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Claus Gerhardt

The Quantization of Gravity

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Preface

A unified quantum theory incorporating the four fundamental forces of nature is one of the major open problems in physics. The Standard Model combines electromagnetism, the strong force and the weak force, but ignores gravity. The quantization of gravity is, therefore, a necessary first step to achieve a unified quantum theory.

The Einstein equations are the Euler–Lagrange equations of the Einstein–Hilbert functional, and quantization of a Lagrangian theory requires to switch from a Lagrangian view to a Hamiltonian view. In a groundbreaking paper, Arnowitt, Deser and Misner [1] expressed the Einstein–Hilbert Lagrangian in a form which allowed to derive a corresponding Hamilton function by applying the Legendre transformation. However, since the Einstein–Hilbert Lagrangian is singular, the Hamiltonian description of gravity is only correct if two additional constraints are satisfied, namely the Hamilton constraint, which is expressed by the equation $H = 0$, where H is the Hamilton function, and the diffeomorphism constraint. Dirac [8] proved how to quantize a constrained Hamiltonian system—at least in principle—and his method has been applied to the Hamiltonian setting of gravity, cf. the paper of DeWitt [6] and the monographs by Kiefer [36] and Thiemann [42]. In the general case, when arbitrary globally hyperbolic spacetime metrics are allowed, the problem turned out to be extremely difficult and solutions could only be found by assuming a high degree of symmetry, cf., e.g. [14].

However, in [15, 17, 18] we proposed a model for the quantization of gravity for general hyperbolic spacetimes, in which we eliminated the diffeomorphism constraint by reducing the number of variables and proving that the Euler–Lagrange equations for this special class of metrics were still the full Einstein equations. The Hamiltonian description of the Einstein–Hilbert functional then allowed a canonical quantization. We quantized the action by looking at the Wheeler–DeWitt equation in a fibre bundle E, where the base space is a Cauchy hypersurface of the spacetime which has been quantized and the elements of the fibres are Riemannian metrics. The fibres of E are equipped with a Lorentzian metric such that they are globally hyperbolic, and the transformed Hamiltonian, which is now a hyperbolic operator,

$\hat{H}$, is a normally hyperbolic operator acting only in the fibres. The Wheeler–DeWitt equation has the form $\hat{H}u = 0$ with $u \in C^\infty(E, \mathbb{C})$ and we defined with the help of the Green's operator a symplectic vector space and a corresponding Weyl system.

The Wheeler–DeWitt equation seems to be the obvious quantization of the Hamilton condition. However, $\hat{H}$ acts only in the fibres and not in the base space which is due to the fact that the derivatives are only ordinary covariant derivatives and not functional derivatives, though they are supposed to be functional derivatives, but this property is not really invoked when a functional derivative is applied to u, since the result is the same as applying a partial derivative.

Therefore, we discarded the Wheeler–DeWitt equation in [19] and expressed the Hamilton condition differently by looking at the evolution equation of the mean curvature of the foliation hypersurfaces $M(t)$ and implementing the Hamilton condition on the right-hand side of this evolution equation. The left-hand side, a time derivative, we replaced by the corresponding Poisson brackets. After canonical quantization, the Poisson brackets became a commutator and now we could employ the fact that the derivatives are functional derivatives, since we had to differentiate the scalar curvature of a metric. As a result, we obtained an elliptic differential operator in the base space, the main part of which was the Laplacian of the metric.

On the right-hand side of the evolution equation, the interesting term was H^2, the square of the mean curvature. It transformed to a second-time derivative, the only remaining derivative with respect to a fibre variable, since the differentiations with respect to the other variables cancelled each other. The resulting quantized equation is then a wave equation in a globally hyperbolic spacetime.

$$Q = (0, \infty) \times \mathcal{S}_0,$$

where $\mathcal{S}_0$ is the Cauchy hypersurface. When $\mathcal{S}_0$ is a space of constant curvature, then the wave equation, considered only for functions u which do not depend on x, is identical to the equation obtained by quantizing the Hamilton constraint in a Friedmann universe without matter but including a cosmological constant.

There also exist temporal and spatial self-adjoint operators H_0 resp. H_1 such that the hyperbolic equation is equivalent to

$$H_0 u - H_1 u = 0,$$

where $u = u(t, x)$, and H_0 has a pure point spectrum with eigenvalues λ_i while, for H_1, it is possible to find corresponding eigendistributions for each of the eigenvalues λ_i, if $\mathcal{S}_0$ is asymptotically Euclidean or if the quantized spacetime is a black hole with a negative cosmological constant, cf. [22–24]. The hyperbolic equation then has a sequence of smooth solutions which are products of temporal eigenfunctions and spatial eigendistributions. Due to this "spectral resolution" of the wave equation, we were also able to apply quantum statistics to the quantized systems, cf. [25]. These quantum statistical results could help to explain the nature of dark matter and dark energy.

We believe that the wave equation model in the spacetime Q is a very promising model for describing quantum gravity.

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Chapter 1
The Quantization of a Globally Hyperbolic Spacetime

1.1 Introduction

The quantization of gravity is hampered by the fact that the Einstein–Hilbert Lagrangian is singular. Switching to a Hamiltonian setting requires to impose two constraints, the Hamilton constraint and the diffeomorphism constraint. Though we were able to eliminate the diffeomorphism constraint in a recent paper [16], the Hamilton constraint is a serious obstacle. Quantization of a Hamiltonian setting requires a model in which the quantized variables, which turn into operators, act, and, in case of constraints, preferably given as an equation, to quantize this equation.

In the former paper, we proposed a quantization of gravity by working in a fiber bundle E with base space $\mathcal{S}_0$ after quantization, the Hamilton function H was transformed to an hyperbolic operator $\widehat{H}$, and the Hamilton condition, which could be expressed by

$$H = 0, \tag{1.1.1}$$

was transformed to the Wheeler–DeWitt equation

$$\widehat{H} u = 0 \tag{1.1.2}$$

in the bundle E. However, the operator $\widehat{H}$ acts only in the fibers and there is no differentiation in the base space $\mathcal{S}_0$, though the solutions are defined in E. This seems to be unsatisfactory.

We therefore use a better quantization model, cf. [18]: We are still working in the bundle E, but we discard the Wheeler–DeWitt equation; i.e., we do not express the Hamilton constraint by Eq. (1.1.1) but differently using the Hamilton equations. The second Hamilton equation has the form

$$\dot{\pi}^{ij} = -\frac{\delta \mathcal{H}}{\delta g_{ij}}, \tag{1.1.3}$$

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or equivalently,

$$\dot{\pi}^{ij} = \{\pi^{ij}, \mathcal{H}\}, \tag{1.1.4}$$

where we use a Hamiltonian density at the moment. Hence, we have the identity

$$g_{ij}\{\pi^{ij}, \mathcal{H}\} = -g_{ij}\frac{\delta \mathcal{H}}{\delta g_{ij}} \tag{1.1.5}$$

which is a scalar equation.

The Hamilton constraint can be expressed in the form

$$|A|^2 - H^2 = (R - 2\Lambda). \tag{1.1.6}$$

Looking at the right-hand side of (1.1.5), the term $|A|^2 - H^2$, which will be transformed to be the main part of the hyperbolic operator, occurs on the right-hand side in two places. Replacing $|A|^2 - H^2$ on the right side by $(R - 2\Lambda)$ will give an equation that defines the Hamilton constraint.

We developed two models: In the first model, we replaced $|A|^2 - H^2$ partially in (1.1.5). The quantization of the modified equation then leads to a hyperbolic equation

$$Pu = 0 \tag{1.1.7}$$

in E, where P acts in the fibers as well as in $\mathcal{S}_0$. P is a symmetric operator, and with the help of its Green's operator, one can define a symplectic vector space and then a Weyl system, or a quantum field.

In the second model, we use a geometric evolution equation to express the Hamilton constraint by replacing $|A|^2 - H^2$ completely in the evolution equation. After quantization, we then obtain a wave equation in E

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^2(t^{-\frac{4}{n}}R - 2\Lambda)u = 0 \tag{1.1.8}$$

in points $(x, t, \xi) \in E$, where a metric g_{ij} in the fiber over $x \in \mathcal{S}_0$ has the form

$$g_{ij} = t^{\frac{4}{n}}\sigma_{ij}(x, \xi) \tag{1.1.9}$$

and the Laplacian in (1.1.8) is defined with respect to σ_{ij}. Hence, for any ξ we have a wave equation in

$$\mathcal{S}_0 \times \mathbb{R}^*_+ \tag{1.1.10}$$

with solutions $u = u(x, t, \xi)$. We prove that solutions of the corresponding Cauchy problems exist and are smooth in all variables.

This second model seems to be the right model since it contains the quantization of a cosmological Friedmann universe, without matter but with a cosmological constant, as a special case by choosing σ_{ij} to be the metric of a space of constant curvature and by assuming $u = u(t)$. Equation (1.1.8) is in this case identical to the quantized Friedmann equation up to the last constant.

Moreover, assuming $\mathcal{S}_0$ to be compact we also derive a spectral resolution of Eq. (1.1.8), by constructing a countable basis of solutions of the form

$$u = w(t)v(x), \tag{1.1.11}$$

where v is an eigenfunction of the problem

$$-(n-1)\Delta v - \frac{n}{2}Rv = \mu v \tag{1.1.12}$$

in $\mathcal{S}_0$ with $\mu > 0$ and w an eigenfunction of an ODE. These solutions have finite energy, cf. (1.6.73) on page 49.

However, the most satisfying spectral resolution we shall obtain later by defining *temporal* resp. *spatial* Hamiltonians H_0 resp. H_1 such that the wave equation (1.1.8) can be expressed in the form

$$t^{2-\frac{n}{4}}(H_1 - H_0)u = 0, \tag{1.1.13}$$

where H_0 has a pure point spectrum with positive eigenvalues λ_i and where it is possible to find corresponding eigendistributions of H_1 for each of the eigenvalues λ_i. Let w_i be a complete mutually orthogonal sequence of eigenvectors of H_0 and v_{ij} a sequence of smooth eigendistributions of H_1 satisfying

$$H_1 v_{ij} = \lambda_i v_{ij} \qquad \forall\, j, \tag{1.1.14}$$

then

$$u_{ij} = w_i v_{ij} \tag{1.1.15}$$

will be solutions of the wave equation. This approach will require that the λ_i belong to the continuous spectrum of H_1 which will be the case if the positive real numbers are part of the continuous spectrum. Fortunately, this will be true for important classes of open spacetimes, i.e. spacetimes with non-compact Cauchy hypersurfaces $\mathcal{S}_0$, cf. the examples in the chapters [3–6].

The results for the first model are proved and described in detail in Sects. 1.4 and 1.5. The results for the second model are proved in Sect. 1.6. Here is a more formal summary of the results of the second model:

Theorem 1.1.1 *Let* $(\mathcal{S}_0, \sigma_{ij})$ *be a given connected, smooth and complete* n*-dimensional Riemannian manifold and let*

$$Q = \mathcal{S}_0 \times \mathbb{R}_+^* \tag{1.1.16}$$

be the associated globally hyperbolic spacetime equipped with the Lorentzian metric (1.6.41) *or, if necessary, with* (1.6.42)*, then the hyperbolic equation*

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0, \tag{1.1.17}$$

where the Laplacian and the scalar curvature correspond to the metric σ_{ij}, describes a model for quantum gravity. If $\mathcal{S}_0$ is compact, a spectral resolution of this equation has been proved in the theorem below.

Theorem 1.1.2 *Assume $n \geq 2$ and $\mathcal{S}_0$ to be compact and let (v, μ) be a solution of the eigenvalue problem (1.1.12) with $\mu > 0$, then there exist countably many solutions (w_i, Λ_i) of the implicit eigenvalue problem (1.6.57) such that*

$$\Lambda_i < \Lambda_{i+1} < \cdots < 0, \tag{1.1.18}$$

$$\lim_i \Lambda_i = 0, \tag{1.1.19}$$

and such that the functions

$$u_i = w_i v \tag{1.1.20}$$

are solutions of the wave equations (1.1.8). The transformed eigenfunctions

$$\tilde{w}_i(t) = w_i(\lambda_i^{\frac{n}{4(n-1)}} t), \tag{1.1.21}$$

where

$$\lambda_i = (-\Lambda_i)^{-\frac{n-1}{n}} \tag{1.1.22}$$

form a basis of the corresponding Hilbert space H and also of $L^2(\mathbb{R}_+^*, \mathbb{C})$.

1.2 Definitions and Notations

The main objective of this section is to state the equations of Gauß, Codazzi and Weingarten for spacelike hypersurfaces M in a (n + 1)-dimensional Lorentzian manifold N. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\overline{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by (g_{ij}), (R_{ijkl}), etc. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i). Covariant differentiation will simply be indicated by indices, and only in case of possible ambiguity they will be preceded by a semicolon; i.e., for a function u in N, (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but, e.g., the covariant derivative of the curvature tensor will be abbreviated by $\overline{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$\overline{R}_{\alpha\beta\gamma\delta;i} = \overline{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon \tag{1.2.1}$$

with obvious generalizations to other quantities.

Let M be a *spacelike* hypersurface; i.e., the induced metric is Riemannian, with a differentiable normal ν which is timelike.

In local coordinates, (x^α) and (ξ^i), the geometric quantities of the spacelike hypersurface M are connected through the following equations

$$x^\alpha_{ij} = h_{ij}\nu^\alpha \tag{1.2.2}$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.

$$x^\alpha_{ij} = x^\alpha_{,ij} - \Gamma^k_{ij}x^\alpha_k + \overline{\Gamma}^\alpha_{\beta\gamma}x^\beta_i x^\gamma_j. \tag{1.2.3}$$

The comma indicates ordinary partial derivatives.

In this implicit definition, the *second fundamental form* (h_{ij}) is taken with respect to ν.

The second equation is the *Weingarten equation*

$$\nu^\alpha_i = h^k_i x^\alpha_k, \tag{1.2.4}$$

where we remember that ν^α_i is a full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_j x^\delta_k \tag{1.2.5}$$

and the *Gauß equation*

$$R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \overline{R}_{\alpha\beta\gamma\delta}x^\alpha_i x^\beta_j x^\gamma_k x^\delta_l. \tag{1.2.6}$$

Now, let us assume that N is a globally hyperbolic Lorentzian manifold with a Cauchy hypersurface. N is then a topological product $I \times \mathcal{S}_0$, where I is an open interval, $\mathcal{S}_0$ is a Riemannian manifold, and there exists a Gaussian coordinate system (x^α), such that the metric in N has the form

$$d\bar{s}^2_N = e^{2\psi}\{-dx^{0^2} + \sigma_{ij}(x^0, x)dx^i dx^j\}, \tag{1.2.7}$$

where σ_{ij} is a Riemannian metric, ψ a function on N, and x an abbreviation for the spacelike components (x^i). We also assume that the coordinate system is *future oriented*; i.e., the time coordinate x^0 increases on future directed curves. Hence, the *contravariant* timelike vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future directed as is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Let $M = \operatorname{graph} u_{|\mathcal{S}_0}$ be a spacelike hypersurface

$$M = \{\, (x^0, x) \colon x^0 = u(x),\ x \in \mathcal{S}_0 \,\}, \tag{1.2.8}$$

then the induced metric has the form

$$g_{ij} = e^{2\psi}\{-u_i u_j + \sigma_{ij}\} \tag{1.2.9}$$

where σ_{ij} is evaluated at (u, x), and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$g^{ij} = e^{-2\psi}\{\sigma^{ij} + \frac{u^i}{v}\frac{u^j}{v}\}, \tag{1.2.10}$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$\begin{aligned} u^i &= \sigma^{ij}u_j \\ v^2 &= 1 - \sigma^{ij}u_i u_j \equiv 1 - |Du|^2. \end{aligned} \tag{1.2.11}$$

Hence, graph u is spacelike if and only if $|Du| < 1$.

The covariant form of a normal vector of a graph looks like

$$(\nu_\alpha) = \pm v^{-1}e^{\psi}(1, -u_i). \tag{1.2.12}$$

and the contravariant version is

$$(\nu^\alpha) = \mp v^{-1}e^{-\psi}(1, u^i). \tag{1.2.13}$$

Thus, we have

Remark 1.2.1 Let M be spacelike graph in a future oriented coordinate system. Then, the contravariant future directed normal vector has the form

$$(\nu^\alpha) = v^{-1}e^{-\psi}(1, u^i) \tag{1.2.14}$$

and the past directed

$$(\nu^\alpha) = -v^{-1}e^{-\psi}(1, u^i). \tag{1.2.15}$$

In the Gauß formula (1.2.2), we are free to choose the future or past directed normal, but we stipulate that we always use the past directed normal. Look at the component $\alpha = 0$ in (1.2.2) and obtain in view of (1.2.15)

$$e^{-\psi}v^{-1}h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00}u_i u_j - \bar{\Gamma}^0_{0j}u_i - \bar{\Gamma}^0_{0i}u_j - \bar{\Gamma}^0_{ij}. \tag{1.2.16}$$

Here, the covariant derivatives are taken with respect to the induced metric of M, and

$$-\bar{\Gamma}^0_{ij} = e^{-\psi}\bar{h}_{ij}, \tag{1.2.17}$$

where $(\bar{h}_{ij})$ is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

An easy calculation shows

$$\bar{h}_{ij}e^{-\psi} = -\frac{1}{2}\dot{\sigma}_{ij} - \dot{\psi}\sigma_{ij}, \tag{1.2.18}$$

where the dot indicates differentiation with respect to x^0.

1.3 Combining the Hamilton Equations with the Hamilton Constraint

Let $N = N^{n+1}$ be a globally hyperbolic spacetime. We consider the functional

$$J = \int_N (\overline{R} - 2\Lambda), \tag{1.3.1}$$

where $\overline{R}$ is the scalar curvature and Λ a cosmological constant. The integration over N is to be understood only symbolically since we are only interested in the first variation of the functional; i.e., when a metric $\bar{g} = (\bar{g}_{\alpha\beta})$ in N is given, we are only interested in the first variation of J with respect to *compact* variations of $\bar{g}$; hence, it suffices to integrate only over open and precompact subsets $\Omega \subset N$ such that

$$J = \int_\Omega (\overline{R} - 2\Lambda). \tag{1.3.2}$$

It is well known that, when the first variation of J with respect to arbitrary compact variations of $\bar{g}$ vanishes, the metric $\bar{g}$ satisfies the Einstein equations with cosmological constant Λ, namely

$$G_{\alpha\beta} + \Lambda\bar{g}_{\alpha\beta} = 0, \tag{1.3.3}$$

where $G_{\alpha\beta}$ is the Einstein tensor.

When N, endowed with a metric $\bar{g}$, is globally hyperbolic, there exists a global time function $f \in C^\infty(N)$ such that

$$\|Df\|^2 = \bar{g}^{\alpha\beta} f_\alpha f_\beta < 0, \tag{1.3.4}$$

and N can be written as a topological product

$$N = I \times \mathcal{S}_0, \quad I = (a, b) \subset \mathbb{R}, \tag{1.3.5}$$

where

$$\mathcal{S}_0 = f^{-1}(c), \quad a < c < b, \tag{1.3.6}$$

is a Cauchy hypersurface and there exists a Gaussian coordinate system (x^α), $0 \le \alpha \le n$, such that $x^0 = f$ and the metric $\bar{g}$ splits according to

$$d\bar{s}^2 = -w^2(dx^0)^2 + \bar{g}_{ij}dx^i dx^j, \tag{1.3.7}$$

where (x^i), $1 \le i \le n$, are local coordinates of $\mathcal{S}_0$ and

$$\bar{g}_{ij} = \bar{g}_{ij}(x^0, x), \quad x \in \mathcal{S}_0, \tag{1.3.8}$$

are Riemannian and $w > 0$ is function.

Without loss of generality, we may always assume that $0 \in I$ and that $c = 0$. When there exists a time function and an associated Gaussian coordinate system such that (1.3.7) is valid we also say that x^0 splits the metric.

Lemma 1.3.1 *Let $(N, \bar{g})$ be a globally hyperbolic spacetime and let f be a time function that splits $\bar{g}$. Let $\omega = (\omega_{\alpha\beta})$ be an arbitrary smooth symmetric tensor field with compact support and define*

$$\bar{g}(\epsilon) = \bar{g} + \epsilon\omega \tag{1.3.9}$$

for small values of ϵ

$$|\epsilon| < \epsilon_0. \tag{1.3.10}$$

If ϵ_0 is small enough, the tensor fields $\bar{g}(\epsilon)$ will also be Lorentzian metrics that will be split by f.

Proof We shall only prove that the $\bar{g}(\epsilon)$ will be split by *f*, since the other claim is obvious.

Define the conformal *covariant* metrics

$$g(\epsilon) = |\langle Df, Df\rangle|\bar{g}(\epsilon), \tag{1.3.11}$$

then Df is a unit gradient field for each $g(\epsilon)$. Let $\mathcal{S}_0 = f^{-1}(0)$ and consider the flow $x = x(t, \xi)$ satisfying

$$\begin{aligned} \dot{x} &= -Df, \\ x(0, \xi) &= \xi, \end{aligned} \tag{1.3.12}$$

where $\xi \in \mathcal{S}_0$. For fixed ξ, the flow is defined on a maximal time interval $J = (a_0, b_0)$. If we can prove that $J = I = (a, b) = f(N)$, then we would have proved that each metric $g(\epsilon)$ satisfies

$$d\bar{s}^2 = -(dx^0)^2 + g_{ij}dx^d x^j, \tag{1.3.13}$$

where the g_{ij} are Riemannian and depend smoothly on ϵ, cf. the arguments in [12, p. 27].

It suffices to prove $b_0 = b$. Assume that

$$b_0 < b, \tag{1.3.14}$$

and let K be the support of ω. Then, there exists $t_0 < b_0$ such that

$$x(t, \xi) \notin K \qquad \forall t > t_0, \tag{1.3.15}$$

for otherwise there would exist a sequence (t_k)

$$t_k \to b_0 \quad \wedge \quad x(t_k, \xi) \in K \tag{1.3.16}$$

contradicting the maximality of J, since there has to be a "singularity" for the flow in b_0.

Thus, choose

$$t_0 < t_1 < b_0, \tag{1.3.17}$$

then

$$x(t_1, \xi) \in M(t_1) = f^{-1}(t_1), \tag{1.3.18}$$

because

$$f(x(t, \xi)) = t \tag{1.3.19}$$

as one easily checks.

Let $y = y(t, \xi)$ be the flow corresponding to $\epsilon = 0$, then y covers N by assumption and hence there exists $\zeta \in \mathcal{S}_0$ such that

$$y(t_1, \zeta) = x(t_1, \xi). \tag{1.3.20}$$

Then the integral curve

$$\begin{aligned} \dot{y} &= -Df, \\ y(t_1, \zeta) &= x(t_1, \xi), \end{aligned} \tag{1.3.21}$$

where the contravariant vector is now defined with the help $\bar{g} = \bar{g}(0)$

$$D^\alpha f = \bar{g}^{\alpha\beta} f_\beta, \tag{1.3.22}$$

would be a smooth continuation of $x(t, \xi)$ past b_0, a contradiction. □

The preceding lemma will enable us to eliminate the so-called diffeomorphism constraint when switching from a Lagrangian to a Hamiltonian view of gravity.

Theorem 1.3.2 *Let $(N, \bar{g})$ be a globally hyperbolic spacetime, f a time function that splits $\bar{g}$ with Cauchy hypersurface $\mathcal{S}_0$. Let $\tilde{\Omega} \Subset N$ be open and precompact and assume that the first variation of the functional*

$$J = \int_{\tilde{\Omega}} (\overline{R} - 2\Lambda) \tag{1.3.23}$$

vanishes in $\bar{g}$ for those compact variations of $\bar{g}$ which can be expressed in the form

$$d\bar{s}^2 = -w^2 (dx^0)^2 + g_{ij} dx^i dx^j, \tag{1.3.24}$$

where $(g_{ij}(x^0, x))$ *is Riemannian, then the first variation of* J *in* $\bar{g}$ *also vanishes for arbitrary compact variations.*

Proof Let $\omega = (\omega_{\alpha\beta})$ be an arbitrary smooth symmetric tensor with compact support in $\tilde{\Omega}$. The metrics

$$\bar{g}(\epsilon) = \bar{g} + \epsilon\,\omega, \qquad |\epsilon| < \epsilon_0, \tag{1.3.25}$$

then satisfy (1.3.24) for small ϵ_0, in view of the preceding lemma, hence

$$\delta J(\bar{g}; \dot{\bar{g}}(0)) = 0. \tag{1.3.26}$$

But the first variation is a scalar, hence

$$\delta J(\bar{g}; \dot{\bar{g}}(0)) = \delta J(\bar{g}; \omega), \tag{1.3.27}$$

where at the right-hand side we used an arbitrary coordinate system to express the tensors. □

We are now ready to look at the Hamiltonian form of the Einstein–Hilbert action following [1].

Let $\tilde{\Omega} \Subset N$ be an arbitrary open, precompact set. Then, we consider the functional

$$J = \int_{\tilde{\Omega}} (\overline{R} - 2\Lambda), \tag{1.3.28}$$

where Λ is a negative constant and we assume that there exists a time function $f = x^0$ in N with Cauchy hypersurface $\mathcal{S}_0 = f^{-1}(0)$ and where, in view of Theorem 1.3.2, we only consider metrics of the form

$$d\bar{s}^2 = -w^2 (dx^0)^2 + g_{ij} dx^i dx^j, \tag{1.3.29}$$

where w is an arbitrary smooth positive function and $g_{ij} = g_{ij}(x^0, x)$, $x \in \mathcal{S}_0$, Riemannian metrics. Let us fix a metric $\bar{g} = (\bar{g}_{\alpha\beta})$ as in (1.3.29), then we deduce from the Gauß equation

$$\overline{R} = H^2 - |A|^2 + R - 2\overline{R}_{\alpha\beta}\nu^\alpha\nu^\beta, \tag{1.3.30}$$

where R is the Scalar curvature of the slices

$$M(t) = \{x^0 = t\}, \tag{1.3.31}$$

H the Mean curvature of $M(t)$,

$$H = g^{ij} h_{ij} = \sum_{i=1}^{n} \kappa_i, \tag{1.3.32}$$

where κ_i are the principal curvatures, $|A|^2$ is defined by

$$|A|^2 = h_{ij}h^{ij} = \sum_{i=1}^{n} \kappa_i^2, \tag{1.3.33}$$

and where the second fundamental form h_{ij} of $M(t)$ can be expressed as

$$h_{ij} = -\frac{1}{2}\dot{g}_{ij} w^{-1}, \tag{1.3.34}$$

where

$$\dot{g}_{ij} = \frac{\partial g_{ij}}{\partial t}, \tag{1.3.35}$$

when we identify t with x^0.

The last term on the right-hand side of (1.3.30) can be written as

$$-2\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta = -2(H^2 - |A|^2) + D_\alpha a^\alpha, \tag{1.3.36}$$

cf. Lemma 1.3.6 below. Since the divergence term can be neglected, the functional J is equal to

$$J = \int_a^b \int_\Omega \{|A|^2 - H^2 + R - 2\Lambda\} w\sqrt{g}, \tag{1.3.37}$$

where we may assume that

$$\tilde{\Omega} = (a, b)\times\Omega, \quad \Omega \Subset \mathcal{S}_0, \tag{1.3.38}$$

and

$$(a, b) \Subset x^0(N) = I. \tag{1.3.39}$$

This way of expressing the Einstein–Hilbert functional is known as the ADM approach, see [1].

Let $F = F(h_{ij})$ be the scalar curvature operator

$$F = \frac{1}{2}(H^2 - |A|^2) \tag{1.3.40}$$

and let

$$F^{ij,kl} = g^{ij}g^{kl} - \frac{1}{2}\{g^{ik}g^{jl} + g^{il}g^{jk}\} \tag{1.3.41}$$

be its Hessian, then

$$F^{ij,kl}h_{ij}h_{kl} = 2F = H^2 - |A|^2 \tag{1.3.42}$$

and

$$F^{ij} = F^{ij,kl}h_{kl} = Hg^{ij} - h^{ij}. \tag{1.3.43}$$

In physics

$$G^{ij,kl} = -F^{ij,kl} \tag{1.3.44}$$

is known as the DeWitt metric, or more precisely, a conformal metric, where the conformal factor is even a density, is known as the DeWitt metric, but we prefer the above definition.

Combining (1.3.34) and (1.3.42) J can be expressed in the form

$$J = \int_a^b \int_\Omega \{\frac{1}{4}G^{ij,kl}\dot{g}_{ij}\dot{g}_{kl}w^{-2} + (R - 2\Lambda)\}w\sqrt{g}. \tag{1.3.45}$$

The Lagrangian density $\mathcal{L}$ is a regular Lagrangian with respect to the variables g_{ij}. Define the conjugate momenta

$$\begin{aligned} \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}} &= \frac{1}{2}G^{ij,kl}\dot{g}_{kl}w^{-1}\sqrt{g} \\ &= -G^{ij,kl}h_{kl}\sqrt{g} \end{aligned} \tag{1.3.46}$$

and the Hamiltonian density

$$\begin{aligned} \mathcal{H} &= \pi^{ij}\dot{g}_{ij} - \mathcal{L} \\ &= \frac{1}{\sqrt{g}}wG_{ij,kl}\pi^{ij}\pi^{kl} - (R - 2\Lambda)w\sqrt{g}, \end{aligned} \tag{1.3.47}$$

where

$$G_{ij,kl} = \frac{1}{2}\{g_{ik}g_{jk} + g_{il}g_{jk}\} - \frac{1}{n-1}g_{ij}g_{kl} \tag{1.3.48}$$

is the inverse of $G^{ij,kl}$.

Let us now consider an arbitrary variation of g_{ij} with compact support

$$g_{ij}(\epsilon) = g_{ij} + \epsilon\omega_{ij}, \tag{1.3.49}$$

where $\omega_{ij} = \omega_{ij}(t, x)$ is an arbitrary smooth, symmetric tensor with compact support in Ω. The vanishing of the first variation leads to the Euler–Lagrange equations

$$G_{ij} + \Lambda g_{ij} = 0, \tag{1.3.50}$$

i.e. to the tangential Einstein equations. We obtain these equations by either varying (1.3.28) or (1.3.37).

To obtain the full Einstein equations we impose the Hamilton constraint, namely that the Hamiltonian density vanishes or, equivalently, that the normal component of the Einstein equations are satisfied

$$G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda = 0. \tag{1.3.51}$$

We then conclude that any metric $(\bar{g}_{\alpha\beta})$ satisfying (1.3.29), (1.3.50) and (1.3.51) has the property that it is a stationary point for the functional (1.3.28) in the class of metrics which can be split according to (1.3.29). Applying then the result of Theorem 1.3.2, we deduce that $\bar{g}_{\alpha\beta}$ satisfies the full Einstein equations.

The Lagrangian density $\mathcal{L}$ in (1.3.45) is regular with respect to the variables g_{ij}; hence, the tangential Einstein equations are equivalent to the Hamilton equations

$$\dot{g}_{ij} = \frac{\delta\mathcal{H}}{\delta\pi^{ij}} \tag{1.3.52}$$

and

$$\dot{\pi}^{ij} = -\frac{\delta\mathcal{H}}{\delta g_{ij}}, \tag{1.3.53}$$

where the differentials on the right-hand side of these equations are variational or functional derivatives; i.e., they represent the Euler–Lagrange operator of the corresponding functionals, or more precisely, of the corresponding Lagrangians, with respect to the indicated variables, in this case, the functional is

$$\int_\Omega \mathcal{H}, \tag{1.3.54}$$

where $\mathcal{S}_0$ is locally parameterized over $\Omega \subset \mathbb{R}^n$. Occasionally, we shall also write

$$\int_{\mathcal{S}_0} \mathcal{H} \tag{1.3.55}$$

by considering $\mathcal{S}_0$ simply to be a parameter domain without any intrinsic volume element.

We have therefore proved:

Theorem 1.3.3 *Let $N = N^{n+1}$ be a globally hyperbolic spacetime and let the metric $\bar{g}_{\alpha\beta}$ be expressed as in* (1.3.29)*. Then, the metric satisfies the full Einstein equations if and only if the metric is a solution of the Hamilton equations* (1.3.52) *and* (1.3.53) *and of the Eq.* (1.3.51) *which is equivalent to*

$$\mathcal{H} = 0 \tag{1.3.56}$$

and is called the Hamilton constraint. These equations are equations for the variables g_{ij}. The function w is merely part of the equations and not looked at as a variable though it is of course specified in the component $\bar{g}_{00}$.

We define the Poisson brackets

$$\{u, v\} = \frac{\delta u}{\delta g_{kl}} \frac{\delta v}{\delta \pi^{kl}} - \frac{\delta u}{\delta \pi^{kl}} \frac{\delta v}{\delta g_{kl}} \tag{1.3.57}$$

and obtain

$$\{g_{ij}, \pi^{kl}\} = \delta_{ij}^{kl}, \tag{1.3.58}$$

where

$$\delta_{ij}^{kl} = \frac{1}{2}\{\delta_i^k \delta_j^l + \delta_i^l \delta_j^k\}. \tag{1.3.59}$$

Then, the second Hamilton equation can also be expressed as

$$\dot{\pi}^{ij} = \{\pi^{ij}, \mathcal{H}\}. \tag{1.3.60}$$

In the next section, we want to quantize this Hamiltonian setting and especially the Hamiltonian constraint. In order to achieve this, we shall express equations (1.3.53), (1.3.52) and (1.3.51) by a set of equivalent equations, namely (1.3.53), (1.3.52) and (1.3.61)

$$\begin{aligned} g_{ij}\{\pi^{ij}, \mathcal{H}\} = {} & (n-1)(R - 2\Lambda)w\sqrt{g} - Rw\sqrt{g} - (n-1)\tilde{\Delta} w\sqrt{g} \\ & - \frac{1}{\sqrt{g}} G_{rs,kl}\pi^{rs}\pi^{kl} w, \end{aligned} \tag{1.3.61}$$

where $\tilde{\Delta}$ is the Laplacian with respect to the metric g_{ij}. Let us formulate this claim as a theorem:

Theorem 1.3.4 *Let $N = N^{n+1}$ be a globally hyperbolic spacetime and let the metric $\bar{g}_{\alpha\beta}$ be expressed as in* (1.3.29)*. Then, the metric satisfies the full Einstein equations if and only if the metric is a solution of the Hamilton equations* (1.3.52) *and* (1.3.53) *and of equation* (1.3.61).

Proof The second Hamilton equation states

$$\dot{\pi}^{ij} = -\frac{\delta \mathcal{H}}{\delta g_{ij}}, \tag{1.3.62}$$

which is of course equal to (1.3.60), and

$$-\frac{\delta \mathcal{H}}{\delta g_{ij}} = -\frac{\partial}{\partial g_{ij}}\left(\frac{1}{\sqrt{g}} G_{rs,kl}\pi^{rs}\pi^{kl}\right)w + \frac{\delta((R - 2\Lambda)w\sqrt{g})}{\delta g_{ij}}. \tag{1.3.63}$$

In the lemma below, we shall prove

$$\frac{\delta((R-2\Lambda)w\sqrt{g})}{\delta g_{ij}} = \frac{1}{2}Rg^{ij}w\sqrt{g} - R^{ij}w\sqrt{g} \\ + \{w^{ij} - \tilde{\Delta}wg^{ij} - \Lambda g^{ij}w\}\sqrt{g} \tag{1.3.64}$$

and a simple but somewhat lengthy computation will reveal

$$-\frac{\partial}{\partial g_{ij}}(\frac{1}{\sqrt{g}}G_{rs,kl}\pi^{rs}\pi^{kl})w = \frac{1}{2}(|A|^2 - H^2)g^{ij}w\sqrt{g} \\ - 2\pi^i_r\pi^{rj}w\frac{1}{\sqrt{g}} + \frac{2}{n-1}\pi^{ij}\pi^r_r w\frac{1}{\sqrt{g}}, \tag{1.3.65}$$

where the indices are lowered with the help of g_{ij}, and we further conclude

$$\begin{aligned} &- g_{ij}\frac{\partial}{\partial g_{ij}}(\frac{1}{\sqrt{g}}G_{rs,kl}\pi^{rs}\pi^{kl})w \\ &= \frac{n}{2}(|A|^2 - H^2)w\sqrt{g} - 2(|A|^2 - H^2)w\sqrt{g} \\ &= (\frac{n}{2} - 1)(|A|^2 - H^2)w\sqrt{g} - \frac{1}{\sqrt{g}}G_{rs,kl}\pi^{rs}\pi^{kl}w \end{aligned} \tag{1.3.66}$$

On the other hand, the Hamilton density is equal to

$$\mathcal{H} = -2\{G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda\}w\sqrt{g} \tag{1.3.67}$$

because of the Gauß equation. Hence,

$$\frac{1}{2}\{|A|^2 - H^2\}w\sqrt{g} = \frac{1}{2}(R - 2\Lambda)w\sqrt{g} \tag{1.3.68}$$

iff the Hamilton constraint is valid, from which the proof of the theorem immediately follows. □

Lemma 1.3.5 *Let M be a Riemannian manifold with metric g_{ij}, scalar curvature R and let $w \in C^2(M)$ and $\Lambda \in \mathbb{R}$, then the Eq.* (1.3.64) *is valid.*

Proof It suffices to consider the term

$$\frac{\delta(Rw\sqrt{g})}{\delta g_{ij}}, \tag{1.3.69}$$

since the result for the second term is trivial.

Let $\Omega \subset M$ be open and bounded and define the functional

$$J = \int_\Omega Rw\sqrt{g}. \tag{1.3.70}$$

Let $g_{ij}(\epsilon)$ be a variation of g_{ij} with support in Ω such that

$$g_{ij} = g_{ij}(0) \tag{1.3.71}$$

and denote differentiation with respect to ϵ by a dot or prime, then the first variation of J with respect to this variation is equal to

$$\dot{J}(0) = \int_\Omega \{\dot{g}^{ij} R_{ij} + g^{ij} \dot{R}_{ij}\} w \sqrt{g} + \int_\Omega R w \sqrt{g}'. \tag{1.3.72}$$

Again we only consider the non-trivial term

$$\int_\Omega g^{ij} \dot{R}_{ij} w \sqrt{g}. \tag{1.3.73}$$

It is well known that

$$\dot{R}_{ij} = -(\dot{\Gamma}^k_{ik})_{;j} + (\dot{\Gamma}^k_{ij})_{;k}, \tag{1.3.74}$$

where the semicolon indicates covariant differentiation, $\dot{\Gamma}^k_{ij}$ is a tensor. Hence, we deduce that (1.3.73) is equal to

$$\int_\Omega \{g^{ij} \dot{\Gamma}^k_{ik} w_j - g^{ij} \dot{\Gamma}^k_{ij} w_k\} \sqrt{g} \tag{1.3.75}$$

which in turn can be expressed as

$$\begin{aligned} &\int_\Omega g^{ij} g^{kl} \frac{1}{2} (\dot{g}_{il;k} + \dot{g}_{kl;i} - \dot{g}_{ik;l}) w_j \\ &- \int_\Omega g^{ij} g^{kl} \frac{1}{2} (\dot{g}_{il;j} + \dot{g}_{jl;i} - \dot{g}_{ij;l}) w_k, \end{aligned} \tag{1.3.76}$$

where we omitted the notation of the density $\sqrt{g}$. Let us agree that each row of the preceding expression contains three integrals. Then, the first integrals in each row cancel each other, the second in the first row is equal to the third integral in the second row, and the third integral in the first row is equal to the second integral in the second row. Therefore, we obtain by integrating by parts

$$-\int_\Omega \tilde{\Delta} w g^{kl} \dot{g}_{kl} + \int_\Omega w^l_i \dot{g}^i_l = \int_\Omega \{-\tilde{\Delta} w g^l_i + w^l_i\} \dot{g}^i_l \tag{1.3.77}$$

and conclude

$$\frac{\delta(R w \sqrt{g})}{\delta g_{ij}} = (\frac{1}{2} R g^{ij} - R^{ij}) w \sqrt{g} + (w^{ij} - \tilde{\Delta} w g^{ij}) \sqrt{g}. \tag{1.3.78}$$

□

Let us now prove the relation (1.3.36).

Lemma 1.3.6 *Let $M \subset N$ be a spacelike hypersurface and let $\nu = (\nu^\alpha)$ be a smooth field of unit normal vectors of M, then*

$$\overline{R}_{\alpha\beta}\nu^\alpha\nu^\beta = (H^2 - |A|^2) + D_\alpha a^\alpha, \tag{1.3.79}$$

where a^α is a smooth vector field.

Proof Without loss of generality we may assume that the vector field ν is defined in a neighbourhood of M by looking at a tubular neighbourhood of M, cf. [12, Theorem 1.3.13]. Applying the Ricci identities, we then deduce

$$\nu^\delta_{;\alpha\delta} = \nu^\delta_{;\delta\alpha} - \overline{R}^{\,\delta}_{\beta\alpha\delta}\nu^\beta = \nu^\delta_{;\delta\alpha} + \overline{R}_{\alpha\beta}\nu^\beta. \tag{1.3.80}$$

Next, we have the identities

$$(\nu^\delta_{;\alpha}\nu^\alpha)_{;\delta} = \nu^\delta_{;\alpha\delta}\nu^\delta + \nu^\delta_{;\alpha}\nu^\alpha_{;\delta} \tag{1.3.81}$$

and

$$(\nu^\delta_{;\delta}\nu^\alpha)_{;\alpha} = \nu^\delta_{;\delta\alpha}\nu^\alpha + \nu^\delta_{;\delta}\nu^\alpha_{;\alpha}, \tag{1.3.82}$$

from which we infer, by subtracting the last equation from the preceding one,

$$(\nu^\delta_{;\alpha}\nu^\alpha)_{;\delta} - (\nu^\delta_{;\delta}\nu^\alpha)_{;\alpha} = \nu^\delta_{;\alpha\delta}\nu^\delta - \nu^\delta_{;\delta\alpha}\nu^\alpha + |A|^2 - H^2, \tag{1.3.83}$$

where we also used that

$$\nu^\alpha_{;\alpha} = \pm H \quad \wedge \quad \nu^\delta_{;\alpha}\nu^\alpha_{;\delta} = |A|^2. \tag{1.3.84}$$

Combining (1.3.83) and (1.3.80), we obtain (1.3.79). □

1.4 The Quantization

For the quantization of the Hamiltonian setting we first replace all densities by tensors, by choosing a fixed Riemannian metric in $\mathcal{S}_0$

$$\chi = (\chi_{ij}(x)), \tag{1.4.1}$$

and, for a given metric $g = (g_{ij}(t, x))$, we define

$$\varphi = \varphi(x, g_{ij}) = \Big(\frac{\det g_{ij}}{\det \chi_{ij}}\Big)^{\frac{1}{2}} \tag{1.4.2}$$

such that the Einstein–Hilbert functional J in (1.3.45) on page 12 can be written in the form

$$J = \int_a^b \int_\Omega \{\frac{1}{4} G^{ij,kl} \dot{g}_{ij} \dot{g}_{kl} w^{-2} + (R - 2\Lambda)\} w\varphi\sqrt{\chi}. \tag{1.4.3}$$

The Hamilton density $\mathcal{H}$ is then replaced by the function

$$H = \{\varphi^{-1} G_{ij,kl} \pi^{ij} \pi^{kl} - (R - 2\Lambda)\varphi\} w, \tag{1.4.4}$$

where now

$$\pi^{ij} = -\varphi G^{ij,kl} h_{kl} \tag{1.4.5}$$

and

$$h_{ij} = -\varphi^{-1} G_{ij,kl} \pi^{kl}. \tag{1.4.6}$$

The effective Hamiltonian is of course

$$w^{-1} H. \tag{1.4.7}$$

Fortunately, we can, at least locally, assume

$$w = 1 \tag{1.4.8}$$

by choosing an appropriate coordinate system: Let $(t_0, x_0) \in N$ be an arbitrary point, then consider the Cauchy hypersurface

$$M(t_0) = \{t_0\} \times \mathcal{S}_0 \tag{1.4.9}$$

and look at a tubular neighbourhood of $M(t_0)$; i.e., we define new coordinates (t, x^i), where (x^i) are coordinates for $\mathcal{S}_0$ near x_0 and t is the signed Lorentzian distance to $M(t_0)$ such that the points

$$(0, x^i) \in M(t_0). \tag{1.4.10}$$

The Lorentzian metric of the ambient space then has the form

$$d\bar{s}^2 = -dt^2 + g_{ij} dx^i dx^j. \tag{1.4.11}$$

Secondly, we use the same model as in [16, Sect. 3]: The Riemannian metrics $g_{ij}(t, \cdot)$ are elements of the bundle $T^{0,2}(\mathcal{S}_0)$. Denote by E the fiber bundle with base $\mathcal{S}_0$ where the fibers consist of the Riemannian metrics (g_{ij}). We shall consider each fiber to be a Lorentzian manifold equipped with the DeWitt metric. Each fiber F has dimension

$$\dim F = \frac{n(n+1)}{2} \equiv m + 1. \tag{1.4.12}$$

Let (ξ^a), $0 \le a \le m$, be coordinates for a local trivialization such that

$$g_{ij}(x, \xi^a) \tag{1.4.13}$$

is a local embedding. The DeWitt metric is then expressed as

$$G_{ab} = G^{ij,kl} g_{ij,a} g_{kl,b}, \tag{1.4.14}$$

where a comma indicates partial differentiation. In the new coordinate system, the curves

$$t \to g_{ij}(t, x) \tag{1.4.15}$$

can be written in the form

$$t \to \xi^a(t, x) \tag{1.4.16}$$

and we infer

$$G^{ij,kl} \dot{g}_{ij} \dot{g}_{kl} = G_{ab} \dot{\xi}^a \dot{\xi}^b. \tag{1.4.17}$$

Hence, we can express (1.3.37) as

$$J = \int_a^b \int_\Omega \{\frac{1}{4} G_{ab} \dot{\xi}^a \dot{\xi}^b \varphi + (R - 2\Lambda)\varphi\}, \tag{1.4.18}$$

where we now refrain from writing down the density $\sqrt{\chi}$ explicitly, since it does not depend on (g_{ij}) and therefore should not be part of the Legendre transformation. We also emphasize that we are now working in the gauge $w = 1$. Denoting the Lagrangian *function* in (1.4.18) by L, we define

$$\pi_a = \frac{\partial L}{\partial \dot{\xi}^a} = \varphi G_{ab} \frac{1}{2} \dot{\xi}^b \tag{1.4.19}$$

and we obtain for the Hamiltonian function H

$$\begin{aligned} H &= \dot{\xi}^a \frac{\partial L}{\partial \dot{\xi}^a} - L \\ &= \varphi G_{ab} \big(\frac{1}{2} \dot{\xi}^a\big)\big(\frac{1}{2} \dot{\xi}^b\big) - (R - 2\Lambda)\varphi \\ &= \varphi^{-1} G^{ab} \pi_a \pi_b - (R - 2\Lambda)\varphi, \end{aligned} \tag{1.4.20}$$

where G^{ab} is the inverse metric.

The fibers equipped with the metric

$$(\varphi G_{ab}) \tag{1.4.21}$$

are then globally hyperbolic Lorentzian manifolds as we shall now prove.

DeWitt already analysed the fibers in [6], though he did not look at them as fibers. Some of the ideas that we shall use in the proofs below can already be found in DeWitt's paper.

Lemma 1.4.1 *Let F be a fiber, then F is connected and*

$$\tau = \log \varphi \tag{1.4.22}$$

is a time function satisfying

$$\varphi^{-1} G^{ab} \tau_a \tau_b = -\frac{n}{4(n-1)} \varphi^{-1}. \tag{1.4.23}$$

Proof F is obviously connected, since F is a convex cone in the vector space defined by the symmetric covariant tensors of order two.

To prove (1.4.23), we use the original coordinate representation g_{ij} and conclude

$$\tau^{ij} = \frac{\partial \tau}{\partial g_{ij}} = \frac{1}{2} g^{ij}, \tag{1.4.24}$$

and hence

$$G_{ij,kl} \tau^{ij} \tau^{kl} = -\frac{n}{4(n-1)}, \tag{1.4.25}$$

where

$$G_{ij,kl} = \frac{1}{2}\{g_{ik} g_{jl} + g_{il} g_{jk}\} - \frac{1}{n-1} g_{ij} g_{kl} \tag{1.4.26}$$

is the inverse of $G^{ij,kl}$, hence the result. □

Theorem 1.4.2 *Each fiber F is globally hyperbolic, the hypersurface*

$$M = \{\varphi = 1\} = \{\tau = 0\} \tag{1.4.27}$$

is a Cauchy hypersurface and in the corresponding Gaussian coordinate system (ξ^a) the metric φG_{ab} can be expressed as

$$ds^2 = \frac{4(n-1)}{n} \varphi \{-d\tau^2 + G_{AB} d\xi^A d\xi^B\}, \tag{1.4.28}$$

where

$$\tau = \xi^0 \quad \wedge \quad -\infty < \tau < \infty \tag{1.4.29}$$

and (ξ^A), $1 \le A \le m$, are local coordinates for M. The metric G_{AB} is also static, i.e., it does not depend on τ.

Proof (i) Let τ be as in Lemma 1.4.1, then $\tau(F) = \mathbb{R}$ and in the conformal metric

$$\widetilde{G}_{ab} = \varphi^{-1}\frac{n}{4(n-1)}(\varphi G_{ab}) \tag{1.4.30}$$

τ_a is a unit gradient field in view of (1.4.23).

(ii) The hypersurface M in (1.4.27) is therefore spacelike and has at most countably many connected components.

Consider the flow

$$\begin{aligned}\dot{\xi} &= -D\tau = -(\widetilde{G}^{ab}\tau_b)\\ \xi(0,\zeta) &= \zeta, \quad \zeta \in M.\end{aligned} \tag{1.4.31}$$

It will be convenient to express the flow in the original coordinate system, i.e.,

$$\begin{aligned}\dot{g}_{ij} &= -\frac{4(n-1)}{n}G_{ij,kl}\tau^{kl},\\ g_{ij}(0,\zeta) &= \zeta = \bar{g}_{ij},\end{aligned} \tag{1.4.32}$$

where $G_{ij,kl}$ is the metric in (1.4.26). The flow exists on a maximal time interval J_ζ.

From (1.4.26), we obtain

$$\begin{aligned}G_{ij,kl}\tau^{kl} &= \frac{1}{2}G_{ij,kl}g^{kl}\\ &= \frac{1}{2}g_{ij}\Big(1 - \frac{n}{n-1}\Big) = -\frac{1}{2(n-1)}g_{ij},\end{aligned} \tag{1.4.33}$$

hence

$$\dot{g}_{ij} = \frac{2}{n}g_{ij}. \tag{1.4.34}$$

Let $(\eta^i) \in T^{1,0}_x(\mathcal{S}_0)$ be an arbitrary unit vector with respect to the metric χ_{ij}, then

$$(g_{ij}\eta^i\eta^j)' = \frac{2}{n}g_{ij}\eta^i\eta^j \tag{1.4.35}$$

leading to

$$g_{ij}\eta^i\eta^j = \bar{g}_{ij}\eta^i\eta^j e^{\frac{2}{n}t}, \tag{1.4.36}$$

and thus, the eigenvalues of g_{ij} with respect to χ_{ij} are uniformly bounded from above and strictly bounded against zero when $|t| \le$ const. Moreover,

$$\tau(g_{ij}) = t \tag{1.4.37}$$

from which we conclude

$$J_\zeta = \mathbb{R}. \tag{1.4.38}$$

If M would be connected, then we would have proved that F is product

$$F = \mathbb{R} \times M \tag{1.4.39}$$

and that the metric would split as (1.4.28). However, if M had more then one connected component, then the corresponding cylinders defined by the flow would be disjoint and hence F would not be connected.

