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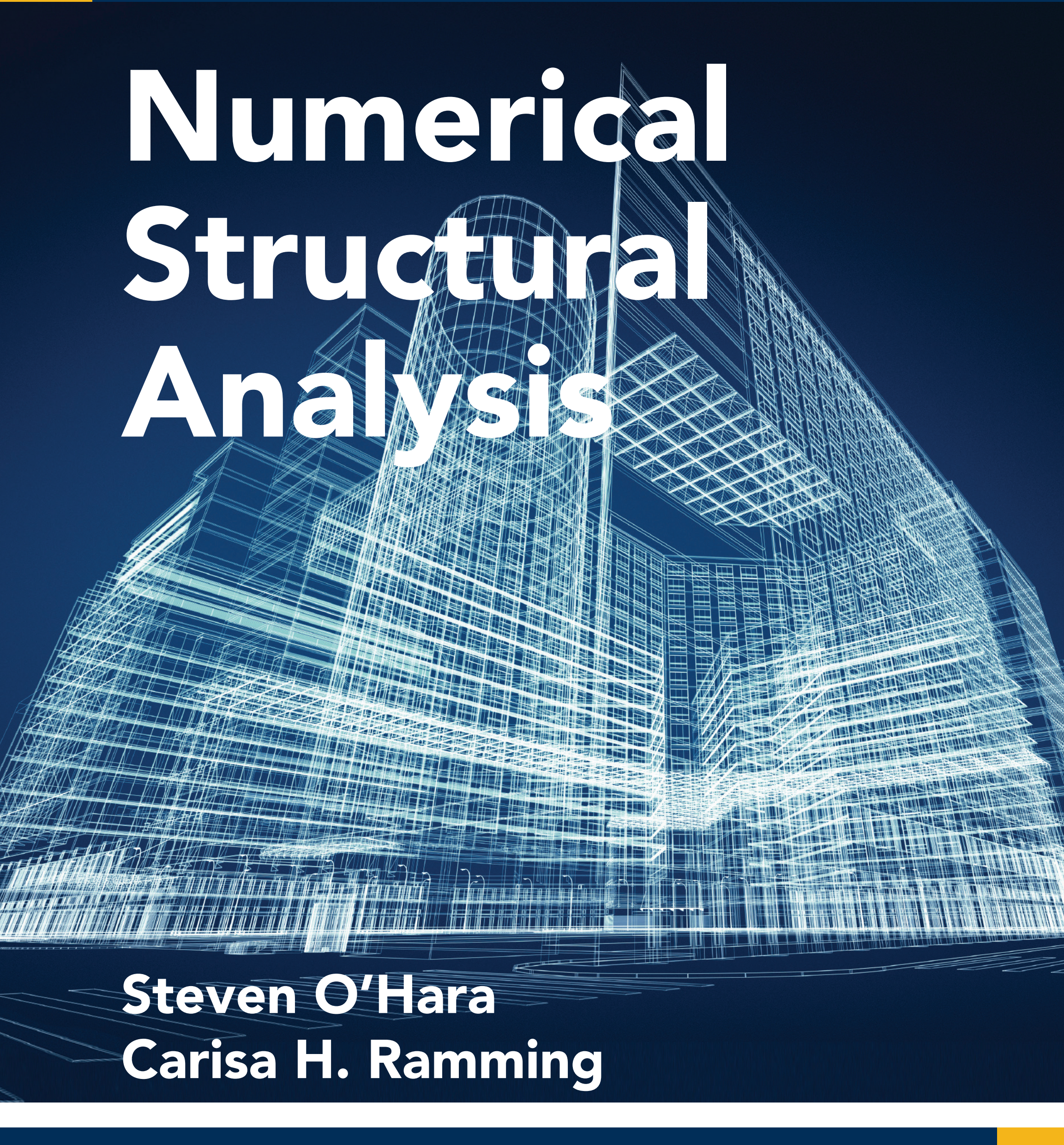
Numerical Structural Analysis



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Abstract

As structural engineers move further into the age of digital computation and rely more heavily on computers to solve problems, it remains paramount that they understand the basic mathematics and engineering principles used to design and analyze building structures. The analysis of complex structural systems involves the knowledge of science, technology, engineering, and math to design and develop efficient and economical buildings and other structures. The link between the basic concepts and application to real world problems is one of the most challenging learning endeavors that structural engineers face. A thorough understanding of the analysis procedures should lead to successful structures.

The primary purpose of this book is to develop a structural engineering student's ability to solve complex structural analysis problems that they may or may not have ever encountered before. The book will cover and review numerical techniques to solve mathematical formulations. These are the theoretical math and science principles learned as prerequisites to engineering courses, but will be emphasized in numerical formulation. A basic understanding of elementary structural analysis is important and many methods will be reviewed. These formulations are necessary in developing the analysis procedures for structural engineering. Once the numerical formulations are understood, engineers can then develop structural analysis methods that use these techniques. This will be done primarily with matrix structural stiffness procedures. Both of these will supplement both numerical and computer solutions. Finally, advanced stiffness topics will be developed and presented to solve unique structural problems. These include member end releases, nonprismatic, shear, geometric, and torsional stiffness.

KEY WORDS

adjoint matrix, algebraic equations, area moment, beam deflection, carry-over factor, castigliano's theorems, cofactor matrix, column matrix,

complex conjugate pairs, complex roots, conjugate beam, conjugate pairs, convergence, diagonal matrix, differentiation, distinct roots, distribution factor, eigenvalues, elastic stiffness, enke roots, extrapolation, flexural stiffness, geometric stiffness, homogeneous, identity matrix, integer, integration, interpolation, inverse, joint stiffness factor, linear algebraic equations, lower triangular matrix, matrix, matrix minor, member end release, member relative stiffness factor, member stiffness factor, moment-distribution, non-homogeneous, non-prismatic members, partial pivoting, pivot coefficient, pivot equation, polynomials, principal diagonal, roots, rotation, rotational stiffness, row matrix, second-order stiffness, shear stiffness, slope-deflection, sparse matrix, square matrix, stiffness matrix, structural flexibility, structural stiffness, symmetric transformation, torsional stiffness, transcendental equations, transformations, transmission, transposed matrix, triangular matrix, upper triangular matrix, virtual work, visual integration

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CHAPTER 1

Roots of Algebraic and Transcendental Equations

In structural engineering, it is important to have a basic knowledge of how computers and calculators solve equations for unknowns. Some equations are solved simply by algebra while higher order equations will require other methods to solve for the unknowns. In this chapter, methods of finding roots to various equations are explored. The *roots* of an equation are defined as values of x where the solution of an equation is true. The most common roots are where the value of the function is zero. This would indicate where a function crosses an axis. Roots are sometimes *complex roots* where they contain both a real number and an imaginary unit.

1.1 EQUATIONS

Equations are generally grouped into two main categories, algebraic equations and transcendental equations. The first type, an *algebraic equation*, is defined as an equation that involves only powers of x. The powers of x can be any real number whether positive or negative. The following are examples of algebraic equations:

$$8x^3 - 3x^2 + 5x - 6 = 0$$

$$\frac{1}{x} + 2\sqrt{x} = 0$$

$$x^{1.25} - 3\pi = 0$$

The second type is *transcendental equations*. These are non-algebraic equations or functions that transcend, or cannot be expressed in terms of algebra. Examples of such are exponential functions, trigonometric functions, and the inverses of each. The following are examples of transcendental equations:

$$cos(x) + sin(x) = 0$$
$$e^x + 15 = 0$$

Transcendental functions may have an infinite number of roots or may not have any roots at all. For example, the function $sin(x) = 0$ has an infinite number of roots $x = \pm kx$ and $k = 0, 1, 2....$

The solution of algebraic or transcendental equations is rarely carried out from the beginning to end by one method. The roots of the equation can generally be determined by one method with some small accuracy, and then made more accurate by other methods. For the intent and purpose of this text, only a handful of the available methods are discussed. These methods include: Descartes' Rule, Synthetic Division, Incremental Search, Refined Incremental Search, Bisection, False Position, Secant, Newton–Raphson, Newton's Second Order, Graeffe's Root Squaring, and Bairstow's methods. Some of these methods are used to solve specific types of equations, while others can be used for both equation types.

1.2 POLYNOMIALS

A *polynomial* is defined as an algebraic equation involving only positive *integer* (whole number) powers of x. Polynomials are generally expressed in the following form:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x^1 + a_n = 0$$

In most cases, the polynomial form is revised by dividing the entire equation by the coefficient of the highest power of a, a_0, resulting in the following form:

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x^1 + a_n = 0$$

For these polynomials, the following apply:

- The order or degree of the polynomial is equal to the highest power of x and the number of roots is directly equal to the degree or n, where a^n is not equal to 0. For example, a sixth degree polynomial, or a polynomial with $n = 6$ has six roots.
- The value of n must be a non-negative integer. In other words, it must be a whole number that is equal to zero or a positive integer.

- The coefficients (a_0, a_1, a_2 …, a_{n-1}, a_n) are real numbers.
- There will be at least one real root if n is an odd integer.
- It is possible that equal roots exist.
- When complex roots exist, they occur in *conjugate pairs*. For example:

$$x = u \pm vi = u \pm v\sqrt{-1}$$

1.3 DESCARTES' RULE

Descartes' rule is a method of determining the maximum number of positive and negative real roots of a polynomial. This method was published by René Descartes in 1637 in his work La Géométrie (Descartes 1637). This rule states that the number of positive real roots is equal to the number of sign changes of the coefficients or is less than this number by an even integer. For positive roots, start with the sign of the coefficient of the lowest (or highest) power and count the number of sign changes from the lowest to the highest power (ignore powers that do not appear). The number of sign changes proves to be the number of positive roots. Using $x = 1$ in evaluating $f(x) = 0$ is the easiest way to look at the coefficients.

For negative roots, begin by transforming the polynomial to $f(x) = 0$. The signs of all the odd powers are reversed while the even powers remain unchanged. Once again, the sign changes can be counted from either the highest to lowest power, or vice versa. The number of negative real roots is equal to the number of sign changes of the coefficients, or less than by an even integer. Using $x = -1$ in evaluating $f(x) = 0$ is the easiest way to look at the coefficients.

When considering either positive or negative roots, the statement "less than by an even integer" is included. This statement accounts for *complex conjugate pairs* that could exist. Complex conjugates change the sign of the imaginary part of the complex number. Descartes' rule is valid as long as there are no zero coefficients. If zero coefficients exist, they are ignored in the count. Also, one could find a root and divide it out to form a new polynomial of degree "$n - 1$" and apply Descartes' rule again.

Example 1.1 Descartes' rule

Find the possible number of positive, negative, and complex roots for the following polynomial:

$$x^3 - 6x^2 + 11x - 6 = 0$$

Find possible positive roots for $f(x) = 0$:

$$x^3 - 6x^2 + 11x - 6 = 0$$

$$1 \quad 2 \quad 3 \quad = \quad 3 \text{ sign changes}$$

Since there are three sign changes, there is a maximum of three positive roots. Three positive real roots exist or one positive real root plus two imaginary roots.

Find possible negative roots by rewriting the function for $f(-x) = 0$:

$$(-x)^3 - 6(-x)^2 + 11(-x) - 6 = -x^3 - 6x^2 - 11x - 6 = 0$$

$$-x^3 - 6x^2 - 11x - 6 = 0$$

$$0 \quad 0 \quad 0 \quad = \quad 0 \text{ sign changes}$$

Notice the signs of all the odd powers reverse while the signs of the even powers remain unchanged. Count the number of sign changes, n. This number is the maximum possible negative roots. Since there is no sign change, zero negative roots exist.

Possible complex roots:

Complex roots appear in conjugate pairs. Therefore, either zero or two complex roots exist. In this example the roots are $x = 1, 2, 3$.

Example 1.2 Descartes' rule

Find the possible number of positive, negative, and complex roots for the following polynomial:

$$x^3 - 7x^2 + 6 = 0$$

Find possible positive roots for $f(x) = 0$:

$$x^3 - 7x^2 + 6 = 0$$

$$1 \quad 2 \quad = \quad 2 \text{ sign changes}$$

Since there are two sign changes, there is a maximum of two positive roots. Two or zero positive real roots exist.

Find possible negative roots by rewriting the function for $f(-x) = 0$:

$$(-x)^3 - 7(-x)^2 + 6 = -x^3 - 7x^2 + 6 = 0$$

$$-x^3 - 7x^2 + 6 = 0$$

0 1 = 1 sign change

Again, the signs of all the odd powers reverse while the signs of the even powers remain unchanged. Count the number of sign changes, n. This number is the maximum possible negative roots. Since there is one sign change, one negative root exists.

Possible complex roots:

Complex roots appear in conjugate pairs. Therefore, either zero or two complex roots exist. In this example, the roots are $x = 1, 2, -3$.

Example 1.3 Descartes' rule

Find the possible number of positive, negative, and complex roots for the following polynomial:

$$x^3 - 3x^2 + 4x - 6 = 0$$

Find possible positive roots for $f(x) = 0$:

$$x^3 - 3x^2 + 4x - 6 = 0$$

1 2 3 = 3 sign changes

Since there are three sign changes, there is a maximum of three positive roots. Three or one positive real roots exist.

Find possible negative roots by rewriting the function for $f(-x) = 0$:

$$(-x)^3 - 3(-x)^2 + 4(-x) - 6 = -x^3 - 3x^2 - 4x - 6 = 0$$

$$-x^3 - 3x^2 - 4x - 6 = 0$$

0 0 0 = 0 sign changes

Count the number of sign changes. Since there is no sign change, zero negative roots exist.

Possible complex roots:

Complex roots appear in conjugate pairs. Therefore, either zero or two complex roots exist. In this example the roots are $x = 1, 1 + i, 1 - i$. It should be noted that the existence of complex conjugate pairs cannot be readily known. Examples 1.1 and 1.3 had the same number of sign change count, but the latter had a complex pair of roots.

Example 1.4 Descartes' rule

Find the possible number of positive, negative, and complex roots for the following polynomial:

$$x^3 - x^2 + 2x = 0$$

Find possible positive roots for $f(x) = 0$:

$$x^3 - x^2 + 2x = 0$$

1 2 = 2 sign changes

Since there are two sign changes, there is a maximum of two positive roots. Two or zero positive real roots exist.

Find possible negative roots by rewriting the function for $f(-x) = 0$:

$$(-x)^3 - (-x)^2 + 2(-x) = -x^3 - x^2 - 2x = 0$$
$$-x^3 - x^2 - 2x = 0$$

0 0 = 0 sign changes

Count the number of sign changes, n. Since there is no sign change, zero negative roots exist.

Possible complex roots:

Complex roots appear in conjugate pairs. Therefore, either zero or two complex roots exist. In this example, the roots are $x = 0, 1 + i, 1 - i$. The existence of zero as a root could have been discovered by noticing that there was not a constant term in the equation. Therefore, dividing the equation by x yields the same as $x = 0$.

1.4 SYNTHETIC DIVISION

Synthetic division is taught in most algebra courses. The main outcome is to divide a polynomial by a value, r. This is in fact the division of a polynomial, $f(x) = 0$ by the linear equation $x - r$. The general polynomial can be divided by $x - r$ as follows:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x^1 + a_n = 0$$

Table 1.1. Synthetic division

r	a_0	a_1	a_2	…….	a_{n-1}	a_n
	0	rb_1	rb_2	…….	rb_{n-1}	rb_n
	b_1	b_2	b_3	…….	b_n	R

The results, b, are the sum of the rows above (i.e., $b_1 = a_0 + 0$ or $b_n = a_{n-1} + rb_{n-1}$). If r is a root, then the remainder, R, will be zero. If r is not a root, then the remainder, R, is the value of the polynomial for $f(x)$ at $x = r$.

Furthermore, after the first division of a polynomial, divide again to find the value of the first derivative equal to the remainder times one factorial, *R*1!*. After the second division of a polynomial, divide again to find the value of the second derivative equal to the remainder times two factorial, *R*2!*. Continuing this process, and after the third division of a polynomial, divide again to find the value of the third derivative equal to the remainder times three factorial, *R*3!*. Basically, two synthetic divisions yield the first derivative, three synthetic divisions yield the second derivative, four synthetic divisions yield the third derivative, and so on.

Example 1.5 Synthetic division

Find $f(1)$ or divide the following polynomial by $x - 1$.

$$x^3 - 6x^2 + 11x - 6 = 0$$

Set up the equation as shown below by writing the divisor, r, and coefficient, a, in the first row.

Table 1.2. Example 1.5 Synthetic division

1	1	−6	11	−6

Add the columns by starting at the left. Multiply each result by $r = 1$ and add this to the next column.

Table 1.3. Example 1.5 Synthetic division

1	1	–6	11	–6
	0	1	–5	6
	1	–5	6	0

Since the remainder, R, is zero, $f(r) = 0$ and $r = 1$ is a root. The polynomial can now be written as a linear equation, $x - r$ or $x - 1$, and the resulting reduced polynomial with coefficient of the resultants as follows:

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6) = 0$$

Use the quadratic equation to reduce the remaining polynomial as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)6}}{2(1)} = \frac{5 \pm 1}{2} = 2,\ 3$$

If the remaining polynomial was divided by $x - 2$, $r = 2$ is a root as follows:

Table 1.4. Example 1.5 Synthetic division

2	1	–5	6
	0	2	–6
	1	–3	0

Since the remainder, R, is zero, $f(2) = 0$ and $r = 2$ is a root. The resulting polynomial is $x - 3$, thus $x = r$ is the third root. This can also be shown by repeating division with $x - 3$.

Table 1.5. Example 1.5 Synthetic division

3	1	–3
	0	3
	1	0

Since the remainder, R, is zero, $f(3) = 0$ and $r = 3$ is a root. The polynomial is now written as:

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6) = (x-1)(x-2)(x-3) = 0$$

The roots are $x = 1, 2, 3$.

Example 1.6 Synthetic division

Find $f(-1)$, $f'(-1)$, and $f''(-1)$ or perform three divisions of the following polynomial by $x + 1$:

$$x^3 - 6x^2 + 11x - 6 = 0$$

Set up the equation as shown in the following by writing the divisor, r, and coefficient, a, in the first row.

Table 1.6. Example 1.6 Synthetic division

–1	1	–6	11	–6

Add the columns by starting at the left. Multiply each result by $r = 1$ and add this to the next column.

Table 1.7. Example 1.6 Synthetic division

–1	1	–6	11	–6
	0	–1	7	–18
	1	–7	18	–24

Since the remainder, R, is –24, $f(-1) = -24$, the polynomial evaluated at –1 is –24. Performing a check as follows:

$$f(-1) = (-1)^3 - 6(-1)^2 + 11(-1) - 6 = -24$$

Now divide the remaining polynomial again by –1 to find $f'(-1)$.

Table 1.8. Example 1.6 Synthetic division

–1	1	–6	11	–6
	0	–1	7	–18
–1	1	–7	18	–24
	0	–1	8	
	1	–8	26	

Since the remainder, R, is 26, $f'(-1)=R*1!=26$. The first derivative of the polynomial evaluated at –1 is 26. Performing a check as follows:

$$f'(x)=3x^2-12x+11=0$$

$$f'(-1)=3(-1)^2-12(-1)+11=26$$

Divide the remaining polynomial again by –1 to find $f''(-1)$.

Table 1.9. Example 1.6 Synthetic division

–1	1	–6	11	–6
	0	–1	7	–18
–1	1	–7	18	–24
	0	–1	8	
–1	1	–8	26	
	0	–1		
	1	–9		

Since the remainder, R, is –9, $f''(-1)=R*2!=$ –9(1)(2)= –18 and the second derivative of the polynomial evaluated at –1 is –18. Performing a check as follows:

$$f''(x)=6x-12=0$$

$$f''(-1)=6(-1)-12=-18$$

1.5 INCREMENTAL SEARCH METHOD

The incremental search method is a simple and quick way to find the approximate location of real roots to algebraic and transcendental equations. A search is performed over a given range of values for x usually denoted as minimum and maximum values, x_{min} and x_{max}. An increment on x of Δx is used to determine the successive values of $f(x)$. Each consecutive pair of functions of x are compared and when their signs are different a root of x has been bounded by the two values of x. Written in algorithmic form, a sign change occurs between x_i and x_{i+1} if $f(x_i)f(x_{i+1}) \leq 0$. The sign change generally indicates a root has been passed but could also indicate a discontinuity in the function. This process is illustrated graphically in Figure 1.1.

Example 1.7 Incremental search method

Determine the first approximate root of the following function starting at $x_{min} = 0$ and using an increment $\Delta x = 0.25$:

$$x^3 - 8.4x^2 + 20.16x - 13.824 = 0$$

Table 1.10. Example 1.7 Incremental search method

x	0	0.25	0.5	0.75	**1**	**1.25**
f(x)	–13.8240	–9.2934	–5.7190	–3.0071	**–1.0640**	**0.2041**

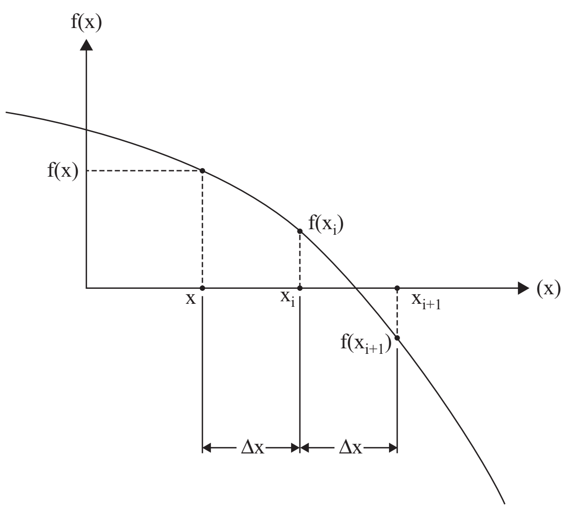

Figure 1.1. Incremental search method.

Since the sign of $f(x)$ changed between $x = 1$ and $x = 1.25$, it is assumed that a root was passed between those values. The actual root occurs at $x = 1.2$. In this example, five digits of precision were used, but in most cases it is a good rule to carry one more digit in the calculations than in the desired accuracy of the answer.

Once the roots have been bounded by the incremental search method, other methods can be utilized in finding more accurate roots: The following sections will cover the refined incremental search, bisection, false position, secant, Newton–Raphson, and Newton's second order methods to determine more accurate roots of algebraic and transcendental equations.

1.6 REFINED INCREMENTAL SEARCH METHOD

Closer approximations of the root may be obtained by the refined incremental search method. This method is a variation of the incremental search method. Once a root has been bounded by a search, the last value of x preceding the sign change is used to perform another search using a smaller increment such as $\Delta x/10$ as shown in Figure 1.2 until the sign changes again.

This process can be repeated with smaller increments of x until the desired accuracy of the root is obtained. Usually the accuracy on the

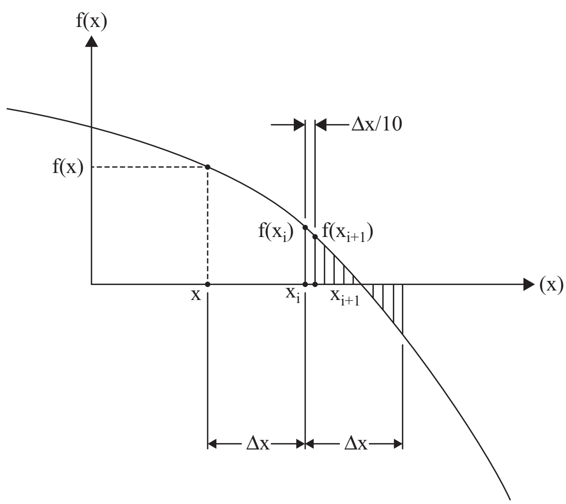

Figure 1.2. Refined incremental search method.

function of x is represented by epsilon, ε, where $|f(x)| < \varepsilon$. Care must be taken in the selection of the starting point and the increment so that a root is not missed. This could happen if two roots occur within an increment and the sign of the function does not change at the successive values of x.

Example 1.8 Refined incremental search method

Refine the search of the function from Example 1.7 between 1.0 and 1.25 using an increment of $\Delta x = 0.25/10$ or 1/10th the original increment.

$$x^3 - 8.4x^2 + 20.16x - 13.824 = 0$$

Table 1.11. Example 1.8 Refined incremental search method

x	1	1.025	1.05	1.075
f(x)	–1.0640	–0.9084	–0.7594	–0.6170

x	1.1	1.125	1.15	**1.175**	**1.2**
f(x)	–0.4810	–0.3514	–0.2281	**–0.1110**	**0.0000**

Since the sign of $f(x)$ changed between $x = 1.175$ and $x = 1.2$, it is assumed that a root was passed between those values. The actual root occurs at $x = 1.2$.

1.7 BISECTION METHOD

After a sign change has occurred in a search method, another way to rapidly *converge* (become closer and closer to the same number) on a root is the bisection method, also known as the half-interval method or the Bolzano method developed in 1817 by Bernard Bolzano. This method takes the bounded increment between two points x_i and x_{i+1} where $f(x_i)f(x_{i+1}) \leq 0$ and divides it in two equal halves or "bisects" the increment. The two subintervals have the first interval from x_i to $x_{i+½}$ and the second interval from $x_{i+½}$ to x_{i+1} as seen in Figure 1.3.

Next, the subinterval containing the root can be found by the following algorithm:

$f(x_i)f(x_{i+½}) < 0$, first interval contains the root
$f(x_i)f(x_{i+½}) > 0$, second interval contains the root
$f(x_i)f(x_{i+½}) = 0$, $x_{i+½}$ is the root

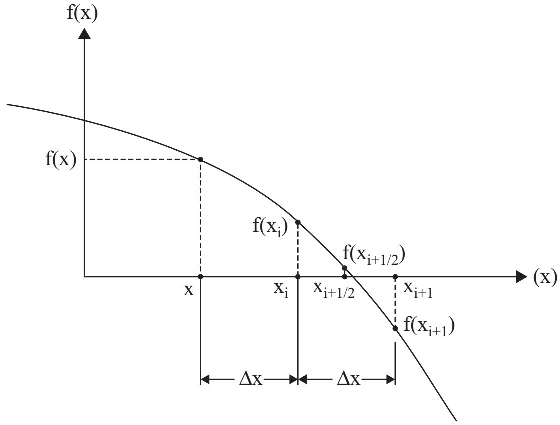

Figure 1.3. Bisection method.

The process is continued by bisecting the subinterval containing the root and repeating the procedure until the desired accuracy is achieved. After n bisections, the size of the original interval has been reduced by a factor of 2^n.

Example 1.9 Bisection method

Refine the search of the function from Example 1.7 between 1.0 and 1.25 using the bisection method to increase the accuracy. Use $\varepsilon = 0.01$ that is $|f(x)| < \varepsilon$:

$$x^3 - 8.4x^2 + 20.16x - 13.824 = 0$$

Begin by solving the equation for 1 and 1.25, which was done in Example 1.7. The sign changes, so a root lies between the two. We also know from the refined incremental search method the root should fall between 1.175 and 1.2. Next, bisect the increment between 1 and 1.25, which is a value of 1.125. Evaluate the function at that point and compare the two subintervals for the sign changes. Also, check to see if the desired accuracy on $f(x)$ is achieved. This occurs between 1.125 and 1.25, so that interval is subdivided again at 1.1875. Continue the bisections until the desired accuracy is achieved. Note in Table 1.12 this occurs at $x = 1.1992$ where $f(x) = -0.0034$. This is the last bisection between 1.1953 and 1.2031.

Table 1.12. Example 1.9 Bisection method

	1	2	3	4
x	1	1.25	1.125	1.1875
interval			1&2	2&3
f(x)	−1.0640	0.2041	−0.3514	−0.0548

	5	6	7	
x	1.2188	1.2031	1.1953	**1.1992**
interval	2&4	4&5	4&6	6&7
f(x)	0.0793	0.0135	−0.0204	**−0.0034**

1.8 METHOD OF FALSE POSITION OR LINEAR INTERPOLATION

Although the bisection method can be used to reach convergence, other methods such as false position provide the same accuracy more rapidly. The process is similar to the bisection method in that between x_i and x_{i+1} where $f(x_i)f(x_{i+1}) \le 0$ a root exists. Refer to x_i and x_{i+1} as x_1 and x_2, respectively. A straight line connecting x_1 and x_2 intersects the x-axis at a new value, say x_3, which is closer to the root than either x_1 or x_2. Thus, by similar triangles, the value of x_3 can be found.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{0-f(x_1)}{x_3-x_1} \therefore x_3 = x_1 - f(x_1)\frac{x_2-x_1}{f(x_2)-f(x_1)}$$

This equation can also be rewritten as follows:

$$x_3=\frac{x_1 f(x_2)-x_2 f(x_1)}{f(x_2)-f(x_1)} \tag{1.1}$$

The relationship between x_1, x_2, and x_3 can be seen in Figure 1.4.

$f(x_1)f(x_3) < 0$, first interval contains the root
$f(x_1)f(x_3) > 0$, second interval contains the root
$f(x_1)f(x_3) = 0$, x_3 is the root

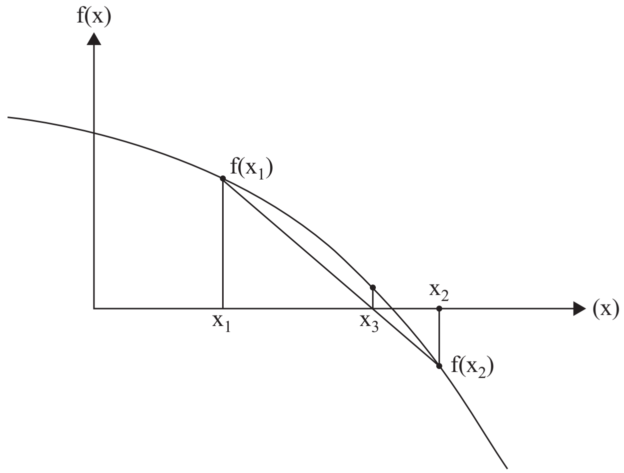

Figure 1.4. Method of false position or linear interpolation.

If the first interval contains the root, the values for the next cycle for x_1 and x_2 and the corresponding functions $f(x_1)$ and $f(x_2)$ are as follows:

x_1 and $f(x_1)$ remain unchanged
$x_2 = x_3$
$f(x_2) = f(x_3)$

If the second interval contains the root, then the values are used for x_1 and x_2 and the corresponding functions $f(x_1)$ and $f(x_2)$ are as follows:

x_2 and $f(x_2)$ remain unchanged
$x_1 = x_3$
$f(x_1) = f(x_3)$

The process is continued until the desired accuracy is obtained.

Example 1.10 Method of false position

Refine the search of the function from Example 1.7 between 1.0 and 1.25 using the false position method to increase the accuracy of the approximate root. For the accuracy test use $\varepsilon = 0.01$ that is $|f(x)| < \varepsilon$:

$$x^3 - 8.4x^2 + 20.16x - 13.824 = 0$$

Using Equation 1.1 to solve for a closer point between $x_1 = 1$ and $x_2 = 1.25$.

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{1(0.2041) - 1.25(-1.064)}{0.2041 - (-1.064)} = 1.2098$$

Repeat this process until the desired accuracy of the root is achieved as shown in Table 1.13.

Table 1.13. Example 1.10 Method of false position

	1	2	3	4
x	1	1.25	1.2098	**1.2018**
interval			1&2	1&3
f(x)	−1.064	0.2041	0.0417	**0.0080**

1.9 SECANT METHOD

The secant method is similar to the false position method except that the two most recent values of x (x_2 and x_3) and their corresponding function values [$f(x_2)$ and $f(x_3)$] are used to obtain a new approximation to the root instead of checking values that bound the root. This eliminates the need to check which subinterval contains the root. The variable renaming process for iteration is as follows:

$$x_1 = x_2 \quad \text{and} \quad x_2 = x_3$$
$$f(x_1) = f(x_2) \quad \text{and} \quad f(x_2) = f(x_3)$$

In some instances *interpolation* occurs, this is when the new value is between the previous two values. In others, *extrapolation* occurs, meaning the new value is not between the previous two values. Interpolation was shown in Figure 1.4 and extrapolation is shown in Figure 1.5.

Example 1.11 Secant method

Refine the search of the function from Example 1.7 between 1.0 and 1.25 using the secant method to increase the accuracy of the approximate root. For the accuracy test use $\varepsilon = 0.01$.

$$x^3 - 8.4x^2 + 20.16x - 13.824 = 0$$

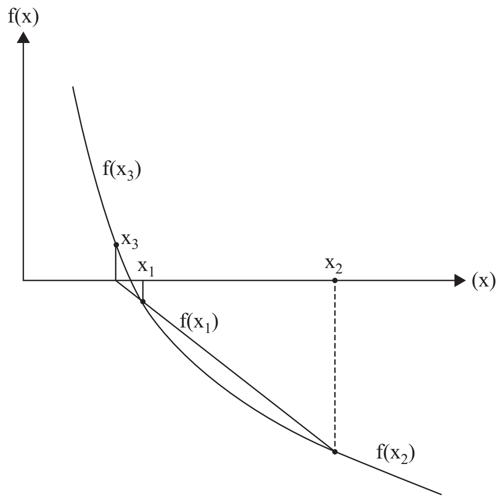

Figure 1.5. Secant method.

The process of finding the new value is the same as linear interpolation using Equation 1.1 to solve for a closer point between $x_1 = 1$ and $x_2 = 1.25$.

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{1(0.2041) - 1.25(-1.064)}{0.2041 - (-1.064)} = 1.2098$$

The reassignment of the values simply uses the last two values and their corresponding functions as shown in Table 1.14.

Table 1.14. Example 1.11 Secant method

	1	2	3	4
x	1	1.25	1.2098	**1.1994**
			1&2	2&3
f(x)	−1.0640	0.2041	0.0417	**−0.0025**

This happens to be similar to the false position Example 1.10 as only interpolations occur, but with different sub-intervals.

1.10 NEWTON–RAPHSON METHOD OR NEWTON'S TANGENT

The Newton–Raphson method uses more information about the function to speed up convergence. It was originally developed by Issac Newton in

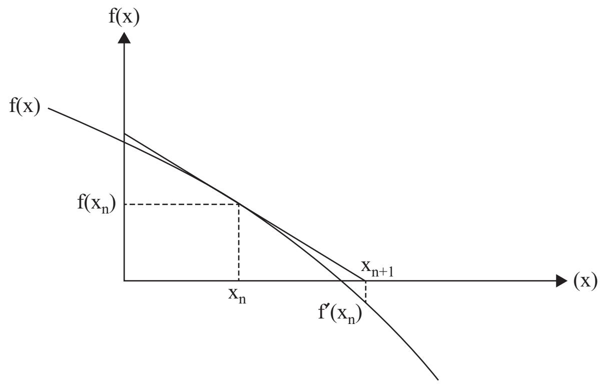

Figure 1.6. Newton–Raphson method or Newton's tangent.

1669 (Newton 1669). Once an approximate root x_n has been found, not only is the function, $f(x_n)$, used, but the slope of the function at that point, $f'(x_n)$, is also incorporated to converge to the root more rapidly. The slope of the function is found from the first derivative of the function evaluated at a point. This only requires the use of one value to be known. The slope intersects the x-axis at a value x_{n+1} as shown in Figure 1.6 and the relationship is given in Equation 1.2.

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}} \therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.2}$$

Repeat the process using a new value until convergence occurs. Convergence may not occur in the following two cases:

- $f''(x_n)$, (curvature) changes sign near a root, shown in Figure 1.7.
- Initial approximation is not sufficiently close to the true root and the slope at that point has a small value, shown in Figure 1.8.

Example 1.12 Newton–Raphson method

Refine the search from Example 1.7 with a starting value of 1.25 using the Newton–Raphson method to increase the accuracy of the approximate root. For the accuracy test use $\varepsilon = 0.01$.

$$x^3 - 8.4x^2 + 20.16x - 13.824 = 0$$

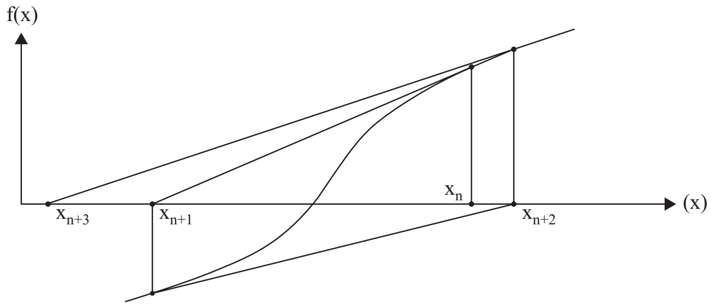

Figure 1.7. Newton–Raphson method or Newton's tangent.

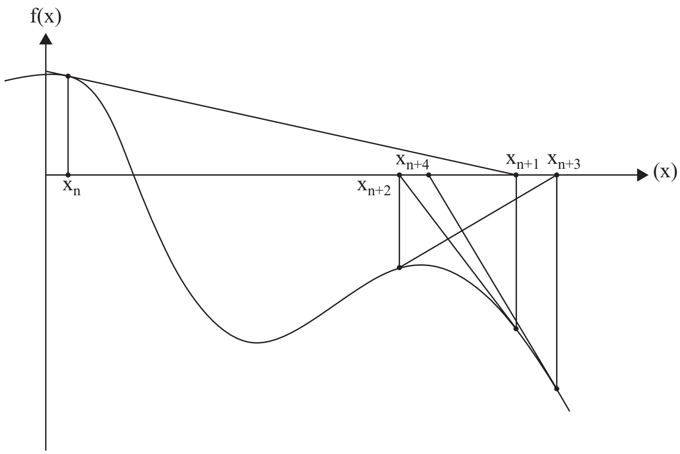

Figure 1.8. Newton–Raphson method or Newton's tangent.

The derivative of the function must be obtained to find the slope at any given value.

$$f(x) = x^3 - 8.4x^2 + 20.16x - 13.824$$
$$f'(x) = 3x^2 - 16.8x + 20.16$$

Beginning with $x_n = 1.25$, use Equation 1.2 to determine the next value.

$$f(1.25) = 1.25^3 - 8.4(1.25)^2 + 20.16(1.25) - 13.824 = 0.2041$$
$$f'(1.25) = 3(1.25)^2 - 16.8(1.25) + 20.16 = 3.8475$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1.25 - \frac{0.2041}{3.8475} = 1.1969$$

Repeat the process until the desired accuracy is obtained in Table 1.15.

