valuc of P is

$$\bar{P} = \frac{1.484 \times 10^4}{1.507 \times 10^4} = 0.9848$$

For the known or assumed δ_P, median of P and standard deviation of ℓn P can be calculated. They are given in Table 12.2. The procedure of further calculations is same as given in the previous example. The median values of R and Q are

$$R = (0.999971) \, [\,(0.985) \ (1.366)]^3$$

$$= 2.397$$

$$\tilde{Q} = 0.995 \times 0.99994 \ (0.51 \times 0.989 \times 0.990)^3$$

$$= 0.124$$

Value of $\sigma_{\ell n \ R}$ is the same as calculated in the previous example i.e.

$$\sigma_{\ell n \ R} = 0.596$$

Using Eq. 12.70,

$$\sigma_{\ell n \ Q} = \ell n[(1{+}0.099751^2)\,(1{+}0.00647^2)\,(1{+}0.1^2)^9\,(1{+}0.15^2)^9 \times (1{+}0.099^2)^9]$$

$$\sigma_{\ell n \ Q} = 0.623$$

Hence the reliability index is

$$\beta = \frac{\ell n(\dfrac{2.397}{0.124})}{\sqrt{0.596^2 + 0.623^2}} = 3.435$$

This is the value of fatigue reliability index for the member L_2 L_3 of the riveted railway truss bridge.

12.5 APPLICATIONS IN OFFSHORE AND SHIP STRUCTURES

Lognormal format

Wirsching (12.12) has formulated the fatigue reliability problem of welded joints in offshore structures and given a closed form expression to compute p_f assuming lognormal format. If f_o is defined as the average frequency of the cycle, that is

$$f_o = \frac{N_T}{T} \qquad (12.71)$$

then cumulative damage D, using Eq. 12.13, can be written as

$$D = \left(\frac{f_o T}{K}\right) E(S^m) \qquad (12.72)$$

where N_T is the total number of cycles in time T. If spectral approach is used for analysis of random process and if it is assumed that the process is stationary, Gaussian and narrow band, then (12.13),

$$f_o\, E(S^m) = \lambda(m)\,(2\sqrt{2})^m \left(\frac{m}{2}+1\right) \sum_{i=1}^{b} r_i\, f_i\, \sigma_i^m \qquad (12.73)$$

where f_i is the frequency of wave loading in i th sea-state and σ_i is the root mean square (RMS) stress process in the i th sea-state. f_i and σ_i can be calculated from the given spectral density function $W_i(f)$ for the fatigue stress range. r_i is percent of time in the i th sea-state, and λ_i is a correction factor to be used for the narrow band assumption. It is computed by calculating D_i from rain flow analysis and comparing it to the narrow band assumption. Wirsching (12.13) has found that $\lambda(m) \approx 0.86$ for m = 3 and $\lambda(m) \approx 0.76$ for m = 4.38.

Instead of spectral approach, if Weibull model is assumed for long term distribution of stress range S, then

$$F_S(s) = 1 - \exp\left[-\left(\frac{s}{u}\right)^k\right] \qquad s \geq 0 \qquad (12.74)$$

where u and k are parameters of the distribution. The weibull shape parameter k varies from 0.5 to 1.4 for offshore platforms and is equal to one for ship structures (12.12). If N_T is the total number of cycles in service life T, long term design stress range, S_{rd}, is defined as

$$P[S > S_{rd}] = \frac{1}{N_T} \qquad (12.75)$$

This is the stress S_{rd} that is exceeded, on the average, once every N_T cycles. S_{rd} is also called as "once in a life time" stress.. Hence using Eq.12.74,

$$\begin{aligned} F_S(S_{rd}) &= P[S \leq S_{rd}] \\ &= 1 - \exp\left[-\left(\frac{S_{rd}}{u}\right)^k\right] \end{aligned} \qquad (12.76)$$

Using Eq. 12.75, it can be written as

$$1 - \exp\left[-\left(\frac{S_{rd}}{u}\right)^k\right] = 1 - \frac{1}{N_T}$$

$$S_{rd} = u\,[\ell n\, N_T]^{1/k}$$

Or $$u = S_{rd}\,[\ell n\, N_T]^{-1/k}$$

If S follows Type 3 extremal (smallest) distribution(Weibull), S^m also follows the same distribution with mean,

$$E(S^m) = u^m\ \Gamma\left(\frac{m}{k}+1\right) \tag{12.78}$$

Using Eq. 12.77 in the above equation,

$$E(S^m) = (S_{rd})^m\,[\ell n\, N_T]^{-m/k}\ \Gamma\left(\frac{m}{k}+1\right)$$

$$E(S^m) = \lambda(m)\,(S_{rd})^m\ [\ell n\, N_T]^{-m/k}\ \Gamma\left(\frac{m}{k}+1\right) \tag{12.79}$$

Miner's rule states that failure under variable stress range occurs when $D \geq 1$. But random fatigue experimental results show that the critical value of the cumulative damage at failure, D_f, is not always close to 1.0 ; but in fact varies widely. Therefore, D_f is taken as a random variable which quantifies modelling error associated with Miner's rule. Failure can be defined as the event $D > D_f$.

If T denotes time to fatigue failure and letting $D = D_f$, the basic damage expression Eq. 12.72 can be rewritten as

$$T = \frac{D_f\ \ K}{B^m\, f_o\, E(S^m)} \tag{12.80}$$

where B is model error in estimated stress range. That is, if S is the estimated stress range, actual stress range = B S Since D_f, k and B are random variables, T is also a random variable. If T_s is the service life of structure, fatigue failure of a joint occurs when $T < T_s$. Then

$$p_f = P(T < T_s) \tag{12.81}$$

Failure function is

$$g(\ ) = T - T_s \tag{12.82}$$

Here, T_s is deterministic. If statistics of random variables D_f, K and B are known, β can be calculated using Level 2 reliability method. If D_f, K and B are lognormally distributed, then

$$\beta = \frac{\ell n\left(\frac{\tilde{T}}{T_s}\right)}{\sigma_{\ell n\,T}} \tag{12.83}$$

where

$$\sigma_{\ell n\,T} = \left[\ell n\left\{(1+\delta_{D_f}^2)(1+\delta_K^2)(1+\delta_B^2)^{m^2}\right\}\right]^{1/2} \tag{12.84}$$

$$\tilde{T} = \frac{\tilde{D}_f \tilde{K}}{\tilde{B}^m f_o E(S^m)} \tag{12.85}$$

Using the above approach, Wirsching (12.12) demonstrated the computation of fatigue reliability of welded joints in offshore structures.

The model parameter B which is a random variable can be split into several factors, as given below, which contribute to the overall variation (uncertainty) in B. Let

$$B = B_F \; B_S \; B_W \; B_N \; B_H \tag{12.86}$$

where

B_F = uncertainty due to fabrication and workmanship

B_S = uncertainty due to sea state description

B_W = uncertainty due to wave load prediction

B_N = uncertainty in predicting nominal loads

B_H = uncertainty in estimation of hot spot stress concentration factor

The above factors are the sources which contribute to the overall uncertainty in the estimation of fatigue stress. Any other factor can be included. If the coeffocient of variation of each variable is known, the overall variation in B can be computed.

$$\delta_B^2 = \delta_{B_F}^2 + \delta_{B_S}^2 + \delta_{B_W}^2 + \delta_{B_N}^2 + \delta_{B_H}^2 \tag{12.87}$$

If the variables are assumed lognormally distributed, the parameters $\tilde{B}$ and $\sigma_{\ell n\, B}$ of the lognormally distributed B can be found out as follows.

$$\tilde{B} = \tilde{B}_F \; \tilde{B}_S \; \tilde{B}_W \; \tilde{B}_N \; \tilde{B}_H \tag{12.88}$$

$$\sigma_{\ell n B} = \left[\sigma_{\ell n\, B_F}^2 + \sigma_{\ell n\, B_S}^2 + \sigma_{\ell n\, B_W}^2 + \sigma_{\ell n\, B_N}^2 + \sigma_{\ell n\, B_H}^2\right]^{1/2} \tag{12.89}$$

Or $$\delta_B^2 = \left[(1+\delta_{B_F}^2)(1+\delta_{B_S}^2)(1+\delta_{B_W}^2)(1+\delta_{B_N}^2)(1+\delta_{B_H}^2) - 1\right]^{1/2} \tag{12.90}$$

$\tilde{B}_i$ is the median of B_i. Wirsching (12.12) has suggested $\tilde{B}$ about 0.7 and δ_B about 0.5 in evaluating fatigue reliability of joints in offshore platform. Values of $\tilde{D}_f$ and δ_{D_f} equal to 1.0 and 0.3 respectively have been recommended. The procedure of computation of fatigue reliability of a joint in offshore structure is illustrated with an example.

EXAMPLE 12.7 Determine the reliability of a welded joint in an offshore platform using Wirsching's approach assuming all variables are lognormally

distributed. It is given that for a 20 year life, long term stress range is 383.3 N/mm^2 (That is S_{rd} = 383.3 N/mm^2) and long term stress range follows Weibull distribution. Following data are also given.