(iii) Let (ξ^a), $0 \le a \le m$, be the corresponding Gaussian coordinate system such that

$$\xi^0 = \tau = t \tag{1.4.40}$$

and (ξ^A), $1 \le A \le m$, are local coordinates for M. Let $g_{ij}(\xi^a)$ be a local embedding in the new coordinate system, where the ambient metric should be the conformal metric up to a multiplicative constant; i.e., we consider

$$G^{ij,kl} = \frac{1}{2}\{g^{ij}g^{kl} + g^{il}g^{jk}\} - g^{ij}g^{kl} \tag{1.4.41}$$

to be the ambient metric such that

$$G_{ab} = G^{ij,kl}g_{ij,a}g_{kl,b}. \tag{1.4.42}$$

The metric splits, and we claim that

$$G_{AB} = G^{ij,kl}g_{ij,A}g_{kl,B} \tag{1.4.43}$$

is stationary

$$\frac{d}{dt}G_{AB} = 0. \tag{1.4.44}$$

To prove this equation, we observe that the normal to $M(t) = \{\tau = t\}$ is a multiple of g^{ij}, cf.(1.4.24), hence

$$g^{ij}g_{ij,A} = 0 \tag{1.4.45}$$

for $g_{ij}(t, \xi^A)$ is a local embedding of $M(t)$ from which we deduce

$$G_{AB} = \frac{1}{2}\{g^{ik}g^{jl} + g^{il}g^{jk}\}g_{ij,A}g_{kl,B}. \tag{1.4.46}$$

Differentiating this equation with respect to t we infer, in view of (1.4.34),

$$\begin{aligned}\frac{d}{dt}G_{AB} &= -\frac{2}{n}\{g^{ik}g^{jl} + g^{il}g^{jk}\}g_{ij,A}g_{kl,B} \\ &\quad + \frac{2}{n}\{g^{ik}g^{jl} + g^{il}g^{jk}\}g_{ij,A}g_{kl,B} \\ &= 0\end{aligned} \tag{1.4.47}$$

where we also used

$$\dot{g}^{ij} = -\frac{2}{n} g^{ij}. \tag{1.4.48}$$

(iv) Finally, we want to prove that $M = M(0)$ is a Cauchy hypersurface and hence F globally hyperbolic, cf. [40, Corollary 39, p. 422]. It suffices to prove this result for a conformal metric G_{ab} where

$$d\bar{s}^2 = -d\tau^2 + G_{AB} d\xi^A d\xi^B \tag{1.4.49}$$

and G_{AB} is stationary.

G_{AB} is the metric of M. In case $n = 3$, DeWitt proved in [6, Remarks past equation (5.15)] that M is a symmetric space and hence complete. DeWitt's proof in [6, Appendix A] remains valid for $n > 3$. We shall only use the fact that M is complete; in Lemma 1.4.3 below we shall give a second proof which does not rely on DeWitt's result.

Let $\gamma(s) = (\gamma^a(s))$, $s \in I$, be an inextendible future directed causal curve in F and assume that γ does not intersect M. We shall show that this will lead to a contradiction. It is also obvious that γ can meet M at most once.

Assume that there exists $s_0 \in I$ such that

$$\tau(\gamma(s_0)) < 0 \tag{1.4.50}$$

and assume from now on that s_0 is the left endpoint of I. Since τ is continuous, the whole curve γ must be contained in the past of M.

γ is causal, i.e.,

$$G_{AB} \dot{\gamma}^A \dot{\gamma}^B \le |\dot{\gamma}^0|^2 \tag{1.4.51}$$

and thus

$$\sqrt{G_{AB} \dot{\gamma}^A \dot{\gamma}^B} \le \dot{\gamma}^0 \tag{1.4.52}$$

since γ is future directed. Let

$$\tilde{\gamma} = (\gamma^A) \tag{1.4.53}$$

be the projection of γ to M, then the length of $\tilde{\gamma}$ is bounded

$$L(\tilde{\gamma}) = \int_I \sqrt{G_{AB} \dot{\gamma}^A \dot{\gamma}^B} \le \int_I \dot{\gamma}^0 \le -\gamma^0(s_0). \tag{1.4.54}$$

Hence, $\tilde{\gamma}$ stays in a compact set since M is complete and the timelike coefficient is also bounded

$$\gamma^0(s_0) \le \gamma^0(s) < 0 \qquad \forall s \in I, \tag{1.4.55}$$

which is a contradiction since γ should be inextendible but stays in a compact set of F. □

Lemma 1.4.3 *The hypersurface $M = M(x)$ is a Cauchy hypersurface in $F(x)$.*

Proof As in the proof above we consider an inextendible causal curve γ and look at the projection $\tilde{\gamma}$ given in (1.4.53) which has finite length, cf. (1.4.54). Then, it suffices to prove that $\tilde{\gamma}$ stays in a compact subset of M.

Representing $\tilde{\gamma} = \tilde{\gamma}(s)$, $s \in I = [s_0, b)$, in the original coordinate system (g_{ij})

$$\tilde{\gamma} = (g_{ij}(x, s)) \equiv (g_{ij}(s)) \tag{1.4.56}$$

we use (1.4.46) to deduce

$$L(\tilde{\gamma}) = \int_I \|\dot{g}_{ij}\| \le -\gamma^0(s_0), \tag{1.4.57}$$

where

$$\|\dot{g}_{ij}\|^2 = g^{ik} g^{jl} \dot{g}_{ij} \dot{g}_{kl}, \tag{1.4.58}$$

from which we infer, in view of [28, Lemma 14.2], that the metrics $(g_{ij}(s))$ are all uniformly equivalent in I and converge to a positive definite metric when $s \to b$. Hence, the limit metric belongs to M and $\tilde{\gamma}$ stays in a compact subset of M. □

When we work in a local trivialization of E, the coordinates ξ^A are independent of x.

Lemma 1.4.4 *The function φ is independent of x.*

Proof Let

$$g_{ij}(x, \tau, \xi^A) \tag{1.4.59}$$

be the local embedding in E, then we have

$$\dot{g}_{ij} = \frac{\partial g_{ij}}{\partial \tau} = \frac{2}{n} g_{ij}, \tag{1.4.60}$$

cf. (1.4.34), hence we conclude

$$\begin{aligned} g_{ij} &= e^{\frac{2}{n}\tau} g_{ij}(x, 0, \xi^A) \\ &\equiv e^{\frac{2}{n}\tau} \sigma_{ij}(x, \xi^A), \end{aligned} \tag{1.4.61}$$

where

$$\sigma_{ij} = g_{ij}(0) \in M \tag{1.4.62}$$

and we further deduce

$$\varphi^2 = \frac{\det g_{ij}}{\det \chi_{ij}} = e^{2\tau} \frac{\det \sigma_{ij}}{\det \chi_{ij}}. \tag{1.4.63}$$

In the embedding (1.4.59), τ is considered to be independent of x being the time component of a coordinate system (x, ξ^a) describing a local trivialization of the bundle E. Therefore, we infer from (1.4.63)

$$\det \sigma_{ij} = \det \chi_{ij}, \tag{1.4.64}$$

proving the lemma. □

We can now quantize the Hamiltonian setting using the original variables g_{ij} and π^{ij}. We consider the bundle E equipped with the metric (1.4.28), or equivalently,

$$(\varphi G^{ij,kl}), \tag{1.4.65}$$

which is the *covariant* form, in the fibers and with the Riemannian metric χ in $\mathcal{S}_0$. Furthermore, let

$$C_c^\infty(E) \tag{1.4.66}$$

be the space of real-valued smooth functions with compact support in E.

In the quantization process, where we choose $\hbar = 1$, the variables g_{ij} and π^{ij} are then replaced by operators $\hat{g}_{ij}$ and $\hat{\pi}^{ij}$ acting in $C_c^\infty(E)$ satisfying the commutation relations

$$[\hat{g}_{ij}, \hat{\pi}^{kl}] = i\delta_{ij}^{kl}, \tag{1.4.67}$$

while all the other commutators vanish. These operators are realized by defining $\hat{g}_{ij}$ to be the multiplication operator

$$\hat{g}_{ij} u = g_{ij} u \tag{1.4.68}$$

and $\hat{\pi}^{ij}$ to be the *functional* differentiation

$$\hat{\pi}^{ij} = \frac{1}{i}\frac{\delta}{\delta g_{ij}}, \tag{1.4.69}$$

i.e., if $u \in C_c^\infty(E)$, then

$$\frac{\delta u}{\delta g_{ij}} \tag{1.4.70}$$

is the Euler–Lagrange operator of the functional

$$\int_{\mathcal{S}_0} u\sqrt{\chi} \equiv \int_{\mathcal{S}_0} u. \tag{1.4.71}$$

Hence, if u only depends on (x, g_{ij}) and not on derivatives of the metric, then

$$\frac{\delta u}{\delta g_{ij}} = \frac{\partial u}{\partial g_{ij}}. \tag{1.4.72}$$

Therefore, the transformed Hamiltonian $\widehat{H}$ can be looked at as the hyperbolic differential operator

$$\widehat{H} = -\Delta - (R - 2\Lambda)\varphi, \tag{1.4.73}$$

where Δ is the Laplacian of the metric in (1.4.65) acting on functions

$$u = u(x, g_{ij}). \tag{1.4.74}$$

We used this approach in [16] to transform the Hamilton constraint to the Wheeler–DeWitt equation

$$\hat{H}u = 0 \qquad \text{in } E \tag{1.4.75}$$

which can be solved with suitable Cauchy conditions. However, the Hamiltonian in the Wheeler–DeWitt equation is a differential operator that only acts in the fibers of E and not in the base space $\mathcal{S}_0$ which seems to be insufficient. This shortcoming will be eliminated when, instead of the explicit Hamilton constraint, its equivalent implicit version, Eq, (1.3.61) on page 14,[1] is quantized: Following Dirac, the Poisson brackets are replaced by $\frac{1}{i}$ times the commutators in the quantization process, $\hbar = 1$, i.e., we obtain

$$\{\pi^{ij}, H\} \rightarrow i[\hat{H}, \hat{\pi}^{ij}]. \tag{1.4.76}$$

Dropping the hats in the following to improve the readability equation (1.3.61) is transformed to

$$i g_{ij}[H, \pi^{ij}] = (n-1)(R - 2\Lambda)\varphi - R\varphi + \Delta, \tag{1.4.77}$$

where Δ is the Laplace operator with respect to the fiber metric.

Now, we have

$$\begin{aligned} i[H, \pi^{ij}] &= [H, \frac{\delta}{\delta g_{ij}}] \\ &= [-\Delta, \frac{\delta}{\delta g_{ij}}] - [(R - 2\Lambda)\varphi, \frac{\delta}{\delta g_{ij}}], \end{aligned} \tag{1.4.78}$$

cf. (1.4.73). Since we apply both sides to functions $u \in C_c^\infty(E)$

$$[-\Delta, \frac{\delta}{\delta g_{ij}}]u = [-\Delta, \frac{\partial}{\partial g_{ij}}]u = -R^{ij}{}_{,kl}u^{kl}, \tag{1.4.79}$$

because of the Ricci identities, where

[1] Note that the left-hand side of this equation is a variant of the evolution equation of the mean curvature of the foliation hypersurfaces, cf. (1.6.16) on page 42, i.e., the implementation of the Hamilton constraint is very similar for these two models.

$$R^{ij}{}_{,kl} \tag{1.4.80}$$

is the Ricci tensor of the fiber metric (1.4.65) and

$$u^{kl} = \frac{\partial u}{\partial g_{kl}} \tag{1.4.81}$$

is the gradient of u.

For the second commutator on the right-hand side of (1.4.78), we obtain

$$-[(R-2\Lambda)\varphi, \frac{\delta}{\delta g_{ij}}]u = -(R-2\Lambda)\varphi\frac{\partial u}{\partial g_{ij}} + \frac{\delta}{\delta g_{ij}}\{(R-2\Lambda)u\varphi\}, \tag{1.4.82}$$

where the last term is the Euler–Lagrange operator of the functional

$$\begin{aligned}\int_{\mathcal{S}_0}(R-2\Lambda)u\varphi &\equiv \int_{\mathcal{S}_0}(R-2\Lambda)u\varphi\sqrt{\chi}\\ &= \int_{\mathcal{S}_0}(R-2\Lambda)u\sqrt{g}\end{aligned} \tag{1.4.83}$$

with respect to the variable g_{ij}, since the scalar curvature R depends on the derivatives of g_{ij}. From (1.3.64) and the proof of Lemma 1.3.5 on page 15 we infer

$$\begin{aligned}\frac{\delta}{\delta g_{ij}}\{(R-2\Lambda)u\varphi\} &= \frac{1}{2}(R-2\Lambda)g^{ij}u\varphi - R^{ij}u\varphi\\ &\quad + \varphi\{u^{ij}_{;} - \tilde{\Delta}ug^{ij}\} + (R-2\Lambda)\varphi\frac{\partial u}{\partial g_{ij}},\end{aligned} \tag{1.4.84}$$

where the semicolon indicates covariant differentiation in $\mathcal{S}_0$ with respect to the metric g_{ij}, $\tilde{\Delta}$ is the corresponding Laplacian, and where we observe that the fundamental lemma of the calculus of variations has been applied to functions in $L^2(\mathcal{S}_0, \sqrt{\chi})$, i.e.,

$$\int_{\mathcal{S}_0} f\eta\sqrt{g} = \int_{\mathcal{S}_0} f\eta\varphi\sqrt{\chi}; \tag{1.4.85}$$

here, we have

$$f \in C^0(\mathcal{S}_0), \quad \eta \in C_c^\infty(\mathcal{S}_0). \tag{1.4.86}$$

We also note that

$$\begin{aligned}D_k u &= \frac{\partial u}{\partial x^k} + \frac{\partial u}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial x^k}\\ &= \frac{\partial u}{\partial x^k}\end{aligned} \tag{1.4.87}$$

in Riemannian normal coordinates.

Hence, we conclude that Eq. (1.4.77) is equivalent to

$$-\Delta u - (n-1)\varphi\tilde{\Delta}u - \frac{n-2}{2}\varphi(R-2\Lambda)u = 0 \tag{1.4.88}$$

in E, since

$$g_{ij}R^{ij}{}_{,kl} = 0 \tag{1.4.89}$$

for

$$\frac{1}{\sqrt{n(n-1)\varphi}}g_{ij} \tag{1.4.90}$$

is the future directed unit normal of the Cauchy hypersurfaces $\{\varphi = \text{const}\}$: The gradient of φ

$$\frac{\partial\varphi}{\partial g_{ij}} = \frac{1}{2}\varphi g^{ij} \tag{1.4.91}$$

is a past directed normal in *covariant* notation. Its contravariant version has the form

$$\varphi^{-1}G_{ij,kl}g^{kl}\frac{1}{2}\varphi = -\frac{1}{2(n-1)}g_{ij}. \tag{1.4.92}$$

Therefore, the vector in (1.4.90) is future directed and has unit length as can easily be checked.

Now, let us choose a coordinate system (τ, ξ^A) associated with the Cauchy hypersurface

$$M = \{\varphi = 1\} \tag{1.4.93}$$

and express the metric as in (1.4.28). The time coordinate τ is defined as

$$\tau = \log\varphi. \tag{1.4.94}$$

Let t be the time function

$$t = \sqrt{\varphi} = e^{\frac{1}{2}\tau}, \tag{1.4.95}$$

then

$$dt^2 = \frac{1}{4}\varphi d\tau^2 \tag{1.4.96}$$

and we conclude that the Fiber metric can be expressed as

$$ds^2 = -\frac{16(n-1)}{n}dt^2 + \frac{4(n-1)}{n}t^2 G_{AB}d\xi^A d\xi^B, \tag{1.4.97}$$

where G_{AB} is independent of t. We also emphasize that t is independent of x, cf. Lemma 1.4.4.

Let $(\xi^a) = (t, \xi^A)$, $0 \le a \le m$, be the coordinates such that

$$\xi^0 = t \quad \wedge \quad 1 \le A \le m, \tag{1.4.98}$$

then we immediately deduce from (1.4.97) or (1.4.28) that the Ricci tensor satisfies

$$R_{0a} = 0 \qquad \forall\, 0 \le a \le m, \tag{1.4.99}$$

cf. also the arguments following (2.5.14) on page 69, where a more detailed proof is given.

Since the determinant of the metric in (1.4.97) is equal to

$$|\det(G_{ab})| = 16(\frac{n-1}{n})\{4(\frac{n-1}{n})\}^m t^{2m} \det(G_{AB}) \tag{1.4.100}$$

we conclude that Eq. (1.4.88) can be expressed in the form

$$\begin{aligned} &\frac{1}{16}\frac{n}{n-1}t^{-m}\frac{\partial(t^m \dot{u})}{\partial t} - \frac{1}{4}\frac{n}{n-1}t^{-2}\Delta_G u \\ &\qquad - (n-1)t^2\tilde{\Delta}u - \frac{n-2}{2}t^2(R - 2\Lambda)u = 0, \end{aligned} \tag{1.4.101}$$

where Δ_G is the Laplacian with respect to the metric G_{AB}.

For any point

$$(x, g_{ij}) \in E \tag{1.4.102}$$

the metric can be written in the form

$$g_{ij} = t^{\frac{4}{n}}\sigma_{ij}, \tag{1.4.103}$$

where σ_{ij} is independent of t and

$$\det \sigma_{ij} = \det \chi_{ij}, \tag{1.4.104}$$

cf. (1.4.61) and (1.4.64). Hence, we can write

$$\tilde{\Delta}u = t^{-\frac{4}{n}}\tilde{\Delta}_{\sigma_{ij}}u. \tag{1.4.105}$$

Thus, let us equip E with the metric

$$\begin{aligned} d\bar{s}^2 &= -\frac{16(n-1)}{n}dt^2 + \frac{4(n-1)}{n}t^2 G_{AB}d\xi^A d\xi^B + \frac{1}{n-1}\sigma_{ij}dx^i dx^j \\ &\equiv G_{ab}d\xi^a d\xi^b + \frac{1}{n-1}\sigma_{ij}dx^i dx^j \\ &\equiv G_{\alpha\beta}d\zeta^\alpha d\zeta^\beta, \end{aligned} \tag{1.4.106}$$

where $0 \le a \le m$ and $\xi^0 = t$. We call G_{ab} the Fiber metric and σ_{ij} the Base metric, which are to be evaluated at the points

$$(x, \xi^a) \equiv (x, g_{ij}) = (x, t^{\frac{4}{n}} \sigma_{ij}). \tag{1.4.107}$$

Beware that

$$\sigma_{ij} = \sigma_{ij}(x, \xi^A) \in E_1, \tag{1.4.108}$$

where E_1 is the subbundle

$$E_1 = \{t = 1\}. \tag{1.4.109}$$

The operator P in (1.4.101) is a symmetric hyperbolic differential operator

$$Pu = -D_\alpha(a^{\alpha\beta} D_\beta u), \tag{1.4.110}$$

where the derivatives are covariant derivatives with respect to the metric in (1.4.106) and the coefficients $a^{\alpha\beta}$ represent a Lorentzian metric. However, it is not normally hyperbolic; i.e., its main part is not identical with the Laplacian of the ambient metric. Nevertheless, we can consider P as a normally hyperbolic operator by equipping E with the metric

$$\begin{aligned} d\tilde{s}^2 = &-\frac{16(n-1)}{n} dt^2 + \frac{4(n-1)}{n} t^2 G_{AB} d\xi^A d\xi^B \\ &\qquad + \frac{1}{n-1} t^{\frac{4}{n}-2} \sigma_{ij} dx^i dx^j \\ \equiv &\ \widetilde{G}_{\alpha\beta} d\zeta^\alpha d\zeta^\beta, \end{aligned} \tag{1.4.111}$$

though, of course, P is not symmetric in this metric.

Let E, $\widetilde{E}$ be the bundles

$$(E, G_{\alpha\beta}) \quad \wedge \quad (E, \widetilde{G}_{\alpha\beta}) \tag{1.4.112}$$

respectively, and E_1 resp. $\widetilde{E}_1$ the corresponding subbundles defined by

$$\{t = 1\}. \tag{1.4.113}$$

We shall now prove that E and $\widetilde{E}$ are both globally hyperbolic manifolds and the subbundles E_1 resp. $\widetilde{E}_1$, or more generally, the subbundles $E_1(\tau)$ resp. $\widetilde{E}_1(\tau)$, defined by

$$\{t = \tau\}, \quad \tau > 0, \tag{1.4.114}$$

Cauchy hypersurfaces provided the base space $\mathcal{S}_0$ is either compact or a homogeneous space for a suitable metric ρ_{ij}.

Lemma 1.4.5 *The bundles E and $\widetilde{E}$ are both globally hyperbolic manifolds, if $\mathcal{S}_0$ is either compact or a homogeneous space for a suitable metric ρ_{ij}, and the hypersurfaces $E_1(\tau)$ resp. $\widetilde{E}_1(\tau)$ are Cauchy hypersurfaces.*

Proof We shall only prove that E is globally hyperbolic, since the proof for $\widetilde{E}$ is essentially identical. We shall show that E_1 is a Cauchy hypersurface. The arguments will then also apply in case of the hypersurfaces $E_1(\tau)$. The proof will be similar to the proof of Lemma 1.4.3, where we proved that the fibers of E are globally hyperbolic. The fact that we now consider the whole bundle creates a small complication which will be handled by the additional assumption on $\mathcal{S}_0$.

We shall now prove that E_1 is a Cauchy hypersurface implying that E is globally hyperbolic. Let us argue by contradiction. Thus, let

$$\gamma(s) = (\gamma^\alpha(s)), \quad s \in I = (a, b), \tag{1.4.115}$$

be an inextendible future directed causal curve in E and assume that γ does not intersect E_1. We shall show that this will lead to a contradiction. It is also obvious that γ can meet E_1 at most once.

Assume that there exists $s_0 \in I$ such that

$$t(\gamma(s_0)) < 1 \tag{1.4.116}$$

and assume from now on that s_0 is the left end point of I. Since t is continuous, the whole curve γ must be contained in the past of E_1.

γ is causal, i.e.,

$$\frac{1}{n-1}\sigma_{ij}\dot{x}^i\dot{x}_j + \frac{4(n-1)}{n}t^2 G_{AB}\dot{\gamma}^A\dot{\gamma}^B \le \frac{16(n-1)}{n}|\dot{\gamma}^0|^2 \tag{1.4.117}$$

and thus

$$\sqrt{\frac{1}{n-1}\sigma_{ij}\dot{x}^i\dot{x}_j + \frac{4(n-1)}{n}t^2 G_{AB}\dot{\gamma}^A\dot{\gamma}^B} \le 4\dot{\gamma}^0, \tag{1.4.118}$$

since γ is future directed.

Let

$$\tilde{\gamma} = (x^i, \gamma^A) \tag{1.4.119}$$

be the projection of γ onto E_1, then the length of $\tilde{\gamma}$ is bounded

$$\begin{aligned} L(\tilde{\gamma}) &\le \int_I \sqrt{\frac{1}{n-1}\sigma_{ij}\dot{x}^i\dot{x}_j + \frac{4(n-1)}{n}G_{AB}\dot{\gamma}^A\dot{\gamma}^B} \\ &\le 4(1 - t(s_0)) < 4. \end{aligned} \tag{1.4.120}$$

Expressing the quadratic form

$$G_{AB}\dot{\gamma}^A\dot{\gamma}^B \tag{1.4.121}$$

in E_1 in the coordinates $(g_{ij}) = (\sigma_{ij})$, we have

$$\begin{aligned} G_{AB}\dot\gamma^A\dot\gamma^B &= \sigma^{ik}\sigma^{jl}\dot\sigma_{ij}\dot\sigma_{kl} \\ &\equiv \|\dot\sigma_{ij}\|^2, \end{aligned} \tag{1.4.122}$$

since the right-hand side is exactly

$$G^{ij,kl}\dot\sigma_{ij}\dot\sigma_{kl}, \tag{1.4.123}$$

if

$$\dot\sigma_{ij} \in T(E_1). \tag{1.4.124}$$

Hence, we infer, in view of [28, Lemma 14.2], that the metrics $(\sigma_{ij}(s))$ are all uniformly equivalent in I and converge to a positive definite metric when s tends to b. It remains to prove that the points $(x^i(s))$ are precompact in $\mathcal{S}_0$ and then we would have derived a contradiction.

If $\mathcal{S}_0$ is compact, then the precompactness of $(x^i(s))$ is trivial; thus, let us assume that $(\mathcal{S}_0, \rho_{ij})$ is a homogeneous space. Then, $\sigma_{ij}(s_0)$ is equivalent to $\rho_{ij}(x(s_0))$, and hence, in view of the homogeneity, $\sigma_{ij}(s)$ is uniformly equivalent to $\rho_{ij}(x(s))$ for all $s \in I$, and we conclude

$$\int_I \sqrt{\rho_{ij}\dot x^i \dot x^j} \le \text{const} \tag{1.4.125}$$

proving the precompactness. E_1 is therefore a Cauchy hypersurface and E is globally hyperbolic. $\square$

Remark 1.4.6 Since $\widetilde{E}$ is globally hyperbolic and P is a normally hyperbolic differential operator, the Cauchy problems

$$\begin{aligned} Pu &= f, \\ u_{|_{\widetilde{E}_1(\tau)}} &= u_0, \\ u_\alpha\tilde\nu^\alpha{}_{|_{\widetilde{E}_1(\tau)}} &= u_1 \end{aligned} \tag{1.4.126}$$

have unique solutions

$$u \in C^\infty(\widetilde{E}) \tag{1.4.127}$$

for given values $u_0, u_1 \in C_c^\infty(\widetilde{E}_1(\tau))$, and $f \in C_c^\infty(\widetilde{E})$ such that

$$\operatorname{supp} u \subset J^{\widetilde{E}}(K), \tag{1.4.128}$$

where

$$K = \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} f, \tag{1.4.129}$$

cf. [2, 26, 27].

Since E, $\widetilde{E}$, and $E_1(\tau)$ resp. $\widetilde{E}_1(\tau)$ coincide as sets and the normals (ν^α) resp. $\tilde{\nu}^\alpha)$ are also identical

$$\tilde{\nu} = \nu \tag{1.4.130}$$

we immediately deduce that the Cauchy problems (1.4.126) are also uniquely solvable in E. Using this information, we then could derive the existence of the fundamental solutions $F_\pm$ for P in E and also the existence of the advanced resp. retarded Green's operators $G_\pm$ of P, cf. [26, Theorem 4].

However, we would like to show how the fundamental solutions $\widetilde{F}_\pm$ of P in $\widetilde{E}$ can easily be transformed to yield fundamental solutions of P in E and similarly Green's functions $\widetilde{G}_\pm$. This process is valid in general pseudo-riemannian manifolds and thus also valid for elliptic operators; however, we shall only consider Lorentzian manifolds. The notations N resp. $\widetilde{N}$ refer to the same manifold N equipped with the metrics $g_{\alpha\beta}$ resp. $\widetilde{g}_{\alpha\beta}$.

Definition 1.4.7 Let $T \in \mathcal{D}'(N)$ be a distribution and let $\sqrt{|g|}$ be the volume element in N, where

$$g = \det g_{\alpha\beta}, \tag{1.4.131}$$

then we use the notation

$$\langle T, \eta\sqrt{|g|}\rangle \tag{1.4.132}$$

or

$$T[\eta\sqrt{|g|}] \tag{1.4.133}$$

to refer to "T acts on η" instead of the usual symbols

$$\langle T, \eta\rangle \tag{1.4.134}$$

or

$$T[\eta]. \tag{1.4.135}$$

If P is a differential operator in N and P^* its formal adjoint, then

$$\langle PT, \eta\sqrt{|g|}\rangle = \langle T, (P^*\eta)\sqrt{|g|}\rangle. \tag{1.4.136}$$

We found this notation in [10, Definition 2.8.1, p. 60].

Lemma 1.4.8 *Let $T \in \mathcal{D}'(N, \widetilde{g})$ and let g be a another smooth metric in N and set*

$$\psi = \frac{\sqrt{|\widetilde{g}|}}{\sqrt{|g|}}, \tag{1.4.137}$$

then

$$\psi T \in \mathcal{D}'(N, g) \tag{1.4.138}$$

and

$$\langle \psi T, \eta\sqrt{|g|}\rangle = \langle T, \eta\sqrt{|\widetilde{g}|}\rangle \qquad \forall\, \eta \in C_c^\infty(N). \tag{1.4.139}$$

Proof Follows immediately from the definition of ψT

$$\langle \psi T, \eta\sqrt{|g|}\rangle = \langle T, \psi\eta\sqrt{|g|}\rangle = \langle T, \eta\sqrt{|\widetilde{g}|}\rangle. \tag{1.4.140}$$

□

As an application we obtain:

Corollary 1.4.1 *Let $\widetilde{F}_\pm$ resp. $\widetilde{G}_\pm$ be the fundamental solutions of P in $\widetilde{E}$ resp. the advanced and retarded Green's operators, and define*

$$\psi = \frac{\sqrt{|\widetilde{G}|}}{\sqrt{|G|}} = t^{2-n}, \tag{1.4.141}$$

then

$$F_\pm = \psi \widetilde{F}_\pm \tag{1.4.142}$$

are fundamental solutions of P in E and

$$G_\pm = \psi \widetilde{G}_\pm \tag{1.4.143}$$

the advanced and retarded Green's operators.

Proof "(1.4.142)" We have

$$\begin{aligned} F_\pm[\eta\sqrt{|G|}\,] &= \psi \widetilde{F}_\pm[\eta\sqrt{|G|}\,] \\ &= \widetilde{F}_\pm[\eta\sqrt{|\widetilde{G}|}\,] \end{aligned} \tag{1.4.144}$$

and

$$P F_\pm[\eta\sqrt{|G|}\,] = P \widetilde{F}_\pm[\eta\sqrt{|\widetilde{G}|}\,] = \eta. \tag{1.4.145}$$

"(1.4.143)" To prove the second claim, we note that Green's operators are defined as maps

$$C_c^\infty(E) \to C^\infty(E) \tag{1.4.146}$$

by the definition

$$G_\pm[\eta\sqrt{|G|}\,](p) = F_\pm(p)[\eta\sqrt{|G|}\,], \qquad p \in E. \tag{1.4.147}$$

Now, from (1.4.144), we deduce

$$\begin{aligned} F_\pm(p)[\eta\sqrt{|G|}\,] &= \widetilde{F}_\pm(p)[\eta\sqrt{|\widetilde{G}|}\,] \\ &= \widetilde{G}_\pm[\eta\sqrt{|\widetilde{G}|}\,](p) \\ &= \psi\widetilde{G}_\pm[\eta\sqrt{|G|}\,](p). \end{aligned} \tag{1.4.148}$$

□

Remark 1.4.9 Let G be the Green's operator of P in E

$$G = G_+ - G_-, \tag{1.4.149}$$

then

$$N(P) = \{Gu : u \in C_c^\infty(E)\} \tag{1.4.150}$$

is the kernel of P. Its elements are smooth functions which are spacelike compact; however, this condition is strictly correct only in $\widetilde{E}$, since the light cones in $\widetilde{E}$ and E are different. Fortunately, we only need one special property of spacelike compact functions, namely that their restrictions to Cauchy hypersurfaces have compact support. This will be case in E, if we only consider the Cauchy hypersurfaces $E_1(\tau)$, as we shall prove in the lemma below.

Lemma 1.4.10 *The compact subsets of $\widetilde{E}_1(\tau)$ are also compact in $E_1(\tau)$ and vice versa.*

Proof The Cauchy hypersurfaces $E_1(\tau)$ resp. $\widetilde{E}_1(\tau)$ carry the same topology, since their induced metrics are uniformly equivalent as one easily checks. □

1.5 The Second Quantization

Let us first summarize some facts about Green's operators $G_\pm$ of P in E which are still valid even though P is not normally hyperbolic.

Lemma 1.5.1 *Let $G_\pm$ resp. $\widetilde{G}_\pm$ be Green's operators of P in E resp. $\widetilde{E}$, then*

$$G_\pm : C_c^\infty(E) \to C^\infty(E) \tag{1.5.1}$$

$$P \circ G_\pm = G_\pm \circ P_{|_{C_c^\infty(E)}} = \mathrm{id}_{|_{C_c^\infty(E)}} \tag{1.5.2}$$

$$supp(G_\pm u) = supp(\widetilde{G}_\pm u) \quad \forall\, u \in C_c^\infty(E) \tag{1.5.3}$$

$$supp\, G_+ u \subset J_+^{\widetilde{E}}(supp\, u) \quad \forall\, u \in C_c^\infty(E) \tag{1.5.4}$$

$$supp\, G_- u \subset J_-^{\widetilde{E}}(supp\, u) \quad \forall\, u \in C_c^\infty(E) \tag{1.5.5}$$

$$\textit{supp}\, G_+u \cap \textit{supp}\, G_-v \quad \textit{is compact} \tag{1.5.6}$$

for all $u, v \in C_c^\infty(E)$ *and*

$$G_\pm^* = G_\mp. \tag{1.5.7}$$

Proof The properties (1.5.1) and (1.5.2) immediately follow from the corresponding relations for $\widetilde{G}_\pm$ of P in $\widetilde{E}$ and the fact that

$$G_\pm = t^{2-n}\widetilde{G}_\pm, \tag{1.5.8}$$

cf. Corollary 1.4.1 on page 34. The preceding relation also proves the properties (1.5.3)–(1.5.6), since the topologies of E and $\widetilde{E}$ are identical.

It remains to prove (1.5.7). Let $u, v \in C_c^\infty(E)$, then

$$\begin{aligned}\int_E \langle G_\pm u, v\rangle &= \int_E \langle G_\pm u, PG_\mp v\rangle \\ &= \int_E \langle PG_\pm u, G_\mp v\rangle \\ &= \int_E \langle u, G_\mp v\rangle,\end{aligned} \tag{1.5.9}$$

where the partial integration is justified because of (1.5.6), and the scalar product is just normal multiplication. □

Lemma 1.5.2 *Let* $E_1(\tau)$ *be one of the special Cauchy hypersurfaces in* E*, then*

$$\int_E \langle u, Gv\rangle = \int_{E_1(\tau)} \{\langle D_\nu(Gu), Gv\rangle - \langle Gu, D_\nu Gv\rangle\}, \tag{1.5.10}$$

for all $u, v \in C_c^\infty(E)$*, where* ν *is the future directed normal of* $E_1(\tau)$*.*

Proof Let E_+, E_- be defined by

$$E_+ = \{t > \tau\} \tag{1.5.11}$$

and

$$E_- = \{t < \tau\}, \tag{1.5.12}$$

then

$$\int_E \langle u, Gv\rangle = \int_{E_+} \langle u, Gv\rangle + \int_{E_-} \langle u, Gv\rangle. \tag{1.5.13}$$

Now, in E_+ we have

$$PG_-u = u \tag{1.5.14}$$

and

$$PGv = 0 = GPv. \tag{1.5.15}$$

Moreover,

$$\operatorname{supp}(G_{-}u) \cap E_{+} \quad \text{is compact,} \tag{1.5.16}$$

since

$$\operatorname{supp}(\widetilde{G}_{-}u) \cap \widetilde{E}_{+} \quad \text{is compact,} \tag{1.5.17}$$

hence we obtain by partial integration

$$\int_{E_{+}} \langle PG_{-}u, Gv\rangle = -\int_{E_1(\tau)} \langle D_\nu G_{-}u, Gv\rangle + \int_{E_1(\tau)} \langle G_{-}u, D_\nu Gv\rangle. \tag{1.5.18}$$

A similar argument applies to E_{-} by looking at

$$PG_{+}u = 0 \tag{1.5.19}$$

leading to

$$\int_{E_{-}} \langle PG_{+}u, Gv\rangle = \int_{E_1(\tau)} \langle D_\nu G_{+}u, Gv\rangle - \int_{E_1(\tau)} \langle G_{+}u, D_\nu Gv\rangle. \tag{1.5.20}$$

Adding these two equations implies the result. □

We shall now construct a CCR representation or a Weyl system for P and its kernel

$$N(P) = \{u \in C^\infty(E) : Pu = 0\} = \{Gu : u \in C_c^\infty(E)\}. \tag{1.5.21}$$

This characterization of $N(P)$ is correct, since it is valid in $\widetilde{E}$ and because of

$$PG[u\sqrt{|G|}\,] = P\widetilde{G}[u\sqrt{|\widetilde{G}|}\,], \tag{1.5.22}$$

cf. (1.4.143) on page 34.

There are two ways to construct a Weyl system given a formally self-adjoint, normally hyperbolic operator in a globally hyperbolic spacetime which are also applicable in our case, though P is not normally hyperbolic. One possibility is to define a symplectic vector space

$$V = C_c^\infty(e)/N(G), \tag{1.5.23}$$

where G is Green's operator of P

$$G = G_{+} - G_{-}. \tag{1.5.24}$$

Since

$$G^* = -G \tag{1.5.25}$$

the bilinear form

$$\omega(u, v) = \int_E \langle u, Gv \rangle, \qquad u, v \in V, \tag{1.5.26}$$

is skew-symmetric, non-degenerate by definition, and hence symplectic. Then, there is a canonical way to construct a corresponding Weyl system.

The second method is to pick a Cauchy hypersurface E_1 in E and then define a quantum field Φ with values in the space of essentially self-adjoint operators in a corresponding symmetric Fock space.

We pick a Cauchy hypersurface $E_1 = E_1(\tau)$ in E and define the complex Hilbert space

$$H_{E_1} = L^2(E_1) \otimes \mathbb{C} = L^2(E_1, \mathbb{C}) \tag{1.5.27}$$

the complexification of the real Hilbert space $L^2(E_1)$ with complexified scalar product

$$\langle u, v \rangle_{E_1} = \int_{E_1} \langle u, v \rangle_{\mathbb{C}}. \tag{1.5.28}$$

We denote the symmetric Fock space of H_{E_1} by $\mathcal{F}_+(H_{E_1})$. Let Θ be the corresponding Segal field. Since $G^* = -G$, we deduce from (1.5.4), (1.5.6) and Remark 1.4.9 on page 35 that

$$G^* u_{|_{E_1}} \in C_c^\infty(E_1) \subset H_{E_1} \quad \forall\, u \in C_c^\infty(E). \tag{1.5.29}$$

We can therefore define

$$\Phi_{E_1}(u) = \Theta(i(G^*u)_{|_{E_1}} - D_\nu (G^*u)_{|_{E_1}}). \tag{1.5.30}$$

From the proof of [2, Lemma 4.6.8], we conclude that the right-hand side of (1.5.30) is an essentially self-adjoint operator in $\mathcal{F}_+(H_{E_1})$. We therefore call the map Φ_{E_1} from $C_c^\infty(E)$ to the set of self-adjoint operators in $\mathcal{F}_+(H_{E_1})$ a quantum field defined in E_1.

Lemma 1.5.3 *The quantum field Φ_{E_1} satisfies the equation*

$$P\Phi_{E_1} = 0 \tag{1.5.31}$$

in the distributional sense, i.e.

$$\begin{aligned} \langle P\Phi_{E_1}, u \rangle &= \langle \Phi_{E_1}, Pu \rangle \\ &= \Phi_{E_1}(Pu) = 0 \qquad \forall\, u \in C_c^\infty(E). \end{aligned} \tag{1.5.32}$$

Proof In view of (1.5.25), we have

$$G^*(Pu) = 0. \tag{1.5.33}$$

□

With the help of the quantum field Φ_{E_1}, we shall construct a Weyl system and hence a CCR representation of the symplectic vector space (V, ω) which we defined in (1.5.23) and (1.5.26).

From (1.5.30), we conclude the commutator relation

$$[\Phi_{E_1}(u), \Phi_{E_1}(v)] = i\operatorname{Im}\langle iG^*u - D_\nu(G^*u), iG^*v - D_\nu(G^*v)\rangle_{E_1} I, \tag{1.5.34}$$

for all $u, v \in C_c^\infty(E)$, cf. [3, Proposition 5.2.3], where both sides are defined in the algebraic Fock space $\mathcal{F}_{+\text{alg}}(H_{E_1})$.

On the other hand

$$\begin{aligned}\operatorname{Im}\langle iG^*u - D_\nu(G^*u), & iG^*v - D_\nu(G^*v)\rangle_{E_1} \\ &= -\operatorname{Im}\langle iG^*u, D_\nu(G^*v)\rangle_{E_1} - \operatorname{Im}\langle D_\nu(G^*u), iG^*v\rangle_{E_1} \\ &= \int_{E_1} \{\langle G^*u, D_\nu(G^*v)\rangle - \langle D_\nu(G^*u), G^*v\rangle\} \\ &= \int_E \langle u, Gv\rangle\end{aligned} \tag{1.5.35}$$

in view of (1.5.10) and (1.5.25).

As a corollary, we conclude

$$[\Phi_{E_1}(u), \Phi_{E_1}(v)] = i\int_{E_1} \langle u, Gv\rangle I \quad \forall u, v \in C_c^\infty(E). \tag{1.5.36}$$

From [3, Proposition 5.2.3] and (1.5.35), we immediately infer

Theorem 1.5.4 *Let (V, ω) be the symplectic vector space in (1.5.23) and (1.5.26) and denote by $[u]$ the equivalence classes in V, then*

$$W([u]) = e^{i\Phi_{E_1}(u)} \tag{1.5.37}$$

defines a Weyl system for (V, ω), where $\Phi_{E_1}(u)$ is now supposed to be the closure of $\Phi_{E_1}(u)$ in $\mathcal{F}_+(H_{E_1})$, i.e., $\Phi_{E_1}(u)$ is a self-adjoint operator. The Weyl system generates a C^*-algebra with unit which we call a CCR representation of (V, ω).

Remark 1.5.5 Since all CCR representations of (V, ω) are *-isomorphic, where the isomorphism maps Weyl systems to Weyl systems, cf. [3, Theorem 5.2.8], this especially applies to the CCR representations corresponding to different Cauchy hypersurfaces $E_1 = E_1(\tau)$ and ${E_1}' = E_1(\tau')$; i.e., there exists a *-isomorphism T such that

$$T(e^{i\Phi_{E_1}(u)}) = e^{i\Phi_{{E_1}'}(u)} \qquad \forall\,[u] \in V. \tag{1.5.38}$$

1.6 The Gravitational Waves Model

In the previous sections, we saw that the quantization of the Hamilton constraint does not yield a unique result but depends on the equation by which the Hamilton constraint is expressed. In [16], we obtain the Wheeler–DeWitt equation after quantization and in the previous sections the Eq. (1.4.101) on page 29 which differs significantly. In this section, we shall propose another model by replacing any occurrence of the term

$$|A|^2 - H^2 \tag{1.6.1}$$

by

$$(R - 2\Lambda). \tag{1.6.2}$$

However, when we do this on the right-hand side of (1.3.61) on page 14, then, after quantization, we would obtain an elliptic equation instead of an hyperbolic equation, namely

$$-(n-1)\tilde{\Delta} u + \frac{n-4}{2}(R - 2\Lambda)u = 0 \tag{1.6.3}$$

valid in E, which, for fixed (t, g_{ij}), can be looked at as an eigenvalue equation, where Λ would be a constant multiple of the eigenvalue provided $n \neq 4$. In case $\mathcal{S}_0$ is compact, a spectral resolution of equation (1.6.3) would be possible.

However, we believe that a hyperbolic and not an elliptic equation should define the possible states of quantum gravity. In order to obtain a hyperbolic equation while eliminating any occurrences of the term in (1.6.1) we have to express the Hamilton constraint by a different equation. In Sect. 1.3, the Hamilton equations only yielded the Tangential Einstein equations (1.3.50) on page 12, or equivalently,

$$\overline{R}_{ij} - \frac{1}{2}\overline{R} g_{ij} + \Lambda g_{ij} = 0. \tag{1.6.4}$$

The Hamilton constraint expresses the normal component of the Einstein equations, where the terms *tangential* und *normal* refer to the foliation $M(t)$ of the spacetime N. This foliation is also the solution set of the geometric flow equation

$$\dot{x} = -w\nu \tag{1.6.5}$$

with initial hypersurface

$$M_0 = \mathcal{S}_0, \tag{1.6.6}$$

where ν is the past directed normal ν of the solution hypersurfaces $M(t)$, cf. [12, Eq. (2.3.25)]. We shall use the evolution equation of the mean curvature $H(t)$ of the $M(t)$ to define the Hamilton constraint.

The mean curvature satisfies the evolution equation

$$\dot{H} = -\tilde{\Delta} w + \{|A|^2 + \overline{R}_{\alpha\beta}\nu^\alpha\nu^\beta\}w, \tag{1.6.7}$$

where we embellished the Laplacian with a tilde, cf. [12, Eq. (2.3.27)] observing that in that reference

$$e^\psi = w. \tag{1.6.8}$$

To exploit this evolution equation, we need the following lemma:

Lemma 1.6.1 *Assume that the Eq.* (1.6.4) *is valid, then*

$$\frac{1}{2}\overline{R} = \frac{1}{n-1}\{G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda\} + \frac{n+1}{n-1}\Lambda \tag{1.6.9}$$

and

$$\overline{R}_{\alpha\beta}\nu^\alpha\nu^\beta = \frac{n-2}{n-1}\{G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda\} - \frac{2}{n-1}\Lambda. \tag{1.6.10}$$

Proof "(1.6.9)" There holds

$$\overline{R} = g^{ij}\overline{R}_{ij} - \overline{R}_{\alpha\beta}\nu^\alpha\nu^\beta \tag{1.6.11}$$

and hence

$$\overline{R}_{\alpha\beta}\nu^\alpha\nu^\beta + \frac{1}{2}\overline{R} = \frac{n-1}{2}\overline{R} - n\Lambda \tag{1.6.12}$$

or, equivalently,

$$\frac{1}{n-1}\{G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda\} = \frac{1}{2}\overline{R} - \frac{n+1}{n-1}\Lambda. \tag{1.6.13}$$

"(1.6.10)" Combining (1.6.12) and (1.6.13), we deduce

$$\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta = \frac{n-2}{n-1}\{G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda\} - \frac{2}{n-1}\Lambda. \tag{1.6.14}$$

□

We note that

$$\pi^{ij} = (Hg^{ij} - h^{ij})\varphi, \tag{1.6.15}$$

where (h^{ij}) is the contravariant version of the second fundamental form and where we also point out that, as before, we introduced the function φ to replace the density $\sqrt{g}$ in order to deal with tensors instead of densities.

Hence, we have

$$(n-1)H\varphi = g_{ij}\pi^{ij} \tag{1.6.16}$$

and we shall use the evolution equation of

$$(n-1)H\varphi^{\frac{1}{2}} \tag{1.6.17}$$

to express the Hamilton constraint.

We immediately deduce

$$\begin{aligned}(\varphi^{\frac{1}{2}})' &= \frac{1}{4}\varphi^{\frac{1}{2}} g^{ij}\dot{g}_{ij} \\ &= -\frac{1}{2}\varphi^{\frac{1}{2}} H w,\end{aligned} \tag{1.6.18}$$

cf. (1.3.34) on page 11, and obtain, in view of (1.6.7) and (1.6.10),

$$\begin{aligned}(n-1)(H\varphi^{\frac{1}{2}})' &= -(n-1)\tilde{\Delta} w\varphi^{\frac{1}{2}} \\ &\quad + (n-1)\{|A|^2 + \overline{R}_{\alpha\beta}\nu^\alpha\nu^\beta\}w\varphi^{\frac{1}{2}} - \frac{n-1}{2}H^2\varphi^{\frac{1}{2}}w \\ &= -(n-1)\tilde{\Delta} w\varphi^{\frac{1}{2}} + (n-1)(|A|^2 - H^2)\varphi^{\frac{1}{2}}w \\ &\quad + \frac{n-1}{2}H^2\varphi^{\frac{1}{2}}w + (n-2)\{G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda\}\varphi^{\frac{1}{2}}w \\ &\quad - 2\Lambda\varphi^{\frac{1}{2}}w.\end{aligned} \tag{1.6.19}$$

Employing now the Hamilton condition and observing that

$$\frac{1}{2}\{|A|^2 - H^2 - (R - 2\Lambda)\} = -\{G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda\}, \tag{1.6.20}$$

cf. [12, Eq. (1.1.43)], we conclude that the evolution equation

$$\begin{aligned}(n-1)(H\varphi^{\frac{1}{2}})' = -(n-1)\tilde{\Delta} w\varphi^{\frac{1}{2}} + (n-1)(R-2\Lambda)\varphi^{\frac{1}{2}}w \\ - 2\Lambda\varphi^{\frac{1}{2}}w + \frac{n-1}{2}H^2\varphi^{\frac{1}{2}}w\end{aligned} \tag{1.6.21}$$

is equivalent to the Hamilton condition provided the tangential Einstein equations are valid.

Finally, expressing the time derivative on the left-hand side by the Poisson brackets such that

$$(n-1)\{H\varphi^{\frac{1}{2}}, \mathcal{H}\} = -(n-1)\tilde{\Delta} w\varphi^{\frac{1}{2}} + (n-1)(R-2\Lambda)\varphi^{\frac{1}{2}} w \\ - 2\Lambda\varphi^{\frac{1}{2}} w + \frac{n-1}{2} H^2 \varphi^{\frac{1}{2}} w \tag{1.6.22}$$

we conclude that the Hamilton equations and the geometric evolution equation (1.6.22) are equivalent to the full Einstein equation, cf. the proof of Theorem 1.3.3 on page 13.