Table 1.15. Example 1.12 Newton–Raphson method

x	1.25	1.1969	**1.19999**
f(x)	0.2041	–0.0132	–0.00004
f′(x)	3.8475	4.3493	**4.32010**

1.11 NEWTON'S SECOND ORDER METHOD

Newton's second order method is often a preferred method to determine the value of a root due to its rapid convergence and extremely close approximation. This method also includes the second derivative of the function or the curvature to find the approximate root. The equation $f(x) = 0$ is considered as the target for the root. Figure 1.9 shows the plot of the actual function.

The Taylor series expansion was discovered by James Gregory and introduced by Brook Taylor in 1715 (Taylor 1715). The following is a Taylor series expansion of $f(x)$ about $x = x_n$:

$$f(x_{n+1}) = f(x_n) + f'(x_n)(\Delta x) + \frac{f''(x_n)(\Delta x)^2}{2!} + \frac{f'''(x_n)(\Delta x)^3}{3!} + \ldots$$

For a means of determining a value of Δx that will make the Taylor series expansion go to zero, the first three terms of the right hand side of the equation are set equal to zero to obtain an approximate value.

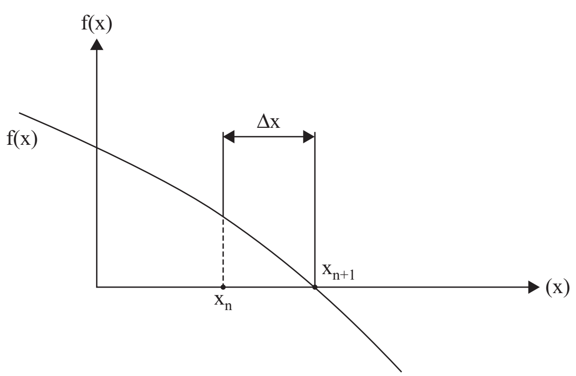

Figure 1.9. Newton's second order method.

$$f(x_n)+(\Delta x)\left[f'(x_n)+\frac{f''(x_n)(\Delta x)}{2}\right]=0$$

The exact value of Δx cannot be determined from this equation since only the first three terms of the infinite series were used in the calculation. However, a close approximation of the root is a result. When using this equation to calculate Δx, a quadratic must be solved yielding two possible roots. In order to avoid this problem, $\Delta x = -f(x_n)/f'(x_n)$ from Newton's tangent may be substituted into the bracketed term only.

$$f(x_n)+(\Delta x)\left[f'(x_n)-\frac{f''(x_n)f(x_n)}{2f'(x_n)}\right]=0$$

Solving for Δx we obtain the following:

$$\Delta x=-\left[\frac{f(x_n)}{f'(x_n)-\frac{f''(x_n)f(x_n)}{2f'(x_n)}}\right]$$

Observing Figure 1.9 we see that $\Delta x = x_{n+1} - x_n$. Substituting into the previous equation, Equation 1.3 is obtained as follows:

$$x_{n+1}=x_n-\left[\frac{f(x_n)}{f'(x_n)-\frac{f''(x_n)f(x_n)}{2f'(x_n)}}\right] \quad (1.3)$$

If the first derivative is small, the slope is close to zero near the value and the next approximation may be inaccurate. Therefore, use the second derivative term as follows:

$$f'(x_n)=0$$

$$f(x_n)+(\Delta x)\left[\frac{f''(x_n)(\Delta x)}{2}\right]=0$$

$$-f(x_n)=(\Delta x)\left[\frac{f''(x_n)(\Delta x)^2}{2}\right]$$

$$\Delta x^2+\frac{f(x_n)}{\frac{f''(x_n)}{2}}=0$$

Solving by the quadratic equation, two roots are obtained:

$$\Delta x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(0) \pm \sqrt{0^2 - 4(1)\dfrac{f(x_n)}{\dfrac{f''(x_n)}{2}}}}{2(1)} = \pm \frac{2\sqrt{-\dfrac{f(x_n)}{\dfrac{f''(x_n)}{2}}}}{2}$$

$$\Delta x = \pm \sqrt{-\frac{f(x_n)}{\dfrac{f''(x_n)}{2}}}$$

With $\Delta x = x_{n+1} - x_n$, Equation 1.4 is as follows:

$$x_{n+1} = x_n \pm \sqrt{-\frac{f(x_n)}{\dfrac{f''(x_n)}{2}}} \qquad (1.4)$$

This process is a good tool for finding two roots that are near each other. This will happen when the slope is close to zero near a root. Double roots occur when the first derivative is zero, triple roots occur when the first and second derivatives are zero, and so on. These are shown graphically in Figure 1.10.

Example 1.13 Newton's second order method

Refine the search from Example 1.7 with a starting value of 1.25 using the Newton's second order method to increase the accuracy of the approximate root. For the accuracy test use $\varepsilon = 0.01$.

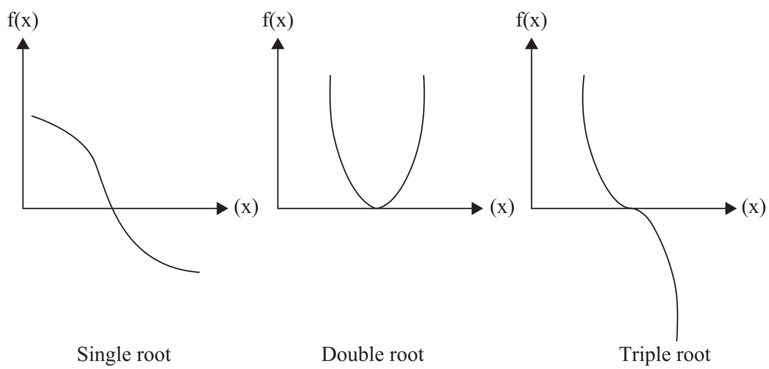

Figure 1.10. Newton's second order method.

$$x^3 - 8.4x^2 + 20.16x - 13.824 = 0$$

The first and second derivatives of the function must be obtained to find the slope and curvature at any given value.

$$f(x) = x^3 - 8.4x^2 + 20.16x - 13.824$$
$$f'(x) = 3x^2 - 16.8x + 20.16$$
$$f''(x) = 6x - 16.8$$

Beginning with x_n=1.25, use Equation 1.3 to determine the next value.

$$f(1.25) = 1.25^3 - 8.4(1.25)^2 + 20.16(1.25) - 13.824 = 0.2041$$
$$f'(1.25) = 3(1.25)^2 - 16.8(1.25) + 20.16 = 3.8475$$
$$f''(1.25) = 6(1.25) - 16.8 = -9.30$$

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n) - \frac{f''(x_n)f(x_n)}{2f'(x_n)}}\right] = 1.25 - \left[\frac{0.2401}{3.8475 - \frac{(-9.3)0.2041}{2(3.8475)}}\right]$$
$$= 1.2001$$

Repeat the process until the desired accuracy is obtained in Table 1.16.

Table 1.16. Example 1.13 Newton's second order method

x	1.25	1.200143	1.2
f(x)	0.2041	0.00062	0.000000
f′(x)	3.8475	4.31863	4.320000
f″(x)	–9.3000	–9.59914	–9.600000

1.12 GRAEFFE'S ROOT SQUARING METHOD

Graeffe's root squaring method is a root-finding method that was among the most popular methods for finding roots of polynomials in the 19th and 20th centuries. This method was developed independently by

Germinal Pierre Dandelin in 1826 and Karl Heinrich Gräffe in 1837. The Graeffe's root squaring method is especially effective if all roots are real. The derivation of this method proceeds by multiplying a polynomial $f(x)$ by $f(-x)$ using the following polynomial equations in factored form:

$$f(x) = (x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n)$$
$$f(-x) = (-1)(x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n)$$
$$f(x)\,f(-x) = (-1)^n\left(x^2 - a_1^2\right)\left(x^2 - a_2^2\right)\left(x^2 - a_3^2\right)\ldots\left(x^2 - a_n^2\right)$$

For example, use a third degree polynomial with roots x_1, x_2, and x_3 as follows:

$$f(x) = 0 = x^3 + a_1x^2 + a_2x + a_3$$

A polynomial with roots $-x_1$, x_2, and $-x_3$ follows:

$$f(-x) = 0 = -x^3 + a_1x^2 - a_2x + a_3$$

Multiplying the two equations together yields the following:

$$f(x)\,f(-x) = 0 = -x^6 + \left(a_1^2 - 2a_2\right)x^4 + \left(-a_2^2 + 2a_1a_3\right)x + a_3^2$$

Letting $y = -x^2$ this equation may be written as follows:

$$0 = y^3 + \left(a_1^2 - 2a_2\right)y^2 + \left(a_2^2 - 2a_1a_3\right)y + a_3^2$$

This equation has roots of $-x_1^2$, $-x_2^2$, and $-x_3^2$. If the procedure was applied again, another polynomial would be derived with roots of $-x_1^4$, $-x_2^4$, and $-x_3^4$. If computed a third time, they would be of $-x_1^8$, $-x_2^8$, and $-x_3^8$. The pattern of the roots becomes clear with continuing cycles. The general process of an nth degree polynomial would be in the following forms:

$$0 = y^n + \left(a_1^2 - 2a_2\right)y^{n-1} + \left(a_2^2 - 2a_1a_3 + 2a_4\right)y^{n-2}$$
$$+ \left(a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_6\right)y^{n-3} + \cdots + a_n^2$$

Table 1.17. Graeffe's root squaring method

m	a_0x^n	a_1x^{n-1}	a_2x^{n-2}	a_3x^{n-3}	....	$a_{n-1}x$	a_nx^0	cycle
1	a_0	a_1	a_2	a_3	....	a_{n-1}	a_n	1
	a_0^2	a_1^2	a_2^2	a_3^2	....	a_{n-1}^2	a_n^2	
		$-2a_0a_2$	$-2a_1a_3$	$-2a_2a_4$	....	....	....	
			$+2a_0a_4$	$+2a_1a_5$	....	....	....	
				$-2a_0a_6$	....	....	....	
2	b_0	b_1	b_2	b_3	....	b_{n-1}	b_n	2
	b_0^2	b_1^2	b_2^2	b_3^2	....	b_{n-1}^2	b_n^2	
		$-2b_0b_2$	$-2b_1b_3$	$-2b_2b_4$	....	....	....	
			$+2b_0b_4$	$+2b_1b_5$	....	....	....	
				$-2b_0b_6$	....	....	....	
4	c_0	c_1	c_2	c_3	....	c_{n-1}	c_n	3
r		$(c_1/c_0)^{1/m}$	$(c_2/c_1)^{1/m}$	$(c_3/c_2)^{1/m}$	....	$(c_{n-1}/c_{n-2})^{1/m}$	$(c_n/c_{n-1})^{1/m}$	

This can be written with the coefficients in a vertical format as follows:

$$0 = y^n + \begin{Bmatrix} a_1^2 \\ -2a_2 \end{Bmatrix} y^{n-1} + \begin{Bmatrix} a_2^2 \\ -2a_1a_3 \\ +2a_4 \end{Bmatrix} y^{n-2} + \begin{Bmatrix} a_3^2 \\ -2a_2a_4 \\ +2a_1a_5 \\ -2a_6 \end{Bmatrix} y^{n-3}$$

$$+ \begin{Bmatrix} a_4^2 \\ -2a_3a_5 \\ +2a_2a_6 \\ -2a_1a_7 \\ +2a_8 \end{Bmatrix} y^{n-4} + \cdots + a_n^2$$

A tabular solution may be set up considering the following general polynomial:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x^1 + a_n = 0$$

Carefully inspect the coefficient of the polynomial for a pattern. The solution of the original polynomial can take three different forms. These are shown in Figure 1.11.

- Real and distinct roots
- Real and equal roots
- Complex roots

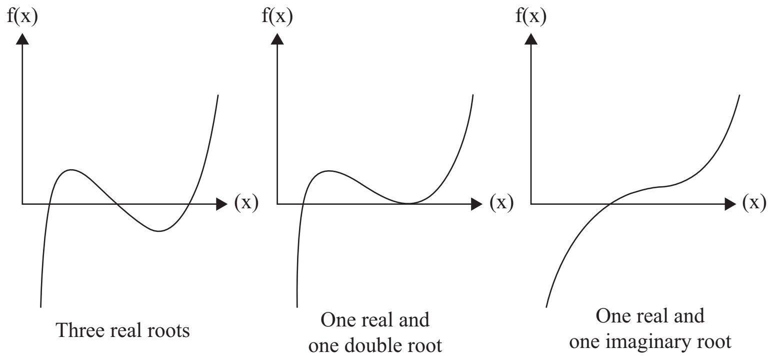

Figure 1.11. Graeffe's root squaring method.

1.12.1 REAL AND DISTINCT ROOTS

The first possible solution type will occur after many cycles of squaring the polynomial; the coefficients of the derived polynomial are the squares of the terms from the preceding cycle. This is known as the regular solution and yields real and *distinct roots* (not equal). The roots of the polynomial or the derived polynomials can be determined from the factored form of the polynomials. The Enke roots of a polynomial are the negatives of the roots of the polynomial. If r is denoted as the Enke root designation, then $x_1 = -r_1$, $x_2 = -r_2$ ... $x_n = -r_n$. The third degree polynomial is shown in factored form:

$$f(x) = 0 = x^3 + a_1x^2 + a_2x + a_3$$
$$f(x) = 0 = (x - x_1)(x - x_2)(x - x_3)$$

If the previous equation is multiplied out, the following is the result:

$$f(x) = 0 = x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3$$

Therefore, for the polynomial, the original coefficients are as follows:

$$a_1 = -(x_1 + x_2 + x_3)$$
$$a_2 = x_1x_2 + x_1x_3 + x_2x_3$$
$$a_3 = -x_1x_2x_3$$

The Enke roots of $x_1 = -r_1$, $x_2 = -r_2$, and $x_3 = -r_3$ are substituted in. The sign has been lost so the Enke roots are used as the basis ($x_1 = -r_1$, $x_2 = -r_2$, etc.), then the following is true:

$$a_1 = r_1 + r_2 + r_3$$
$$a_2 = r_1r_2 + r_1r_3 + r_2r_3$$
$$a_3 = r_1r_2r_3$$

As the cycles (m) continue, the derived polynomial becomes the following:

$$f(x) = 0 = y^3 + b_1y^2 + b_2y + b_3$$

The Enke root relationship is then as follows:

$$b_1 = r_1^m + r_2^m + r_3^m$$
$$b_2 = r_1^m r_2^m + r_1^m r_3^m + r_2^m r_3^m$$
$$b_3 = r_1^m r_2^m r_3^m$$

If we consider only the dominant terms in each expression, the following occurs:

$$b_1 \cong r_1^m$$
$$b_2 \cong r_1^m r_2^m$$
$$b_3 \cong r_1^m r_2^m r_3^m$$

These become the following:

$$b_1 \cong r_1^m \therefore r_1 \cong b_1^{1/m}$$
$$b_2 \cong r_1^m r_2^m \therefore r_2^m \cong \frac{b_2}{r_1^m} \cong \frac{b_2}{b_1} \therefore r_2 = \left(\frac{b_2}{b_1}\right)^{1/m}$$
$$b_3 \cong r_1^m r_2^m r_3^m \therefore r_3^m \cong \frac{b_3}{r_1^m r_2^m} \cong \frac{b_3}{b_2} \therefore r_3 = \left(\frac{b_3}{b_2}\right)^{1/m}$$

The general expression for this regular solution is the following:

$$r_n^m \cong \frac{b_n}{b_{n-1}} \therefore r_n = \left(\frac{b_n}{b_{n-1}}\right)^{1/m}$$

The Enke roots only lack the proper sign and either positive or negative may be correct, so a check is necessary.

Example 1.14 Graeffe's root squaring method—real and distinct roots.

Find the root of the following polynomial using Graeffe's root squaring method.

$$f(x) = 0 = x^4 - 10x^3 + 35x^2 - 50x + 24$$

Table 1.18. Example 1.14 Graeffe's root squaring method—real and distinct roots

m	x^4	a_1x^3	a_2x^2	a_3x	a_4x^0	cycle
1	1	–10	35	–50	24	1
	1	100	1225	2500	576	
		–70	–1000	–1680		
			48			
2	1	30	273	820	576	2
	1	900	74529	672400	331776	
		–546	–49200	–314496		
			1152			
4	1	354	26481	357904	331776	3
	1	1.253E+05	7.012E+08	1.281E+11	1.101E+11	
		–5.296E+04	–2.534E+08	–1.757E+10		
			6.636E+05			
8	1	7.235E+04	4.485E+08	1.105E+11	1.101E+11	4
	1	5.235E+09	2.012E+17	1.222E+22	1.212E+22	
		–8.970E+08	–1.599E+16	–9.874E+19		
			2.202E+11			
16	1	4.338E+09	1.852E+17	1.212E+22	1.212E+22	5
	1	1.882E+19	3.429E+34	1.468E+44	1.468E+44	
		–3.703E+17	–1.051E+32	–4.487E+39		
			2.423E+22			
32	1	1.845E+19	3.418E+34	1.468E+44	1.468E+44	6
	1	3.404E+38	1.168E+69	2.155E+88	2.155E+88	
		–6.836E+34	–5.417E+63	–1.004E+79		
			2.936E+44			
64	1	3.403E+38	1.168E+69	2.155E+88	2.155E+88	7
	1	1.158E+77	1.365E+138	4.646E+176	4.646E+176	
		–2.337E+69	–1.467E+127	–5.037E+157		
			4.311E+88			
128	1	1.158E+77	1.365E+138	4.646E+176	4.646E+176	8
r		4	3	2	1	

Refer to Table 1.17 for the basic procedure for the root squaring. Table 1.18 shows the process for this polynomial.

To determine the proper sign of the roots from the Enke roots, a check is required.

$$r_1 \cong b_1^{1/m} = \left[1.158(10)^{77}\right]^{1/128} = \pm 4.000$$

$$r_2 = \left(\frac{b_2}{b_1}\right)^{1/m} = \left[\frac{1.365(10)^{138}}{1.158(10)^{77}}\right]^{1/128} = \pm 3.000$$

$$r_3 = \left(\frac{b_3}{b_2}\right)^{1/m} = \left[\frac{4.646(10)^{176}}{1.365(10)^{138}}\right]^{1/128} = \pm 2.000$$

$$r_4 = \left(\frac{b_4}{b_3}\right)^{1/m} = \left[\frac{4.646(10)^{176}}{4.646(10)^{176}}\right]^{1/128} = \pm 1.000$$

Substituting the Enke roots into the original equations yields $x_1 = 4.000$, $x_2 = 3.000$, $x_3 = 2.000$, and $x_4 = 1.000$.

1.12.2 REAL AND EQUAL ROOTS

After many cycles of squaring the polynomial, the second possible solution type will occur when the coefficients of the derived polynomial are the squares of the terms in the preceding cycle with the exception of one term that is ½ the square of the term in the preceding cycle. This indicates that two of the roots are equal, the one with the ½ squared term and the next one to the right. Furthermore, if one term is ⅓ the square of the term in the proceeding cycle, three of the roots are equal—the term with the ⅓ squared term and the next two to the right. A similar relationship will occur if four or more roots are equal. The roots (Enke roots) will have a relationship similar to the following assuming $r_1 = r_2$ and considering only the dominant terms in each expression:

$$b_1 = r_1^m + r_2^m + r_3^m = r_1^m + r_1^m \therefore b_1 \cong 2r_1^m$$
$$b_2 = r_1^m r_2^m + r_1^m r_3^m + r_2^m r_3^m = r_1^m r_1^m \therefore b_2 \cong r_1^{2m}$$
$$b_3 = r_1^m r_2^m r_3^m = r_1^m r_1^m r_3^m \therefore b_3 \cong r_1^{2m} r_3^m$$

These become the following:

$$b_1 \cong 2r_1^m \therefore r_1 \cong \left(\frac{b_1}{2}\right)^{1/m}$$
$$b_2 \cong r_1^{2m} \therefore r_1 = (b_2)^{1/2m} = r_2$$
$$b_3 \cong r_1^{2m} r_3^m = b_2 r_3^m \therefore r_3 = \left(\frac{b_3}{b_2}\right)^{1/m}$$

After the multiple roots have been passed, the rest of the terms have the regular solution relationship and will appear as follows:

$$r_n^m \cong \frac{b_n}{b_{n-1}} \therefore r_n = \left(\frac{b_n}{b_{n-1}}\right)^{1/m}$$

If the second term was ½ the square of the term in the previous cycle, then the solution would appear as follows, assuming $r_2 = r_3$ and considering only the dominant terms in each expression:

$$b_1 = r_1^m + r_2^m + r_3^m = r_1^m \therefore b_1 \cong r_1^m$$
$$b_2 = r_1^m r_2^m + r_1^m r_3^m + r_2^m r_3^m = r_1^m r_2^m + r_1^m r_2^m \therefore b_2 \cong 2r_1^m r_2^m$$
$$b_3 = r_1^m r_2^m r_3^m = r_1^m r_2^m r_2^m \therefore b_3 \cong r_1^m r_2^{2m}$$

These become the following:

$$b_1 \cong r_1^m \therefore r_1 \cong (b_1)^{1/m}$$
$$b_2 \cong 2r_1^m r_2^m = 2b_1 r_2^m \therefore r_2 = \left(\frac{b_2}{2b_1}\right)^{1/m} = r_3$$
$$b_3 \cong r_1^m r_2^{2m} = b_1 r_2^{2m} \therefore r_2 = \left(\frac{b_3}{b_1}\right)^{1/2m} = r_3$$

Similar to the previous case where $r_1 = r_2$, after the multiple roots have been passed, the rest of the terms have the regular solution relationship and will appear as follows:

$$r_n^m \cong \frac{b_n}{b_{n-1}} \therefore r_n = \left(\frac{b_n}{b_{n-1}}\right)^{1/m}$$

Note the pattern of the powers and the relationship for any other variations of two roots may be found. Derive one more case for a triple root. If the first term was ⅓ the square of the term in the previous cycle, it would indicate a triple root or $r_1 = r_2 = r_3$. If we consider only the dominant terms in each expression, the following relationships occur:

$$b_1 = r_1^m + r_2^m + r_3^m + r_4^m = r_1^m + r_1^m + r_1^m \therefore b_1 \cong 3r_1^m$$
$$b_2 = r_1^m r_2^m + r_1^m r_3^m + r_1^m r_4^m + r_2^m r_3^m + r_2^m r_4^m + r_3^m r_4^m = r_1^m r_1^m$$
$$+ r_1^m r_1^m + r_1^m r_1^m \therefore b_2 \cong 3r_1^{2m}$$
$$b_3 = r_1^m r_2^m r_3^m + r_1^m r_2^m r_4^m + r_1^m r_3^m r_4^m + r_2^m r_3^m r_4^m = r_1^m r_1^m r_1^m \therefore b_3 \cong r_1^{3m}$$

These become the following:

$$b_1 \cong 3r_1^m \therefore r_1 \cong \left(\frac{b_1}{3}\right)^{1/m}$$
$$b_2 \cong 3r_1^{2m} \therefore r_1 = \left(\frac{b_2}{2}\right)^{1/2m} = r_2 = r_3$$
$$b_3 \cong r_1^{3m} = b_2 r_3^m \therefore r_1 = (b_3)^{1/3m} = r_2 = r_3$$

After the multiple roots have been passed, the rest of the terms have the regular solution relationship and will appear as follows:

$$r_n^m \cong \frac{b_n}{b_{n-1}} \therefore r_n = \left(\frac{b_n}{b_{n-1}}\right)^{1/m}$$

Just like with the regular solution for real and distinct roots, the Enke roots only lack the proper sign and either + or – must be checked.

Example 1.15 Graeffe's root squaring method—real and equal roots

Find the root of the following polynomial using Graeffe's root squaring method.

$$f(x) = 0 = x^3 + 3x^2 - 4$$

Refer to Table 1.17 for the basic procedure for the root squaring. Table 1.19 shows the process for this polynomial.

Table 1.19. Example 1.15 Graeffe's root squaring method—real and equal roots

m	x^3	a_1x^2	a_2x^1	a_3x^0	cycle
1	1	3	0	–4	1
	1	9	0	16	
		0	24		
2	1	9	24	16	2
	1	81	576	256	
		–48	–288		
4	1	33	288	256	3
	1	1089	82944	65536	
		–576	–16896		
8	1	513	66048	65536	4
	1	2.632E+05	4.362E+09	4.295E+09	
		–1.321E+05	–6.724E+07		
16	1	1.311E+05	4.295E+09	4.295E+09	5
	1	1.718E+10	1.845E+19	1.845E+19	
		–8.590E+09	–1.126E+15		
32	1	8.590E+09	1.845E+19	1.845E+19	6
	1	7.379E+19	3.403E+38	3.403E+38	
		–3.689E+19	–3.169E+29		
64	1	3.689E+19	3.403E+38	3.403E+38	7
	1	1.361E+39	1.158E+77	1.158E+77	
		–6.806E+38	–2.511E+58		
128	1	6.806E+38	1.158E+77	1.158E+77	8
	1	4.632E+77	1.341E+154	1.341E+154	
		–2.316E+77	–1.576E+116		
256	1	2.316E+77	1.341E+154	1.341E+154	9
r		2	2	1	

Notice that the first term in the table for cycle 9 is ½ the square of the term in the previous cycles and the following solution applies:

$$r_1 \cong \left(\frac{b_1}{2}\right)^{1/m} = \left[\frac{2.316(10)^{77}}{2}\right]^{1/256} = \pm 2.000$$

$$r_2 \cong (b_2)^{1/2m} = \left[1.341(10)^{154}\right]^{1/2(256)} = \pm 2.000 = r_1$$

$$r_3 \cong \left(\frac{b_3}{b_2}\right)^{1/m} = \left[\frac{1.341(10)^{154}}{1.341(10)^{154}}\right]^{1/256} = \pm 1.000$$

Substituting the Enke roots into the original equations yields $x_1 = 2.000$, $x_2 = 2.000$, and $x_3 = 1.000$.

1.12.3 REAL AND COMPLEX ROOTS

The third possible solution type will occur after many cycles of squaring the polynomial; the coefficients of the derived polynomial are the squares of the terms in the preceding cycle, except if one or more terms have a sign fluctuation, then two of the roots are complex—the one with the sign fluctuation term and the next one to the right constitutes the complex conjugate pair of roots. The roots (Enke roots) will have a relationship similar to the following assuming r_3 and r_4 are the complex conjugate pair of roots and considering only the dominant terms in each expression:

$$x_3 = Re^{i\theta} = \left(cos\theta + isin\theta\right) = u + iv$$
$$x_4 = Re^{-i\theta} = \left(cos\theta - isin\theta\right) = u - iv$$

The values i and R for the complex form in polar or Cartesian simple form are:

$$i = \sqrt{-1} \quad \text{and} \quad R = \sqrt{u^2 + v^2}$$

The form of the coefficients will become the following:

$$b_1 = r_1^m + r_2^m + R^m\left(e^i\theta^m + e^{-i}\theta^m\right)$$
$$b_2 = r_1^m r_2^m + \left(r_1 Re^i\theta\right)^m + \left(r_1 Re^{-i}\theta\right)^m + \left(r_2 Re^i\theta\right)^m + \left(r_2 Re^{-i}\theta\right)^m + R^{2m}$$
$$b_3 = \left(r_1 r_2 Re^i\theta\right)^m + \left(r_1 r_2 Re^{-i}\theta\right)^m + \left(r_1 R^2\right)^m + \left(r_2 R^2\right)^m$$
$$b_4 = \left(r_1 r_2 R^2\right)^m$$

These become the following using polar transformations:

$$b_1 = r_1^m + r_2^m + 2R^m\left(cos\ m\theta\right)$$
$$b_2 = r_1^m r_2^m + 2R^m\left(r_1^m + r_2^m + cos\ m\theta\right) + R^{2m}$$
$$b_3 = 2\left(r_1 r_2 R\right)^m cos\ m\theta + R^{2m}\left(r_1^m + r_2^m\right)$$
$$b_4 = \left(r_1 r_2 R^2\right)^m$$

If we consider only the dominant terms in each expression, the following occurs:

$$b_1 = r_1^m \therefore r_1 = (b_1)^{1/m}$$

$$b_2 = r_1^m r_2^m = b_1 r_2^m \therefore r_2 = \left(\frac{b_2}{b_1}\right)^{1/m}$$

$$b_3 = 2(r_1 r_2 R)^m \cos m\theta$$

$$b_4 = (r_1 r_2 R^2)^m$$

Dividing the second and fourth equations:

$$\frac{b_4}{b_2} = R^{2m} \therefore R \cong \left(\frac{b_4}{b_2}\right)^{1/2m}$$

Using the fact that $R^2 = u^2 + v^2$ the following is used to find u and v:

$$a_1 = -(x_1 + x_2 + x_3 + x_4)$$

$$a_1 = -(x_1 + x_2 + (u + vi) + (u - vi))$$

$$a_1 = -(x_1 + x_2 + 2u)$$

Use b_1 and b_2 to find r_1 and r_2, then x_1 and x_2. Use b_4 and b_2 to find R then use a_1 to find u and R to find v. The Enke roots, once again, only lack the proper sign and either + or – may be correct.

Example 1.16 Graeffe's root squaring method—real and complex roots

Find the root of the following polynomial using Graeffe's root squaring method.

$$f(x) = 0 = x^4 + x^3 - 6x^2 - 14x - 12$$

Refer to Table 1.17 for the basic procedure for root squaring. Table 1.20 shows the process for this polynomial.

The third term has a sign fluctuation thus the previously derived relationships apply and the following is the solution:

Table 1.20. Example 1.16 Graeffe's root squaring method—real and complex roots

m	x^4	a_1x^3	a_2x^2	a_3x	a_3x^0	cycle
1	1	1	−6	−14	−12	1
	1	1	36	196	144	
		12	28	−144		
			−24			
2	1	13	40	52	144	2
	1	169	1600	2704	20736	
		−80	−1352	−11520		
			288			
4	1	89	536	−8816	20736	3
	1	7921	287296	77721856	429981696	
		−1072	1569248	−22228992		
			41472			
8	1	6.849E+03	1.898E+06	5.549E+07	4.300E+08	4
	1	4.691E+07	3.602E+12	3.079E+15	1.849E+17	
		−3.796E+06	−7.601E+11	−1.632E+15		
			8.600E+08			
16	1	4.311E+07	2.843E+12	1.447E+15	1.849E+17	5
	1	1.859E+15	8.084E+24	2.094E+30	3.418E+34	
		−5.686E+12	−1.248E+23	−1.051E+30		
			3.698E+17			
32	1	1.853E+15	7.959E+24	1.043E+30	3.418E+34	6
	1	3.434E+30	6.334E+49	1.088E+60	1.168E+69	
		−1.592E+25	−3.866E+45	−5.441E+59		
			6.836E+34			
64	1	3.434E+30	6.334E+49	5.441E+59	1.168E+69	7
	1	1.179E+61	4.012E+99	2.960E+119	1.365E+138	
		−1.267E+50	−3.736E+90	−1.480E+119		
			2.337E+69			
128	1	1.179E+61	4.012E+99	1.480E+119	1.365E+138	8
r		3	2	1.421892602	1.40657599	

$$r_1 = (b_1)^{1/m} = \left(1.179(10)^{61}\right)^{1/128} = \pm 3.000$$

$$r_2 = \left(\frac{b_2}{b_1}\right)^{1/m} = \left(\frac{4.012(10)^{99}}{1.179(10)^{61}}\right)^{1/128} = \pm 2.000$$

Substituting the Enke roots into the original equations yields x_1 = 3.000 and x_2 = –2.000.

$$R = \left(\frac{b_4}{b_2}\right)^{\frac{1}{2m}} = \left(\frac{1.3652(10)^{138}}{4.012(10)^{99}}\right)^{\frac{1}{256}} = \pm 1.414 = \pm\sqrt{2}$$

Using the fact that $R^2 = u^2 + v^2$, the following is used to find u and v:

$$1 = a_1 = -(x_1 + x_2 + 2u) = -(3 - 2 + 2u) \therefore u = -1$$
$$R = \sqrt{u^2 + v^2} \therefore v = \sqrt{R^2 - u^2} = 2 - 1 = 1$$

Thus results are $x_3 = -1 + i$ and $x_4 = -1 - i$.

1.13 BAIRSTOW'S METHOD

Bairstow's method was first published by Leonard Bairstow in 1920 (Bairstow 1920). If we divided a polynomial of nth degree by a quadratic equation, the result will be a polynomial of n–2 degree plus some remainder. This remainder can be used to give a closer approximation of the root quadratic equation. When the remainder is zero, the quadratic is a root equation. Bairstow's method involves using the remainders from double synthetic division to approximate the error in an assumed quadratic root equation of a polynomial. The derivation is omitted from this text, but may be found in "Applied Numerical Methods for Digital Computations," by James, Smith and Wolford (1977). Look at the process of factoring a polynomial into a quadratic equation times a polynomial of two degrees less than the original polynomial as follows:

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x^1 + a_n = 0$$
$$\left(x^2 + ux + v\right)\left(x^{n-2} + b_1x^{n-3} + b_2x^{n-4} + \cdots + b_{n-3}x^1 + b_{n-2} + remainder\right) = 0$$

The derived polynomial follows with the terms in the brackets being the remainder:

$$\left(x^{n-2} + b_1x^{n-3} + b_2x^{n-4} + \cdots + b_{n-3}x^1 + b_{n-2} + \left[b_{n-1} + b_n\right]\right) = 0$$

Divide the resulting polynomial by the quadratic equation in order to derive an equation that has something to do with the derivative of the original equation. This will be a polynomial four degrees less than the original polynomial. The following is the form of the second polynomial with the terms in the brackets being the remainder:

$$\left(x^{n-4} + c_1 x^{n-5} + c_2 x^{n-6} + \cdots + c_{n-5} x^1 + c_{n-4} + \left[c_{n-3} + c_{n-2} + c_{n-1}\right]\right) = 0$$

The solution may be set up in synthetic division form shown in Table 1.21:

Table 1.21. Bairstow's method

$-u$	a_0	a_1	a_2	…….	a_{n-3}	a_{n-2}	a_{n-1}	a_n
	0	$-ub_0$	$-ub_1$		$-ub_{n-4}$	$-ub_{n-3}$	$-ub_{n-2}$	$-ub_{n-1}$
$-v$	0	0	$-vb_0$	…….	$-vb_{n-5}$	$-vb_{n}{-}_{4}$	$-vb_{n-3}$	$-vb_{n-2}$
$-u$	b_0	b_1	b_2	…….	b_{n-3}	b_{n-2}	b_{n-1}	b_n
	0	$-uc_0$	$-uc_1$		$-uc_{n-4}$	$-uc_{n-3}$	$-uc_{n-2}$	
$-v$	0	0	$-vc_0$	…….	$-vc_{n-5}$	$-vc_{n-4}$	$-vc_{n-3}$	
	c_0	c_1	c_2	…….	c_{n-3}	c_{n-2}	c_{n-1}	

Using the preceding values, the approximations for the change in u and v values denoted Δu and Δv are as follows:

$$\Delta u = \frac{\begin{vmatrix} b_n & c_{n-2} \\ b_{n-1} & c_{n-3} \end{vmatrix}}{\begin{vmatrix} c_{n-1} & c_{n-2} \\ c_{n-2} & c_{n-3} \end{vmatrix}} \quad and \quad \Delta v = \frac{\begin{vmatrix} c_{n-1} & b_n \\ c_{n-2} & b_{n-1} \end{vmatrix}}{\begin{vmatrix} c_{n-1} & c_{n-2} \\ c_{n-2} & c_{n-3} \end{vmatrix}}$$

$$u_2 = u + \Delta u \quad and \quad v_2 = v + \Delta v$$

Continue the process until Δu and Δv are equal to zero. The two roots are as follows by the quadratic equation:

$$\left(x^2 + ux + v\right) \quad with\ x_{1,2} = \frac{-u \pm \sqrt{u^2 - 4v}}{2}$$

Example 1.17 Bairstow's method

Find all the roots of the following polynomial using Bairstow's method.

$$f(x) = 0 = x^5 - 3x^4 - 10x^3 + 10x^2 + 44x + 48$$

Begin by assuming $u = 1.5$ and $v = 1.5$ to perform the synthetic division shown in Table 1.22.

Table 1.22. Example 1.17 Bairstow's method

–1.5	1	–3	–10	10	44	48
		–1.5	6.75	7.125	–35.813	–22.969
–1.5			–1.5	6.75	7.125	–35.813
–1.5	1	–4.5	–4.75	23.875	**15.313**	**–10.781**
		–1.5	9	–4.125	–43.125	
–1.5			–1.5	9	–4.125	
	1	–6	**2.75**	**28.75**	**–31.938**	

$$\Delta u = \frac{\begin{vmatrix} -10.781 & 28.750 \\ 15.313 & 2.750 \end{vmatrix}}{\begin{vmatrix} -31.938 & 28.750 \\ 28.750 & 2.750 \end{vmatrix}} = \frac{(-10.781)(2.750)-(28.750)(15.313)}{(-31.935)(2.750)-(28.750)(28.750)}$$

$$= \frac{-469.88}{-914.39} = 0.5139$$

$$\Delta v = \frac{\begin{vmatrix} -31.938 & -10.781 \\ 28.750 & 15.313 \end{vmatrix}}{\begin{vmatrix} -31.938 & 28.750 \\ 28.750 & 2.750 \end{vmatrix}} = \frac{(-31.938)(15.313)-(-10.781)(28.750)}{(-31.935)(2.750)-(28.750)(28.750)}$$

$$= \frac{-179.08}{-914.39} = 0.1958$$

$$u_2 = u + \Delta u = 1.5 + 0.5139 = 2.0139$$

$$v_2 = v + \Delta v = 1.5 + 0.1958 = 1.6958$$

Now, repeat the process using the revised values for u and v shown in Table 1.23.