T_s	=	20 yr	;	k = 0.69
m	=	3	;	f_o = 0.25 hertz
$\tilde{K}$	=	1.9365×10^{13}	;	δ_K = 0.73
$\tilde{D}_f$	=	1.0	;	δ_{D_f} = 0.30
$\tilde{B}$	=	0.7	;	δ_B = 0.50
$\lambda(m)$	=	0.86	;	

Mean value of S^m is first calculated using Eq. 12.79

$$E(S^m) = \lambda(m)(S_{rd})^m \, [\ell n \, N_T]^{-m/k} \, \Gamma\left(\frac{m}{k}+1\right)$$

Substituting the given values, each term in the above equation, is calculated as follows.

$$\Gamma\left(\frac{m}{k}+1\right) = \Gamma\left(\frac{3}{0.69}+1\right)$$

$$= 41$$

$$N_T = f_o \, T_S$$

$$= 0.25 \times 20 \times 365 \times 24 \times 3600$$

$$= 1.575 \times 10^8 \text{ cycles in 20 years}$$

$$[\ell n \, N_T]^{-m/k} = \left[\ell n\left(1.575 x 10^8\right)\right]^{-3/0.69}$$

$$= 2.836 \times 10^{-6}$$

$$(S_{rd})^m = (383.3)^3 = 56314010$$

Hence mean value of S^m is

$$E(S^m) = (0.86)(56314010)(2.836 \times 10^{-6}) \times 41$$

$$= 5629.3$$

$$f_o \, E(S^m) = 0.25 \times 5629.3 = 1407.33$$

Since all variables are lognormally distributed, using Eq. 12.85,

$$\tilde{T} = \frac{\tilde{D}_f \, \tilde{K}}{\tilde{B}^m \, f_o \, E(S^m)}$$

$$= \frac{(1.0)(1.9365 x 10^{13})}{(0.7)^3 \, (1407.33)} = 1291.39 \; yr$$

Using Eq. 12.84

$$\sigma_{\ell nT} = \left[\ell n \left\{ \left(1+\delta_{D_f}^2\right) \left(1+\delta_K^2\right) \left(1+\delta_B^2\right)^{m^2} \right\} \right]^{1/2}$$

$$\sigma_{\ell nT} = \left[\ell n \left\{ \left(1+0.3^2\right) \left(1+0.73^2\right) \left(1+0.5^2\right)^9 \right\}\right]^{1/2}$$

$$= 1.588$$

Using Eq. 12.83, the reliability index is calculated.

$$\beta = \frac{\ell n \left(\frac{1291.39}{20} \right)}{0.588}$$

$$= 2.62$$

Using the lognormal format explained above, it is also possible to determine the allowable (design) stress range for required service life of the structure and target reliability level. This is illustrated in the following example.

EXAMPLE 12.8 Determine the minimum allowable stress range for 20 year life, for the design of a welded joint in an offshore platform for a reliability level of $\beta_o = 3$ against fatigue. All the variables are lognormally distributed. Long term stress range follows Weibull distribution. Following data are given.

T_S	=	20 yr.	;	k	=	0.69
m	=	3				
F_o	=	0.25 hertz	;	$\lambda(m)$	=	0.86
$\widetilde{K}$	=	1.9×10^{13}	;	δ_K	=	0.7
$\widetilde{D}_f$	=	1.0	;	δ_{D_f}	=	0.3
$\widetilde{B}$	=	0.7	;	δ_B	=	0.5

Using Eq. 12.83

$$\widetilde{T} = T_S \exp\left[\beta_o \, \sigma_{\ell nT}\right]$$

Using the same in Eq. 12.85

$$E\left(S^m\right) = \frac{\widetilde{D}_f \, \widetilde{K}}{\widetilde{B}^m \left(f_o T_S\right) \exp\left(\beta_o \, \sigma_{\ell nT}\right)}$$

For long term stress range following Weibull distribution, Eq. 12.79 gives $E(S^m)$. Using the same, the expression for design (allowable) stress range for given β_o becomes

$$S_{rd} = [\ell n(f_o T_S)]^{1/k} \left[\frac{\tilde{D}_f \tilde{K}}{(f_o T_S) \lambda_m \tilde{B}^m \exp(\beta_o \sigma_{\ell nT}) \Gamma\left(\frac{m}{k}+1\right)} \right]^{1/m} \tag{12.91}$$

The various terms in the above equation are first calculated. The value of $\sigma_{\ell nT}$, using Eq. 12.84, is

$$\sigma_{\ell nT} = \left[\ell n \left\{(1+0.3^2)(1+0.7^2)(1+0.5^2)^9\right\}\right]^{1/2}$$
$$= 1.579$$
$$\Gamma\left(\frac{m}{k}+1\right) = \Gamma\left(\frac{3}{0.69}+1\right) = 41$$
$$\exp(\beta_o \sigma_{\ell nT}) = [(3)(1.579)] = 114.09$$
$$f_o T_S = 0.25 \times 20 \times 365 \times 24 \times 3600$$
$$= 1.575 \times 10^8$$

Substituting the above values and other given data in Eq. 12.91,

$$S_{rd'} = \left[\ell n\left(1.575 \times 10^8\right)\right]^{1/0.69} \left[\frac{1 \times 1.9 \times 10^{13}}{\left(1.575 \times 10^8\right)(0.86)(0.7)^3(23.52)(41)}\right]^{1/3}$$
$$= (70.592)\ (4.437)$$
$$= 313.2\ \text{N/mm}^2$$

This is the design stress range or allowable stress range for 20 year service period for the required reliability level $\beta_o = 3$.

Weibull format

Here N and long term stress range are assumed to follow Weibull distribution. If N is a random variable denoting the number of cycles to failure in variable amplitude fatigue loading and if it is assumed that N follows Weibull distribution (Type 3 extremal smallest distribution - refer Chapter 3) with parameters u and k_N, then (12.14)

$$k_N \approx (\delta_N)^{-1.08} \tag{12.92}$$

$$\mu_N = u\,\Gamma\left(\frac{1}{k}+1\right) \tag{12.93}$$

It is to be noted that Eq. 3.133 is approximated by Eq. 12.92 and Eq. 3.131 and Eq. 12.93 are same. Cumulative distribution of N is given by Eq. 3.130.

$$F_N(n) = 1 - \exp\left[-\left(\frac{n}{u}\right)^{k_N}\right] \qquad n \geq 0 \tag{12.94}$$

If $n = N_T$, failure occurs when $N < N_T$. Hence

$$p_f = P[N < N_T]$$

$$= 1 - \exp\left[-\left(\frac{N_T}{u}\right)^{k_N}\right]$$

If $p_f << 1$, the above equation can be approximated. For $p_f << 1$,

$$\left(\frac{N_T}{u}\right)^{k_N} = p_f$$

Or

$$u = \frac{N_T}{\left(p_f\right)^{1/k_N}}$$

Using the above equation in Eq. 12.93, the mean value of N is given by

$$\mu_N = \frac{N_T \Gamma\left(\frac{1}{k_N} + 1\right)}{\left(p_f\right)^{1/k_N}} \tag{12.94}$$

Assuming Miner's rule is applicable and D = 1 at failure, for safety

$$E(S)^m \leq \frac{\mu_K}{\mu_N} \tag{12.95}$$

where parameters K and N in S-N curve, are random variables with mean μ_K and μ_N respectively. Using Eq. 12.94 in Eq. 12.95

$$p_f = \left[\frac{N_T\, E(S^m)\, \Gamma\left(1 + \frac{1}{k_N}\right)}{\mu_K}\right]^{k_N} \tag{12.96}$$

If $k_N \approx (\delta_N)^{1.08}$, then

$$p_f = \left[\frac{N_T\, E(S^m)\, \Gamma\left(1 + \frac{1}{k_N}\right)}{\mu_K}\right]^{(\delta_N)^{-1.08}} \tag{12.97}$$

Expression for $E(S^m)$ is given by Eq. 12.79 assuming Weibull distribution for S. In the above treatment, k_N is a function of δ_N. To compute δ_N, let the fatigue model be

$$N = f(K\, S^{-m}) \tag{12.98}$$

where the parameter f accounts for the scatter in the constant amplitude S-N data. Using Taylor's series expansion, approximate value for δ_N is given by

$$\delta_N^2 = \delta_f^2 + (\mu_m)^2\, (\delta_s)^2 + \delta_K^2 + (\mu_m\, \ell n\, \mu_s)^2\, \delta_m^2 \tag{12.99}$$

Uncertainties in workmanship and fabrication are also included in δ_f . Once overall variation in N is determined, probability of failure can be evaluated using Eq. 12.97. Munse et al (12.15) analysed fatigue reliability of ship details using Weibull format.