Switching to the gauge $w = 1$, we then quantize equation (1.6.22). Because of the relation (1.6.16), the left-hand side of (1.6.22) is transformed to

$$i[\widehat{H}, \varphi^{-\frac{1}{2}} \hat{g}_{ij} \hat{\pi}^{ij}] = [\hat{H}, \varphi^{-\frac{1}{2}} g_{ij} \frac{\delta}{\delta g_{ij}}], \tag{1.6.23}$$

where $\widehat{H}$ is the transformed Hamiltonian. On the other hand,

$$\varphi^{-\frac{1}{2}} g_{ij} \frac{\delta}{\delta g_{ij}} = \sqrt{n(n-1)}\nu^a D_a = \sqrt{n(n-1)}\nu^0 D_0 \\ = \frac{n}{4}\frac{\partial}{\partial t}, \tag{1.6.24}$$

where ν^a is the future unit normal of the hypersurfaces

$$M(t) = \{\xi^0 = t\}, \tag{1.6.25}$$

i.e., the left-hand side of (1.6.24) is a constant multiple of the covariant derivative with respect to t in the fiber when the differential operator is applied to functions $u = u(x, g_{ij})$. Hence,

$$[\widehat{H}, \varphi^{-\frac{1}{2}} g_{ij} \frac{\delta}{\delta g_{ij}}]u \\ = \varphi^{-\frac{1}{2}} g_{ij} \frac{\delta}{\delta g_{ij}} \{(R-2\Lambda)u\varphi\} - \varphi^{-\frac{1}{2}}(R-2\Lambda)\varphi g_{ij} \frac{\partial u}{\partial g_{ij}} \\ = \varphi^{-\frac{1}{2}} \{\frac{n}{2}(R-2\Lambda)u\varphi - Ru\varphi - (n-1)\tilde{\Delta} u\varphi\}, \tag{1.6.26}$$

in view of (1.4.84) on page 27. The transformation of the right-hand side of (1.6.22), note that $w = 1$, yields

$$(n-1)(R-2\Lambda)u\varphi^{\frac{1}{2}} - 2\Lambda u\varphi^{\frac{1}{2}} + \varphi^{\frac{1}{2}} \frac{n-1}{2} H^2 u, \tag{1.6.27}$$

where

$$\begin{aligned}\varphi^{\frac{1}{2}}\frac{n-1}{2}H^2u &= -\frac{n}{2}\varphi^{-\frac{1}{2}}\{\frac{1}{n(n-1)}\varphi^{-1}g_{ij}g_{kl}\frac{\delta}{\delta gij}\frac{\delta}{\delta g_{kl}}\}u \\ &= -\frac{n}{2}\varphi^{-\frac{1}{2}}(\nu^a\nu^b D_aD_bu)\end{aligned} \tag{1.6.28}$$

or

$$\varphi^{\frac{1}{2}}\frac{n-1}{2}H^2u = -\frac{n}{2}\varphi^{-\frac{1}{2}}D_a(\nu^a\nu^b D_bu) \tag{1.6.29}$$

depending on the ordering of the derivatives.

Observing that

$$\nu = (\nu^0, 0, \dots, 0) \tag{1.6.30}$$

and

$$\nu^0 = \frac{1}{4}\sqrt{\frac{n}{n-1}} \tag{1.6.31}$$

we obtain, after multiplying both sides with $\varphi^{\frac{1}{2}}$, the hyperbolic equations

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^2\tilde{\Delta}u - \frac{n}{2}Rt^2u + n\Lambda t^2u = 0 \tag{1.6.32}$$

or

$$\frac{1}{32}\frac{n^2}{n-1}t^{-m}\frac{\partial}{\partial t}(t^m\dot{u}) - (n-1)t^2\tilde{\Delta}u - \frac{n}{2}Rt^2u + n\Lambda t^2u = 0 \tag{1.6.33}$$

where we recall that $\varphi = t^2$, cf. (1.4.95) and (1.4.101) on page 29.

These equations can be rewritten, as before, by observing that

$$g_{ij} = t^{\frac{4}{n}}\sigma_{ij}, \tag{1.6.34}$$

such that

$$\tilde{\Delta}u = t^{-\frac{4}{n}}\tilde{\Delta}_{\sigma_{ij}}u \tag{1.6.35}$$

and

$$R = t^{-\frac{4}{n}}R_{\sigma_{ij}}, \tag{1.6.36}$$

where $R_{\sigma_{ij}}$ is the scalar curvature of the metric σ_{ij}. Both equations are hyperbolic equations in E, where $u = u(x, t, \xi^A)$, $1 \le A \le m$, and $\sigma_{ij} = \sigma_{ij}(x, \xi^A)$. However, for fixed (ξ^A), we may consider these equations as hyperbolic equations in

$$\mathcal{S}_0 \times \mathbb{R}^*_+, \tag{1.6.37}$$

where the solutions as well as the metric depend on an additional parameter (ξ^A). To simplify the notation, let us drop the tilde over the Laplacian and stipulate that the Laplacian as well as the scalar curvature refer to the metric σ_{ij}. Then, we can rewrite the equations as

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0 \tag{1.6.38}$$

and

$$\frac{1}{32}\frac{n^2}{n-1}t^{-m}\frac{\partial(t^m\dot{u})}{\partial t} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0, \tag{1.6.39}$$

respectively. We also note that

$$\det \sigma_{ij} = \det \chi_{ij} \tag{1.6.40}$$

and that $\sigma_{ij} \in E_1$ is arbitrary but fixed.

Lemma 1.6.2 *Both operators are symmetric with respect to the Lorentzian metric*

$$d\bar{s}^2 = -\frac{32(n-1)}{n^2}dt^2 + \sigma_{ij}dx^i dx^j \tag{1.6.41}$$

and they are normally hyperbolic with respect to the metric

$$d\tilde{s}^2 = -\frac{32(n-1)}{n^2}dt^2 + \frac{1}{n-1}t^{\frac{4}{n}-2}\sigma_{ij}dx^i dx^j. \tag{1.6.42}$$

Thus, if

$$Q = \mathcal{S}_0 \times \mathbb{R}^*_+ \tag{1.6.43}$$

is globally hyperbolic with respect to these metrics, and if we denote Q equipped with the metric (1.6.42) by $\tilde{Q}$ and stipulate that Q is equipped with the metric (1.6.41), then the results from Sects. 1.4 and 1.5 can be applied to the present setting.

Lemma 1.6.3 *Assume that the metric*

$$\sigma_{ij}(x,\xi) \in E_1, \tag{1.6.44}$$

where $\xi = (\xi^A)$ is fixed, is complete, then the Lorentzian manifolds Q and $\tilde{Q}$ are globally hyperbolic, and the hypersurfaces

$$M_\tau = \{t = \tau\} \subset Q \tag{1.6.45}$$

are Cauchy hypersurfaces.

Proof Let us only consider Q. From the proof of Lemma 1.4.5 on page 30, we infer that the claims are correct if a bounded curve

$$\gamma(s) \subset \mathcal{S}_0, \qquad s \in I, \tag{1.6.46}$$

where bounded means, bounded relative to σ_{ij}, is relatively compact which is the case, if $(\mathcal{S}_0, \sigma_{ij})$ is complete. □

In the next theorem, we would like to prove that the solutions depend smoothly on ξ. In order to achieve this, the Cauchy values have to be prescribed on $E_1(\tau)$ and not only on M_τ.

Theorem 1.6.4 *Let P be one of the hyperbolic operators in (1.6.39) or (1.6.38), and let $E_1(\tau)$ be given as well as functions $f \in C_c^\infty(E)$ and $u_0, u_1 \in C_c^\infty(E_1(\tau))$. These functions depend on (x, t, ξ). Since f, u_0, u_1 have compact support, the corresponding ξ, such that $f(\xi), u_0(\xi), u_1(\xi)$ do not identically vanish in Q, are contained in a relatively compact, open set U. Assume that the metrics*

$$\sigma_{ij}(x, \xi), \quad \xi \in U, \tag{1.6.47}$$

are all complete, then the Cauchy problems

$$\begin{aligned} Pu &= f \\ u_{|E_1(\tau)} &= u_0 \\ \dot{u}_{|E_1(\tau)} &= u_1 \end{aligned} \tag{1.6.48}$$

are uniquely solvable in (Q, σ_{ij}) for all $\xi \in U$ such that

$$u = u(x, t, \xi) \in C^\infty(E_{|U}), \tag{1.6.49}$$

where

$$E_{|U} = \{(x, t, \xi) : \xi \in U\}. \tag{1.6.50}$$

Proof First, we apply the results in Sect. 1.4 to the operator P and the globally hyperbolic spaces Q and $\tilde{Q}$ for each $\xi \in U$ to conclude that, for fixed $\xi \in U$, the solutions exist, are uniquely determined, and are smooth in (x, t). Arguing then as in the proof of [16, Theorem 5.4], where we considered solutions of hyperbolic problems in the fibers of E, where the solutions and the data were depending on the parameter $x \in \mathcal{S}_0$, we can prove, by considering the problems in $\tilde{Q}$, so that P is normally hyperbolic, that the solutions are also smooth in ξ. Moreover, for each $\xi \in U$, the solution $u(\xi)$ satisfies the known support properties of solutions in $\tilde{Q}$. □

Equations (1.6.39) or (1.6.38) can be looked at as being gravitational wave equations and some of the solutions $u = u(x, \xi)$ can be considered to be gravitons. Note that $\xi = (\xi^A)$ are coordinates for the metrics in the fibers, and the pair (x, ξ) represents the metric $\sigma_{ij}(x, \xi)$ in $\mathcal{S}_0$.

If $\mathcal{S}_0$ is compact, then we shall construct variational solutions of Eq. (1.6.38) with finite energy which may be considered to provide a spectral resolution of the problem for fixed ξ.

Let us start with the following well-known lemma:

Lemma 1.6.5 *Let $\mathcal{S}_0$ be compact equipped with the metric $\sigma_{ij} = \sigma_{ij}(\xi)$. Then the eigenvalue problem*

$$-(n-1)\Delta v - \frac{n}{2} R v = \mu v \tag{1.6.51}$$

has countably many solutions (v_i, μ_i) such that

$$\mu_0 < \mu_1 \le \mu_2 \le \cdots, \tag{1.6.52}$$

$$\lim_i \mu_i = \infty, \tag{1.6.53}$$

and

$$\int_{\mathcal{S}_0} \bar{v}_i v_j = \delta_{ij}, \tag{1.6.54}$$

where we now consider complex-valued functions. The eigenfunctions are a basis for $L^2(\mathcal{S}_0, \mathbb{C})$ and are smooth.

Now, we argue similarly as in [13, Sect. 6.7]: Choose any eigenfunction $v = v_i$ with positive eigenvalue $\mu = \mu_i$ and then we look at solutions u of (1.6.38) of the form

$$u(x,t) = w(t)v(x). \tag{1.6.55}$$

Inserting u in the equation, we deduce

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + \mu t^{2-\frac{4}{n}} w + n t^2 \Lambda w = 0, \tag{1.6.56}$$

or equivalently,

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} - \mu t^{2-\frac{4}{n}} w - n t^2 \Lambda w = 0. \tag{1.6.57}$$

This equation can be considered to be an implicit eigenvalue problem with eigenvalue Λ.

To solve (1.6.57), we first solve

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n t^2 w = \lambda \mu t^{2-\frac{4}{n}} w, \tag{1.6.58}$$

where λ is the eigenvalue. Let $I = \mathbb{R}_+^*$ and H be the embedded subspace of the Sobolev space $H_0^{1,2}(I)$

$$H \hookrightarrow H_0^{1,2}(I, \mathbb{C}) \tag{1.6.59}$$

defined as the completion of $C_c^\infty(I, \mathbb{C})$ under the norm of the scalar product

$$\langle w, \tilde{w}\rangle_1 = \int_I \{\bar{w}'\tilde{w}' + t^2\bar{w}\tilde{w}\}, \tag{1.6.60}$$

where a prime or a dot denotes differentiation with respect to t. Moreover, let B, K be the symmetric forms

$$B(w, \tilde{w}) = \int_I \{\frac{1}{32}\frac{n^2}{n-1}\bar{w}'\tilde{w}' + nt^2\bar{w}\tilde{w}\} \tag{1.6.61}$$

and

$$K(w, \tilde{w}) = \int_I \mu t^{2-\frac{4}{n}}\bar{w}\tilde{w}, \tag{1.6.62}$$

then the eigenvalue equation (1.6.58) is equivalent to

$$B(w, \varphi) = \lambda K(w, \varphi) \qquad \forall \varphi \in H \tag{1.6.63}$$

as one easily checks.

Lemma 1.6.6 *The quadratic form $K(w) = K(w, w)$ is compact relative to the quadratic form B, i.e. if $w_k \in H$ converges weakly to $w \in H$*

$$w_k \rightharpoonup w \qquad \text{in } H, \tag{1.6.64}$$

then

$$K(w_k) \to K(w). \tag{1.6.65}$$

Proof The proof is essentially the same as the proof of [13, Lemma 6.8] and will be omitted. □

Hence, the eigenvalue problem (1.6.63) has countably many solutions $(\tilde{w}_i, \lambda_i)$ such that

$$0 < \lambda_0 < \lambda_1 < \cdots, \tag{1.6.66}$$

$$\lim \lambda_i = \infty \tag{1.6.67}$$

and

$$K(\tilde{w}_i, \tilde{w}_j) = \delta_{ij}. \tag{1.6.68}$$

For a proof of this well-known result, except the strict inequalities in (1.6.66), see e.g. [15, Theorem 1.6.3, p. 37]. Each eigenvalue has multiplicity one since we have a linear ODE of order two and all solutions satisfy the boundary condition

$$\tilde{w}_i(0) = 0. \tag{1.6.69}$$

The kernel is two-dimensional, and the condition (1.6.69) defines a one-dimensional subspace. Note, that we considered only real-valued solutions to apply this argument.

Finally, the functions

$$w_i(t) = \tilde{w}_i(\lambda_i^{-\frac{n}{4(n-1)}} t) \tag{1.6.70}$$

then satisfy (1.6.57) with eigenvalue

$$\Lambda_i = -\lambda_i^{-\frac{n}{n-1}} \tag{1.6.71}$$

and

$$u_i = w_i v \tag{1.6.72}$$

is a solution of the wave equation (1.6.38) with finite energy

$$\|u_i\|^2 = \int_Q \{|\dot{u}|^2 + (1+t^2)\sigma^{ij}\bar{u}_i u_j + \mu t^{2-\frac{4}{n}}|u|^2\} < \infty. \tag{1.6.73}$$

Note that the actual energy is defined by a weaker norm

$$\int_Q \{|\dot{u}|^2 + t^{2-\frac{4}{n}}\sigma^{ij}\bar{u}_i u_j + \mu t^{2-\frac{4}{n}}|u|^2\} \tag{1.6.74}$$

which is of course bounded too.

Let us summarize these results:

Theorem 1.6.7 *Assume $n \geq 2$ and $\mathcal{S}_0$ to be compact and let (v, μ) be a solution of the eigenvalue problem (1.6.51) with $\mu > 0$, then there exist countably many solutions (w_i, Λ_i) of the implicit eigenvalue problem (1.6.57) such that*

$$\Lambda_i < \Lambda_{i+1} < \cdots < 0, \tag{1.6.75}$$

$$\lim_i \Lambda_i = 0, \tag{1.6.76}$$

and such that the functions

$$u_i = w_i v \tag{1.6.77}$$

are solutions of the wave equations (1.6.38). The transformed eigenfunctions

$$\tilde{w}_i(t) = w_i(\lambda_i^{\frac{n}{4(n-1)}} t), \tag{1.6.78}$$

where

$$\lambda_i = (-\Lambda_i)^{-\frac{n-1}{n}} \tag{1.6.79}$$

form a basis of the Hilbert space H and also of $L^2(\mathbb{R}_+^*, \mathbb{C})$.

Remark 1.6.8 Let σ_{ij} be a smooth and complete Riemannian metric in $\mathcal{S}_0$, then σ_{ij} is in general only a section of E but not an element. However, the metric χ_{ij} in

(1.4.1) on page 17, which we used to define φ in order to replace the density $\sqrt{g}$, can certainly be assumed to belong to E, and hence to the subbundle E_1, because we can easily define a covering of local trivializations where χ is always part of the generating local frames. Since χ is chosen arbitrarily, we may just as well assume that

$$\chi_{ij} = \sigma_{ij}. \tag{1.6.80}$$

Hence, the hyperbolic equations (1.6.38) or (1.6.39), which are supposed to describe a model for quantum gravity, can be applied to any given smooth and complete metric σ_{ij}, or more precisely, to any complete Riemannian manifold $(\mathcal{S}_0, \sigma_{ij})$.

Let us formulate this result in case of Eq. (1.6.38) as a theorem:

Theorem 1.6.9 *Let $(\mathcal{S}_0, \sigma_{ij})$ be a given connected, smooth and complete n-dimensional Riemannian manifold and let*

$$Q = \mathcal{S}_0 \times \mathbb{R}_+^* \tag{1.6.81}$$

be the corresponding globally hyperbolic spacetime equipped with the Lorentzian metric (1.6.41) *or, if necessary, with* (1.6.42)*, then the hyperbolic equation*

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0, \tag{1.6.82}$$

where the Laplacian and the scalar curvature correspond to the metric σ_{ij}, describes a model of quantum gravity. If $\mathcal{S}_0$ is compact, a spectral resolution of this equation has been proved in Theorem 1.6.7.

Remark 1.6.10 If $\mathcal{S}_0$ is not compact, then we shall prove in later chapters that a spectral resolution is possible if either $\mathcal{S}_0$ is an asymptotically Euclidean Cauchy hypersurface of a globally hyperbolic spacetime N, or, if N is a black hole, if $\mathcal{S}_0$ is the smooth limit of Cauchy hypersurfaces representing the event horizon though with a different metric.

Remark 1.6.11 When σ_{ij} is the metric of a space of constant curvature then equation (1.6.38), considered only for functions u which do not depend on x, is identical to the equation obtained by quantizing the Hamilton constraint in a Friedmann universe without matter but including a cosmological constant. The quantized Friedmann equation is the ODE

$$\frac{1}{16}\frac{n}{n-1}\ddot{u} - Rr^{2-\frac{4}{n}}u + 2r^2\Lambda u = 0, \qquad 0 < r < \infty, \tag{1.6.83}$$

cf. [13, Eq. (3.37)], though the equation there looks differently, since in that paper we divided the Lagrangian by $n(n-1)$.

Chapter 2
Interaction of Gravity with Yang-Mills and Higgs Fields

2.1 Gravity Interacting with Other Fields

The quantization of gravity interacting with Yang-Mills and Higgs fields poses no additional greater challenges—at least in principle. The number of variables will be increased, the combined Hamiltonian is the sum of several individual Hamiltonians, and, since gravity is involved, we have the Hamilton constraint as a side condition. Deriving the Einstein equations by a Hamiltonian setting requires a global time function x^0 and foliation of spacetime by its level hypersurfaces as we have seen in the previous chapter. Thus, we consider a spacetime $N = N^{n+1}$ with metric $(\bar{g}_{\alpha\beta})$, $0 \le \alpha, \beta \le n$, assuming the existence of a global time function x^0 which will also define the time coordinate. Furthermore, we only consider metrics that can be split by the time function, i.e., the metrics can be expressed in the form

$$d\bar{s}^2 = -w^2(dx^0)^2 + g_{ij}dx^i dx^j, \tag{2.1.1}$$

where $w > 0$ is a smooth function and $g_{ij}(x^0, x)$ are Riemannian metrics. Let

$$M(t) = \{x^0 = t\}, \qquad t \in x^0(N) \equiv I, \tag{2.1.2}$$

be the coordinate slices, then the g_{ij} are the induced metrics. Moreover, let $\mathcal{G}$ be a compact, semi-simple, connected Lie group with Lie algebra $\mathfrak{g}$, and letE_2

$$E_2 = (N, \mathfrak{g}, \pi, \operatorname{Ad}(\mathcal{G})) \tag{2.1.3}$$

be the corresponding adjoint bundle with base space N. Then we consider the functional

$$J = \int_N (\bar{R} - 2\Lambda) + \int_N (\alpha_1 L_{YM} + \alpha_2 L_H), \tag{2.1.4}$$

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where the α_i, $i = 1, 2$, are positive coupling constants, $\bar{R}$ the scalar curvature, Λ a cosmological constant, L_{YM} the energy of a connection in E_2 and L_H the energy of a Higgs field with values in $\mathfrak{g}$. The integration over N is to be understood symbolically, since we shall always integrate over an open precompact subset $\tilde{\Omega} \subset N$.

In [17] we already considered a canonical quantization of the above action and proved that it will be sufficient to only consider connections A^a_μ satisfying the Hamilton gauge

$$A^a_0 = 0, \tag{2.1.5}$$

thereby eliminating the Gauß constraint, such that the only remaining constraint is the Hamilton constraint, cf. [17, Theorem 2.3].

Using the ADM partition (2.1.2) of N, cf. [1] and Sect. 1.3 on page 7, such that

$$N = I \times \mathcal{S}_0, \tag{2.1.6}$$

where $\mathcal{S}_0$ is the Cauchy hypersurface $M(0)$ and applying canonical quantization we obtained a Hamilton operator $\mathcal{H}$ which was a normally hyperbolic operator in a fiber bundle E with base space $\mathcal{S}_0$ and fibers

$$F(x) \times (\mathfrak{g} \otimes T^{0,1}_x(\mathcal{S}_0)) \times \mathfrak{g}, \qquad x \in \mathcal{S}_0, \tag{2.1.7}$$

where $F(x)$ is the space of Riemannian metrics. We quantized the action by looking at the Wheeler–DeWitt equation in this bundle. The fibers of E are equipped with a Lorentzian metric such that they are globally hyperbolic and the transformed Hamiltonian $\mathcal{H}$, which is now a hyperbolic operator, is a normally hyperbolic operator acting only in the fibers.

The Wheeler-DeWitt equation has the form

$$\mathcal{H}u = 0, \tag{2.1.8}$$

with $u \in C^\infty(E, \mathbb{C})$ and we defined with the help of the Green's operator a symplectic vector space and a corresponding Weyl system.

The Wheeler-DeWitt equation seems to be the obvious quantization of the Hamilton condition. However, $\mathcal{H}$ acts only in the fibers and not in the base space which is due to the fact that the derivatives are only ordinary covariant derivatives and not functional derivatives, though they are supposed to be functional derivatives, but this property is not really invoked when a functional derivative is applied to u, since the result is the same as applying a partial derivative.

Therefore, we shall use the same approach as in Sect. 1.6 on page 40 by discarding the Wheeler-DeWitt equation and, instead, express the Hamilton condition differently by looking at the evolution equation of the mean curvature of the foliation hypersurfaces $M(t)$ and implementing the Hamilton condition on the right-hand side of this evolution equation. The left-hand side, a time derivative, we shall express with the help of the Poisson brackets. After canonical quantization the Poisson brackets become a commutator and now we can employ the fact that the derivatives are functional derivatives, since we have to differentiate the scalar curvature of a metric. As a result we obtain an elliptic differential operator in the base space, the main part of which is the Laplacian of the metric.

On the right-hand side of the evolution equation the interesting term is H^2, the square of the mean curvature. It will be transformed to a second time derivative and will be the only remaining derivative with respect to a fiber variable, since the differentiations with respect to the other variables cancel each other.

The resulting quantized equation is then a wave equation

$$\begin{aligned} &\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + \alpha_1\frac{n}{8}t^{2-\frac{4}{n}}F_{ij}F^{ij}u \\ &+ \alpha_2\frac{n}{4}t^{2-\frac{4}{n}}\gamma_{ab}\sigma^{ij}\Phi_i^a\Phi_j^b u + \alpha_2\frac{n}{2}mt^{2-\frac{4}{n}}V(\Phi)u + nt^2\Lambda u = 0, \end{aligned} \tag{2.1.9}$$

in a globally hyperbolic spacetime

$$Q = (0,\infty)\times\mathcal{S}_0 \tag{2.1.10}$$

describing the interaction of a given complete Riemannian metric σ_{ij} in $\mathcal{S}_0$ with a given Yang-Mills and Higgs field; V is the potential of the Higgs field and m a positive constant. The existence of the time variable, and its range, is due to the Lorentzian metric in the fibers of E.

Remark 2.1.1 For the results and arguments in this chapter it is completely irrelevant that the values of the Higgs field Φ lie in a Lie algebra, i.e., Φ could also be just an arbitrary scalar field, or we could consider a Higgs field as well as an another arbitrary scalar field. Hence, let us stipulate that the Higgs field could also be just an arbitrary scalar field. It will later be used to produce a Mass gap simply by interacting with the gravitation ignoring the Yang-Mills field.

If $\mathcal{S}_0$ is compact we also prove a spectral resolution of Equation (2.1.9) by first considering a stationary version of the hyperbolic equation, namely, the elliptic eigenvalue equation

$$\begin{aligned} &-(n-1)\Delta v - \frac{n}{2}Rv + \alpha_1\frac{n}{8}F_{ij}F^{ij}v \\ &+ \alpha_2\frac{n}{4}\gamma_{ab}\sigma^{ij}\Phi_i^a\Phi_j^b v + \alpha_2\frac{n}{2}mV(\Phi)v = \mu v. \end{aligned} \tag{2.1.11}$$

It has countably many solutions (v_i,μ_i) such that

$$\mu_0 < \mu_1 \le \mu_2 \le \cdots, \tag{2.1.12}$$

$$\lim \mu_i = \infty. \tag{2.1.13}$$

Let v be an eigenfunction with eigenvalue $\mu > 0$, then we look at solutions of (2.1.9) of the form

$$u(x,t) = w(t)v(x). \tag{2.1.14}$$

u is then a solution of (2.1.9) provided w satisfies the implicit eigenvalue equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} - \mu t^{2-\frac{4}{n}} w - nt^2 \Lambda w = 0, \tag{2.1.15}$$

where Λ is the eigenvalue.

This eigenvalue problem we also considered in the previous chapter and we proved that it has countably many solutions (w_i, Λ_i) with finite energy, i.e.,

$$\int_0^\infty \{|\dot{w}_i|^2 + (1 + t^2 + \mu t^{2-\frac{4}{n}})|w_i|^2\} < \infty, \tag{2.1.16}$$

cf. Theorem 1.6.7 on page 49.

2.2 The Yang-Mills Functional

Let $N = N^{n+1}$ be a globally hyperbolic spacetime with metric $(\bar{g}_{\alpha\beta})$, $\mathcal{G}$ a compact, semi-simple, connected Lie group, $\mathfrak{g}$ its Lie algebra and $E_2 = (N, \mathfrak{g}, \pi, \operatorname{Ad}(\mathcal{G}))$ the corresponding adjoint bundle with base space N. The Yang-Mills functional is then defined by

$$J_{YM} = \int_N -\frac{1}{4} F_{\mu\lambda} F^{\mu\lambda} = \int_N -\frac{1}{4}\gamma_{ab}\bar{g}^{\mu\rho_2}\bar{g}^{\lambda\rho_1} F^a_{\mu\rho_1} F^b_{\rho_2\lambda}, \tag{2.2.1}$$

where γ_{ab} is the Cartan-Killing metric in $\mathfrak{g}$,

$$F^a_{\mu\lambda} = A^a_{\lambda,\mu} - A^a_{\mu,\lambda} + f^a_{bc} A^b_\mu A^c_\lambda \tag{2.2.2}$$

is the curvature of a connection

$$A = (A^a_\mu) \tag{2.2.3}$$

in E_2 and f_c

$$f_c = (f^a_{cb}) \tag{2.2.4}$$

are the Structural constants of $\mathfrak{g}$. The integration over N is to be understood symbolically since we shall always integrate over an open precompact subset $\tilde{\Omega}$ of N.

Definition 2.2.1 The adjoint bundle E_2 is vector bundle; let E_2^* be the dual bundle, then we denote by

$$T^{r,s}(E_2) = \underbrace{E_2 \otimes \cdots \otimes E_2}_{r} \otimes \underbrace{E_2^* \otimes \cdots \otimes E_2^*}_{s} \tag{2.2.5}$$

the corresponding tensor bundle and by

$$\Gamma(T^{r,s}(E_2)), \tag{2.2.6}$$

or more precisely,

$$\Gamma(N, T^{r,s}(E_2)), \tag{2.2.7}$$

the sections of the bundle, where N is the base space. Especially we have

$$T^{1,0}(E_2) = E_2. \tag{2.2.8}$$

Thus, we have

$$F^a_{\mu\lambda} \in \Gamma(T^{1,0}(E_2) \otimes T^{0,2}(N)). \tag{2.2.9}$$

When we fix a connection $\bar{A}$ in E_2, then a general connection A can be written in the form

$$A^a_\mu = \bar{A}^a_\mu + \tilde{A}^a_\mu, \tag{2.2.10}$$

where $\tilde{A}^a_\mu$ is a tensor

$$\tilde{A}^a_\mu \in \Gamma(T^{1,0}(E_2) \otimes T^{0,1}(N)). \tag{2.2.11}$$

To be absolutely precise a connection in E_2 is of the form

$$f_c A^c_\mu, \tag{2.2.12}$$

where f_c is defined in (2.2.4); A^a_μ is merely a coordinate representation.

Definition 2.2.2 A connection A of the form (2.2.10) is sometimes also denoted by $(\bar{A}^a_\mu, \tilde{A}^a_\mu)$.

Since we assume that there exists a globally defined time function x^0 in N we may consider globally defined tensors $(\tilde{A}^a_\mu)$ satisfying

$$\tilde{A}^a_0 = 0. \tag{2.2.13}$$

These tensors can be written in the form $(\tilde{A}^a_i)$ and they can be viewed as maps

$$(\tilde{A}^a_i) : N \to \mathfrak{g} \otimes T^{0,1}(\mathcal{S}_0), \tag{2.2.14}$$

where $\mathcal{S}_0$ is a Cauchy hypersurface of N, e.g., a coordinate slice

$$\mathcal{S}_0 = \{x^0 = \text{const}\}. \tag{2.2.15}$$

It is well-known that the Yang-Mills Lagrangian is singular and requires a local gauge fixing when applying canonical quantization. We impose a local gauge fixing by stipulating that the connection $\bar{A}$ satisfies

$$\bar{A}^a_0 = 0. \tag{2.2.16}$$

Hence, all connections in (2.2.10) will obey this condition since we also stipulate that the tensor fields $\tilde{A}^a_\mu$ have vanishing temporal components as in (2.2.13). The gauge (2.2.16) is known as the *Hamilton gauge*, cf. [9, p. 82]. However, this gauge fixing leads to the so-called Gauß constraint, since the first variation in the class of these connections will not formally yield the full Yang-Mills equations.

In [17, Theorem 2.3], we proved that the Gauß constraint does not exist and that it suffices to consider connections of the form (2.2.10) satisfying (2.2.13) and (2.2.16) in the Yang-Mills functional J_{YM}:

Theorem 2.2.3 *Let $\tilde{\Omega} \Subset N$ be open and precompact such that there exists a local trivialization of E_2 in $\tilde{\Omega}$. Let $A = (\bar{A}^a_\mu, \tilde{A}^a_\mu)$ be a connection satisfying (2.2.13) and (2.2.16) in $\tilde{\Omega}$, and suppose that the first variation of J_{YM} vanishes at A with respect to compact variations of $\tilde{A}^a_\mu$ all satisfying (2.2.13). Then A is a Yang-Mills connection, i.e., the Yang-Mills equation*

$$F^{a\mu}_{\lambda\ ;\mu} = 0 \tag{2.2.17}$$

is valid in $\tilde{\Omega}$.

Proof Let η^a_μ be an arbitrary tensor field with compact support in $\tilde{\Omega}$ satisfying

$$\eta^a_0 = 0 \tag{2.2.18}$$

and define the connections

$$A(\epsilon) = (\bar{A}^a_\mu, \tilde{A}^a_\mu + \epsilon\eta^a_\mu). \tag{2.2.19}$$

Differentiating the functional

$$J_{YM}(\epsilon) = \int_{\tilde{\Omega}} -\frac{1}{4} F_{\mu\lambda}(\epsilon) F^{\mu\lambda}(\epsilon) \tag{2.2.20}$$

with respect to ϵ and evaluating in $\epsilon = 0$ yields

$$\frac{dJ_{YM}}{d\epsilon} = -\int_{\tilde{\Omega}} \gamma_{ab} F^{a\mu\lambda} \eta^b_{\lambda;\mu} = \int_{\tilde{\Omega}} \gamma_{ab} F^{a\mu\lambda}{}_{;\mu} \eta^b_\lambda. \tag{2.2.21}$$

Assuming that the first variation of the functional vanishes we deduce

$$F^{ai\mu}{}_{;\mu} = 0 \tag{2.2.22}$$

which is equivalent to

$$F^{a\ \mu}_{\ i\ ;\mu} = 0 \tag{2.2.23}$$

since we only consider spacetime metrics $(\bar{g}_{\alpha\beta})$ that splits, i.e.,

$$d\bar{s}^2 = -w^2(dx^0)^2 + g_{ij}(x^0, x)dx^i dx^j \tag{2.2.24}$$

in view of the result in Theorem 1.3.2 on page 9. Similarly, the conditions

$$F^{a0\mu}{}_{;\mu} = 0 \tag{2.2.25}$$

and

$$F^{a}{}_{0}{}^{\mu}{}_{;\mu} = 0 \tag{2.2.26}$$

are equivalent.

To prove that A also satisfies

$$F^{a0\mu}{}_{;\mu} = 0 \tag{2.2.27}$$

in $\tilde{\Omega}$, we argue by contradiction supposing there exists $(t_0, x_0) \in \tilde{\Omega}$ such that

$$F^{a0\mu}{}_{;\mu}(t_0, x_0) \neq 0. \tag{2.2.28}$$

Define

$$\xi^a = F^{a0\mu}{}_{;\mu}\bar{g}_{00} \tag{2.2.29}$$

so that

$$\gamma_{ab}\xi^a F^{b0\mu}{}_{;\mu} < 0 \tag{2.2.30}$$

in (t_0, x_0). Choosing a cut-off function $\varphi = \varphi(t, x)$ satisfying $\varphi(t_0, x_0) = 1$ we then infer

$$\gamma_{ab}\tilde{\xi}^a F^{b0\mu}{}_{;\mu} \leq 0 \tag{2.2.31}$$

in N and strictly negative in (t_0, x_0), where

$$\tilde{\xi}^a = \xi^a\varphi. \tag{2.2.32}$$

Next we consider the gauge transformation $\omega = \omega(t, x)$ where ω is the flow

$$\begin{aligned} \dot{\omega} &= -\omega\epsilon f_c\tilde{\xi}^c, \\ \omega(t_0, x) &= \mathrm{id}, \end{aligned} \tag{2.2.33}$$

which is well defined in a neighbourhood of supp φ. After the gauge transformation the connections $A(\epsilon)$ in (2.2.19) look like

$$\omega f_c A^c_\mu(\epsilon)\omega^{-1} - \omega_\mu\omega^{-1} \tag{2.2.34}$$

and the component $\mu = 0$ is equal to

$$-\dot{\omega}\omega^{-1} = \epsilon\omega f_c \tilde{\xi}^c \omega^{-1}. \tag{2.2.35}$$

Since the Yang-Mills functional is gauge invariant its first variation still vanishes after the gauge transformation and we deduce from (2.2.21) and (2.2.22)

$$0 = \int_{\tilde{\Omega}} \gamma_{ab} F^{a0\mu}{}_{;\mu} \tilde{\xi}^b \tag{2.2.36}$$

contradicting (2.2.31). □

Remark 2.2.4 Gauge fixing is an appropriate method for reducing the number of independent variables, but in the context of canonical quantization it is only legitimate if it is also used before deriving the Euler-Lagrange equation and if in addition it is proved that the correct Euler-Lagrange equation is still valid.

Let $(B_{\rho_k}(x_k))_{k\in\mathbb{N}}$ be a covering of $\mathcal{S}_0$ by small open balls such that each ball lies in a coordinate chart of $\mathcal{S}_0$. Then the cylinders

$$U_k = I \times B_{\rho_k}(x_k) \tag{2.2.37}$$

are a covering of N such that each U_k is contractible, hence each bundle $\pi^{-1}(U_k)$ is trivial and the connection $\bar{A}$ can be expressed in coordinates in each U_k

$$\bar{A} = (\bar{A}^a_\mu) = f_a A^a_\mu dx^\mu. \tag{2.2.38}$$

We shall prove:

Lemma 2.2.5 *In each cylinder U_k there exists a gauge transformation $\omega = \omega(t, x)$ such that*

$$\bar{A}^a_0(t, x) = 0 \qquad \forall\, (t, x) \in U_k \tag{2.2.39}$$

after applying the gauge transformation.

Proof For fixed k we consider the flow

$$\begin{aligned} \dot{\omega} &= \omega f_c \bar{A}^c_0, \\ \omega(0, x) &= \mathrm{id}, \qquad\qquad x \in B_{\rho_k}(x_k). \end{aligned} \tag{2.2.40}$$

For fixed $x \in B_{\rho_k}(x_k)$ the integral curve exists on a maximal interval J_x. If we can show $J_x = I$, then the lemma is proved.

The claim is obvious, since the integral curve cannot develop singularities, for let $\langle\cdot,\cdot\rangle$ be the negative of the Killing metric, then

$$\begin{aligned} \langle\dot{\omega}, \dot{\omega}\rangle &= -\operatorname{tr}(\omega A_0 \omega A_0) \\ &= -\operatorname{tr}(A_0 A_0) = \gamma_{ab} A^a_0 A^b_0 \end{aligned} \tag{2.2.41}$$

from which the result immediately follows. □

Lemma 2.2.6 *Let U_k, U_l be overlapping cylinders and let $\omega = \omega(t, x)$ be a gauge transformation relating the respective representations of the connection $\bar{A}$ in the overlap $U_k \cap U_l$ where both representations use the Hamilton gauge, then ω does not depend on t, i.e.,*

$$\dot{\omega} = 0. \tag{2.2.42}$$

Proof Let $(\hat{\bar{A}}^a_\mu)$ resp. $(\bar{A}^a_\mu)$ be the representations of $\bar{A}$ in U_k resp. U_l such that

$$\hat{\bar{A}}^a_0 = \bar{A}^a_0 = 0, \tag{2.2.43}$$

then

$$\hat{\bar{A}}_0 = \omega \bar{A}_0 \omega^{-1} - \dot{\omega}\omega^{-1}, \tag{2.2.44}$$

hence

$$\dot{\omega} = 0 \quad \text{in} \quad U_k \cap U_l. \tag{2.2.45}$$

□

Let E_0 be the adjoint bundle

$$E_0 = (\mathcal{S}_0, \mathfrak{g}, \pi, \mathrm{Ad}(\mathcal{G})) \tag{2.2.46}$$

with base space $\mathcal{S}_0$, where the gauge transformations only depend on the spatial variables $x = (x^i)$. For fixed t $A^a_{i,0}$ are elements of $T^{1,0}(E_0) \otimes T^{0,1}(\mathcal{S}_0)$

$$A^a_{i,0} \in T^{1,0}(E_0) \otimes T^{0,1}(\mathcal{S}_0), \tag{2.2.47}$$

but the vector potentials $A^a_i(t, \cdot)$ are connections in E_0 for fixed t and therefore cannot be used as independent variables, since the variables should be the components of a tensor. However, in view of the results in Lemma 2.2.5 and Lemma 2.2.6 the difference

$$\tilde{A}^a_i(t, \cdot) = A^a_i(t, \cdot) - \bar{A}^a_i(0, \cdot) \in T^{1,0}(E_0) \otimes T^{0,1}(\mathcal{S}_0). \tag{2.2.48}$$

Hence, we shall define $\tilde{A}^a_i$ to be the independent variables such that

$$A^a_i = \bar{A}^a_i(0, \cdot) + \tilde{A}^a_i \tag{2.2.49}$$

and we infer

$$A^a_{i,0} = \tilde{A}^a_{i,0}. \tag{2.2.50}$$

In the Hamilton gauge we therefore have

$$F^a_{0i} = \tilde{A}^a_{i,0} \tag{2.2.51}$$

and hence we conclude

$$-\frac{1}{4}F_{\mu\lambda}F^{\mu\lambda}=\frac{1}{2}w^{-2}g^{ij}\gamma_{ab}\tilde{A}^a_{i,0}\tilde{A}^b_{j,0}-\frac{1}{4}F_{ij}F^{ij}, \tag{2.2.52}$$

where we used (2.1.1).

Writing the density

$$\sqrt{g}=\sqrt{\det g_{ij}} \tag{2.2.53}$$

in the form

$$\sqrt{g}=\varphi\sqrt{\det \chi_{ij}}, \tag{2.2.54}$$

where χ is a fixed Riemannian metric in S_0, $\chi_{ij}=\chi_{ij}(x)$, such that $0<\varphi=\varphi(x,g_{ij})$ is a function, we obtain as Lagrangian function

$$L_{YM}=\frac{1}{2}\gamma_{ab}g^{ij}\tilde{A}^a_{i,0}\tilde{A}^b_{j,0}w^{-1}\varphi-\frac{1}{4}F_{ij}F^{ij}w\varphi. \tag{2.2.55}$$

In order to prove a spectral resolution of the combined Hamilton operator after quantization we need to modify the Yang-Mills Lagrangian slightly. We shall call this modification process *renormalization* though the renormalization is different from the usual renormalization in quantum field theory.

Remark 2.2.7 The renormalization is necessary since the Yang-Mills energy depends quadratically on the inverse of the metric, and hence shows a wrong scaling behaviour with respect to the metric. The appropriate scaling behaviour would be linear.

Definition 2.2.8 When we only consider metrics $\bar{g}_{\alpha\beta}$ that can be split by a given time function x^0, such that the Yang-Mills Lagrangian is expressed as in (2.2.55), then we define the renormalized Lagrangian by

$$L_{YMmod}=\frac{1}{2}\gamma_{ab}g^{ij}\tilde{A}^a_{i,0}\tilde{A}^b_{j,0}w^{-1}\varphi^p\varphi-\frac{1}{4}F_{ij}F^{ij}w\varphi^p\varphi, \tag{2.2.56}$$

where $p\in\mathbb{R}$ is real. We shall choose

$$p=\frac{2}{n}. \tag{2.2.57}$$

An equivalent description is, that we have replaced

$$F^2=F_{\alpha\beta}F^{\alpha\beta} \tag{2.2.58}$$

by

$$F^2\varphi^p \tag{2.2.59}$$

though this always requires that the metric is split by a time function otherwise the definition of φ makes no sense.

The $\tilde{A}_i^a(t,\cdot)$ can be looked at to be mappings from $\mathcal{S}_0$ to $T^{1,0}(E_0)\otimes T^{0,1}(\mathcal{S}_0)$

$$\tilde{A}_i^a(t,\cdot):\mathcal{S}_0\to T^{1,0}(E_0)\otimes T^{0,1}(\mathcal{S}_0). \tag{2.2.60}$$

The fibers of $T^{1,0}(E_0)\otimes T^{0.1}(\mathcal{S}_0)$ are the tensor products

$$\mathfrak{g}\otimes T_x^{0,1}(\mathcal{S}_0),\qquad x\in\mathcal{S}_0, \tag{2.2.61}$$

which are vector spaces equipped with metric

$$\gamma_{ab}\otimes g^{ij}. \tag{2.2.62}$$

For our purposes it is more convenient to consider the fibers to be Riemannian manifolds endowed with the above metric. Let (ζ^p), $1\le p\le n_1 n$, where $n_1=\dim\mathfrak{g}$, be local coordinates and

$$(\zeta^p)\to\tilde{A}_i^a(\zeta^p)\equiv\tilde{A}(\zeta) \tag{2.2.63}$$

be a local embedding, then the metric has the coefficients

$$G_{pq}=\langle\tilde{A}_p,\tilde{A}_q\rangle=\gamma_{ab}g^{ij}\tilde{A}^a_{i,p}\tilde{A}^b_{j,q}. \tag{2.2.64}$$

Hence, the Lagrangian L_{YMmod} in (2.2.56) can be expressed in the form

$$L_{YMmod}=\frac{1}{2}G_{pq}\dot{\zeta}^p\dot{\zeta}^q w^{-1}\varphi^{1+\frac{2}{n}}-\frac{1}{4}F_{ij}F^{ij}w\varphi^{1+\frac{2}{n}} \tag{2.2.65}$$

and we deduce

$$\tilde{\pi}_p=\frac{\partial L_{YMmod}}{\partial\dot{\zeta}^p}=G_{pq}\dot{\zeta}^q w^{-1}\varphi^{1+\frac{2}{n}} \tag{2.2.66}$$

yielding the Hamilton function

$$\begin{aligned}\hat{H}_{YMmod}&=\pi_p\dot{\zeta}^p-L_{YMmod}\\&=\frac{1}{2}G_{pq}(\dot{\zeta}^p w^{-1}\varphi^{1+\frac{2}{n}})(\dot{\zeta}^q w^{-1}\varphi^{1+\frac{2}{n}})w\varphi^{-(1+\frac{2}{n})}+\frac{1}{4}F_{ij}F^{ij}w\varphi^{1+\frac{2}{n}}\\&=\frac{1}{2}G^{pq}\tilde{\pi}_p\tilde{\pi}_q w\varphi^{-(1+\frac{2}{n})}+\frac{1}{4}F_{ij}F^{ij}w\varphi^{1+\frac{2}{n}}\\&\equiv H_{YMmod}w.\end{aligned} \tag{2.2.67}$$

Thus, the effective Hamiltonian that will enter the Hamilton constraint equation is

$$H_{YMmod} = \frac{1}{2}\varphi^{-(1+\frac{2}{n})}G^{pq}\tilde{\pi}_p\tilde{\pi}_q + \frac{1}{4}F_{ij}F^{ij}\varphi^{1+\frac{2}{n}}. \tag{2.2.68}$$

If the Yang-Mills Lagrangian is multiplied by a coupling constant α_1, then the effective Hamiltonian is H_{YMmod}

$$H_{YMmod} = \alpha_1^{-1}\frac{1}{2}\varphi^{-(1+\frac{2}{n})}G^{pq}\tilde{\pi}_p\tilde{\pi}_q + \alpha_1\frac{1}{4}F_{ij}F^{ij}\varphi^{1+\frac{2}{n}}. \tag{2.2.69}$$

2.3 The Higgs Functional

Let Φ be a scalar field, a map from N to E_2,

$$\Phi : N \to E_2, \tag{2.3.1}$$

i.e., Φ is a section of E_2. The Higgs Lagrangian is defined by

$$L_H = -\frac{1}{2}\bar{g}^{\alpha\beta}\gamma_{ab}\Phi^a_\alpha\Phi^b_\beta - mV(\Phi), \tag{2.3.2}$$

where $V \geq 0$ is a smooth potential and $m > 0$ a constant. Given a global time function with corresponding foliation of N we also consider a renormalized potential, namely, we replace V by

$$V\varphi^q, \qquad q = -\frac{2}{n}, \tag{2.3.3}$$

such that

$$L_{Hmod} = -\frac{1}{2}\bar{g}^{\alpha\beta}\gamma_{ab}\Phi^a_\alpha\Phi^b_\beta - mV(\Phi)\varphi^q. \tag{2.3.4}$$

Let us note that V does not depend on the metric and hence has also the wrong scaling behaviour.

We assume for simplicity that in a local coordinate system Φ has real coefficients. The covariant derivatives of Φ are defined by a connection $A = (A^a_\mu)$ in E_2

$$\Phi^a_\mu = \Phi^a_{,\mu} + f^a_{cb}A^c_\mu\Phi^b. \tag{2.3.5}$$

As in the preceding section we work in a local trivialization of E_2 using the Hamilton gauge, i.e.,

$$A^a_0 = 0, \tag{2.3.6}$$

hence, we conclude

$$\Phi^a_0 = \Phi^a_{,0}. \tag{2.3.7}$$

Moreover, let

$$\bar{\Phi} : \mathcal{S}_0 \to E_2 \tag{2.3.8}$$

be an arbitrary but fixed smooth section of E_2 depending only on $x \in \mathcal{S}_0$ and let

$$\tilde{\Phi} : N \to E_2 \tag{2.3.9}$$

be an arbitrary smooth section, then we define

$$\Phi = \bar{\Phi} + \tilde{\Phi} \tag{2.3.10}$$

to be the argument that enters in the Higgs Lagrangian but stipulate that $\tilde{\Phi}$ will the variable.

Expressing the density g as in (2.2.54) on page 60 we obtain the LagrangianL_{Hmod}

$$L_{Hmod} = \frac{1}{2}\gamma_{ab}\tilde{\Phi}^a_{,0}\tilde{\Phi}^b_{,0}w^{-1}\varphi - \frac{1}{2}g^{ij}\gamma_{ab}\Phi^a_i\Phi^b_j w\varphi - mV(\Phi)w\varphi^{(1+q)} \tag{2.3.11}$$

which we have to use for the Legendre transformation. Before applying the Legendre transformation we again consider the vector space $\mathfrak{g}$ to be a Riemannian manifold with metric γ_{ab}. The representation of $\tilde{\Phi}$ in the form $(\tilde{\Phi}^a)$ can be looked at, in a local trivialization, to be the representation of the local coordinates (Θ^a) such that the metric γ_{ab} now depends on x.

Let us define

$$p_a = \frac{\partial L_{Hmod}}{\partial \dot{\Theta}^a}, \qquad \dot{\Theta}^a = \Theta^a_{,0}, \tag{2.3.12}$$

then we obtain the Hamiltonian

$$\begin{aligned} \hat{H}_{Hmod} &= p_a\dot{\Theta}^a - L_{Hmod} \\ &= \frac{1}{2}\varphi^{-1}\gamma^{ab}p_a p_b + \frac{1}{2}g^{ij}\gamma_{ab}\Phi^a_i\Phi^b_j w\varphi + mV(\Phi)w\varphi^{(1+q)} \\ &\equiv H_{Hmod}w. \end{aligned} \tag{2.3.13}$$

Thus, the Hamiltonian which will enter the Hamilton constraint isH_{Hmod}

$$H_{Hmod} = \frac{1}{2}\varphi^{-1}\gamma^{ab}p_a p_b + \frac{1}{2}g^{ij}\gamma_{ab}\Phi^a_i\Phi^b_j\varphi + mV(\Phi)\varphi^{(1+q)}. \tag{2.3.14}$$

If the Higgs Lagrangian is multiplied by a coupling constant α_2, then

$$H_{Hmod} = \alpha_2^{-1}\frac{1}{2}\varphi^{-1}\gamma^{ab}p_a p_b + \alpha_2\frac{1}{2}g^{ij}\gamma_{ab}\Phi^a_i\Phi^b_j\varphi + \alpha_2 mV(\Phi)\varphi^{(1+q)}. \tag{2.3.15}$$

2.4 The Hamilton Condition

Considering the foliation given by the time function t the Einstein-Hilbert functional with cosmological constant Λ can be expressed in the form

$$J_G = \int_a^b \int_\Omega \{\frac{1}{4} G^{ij,kl} \dot{g}_{ij} \dot{g}_{kl} w^{-2} + (R - 2\Lambda)\} w \varphi \sqrt{\chi}, \tag{2.4.1}$$

where we already replaced the density $\sqrt{g}$ by $\varphi\sqrt{\chi}$, cf. (1.4.3) on page 18. The metric $G^{ij,kl}$ is defined by

$$G^{ij,kl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl} \tag{2.4.2}$$

and its inverse is given by

$$G_{ij,kl} = \frac{1}{2}\{g_{ik} g_{jk} + g_{il} g_{jk}\} - \frac{1}{n-1} g_{ij} g_{kl}. \tag{2.4.3}$$

R is the scalar curvature of the metric g_{ij}.