Table 1.23. Example 1.17 Bairstow's method

–2.0139	1	–3	–10	10	44	48
		–2.014	10.097	3.219	–43.745	–5.972
–1.6958			–1.696	8.503	2.711	–36.837
–2.0139	1	–5.014	–1.599	21.722	**2.965**	**5.191**
		–2.014	14.153	–21.868	–23.707	
–1.6958			–1.696	11.918	–18.415	
	1	–7.028	**10.859**	**11.772**	**–39.157**	

$$\Delta u = \frac{\begin{vmatrix} 5.191 & 11.772 \\ 2.965 & 10.859 \end{vmatrix}}{\begin{vmatrix} -39.157 & 11.772 \\ 11.772 & 10.859 \end{vmatrix}} = \frac{(5.191)(10.859)-(11.772)(2.965)}{(-39.157)(10.859)-(11.772)(11.772)}$$

$$= \frac{21.45}{-563.77} = -0.0381$$

$$\Delta v = \frac{\begin{vmatrix} -39.157 & 5.191 \\ 11.772 & 2.965 \end{vmatrix}}{\begin{vmatrix} -39.157 & 11.772 \\ 11.772 & 10.859 \end{vmatrix}} = \frac{(-39.157)(2.965)-(5.191)(11.772)}{(-39.157)(10.859)-(11.772)(11.772)}$$

$$= \frac{-177.22}{-563.77} = 0.3145$$

$$u_2 = u + \Delta u = 2.0139 + -0.0381 = 1.9758$$
$$v_2 = v + \Delta v = 1.6958 + 0.3145 = 2.0102$$

Now, repeat the process using the revised values for u and v shown in Table 1.24.

Table 1.24. Example 1.17 Bairstow's method

−1.9758	1	−3	−10	10	44	48
		−1.976	9.831	4.305	−48.027	−0.698
−2.0102			−2.010	10.002	4.380	−48.863
−1.9758	1	−4.976	−2.179	24.308	**0.353**	**−1.561**
		−1.976	13.735	−18.861	−38.373	
−2.0102			−2.010	13.974	−19.189	
	1	−6.952	**9.546**	**19.421**	**−57.208**	

$$\Delta u = \frac{\begin{vmatrix} -1.561 & 19.421 \\ 0.353 & 9.546 \end{vmatrix}}{\begin{vmatrix} -57.208 & 19.421 \\ 19.421 & 9.546 \end{vmatrix}} = \frac{(-1.561)(9.546)-(19.421)(0.353)}{(-57.208)(9.546)-(19.421)(19.421)}$$

$$= \frac{-21.74}{-923.27} = 0.0236$$

$$\Delta v = \frac{\begin{vmatrix} -57.208 & -1.561 \\ 19.421 & 0.353 \end{vmatrix}}{\begin{vmatrix} -57.208 & 19.421 \\ 19.421 & 9.546 \end{vmatrix}} = \frac{(-57.208)(0.353)-(-1.561)(19.421)}{(-57.208)(9.546)-(19.421)(19.421)}$$

$$= \frac{10.11}{-923.27} = -0.0110$$

$$u_2 = u + \Delta u = 1.9758 + 0.0236 = 1.999$$
$$v_2 = v + \Delta v = 2.0102 - 0.0110 = 1.999$$

It appears the values are $u = 2$ and $v = 2$. Repeat the process using the revised values for u and v shown in Table 1.25.

Table 1.25. Example 1.17 Bairstow's method

–2	1	–3	–10	10	44	48
		–2	10	4	–48	0
–2			–2	10	4	–48
	1	–5	–2	24	0	0

Since the remainders of the first division b_{n-1} and b_n are both zero, $u = 2$ and $v = 2$ are the coefficients of the root quadratic. Substitute into the quadratic equation to find the roots.

$$\left(x^2 + 2x + 2\right) \; with \; x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(2)}}{2} = -1 \pm \sqrt{-1} = -1 \pm i$$

The first two roots are $x_1 = -1 + i$ and $x_2 = -1 - i$. The remaining values are the coefficients of the factored polynomial.

$$f(x) = \left(x^2 + 2x + 2\right)\left(x^3 - 5x^2 - 2x + 24\right)$$

The remaining polynomial may be solved using the same method. This time begin with $u = 0$ and $v = 0$ in Table 1.26.

Table 1.26. Example 1.17 Bairstow's method

0	1	–5	–2	24
		0	0	0
0			0	0
0	1	–5	**–2**	**24**
		0	0	
0			0	
	1	**–5**	**–2**	

$$\Delta u = \frac{\begin{vmatrix} 24 & -5 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} -2 & -5 \\ -5 & 1 \end{vmatrix}} = \frac{(24)(1)-(-5)(-2)}{(-2)(1)-(-5)(-5)} = \frac{14}{-27} = -0.5185$$

$$\Delta v = \frac{\begin{vmatrix} -2 & 24 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & -5 \\ -5 & 1 \end{vmatrix}} = \frac{(-2)(-2)-(24)(5)}{(-2)(1)-(-5)(-5)} = \frac{124}{-27} = -4.5626$$

$$u_2 = u + \Delta u = 0 - 0.0236 = -0.5185$$
$$v_2 = v + \Delta v = 0 - 4.5626 = -4.5626$$

Repeat the process using the revised values for u and v shown in Table 1.27.

Table 1.27. Example 1.17 Bairstow's method

0.5185	1	–5	–2	24
		0.519	–2.324	0.139
4.5926			4.593	–20.582
0.5185	1	–4.481	**0.269**	**3.558**
		0.519	–2.055	
4.5926			4.593	
	1	**–3.963**	**2.807**	

$$\Delta u = \frac{\begin{vmatrix} 3.558 & -3.963 \\ 0.269 & 1 \end{vmatrix}}{\begin{vmatrix} 2.807 & -3.963 \\ -3.963 & 1 \end{vmatrix}} = \frac{(3.558)(1)-(-3.963)(0.269)}{(2.807)(1)-(-3.963)(-3.963)}$$

$$= \frac{4.623}{-12.900} = -0.3584$$

$$\Delta v = \frac{\begin{vmatrix} 2.807 & 3.558 \\ -3.963 & 0.269 \end{vmatrix}}{\begin{vmatrix} 2.807 & -3.963 \\ -3.963 & 1 \end{vmatrix}} = \frac{(2.807)(0.269)-(3.558)(-3.963)}{(2.807)(1)-(-3.963)(-3.963)}$$

$$= \frac{14.844}{-12.9007} = -1.1516$$

$$u_2 = u + \Delta u = -0.5185 - 0.3584 = -0.8769$$
$$v_2 = v + \Delta v = -4.5926 - 1.1516 = -5.7742$$

Repeat the process using the revised values for u and v shown in Table 1.28.

Table 1.28. Example 1.17 Bairstow's method

0.8769	1	–5	–2	24
		0.877	–3.616	0.113
5.7442			5.744	–23.684
0.8769	1	–4.123	**0.129**	**0.429**
		0.877	–2.847	
5.7442			5.744	
	1	**-3.246**	**3.026**	

$$\Delta u = \frac{\begin{vmatrix} 0.429 & -3.246 \\ 0.129 & 1 \end{vmatrix}}{\begin{vmatrix} 3.026 & -3.246 \\ -3.246 & 1 \end{vmatrix}} = \frac{(0.429)(1) - (-3.246)(0.129)}{(3.026)(1) - (-3.246)(-3.246)}$$
$$= \frac{0.848}{-7.511} = -0.1129$$

$$\Delta v = \frac{\begin{vmatrix} 3.026 & 0.429 \\ -3.246 & 0.129 \end{vmatrix}}{\begin{vmatrix} 3.026 & -3.246 \\ -3.246 & 1 \end{vmatrix}} = \frac{(3.026)(0.129) - (0.429)(-3.246)}{(3.026)(1) - (-3.246)(-3.246)}$$
$$= \frac{1.783}{-7.511} = -0.2374$$

$$u_2 = u + \Delta u = -0.8769 - 0.1129 = -0.9898$$
$$v_2 = v + \Delta v = -5.7442 - 0.2374 = -5.9816$$

Repeat the process using the revised values for u and v shown in Table 1.29.

Table 1.29. Example 1.17 Bairstow's method

0.9898	1	–5	–2	24
		0.990	–3.969	0.012
5.9816			5.982	–23.987
0.9898	1	–4.010	**0.012**	**0.025**
		0.990	–2.990	
5.9816			5.982	
	1	**–3.020**	**3.004**	

$$\Delta u = \frac{\begin{vmatrix} 0.025 & -3.020 \\ 0.012 & 1 \end{vmatrix}}{\begin{vmatrix} 3.004 & -3.020 \\ -3.020 & 1 \end{vmatrix}} = \frac{(0.025)(1)-(-3.020)(0.012)}{(3.004)(1)-(-3.020)(-3.020)}$$

$$= \frac{0.061}{-6.116} = -0.0100$$

$$\Delta v = \frac{\begin{vmatrix} 3.004 & 0.025 \\ -3.020 & 0.012 \end{vmatrix}}{\begin{vmatrix} 3.004 & -3.020 \\ -3.020 & 1 \end{vmatrix}} = \frac{(3.004)(0.012)-(0.025)(-3.020)}{(3.004)(1)-(-3.020)(-3.020)}$$

$$= \frac{0.111}{-6.116} = -0.0181$$

$$u_2 = u + \Delta u = -0.9898 - 0.0100 = -0.9998$$

$$v_2 = v + \Delta v = -5.9816 - 0.0181 = -5.9997$$

It appears the values are $u =\ \ 1$ and $v =\ \ 6$. Repeat the process using the revised values for u and v as shown in Table 1.30.

Table 1.30. Example 1.17 Bairstow's method

1	1	–5	–2	24
		1	–4	0
6			6	–24
	1	–4	**0**	**0**

Since the remainders of the first division b_{n-1} and b_n are both zero, $u = -1$ and $v = -6$ are the coefficients of the root quadratic. Substitute them into the quadratic equation to find the roots.

$$\left(x^2 - 1x - 6\right) \; with \; x_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4(-6)}}{2} = 0.5 \pm 2.5 = -2, 3$$

The first two roots are $x_3 = 2$ and $x_4 = 3$. The remaining values are the coefficients of the factored polynomial.

$$f(x) = \left(x^2 + 2x + 2\right)\left(x^2 - 1x - 6\right)\left(x - 4\right)$$

The last root is $x_5 = 4$, which is the value in the remaining polynomial of degree one $x - r = x - 4$. The final factored form of the original quadratic can be written and the five roots are $x_1 = -1+i$, $x_2 = -1-i$, $x_3 = -2$, $x_4 = 3$, and $x_5 = 4$.

$$f(x) = \left(x^2 + 2x + 2\right)\left(x + 2\right)\left(x - 3\right)\left(x - 4\right)$$

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CHAPTER 2

Solutions of Simultaneous Linear Algebraic Equations Using Matrix Algebra

Matrix algebra is commonly utilized in structural analysis as a method of solving simultaneous equations. Each motion, translation or rotation, at each discrete location in a structure is normally the desired variable. This chapter explores matrix terminology, matrix algebra, and various methods of linear algebra to determine solutions to simultaneous equations.

2.1 SIMULTANEOUS EQUATIONS

The solutions of simultaneous equations in structural analysis normally involve hundreds and even thousands of unknown variables. These solutions are generally *linear algebraic equations*. A typical linear algebraic equation with n unknowns is as follows, where a is the coefficient, x is the unknown, and C is the constant. In some cases, x, y, and z are used in lieu of x_1, x_2, etc.

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n = C$$

Equation sets can be separated into two categories, *homogeneous* and *non-homogeneous*. Homogeneous equation sets are those in which all the Cs are zero and all other equation sets are known as non-homogeneous. A unique solution to a non-homogeneous set exists only if the equations are independent or non-singular (determinant is non-zero), and a non-trivial solution set exists to a homogeneous set only if the equations are not independent (determinant is zero). The determinant is further discussed in

Section 2.3. In comparison to a non-trivial solution, a trivial solution is one where all the unknowns in the equations are equal to zero.

The typical set of *n* equations with *n* unknowns is as follows:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= C_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= C_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= C_3 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= C_n
\end{aligned}
$$

These equations can be written in matrix form, $[A][x]=[C]$ as follows:

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix}
$$

2.2 MATRICES

A *matrix* can be defined as a rectangular array of symbols or numerical quantities arranged in rows and columns. This array is enclosed in brackets and if there are *n* rows and *m* columns, the general form of this matrix is expressed by the following:

$$
[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}
$$

A matrix consisting of *n* rows and *m* columns is defined as a matrix of order $n \times m$. The relationship between the number of rows and the number of columns is arbitrary in a general matrix. Many types of matrices exist, such as row, column, diagonal, square, triangular, identity, and invert. These are discussed in the following sub-sections.

2.2.1 ROW AND COLUMN MATRICES

A *row matrix* is a matrix that reduces to a single row ($n = 1$).

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \end{bmatrix}$$

Similar to a row matrix, a *column matrix* is a matrix that reduces to a single column ($m = 1$).

$$[A] = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix}$$

2.2.2 SQUARE MATRIX

A matrix in which the number of rows is equal to the number of columns ($n = m$) is referred to as a *square matrix*.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Square matrices are unique because they are the only matrix that has a reciprocal or invert as described later in this section. Several types of square matrices exist such as the diagonal matrix, the identity matrix, the triangular matrix, and the invert matrix.

2.2.3 DIAGONAL MATRIX

A *diagonal matrix* is defined as a matrix where all elements outside of the *principal diagonal* are equal to zero. The diagonal running from the upper left corner of the array to the lower right corner is considered the principal diagonal.

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

2.2.4 IDENTITY MATRIX

An *identity matrix* is a diagonal matrix where all of the elements along the principal diagonal are equal to one and is denoted $[I]$.

$$[I] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

2.2.5 TRIANGULAR MATRIX

When all of the elements on one side of the principal diagonal are zero, this matrix is a *triangular matrix*. There are two types of triangular matrices, upper and lower. An *upper triangular matrix*, $[U]$, is when all of the elements below the principal diagonal are zero, and a *lower triangular matrix*, $[L]$, occurs when all of the elements above the principal diagonal are zero.

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$[L] = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

2.2.6 INVERTED MATRIX

Only square matrices where the determinant is not equal to zero ($|A| \neq 0$) can have an *inverse* or reciprocal. These matrices are called non-singular, which implies that reciprocals of rectangular matrices do exist. The inverse of a matrix is defined as follows:

$$[I] = [A][A]^{-1}$$

2.2.7 MATRIX MINOR

The *matrix minor*, $[A_{ij}]$, is found by omitting the ith row and the jth column of a matrix and writing the remaining terms in a matrix of one size smaller in rows and columns. It is used in the computation of the determinant. For example, the minor, $[A_{22}]$ is shown in the following. Note that $i = 2$ and $j = 2$:

$$[A_{22}] = \begin{bmatrix} a_{11} & \cancel{a_{12}} & a_{13} & \cdots & a_{1m} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} & \cdots & \cancel{a_{2m}} \\ a_{31} & \cancel{a_{32}} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cancel{a_{n2}} & a_{n3} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{13} & \cdots & a_{1m} \\ a_{31} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nm} \end{bmatrix}$$

2.2.8 TRANSPOSED MATRIX

The *transposed matrix* is found by writing the a_{ij} elements of a matrix as the a_{ji} elements of the matrix, $[A]^T$.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}$$

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & a_{3m} & \cdots & a_{mn} \end{bmatrix}$$

2.3 MATRIX OPERATIONS

2.3.1 MATRIX ADDITION AND SUBTRACTION

Matrices of the same size can easily be added or subtracted. Addition is achieved by adding the terms with the same row and column position. For example, if matrix [A] and [B] were to be added to obtain matrix [C], the following equations would be valid:

$$
\begin{aligned}
&[A]+[B]=[C]\\
&a_{11}+b_{11}=c_{11}\\
&a_{12}+b_{12}=c_{12}\\
&\quad\vdots\\
&a_{ij}+b_{ij}=c_{ij}
\end{aligned}
$$

Matrix subtraction follows the same form as addition where the matrices are of the same size.

$$
\begin{aligned}
&[A]-[B]=[C]\\
&a_{11}-b_{11}=c_{11}\\
&a_{12}-b_{12}=c_{12}\\
&\quad\vdots\\
&a_{ij}-b_{ij}=c_{ij}
\end{aligned}
$$

Example 2.1 Matrix addition and subtraction

Add matrix [A] and [B] and then subtract matrix [B] from [A].

$$
[A]=\begin{bmatrix}2&4&6\\7&9&3\\6&5&1\end{bmatrix}\quad\text{and}\quad[B]=\begin{bmatrix}1&2&4\\6&5&2\\4&3&0\end{bmatrix}
$$

Addition:

$$
[A]+[B]=\begin{bmatrix}2&4&6\\7&9&3\\6&5&1\end{bmatrix}+\begin{bmatrix}1&2&4\\6&5&2\\4&3&0\end{bmatrix}=\begin{bmatrix}3&6&10\\13&14&5\\10&8&1\end{bmatrix}
$$

Subtraction:

$$[A]-[B]=\begin{bmatrix}2 & 4 & 6\\7 & 9 & 3\\6 & 5 & 1\end{bmatrix}-\begin{bmatrix}1 & 2 & 4\\6 & 5 & 2\\4 & 3 & 0\end{bmatrix}=\begin{bmatrix}1 & 2 & 2\\1 & 4 & 1\\2 & 2 & 1\end{bmatrix}$$

2.3.2 SCALAR MULTIPLICATION

Scalar multiplication consists of multiplying a matrix by a scalar. When this occurs, every entry is multiplied by that scalar as seen in the following:

$$c[A]=c\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1m}\\a_{21} & a_{22} & a_{23} & \cdots & a_{2m}\\a_{31} & a_{32} & a_{33} & \cdots & a_{3m}\\\vdots & \vdots & \vdots & \ddots & \vdots\\a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm}\end{bmatrix}=\begin{bmatrix}ca_{11} & ca_{12} & ca_{13} & \cdots & ca_{1m}\\ca_{21} & ca_{22} & ca_{23} & \cdots & ca_{2m}\\ca_{31} & ca_{32} & ca_{33} & \cdots & ca_{3m}\\\vdots & \vdots & \vdots & \ddots & \vdots\\ca_{n1} & ca_{n2} & ca_{n3} & \cdots & ca_{nm}\end{bmatrix}$$

2.3.3 MATRIX MULTIPLICATION

Matrix multiplication proves a little more complicated than scalar multiplication. In order for two matrices to be multiplied, the number of columns in the first matrix must equal the number of rows in the second matrix. The resulting product consists of the same number of rows as the first matrix and the same number of columns as the second matrix.

$$[A]_{n\times m}[B]_{m\times o}=[C]_{n\times o}$$

Each term of the product matrix (row i and column j) is obtained by multiplying each term in row i of the first matrix by the term in row j of the second matrix and then summing these products.

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$$

Example 2.2 Matrix multiplication

Multiply matrix $[A]$ and $[B]$ to get $[C]$.

$$[A]=\begin{bmatrix}2&4\\3&2\\1&0\\3&4\end{bmatrix} \quad \text{and} \quad [B]=\begin{bmatrix}1&4&2&3&2&4\\2&3&2&4&1&1\end{bmatrix}$$

The product of a 4 × 2 matrix multiplied with a 2 × 6 matrix is a 4 × 6 matrix.

$$[A]_{4\times 2}[B]_{2\times 6}=[C]_{4\times 6}$$

$$[A][B]=\begin{bmatrix}2&4\\3&2\\1&0\\3&4\end{bmatrix}\begin{bmatrix}1&4&2&3&2&4\\2&3&2&4&1&1\end{bmatrix}=[C]=\begin{bmatrix}10&20&12&22&8&12\\7&18&10&17&8&14\\1&4&2&3&2&4\\11&24&14&25&10&16\end{bmatrix}$$

The first two elements are computed as follows:

$$c_{11}=2(1)+4(2)=2+8=10$$
$$c_{12}=2(4)+4(3)=8+12=20$$

2.3.4 MATRIX DETERMINANTS

A determinant is only defined for square matrices and can be easily achieved through expansion by minors of a row or a column when dealing with small matrices. A row expansion is as follows:

$$|A|=a_{11}|A_{11}|-a_{12}|A_{12}|+\ldots(-1)^{1+j}a_{1j}|A_{1j}|+\ldots(-1)^{n+1}a_{1n}|A_{1n}|$$

$$\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\vdots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nm}\end{vmatrix}=a_{11}\begin{vmatrix}a_{22}&a_{23}&\cdots&a_{2n}\\a_{32}&a_{33}&\cdots&a_{3n}\\\vdots&\vdots&\vdots&\vdots\\a_{n2}&a_{n3}&\cdots&a_{nm}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{23}&\cdots&a_{2n}\\a_{31}&a_{33}&\cdots&a_{3n}\\\vdots&\vdots&\vdots&\vdots\\a_{n1}&a_{n3}&\cdots&a_{nm}\end{vmatrix}$$

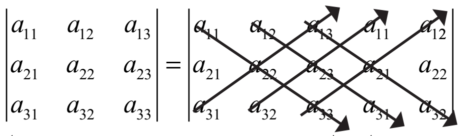
$$+\ldots+(-1)^{1+j}a_{1j}|A_{1j}|+\ldots(-1)^{n+1}a_{1n}\begin{vmatrix}a_{21}&a_{22}&\cdots&a_{2,n-1}\\a_{31}&a_{32}&\cdots&a_{3,n-1}\\\vdots&\vdots&&\vdots\\a_{n1}&a_{n2}&\cdots&a_{n,n-1}\end{vmatrix}$$

If $|A|$ is the determinant of the matrix $[A]$, then the following equations are valid when row k and column k are expanded:

$$|A| = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \left|A_{kj}\right| \quad \text{and} \quad |A| = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \left|A_{ik}\right|$$

The basket weave method may be used for a three-by-three determinant only. Take the sum of the products of the three down-right diagonals minus the sum of the product of the three up-right diagonals shown as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$



$$\left(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\right) - \left(a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}\right)$$

Also, a determinant may be found by the product of the diagonal of any triangular matrix.

Example 2.3 Matrix determinants

Find the determinant of the following matrix, $[A]$, by expansion of minors and by the basket weave method.

$$[A] = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix}$$

Expansion of row 1 yields:

$$|A| = 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 6\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$|A| = 3(4-3) - 4(2-6) + 6(1-4)$$

$$|A| = 3 + 16 - 18 = 1$$

Basket weave method yields:

$$|A| = \begin{vmatrix} 3 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 6 & 3 & 4 \\ 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \end{vmatrix}$$

$$|A| = [3(2)(2)+4(3)(2)+6(1)(1)] - [4(1)(2)+3(3)(1)+6(2)(2)]$$

$$|A| = 42 - 41 = 1$$

2.4 CRAMER'S RULE

Two common methods to solve simultaneous equations exist. One is the elimination of unknowns by elementary row operations and the second involves the use of determinates. One of the methods involving determinates is known as Cramer's rule. This method was published in 1750 by Gabriel Cramer (1750). The procedure for Cramer's rule in the solution to n linear equations with n unknowns is as follows:

$$x_1 = \frac{|A_1|}{|A|},\ x_2 = \frac{|A_2|}{|A|},\ x_3 = \frac{|A_3|}{|A|}, \ldots,\ x_n = \frac{|A_n|}{|A|}$$

$$|A_1| = \begin{vmatrix} c_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ c_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ c_3 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}, |A_2| = \begin{vmatrix} a_{11} & c_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & c_2 & a_{23} & \cdots & a_{2n} \\ a_{31} & c_3 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & c_n & a_{n3} & \cdots & a_{nn} \end{vmatrix},$$

$$|A_3| = \begin{vmatrix} a_{11} & a_{12} & c_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & c_2 & \cdots & a_{2n} \\ a_{31} & a_{32} & c_3 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & c_n & \cdots & a_{nn} \end{vmatrix}, \ldots, |A_n| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & c_1 \\ a_{21} & a_{22} & a_{23} & \cdots & c_2 \\ a_{31} & a_{32} & a_{33} & \cdots & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & c_n \end{vmatrix}$$

As you might see, $|A_1|$ is the original coefficient matrix, $[A]$, with column one replaced with the constant column matrix, $[c]$. The solution to n simultaneous equations by Cramer's rule requires $(n-1)*(n+1)!$ multiplications. In other words, the solution of ten simultaneous equations by determinants would require (9)*(11!) =359,251,200 multiplications.

Example 2.4 Cramer's rule

Find the solution set to the following non-homogeneous linear algebraic equations using Cramer's rule.

$$2x_1 + 8x_2 + 2x_3 = 14$$
$$x_1 + 6x_2 - x_3 = 13$$
$$2x_1 - x_2 + 2x_3 = 5$$

$$|A| = \begin{vmatrix} 2 & 8 & 2 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{vmatrix} = -36$$

$$|A_1| = \begin{vmatrix} 14 & 8 & 2 \\ 13 & 6 & -1 \\ 5 & -1 & 2 \end{vmatrix} = -180$$

$$|A_2| = \begin{vmatrix} 2 & 14 & 2 \\ 1 & 13 & -1 \\ 2 & 5 & 2 \end{vmatrix} = -36$$

$$|A_3| = \begin{vmatrix} 2 & 8 & 14 \\ 1 & 6 & 13 \\ 2 & -1 & 5 \end{vmatrix} = 72$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-180}{-36} = 5$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-36}{-36} = 1$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{72}{-36} = -2$$

The determinants are shown by row expansion in Table 2.1.

2.5 METHOD OF ADJOINTS OR COFACTOR METHOD

The solution to a set of linear algebraic equations can be achieved by using the invert of a matrix. The cofactor and adjoint matrices are very helpful in finding this invert by utilizing determinants.The *cofactor matrix* is one where the elements of the matrix are cofactors. Each term in the cofactor

Table 2.1. Example 2.4 Cramer's rule

$$|A| = \begin{vmatrix} 2 & 8 & 2 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$|A| = 2\begin{vmatrix} 6 & -1 \\ -1 & 2 \end{vmatrix} - 8\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 6 \\ 2 & -1 \end{vmatrix}$$

$$|A| = 22 - 32 + -26 = \mathbf{-36}$$

$$|A_1| = \begin{vmatrix} 14 & 8 & 2 \\ 13 & 6 & -1 \\ 5 & -1 & 2 \end{vmatrix}$$

$$|A_1| = 14\begin{vmatrix} 6 & -1 \\ -1 & 2 \end{vmatrix} - 8\begin{vmatrix} 13 & -1 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 13 & 6 \\ 5 & -1 \end{vmatrix}$$

$$|A_1| = 154 - 248 + -86 = \mathbf{-180}$$

$$|A_2| = \begin{vmatrix} 2 & 14 & 2 \\ 1 & 13 & -1 \\ 2 & 5 & 2 \end{vmatrix}$$

$$|A_2| = 2\begin{vmatrix} 13 & -1 \\ 5 & 2 \end{vmatrix} - 14\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 13 \\ 2 & 5 \end{vmatrix}$$

$$|A_2| = 62 - 56 + -42 = \mathbf{-36}$$

$$|A_3| = \begin{vmatrix} 2 & 8 & 14 \\ 1 & 6 & 13 \\ 2 & -1 & 5 \end{vmatrix}$$

$$|A_3| = 2\begin{vmatrix} 6 & 13 \\ -1 & 5 \end{vmatrix} - 8\begin{vmatrix} 1 & 13 \\ 2 & 5 \end{vmatrix} + 14\begin{vmatrix} 1 & 6 \\ 2 & -1 \end{vmatrix}$$

$$|A_3| = 86 - -168 + -182 = \mathbf{72}$$

of a matrix, C_{ij}, is given in the following equation where $|A_{ij}|$ is the determinant of the minor as defined in Section 2.2.

$$C_{ij} = (-1)^{i+j} \left|A_{ij}\right|$$

Therefore, given the matrix [A], the cofactor matrix is shown as follows. The matrix of cofactors should not be confused with the constant matrix of the original linear algebraic equation, although they have the same variable, [C]:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}$$

$$[C] = [A]_{cofactor} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1m} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2m} \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nm} \end{bmatrix}$$

Once the cofactor matrix is known, the invert can be easily calculated:

$$\left[A^{-1}\right] = \frac{[C]^T}{|A|}$$

The solutions to the simultaneous equations are now found from matrix multiplication.

$$[A][x] = [C] \quad \text{Or} \quad [A^{-1}][C] = [x]$$

Similar to the cofactor method, the method of adjoints, $Adj[A]$, is another common way to solve for the invert of a matrix. This method is as follows:

$$\left[A^{-1}\right] = \frac{Adj[A]}{|A|}$$

The *adjoint matrix*, $Adj[A]$, is simply the transpose of the cofactor matrix. This can be expressed in a few ways.

$$Adj[A] = [C]^T = [A_{cofactor}]^T \quad \text{Or} \quad c_{ij} = (-1)^{(i+j)}[A_{ji}]$$

$$Adj[A] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1m} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2m} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nm} \end{bmatrix}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{n1} \\ c_{12} & c_{22} & c_{32} & \cdots & c_{n2} \\ c_{13} & c_{23} & c_{33} & \cdots & c_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1m} & c_{2m} & c_{3m} & \cdots & c_{mn} \end{bmatrix}$$

It is noted that the subscripts of the adjoint matrix are the reverse of the cofactor matrix. The main difference is that the transpose is performed during the operation of taking the adjoint, while in the cofactor method is done at the end.

Example 2.5 Cofactor method

Find the solution set to the following nonhomogeneous linear algebraic equations using the cofactor method.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 10 \\ 8x_1 + 4x_2 + 2x_3 + x_4 &= 26 \\ -x_1 + x_2 - x_3 + x_4 &= 2 \\ 3x_1 + 2x_2 + 1x_3 + 0x_4 &= 10 \end{aligned}$$

$$\left[A^{-1}\right] = \frac{[C]^T}{|A|} \quad \text{And} \quad C_{ij} = (-1)^{i+j}\left|A_{ij}\right|$$

The determinants are shown by row expansion for the 4 × 4 matrix and by the basket weave for all the 3 × 3 matrices in Table 2.2.

The last step in Table 2.2 is to multiply the invert of A, $[A]^{-1}$, times the constant vector, $[C]$, to get the final solution vector, $[x]$.

Example 2.6 Method of adjoints

Determine the solution to the following set of equations using the adjoint method. Use the basket weave method for determinants.

Table 2.2. Example 2.5 Cofactor method

$$|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ -1 & 1 & -1 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix} \qquad |C| = \begin{vmatrix} 10 \\ 26 \\ 2 \\ 10 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 8 & 2 & 1 \\ -1 & -1 & 1 \\ 3 & 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 8 & 4 & 1 \\ -1 & 1 & 1 \\ 3 & 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 8 & 4 & 2 \\ -1 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$|A| = 3 + 0 + -9 + -6 \qquad |A| = \mathbf{-12}$$

$$c_{11} = \begin{vmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix}(-1)^{1+1} \quad c_{12} = \begin{vmatrix} 8 & 2 & 1 \\ -1 & -1 & 1 \\ 3 & 1 & 0 \end{vmatrix}(-1)^{1+2} \quad c_{13} = \begin{vmatrix} 8 & 4 & 1 \\ -1 & 1 & 1 \\ 3 & 2 & 0 \end{vmatrix}(-1)^{1+3} \quad c_{14} = \begin{vmatrix} 8 & 4 & 2 \\ -1 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}(-1)^{1+4}$$

$$c_{11} = 3 \qquad c_{12} = 0 \qquad c_{13} = -9 \qquad c_{14} = -6$$

$$c_{21} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix}(-1)^{2+1} \quad c_{22} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 3 & 1 & 0 \end{vmatrix}(-1)^{2+2} \quad c_{23} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 3 & 2 & 0 \end{vmatrix}(-1)^{2+3} \quad c_{24} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}(-1)^{2+4}$$

$$c_{21} = -4 \qquad c_{22} = 4 \qquad c_{23} = 4 \qquad c_{24} = -4$$

$$c_{31} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix}(-1)^{3+1} \quad c_{32} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix}(-1)^{3+2} \quad c_{33} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 1 \\ 3 & 2 & 0 \end{vmatrix}(-1)^{3+3} \quad c_{34} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 3 & 2 & 1 \end{vmatrix}(-1)^{3+4}$$

$$c_{31} = 1 \qquad c_{32} = -4 \qquad c_{33} = 5 \qquad c_{34} = -2$$

$$c_{41} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}(-1)^{4+1} \quad c_{42} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 2 & 1 \\ -1 & -1 & 1 \end{vmatrix}(-1)^{4+2} \quad c_{43} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 1 \\ -1 & 1 & 1 \end{vmatrix}(-1)^{4+3} \quad c_{44} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ -1 & 1 & -1 \end{vmatrix}(-1)^{4+4}$$

$$c_{41} = 6 \qquad c_{42} = -12 \qquad c_{43} = -6 \qquad c_{44} = 12$$

$$|C| = \begin{vmatrix} 3 & 0 & -9 & -6 \\ -4 & 4 & 4 & -4 \\ 1 & -4 & 5 & -2 \\ 6 & -12 & -6 & 12 \end{vmatrix} \qquad |C^T| = \begin{vmatrix} 3 & -4 & 1 & 6 \\ 0 & 4 & -4 & -12 \\ -9 & 4 & 5 & -6 \\ -6 & -4 & -2 & 12 \end{vmatrix}$$

$$|A^{-1}| = \begin{vmatrix} -0.2500 & 0.3333 & -0.0833 & -0.5000 \\ 0.0000 & -0.3333 & 0.3333 & 1.0000 \\ 0.7500 & -0.3333 & -0.4167 & 0.5000 \\ 0.5000 & 0.3333 & 0.1667 & -1.0000 \end{vmatrix} \qquad |x| = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 4 \end{vmatrix}$$

Table 2.3. Example 2.6 Method of adjoints

$$|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad |C| = \begin{vmatrix} 10 \\ 26 \\ 2 \\ 4 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 8 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 8 & 4 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 8 & 4 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$|A| = -6 + 6 + 12 + 0 \qquad |A| = \mathbf{12}$$

$$c_{11} = \begin{vmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{1+1} \quad c_{21} = \begin{vmatrix} 8 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{1+2} \quad c_{31} = \begin{vmatrix} 8 & 4 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{1+3} \quad c_{41} = \begin{vmatrix} 8 & 4 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix}(-1)^{1+4}$$

$$c_{11} = -6 \qquad c_{21} = 6 \qquad c_{31} = 12 \qquad c_{41} = 0$$

$$c_{12} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{2+1} \quad c_{22} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{2+2} \quad c_{32} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{2+3} \quad c_{42} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix}(-1)^{2+3}$$

$$c_{12} = 2 \qquad c_{22} = 0 \qquad c_{32} = -2 \qquad c_{42} = 0$$

$$c_{13} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{3+1} \quad c_{23} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{3+2} \quad c_{33} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix}(-1)^{3+3} \quad c_{43} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 0 & 0 & 0 \end{vmatrix}(-1)^{3+4}$$

$$c_{13} = -2 \qquad c_{23} = 6 \qquad c_{33} = -4 \qquad c_{43} = 0$$

$$c_{14} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}(-1)^{4+1} \quad c_{24} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 2 & 1 \\ -1 & -1 & 1 \end{vmatrix}(-1)^{4+2} \quad c_{34} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 1 \\ -1 & 1 & 1 \end{vmatrix}(-1)^{4+3} \quad c_{44} = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ -1 & 1 & -1 \end{vmatrix}(-1)^{4+4}$$

$$c_{14} = 6 \qquad c_{24} = -12 \qquad c_{34} = -6 \qquad c_{44} = 12$$

$$|c| = \begin{vmatrix} -6 & 2 & -2 & 6 \\ 6 & 0 & 6 & -12 \\ 12 & -2 & -4 & -6 \\ 0 & 0 & 0 & 12 \end{vmatrix}$$

$$|A^{-1}| = \begin{vmatrix} -0.5000 & 0.1667 & -0.1667 & 0.5000 \\ 0.5000 & 0.0000 & 0.5000 & -1.0000 \\ 1.0000 & -0.1667 & -0.3333 & -0.5000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{vmatrix} \qquad |x| = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 4 \end{vmatrix}$$

$$x_1 + x_2 + x_3 + x_4 = 10$$
$$8x_1 + 4x_2 + 2x_3 + x_4 = 26$$
$$-x_1 + x_2 - x_3 + x_4 = 2$$
$$0x_1 + 0x_2 + 0x_3 + x_4 = 4$$
$$[A^{-1}] = \frac{[C]^T}{|A|} \quad \text{and} \quad c_{ij} = (-1)^{i+j}\left|A_{ji}\right|$$

The determinants are shown by row expansion for the 4 × 4 matrix and by the basket weave for all the 3 × 3 matrices in Table 2.3.

The last step in Table 2.3 is to multiply the invert of A, $[A]^{-1}$, by the constant vector, $[C]$, to get the final solution vector, $[x]$.

2.6 GAUSSIAN ELIMINATION METHOD

This method is named for Carl Friedrich Gauss who developed it in 1670 (Newton 1707). It was referenced by the Chinese as early as 179 (Anon 179). Gaussian elimination is a method for solving matrix equations by composing an augmented matrix, $[A|C]$, and then utilizing elementary row operations to reduce this matrix into upper triangular form, $[U|D]$. The elementary row operations used to reduce the matrix into upper triangular form consist of multiplying an equation by a scalar, or adding two equations to form another equation. The equation used to eliminate terms in other equations is referred to as the *pivot equation*. The coefficient of the pivot equation that lies in the column of terms to be eliminated is called the *pivot coefficient* or *pivot element*.

If this coefficient of the pivot element is zero, the pivot row must be exchanged with another row. This exchange is called *partial pivoting*. If the row with the largest element in the pivot column is exchanged to the pivot row, accuracy is increased. Partial pivoting is when the rows are interchanged and full pivoting is when both the rows and columns are reordered to place a particular element in the diagonal position prior to a particular operation. Whenever partial pivoting is utilized, the determinant changes sign with each pivot unless done before reduction starts. However, the value of the determinant of the matrix is not affected by elementary row operations.

Once the matrix is reduced to an equivalent upper triangular matrix, the solutions are found by solving equations by back substitution. The following is the reduction procedure in algorithmic form:

$$a_{ij}^{k} = a_{ij}^{k-1} - \frac{a_{kj}^{k-1}}{a_{kk}^{k-1}}\left(a_{ik}^{k-1}\right) \quad \text{where} \quad \begin{Bmatrix} k+1 \le j \le m \\ k+1 \le i \le n \end{Bmatrix}$$

With the following variables:

- a^{k-1} original elements
- a^k new elements
- i row (n)
- j column (m)
- k pivotal row number

The following is the back substitution procedure in algorithmic form:

$$x_n = \frac{a_{nm}}{a_{nn}}$$

$$x_i = \frac{a_{im} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \quad \text{where } i = n-1, n-2, \ldots, 1$$

Example 2.7 Gaussian elimination method

Find the solution set to the following nonhomogeneous linear algebraic equations using Gaussian elimination.