Using Weibull format an expression for design stress range S_{rd} can also be written. If p_f, which is equal to P [N < N_T], is specified, then Eq. 12.96 can be rewritten as,

$$\frac{N_T\, E(S^m)\, \Gamma\left(1+\frac{1}{k_N}\right)}{\mu_K} = (p_f)^{1/k_N}$$

Assuming long term stress range follows Weibull distribution, the expression for $E(S^m)$ given by Eq. 12.79, can be used in the above equation. Hence

$$\frac{N_T\, \Gamma\left(1+\frac{1}{k_N}\right)}{\mu_K}\left[\lambda(m)(S_{rd})^m\,(\ell n\, N_T)^{-m/k}\,\Gamma\left(\frac{m}{k}+1\right)\right] = (p_f)^{1/k_N}$$

Rewriting the same equation for S_{rd} ,

$$S_{rd} = \left[\frac{\mu_K\,(p_f)^{1/k_N}}{N_T\,\Gamma\left(1+\frac{1}{k_N}\right)}\right]^{1/m}\left[\frac{1}{\lambda(m)\left[\ell n\, N_T\right]^{-1/k}\,\Gamma\left(\frac{m}{k}+1\right)^{1/m}}\right] \tag{12.100}$$

But from the mean S-N curve,

$$\frac{\mu_K}{N_T} = \left(\bar{S}\right)^m \tag{12.101}$$

Here $\bar{S}$ is the value of stress obtained from the constant amplitude mean S-N curve (from the test results). Using $\lambda(m) = 1$, Eq. 12.100, can be rewritten as

$$S_{rd} = \left[\frac{(p_f)^{1/k_N}}{\Gamma\left(1+\frac{1}{k_N}\right)}\right]^{1/m}\left[\frac{1}{(\ell n\, N_T)^{-1/k}\,\Gamma\left(\frac{m}{k}+1\right)^{1/m}}\right]\left(\bar{S}\right)$$

Let

$$R_f = \left[\frac{(p_f)^{1/k_N}}{\Gamma\left(1 + \frac{1}{k_N}\right)} \right]^{1/m} \tag{12.102}$$

$$\xi = (\ell n\, N_T)^{1/k}\, \Gamma\left(\frac{m}{k} + 1\right)^{-1/m} \tag{12.103}$$

Then

$$S_{rd} = R_f\ \xi\ \bar{S} \tag{12.104}$$

Munse (12.15) calls R_f as reliability factor and ξ as random load factor. Hence to get the design stress range, the stress range obtained from mean S-N curve is to be multiplied by R_f and ξ. Here, the idea is to reduce the equivalent stress range by reliability factor. The equivalent stress range is found by using the mean value of the fatigue life for calculating stress range from the S-N curve. The reliability factor contains the term δ_N (k_N is related to δ_N) which covers the uncertainty of all the factors in resistance and the term p_f which contains the desired level of the exceedance of design life. The random load factor connects the constant amplitude equivalent stress range for the loading to the once in a lifetime design stress. For ship structures, k is generally found to be 1. If the same value is used,

$$\xi = (\ell n\, N_T)\,\Gamma(m+1)^{-1/m}$$

Using the same, White and Ayyub (12.67) have determined the design stress ranges for details of ship structures.

EXAMPLE 12.9 The design stress range is to be suggested for the fatigue design of a welded structural detail in a ship. Determine the design stress range using Munse's approach based on Weibull format for the desired reliability level of 0.999 for a design life, N_d, of 10^8 cycles. It is given:

$\delta_N = 1.137$: $m = 7.0$

$K = 7.4 \times 10^{21}$ (Mpa units) for mean S-N curve.

Since design life is given as 10^8 cycles, $N_T = N_d = 10^8$ in Eq. 12.103. Required reliability level = 0.999. Hence $P_f = 1 - 0.999 = 0.001$. It is known

$$N\ S^m = K$$

Using the given values of N_T and K,

$$S = \left[\frac{7.4 \times 10^{21}}{10^8} \right]^{1/7}$$

$$= 95.79\ \text{N/mm}^2$$

The value of k_N is calculated using Eq. 12.92

$$k_N = (\delta_N)^{-1.08}$$
$$\frac{1}{k_N} = (1.137)^{1.08} = 1.1487$$

Using Eq. 12.102, the value of reliability factor is calculated.

$$R_f = \left[\frac{(p_f)^{1/k_N}}{\Gamma\left(1+\frac{1}{k_N}\right)}\right]^{1/m} = \left[\frac{(0.001)^{1.1487}}{\Gamma\left(1+\frac{1}{1.1487}\right)}\right]^{1/7} = 0.3187$$

Taking the value of k as 1 in Eq. 12.103, the random load factor is

$$\xi = \ell n\,(N_T)\,[\Gamma\,(m+1)]^{-1/m}$$
$$= \ell n\left(10^8\right)\{\Gamma\,[(7)+1]\,\}^{-1/7}$$
$$= (18.42)\ (0.2959) = 5.45$$

Hence the design stress range for reliability level of 0.999 for a life of 10^8 cycles is

$$S_{rd} = \left(\bar{S}\right)(R_f)(\xi)$$
$$= (95.79)\ (0.3187)\ (5.45)$$
$$= 166.38\ \text{N/mm}^2$$

12.6 FRACTURE MECHANICS APPROACH

Application of Fracture Mechanics for modeling fatigue crack growth propagation is well established (12.2, 12.16, 12.17). Fracture mechanics provides the methods by which techniques of applied mechanics can be applied to structures in the presence of a crack. In majority of fatigue situations, the crack will occur under elastic conditions. Hence the size of the plastic zone at the crack tip would be small compared to the crack size, thus making way for using Linear Elastic Fracture Mechanics (LEFM) concept. Inherent assumptions are small displacements and general linearity between stresses and strains. The behaviour of a cracked component is characterized by stress, crack size and structural dimensions. The effect of these parameters is modelled by defining Stress Intensity Factor (SIF), which is determined as

$$k = Y(a)\,S\,\sqrt{\pi\,a} \tag{12.105}$$

in which a is the crack size, S is the stress acting on the component and Y(a) is a geometric function depending on the shape of the specimen and crack geometry.

There are generally three models of loading which involve different crack surface displacements (12.2) in fracture mechanics study. They are

- Mode I : Opening or tearing mode,
- Mode II : Sliding or in-plane shear, and
- Mode III : Tearing or out-of-plane shear.

Mode I is the predominant loading mode in most of the structures (12.2). For an infinite plate subjected to uniform tensile stress (Mode I), SIF is given by

$$k = S\sqrt{\pi a} \tag{12.106}$$

At the moment of failure, the value of SIF reaches a critical value known as fracture toughness which is a material parameter. Fracture toughness represents the ultimate ability of a material to resist progressive crack extension. This property of a material has to be determined experimentally. It is seen that fracture toughness decreases with increase in specimen thickness upto a certain limit beyond which it almost becomes a constant.

One of the important parameters required for application of fracture mechanics is the crack size which can be suitably assumed or obtained by field measurements. The parameters involved in fracture mechanics studies, like fracture toughness, stress range, crack size, cannot be quantified exactly. There is always a certain amount of uncertainty in these parameters. Hence the principles of structural reliability can be made use of for estimating the probability of failure of a structure. Here a method for finding fatigue life is explained using principles of LEFM as applied to fatigue.

It is well known that fracture mechanics gives a better picture of fatigue crack growth than empirical S-N curve approach. In FM approach, Paris law (12.18) is used for modelling crack growth. The concept of equivalent stress range for representing the variable amplitude stress history is used.

Fatigue crack propagation is modelled using the concepts of LEFM. The crack growth rate is a function of stress intensity factor range which is given by

$$\Delta k = k_{max} - k_{min} \tag{12.107}$$

where k_{max} is the maximum SIF and k_{min} is the minimum SIF. The rate of fatigue crack propagation follows Paris crack growth law (12.18) given by,

$$\frac{da}{dN} = C\,(\Delta k)^n \tag{12.108}$$

in which a is the crack size, N is the number of cycles, C and n are crack growth parameters. C and n have to be determined experimentally. Figure 12.11 represents the typical crack growth rate curve. The curve has three distinct regions. Region I begins with a threshold value of SIF range, Δk_{th}, below which crack does not propagate. Region II is the zone in which the plot is linear where Paris law holds good. Region III has a steep slope and the curve approaches the maximum stress intensity factor range which is equal to the fracture toughness of the material. The steep gradient indicates unstable crack extension.