The corresponding Hamiltonian H_G has the form

$$H_G = \{\varphi^{-1} G_{ij,kl} \pi^{ij} \pi^{kl} - (R - 2\Lambda)\varphi\} w, \tag{2.4.4}$$

cf. (1.4.4) on page 18. Hence, the Hamiltonian of the combined Lagrangian is

$$\mathcal{H} = H_G + H_{YMmod} + H_{Hmod}, \tag{2.4.5}$$

where coupling constants are already integrated in the Hamiltonians and the Hamilton equations

$$\dot{g}_{ij} = \frac{\delta \mathcal{H}}{\delta \pi^{ij}}, \tag{2.4.6}$$

$$\dot{\pi}^{ij} = -\frac{\delta \mathcal{H}}{\delta g_{ij}} \tag{2.4.7}$$

are equivalent to the Tangential Einstein equations

$$G_{ij} + \Lambda g_{ij} - T_{ij} = 0, \tag{2.4.8}$$

where $T_{\alpha\beta}$ is the stress-energy tensor comprised of the modified Yang-Mills and Higgs Lagrangians.

The normal component of the Einstein equations

$$G_{\alpha\beta} \nu^\alpha \nu^\beta - \Lambda - T_{\alpha\beta} \nu^\alpha \nu^\beta = 0 \tag{2.4.9}$$

cannot be derived from the Hamilton equations and this equation has to be stipulated as an extra condition, the so-called Hamilton condition, cf. (1.3.51) on page 13.

In Theorem 1.3.2 on page 9 we proved that any metric $(\bar{g}_{\alpha\beta})$ which splits according to (2.1.1) on page51 satisfying (2.4.8) and (2.4.9) also solves the full Einstein equations, i.e., it also satisfies the Einstein equations for the mixed components

$$G_{0j} + \Lambda g_{0j} - T_{0j} = 0. \tag{2.4.10}$$

The Hamilton condition is equivalent to the equation

$$\mathcal{H} = 0 \tag{2.4.11}$$

and after quantization, when the quantized Hamiltonian, still denoted by $\mathcal{H}$, is a differential operator in a fiber bundle, the quantum equivalent of Eq. (2.4.11) is usually considered to be

$$\mathcal{H}u = 0, \tag{2.4.12}$$

i.e., the elements of the kernel of $\mathcal{H}$ are supposed to be the physical interesting solutions. The Eq. (2.4.12) is known as the Wheeler–DeWitt equation. In [16, 17] we used this approach and solved the Wheeler-DeWitt equation in a fiber bundle E. The Hamilton operator is then a hyperbolic operator acting only in the fibers of the bundle as a differential operator and not in the base space $\mathcal{S}_0$, which is unsatisfactory. Therefore we shall express the Hamilton condition differently employing our approach in Sect. 1.4 on page 17, or, more precisely, in Sect. 1.6 on page 40, since we believe that the wave equation model is more appropriate to express the quantization of the Hamilton condition.

The foliation $M(t)$ is also the solution set of the geometric flow

$$\dot{x} = -w\nu \tag{2.4.13}$$

with initial hypersurface

$$M_0 = \mathcal{S}_0, \tag{2.4.14}$$

where ν is the past directed normal, cf. (1.6.5) on page 40. Let h_{ij} be the second fundamental form of $M(t)$, then π^{ij} and h_{ij} are related by the equation

$$h_{ij} = -\varphi^{-1} G_{ij,kl}\pi^{kl}, \tag{2.4.15}$$

cf. (1.4.6) on page 18, and the second Hamilton equation

$$\dot{\pi}^{ij} = -\frac{\delta\mathcal{H}}{\delta g_{ij}} \tag{2.4.16}$$

is equivalent to the evolution equation of the h_{ij} if the tangential Einstein equations (2.4.8) are supposed to be satisfied. In Sect. 1.6 on page 40 we used the evolution equation of the mean curvature

$$H = g^{ij} h_{ij} \tag{2.4.17}$$

to express the Hamilton condition, i.e., we modified this equation such that it was equivalent to the Hamilton condition and we shall use this approach again in the present situation.

We recall that

$$\pi^{ij} = (Hg^{ij} - h^{ij})\varphi, \tag{2.4.18}$$

and hence

$$(n-1)H\varphi = g_{ij}\pi^{ij}. \tag{2.4.19}$$

We shall modify the evolution equation

$$\begin{aligned}(\varphi^{-\frac{1}{2}} g_{ij}\pi^{ij})' &= -\frac{1}{4}\varphi^{-\frac{1}{2}} g^{kl}\dot{g}_{kl} g_{ij}\pi^{ij} + \varphi^{-\frac{1}{2}}\dot{g}_{ij}\pi^{ij} + \varphi^{-\frac{1}{2}} g_{ij}\dot{\pi}^{ij} \\ &= \frac{n-1}{2} H^2 \varphi^{\frac{1}{2}} w - 2\varphi^{-\frac{1}{2}} h_{ij}\pi^{ij} w + \varphi^{-\frac{1}{2}} g_{ij}\dot{\pi}^{ij},\end{aligned} \tag{2.4.20}$$

where we used that

$$h_{ij} = -\frac{1}{2}\dot{g}_{ij} w^{-1}, \tag{2.4.21}$$

and where we emphasize that the symbol H represents the mean curvature and $\mathcal{H}$ the Hamilton function. The Hamilton function is the sum of three Hamiltonians

$$\mathcal{H} = H_0 + H_1 + H_2, \tag{2.4.22}$$

where H_0 is the gravitational, H_1 the renormalized Yang-Mills and H_2 the renormalized Higgs Hamiltonian. Thus, we infer

$$g_{ij}\dot{\pi}^{ij} = -g_{ij}\frac{\delta\mathcal{H}}{\delta g_{ij}} = -g_{ij}\frac{\delta(H_0 + H_1 + H_2)}{\delta g_{ij}} \tag{2.4.23}$$

and we deduce further

$$\begin{aligned}-g_{ij}\frac{\delta H_0}{\delta g_{ij}} &= (\frac{n}{2} - 2)\varphi^{-1} G_{ij,kl}\pi^{ij}\pi^{kl} w + \frac{n}{2}(R - 2\Lambda)\varphi w \\ &- \frac{1}{2} R\varphi w - (n-1)\tilde{\Delta} w\varphi,\end{aligned} \tag{2.4.24}$$

where the scalar curvature and the Laplacian are defined by the metric g_{ij}; for a proof see the proof of Theorem 1.3.4 on page 14.

Writing

$$H_1 = \alpha_1^{-1}\frac{1}{2} G^{pq}\tilde{\pi}_p\tilde{\pi}_q\varphi^{-(1+\frac{2}{n})} w + C_1 \tag{2.4.25}$$

and

$$H_2 = \alpha_2^{-1}\frac{1}{2}\gamma^{ab}p_a p_b \varphi^{-1}w + C_2 \tag{2.4.26}$$

we infer

$$-g_{ij}\frac{\delta H_1}{\delta g_{ij}} = \frac{n}{2}\alpha_1^{-1}\frac{1}{2}G^{pq}\tilde{\pi}_p\tilde{\pi}_q\varphi^{-(1+\frac{2}{n})}w - g_{ij}\frac{\delta C_1}{\delta g_{ij}} \tag{2.4.27}$$

and

$$-g_{ij}\frac{\delta H_2}{\delta g_{ij}} = \frac{n}{2}\alpha_2^{-1}\frac{1}{2}\gamma^{ab}p_a p_b\varphi^{-1}w - g_{ij}\frac{\delta C_2}{\delta g_{ij}}. \tag{2.4.28}$$

Hence, we conclude

$$\begin{aligned}(\varphi^{-\frac{1}{2}}g_{ij}\pi^{ij})' = \;& \frac{1}{2(n-1)}g_{ij}\pi^{ij}g_{kl}\pi^{kl}\varphi^{\frac{1}{2}}w \\ &+ \frac{n}{2}\varphi^{-1}G_{ij,kl}\pi^{ij}\pi^{kl}\varphi^{-\frac{1}{2}}w + \frac{n}{2}(R-2\Lambda)\varphi^{\frac{1}{2}}w \\ &- \frac{1}{2}R\varphi^{\frac{1}{2}}w - (n-1)\tilde{\Delta}w\varphi^{\frac{1}{2}} \\ &+ \frac{n}{2}\{\alpha_1^{-1}\frac{1}{2}G^{pq}\tilde{\pi}_p\tilde{\pi}_q\varphi^{-(1+\frac{2}{n})} + \alpha_2^{-1}\frac{1}{2}g^{ab}p_a p_b\varphi^{-1}\}\varphi^{-\frac{1}{2}}w \\ &- g_{ij}\{\frac{\delta C_1}{\delta g_{ij}} + \frac{\delta C_2}{\delta g_{ij}}\}\varphi^{-\frac{1}{2}}.\end{aligned} \tag{2.4.29}$$

On the right-hand side of this evolution equation we now implement the Hamilton condition by replacing

$$\varphi^{-1}G_{ij,kl}\pi^{ij}\pi^{kl}w \tag{2.4.30}$$

by

$$(R-2\Lambda)\varphi w - H_1 - H_2. \tag{2.4.31}$$

Expressing the time derivative on the left-hand side of (2.4.29) with the help of the Poisson brackets, we finally obtain

$$\begin{aligned}\{\varphi^{-\frac{1}{2}}g_{ij}\pi^{ij}, \mathcal{H}\} = \;& \frac{1}{2(n-1)}g_{ij}\pi^{ij}g_{kl}\pi^{kl}\varphi^{\frac{1}{2}}w \\ &+ \frac{n}{2}(R-2\Lambda)\varphi^{\frac{1}{2}}w - \frac{n}{2}(C_1+C_2)\varphi^{-\frac{1}{2}} \\ &+ \frac{n}{2}(R-2\Lambda)\varphi^{\frac{1}{2}}w - \frac{1}{2}R\varphi^{\frac{1}{2}}w - (n-1)\tilde{\Delta}w\varphi^{\frac{1}{2}} \\ &- g_{ij}\{\frac{\delta C_1}{\delta g_{ij}} + \frac{\delta C_2}{\delta g_{ij}}\}\varphi^{-\frac{1}{2}}.\end{aligned} \tag{2.4.32}$$

which is equivalent to the Hamilton condition if the Hamilton equations are valid.

Thus, we have proved:

Theorem 2.4.1 *Let $N = N^{n+1}$ be a globally hyperbolic spacetime and let the metric $\bar{g}_{\alpha\beta}$ be expressed as in (2.1.1) on page 51. Then, the metric satisfies the full Einstein equations if and only if the metric is a solution of the Hamilton equations and of the Eq.* (2.4.32).

2.5 The Quantization

For the quantization we use a similar model as in Sect. 1.4 on page 40. First, we switch to the gauge $w = 1$. Previously, we considered a bundle with base space $\mathcal{S}_0$ and fibers $F(x)$, $x \in \mathcal{S}_0$, the elements of which were the Riemannian metrics $(g_{ij}(x))$. The fibers were equipped with the Lorentzian metric

$$(\varphi^{-1} G_{ij,kl}) \tag{2.5.1}$$

which, in a suitable coordinate system

$$(t, \xi^A), \qquad t = \varphi^{\frac{1}{2}}, \tag{2.5.2}$$

has the form

$$ds^2 = -\frac{16(n-1)}{n} dt^2 + \frac{4(n-1)}{n} t^2 G_{AB} d\xi^A d\xi^B, \tag{2.5.3}$$

where G_{AB} is independent of t and the coordinates (t, ξ^A) are independent of x, cf. (1.4.97) on page 28.

In the present situation we consider a bundle E with base space $\mathcal{S}_0$ and the fibers over $x \in \mathcal{S}_0$ are

$$F(x) \times (\mathfrak{g} \otimes T_x^{01,}(\mathcal{S}_0)) \times \mathfrak{g}, \tag{2.5.4}$$

where the additional components are due to the Yang-Mills fields $(\tilde{A}_i^a)$ and the Higgs field $(\tilde{\Phi}^a)$. Let us emphasize that the elements of the fibers are tensors and that a fixed connection $\bar{A} = (\bar{A}_i^a(x))$ and fixed Higgs field $\bar{\Phi}^a$ are used to define the connections

$$A_i^a = \bar{A}_i^a + \tilde{A}_i^a \tag{2.5.5}$$

resp. the Higgs fields

$$\Phi^a = \bar{\Phi}^a + \tilde{\Phi}^a \tag{2.5.6}$$

the terms in the Hamiltonian will depend on. After the quantization is finished and we have obtained the final equation governing the interaction of a Riemannian metric with Yang-Mills and Higgs fields, we shall choose $\tilde{A}_i^a = 0$ and $\tilde{\Phi}^a = 0$ such that only the arbitrary *sections* $\bar{A}_i^a$ and $\bar{\Phi}^a$ are involved and not any *elements* of the bundle.

The fibers in (2.5.4) are equipped with the metric

$$
\begin{aligned}
ds^2 = -\frac{16(n-1)}{n}dt^2 + \frac{4(n-1)}{n}t^2 G_{AB} d\xi^A d\xi^B \\
+ t^2\alpha_1 \tilde{G}_{pq} d\zeta^p d\zeta^q + t^2 \alpha_2 \gamma_{ab} d\Theta^a d\Theta^b,
\end{aligned}
\tag{2.5.7}
$$

where the metrics $\tilde{G}_{pq}$ and γ_{ab} are independent of t. The metric G_{pq} in (2.2.69) on page 62 is related with $\tilde{G}_{pq}$ by

$$
G_{pq} = t^{-\frac{4}{n}} \tilde{G}_{pq}. \tag{2.5.8}
$$

Here, we used that a metric

$$
g_{ij}(x) \in F(x) \tag{2.5.9}
$$

can be expressed in the form

$$
g_{ij} = t^{\frac{4}{n}} \sigma_{ij}, \tag{2.5.10}
$$

where σ_{ij} is independent of t satisfying

$$
\det \sigma_{ij} = \det \chi_{ij}, \tag{2.5.11}
$$

cf. (1.4.61) and (1.4.64) on page 25.

Let us abbreviate the fiber metric in (2.5.7) by

$$
ds^2 = \bar{g}_{\alpha\beta} d\xi^\alpha \xi^\beta, \qquad 0 \le \alpha, \beta \le n_2, \tag{2.5.12}
$$

such that

$$
\xi^0 = t, \tag{2.5.13}
$$

and let $\bar{R}_{\alpha\beta}$ be the corresponding Ricci tensor, then

$$
\bar{R}_{0\beta} = 0 \qquad \forall\, \beta \tag{2.5.14}
$$

as can be easily derived by introducing a conformal time

$$
\tau = \log t \tag{2.5.15}
$$

such that

$$
\bar{g}_{\alpha\beta} = e^{2\psi} g_{\alpha\beta}, \tag{2.5.16}
$$

where the coefficients $g_{\alpha\beta}$ are independent of τ,

$$
g_{00} = -1, \tag{2.5.17}
$$

and

$$\psi = \tau + c, \qquad c = \text{const} \tag{2.5.18}$$

and using the well-known formula

$$\bar{R}_{\alpha\beta} = R_{\alpha\beta} - (n_2 - 1)[\psi_{\alpha\beta} - \psi_\alpha\psi_\beta] - g_{\alpha\beta}[\Delta\psi + (n_2 - 1)\|D\psi\|^2] \tag{2.5.19}$$

connecting the Ricci tensors of conformal metrics. Norms and derivatives on the right-hand side are all with respect to the metric $g_{\alpha\beta}$. The index 0 now refers to the variable τ.

We can now quantize the Hamiltonian setting using the original variables $(g_{ij}, \pi^{kl}, \ldots)$. We consider the bundle E equipped with the metric (2.5.7) in the fibers and with the Riemannian metric χ in $\mathcal{S}_0$. Furthermore, let

$$C_c^\infty(E) \tag{2.5.20}$$

be the space of real valued smooth functions with compact support in E.

In the quantization process, where we choose $\hbar = 1$, the variables g_{ij}, π^{ij}, etc. are then replaced by operators $\hat{g}_{ij}$, $\hat{\pi}^{ij}$, etc. acting in $C_c^\infty(E)$ and satisfying the commutation relations

$$[\hat{g}_{ij}, \hat{\pi}^{kl}] = i\delta_{ij}^{kl}, \tag{2.5.21}$$

for the gravitational variables,

$$[\hat{\zeta}^p, \hat{\tilde{\pi}}_q] = i\delta_q^p \tag{2.5.22}$$

for the Yang-Mills variables, and

$$[\hat{\theta}^a, \hat{p}_b] = i\delta_b^a \tag{2.5.23}$$

for the Higgs variables, while all the other commutators vanish. These operators are realized by defining $\hat{g}_{ij}$ to be the multiplication operator

$$\hat{g}_{ij}u = g_{ij}u \tag{2.5.24}$$

and $\hat{\pi}^{ij}$ to be the *functional* derivative

$$\hat{\pi}^{ij} = \frac{1}{i}\frac{\delta}{\delta g_{ij}}, \tag{2.5.25}$$

i.e., if $u \in C_c^\infty(E)$, then

$$\frac{\delta u}{\delta g_{ij}} \tag{2.5.26}$$

is the Euler-Lagrange operator of the functional

$$\int_{\mathcal{S}_0} u\sqrt{\chi} \equiv \int_{\mathcal{S}_0} u. \tag{2.5.27}$$

Hence, if u only depends on (x, g_{ij}) and not on derivatives of the metric, then

$$\frac{\delta u}{\delta g_{ij}} = \frac{\partial u}{\partial g_{ij}}. \tag{2.5.28}$$

The same definitions and reasonings are also valid for the other variables. Therefore, the transformed Hamiltonian $\hat{\mathcal{H}}$ can be looked at as the hyperbolic differential operator

$$\hat{\mathcal{H}} = -\Delta + C_0 + C_1 + C_2, \tag{2.5.29}$$

where Δ is the Laplacian of the metric in (2.5.7) acting on functions $u \in C_c^\infty(E)$ and the symbols C_i, $i = 0, 1, 2$, represent the lower order terms of the respective Hamiltonians H_0, H_1 and H_2.

Following Dirac the Poisson brackets on the left-hand side of (2.4.32) on page 67 are replaced by $\frac{1}{i}$ times the commutators of the transformed quantities in the quantization process, since $\hbar = 1$. Dropping the hats in the following to improve the readability the left-hand side of Eq. (2.4.32) is transformed to

$$i[\mathcal{H}, \varphi^{-\frac{1}{2}} g_{ij}\pi^{ij}] = [\mathcal{H}, \varphi^{-\frac{1}{2}} g_{ij}\frac{\delta}{\delta g_{ij}}]. \tag{2.5.30}$$

Using the relation in (1.6.24) on page 43

$$\varphi^{-\frac{1}{2}} g_{ij}\frac{\delta}{\delta g_{ij}} = \frac{n}{4}\frac{\partial}{\partial t} \tag{2.5.31}$$

when applied to functions u, we conclude

$$[-\Delta, \frac{n}{4}\frac{\partial}{\partial t}]u = 0, \tag{2.5.32}$$

in view of (2.5.14), and

$$[C_0 + C_1 + C_2, \varphi^{-\frac{1}{2}} g_{ij}\frac{\delta}{\delta g_{ij}}]u = -(n-1)\varphi^{-\frac{1}{2}}\tilde{\Delta}u\varphi - \varphi^{-\frac{1}{2}}(\sum_{k=0}^{2}\frac{\delta}{\delta g_{ij}}C_k)u, \tag{2.5.33}$$

cf. (1.6.26) on page 43, where $\tilde{\Delta}$ is the Laplace operator with respect to the metric g_{ij}. Here, we evaluate the Eq. (2.5.33) at an arbitrary point

$$(x, g_{ij}, \tilde{A}^a_k, \tilde{\Phi}^b) \equiv (x, t, \zeta^A) \tag{2.5.34}$$

in E, where we used the abbreviation

$$(\zeta^\alpha) = (\zeta^0, \zeta^A) \equiv (t, \zeta^A) \tag{2.5.35}$$

to denote the fiber coordinates in a local trivialization. The spatial fiber coordinates (ζ^A) are the coordinates for the fibers of the subbundle

$$E_1 = \{t = 1\} \tag{2.5.36}$$

which is a Cauchy hypersurface, since the fibers of E are globally hyperbolic, cf. [17, Theorem 4.1].

Remark 2.5.1 If we consider u to depend on the left-hand side of (2.5.34), then $\tilde{\Delta} u$ has to be evaluated by applying the chain rule. However, if we consider u to depend on (x, t, ζ^A), which are independent variables, then $\tilde{\Delta} u$ is the Laplacian of

$$u(\cdot, t, \zeta^A). \tag{2.5.37}$$

We shall adopt the latter view. Indeed, after having derived the quantized version of (2.4.32) on page 67 we shall consider u to depend on (x, t) and only implicitly on a fixed ζ^A, i.e., on a given $(\tilde{A}^a_i)$ and $(\tilde{\Phi}^a)$, especially since we shall then specify

$$\tilde{A}^a_i = 0 \quad \wedge \quad \tilde{\Phi}^a = 0. \tag{2.5.38}$$

Let us now transform the right-hand side of (2.4.32) on page 67 by having in mind that $w = 1$ and by multiplying all terms with $\varphi^{\frac{1}{2}}$ before applying them to a function u. Later, when we compare the left and right-hand sides, we of course multiply the left-hand side by the same factor $\varphi^{\frac{1}{2}}$.

The only non-trivial term on the right-hand side of (2.4.32) is the first one with the second derivatives. We arrange the covariant derivatives such that we obtain

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{u}, \tag{2.5.39}$$

where the derivatives are ordinary partial derivatives with respect to t, cf. the arguments in (1.6.27)–(1.6.32) on page 44. The other terms are trivial and we infer that the right-hand side is transformed to

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - \frac{n}{2}(C_0 + C_1 + C_2)u - (g_{ij}\frac{\delta}{\delta g_{ij}}(C_0 + C_1 + C_2))u. \tag{2.5.40}$$

Now, multiplying (2.5.33) by $\varphi^{\frac{1}{2}}$ and observing that it equals (2.5.40), we finally obtain the hyperbolic equation

$$\begin{aligned}&\frac{1}{32}\frac{n^2}{n-1}\ddot{u}-(n-1)\varphi\tilde{\Delta}u-\frac{n}{2}(R-2\Lambda)\varphi u+\alpha_1\frac{n}{8}F_{ij}F^{ij}\varphi^{1+\frac{2}{n}}\\&+\alpha_2\frac{n}{4}\gamma_{ab}g^{ij}\Phi_i^a\Phi_i^b\varphi u+\alpha_2\frac{n}{2}mV(\Phi)\varphi^{1-\frac{2}{n}}u=0,\end{aligned}\tag{2.5.41}$$

where

$$(g_{ij},\tilde{A}_k^a,\tilde{\Phi}^b)\tag{2.5.42}$$

are arbitrary but fixed elements of the bundle.

Citing (2.5.10) and (2.5.11) we have

$$g_{ij}(x,t)=t^{\frac{4}{n}}\sigma_{ij}(x),\tag{2.5.43}$$

where

$$\det\sigma_{ij}=\det\chi_{ij},\tag{2.5.44}$$

such that

$$(\sigma_{ij},\tilde{A}_k^a,\tilde{\Phi}^b)\tag{2.5.45}$$

belong to the subbundle E_1. Observing that

$$\tilde{\Delta}u=t^{-\frac{4}{n}}\tilde{\Delta}_{\sigma_{ij}}u,\tag{2.5.46}$$

and

$$R=t^{-\frac{4}{n}}R_{\sigma_{ij}},\tag{2.5.47}$$

where $R_{\sigma_{ij}}$ is the scalar curvature of the metric σ_{ij}, we can express (2.5.41) in the form

$$\begin{aligned}&\frac{1}{32}\frac{n^2}{n-1}\ddot{u}-(n-1)t^{2-\frac{4}{n}}\Delta u-\frac{n}{2}t^{2-\frac{4}{n}}Ru+\alpha_1\frac{n}{8}t^{2-\frac{4}{n}}F_{ij}F^{ij}u\\&+\alpha_2\frac{n}{4}t^{2-\frac{4}{n}}\gamma_{ab}\sigma^{ij}\Phi_i^a\Phi_j^bu+\alpha_2\frac{n}{2}mt^{2-\frac{4}{n}}V(\Phi)u+nt^2\Lambda u=0,\end{aligned}\tag{2.5.48}$$

where we dropped the tilde from $\tilde{\Delta}u$ and where the Laplacian, the scalar curvature and the raising and lowering of indices are defined with respect to the metric σ_{ij}.

In Remark 1.6.8 on page 49 we have proved that we may choose $\sigma_{ij}=\chi_{ij}$, and since χ_{ij} has been an arbitrary Riemannian metric on $\mathcal{S}_0$, we can therefore prove:

Theorem 2.5.2 *Let $(\mathcal{S}_0,\sigma_{ij})$ be a connected, complete, and smooth n-dimensional Riemann manifold and let $E_0=(\mathcal{S}_0,\mathfrak{g},\pi,Ad(\mathcal{G}))$ be the adjoint bundle defined in* (2.2.46) *on page 59, and let*

$$A=(A_i^a)\tag{2.5.49}$$

be an arbitrary smooth connection in E_0, i.e., an arbitrary smooth section, and let

$$\Phi = (\Phi^a) \tag{2.5.50}$$

be an arbitrary smooth Higgs field, then the hyperbolic equation (2.5.48) *in*

$$Q = \mathbb{R}_+^* \times \mathcal{S}_0 \tag{2.5.51}$$

describes the quantized version of the interaction of $(\mathcal{S}_0, \sigma_{ij})$ *with these bosonic fields.*

Proof We only have to prove that we may choose the connection (A_i^a) and the Higgs field (Φ^a) as arbitrary smooth sections. This follows immediately by evaluating (2.5.48) at the bundle elements

$$\tilde{A}_i^a = 0 \quad \wedge \quad \tilde{\Phi}^a = 0, \tag{2.5.52}$$

then the connection A_i^a and the Higgs field Φ^a coincide with $\bar{A}_i^a$ resp. $\bar{\Phi}^a$ which are arbitrary smooth sections. □

Remark 2.5.3 If we define in Q the Lorentz metric

$$d\bar{s}^2 = -32\frac{n-1}{n^2}dt^2 + \frac{1}{n-1}\sigma_{ij}dx^i dx^j, \tag{2.5.53}$$

then Q is globally hyperbolic and the operator in (2.5.48) is symmetric. If we equip Q with the metric

$$d\bar{s}^2 = -32\frac{n-1}{n^2}dt^2 + \frac{1}{n-1}t^{\frac{4}{n}-2}\sigma_{ij}dx^i dx^j, \tag{2.5.54}$$

then Q is also globally hyperbolic, the operator in (2.5.48) normally hyperbolic but not symmetric, and Q has a Big bang singularity in $t = 0$ if $n \geq 3$.

Proof Since σ_{ij} is complete it suffices to prove the big bang assertion. Let

$$M(t) = \{x^0 = t\} \tag{2.5.55}$$

be the Cauchy hypersurfaces and h_{ij} their second fundamental form with respect to the past directed normal, then

$$h_{ij} = -\frac{1}{2(n-1)}(t^{\frac{4}{n}-2})'\sigma_{ij} = p\frac{1}{2(n-1)}t^{-(p+1)}\sigma_{ij}, \tag{2.5.56}$$

where

$$p = 2 - \frac{4}{n}. \tag{2.5.57}$$

Hence the $M(t)$ are all umbilical. Let H be the mean curvature, then

$$H = \frac{np}{2} t^{-1}. \tag{2.5.58}$$

Moreover, let $\tilde{R}$ be the scalar curvature of the $M(t)$ and R the scalar curvature of σ_{ij}, then

$$\tilde{R} = (n-1)t^p R \tag{2.5.59}$$

and we deduce

$$\lim_{t \to 0} \tilde{R} = 0 \tag{2.5.60}$$

and

$$\lim_{t \to 0} H^2 = \infty. \tag{2.5.61}$$

Hence, some sectional curvatures of the ambient metric must also get unbounded in view of the Gauß equation and the fact that the $M(t)$ are umbilical. □

2.6 The Spectral Resolution

In case $\mathcal{S}_0$ is compact we can prove a spectral resolution for the Eq. (2.5.48) on page 73, where Λ will act as an implicit eigenvalue. The proof is similar as in Sect. 1.6 on page 40. First, let us consider an elliptic eigenvalue problem which can be looked at to be the stationary version of Eq. (2.5.48).

Lemma 2.6.1 *Let $\mathcal{S}_0$ be compact equipped with the metric σ_{ij}. Then, the eigenvalue problem*

$$\begin{aligned} &-(n-1)\Delta v - \frac{n}{2} R v + \alpha_1 \frac{n}{8} F_{ij} F^{ij} v \\ &+ \alpha_2 \frac{n}{4} \gamma_{ab} \sigma^{ij} \Phi_i^a \Phi_j^b v + \alpha_2 \frac{n}{2} m V(\Phi) v = \mu v \end{aligned} \tag{2.6.1}$$

has countably many solutions (v_i, μ_i) such that

$$\mu_0 < \mu_1 \le \mu_2 \le \cdots, \tag{2.6.2}$$

$$\lim \mu_i = \infty \tag{2.6.3}$$

and

$$\int_{\mathcal{S}_0} \bar{v}_i v_j = \delta_{ij}, \tag{2.6.4}$$

where now we consider complex valued functions. The solutions are smooth in $\mathcal{S}_0$ and form a basis in $L^2(\mathcal{S}_0, \mathbb{C})$.

This result is well-known, see also Lemma (1.6.5) on page 47. For clarification let us recall R is the scalar curvature of σ_{ij}, and the other coefficients depend on a given smooth Yang-Mills field and a Higgs field. There is no sign condition on the potential V, but later, when establishing assumptions guaranteeing that

$$\mu_0 > 0, \tag{2.6.5}$$

we shall require that

$$V \geq 0, \tag{2.6.6}$$

or even

$$V > 0 \qquad \text{a.e.}, \tag{2.6.7}$$

i.e., V is strictly positive except on a Lebesgue null set. The constant m is always supposed to be non-negative.

To prove a spectral resolution of the hyperbolic equation (2.5.48) we choose an eigenfunction $v = v_i$ with positive eigenvalue $\mu = \mu_i$ and look at solutions of (2.5.48) of the form

$$u(x,t) = w(t)v(x). \tag{2.6.8}$$

u is then a solution of (2.5.48) provided w satisfies the implicit eigenvalue equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} - \mu t^{2-\frac{4}{n}}w - nt^2 \Lambda w = 0, \tag{2.6.9}$$

where Λ is the eigenvalue.

This eigenvalue problem we also considered in the previous chapter and proved that it has countably many solutions (w_i, Λ_i) with finite energy, i.e.,

$$\int_0^\infty \{|\dot{w}_i|^2 + (1 + t^2 + \mu t^{2-\frac{4}{n}})|w_i|^2\} < \infty, \tag{2.6.10}$$

cf. Theorem 1.6.7 on page 49.

Remark 2.6.2 A different, but similar, approach for non-compact $\mathcal{S}_0$ will be used in the next chapter under the assumption that $(\mathcal{S}_0, \sigma_{ij})$ is asymptotically Euclidean.

Finally, let us consider under which assumptions the lowest eigenvalue μ_0 of the eigenvalue problem (2.6.1) is strictly positive. This property can also be called a mass gap. We prove the existence of a mass gap in two cases.

In the first case we assume that V satisfies the condition (2.6.7).

Theorem 2.6.3 *Let $\mathcal{S}_0$ be compact and let V satisfy (2.6.7), then there exists m_0 such that for all $m \geq m_0$ the first eigenvalue μ_0 of Eq. (2.6.1) is strictly positive with an a priori bound from below depending on the data.*

The theorem immediately follows from a well-known compactness lemma:

Lemma 2.6.4 *Under the assumptions of the previous theorem there exists for any $\epsilon > 0$ a constant c_ϵ such that*

$$\int_{\mathcal{S}_0} |u|^2 \le \epsilon \int_{\mathcal{S}_0} |Du|^2 + c_\epsilon \int_{\mathcal{S}_0} V|u|^2 \qquad \forall u \in C^1(\mathcal{S}_0). \tag{2.6.11}$$

Proof We prove the estimate (2.6.11) in the Sobolev space $H^{1,2}(\mathcal{S}_0)$ instead of $C^1(\mathcal{S}_0)$, since this is the appropriate function space, and argue by contradiction.

If the estimate (2.6.11) would be false, then there would exist $\epsilon > 0$ and a sequence of functions

$$u_k \in H^{1,2}(\mathcal{S}_0) \tag{2.6.12}$$

such that

$$\int_{\mathcal{S}_0} |u_k|^2 > \epsilon \int_{\mathcal{S}_0} |Du_k|^2 + k \int_{\mathcal{S}_0} V|u_k|^2. \tag{2.6.13}$$

Without loss of generality we may assume

$$\int_{\mathcal{S}_0} |u_k|^2 = 1. \tag{2.6.14}$$

Hence, the u_k are bounded in $H^{1,2}(\mathcal{S}_0)$ and a subsequence, not relabeled, will weakly converge in $H^{1,2}(\mathcal{S}_0)$ to a function u such that

$$u_k \to u \qquad \text{in} \quad L^2(\mathcal{S}_0), \tag{2.6.15}$$

since the embedding from $H^{1,2}(\mathcal{S}_0)$ into $L^2(\mathcal{S}_0)$ is compact, and we would deduce

$$\int_{\mathcal{S}_0} |u|^2 = 1 \tag{2.6.16}$$

and also

$$\int_{\mathcal{S}_0} V|u|^2 = 0, \tag{2.6.17}$$

a contradiction. □

In the second case, we only assume $V \ge 0$ such that we may ignore the contribution of the Higgs field to the quadratic form defined by the elliptic operator in Eq. (2.6.1) completely, since its contribution is non-negative, and only look at the smaller operator

$$-(n-1)\Delta v - \frac{n}{2}Rv + \alpha_1 \frac{n}{8} F_{ij}F^{ij} v. \tag{2.6.18}$$

If we can prove that the eigenvalues of this operator are strictly positive, then the eigenvalues of Eq. (2.6.1) are also strictly positive.

Theorem 2.6.5 *Let $\mathcal{S}_0$ be compact, $R \le 0$, then the smallest eigenvalue of the operator (2.6.18) is strictly positive provided either R or $F_{ij}F^{ij}$ do not vanish everywhere.*

Proof Under the assumptions the eigenvalues are always non-negative and the spectral resolution described in Lemma 2.6.1 is valid. Therefore, assume that $\mu_0 = 0$ and let u be a corresponding eigenfunction, then

$$0 = \int_{\mathcal{S}_0} |Du|^2 - \frac{n}{2} \int_{\mathcal{S}_0} R|u|^2 + \alpha_1 \frac{n}{8} \int_{\mathcal{S}_0} F_{ij} F^{ij} |u|^2. \qquad (2.6.19)$$

Hence, each of the integrals will vanish and we conclude that

$$u = \text{const} \qquad (2.6.20)$$

and

$$-R + F_{ij} F^{ij} = 0, \qquad (2.6.21)$$

contradicting the assumptions. □

Chapter 3
The Quantum Development of an Asymptotically Euclidean Cauchy Hypersurface

3.1 Spectral Resolution of a Hyperbolic Equation

In the preceding chapters, we obtained as a result of the canonical quantization of gravity a quantized version of the Hamilton constraint which is the wave equation

$$\begin{aligned}&\frac{1}{32}\frac{n^2}{n-1}\ddot{u}-(n-1)t^{2-\frac{4}{n}}\Delta u-\frac{n}{2}t^{2-\frac{4}{n}}Ru+\alpha_1\frac{n}{8}t^{2-\frac{4}{n}}F_{ij}F^{ij}u\\&+\alpha_2\frac{n}{4}t^{2-\frac{4}{n}}\gamma_{ab}\sigma^{ij}\Phi^a_i\Phi^b_j u+\alpha_2\frac{n}{2}mt^{2-\frac{4}{n}}V(\Phi)u+nt^2\Lambda u=0\end{aligned} \tag{3.1.1}$$

in a globally hyperbolic spacetime

$$Q=(0,\infty)\times\mathcal{S}_0, \tag{3.1.2}$$

where $\mathcal{S}_0=(\mathcal{S}_0,g_{ij})$ is a Cauchy hypersurface of the globally hyperbolic spacetime that had been quantized. This hyperbolic equation is sometimes considered to be the result of a *first quantization*. The next step would be the *second quantization*. If the result of the first quantization would have been a self-adjoint operator H acting in a Hilbert space $\mathcal{H}$, then for the second quantization one would consider the corresponding Fock space and the extension of H to that Fock space.

When the first quantization leads to a hyperbolic equation, then one usually tries to construct a Weyl system and to apply the techniques of algebraic quantum field theory. We used the latter approach when we looked at the Wheeler–DeWitt equation, cf. [16, 17], or, the more general hyperbolic equation (1.4.101 on page 28, cf. Sect. 1.5 on page 35. But when we considered our preferred model, namely the wave equation above, to be the result of a first quantization, we tried, at least for compact $\mathcal{S}_0$, to find solutions of the wave equation which are products of temporal and spatial eigenfunctions or eigendistributions of self-adjoint operators associated with the hyperbolic operator. We, tentatively, called this approach a spectral resolution of the wave equation.

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Now, we want to formalize this approach such that unbounded $\mathcal{S}_0$ can also be allowed. For simplicity, we still assume the time interval I to be $(0, \infty)$ but, of course, I could be an arbitrary open interval.

Definition 3.1.1 Let $(\mathcal{S}_0, g_{ij})$ be a complete n-dimensional Riemannian manifold and

$$\mathscr{D}u = 0 \tag{3.1.3}$$

a second-order hyperbolic differential equation in

$$Q = (0, \infty) \times \mathcal{S}_0. \tag{3.1.4}$$

Suppose that there exist temporal and spatial self-adjoint operators H_0 resp. H_1 such that the hyperbolic equation is equivalent to

$$H_0 u - H_1 u = 0, \tag{3.1.5}$$

where $u = u(t, x)$, and that one of the operators has a pure point spectrum with eigenvalues λ_i while, for the other operator, it is possible to find corresponding eigendistributions for each of the eigenvalues λ_i. Assuming, e.g., that H_0 has a pure point spectrum with corresponding mutually orthogonal eigenfunctions w_i and H_1 has smooth eigendistributions v_{ij} satisfying

$$H_1 v_{ij} = \lambda_i v_{ij} \qquad \forall\, j \tag{3.1.6}$$

then

$$u_{ij} = w_i v_{ij} \tag{3.1.7}$$

would be solutions of the hyperbolic equation. We call the triple (H_0, H_1, u_{ij}) a *spectral resolution* of the hyperbolic equation (3.1.3).

Weyl already used this approach to analyse the radiation of a black body, cf. [43, Kap. 6], though in this case the spatial Hamiltonian H_1 had a pure point spectrum and the temporal Hamiltonian H_0, which was just the classical harmonic oscillator,

$$H_0 w = -\ddot{w}, \tag{3.1.8}$$

had only a continuous spectrum.

In case of the wave equation (3.1.1), the temporal operator H_0 is given by

$$H_0 w = \varphi_0^{-1}\left(-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + nt^2|\Lambda| w\right), \tag{3.1.9}$$

where $\Lambda < 0$ is fixed and

$$\varphi_0(t) = t^{2-\frac{4}{n}}. \tag{3.1.10}$$

H_0 is self-adjoint in $L^2(\mathbb{R}_+^*, \varphi_0 dt)$ with a pure positive point spectrum

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{3.1.11}$$

and there exists a basis of mutually orthogonal eigenfunctions w_i. The spatial operator H_1 is defined as the closure of the elliptic operator

$$\begin{aligned} Av = &- (n-1)\Delta v - \frac{n}{2} R v + \alpha_1 \frac{n}{8} F_{ij} F^{ij} v \\ &+ \alpha_2 \frac{n}{4} \gamma_{ab} \sigma^{ij} \Phi_i^a \Phi_j^b v + \alpha_2 \frac{n}{2} m V(\Phi) v. \end{aligned} \tag{3.1.12}$$

In order to obtain a spectral resolution for the hyperbolic equation, we then must prove that for each λ_i there exist corresponding eigendistributions v_{ij} satisfying (3.1.6).

In this chapter, we shall prove that this is indeed the case if $\mathcal{S}_0$ is asymptotically Euclidean and the coefficients of A satisfy some natural assumptions. If the coefficients of A are smooth and bounded in any

$$C^m(\mathcal{S}_0), \qquad m \in \mathbb{N}, \tag{3.1.13}$$

then A is essentially self-adjoint in $L^2(\mathcal{S}_0, \mathbb{C})$, and if $\mathcal{S}_0$ is asymptotically Euclidean, i.e. if it satisfies the very mild conditions in Assumption 3.3.1 on page 93, then the Schwartz space $\mathscr{S}$ of rapidly decreasing functions can also be defined in $\mathcal{S}_0$,

$$\mathscr{S} = \mathscr{S}(\mathcal{S}_0), \tag{3.1.14}$$

such that

$$\mathscr{S} \subset L^2(\mathcal{S}_0) \subset \mathscr{S}' \tag{3.1.15}$$

is a Gelfand triple and the eigenvalue problem in $\mathscr{S}'$

$$Af = \lambda f \tag{3.1.16}$$

has a solution for any $\lambda \in \sigma(A)$, cf. Theorem 3.2.5 on page 87. Let

$$(\mathscr{E}_\lambda)_{\lambda \in \sigma(A)} \tag{3.1.17}$$

be the set of eigendistributions in $\mathscr{S}'$ satisfying

$$Af(\lambda) = \lambda f(\lambda), \qquad f(\lambda) \in \mathscr{E}_\lambda, \tag{3.1.18}$$

then the $f(\lambda)$ are actually smooth functions in $\mathcal{S}_0$ with polynomial growth, cf. [20, Theorem 3] and Sect. 7.1 on page 187. Moreover, due to a result of Donnelly [8], we know that

$$[0, \infty) \subset \sigma_{\text{ess}}(A), \tag{3.1.19}$$

and hence, any temporal eigenvalue λ_i of H_0 is also a spatial eigenvalue of A in $\mathscr{S}'$

$$Af(\lambda_i) = \lambda_i f(\lambda_i). \tag{3.1.20}$$

Since the eigenspaces $\mathscr{E}_{\lambda_i}$ are separable, we deduce that for each i there is an at most countable basis of eigendistributions in $\mathscr{E}_{\lambda_i}$

$$v_{ij} \equiv f_j(\lambda_i), \qquad 1 \le j \le n(i) \le \infty, \tag{3.1.21}$$

satisfying

$$Av_{ij} = \lambda_i v_{ij}, \tag{3.1.22}$$

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{3.1.23}$$

The functions

$$u_{ij} = w_i v_{ij} \tag{3.1.24}$$

are then smooth solutions of the wave equations. They are considered to describe the *quantum development* of the Cauchy hypersurface $\mathcal{S}_0$.

Let us summarize this result as a theorem:

Theorem 3.1.2 *Let A and $\mathcal{S}_0$ satisfy the conditions in* (3.1.13) *and Assumption 3.3.1, and let w_i resp. v_{ij} be the countably many solutions of the temporal resp. spatial eigenvalue problems, then*

$$u_{ij} = w_i v_{ij} \tag{3.1.25}$$

are smooth solutions of the wave equation (3.1.1). *They are considered to describe the quantum development of the Cauchy hypersurface $\mathcal{S}_0$.*

3.2 Existence of a Complete Set of Eigendistributions

Let H be a separable Hilbert space, $\mathcal{S}$ a complete nuclear space and

$$j : \mathcal{S} \hookrightarrow H \tag{3.2.1}$$

an embedding such that $j(\mathcal{S})$ is dense in H. The triple

$$\mathcal{S} \subset H \subset \mathcal{S}' \tag{3.2.2}$$

is then called a Gelfand triple and H a *rigged Hilbert space*. Moreover, we require that the semi-norms $\|\cdot\|_p$ defining the topology of $\mathcal{S}$ are a countable family. In view of the Assumption (3.2.1) at least one of the semi-norms is already a norm, since there exist a constant c and a semi-norm $\|\cdot\|_p$ such that

$$\|j(\varphi)\| \leq c\|\varphi\|_p \qquad \forall\, \varphi \in \mathcal{S}, \tag{3.2.3}$$

and hence, $\|\cdot\|_p$ is a norm since j is injective. But then there exists an equivalent sequence of norms generating the topology of $\mathcal{S}$. Since $\mathcal{S}$ is nuclear, we may also assume that the norms are derived from a scalar product, cf. [44, Theorem 2, p. 292].

Let $\mathcal{S}_p$ be the completion of $\mathcal{S}$ with respect to $\|\cdot\|_p$, then

$$\mathcal{S} = \bigcap_{p=1}^{\infty} \mathcal{S}_p \tag{3.2.4}$$

and

$$\mathcal{S}' = \bigcup_{p=1}^{\infty} \mathcal{S}'_p. \tag{3.2.5}$$

A nuclear space $\mathcal{S}$ having these properties is called a nuclear countably Hilbert space or a nuclear Fréchet–Hilbert space.

Let A be a self-adjoint operator in H with spectrum

$$\Lambda = \sigma(A). \tag{3.2.6}$$

Identifying $\mathcal{S}$ with $j(\mathcal{S})$, we assume

$$A(\mathcal{S}) \subset \mathcal{S} \tag{3.2.7}$$

and we want to prove that for any $\lambda \in \Lambda$ there exists

$$0 \neq f(\lambda) \in \mathcal{S}' \tag{3.2.8}$$

satisfying

$$\langle f(\lambda), A\varphi\rangle = \lambda\langle f(\lambda), \varphi\rangle \quad \forall\, \varphi \in \mathcal{S}. \tag{3.2.9}$$

$f(\lambda)$ is then called a generalized eigenvector, or an eigendistribution, if $\mathcal{S}'$ is a space of distributions. The crucial point is that we need to prove the existence of a generalized eigenvector for any $\lambda \in \Lambda$.

Definition 3.2.1 We define

$$\mathscr{E}_\lambda = \{\, f \in \mathscr{S}' : Af = \lambda f \,\} \tag{3.2.10}$$

to be the generalized eigenspace of A with eigenvalue $\lambda \in \Lambda$ provided

$$\mathscr{E}_\lambda \neq \{0\}. \tag{3.2.11}$$

If (3.2.11) is valid for all $\lambda \in \Lambda$, then we call

$$(\mathscr{E}_\lambda)_{\lambda\in\Lambda} \tag{3.2.12}$$

a complete system of generalized eigenvectors of A in $\mathcal{S}'$.

Lemma 3.2.2 *If $\mathcal{S}$ is separable, then each $\mathscr{E}_\lambda \neq \{0\}$ is also separable in the inherited strong topology of $\mathcal{S}'$.*

Proof The Hilbert spaces $\mathcal{S}_p$ are all separable by assumption, so are their duals $\mathcal{S}'_p$. Let $\mathcal{B}_p$ be a countable dense subset of $\mathcal{S}'_p$ and set

$$\mathcal{B} = \bigcup_{p=1}^{\infty} \mathcal{B}_p, \tag{3.2.13}$$

Then, $\mathcal{B}$ is dense in $\mathcal{S}'$ in the strong topology. Indeed, consider $f \in \mathcal{S}'$ and a bounded subset $B \subset \mathcal{S}$, then there exists p such that $f \in \mathcal{S}'_p$, in view of (3.2.5), and for any $g \in \mathcal{B}_p$ we obtain

$$\sup_{\varphi\in B}|\langle f-g,\varphi\rangle| \leq \|f-g\|_{-p} \sup_{\varphi\in B}\|\varphi\|_p \leq c_B\|f-g\|_{-p} \tag{3.2.14}$$

proving the claim. □

Let E be the spectral measure of A mapping Borel sets of Λ to projections in H, then we can find an at most countable family of mutually orthogonal unit vectors

$$v_i \in H, \qquad 1 \leq i \leq m \leq \infty, \tag{3.2.15}$$

and mutually orthogonal subspaces

$$H_i \in H \tag{3.2.16}$$

which are generated by the vectors

$$E(\Omega)v_i, \qquad \Omega \in \mathscr{B}(\Lambda), \tag{3.2.17}$$

where Ω is an arbitrary Borel set in Λ, such that

$$H = \bigoplus_{i=1}^{m} H_i. \tag{3.2.18}$$

Each subspace H_i is isomorphic to the function space

$$\hat{H}_i = L^2(\Lambda, \mathbb{C}, \mu_i) \equiv L^2(\Lambda, \mu_i), \tag{3.2.19}$$

where μ_i is the positive Borel measure

$$\mu_i = \langle E v_i, v_i \rangle. \tag{3.2.20}$$

We have

$$\mu_i(\Lambda) = 1 \tag{3.2.21}$$

and there exists a unitary map U from H_i onto $\hat{H}_i$ such that

$$\langle u, v \rangle = \int_\Lambda \bar{\hat{u}}(\lambda)\hat{v}(\lambda) d\mu_i \qquad \forall u, v \in H_i \tag{3.2.22}$$

where we have set

$$\hat{u} = Uu \qquad \forall u \in H_i. \tag{3.2.23}$$

Hence, there exists a unitary surjective operator, also denoted by U,

$$U : H \to \hat{H} = \bigoplus_{i=1}^{m} \hat{H}_i \tag{3.2.24}$$

such that $u = (u^i)$ is mapped to

$$\hat{u} = Uu = (Uu^i) = (\hat{u}^i) \tag{3.2.25}$$

and

$$\hat{u}^i = \hat{u}^i(\lambda) \in L^2(\Lambda, \mu_i). \tag{3.2.26}$$

Moreover, if $u \in D(A)$, then

$$\widehat{Au} = (\widehat{Au}^i(\lambda)) = (\lambda \hat{u}^i) = \lambda \hat{u}. \tag{3.2.27}$$

For a proof of these well-known results, see e.g. [11, Chap. I, Appendix, p. 127].

Remark 3.2.3 We define the positive measure

$$\mu = \sum_{i=1}^{m} 2^{-i} \mu_i \tag{3.2.28}$$

in Λ, and we shall always have this measure in mind when referring to null sets in Λ. Moreover, applying the Radon–Nikodym theorem, we conclude that there are nonnegative Borel functions, which we express in the form h_i^2, $0 \le h_i$, such that

$$h_i^2 \in L^1(\Lambda, \mu) \tag{3.2.29}$$

and

$$d\mu_i = h_i^2 d\mu. \tag{3.2.30}$$

The map

$$v \in L^2(\Lambda, \mu_i) \to h_i v \in L^2(\Lambda, \mu) \tag{3.2.31}$$

is a unitary embedding.

Lemma 3.2.4 *The functions h_i satisfy the following relations*

$$\sum_{i=1}^{m} 2^{-2i} h_i^2 < \infty \qquad \mu \text{ a.e.} \tag{3.2.32}$$

and

$$\sum_{i=1}^{m} 2^{-2i} h_i^2 \neq 0 \qquad \mu \text{ a.e.} \tag{3.2.33}$$

Replacing the values of h_i on the exceptional null sets by 2^{-i}, the two previous relations are valid everywhere in Λ.

Proof (i) We first prove that, for a fixed i, h_i cannot vanish on a Borel set G with positive μ_i measure, $\mu_i(G) > 0$. We argue by contradiction assuming that h_i would vanish on a Borel set G with $\mu_i(G) > 0$. Let $v \in H$ be arbitrary and let v^i be the component belonging to H_i, then

$$\begin{aligned} \int_G |\hat{v}^i|^2 d\mu_i &= \int_\Lambda \chi_G |\hat{v}^i|^2 d\mu_i \\ &= \int_\Lambda \chi_G h_i^2 |\hat{v}^i|^2 d\mu = 0, \end{aligned} \tag{3.2.34}$$

and we deduce

$$\hat{v}^i = 0 \qquad \mu_i \text{ a.e. in } G \qquad \forall\, v \in H, \tag{3.2.35}$$

a contradiction, since the $\hat{v}^i$ generate $L^2(\Lambda, \mu_i)$.