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 10 \\
8x_1 + 4x_2 + 2x_3 + x_4 &= 26 \\
-x_1 + x_2 - x_3 + x_4 &= 2 \\
0x_1 + 0x_2 + 0x_3 + x_4 &= 4
\end{aligned}$$

The reduction process is shown in Table 2.4. The numbers to the right of each row outside the augmented matrix, $[A|C]$, are the reduction multipliers. The pivot row is multiplied by these numbers to reduce the rows below the pivot row. As an example, the first row is multiplied by −8 and added to the second row, producing a zero in the first column of the second row. Then, the first row is multiplied by 1 and added to the third row, producing a zero in the first column of the third row. Last, the first row is multiplied by 0 and added to the last row, producing a zero in the first column of the last row. The result is the reduction of all the values below the pivot element in the first column to zero. The second column is then reduced to zeros below the pivot element, and lastly the third column is reduced to zeros below the pivot element. The solution vector, $[x]$, is also shown.

These values for x are solved after the final reduction. From row four, the following equation can be written:

$$1x_4 = 4 \therefore x_4 = 4$$

Table 2.4. Example 2.7 Gaussian elimination

$$|A|C| = \left|\begin{array}{cccc|c|} 1 & 1 & 1 & 1 & 10 \\ 8 & 4 & 2 & 1 & 26 \\ -1 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right.$$

$$|A|C| = \left|\begin{array}{cccc|c|c} 1 & 1 & 1 & 1 & 10 & \\ 0 & -4 & -6 & -7 & -54 & -8 \\ 0 & 2 & 0 & 2 & 12 & 1 \\ 0 & 0 & 0 & 1 & 4 & 0 \end{array}\right.$$

$$|A|C| = \left|\begin{array}{cccc|c|c} 1 & 1 & 1 & 1 & 10 & \\ 0 & -4 & -6 & -7 & -54 & \\ 0 & 0 & -3 & -1.5 & -15 & 0.5 \\ 0 & 0 & 0 & 1 & 4 & 0 \end{array}\right.$$

$$|A|C| = \left|\begin{array}{cccc|c|c} 1 & 1 & 1 & 1 & 10 & \\ 0 & -4 & -6 & -7 & -54 & \\ 0 & 0 & -3 & -1.5 & -15 & \\ 0 & 0 & 0 & 1 & 4 & 0 \end{array}\right.$$

$$|x| = \left|\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\right|$$

From row three, the following equation is valid:

$$-3x_3 - 1.5x_4 = -15$$

Substituting $x_4 = 4$ into this equation yields:

$$-3x_3 - 1.5(4) = -15 \therefore -3x_3 = -9 \therefore x_3 = 3$$

Row two produces the following equation:

$$-4x_2 - 6x_3 - 7x_4 = -54$$

Substituting $x_3 = 3$ and $x_4 = 4$ into this equation yields:

$$-4x_2 - 6(3) - 7(4) = -54 \therefore -4x_2 = -8 \therefore x_2 = 2$$

And, from row one the following equation:

$$1x_1 + 1x_2 + 1x_3 + 1x_4 = 10$$

Substituting $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$ into this equation yields:

$$1x_1 + 1(2) + 1(3) + 1(4) = 10 \therefore x_1 = 1$$

It should be noted that the product the diagonal values of any triangular matrix is the determinant. In this case, the determinant is $|A| = (1)(-4)(-3)(1) = 12$.

Example 2.8 Gaussian elimination method

Determine the solution to the following set of equations using Gaussian elimination and back substitution using partial pivoting. Include a determinant check for uniqueness.

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 = 10 \\ &8x_1 + 4x_2 + 2x_3 + x_4 = 26 \\ &-x_1 + x_2 - x_3 + x_4 = 2 \\ &3x_1 + 2x_2 + 1x_3 + 0x_4 = 10 \end{aligned}$$

The reduction process is shown in Table 2.5. Again, the numbers to the right of each row outside the augmented matrix, $[A|C]$, are the reduction multipliers. Also noted is the partial pivoting. Note that the second row has the largest number in the first column. Therefore, that row is swapped with the first row, placing the largest element in the pivot element position. Reduction is then performed on the first column. After reduction of the first column, the largest number in the second column is in the third row. That row is swapped with the second row, placing the largest number in the pivot position. Reduction is then performed on the second column. The third column does not require partial pivoting, since the largest number in the third column is already in the pivot position.

Since two partial pivots were performed, the product of the diagonal must be multiplied by $(-1)^2$ to achieve the correct sign on the determinant, $|A| = (8)(1.5)(1)(-1)(-1)^2 = -12$. Back substitution is performed to determine the solution vector, $[x]$, which is shown in Table 2.5.

Table 2.5. Example 2.8 Gaussian elimination

		1	*1*	*1*	*1*	*10*	
|A|	=	*8*	*4*	*2*	*1*	*26*	
		–1	1	–1	1	2	
		3	2	1	0	10	

		8	4	2	1	26	Swap
|A|	=	1	1	1	1	10	Pivot
		–1	1	–1	1	2	
		3	2	1	0	10	

		8	4	2	1	26	
|A|	=	*0*	*0.5*	*0.75*	*0.88*	*6.75*	–0.125
		0	*1.5*	*–0.8*	*1.13*	*5.25*	0.125
		0	0.5	0.25	–0.4	0.25	–0.375

		8	4	2	1	26	
|A|	=	0	1.5	–0.8	1.13	5.25	Swap
		0	0.5	0.75	0.88	6.75	Pivot
		0	0.5	0.25	–0.4	0.25	

		8	4	2	1	26	
|A|	=	0	1.5	–0.8	1.13	5.25	
		0	0	1	0.5	5	–0.333
		0	0	0.5	–0.8	–1.5	–0.333

|A|	=	8	4	2	1	26	
		0	1.5	–0.8	1.13	5.25	
		0	0	1	0.5	5	
		0	0	0	–4	–4	–0.500

		1				
|x|	=	2	|A|	=	–12	$* (-1)^2$
		3				
		4				

2.7 GAUSS–JORDAN ELIMINATION METHOD

The Gauss–Jordan elimination method is a variation of the Gaussian elimination method in which an unknown is eliminated from all equations except the pivot during the elimination process. This method was described by Wilhelm Jordan in 1887 (Clasen 1888). When an unknown is eliminated, it is eliminated from equations preceding the pivot equation as well as those following the pivot equation. The result is a diagonal matrix and eliminates the need for back substitution. In fact, if the pivot elements are changed to "ones" by dividing each row by the pivot element, the last column will contain the solution.

The disadvantage to the Gauss–Jordan elimination method is that two matrices are required for elimination, however, they do get smaller as the process continues. The previous pivot is moved to the bottom, with a new pivot on top. The Gauss–Jordan process is shown in Example 2.9.

Example 2.9 Gauss–Jordan elimination method

Determine the solution to the following set of equations using Gauss–Jordan elimination.

$$2x_1 - 2x_2 + 5x_3 = 13 \tag{2.1a}$$

$$2x_1 + 3x_2 + 4x_3 = 20 \tag{2.1b}$$

$$3x_1 - x_2 + 3x_3 = 10 \tag{2.1c}$$

The first step is to divide the first equation of the set by the coefficient of the first unknown in that equation, 2. Equation 2.2a is then multiplied by the corresponding coefficient of that unknown of Equations 2.1b and 2.1c to give the following:

$$x_1 - x_2 + \frac{5}{2}x_3 = \frac{13}{2} \tag{2.2a}$$

$$2x_1 - 2x_2 + 5x_3 = 13 \tag{2.2b}$$

$$3x_1 - 3x_2 + \frac{15}{2}x_3 = \frac{39}{2} \tag{2.2c}$$

Next, Equation 2.2b is subtracted from 2.1b and becomes 2.3a. Equation 2.2c is subtracted from 2.1c and becomes 2.3b. Equation 2.2a now becomes 2.3c, thereby moving to the bottom.

$$5x_2 - x_3 = 7 \tag{2.3a}$$

$$2x_2 - \frac{9}{2}x_3 = -\frac{19}{2} \tag{2.3b}$$

$$x_1 - x_2 + \frac{5}{2}x_3 = \frac{13}{2} \tag{2.3c}$$

Now, Equation 2.3a is divided by the first unknown in that equation, 5, and the new Equation 2.4a is multiplied by the corresponding coefficient of that unknown from the two other equations (2.3b and 2.3c) to yield Equations 2.4b and 2.4c.

$$x_2 - \frac{1}{5}x_3 = \frac{7}{5} \tag{2.4a}$$

$$2x_2 - \frac{2}{5}x_3 = \frac{14}{5} \tag{2.4b}$$

$$-x_2 + \frac{1}{5}x_3 = -\frac{7}{5} \tag{2.4c}$$

Just as the previous cycle, Equation 2.4b is subtracted from 2.3b and becomes 2.5a. Equation 2.4c is subtracted from 2.3c and becomes 2.5b. Equation 2.4a now becomes 2.5c.

$$-\frac{41}{10}x_3 = -\frac{123}{10} \tag{2.5a}$$

$$x_1 + \frac{23}{10}x_3 = \frac{79}{10} \tag{2.5b}$$

$$x_2 - \frac{1}{5}x_3 = \frac{7}{5} \tag{2.5c}$$

Equation 2.5a is divided by the first unknown in that equation, –41/10 and the new Equation 2.6a is multiplied by the corresponding coefficient of that unknown from the two other equations (2.5b and 2.5c) to yield Equations 2.6b and 2.6c.

$$x_3 = 3 \tag{2.6a}$$

$$\frac{23}{10}x_3 = \frac{69}{10} \tag{2.6b}$$

$$-\frac{1}{5}x_3 = -\frac{3}{5} \tag{2.6c}$$

Equation 2.6b is subtracted from 2.5b and becomes 2.7a. Equation 2.6c is subtracted from 2.5c and becomes 2.7b. Equation 2.6a now becomes 2.7c. All the unknowns are found with the following solution:

$$x_1 = 1 \tag{2.7a}$$

$$x_2 = 2 \tag{2.7b}$$

$$x_3 = 3 \tag{2.7c}$$

The same result may be obtained by working with just the coefficients and constants of the equations. Given the same equations, the following augmented matrix is valid:

$$2x_1 - 2x_2 + 5x_3 = 13$$

$$2x_1 + 3x_2 + 4x_3 = 20$$

$$3x_1 - x_2 + 3x_3 = 10$$

$$[A] = \begin{bmatrix} 2 & -2 & 5 & 13 \\ 2 & 3 & 4 & 20 \\ 3 & -1 & 3 & 10 \end{bmatrix}$$

An augmented matrix, [*B*], is established from the following algorithm:

$$b_{i-1,j-1} = a_{ij} - \frac{a_{1j}a_{i1}}{a_{11}} \quad \text{where} \begin{Bmatrix} 1 < i \le n \\ 1 < j \le m \\ a_{11} \ne 0 \end{Bmatrix}$$

Also, the final row of the new matrix is found by:

$$b_{n,j-1} = \frac{a_{1j}}{a_{11}} \quad \text{where} \begin{Bmatrix} 1 < j \le m \\ a_{11} \neq 0 \end{Bmatrix}$$

where,

a	elements in old matrix A
b	elements in new matrix B
i	row number in old matrix A (*n*)
j	column number of old matrix A (*m*)

For example,

$$b_{2,2} = a_{2,2} - \frac{a_{1,2}a_{2,1}}{a_{11}} = 3 - \frac{-2(2)}{2} = 5$$

From these equations, column 1 is reduced.

$$[B] = \begin{bmatrix} 5 & -1 & 7 \\ 2 & -\frac{9}{2} & -\frac{19}{2} \\ -1 & \frac{5}{2} & \frac{13}{2} \end{bmatrix}$$

Following the process again to reduce column 2:

$$[C] = \begin{bmatrix} -\frac{41}{10} & -\frac{123}{10} \\ \frac{23}{10} & \frac{79}{10} \\ -\frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

Finally, one more cycle reduces column 3 and obtains the solution:

$$[D] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

It is known that applying these equations to a number of simultaneous equations, *n*, will produce the same number of matrices, *n*, to achieve the solution.

2.8 IMPROVED GAUSS–JORDAN ELIMINATION METHOD

In comparison to the Gauss–Jordan elimination method, the improved Gauss–Jordan elimination method uses the same space for both the A and B arrays. This is beneficial if the amount of space available on the computer is limited. The algorithm for this improved method is as follows:

$$\left\{\begin{matrix} a'_{kj} = \dfrac{a_{kj}}{a_{kk}} \\ a'_{ij} = a_{ij} - a_{ik}\, a'_{kj} \\ 1 \le i \le n \\ k+1 \le j \le n+1 \end{matrix}\right\} \quad k = 1,\ 2,\ 3 \ldots n \text{ except } i \ne k$$

where,

a'	original elements
a	new elements
i	row (n)
j	column (m)
k	pivotal row number

In other words, normalize the matrix then utilize partial pivoting to reduce the matrix. However, there is no need to reduce the elements under the pivot. Reducing up and down is easier without a need to reorder the rows. This is how Example 2.10 is performed.

Example 2.10 Improved Gauss–Jordan elimination method

Determine the solution to the following set of equations using improved Gaussian–Jordan elimination. Include a determinant check for uniqueness.

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 + x_5 = 8 \\ &16x_1 + 8x_2 + 4x_3 + 2x_4 + 1x_5 = 44 \\ &x_1 - x_2 + x_3 - x_4 + x_5 = 2 \\ &81x_1 - 27x_2 + 9x_3 - 3x_4 + x_5 = 44 \\ &16x_1 - 8x_2 + 4x_3 - 2x_4 + x_5 = 8 \end{aligned}$$

$$\left[A|C\right]=\left[\begin{array}{ccccc|c}1 & 1 & 1 & 1 & 1 & 8\\16 & 8 & 4 & 2 & 1 & 44\\1 & -1 & 1 & -1 & 1 & 2\\81 & -27 & 9 & -3 & 1 & 44\\16 & -8 & 4 & -2 & 1 & 8\end{array}\right]$$

From partial pivoting, the first row can be swapped with the fourth row to form the following matrix:

$$\left[A|C\right]=\left[\begin{array}{ccccc|c}81 & -27 & 9 & -3 & 1 & 44\\16 & 8 & 4 & 2 & 1 & 44\\1 & -1 & 1 & -1 & 1 & 2\\1 & 1 & 1 & 1 & 1 & 8\\16 & -8 & 4 & -2 & 1 & 8\end{array}\right]$$

Now elimination may be performed as shown in Table 2.6. Note that for each column reduction, elements are reduced to zero below and above the pivot position. Once it reduces to a diagonal matrix, the solution is found by dividing each row by the pivot element. The determinant is found as $|A| = (81)(13.3333)(1)(1.3333)\,(-2) = -2880$.

2.9 CHOLESKY DECOMPOSITION METHOD

Cholesky decomposition is also known as Crout's method or matrix factorization. This method was discovered by André-Louis Cholesky (Commandant Benoit 1924). Cholesky decomposition changes the original augmented equation to an equivalent upper and lower triangular set. If a set of three simultaneous equations exist, they can be represented as follows:

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

If $[A]$ represents the coefficient matrix, $[x]$ represents the column matrix of the unknowns, and $[C]$ represents the column matrix of the constants, the previous can be expressed as the following equation:

Table 2.6. Example 2.10 Improved Gaussian–Jordan elimination method

	81	-27	9	-3	1	44	
	16	8	4	2	1	44	
|A| =	1	-1	1	-1	1	2	
	1	1	1	1	1	8	
	16	-8	4	-2	1	8	

	81	-27	9	-3	1	44	
	0	13.3333	2.2222	2.5926	0.8025	35.3086	-0.1975
|A| =	0	-0.6667	0.8889	-0.9630	0.9877	1.4568	-0.0123
	0	1.3333	0.8889	1.0370	0.9877	7.4568	-0.0123
	0	-2.6667	2.2222	-1.4074	0.8025	-0.6914	-0.1975

	81	0	13.5	2.25	2.625	115.5	2.0250
	0	13.3333	2.2222	2.5926	0.8025	35.3086	
|A| =	0	0	1	-0.8333	1.0278	3.2222	0.0500
	0	0	0.6667	0.7778	0.9074	3.9259	-0.1000
	0	0	2.6667	-0.8889	0.9630	6.3704	0.2000

	81	0	0	13.5	-11.25	72	-13.5000
	0	13.3333	0	4.4444	-1.4815	28.1481	-2.2222
|A| =	0	0	1	-0.8333	1.0278	3.2222	
	0	0	0	1.3333	0.2222	1.7778	-0.6667
	0	0	0	1.3333	-1.7778	-2.2222	-2.6667

	81	0	0	0	-13.5	54	-10.1250
	0	13.3333	0	0	-2.2222	22.2222	-3.3333
|A| =	0	0	1	0	1.1667	4.3333	0.6250
	0	0	0	1.33333	0.2222	1.7778	
	0	0	0	0	-2	-4	-1.0000

	81	0	0	0	0	81	-6.7500
	0	13.3333	0	0	0	26.6667	-1.1111
|A| =	0	0	1	0	0	2	0.5833
	0	0	0	1.33333	0	1.33333	0.1111
	0	0	0	0	-2	-4	

	1			
	2			
|x| =	2	|A|	=	-2880.00
	1			
	2			

$$[A][x]=[C] \text{ or } [A][x]-[C]=0$$

If the original system of equations is reduced into an equivalent system in upper triangular form, the following is true:

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$[U][x]-[D]=0$$

Also, a lower triangular matrix exists, such that, when the first set is pre-multiplied by $[L]$, the result is the second set as follows:

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$[L]([U][x]-[D])=[A][x]-[C]$$

$$[L][U]=[A] \text{ and } [L][D]=[C]$$

In matrix form, it looks as follows:

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & d_1 \\ 0 & 1 & u_{23} & d_2 \\ 0 & 0 & 1 & d_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{bmatrix}$$

The order of the solution process is as follows with each producing an equation involving only one unknown:

1. Obtain column 1 of $[L]$ by multiplying each row of $[L]$ by column 1 of $[U]$ to get column 1 of $[A]$. That is, use a_{11}, a_{21}, a_{31} to get l_{11}, l_{21}, l_{31}.
2. Obtain row 1 of $[U]$ by multiplying row 1 of $[L]$ times each column of $[U]$ to get row 1 of $[A]$, excluding column 1 of $[U]$. That is, use a_{12}, a_{13}, c_1 to get u_{12}, u_{13}, d_1.

3. Obtain column 2 of $[L]$ by multiplying each row of $[L]$ times column 2 of $[U]$ to get column 2 of $[A]$, excluding row 1 of $[L]$, That is, use a_{22}, a_{32} to get l_{22}, l_{32}.
4. Obtain row 2 of $[U]$ by multiplying row 2 of $[L]$ times each column of $[U]$ to get row 2 of $[A]$, excluding columns 1 and 2 of $[U]$. That is, use a_{23}, c_2 to get u_{23}, d_2.
5. Obtain column 3 of $[L]$ by multiplying each row of $[L]$ times column 3 of $[U]$ to get column 3 of $[A]$, excluding row 1 and 2 of $[L]$. That is, use a_{33} to get l_{33}.
6. Obtain row 3 of $[U]$ by multiplying row 3 of $[L]$ times each column of $[U]$ to get row 3 of $[A]$, excluding columns 1, 2, and 3 of $[U]$. That is, use c_3 to get d_3.

All of these arithmetic operations can be done with an algorithm as follows:

$$l_{i1} = a_{i1} \quad \text{For} \begin{Bmatrix} i = 1, 2, 3, \ldots, n \\ j = 1 \end{Bmatrix}$$

$$u_{1j} = \frac{a_{1j}}{a_{11}} \quad \text{For} \begin{Bmatrix} j = 2, 3, 4, \ldots, n+1 \\ i = 1 \end{Bmatrix}$$

$$l_{i,j} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \quad \text{For} \begin{Bmatrix} j = 2, 3, 4, \ldots, n \\ i = j, j+1, j+2, \ldots, n \\ (\textit{for each value of } j) \end{Bmatrix}$$

$$u_{i,j} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}}{l_{ii}} \quad \text{For} \begin{Bmatrix} i = 2, 3, 4, \ldots, n \\ j = i, i+1, i+2, \ldots, n+1 \\ (\textit{for each value of } i) \end{Bmatrix}$$

$$x_n = u_{n,n+1}$$

$$x_i = u_{i,n+1} - \sum_{j=i+1}^{n} u_{ij} x_j \quad \text{For} \{ i = n-1,\ n-2,\ n-3, \ldots, 1 \}$$

An improved Cholesky decomposition scheme can also be used with only one matrix in the process as follows:

$$a_{1j} = \frac{a_{1j}}{a_{11}} \quad \text{For } \{ j = 2, 3, 4, \ldots, n+1 \}$$

$$a_{i,j} = a_{ij} - \sum_{k=1}^{j-1} a_{ik} a_{kj} \quad \text{For} \begin{Bmatrix} j = 2, 3, 4, \ldots, n \\ i = j, j+1, j+2, \ldots, n \\ (\textit{for each value of } j) \end{Bmatrix}$$

$$a_{i,j} = \frac{a_{ij} - \sum_{k=1}^{i-1} a_{ik} a_{kj}}{a_{ii}} \quad \text{For} \begin{Bmatrix} i = 2, 3, 4, \ldots, n \\ j = i, i+1, i+2, \ldots, n+1 \\ (\textit{for each value of } i) \end{Bmatrix}$$

Example 2.11 Cholesky decomposition method

Find the solution set to the following non-homogeneous linear algebraic equations using Cholesky decomposition.

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 + x_5 = 8 \\ &16x_1 + 8x_2 + 4x_3 + 2x_4 + 1x_5 = 44 \\ &x_1 - x_2 + x_3 - x_4 + x_5 = 2 \\ &81x_1 - 27x_2 + 9x_3 - 3x_4 + x_5 = 44 \\ &16x_1 - 8x_2 + 4x_3 - 2x_4 + x_5 = 8 \end{aligned}$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & d_1 \\ 0 & 1 & u_{23} & d_2 \\ 0 & 0 & 1 & d_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{bmatrix}$$

The matrix in augmented form is shown in Table 2.7.

Table 2.7. Example 2.11 Cholesky decomposition method

$$|A| = \left|\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 8 \\ 16 & 8 & 4 & 2 & 1 & 44 \\ 1 & -1 & 1 & -1 & 1 & 2 \\ 81 & -27 & 9 & -3 & 1 & 44 \\ 16 & -8 & 4 & -2 & 1 & 8 \end{array}\right|$$

The reduced lower triangular matrix is shown in Table 2.8.

Table 2.8. Example 2.11 Cholesky decomposition method

		1	0	0	0	0
		16	–8	0	0	0
| L |	=	1	–2	3	0	0
		81	–108	90	60	0
		16	–24	24	12	–2

The upper triangular matrix is found at the same time and is shown in Table 2.9.

Table 2.9. Example 2.11 Cholesky decomposition method

		1	1	1	1	1	8
		0	1	1.5	1.75	1.875	10.5
| U|D |	=	0	0	1	0.5	1.25	5
		0	0	0	1	0.167	1.333
		0	0	0	0	1	2

Finally, the solution is calculated from the [U|D] matrix using back substitution and shown in Table 2.10.

Table 2.10. Example 2.11 Cholesky decomposition method

		1
		2
| x |	=	2
		1
		2

2.10 ERROR EQUATIONS

Error equations are intended to increase the accuracy in which the roots of simultaneous equations are determined by reducing the error due to rounding off. Consider the set of equations as follows:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = C_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = C_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = C_3$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = C_n$$

If the approximate roots x'_1, x'_2, $x'_3 \ldots$, x'_n have been obtained by elimination, upon substitution into the equations the constants C'_1, C'_2, $C'_3 \ldots$, C'_n are found as follows:

$$a_{11}x'_1 + a_{12}x'_2 + a_{13}x'_3 + \ldots + a_{1n}x'_n = C'_1$$
$$a_{21}x'_1 + a_{22}x'_2 + a_{23}x'_3 + \ldots + a_{2n}x'_n = C'_2$$
$$a_{31}x'_1 + a_{32}x'_2 + a_{33}x'_3 + \ldots + a_{3n}x'_n = C'_3$$
$$\vdots$$
$$a_{n1}x'_1 + a_{n2}x'_2 + a_{n3}x'_3 + \ldots + a_{nn}x'_n = C'_n$$

If Δx_1, Δx_2, Δx_3 …, Δx_n are the corrections that must be added to the approximate root to obtain the exact root values x_1, x_2, $x_3, \ldots$, x_n, the following is utilized:

$$x_1 = x'_1 + \Delta x_1$$
$$x_2 = x'_2 + \Delta x_2$$
$$x_3 = x'_3 + \Delta x_3$$
$$\vdots$$
$$x_n = x'_n + \Delta x_n$$

If we substitute these expressions for the exact root, we obtain the following:

$$a_{11}\left(x'_1 + \Delta x_1\right) + a_{12}\left(x'_2 + \Delta x_2\right) + a_{13}\left(x'_3 + \Delta x_3\right) + \cdots + a_{1n}\left(x'_n + \Delta x_n\right) = C_1$$
$$a_{21}\left(x'_1 + \Delta x_1\right) + a_{22}\left(x'_2 + \Delta x_2\right) + a_{23}\left(x'_3 + \Delta x_3\right) + \cdots + a_{2n}\left(x'_n + \Delta x_n\right) = C_2$$
$$a_{31}\left(x'_1 + \Delta x_1\right) + a_{32}\left(x'_2 + \Delta x_2\right) + a_{33}\left(x'_3 + \Delta x_3\right) + \cdots + a_{3n}\left(x'_n + \Delta x_n\right) = C_3$$
$$\vdots$$
$$a_{n1}\left(x'_1 + \Delta x_1\right) + a_{n2}\left(x'_2 + \Delta x_2\right) + a_{n3}\left(x'_3 + \Delta x_3\right) + \cdots + a_{nn}\left(x'_n + \Delta x_n\right) = C_n$$

If these equations are subtracted from the approximate equations, the following is obtained:

$$
\begin{aligned}
a_{11}\Delta x_1 + a_{12}\Delta x_2 + a_{13}\Delta x_3 + \cdots + a_{1n}\Delta x_n &= C_1 - C_1' = e_1 \\
a_{21}\Delta x_1 + a_{22}\Delta x_2 + a_{23}\Delta x_3 + \cdots + a_{2n}\Delta x_n &= C_2 - C_2' = e_2 \\
a_{31}\Delta x_1 + a_{32}\Delta x_2 + a_{33}\Delta x_3 + \cdots + a_{3n}\Delta x_n &= C_3 - C_3' = e_3 \\
&\vdots \\
a_{n1}\Delta x_1 + a_{n2}\Delta x_2 + a_{n3}\Delta x_3 + \cdots + a_{nn}\Delta x_n &= C_n - C_n' = e
\end{aligned}
$$

This shows that the corrections, $\Delta x'$s, can be obtained by replacing the constant vector of the solution with the difference of the constant vectors, (C–C's)'s, and applying reduction to find the error. These are then added to the approximate solution and the process is repeated until accuracy is achieved.

Example 2.12 Error equations

Determine the solution to the following set of equations using any Gauss–Jordan elimination, but only carry two decimals of accuracy (i.e., *x.xx*) then apply error equations to increase accuracy.

$$
\begin{aligned}
2.11x_1 + 2.11x_2 - 3.04x_3 + 1.11x_4 &= 1.65 \\
-0.02x_1 + 1.23x_2 + 2.22x_3 + 1.02x_4 &= 13.18 \\
0.14x_1 - 0.06x_2 + 1.21x_3 - 1.08x_4 &= -0.67 \\
1.32x_1 + 0.20x_2 + 0.00x_3 + 3.90x_4 &= 17.32
\end{aligned}
$$

The process of three complete cycles is shown in Tables 2.11–2.13.

2.11 MATRIX INVERSION METHOD

The solution to a set of linear equations can be achieved by using any reduction technique on the coefficient matrix augmented with the identity matrix.

$$
[A\,|\,I] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & 1 & 0 & 0 & \cdots & 0 \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & 0 & 1 & 0 & \cdots & 0 \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
$$

From here, the coefficient matrix is reduced until the identity matrix is on the left and the original identity on the right becomes the invert of A.

Table 2.11. Example 2.12 Error equations

|A|C|	=	2.11	2.11	–3.04	1.11	1.65
		–0.02	1.23	2.22	1.02	13.18
		0.14	–0.06	1.21	–1.08	–0.67
		1.32	0.20	0.00	3.90	17.32

|A|C|	=	2.11	2.11	–3.04	1.11	1.65
		0.00	1.25	2.19	1.03	13.20
		0.00	–0.20	1.41	–1.15	–0.78
		0.00	–1.12	1.90	3.21	16.29

|A|C|	=	2.11	0.00	–6.74	–0.63	–20.63
		0.00	1.25	2.19	1.03	13.20
		0.00	0.00	1.76	–0.99	1.33
		0.00	0.00	3.86	4.13	28.12

|A|C|	=	2.11	0.00	0.00	–4.42	–15.54
		0.00	1.25	0.00	2.26	11.55
		0.00	0.00	1.76	–0.99	1.33
		0.00	0.00	0.00	6.30	25.20

|A|C|	=	2.11	0.00	0.00	0.00	2.14
		0.00	1.25	0.00	0.00	2.51
		0.00	0.00	1.76	0.00	5.29
		0.00	0.00	0.00	6.30	25.20

|x|	=	1.01	$\|C_1\|$	=	1.66
		2.01			13.21
		3.01			–0.66
		4.00			17.34

$\|e_1\|$	=	**–0.01**
		–0.03
		–0.01
		–0.02

Table 2.12. Example 2.12 Error equations

$$\left|A|C\right| = \left|\begin{array}{cccc|c} 2.11 & 2.11 & -3.04 & 1.11 & -0.01 \\ -0.02 & 1.23 & 2.22 & 1.02 & -0.03 \\ 0.14 & -0.06 & 1.21 & -1.08 & -0.01 \\ 1.32 & 0.20 & 0.00 & 3.90 & -0.02 \end{array}\right|$$

$$\left|A|C\right| = \left|\begin{array}{cccc|c} 2.11 & 2.11 & -3.04 & 1.11 & -0.01 \\ 0.00 & 1.25 & 2.19 & 1.03 & -0.03 \\ 0.00 & -0.20 & 1.41 & -1.15 & -0.01 \\ 0.00 & -1.12 & 1.90 & 3.21 & -0.01 \end{array}\right|$$

$$\left|A|C\right| = \left|\begin{array}{cccc|c} 2.11 & 0.00 & -6.74 & -0.63 & 0.04 \\ 0.00 & 1.25 & 2.19 & 1.03 & -0.03 \\ 0.00 & 0.00 & 1.76 & -0.99 & -0.01 \\ 0.00 & 0.00 & 3.86 & 4.13 & -0.04 \end{array}\right|$$

$$\left|A|C\right| = \left|\begin{array}{cccc|c} 2.11 & 0.00 & 0.00 & -4.42 & 0.00 \\ 0.00 & 1.25 & 0.00 & 2.26 & -0.02 \\ 0.00 & 0.00 & 1.76 & -0.99 & -0.01 \\ 0.00 & 0.00 & 0.00 & 6.30 & -0.02 \end{array}\right|$$

$$\left|A|C\right| = \left|\begin{array}{cccc|c} 2.11 & 0.00 & 0.00 & 0.00 & -0.01 \\ 0.00 & 1.25 & 0.00 & 0.00 & -0.01 \\ 0.00 & 0.00 & 1.76 & 0.00 & -0.01 \\ 0.00 & 0.00 & 0.00 & 6.30 & -0.02 \end{array}\right|$$

$$\left|x\right| = \left|\begin{array}{c} 0.00 \\ -0.01 \\ -0.01 \\ 0.00 \end{array}\right| \qquad \left|\Delta x_1\right| = \left|\begin{array}{c} 1.01 \\ 2.00 \\ 3.00 \\ 4.00 \end{array}\right|$$

$$\left|C_2\right| = \left|\begin{array}{c} 1.67 \\ 13.18 \\ -0.67 \\ 17.33 \end{array}\right| \qquad \left|e_2\right| = \left|\begin{array}{c} \mathbf{-0.02} \\ \mathbf{0.00} \\ \mathbf{0.00} \\ \mathbf{-0.01} \end{array}\right|$$

Table 2.13. Example 2.12 Error equations

$$\left| A|C \right| = \left|\begin{array}{cccc|c} 2.11 & 2.11 & -3.04 & 1.11 & -0.02 \\ -0.02 & 1.23 & 2.22 & 1.02 & 0.00 \\ 0.14 & -0.06 & 1.21 & -1.08 & 0.00 \\ 1.32 & 0.20 & 0.00 & 3.90 & -0.01 \end{array}\right|$$

$$\left| A|C \right| = \left|\begin{array}{cccc|c} 2.11 & 2.11 & -3.04 & 1.11 & -0.02 \\ 0.00 & 1.25 & 2.19 & 1.03 & 0.00 \\ 0.00 & -0.20 & 1.41 & -1.15 & 0.00 \\ 0.00 & -1.12 & 1.90 & 3.21 & 0.00 \end{array}\right|$$

$$\left| A|C \right| = \left|\begin{array}{cccc|c} 2.11 & 0.00 & -6.74 & -0.63 & -0.02 \\ 0.00 & 1.25 & 2.19 & 1.03 & 0.00 \\ 0.00 & 0.00 & 1.76 & -0.99 & 0.00 \\ 0.00 & 0.00 & 3.86 & 4.13 & 0.00 \end{array}\right|$$

$$\left| A|C \right| = \left|\begin{array}{cccc|c} 2.11 & 0.00 & 0.00 & -4.42 & -0.02 \\ 0.00 & 1.25 & 0.00 & 2.26 & 0.00 \\ 0.00 & 0.00 & 1.76 & -0.99 & 0.00 \\ 0.00 & 0.00 & 0.00 & 6.30 & 0.00 \end{array}\right|$$

$$\left| A|C \right| = \left|\begin{array}{cccc|c} 2.11 & 0.00 & 0.00 & 0.00 & -0.02 \\ 0.00 & 1.25 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.76 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 6.30 & 0.00 \end{array}\right|$$

$$\left| x \right| = \left|\begin{array}{c} -0.01 \\ 0.00 \\ 0.00 \\ 0.00 \end{array}\right| \qquad \left| \Delta x_2 \right| = \left|\begin{array}{c} 1.00 \\ 2.00 \\ 3.00 \\ 4.00 \end{array}\right|$$

$$\left| C_3 \right| = \left|\begin{array}{c} 1.67 \\ 13.18 \\ -0.67 \\ 17.32 \end{array}\right| \qquad \left| e_3 \right| = \left|\begin{array}{c} \mathbf{0.00} \\ \mathbf{0.00} \\ \mathbf{0.00} \\ \mathbf{0.00} \end{array}\right|$$

$$\left[I\,|\,A^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} & \cdots & a_{1n}^{-1} \\ 0 & 1 & 0 & \cdots & 0 & a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} & \cdots & a_{2n}^{-1} \\ 0 & 0 & 1 & \cdots & 0 & a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1} & \cdots & a_{3n}^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{n1}^{-1} & a_{n2}^{-1} & a_{n3}^{-1} & \cdots & a_{nn}^{-1} \end{bmatrix}$$

If partial pivoting is used during the reduction, the columns in the invert must be swapped back in reverse order as the rows were swapped during reduction. The constant vector must also be reordered in the same way.

Example 2.13 Matrix inversion

Determine the solution to the following set of equations using inversion-in-place (improved Gauss–Jordan). Include partial pivoting during the reduction.

$$\begin{aligned} 2.11x_1 + 2.11x_2 - 3.04x_3 + 1.11x_4 &= 1.65 \\ -0.02x_1 + 1.23x_2 + 2.22x_3 + 1.02x_4 &= 13.18 \\ 0.14x_1 - 0.06x_2 + 1.21x_3 - 1.08x_4 &= -0.67 \\ 1.32x_1 + 0.20x_2 + 0.00x_3 + 3.90x_4 &= 17.32 \end{aligned}$$

The matrix is shown in Table 2.14 in augmented form and Gauss–Jordan elimination is performed for the first two columns.

Table 2.14. Example 2.13 Matrix inversion method

|A|C| =	2.11	2.11	−3.04	1.11	1.65
	−0.02	1.23	2.22	1.02	13.18
	0.14	−0.06	1.21	−1.08	−0.67
	1.32	0.20	0.00	3.90	17.32

|A|I| =	2.11	2.11	−3.04	1.11	1.00	0.00	0.00	0.00
	−0.02	1.23	2.22	1.02	0.00	1.00	0.00	0.00
	0.14	−0.06	1.21	−1.08	0.00	0.00	1.00	0.00
	1.32	0.20	0.00	3.90	0.00	0.00	0.00	1.00

(*Continued*)

Table 2.14. (*Continued*)

|A|I| =	2.110	2.110	–3.040	1.110	1.000	0.000	0.000	0.000
	0.000	1.250	2.191	1.031	0.009	1.000	0.000	0.000
	0.000	–0.200	1.412	–1.154	–0.066	0.000	1.000	0.000
	0.000	–1.120	1.902	3.206	–0.626	0.000	0.000	1.000

|A|I| =	2.1100	0.0000	–6.7387	–0.6295	0.9840	–1.6880	0.0000	0.0000
	0.0000	1.2500	2.1912	1.0305	0.0095	1.0000	0.0000	0.0000
	0.0000	0.0000	1.7623	–0.9888	–0.0648	0.1600	1.0000	0.0000
	0.0000	0.0000	3.8651	4.1289	–0.6171	0.8960	0.0000	1.0000

Note that partial pivoting should be performed and row three is now swapped with row four. Columns 3 and 4 are then eliminated as shown in Table 2.15.