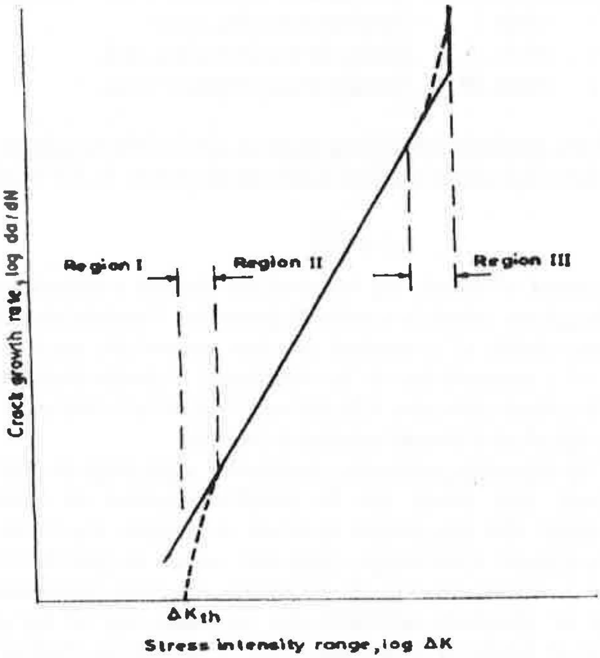


Fig. 12.11 Regions of fatigue crack growth

The general expression for stress intensity factor range is

$$\Delta\ k = Y(a)\, S \sqrt{\pi\, a} \tag{12.109}$$

in which S is the far field stress range from applied load. In actual situations, the stress range is not of constant amplitude, but of variable amplitude and frequency. For such a case equivalent static stress range, S_e is determined, and the same is used in Eq. 12.109. Hence stress intensity factor range, Δ k, becomes

$$\Delta\ k = Y(a)\, S_e \sqrt{\pi\, a} \tag{12.110}$$

Y(a) depends on the dimensions of the component. For various shapes and crack configurations, equations for determination of SIF are available (12.2, 12.19, 12.20). Once the expression for SIF is known, fatigue propagation life can be determined from Eq. 12.108 by separation of variables and adopting numerical integration. The fatigue life

$$N = \int_{a_i}^{a_f} \frac{da}{C\,(\Delta k)^n} \tag{12.111}$$

where a_i is the initial crack size and a_f is the final crack size. N is the number of cycles required for the crack to grow from a_i to a_f. Using Eq. 12.109 in the above equation,

$$\int_{a_i}^{a_f} \frac{da}{\left[Y(a)\sqrt{\pi a}\right]^m} = C\,N\,S_e^m \qquad (12.112)$$

For constant stress range S_e and Y(a) constant (that is Y(a) = Y) during crack' growth from a_i to a_f over N cycles, the above equation simplifies to

$$N S_e^m = \frac{1}{\left(\frac{m}{2}-1\right) C\, Y^m\, \pi^{m/2}} \left[\frac{1}{a_i^{\frac{m}{2}-1} - a_f^{\frac{m}{2}-1}}\right] \qquad (12.113)$$

This correspond to an S-N curve N S^m = K and suggests that the constant K can be expressed as a function of more basic quantities. Final crack size using the above equation becomes,

$$a_f = \left[a_i^{\left(1-\frac{m}{2}\right)} + \left(1-\frac{m}{2}\right) C \left(Y S_e \sqrt{\pi}\right)^m N\right]^{2/(2-m)} \qquad (12.114)$$

For stress cycles of varying amplitude, Eq. 12.113 may be used as S-N curve equation and reliability analysis can be carried out as explained earlier under S-N curve approach.

For reliability analysis two separate types of failure criteria can be used.

i) Failure occurs when the crack developed exceeds the predetermined or specified critical size a_c. The limit state function is written as

$$Z = a_c - a \qquad (12.115)$$

This criterion is based on the concept that when the crack has developed to the size a_c, it becomes unstable and the component is assumed to fail. This comes under serviceability limit state.

ii) Failure occurs when the stress intensity factor K at the leading edge of the crack exceeds the fracture toughness K_c. The limit state function is

$$\begin{aligned} Z &= K_c - K \\ &= K_c - Y(a)\ S\sqrt{\pi a} \end{aligned} \qquad (12.116)$$

The resistance is characterized by the material parameter K_c . The criterion comes under ultimate limit state. If small variance approximation is used, reliability index is given by (12.1),

$$\beta = \frac{\ell n \left[\frac{\overline{K_c}}{0.637\, \overline{S}\, (\pi \overline{a})^{1/2}} \right]}{\left[\delta_{K_c}^2 + \delta_S^2 + \frac{\delta_a^2}{4} \right]^{1/2}} \qquad (12.117)$$

Hence β can be calculated if statistics of K_c , S and a are known. Here $\overline{K_c}$ means the sample mean value of K_c .

For the development of inspection strategy and maintenance, it is necessary to know the number of cycles required to propagate the crack from a to a crack size a_f . The general expression for stress intensity factor range Δk is given by Eq. 12.109. Y(a) depends on crack shape, size and other factors. Sometimes Δk is generally written as

$$\Delta k = k_1\ k_2\ k_3\ k_4 \ldots\ldots S \sqrt{\pi a} \qquad (12.118)$$

where k_i are correction factors for crack shape, free surface effect, finite width effect, stress gradient effect etc. Equations for stress intensity factors are available for a variety of problems (12.2, 12.19, 12.20). The expression for SIF being known, the fatigue propagation life can be determined from Eq. 12.108 by separation of variables and adopting numerical integration. Hence fatigue life, N, is given by

$$N = \int_{a_i}^{a_f} \frac{da}{C\,(\Delta k)^n} \qquad (12.119)$$

The final crack size is calculated using Eq. 12.110

$$a_f = \frac{1}{\pi} \left[\frac{K_c}{Y(a)\, S_e} \right]^2 \qquad (12.120)$$

where K_c is the fracture toughness. Equation 12.120 is to be numerically solved since Y(a) is a function of a . Newton Raphson method can be used. The scheme of computation for a_f is as follows.

i) For the problem on hand, appropriate expression for SIF is selected.

ii) Knowing the initial cracks size and the fracture toughness, the final crack size is computed using Eq. 12.120. In the expression for SIF, a_i is substituted for a.

Monte Carlo Simulation

The variables involved in the scheme of computation for N are random variables in nature. Hence the number of cycles to fatigue crack propagation will also be a random variable which brings the concept of probability of failure. Monte Carlo technique is generally used for computing probability of failure for various desired number of cycles. The scheme of computation is as follows:

i) Knowing the distribution and parameters of random variables considered (say a_i, m, K_c, model parameter attached to the calculated stress range etc.), random values are generated for each of the variables.

ii) The final crack size, a_f, is computed using the generated values at the given stress range level and using Eq. 12.120.

iii) Knowing a_i and a_f, number of cycles elapsed for the crack propagating from a_i to a_f is determined from Eq. 12.119.

iv) The desired life in terms of cycles, N_s, being given, the limit state function is

$$Z = N - N_s$$

in which N is the number of cycles computed in step (iii).

v) Steps (i) to (iv) are repeated for a number of times say, n_s, to get an ensemble of realizations for Z.

vi) The probability of failure is then calculated as

$$p_f = \frac{n_f}{n_s} \tag{12.121}$$

where n_f is the number of times $Z < 0$ during simulation. Reliability index is taken as

$$\beta = -\Phi^{-1}\left(p_f\right)$$

The number of simulations, n_s, is fixed on the Schooman's error criterion (Refer Eq. 7.37 in Chapter 7).

Considerable work has been done on the fatigue reliability evaluation of riveted railway steel bridges in India (12.23, 12.24), welded steel bridges in U.S.A. (12.22) and marine structures (12.21, 12.25, 12.26) and application of fatigue reliability to offshore platform inspection (12.27, 12.28).

REFERENCES

12.1 ASCE Committee on Fatigue and Fracture Reliability,"(i) Fatigue Reliability: Introduction, (ii) Fatigue Reliability : Quality Assurance and Maintainability, (iii) Fatigue Reliability : Variable Amplitude Loading and (iv) Fatigue Reliability : Development of Criteria for Design", *Journal of Structural Engineering, ASCE*, Vol. 108, 1982, pp.3-88.

12.2 Bannantine, J.A. Comer, J.J. and Handrock, J.L., *Fundamentals of Metal Fatigue Analysis,* Prentice Hall, Englewood Cliffs, New Jersey, 1990

12.3 Albrecht, P., " Fatigue Reliability Analysis of Highway Bridges", *Probabilistic Fracture Mechanics and Fatigue Methods : Applications for Structural Design and Maintenance, ASTM - STP 798*, J.M. Bloom and J.C.Ekvall, Eds., ASTM 1983, pp. 184-204.

12.4 Moses, F., Schillinq, C.G. and Raju, K.S., "Fatigue Evaluation Procedures for Steel Bridges', *NCHRP 299, TRB*, Washington D.C., 1987

12.5 "Standard Specifications for Highway Bridges", 12th edition, *American Association of State Highway and Transportation Officials (AASHTO)*, Washington, D.C., 1977.

12.6 White, G.J. and Ayyub, B.M., "Reliability Based Fatigue Design", *Naval Engineers Journal, ASNE*, Vol. 99, pp.135 - 149.

12.7 BS : 5400 : PART 10 : 1980, "Steel, Concrete and Composite Bridges-Part 10 : Code of Practice for Fatigue", *British Standards Institution*, 1980.

12.8 I.F.C. Smith and M.A.Hirt, "Fatigue Reliability: ECCS Safety Factors", *Journal of Structural Engineering*, ASCE, Vol. 113, ST-3, PP.623-628.

12.9 Nyman W.E. and Moses, F., "Calibration of a Bridge Fatigue Design Model", *Journal of Structural Engineering, ASCE*, Vol. 111, ST 6, June 1985, pp. 1251-1266.