(ii) Now, let $G \subset \Lambda$ be an arbitrary Borel set satisfying $\mu(G) > 0$ and define $\hat{\psi} = (\hat{\psi}^i)$ by setting

$$\hat{\psi}^i = \chi_G 2^{-i}, \tag{3.2.36}$$

then we obtain

$$\begin{aligned} \|\hat{\psi}\|^2 &= \sum_{i=1}^{m} \int_\Lambda \chi_G 2^{-2i} d\mu_i \\ &= \sum_{i=1}^{m} \int_\Lambda \chi_G 2^{-2i} h_i^2 d\mu < \infty \end{aligned} \tag{3.2.37}$$

concluding

$$\sum_{i=1}^{m} 2^{-2i} h_i^2 < \infty \qquad \mu \text{ a.e.} \tag{3.2.38}$$

as well as

$$\sum_{i=1}^{m} 2^{-2i} h_i^2 \neq 0 \qquad \mu \text{ a.e.,} \tag{3.2.39}$$

where the last conclusion is due to the result proved in (i), since for any Borel set G with $\mu(G) > 0$ there must exist an i such that $\mu_i(G) > 0$. □

Now we can prove:

Theorem 3.2.5 *Let H be a separable rigged Hilbert space as above assuming that the nuclear space $\mathcal{S}$ is a Fréchet–Hilbert space, and let A be a self-adjoint operator in H satisfying (3.2.7). Then, there exists a complete system of generalized eigenvectors $(\mathscr{E}_\lambda)_{\lambda \in \Lambda}$. If $\mathcal{S}$ is separable, then each eigenspace $\mathscr{E}_\lambda$ is separable.*

Proof Since $\mathcal{S}$ is nuclear, there exists a norm $\|\cdot\|_p$ such that the embedding

$$j : \mathcal{S}_p \hookrightarrow H \tag{3.2.40}$$

is nuclear; i.e. we can write

$$j(\varphi) = \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle u_k \qquad \forall \varphi \in \mathcal{S}, \tag{3.2.41}$$

where

$$0 \le \lambda_k \quad \wedge \quad \sum_{k=1}^{\infty} \lambda_k < \infty, \tag{3.2.42}$$

$$f_k \in \mathcal{S}_p' \quad \wedge \quad \|f_k\| = 1, \tag{3.2.43}$$

and $u_k \in H$ is an orthonormal sequence. We may, and shall, also assume

$$u_k \in D(A), \tag{3.2.44}$$

since $D(A)$ is dense in H: Let

$$v_k \in D(A) \tag{3.2.45}$$

be a sequence of linearly independent vectors generating a dense subspace in H, then we can define an orthonormal basis $(\tilde{v}_k)$ in H which spans the same subspace. Hence, there exists a unitary map T such that

$$\tilde{v}_k = T u_k \qquad \forall k \in \mathbb{N}. \tag{3.2.46}$$

Instead of the embedding j, we can then consider the embedding

$$T \circ j \tag{3.2.47}$$

proving our claim. Thus, we shall assume (3.2.44) which is convenient but not necessary.

From the assumption that $j(\mathcal{S})$ is dense in H we immediately draw the following conclusions:

$$\text{The } (u_k) \text{ are complete in H,} \tag{3.2.48}$$

$$0 < \lambda_k \qquad \forall\, k, \tag{3.2.49}$$

and

$$\text{for all } k \text{ there exists } \varphi \in \mathcal{S} \text{ such that } \langle f_k, \varphi\rangle \neq 0. \tag{3.2.50}$$

Let U be the unitary operator in (3.2.24), then we define

$$\hat{\varphi} = U \circ j(\varphi) = \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi\rangle \hat{u}_k \tag{3.2.51}$$

such that

$$\hat{u}_k = (\hat{u}_k^i(\lambda))_{1\le i\le m} \tag{3.2.52}$$

$$\hat{u}_k^i \in L^2(\Lambda, \mu_i). \tag{3.2.53}$$

Applying the embedding in (3.2.31), we can also express $\hat{u}_k$ in the form

$$\hat{u}_k = (h_i \hat{u}_k^i(\lambda))_{1\le i\le m} \tag{3.2.54}$$

$$h_i \hat{u}_k^i \in L^2(\Lambda, \mu). \tag{3.2.55}$$

Similarly, we have

$$\hat{\varphi} = (h_i \hat{\varphi}^i) \tag{3.2.56}$$

and

$$\widehat{A\varphi} = (\lambda h_i \hat{\varphi}^i), \tag{3.2.57}$$

in view of (3.2.27). Here, we identify φ and $j\varphi$, i.e.

$$A\varphi \equiv A(j\varphi). \tag{3.2.58}$$

We want to prove that

$$\begin{aligned}\widehat{A(j\varphi)} &= \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle \widehat{Au_k} \\ &= \lambda \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle \hat{u}_k.\end{aligned} \tag{3.2.59}$$

Indeed, for any bounded Borel set $\Omega \subset \Lambda$

$$AE(\Omega) \tag{3.2.60}$$

is a self-adjoint bounded operator in H such that

$$\|AE(\Omega)\| \leq \sup_{\lambda \in \Omega} |\lambda|. \tag{3.2.61}$$

Hence, we deduce

$$AE(\Omega)(j\varphi) = \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle AE(\Omega) u_k \tag{3.2.62}$$

and

$$A\widehat{E(\Omega)}u_k = \lambda \chi_\Omega \hat{u}_k \tag{3.2.63}$$

and we infer

$$\begin{aligned}\chi_\Omega \widehat{A(j\varphi)} &= \chi_\Omega \lambda \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle \hat{u}_k \\ &= \chi_\Omega \lambda \hat{\varphi}.\end{aligned} \tag{3.2.64}$$

Since $\Omega \subset \Lambda$ is an arbitrary bounded Borel set, we conclude

$$\begin{aligned}\widehat{A(j\varphi)} &= \lambda \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle \hat{u}_k \\ &= \lambda \hat{\varphi}.\end{aligned} \tag{3.2.65}$$

The right-hand side of the second equation is square integrable and therefore the right-hand side of the first equation too.

Let us set

$$\hat{\varphi}(\lambda) = (h_i \hat{\varphi}^i(\lambda)). \tag{3.2.66}$$

$h_i \hat{\varphi}^i$ is an equivalence class, and to define $h_i \hat{\varphi}^i(\lambda)$ as a complex number for a fixed $\lambda \in \Lambda$ requires to pick a representative of the equivalence class. It is well known that for a given representative $h_i \hat{\varphi}^i(\lambda)$ is well defined for almost every $\lambda \in \Lambda$, i.e. apart from a null set. We shall show that $\hat{\varphi}(\lambda)$ can be well defined for *any* $\lambda \in \Lambda$ and any $\varphi \in \mathcal{S}$. The choices we shall have to make will be independent of φ.

Firstly, let us define the product

$$h_i \hat{u}_k^i \tag{3.2.67}$$

unambiguously. In view of Lemma 3.2.4, h_i is everywhere finite; i.e., we only have to consider the case when $h_i = 0$ and $|\hat{u}_k^i| = \infty$. In this case, we stipulate that

$$h_i \hat{u}_k^i = 0. \tag{3.2.68}$$

This definition insures that the integrals, e.g.

$$\int_\Lambda |h_i \hat{u}_k^i|^2 d\mu \tag{3.2.69}$$

will give the correct values, because of Lebesgue's monotone convergence theorem: approximate $|\hat{u}_k^i|$ by

$$\min(|\hat{u}_k^i|, r), \qquad r \in \mathbb{N}. \tag{3.2.70}$$

Secondly, we observe that

$$1 = \|\hat{u}_k\|^2 = \sum_{i=1}^m \int_\Lambda |h_i \hat{u}_k^i(\lambda)|^2, \tag{3.2.71}$$

and hence

$$\sum_{i=1}^m |h_i \hat{u}_k^i(\lambda)|^2 < \infty \qquad \text{a.e. in } \Lambda. \tag{3.2.72}$$

Thirdly, we have

$$\sum_{k=1}^\infty \sum_{i=1}^m |h_i \hat{u}_k^i(\lambda)|^2 \neq 0 \qquad \text{a.e. in } \Lambda. \tag{3.2.73}$$

Indeed, suppose there were a Borel set

$$G \subset \Lambda \tag{3.2.74}$$

such that

$$0 < \mu(G) = \sum_i 2^{-i} \mu_i(G) \tag{3.2.75}$$

and

$$\sum_{k=1}^\infty \sum_{i=1}^m |h_i \hat{u}_k^i(\lambda)|^2 = 0 \qquad \text{in } G, \tag{3.2.76}$$

then there would exist j such that

$$\mu_j(G) > 0 \tag{3.2.77}$$

and we would deduce

$$0 = \sum_{k=1}^{\infty} \int_G |h_j \hat{u}_k^j|^2 d\mu = \sum_{k=1}^{\infty} \int_G |\hat{u}_k^j|^2 d\mu_j, \tag{3.2.78}$$

contradicting the fact that the $(\hat{u}_k^j)$ are a basis for $L^2(\Lambda, \mu_j)$.

Fourthly, we have

$$\sum_k \sum_i \int_\Lambda \lambda_k |h_i \hat{u}_k^i(\lambda)|^2 d\mu = \sum_k \lambda_k \|\hat{u}_k\|^2 = \sum_k \lambda_k < \infty, \tag{3.2.79}$$

and hence we deduce

$$\sum_k \sum_i \lambda_k |h_i \hat{u}_k^i(\lambda)|^2 < \infty \qquad \text{a.e. in } \Lambda. \tag{3.2.80}$$

Now, for any (i, k) we choose a particular representative of $h_i \hat{u}_k^i$ by first picking the representative of h_i we defined in Lemma 3.2.4 and a representative of $\hat{u}_k^i$ satisfying the relations in (3.2.72), (3.2.73) and (3.2.80) and then defining the values of these particular representatives in the exceptional null sets occurring in the just mentioned relations by

$$h_i = 2^{-i} \quad \wedge \quad \hat{u}_k^i = 2^{-i} 2^{-k}. \tag{3.2.81}$$

Then, $h_i \hat{u}_k^i(\lambda)$ is well defined for any $\lambda \in \Lambda$ and the relations in (3.2.72), (3.2.73) and (3.2.80) are valid for any $\lambda \in \Lambda$.

Moreover, the series

$$h_i \hat{\varphi}^i(\lambda) = \sum_k \lambda_k \langle f_k, \varphi \rangle h_i \hat{u}_k^i(\lambda) \tag{3.2.82}$$

converges absolutely, since

$$\begin{aligned} \sum_k \lambda_k |\langle f_k, \varphi \rangle| |h_i \hat{u}_k^i(\lambda)| &\le \|\varphi\|_p \sum_k \lambda_k |h_i \hat{u}_k^i(\lambda)| \\ &\le \|\varphi\|_p (\sum_k \lambda_k)^{\frac{1}{2}} (\sum_k \lambda_k |h_i \hat{u}_k^i(\lambda)|^2)^{\frac{1}{2}} < \infty, \end{aligned} \tag{3.2.83}$$

in view of (3.2.80).

Definition 3.2.6 Let us define the sequence space

$$l_2 = \{\, (a_k^i) \colon \sum_k \sum_i |a_k^i|^2 < \infty \,\} \tag{3.2.84}$$

with scalar product

$$\langle (a_k^i), (b_k^i) \rangle = \sum_k (\sum_i \bar{a}_k^i b_k^i). \tag{3.2.85}$$

Thus, we have

$$(\lambda_k \langle f_k, \varphi \rangle h_i \hat{u}_k^i(\lambda)) \in l_2, \tag{3.2.86}$$

since

$$\lambda_k^2 < \lambda_k \tag{3.2.87}$$

for k large. By a slight abuse of language, we shall also call this sequence $\hat{\varphi}(\lambda)$,

$$\hat{\varphi}(\lambda) = (\lambda_k \langle f_k, \varphi \rangle h_i \hat{u}_k^i(\lambda)). \tag{3.2.88}$$

We are now ready to complete the proof of the theorem. Let $\lambda \in \Lambda$ be arbitrary, then there exists a pair (i_0, k_0) such that

$$h_{i_0} \hat{u}_{k_0}^{i_0}(\lambda) \neq 0, \tag{3.2.89}$$

in view of (3.2.73), which is now valid for any $\lambda \in \Lambda$. Define

$$f(\lambda) = (h_{i_0} \hat{u}_{k_0}^{i_0}(\lambda)) \in l_2 \tag{3.2.90}$$

to be the sequence with just one non-trivial term. We may consider

$$f(\lambda) \in \mathcal{S}_p' \subset \mathcal{S}' \tag{3.2.91}$$

by defining

$$\langle f(\lambda), \varphi \rangle = \langle f(\lambda), \hat{\varphi}(\lambda) \rangle \qquad \forall \varphi \in \mathcal{S}, \tag{3.2.92}$$

where the right-hand side is the scalar product in l_2. Indeed, we obtain

$$\begin{aligned} |\langle f(\lambda), \varphi \rangle| &= \lambda_{k_0} |\langle f_{k_0}, \varphi \rangle| |h_{i_0} \hat{u}_{k_0}^{i_0}(\lambda)|^2 \\ &\le \lambda_{k_0} |h_{i_0} \hat{u}_{k_0}^{i_0}(\lambda)|^2 \|\varphi\|_p \quad \forall \varphi \in \mathcal{S} \end{aligned} \tag{3.2.93}$$

yielding

$$f(\lambda) \in \mathcal{S}_p'. \tag{3.2.94}$$

Furthermore,

$$f(\lambda) \neq 0, \tag{3.2.95}$$

since there exists $\varphi \in \mathcal{S}$ such that

$$\langle f_{k_0}, \varphi \rangle \neq 0, \tag{3.2.96}$$

in view of (3.2.50).

$f(\lambda)$ is also a generalized eigenvector of A with eigenvalue λ, since

$$\langle f(\lambda), A\varphi \rangle = \langle f(\lambda), \widehat{A\varphi}(\lambda) \rangle = \langle f(\lambda), \lambda\hat{\varphi}(\lambda) \rangle = \lambda \langle f(\lambda), \varphi \rangle \tag{3.2.97}$$

because of (3.2.57) and (3.2.59). The final conclusions are derived from Lemma 3.2.2. □

3.3 Properties of $\sigma(A)$ in the Asymptotically Euclidean Case

Let A be the elliptic operator

$$\begin{aligned} &-(n-1)\Delta v - \frac{n}{2}Rv + \alpha_1 \frac{n}{8} F_{ij} F^{ij} v \\ &+ \alpha_2 \frac{n}{4} \gamma_{ab} g^{ij} \Phi_i^a \Phi_j^b v + \alpha_2 \frac{n}{2} m V(\Phi) v. \end{aligned} \tag{3.3.1}$$

We want to prove that

$$[0, \infty) \subset \sigma(A), \tag{3.3.2}$$

in order to be able to quantize the wave equation (3.1.1) on page 79. Using the results in [8], we shall show that (3.3.2) or even the stronger result

$$[0, \infty) \subset \sigma_{\text{ess}}(A), \tag{3.3.3}$$

where $\sigma_{\text{ess}}(A)$ is the essential spectrum, is valid provided the following assumptions are satisfied:

Assumption 3.3.1 We assume there exists a compact $K \subset \mathcal{S}_0$ and a coordinate system (x^i) covering $\mathcal{S}_0 \backslash K$ such that $\mathcal{S}_0 \backslash K$ is diffeomorphic with an exterior region

$$\Omega \subset \mathbb{R}^n \tag{3.3.4}$$

and

$$x = (x^i) \in \Omega. \tag{3.3.5}$$

The metric (g_{ij}) then has to satisfy

$$\lim_{|x|\to\infty} g_{ij}(x) = \delta_{ij}, \tag{3.3.6}$$

$$\lim_{|x|\to\infty} g_{ij,k}(x) = 0, \tag{3.3.7}$$

where a comma indicates partial differentiation, and there is a constant c such that

$$cr \le |x| \le c^{-1} r \qquad \forall\, x \in \Omega, \tag{3.3.8}$$

where r is the geometric distance to a base point $p \in K$.

Furthermore, we require that the lower-order terms of A vanish at infinity, i.e.

$$\lim_{|x|\to\infty} \{|R| + |F_{ij}F^{ij}| + |\gamma_{ab} g^{ij} \Phi_i^a \Phi_i^b| + |V(\Phi)|\} = 0. \tag{3.3.9}$$

Let us refer the lower-order terms with the symbol $V = V(x)$ such that

$$A = (n-1)\{-\Delta + V\}, \tag{3.3.10}$$

then we shall prove

Theorem 3.3.2 *The operator A in (3.3.10) has the property*

$$[0, \infty) \subset \sigma_{\text{ess}}(A). \tag{3.3.11}$$

Proof We first prove the result for the operator $(-\Delta + V)$. Let us define a positive function

$$b \in C^\infty(\mathcal{S}_0), \tag{3.3.12}$$

such that

$$b(x) = |x| \qquad \forall\, x \notin B_R(p), \tag{3.3.13}$$

where $B_R(p)$ is a large geodesic ball containing the compact set K. In view of the assumptions (3.3.6), (3.3.7) and (3.3.8), b satisfies the conditions (i), (ii) and (iii) in [8, Properties 2.1]. Moreover, the assumption (3.3.9), which implies

$$\lim_{|x|\to\infty} |V| = 0, \tag{3.3.14}$$

insures that the condition (iv) in [8, Theorem 2.4] can be applied yielding

$$[0, \infty) = \sigma_{\text{ess}}(-\Delta + V). \tag{3.3.15}$$

However, since only the inclusion

$$[0, \infty) \subset \sigma_{\text{ess}}(-\Delta + V). \tag{3.3.16}$$

is proved while the reverse inclusion is merely referred to, and we could not look at the given references, we shall only use (3.3.16). This relation is proved by constructing, for each $\epsilon > 0$ and $\lambda > 0$, an infinite-dimensional subspace G_ϵ of $C_c^2(\mathcal{S}_0)$ such that

$$\int_M |(-\Delta + V - \lambda^2)v|^2 \le \epsilon^2 \int_M |v|^2 \qquad \forall\, v \in G_\epsilon. \tag{3.3.17}$$

Multiplying this inequality by $(n-1)^2$, we infer that (3.3.16) is also valid when the operator $(-\Delta + V)$ is replaced by

$$A = (n-1)(-\Delta + V) \tag{3.3.18}$$

proving the theorem. □

3.4 The Quantization of the Wave Equation

The quantization of the hyperbolic equation (3.1.1) on page 79 will be achieved by splitting the equation into two equations: A temporal eigenvalue equation, an ODE, and a spatial elliptic eigenvalue equation.

Let us first consider the temporal eigenvalue equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n|\Lambda|t^2 w = \lambda t^{2-\frac{4}{n}} w, \tag{3.4.1}$$

where

$$\Lambda < 0 \tag{3.4.2}$$

is a cosmological constant.

The eigenvalue problem (3.4.1) can be solved by considering the generalized eigenvalue problem for the bilinear forms

$$B(w, \tilde{w}) = \int_{\mathbb{R}_+^*} \{\frac{1}{32}\frac{n^2}{n-1}\bar{w}'\tilde{w}' + n|\Lambda|t^2\bar{w}\tilde{w}\} \tag{3.4.3}$$

and

$$K(w, \tilde{w}) = \int_{\mathbb{R}_+^*} t^{2-\frac{4}{n}}\bar{w}\tilde{w} \tag{3.4.4}$$

in the Sobolev space $\mathcal{H}$ which is the completion of

$$C_c^\infty(\mathbb{R}_+^*, \mathbb{C}) \tag{3.4.5}$$

in the norm defined by the first bilinear form.

We then look at the generalized eigenvalue problem

$$B(w,\varphi)=\lambda K(w,\varphi)\quad \forall \varphi\in\mathcal{H} \tag{3.4.6}$$

which is equivalent to (3.4.1).

Theorem 3.4.1 *The eigenvalue problem* (3.4.6) *has countably many solutions* (w_i,λ_i) *such that*

$$0<\lambda_0<\lambda_1<\lambda_2<\cdots, \tag{3.4.7}$$

$$\lim\lambda_i=\infty, \tag{3.4.8}$$

and

$$K(w_i,w_j)=\delta_{ij}. \tag{3.4.9}$$

The w_i *are complete in* $\mathcal{H}$ *as well as in* $L^2(\mathbb{R}_+^*)$.

Proof The quadratic form K is compact with respect to the quadratic form B as one can easily prove, cf. [13, Lemma 6.8], and hence a proof of the result, except for the strict inequalities in (3.4.7), can be found in [15, Theorem 1.6.3, p. 37]. Each eigenvalue has multiplicity one since we have a linear ODE of order two and all solutions satisfy the boundary condition

$$w_i(0)=0. \tag{3.4.10}$$

The kernel is two-dimensional, and the condition (3.4.10) defines a one-dimensional subspace. Note that we considered only real-valued solutions to apply this argument. □

Remark 3.4.2 In [15, Theorem 1.6.3, p. 37], we gave a proof of the more general problem of solving abstract eigenvalue problems in a Hilbert space by variational methods which can be applied to eigenvalue problems for elliptic linear operators in Euclidean space or Riemannian manifolds as well as to ordinary differential operators as above.

The elliptic eigenvalue equation has the form

$$Av=\lambda v, \tag{3.4.11}$$

where A is the elliptic operator in (3.3.1) on page 93 and $v\in C^\infty(\mathcal{S}_0)$. A is a self-adjoint operator in $L^2(\mathcal{S}_0,\mathbb{C})$. Let

$$\mathscr{S}=\mathscr{S}(\mathcal{S}_0) \tag{3.4.12}$$

be the Schwartz space of rapidly decreasing smooth functions, then $\mathscr{S}$ is a separable nuclear Fréchet–Hilbert space and

$$\mathscr{S} \subset L^2(\mathcal{S}_0, \mathbb{C}) \subset \mathscr{S}' \tag{3.4.13}$$

a Gelfand triple. Applying the results of Theorem 3.2.5 on page 87, we infer that there exists a complete system of eigendistributions

$$(\mathscr{E}_\lambda)_{\lambda \in \sigma(A))} \tag{3.4.14}$$

in $\mathscr{S}'$, i.e.

$$Af(\lambda) = \lambda f(\lambda) \qquad \forall\, f(\lambda) \in \mathscr{E}_\lambda. \tag{3.4.15}$$

These eigendistributions are actually smooth functions in $\mathcal{S}_0$ with polynomial growth as we proved in [20, Theorem 3]; see also Sect. 7.1 on page 187, where a proof is presented for the convenience of the reader. Assuming, furthermore, that the conditions in Assumption 3.3.1 on page 93 are satisfied, we conclude that

$$[0, \infty) \subset \sigma_{\mathrm{ess}}(A), \tag{3.4.16}$$

in view of Theorem 3.3.2 on page 93; i.e., the equation (3.4.11) is valid for all $\lambda \in \mathbb{R}_+$, and we conclude further that each temporal eigenvalue λ_i of the equation (3.4.1) can also be looked at as a spatial eigenvalue of the equation (3.4.11). Since the eigenspaces $\mathscr{E}_{\lambda_i}$ are separable, we deduce that for each i there is an at most countable basis of eigendistributions in $\mathscr{E}_{\lambda_i}$

$$v_{ij} \equiv f_j(\lambda_i), \qquad 1 \le j \le n(i) \le \infty, \tag{3.4.17}$$

satisfying

$$Av_{ij} = \lambda_i v_{ij}, \tag{3.4.18}$$

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{3.4.19}$$

The functions

$$u_{ij} = w_i v_{ij} \tag{3.4.20}$$

are then smooth solutions of the wave equation. They are considered to describe the quantum development of the Cauchy hypersurface $\mathcal{S}_0$.

Let us summarize this result as a theorem:

Theorem 3.4.3 *Let $\mathcal{S}_0$ satisfy the conditions in Assumption 3.3.1 and let w_i resp. v_{ij} be the countably many solutions of the temporal resp. spatial eigenvalue problems, then*

$$u_{ij} = w_i v_{ij} \tag{3.4.21}$$

are smooth solutions of the wave equation. They describe the quantum development of the Cauchy hypersurface $\mathcal{S}_0$.

Chapter 4
The Quantization of a Schwarzschild-AdS Black Hole

4.1 The Quantum Model

In the previous chapter, we looked at the quantum model for the interaction of gravity with Yang–Mills and Higgs fields and proved a spectral resolution of the underlying hyperbolic equation provided the Cauchy hypersurface $\mathcal{S}_0$ was asymptotically Euclidean. In this chapter, we shall consider a Schwarzschild-AdS black hole N of dimension $n+1$, $n \geq 3$. Picking a Cauchy hypersurface $\mathcal{S}_0$ with induced metric g_{ij} in N, then its quantum development would be governed by the hyperbolic equation

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0 \tag{4.1.1}$$

defined in the spacetime

$$Q = (0,\infty) \times \mathcal{S}_0. \tag{4.1.2}$$

The Laplacian is the Laplacian with respect to g_{ij}, R is the scalar curvature of the metric, $0 < t$ is the time coordinate defined by the derivation process of the equation and $\Lambda < 0$ a cosmological constant.

We shall especially choose a Cauchy hypersurface in the black hole region of the form

$$\{r = \text{const} < r_0\}, \tag{4.1.3}$$

where r_0 is the radius of the event horizon. It turns out that the induced metric of the Cauchy hypersurface can be expressed in the form

$$ds^2 = d\tau^2 + r^2\sigma_{ij}dx^i dx^j, \tag{4.1.4}$$

where

$$-\infty < \tau < \infty, \tag{4.1.5}$$

C. Gerhardt, *The Quantization of Gravity*, Fundamental Theories of Physics 194,
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$r = \text{const}$ and σ_{ij} is the metric of a spaceform $M = M^{n-1}$ with curvature $\tilde{\kappa}$,

$$\tilde{\kappa} \in \{-1, 0, 1\}. \tag{4.1.6}$$

The metric in (4.1.4) is free of any coordinate singularity; hence, we can let r tend to r_0 such that $\mathcal{S}_0$ represents the event horizon at least topologically. Furthermore, the Laplacian of the metric in (4.1.4) comprises a harmonic oscillator with respect to τ which enables us to write the stationary eigenfunctions v_j in the form

$$v_j(\tau, x) = \zeta(\tau)\varphi_j(x), \tag{4.1.7}$$

where φ_j is an eigenfunction of the Laplacian of M and ζ a harmonic oscillator the frequency of which are still to be determined.

The temporal eigenvalue problem is described by the equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n|\Lambda|t^2 w = \lambda t^{2-\frac{4}{n}} w \tag{4.1.8}$$

with a fixed $\Lambda < 0$, where we choose Λ to be the cosmological constant of the AdS spacetime.

In view of Theorem 3.4.1 on page 96, the eigenvalue problem (4.1.8) has a complete sequence (w_i, λ_i) of eigenfunctions with finite energies λ_i such that

$$0 < \lambda_0 < \lambda_1 < \cdots \tag{4.1.9}$$

and by choosing the frequencies of ζ appropriately we can arrange that the stationary eigenvalues μ_j of v_j agree with the temporal eigenvalues λ_i. If this is the case, then the eigenfunctions

$$u = w_i v_j \tag{4.1.10}$$

will be a solution of the wave equation. More precisely, we shall prove:

Theorem 4.1.1 *Let $(\varphi_j, \tilde{\mu}_j)$ resp. (w_i, λ_i) be eigenfunctions of*

$$-\tilde{\Delta} = -\Delta_M \tag{4.1.11}$$

resp. the temporal eigenfunctions and set

$$\hat{\mu}_j = (n-1)r_0^{-2}\tilde{\mu}_j - \frac{n}{2}(n-1)(n-2)r_0^{-2}\tilde{\kappa}. \tag{4.1.12}$$

Let λ_{i_0} be the smallest eigenvalue of the (λ_i) with the property

$$\lambda_{i_0} \geq \hat{\mu}_j, \tag{4.1.13}$$

then, for any $i \geq i_0$*, there exists*

$$\omega = \omega_{ij} \geq 0 \tag{4.1.14}$$

and corresponding ζ_{ij} *satisfying*

$$-\ddot{\zeta}_{ij} = r_0^{-2}\omega_{ij}^2\zeta_{ij} \tag{4.1.15}$$

such that

$$\lambda_i = \mu_{ij} = (n-1)r_0^{-2}\omega_{ij}^2 + \hat{\mu}_j \quad \forall i \geq i_0. \tag{4.1.16}$$

The functions

$$u_{ij} = w_i\zeta_{ij}\varphi_j \tag{4.1.17}$$

are then solutions of the wave equation with bounded energies satisfying

$$\lim_{t\to 0} u_{ij}(t) = \lim_{t\to\infty} u_{ij}(t) = 0 \tag{4.1.18}$$

and

$$u_{ij} \in C^\infty(\mathbb{R}_+^* \times \mathcal{S}_0) \cap C^{2,\alpha}(\bar{\mathbb{R}}_+^* \times \mathcal{S}_0) \tag{4.1.19}$$

for some

$$\frac{2}{3} \leq \alpha < 1. \tag{4.1.20}$$

Moreover, we have

$$\omega_{ij} > 0 \quad \forall i > i_0. \tag{4.1.21}$$

If

$$\lambda_{i_0} = \hat{\mu}_j, \tag{4.1.22}$$

then we define

$$\zeta_{i_0 j} \equiv 1. \tag{4.1.23}$$

In case $j = 0$ *and* $\tilde{\kappa} \neq -1$ *we always have*

$$\hat{\mu}_0 \leq 0 \tag{4.1.24}$$

and

$$\varphi_0 = \text{const} \neq 0 \tag{4.1.25}$$

and hence

$$\omega_{i0} > 0 \quad \forall i \geq 0. \tag{4.1.26}$$

Remark 4.1.2 (i) The event horizon corresponds to the Cauchy hypersurface $\{t = 1\}$ in Q and the open black hole region to the region

$$(0, 1) \times \mathcal{S}_0, \tag{4.1.27}$$

while the open exterior of the black hole region is represented by

$$(1, \infty) \times \mathcal{S}_0. \tag{4.1.28}$$

The black hole singularity corresponds to $\{t = 0\}$ which is also a curvature singularity in the quantum spacetime provided we equip Q with a metric such that the hyperbolic operator is normally hyperbolic, cf. Remark 2.5.3 on page 74. Moreover, in the quantum spacetime, the Cauchy hypersurface $\mathcal{S}_0$ can be crossed by causal curves in both directions; i.e., the information paradox does not occur.

(ii) The stationary eigenfunctions can be looked at as being radiation because they comprise the harmonic oscillator, while we consider the temporal eigenfunctions to be gravitational waves.

As it is well known, the Schwarzschild black hole or more specifically the extended Schwarzschild space has already been analyzed by Hawking [30] and Hartle and Hawking [29], see also the book by Wald [42], using quantum field theory, but not quantum gravity, to prove that the black hole emits radiation.

4.2 The Quantization

The metric in the interior of the black hole can be expressed in the form

$$d\bar{s}^2 = -\tilde{h}^{-1}dr^2 + \tilde{h}dt^2 + r^2\sigma_{ij}dx^i dx^j, \tag{4.2.1}$$

where (σ_{ij}) is the metric of an $(n-1)$-dimensional space form M and $\tilde{h}(r)$ is defined by

$$\tilde{h} = mr^{-(n-2)} + \tfrac{2}{n(n-1)}\Lambda r^2 - \tilde{\kappa}, \tag{4.2.2}$$

where $m > 0$ is the mass of the black hole (or a constant multiple of it), $\Lambda < 0$ a cosmological constant, and $\tilde{\kappa} \in \{-1, 0, 1\}$ is the curvature of $M = M^{n-1}$, $n \geq 3$. We also stipulate that M is compact in the cases $\tilde{\kappa} \neq 1$. If $\tilde{\kappa} = 1$, we shall assume

$$M = \mathbb{S}^{n-1} \tag{4.2.3}$$

which is of course the important case. By assuming M to be compact, we can use eigenfunctions instead of eigendistributions when we consider spatial eigenvalue problems.

The radial variable r ranges between

$$0 < r \leq r_0, \tag{4.2.4}$$

where r_0 is the radius of the *unique* event horizon.

The interior region of the black hole is a globally hyperbolic $(n+1)$-dimensional spacetime and the hypersurfaces

$$S_r = \{r = \text{const} < r_0\} \tag{4.2.5}$$

are Cauchy hypersurfaces with induced metric

$$ds^2 = \tilde{h}dt^2 + r^2\sigma_{ij}dx^i dx^j, \tag{4.2.6}$$

where

$$-\infty < t < \infty. \tag{4.2.7}$$

Note that $r = \text{const}$ and hence

$$0 < \tilde{h} = \text{const}. \tag{4.2.8}$$

The coordinate transformation

$$\tau = \tilde{h}^{\frac{1}{2}}t \tag{4.2.9}$$

yields

$$ds^2 = d\tau^2 + r^2\sigma_{ij}dx^i dx^j, \tag{4.2.10}$$

where $\tau \in \mathbb{R}$. Since we have removed the coordinate singularity, we can now let r converge to r_0 such the resulting manifold $\mathcal{S}_0$ represents the event horizon topologically but with different metric. However, by a slight abuse of language, we shall call $\mathcal{S}_0$ to be a Cauchy hypersurface though it is only the geometric limit of Cauchy hypersurfaces.

However, $\mathcal{S}_0$ is a genuine Cauchy hypersurface in the quantum model which is defined by the equation (4.1.1) on page 99.

Let us now look at the stationary eigenvalue equation

$$-(n-1)\Delta v - \frac{n}{2}Rv = \mu v \tag{4.2.11}$$

in $\mathcal{S}_0$, where

$$-(n-1)\Delta v = -(n-1)\ddot{v} - (n-1)r_0^{-2}\tilde{\Delta}v. \tag{4.2.12}$$

Using separation of variables, let us write

$$v(\tau, x) = \zeta(\tau)\varphi(x) \tag{4.2.13}$$

to conclude that the left-hand side of (4.2.11) can be expressed in the form

$$-(n-1)\ddot{\zeta}\varphi + \zeta\{-(n-1)r_0^{-2}\tilde{\Delta}\varphi - \frac{n}{2}(n-1)(n-2)r_0^{-2}\tilde{\kappa}\varphi\}, \tag{4.2.14}$$

since the scalar curvature R of the metric (4.2.10) is

$$R = (n-1)(n-2)r_0^{-2}\tilde{\kappa}. \tag{4.2.15}$$

Hence, the eigenvalue problem (4.2.11) can be solved by setting

$$v = \zeta\varphi_j, \tag{4.2.16}$$

where φ_j, $j \in \mathbb{N}$, is an eigenfunction of $-\tilde{\Delta}$ such that

$$-\tilde{\Delta}\varphi_j = \tilde{\mu}_j\varphi_j, \tag{4.2.17}$$

$$0 = \tilde{\mu}_0 < \tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots \tag{4.2.18}$$

and ζ is an eigenfunction of the harmonic oscillator. The eigenvalue of the harmonic oscillator can be arbitrarily positive or zero. We define it at the moment as

$$r_0^{-2}\omega^2 \tag{4.2.19}$$

where $\omega \ge 0$ will be determined later. For $\omega > 0$, we shall consider the real eigenfunction

$$\zeta = \sin r_0^{-1}\omega\tau \tag{4.2.20}$$

which represents the ground state in the interval

$$I_0 = (0, \frac{\pi}{r_0^{-1}\omega}) \tag{4.2.21}$$

with vanishing boundary values. ζ is a solution of the variational problem

$$\frac{\int_{I_0}|\dot{\vartheta}|^2}{\int_{I_0}|\vartheta|^2} \to \min \quad \forall\, 0 \ne \vartheta \in H_0^{1,2}(I_0) \tag{4.2.22}$$

in the Sobolev space $H_0^{1,2}(I_0)$.

Multiplying ζ by a constant, we may assume

$$\int_{I_0} |\zeta|^2 = 1. \tag{4.2.23}$$

Obviously,

$$\mathcal{S}_0 = \mathbb{R} \times M \tag{4.2.24}$$

and though ζ is defined in $\mathbb{R}$ and is even an eigenfunction, it has infinite norm in $L^2(\mathbb{R})$. However, when we consider a finite disjoint union of N open intervals I_j

$$\Omega = \bigcup_{j=1}^{N} I_j, \tag{4.2.25}$$

where

$$I_j = \left(k_j \frac{\pi}{r_0^{-1}\omega}, (k_j + 1)\frac{\pi}{r_0^{-1}\omega} \right), \quad k_j \in \mathbb{Z}, \tag{4.2.26}$$

then

$$\zeta_N = N^{-\frac{1}{2}}\zeta \tag{4.2.27}$$

is a unit eigenfunction in Ω with vanishing boundary values having the same energy as ζ in I_0. Hence, it suffices to consider ζ only in I_0 since this configuration can immediately be generalized to arbitrarily large bounded open intervals

$$\Omega \subset \mathbb{R}. \tag{4.2.28}$$

We then can state:

Lemma 4.2.1 *There exists a complete sequence of unit eigenfunctions φ_j of $-\tilde{\Delta}$ with eigenvalues $\tilde{\mu}_j$ such that the functions*

$$v_j = \zeta\varphi_j, \tag{4.2.29}$$

where ζ is a constant multiple of the function in (4.2.20) with unit $L^2(I_0)$ norm, are solutions of the eigenvalue problem (4.2.11) with eigenvalue

$$\mu_j = (n-1)r_0^{-2}\omega^2 + (n-1)r_0^{-2}\tilde{\mu}_j - \frac{n}{2}(n-1)(n-2)r_0^{-2}\tilde{\kappa}. \tag{4.2.30}$$

The eigenfunctions v_j are mutually orthogonal in $L^2(I_0 \times M, \mathbb{C})$. The eigenvector

$$v_0 = \zeta\varphi_0, \quad \varphi_0 = \text{const}, \tag{4.2.31}$$

is a ground state with spatial energy

$$(n-1)\int_{I_0\times M}|Dv_0|^2=(n-1)\int_{I_0}|\dot{\zeta}|^2|\varphi_0|^2=(n-1)r_0^{-2}\omega^2|\varphi_0|^2. \tag{4.2.32}$$

The energy of the stationary Hamiltonian, i.e., the operator on the left-hand side of (4.2.11)*, evaluated at an eigenfunction v_j is equal to the eigenvalue μ_j in* (4.2.30).

To solve the wave equation (4.1.1) on page 99, let us first consider the following eigenvalue problem

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w}+n|\Lambda|t^2w=\lambda t^{2-\frac{4}{n}}w \tag{4.2.33}$$

in the Sobolev space

$$H_0^{1,2}(\mathbb{R}_+^*). \tag{4.2.34}$$

Here,

$$\Lambda<0 \tag{4.2.35}$$

can in principle be an arbitrary negative parameter, but in the case of an AdS black hole, it seems reasonable to choose the cosmological constant of the AdS spacetime. However, if the cosmological constant is equal to zero, i.e., if we consider a pure Schwarzschild spacetime, then we have either to pick an arbitrary negative constant, if we still want to consider an explicit eigenvalue problem, or we have to consider an implicit eigenvalue problem, where Λ plays the role of an eigenvalue, cf. [18, Theorem 6.7], [19, eq. (7.9)] or Theorem 1.6.7 on page 49. Since our stationary Hamiltonian comprises a harmonic oscillator, the frequency of which is still at our disposal, we would consider an explicit eigenvalue problem with a fixed negative Λ, e.g.,

$$\Lambda=-1 \tag{4.2.36}$$

if we wanted to quantize a Schwarzschild black hole.

The eigenvalue problem (4.2.33) has already been solved in the previous chapter, cf. Theorem 3.4.1 on page 96. Let us summarize the results: define the Hilbert space $\mathcal{H}$ to be the completion of

$$C_c^\infty(\mathbb{R}_+^*,\mathbb{C}) \tag{4.2.37}$$

with respect to the bilinear form

$$B(w,\tilde{w})=\int_{\mathbb{R}_+^*}\left\{\frac{1}{32}\frac{n^2}{n-1}\bar{w}'\tilde{w}'+n|\Lambda|t^2\bar{w}\tilde{w}\right\}, \tag{4.2.38}$$

then we have proved:

Theorem 4.2.2 *The eigenvalue problem* (4.2.33) *has countably many solutions* (w_i,λ_i) *such that*

$$0<\lambda_0<\lambda_1<\lambda_2<\cdots, \tag{4.2.39}$$

$$\lim \lambda_i = \infty, \tag{4.2.40}$$

and

$$\int_{\mathbb{R}_+^*} \bar{w}_i w_j = \delta_{ij}. \tag{4.2.41}$$

The w_i are complete in $\mathcal{H}$ as well as in $L^2(\mathbb{R}_+^*)$.

We are now ready to define the solutions of the wave equation (4.1.1).

Theorem 4.2.3 *Let $(\varphi_j, \tilde{\mu}_j)$ resp. (w_i, λ_i) be eigenfunctions of*

$$-\tilde{\Delta} = -\Delta_M \tag{4.2.42}$$

resp. the temporal eigenfunctions and set

$$\hat{\mu}_j = (n-1)r_0^{-2}\tilde{\mu}_j - \frac{n}{2}(n-1)(n-2)r_0^{-2}\tilde{\kappa}. \tag{4.2.43}$$

Let λ_{i_0} be the smallest eigenvalue of the (λ_i) with the property

$$\lambda_{i_0} \ge \hat{\mu}_j, \tag{4.2.44}$$

then, for any $i \ge i_0$, there exists

$$\omega = \omega_{ij} \ge 0 \tag{4.2.45}$$

and corresponding ζ_{ij} satisfying

$$-\ddot{\zeta}_{ij} = r_0^{-2}\omega_{ij}^2\zeta_{ij} \tag{4.2.46}$$

such that

$$\lambda_i = \mu_{ij} = (n-1)r_0^{-2}\omega_{ij}^2 + \hat{\mu}_j \quad \forall\, i \ge i_0. \tag{4.2.47}$$

The functions

$$u_{ij} = w_i\zeta_{ij}\varphi_j \tag{4.2.48}$$

are then solutions of the wave equation with bounded energies satisfying

$$\lim_{t\to 0} u_{ij}(t) = \lim_{t\to\infty} u_{ij}(t) = 0 \tag{4.2.49}$$

and

$$u_{ij} \in C^\infty(\mathbb{R}_+^* \times \mathcal{S}_0) \cap C^{2,\alpha}(\bar{\mathbb{R}}_+^* \times \mathcal{S}_0) \tag{4.2.50}$$

for some

$$\frac{2}{3} \le \alpha < 1. \tag{4.2.51}$$

Moreover, we have

$$\omega_{ij} > 0 \qquad \forall\, i > i_0. \tag{4.2.52}$$

If

$$\lambda_{i_0} = \hat{\mu}_j, \tag{4.2.53}$$

then we define

$$\zeta_{i_0 j} \equiv 1. \tag{4.2.54}$$

In case $j = 0$ *and* $\tilde{\kappa} \neq -1$ *we always have*

$$\hat{\mu}_0 \leq 0 \tag{4.2.55}$$

and

$$\varphi_0 = \text{const} \neq 0 \tag{4.2.56}$$

and hence

$$\omega_{i0} > 0 \qquad \forall\, i \geq 0. \tag{4.2.57}$$

Proof The proof is obvious.

□

Remark 4.2.4 (i) By construction, the temporal and spatial energies of the solutions of the wave equation have to be equal.

(ii) The stationary solutions comprising a harmonic oscillator can be looked at a being radiation while we consider the temporal solutions to be gravitational waves.

(iii) If one wants to replace the bounded Interval I_0 by $\mathbb{R}$, then the eigenfunctions ζ_{ij} have to be replaced by eigendistributions. An appropriate choice would be

$$\zeta_{ij} = e^{i r_0^{-1} \omega_{ij} \tau}. \tag{4.2.58}$$

The hyperbolic operator defined by the wave equation (4.1.1) on page 99 can be defined in the spacetime

$$Q = \mathbb{R}_+^* \times \mathcal{S}_0 \tag{4.2.59}$$

which can be equipped with the Lorentzian metrics

$$d\bar{s}^2 = -\frac{32(n-1)}{n^2} dt^2 + g_{ij} dx^i dx^j \tag{4.2.60}$$

as well as with the metric

$$d\tilde{s}^2 = -\frac{32(n-1)}{n^2} dt^2 + \frac{1}{n-1} t^{\frac{4}{n}-2} g_{ij} dx^i dx^j, \tag{4.2.61}$$

where g_{ij} is the metric defined on $\mathcal{S}_0$ and the indices now have the range $1 \leq i, j \leq n$. In both metrics, Q is globally hyperbolic provided $\mathcal{S}_0$ is complete, which is the case for the metric defined in (4.2.10). The hyperbolic operator is symmetric in the first metric but not normally hyperbolic while it is normally hyperbolic but not symmetric in the second metric. Normally hyperbolic means that the main part of the operator is identical to the Laplacian of the spacetime metric.

Hence, if we want to describe quantum gravity not only by an equation but also by the metric of a spacetime, then the metric in (4.2.61) has to be chosen. In this metric, Q has a curvature singularity in $t = 0$, cf. Remark 2.5.3 on page 74. The Cauchy hypersurface $\mathcal{S}_0$ then corresponds to the hypersurface

$$\{t = 1\} \tag{4.2.62}$$

which also follows from the derivation of the quantum model where we consider a fiber bundle E with base space $\mathcal{S}_0$ and the elements of the fibers were Riemann metrics of the form

$$g_{ij}(t, x) = t^{\frac{4}{n}} \sigma_{ij}(x) \tag{4.2.63}$$

where σ_{ij} were metrics defined in $\mathcal{S}_0$ and t is the time coordinate that we use in Q, i.e.,

$$g_{ij}(1, x) = \sigma_{ij}(x). \tag{4.2.64}$$

In the present situation, we used the symbol g_{ij} to denote the metric on $\mathcal{S}_0$ since σ_{ij} is supposed to be the metric of the spaceform M.

Thus, the event horizon is characterized by the Cauchy hypersurface

$$\{t = 1\} \tag{4.2.65}$$

and obviously, we shall assume that the black hole singularity

$$\{r = 0\} \tag{4.2.66}$$

corresponds to the curvature singularity

$$\{t = 0\} \tag{4.2.67}$$

of Q; i.e., the open black hole region is described in the quantum model by

$$(0, 1) \times \mathcal{S}_0 \tag{4.2.68}$$

and the open exterior region by

$$(1, \infty) \times \mathcal{S}_0. \tag{4.2.69}$$

Stipulating that the time orientation in the quantum model should be the same as in the AdS spacetime we conclude that the curvature singularity $t = 0$ is a future singularity; i.e., the present time function is not future directed. To obtain a future directed coordinate system, we have to choose t negative, i.e.,

$$Q = (-\infty, 0) \times \mathcal{S}_0. \tag{4.2.70}$$

In the metric (4.2.61), we then have to replace

$$t^{\frac{4}{n}-2} \tag{4.2.71}$$

by

$$|t|^{\frac{4}{n}-2} \tag{4.2.72}$$

and similarly in the wave equation, which is then invariant with respect to the reflection

$$t \to -t. \tag{4.2.73}$$

As a final remark in this section, let us state:

Remark 4.2.5 In the quantum model of the black hole, the event horizon is a regular Cauchy hypersurface and can be crossed in both directions by causal curves; hence, no information paradox can occur.

4.3 Transition from the Black Hole to the White Hole

We shall choose the time variable t negative to have a future-oriented coordinate system. The quantum model of the white hole will then be described by a positive t variable and the transition from black to white hole would be future oriented. Obviously, we only have to consider the temporal eigenvalue equation (4.2.33) on page 106 to define a transition, where of course (4.2.72) on page 110 and its inverse relation have to be observed.

Since the coefficients of the ODE in (4.2.33) are at least Hölder continuous in $\mathbb{R}$, a solution w defined on the negative axis has a natural extension to $\mathbb{R}$ since we know that $w(0) = 0$. Denote the fully extended function by w too, then

$$w \in C^{2,\alpha}(\mathbb{R}), \tag{4.3.1}$$

where we now, without loss of generality, only consider a real solution.

Theorem 4.3.1 *A naturally extended solution w of the temporal eigenvalue equation* (4.2.33) *is antisymmetric in t,*

$$w(-t) = -w(t) \tag{4.3.2}$$

and the restriction of w to the positive axis is also a variational solution as defined in Theorem 3.4.1 *on page 96.*

Proof It suffices to prove (4.3.2). Let $t > 0$ and define

$$\tilde{w}(t) = -w(-t), \tag{4.3.3}$$

then $\tilde{w}$ solves the ODE in $\mathbb{R}_+^*$ and

$$\tilde{w}(0) = w(0) = 0 \tag{4.3.4}$$

as well as

$$\dot{\tilde{w}}(0) = \dot{w}(0), \tag{4.3.5}$$

hence, we deduce

$$\tilde{w}(t) = w(t) \qquad \forall t > 0 \tag{4.3.6}$$

because the solutions of a second-order ODE are uniquely determined by the initial values of the function and its derivative. □

Remark 4.3.2 This transition result is also valid in the general case when the curvature singularity in $t = 0$ does not necessarily correspond to the singularity of a black or white hole. The quantum evolution of any Cauchy hypersurface $\mathcal{S}_0$ in a globally hyperbolic spacetime will always have a curvature singularity either in the past or in the future of $\mathcal{S}_0$ and the evolution can be extended past this singularity. Note also that the quantum Lorentzian distance to that past or future singularity is always finite.

Chapter 5
The Quantization of a Kerr-AdS Black Hole

5.1 Rotating Black Holes

In the previous chapter, we looked at the quantum development of a Schwarzschild-AdS spacetime and derived a spectral resolution of the underlying wave equation

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0 \tag{5.1.1}$$

defined in the spacetime

$$Q = (0, \infty) \times \mathcal{S}_0, \tag{5.1.2}$$

where $\mathcal{S}_0 = (\mathcal{S}_0, g_{ij})$ is a Cauchy hypersurface. In the present chapter, we want to show that similar arguments can also be used to quantize a Kerr-AdS spacetime with a rotating black hole.

We consider an odd-dimensional Kerr-AdS spacetime N, $\dim N = 2m + 1$, $m \ge 2$, where all rotational parameters are equal

$$a_i = a \qquad \forall\, 1 \le i \le m, \tag{5.1.3}$$

and where we also set

$$n = 2m. \tag{5.1.4}$$

Replacing the r coordinate in a generalized Boyer-Lindquist coordinate system by

$$\rho = r^2 \tag{5.1.5}$$

we shall prove that in the new coordinate system the metric is smooth in the interval

$$-a^2 < \rho < \infty \tag{5.1.6}$$

C. Gerhardt, *The Quantization of Gravity*, Fundamental Theories of Physics 194,
https://doi.org/10.1007/978-3-319-77371-1_5

and that the extended spacetime N has a timelike curvature singularity in $\rho = -a^2$, cf. Lemma 5.3.5 on page 129.