Table 2.15. Example 2.13 Matrix inversion method

|A|I| =	2.1100	0.0000	–6.7387	–0.6295	0.9840	–1.6880	0.0000	0.0000
	0.0000	1.2500	2.1912	1.0305	0.0095	1.0000	0.0000	0.0000
	0.0000	*0.0000*	*3.8651*	*4.1289*	*–0.6171*	*0.8960*	*0.0000*	*1.0000*
	0.0000	*0.0000*	*1.7623*	*–0.9888*	*–0.0648*	*0.1600*	*1.0000*	*0.0000*

|A|I| =	2.1100	0.0000	0.0000	6.5692	–0.0919	–0.1258	0.0000	1.7435
	0.0000	1.2500	0.0000	–1.3102	0.3593	0.4920	0.0000	–0.5669
	0.0000	0.0000	3.8651	4.1289	–0.6171	0.8960	0.0000	1.0000
	0.0000	0.0000	0.0000	–2.8714	0.2165	–0.2485	1.0000	–0.4560

|A|I| =	2.1100	0.0000	0.0000	0.0000	0.4035	–0.6944	2.2878	0.7003
	0.0000	1.2500	0.0000	0.0000	0.2605	0.6055	–0.4563	–0.3589
	0.0000	0.0000	3.8651	0.0000	–0.3057	0.5386	1.4380	0.3444
	0.0000	0.0000	0.0000	–2.8714	0.2165	–0.2485	1.0000	–0.4560

Now the third and fourth columns are swapped back for the inverse matrix shown in Table 2.16.

Table 2.16. Example 2.13 Matrix inversion method

$\|I\|A^{-1}\|$ =	1.0000	0.0000	0.0000	0.0000	0.1912	–0.3291	*1.0843*	*0.3319*
	0.0000	1.0000	0.0000	0.0000	0.2084	0.4844	*–0.3651*	*–0.2871*
	0.0000	0.0000	1.0000	0.0000	–0.0791	0.1394	*0.3720*	*0.0891*
	0.0000	0.0000	0.0000	1.0000	–0.0754	0.0866	*–0.3483*	*0.1588*

(*Continued*)

Table 2.16. (*Continued*)

|A^{-1}|	=	0.1912	–0.3291	0.3319	1.0843
		0.2084	0.4844	–0.2871	–0.3651
		–0.0791	0.1394	0.0891	0.3720
		–0.0754	0.0866	0.1588	–0.3483

Now the third and fourth rows are swapped in coefficient matrix and the solution is found as $[x]=[A]^{-1}[C]$ in Table 2.17:

Table 2.17. Example 2.13 Matrix inversion method

|C|	=	1.65	|C|	=	1.65	|x|	=	1
		13.18			13.18			2
		-0.67			17.32			3
		17.32			–0.67			4

2.12 GAUSS–SEIDEL ITERATION METHOD

Some methods such as the Gaussian elimination are not appropriate when a *sparse matrix* exists. A matrix normally can be considered sparse if approximately two-thirds or more of the entries in a matrix are zero. The Gauss–Seidel iteration method was developed for such systems. The method is named for Carl Friedrich Gauss and Philipp Ludwig von Seidel (Gauss 1903). This method is an iteration method in which the last calculated values are used to determine a more accurate solution. Typically, all unknown x values are assumed to be zero to begin the iteration. This method mainly works best with a diagonal system in which the largest values lie on the diagonal. The elastic stiffness matrix used to analyze structures is a typical example of a diagonal system and will be presented in Chapter 4. A diagonal system is sufficient, but not necessary to provide convergence. During the process, each row is used to find a better approximation of the variable corresponding to the row using all other variables as known.

Example 2.14 Gauss–Seidel Iteration method

Determine the solution to the following set of equations using the Gauss–Seidel iteration method with $\varepsilon = 0.01$ and assume $x_1 = x_2 = x_3 = x_4 = 0$.

$$
\begin{aligned}
10x_1 + x_2 + 2x_3 + x_4 &= 50 \\
2x_1 + 10x_2 + x_3 + 2x_4 &= 63 \\
x_1 + 2x_2 + 10x_3 + x_4 &= 67 \\
2x_1 + x_2 + x_3 + 10x_4 &= 75
\end{aligned}
$$

Note that this is a diagonal system with 10's on the diagonal and all other coefficients are much less. Begin the iteration by setting $x_1 = x_2 = x_3 = x_4 = 0$ and solving for each of the unknowns using the corresponding equation in a top down order.

$$
\begin{aligned}
&10x_1 + 0 + 2(0) + 0 = 50 \therefore x_1 = 5.000 \\
&2(5.000) + 10x_2 + 0 + 2(0) = 63 \therefore x_2 = 5.300 \\
&5.000 + 2(5.300) + 10x_3 + 0 = 67 \therefore x_3 = 5.140 \\
&2(5.000) + 5.300 + 5.140 + 10x_4 = 75 \therefore x_4 = 5.456
\end{aligned}
$$

After completing the first cycle, start with the first equation using the new values and find a closer approximation for each unknown. Also, check the difference between the new values and the previous values to determine if the desired accuracy is achieved.

$$
\begin{aligned}
&10x_1 + 5.300 + 2(5.140) + 5.456 = 50 \therefore x_1 = 2.896 \text{ and } \Delta x_1 = -2.104 \\
&2(2.896) + 10x_2 + 5.140 + 2(5.456) = 63 \therefore x_2 = 4.116 \text{ and } \Delta x_2 = -1.184 \\
&2.896 + 2(4.116) + 10x_3 + 5.456 = 67 \therefore x_3 = 5.042 \text{ and } \Delta x_3 = -0.098 \\
&2(2.896) + 4.116 + 5.042 + 10x_4 = 75 \therefore x_4 = 6.005 \text{ and } \Delta x_4 = 0.549
\end{aligned}
$$

None of the values of Δx are less than $\varepsilon = 0.01$, so the process is repeated. Table 2.18 shows the entire process to convergence. The process can be stopped when each value has changed less than ε or when a cycle results in each value changing less than ε.

2.13 EIGENVALUES BY CRAMER'S RULE

A homogeneous equation is one where all the constants on the right-hand side of the equal sign are zero. The typical set of n homogeneous equations with n unknown solution sets is as follows:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= 0
\end{aligned}
$$

This can be written in matrix form, $[A][x]=[0]$, as follows:

Table 2.18. Example 2.14 Gauss–Seidel iteration method

x_1=	5.000	Δx_1=	5.000	
x_2=	5.300	Δx_2=	5.300	
x_3=	5.140	Δx_3=	5.140	
x_4=	5.456	Δx_4=	5.456	
x_1=	2.896	Δx_1=	-2.104	
x_2=	4.116	Δx_2=	-1.184	
x_3=	5.042	Δx_3=	-0.098	
x_4=	6.005	Δx_4=	0.549	
x_1=	2.980	Δx_1=	0.083	
x_2=	3.999	Δx_2=	-0.117	
x_3=	5.002	Δx_3=	-0.040	
x_4=	6.004	Δx_4=	-0.001	<epsilon
x_1=	2.999	Δx_1=	0.020	
x_2=	3.999	Δx_2=	0.000	<epsilon
x_3=	5.000	Δx_3=	-0.002	<epsilon
x_4=	6.000	Δx_4=	-0.004	<epsilon
x_1=	3.000	Δx_1=	0.001	<epsilon
x_2=	4.000	Δx_2=	0.001	<epsilon
x_3=	5.000	Δx_3=	0.000	<epsilon
x_4=	6.000	Δx_4=	0.000	<epsilon

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let us consider the solution of eigenvalue problems. For any square matrix, $[A]$, the determinant equation $|A-\lambda I|=0$ is a polynomial equation of degree n unknowns in the variable λ. In other words, there are exactly n roots that satisfy this equation. These roots are known as *eigenvalues* of A.

$$\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + (a_{33}-\lambda)x_3 + \cdots + a_{3n}x_n &= 0 \\ \vdots\vdots\vdots\vdots\vdots\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}$$

Converting these equations into matrix form, $[A-\lambda I][x]=0$:

$$\begin{bmatrix} (a_{11}-\lambda) & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & (a_{33}-\lambda) & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & (a_{nn}-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The non-trivial solution exists if the determinant of the coefficient matrix is zero. We use this so that Cramer's rule can be used to find the eigenvalues. Example 2.15 shows the process of determining the eigenvalues by Cramer's rule.

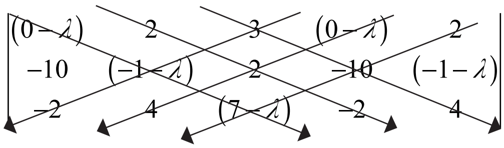
Example 2.15 Eigenvalues by Cramer's rule

Determine the eigenvalues for the following set of equations using Cramer's rule.

$$\begin{aligned} 0x_1 + 2x_2 + 3x_3 &= 0 \\ -10x_1 - 1x_2 + 2x_3 &= 0 \\ -2x_1 + 4x_2 + 7x_3 &= 0 \end{aligned}$$

$$\begin{vmatrix} (0-\lambda) & 2 & 3 \\ -10 & (-1-\lambda) & 2 \\ -2 & 4 & (7-\lambda) \end{vmatrix} = 0$$

Solve the determinant by the basket weave method.



$$\begin{aligned} &[(0-\lambda)(-1-\lambda)(7-\lambda)]+[(2)(2)(-2)]+[(3)(-10)(4)] \\ &\qquad -[(3)(-1-\lambda)(-2)]-[(0-\lambda)(2)(4)]-[(2)(-10)(7-\lambda)] \\ &\lambda^3-6\lambda^2+11\lambda-6=0 \end{aligned}$$

The solution to the cubic equation can be found by many of the methods from Chapter 1 and represent the eigenvalues $\lambda = 1$, 2, and 3.

2.14 FADDEEV–LEVERRIER METHOD

The Faddeev–Leverrier method is a polynomial method used to find the eigenvalues. The method is named for Dmitrii Konstantinovich Faddeev and published by Urbain Jean Joseph Le Verrier in 1840 (Le Verrier 1839). From linear algebra, the trace of a matrix is the sum of the diagonal terms. The process for determining the characteristic polynomial is as follows:

$$(-1)^n\left(\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - p_3\lambda^{n-3} - \cdots - p_n\right) = 0$$

$$\begin{aligned} &[B_1]=[A] && p_1 = tr[B_1] \\ &[B_2]=[A]([B_1]-p_1[I]) && p_2 = \frac{1}{2}tr[B_2] \\ &[B_3]=[A]([B_2]-p_2[I]) && p_3 = \frac{1}{3}tr[B_3] \\ &\qquad\vdots && \vdots \\ &[B_k]=[A]([B_{k-1}]-p_{k-1}[I]) && p_k = \frac{1}{k}tr[B_k] \\ &\qquad\vdots && \vdots \\ &[B_n]=[A]([B_{n-1}]-p_{n-1}[I]) && p_n = \frac{1}{n}tr[B_n] \end{aligned}$$

Example 2.16 shows the process of determining the characteristic polynomial and the eigenvalues by the Faddeev–Leverrier method.

Example 2.16 Faddeev–Leverrier method

Determine the eigenvalues for the following set of equations using the Faddeev–Leverrier method.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -10x_1 + 0x_2 + 2x_3 &= 0 \\ -2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

The matrix operations are shown in Table 2.19.
The characteristic polynomial is found from the trace values.

$$\begin{aligned} &(-1)^3\left(\lambda^3 - 9\lambda^2 - (-26)\lambda - 24\right) = 0 \\ &-\lambda^3 + 9\lambda^2 - 26\lambda + 24 = 0 \end{aligned}$$

The solution to the cubic equation can be found by many of the methods from Chapter 1 and represent the eigenvalues $\lambda = 2$, 3, and 4.

2.15 POWER METHOD OR ITERATION METHOD

The power method is an iterative method used when only the smallest or largest eigenvalues and eigenvectors are desired. It may also be used to find intermediate eigenvalues and eigenvectors using a sweeping technique. The sweeping technique can be found in "Applied Numerical Methods for Digital Computations," By M.L. James, G.M. Smith, and J.C. Wolford. The largest eigenvalue is found by iterating on the equation $[A][x]= \lambda[x]$.

The steps of procedure are as follows:

1. Assume values for the components of the eigenvector $[x]=1$.
2. Multiply the coefficient matrix times the vector $[A][x]$.
3. Normalize the right hand side of the equation as follows $\lambda[x]$:

Table 2.19. Example 2.16 Faddev–Leverrier method

$$[B_1]=[A]=\begin{vmatrix}1&2&3\\-10&0&2\\-2&4&8\end{vmatrix}\qquad\begin{vmatrix}1&2&3\\-10&0&2\\-2&4&8\end{vmatrix}$$

$[B_2]=([A]([B_1]-p_1I)$ $\qquad p_1=tr[B_1]=9$

$$\begin{vmatrix}1&2&3\\-10&0&2\\-2&4&8\end{vmatrix} * \begin{vmatrix}1&2&3\\-10&0&2\\-2&4&8\end{vmatrix} - \begin{vmatrix}9&&\\&9&\\&&9\end{vmatrix}$$

$$\begin{vmatrix}1&2&3\\-10&0&2\\-2&4&8\end{vmatrix} * \begin{vmatrix}-8&2&3\\-10&-9&2\\-2&4&-1\end{vmatrix} = \begin{vmatrix}-34&-4&4\\76&-12&-32\\-40&-8&-6\end{vmatrix}$$

$[B_3]=([A]([B_2]-p_2I)$ $\qquad p_2=(1/2)tr[B_2]=-26$

$$\begin{vmatrix}1&2&3\\-10&0&2\\-2&4&8\end{vmatrix} * \begin{vmatrix}-34&-4&4\\76&-12&-32\\-40&-8&-6\end{vmatrix} - \begin{vmatrix}-26&&\\&-26&\\&&-26\end{vmatrix}$$

$$\begin{vmatrix}1&2&3\\-10&0&2\\-2&4&8\end{vmatrix} * \begin{vmatrix}-8&-4&4\\76&14&-32\\-40&-8&20\end{vmatrix} = \begin{vmatrix}24&0&0\\0&24&0\\0&0&24\end{vmatrix}$$

$p3=(1/3)tr[B_3]=24$

a. Divide all the x's by first x value.
b. Divide all the x's by the largest x.
c. Normalize to a unit length.

4. Use the components of the normalized vector as improved values of x.
5. Repeat steps 2 through 4 until the previous values differ from the new values by less than some small value (ε).

The smallest eigenvalue is found by iterating on the equation $[A^{-1}][x]= \lambda^{-1}[x]$ in the same manner as the largest value. A common structural problem is the modal node analysis of a multi-story frame. The general steps are as follows:

$$[K][x] - \lambda[M][x] = 0$$

The value $\lambda = \omega^2$, where ω is the frequency of the building. The equation can be rewritten as follows:

$$[B][M][x] = \frac{1}{\lambda}[x]$$

The values of the matrices are $[K]$ for stiffness, $[M]$ for mass, and $[B]$ for flexibility.

Example 2.17 Power method

Determine the first mode shape (lowest eigenvector) for the following set of equations using the power method with $\varepsilon = 0.001$ and assuming $[x] = [1]$ as an initial value. The $[B]$ and $[M]$ matrices are for a four-story single mass structural model.

$$\begin{bmatrix} 1/12 & 1/12 & 1/12 & 1/12 \\ 1/12 & 11/60 & 11/60 & 11/60 \\ 1/12 & 11/60 & 37/120 & 37/120 \\ 1/12 & 11/60 & 37/120 & 57/120 \end{bmatrix}^{10^{-6}} \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}^{10^{3}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The $[B]$ and $[M]$ matrices can be combined by matrix multiplication.

$$\begin{bmatrix} 0.000500 & 0.000417 & 0.000333 & 0.000250 \\ 0.000500 & 0.000917 & 0.000733 & 0.000550 \\ 0.000500 & 0.000917 & 0.001233 & 0.000925 \\ 0.000500 & 0.000917 & 0.001233 & 0.001425 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The iteration process and the final solution are shown in Table 2.20. The x vector was normalized to the top value.

Table 2.20. Example 2.17 Power method

0.0005	0.0004167	0.0003333	0.00025	*	1
0.0005	0.0009167	0.0007333	0.00055		1
0.0005	0.0009167	0.0012333	0.000925		1
0.0005	0.0009167	0.0012333	0.001425		1
1		**1**			
0.0015	=	1	0.0015		
0.0027		1.8			
0.003575		2.3833333			
0.004075		2.7166667			
2		**2**			
0.002724	=	1	0.0027236	ε_{max} = 0.5733384	
0.005392		1.9797042			
0.007602		2.79128			
0.008961		3.2900051			
3		**3**			
0.003078	=	1	0.0030778	ε_{max} = 0.103831	
0.006171		2.0050559			
0.008801		2.8593634			
0.010446		3.3938361			
4		**4**			
0.003137	=	1	0.003137	ε_{max} = 0.0172777	
0.006301		2.0087357			
0.009004		2.8701807			
0.010701		3.4111138			
5		**5**			
0.003146	=	1	0.0031465	ε_{max} = 0.002883	
0.006322		2.0093106			
0.009037		2.8719442			
0.010742		3.4139968			
6		**6**			
0.003148	=	1	0.003148	**ε_{max} = 0.000484**	
0.006326		2.0094044		**ε = 0.001**	
0.009042		2.87223733		**λ_1 = 317.6593**	
0.010749		3.4144812			

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CHAPTER 3

NUMERICAL INTEGRATION AND DIFFERENTIATION

The *integration* of a continuous function is used to find the area under the function and to evaluate integral relationships of functions. *Differentiation* evaluates the rate of change of one variable with respect to another. Examples of structural engineering problems involving integration and differentiation include geometrical properties of centroids of areas and volumes; moment of inertia; relationships between load, shear, moment, rotation, and deflection of beams using the equation of the elastic curve; and other strain energy relationships of structures involving shear, torsion, and axial forces. Many methods exist to solve such types of problems with varying levels of exactness. These and other problems will be covered in the following chapters.

3.1 TRAPEZOIDAL RULE

Consider a function $f(x)$ graphed between points a and b along the x-axis as shown in Figure 3.1. One approximation of the area under the curve is to apply the trapezoidal rule by dividing the area into n strips of width Δx. Then, approximate the area of each strip as a trapezoid.

Calling the ordinates $f(x_i) = y_i$ ($i = 1, 2, 3, \ldots, n, n+1$), the areas of each strip are as follows:

$$A_1 = \Delta x\left(\frac{y_1 + y_2}{2}\right), A_2 = \Delta x\left(\frac{y_2 + y_3}{2}\right), A_3 = \Delta x\left(\frac{y_3 + y_4}{2}\right), \ldots A_n$$
$$= \Delta x\left(\frac{y_n + y_{n+1}}{2}\right)$$

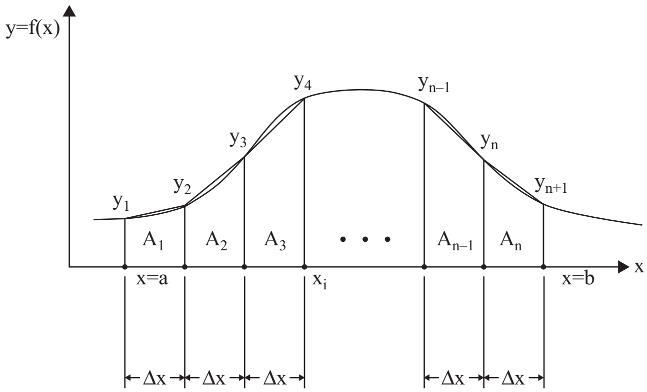

Figure 3.1. Trapezoidal rule.

$$A = \int_a^b f(x)\,dx = A_1 + A_2 + A_3 + \cdots + A_n$$

$$A = \int_a^b f(x)\,dx = \frac{\Delta x}{2}\left(y_1 + 2y_2 + 2y_3 + \cdots + 2y_n + y_{n+1}\right)$$

$$A = \frac{\Delta x}{2}\left(y_1 + \sum_{i=2}^{n} 2y_i + y_{n+1}\right)$$

Example 3.1 Trapezoidal rule

Determine the area under the curve from 0 to π for $y=sin(x)$ using the trapezoidal rule with 2 and 4 strips.

$$A = \frac{\Delta x}{2}\left(y_1 + \sum_{i=2}^{n} 2y_i + y_{n+1}\right)$$

Two strips are shown in Table 3.1:

Table 3.1. Example 3.1 Trapezoidal rule

x	0	π/2	π
x	0	1.5708	3.1416
y=sin(x)	0	1	1.23E-16

$$A = \frac{\pi}{4}\left(0 + 2(1) + 0\right) = \frac{\pi}{2} = 1.5708$$

Four strips are shown in Table 3.2:

Table 3.2. Example 3.1 Trapezoidal rule

x	0	π/4	π/2	3π/4	π
x	0	0.7854	1.5708	2.3562	3.1416
y=sin(x)	0	0.7071	1	0.7071	1.23E-16

$$A=\frac{\pi}{8}\big(0+2(0.7071)+2(1)+2(0.7071)+0\big)=\frac{\pi}{8}(4.8284)=1.8961$$

The exact solution may be found by the integral:

$$\int_0^{\pi} sin(x)\,dx=-cos(x)|_0^{\pi}=-\left[cos(\pi)-cos(0)\right]=1+1=2$$

3.2 ROMBERG INTEGRATION

A more accurate integral can be obtained using Romberg's method (Romberg 1955). If a function can be defined as a continuous mathematical expression having continuous derivative $f'(x)$ and $f''(x)$, the error of the trapezoidal rule is shown in Figure 3.2 and can be found as follows:

Expanding y_{i+1} in a Taylor series about x_i and letting $\Delta x=h$ as follows:

$$y_{i+1}=y_i+y_i'h+\frac{y_i''h^2}{2!}+\frac{y_i'''h^3}{3!}+Higher\ order\ terms$$

The change in y between points i and $i+1$ is equal to the area under the y' curve between those two points, therefore the exact area in the strip is as follows:

$$y_{i+1}-y_i=y_i'h+\frac{y_i''h^2}{2!}+\frac{y_i'''h^3}{3!}+Higher\ order\ terms$$

$$y_{i+1}-y_i=f_ih+\frac{f_i'h^2}{2!}+\frac{f_i''h^3}{3!}+Higher\ order\ terms \tag{3.1}$$

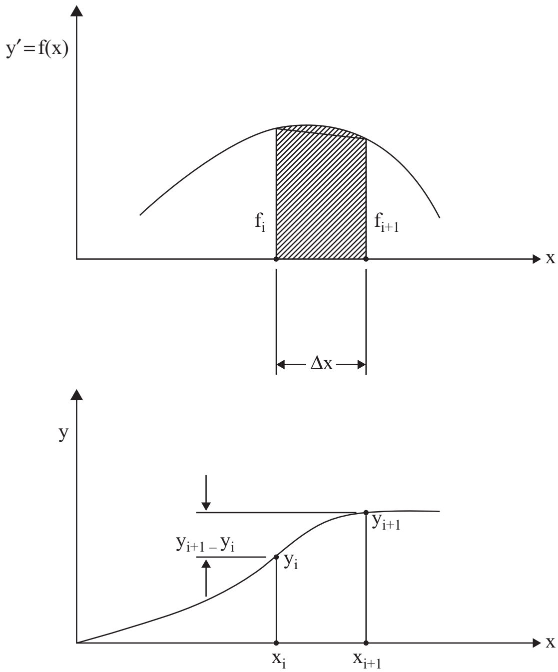

Figure 3.2. Romberg integration.

The Taylor series for f_{i+1} expanded about x_i is as follows:

$$f_{i+1} = f_i + f_i' h + \frac{f_i'' h^2}{2!} + Higher\ order\ terms$$

$$f_i' h = f_{i+1} - f_i - \frac{f_i'' h^2}{2!} - Higher\ order\ terms \tag{3.2}$$

Substituting Equation 3.2 into Equation 3.1, the following can be derived:

$$y_{i+1} - y_i = f_i h + \frac{\left[f_{i+1} - f_i - \frac{f_i'' h^2}{2!} - Higher\ order\ terms\right] h}{2!}$$

$$+ \frac{f_i'' h^3}{3!} + Higher\ order\ terms$$

$$y_{i+1} - y_i = exact\ area = \frac{(f_i + f_{i+1})}{2} h - \frac{f_i'' h^3}{12} + Higher\ order\ terms \tag{3.3}$$

The first term on the right side in Equation 3.3 is the area or the trapezoid and the rest is the error as follows:

$$E_T = -\frac{f_i'' h^3}{12} + Higher\ order\ terms$$

The exact integral, I, can be derived using this error relation from two separate approximate integrals. The derivation is omitted from this text, but may be found in “Applied Numerical Methods for Digital Computations,” by James, Smith, and Wolford (1977). The improved integral is based on two approximate integrals with a strip where $h_2 < h_1$ as follows:

$$I \cong I_{h2} + \left[\frac{I_{h1} - I_{h2}}{h_2^2 - h_1^2}\right] h_2^2 = I_{h2} + \frac{I_{h1} - I_{h2}}{\left(h_1/h_2\right)^2 - 1}$$

$$I \cong \frac{I_{h2}\left(h_1/h_2\right)^2 - I_{h1}}{\left(h_1/h_2\right)^2 - 1}$$

If the second integration uses a strip one-half that of the first with $h_1/h_2=2$, the equation becomes the following:

$$I \cong \frac{I_{h2}(2)^2 - I_{h1}}{(2)^2 - 1}$$

This is a defined as a first-order extrapolation. If two first-order extrapolations are performed, then their results can be combined into a second-order relationship with the following:

$$I \cong \frac{I_{h2}(2)^4 - I_{h1}}{(2)^4 - 1}$$

The general nth order extrapolation would take the following form with n being the order of extrapolation:

$$I \cong \frac{I_{h2}4^n - I_{h1}}{4^n - 1}$$

Example 3.2 Romberg integration

Determine the third-order extrapolation for the area under the curve from 0 to 1 for $y=10^x$ using Romberg integration along with the trapezoidal rule.

$$A=\frac{\Delta x}{2}\left(y_1+\sum_{i=2}^{n}2y_i+y_{n+1}\right)$$

The trapezoidal integration for one strip is as follows in Table 3.3:

Table 3.3. Example 3.2 Romberg integration

x	0	1.000
y	1.0000	10.0000

$$A=\frac{1}{2}(1+10)=5.5$$

Two strips are shown in Table 3.4:

Table 3.4. Example 3.2 Romberg integration

x	0	0.500	1.000
y	1.0000	3.1623	10.0000

$$A=\frac{1}{4}(1+2(3.1623)+10)=4.33114$$

Four strips are shown in Table 3.5:

Table 3.5. Example 3.2 Romberg integration

x	0	0.250	0.500	0.750	1.000
y	1.0000	1.7783	3.1623	5.6234	10.0000

$$A=\frac{1}{8}(1+2(1.7783)+2(3.1623)+2(5.6234)+10)=4.01599$$

Eight strips are shown in Table 3.6:

Table 3.6. Example 3.2 Romberg integration

x	0	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
y	1.0000	1.3335	1.7783	2.3714	3.1623	4.2170	5.6234	7.4989	10.0000

$$A = \frac{1}{16}\big(1 + 2(1.3335) + 2(1.7783) + 2(2.3724) + 2(3.1623)$$
$$+ 2(4.2170) + 2(5.6234) + 2(7.4989) + 10\big) = 3.93560$$

From Romberg integration,

$$I \cong \frac{I_{h2}4^n - I_{h1}}{4^n - 1}$$

First order:

$$I \cong \frac{4.33114(4)^1 - 5.5}{(4)^1 - 1} = 3.941518$$

$$I \cong \frac{4.01599(4)^1 - 4.33114}{(4)^1 - 1} = 3.910944$$

$$I \cong \frac{3.93560(4)^1 - 4.01599}{(4)^1 - 1} = 3.908798$$

Second order:

$$I \cong \frac{3.910944(4)^2 - 3.941518}{(4)^2 - 1} = 3.9089055$$

$$I \cong \frac{3.908798(4)^2 - 3.910943}{(4)^2 - 1} = 3.9808648$$

Third order:

$$I \cong \frac{3.9086548(4)^3 - 3.9089055}{(4)^3 - 1} = 3.98086509$$

5.5 4.33114 4.01599 3.9560

3.941518 3.910944 3.908798

3.9089055 3.6086548

3.98086509

The exact solution may be found from the following integral:

$$\int_0^1 10^x \, dx = \frac{10^x}{\ln 10}\Big|_0^1 = 4.342944819 - 0.434294482 = 3.9808650337$$

3.3 SIMPSON'S RULE

More accurate integration can be achieved by Simpson's rules credited to Simpson (1750). Consider a function $f(x)$ graphed between $x=-\Delta x$ and $x=\Delta x$ as shown in Figure 3.3. An approximation of the area under the curve between these two points would be to pass a parabola through the points and zero (three points). The general second-degree parabola connecting the three points is as follows:

$$y = f(x) = ax^2 + bx + c$$

$$A = \int_{-\Delta x}^{\Delta x} \left(ax^2 + bx + c\right) dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-\Delta x}^{\Delta x}$$

$$A = \frac{2}{3} a(\Delta x)^3 + 2c(\Delta x) \tag{3.4}$$

The constants a, b, and c are found using the three points $(-\Delta x, y_i)$, $(0, y_{i+1})$, and $(\Delta x, y_{i+2})$ as follows:

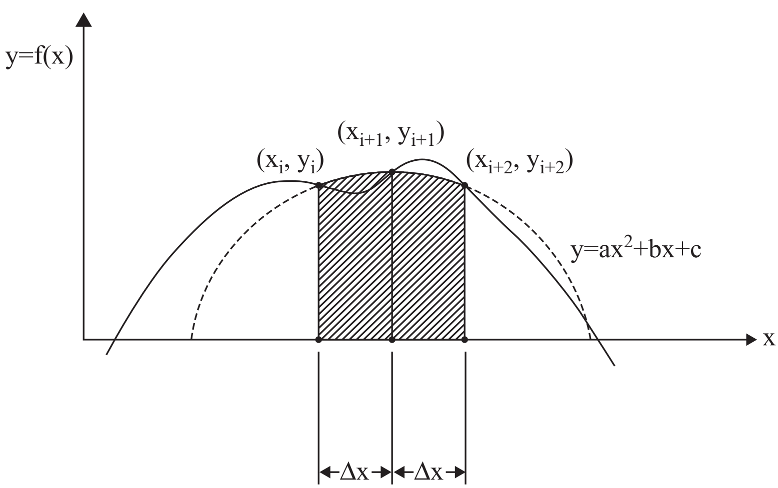

Figure 3.3. Simpson's rule.

$$y_i = a(-\Delta x)^2 + b(-\Delta x) + c$$
$$y_{i+1} = c$$
$$y_{i+2} = f(x) = a(\Delta x)^2 + b(\Delta x) + c$$

Solving the three equations with three unknowns we obtain the following:

$$a = \frac{y_i - 2y_{i+1} + y_{i+2}}{2(\Delta x)^2}$$
$$b = \frac{-y_i + y_{i+2}}{2(\Delta x)}$$
$$c = y_{i+1}$$

Substituting the expressions for a and c into Equation 3.4, the following is achieved:

$$A = \frac{\Delta x}{3}\left(y_i + 4y_{i+1} + y_{i+2}\right)$$

If we apply this to n even numbered strips, the following occurs:

$$A = \frac{\Delta x}{3}\left(y_1 + 4y_2 + y_3\right)$$
$$A = \frac{\Delta x}{3}\left(y_3 + 4y_4 + y_5\right)$$
$$A = \frac{\Delta x}{3}\left(y_5 + 4y_6 + y_7\right)$$
$$A = \frac{\Delta x}{3}\left(y_{n-1} + 4y_n + y_{n+1}\right)$$

In general form, this is the following:

$$A = \frac{\Delta x}{3}\left(y_1 + 4\sum_{i=2,4,6}^{n} y_i + 2\sum_{i=3,5,7}^{n-1} y_i + y_{n+1}\right)$$

If we performed the error truncation to obtain Romberg's integration, the following occurs:

$$I \cong \frac{I_{h2}16^n - I_{h1}}{16^n - 1}$$

Similarly, Simpson's three-eighths rule can be derived using three strips and a third-degree parabola. The following is the solution:

$$y = f(x) = ax^3 + bx^2 + cx + d$$

$$A = \int_{\frac{-3\Delta x}{2}}^{\frac{3\Delta x}{2}} \left(ax^3 + bx^2 + cx + d\right) dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{\frac{-3\Delta x}{2}}^{\frac{3\Delta x}{2}}$$

$$A = \frac{9}{4} b(\Delta x)^3 + 3d(\Delta x) \tag{3.5}$$

The constants a, b, c, and d are found using the four points $(-3\Delta x/2, y_i)$, $(-\Delta x/2, y_{i+1})$, $(\Delta x/2, y_{i+2})$, and $(3\Delta x/2, y_{i+3})$ as follows:

$$y_i = a\left(\frac{-3\Delta x}{2}\right)^3 + b\left(\frac{-3\Delta x}{2}\right)^2 + c\left(\frac{-3\Delta x}{2}\right) + d$$

$$y_{i+1} = a\left(\frac{-\Delta x}{2}\right)^3 + b\left(\frac{-\Delta x}{2}\right)^2 + c\left(\frac{-\Delta x}{2}\right) + d$$

$$y_{i+2} = a\left(\frac{\Delta x}{2}\right)^3 + b\left(\frac{\Delta x}{2}\right)^2 + c\left(\frac{\Delta x}{2}\right) + d$$

$$y_{i+3} = a\left(\frac{3\Delta x}{2}\right)^3 + b\left(\frac{3\Delta x}{2}\right)^2 + c\left(\frac{3\Delta x}{2}\right) + d$$

Solve the four equations with four unknowns and then substitute these back into Equation 3.5 to achieve the following:

$$A = \frac{3\Delta x}{8}\left(y_i + 3y_{i+1} + 3y_{i+2} + y_{i+3}\right)$$

The general form with n strips is as follows:

$$A = \frac{3\Delta x}{8}\left(y_1 + 3\sum_{i=2,3,5,6}^{n-1,n} y_i + 2\sum_{i=4,7}^{n-2} y_i + y_{n+1}\right)$$

For an odd number of strips, both the one-third and three-eighths rules must be used. The three-eighths rule is used to obtain the area contained in three strips under the curve and then the one-third rule is used for the remaining n–3 strips.

Example 3.3. Simpson's one-third rule

Determine the area under the curve from $-\pi/2$ *to* $\pi/2$ for $y = x^2cos(x)$ using Simpson's one-third rule with four and eight strips.

$$A = \frac{\Delta x}{3}\left(y_1 + 4\sum_{i=2,4,6}^{n} y_i + 2\sum_{i=3,5,7}^{n-1} y_i + y_{n+1} \right)$$

Four strips are shown in Table 3.7:

Table 3.7. Example 3.3 Simpson's one-third rule

	-π/2	-π/4	0	π/4	π/2
x	-1.57080	-0.78540	0.00000	0.78540	1.57080
y	0.00000	0.43618	0.00000	0.43618	0.00000

$$A = \frac{\pi}{12}\left(0 + 4(0.43618) + 2(0) + 4(0.43618) + 0\right) = \frac{\pi}{12}(3.48944)$$
$$= 0.913533$$

Eight strips are shown in Table 3.8:

Table 3.8. Example 3.3 Simpson's one-third rule

	-π/2	-3π/8	-π/4	-π/8	0	π/8	π/4	3π/8	π/2
x	-1.57080	-1.17810	-0.78540	-0.39270	0.00000	0.39270	0.78540	1.17810	1.57080
y	0.00000	0.53113	0.43618	0.14247	0.00000	0.14247	0.43618	0.53113	0.00000

$$A = \frac{\pi}{24}\left(0 + 4(0.53113) + 2(0.43618) + 4(0.14247) + 2(0) + 4(0.14247)\right.$$
$$\left. + 2(0.43618) + 4(0.53113) + 0\right) = \frac{\pi}{24}(7.13352) = 0.933776$$

Perform Romberg extrapolation with these two integrations to get a more exact solution as follows:

$$I \cong \frac{I_{h2}16^n - I_{h1}}{16^n - 1} = \frac{(0.933776)16^1 - 0.913533}{16^1 - 1} = 0.935126$$

The exact solution may be found by the integral:

$$\int_{-\pi/2}^{\pi/2} x^2\cos(x)\,dx = \left[2x\cos(x) + \left(x^2 - 2\right)\sin(x)\right]_{-\pi/2}^{\pi/2}$$

$$= (0 + 0.4674011003) - (0 - 0.4674011003) = 0.9348022005$$

Example 3.4. Simpson's one-third and three-eighths rules

Determine the area under the curve from 0 to $\pi/2$ for $y = sin^3x + cos^3x$ using Simpson's three-eighths and one-third rules (in that order) with five strips.

Simpson's three-eighths rule is set up in Table 3.9:

$$A = \frac{3\Delta x}{8}\left(y_i + 3y_{i+1} + 3y_{i+2} + y_{i+3}\right)$$

Table 3.9. Example 3.4 Simpson's one-third and three-eighths rules

	0	π/10	π/5	3π/10	2π/5	π/2
x	0.00000	0.31416	0.62832	0.94248	1.25664	1.57080
y	**1.00000**	**0.88975**	**0.73258**	**0.73258**	0.88975	1.00000

$$A = \frac{3\pi}{80}\left(1 + 3(0.88975) + 3(0.73258) + 0.73258\right) = \frac{3\pi}{80}(6.59957) = 0.777494$$

Simpson's one-third rule is set up in Table 3.10:

$$A = \frac{\Delta x}{3}\left(y_i + 4y_{i+1} + y_{i+2}\right)$$

Table 3.10. Example 3.4 Simpson's one-third and three-eighths rules

0	π/10	π/5	3π/10	2π/5	π/2
0.00000	0.31416	0.62832	0.94248	1.25664	1.57080
1.00000	0.88975	0.73258	**0.73258**	**0.88975**	**1.00000**

$$A = \frac{\pi}{30}\left(0.73258 + 4(0.88975) + 1\right) = \frac{\pi}{30}(5.29158) = 0.554133$$

Adding the two together for a total area:

$$A = 0.777494 + 0.554133 = 1.331627$$

The exact solution may be found by the integral:

$$\int_0^{\pi/2} \left(sin^2 x + cos^2 x\right) dx = \int_0^{\pi/2} \left(sin^{2n-1} x + cos^{2n-1} x\right) dx$$

$$= \frac{2(4)6\ldots(2n)}{3(5)7\ldots(2n+1)} + \frac{2(4)6\ldots(2n)}{3(5)7\ldots(2n+1)} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

3.4 GAUSSIAN QUADRATURE

The main difference of Gaussian quadrature from the previous methods is that the interval to be integrated is not divided into strips. Instead, a central point is used to determine the best places to evaluate the function. The Gauss points indicate how far from the central point to go and then each point is weighted. The derivation of this method is not included here, but can be found in many advanced mathematics textbooks. The method is named for Carl Friedrich Gauss (1801). The following is a general equation that shows the process for n Gauss points for integration:

$$A = \int_a^b f(x)\,dx = s\sum_{i=1}^{n} f\left(s_1 \pm s x_i\right) w_i$$

$$s = \frac{b-a}{2}$$

$$s_1 = \frac{b+a}{2}$$

The number of points used should closely match the degree of the equation to integrate. Table 3.11 shows some of the Gaussian quadrature points, x_i, and their weights, w_i.