12.10 Ranganathan, R and Ravi G., "Fatigue Reliability Analysis of Riveted Steel Bridges", *Proceedings of National Conference on Civil Engineering. Materials and Structures (NC-CEMS)*, Hyderabad, Jan. 1995, pp. 424-431.

12.11 Bridge Rules (in SI units) incorporating correction slips 1 -16, Revised 1964, *Ministry of Railways, Government of India*

12.12 Wirsching, P.H., 'Fatigue Reliability of Offshore Structures', *Journal of Structure Engineering, ASCE*, Vol. 110, ST 8, October 1984, pp. 2340 -2356.

12.13 Wirsching, P.H., "Fatigue Reliability in Welded Joints of Offshore Structures", *Proceedings of Offshore Technology Conference*, Paper 3380, Houston, Texas, 1979, pp. 197 - 206.

12.14 Wirsching, P.H., "A Review of Modern approaches to Fatigue Reliability Analysis and Design"., *Proceedings of fourth National Congress on Pressure Vessels and piping Technology on Random Fatigue Prediction*, Portland Oregon, June 1983, Ed. Y.S Shin and M.K. Au Yang, Published by ASME, New York, 1983, pp. 107 - 120.

12.15 Munse, W.H., Wilbur, T.W., Tellalian, M.L., Nicoll, K and Wilson, K., "Fatigue Characterization of Fabricated Ship Details for Design", *Ship Structuress Committee Report* - 318, 1983.

12.16 Rolfe, S.T. and Barsom, J.M., *Fracture and Fatigue Control in Structures - Applications of Fracture Mechanics*, Prentice Hall, Englewood Cliffs, 1977.

12.17 Parker, A.P., *The Mechanics of Fracture and Fatigue*, E and F.N. Spon Ltd. New York, 1981.

12.18 Paris, P. and Erdogan, F., "A Critical Analysis of Crack Growth Propagation Laws", *Journal of Basic Engineering*, Vol. 85, 1945, pp. 528 - 534.

12.19 Rooke, D.P. and Cartwright, D.J., "Compendium of Stress Intensity Factors", Her Majesty's Stationery Office, London 1976.

12.20 *Stress Intensity Factors Handbook*, Vol. I, Ed. Y. Murakami, Pergamon Press, Oxford, 1988.

12.21 Rackwitz, R, "Probabilistic Deterioration Models and Optimization of Inspection", *Bridge Evaluation, Repair and Rehabilitation Proceedings of US European Workshop*, Eds. A.S. Nowak and E. Absi, 1987, pp. 359 - 364.

12.22 Yazdani, N. and Albrecht, P. "Risk Analysis of Fatigue Failure of Highway Steel Bridges" *Journal of Structural Engineering, ASCE*, Vol. 113, ST 3, March 1987, pp. 483 - 500.

12.23 Ravi, G. and Ranganathan, R., "Fatigue Crack Growth Reliability of Riveted Bridges", *International Journal of Structures,* Vol. 14, No.2, 1994, pp. 103-104.

12.24 Ravi G., 'Fatique Reliability Analysis and Design Approach to Riveted Steel Railway Bridges', *Ph.D. Thesis,* Indian Institute of Technology, Bombay, 1993

12.25 Wirsching, P.H. and Torng, T.Y., " Fatigue Reliability and Maintainability of Marine Structures", *Marine Structures,* No.3, 1990, pp. 265 -284.

12.26 Ayyub, B.M. White, G.J. and Purcell, E.S., "Estimation of Structural Service Life of Ships", *Naval Engineers Journal,* Vol. 101, May 1989, pp. 156 - 166.

12.27 Ashok Kumar and Karsen, D.I. "Fatigue Reliability of Parallel Systems", *Journal of Structural Engineering, ASCE,* Vol. 116, ST 3, March 1990, pp. 719 - 729.

12.28 Karsan, D.I. and Ashok Kumar, "Fatigue Failure Paths for Offshore Platform Inspection", *Journal of Structural Engineering, ASCE,* Vol. 116, ST 6, June 1990, pp. 1679 - 1695.

EXERCISE

12.1 Determine the fatigue reliability of a detail in a bridge designed as per LRFD format. It is given:

$\delta_R = 0.36$; $\delta_{S_e} = 0.15$; $m = 3$

$\gamma_R = 1.0$; $\gamma_S = 1.536$

(Ans. $\beta = 3.5$)2

12.2 A detail is to be designed for a reliability level of $\beta_0 = 3.5$. Determine γ_S fixing $\gamma_R = 1.0$. It is given :

$\delta_{S_e} = 0.25$; $\delta_R = 0.36$; $m = 3$

(Ans. $\gamma_S = 2.056$)

12.3 Determine the fatigue reliability of a welded joint in an offshore platform using Wirsching's approach for a 20 year life and long term design stress range $S_{rd} =$ 383.3 N/mm^2. It is given:

T_S = 20 yr ; k = 0.69 ; $\lambda(m)$ = 0.79

m = 4.42 ; f_0 = 0.25 hertz

$\tilde{K}$ = 9.22×10^{15} ; δ_K = 1.35

$\tilde{B}$ = 0.7 ; δ_B = 0.5

$\tilde{D}_f$ = 1.0 ; δ_{D_f} = 0.3

Long term stress range follows Weibull distribution and all other variables are lognormally distributed.

(Ans. $\beta = 2.09$)

12.4 Determine the design stress range of a welded detail in a ship using Munse's approach based on Weibull format for the desired fatigue reliability level of 0.999 and for a design life of 10^8 cycles. It is given:

$\delta_N = 0.78$; $m = 3.71$

$K = 2.53 \times 10^{14}$ (Mpa units)

(Ans. 109.7 N/mm^2)

APPENDIX A

Standard Normal Tables

TABLE A 1 *Cumulative probability of standard normal variate* $= \Phi(u)$

u	$\Phi(u)$	u	$\Phi(u)$	u	$\Phi(u)$
0	.50000				
−.01	.49601	−.37	.35569	−.73	.23270
−.02	.49202	−.38	.35197	−.74	.22965
−.03	.48803	−.39	.34827	−.75	.22663
−.04	.48405	−.40	.34458	−.76	.22363
−.05	.48006	−.41	.34090	−.77	.22065
−.06	.47608	−.42	.33724	−.78	.21770
−.07	.47210	−.43	.33360	−.79	.21476
−.08	.46812	−.44	.32997	−.80	.21186
−.09	.46414	−.45	.32636	−.81	.20897
−.10	.46017	−.46	.32276	−.82	.20611
−.11	.45620	−.47	.31918	−.83	.20327
−.12	.45224	−.48	.31561	−.84	.20045
−.13	.44828	−.49	.31207	−.85	.19766
−.14	.44433	−.50	.30854	−.86	.19489
−.15	.44038	−.51	.30503	−.87	.19215
−.16	.43644	−.52	.30153	−.88	.18943
−.17	.43251	−.53	.29806	−.89	.18673
−.18	.42858	−.54	.29460	−.90	.18406
−.19	.42465	−.55	.29116	−.91	.18141
−.20	.42074	−.56	.28774	−.92	.17879
−.21	.41683	−.57	.28434	−.93	.17619
−.22	.41294	−.58	.28096	−.94	.17361
−.23	.40905	−.59	.27760	−.95	.17106
−.24	.40517	−.60	.27425	−.96	.16853
−.25	.40129	−.61	.27093	−.97	.16602
−.26	.39743	−.62	.26763	−.98	.16354
−.27	.39358	−.63	.26435	−.99	.16109
−.28	.38974	−.64	.26109	−1.00	.15866
−.29	.38591	−.65	.25785	−1.01	.15625
−.30	.38209	−.66	.25463	−1.02	.15386
−.31	.37828	−.67	.25143	−1.03	.15151
−.32	.37448	−.68	.24825	−1.04	.14917
−.33	.37070	−.69	.24510	−1.05	.14686
−.34	.36693	−.70	.24196	−1.06	.14457
−.35	.36317	−.71	.23885	−1.07	.14231
−.36	.35942	−.72	.23576	−1.08	.14007