For the quantization, we first assume that there is a non-empty interior black hole region B which is bounded by two horizons

$$B = \{r_1 < r < r_2\} \tag{5.1.7}$$

where the outer horizon is the event horizon. Picking a Cauchy hypersurface in B of the form

$$\{r = \text{const}\}, \tag{5.1.8}$$

we shall prove that the induced metric of the Cauchy hypersurface can be expressed in the form

$$ds^2 = d\tau^2 + \sigma_{ij}dx^i dx^j, \tag{5.1.9}$$

where

$$-\infty < \tau < \infty, \tag{5.1.10}$$

$r =$ const and σ_{ij} is a smooth Riemannian metric on $\mathbb{S}^{2m-1}$ depending on r, a and the cosmological constant $\Lambda < 0$. The metric in (5.1.9) is free of any coordinate singularity; hence, we can let r tend to r_2 such that the Cauchy hypersurfaces converge to a Riemannian manifold $\mathcal{S}_0$ which represents the event horizon at least topologically. Furthermore, the Laplacian of the metric in (5.1.9) comprises a harmonic oscillator with respect to τ which enables us to write the stationary eigenfunctions v_j in the form

$$v_j(\tau, x) = \zeta(\tau)\varphi_j(x), \tag{5.1.11}$$

where φ_j is an eigenfunction of the elliptic operator

$$-(n-1)\tilde{\Delta} - \frac{n}{2}R, \tag{5.1.12}$$

where

$$\tilde{\Delta} = \Delta_M, \tag{5.1.13}$$

$M = (\mathbb{S}^{2m-1}, \sigma_{ij})$, and ζ an eigenfunction of the harmonic oscillator the frequency of which is still to be determined.

Due to the presence of the harmonic oscillator, we can now consider an *explicit* temporal eigenvalue problem; i.e., we consider the eigenvalue problem

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n|\Lambda|t^2 w = \lambda t^{2-\frac{4}{n}} w \tag{5.1.14}$$

with a fixed $\Lambda < 0$, where we choose Λ to be the cosmological constant of the Kerr-AdS spacetime.

The eigenvalue problem (5.1.14) has a complete sequence (w_i, λ_i) of eigenfunctions with finite energies λ_i such that

$$0 < \lambda_0 < \lambda_1 < \cdots \tag{5.1.15}$$

and by choosing the frequencies of ζ appropriately we can arrange that the stationary eigenvalues μ_j of v_j agree with the temporal eigenvalues λ_i. If this is the case, then the eigenfunctions

$$u = w_i v_j \tag{5.1.16}$$

will be a solution of the wave equation. More precisely, we shall prove:

Theorem 5.1.1 *Let $(\varphi_j, \tilde{\mu}_j)$ resp. (w_i, λ_i) be eigenfunctions of the elliptic operator in (5.1.12) resp. the temporal eigenfunctions and, for a given index j, let λ_{i_0} be the smallest eigenvalue of the (λ_i) with the property*

$$\lambda_{i_0} \geq \tilde{\mu}_j, \tag{5.1.17}$$

then, for any $i \geq i_0$, there exists

$$\omega = \omega_{ij} \geq 0 \tag{5.1.18}$$

and corresponding ζ_{ij} satisfying

$$-\ddot{\zeta}_{ij} = \omega_{ij}^2 \zeta_{ij} \tag{5.1.19}$$

such that

$$\lambda_i = \mu_{ij} = (\omega - 1)\omega_{ij}^2 + \tilde{\mu}_j \quad \forall i \geq i_0. \tag{5.1.20}$$

The functions

$$u_{ij} = w_i \zeta_{ij} \varphi_j \tag{5.1.21}$$

are then solutions of the wave equation with bounded energies satisfying

$$\lim_{t \to 0} u_{ij}(t) = \lim_{t \to \infty} u_{ij}(t) = 0 \tag{5.1.22}$$

and

$$u_{ij} \in C^\infty(\mathbb{R}_+^* \times \mathcal{S}_0) \cap C^{2,\alpha}(\bar{\mathbb{R}}_+^* \times \mathcal{S}_0) \tag{5.1.23}$$

for some

$$\frac{2}{3} \leq \alpha < 1. \tag{5.1.24}$$

Moreover, we have

$$\omega_{ij} > 0 \quad \forall i > i_0. \tag{5.1.25}$$

If

$$\lambda_{i_0} = \tilde{\mu}_j, \tag{5.1.26}$$

then we define

$$\zeta_{i_0 j} \equiv 1. \tag{5.1.27}$$

Remark 5.1.2 (i) The event horizon corresponds to the Cauchy hypersurface $\{t = 1\}$ in Q and the open set

$$\{-a^2 < \rho < \rho_2\} \tag{5.1.28}$$

in N, where

$$\rho_2 = r_2^2, \tag{5.1.29}$$

to the region

$$(0, 1) \times \mathcal{S}_0, \tag{5.1.30}$$

while the part

$$\{\rho_2 < \rho < \infty\} \tag{5.1.31}$$

is represented by

$$(1, \infty) \times \mathcal{S}_0. \tag{5.1.32}$$

The *timelike* black hole singularity corresponds to $\{t = 0\}$ which is a *spacelike* curvature singularity in the quantum spacetime provided we equip Q with a metric such that the hyperbolic operator is normally hyperbolic, cf. Remark 2.5.3 on page 74. Moreover, in the quantum spacetime, the Cauchy hypersurface $\mathcal{S}_0$ can be crossed by causal curves in both directions; i.e., the information paradox does not occur.

(ii) The stationary eigenfunctions can be looked at as being radiation because they comprise the harmonic oscillator, while we consider the temporal eigenfunctions to be gravitational waves.

The metric describing a rotating black hole in a four-dimensional vacuum spacetime was first discovered by Kerr [34]. Carter [4] generalized the Kerr solution by describing a rotating black hole in a four-dimensional de Sitter or anti-de Sitter background. Higher dimensional solutions for a rotating black hole were given by Myers and Perry [39] in even-dimensional Ricci flat spacetimes and by Hawking, Hunter and Taylor [31] in five-dimensional spacetimes satisfying the Einstein equations with cosmological constant.

A general solution in all dimension was given in [25] by Gibbons et al., and we shall use their metric in odd dimensions, with all rotational parameters supposed to be equal, to define our spacetime N, though we shall maximally extend it.

5.1.1 Notations

We apply the summation convention and label coordinates with contravariant indices, e.g., μ^i. However, for better readability, we shall usually write

$$\mu_i^2 \tag{5.1.33}$$

instead of

$$(\mu^i)^2. \tag{5.1.34}$$

5.2 Preparations

We consider odd-dimensional Kerr-AdS spacetimes N, $\dim N = 2m + 1$, $m \geq 2$, assuming that all rotational parameters are equal

$$a_i = a \neq 0, \qquad \forall\, 1 \leq i \leq m. \tag{5.2.1}$$

The Kerr-Schild form of the metric can then be expressed as

$$\begin{aligned} d\bar{s}^2 = &-\frac{1 + l^2r^2}{1 - a^2l^2}dt^2 + \frac{r^2dr^2}{(1 + l^2r^2)(r^2 + a^2)} \\ &+ \frac{r^2 + a^2}{1 - a^2l^2}\sum_{i=1}^{m}(d\mu_i^2 + \mu_i^2 d\varphi_i^2) \\ &+ \frac{2m_0}{U}\Big(\frac{1}{1 - a^2l^2}(dt - a\sum_{i=1}^{m}\mu_i^2 d\varphi^i) + \frac{r^2dr}{(1 + l^2r^2)(r^2 + a^2)}\Big)^2, \end{aligned} \tag{5.2.2}$$

where

$$l^2 = -\frac{1}{m(2m - 1)}\Lambda \tag{5.2.3}$$

and $\Lambda < 0$ is the cosmological constant such that the Einstein equations

$$G_{\alpha\beta} + \Lambda\bar{g}_{\alpha\beta} = 0 \tag{5.2.4}$$

are satisfied in N, m_0 is the mass of the black hole,

$$U = (r^2 + a^2)^{m-1}, \tag{5.2.5}$$

$$\sum_{i=1}^{m}(d\mu_i^2 + \mu_i^2 d\varphi_i^2) \tag{5.2.6}$$

is the standard metric of $\mathbb{S}^{2m-1}$, where the φ^i are the azimuthal coordinates, the values of which have to be identified modulo 2π, and the μ^i are the latitudinal coordinates subject to the side condition

$$\sum_{i=1}^{m} \mu_i^2 = 1. \tag{5.2.7}$$

The μ^i also satisfy

$$0 \le \mu^i \le 1 \qquad \forall\, 1 \le i \le m. \tag{5.2.8}$$

The coordinates (t, r) are defined in

$$-\infty < t < \infty \tag{5.2.9}$$

and

$$0 < r < \infty \tag{5.2.10}$$

respectively, cf. [25, Sect. 2 and Appendix B].

The horizons are hypersurfaces $\{r = \text{const}\}$, where $\rho = r^2$ satisfies the equation

$$(1 + l^2\rho)(\rho + a^2)^m - 2m_0\rho = 0. \tag{5.2.11}$$

Let

$$\Phi = \Phi(\rho) \tag{5.2.12}$$

be the polynomial on the left-hand side of (5.2.11), then Φ is strictly convex in $\mathbb{R}_+$ and we have

$$\Phi(0) > 0 \tag{5.2.13}$$

and

$$\lim_{\rho \to \infty} \Phi(\rho) = \infty, \tag{5.2.14}$$

from which we deduce that the Eq. (5.2.11) is satisfied if and only if

$$\inf_{\mathbb{R}_+} \Phi \le 0, \tag{5.2.15}$$

and in case

$$\inf_{\mathbb{R}_+} \Phi < 0 \tag{5.2.16}$$

we have exactly two solutions otherwise only one. If there are two solutions r_i, $i = 1, 2$, such that

$$0 < r_1 < r_2, \tag{5.2.17}$$

then the outer horizon is called *event horizon* and the black hole has an interior region

$$B = \{r_1 < r < r_2\} \tag{5.2.18}$$

in which the variable r is a time coordinate. If there is only one solution r_0, then B is empty and the black hole is called *extremal*.

We shall first quantize a black hole with $B \neq \emptyset$; the quantization of an extremal black hole is then achieved by approximation.

Thus, let us consider a non-extremal black hole and let $S \subset B$ be a spacelike coordinate slice

$$S = S(r) = \{r = \text{const}\}, \tag{5.2.19}$$

where r also denotes the constant value.

In view of (5.2.2), the induced metric can be expressed as

$$\begin{aligned} ds_S^2 = &\Big(\frac{2m_0}{U}\frac{1}{(1-a^2l^2)^2} - \frac{1+l^2r^2}{1-a^2l^2}\Big)dt^2 - \frac{2m_0}{U}\frac{2a}{(1-a^2l^2)^2}\mu_i^2 dt d\varphi^i \\ &+ \Big(\frac{2m_0}{U}\frac{a^2}{(1-a^2l^2)^2}\mu_i^2\mu_j^2 + \frac{r^2+a^2}{1-a^2l^2}\mu_i^2\delta_{ij}\Big)d\varphi^i d\varphi^j \\ &+ \frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^m d\mu_i^2, \end{aligned} \tag{5.2.20}$$

from which we deduce

$$g_{tt} = \frac{2m_0}{U}\frac{1}{(1-a^2l^2)^2} - \frac{1+l^2r^2}{1-a^2l^2}, \tag{5.2.21}$$

$$g_{t\varphi^i} = g_{\varphi^i t} = -\frac{2m_0}{U}\frac{a}{(1-a^2l^2)^2}\mu_i^2, \tag{5.2.22}$$

and

$$g_{\varphi^i\varphi^j} = \frac{2m_0}{U}\frac{a^2}{(1-a^2l^2)^2}\mu_i^2\mu_j^2 + \frac{r^2+a^2}{1-a^2l^2}\mu_i^2\delta_{ij}. \tag{5.2.23}$$

The other components of the metric are either 0 or are represented by the line element

$$\frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^m d\mu_i^2, \tag{5.2.24}$$

note the constraint (5.2.7).

To eliminate the $g_{t\varphi^i}$, we shall introduce new coordinates. First, let us make the simple change by defining t' through

$$ct' = t, \tag{5.2.25}$$

where $c \neq 0$ is a constant which will be specified later, and dropping the prime in the sequel, resulting in a replacement of the components in (5.2.21) and (5.2.22) by

$$g_{tt} = c^2\Big(\frac{2m_0}{U}\frac{1}{(1-a^2l^2)^2} - \frac{1+l^2r^2}{1-a^2l^2}\Big) \tag{5.2.26}$$

respectively,

$$g_{t\varphi^i} = g_{\varphi^i t} = -c\frac{2m_0}{U}\frac{a}{(1-a^2l^2)^2}\mu_i^2. \tag{5.2.27}$$

Next, we define new coordinates $(\tilde{t}, \tilde{\varphi}^i)$ by

$$\alpha\tilde{t} = t \tag{5.2.28}$$

and

$$\tilde{\varphi}^i = \varphi^i - a\gamma t, \tag{5.2.29}$$

where α, γ are non-vanishing constants to specified later, such that

$$\varphi^i = \tilde{\varphi}^i + a\alpha\gamma\tilde{t}. \tag{5.2.30}$$

In the new coordinates, the only interesting new components are

$$\begin{aligned} g_{\tilde{t}\tilde{t}} &= g_{tt}\frac{\partial t}{\partial\tilde{t}}\frac{\partial t}{\partial\tilde{t}} + 2g_{t\varphi^i}\frac{\partial t}{\partial\tilde{t}}\frac{\partial\varphi^i}{\partial\tilde{t}} + g_{\varphi^i\varphi^j}\frac{\partial\varphi^i}{\partial\tilde{t}}\frac{\partial\varphi^j}{\partial\tilde{t}} \\ &= \alpha^2\big(g_{tt} + 2a\gamma\sum_i g_{t\varphi^i} + a^2\gamma^2\sum_{i,j} g_{\varphi^i\varphi^j}\big) \end{aligned} \tag{5.2.31}$$

and

$$\begin{aligned} g_{\tilde{t}\tilde{\varphi}^i} &= g_{t\varphi^j}\frac{\partial t}{\partial\tilde{t}}\frac{\partial\varphi^j}{\partial\tilde{\varphi}^i} + g_{\varphi^k\varphi^l}\frac{\partial\varphi^k}{\partial\tilde{t}}\frac{\partial\varphi^l}{\partial\tilde{\varphi}^i} \\ &= \alpha\big(g_{t\varphi^i} + a\gamma\sum_k g_{\varphi^k\varphi^i}\big). \end{aligned} \tag{5.2.32}$$

We therefore deduce, in view of (5.2.23), (5.2.26) and (5.2.27),

$$g_{\tilde{t}\tilde{t}} = \alpha^2\Big(\frac{2m_0}{U}\frac{1}{(1-a^2l^2)^2}(c-a^2\gamma)^2 + \frac{r^2+a^2}{1-a^2l^2}a^2\gamma^2 - \frac{1+l^2r^2}{1-a^2l^2}c^2\Big) \tag{5.2.33}$$

and

$$g_{\tilde{t}\tilde{\varphi}^i} = \alpha\Big(\frac{2m_0}{U}\frac{a}{(1-a^2l^2)^2}(a^2\gamma - c) + \frac{r^2+a^2}{1-a^2l^2}a\gamma\Big)\mu_i^2. \tag{5.2.34}$$

Choosing now

$$c = \big(a^2 + \frac{U}{2m_0}(r^2 + a^2)(1 - a^2 l^2)\big)\gamma \tag{5.2.35}$$

we conclude

$$g_{\tilde{t}\tilde{\varphi}^i} = 0. \tag{5.2.36}$$

Combining then (5.2.35) and (5.2.33) by setting $\gamma = 1$, we obtain

$$\begin{aligned} g_{\tilde{t}\tilde{t}} &= \alpha^2\big(\frac{U}{2m_0}(r^2 + a^2)^2 + \frac{r^2 + a^2}{1 - a^2 l^2} a^2 \\ &\quad - \frac{1 + l^2 r^2}{1 - a^2 l^2}(a^2 + \frac{U}{2m_0}(r^2 + a^2)(1 - a^2 l^2))^2\big). \end{aligned} \tag{5.2.37}$$

Define

$$\beta = \frac{U}{2m_0}(r^2 + a^2) - \frac{r^2}{1 + l^2 r^2}, \tag{5.2.38}$$

then

$$\beta < 0 \qquad \text{in } B, \tag{5.2.39}$$

since the function Φ in (5.2.12) is negative in B. Writing

$$\begin{aligned} a^2 + \frac{U}{2m_0}(r^2 + a^2)(1 - a^2 l^2) &= a^2 + \frac{r^2}{1 + l^2 r^2}(1 - a^2 l^2) \\ &\quad + \beta(1 - a^2 l^2) \\ &= \frac{r^2 + a^2}{1 + l^2 r^2} + \beta(1 - a^2 l^2), \end{aligned} \tag{5.2.40}$$

we infer

$$g_{\tilde{t}\tilde{t}} = \alpha^2(-\beta(r^2 + a^2) - \beta^2(1 + l^2 r^2)(1 - a^2 l^2)). \tag{5.2.41}$$

The term in the brackets vanishes on the event horizon and is strictly positive in B, in view of (5.2.39) and the identity

$$\begin{aligned} &(r^2 + a^2) + \beta(1 + l^2 r^2)(1 - a^2 l^2) \\ &= (r^2 + a^2) + \frac{(r^2 + a^2)^m}{2m_0}(1 + l^2 r^2)(1 - a^2 l^2) - r^2(1 - a^2 l^2) \\ &= a^2(1 + l^2) + \frac{(r^2 + a^2)^m}{2m_0}(1 + l^2 r^2)(1 - a^2 l^2) > 0. \end{aligned} \tag{5.2.42}$$

Hence, for any r satisfying

$$r_1 < r < r_2 \tag{5.2.43}$$

we can choose $\alpha > 0$ such that

$$g_{\tilde{t}\tilde{t}} = 1. \tag{5.2.44}$$

Writing (τ, φ^i) instead of $(\tilde{t}, \tilde{\varphi}^i)$, we can then state

Lemma 5.2.1 *For any hypersurface*

$$S = S(r) \subset B \tag{5.2.45}$$

the induced metric can be expressed in the form

$$\begin{aligned} ds_S^2 &= d\tau^2 + \Big(\frac{2m_0}{U}\frac{a^2}{(1-a^2l^2)^2}\mu_i^2\mu_j^2 + \frac{r^2+a^2}{1-a^2l^2}\mu_i^2\delta_{ij}\Big)d\varphi^i d\varphi^j \\ &\quad + \frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^m d\mu_i^2 \\ &\equiv d\tau^2 + \sigma_{ij}dx^i dx^j, \end{aligned} \tag{5.2.46}$$

where

$$\sigma_{ij} = \sigma_{ij}(r, a, l) \tag{5.2.47}$$

is a smooth Riemannian metric on $\mathbb{S}^{2m-1}$ *and* τ *ranges in* $\mathbb{R}$, *while in case*

$$S = S(r) \subset N \backslash \bar{B}, \tag{5.2.48}$$

the induced metric is Lorentzian of the form

$$\begin{aligned} ds_S^2 &= -d\tau^2 + \Big(\frac{2m_0}{U}\frac{a^2}{(1-a^2l^2)^2}\mu_i^2\mu_j^2 + \frac{r^2+a^2}{1-a^2l^2}\mu_i^2\delta_{ij}\Big)d\varphi^i d\varphi^j \\ &\quad + \frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^m d\mu_i^2 \\ &\equiv -d\tau^2 + \sigma_{ij}dx^i dx^j. \end{aligned} \tag{5.2.49}$$

If $r < r_2$ *tends to* r_2, *then the hypersurfaces* $S(r)$ *converge topologically to the event horizon and the induced metrics to the Riemannian metric*

$$\begin{aligned} ds_S^2 &= d\tau^2 + \frac{r_2^2+a^2}{1-a^2l^2}\big(\delta_{ij}d\mu^i d\mu^j + \mu_i^2\delta_{ij}d\varphi^i d\varphi^j\big) \\ &\quad + a^2\frac{(1+l^2r_2^2)(r_2^2+a^2)}{r_2^2(1-a^2l^2)^2}\mu_i^2\mu_j^2 d\varphi^i d\varphi^j. \end{aligned} \tag{5.2.50}$$

Proof We only have to prove the case (5.2.48). However, the proof of this case is identical to the proof when (5.2.45) is valid by observing that then the term β in (5.2.41) is strictly positive. □

5.3 The Quantization

We are now in a position to argue very similar as in the previous chapter. For the convenience of the reader, we shall repeat some of the arguments so that the results can be understood directly without having to look up the details.

The interior of the black hole is a globally hyperbolic spacetime and the slices $S(r)$ with

$$r_1 < r < r_2 \tag{5.3.1}$$

are Cauchy hypersurfaces. Let r tend to r_2 and let $\mathcal{S}_0$ be the resulting limit Riemannian manifold; i.e., topologically, it is the event horizon but equipped with the metric in (5.2.50) which we shall write in the form

$$ds^2 = d\tau^2 + \sigma_{ij}dx^i dx^j \tag{5.3.2}$$

as in (5.2.46) on page 122. By a slight abuse of language, we shall also call $\mathcal{S}_0$ to be a Cauchy hypersurface though it is only the geometric limit of Cauchy hypersurfaces. However, $\mathcal{S}_0$ is a genuine Cauchy hypersurface in the quantum model which is defined by the Eq. (5.1.1) on page 113.

Let us now look at the stationary eigenvalue equation, where we recall that $n = 2m$,

$$-(n-1)\Delta v - \frac{n}{2}Rv = \mu v \tag{5.3.3}$$

in $\mathcal{S}_0$, where

$$-(n-1)\Delta v = -(n-1)\ddot{v} - (n-1)\tilde{\Delta} v \tag{5.3.4}$$

and $\tilde{\Delta}$ is the Laplacian in the Riemannian manifold

$$M = (\mathbb{S}^{n-1}, \sigma_{ij}); \tag{5.3.5}$$

moreover the scalar curvature R is also the scalar curvature with respect to σ_{ij} in view of (5.3.2). Using separation of variables, let us write

$$v(\tau, x) = \zeta(\tau)\varphi(x) \tag{5.3.6}$$

to conclude that the left-hand side of (5.3.3) can be expressed in the form

$$-(n-1)\ddot{\zeta}\varphi + \zeta\{-(n-1)\tilde{\Delta}\varphi - \frac{n}{2}R\varphi\}. \tag{5.3.7}$$

Hence, the eigenvalue problem (5.3.3) can be solved by setting

$$v = \zeta\varphi_j, \tag{5.3.8}$$

where φ_j, $j \in \mathbb{N}$, is an eigenfunction of the elliptic operator

$$-(n-1)\tilde{\Delta} - \frac{n}{2}R \tag{5.3.9}$$

such that

$$-(n-1)\tilde{\Delta}\varphi_j - \frac{n}{2}R\varphi_j = \tilde{\mu}_j\varphi_j, \tag{5.3.10}$$

$$\tilde{\mu}_0 < \tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots \tag{5.3.11}$$

and ζ is an eigenfunction of the harmonic oscillator. The eigenvalue of the harmonic oscillator can be arbitrarily positive or zero. We define it at the moment as

$$\omega^2 \tag{5.3.12}$$

where $\omega \ge 0$ will be determined later. For $\omega > 0$, we shall consider the real eigenfunction

$$\zeta = \sin \omega\tau \tag{5.3.13}$$

which represents the ground state in the interval

$$I_0 = (0, \frac{\pi}{\omega}) \tag{5.3.14}$$

with vanishing boundary values. ζ is a solution of the variational problem

$$\frac{\int_{I_0} |\dot{\vartheta}|^2}{\int_{I_0} |\vartheta|^2} \to \min \quad \forall\, 0 \ne \vartheta \in H_0^{1,2}(I_0) \tag{5.3.15}$$

in the Sobolev space $H_0^{1,2}(I_0)$.

Multiplying ζ by a constant, we may assume

$$\int_{I_0} |\zeta|^2 = 1. \tag{5.3.16}$$

Obviously,

$$\mathcal{S}_0 = \mathbb{R} \times M \tag{5.3.17}$$

and though ζ is defined in $\mathbb{R}$ and is even an eigenfunction it has infinite norm in $L^2(\mathbb{R})$. However, when we consider a finite disjoint union of N open intervals I_j

$$\Omega = \bigcup_{j=1}^{N} I_j, \tag{5.3.18}$$

where

$$I_j = (k_j \frac{\pi}{\omega}, (k_j + 1)\frac{\pi}{\omega}), \quad k_j \in \mathbb{Z}, \tag{5.3.19}$$

then

$$\zeta_N = N^{-\frac{1}{2}} \zeta \tag{5.3.20}$$

is a unit eigenfunction in Ω with vanishing boundary values having the same energy as ζ in I_0. Hence, it suffices to consider ζ only in I_0 since this configuration can immediately be generalized to arbitrary large bounded open intervals

$$\Omega \subset \mathbb{R}. \tag{5.3.21}$$

We then can state:

Lemma 5.3.1 *There exists a complete sequence of unit eigenfunctions of the operator in* (5.3.9) *with eigenvalues* $\tilde{\mu}_j$ *such that the functions*

$$v_j = \zeta \varphi_j, \tag{5.3.22}$$

where ζ *is a constant multiple of the function in* (5.3.13) *with unit* $L^2(I_0)$ *norm, are solutions of the eigenvalue problem* (5.3.3) *with eigenvalue*

$$\mu_j = (n-1)\omega^2 + \tilde{\mu}_j. \tag{5.3.23}$$

The eigenfunctions v_j *are mutually orthogonal in* $L^2(I_0 \times M, \mathbb{C})$.

To solve the wave Eq. (5.1.1) on page 113, let us first consider the following eigenvalue problem

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n|\Lambda|t^2 w = \lambda t^{2-\frac{4}{n}} w \tag{5.3.24}$$

in the Sobolev space

$$H_0^{1,2}(\mathbb{R}_+^*). \tag{5.3.25}$$

Here,

$$\Lambda < 0 \tag{5.3.26}$$

can in principle be an arbitrary negative parameter, but in the case of a Kerr-AdS black hole, it seems reasonable to choose the cosmological constant of the Kerr-AdS spacetime.

The eigenvalue problem (5.3.24) can be solved by considering the generalized eigenvalue problem for the bilinear forms

$$B(w, \tilde{w}) = \int_{\mathbb{R}_+^*} \{\frac{1}{32}\frac{n^2}{n-1}\bar{w}'\tilde{w}' + n|\Lambda|t^2 \bar{w}\tilde{w}\} \tag{5.3.27}$$

and

$$K(w,\tilde{w}) = \int_{\mathbb{R}_+^*} t^{2-\frac{4}{n}} \bar{w}\tilde{w} \tag{5.3.28}$$

in the Sobolev space $\mathcal{H}$ which is the completion of

$$C_c^\infty(\mathbb{R}_+^*, \mathbb{C}) \tag{5.3.29}$$

in the norm defined by the first bilinear form.

We then look at the generalized eigenvalue problem

$$B(w,\varphi) = \lambda K(w,\varphi) \quad \forall \varphi \in \mathcal{H} \tag{5.3.30}$$

which is equivalent to (5.3.24).

Theorem 5.3.2 *The eigenvalue problem* (5.3.30) *has countably many solutions* (w_i, λ_i) *such that*

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \tag{5.3.31}$$

$$\lim \lambda_i = \infty, \tag{5.3.32}$$

and

$$K(w_i, w_j) = \delta_{ij}. \tag{5.3.33}$$

The w_i *are complete in* $\mathcal{H}$ *as well as in* $L^2(\mathbb{R}_+^*)$.

The above theorem is a mere restatement of Theorem 3.4.1 on page 96.

Combining the temporal und spatial eigenfunctions, we are now ready to define the solutions of the wave Eq. (5.1.1).

Theorem 5.3.3 *Let* $(\varphi_j, \tilde{\mu}_j)$ *resp.* (w_i, λ_i) *be eigenfunctions of the elliptic operator in* (5.3.9) *resp. the temporal eigenfunctions and let* λ_{i_0} *be the smallest eigenvalue of the* (λ_i) *with the property*

$$\lambda_{i_0} \geq \tilde{\mu}_j, \tag{5.3.34}$$

then, for any $i \geq i_0$*, there exists*

$$\omega = \omega_{ij} \geq 0 \tag{5.3.35}$$

and corresponding ζ_{ij} *satisfying*

$$-\ddot{\zeta}_{ij} = \omega_{ij}^2 \zeta_{ij} \tag{5.3.36}$$

such that

$$\lambda_i = \mu_{ij} = (n-1)\omega_{ij}^2 + \tilde{\mu}_j \quad \forall i \geq i_0. \tag{5.3.37}$$

The functions

$$u_{ij} = w_i \zeta_{ij} \varphi_j \tag{5.3.38}$$

are then solutions of the wave equation with bounded energies satisfying

$$\lim_{t\to 0} u_{ij}(t) = \lim_{t\to\infty} u_{ij}(t) = 0 \tag{5.3.39}$$

and

$$u_{ij} \in C^\infty(\mathbb{R}_+^* \times \mathcal{S}_0) \cap C^{2,\alpha}(\bar{\mathbb{R}}_+^* \times \mathcal{S}_0) \tag{5.3.40}$$

for some

$$\frac{2}{3} \le \alpha < 1. \tag{5.3.41}$$

Moreover, we have

$$\omega_{ij} > 0 \qquad \forall i > i_0. \tag{5.3.42}$$

If

$$\lambda_{i_0} = \tilde{\mu}_j, \tag{5.3.43}$$

then we define

$$\zeta_{i_0 j} \equiv 1. \tag{5.3.44}$$

Proof The proof is obvious. □

Remark 5.3.4 (i) By construction, the temporal and spatial energies of the solutions of the wave equation have to be equal.
(ii) The stationary solutions comprising a harmonic oscillator can be looked at a being radiation while we consider the temporal solutions to be gravitational waves.
(iii) If one wants to replace the bounded Interval I_0 by $\mathbb{R}$, then the eigenfunctions ζ_{ij} have to be replaced by eigendistributions. An appropriate choice would be

$$\zeta_{ij} = e^{i\omega_{ij}\tau}. \tag{5.3.45}$$

The hyperbolic operator defined by the wave Eq. (5.1.1) on page 113 can be defined in the spacetime

$$Q = \mathbb{R}_+^* \times \mathcal{S}_0 \tag{5.3.46}$$

which can be equipped with the Lorentzian metric

$$d\bar{s}^2 = -\frac{32(n-1)}{n^2} dt^2 + g_{ij} dx^i dx^j \tag{5.3.47}$$

as well as with the metric

$$d\tilde{s}^2 = -\frac{32(n-1)}{n^2} dt^2 + \frac{1}{n-1} t^{\frac{4}{n}-2} g_{ij} dx^i dx^j, \tag{5.3.48}$$

where g_{ij} is the metric defined on $\mathcal{S}_0$ and the indices now have the range $1 \le i, j \le n$. In both metrics, Q is globally hyperbolic provided $\mathcal{S}_0$ is complete, which is the case for the metric defined in (5.3.2). The hyperbolic operator is symmetric in the first metric but not normally hyperbolic while it is normally hyperbolic but not symmetric in the second metric. Normally hyperbolic means that the main part of the operator is identical to the Laplacian of the spacetime metric.

Hence, if we want to describe quantum gravity not only by an equation but also by the metric of a spacetime then the metric in (5.3.48) has to be chosen. In this metric, Q has a curvature singularity in $t = 0$, cf. Remark 2.5.3 on page 74. The Cauchy hypersurface $\mathcal{S}_0$ then corresponds to the hypersurface

$$\{t = 1\} \tag{5.3.49}$$

which also follows from the derivation of the quantum model where we consider a fiber bundle E with base space $\mathcal{S}_0$ and the elements of the fibers were Riemann metrics of the form

$$g_{ij}(t, x) = t^{\frac{4}{n}} \sigma_{ij}(x) \tag{5.3.50}$$

where σ_{ij} were metrics defined in $\mathcal{S}_0$ and t is the time coordinate that we use in Q, i.e.,

$$g_{ij}(1, x) = \sigma_{ij}(x). \tag{5.3.51}$$

In the present situation, we used the symbol g_{ij} to denote the metric on $\mathcal{S}_0$ since σ_{ij} is supposed to be a metric on $\mathbb{S}^{2m-1}$.

Thus, the event horizon is characterized by the Cauchy hypersurface

$$\{t = 1\}. \tag{5.3.52}$$

If $a = 0$, i.e., in case we consider a Schwarzschild-AdS black hole, then we shall obviously assume that the black hole singularity

$$\{r = 0\} \tag{5.3.53}$$

corresponds to the curvature singularity

$$\{t = 0\} \tag{5.3.54}$$

of Q; i.e., the open black hole region is described in the quantum model by

$$(0, 1) \times \mathcal{S}_0 \tag{5.3.55}$$

and the open exterior region by

$$(1, \infty) \times \mathcal{S}_0. \tag{5.3.56}$$

If $a \neq 0$, then there is no curvature singularity in $r = 0$, only a coordinate singularity in our present coordinate system. Indeed, if we choose generalized Boyer-Lindquist coordinates, cf. [25, Eq. (3.1)], the metric has the form

$$\begin{aligned} d\bar{s}^2 = &-\frac{1+l^2r^2}{1-a^2l^2}d\tau^2 + \frac{Ur^2dr^2}{(1+l^2r^2)(r^2+a^2)U - 2m_0r^2} \\ &+ \frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^{m}(d\mu_i^2 + \mu_i^2(d\varphi_i + l^2 d\tau)^2) \\ &+ \frac{2m_0}{U}\big(d\tau - \frac{a}{1-a^2l^2}\sum_{i=1}^{m}\mu_i^2 d\varphi^i\big)^2. \end{aligned} \tag{5.3.57}$$

Then, defining

$$\rho = r^2, \tag{5.3.58}$$

such that

$$d\rho = 2rdr \tag{5.3.59}$$

we obtain new coordinates in which the metric is smooth up to $\rho = 0$; indeed, the metric is even smooth in the interval

$$-a^2 < \rho < \infty. \tag{5.3.60}$$

In $\rho = -a^2$, there is curvature singularity:

Lemma 5.3.5 *The extended spacetime N has a timelike curvature singularity in $\rho = -a^2$.*

Proof The fact that the curvature singularity is timelike follows immediately from (5.2.49) on page 122, where we proved that outside the black hole region the hypersurfaces

$$\{\rho = \text{const}\} \tag{5.3.61}$$

are timelike.

To prove the existence of a curvature singularity, we first consider the case $m \geq 3$. Looking at the metric in (5.3.57), we observe that the components with respect to the coordinates μ_i form a diagonal matrix without any cross terms with the other coordinates, namely

$$\frac{\rho + a^2}{1-a^2l^2}\sum_{i=1}^{m} d\mu_i^2, \tag{5.3.62}$$

where the μ_i are subject to the side condition

$$\sum_{i=1}^{m}\mu_i^2 = 1, \tag{5.3.63}$$

i.e., (5.3.62) represents the metric of a sphere of radius

$$\sqrt{\frac{\rho + a^2}{1 - a^2 l^2}} \tag{5.3.64}$$

embedded in $\mathbb{R}^m$ and the corresponding sectional curvatures in N are defined independently of the other components of the metric in N and they obviously become unbounded when ρ tends to $-a^2$, since the sectional curvature σ_p in a point $p \in N$ of a plane spanned by two linearly independent vectors in

$$T_p(\mathbb{S}^{m-1}) \hookrightarrow T_p(N) \tag{5.3.65}$$

is equal to

$$\frac{1 - a^2 l^2}{\rho + a^2}. \tag{5.3.66}$$

Secondly, in case $m = 2$, we used the package GREAT [33] in Mathematica to compute the squared Riemannian curvature tensor in dimension 5 and obtained

$$\bar{R}_{\alpha\beta\gamma\delta}\bar{R}^{\alpha\beta\gamma\delta} = \frac{96 m_0^2 \left(3a^2 - \rho\right)\left(a^2 - 3\rho\right)}{\left(\rho + a^2\right)^6} + 40 l^4 \tag{5.3.67}$$

completing the proof of the lemma. □

Since the curvature singularity is *timelike* and not *spacelike* as the singularity of a Schwarzschild-AdS spacetime or the singularity in our quantum spacetime, equipped with the metric in (5.3.48), it is easily avoidable. Despite this difference, we stipulate that the region in (5.3.56) corresponds to

$$\{\rho_2 < \rho < \infty\}, \tag{5.3.68}$$

where

$$r_2^2 = \rho_2, \tag{5.3.69}$$

and the region in (5.3.55) to

$$\{-a^2 < \rho < \rho_2\}. \tag{5.3.70}$$

Remark 5.3.6 The time coordinate τ in a generalized Boyer-Lindquist coordinate system is a time function in

$$N \backslash \bar{B}, \tag{5.3.71}$$

where N is the extended Kerr-AdS spacetime. We proved it directly with the help of Mathematica, if $\dim N = 5$, by proving

$$\bar{g}^{\alpha\beta}\tau_\alpha\tau_\beta = \bar{g}^{00} < 0. \tag{5.3.72}$$

For a proof in any odd dimension, it will be sufficient to prove that the slices

$$\{\tau = \text{const}\} \tag{5.3.73}$$

are spacelike in the region specified in (5.3.71). Looking at the metric (5.3.57), we immediately see, by setting $d\tau = 0$, that the induced metric is Riemannian.

Remark 5.3.7 When we have an extremal black hole with mass m_0' and corresponding radius r_0 for the event horizon, then the function $\Phi = \Phi(\rho)$ in (5.2.12), where

$$\rho = r^2, \tag{5.3.74}$$

satisfies

$$0 = \Phi(\rho_0) = \inf \Phi, \tag{5.3.75}$$

hence

$$\Phi'(\rho_0) = 0. \tag{5.3.76}$$

From the definition of Φ, we then conclude that any black hole with mass

$$m_0 > m_0', \tag{5.3.77}$$

while the other parameters remain equal, will have an interior region. Hence, our previous arguments could then be applied to yield a quantum model depending on the Riemannian metric in (5.2.50) on page 122. Letting m_0 tends to m_0' the corresponding radii of the event horizons will then converge to r_0 leading to a quantum model for an extremal black hole.

Remark 5.3.8 In the quantum model of the black hole, the event horizon is a regular Cauchy hypersurface and can be crossed in both directions by causal curves; hence, no Information paradox can occur.

Chapter 6
A Partition Function for Quantized Globally Hyperbolic Spacetimes with a Negative Cosmological Constant

6.1 Trace Class Operators

Consider a physical system that can be described by a separable Hilbert space $\mathcal{H}$ and a self-adjoint operator H assuming that H has a pure point spectrum. If one wants to apply quantum statistics to this system, then, for any $\beta > 0$, the operator

$$e^{-\beta H} \tag{6.1.1}$$

has to be of trace class in $\mathcal{H}$, or, if H is extended to the corresponding symmetric Fock space, the extended operator in (6.1.1) has to be of trace class in $\mathcal{F}_+(\mathcal{H})$. In case H is a Schrödinger operator or, more generally, a self-adjoint elliptic operator in a bounded domain of $\mathbb{R}^n$ with homogenous boundary conditions, it is well known that the operator in (6.1.1) is of trace class because of Weyl's asymptotic behaviour formula for the eigenvalues λ_j,

$$\lambda_j \sim C_n(\frac{j}{V})^{\frac{2}{n}}, \tag{6.1.2}$$

where C_n is the so-called Weyl constant, V the Euclidean volume of the domain and the λ_j are labelled such that

$$\lambda_1 \le \lambda_2 \le \cdots \tag{6.1.3}$$

We prefer to start the numbering with $j = 0$ instead of $j = 1$, though this is of course irrelevant as far as the asymptotic formulas are concerned, but it might become relevant if more precise estimates are considered. Hence, when citing estimates the labelling in (6.1.3) will always be assumed.

Weyl used variational methods and properties of the Green's function to obtain the asymptotic estimates, cf. [43] and also [5, Kap. VI.4]. Li and Yau proved a lower bound

$$\lambda_j \ge \frac{nC_n}{n+2}(\frac{j}{V})^{\frac{2}{n}} \tag{6.1.4}$$

C. Gerhardt, *The Quantization of Gravity*, Fundamental Theories of Physics 194,
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assuming the eigenvalues to be positive; they used the heat kernel for this estimate, cf. [36].

In case of unbounded domains, we do not know of any asymptotic or lower estimates which would imply the operator in (6.1.1) to be of trace class—apart from special cases, when the eigenvalues are explicitly known.

In this chapter, we shall consider self-adjoint elliptic differential operators defined in $\mathbb{R}_+$ or $\mathbb{R}^n$, $n \geq 2$, and shall prove, by imposing reasonable assumptions, that the operator in (6.1.1) is of trace class. The proof will not rely on showing either asymptotic or explicit lower estimates but we shall instead construct explicit majorants from the existence of which we will infer

$$\operatorname{tr}(e^{-\beta H}) < \infty. \tag{6.1.5}$$

One crucial ingredient in the proof is a generalization of Maurin's Hilbert–Schmidt-type embedding theorem, cf. [38, Theorem 1, p. 336], to unbounded domains with special weighted measures combined with an interpolation inequality involving the norm of the target space of the Hilbert–Schmidt embedding.

These new trace class estimates can especially be applied when the physical system is defined by a wave equation, which is either obtained by a classical description or is the result of a (first) quantization process. In either case, it is worthwhile to use, if possible, a separation of variables to split a solution u of the wave equation into a product

$$u(t, x) = w(t)v(x) \tag{6.1.6}$$

and then finding temporal and spatial self-adjoint operators H_0 resp. H_1 such that one of them has a pure point spectrum with eigenvalues λ_i while, for the other operator, it is possible to find corresponding eigendistributions for each of the eigenvalues λ_i. Assuming, e.g. that H_0 has a pure point spectrum with corresponding mutually orthogonal eigenfunctions w_i and H_1 has smooth eigendistributions v_{ij} satisfying

$$H_1 v_{ij} = \lambda_i v_{ij} \qquad \forall\, j \tag{6.1.7}$$

then

$$u_{ij} = w_i v_{ij} \tag{6.1.8}$$

would be solutions of the wave equation.

We shall especially look at quantum systems governed by the wave equation

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0, \tag{6.1.9}$$

defined in a quantum spacetime

$$N = \mathbb{R}_+ \times \mathcal{S}_0, \tag{6.1.10}$$

where $\mathcal{S}_0$ is a n-dimensional, $n \geq 3$, Cauchy hypersurface of the original spacetime, or, in case of black holes, the smooth limit of Cauchy hypersurfaces. The Laplacian and the scalar curvature correspond to the metric σ_{ij} in $\mathcal{S}_0$. The cosmological constant Λ is supposed to be negative. In the previous chapters, we applied this model to a Schwarzschild-AdS resp. Kerr-AdS black hole and to a globally hyperbolic spacetime with an asymptotic Euclidean Cauchy hypersurface. In all three cases, we obtained a sequence of smooth functions as solutions of the wave equation which are a product of temporal eigenfunctions and spatial eigendistributions.

In case of the globally hyperbolic spacetime with an asymptotically Euclidean Cauchy hypersurface, the solutions to the wave equation can be expressed in the form

$$u_{ij} = w_i v_{ij}, \qquad i \in \mathbb{N},\ 1 \leq j \leq m \leq \infty, \tag{6.1.11}$$

where the w_i are the eigenfunctions of a temporal Hamilton operator H_0

$$H_0 w_i = \lambda_i w_i \tag{6.1.12}$$

and the λ_i have multiplicity one such that

$$0 < \lambda_0 < \lambda_1 < \cdots \tag{6.1.13}$$

and for each fixed i, the at most countably many v_{ij} generate an eigenspace

$$\mathscr{E}_{\lambda_i} \subset \mathscr{S}'(\mathcal{S}_0) \tag{6.1.14}$$

of a spatial Hamiltonian H_1, i.e.

$$H_1 v_{ij} = \lambda_i v_{ij}. \tag{6.1.15}$$

We have

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0), \tag{6.1.16}$$

cf. Theorem 3.1.2 on page 82. A similar spectral resolution has been proved for black holes in the preceding two chapters.

Let us now give a more detailed summary of our results. First, for the general trace class estimates. We consider eigenvalue problems in $\mathbb{R}^n$, $n \geq 2$. Let A be the linear elliptic operator

$$Au = -D_i(a^{ij} D_j u) + b(x)u, \tag{6.1.17}$$

where

$$a^{ij}, b \in L^\infty_{\text{loc}}(\mathbb{R}^n), \tag{6.1.18}$$

a^{ij} is symmetric and we assume there exists $a_0 > 0$ such that

$$a_0|\xi|^2 \le a^{ij}\xi_i\xi_j \qquad \forall\, \xi \in \mathbb{R}^n \tag{6.1.19}$$

and that there exists $R_0 > 1$ and positive p, c_1 such that

$$c_1|x|^p \le b(x) \qquad \forall\, |x| \ge R_0. \tag{6.1.20}$$

Then, we shall prove:

Theorem 6.1.1 *The operator A is essentially self-adjoint in $\mathcal{H} = L^2(\mathbb{R}^n)$ with a pure point spectrum*

$$0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \tag{6.1.21}$$

Let H be its self-adjoint extension then, for any $\beta > 0$,

$$e^{-\beta H} \tag{6.1.22}$$

is of trace class in $\mathcal{H}$.

Next, let us consider a Sturm–Liouville operator A in $\mathbb{R}_+$ of the form

$$Au = -(au')' + bu, \tag{6.1.23}$$

where a dot or a prime indicates differentiation, and corresponding eigenvalue problems

$$Au = \lambda\varphi_0 u, \tag{6.1.24}$$

where the coefficients a, b and the function φ_0 are all measurable and locally bounded in $\mathbb{R}_+$, and b is even locally bounded in $[0, \infty)$, and they satisfy

$$a(t) \ge a_0 > 0 \qquad \forall\, t \in \mathbb{R}_+, \tag{6.1.25}$$

and there exist positive constants c_1, c_2, p, r and $t_0 > 1$ such that

$$b(t) \ge c_1 t^p \qquad \forall\, t \ge t_0, \tag{6.1.26}$$

$$\varphi_0(t) \le c_2 t^r \qquad \forall\, t \ge t_0, \tag{6.1.27}$$

and

$$0 < r < p, \tag{6.1.28}$$

where the function φ_0 is also positive almost everywhere. Then we proved:

Theorem 6.1.2 *The eigenvalue problem*

$$Au = \lambda\varphi_0 u \tag{6.1.29}$$

has countably many solutions (λ_i, w_i) *such that*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{6.1.30}$$

and the w_i *form an ONB in*

$$\mathcal{H} = L^2(\mathbb{R}_+, d\mu), \tag{6.1.31}$$

$$d\mu = \varphi_0 dt. \tag{6.1.32}$$

The operator

$$\varphi_0^{-1} A \tag{6.1.33}$$

is essentially self-adjoint in $\mathcal{H}$*. Let* H_0 *be its self-adjoint extension then, for any* $\beta > 0$*,*

$$e^{-\beta H_0} \tag{6.1.34}$$

is of trace class in $\mathcal{H}$*.*

Finally, let us describe the results with respect to the wave Eq. (6.1.9). In Sect. 6.4, we shall prove that the wave equation can be expressed in the form

$$\varphi_0(H_0 u - H_1 u) = 0, \tag{6.1.35}$$

where $u = u(t, x)$ is a smooth function, $x \in \mathcal{S}_0$ and

$$\varphi_0(t) = t^{2-\frac{4}{n}}. \tag{6.1.36}$$

H_0 is an operator which satisfies the assumptions in the previous theorem, and in Sect. 6.5, we shall define an abstract Hilbert space $\mathcal{H}$, where the eigendistributions of H_1 form an ONB, such that H_0 and H_1 have the same eigenvalues but with different multiplicities. H_1 is also essentially self-adjoint in $\mathcal{H}$. Let $\tilde{H}_1$ be the unique self-adjoint extension of H_1, namely its closure, then we shall prove that for any $\beta > 0$

$$e^{-\beta \tilde{H}_1} \tag{6.1.37}$$

is of trace class in $\mathcal{H}$. In addition, $\tilde{H}_1$ satisfies

$$\tilde{H}_1 \geq \lambda_0 I, \qquad \lambda_0 > 0. \tag{6.1.38}$$

Let

$$H \equiv d\Gamma(\tilde{H}_1) \tag{6.1.39}$$

be the canonical extension of $\tilde{H}_1$ to the symmetric Fock space

$$\mathscr{F} = \mathscr{F}_+(\mathcal{H}), \tag{6.1.40}$$

then

$$e^{-\beta H} \tag{6.1.41}$$

is of trace class in $\mathscr{F}$ because of (6.1.37) and (6.1.38), cf. [3, Prop. 5.2.27]. Hence we can define the partition function

$$Z = \operatorname{tr}(e^{-\beta H}), \tag{6.1.42}$$

the density operator

$$\rho = Z^{-1} e^{-\beta H} \tag{6.1.43}$$

and the von Neumann entropy

$$S = -\operatorname{tr}(\rho \log \rho) = \log Z + \beta E, \tag{6.1.44}$$

where E is the average energy and $\beta > 0$ the inverse temperature

$$\beta = T^{-1}. \tag{6.1.45}$$

Here is a summary of some of the results derived in Sect. 6.5.

Theorem 6.1.3 (i) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{6.1.46}$$

we have

$$\lim_{\Lambda \to 0} E = \infty \tag{6.1.47}$$

as well as

$$\lim_{\Lambda \to 0} S = \infty, \tag{6.1.48}$$

where the limits are uniform in β.

(ii) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$\beta \ge \beta_0, \tag{6.1.49}$$

we have

$$\lim_{|\Lambda| \to \infty} E = 0 \tag{6.1.50}$$

as well as

$$\lim_{|\Lambda|\to\infty} S = 0, \tag{6.1.51}$$

where the limits are uniform in β.

The behaviour of Z with respect to Λ is described in the theorem:

Theorem 6.1.4 *Let* $\beta_0 > 0$ *be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{6.1.52}$$

we have

$$\lim_{\Lambda\to 0} Z = \infty \tag{6.1.53}$$

and for any

$$\beta_0 \le \beta \tag{6.1.54}$$

the relation

$$\lim_{|\Lambda|\to\infty} Z = 1 \tag{6.1.55}$$

is valid. The convergence in both limits is uniform in β.

Remark 6.1.5 The first part of Theorem 6.1.3 reveals that the energy becomes very large for small values of $|\Lambda|$. Since this is the energy obtained by applying quantum statistics to the quantized version of a black hole or of a globally hyperbolic spacetime—assuming its Cauchy hypersurfaces are asymptotically Euclidean—a small negative cosmological constant might be responsible for the dark matter, where we equate the energy of the quantized universe with matter. As source for the dark energy density, we consider the eigenvalue of the density operator ρ with respect to the vacuum vector η

$$\rho\eta = Z^{-1}\eta, \tag{6.1.56}$$

i.e. the dark energy density should be proportional to Z^{-1}.