Example 3.5. Gaussian quadrature

Determine the area under the curve from 1 to 10 for $y = \log_{10} x$ using Gaussian quadrature with 2, 3, and 4 points.

$$A = \int_a^b f(x)\,dx = s\sum_{i=1}^{n} f\left(s_1 \pm s x_i\right) w_i$$

$$s = \frac{b-a}{2} = \frac{10-1}{2} = 4.5$$

$$s_1 = \frac{b+a}{2} = \frac{10+1}{2} = 5.5$$

Table 3.11. Gaussian quadrature

	x_i	w_i
One point	0	2
Two points	-0.577350269	1
	0.577350269	1
Three points	-0.774596669	0.555555556
	0	0.888888889
	0.774596669	0.555555556
Four points	-0.861136312	0.347854845
	-0.339981044	0.652145155
	0.339981044	0.652145155
	0.861136312	0.347854845
Five points	-0.906179846	0.236926885
	-0.538469310	0.478628670
	0	0.568888889
	0.538469310	0.478628670
	0.906179846	0.236926885
Six points	-0.932469514	0.171324492
	-0.661209386	0.360761573
	-0.238619186	0.467913935
	0.238619186	0.467913935
	0.661209386	0.360761573
	0.932469514	0.171324492

Two points are shown in Table 3.12:

Table 3.12. Example 3.5 Gaussian quadrature

x_i	-0.577350269
2-points	0.577350269
w_i	1
	1
s_1+sx_i	2.90192379
	8.098076211
$f(s_1+sx_i)$	0.462686003
	0.90838186
$f(s_1+sx_i)w_i$	0.462686003
	0.90838186
Σ	1.371067862
$\Sigma*s$	6.169805381

Three points are shown in Table 3.13:

Table 3.13. Example 3.5 Gaussian quadrature

x_i	-0.774596669
3-point	0
	0.774596669
w_i	0.555555556
	0.888888889
	0.555555556
s_1+sx_i	2.01431499
	5.5
	8.985685011
$f(s_1+sx_i)$	0.304127385
	0.740362689
	0.953551191
$f(s_1+sx_i)w_i$	0.168959658
	0.658100169
	0.529750662
Σ	1.356810489
Σ*s	6.105647198

Four points are shown in Table 3.14:

Table 3.14. Example 3.5 Gaussian quadrature

x_i	-0.861136312
4-points	-0.339981044
	0.339981044
	0.861136312
w_i	0.347854845
	0.652145155
	0.652145155
	0.347854845
s_1+sx_i	1.624886596
	3.970085302
	7.029914698
	9.375113404
$f(s_1+sx_i)$	0.210823056
	0.598799838
	0.846950055
	0.97197653
$f(s_1+sx_i)w_i$	0.073335822
	0.390504413
	0.552334375
	0.338106745
Σ	1.354281355
Σ*s	6.094266097

The exact solution may be found by the integral:

$$\int_1^{10} \log(x)\,dx = \left[x\log x - x\log e\right]_1^{10} = 5.65706 + 0.43429 = 6.09135$$

3.5 DOUBLE INTEGRATION BY SIMPSON'S ONE-THIRD RULE

When using double integration by Simpson's one-third rule, weighting is applied in both directions and is then multiplied by the spacing in both directions. The following is a general weighting array for four strips:

1	4	2	4	1
4	16	8	16	4
2	8	4	8	2
4	16	8	16	4
1	4	2	4	1

For two strips, the weighting array is the following:

1	4	1
4	16	4
1	4	1

Any even set of strips will follow the same pattern and this could also be done using any other type of integration. Using the trapezoidal rule would be less accurate, but could do any number of strips and Simpson's three-eighths rule would require a multiple of three strips in each direction. The summation of the weighting array multiplied by $f(x,y)$ is used in the following equation to obtain the volume.

$$\frac{hxhy}{9}\sum = \iint f(x,y) = V$$

The terms hx and hy are the spacing in the x and y directions, respectively.

Example 3.6 Double integration by Simpson's rule

Determine the volume under the hyperbolic paraboloid from $x = 0$ to 8 and $y = 0$ to 8 for $0 = 16z - xy$ using Simpson's one-third rule with four strips in each direction.

$x = 0$ to $8 \quad \Delta x = 2 = h_x \quad$ and $\quad y = 0$ to $8 \quad \Delta y = 2 = h_y$

$$\frac{hxhy}{9}\sum = \iint f(x,y) = V$$

Weighting operator:

$$\begin{matrix} 1 & 4 & 2 & 4 & 1 \\ 4 & 16 & 8 & 16 & 4 \\ 2 & 8 & 4 & 8 & 2 \\ 4 & 16 & 8 & 16 & 4 \\ 1 & 4 & 2 & 4 & 1 \end{matrix}$$

Solving for z:

$$z = \frac{xy}{16}$$

Table 3.15 shows the set up and summation as follows:

$$\frac{hxhy}{9}\sum = \iint f(x,y) = V$$

$$\frac{2(2)}{9}144 = 64 = V$$

Table 3.15. Example 3.6 Double integration by Simpson's one-third rule

x	y	weight	f(x,y)	weight*f(x,y)
0	0	1	0	0
0	2	4	0	0
0	4	2	0	0
0	6	4	0	0
0	8	1	0	0
2	0	4	0	0
2	2	16	0.25	4
2	4	8	0.5	4
2	6	16	0.75	12

(*Continued*)

Table 3.15. (*Continued*)

x	y	weight	f(x,y)	weight*f(x,y)
2	8	4	1	4
4	0	2	0	0
4	2	8	0.5	4
4	4	4	1	4
4	6	8	1.5	12
4	8	2	2	4
6	0	4	0	0
6	2	16	0.75	12
6	4	8	1.5	12
6	6	16	2.25	36
6	8	4	3	12
8	0	1	0	0
8	2	4	1	4
8	4	2	2	4
8	6	4	3	12
8	8	1	4	4
			Σ	144

3.6 DOUBLE INTEGRATION BY GAUSSIAN QUADRATURE

Double integration by Gaussian quadrature is very similar to the single integration process. In this case, the Gauss equation is applied in both directions and then multiplied by the weighting factors.

$$V = \iint_{a_x,b_x}^{b_x,b_y} f(x,y)\,dxdy = s_x s_y \sum_{i=1}^{n} f\left(s_{x1} \pm s_x x_i, s_{y1} \pm s_x y_i\right) w_{xi} w_{yi}$$

$$s_x = \frac{b_x - a_x}{2} \text{ and } s_{x1} = \frac{b_x + a_x}{2}$$

$$s_y = \frac{b_y - a_y}{2} \text{ and } s_{y1} = \frac{b_y + a_y}{2}$$

The number of points used should closely match the degree of the equation to integrate. The same Gauss points and weights from Section 3.4 are used in each direction.

Example 3.7 Double integration by Gaussian quadrature

Determine the volume under the hemisphere from $x = -4$ to 4 and $y = -4$ to 4 for $64 = x^2+y^2+z^2$ using Gaussian quadrature with three points in each direction.

Solving for z and the points:

$$z = \sqrt{64 - x^2 - y^2}$$
$$s_x = \frac{b_x - a_x}{2} = \frac{4-(-4)}{2} = 4$$
$$s_{x1} = \frac{b_x + a_x}{2} = \frac{4+(-4)}{2} = 0$$
$$s_y = \frac{b_y - a_y}{2} = \frac{4-(-4)}{2} = 4$$
$$s_{y1} = \frac{b_y + a_y}{2} = \frac{4+(-4)}{2} = 0$$

Table 3.16 shows the set up and summation as follows with x_i, y_i given values and w_x, w_y corresponding weights when using three points:

Table 3.16. Example 3.7 Double integration by Gaussian quadrature

x_i	y_i	$x = s_{x1} + s_x x_i$	$y = s_{y1} + s_y y_i$	w_x	w_y	f(x,y)	$w_x * w_y * f(x,y)$
-0.774597	-0.774597	-3.098387	-3.098387	0.555556	0.555556	6.69328	2.065827
-0.774597	0	-3.098387	0	0.555556	0.888889	7.375636	3.642289
-0.774597	0.774597	-3.098387	3.098387	0.555556	0.555556	6.69328	2.065827
0	-0.774597	0	-3.098387	0.888889	0.555556	7.375636	3.642289
0	0	0	0	0.888889	0.888889	8	6.320988
0	0.774597	0	3.098387	0.888889	0.555556	7.375636	3.642289
0.774597	-0.774597	3.098387	-3.098387	0.555556	0.555556	6.69328	2.065827
0.774597	0	3.098387	0	0.555556	0.888889	7.375636	3.642289
0.774597	0.774597	3.098387	3.098387	0.555556	0.555556	6.69328	2.065827
						Σ	29.153453

$$V = s_x s_y \sum = 4(4)29.153 = 466.45$$

3.7 TAYLOR SERIES POLYNOMIAL EXPANSION

The differentiation of a continuous function is used to find slopes, curvatures, and values for a function but can also be used to find other relationships of functions. Often in structural engineering, there is a need to find differential relationships. One simple way to easily evaluate transcendental equations is to use polynomial expansion developed for the Taylor series. This is often referred to as the power series. The general Taylor series polynomial expansion of a function is as follows:

$$y = f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n$$

Successive derivatives of the function evaluated at zero can yield the coefficients, b.

$$\begin{aligned}
&f(0) = b_0 \\
&f'(0) = 1b_1 \\
&f''(0) = 1(2)b_2 \\
&f'''(0) = 1(2)3b_3 = 3!b_3 \\
&\qquad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \\
&f^i(0) = 1(2)3\ldots(i)b_i = i!b_i \\
&\qquad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \\
&f^n(0) = 1(2)3\ldots(n)b_n = n!b_n
\end{aligned}$$

By taking successive derivatives of the function then evaluating them, the coefficients of the polynomial may be found. This is how most digital equipment like computers and calculators find values for transcendental equations.

Example 3.8 Taylor series polynomial expansion

Expand $y=sin(x)$ into a polynomial using Taylor series including up to the ninth degree term. Check by calculating sin45°.

$$y = sin(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_9 x^9$$

Evaluate the successive derivatives as follows:

$$
\begin{aligned}
&f(0) = b_0 = sin(0) = 0 \therefore b_0 = 0\\
&f'(0) = 1b_1 = cos(0) = 1 \therefore b_1 = 1\\
&f''(0) = 2b_2 = -sin(0) = 0 \therefore b_2 = 0\\
&f'''(0) = 3!b_3 = -cos(0) = -1 \therefore b_3 = -\frac{1}{3!}\\
&f''''(0) = 4!b_4 = sin(0) = 0 \therefore b_4 = 0\\
&f'''''(0) = 5!b_5 = cos(0) = 1 \therefore b_5 = \frac{1}{5!}\\
&f''''''(0) = 6!b_6 = -sin(0) = 0 \therefore b_6 = 0\\
&f'''''''(0) = 7!b_7 = -cos(0) = -1 \therefore b_7 = -\frac{1}{7!}\\
&f''''''''(0) = 8!b_8 = sin(0) = 0 \therefore b_8 = 0\\
&f'''''''''(0) = 9!b_9 = cos(0) = 1 \therefore b_9 = \frac{1}{9!}
\end{aligned}
$$

The polynomial can then be written with the coefficients.

$$y = \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9$$

The value of sin45°=sin(π/4) can be evaluated to check the accuracy of the approximation.

$$
\begin{aligned}
&y = \sin\left(\pi/4\right) = \pi/4 - \frac{1}{3!}\left(\pi/4\right)^3 + \frac{1}{5!}\left(\pi/4\right)^5 - \frac{1}{7!}\left(\pi/4\right)^7 + \frac{1}{9!}\left(\pi/4\right)^9\\
&y = 0.7853981634 - 0.0808455122 + 0.0024903946 - 0.0000365762\\
&\quad + 0.0000003134\\
&y = 0.7071067829
\end{aligned}
$$

The exact solution to the same accuracy is y=0.7071067812.

Example 3.9. Taylor series polynomial expansion

Expand $y = e^{-x}$ into a polynomial using Taylor series including up to the fifth degree term. Check by calculating e^{-1}.

$$y = e^{-x} = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_5x^5$$

Evaluate the successive derivatives as follows:

$$f(0) = b_0 = e^{-0} = 1 \therefore b_0 = 1$$
$$f'(0) = b_1 = -e^{-0} = -1 \therefore b_1 = -1$$
$$f''(0) = 2b_2 = e^{-0} = 1 \therefore b_2 = \frac{1}{2}$$
$$f'''(0) = 3!b_3 = -e^{-0} = -1 \therefore b_3 = -\frac{1}{3!}$$
$$f''''(0) = 4!b_4 = e^{-0} = 1 \therefore b_4 = \frac{1}{4!}$$
$$f'''''(0) = 5!b_5 = -e^{-0} = -1 \therefore b_5 = -\frac{1}{5!}$$

The polynomial can then be written with the coefficients.

$$y = e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5$$

The value of the e^{-1} can be evaluated to check the accuracy of the approximation.

$$y = e^{-x} = 1 - 1 + \frac{1}{2!}(1)^2 - \frac{1}{3!}(1)^3 + \frac{1}{4!}(1)^4 - \frac{1}{5!}(1)^5$$
$$y = 1 - 1 + 0.5 - 0.166667 + 0.041667 - 0.008333$$
$$y = 0.366667$$

The exact solution to the same accuracy is $y = 0.367879$. If the polynomial was calculated up to the x^9 term, the value would be $y = 0.3678791$ versus the exact value of $y = 0.3678794$.

3.8 DIFFERENCE OPERATORS BY TAYLOR SERIES EXPANSION

The numerical differential equation relationships can be found using the Taylor series expansion. Expanding the Taylor series for a function $y=f(x)$ at $x=(x_i+h)$ gives the following equation:

$$y(x_i + h) = y_i + y_i' h + \frac{y_i'' h^2}{2!} + \frac{y_i''' h^3}{3!} + \cdots \tag{3.6}$$

Next, expanding the Taylor series for a function $y=f(x)$ at $x=(x_i-h)$ gives the following equation:

$$y(x_i - h) = y_i - y_i' h + \frac{y_i'' h^2}{2!} - \frac{y_i''' h^3}{3!} + \cdots \tag{3.7}$$

If Equation 3.7 is subtracted from Equation 3.6, a first derivative relationship is as follows:

$$y_i' = \frac{y(x_i + h) - y(x_i - h)}{2h} - \left(-\frac{y_i''' h^2}{6} + \cdots \right) \tag{3.8}$$

This may be written as follows with successive values of y and the higher order terms omitted:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

This equation is known as the central-difference approximation of y_i' at x_i with errors, order of h^2. If Equation 3.7 is added to Equation 3.6, a second derivative relationship is as follows:

$$y_i'' = \frac{y(x_i + h) - 2y_i + y(x_i - h)}{h^2} - \left(\frac{y_i'''' h^2}{12} + \cdots \right) \tag{3.9}$$

This may be written as follows with successive values of y and the higher order terms omitted:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

This equation is known as the central-difference approximation of y_i'' at x_i with errors, order of h^2. Next, expanding the Taylor series for a function $y=f(x)$ at $x=(x_i+2h)$ gives the following equation:

$$y(x_i + 2h) = y_i + y_i' \, 2h + \frac{y_i''(2h)^2}{2!} + \frac{y_i'''(2h)^3}{3!} + \frac{y_i''''(2h)^4}{4!} + \cdots \tag{3.10}$$

Finally, expanding the Taylor series for a function $y=f(x)$ at $x=(x_i-2h)$ gives the following equation:

$$y(x_i-2h)=y_i-y_i'2h+\frac{y_i''(2h)^2}{2!}-\frac{y_i'''(2h)^3}{3!}+\frac{y_i''''(2h)^4}{4!}\cdots \quad (3.11)$$

If Equation 3.11 is subtracted from Equation 3.10 and the equation for the first derivative is substituted into the result, a third derivative relationship is as follows, with an order of error h^2:

$$y_i'''=\frac{y_{i+2}-2y_{i+1}+2y_{i-1}-y_{i-2}}{2h^3}$$

If Equation 3.11 is added to Equation 3.10 and the equation for the second derivative is substituted into the result, a fourth derivative relationship is as follows with an order of error h^2:

$$y_i''''=\frac{y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}}{h^4}$$

These are the central difference expressions with error order h^2. Higher order expressions can be derived if we include more terms in each expansion. Forward difference expressions can be derived by using Taylor series expansion of $x=(x_i+h)$, $x=(x_i+2h)$, $x=(x_i+3h)$, and so forth. Backward difference expressions may also be derived by Taylor series expansion of $x=(x_i-h)$, $x=(x_i-2h)$, $x=(x_i-3h)$, and so forth. The following are the central, forward, and backward difference expressions of varying error order. These were compiled from "Applied Numerical Methods for Digital Computations," by James, Smith, and Wolford (1977). They can also be written in a reverse graphical form that is sometimes used and compiled from "Numerical Methods in Engineering," by Salvadori and Baron (1961).

Central difference expressions with error order h^2:

$$y_i'=\frac{y_{i+1}-y_{i-1}}{2h}$$

$$y_i''=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$$

$$y_i'''=\frac{y_{i+2}-2y_{i+1}+2y_{i-1}-y_{i-2}}{2h^3}$$

$$y_i''''=\frac{y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}}{h^4}$$

	$i-2$	$i-1$	i	$i+1$	$i+2$
$2hD$		–1	0	1	
h^2D^2		1	–2	1	
$2h^3D^3$	–1	2	0	–2	1
h^4D^4	1	–4	6	–4	1

Central difference expressions with error order h^4:

$$y_i' = \frac{-y_{i+2} + 8y_{i+1} - 8y_{i-1} + y_{i-2}}{12h}$$

$$y_i'' = \frac{-y_{i+2} + 16y_{i+1} - 30y_i + 16y_{i-1} - y_{i-2}}{12h^2}$$

$$y_i''' = \frac{-y_{i+3} + 8y_{i+2} - 13y_{i+1} + 13y_{i-1} - 8y_{i-2} + y_{i-3}}{8h^3}$$

$$y_i'''' = \frac{-y_{i+3} + 12y_{i+2} - 39y_{i+1} + 56y_i - 39y_{i-1} + 12y_{i-2} - y_{i-3}}{6h^4}$$

	$i-3$	$i-2$	$i-1$	i	$i+1$	$i+2$	$i+3$
$12hD$		1	–8	0	8	–1	
$12h^2D^2$		–1	16	–30	16	–1	
$8h^3D^3$	1	–8	13	0	–13	8	–1
$6h^4D^4$	–1	12	–39	56	–39	12	–1

Forward difference expressions with error order h:

$$y_i' = \frac{y_{i+1} - y_i}{h}$$

$$y_i'' = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2}$$

$$y_i''' = \frac{y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i}{h^3}$$

$$y_i'''' = \frac{y_{i+4} - 4y_{i+3} + 6y_{i+2} - 4y_{i+1} + y_i}{h^4}$$

	i	$i+1$	$i+2$	$i+3$	$i+4$
hD	–1	1			
h^2D^2	1	–2	1		
h^3D^3	–1	3	–3	1	
h^4D^4	1	–4	6	–4	1

Forward difference expressions with error order h^2:

$$y_i' = \frac{-y_{i+2} + 4y_{i+1} - 3y_i}{2h}$$

$$y_i'' = \frac{-y_{i+3} + 4y_{i+2} - 5y_{i+1} + 2y_i}{h^2}$$

$$y_i''' = \frac{-3y_{i+4} + 14y_{i+3} - 24y_{i+2} + 18y_{i+1} - 5y_i}{2h^3}$$

$$y_i'''' = \frac{-2y_{i+5} + 11y_{i+4} - 24y_{i+3} + 26y_{i+2} - 14y_{i+1} + 3y_i}{h^4}$$

	i	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$
$2hD$	–3	4	–1			
h^2D^2	2	–5	4	–1		
$2h^3D^3$	–5	18	–24	14	–3	
h^4D^4	3	–14	26	–24	11	–2

Backward difference expressions with error order h:

$$y_i' = \frac{y_i - y_{i-1}}{h}$$

$$y_i'' = \frac{y_i - 2y_{i-1} + y_{i-2}}{h^2}$$

$$y_i''' = \frac{y_i - 3y_{i-1} + 3y_{i-2} - y_{i-3}}{h^3}$$

$$y_i'''' = \frac{y_i - 4y_{i-1} + 6y_{i-2} - 4y_{i-3} + y_{i-4}}{h^4}$$

	$i-4$	$i-3$	$i-2$	$i-1$	i
hD				–1	1
h^2D^2			1	–2	1
h^3D^3		–1	3	–3	1
h^4D^4	1	–4	6	–4	1

Backward difference expressions with error order h^2:

$$y_i' = \frac{3y_i - 4y_{i-1} + y_{i-2}}{2h}$$

$$y_i'' = \frac{2y_i - 5y_{i-1} + 4y_{i-2} - y_{i-3}}{h^2}$$

$$y_i''' = \frac{5y_i - 18y_{i-1} + 24y_{i-2} - 14y_{i-3} + 3y_{i-4}}{2h^3}$$

$$y_i'''' = \frac{3y_i - 14y_{i-1} + 26y_{i-2} - 24y_{i-3} + 11y_{i-4} - 2y_{i-5}}{h^4}$$

	$i-5$	$i-4$	$i-3$	$i-2$	$i-1$	i
$2hD$				1	–4	3
h^2D^2			–1	4	–5	2
$2h^3D^3$		3	–14	24	–18	5
h^4D^4	–2	11	–24	26	–14	3

3.9 NUMERIC MODELING WITH DIFFERENCE OPERATORS

The difference operators, sometimes referred to as finite difference operators, can be used to solve many structural engineering problems involving differential equation relationships. One common relationship is that of the equation of the elastic curve. The equation of the elastic curve relates the deflected shape of a beam to the rotation, moment, shear, and load on the beam. This is a typical strength of materials topic and the following are the basic relationships based on the deflection equation in y and θ, M, V, and q:

$$y_i' = \theta$$

$$y_i'' = -\frac{M}{EI}$$

$$y_i''' = -\frac{V}{EI}$$

$$y_i'''' = -\frac{q}{EI}$$

Example 3.10 Simple beam with difference operator

Calculate the shear, moment, rotation, and deflection for a 25 foot long, simply supported beam with a uniformly distributed load of 4 k/ft using central difference operator of order of error h^2 at 1/6th points. The beam has $E = 40{,}000$ ksi (modulus of elasticity) and $I = 1000$ in^4 (moment of inertia).

To solve the problem, a sketch of the beam and the assumed deflected shape is created. To use central difference operators, the model must go beyond the boundaries of the physical beam. The deflected shape must also be modeled beyond those boundaries with some confidence. If the general equation is continuous at the boundaries, then this type of model is appropriate. If not, then the model should end at the boundaries and forward or central difference operators must be used. Figure 3.4 shows the beam and the assumed deflections at 1/6th points of the beam. By symmetry of the model, only four specific values of the deflection are unknown.

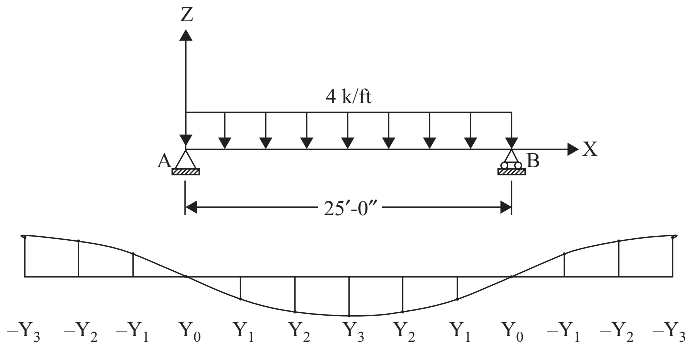

Figure 3.4. Example 3.10 Simple beam with difference operator.

The central-difference expressions with error of order h^2 will be used to solve for the values. Since the load is known, we will use the fourth derivative relationship between load and deflection.

$$y_i'''' = -\frac{q}{EI} = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4}$$

$$q = -\frac{EI}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right)$$

Placing the central difference operator on y_0, the first equation can be written from Figure 3.5:

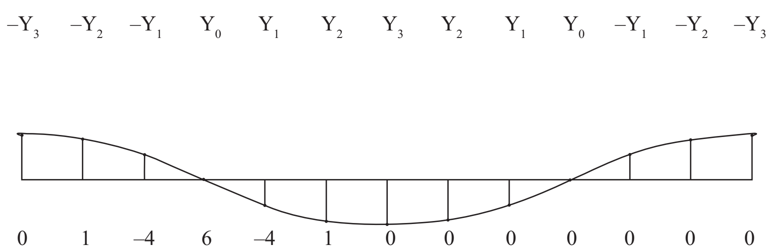

Figure 3.5. Example 3.10 Simple beam with difference operator.

$$q_0 = -\frac{EI}{h^4}\left((-y_2) - 4(-y_1) + 6y_0 - 4y_1 + y_2\right) = -\frac{EI}{h^4}(6y_0)$$

For the second equation, the central difference operator is placed on y_1 and is shown in Figure 3.6.

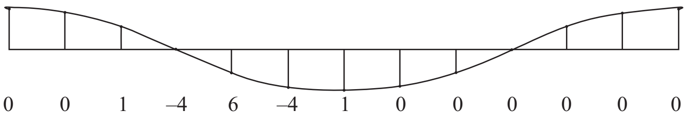

Figure 3.6. Example 3.10 Simple beam with difference operator.

$$q_1 = -\frac{EI}{h^4}\left(\left(-y_1\right) - 4y_0 + 6y_1 - 4y_2 + y_3\right) = -\frac{EI}{h^4}\left(-4y_0 + 5y_1 - 4y_2 + y_3\right)$$

The third and fourth equations can be written by placing the central difference operator on y_2 and y_3.

$$q_2 = -\frac{EI}{h^4}\left(y_0 - 4y_1 + 6y_2 - 4y_3 + y_2\right) = -\frac{EI}{h^4}\left(y_0 - 4y_1 + 7y_2 - 4y_3\right)$$
$$q_3 = -\frac{EI}{h^4}\left(y_1 - 4y_2 + 6y_3 - 4y_2 + y_1\right) = -\frac{EI}{h^4}\left(2y_1 - 8y_2 + 6y_3\right)$$

These four equations constitute a non-homogeneous linear algebraic set and can be written in matrix form.

$$-\frac{EI}{h^4}\begin{bmatrix} 6 & 0 & 0 & 0 \\ -4 & 5 & -4 & 1 \\ 1 & -4 & 7 & -4 \\ 0 & 2 & -8 & 6 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

From the conditions of the beam, two simplifications can be made. First, the load is uniform and all the values of q are the same. Second, the deflection at point 0 is known to be zero, so the first row and column can be eliminated since they correspond to those values.

$$-\frac{EI}{h^4}\begin{bmatrix} 5 & -4 & 1 \\ -4 & 7 & -4 \\ 2 & -8 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} q \\ q \\ q \end{bmatrix}$$

This can be solved by many of the methods presented in Chapter 2. The method of cofactors is used here, since the solution is small enough to solve determinants directly.

$$\frac{qh^4}{EI}\begin{bmatrix} 2.50 & 4.00 & 2.25 \\ 4.00 & 7.00 & 4.00 \\ 4.50 & 8.00 & 4.75 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

When the value of $h = L/6$ is substituted in and the matrix multiplication is performed, the following is a general solution for a simply supported beam with a uniformly distributed load:

$$\frac{qL^4}{1296EI}\begin{bmatrix} 8.75 \\ 15.00 \\ 17.25 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The particular solution for q=4 k/ft, L=25 ft, E=40,000 ksi, and I=1000 in^4 is then obtained. The exact solution at the center is y_3= 0.87891 inches, which is a 2.22% error.

$$\begin{bmatrix} 0.44573 \\ 0.78125 \\ 0.89855 \end{bmatrix} \text{inches} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Once the deflections are found, the other three desired values, θ, M, and V, can each be found from the corresponding central difference operators using the same order of error. The procedure is the same as the operator is laid upon each of the values that are unknown then the corresponding equation may be written. The relationship for rotation is as follows:

$$y_i' = \theta = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$\frac{1}{2h}\begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\frac{3}{L}\begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\frac{3}{L}\begin{bmatrix} 2y_1 \\ -y_0 + y_2 \\ -y_1 + y_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\begin{bmatrix} 0.00912 \\ 0.00781 \\ 0.00443 \\ 0.0000 \end{bmatrix} \text{radians} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

The exact solution at the end is θ_0=0.00936 radians, which is a 2.51% error, while the value in the center is exact. The relationship for moment is as follows:

$$y_i'' = -\frac{M}{EI} = \frac{y_{i+1} - 2y_i - y_{i-1}}{h^2}$$

$$-\frac{EI}{h^2}\begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$-\frac{36EI}{L^2}\begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$-\frac{36EI}{L^2}\begin{bmatrix} -2y_0 \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ 2y_2 - 2y_3 \end{bmatrix} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$\begin{bmatrix} 0.00 \\ 2083.4 \\ 3333.3 \\ 3750.0 \end{bmatrix} \text{kip-inches} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

The exact solution at the center is M_3=3750 k-in which is exact, while the values at the ends is also exact. The relationship for shear is as follows:

$$y_i''' = -\frac{V}{EI} = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3}$$

$$-\frac{EI}{2h^3}\begin{bmatrix} 0 & -4 & 2 & 0 \\ 2 & 1 & -2 & 1 \\ -1 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$-\frac{108EI}{L^3}\begin{bmatrix} 0 & -4 & 2 & 0 \\ 2 & 1 & -2 & 1 \\ -1 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$-\frac{108EI}{L^3}\begin{bmatrix} -4y_1 + 2y_2 \\ 2y_0 + y_1 - 2y_2 + y_3 \\ -y_0 + 2y_1 + y_2 - 2y_3 \\ 0 \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\begin{bmatrix} 35.27 \\ 34.92 \\ 19.90 \\ 0 \end{bmatrix} \text{kips} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

The exact solution at the end is $V_0 = 50.00\ k$, which is a 16.67% error, while the value in the center is exact. The large error at the end is due to the fact that the shear drops there, which creates a discontinuity in the equation. Using more segments would reduce the error, but it would still be more inaccurate than the other values.

The next example is similar to the previous one, but is included to show differences in modeling and accuracy. It also uses the higher order (smaller error) of error equations for more accuracy.

Example 3.11 Fixed beam with difference operator

Calculate the shear, moment, rotation, and deflection for a 30 ft long fixed end beam with a uniformly distributed load of 5 k/ft using central difference operator of order of error h^4 at 1/6th points. The beam has $E = 29{,}000$ ksi and $I = 1000\ \text{in}^4$.

The primary difference in the simply supported beam in Example 3.10 and the fixed end beam in this example is the model of the deflected curve beyond the boundary as shown in Figure 3.7.

The solution to this example is very similar to Example 3.10 and only the setup and solutions are presented. The central difference expressions with error of order h^4 will be used to solve for the values. Since the load is known, we will use the fourth derivative relationship between load and deflection.

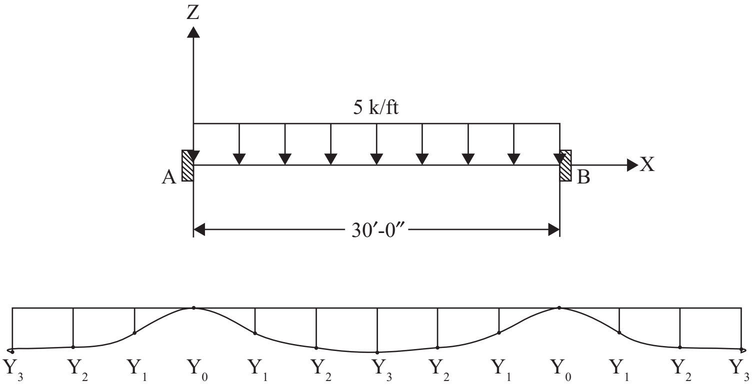

Figure 3.7. Example 3.11 Fixed beam with difference operator.

$$y_i'''' = -\frac{q}{EI} = \frac{-y_{i+3} + 12y_{i+2} - 39y_{i+1} + 56y_i - 39y_{i-1} + 12y_{i-2} - y_{i-3}}{6h^4}$$

$$q = -\frac{EI}{6h^4}\left(-y_{i+3} + 12y_{i+2} - 39y_{i+1} + 56y_i - 39y_{i-1} + 12y_{i-2} - y_{i-3}\right)$$

Placing the central difference operator on all four unknown points in the model, the linear non-homogeneous solution set can be obtained and solved.

$$-\frac{EI}{6h^4}\begin{bmatrix} 56 & -78 & 24 & -2 \\ -39 & 68 & -41 & 12 \\ 12 & -41 & 68 & -39 \\ -2 & 24 & -78 & 56 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$-\frac{EI}{6h^4}\begin{bmatrix} 68 & -41 & 12 \\ -41 & 68 & -39 \\ 24 & -78 & 56 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} q \\ q \\ q \end{bmatrix}$$

$$\frac{6qh^4}{EI}\begin{bmatrix} 766 & 1360 & 783 \\ 1360 & 3520 & 2160 \\ 1566 & 4320 & 2943 \end{bmatrix}\frac{1}{15120}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\frac{qL^4}{3265920EI}\begin{bmatrix} 2909 \\ 7040 \\ 8829 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 0.2150 \\ 0.5202 \\ 0.6524 \end{bmatrix} \text{inches} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The exact solution at the center is y_3=0.6284 inches, which is a 3.81% error. This is more error than the deflection found in Example 3.10, even though a more accurate operator was used. This is due to the fact that this physical model has more variation in deflection than that of the simply supported beam. The relationship for rotation is as follows:

$$y_i' = \theta = \frac{-y_{i+2} + 8y_{i+1} - 8y_{i-1} + y_{i-2}}{12h}$$

$$\frac{1}{12h}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -8 & 1 & 8 & -1 \\ 1 & -8 & -1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\frac{1}{2L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -8 & 1 & 8 & -1 \\ 1 & -8 & -1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\frac{1}{2L}\begin{bmatrix} 0 \\ -8y_0 + y_1 + 8y_2 - y_3 \\ y_0 - 8y_1 - y_2 + 8y_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\begin{bmatrix} 0.0000 \\ 0.0052 \\ 0.0041 \\ 0.0000 \end{bmatrix} radians = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

The solution at the end is θ_0=0.00000 *radians*, which is exact, while the value in the center is also exact. The relationship for moment is as follows:

$$y_i'' = -\frac{M}{EI} = \frac{-y_{i+2} + 16y_{i+1} - 30y_i + 16y_{i-1} - y_{i-2}}{12h^2}$$

$$-\frac{EI}{12h^2}\begin{bmatrix} -30 & 32 & -2 & 0 \\ 16 & -31 & 16 & -1 \\ -1 & 16 & -31 & 16 \\ 0 & -2 & 32 & -30 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$-\frac{3EI}{L^2}\begin{bmatrix} -30 & 32 & -2 & 0 \\ 16 & -31 & 16 & -1 \\ -1 & 16 & -31 & 16 \\ 0 & -2 & 32 & -30 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$-\frac{3EI}{L^2}\begin{bmatrix} -30y_0 + 32y_1 - 2y_2 \\ 16y_0 - 31y_1 + 16y_2 - y_3 \\ -y_0 + 16y_1 - 31y_2 + 16y_3 \\ -2y_1 + 32y_2 - 30y_3 \end{bmatrix} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$\begin{bmatrix} -3920 \\ -676 \\ 1510 \\ 2252 \end{bmatrix}\text{kip-inches} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

The exact solution at the end is $M_0 = -4500$ k-in, which is a 12.89% error, while the value in the center is 2250 k-in, which is a 0.11% error. The large error at the end is due to the fact that the moment drops there, which creates a discontinuity in the equation. Using more segments would reduce the error, but it would still be more inaccurate than the other values. The relationship for shear is as follows:

$$y_i''' = -\frac{V}{EI} = \frac{-y_{i+3} + 8y_{i+2} - 13y_{i+1} + 13y_{i-1} - 8y_{i-2} + y_{i-3}}{8h^3}$$

$$-\frac{EI}{8h^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 13 & -8 & -13 & 8 \\ -8 & 13 & 8 & -13 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$-\frac{27EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 13 & -8 & -13 & 8 \\ -8 & 13 & 8 & -13 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$-\frac{27EI}{L^3}\begin{bmatrix} 0 \\ 13y_0 - 8y_1 - 13y_2 + 8y_3 \\ -8y_0 + 13y_1 + 8y_2 - 13y_3 \\ 0 \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 54.76 \\ 25.60 \\ 0 \end{bmatrix} \text{kips} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

The exact solution at the end is V_0=75.00 *k*, which is a huge error, while the value in the center is exact. The large error at the end is due to the fact that the shear and moment drop dramatically at that point, which creates a discontinuity in the equation.

The final example of using difference operators to solve differential equations is the critical buckling load of a column. The critical buckling load of a pinned end column is sometimes included in strength of materials, but will be derived here. The derivations start with the differential equation of the elastic curve similar to a beam. The deflected column under the critical load is shown in Figure 3.8.

$$\frac{d^2y}{dx^2} = y'' = -\frac{M}{EI}$$

At any point *x* along the column, there is a deflection *y* that will produce an eccentric moment in the column equal to $P_{cr}y$. This is used in the equation of the elastic curve as follows:

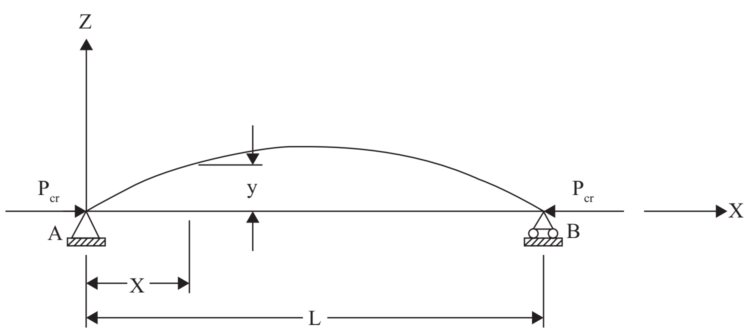

Figure 3.8. Numeric modeling with difference operators.

$$\frac{d^2y}{dx^2} = -\frac{P_{cr}}{EI}y$$

$$y'' + \frac{P_{cr}}{EI}y = 0$$

This is a second-order, linear, ordinary differential equation with the following general solution.

$$y = Acos(kx) + Bsin(kx)$$

$$\text{where } k = \sqrt{\frac{P_{cr}}{EI}}$$

The constants A and B can be evaluated using the boundary condition. At x=0, y=0, the equation becomes:

$$0 = A\cos(k0) + B\sin(k0)$$

This equation yields A=0 and at x=L, y=0, the equation becomes:

$$0 = B\sin(kL)$$

This condition is true when $B\sin(kL)=0$, which can only be true for three conditions as follows:

$B = 0$	No deflection
$kL = 0$	No load
$kL = \pi, 2\pi, 3\pi, \ldots, = n\pi$	where $n = 1, 2, 3, \ldots$

Therefore, the following can be found and is the critical buckling load:

$$kL = \sqrt{\frac{P_{cr}}{EI}}L = n\pi$$

$$\frac{P_{cr}}{EI}L^2 = n^2\pi^2$$

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2}$$

The lowest buckling mode corresponds to a value of $n = 1$, which is the case of single curvature shown. The other (higher) modes can also be found with the other values of n.