u	$\Phi(u)$	u	$\Phi(u)$	u	$\Phi(u)$
−1.09	.13786	−1.60	.05480	−2.11	.01743
−1.10	.13567	−1.61	.05370	−2.12	.01700
−1.11	.13350	−1.62	.05262	−2.13	.01659
−1.12	.13136	−1.63	.05155	−2.14	.01618
−1.13	.12924	−1.64	.05050	−2.15	.01578
−1.14	.12714	−1.65	.04947	−2.16	.01539
−1.15	.12507	−1.66	.04846	−2.17	.01500
−1.16	.12302	−1.67	.04746	−2.18	.01463
−1.17	.12100	−1.68	.04648	−2.19	.01426
−1.18	.11900	−1.69	.04551	−2.20	.01390
−1.19	.11702	−1.70	.04457	−2.21	.01355
−1.20	.11507	−1.71	.04363	−2.22	.01321
−1.21	.11314	−1.72	.04272	−2.23	.01287
−1.22	.11123	−1.73	.04182	−2.24	.01255
−1.23	.10935	−1.74	.04093	−2.25	.01222
−1.24	.10749	−1.75	.04006	−2.26	.01191
−1.25	.10565	−1.76	.03920	−2.27	.01160
−1.26	.10383	−1.77	.03836	−2.28	.01130
−1.27	.10204	−1.78	.03754	−2.29	.01101
−1.28	.10027	−1.79	.03673	−2.30	.01072
−1.29	.09853	−1.80	.03593	−2.31	.01044
−1.30	.09680	−1.81	.03515	−2.32	.01017
−1.31	.09510	−1.82	.03438	−2.33	.00990
−1.32	.09342	−1.83	.03362	−2.34	.00964
−1.33	.09176	−1.84	.03288	−2.35	.00939
−1.34	.09012	−1.85	.03216	−2.36	.00914
−1.35	.08851	−1.86	.03144	−2.37	.00889
−1.36	.08691	−1.87	.03074	−2.38	.00866
−1.37	.08534	−1.88	.03005	−2.39	.00842
−1.38	.08379	−1.89	.02938	−2.40	.00820
−1.39	.08226	−1.90	.02872	−2.41	.00798
−1.40	.08076	−1.91	.02807	−2.42	.00776
−1.41	.07927	−1.92	.02743	−2.43	.00755
−1.42	.07780	−1.93	.02680	−2.44	.00734
−1.43	.07636	−1.94	.02619	−2.45	.00714
−1.44	.07493	−1.95	.02559	−2.46	.00695
−1.45	.07353	−1.96	.02500	−2.47	.00676
−1.46	.07215	−1.97	.02442	−2.48	.00657
−1.47	.07078	−1.98	.02385	−2.49	.00639
−1.48	.06944	−1.99	.02330	−2.50	.00621
−1.49	.06811	−2.00	.02275	−2.51	.00604
−1.50	.06681	−2.01	.02222	−2.52	.00587
−1.51	.06552	−2.02	.02169	−2.53	.00570
−1.52	.06426	−2.03	.02118	−2.54	.00554
−1.53	.06301	−2.04	.02068	−2.55	.00539
−1.54	.06178	−2.05	.02018	−2.56	.00523
−1.55	.06057	−2.06	.01970	−2.57	.00508
−1.56	.05938	−2.07	.01923	−2.58	.00494
−1.57	.05821	−2.08	.01876	−2.59	.00480
−1.58	.05705	−2.09	.01831	−2.60	.00466
−1.59	.05592	−2.10	.01786	−2.61	.00453

(Contd.)

TABLE A 1 (*Contd.*)

u	$\Phi(u)$	u	$\Phi(u)$	u	$\Phi(u)$
−2.62	.00440	−3.13	.87403E—03	−3.64	.13632E—03
−2.63	.00427	−3.14	.84474E—03	−3.65	.13112E—03
−2.64	.00415	−3.15	.81635E—03	−3.66	.12611E—03
−2.65	.00402	−3.16	.78885E—03	−3.67	.12128E—03
−2.66	.00391	−3.17	.76219E—03	−3.68	.11662E—03
−2.67	.00379	−3.18	.73638E—03	−3.69	.11213E—03
−2.68	.00368	−3.19	.71136E—03	−3.70	.10780E—03
−2.69	.00357	−3.20	.68714E—03	−3.71	.10363E—03
−2.70	.00347	−3.21	.66367E—03	−3.72	.99611E—04
−2.71	.00336	−3.22	.64095E—03	−3.73	.95740E—04
−2.72	.00326	−3.23	.61895E—03	−3.74	.92010E—04
−2.73	.00317	−3.24	.59765E—03	−3.75	.88417E—04
−2.74	.00307	−3.25	.57703E—03	−3.76	.84957E—04
−2.75	.00298	−3.26	.55706E—03	−3.77	.81624E—04
−2.76	.00289	−3.27	.53774E—03	−3.78	.78414E—04
−2.77	.00280	−3.28	.51904E—03	−3.79	.75324E—04
−2.78	.00272	−3.29	.50094E—03	−3.80	.72348E—04
−2.79	.00264	−3.30	.48342E—03	−3.81	.69483E—04
−2.80	.00256	−3.31	.46648E—03	−3.82	.66726E—04
−2.81	.00248	−3.32	.45009E—03	−3.83	.64072E—04
−2.82	.00240	−3.33	.43423E—03	−3.84	.61517E—04
−2.83	.00233	−3.34	.41889E—03	−3.85	.59059E—04
−2.84	.00226	−3.35	.40406E—03	−3.86	.56694E—04
−2.85	.00219	−3.36	.38971E—03	−3.87	.54418E—04
—2.86	.00212	−3.37	.37584E—03	−3.88	.52228E—04
−2.87	.00205	−3.38	.36243E—03	−3.89	.50122E—04
−2.88	.00199	−3.39	.34946E—03	−3.90	.48096E—04
−2.89	.00193	−3.40	.33693E—03	−3.91	.46148E—04
−2.90	.00187	—3.41	.32481E—03	−3.92	.44274E—04
−2.91	.00181	−3.42	.31311E—03	−3.93	.42473E—04
−2.92	.00175	−3.43	.30179E—03	−3.94	.40741E—04
−2.93	.00169	−3.44	.29086E—03	−3.95	.39076E—04
−2.94	.00164	−3.45	.28029E—03	−3.96	.37475E—04
−2.95	.00159	−3.46	.27009E—03	−3.97	.35936E—04
−2.96	.00154	−3.47	.26023E—03	−3.98	.34458E—04
−2.97	.00149	−3.48	.25071E—03	−3.99	.33037E—04
−2.98	.00144	−3.49	.24151E—03	−4.00	.31671E—04
−2.99	.00139	−3.50	.23263E—03	−4.01	.30359E—04
−3.00	.00135	−3.51	.22405E—03	−4.02	.29099E—04
−3.01	.13062E—02	−3.52	.21577E—03	−4.03	.27888E—04
−3.02	.12639E—02	−3.53	.20778E—03	−4.04	.26726E—04
−3.03	.12228E—02	−3.54	.20006E—03	−4.05	.25609E—04
−3.04	.11829E—02	−3.55	.19262E—03	−4.06	.24536E—04
−3.05	.11442E—02	−3.56	.18543E—03	−4.07	.23507E—04
−3.06	.11067E—02	−3.57	.17849E—03	−4.08	.22518E—04
−3.07	.10703E—02	−3.58	.17180E—03	−4.09	.21569E—04
−3.08	.10350E—02	−3.59	.16534E—03	−4.10	.20658E—04
−3.09	.10008E—02	−3.60	.15911E—03	−4.11	.19783E—04
−3.10	.96760E—03	−3.61	.15310E—03	−4.12	.18944E—04
−3.11	.93544E—03	−3.62	.14730E—03	−4.13	.18138E—04
−3.12	.90426E—03	−3.63	.14171E—03	−4.14	.17365E—04