In Sect. 6.4, we also applied quantum statistics to the quantized version of a Friedmann universe and proved:

Theorem 6.1.6 *The results in the last two theorems and the conjectures in the remark above are also valid, if the quantized spacetime* $N = N^{n+1}$, $n \ge 3$, *is a Friedmann universe without matter but with a negative cosmological constant* Λ *and with vanishing spatial curvature. The eigenvalues of the spatial Hamiltonian* H_1 *all have multiplicity one.*

Remark 6.1.7 Let us also mention that we use Planck units in this book, i.e.

$$c = G = \hbar = K_B = 1. \tag{6.1.57}$$

6.2 Trace Class Estimates in $\mathbb{R}_+$

Let us first consider a Sturm–Liouville operator A in $\mathbb{R}_+$ of the form

$$Au = -(au')' + bu, \tag{6.2.1}$$

where a dot or a prime indicates differentiation, and corresponding eigenvalue problems

$$Au = \lambda\varphi_0 u, \tag{6.2.2}$$

where the coefficients a, b and the function φ_0 are all measurable and locally bounded in $\mathbb{R}_+$, and b is even locally bounded in $[0, \infty)$, and they satisfy

$$a(t) \ge a_0 > 0 \qquad \forall\, t \in \mathbb{R}_+, \tag{6.2.3}$$

and there exist positive constants c_1, c_2, p, r and $t_0 > 1$ such that

$$b(t) \ge c_1 t^p \qquad \forall\, t \ge t_0, \tag{6.2.4}$$

$$\varphi_0(t) \le c_2 t^r \qquad \forall\, t \ge t_0, \tag{6.2.5}$$

and

$$0 < r < p, \tag{6.2.6}$$

where φ_0 is also assumed to be positive almost everywhere (a.e.), and where the specification

$$\forall\, t \ge t_0 \tag{6.2.7}$$

means

$$\text{a.e. in } \{t \ge t_0\} \tag{6.2.8}$$

when used in connection with measurable functions which are not assumed to be continuous.

We define the bilinear forms

$$B(u, v) = \langle Au, v\rangle = \int_{\mathbb{R}_+} \{a\bar{u}'v' + b\bar{u}v\} \tag{6.2.9}$$

and

$$K(u, v) = \int_{\mathbb{R}_+} \varphi_0 \bar{u} v \tag{6.2.10}$$

for

$$u, v \in C_c^\infty(\mathbb{R}_+, \mathbb{C}), \tag{6.2.11}$$

and we denote the corresponding quadratic forms by $B(u)$ resp. $K(u)$.

Lemma 6.2.1 *Define*

$$b_0(t) = \begin{cases} 0, & 0 \le t < t_0, \\ b(t), & t_0 \le t, \end{cases} \tag{6.2.12}$$

and

$$B_0(u) = \int_{\mathbb{R}_+} \{a|u'|^2 + b_0|u|^2\}, \tag{6.2.13}$$

then, for any $\epsilon > 0$*, there exists* c_ϵ *such that*

$$\|u\|_2^2 = \int_{\mathbb{R}_+} |u|^2 \le \epsilon B_0(u) + c_\epsilon K(u) \qquad \forall\, u \in C_c^\infty(\mathbb{R}_+). \tag{6.2.14}$$

Proof This compactness lemma is well known. However, we give a short proof for the convenience of the reader. We argue by contradiction and assume there would exist $\epsilon > 0$ and a sequence

$$u_k \in C_c^\infty(\mathbb{R}_+) \tag{6.2.15}$$

such that

$$\|u_k\|_2^2 > \epsilon B_0(u_k) + kK(u_k). \tag{6.2.16}$$

Without loss of generality, we may assume that

$$\|u_k\|_2^2 = 1. \tag{6.2.17}$$

Hence the u_k would be uniformly bounded in the Sobolev space

$$H^{1,2}(J) \tag{6.2.18}$$

with norm

$$\|u\|_{1,2}^2 = \int_J (|u'|^2 + |u|^2), \tag{6.2.19}$$

for any bounded interval

$$J \Subset [0, \infty), \tag{6.2.20}$$

and we would deduce

$$\lim_{k \to \infty} K(u_k) = 0. \tag{6.2.21}$$

Moreover, by applying the Sobolev embedding theorem, we would know that a subsequence, not relabelled, would converge strongly in any

$$L^2(J, \mathbb{C}) \tag{6.2.22}$$

to a function u. In view of Fatou's lemma, we would also infer

$$K(u) \le \lim K(u_k) = 0 \tag{6.2.23}$$

and thus

$$u \equiv 0. \tag{6.2.24}$$

But this would lead to a contradiction, since, for any $m > t_0$, we would have

$$\begin{aligned} 1 &= \int_0^m |u_k|^2 + \int_m^\infty |u_k|^2 \\ &\le \int_0^m |u_k|^2 + c_1^{-1} m^{-p} \int_m^\infty b_0 |u_k|^2 \\ &\le \int_0^m |u_k|^2 + c_1^{-1} m^{-p} \limsup B_0(u_k) \end{aligned} \tag{6.2.25}$$

yielding

$$1 \le c_1^{-1} m^{-p} \limsup B_0(u_k) \le c_1^{-1} m^{-p} \epsilon^{-1} \qquad \forall\, m \ge t_0, \tag{6.2.26}$$

in view of (6.2.16) and (6.2.17). □

As an immediate corollary, we obtain

Corollary 6.2.2 *There exists a positive constant* c_0 *such that*

$$\|u\|^2 \equiv \|u\|_2^2 \le B(u) + c_0 K(u) \qquad \forall\, u \in C_c^\infty(\mathbb{R}_+) \tag{6.2.27}$$

and

$$\frac{1}{2} B_0(u) \le B(u) + c_0 K(u) \qquad \forall\, u \in C_c^\infty(\mathbb{R}_+). \tag{6.2.28}$$

Proof Since b is bounded in $I = [0, t_0]$ we conclude, in view of (6.2.14),

$$\begin{aligned} B(u) &\ge B_0(u) - c\|u\|_2^2 \\ &\ge B_0(u) - c\epsilon B_0(u) - cc_\epsilon K(u) \\ &= (1 - c\epsilon) B_0(u) - cc_\epsilon K(u) \\ &\ge \|u\|_2^2 - c_0 K(u), \end{aligned} \tag{6.2.29}$$

if we choose

$$\epsilon = \frac{1}{2c} \tag{6.2.30}$$

and c_0 appropriately, proving both estimates. □

In view of the Poincaré inequality on bounded intervals, we also conclude that there exists $c > 0$ such that

$$\|u\|_{1,2}^2 \le cB_0(u) \qquad \forall\, u \in C_c^\infty(\mathbb{R}_+). \tag{6.2.31}$$

Definition 6.2.3 We define the Hilbert space $\mathcal{H}_1$ as the completion of $C_c^\infty(\mathbb{R}_+)$ with respect to the scalar product defined by the bilinear form

$$B + c_0 K, \tag{6.2.32}$$

cf. Corollary 6.2.2, and we denote this scalar product by the symbol

$$\langle \cdot, \cdot \rangle_1 \tag{6.2.33}$$

and corresponding norm

$$\|\cdot\|_1. \tag{6.2.34}$$

The Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{H} = L^2(\mathbb{R}_+, d\mu), \tag{6.2.35}$$

where

$$d\mu = \varphi_0(t)dt. \tag{6.2.36}$$

The corresponding scalar product is K and it is also characterized by the symbol

$$\langle \cdot, \cdot \rangle \tag{6.2.37}$$

and corresponding norm

$$\|\cdot\|. \tag{6.2.38}$$

Using the arguments in the proof of Lemma 6.2.1, the results of Corollary 6.2.2 and the Assumptions (6.2.5) and (6.2.6), we immediately obtain:

Lemma 6.2.4 *The embedding*

$$j : \mathcal{H}_1 \hookrightarrow \mathcal{H} \tag{6.2.39}$$

is compact, i.e. if $u_k \in \mathcal{H}_1$ converges weakly to u

$$u_k \rightharpoonup u, \tag{6.2.40}$$

then

$$j(u_k) \to j(u). \tag{6.2.41}$$

We conclude further that the generalized eigenvalue problem

$$B(u, v) = \lambda K(u, v) \qquad \forall\, v \in \mathcal{H}_1 \tag{6.2.42}$$

can be solved by a variational process which goes back to Courant–Hilbert [5, Kap. 6]. We describe it in the following theorem:

Theorem 6.2.5 *Let $\mathcal{H}$ be a complex, separable Hilbert space, B and K sesquilinear, symmetric forms on $\mathcal{H}$ and assume there exists a positive constant c_0 such that*

$$B + c_0 K \tag{6.2.43}$$

is an equivalent scalar product in $\mathcal{H}$. K is also supposed to be a positive definite and compact in $\mathcal{H}$, i.e.

$$u_k \rightharpoonup u \implies K(u_k) \to K(u). \tag{6.2.44}$$

Then the eigenvalue problem

$$B(u, v) = \lambda K(u, v) \qquad \forall v \in \mathcal{H}_1 \tag{6.2.45}$$

has countably many eigenvalues with finite multiplicities. If we label the eigenvectors such that

$$\lambda_0 \le \lambda_1 \le \cdots \tag{6.2.46}$$

then

$$\lim_{i \to \infty} \lambda_i = \infty, \tag{6.2.47}$$

and

$$-c_0 < \lambda_0. \tag{6.2.48}$$

There exists a sequence of corresponding eigenvectors u_i which are complete in $\mathcal{H}$ satisfying

$$K(u_i, u_j) = \delta_{ij} \tag{6.2.49}$$

and

$$B(u_i, u_j) = \lambda_i K(u_i, u_j) \tag{6.2.50}$$

as well as the expansion

$$B(u, v) = \sum_i \lambda_i K(u, u_i) K(u_i, v) \tag{6.2.51}$$

and

$$K(u, v) = \sum_i K(u, u_i) K(u_i, v). \tag{6.2.52}$$

The pairs (λ_i, u_i) are defined by the variational problems

$$\begin{aligned}\lambda_i &= \inf\{\frac{B(u)}{K(u)} : 0 \neq u \in \mathcal{H},\ K(u, u_j) = 0 \quad \forall\, 0 \leq j \leq i-1\} \\ &= B(u_i, u_i).\end{aligned} \tag{6.2.53}$$

This theorem is well known. A proof can be found in [15, Theorem 1.6.3].

We apply this theorem to the previously defined Hilbert space $\mathcal{H}_1$ and the bilinear (sesquilinear) forms B and K. Let (λ_i, w_i) be the corresponding pairs of eigenvalues and eigenvectors, then the w_i satisfy the ODE

$$Aw_i = \lambda_i \varphi_0 w_i \tag{6.2.54}$$

in the weak sense. The operator

$$\tilde{A} = \varphi_0^{-1} A \tag{6.2.55}$$

is symmetric in

$$\mathcal{H} = L^2(\mathbb{R}_+, d\mu), \qquad d\mu = \varphi_0 dt, \tag{6.2.56}$$

and the w_i are eigenfunctions of $\tilde{A}$

$$\tilde{A} w_i = \lambda_i w_i. \tag{6.2.57}$$

Equation (6.2.54) is equivalent to

$$\varphi_0 \tilde{A} w_i = \lambda_i \varphi_0 w_i \tag{6.2.58}$$

and $\tilde{A}$, with domain

$$D(\tilde{A}) = \langle w_i : i \in \mathbb{N} \rangle \subset \mathcal{H}, \tag{6.2.59}$$

is essentially self-adjoint as will be proved later, Lemma 6.5.1 on page 174, in a more general setting. We denote its unique self-adjoint extension by H_0.

We shall now prove that

$$e^{-\beta H_0}, \qquad \beta > 0, \tag{6.2.60}$$

is of trace class in $\mathcal{H}$.

First, we need two lemmata:

Lemma 6.2.6 *The embedding*

$$j : \mathcal{H}_1 \hookrightarrow \mathcal{H}_0 = L^2(\mathbb{R}_+, d\tilde{\mu}), \tag{6.2.61}$$

where

$$d\tilde{\mu} = (1+t)^{-2} dt, \tag{6.2.62}$$

is Hilbert–Schmidt.

Proof Maurin was the first to prove that the embedding

$$H^{m,2}(\Omega) \hookrightarrow L^2(\Omega), \tag{6.2.63}$$

where

$$\Omega \subset \mathbb{R}^n \tag{6.2.64}$$

is a bounded domain, is Hilbert–Schmidt provided

$$m > \frac{n}{2}, \tag{6.2.65}$$

cf. [38, Theorem 1, p. 336]. We adapt his proof to the present situation.

Let $w \in \mathcal{H}_1$, then, assuming w is real valued,

$$\begin{aligned} |w(t)|^2 = 2\int_0^t \dot{w}w &\le 2\int_o^\infty |\dot{w}|^2 + \frac{1}{2}\int_0^\infty |w|^2 \\ &\le c\|w\|_1^2 \end{aligned} \tag{6.2.66}$$

for all $t > 0$, where $\|\cdot\|_1$ is the norm in $\mathcal{H}_1$. To derive the last inequality in (6.2.66), we used Corollary 6.2.2. The estimate

$$|w(t)| \le c\|w\|_1 \qquad \forall t > 0 \tag{6.2.67}$$

is of course also valid for complex-valued functions from which infer that, for any $t > 0$, the linear form

$$w \to w(t), \qquad w \in \mathcal{H}_1, \tag{6.2.68}$$

is continuous, hence it can be expressed as

$$w(t) = \langle \varphi_t, w \rangle, \tag{6.2.69}$$

where

$$\varphi_t \in \mathcal{H}_1 \tag{6.2.70}$$

and

$$\|\varphi_t\|_1 \le c. \tag{6.2.71}$$

Now, let

$$e_i \in \mathcal{H}_1 \tag{6.2.72}$$

be an ONB, then

$$\sum_{i=0}^\infty |e_i(t)|^2 = \sum_{i=0}^\infty |\langle \varphi_t, e_i \rangle|^2 = \|\varphi_t\|_1^2 \le c^2. \tag{6.2.73}$$

Integrating this inequality over $\mathbb{R}_+$ with respect to $d\tilde{\mu}$, we infer

$$\sum_{i=0}^{\infty}\int_0^{\infty}|e_i(t)|^2 d\tilde{\mu} \le c^2 \tag{6.2.74}$$

completing the proof of the lemma. □

Lemma 6.2.7 *Let w_i be the eigenfunctions of H_0, then there exist positive constants c and γ such that*

$$\|w_i\|_1 \le c|\lambda_i + c_0|^{\gamma}\|w_i\|_0 \qquad \forall\, i \in \mathbb{N}, \tag{6.2.75}$$

where $\|\cdot\|_0$ is the norm in $\mathcal{H}_0$.

Proof We have, in view of (6.2.32) and (6.2.5),

$$\begin{aligned}\|w_i\|_1^2 &= (\lambda_i + c_0)\int_0^{\infty}\varphi_0(t)|w_i|^2 \\ &\le (\lambda_i + c_0)\left\{\int_0^{t_0}\varphi_0(t)|w_i|^2 + c_2\int_{t_0}^{\infty} t^r|w_i|^2\right\}.\end{aligned} \tag{6.2.76}$$

To estimate the second integral in the braces, we exploit the Assumptions (6.2.4) and (6.2.6) and choose m so large that

$$r \le p - \frac{p}{m}, \tag{6.2.77}$$

and hence,

$$t^r \le t^{p-\frac{p}{m}} \qquad \forall\, t \ge t_0 > 1. \tag{6.2.78}$$

Then, choosing small positive constants δ and ϵ, we apply Young's inequality, with

$$q = \frac{p}{p - p\delta} = \frac{1}{1-\delta} \tag{6.2.79}$$

and

$$q' = \delta^{-1} \tag{6.2.80}$$

to estimate the integral from above by

$$\begin{aligned}&\frac{1}{q}\epsilon^q\int_{t_0}^{\infty}\left\{t^{p-\frac{p}{m}}(1+t)^{\frac{p}{m}-p\delta}\right\}^q|w_i|^2 \\ &\qquad + \frac{1}{q'}\epsilon^{-q'}\int_{t_0}^{\infty}(1+t)^{-(\frac{p}{m}-p\delta)q'}|w_i|^2.\end{aligned} \tag{6.2.81}$$

Choosing, now, δ so small such that

$$(\frac{p}{m} - p\delta)\delta^{-1} > 2 \tag{6.2.82}$$

the preceding integrals can be estimated from above by

$$\frac{1}{q}\epsilon^q \int_{t_0}^{\infty} (1+t)^p |w_i|^2 + \frac{1}{q'}\epsilon^{-q'} \int_0^{\infty} (1+t)^{-2} |w_i|^2 \tag{6.2.83}$$

which in turn can be estimated by

$$\frac{1}{q}\epsilon^q c \|w_i\|_1^2 + \frac{1}{q'}\epsilon^{-q'} \|w_i\|_0^2, \tag{6.2.84}$$

in view of (6.2.27).

The first integral in the braces on the right-hand side of (6.2.76) can be estimated by

$$\begin{aligned}\int_0^{t_0} \varphi_0(t)|w_i|^2 &\le \frac{1}{2}c(1+t_0)^2\epsilon^2 \int_0^{\infty} |w_i|^2 \\ &\quad + \frac{1}{2}\epsilon^{-2} \int_0^{\infty} (1+t)^{-2}|w_i|^2 \\ &\le \tilde{c}\epsilon^2 \|w_i\|_1^2 + \frac{1}{2}\epsilon^{-2}\|w_i\|_0^2,\end{aligned} \tag{6.2.85}$$

because of (6.2.27).

Choosing now ϵ, γ and c appropriately, the result follows. □

We are now ready to prove:

Theorem 6.2.8 *Let $\beta > 0$, then the operator*

$$e^{-\beta H_0} \tag{6.2.86}$$

is of trace class in $\mathcal{H}$, i.e.

$$tr(e^{-\beta H_0}) = \sum_{i=0}^{\infty} e^{-\beta\lambda_i} = c(\beta) < \infty. \tag{6.2.87}$$

Proof In view of Lemma 6.2.6, the embedding

$$j : \mathcal{H}_1 \hookrightarrow \mathcal{H}_0 \tag{6.2.88}$$

is Hilbert–Schmidt. Let

$$w_i \in \mathcal{H} \tag{6.2.89}$$

be an ONB of eigenfunctions, then

$$\begin{aligned} e^{-\beta\lambda_i} &= e^{-\beta\lambda_i}\|w_i\|^2 = e^{-\beta\lambda_i}|\lambda_i + c_0|^{-1}\|w_i\|_1^2 \\ &\le e^{\beta c_0}e^{-\beta(\lambda_i+c_0)}|\lambda_i + c_0|^{-1}c|\lambda_i + c_0|^{2\gamma}\|w_i\|_0^2, \end{aligned} \tag{6.2.90}$$

in view of (6.2.75), but

$$\|w_i\|_0^2 = \|w_i\|_1^2\,\|\tilde{w}_i\|_0^2 = (\lambda_i + c_0)\|\tilde{w}_i\|_0^2, \tag{6.2.91}$$

where

$$\tilde{w}_i = w_i\|w_i\|_1^{-1} \tag{6.2.92}$$

is an ONB in $\mathcal{H}_1$, yielding

$$\sum_{i=0}^{\infty} e^{-\beta\lambda_i} \le c_\beta \sum_{i=0}^{\infty}\|\tilde{w}_i\|_0^2 < \infty, \tag{6.2.93}$$

since j is Hilbert–Schmidt. □

6.3 Trace Class Estimates in $\mathbb{R}^n$

Let us now consider eigenvalue problems in $\mathbb{R}^n$, $n \ge 2$, and let A be the linear elliptic operator

$$Au = -D_i(a^{ij}D_ju) + b(x)u, \tag{6.3.1}$$

where

$$a^{ij}, b \in L^\infty_{\text{loc}}(\mathbb{R}^n), \tag{6.3.2}$$

a^{ij} is symmetric and there exists $a_0 > 0$ such that

$$a_0|\xi|^2 \le a^{ij}\xi_i\xi_j \qquad \forall\, \xi \in \mathbb{R}^n \tag{6.3.3}$$

and there exists $R_0 > 1$ and positive p, c_1 such that

$$c_1|x|^p \le b(x) \qquad \forall\, |x| \ge R_0. \tag{6.3.4}$$

Then, we look at the eigenvalue problem

$$Au = \lambda u. \tag{6.3.5}$$

This eigenvalue problem can be solved by similar, if not identical, arguments as in the case of the Sturm–Liouville operator.

We define the bilinear forms

$$B(u, v) = \int_{\mathbb{R}^n} a^{ij} D_i \bar{u} D_j v \tag{6.3.6}$$

and

$$K(u, v) = \int_{\mathbb{R}^n} \bar{u} v \tag{6.3.7}$$

in $C_c^\infty(\mathbb{R}^n, \mathbb{C})$, and one can easily prove the analogues of Corollary 6.2.2 on page 142 and Theorem 6.2.5 on page 144, i.e. there exists $c_0 > 0$ such that

$$B + c_0 K \geq K, \tag{6.3.8}$$

K is compact relative to $B + c_0 K$, and there exists countably many pairs (λ_i, u_i) of eigenvalues with corresponding eigenfunctions satisfying the properties specified in Theorem 6.2.5, and we shall now prove that

$$e^{-\beta H}, \qquad \beta > 0, \tag{6.3.9}$$

is of trace class, where

$$H = \bar{A} \tag{6.3.10}$$

is the unique self-adjoint extension of A. We recall that A satisfies the estimate (6.2.28) on page 142 which can be rephrased as

$$A + c_0 \geq \frac{1}{2}\{-D_i(a^{ij} D_j) + b_0\}, \tag{6.3.11}$$

where

$$b_0(x) = \begin{cases} 0, & |x| \leq R_0, \\ b(x), & |x| > R_0. \end{cases} \tag{6.3.12}$$

The right-hand side of (6.3.11) is a strictly positive operator. Since eigenvalues, obtained by the variational process described in Theorem 6.2.5, also satisfy a minimax principle, cf. e.g. [15, Theorem 1.6.4], we conclude that

$$\mu_i \leq \tilde{\lambda}_i \qquad \forall\, i \in \mathbb{N}, \tag{6.3.13}$$

where μ_i are the ordered eigenvalues of the operator on the right-hand side of (6.3.11) and $\tilde{\lambda}_i$ the ordered eigenvalues of $A + c_0$. Hence, it suffices to prove that

$$\sum_{i=0}^{\infty} e^{-\beta \mu_i} < \infty. \tag{6.3.14}$$

For reasons that will become apparent later, we shall derive trace class estimates for the operator

$$\tilde{A}u = -\alpha_0 \Delta u + \Theta u, \tag{6.3.15}$$

where

$$\alpha_0 = \frac{a_0}{2,} \tag{6.3.16}$$

$$\Theta(x) = \frac{c_1}{2}\eta_0 |x|^{p_0}, \tag{6.3.17}$$

$$p_0 = \min(p, 1) \tag{6.3.18}$$

and η_0 is a cut-off function such that

$$\eta_0(x) = \begin{cases} 0, & |x| \le R_0, \\ 1, & |x| \ge 2R_0. \end{cases} \tag{6.3.19}$$

We emphasize that

$$\Theta \le \frac{1}{2} b_0 \tag{6.3.20}$$

and hence, due to the inequalities (6.3.3) and (6.3.11),

$$A + c_0 \ge \tilde{A}. \tag{6.3.21}$$

Therefore, it will suffice to prove that $\tilde{A}$ is a trace class operator. To simplify notations, let us also drop the tilde and let us write A for the operator in (6.3.15), i.e.

$$Au = -\alpha_0 \Delta u + \Theta u. \tag{6.3.22}$$

Furthermore, the previous definitions of the bilinear form B and the Hilbert space $\mathcal{H}_1$ are also adopted while the Hilbert space $\mathcal{H}$ is now $L^2(\mathbb{R}^n)$. A is essentially self-adjoint in $\mathcal{H}$ with domain

$$D(A) = \langle u_i : i \in \mathbb{N} \rangle, \tag{6.3.23}$$

where u_i are a sequence of mutually orthogonal eigenfunctions of A

$$Au_i = \lambda_i u_i. \tag{6.3.24}$$

Note that

$$0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \tag{6.3.25}$$

We shall first prove that the eigenfunctions of A are smooth with uniformly bounded norms

$$\|u_i\|_{m,2}^2 = \sum_{|\alpha|\leq m} \int_{\mathbb{R}^n} |D^\alpha u|^2 \tag{6.3.26}$$

in the usual Sobolev spaces $H^{m,2}(\mathbb{R}^n)$.

Theorem 6.3.1 *Let $u \in H^{m-1,2}(\mathbb{R}^n) \cap \mathcal{H}_1$ be a weak solution of the equation*

$$-\alpha_0 \Delta u + \Theta u = f, \tag{6.3.27}$$

where $f \in H^{m-2,2}(\mathbb{R}^n)$, $m \geq 2$, and assume that

$$\|u\|_{m-1,2}^2 + \sum_{|\alpha|\leq m-2} \int_{\mathbb{R}^n} \Theta |D^\alpha u|^2 \leq c\|f\|_{m-3,2}^2, \tag{6.3.28}$$

then $u \in H^{m,2}(\mathbb{R}^n)$ and

$$\|u\|_{m,2}^2 + \sum_{|\alpha|\leq m-1} \int_{\mathbb{R}^n} \Theta |D^\alpha u|^2 \leq c\|f\|_{m-2,2}^2, \tag{6.3.29}$$

where the constants c depend on m, Θ, p_0, n and α_0.

Proof We shall prove the theorem by induction. First, in the lemma below we shall prove that the theorem is valid for $m = 2$. Thus, let us assume that the theorem is correct for $m = q \geq 2$ and show that it is then also valid for $m = q + 1$.

Fix $1 \leq k \leq n$ and define

$$v = D_k u. \tag{6.3.30}$$

Differentiating (6.3.27), we obtain

$$-\alpha_0 \Delta v + \Theta v = D_k f - D_k \Theta u \equiv \tilde{f}. \tag{6.3.31}$$

We observe that

$$\tilde{f} \in H^{q-2,2}(\mathbb{R}^n) \tag{6.3.32}$$

and that

$$\|\tilde{f}\|_{q-2,2}^2 \leq c\|f\|_{q-1,2}^2, \tag{6.3.33}$$

because

$$\begin{aligned}\|D_k \Theta u\|_{q-2,2}^2 &\leq c\{\|u\|_{q-2,2} + \sum_{|\alpha|\leq q-2} \Theta |D^\alpha u|^2\} \\ &\leq c\{\|u\|_{q,2} + \sum_{|\alpha|\leq q-1} \int_{\mathbb{R}^n} \Theta |D^\alpha u|^2\} \\ &\leq c\|f\|_{q-2,2}^2\end{aligned} \tag{6.3.34}$$

in view of the definition of Θ and (6.3.29). Applying then the induction hypothesis for $m = q$, we conclude that the theorem is also valid for $m = q + 1$. □

Lemma 6.3.2 *The preceding theorem is valid for $m = 2$, i.e. any weak solution $u \in \mathcal{H}_1$ of*

$$-\alpha_0 \Delta u + \Theta u = f \tag{6.3.35}$$

satisfies the estimates (6.3.28) *and* (6.3.29)*, where we note that*

$$H^{-1,2}(\mathbb{R}^n) = \{ D_i g^i + g_0 \colon g_0, g^i \in L^2(\mathbb{R}^n) \} \tag{6.3.36}$$

is the dual space of $H^{1,2}(\mathbb{R}^n)$ and

$$L^2(\mathbb{R}^n) \hookrightarrow H^{-1,2}(\mathbb{R}^n) \subset \mathcal{H}_1'. \tag{6.3.37}$$

Equation (6.3.35) *has also a unique solution which can be found by minimizing a functional if we consider f and u to be real valued. Of course we then also obtain a solution for complex-valued f.*

Proof First, the existence of a solution $u \in \mathcal{H}_1$ of (6.3.35) satisfying

$$B(u) = \langle Au, u \rangle \le c \|f\|^2 \tag{6.3.38}$$

is obvious, since

$$K(v) = \|v\|^2 \tag{6.3.39}$$

is compact relative to B, and for real-valued f and v and $\epsilon > 0$, we have

$$\begin{aligned} |\langle f, v \rangle| &\le \frac{1}{2}\epsilon \|v\|^2 + \frac{1}{2}\epsilon^{-1}\|f\|^2 \\ &\le \frac{1}{2}\epsilon \lambda_0^{-1} B(v) + \frac{1}{2}\epsilon^{-1}\|f\|^2, \end{aligned} \tag{6.3.40}$$

where $0 < \lambda_0$ is the smallest eigenvalue of A. It then immediately follows that the variational problem

$$J(v) = B(v) - 2\langle f, v \rangle \to \min \qquad \forall\, v \in \mathcal{H}_1 \tag{6.3.41}$$

has a unique solution u, which is also a weak solution of the corresponding Euler–Lagrange equation, and that u satisfies (6.3.38) which is equivalent to (6.3.28) for $m = 2$.

Secondly, to prove (6.3.29) for $m = 2$ we note that

$$u \in C^\infty(\mathbb{R}^n), \tag{6.3.42}$$

in view of the interior L^2-estimates, since A is uniformly elliptic with smooth coefficients. Hence, choosing a cut-off function η

$$0 \le \eta \in C_c^\infty(\mathbb{R}^n) \tag{6.3.43}$$

such that

$$|D\eta| \le 2 \tag{6.3.44}$$

and $1 \le k \le n$, we have

$$D_k u\eta^2 \in H^{1,2}(\mathbb{R}^n). \tag{6.3.45}$$

Multiplying (6.3.35) by

$$-D_k(D^k u\eta^2), \tag{6.3.46}$$

where we use summation convention, integrating by parts and employing some trivial estimates, we deduce

$$\begin{aligned} &\frac{\alpha_0}{2}\int_{\mathbb{R}^n} |D^2u|^2\eta^2 + \frac{1}{2}\int_{\mathbb{R}^n} \Theta|Du|^2\eta^2 \\ &\le c\{\|f\|^2 + \|u\|_{1,2}^2 + \int_{\mathbb{R}^n}\Theta|u|^2\} \le c\|f\|^2, \end{aligned} \tag{6.3.47}$$

where we also used (6.3.38), (6.3.44) and where the symbol c may represent different constants. Since η is an arbitrary cut-off function, only subject to (6.3.44), the result follows. □

As a corollary to Theorem 6.3.1 and Lemma 6.3.2, we obtain

Theorem 6.3.3 *Let $f \in H^{m-2,2}(\mathbb{R}^n)$, $m \ge 2$, then the equation*

$$Au = -\alpha_0\Delta u + \Theta u = f \tag{6.3.48}$$

has a unique solution $u \in H^{m,2}(\mathbb{R}^n) \cap \mathcal{H}_1$ satisfying

$$\|u\|_{m,2}^2 + \sum_{|\alpha|\le m-1}\int_{\mathbb{R}^n}\Theta|D^\alpha u|^2 \le c\|f\|_{m-2}^2, \tag{6.3.49}$$

where c depends on m, n, Θ, p_0 and α_0.

Moreover, the eigenfunctions u satisfying

$$Au = \lambda u \tag{6.3.50}$$

are smooth and the $H^{m,2}$-norm can be estimated by

$$\|u\|_{m,2}^2 \le c_m\lambda^m\|u\|^2 \qquad \forall m \ge 1, \tag{6.3.51}$$

where c_m also depends on the smallest eigenvalue λ_0 of A.

Proof It suffices to prove the last estimate, which can be deduced from (6.3.49) by induction

$$\|u\|^2_{m,2} \le c\lambda^2 \|u\|^2_{m-2} \le c\lambda^2\lambda^{m-2}\|u\|^2 = c\lambda^m\|u\|^2. \tag{6.3.52}$$

The proof for $m = 1$ follows from

$$\|u\|^2_{1,2} \le c(1 + \lambda_0^{-1})B(u) = c(1 + \lambda_0^{-1})\lambda\|u\|^2. \tag{6.3.53}$$

□

Lemma 6.3.4 *Let $\mathcal{H}_{2m}(\mathbb{R}^n)$, $m \ge 1$, be the completion of $C_c^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the scalar product*

$$\langle A^m u, A^m v\rangle = \int_{\mathbb{R}^n} A^m \bar{u} A^m v, \tag{6.3.54}$$

then

$$\|u\|^2_{2m,2} \le c\|A^m u\|^2 \qquad \forall u \in \mathcal{H}_{2m}(\mathbb{R}^n), \tag{6.3.55}$$

$$\|A^{m-1}u\|^2 \le c\|A^m u\|^2 \qquad \forall u \in \mathcal{H}_{2m}(\mathbb{R}^n), \tag{6.3.56}$$

and the eigenfunctions of A are complete in $\mathcal{H}_{2m}(\mathbb{R}^n)$ for any $m \ge 1$. Furthermore, if the eigenfunctions are mutually orthogonal in $L^2(\mathbb{R}^n)$ then they are also mutually orthogonal in $\mathcal{H}_{2m}(\mathbb{R}^n)$ and vice versa.

Proof We prove the first estimate by induction.

"(6.3.55)" The estimate is valid for $m = 1$, in view of Theorem 6.3.3.

Suppose the estimate is valid for $q \ge 1$ and let u be test function, then

$$\begin{aligned} \|u\|^2_{2(q+1),2} &\le c\|Au\|^2_{2q,2} \\ &\le c\|A^q(Au)\|^2 \\ &= c\|A^{q+1}u\|^2, \end{aligned} \tag{6.3.57}$$

where we used Theorem 6.3.3 in the first inequality and the induction hypothesis in the second.

"(6.3.56)" Let $m \ge 1$, then

$$\|A^{m-1}u\|^2 \le \lambda_0^{-1}\langle AA^{m-1}u, A^{m-1}u\rangle \le \lambda_0^{-1}\|A^m u\|\,\|A^{m-1}u\|. \tag{6.3.58}$$

It remains to prove the completeness of the eigenfunctions u_i obtained in Theorem 6.2.5 on page 144. They are complete in $\mathcal{H}_1$ but also in $L^2(\mathbb{R}^n)$ because of the Parseval's identity (6.2.52).

If they were not complete in $H_{2m}(\mathbb{R}^n)$ for some m, then there would exist $0 \ne u \in H_{2m}(\mathbb{R}^n)$ such that

$$0 = \langle A^m u, A^m u_i \rangle = \langle u, A^{2m} u_i \rangle = \lambda_i^{2m} \langle u, u_i \rangle \qquad \forall i \in \mathbb{N}, \tag{6.3.59}$$

hence we would infer

$$u = 0; \tag{6.3.60}$$

a contradiction. □

The elliptic operator A with

$$D(A) = C_c^\infty(\mathbb{R}^n) \subset \mathcal{H} = L^2(\mathbb{R}^n) \tag{6.3.61}$$

is essentially self-adjoint; for a proof, see Lemma 6.5.1 on page 174. Let us denote its unique self-adjoint extension by the same symbol since the domain of the extension is $\mathcal{H}_2(\mathbb{R}^n)$. We are almost ready to prove the trace class estimates for A but we need two additional lemmata.

Lemma 6.3.5 *Let $\mathcal{H}_0$ be the Hilbert space*

$$\mathcal{H}_0 = L^2(\mathbb{R}^n, d\mu) \tag{6.3.62}$$

where

$$d\mu = (1 + |x|)^{-(n+1)}, \tag{6.3.63}$$

then the embedding

$$j : \mathcal{H}_{2m}(\mathbb{R}^n) \hookrightarrow \mathcal{H}_0 \tag{6.3.64}$$

is Hilbert–Schmidt provided $m > \frac{n}{2}$.

Proof As in the proof of Lemma 6.2.6 on page 145, we adapt Maurin's original proof for bounded subsets of $\mathbb{R}^n$ to the present situation. Let φ be a real-valued test function

$$\varphi \in C_c^\infty(\mathbb{R}^n) \tag{6.3.65}$$

and S the differential operator

$$S = D_1 \circ D_2 \circ \cdots \circ D_n, \tag{6.3.66}$$

then

$$\varphi^2(x) = \int_{-\infty}^{x^1} \cdots \int_{-\infty}^{x^n} S(\varphi^2). \tag{6.3.67}$$

The integrand can be expressed in the form

$$S(\varphi^2) = \sum_{|\alpha|+|\beta|=n} c_{\alpha\beta} D^\alpha \varphi S^\beta \varphi \tag{6.3.68}$$

with multi-indices α, β and constants $c_{\alpha\beta}$, where some constants may be zero. Hence, we deduce

$$|\varphi|^2 \le c\|\varphi\|_{n,2}^2 \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^n). \tag{6.3.69}$$

This estimate is of course also valid for complex-valued $u \in \mathcal{H}_{2m}(\mathbb{R}^n)$.

Now, let $m > \frac{n}{2}$ and let e_i be an ONB in $\mathcal{H}_{2m}(\mathbb{R}^n)$ consisting of eigenfunctions of A, then, for any $x \in \mathbb{R}^n$, the map

$$u \to u(x), \qquad u \in \mathcal{H}_{2m}(\mathbb{R}^n), \tag{6.3.70}$$

is continuous, because of (6.3.69) and (6.3.55), hence it can be expressed in the form

$$u(x) = \langle A^m \varphi_x, A^m u\rangle \qquad \forall\, u \in \mathcal{H}_{2m}(\mathbb{R}^n), \tag{6.3.71}$$

where

$$\varphi_x \in \mathcal{H}_{2m}(\mathbb{R}^n) \tag{6.3.72}$$

and

$$\|A^m \varphi_x\| \le c \tag{6.3.73}$$

are uniformly bounded independent of x. If we choose especially $u = e_i$ then, for any $x \in \mathbb{R}^n$,

$$\sum_{i=0}^{\infty} |e_i(x)|^2 = \sum_{i=0}^{\infty} |\langle A^m \varphi_x, A^m e_i\rangle|^2 = \|A^m \varphi_x\|^2 \le c^2. \tag{6.3.74}$$

Integrating now with respect to measure in (6.3.63) completes the proof of the lemma. □

The next lemma is analogous to Lemma 6.2.7 on page 147.

Lemma 6.3.6 *Let u_i be an eigenfunction of A with eigenvalue λ_i, then there exist positive constants c and γ such that*

$$\|u_i\|_1^2 = B(u_i) = \lambda_i \|u_i\|^2 \le c\lambda_i^\gamma \|u_i\|_0^2, \tag{6.3.75}$$

where c, γ are independent of u_i and $\|\cdot\|_0$ is the norm in $\mathcal{H}_0$.

Proof We have

$$B(u_i) = \int_{\mathbb{R}^n} \{\alpha_0 |Du_i|^2 + \Theta |u_i|^2\} = \lambda_i \|u_i\|^2. \tag{6.3.76}$$

Moreover, we know, in view of (6.3.17) and (6.3.19), that

$$\Theta(x) \ge \frac{1}{2} c_1 |x|^{p_0} \qquad \forall\, |x| \ge 2R_0 > 1, \tag{6.3.77}$$

where $p_0 > 0$. Choosing small positive δ, ϵ and applying Young's inequality with

$$q = \frac{p_0}{p_0 - p_0\delta} = \frac{1}{1-\delta} \tag{6.3.78}$$

and

$$q' = \delta^{-1} \tag{6.3.79}$$

we can estimate the L^2-norm on the right-hand side of (6.3.76) from above by

$$\frac{1}{q}\epsilon^q \int_{\mathbb{R}^n} (1+|x|)^{p_0}|u_i|^2 + \frac{1}{q'}\epsilon^{-q'} \int_{\mathbb{R}^n} (1+|x|)^{-p_0(1-\delta)\delta^{-1}}|u_i|^2. \tag{6.3.80}$$

Choosing δ so small that

$$p_0\delta^{-1} > n+2 \tag{6.3.81}$$

we deduce

$$\|u_i\|^2 \le c\frac{1}{q}\epsilon^q B(u_i) + c\frac{1}{q'}\epsilon^{-q'}\|u_i\|_0^2 \tag{6.3.82}$$

leading immediately to the desired estimate by choosing ϵ appropriately. □

Now we can prove:

Theorem 6.3.7 *Let A be the elliptic differential operator*

$$Au = -\alpha_0 \Delta u + \Theta u, \tag{6.3.83}$$

then

$$e^{-\beta A}, \qquad \beta > 0, \tag{6.3.84}$$

is of trace class in $L^2(\mathbb{R}^n)$, i.e.

$$\sum_{i=0}^{\infty} e^{-\beta\lambda_i} < \infty. \tag{6.3.85}$$

Proof Let (u_i) be an ONB of eigenfunctions of A in $\mathcal{H} = L^2(\mathbb{R}^n)$ and let $m > \frac{n}{2}$, then

$$\begin{aligned} e^{-\beta\lambda_i} &= e^{-\beta\lambda_i}\|u_i\|^2 = e^{-\beta\lambda_i}\lambda_i^{-1}B(u_i) \\ &\le e^{-\beta\lambda_i}\lambda_i^{-1}c\lambda_i^{\gamma}\|u_i\|_0^2 \\ &\le e^{-\beta\lambda_i}\lambda_i^{-1}c\lambda_i^{\gamma}\|A^m u_i\|^2\|\tilde{u}_i\|_0^2 \\ &= ce^{-\beta\lambda_i}\lambda_i^{2m+\gamma-1}\|\tilde{u}_i\|_0^2, \end{aligned} \tag{6.3.86}$$

$$\tilde{u}_i = \frac{u_i}{\|A^m u_i\|} \tag{6.3.87}$$

and where we also used the estimate (6.3.75) to derive the first inequality in (6.3.86).

Hence, we infer

$$e^{-\beta\lambda_i} \le c_\beta \|\tilde{u}_i\|_0^2, \tag{6.3.88}$$

where

$$c_\beta = c \sup_{t>0} e^{-\beta t} t^{2m+\gamma-1}, \tag{6.3.89}$$

and we finally conclude

$$\sum_{i=0}^{\infty} e^{-\beta\lambda_i} \le c_\beta \sum_{i=0}^{\infty} \|\tilde{u}_i\|_0^2 < \infty, \tag{6.3.90}$$

because the embedding

$$j : \mathcal{H}_{2m}(\mathbb{R}^n) \hookrightarrow \mathcal{H}_0 \tag{6.3.91}$$

is Hilbert–Schmidt, in view of Lemma 6.3.5. □

6.4 The Hamiltonians Governing Quantum Gravity

In the Chaps. [3, 4, 5], we applied our model of quantum gravity to a globally hyperbolic spacetime with an asymptotically Euclidean Cauchy hypersurface, a Schwarzschild-AdS and a Kerr-AdS black hole, respectively. In all three cases, the quantized model had the same structure; namely, it consisted of special solutions to a wave equation

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0, \tag{6.4.1}$$

in a quantum spacetime

$$N = \mathbb{R}_+ \times \mathcal{S}_0, \tag{6.4.2}$$

where $\mathcal{S}_0$ is a n-dimensional, $n \ge 3$, Cauchy hypersurface of the original spacetime, or, in case of black holes, the smooth limit of Cauchy hypersurfaces. The Laplacian and the scalar curvature correspond to the metric g_{ij} in $\mathcal{S}_0$.

The special solutions are a sequence of smooth functions which are a product of temporal and spatial eigenfunctions of elliptic operators, where the spatial eigenfunctions are eigendistributions.

In case of the globally hyperbolic spacetime with an asymptotically Euclidean Cauchy hypersurface the solutions to the wave equation can be expressed in the form

$$u_{ij} = w_i v_{ij}, \qquad i \in \mathbb{N},\ 1 \le j \le m \le \infty, \tag{6.4.3}$$

where the w_i are the eigenfunctions of a temporal Hamilton operator H_0

$$H_0 w_i = \lambda_i w_i \tag{6.4.4}$$

and the λ_i have multiplicity one such that

$$0 < \lambda_0 < \lambda_1 < \cdots \tag{6.4.5}$$

and for each fixed i, the at most countably many v_{ij} generate an eigenspace

$$\mathscr{E}_{\lambda_i} \subset \mathscr{S}'(\mathcal{S}_0) \tag{6.4.6}$$

of a spatial Hamiltonian H_1, i.e.

$$H_1 v_{ij} = \lambda_i v_{ij}. \tag{6.4.7}$$

We have

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{6.4.8}$$

In the two remaining cases of the black holes, the special solutions are labelled by three indices

$$u_{ijk} = w_i \zeta_{ijk} \varphi_j, \tag{6.4.9}$$

where the w_i are the same temporal eigenfunctions as before, the φ_j are the eigenfunctions of an elliptic operator A on a smooth compact Riemannian manifold (M, σ_{ij}), where topologically

$$M \simeq \mathbb{S}^{n-1}, \tag{6.4.10}$$

at least in the physically interesting cases, i.e.

$$A\varphi_j = \tilde{\mu}_j \varphi_j, \tag{6.4.11}$$

$$\tilde{\mu}_0 < \tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots \tag{6.4.12}$$

The φ_j form a mutually orthogonal basis of $L^2(M)$. For a Schwarzschild-AdS black hole, we know that

$$\tilde{\mu}_0 \le 0, \tag{6.4.13}$$

and for a Kerr-AdS black hole, this condition can be assured by assuming that the rotational parameter a is small enough such that the scalar curvature of σ_{ij} is positive. Let us emphasize that we considered in Chap. 6 Kerr-AdS black holes of odd dimensions

$$\dim N = 2m + 1, \qquad m \ge 2, \tag{6.4.14}$$

and assumed that all rotational parameters a_i are equal

$$a_i = a \neq 0 \qquad \forall\, 1 \le i \le m. \tag{6.4.15}$$

The ζ_{ijk} are eigendistributions in $\mathscr{S}'(\mathbb{R})$ satisfying

$$-\zeta''_{ijk} = \omega_{ij}^2 \zeta_{ijk}, \qquad k = 1, 2, \tag{6.4.16}$$

where

$$\zeta_{ij1}(\tau) = \frac{1}{\sqrt{2\pi}} e^{i\omega_{ij}\tau} \tag{6.4.17}$$

and

$$\zeta_{ij2}(\tau) = \frac{1}{\sqrt{2\pi}} e^{-i\omega_{ij}\tau}, \tag{6.4.18}$$

where

$$\omega_{ij} \ge 0 \tag{6.4.19}$$

is defined by the relation

$$\lambda_i = \tilde{\mu}_j + (n-1)\omega_{ij}^2, \tag{6.4.20}$$

i.e. for any $i \in \mathbb{N}$, we look for all j satisfying

$$\tilde{\mu}_j \le \lambda_i \tag{6.4.21}$$

and then choose $\omega_{ij} \ge 0$ satisfying (6.4.20). Let N_i be the set of integers such that the $\tilde{\mu}_j$ satisfy (6.4.21), then the smooth functions

$$\zeta_{ijk}\varphi_j \tag{6.4.22}$$

are mutually orthogonal in $L^2(M, \sigma_{ij})$—for fixed i and k; note that we only have two different eigendistributions ζ_{ijk}, if

$$\omega_{ij} > 0, \tag{6.4.23}$$

otherwise we have only one. The eigendistributions ζ_{ij1} and ζ_{ij2} are also considered to be "orthogonal" since their Fourier transforms

$$\hat{\zeta}_{ijk} = \delta_{\pm\omega_{ij}} \tag{6.4.24}$$

have disjoint supports.

Finally, the smooth functions u_{ijk} in (6.4.9) can be considered to be mutually orthogonal since u_{ijk} and $u_{i'j'k'}$ are mutually orthogonal in

$$L^2(\mathbb{R}_+, d\mu) \otimes L^2(M), \tag{6.4.25}$$

where

$$d\mu = t^{2-\frac{4}{n}} dt, \tag{6.4.26}$$

if

$$\omega_{ij} = \omega_{i'j'} \quad \wedge \quad k = k' \tag{6.4.27}$$

and as tempered distributions otherwise.

The u_{ijk} are eigendistributions for both the temporal Hamiltonian H_0 as well as for the spatial Hamiltonian H_1 with the same eigenvalues λ_i, where now the eigenvalues have finite multiplicities different from 1 by definition of the eigendistributions and the u_{ijk} also solve the wave equation, since the wave equation can be expressed as

$$\varphi_0(H_0 u - H_1 u) = 0, \tag{6.4.28}$$

where $u = u(t, x)$ is a smooth function

$$x \in \mathcal{S}_0 = \mathbb{R} \times M \tag{6.4.29}$$

and

$$\varphi_0(t) = t^{2-\frac{4}{n}}. \tag{6.4.30}$$

In Sect. 6.5 on page 173, we shall prove that we can define an abstract Hilbert space $\mathcal{H}$, where the eigendistributions u_{ijk} resp. u_{ij} in (6.4.3) form a basis of mutually orthogonal unit vectors such that the Hamiltonian H_1 can be defined on the dense subspace, which is the algebraic span of the basis vectors, as an essentially self-adjoint operator. Let $\tilde{H}_1$ be its unique self-adjoint extension, namely its closure, then we shall prove that for any $\beta > 0$

$$e^{-\beta \tilde{H}_1} \tag{6.4.31}$$

is of trace class in $\mathcal{H}$. In addition, $\tilde{H}_1$ satisfies

$$\tilde{H}_1 \geq \lambda_0 I, \qquad \lambda_0 > 0. \tag{6.4.32}$$

The temporal eigenfunctions w_i solve the equation

$$H_0 w_i = \lambda_i w_i, \tag{6.4.33}$$

where

$$H_0 w_i = \varphi_0^{-1}\left(-\frac{1}{32}\frac{n^2}{n-1}\ddot{w}_i + nt^2|\Lambda| w_i\right), \tag{6.4.34}$$

which is equivalent to

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w}_i + nt^2|\Lambda| w_i = \lambda_i \varphi_0 w_i, \tag{6.4.35}$$

i.e. it is one of the Sturm–Liouville eigenvalue problems which we considered in (6.2.2) on page 140, where now

$$Au = -\frac{1}{32}\frac{n^2}{n-1}\ddot{u} + nt^2|\Lambda|u, \tag{6.4.36}$$

$$b(t) = nt^2|\Lambda| \tag{6.4.37}$$

and

$$\varphi_0(t) = t^{2-\frac{4}{n}}. \tag{6.4.38}$$

The eigenvalues are obtained by looking at the generalized eigenvalue problem

$$B(u, v) = \lambda K(u, v) \qquad \forall\, v \in \mathcal{H}_1, \tag{6.4.39}$$

where

$$B(u, v) = \langle Au, v\rangle \tag{6.4.40}$$

and

$$K(u, v) = \int_{\mathbb{R}_+} t^{2-\frac{n}{4}}\bar{u}v, \tag{6.4.41}$$

cf. Theorem 6.2.5 on page 144, where now

$$c_0 = 0. \tag{6.4.42}$$

Hence, the assumptions of Theorem 6.2.8 on page 148 are all satisfied and we conclude

Theorem 6.4.1 *Let $\beta > 0$ and let H_0 be the Hamiltonian in (6.4.34), then the operator*

$$e^{-\beta H_0} \tag{6.4.43}$$

is of trace class $L^2(\mathbb{R}_+, d\mu)$.