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

The critical bucking load is sometimes written as a critical buckling stress as follows:

$$F_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 EIr^2}{IL^2} = \frac{\pi^2 E}{\left(L^2/r^2\right)}$$

This is known as the Euler buckling stress and was derived by Leonhard Euler in 1757.

Example 3.12 Column buckling with difference operator

Calculate the critical buckling load, P_{cr}, for a 25 foot long fixed end column using central difference operator of order of error h^2 at 1/6th points. The column has $E = 29{,}000$ ksi and $I = 1000$ in^4.

This problem has the same model as Example 3.11 except an axial load is applied instead of a uniform lateral load and is shown in Figure 3.9.

The central difference expressions with error of order h^2 will be used to solve for the values. Since the load is known, we will use the fourth derivative relationship between load and deflection. This can be found

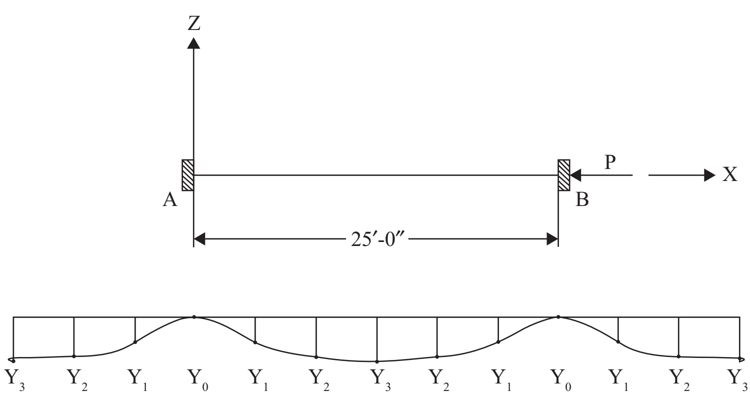

Figure 3.9. Example 3.12 Column buckling with difference operator.

by taking two derivatives from previously used deflection and moment differential equations.

$$y'' + \frac{P_{cr}}{EI} y = 0$$

$$y'''' + \frac{P_{cr}}{EI} y'' = 0$$

$$y'''' + Qy'' = 0 = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} + Q\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$0 = \frac{1}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right) + \frac{Q}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right)$$

The solution process is similar to Example 3.10 and 3.11 except that the two operators are placed on the model at the location of the unknown deflections. The value at point 0 of y_0 is zero and can be eliminated from the solutions. This becomes a homogeneous linear algebraic solution set.

$$\left(\frac{1296}{L^4}\begin{bmatrix} 7 & -4 & 1 \\ -4 & 7 & -4 \\ 2 & -8 & 6 \end{bmatrix} + 36\frac{Q}{L^2}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{bmatrix}\right)\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{252}{L^2} - 2Q & -\frac{144}{L^2} + Q & \frac{36}{L^2} \\ -\frac{144}{L^2} + Q & \frac{252}{L^2} - 2Q & -\frac{144}{L^2} + Q \\ \frac{72}{L^2} & -\frac{288}{L^2} + 2Q & \frac{216}{L^2} - 2Q \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A non-trivial solution to a homogeneous linear algebraic set exists if the determinant of the coefficient matrix is zero. Therefore, we can find Q by setting the determinant of the coefficient matrix equal to zero. This will be done using the basket weave method for a 3×3 matrix.

$$0 = \left(\frac{252}{L^2} - 2Q\right)\left(\frac{252}{L^2} - 2Q\right)\left(\frac{216}{L^2} - 2Q\right) + \left(-\frac{144}{L^2} + Q\right)\left(-\frac{144}{L^2} + Q\right)\left(\frac{72}{L^2}\right)$$
$$+\left(\frac{36}{L^2}\right)\left(-\frac{144}{L^2} + Q\right)\left(-\frac{288}{L^2} + 2Q\right) - \left(\frac{36}{L^2}\right)\left(\frac{252}{L^2} - 2Q\right)\left(\frac{72}{L^2}\right)$$

$$-\left(-\frac{144}{L^2}+Q\right)\left(-\frac{144}{L^2}+Q\right)\left(\frac{216}{L^2}-2Q\right)$$

$$-\left(\frac{252}{L^2}-2Q\right)\left(-\frac{144}{L^2}+Q\right)\left(-\frac{288}{L^2}+2Q\right)$$

$$0=\left(\frac{252}{L^2}-2Q\right)\left(\frac{54432}{L^4}-\frac{936Q}{L^2}+4Q^2-\frac{41472}{L^4}-\frac{576Q}{L^2}-2Q^2\right)$$

Multiplying the values for like terms yields the following:

$$0=\left(-\frac{144}{L^2}-Q\right)\left(-31104+\frac{504Q}{L^2}-2Q^2+\frac{10386}{L^4}-\frac{72Q}{L^2}\right)$$
$$+\left(\frac{36}{L^2}\right)\left(\frac{41472}{L^4}-\frac{576Q}{L^2}+2Q^2-\frac{18144}{L^4}-\frac{144Q}{L^2}\right)$$

Multiplying all the values and combining like terms:

$$0=\frac{1119744}{L^6}-\frac{49248}{L^4}Q+\frac{576}{L^2}Q^2-2Q^3$$

The solution to these cubic equations yields the following general value of Q:

$$Q=\frac{P_{cr}}{EI}=0.0004$$

Substituting in the particular values for the column of E and I gives P_{cr}=11,600 kips. The exact values using the Euler buckling equation with an effective length factor of k=0.5 for a fixed end column is P_{cr}=12,721 kips. This central difference solution has an error of 8.8%.

3.10 PARTIAL DIFFERENTIAL EQUATION DIFFERENCE OPERATORS

More advanced structural analysis problems may require the solution of partial differential equations. One example is the bending of a plate under uniform lateral load. “Theory of Plates and Shells” by Timoshenko and Woinowsky-Krieger (1959) contains an exact solution to general plates. The differential relationship for plate bending uses partial differential equations. Their difference operators can be derived from the basic difference operators using two basic principles. If you add two differential equations you simply add the location of the operator at i, known as the pivot point. To take the product of two differential equations, you must multiply the value of the operator at i of one differential to each of the terms in the other differential. Partial differentials also require both the x and y direction, so they will be written horizontally and vertically. The following are six examples of creating partial difference operators for the equation used to solve plate bending. The partial differential equation is listed first in each of the six examples followed by the basic operators that represent the equation.

Example 3.13 Partial difference operator

$$\frac{\delta^4 z}{dx^4} + 2\frac{\delta^4 z}{dx^2 dy^2} + \frac{\delta^4 z}{dy^4} = \left[\frac{1}{h^4}\begin{pmatrix}1 & -4 & 6 & -4 & 1\end{pmatrix}\right] + 2*\left[\frac{1}{h^2}\begin{pmatrix}1 & -2 & 1\end{pmatrix}*\frac{1}{h^2}\begin{pmatrix}1\\-2\\1\end{pmatrix}\right] + \left[\frac{1}{h^4}\begin{pmatrix}1\\-4\\6\\-4\\1\end{pmatrix}\right]$$

$$= \left[\frac{1}{h^4}\begin{pmatrix}1 & -4 & 6 & -4 & 1\end{pmatrix}\right] + 2*\left[\frac{1}{h^4}\begin{pmatrix}1 & -2 & 1\\-2 & 4 & -2\\1 & -2 & 1\end{pmatrix}\right] + \left[\frac{1}{h^4}\begin{pmatrix}1\\-4\\6\\-4\\1\end{pmatrix}\right]$$

$$= \frac{1}{h^4}\begin{pmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1\\ & 2 & -8 & 2 & \\ & & 1 & & \end{pmatrix}$$

$$\frac{q}{D} = \frac{\delta^4 z}{dx^4} + 2\frac{\delta^4 z}{dx^2 dy^2} + \frac{\delta^4 z}{dy^4} = \frac{1}{h^4}\begin{pmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1\\ & 2 & -8 & 2 & \\ & & 1 & & \end{pmatrix} \tag{3.12}$$

The value D is the flexural rigidity of the plate and is equal to the following:

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

The value of Poisson's ratio, υ, relates the elastic modulus to the shear modulus and is given as the following from strength of materials:

$$G = \frac{E}{2\left(1+\nu\right)}$$

Example 3.14 Partial difference operator

$$\frac{\delta^2 z}{dx^2} + \nu\frac{\delta^2 z}{dy^2} = \left[\frac{1}{h^2}\begin{pmatrix}1 & -2 & 1\end{pmatrix}\right] + \left[\frac{1}{h^2}\begin{pmatrix}\nu \\ -2\nu \\ \nu\end{pmatrix}\right] = \frac{1}{h^2}\begin{pmatrix} & \nu & \\ 1 & -2\left(1+\nu\right) & 1 \\ & \nu & \end{pmatrix}$$

$$M_x = -D\left[\frac{\delta^2 z}{dx^2} + \nu\frac{\delta^2 z}{dy^2}\right] = \frac{1}{h^2}\begin{pmatrix} & \nu & \\ 1 & -2\left(1+\nu\right) & 1 \\ & \nu & \end{pmatrix} \tag{3.13}$$

Example 3.15 Partial difference operator

$$\nu\frac{\delta^2 z}{dx^2} + \frac{\delta^2 z}{dy^2} = \left[\frac{1}{h^2}\begin{pmatrix}\nu & -2\nu & \nu\end{pmatrix}\right] + \left[\frac{1}{h^2}\begin{pmatrix}1 \\ -2 \\ 1\end{pmatrix}\right] = \frac{1}{h^2}\begin{pmatrix} & 1 & \\ \nu & -2\left(1+\nu\right) & \nu \\ & 1 & \end{pmatrix}$$

$$M_y = -D\left[\frac{\delta^2 z}{dx^2} + \frac{\delta^2 z}{dy^2}\right] = \frac{1}{h^2}\begin{pmatrix} & 1 & \\ \nu & -2\left(1+\nu\right) & \nu \\ & 1 & \end{pmatrix} \tag{3.14}$$

Example 3.16 Partial difference operator

$$\frac{\delta^2 z}{dxdy} = \left[\frac{1}{2h}\begin{pmatrix}-1 & 0 & 1\end{pmatrix}\right] * \left[\frac{1}{2h}\begin{pmatrix}1 \\ 0 \\ -1\end{pmatrix}\right] = \frac{1}{4h^2}\begin{pmatrix}-1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1\end{pmatrix}$$

$$M_{xy} = D(1-\nu)\left[\frac{\delta^2 z}{dxdy}\right] = \frac{1}{4h^2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \tag{3.15}$$

$$M_{yx} = -D(1-\nu)\left[\frac{\delta^2 z}{dxdy}\right] = \frac{1}{4h^2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{3.16}$$

Example 3.17 Partial difference operator

$$\frac{\delta^3 z}{dx^3} + \frac{\delta^3 z}{dxdy^2} = \left[\frac{1}{2h^3}\begin{pmatrix} -1 & 2 & 0 & -2 & 1 \end{pmatrix}\right] + \left[\frac{1}{2h}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}\right] * \left[\frac{1}{h^2}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right]$$

$$= \left[\frac{1}{2h^3}\begin{pmatrix} -1 & 2 & 0 & -2 & 1 \end{pmatrix}\right] + \left[\frac{1}{2h^3}\begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{2h^3}\begin{pmatrix} & -1 & 0 & 1 & \\ -1 & 4 & 0 & -4 & 1 \\ & -1 & 0 & 1 & \end{pmatrix}$$

$$Q_x = -D\left[\frac{\delta^3 z}{dx^3} + \frac{\delta^3 z}{dxdy^2}\right] = \frac{1}{2h^3}\begin{pmatrix} & -1 & 0 & 1 & \\ -1 & 4 & 0 & -4 & 1 \\ & -1 & 0 & 1 & \end{pmatrix} \tag{3.17}$$

Example 3.18 Partial difference operator

$$\frac{\delta^3 z}{dx^2 dy} + \frac{\delta^3 z}{dy^3} = \left[\frac{1}{h^2}\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}\right] * \left[\frac{1}{2h}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right] + \left[\frac{1}{2h^3}\begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \\ -1 \end{pmatrix}\right]$$

$$\left[\frac{1}{2h^3}\begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{pmatrix}\right] + \left[\frac{1}{2h^3}\begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \\ -1 \end{pmatrix}\right] = \frac{1}{2h^3}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}$$

$$Q_y = -D\left[\frac{\delta^3 z}{dx^2 dy} + \frac{\delta^3 z}{dy^3}\right] = \frac{1}{2h^3}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix} \tag{3.18}$$

3.11 NUMERIC MODELING WITH PARTIAL DIFFERENCE OPERATORS

The differential equation and the difference operators from Section 3.10 are used in the following example to solve for plate bending. The forces on a middle surface plate are shown in Figure 3.10.

Example 3.19 Plate bending

Calculate the shear, moment, rotation, and deflection for a 1 inch thick, 20 inch square, fixed end steel plate with a uniformly distributed load of 100 psi using central difference operator of order of error h^2 at 1/10th points. The plate has E = 29,000 ksi and υ = 0.25.

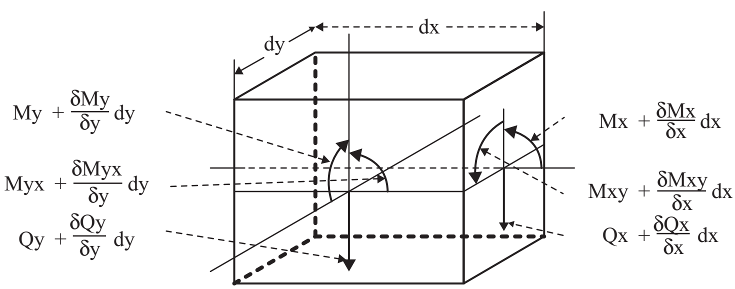

Figure 3.10. Numeric modeling with partial difference operators.

The plate will be divided into 10 segments in the horizontal and vertical directions. Since the plate is square with a uniform load and fixed on all edges, it has horizontal and vertical symmetry. It is also symmetric about both diagonals of the plate. The symmetry will result in only 21 unique points on the plate for the 121 possible locations. This is shown in Figure 3.11.

The difference operator for Equation 3.12 is repeated here and can be shown in graphical form in Figure 3.12.

$$\frac{q}{D} = \frac{\delta^4 z}{dx^4} + 2\frac{\delta^4 z}{dx^2 dy^2} + \frac{\delta^4 z}{dy^4} = \frac{1}{h^4}\begin{pmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{pmatrix}$$

The operator is laid upon each of the 21 unique points on the plate resulting in 21 equations involving the deflections. Figure 3.13 shows the operator laid upon point 1.

The result will be a non-homogenous set of 21 equations involving the unknown deflections. Since the plate is fixed at the boundaries, the

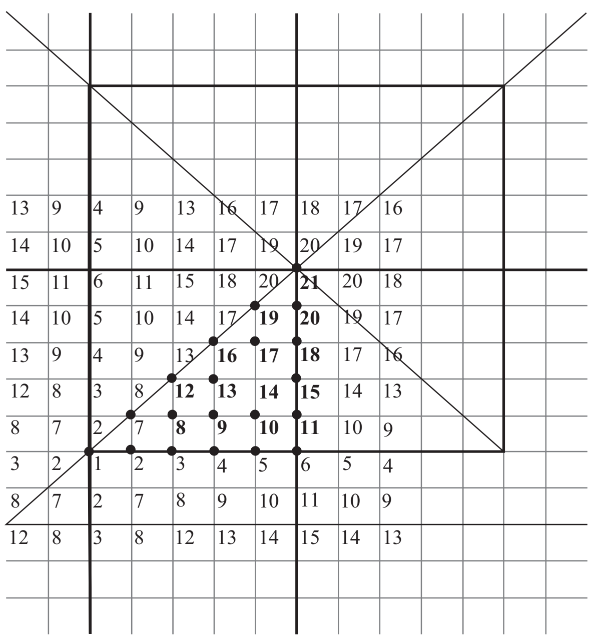

Figure 3.11. Example 3.19 Plate bending.

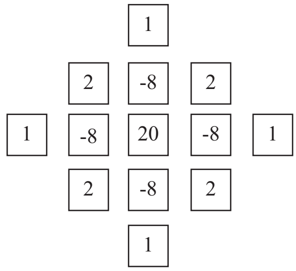

Figure 3.12. Example 3.19 Plate bending.

deflections at points 1 through 6 are zero and may be eliminated. The matrix equation is shown in Table 3.17.

The solution can be found using one of the methods from Chapter 2 on non-homogeneous linear algebraic equations. The deflections are shown in Table 3.18.

All of the other values can be found once the deflections are known. Table 3.19 shows the matrix equation for finding M_x and Table 3.20 shows the solution to M_x and the normal stress.

$$M_x = -D\left[\frac{\delta^2 z}{dx^2} + \nu\,\frac{\delta^2 z}{dy^2}\right] = \frac{1}{h^2}\begin{pmatrix} & \nu & \\ 1 & -2(1+\nu) & 1 \\ & \nu & \end{pmatrix}$$

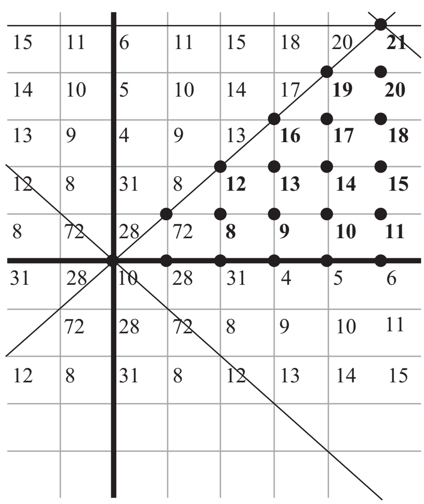

Figure 3.13. Example 3.19 Plate bending.

Table 3.17. Example 3.19 Plate bending

22	-16	2	0	0	2	0	0	0	0	0	0	0	0	0	z_7	=	q_7	*	$\frac{h^4}{D}$
-8	23	-8	1	0	-8	3	0	0	0	0	0	0	0	0	z_8		q_8		
1	-8	21	-8	1	2	-8	2	0	1	0	0	0	0	0	z_9		q_9		
0	1	-8	22	-8	0	2	-8	2	0	1	0	0	0	0	z_{10}		q_{10}		
0	0	2	-16	21	0	0	4	-8	0	0	1	0	0	0	z_{11}		q_{11}		
2	-16	4	0	0	20	-16	2	0	2	0	0	0	0	0	z_{12}		q_{12}		
0	3	-8	2	0	-8	22	-8	1	-8	3	0	0	0	0	z_{13}		q_{13}		
0	0	2	-8	2	1	-8	21	-8	2	-8	2	1	0	0	z_{14}		q_{14}		
0	0	0	4	-8	0	2	-16	20	0	4	-8	0	1	0	z_{15}		q_{15}		
0	0	2	0	0	2	-16	4	0	20	-16	2	2	0	0	z_{16}		q_{16}		
0	0	0	1	0	0	3	-8	2	-8	23	-8	-8	3	0	z_{17}		q_{17}		
0	0	0	0	1	0	0	4	-8	2	-16	20	4	-8	1	z_{18}		q_{18}		
0	0	0	0	0	0	0	2	0	2	-16	4	22	-16	2	z_{19}		q_{19}		
0	0	0	0	0	0	0	0	1	0	6	-8	-16	25	-8	z_{20}		q_{20}		
0	0	0	0	0	0	0	0	0	0	0	4	8	-32	20	z_{21}		q_{21}		

Table 3.18. Example 3.19 Plate bending

z_7	=	0.000291121	inches
z_8		0.000749271	
z_9		0.001156404	
z_{10}		0.001421248	
z_{11}		0.001511939	
z_{12}		0.001945784	
z_{13}		0.003037557	
z_{14}		0.003762980	
z_{15}		0.004014203	
z_{16}		0.004780224	
z_{17}		0.005951021	
z_{18}		0.006358843	
z_{19}		0.007429735	
z_{20}		0.007946380	
z_{21}		0.008501580	

Table 3.19. Example 3.19 Plate bending

-2.5	2.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_1	$* \; -\frac{D}{h^2}$
1.0	-2.5	1.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_2	
0.0	1.0	-2.5	1.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_3	
0.0	0.0	1.0	-2.5	1.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_4	
0.0	0.0	0.0	1.0	-2.5	1.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_5	
0.0	0.0	0.0	0.0	2.0	-2.5	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_6	
0.0	1.3	0.0	0.0	0.0	0.0	-2.5	1.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_7	
0.0	0.0	0.3	0.0	0.0	0.0	1.0	-2.5	1.0	0.0	0.0	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_8	
0.0	0.0	0.0	0.3	0.0	0.0	0.0	1.0	-2.5	1.0	0.0	0.0	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_9	
0.0	0.0	0.0	0.0	0.3	0.0	0.0	0.0	1.0	-2.5	1.0	0.0	0.0	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_{10}	
0.0	0.0	0.0	0.0	0.0	0.3	0.0	0.0	0.0	2.0	-2.5	0.0	0.0	0.0	0.3	0.0	0.0	0.0	0.0	0.0	0.0	z_{11}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.3	0.0	0.0	0.0	-2.5	1.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	z_{12}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	0.0	0.0	1.0	-2.5	1.0	0.0	0.3	0.0	0.0	0.0	0.0	0.0	z_{13}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	0.0	0.0	1.0	-2.5	1.0	0.0	0.3	0.0	0.0	0.0	0.0	z_{14}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	0.0	0.0	2.0	-2.5	0.0	0.0	0.3	0.0	0.0	0.0	z_{15}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.3	0.0	0.0	-2.5	1.3	0.0	0.0	0.0	0.0	z_{16}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	0.0	1.0	-2.5	1.0	0.3	0.0	0.0	z_{17}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	0.0	2.0	-2.5	0.0	0.3	0.0	z_{18}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.3	0.0	-2.5	1.3	0.0	z_{19}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	2.0	-2.5	0.3	z_{20}	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.5	-2.5	z_{21}	

Table 3.20. Example 3.19 Plate bending

$M_x =$	0.000000000	k-in	$\sigma_x =$	0.00000	ksi
	-0.093805542			-0.56283	
	-0.241431854			-1.44859	
	-0.372619154			-2.23571	
	-0.457957773			-2.74775	
	-0.487180224			-2.92308	
	-0.134551926			-0.80731	
	-0.039177480			-0.23506	
	-0.025067632			-0.15041	
	-0.036067778			-0.21641	
	-0.042662612			-0.25598	
	0.084373209			0.50624	
	0.258403727			1.55042	
	0.330357218			1.98214	
	0.349192954			2.09516	
	0.460673396			2.76404	
	0.605975261			3.63585	
	0.647615145			3.88569	
	0.774999710			4.65000	
	0.832219502			4.99332	
	0.894489044			5.36693	

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CHAPTER 4

Matrix Structural Stiffness

Building structures are made up of columns, beams, girders, joists, slabs, walls, shells, and many other components that act together to resist the loads placed on them. These members can be at various orientations, but they must be represented in a common mathematical form. The structural stiffness method is used to represent members, loads, support constraints, and other components of a building structure in a consistent manner. This chapter will focus on linearly elastic members subjected to axial, bending, shear, and torsional forces.

4.1 MATRIX TRANSFORMATIONS AND COORDINATE SYSTEMS

When using the structural stiffness method, the *global* Cartesian right-hand coordinate system will be used to organize the system. This system was developed in 1637 by René Descartes (Descartes 1637). We will denote these three orthogonal axes as X, Y, and Z. You could represent these as your thumb, fore finger, and middle finger on your right hand. Individual members may not be at the same orientation as the global system. All members have their own *local* system represented by x, y, and z. This coordinate system is also a Cartesian right-hand system with the x axis running along the member length. Both of these systems are shown in Figure 4.1.

This chapter provides techniques to reference causes and effects by coordinate systems and how to manipulate these coordinate systems. These manipulations allow changes from one system to another and are called *transformations*. The two basic transformations that are done in

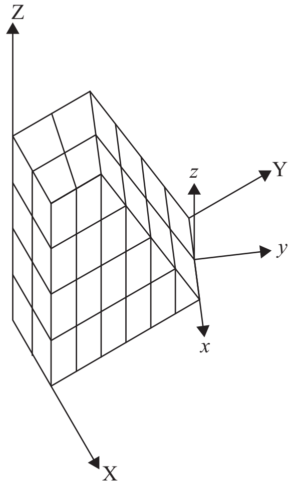

Figure 4.1. Coordinate systems.

the following sections are transmission and rotations. *Transmission* is moving effects from one point to another. *Rotations* are used to re-orient axes about a specific point. These techniques will be used in determining the stiffness of a member (local) and developing the global (joint) stiffness solution.

4.2 ROTATION MATRIX

Rotation can take place about any of the three global axes. The following three examples derive the rotation transformations. The alpha, α, rotation is a rotation about the global Z axis from the global system to the local system. Rotation about the Y axis is a beta, β, rotation and rotation about the X axis is a gamma, γ, rotation.

Example 4.1 Rotation matrix, α

Derive the alpha, α, rotation matrix.

The following variables are represented in Figure 4.2 and are used to develop α. The location of z remains unchanged since rotation is occurring about that axis.

α = rotation about global Z axis from the global to the local system
(x_0, y_0) = global coordinate location
(x_1, y_1) = local coordinate location

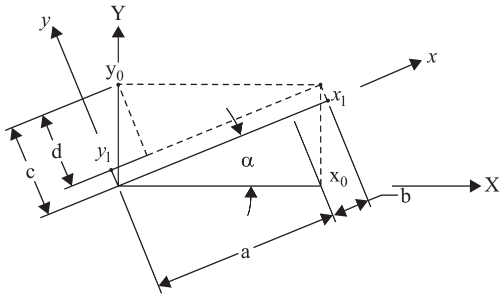

Figure 4.2. Example 4.1 Rotation, *α*.

$$a = x_0 cosa$$
$$b = y_0 sina$$
$$c = y_0 cosa$$
$$d = x_0 sina$$
$$x_l = a + b = x_0 cosa + y_0 sina$$
$$y_l = c - d = y_0 cosa - x_0 sina$$
$$z_l = z_0$$

The equations for *x*, *y*, and *z* can be represented in matrix form as follows:

$$\begin{bmatrix} cosa & sina & 0 \\ -sina & cosa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} \tag{4.1}$$

Example 4.2 Rotation matrix, β

Derive the beta, *β*, rotation matrix.

The following variables are represented in Figure 4.3 and are used to develop *β*. The location of *y* remains unchanged since rotation is occurring about that axis.

β = rotation about global *Y* axis from the global to the local system
(x_0, z_0) = global coordinate location
(x_1, z_1) = local coordinate location

$$a = x_0 cos\beta$$
$$b = z_0 sin\beta$$
$$c = z_0 cos\beta$$
$$d = x_0 sin\beta$$

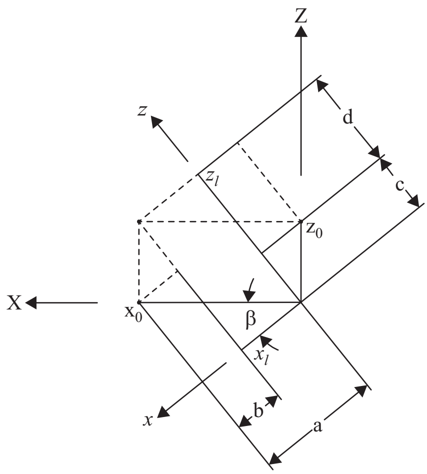

Figure 4.3. Example 4.2 Rotation, β.

$$x_l = a - b = x_0 cos\beta - z_0 sin\beta$$
$$z_l = c + d = z_0 cos\beta + x_0 sin\beta$$
$$y_l = y_0$$

The equations for x, y, and z can be represented in matrix form as follows:

$$\begin{bmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} \tag{4.2}$$

Example 4.3 Rotation matrix, γ

Derive the gamma, γ, rotation matrix.

The following variables are represented in Figure 4.4 and are used to develop γ. The location of x remains unchanged since rotation is occurring about that axis.

γ = rotation about global X axis from the global to the local system
(y_0, z_0) = global coordinate location
(y_l, z_l) = local coordinate location

$$a = y_0 cos\gamma$$
$$b = z_0 sin\gamma$$
$$c = z_0 cos\gamma$$
$$d = y_0 sin\gamma$$

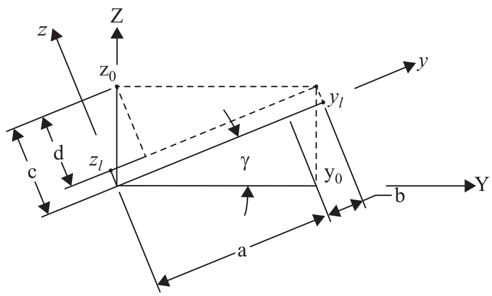

Figure 4.4. Example 4.3 Rotation, γ.

$$y_l = a + b = y_0 cos\gamma + z_0 sin\gamma$$
$$z_l = c - d = z_0 cos\gamma - y_0 sin\gamma$$
$$x_l = x_0$$

The equations for x, y, and z can be represented in matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\gamma & sin\gamma \\ 0 & -sin\gamma & cos\gamma \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} \tag{4.3}$$

The rotation matrix from the global to the local system, $[R_{0l}]$, is found from these rotations about the global Z, Y, and X axes, in that order. The order of the matrix multiplication goes from right to left. Alpha is first multiplied by beta and the resultant is multiplied by gamma.

$$[R_{0l}] = [\gamma][\beta][a]$$

Substituting Equations 4.1 through 4.3 for $[\gamma]$, $[\beta]$, and $[a]$, respectively, results in the following:

$$[R_{0l}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\gamma & sin\gamma \\ 0 & -sin\gamma & cos\gamma \end{bmatrix} \begin{bmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{bmatrix} \begin{bmatrix} cosa & sina & 0 \\ -sina & cosa & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_{0l}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\gamma & sin\gamma \\ 0 & -sin\gamma & cos\gamma \end{bmatrix} \begin{bmatrix} cosacos\beta & sinacos\beta & -sin\beta \\ -sina & cosa & 0 \\ cosasin\beta & sinasin\beta & cos\beta \end{bmatrix}$$

$$[R_{0l}] = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\beta\sin\gamma \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

Rotation about the axes from the local to the global system is simply a reverse operation. It can be shown that the resulting rotation matrices for this transformation are the transpose of the rotations from the global axis to the local axis. This is known as a *symmetric transformation*. The following relationships show those basic expressions:

$$[\alpha_{l0}] = [\alpha_{0l}]^T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\beta_{l0}] = [\beta_{0l}]^T = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$[\gamma_{l0}] = [\gamma_{0l}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

The rotation matrix from local to global system, $[R_{l0}]$, can be found from the individual rotations or directly from $[R_{0l}]$.

$$[R_{l0}] = [R_{0l}]^T = [\alpha]^T [\beta]^T [\gamma]^T$$

$$[R_{l0}] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$[R_{l0}] = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

$$[R_{l0}] = [R_{0l}]^T = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\beta\sin\gamma \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}^T$$

These rotation transformations will be used later to relate the local member stiffness to the global member stiffness.

4.3 TRANSMISSION MATRIX

In addition to knowing the local components of global forces, the effect a force applied to one end of a member has on the other end is important. It is often necessary to state the *effect* at one point in a structural system due to a *cause* known to exist at some other point in the system. This is where the transmission transformation is used. In Figure 4.5, the cause of a force at point 2 is transmitted to the effect at point 1. This is achieved by using an equivalent static force at 1. In the study of rigid body equilibrium, this is stated as $\Sigma F_1 = \Sigma F_2$.

The six orthogonal forces at a point are shown in Figure 4.6. These are the forces and moments in each direction X, Y, and Z. The moments are represented with double arrowheads.

The static equivalent force system is found where the moment arm distances $(x_2 - x_1)$, $(y_2 - y_1)$, and $(z_2 - z_1)$ are measured from the *effect* point, 1, to the *cause* point, 2.

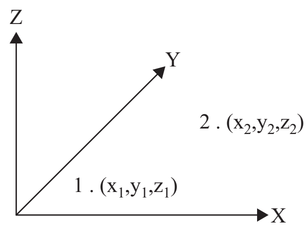

Figure 4.5. Transformation locations.

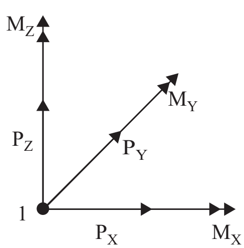

Figure 4.6. Orthogonal forces.

$$\Sigma F_{x1} = \Sigma F_{x2} \therefore P_{x1} = P_{x2}$$
$$\Sigma F_{y1} = \Sigma F_{y2} \therefore P_{y1} = P_{y2}$$
$$\Sigma F_{z1} = \Sigma F_{z2} \therefore P_{z1} = P_{z2}$$
$$\Sigma M_{x1} = \Sigma M_{x2} \therefore M_{x1} = M_{x2} - P_{y2}\left(z_2 - z_1\right) + P_{z2}\left(y_2 - y_1\right)$$
$$\Sigma M_{y1} = \Sigma M_{y2} \therefore M_{y1} = M_{y2} + P_{x2}\left(z_2 - z_1\right) - P_{z2}\left(x_2 - x_1\right)$$
$$\Sigma M_{z1} = \Sigma M_{z2} \therefore M_{z1} = M_{z2} - P_{x2}\left(y_2 - y_1\right) + P_{y2}\left(x_2 - x_1\right)$$

We can simplify the moment arm distances as x, y, and z. The concept of transmitting *cause* to *effect* may be denoted in matrix form as $[T]$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -z & y & 1 & 0 & 0 \\ z & 0 & -x & 0 & 1 & 0 \\ -y & x & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_{x2} \\ P_{y2} \\ P_{z2} \\ M_{x2} \\ M_{y2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} P_{x1} \\ P_{y1} \\ P_{z1} \\ M_{x1} \\ M_{y1} \\ M_{z1} \end{bmatrix} \tag{4.4}$$

Example 4.4 Transmission matrix

Determine the transmission matrix for the coplanar XY system from the origin end, i, to other end, j.

The cause end is i as shown in Figure 4.7 and the effect end is j at the right end a distance L away.

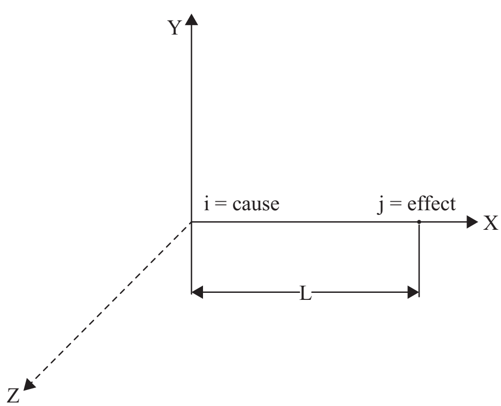

Figure 4.7. Transformation effects.

The distance x is measured from the cause location to the effect location.

$$\text{x} = \left(x_c - x_e\right) = \left(x_i - x_j\right) = -\text{L}$$

This can be substituted into Equation 4.4 and the unneeded distances and forces are removed to get a 3×3 matrix involving only P_x, P_y, and M_z.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & L & 0 & 1 & 0 \\ 0 & -L & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_{x2} \\ P_{y2} \\ P_{z2} \\ M_{x2} \\ M_{y2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} P_{x1} \\ P_{y1} \\ P_{z1} \\ M_{x1} \\ M_{y1} \\ M_{z1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -L & 1 \end{bmatrix} \begin{bmatrix} P_{xi} \\ P_{yi} \\ M_{zi} \end{bmatrix} = \begin{bmatrix} P_{xj} \\ P_{yj} \\ M_{zj} \end{bmatrix}$$

The axial force in the member, P_x, is a direct transmission, whereas the shear, P_y, and bending, M_z, forces are linked. The transmission for just the shear and bending will be used later in the chapter and can be written as follows:

$$\begin{bmatrix} 1 & 0 \\ -L & 1 \end{bmatrix} \begin{bmatrix} P_{yi} \\ M_{zi} \end{bmatrix} = \begin{bmatrix} P_{yj} \\ M_{zj} \end{bmatrix} \tag{4.5}$$

A similar transmission matrix can be derived for shear and bending in the XZ system.

$$\begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} P_{zi} \\ M_{yi} \end{bmatrix} = \begin{bmatrix} P_{zj} \\ M_{yj} \end{bmatrix} \tag{4.6}$$

4.4 AREA MOMENT METHOD

Area moment and the next five sections are review topics on structural analysis. These topics will be used to derive many of the equations and relationships for stiffness methods of analysis. The *area moment method* of analysis is based on the basic equation of beam bending, $\phi = M/EI$, and consists of two theorems. The theorems were developed by Christian Otto Mohr in 1874 (Timoshenko 1953). The first theorem of area moment states the change in slope between two points on the elastic curve, 1 and 2, is equal to the area of the M/EI diagram between the points 1 and 2.

$$\Delta\theta_{12} = \int_1^2 \frac{M}{EI} dx$$

The second theorem of area moment states that the tangential deviation of a point, 1, from a tangent to the elastic curve at point 2 (the tangent) is equal to the moment of the area of the M/EI diagram between points 1 and 2, taken about point 1 (the point). Note that the moment of the area is the area times a distance, $\bar{x}_1$, from point 1.

$$t_{12} = \int_1^2 \frac{M}{EI} \bar{x}_1 dx$$

Example 4.5 Area moment

Find the reactions on the following propped end, uniformly loaded beam shown in Figure 4.8 using the area moment method.