u	$\Phi(u)$	u	$\Phi(u)$	u	$\Phi(u)$
−4.15	.16624E—04	−4.82	.71779E—06	−5.84	.26100E—08
−4.16	.15912E—04	−4.84	.64920E—06	−5.86	.23143E—08
−4.17	.15230E—04	−4.86	.58693E—06	−5.88	.20513E—08
−4.18	.14575E—04	−4.88	.53043E—06	−5.90	.18175E—08
−4.19	.13948E—04	−4.90	.47918E—06	−5.92	.16097E—08
−4.20	.13346E—04	−4.92	.43272E—06	−5.94	.14251E—08
−4.21	.12769E—04	−4.94	.39061E—06	−5.96	.12612E—08
−4.22	.12215E—04	−4.96	.35247E—06	−5.98	.11157E—08
−4.23	.11685E—04	−4.98	.31792E—06	−6.00	.98659E—09
−4.24	.11176E—04	−5.00	.28665E—06	−6.02	.87209E—09
−4.25	.10689E—04	−5.02	.25836E—06	−6.04	.77057E—09
−4.26	.10221E—04	−5.04	.23277E—06	−6.06	.68061E—09
−4.27	.97736E—05	−5.06	.20963E—06	−6.08	.60091E—09
−4.28	.93447E—05	−5.08	.18872E—06	−6.10	.53034E—09
−4.29	.89337E—05	−5.10	.16983E—06	−6.12	.46788E—09
−4.30	.85399E—05	−5.12	.15277E—06	−6.14	.41261E—09
−4.31	.81627E—05	−5.14	.13737E—06	−6.16	.36372E—09
−4.32	.78015E—05	−5.16	.12347E—06	−6.18	.32051E—09
−4.33	.74555E—05	−5.18	.11094E—06	−6.20	.28232E—09
−4.34	.71241E—05	−5.20	.99644E—07	−6.22	.24858E—09
−4.35	.68069E—05	−5.22	.89462E—07	−6.24	.21879E—09
−4.36	.65031E—05	−5.24	.80288E—07	−6.26	.19249E—09
−4.37	.62123E—05	−5.26	.72028E—07	−6.28	.16929E—09
−4.38	.59340E—05	−5.28	.64592E—07	−7.30	.14882E—09
−4.39	.56675E—05	−5.30	.57901E—07	−6.32	.13078E—09
−4.40	.54125E—05	−5.32	.51884E—07	−6.34	.11488E—09
−4.41	.51685E—05	−5.34	.46473E—07	−6.36	.10088E—09
−4.42	.49350E—05	−5.36	.41611E—07	−6.38	.88544E—10
−4.43	.47117E—05	−5.38	.37243E—07	−6.40	.77688E—10
−4.44	.44979E—05	−5.40	.33320E—07	−6.42	.68137E—10
−4.45	.42935E—05	−5.42	.29800E—07	−6.44	.59737E—10
−4.46	.40980E—05	−5.44	.26640E—07	−6.46	.52351E—10
−4.47	.39110E—05	−5.46	.23807E—07	−6.48	.45861E—10
−4.48	.37322E—05	−5.48	.21266E—07	−6.50	.40160E—10
−4.49	.35612E—05	−5.50	.18990E—07	−6.52	.35154E—10
−4.50	.33977E—05	−5.52	.16950E—07	−6.54	.30759E—10
−4.52	.30920E—05	−5.54	.15124E—07	−6.56	.26904E—10
−4.54	.28127E—05	−5.56	.13489E—07	−6.58	.23522E—10
−4.56	.25577E—05	−5.58	.12026E—07	−6.60	.20558E—10
−4.58	.23249E—05	−5.60	.10718E—07	−6.62	.17960E—10
−4.60	.21125E—05	−5.62	.95479E—08	−6.64	.15684E—10
−4.62	.19187E—05	−5.64	.85025E—08	−6.66	.13691E—10
−4.64	.17420E—05	−5.66	.75686E—08	−6.68	.11947E—10
−4.66	.15810E—05	−5.68	.67347E—08	−6.70	.10421E—10
−4.68	.14344E—05	−5.70	.59904E—08	−6.72	.90862E—11
−4.70	.13008E—05	−5.72	.53262E—08	−6.74	.79193E—11
−4.72	.11792E—05	−5.74	.47338E—08	−6.76	.68996E—11
−4.74	.10686E—05	−5.76	.42057E—08	−6.78	.60088E—11
−4.76	.96796E—06	−5.78	.37350E—08	−6.80	.52310E—11
−4.78	.87648E—06	−5.80	.33157E—08	−6.82	.45520E—11
−4.80	.79333E—06	−5.82	.29424E—08	−6.84	.39597E—11

(Contd)

TABLE A 1 (*Contd*)

u	$\Phi(u)$	u	$\Phi(u)$	u	$\Phi(u)$
−6.86	.34430E−11	−7.88	.16369E−14	−8.90	.27923E−18
−6.88	.29926E−11	−7.90	.13945E−14	−8.92	.23314E−18
−6.90	.26001E−11	−7.92	.11876E−14	−8.94	.19459E−18
−6.92	.22582E−11	−7.94	.10109E−14	−8.96	.16234E−18
−6.94	.19605E−11	−7.96	.86020E−15	−8.98	.13538E−18
−6.96	.17014E−11	−7.98	.73167E−15	−9.00	.11286E−18
−6.98	.14759E−11	−8.00	.62210E−15	−9.02	.94045E−19
−7.00	.12798E−11	−8.02	.52873E−15	−9.04	.78336E−19
−7.02	.11093E−11	−8.04	.44919E−15	−9.06	.65225E−19
−7.04	.96120E−12	−8.06	.38147E−15	−9.08	.54287E−19
−7.06	.83251E−12	−8.08	.32383E−15	−9.10	.45166E−19
−7.08	.72077E−12	−8.10	.27480E−15	−9.12	.37562E−19
−7.10	.62378E−12	−8.12	.23309E−15	−9.14	.31226E−19
−7.12	.53964E−12	−8.14	.19764E−15	−9.16	.25949E−19
−7.14	.46665E−12	−8.16	.16751E−15	−9.18	.21555E−19
−7.16	.40339E−12	−8.18	.14192E−15	−9.20	.17897E−19
−7.18	.34856E−12	−8.20	.12019E−15	−9.22	.14855E−19
−7.20	.30106E−12	−8.22	.10175E−15	−9.24	.12325E−19
−7.22	.25994E−12	−8.24	.86105E−16	−9.26	.10222E−19
−7.24	.22434E−12	−8.26	.72836E−16	−9.28	.84739E−20
−7.26	.19355E−12	−8.28	.61588E−16	−9.30	.70223E−20
−7.28	.16691E−12	−8.30	.52056E−16	−9.32	.58170E−20
−7.30	.14388E−12	−8.32	.43982E−16	−9.34	.48197E−20
−7.32	.12399E−12	−8.34	.37145E−16	−9.36	.39868E−20
−7.34	.10680E−12	−8.36	.31359E−16	−9.38	.32986E−20
−7.36	.91955E−13	−8.38	.26464E−16	−9.40	.27282E−20
−7.38	.79145E−13	−8.40	.22324E−16	−9.42	.22554E−20
−7.40	.68092E−13	−8.42	.18824E−16	−9.44	.18639E−20
−7.42	.58560E−13	−8.44	.15867E−16	−9.46	.15397E−20
−7.44	.50343E−13	−8.46	.13369E−16	−9.48	.12714E−20
−7.46	.43261E−13	−8.48	.11260E−16	−9.50	.10495E−20
−7.48	.37161E−13	−8.50	.94795E−17	−9.52	.86590E−21
−7.50	.31909E−13	−8.52	.79777E−17	−9.54	.71416E−21
−7.52	.27388E−13	−8.54	.67111E−17	−9.56	.58878E−21
−7.54	.23499E−13	−8.56	.56434E−17	−9.58	.48522E−21
−7.56	.20153E−13	−8.58	.47437E−17	−9.60	.39972E−21
−7.58	.17278E−13	−8.60	.39858E−17	−9.62	.32916E−21
−7.60	.14807E−13	−8.62	.33477E−17	−9.64	.27094E−21
−7.62	.12684E−13	−8.64	.28107E−17	−9.66	.22293E−21
−7.64	.10861E−13	−8.66	.23588E−17	−9.68	.18336E−21
−7.66	.92967E−14	−8.68	.19788E−17	−9.70	.15075E−21
−7.68	.79544E−14	−8.70	.16594E−17	−9.72	.12389E−21
−7.70	.68033E−14	−8.72	.13910E−17	−9.74	.10178E−21
−7:72	.58165E−14	−8.74	.11656E−17	−9.76	.83578E−22
−7.74	.49708E−14	−8.76	.97625E−18	−9.78	.68605E−22
−7.76	.42465E−14	−8.78	.81737E−18	−9.80	.56293E−22
−7.78	.36262E−14	−8.80	.68408E−18	−9.82	.46172E−22
−7.80	.30954E−14	−8.82	.57230E−18	−9.84	.37855E−22
−7.82	.26412E−14	−8.84	.47859E−18	−9.86	.31025E−22
−7.84	.22527E−14	−8.86	.40007E−18	−9.88	.25416E−22
−7.86	.19207E−14	−8.88	.33430E−18	−9.90	.20814E−22

u	$\Phi(u)$	u	$\Phi(u)$	u	$\Phi(u)$
−9.92	.17038E−22	−2.32	10^{-2}	−5.20	10^{-7}
−9.94	.13941E−22	−3.09	10^{-3}	−5.61	10^{-8}
−9.96	.11403E−22	−3.70	10^{-4}	−6.00	10^{-9}
−9.98	.93233E−23	−4.26	10^{-5}	−6.36	10^{-10}
−10.00	.76199E−23	−4.75	10^{-6}	−6.71	10^{-11}
−1.28	10^{-1}				

APPENDIX B

Partial Safety Factors for RCC Members

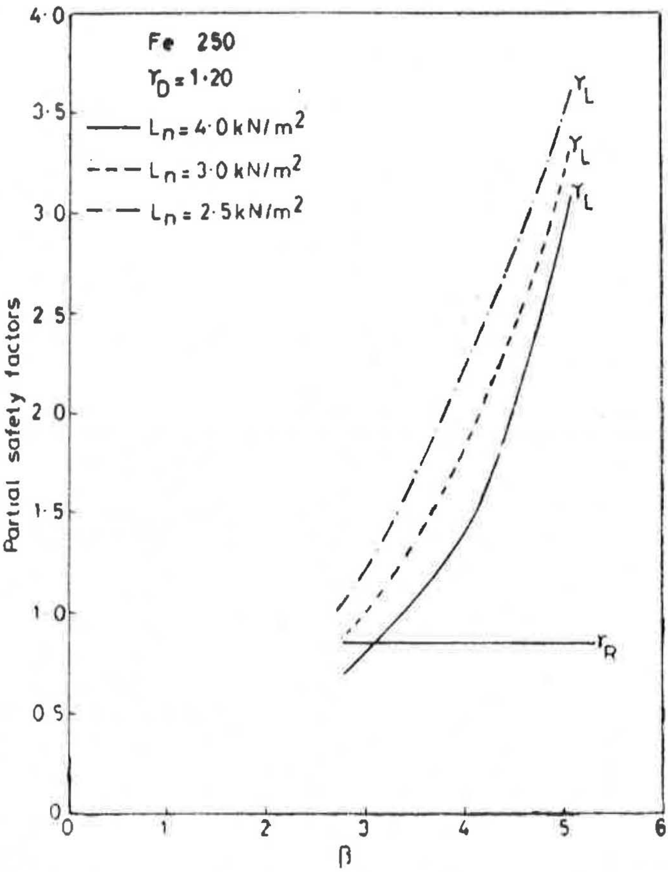


FIG. B1 Optimal values of partial safety factors for RCC slabs in flexure under load $D + L_m$ for steel grade Fe 250