There is also a spatial Hamiltonian H_1, which, in the case of the black holes considered, is a direct product of a classical harmonic oscillator in $\mathbb{R}$ and an elliptic operator A on a compact, smooth Riemannian manifold $M = M^{n-1}$, $n \geq 3$, with metric σ_{ij}, where A has the form

$$A\varphi = -(n-1)\Delta\varphi - \frac{n}{2}R\varphi \tag{6.4.44}$$

and the Laplacian is the Laplacian in M and R the scalar curvature of the metric. A is self-adjoint with domain

$$D(A) = H^{2,2}(M) \subset L^2(M), \tag{6.4.45}$$

where

$$H^{m,2}(M), \qquad m \in M, \tag{6.4.46}$$

are the usual Sobolev spaces with norm

$$\|\varphi\|^2_{m,2} = \sum_{|\alpha|\le m} \int_M |D^\alpha \varphi|^2. \tag{6.4.47}$$

A has a pure point spectrum with countable many eigenvalues $\tilde{\mu}_j$ with finite multiplicities and mutually orthogonal eigenfunctions φ_j such that

$$\tilde{\mu}_0 < \tilde{\mu}_1 \le \cdots \tag{6.4.48}$$

and

$$\lim_j \tilde{\mu}_j = \infty. \tag{6.4.49}$$

We want to prove that

$$e^{-\beta A}, \qquad \beta > 0, \tag{6.4.50}$$

is of trace class in $L^2(M)$.

The proof of this result will follow the arguments in Sect. 6.3 very closely.

Lemma 6.4.2 *Let $m > \frac{n-1}{2}$, then the embedding*

$$j : H^{m,2}(M) \hookrightarrow L^2(M) \tag{6.4.51}$$

is Hilbert–Schmidt.

Proof This result is due to Maurin and its proof is identical with the proof of Lemma 6.2.6 apart from some obvious modifications. □

We also need the lemma:

Lemma 6.4.3 *Let $m \in \mathbb{N}$, then there exists $c_m > 0$ such that*

$$\|\varphi\|^2_{2m,2} \le c_m(\|A^m\varphi\|^2 + \|\varphi\|^2) \tag{6.4.52}$$

and the bilinear form

$$\langle A^m\varphi, A^m\psi\rangle_0 + \langle\varphi,\psi\rangle_0 \tag{6.4.53}$$

defines an equivalent scalar product in $H^{2m,2}(M)$, where

$$\langle\varphi,\psi\rangle_0 = \int_M \bar{\varphi}\psi. \tag{6.4.54}$$

Proof Let

$$f \in H^{m,2}(M) \tag{6.4.55}$$

and

$$\varphi \in H^{2,2}(M) \tag{6.4.56}$$

a solution of

$$A\varphi = f, \tag{6.4.57}$$

then it is well known that

$$\varphi \in H^{m+2,2}(M) \tag{6.4.58}$$

and there exists $\tilde{c}_m$ such that

$$\|\varphi\|_{m+2,2} \le \tilde{c}_m(\|f\|_{m,2} + \|\varphi\|_0). \tag{6.4.59}$$

The constant $\tilde{c}_m$ also depends on A and M. Using this estimate, the relation (6.4.52) can be easily proved by induction. □

Now, we are ready to prove:

Theorem 6.4.4 *Let A be the self-adjoint operator in* (6.4.44)*, then*

$$e^{-\beta A} \tag{6.4.60}$$

is of trace class in $L^2(M)$ for any $\beta > 0$.

Proof Let $m > \frac{n-1}{4}$ and equip $H^{2m,2}(M)$ with the scalar product (6.4.53) such that

$$\|\varphi\|^2_{2m,2} = \langle A^m\varphi, A^m\varphi\rangle_0 + \langle\varphi,\varphi\rangle_0, \tag{6.4.61}$$

then any eigenfunctions φ_i, φ_j of A satisfy

$$\langle\varphi_i,\varphi_j\rangle_0 = 0 \implies \langle\varphi_i,\varphi_j\rangle_{2m,2} = 0. \tag{6.4.62}$$

Let (φ_j) be an ONB of eigenfunctions of A in $L^2(M)$ and define

$$\tilde{\varphi}_j = \varphi_j\|\varphi_j\|^{-1}_{2m,2}, \tag{6.4.63}$$

then the $\tilde{\varphi}_j$ form an ONB in $H^{2m,2}(M)$ and we conclude

$$\begin{aligned} e^{-\beta\tilde{\mu}_j} &= e^{-\beta\tilde{\mu}_j}\|\varphi_j\|^2_0 = e^{-\beta\tilde{\mu}_j}\|\varphi_j\|^2_{2m,2}\|\tilde{\varphi}_j\|^2_0 \\ &= e^{-\beta\tilde{\mu}_j}(1+|\tilde{\mu}_j|^{2m})\|\tilde{\varphi}_j\|^2_0 \le c_\beta\|\tilde{\varphi}_j\|^2_0 \end{aligned} \tag{6.4.64}$$

yielding

$$\sum_{j=0}^{\infty} e^{-\beta\tilde{\mu}_j} \le c_\beta \sum_{j=0}^{\infty} \|\tilde{\varphi}_j\|_0^2 < \infty \tag{6.4.65}$$

in view of Lemma 6.4.2. □

With the help of the preceding lemma, we can now prove that, in case of the black holes, the spatial Hamiltonian H_1 has the property that

$$e^{-\beta H_1} \tag{6.4.66}$$

is of trace class for all $\beta > 0$, where we still have to define an appropriate Hilbert space.

We have

$$H_1 v = -(n-1)\ddot{v} - Av, \tag{6.4.67}$$

where we write v as product

$$v(\tau, x) = \zeta(\tau)\varphi(x) \tag{6.4.68}$$

with

$$\tau \in \mathbb{R} \quad \wedge \quad x \in M = M^{n-1}, \tag{6.4.69}$$

where A is the differential operator in (6.4.44). Let φ_j be the eigenfunctions of A with eigenvalues $\tilde{\mu}_j$, then, for any eigenvalue λ_i, we define

$$N_i = \{j \in \mathbb{N} : \tilde{\mu}_j \le \lambda_i\} \tag{6.4.70}$$

and $\omega_{ij} \ge 0$ such that

$$(n-1)\,\omega_{ij}^2 + \tilde{\mu}_j = \lambda_i. \tag{6.4.71}$$

Note that

$$0 \in N_i \qquad \forall\, i \in \mathbb{N}, \tag{6.4.72}$$

since

$$\tilde{\mu}_0 \le 0. \tag{6.4.73}$$

Let

$$\zeta_{ijk}, \qquad k = 1, 2, \tag{6.4.74}$$

be the tempered distributions

$$\zeta_{ij1} = \frac{1}{\sqrt{2\pi}} e^{i\omega_{ij}\tau} \tag{6.4.75}$$

and

$$\zeta_{ij2} = \frac{1}{\sqrt{2\pi}} e^{-i\omega_{ij}\tau}, \tag{6.4.76}$$

where this distinction only occurs for

$$\omega_{ij} > 0. \tag{6.4.77}$$

Let $\hat{\zeta}_{ijk}$ be the Fourier transform of ζ_{ijk}, then

$$\hat{\zeta}_{ij1} = \delta_{\omega_{ij}} \quad \wedge \quad \hat{\zeta}_{ij2} = \delta_{-\omega_{ij}} \tag{6.4.78}$$

such that these tempered distributions are considered to be mutually "orthogonal". The smooth functions

$$u_{ijk} = \zeta_{ijk}\varphi_j \tag{6.4.79}$$

satisfy

$$H_1 u_{ijk} = \lambda_i u_{ijk}. \tag{6.4.80}$$

Label the eigenvalues of H_1 including their multiplicities and denote them by $\tilde{\lambda}_i$. Then

$$\sum_{i=0}^{\infty} e^{-\beta\tilde{\lambda}_i} \le 2\sum_{i=0}^{\infty} e^{-\beta\lambda_i} n(\lambda_i) = 2\sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\lambda_i} e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i), \tag{6.4.81}$$

where

$$n(\lambda_i) = \# N_i. \tag{6.4.82}$$

Lemma 6.4.5 *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta_0 \le \beta \tag{6.4.83}$$

and for any $i \in \mathbb{N}$, the estimate

$$e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i) \le c(\beta) \le c(\beta_0) \tag{6.4.84}$$

is valid, where $c(\beta_0)$ also depends on A but is independent of $i \in \mathbb{N}$.

Proof Each N_i is the disjoint union

$$N_i' \dot{\cup} N_i'', \tag{6.4.85}$$

where

$$N_i' = \{ j \in \mathbb{N}_i : \tilde{\mu}_j \le 0 \} \tag{6.4.86}$$

and N_i'' is its complement. The operator A has only finitely many eigenvalues which are non-positive, i.e.

$$\#N_i' \le n_0 \qquad \forall\, i \in \mathbb{N}, \tag{6.4.87}$$

hence

$$\begin{aligned} e^{-\frac{\beta}{2}\lambda_i} n_i(\lambda_i) &\le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\lambda_i} \le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\tilde{\mu}_j} \\ &\le n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde{\mu}_j} \\ &= n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde{\mu}_j}(1+|\tilde{\mu}|_j^{2m})\,\|\tilde{\varphi}_j\|_0^2 \\ &\le n_0 + c(\beta)\sum_{j=0}^{\infty}\|\tilde{\varphi}_j\|_0^2 < \infty, \end{aligned} \tag{6.4.88}$$

where we used (6.4.64). The estimate for the Hilbert–Schmidt norm of the embedding

$$j : H^{m,2}(M) \to L^2(M) \tag{6.4.89}$$

depends on A, since we used the equivalent norm given in (6.4.61), and

$$c(\beta) = \sup_{t>0} e^{-\frac{\beta}{2}t}(1+t^{2m}). \tag{6.4.90}$$

□

Corollary 6.4.6 *The sum on the left-hand side of* (6.4.81) *is finite and hence*

$$e^{-\beta H_1}, \qquad \beta > 0, \tag{6.4.91}$$

is of trace class provided we can define a Hilbert space $\mathcal{H}$ such that the eigendistributions form a complete set of eigenvectors in $\mathcal{H}$ and H_1 is essentially self-adjoint in $\mathcal{H}$.

Proof The first claim follows immediately by combining (6.4.88) and Theorem 6.2.8. In Lemma 6.5.1 on page 174, we shall define the Hilbert space $\mathcal{H}$ and shall prove that H_1 is essentially self-adjoint in $\mathcal{H}$ and that the eigendistributions form a complete set of eigenvectors in $\mathcal{H}$. □

The elliptic operator A also depends on Λ, since the underlying Riemannian metric depends on it. The estimates in the preceding lemma remain valid provided $|\Lambda|$ remains in a compact subset of $\mathbb{R}$, since the operator A is then still uniformly elliptic and smooth. However, when

$$|\Lambda| \to \infty, \tag{6.4.92}$$

then the relation (6.4.52) is no longer valid and a more sophisticated analysis is necessary to achieve a corresponding estimate. Let us treat the cases Schwarzschild-AdS and Kerr-AdS black holes separately.

For a Schwarzschild-AdS black hole, the operator A can be written in the form

$$A = r_0^{-2}\tilde{A}, \tag{6.4.93}$$

where r_0 is the black hole radius and

$$\tilde{A}\varphi = -(n-1)\tilde{\Delta}\varphi - \frac{n}{2}\tilde{R}\varphi. \tag{6.4.94}$$

Here, the Laplacian and the scalar curvature $\tilde{R}$ refer to the corresponding quantities of $\mathbb{S}^{n-1}$ with the standard metric, cf. (4.2.12) and (4.2.14) on page 104. The eigenfunctions of A are the eigenfunctions of $\tilde{A}$. Let μ_j be the eigenvalues of $\tilde{A}$ and $\tilde{\mu}_j$ the eigenvalues of A, then

$$\tilde{\mu}_j = r_0^{-2}\mu_j. \tag{6.4.95}$$

From the definition of the black hole radius

$$m r_0^{-(n-2)} = 1 + \frac{2}{n(n-1)}|\Lambda| r_0^2 \tag{6.4.96}$$

it is evident that

$$\lim_{|\Lambda|\to\infty} r_0 = 0 \tag{6.4.97}$$

and also

$$\lim_{|\Lambda|\to\infty} |\Lambda| r_0^2 = \infty, \tag{6.4.98}$$

though the latter result is only needed when we shall treat the Kerr-AdS case.

We can now prove:

Lemma 6.4.7 *Let $\beta_0 > 0$ be arbitrary and $|\Lambda_0|$ so large that*

$$r_0 < 1 \qquad \forall\, |\Lambda| > |\Lambda_0|, \tag{6.4.99}$$

then for any $i \in \mathbb{N}$, any $\beta \ge \beta_0$ and any $|\Lambda| > |\Lambda_0|$

$$e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i) \le c(\beta) \le c(\beta_0), \tag{6.4.100}$$

where $c(\beta_0)$ also depends on $\tilde{A}$ but is independent of $|\Lambda|$ and $i \in \mathbb{N}$.

Proof We follow the proof of Lemma 6.4.5 but use $\tilde{A}$ instead of A to define an equivalent norm in $H^{m,2}(M)$,

$$M = \mathbb{S}^{n-1}. \tag{6.4.101}$$

Then, we infer, cf. (6.4.88),

$$\begin{aligned} e^{-\frac{\beta}{2}\lambda_i} n_i(\lambda_i) &\le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\lambda_i} \le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\tilde{\mu}_j} \\ &\le n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde{\mu}_j} \\ &= n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde{\mu}_j}(1+|\mu|_j^{2m})\,\|\tilde{\varphi}_j\|_0^2 \\ &\le n_0 + c(\beta)\sum_{j=0}^{\infty}\|\tilde{\varphi}_j\|_0^2 < \infty. \end{aligned} \tag{6.4.102}$$

Here, we used

$$\tilde{\mu}_j = r_0^{-2}\mu_j > \mu_j > 0. \tag{6.4.103}$$

□

Let us now look at Kerr-AdS black holes. In 5.2.50 on page 122, we described the metric σ_{ij} on $M = \mathbb{S}^{n-1}$

$$\begin{aligned} ds_M^2 = {} & \frac{r^2+a^2}{1-a^2l^2}\left(\delta_{ij}d\mu^i d\mu^j + \mu_i^2\delta_{ij}d\varphi^i d\varphi^j\right) \\ & + a^2\frac{(1+l^2r^2)(r^2+a^2)}{r^2(1-a^2l^2)^2}\mu_i^2\mu_j^2 d\varphi^i d\varphi^j. \end{aligned} \tag{6.4.104}$$

Here

$$n = 2m, \qquad m \ge 2, \tag{6.4.105}$$

and the coordinates μ_i, $1 \le i \le m$ are subject to the constraint

$$\sum_{i=1}^{m} \mu_i^2 = 1. \tag{6.4.106}$$

They are the latitudinal coordinates of $\mathbb{S}^{n-1}$ and the φ_i, $1 \le i \le m$, are the azimuthal coordinates. The metric

$$\delta_{ij}d\mu^i d\mu^j + \mu_i^2\delta_{ij}d\varphi^i d\varphi^j \tag{6.4.107}$$

is the standard metric of $\mathbb{S}^{n-1}$. The constant r is the radius of the event horizon, $a \ne 0$ the rotational parameter and

$$l^2 = -\frac{1}{m(2m-1)}\Lambda. \tag{6.4.108}$$

The relation

$$a^2 l^2 < 1 \tag{6.4.109}$$

is assumed. We also require that a is small enough such that the scalar curvature R of the metric σ_{ij} is positive. We can write the metric as a conformal metric

$$\sigma_{ij} = \frac{r^2 + a^2}{1 - a^2 l^2} \tilde{\sigma}_{ij}. \tag{6.4.110}$$

Let us also note that the Schwarzschild-AdS black hole is obtained by setting $a = 0$ and that

$$\lim_{a \to 0} r = r_0, \tag{6.4.111}$$

is the Schwarzschild black hole radius.

In order to prove the analogue of Lemma 6.4.7, we assume that, when

$$|\Lambda| \to \infty, \tag{6.4.112}$$

a is supposed to be so small that

$$\lim_{|\Lambda| \to \infty} |\Lambda| a^2 = 0 \tag{6.4.113}$$

and

$$\lim_{|\Lambda| \to \infty} |\Lambda| r^2 = \infty, \tag{6.4.114}$$

and we emphasize that these assumptions are always satisfied if $a = 0$, cf. (6.4.98). If these are satisfied, then the operator A can be expressed in the form

$$A = \frac{1 - a^2 l^2}{r^2 + a^2} \tilde{A}, \tag{6.4.115}$$

where $\tilde{A}$ converges uniformly in $C^\infty(M)$ to the operator $\tilde{A}$ in (6.4.94), i.e. for large $|\Lambda|$, $\tilde{A}$ is uniformly elliptic and smooth such that the number of non-positive eigenvalues $n_0(\tilde{A})$ is bounded from above by the n_0 of the limit operator

$$n_0 \geq \limsup_{|\Lambda| \to \infty} n_0(\tilde{A}), \tag{6.4.116}$$

since n_0 is upper semi-continuous as it is well known.

Lemma 6.4.8 *Under the Assumptions* (6.4.113) *and* (6.4.114) *the results of Lemma 6.4.7 are also valid for the Kerr-AdS black hole, i.e. there exists* $|\Lambda_0| > 0$ *such that for all*

$$|\Lambda| > |\Lambda_0| \tag{6.4.117}$$

and for any β satisfying

$$0 < \beta_0 \le \beta, \tag{6.4.118}$$

where β_0 is arbitrary,

$$e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i) \le c(\beta_0) \tag{6.4.119}$$

uniformly in $i \in \mathbb{N}$, $|\Lambda|$ and β.

Proof The proof is identical to the proof of Lemma 6.4.7 by using the fact that the special $H^{m,2}(M)$-norm

$$\langle \tilde{A}^m \varphi, \tilde{A}^m \varphi \rangle_0 + \langle \varphi, \varphi \rangle_0, \tag{6.4.120}$$

with different m than used to express the dimension of M, is uniformly equivalent to the standard $H^{m,2}(M)$-norm, hence the Hilbert–Schmidt norm of the embedding

$$j : H^{m,2}(M) \hookrightarrow L^2(M) \tag{6.4.121}$$

is uniformly bounded. We also relied on

$$\tilde{\mu}_j = \frac{1 - a^2 l^2}{r^2 + a^2} \mu_j > \mu_j > 0 \tag{6.4.122}$$

for $j \in N_i''$. □

Finally, let us derive the last result in this section.

Lemma 6.4.9 *Let λ_i be the temporal eigenvalues depending on Λ and let $\bar{\lambda}_i$ be the corresponding eigenvalues for*

$$|\Lambda| = 1, \tag{6.4.123}$$

then

$$\lambda_i = \bar{\lambda}_i |\Lambda|^{\frac{n-1}{n}}. \tag{6.4.124}$$

Proof Let B and K be the bilinear forms defined in (6.4.40) resp. (6.4.41), where B corresponds to the cosmological constant Λ, and let B_1 be the form with respect to the value

$$|\Lambda| = 1. \tag{6.4.125}$$

Moreover, let us denote the corresponding quadratic forms by the same symbols, then we have

$$\frac{B(\varphi)}{K(\varphi)} = |\Lambda|^{\frac{n-1}{n}} \frac{B_1(\varphi)}{K(\varphi)} \qquad \forall\, 0 \ne \varphi \in C_c^\infty(\mathbb{R}_+). \tag{6.4.126}$$

To prove (6.4.126), we introduce a new integration variable τ on the left-hand side

$$t = \mu\tau, \qquad \mu > 0, \tag{6.4.127}$$

to conclude

$$\frac{B(\varphi)}{K(\varphi)} = \mu^{-4\frac{n-1}{n}} \frac{B_1(\varphi)}{K(\varphi)} \qquad \forall\, 0 \neq \varphi \in C_c^\infty(\mathbb{R}_+). \tag{6.4.128}$$

provided

$$\mu = |\Lambda|^{-\frac{1}{4}}. \tag{6.4.129}$$

The relation (6.4.126) immediately implies (6.4.124). □

6.5 The Partition Function

We first define the partition function for the black holes and shall later show that the definitions and results are also applicable in case of the quantized globally hyperbolic spacetimes with a negative cosmological constant and asymptotically Euclidean Cauchy hypersurfaces.

We define the partition function by using the spatial Hamiltonian H_1 of the quantized black holes, Kerr or Schwarzschild, which is now defined in the separable Hilbert space $\mathcal{H}$ generated by the eigendistributions

$$u_{ijk} = w_i \zeta_{ijk} \varphi_j \tag{6.5.1}$$

which are smooth functions satisfying the eigenvalue equations

$$H_1 u_{ijk} = \lambda_i u_{ijk} \tag{6.5.2}$$

as well as

$$H_0 u_{ijk} = \lambda_i u_{ijk}, \tag{6.5.3}$$

where H_0 is the temporal Hamiltonian.

In order to explain how the eigendistributions can generate a Hilbert space, let us relabel the eigenfunctions and the eigenvalues by $(u_i, \tilde{\lambda}_i)$ such that

$$H_1 u_i = \tilde{\lambda}_i u_i \tag{6.5.4}$$

and

$$H_0 u_i = \tilde{\lambda}_i u_i, \tag{6.5.5}$$

i.e. the multiplicities of the eigenvalues are now included in the labelling and the ordering is no longer strict

$$\tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots . \tag{6.5.6}$$

To define the Hilbert space $\mathcal{H}$, we simply declare that the eigendistributions are mutually orthogonal unit eigenvectors, hence defining a scalar product in the complex vector space $\mathcal{H}'$ spanned by these eigenvectors. We define the Hilbert space $\mathcal{H}$ to be its completion.

Lemma 6.5.1 *The linear operator H_1 with domain $\mathcal{H}'$ is essentially self-adjoint in $\mathcal{H}$. Let $\bar{H}_1$ be its closure, then the only eigenvectors of $\bar{H}_1$ are those of H_1.*

Proof H_1 is obviously densely defined, symmetric and bounded from below

$$H_1 \geq \tilde{\lambda}_0 I > 0. \tag{6.5.7}$$

Since $\tilde{\lambda}_0 > 0$, the eigenvectors also span $R(H_1)$, i.e. $R(H_1)$ is dense. Let

$$w \in \mathcal{H} \tag{6.5.8}$$

be arbitrary, and let

$$H_1 v_i \in R(H_1) \tag{6.5.9}$$

be a sequence converging to w, then v_i is a Cauchy sequence, because

$$\tilde{\lambda}_0 \|v_i - v_j\|^2 \leq \langle H_1 v_i - H_1 v_j, v_i - v_j \rangle \leq \|H_1 v_i - H_1 v_j\| \, \|v_i - v_j\|, \tag{6.5.10}$$

hence

$$R(\bar{H}_1) = \mathcal{H} \tag{6.5.11}$$

and $\bar{H}_1$ is the unique s.a. extension of H_1.

It remains to prove that $\bar{H}_1$ has no additional eigenvectors. Thus, let u be an eigenvector of $\bar{H}_1$ with eigenvalue λ

$$\bar{H}_1 u = \lambda u, \tag{6.5.12}$$

and let

$$E(\tilde{\lambda}_i) \subset \mathcal{H}', \qquad i \in \mathbb{N}, \tag{6.5.13}$$

be the eigenspaces of H_1. Let us first assume that there exists j such that

$$\lambda = \tilde{\lambda}_j, \tag{6.5.14}$$

but

$$u \notin E(\tilde{\lambda}_j). \tag{6.5.15}$$

Without loss of generality, we may assume

$$u \in E(\tilde{\lambda}_j)^{\perp}. \tag{6.5.16}$$

However, this leads to a contradiction, since then

$$u \in E(\tilde{\lambda}_i)^{\perp} \qquad \forall i \in \mathbb{N}, \tag{6.5.17}$$

and hence

$$u \in \mathcal{H}'^{\perp} \tag{6.5.18}$$

which implies $u = 0$.

Thus, let us assume

$$\lambda \neq \tilde{\lambda}_i \qquad \forall i \in \mathbb{N}, \tag{6.5.19}$$

but then (6.5.17) is again valid leading to the known contradiction. □

Remark 6.5.2 In the following, we shall write H_1 instead of $\bar{H}_1$.

Lemma 6.5.3 *For any $\beta > 0$ the operator*

$$e^{-\beta H_1} \tag{6.5.20}$$

is of trace class in $\mathcal{H}$. Let

$$\mathscr{F} \equiv \mathscr{F}_+(\mathcal{H}) \tag{6.5.21}$$

be the symmetric Fock space generated by $\mathcal{H}$ and let

$$H = d\Gamma(H_1) \tag{6.5.22}$$

be the canonical extension of H_1 to $\mathscr{F}$. Then

$$e^{-\beta H} \tag{6.5.23}$$

is also of trace class in $\mathscr{F}$

$$tr(e^{-\beta H}) = \prod_{i=0}^{\infty}(1 - e^{-\beta\tilde{\lambda}_i})^{-1} < \infty. \tag{6.5.24}$$

Proof The first part of the lemma has already been proved in Corollary 6.4.6 on page 168. This property can now be rephrased as

$$\operatorname{tr}(e^{-\beta H_1}) = \sum_{i=0}^{\infty} e^{-\beta\tilde{\lambda}_i} < \infty. \tag{6.5.25}$$

The second assertion is well known, since

$$H_1 \geq \tilde{\lambda}_0 I > 0, \tag{6.5.26}$$

and the properties (6.5.25) and (6.5.26) imply (6.5.24), cf. [3, Proposition 5.2.7] and [32, Volume II, p. 868], where Eq. (6.5.24) is also proved. □

We then define the partition function Z by

$$Z = \mathrm{tr}(e^{-\beta H}) = \prod_{i=0}^{\infty}(1 - e^{-\beta\tilde{\lambda}_i})^{-1} \tag{6.5.27}$$

and the density operator ρ in $\mathscr{F}$ by

$$\rho = Z^{-1}e^{-\beta H} \tag{6.5.28}$$

such that

$$\mathrm{tr}\rho = 1. \tag{6.5.29}$$

The von Neumann entropy S is then defined by

$$\begin{aligned} S &= -\mathrm{tr}(\rho \log \rho) \\ &= \log Z + \beta Z^{-1}\mathrm{tr}(He^{-\beta H}) \\ &= \log Z - \beta \frac{\partial \log Z}{\partial \beta} \\ &\equiv \log Z + \beta E, \end{aligned} \tag{6.5.30}$$

where E is the average energy

$$E = \mathrm{tr}(H\rho). \tag{6.5.31}$$

E can be expressed in the form

$$E = \sum_{i=0}^{\infty} \frac{\tilde{\lambda}_i}{e^{\beta\tilde{\lambda}_i} - 1}. \tag{6.5.32}$$

Here, we also set the Boltzmann constant

$$K_B = 1. \tag{6.5.33}$$

The parameter β is supposed to be the inverse of the absolute temperature T

$$\beta = T^{-1}. \tag{6.5.34}$$

In view of Lemma 6.4.9 on page 172, we can write the eigenvalues λ_i in the form

$$\lambda_i = \bar{\lambda}_i |\Lambda|^{\frac{n-1}{n}}, \tag{6.5.35}$$

where $\bar{\lambda}_i$ are the eigenvalues corresponding to $|\Lambda| = 1$. Hence, Z, S and E can also be looked at as functions depending on β and Λ, or more conveniently, on (β, τ), where

$$\tau = |\Lambda|^{\frac{n-1}{n}}, \tag{6.5.36}$$

since the $\tilde{\lambda}_i$ can also be expressed as

$$\tilde{\lambda}_i = \lambda_j = \bar{\lambda}_j |\Lambda|^{\frac{n-1}{n}}, \tag{6.5.37}$$

where j is different from i

$$j \le i, \tag{6.5.38}$$

because of the multiplicities of $\tilde{\lambda}_i$. Let emphasize that the multiplicities also depend on Λ, hence it is best to simply note that

$$\tilde{\lambda}_0 = \lambda_0 = \bar{\lambda}_0 |\Lambda|^{\frac{n-1}{n}} \tag{6.5.39}$$

and that the $\tilde{\lambda}_i$ are ordered. We shall never use the relation (6.5.37) explicitly in the proofs of the subsequent theorems and lemmata referring to (6.5.35) instead.

Theorem 6.5.4 (i) *Let* $\beta_0 > 0$ *be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{6.5.40}$$

we have

$$\lim_{\Lambda \to 0} E = \infty \tag{6.5.41}$$

as well as

$$\lim_{\Lambda \to 0} S = \infty, \tag{6.5.42}$$

where the limits are uniform in β.

(ii) *Let* $\beta_0 > 0$ *be arbitrary, then, for any*

$$\beta \ge \beta_0, \tag{6.5.43}$$

we have

$$\lim_{|\Lambda| \to \infty} E = 0 \tag{6.5.44}$$

as well as

$$\lim_{|\Lambda| \to \infty} S = 0, \tag{6.5.45}$$

where the limits are uniform in β.

Proof "(i)" We first observe that

$$E = \sum_{i=0}^{\infty} \frac{\tilde{\lambda}_i}{e^{\beta\tilde{\lambda}_i} - 1} \geq \sum_{i=0}^{\infty} \frac{\lambda_i}{e^{\beta\lambda_i} - 1} \tag{6.5.46}$$

Now, let $m \in \mathbb{N}$ be arbitrary, then

$$E \geq \sum_{i=0}^{m} \frac{\lambda_i}{e^{\beta\lambda_i} - 1} = \sum_{i=0}^{m} \frac{\bar{\lambda}_i \tau}{e^{\beta\bar{\lambda}_i\tau} - 1} \tag{6.5.47}$$

and

$$\begin{aligned} \liminf_{\tau\to 0} E &\geq \lim_{\tau\to 0} \sum_{i=0}^{m} \frac{\bar{\lambda}_i \tau}{e^{\beta\bar{\lambda}_i\tau} - 1} \\ &= (m+1)\beta^{-1} \geq (m+1)\beta_0^{-1} \end{aligned} \tag{6.5.48}$$

yielding

$$\lim_{\Lambda\to 0} E = \infty \tag{6.5.49}$$

uniformly in β.

Since $Z \geq 1$, the relation (6.5.42) follows as well.

"(ii)" We estimate E from above by

$$\begin{aligned} E &= \sum_{i=0}^{\infty} \frac{\tilde{\lambda}_i e^{-\beta\tilde{\lambda}_i}}{1 - e^{-\beta\tilde{\lambda}_i}} = \sum_{i=0}^{\infty} \tilde{\lambda}_i e^{-\frac{\beta}{2}\tilde{\lambda}_i} e^{-\frac{\beta}{2}\tilde{\lambda}_i} (1 - e^{-\beta\tilde{\lambda}_i})^{-1} \\ &\leq (1 - e^{-\beta_0\tilde{\lambda}_0})^{-1} c(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\tilde{\lambda}_i}, \end{aligned} \tag{6.5.50}$$

where we used (6.5.43) and

$$\tilde{\lambda}_i e^{-\frac{\beta}{2}\tilde{\lambda}_i} \leq \sup_{t>0} t e^{-\frac{\beta}{2}t} = c(\beta) \leq c(\beta_0). \tag{6.5.51}$$

Furthermore, we know that

$$\begin{aligned} \sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\tilde{\lambda}_i} &\leq \tilde{c}(\beta) \sum_{i=0}^{\infty} e^{-\frac{\beta}{4}\lambda_i} \\ &\leq \tilde{c}(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta_0}{4}\lambda_i}, \end{aligned} \tag{6.5.52}$$

cf. Lemma 6.4.7 on page 169 and Lemma 6.4.8 on page 171, hence we obtain

$$E \le (1 - e^{-\beta_0 \bar{\lambda}_0 \tau})^{-1} c(\beta_0) \tilde{c}(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta}{4} \bar{\lambda}_i \tau} \tag{6.5.53}$$

deducing further

$$\limsup_{\tau \to \infty} E \le c(\beta_0) \tilde{c}(\beta_0) \lim_{\tau \to \infty} \sum_{i=0}^{\infty} e^{-\frac{\beta}{4} \bar{\lambda}_i \tau} = 0 \tag{6.5.54}$$

uniformly in β and hence

$$\lim_{\tau \to \infty} E = 0. \tag{6.5.55}$$

It remains to prove that S vanishes in the limit. We have

$$\begin{aligned} Z &= \prod_{i=0}^{\infty} (1 - e^{-\beta \tilde{\lambda}_i})^{-1} = \prod_{i=0}^{\infty} (1 + e^{-\beta \tilde{\lambda}_i} (1 - e^{-\beta \tilde{\lambda}_i})^{-1}) \\ &\le \exp\{(1 - e^{\beta_0 \tilde{\lambda}_0})^{-1} \sum_{i=0}^{\infty} e^{-\beta \tilde{\lambda}_i}\}, \end{aligned} \tag{6.5.56}$$

where we used the inequality

$$\log(1 + t) \le t \qquad \forall t \ge 0 \tag{6.5.57}$$

in the last step.

Applying then the arguments preceding the inequality (6.5.54), we conclude

$$\lim_{\tau \to \infty} Z = 1 \tag{6.5.58}$$

uniformly in β. □

Remark 6.5.5 The first part of the preceding theorem reveals that the energy becomes very large for small values of $|\Lambda|$. Since this is the energy obtained by applying quantum statistics to the quantized version of a black hole or of a globally hyperbolic spacetime—assuming its Cauchy hypersurfaces are asymptotically Euclidean—a small negative cosmological constant might be responsible for the dark matter, where we equate the energy of the quantized universe with matter. As source for the dark energy density, we conjecture that the dark energy density should be proportional to the eigenvalue of the density operator ρ with respect to the vacuum vector η

$$\rho \eta = Z^{-1} \eta, \tag{6.5.59}$$

which is Z^{-1}.

The behaviour of Z with respect to Λ is described in the theorem:

Theorem 6.5.6 *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{6.5.60}$$

we have

$$\lim_{\Lambda \to 0} Z = \infty \tag{6.5.61}$$

and for any

$$\beta_0 \le \beta \tag{6.5.62}$$

the relation

$$\lim_{|\Lambda| \to \infty} Z = 1 \tag{6.5.63}$$

is valid. The convergence in both limits is uniform in β.

Proof "(6.5.60)" Let $m \in \mathbb{N}$ be arbitrary, then

$$\begin{aligned} Z &\ge \prod_{i=0}^{\infty}(1 - e^{-\beta\lambda_i})^{-1} = \prod_{i=0}^{\infty}(1 - e^{-\beta\bar{\lambda}_i\tau})^{-1} \\ &\ge \prod_{i=0}^{m}(1 - e^{-\beta_0\bar{\lambda}_i\tau})^{-1} \end{aligned} \tag{6.5.64}$$

and we infer

$$\lim_{\tau \to 0} Z = \liminf_{\tau \to 0} Z = \infty. \tag{6.5.65}$$

"(6.5.63)" This limit relation has already been proved in (6.5.58). □

It is also worthwhile to study the behaviour of S, E and Z if β is varied while keeping Λ fixed. We first observe

Lemma 6.5.7 *Denoting the differentiation with respect to β by a prime we have*

$$S' = \beta E' < 0. \tag{6.5.66}$$

Proof Differentiating the relation

$$S = \log Z - \beta(\log Z)' \tag{6.5.67}$$

we deduce

$$S' = \beta E' \tag{6.5.68}$$

while

$$E' = -\sum_i \frac{\tilde{\lambda}_i^2 e^{\beta\tilde{\lambda}_i}}{(e^{\beta\tilde{\lambda}_i} - 1)^2} < 0 \tag{6.5.69}$$

in view of (6.5.32). □

The corresponding limit relations are expressed in

Theorem 6.5.8 *For fixed $\Lambda < 0$ the following relations are valid*

$$\lim_{\beta\to\infty} Z = 1, \tag{6.5.70}$$

$$\lim_{\beta\to\infty} \beta E = 0, \tag{6.5.71}$$

$$\lim_{\beta\to\infty} S = 0, \tag{6.5.72}$$

and

$$\lim_{\beta\to 0} Z = \infty, \tag{6.5.73}$$

$$\lim_{\beta\to 0} E = \infty, \tag{6.5.74}$$

as well as

$$\lim_{\beta\to 0} S = \infty. \tag{6.5.75}$$

Proof "(6.5.70)" Follows immediately from the estimate (6.5.56).

"(6.5.71)" We have

$$\beta E = \sum_{i=0}^{m} \frac{\beta\tilde{\lambda}_i}{e^{\beta\tilde{\lambda}_i} - 1} + \sum_{i=m}^{\infty} \frac{\beta\tilde{\lambda}_i}{e^{\beta\tilde{\lambda}_i} - 1}. \tag{6.5.76}$$

Denote the second sum by $R(m, \beta)$ and let β_0 satisfy

$$\beta_0\tilde{\lambda}_0 \ge 1, \tag{6.5.77}$$

then

$$R(m, \beta) \le R(m, \beta_0) \qquad \forall\, \beta \ge \beta_0. \tag{6.5.78}$$

Next, let $\epsilon > 0$ be arbitrary, then there exists m such that

$$R(m, \beta_0) < \epsilon \tag{6.5.79}$$

and we conclude

$$\limsup_{\beta\to\infty} \beta E \le \epsilon \tag{6.5.80}$$

proving (6.5.71).

"(6.5.72)" Follows from (6.5.70) and (6.5.71).

The proofs of the remaining relations are either trivial or are similar to the proofs of the corresponding results in Theorems 6.5.4 and 6.5.6. □

Let us now consider the quantized globally hyperbolic spacetimes with an asymptotically Euclidean Cauchy hypersurface. The eigenspaces

$$\mathscr{E}_{\lambda_i} \subset \mathscr{S}'(\mathcal{S}_0) \tag{6.5.81}$$

of H_1 are separable but they are in general not finite dimensional as can be seen by the following counterexample

$$H_1 = -\Delta \tag{6.5.82}$$

in $\mathbb{R}^n$. The eigenspaces

$$\mathscr{E}_{\lambda_i}, \qquad \lambda_i > 0, \tag{6.5.83}$$

contain the tempered distributions

$$e^{i\langle k,x\rangle}, \qquad k \in \mathbb{S}^{n-1}_{\lambda_i}. \tag{6.5.84}$$

As a Hamel basis they generate a vector space, the dimension of which is equal to the cardinality of $\mathbb{S}^{n-1}$. Of course, as a Schauder basis the functions with

$$k \in D \subset \mathbb{S}^{n-1}_{\lambda_i}, \tag{6.5.85}$$

where D is countable and dense, generate a dense subspace.

This example indicates that not all eigendistributions of H_1 might be physically relevant. Contrary to the cases of the black holes, where the selection of eigenvectors and eigendistributions was a natural process, only the temporal eigenvectors are naturally selected in the present situation and of course at least one matching spatial eigendistribution to obtain a solution of the wave equation. Hence, we could use H_0 to define the partition function. However, we believe this choice would be too restrictive, and we shall instead stipulate that we only pick at most

$$c|\lambda_i|^p \tag{6.5.86}$$

spatial eigendistributions in $\mathscr{E}_{\lambda_i}$, where c and p are arbitrary but fixed constants, i.e. we assume that

$$n(\lambda_i) \le c|\lambda_i|^p \qquad \forall\, i \in \mathbb{N}. \tag{6.5.87}$$

With this assumption, it becomes evident that the results and conjectures of Theorem 6.5.4, Remark 6.5.5 and Theorem 6.5.6 are also valid in case of globally hyperbolic spacetimes with asymptotically Euclidean hypersurfaces.

6.6 The Friedmann Universes with Negative Cosmological Constants

In Remark 1.6.11 on page 50, we observed that, if the Cauchy hypersurface $\mathcal{S}_0$ is a space of constant curvature and if the wave Eq. (6.4.1) on page 159 is only considered for functions u which do not depend on x, then this equation is identical to the equation obtained by quantizing the Hamilton constraint in a Friedmann universe without matter but including a cosmological constant. The equation is then the ODE

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - \frac{n}{2}Rt^{2-\frac{4}{n}}u + nt^2\Lambda u = 0, \qquad 0 < t < \infty, \tag{6.6.1}$$

where R is the scalar curvature of $\mathcal{S}_0$. We cannot apply our previous arguments to the solutions of this ODE. However, if we consider instead the more general Eq. (6.4.1), where u is also allowed to depend on x, which certainly is more general and accurate, then the previous arguments can be applied if the curvature $\tilde{\kappa}$ of $\mathcal{S}_0$ vanishes

$$\tilde{\kappa} = 0. \tag{6.6.2}$$

The scalar curvature, which is equal to

$$R = n(n-1)\tilde{\kappa}, \tag{6.6.3}$$

then vanishes too and

$$\mathcal{S}_0 = \mathbb{R}^n. \tag{6.6.4}$$

We are now in the situation which we analysed at the end of the previous section, where now the spatial Hamiltonian is

$$H_1 = -(n-1)\Delta \tag{6.6.5}$$

and some spatial eigendistributions are shown in (6.5.84) on page 182. However, since we consider the quantized version of a Friedmann universe, we shall look for radially symmetric eigendistributions, i.e. we look for smooth functions $v = v(x)$ satisfying

$$v(x) = \varphi(r) \tag{6.6.6}$$

such that

$$\Delta v = \ddot{\varphi} + (n-1)r^{-1}\dot{\varphi} = -\mu^2\varphi \quad \text{in} \quad r > 0, \tag{6.6.7}$$

where $\mu > 0$. Obviously, it is sufficient to assume $\mu = 1$, because, if φ is an eigenfunction for $\mu = 1$, then

$$\tilde{\varphi}(r) = \varphi(\mu r) \tag{6.6.8}$$

is an eigenfunction for the eigenvalue μ^2. Therefore, let us choose $\mu = 1$.

We shall express the solution φ with the help of a Bessel function J_ν. Let ψ be a solution of the Bessel equation

$$\ddot{\psi} + r^{-1}\dot{\psi} + (1 - r^{-2}\nu^2)\psi = 0, \tag{6.6.9}$$

where

$$\nu = \frac{n-2}{2}, \tag{6.6.10}$$

then the function

$$\varphi(r) = r^{-\nu}\psi \tag{6.6.11}$$

satisfies

$$r\ddot{\varphi} + (2\nu + 1)\dot{\varphi} + r\varphi = 0, \tag{6.6.12}$$

which is equivalent to (6.6.7) with $\mu = 1$. The Bessel Eq. (6.6.9) has the two independent solutions J_ν and Y_ν, the Bessel functions of first kind resp. of second kind. It is well known that the functions

$$r^{-\nu} J_\nu \tag{6.6.13}$$

can be expressed as a power series in the variable r^2, cf. [5, Eq. (21), p. 420], i.e. the function

$$v(x) = \varphi(r) = r^{-\nu} J_\nu \tag{6.6.14}$$

is smooth in $\mathbb{R}^n$, while the functions

$$r^{-\nu} Y_\nu \tag{6.6.15}$$

have a singularity in $r = 0$. Hence, there exists exactly one smooth radially symmetric solution v of the eigenvalue equation

$$-\Delta v = \lambda^2 v, \qquad \lambda > 0, \tag{6.6.16}$$

which is given by

$$v = (\lambda r)^{-\nu} J_\nu(\lambda r). \tag{6.6.17}$$

This solution also vanishes at infinity, hence it is uniformly bounded and a tempered distribution.

A solution of the wave Eq. (6.4.1) on page 159, in case of a quantized Friedmann universe, is therefore given by a sequence

$$u_i = w_i(t) v_i(x), \qquad i \in \mathbb{N}, \tag{6.6.18}$$

where w_i is a temporal eigenfunction and v_i a spatial eigenfunction. The u_i are also eigenfunctions for the temporal Hamiltonian as well as for the spatial Hamiltonian. Each eigenvalue has multiplicity one. We have therefore proved:

Theorem 6.6.1 *The results in Theorem 6.5.4, Remark 6.5.5, Theorem 6.5.4, Lemma 6.5.7 and Theorem 6.5.8 are also valid, if the quantized spacetime $N = N^{n+1}$, $n \geq 3$, is a Friedmann universe without matter but with a negative cosmological constant Λ and with vanishing spatial curvature. The eigenvalues of the spatial Hamiltonian H_1 all have multiplicity one.*

Chapter 7
Appendix

7.1 The Eigendistributions are Smooth Functions

We assume that the Cauchy hypersurface $\mathcal{S}_0$ is asymptotically Euclidean and A is an uniformly elliptic linear differential operator with smooth coefficients such that the coefficients are bounded in any

$$C^m(\mathcal{S}_0) \qquad \forall m \in \mathbb{N}. \tag{7.1.1}$$

Then, we can prove:

Theorem 7.1.1 *The solutions $f(\lambda) \in \mathscr{S}'$ of the eigenvalue problem*

$$Af(\lambda) = \mu f(\lambda) \tag{7.1.2}$$

belong to $C^\infty(\mathcal{S}_0)$ and for each $m \in \mathbb{N}$ and $R > 0$ $f(\lambda)$ can be estimated by

$$|f(\lambda)|_{m,B_R(x_0)} \le c_m R^N \|f(\lambda)\|_{-p}, \tag{7.1.3}$$

where $\|\cdot\|_p$ is one of the defining norms in $\mathscr{S}$ such that the dual norm

$$\|f(\lambda)\|_{-p} = \sup_{\|\varphi\|_p=1} |\langle f(\lambda), \varphi\rangle| \tag{7.1.4}$$

and N depends on n, $\|\cdot\|_p$, A and $\mathcal{S}_0$, while c_m depends on m, A the eigenvalue μ and on $\mathcal{S}_0$. $B_R(x_0)$ is a geodesic ball of radius R for a fixed $x_0 \in K \subset \mathcal{S}_0$, where K is the compact set in Assumption 3.3.1 on page 93 and R is so large that $K \subset B_R(x_0)$.

Proof First, we note that we can absorb the right-hand side of the eigenvalue equation into the left-hand side and simply consider the equation

$$Af(\lambda) = 0. \tag{7.1.5}$$

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Hence, it is well known that the distributional solutions is smooth and Eq. (7.1.5) can be understood in the classical sense, see, e.g., [37, Theorem 3.2, p.125].

The important estimate (7.1.3) is due to the fact that $f(\lambda)$ is a Tempered distribution. Since $f(\lambda) \in \mathscr{S}'$, we have

$$|\langle f(\lambda), \varphi\rangle| \leq c \sup_{x \in \mathcal{S}_0} (1 + r(x)^2)^k \sum_{|\alpha| \leq m_0} |D^\alpha \varphi(x)| \equiv c\|\varphi\|_p \tag{7.1.6}$$

and the dual norm

$$\|f(\lambda)\|_{-p} = c. \tag{7.1.7}$$

To prove (7.1.3) we fix $m \in \mathbb{N}$ and assume that

$$|f(\lambda)|_{m, B_{R_1}(x_0)} \leq c_0, \tag{7.1.8}$$

for some sufficiently large radius R_1 such that we only have to prove the estimate in the domain

$$B_R(0) \backslash \bar{B}_{R_0}(0), \tag{7.1.9}$$

where we now consider Euclidean balls. Hence, we may consider Eq. (7.1.5) to be a uniformly elliptic equation in an exterior region of Euclidean space with smooth coefficients.

Let $R > R_0$, then we first prove a priori estimates for $f(\lambda)$ in small balls

$$B_\rho(y) \Subset B_{2R}(0) \backslash B_{R_0}(0), \tag{7.1.10}$$

where

$$2\rho < \rho_0 \leq 1 \tag{7.1.11}$$

and ρ_0 is fixed.

Let

$$H_0^{m,2}(\Omega), \qquad m \in \mathbb{N}, \tag{7.1.12}$$

be the usual Sobolev spaces, where

$$\Omega \subset \mathbb{R}^n \tag{7.1.13}$$

is an open set, to be defined as the completion of $C_c^\infty(\Omega)$ under the norm

$$\|\varphi\|_{m,2}^2 = \int_\Omega \sum_{|\alpha| \leq m} |D^\alpha \varphi|^2. \tag{7.1.14}$$

$H_0^{m,2}(\Omega)$ is a Hilbert space. Its dual space is denoted by

$$H^{-m,2}(\Omega) \tag{7.1.15}$$

and its elements are the distributions $f \in \mathscr{D}'(\Omega)$ which can be written in the form

$$f = \sum_{|\alpha|\le m} D^\alpha u_\alpha, \tag{7.1.16}$$

where

$$u_\alpha \in L^2(\Omega) \tag{7.1.17}$$

and the dual norm of f is equal to

$$\|f\|_{-m,2} = \Big(\sum_{|\alpha|\le m} \|u_\alpha\|_2^2\Big)^{\frac{1}{2}}. \tag{7.1.18}$$

The Sobolev imbedding theorem states that for bounded Ω

$$m > \frac{n}{2} \quad \Longrightarrow \quad H_0^{m,2}(\Omega) \hookrightarrow C^0(\Omega) \tag{7.1.19}$$

such that

$$|u|_0 \le c\|u\|_{m,2} \qquad \forall u \in H_0^{m,2}(\Omega), \tag{7.1.20}$$

where c only depends on $diam\Omega$, m and n.

As a corollary, we deduce

$$m > \frac{n}{2} \quad \Longrightarrow \quad H_0^{m+m_0,2}(\Omega) \hookrightarrow C^{m_0,0}(\Omega) \tag{7.1.21}$$

with a corresponding estimate

$$|u|_{m_0,0} \le c\|u\|_{m+m_0,2}, \tag{7.1.22}$$

where $c = c(diam\Omega n, m, m_0)$.

Hence, for any ball

$$B_{\rho_0}(y) \subset B_{2R}(0) \tag{7.1.23}$$

$f(\lambda)$ can be considered to belong to

$$f(\lambda) \in H^{-(m_0+n),2}(B_{\rho_0}(y)) \tag{7.1.24}$$

with norm

$$\|f(\lambda)\|_{-(n+m_0),2} \le cR^{2k} \tag{7.1.25}$$

in view of the estimate (7.1.6), where we also assume $R_0 > 1$; the constant c depends on n, m_0, k and the constant in (7.1.6).

From the proofs of [37, Theorem 3.1, p. 123] and [37, Theorem 3.2, p. 125], we then deduce that for any $m \in \mathbb{N}$ there exists $\rho < \rho_0$, ρ depending only on the Lipschitz constant of the metric σ_{ij}, m, n and m_0 such that the C^m-norm of the solution $f(\lambda)$ of Eq. (7.1.5) can be estimated by

$$|f(\lambda)|_{m,B_\rho(y)} \le c_\rho R^{2k}, \tag{7.1.26}$$

where c_ρ also depends on the C^m-norms of the coefficients of A and on the ellipticity constants.

Now

$$(4R)^n 2^n \rho^{-n} \tag{7.1.27}$$

balls

$$B_\rho(y) \subset B_{2R}(0) \tag{7.1.28}$$

cover the closed ball $\bar{B}_R(0)$; hence, we conclude

$$|f(\lambda)|_{m,B_R(0)\setminus K_0} \le cR^{2k+n}, \tag{7.1.29}$$

where $c = c(\rho, m, m_0, n, A)$. □

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