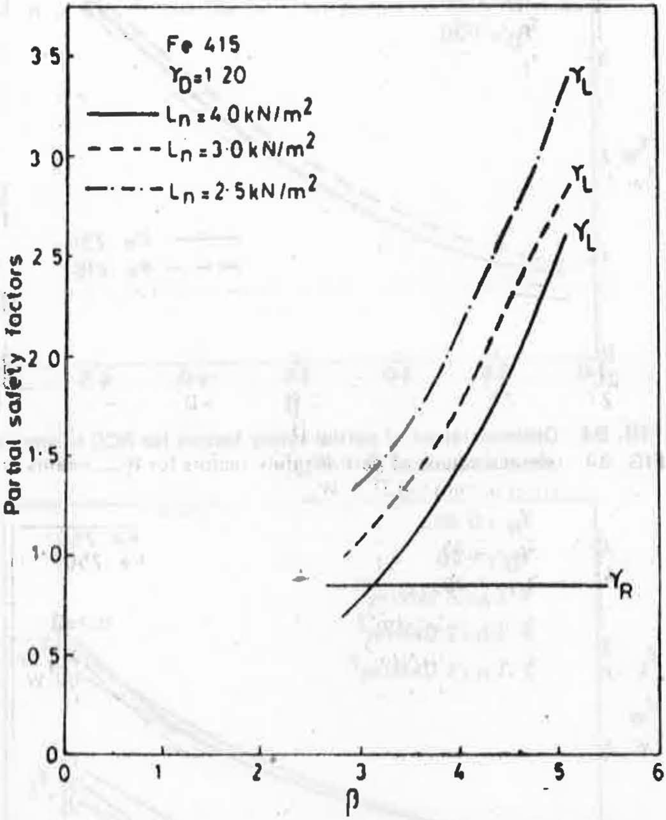


FIG. B2 Optimal values of partial safety factors for RCC slabs in flexure under load $D + L_m$ for steel grade Fe 415

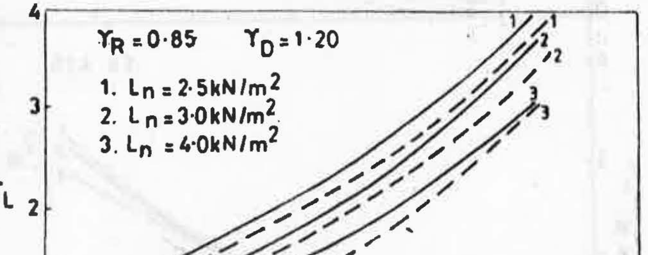


FIG. B3 Optimal values of partial safety factors for RCC beams in shear under load $D + L_m$

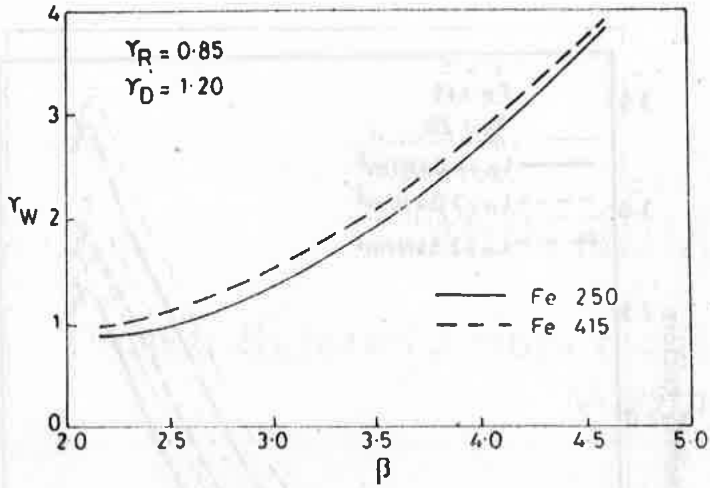


FIG. B4 Optimal values of partial safety factors for RCC beams in shear under load $D + W_m$

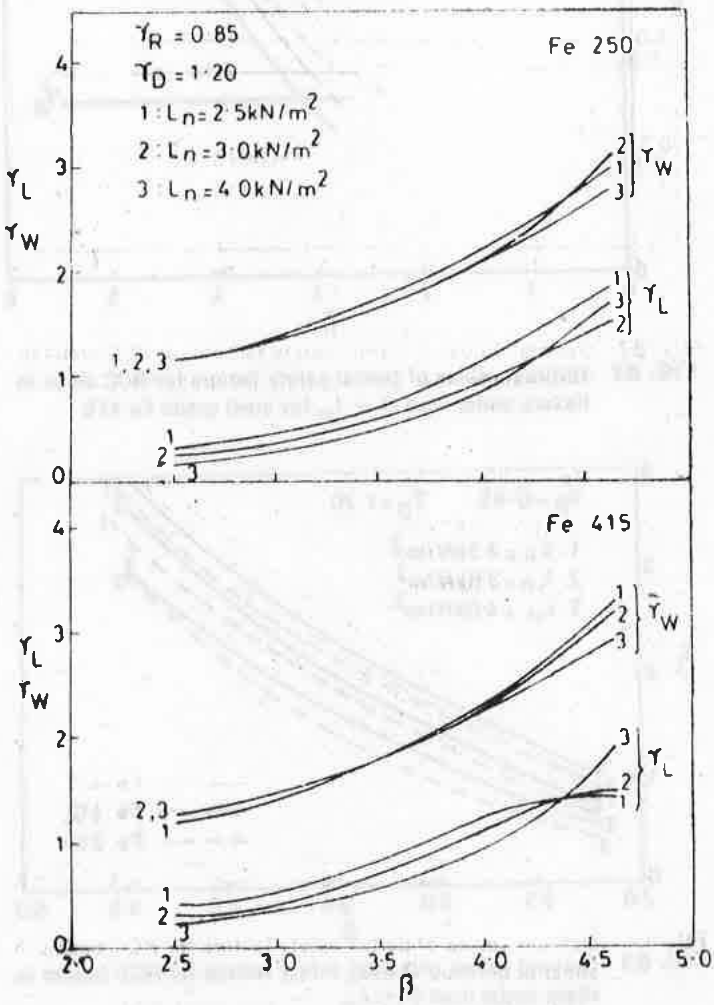


FIG. B5 Optimal values of partial safety factors for RCC beams in shear under load $D + L_{ant} + W_m$

TABLE B1 *Optimal partial safety factors for loads for columns*

$\gamma_D = 1.2$ For Comp. $\gamma_R = 0.725$

For Tens. $\gamma_R = 0.80$

Load Combination	Mix	Failure	β_0	$D + L_m$ γ_L	$D + W_m$ γ_W	$D + L_{apt} + W_m$ γ_L	γ_W
Case (i)							
$L_n = 3$ kN/m²	Design	Comp.	3.0	1.2	1.2	0.30	1.1
			3.5	1.4	1.5	0.27	1.5
			4.0	1.9	2.0	0.24	2.0
		Tension	3.0	1.3	1.5	0.25	1.4
			3.5	1.8	2.0	0.24	1.8
			4.0	2.4	2.5	0.23	2.3
	Nominal	Comp.	3.0	1.4	1.6	0.20	1.6
			3.5	2.5	2.6	0.17	2.6
			4.0	4.0	3.6	0.15	3.6
		Tension	3.0	1.4	1.5	0.22	1.5
			3.5	2.3	2.3	0.20	2.3
			4.0	3.5	3.2	0.18	3.2
Case (ii)							
$L_n = 4$ kN/m²	Design	Comp.	3.0	0.8	1.2	0.20	1.1
			3.5	1.1	1.5	0.18	1.5
			4.0	1.5	2.0	0.17	2.0
		Tension	3.0	1.0	1.5	0.18	1.4
			3.5	1.4	2.0	0.17	1.8
			4.0	1.8	2.5	0.16	2.3
	Nominal	Comp.	3.0	1.1	1.6	0.15	1.6
			3.5	2.1	2.6	0.13	2.5
			4.0	3.9	3.6	0.17	3.5
		Tension	3.0	1.0	1.5	0.17	1.3
			3.5	1.7	2.3	0.15	2.3
			4.0	2.7	3.2	0.12	3.2

Index