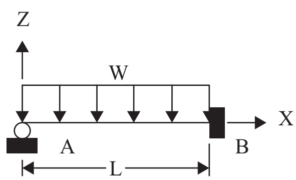

Figure 4.8. Example 4.5 Area moment.

A free-body diagram of the beam is drawn in Figure 4.9.

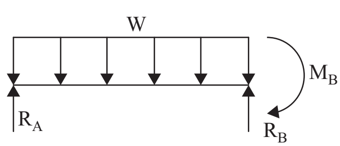

Figure 4.9. Example 4.5 Area moment.

Moment divided by *EI*, *M*/*EI*, diagrams are drawn by parts. Moment diagrams by parts assume each force is acting about point B as if it was the only force. The vertical reaction on the left, R_A, is used as the redundant force. The method of superposition will be used to solve for the assumed redundant, R_A. The reactions at the fixed-end on the right can be written in terms of the load, *w*, as follows from equilibrium on Figure 4.9 (ignoring R_A):

$$\Sigma M_B = 0 = wL\left(\frac{L}{2}\right) - M_B \therefore M_B = \frac{wL^2}{2}$$

$$\Sigma F_y = 0 = -wL + R_B \therefore R_B = wL$$

The *M*/*EI* diagram and the deflected shape for the uniformly distributed load, *w*, is shown in Figure 4.10.

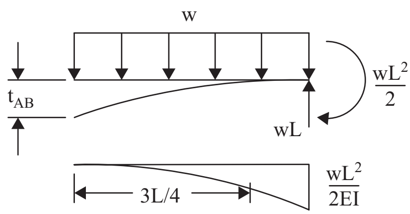

Figure 4.10. Example 4.5 Area moment.

The reactions at the fixed-end on the right can be written in terms of the vertical reaction, R_A, as follows from equilibrium on Figure 4.9 (ignoring the distributed load):

$$\Sigma M_B = 0 = -R_A L - M_B \therefore M_B = -R_A L$$

$$\Sigma F_y = 0 = R_A + R_B \therefore R_B = -R_A$$

The *M*/*EI* diagram and the deflected shape for the vertical reaction, R_A, is shown in Figure 4.11.

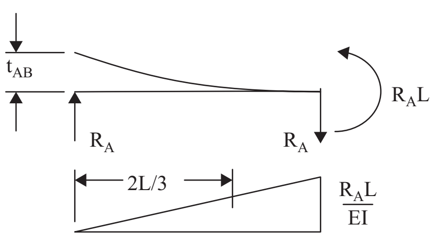

Figure 4.11. Example 4.5 Area moment.

Since point A is pinned, we know that the deflection at point A, Δ_A, is equal to zero. We can superimpose Figures 4.10 and 4.11. Applying the second area moment theorem, the following can be written. It should be noted that the tangent to the deflected shape is at B and therefore the point is at A:

$$\Delta_A = 0 = t_{AB} = \int_A^B \frac{M}{EI}\bar{x}_A dx = \Sigma A\bar{x}_1$$

$$0 = \left(-\frac{1}{3}L\frac{wL^2}{2EI}\right)\left(\frac{3L}{4}\right) + \left(\frac{1}{2}L\frac{R_A L}{EI}\right)\left(\frac{2L}{3}\right) \therefore R_A = \frac{3wL}{8}$$

From statics on the original free-body diagram in Figure 4.9, the other reactions at B can be found.

$$\Sigma F_y = 0 = R_A + R_B - wL \therefore R_B = \frac{5wL}{8}$$

$$\Sigma M_B = 0 = -R_A L - M_B + \frac{wL^2}{2} \therefore M_B = \frac{wL^2}{8}$$

4.5 CONJUGATE BEAM METHOD

The conjugate beam method is based on the equation of beam bending ϕ = M/EI. The method was developed by Heinrich Müller-Breslau in 1865 (Müller-Breslau 1875). A *conjugate beam* can be summarized as an imaginary beam equal in length to the real beam. In the imaginary beam, the shear at the conjugate support is equal to the slope of the real support. Also, the moment at the conjugate support is equal to that of the deflection at the real support. The conjugate beam is loaded with the M/EI diagram from the real beam. A summary of the more common support conditions is shown in Figure 4.12.

Take the pinned condition for example. In the real beam, the rotation is unknown (exist) and deflection is equal to zero. Therefore, in the conjugate beam, the shear is unknown (exist) and the moment is equal, thus creating a pinned connection. This condition does not change from the real beam to the conjugate beam. However, this is not true for all cases. For example, if the real beam is fixed, the rotation is zero and the deflection is zero. The result is zero moment and shear in the conjugate that yields a free end.

Example 4.6 Conjugate beam

Draw the conjugate beam for each of the real beams in Figure 4.13.

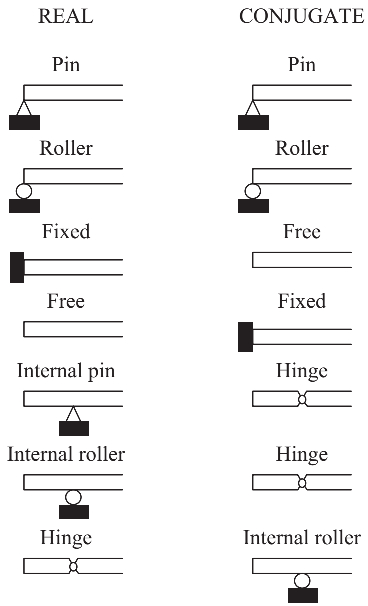

Figure 4.12. Conjugate versus real supports.

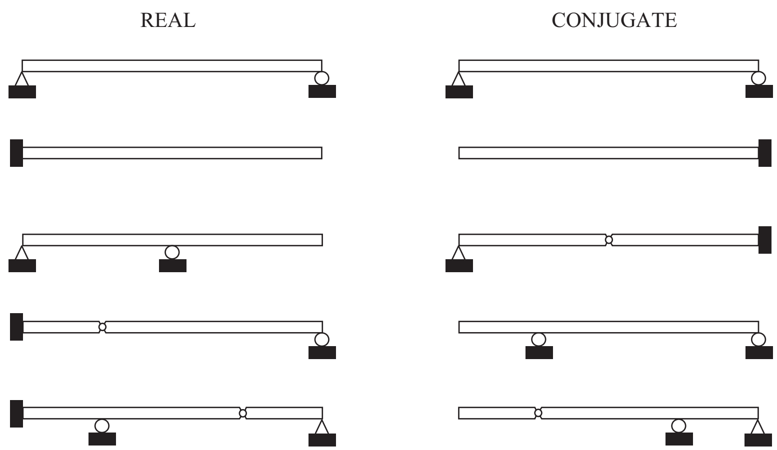

Figure 4.13. Conjugate versus real beams.

Example 4.7 Conjugate beam

Find the slope and deflection at the free end of the cantilever beam in Figure 4.14 by the conjugate beam method.

A free-body diagram of the real beam is drawn with the reactions at the fixed-end found from static equilibrium. The conjugate beam is drawn observing that the shear in the conjugate beam equals the rotations in the real beam and the moment in the conjugate equals the deflection in the real beam. Therefore, the fixed-end of the real beam becomes free in the conjugate beam and the free end in the real beam becomes fixed in the conjugate beam. The conjugate beam is loaded with the *M*/*EI* diagram of the real beam. This is a triangular moment diagram due to the applied load P divided by EI. The last beam in Figure 4.14 is a free-body diagram of the conjugate beam with the area of the *M*/*EI* diagram equated to a point load located at the centroid of the diagram. Static equilibrium can then be performed on the conjugate beam to find shear and moment at point B, which represents rotation and deflection in the real beam shown in Figure 4.15.

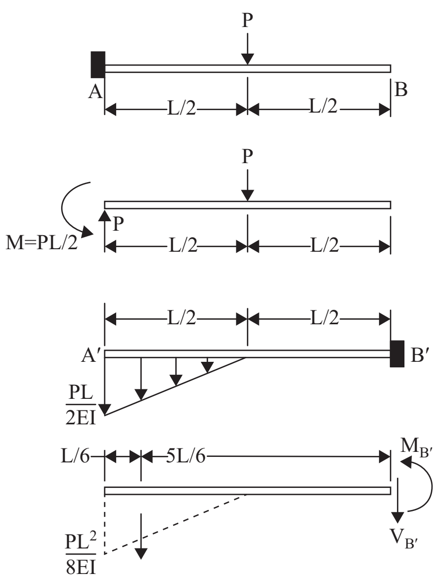

Figure 4.14. Example 4.6 Conjugate beam.

$$\Sigma F_y = 0 = -\frac{PL^2}{8EI} - V_B \therefore V_B = -\frac{PL^2}{8EI}$$

$$\theta_B = -\frac{PL^2}{8EI}$$

$$\Sigma M_B = 0 = \frac{PL^2}{8EI}\frac{5L}{6} + M_B \therefore M_B = -\frac{5PL^3}{48EI}$$

$$\Delta_B = -\frac{5PL^3}{48EI}$$

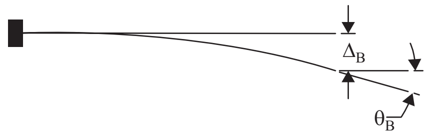

Figure 4.15. Example 4.6 Conjugate beam.

4.6 VIRTUAL WORK

Similar to the conjugate beam method, the virtual work method can be used to obtain the displacement and slope at a specific point on a structure. The method was formulated by Gottfried Leibniz in 1695 (Leibniz 1695). *Virtual work* is based on internal strain energy from the member and external work done by the forces. There are two ways to apply the virtual work method. The first is to apply a virtual (unit) displacement to find a real force. The second is to apply a virtual (unit) force to find a real displacement. The force and the displacement in either case are in the same direction. The basic equation to find a real displacement based on a virtual force is as follows:

$$\theta \text{ or } \Delta = \int_0^L \frac{mM}{EI} dx \tag{4.7}$$

The value of M is the moment equation due to the real loads on the structure. The value of m is the moment equation of the virtual load. Rotation or deflection may be found depending on whether a virtual moment or force is applied at the point under consideration.

Example 4.8 Virtual work

Determine the vertical deflection at the free end of the uniformly loaded, cantilever beam in Figure 4.16 using the virtual work method.

A free-body diagram is drawn of the right-hand side of the beam to determine the internal moment in the beam. The uniformly distributed load is represented as a point load equal to the area under the load and located at the centroid of the area.

From static equilibrium, we can determine the internal moment at any point x measured from the right end of the beam.

$$\text{M} = -\frac{wx^2}{2}$$

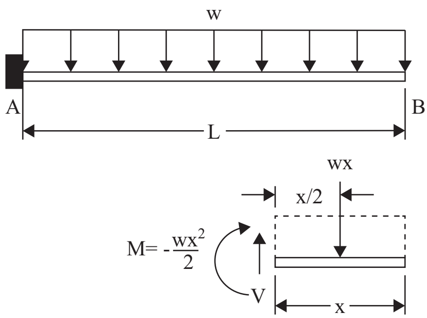

Figure 4.16. Example 4.8 Virtual work.

A virtual force is applied at the point and in the direction that the deflection is desired. This is shown in Figure 4.17 along with a free-body diagram of the right-hand side of the beam. In this case, the internal moment is represented as *m*.

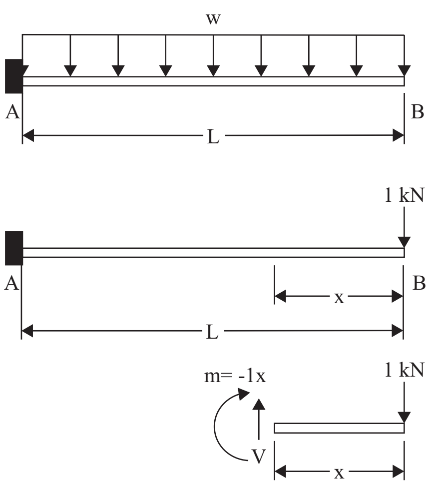

Figure 4.17. Example 4.8 Virtual work.

From static equilibrium, we can determine the internal moment at any point *x* measured from the right end of the beam.

$$m = -1x$$

The values of *M* and *m* can be substituted into Equation 4.7 to determine the deflection at point *B* at the free end of the cantilever.

$$\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^L \frac{(-1x)\left(-\frac{wx^2}{2}\right)}{EI}dx = \int_0^L \frac{wx^3}{2EI}dx = \frac{wL^4}{8EI}$$

4.6.1 VISUAL INTEGRATION

The virtual work method can also be done using the moment diagrams of the real and virtual loads. This is known as *visual integration*. The moment diagrams for the real (Q) and virtual (q) loads are constructed. The integral is equal to the area of the M/EI diagram multiplied by the value of the m diagram taken at the centroid of the M/EI diagram.

Example 4.9 Visual Integration

Determine the deflection at C in inches for the simply supported beam shown in Figure 4.18 using the virtual work method and visual integration. Use $E = 2000$ ksi and $I = 2000$ in^4.

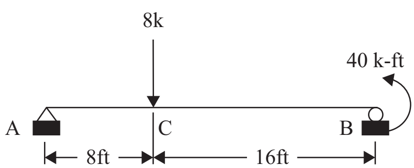

Figure 4.18. Example 4.9 Visual integration.

Solve for the reactions due to the applied loads. The free-body diagram is shown in Figure 4.19.

$$\Sigma M_B = 0 = 8k(16ft) - A_y(24ft) + 40k - ft \therefore A_y = 7k$$
$$\Sigma F_y = 0 = A_y + B_y - 8k \therefore B_y = 1k$$

The shear and moment diagrams are constructed in Figure 4.19. The moment diagram is broken into unique areas that will be continuous over the area of the virtual moment diagrams shown in Figure 4.20.

Apply a virtual force $\delta Q = 1$ at point C as shown in Figure 4.20. Solve for the reactions similar to the real loads. Draw the shear and moment diagram locating the centroid of the real moment diagrams on the virtual moment diagram.

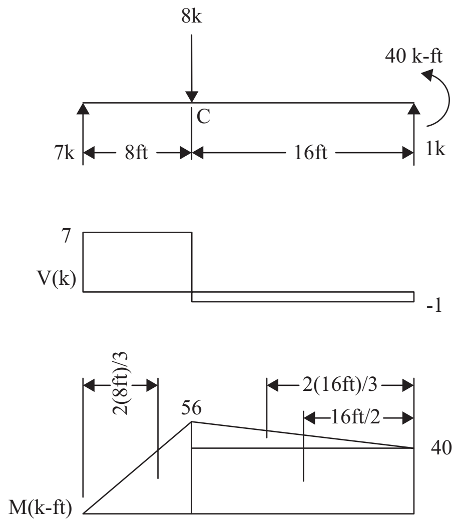

Figure 4.19. Example 4.9 Visual integration.

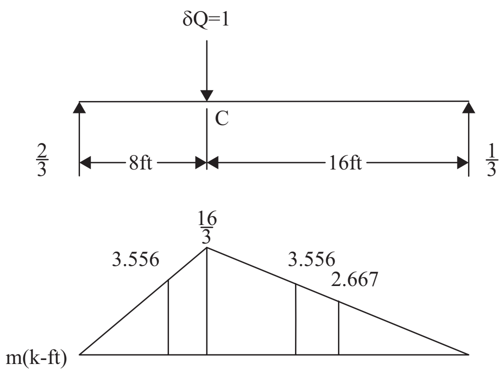

Figure 4.20. Example 4.9 Visual integration.

$$\Sigma M_B = 0 = 1k(16\,ft) - A_y(24\,ft) \therefore A_y = \frac{2}{3}k$$

$$\Sigma F_y = 0 = A_y + B_y - 1k \therefore B_y = \frac{1}{3}k$$

The values of M and m can be substituted into Equation 4.7 to determine the deflection at point C. The integral is the product of the area M/EI diagram and the value of the m diagram at the centroid of the M/EI diagram.

$$\delta Q \Delta_C = \int_0^L \frac{mM}{EI} dx$$

$$1k\Delta_C = \frac{1}{EI}\left[\begin{array}{l}\frac{1}{2}(8ft)56k\text{-}ft(3.556k\text{-}ft)+\frac{1}{2}(16ft)16k\text{-}ft(3.556k\text{-}ft)\\+(16ft)40k\text{-}ft(2.667k\text{-}ft)\end{array}\right]$$

$$1k\Delta_C = \frac{2958k^2\text{-}ft^3}{EI}$$

$$\Delta_C = \frac{2958k\text{-}ft^3}{EI} = \frac{2958\left(1728in^3/ft^3\right)}{2000k/in^2\left(2000in^4\right)} = 1.278in$$

4.7 CASTIGLIANO'S THEOREMS

In 1879 Alberto Castigliano published his two theorems on elastic structures that are known as *Castigliano's Theorems* (Castigliano 1879). The first theorem states that the first partial derivative of strain energy with respect to a particular deflection component is equal to the force applied at the point and in the direction corresponding to that deflection component. This may be written in mathematical terms as shown in Equation 4.8. The second theorem is used more often in statically indeterminate structural analysis and states that the first partial derivative of strain energy with respect to a particular force is equal to the displacement of the point of application of that force in the direction of its line of action. This is shown in Equation 4.9. The equations are written in terms for flexural energy, *M/EI*, of a particular rotation, θ_A, and moment, M_A, relationship and in terms of a particular deflection, Δ_A, and force, P_A, relationship. They can be written for any elastic force and deformation relationship.

$$M_A = \int_0^L M \frac{\partial M}{\partial \theta_A}\frac{dx}{EI} \text{ and } P_A = \int_0^L M \frac{\partial M}{\partial \Delta_A}\frac{dx}{EI} \tag{4.8}$$

$$\theta_A = \int_0^L M \frac{\partial M}{\partial M_A}\frac{dx}{EI} \text{ and } \Delta_A = \int_0^L M \frac{\partial M}{\partial P_A}\frac{dx}{EI} \tag{4.9}$$

Example 4.10 Castigliano's second theorem

Determine the deflection of the beam in Figure 4.21 at point *B* using Castigliano's second theorem.

Since the deflection at point *B* is desired, a force *P* will be placed at *B*. This will be the particular force in the direction of the desired deflection.

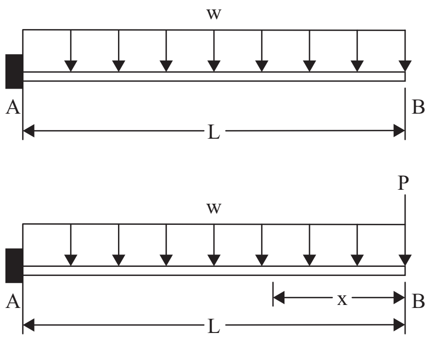

Figure 4.21. Example 4.10 Castigliano's second theorem.

A free-body diagram of the right-hand portion of the beam is shown in Figure 4.22. The moment equation is found from equilibrium on the free-body. Also, the partial derivative of the moment equation is taken with respect to the particular force *P*. These are substituted into Equation 4.9 to find the deflection at point *B*.

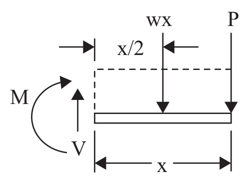

Figure 4.22. Example 4.10 Castigliano's second theorem.

$$\Sigma M_B = 0 = -M - wx\frac{x}{2} - Px \therefore M = -\frac{wx^2}{2} - Px$$

$$\frac{\partial M}{\partial P} = -x$$

$$\Delta_B = \int_0^L M\frac{\partial M}{\partial P}\frac{dx}{EI} = \int_0^L\left(-\frac{wx^2}{2} - Px\right)(-x)\frac{dx}{EI} = \int_0^L\left(\frac{wx^3}{2} + Px^2\right)\frac{dx}{EI}$$

$$\Delta_B = \frac{wL^4}{8EI} + \frac{PL^3}{3EI}$$

Since *P* is a fictitious applied load placed on the beam only to find the deflection, it can be removed and the deflection at *B* becomes the following:

$$\Delta_B = \frac{wL^4}{8EI}$$

Example 4.11 Castigliano's second theorem

Find the reactions for the propped end beam loaded as shown in Figure 4.23.

Since this is a statically indeterminate beam, a redundant force will be applied at point *A*. The force will be represented as *P* but is actually the real vertical reaction on the left end R_A. The last drawing in Figure 4.23 is a free-body diagram of a left-hand section of the beam to find the moment equation by equilibrium.

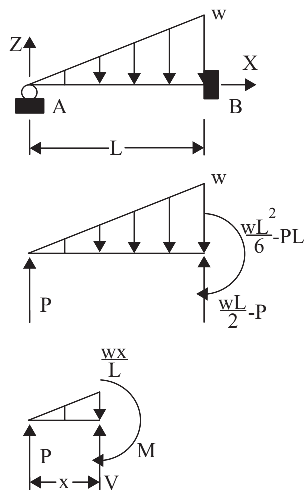

Figure 4.23. Example 4.11 Castigliano's second theorem.

A free-body diagram for the total beam is drawn in Figure 4.23 with reactions at the fixed end written in terms of the redundant force *P*. These reactions are found by equilibrium.

$$\Sigma M_B = 0 = -M_B - PL + \frac{1}{2}wL\frac{L}{3}$$

$$M_B = \frac{wL^2}{6} - PL \tag{4.10}$$

$$\Sigma F_y = 0 = R_B + P - \frac{1}{2}wL$$

$$\Sigma R_B = \frac{wL}{2} - P \tag{4.11}$$

The partial derivative of the moment equation is taken with respect to the particular force P. These are substituted into Equation 4.9 to find the force P, which is the reaction at point A. Note the deflection at point A is set to zero in Castigliano's equation.

$$\Sigma M = 0 = -M - Px + \frac{1}{2}x\frac{wx}{L}\frac{x}{3} \therefore M = \frac{wx^3}{6L} - Px$$

$$\frac{\partial M}{\partial P} = -x$$

$$\Delta_A = 0 = \int_0^L M\frac{\partial M}{\partial P}\frac{dx}{EI} = \int_0^L \left(\frac{wx^3}{6L} - Px\right)(-x)\frac{dx}{EI} = \int_0^L \left(-\frac{wx^4}{6L} + Px^2\right)\frac{dx}{EI}$$

$$0 = -\frac{wL^5}{30L} + \frac{PL^3}{3}$$

$$P = R_A = \frac{wL}{10}$$

This result can be substituted back into Equations 4.10 and 4.11 to find the reactions at A.

$$R_B = \frac{2wL}{5}$$

$$M_B = \frac{wL^2}{15}$$

4.8 SLOPE-DEFLECTION METHOD

The slope-deflection method is a stiffness method that includes flexural or bending stiffness. It was introduced in 1915 by George A. Maney (Maney 1915). In *slope-deflection*, moments at the end of a member are expressed in terms of the rotations at the ends and the fixed-end moments due to the loads. Once the expressions for the moments at the member ends are written, the joint moments are equated to zero and the unknown moments are found from the system of equations. The basic slope-deflection equations are shown in Equations 4.12 and 4.13. These are for the i end and j end of a member, respectively.

$$M_i = FEM_{ij} + \frac{4EI}{L}\phi_i + \frac{2EI}{L}\phi_j - \frac{6EI}{L}\beta \qquad (4.12)$$

$$M_j = FEM_{ji} + \frac{2EI}{L}\phi_i + \frac{4EI}{L}\phi_j - \frac{6EI}{L}\beta \tag{4.13}$$

In these two equations, ϕ is the rotation at the joint and β is the lateral translation between the ends divided by the length of the member. It will be seen later that the values $4EI/L$, $2EI/L$, and $6EI/L$ are flexural stiffness terms. The *FEM* terms are the fixed-end moments due to the loads on the member. A special case may be used if one end of the member is pinned. Equation 4.14 is for the *i* end of a member when the *j* end is pinned.

$$M_i = FEM_{ij} + \frac{FEM_{ji}}{2} + \frac{3EI}{L}(\phi_i - \beta) \tag{4.14}$$

Example 4.12 Slope-deflection

Determine the moments at the ends of the members of the continuous beam in Figure 4.24 using the slope-deflection equations.

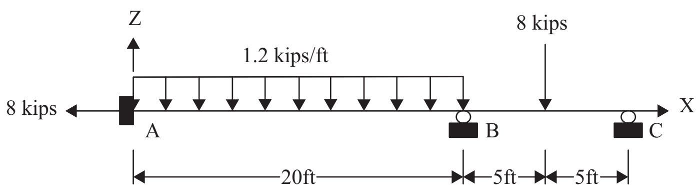

Figure 4.24. Example 4.12 Slope-deflection.

The fixed-end moments due to the loads on each span are computed and can be found in most elementary structural analysis textbooks.

$$FEM_{AB} = -\frac{wL^2}{12} = -\frac{1.2k/ft(20\,ft)^2}{12} = -40k - ft$$

$$FEM_{BA} = \frac{wL^2}{12} = \frac{1.2k/ft(20\,ft)^2}{12} = 40k - ft$$

$$FEM_{BC} = -\frac{Pb^2a}{L^2} = -\frac{8(5\,ft)^2(5\,ft)}{(10\,ft)^2} = -10k - ft$$

$$FEM_{CB} = \frac{Pa^2b}{L^2} = \frac{8(5\,ft)^2(5\,ft)}{(10\,ft)^2} = 10k - ft$$

The slope-deflection equations are applied to each span. In this case, the normal equations (4.12 and 4.13) are used for span *AB* and a special case (4.14) is used for span *BC* since support *C* is a roller. It should be noted that there are no rotations at support *A* and therefore $\phi_A = 0$. Also, since there is no translational movement between the ends, β is zero.

$$M_{AB} = FEM_{AB} + \frac{4EI}{L}\phi_A + \frac{2EI}{L}\phi_B = -40 + \frac{2EI}{20}\phi_B = -40 + \frac{EI}{10}\phi_B$$
$$M_{BA} = FEM_{BA} + \frac{2EI}{L}\phi_A + \frac{4EI}{L}\phi_B = 40 + \frac{4EI}{20}\phi_B = 40 + \frac{EI}{5}\phi_B$$
$$M_{BC} = FEM_{BC} - \frac{FEM_{CB}}{2} + \frac{3EI}{L}\phi_B = -10 - \frac{10}{2} + \frac{3EI}{10}\phi_B = -15 + \frac{3EI}{10}\phi_B$$

Equilibrium equations are written at each joint that has a real rotation. In this case, that is only joint *B*.

$$M_{BA} + M_{BC} = 0 = 40 + \frac{EI}{5}\phi_B - 15 + \frac{3EI}{10}\phi_B$$
$$\phi_B = -\frac{50}{EI}\phi_B$$

Substituting the value of ϕ_B back into the moment equation will result in the final member-end moments.

$$M_{AB} = -40 + \frac{EI}{10}\left(-\frac{50}{EI}\right) = -45k - ft$$
$$M_{BA} = 40 + \frac{EI}{5}\left(-\frac{50}{EI}\right) = 30k - ft$$
$$M_{BC} = -15 + \frac{3EI}{10}\left(-\frac{50}{EI}\right) = -30k - ft$$

4.9 MOMENT-DISTRIBUTION METHOD

The moment-distribution method is an iteration process that uses the same basic assumptions and equations as the slope-deflection method. *Moment-distribution* was developed by Hardy Cross in 1930 (Cross 1930). The difference between the two is that at each joint the fixed-end moments are first summed and distributed to each member in proportion to their *flexural stiffness* (stiffness factor). Then, the member-end moments are carried over to their far ends by the carryover factor. The process is repeated and continues until the amount of moment being distributed becomes significantly small.

Many factors are used with the moment-distribution method. The first is the *member stiffness factor* and is the amount of moment required to rotate the end of a beam 1 radian. This is actually the definition of stiffness, force due to unit motion. The far end of the beam that is rotating is fixed. We will derive this expression in the next section on elastic member stiffness.

$$K = \frac{4EI}{L}$$

The *joint stiffness factor* is the sum of all the member stiffness factors for the members connected at a joint.

$$K_T = \Sigma K$$

The *distribution factor* for each member-end at a joint is the member stiffness factor divided by the joint stiffness factor.

$$D_F = \frac{K}{K_T} = \frac{K}{\Sigma K}$$

If a member is connected to a support and not to other members, the distribution factor is dependent on the support type. If the support is fixed against rotation, then D_F=1. If the support is free to rotate, then D_F=0.

The *member relative stiffness factor* is used when a continuous beam or frame is made from the same material when calculating the distribution factor. This can be used in place of the member stiffness factor for calculation of the other factors.

$$K_R = \frac{I}{L}$$

The final factor is the *carry-over factor*, which represents the fraction of a moment that is carried over from one end of a member to the other. If the member is prismatic, then the ratio of the far end moment to the near end moment of a member is one-half (½).

Finally, if a member is connected to a support, but is free to rotate at the support, a modified member stiffness factor can be used. In this modified method, the support joint is moment balanced then carried over and no further calculations are performed at that joint.

$$K' = \frac{3K}{4}$$

Example 4.13 Moment-distribution method

Determine the moments in the beam shown in Figure 4.25 by the moment-distribution method.

The modulus of elasticity, E, and moment of inertia, I, are constant for the beam.

Since E and I are constant, the member relative stiffness factor can be used and we can use a unit value for I.

$$K_{AB} = \frac{I_{AB}}{L_{AB}} = \frac{1}{20} = 0.05$$
$$K_{BC} = \frac{I_{BC}}{L_{BC}} = \frac{1}{10} = 0.10$$

At fixed support A, the distribution factor for member AB is 0 and at the roller support C, the distribution factor for member BC is 1. For the internal roller, the distribution factor for each member must be calculated.

$$D_{FAB} = \frac{K_{AB}}{K_{AB} + K_{BC}} = \frac{0.05}{0.05 + 0.10} = 0.333$$
$$D_{FBC} = \frac{K_{BC}}{K_{AB} + K_{BC}} = \frac{0.1}{0.05 + 0.10} = 0.667$$

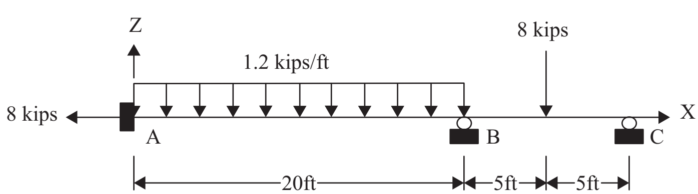

Figure 4.25. Example 4.13 Moment-distribution.

The fixed-end moments are calculated for each member load.

$$FEM_{AB} = -\frac{wL^2}{12} = -\frac{1.2k/ft(20\,ft)^2}{12} = -40k - ft$$

$$FEM_{BA} = \frac{wL^2}{12} = \frac{1.2k/ft(20\,ft)^2}{12} = 40k - ft$$

$$FEM_{BC} = -\frac{Pb^2a}{L^2} = -\frac{8(5\,ft)^2(5\,ft)}{(10\,ft)^2} = -10k - ft$$

$$FEM_{CB} = \frac{Pa^2b}{L^2} = \frac{8(5\,ft)^2(5\,ft)}{(10\,ft)^2} = 10k - ft$$

The moment-distribution is shown in Table 4.1. In the first line, the fixed-end moments are recorded. In the second line, the distribution factor is multiplied times the negative sum of the fixed-end moments. This is a distribution. In each subsequent one-half, the end moment is carried over the opposite end of the members. This is a carry-over. Then, the unbalanced moments at the joint are distributed again and carried over again.

Table 4.1. Example 4.13 Moment-distribution

0.00	0.333	0.667	1.00	Distribution	Factor
–40.00	40.00	–10.00	10.00	Fixed-end	Moment
0.00	–10.00	–20.00	–10.00	Distribution	1
–5.00	0.00	–5.00	–10.00	Carry-over	1
0.00	1.67	3.33	10.00	Distribution	2
0.83	0.00	5.00	1.67	Carry-over	2
0.00	–1.67	–3.33	–1.67	Distribution	3
–0.83	0.00	–0.83	–1.67	Carry-over	3
0.00	0.28	0.56	1.67	Distribution	4
0.14	0.00	0.83	0.28	Carry-over	4
0.00	–0.28	–0.56	–0.28	Distribution	5
0.14	0.00	–0.14	–0.28	Carry-over	5
0.00	0.05	0.09	0.28	Distribution	6
0.02	0.00	0.14	0.05	Carry-over	6
0.00	–0.05	–0.09	–0.05	Distribution	7
–0.02	0.00	–0.02	–0.05	Carry-over	7
–45.00	30.00	–30.02	–0.05	Final Moments	

Example 4.14 Moment-distribution

Determine the moments in the beam shown in Figure 4.25 by the modified moment-distribution method.

There are two primary differences when using this method. First, use a modified member stiffness factor for member BC.

$$K'_{BC} = \frac{3K_{BC}}{4} = \frac{3(0.1)}{4} = 0.075$$

The modified factor must be used to recalculate the distribution factors.

$$D_{FAB} = \frac{K_{AB}}{K_{AB} + K_{BC}} = \frac{0.05}{0.05 + 0.75} = 0.40$$

$$D_{FBC} = \frac{K_{BC}}{K_{AB} + K_{BC}} = \frac{0.075}{0.05 + 0.075} = 0.60$$

Second, the fixed-end moment at the pinned support is balanced then carried over to the far end before the moment-distribution process begins. The process is shown in Table 4.2.

Table 4.2. Example 4.14 Moment-distribution

0.00	0.400	0.600	1.00	Distribution	Factor
–40.00	40.00	–10.00	10.00	Fixed-end	Moment
		–5.00	–10.00	Balance @ C	
0.00	–10.00	–15.00		Distribution	1
–5.00	0.00	0.00		Carry-over	1
–45.00	30.00	–30.00	0.00	Final Moments	

4.10 ELASTIC MEMBER STIFFNESS, X-Z SYSTEM

The stiffness method for analyzing building structures is widely used by engineers and commercial computer structural analysis programs. The method was developed 1934 and 1938 by Arthur Collar (Lewis et al. 1939). The basic definition of *stiffness* is the force due to a unit deformation. *Flexibility* is the reciprocal or inverse of stiffness and is defined as the deformation due to a unit force. Either of these principles can be used to find the behavior of structural members due to motion and loads. In this section, the *elastic member stiffness* for a linear element will be derived.

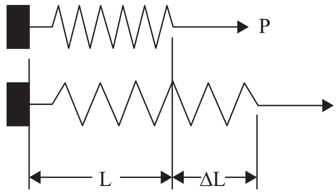

Figure 4.26. Elastic stiffness.

The fixed-end forces on a member can also be found by similar methods. When discussing the flexibility and stiffness of a member, an elastic spring that is axially loaded can be considered. This is equivalent to the axial properties of a linear element used in structures. Figure 4.26 shows an elastic spring in an un-deformed and deformed position.

The displacement of the spring is directly proportional to the applied axial load. This is the basic flexibility equation and is written as follows:

$$\Delta = fP$$

In this equation, f is the flexibility of the spring and P is the applied axial load in the direction of the length of the spring. This can also be written in terms of stiffness using a displacement of 1 unit.

$$P = K\Delta$$

This equation is the general equation for stiffness. P represents all the known forces, Δ represents the unknown rotations and deflections, and K is the stiffness matrix for a member or structure. The entire system of a structure can be modeled into a set of simultaneous equations written in the following form and will be expanded in the next section and chapter:

$$[K][\Delta] = [P]$$

The derivation of elastic member stiffness in the X-Z system will be derived in the two following examples using the conjugate beam method and area moment method.

Example 4.15 θ_{iy} stiffness

Derive the θ_{iy} stiffness using the conjugate beam method for a linear member.

A free-body diagram is shown in Figure 4.27 with an imposed rotation of 1 unit on the *i*-end of the member. The moments are assumed in the

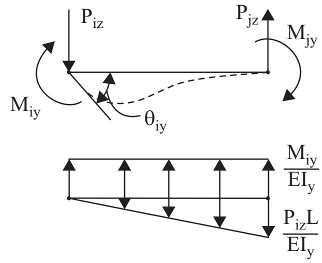

Figure 4.27. Example 4.15 θ_{iy} stiffness.

positive *y* direction using the right-hand rule and the Cartesian right-hand coordinate system. The forces are shown consistent with the deformation. The moment diagram divided by *EI* is shown for the reaction forces on the *i*-end of the member.

The conjugate beam can be constructed for the two basic assumptions. The shear in the conjugate is equal to the slope of the real beam, and the moment of the conjugate is equal to the deflection of the real beam.

$$V_{iconj} = \theta_{ireal} = \theta_{iy}$$
$$M_{iconj} = \Delta_{ireal} = 0$$

Since moment in the conjugate does not exist but shear does, the conjugate beam is pinned on the *i*-end.

$$V_{jconj} = \theta_{jreal} = 0$$
$$M_{jconj} = \Delta_{jreal} = 0$$

Since both the shear and moment in the conjugate do not exist, the conjugate beam is free on the *j*-end. The resulting conjugate beam is shown in Figure 4.28.

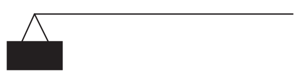

Figure 4.28. Example 4.15 θ_{iy} stiffness.

The conjugate beam method can be applied to find the reactions at the *i*-end of the conjugate beam, which are equal to deformations at the *i*-end of the real beam. The load from Figure 4.27 is applied to the conjugate beam in Figure 4.28.

$$V_{iconj} = \theta_{ireal} = \Delta\theta_{ij} = 0 - \theta_{iy} = -\frac{M_{iy}L}{EI_y} + \frac{P_{iz}L^2}{2EI_y}$$

$$M_{iconj} = \Delta_{ireal} = 0 = -\left(\frac{M_{iy}L}{EI_y}\right)\frac{L}{2} + \left(\frac{P_{iz}L^2}{2EI_y}\right)\frac{2L}{3}$$

Solving the second equation for P_{iz} in terms of M_{iy} and then substituting into the first equation, the stiffness value can be found.

$$P_{iz} = \frac{3M_{iy}}{2L}$$

$$-\theta_{iy} = -\frac{M_{iy}L}{EI_y} + \frac{3M_{iy}L}{4EI_y} = -\frac{M_{iy}L}{4EI_y}$$

$$M_{iy} = \frac{4EI_y}{L}\theta_{iy} \tag{4.15}$$

$$P_{iz} = \frac{6EI_y}{L^2}\theta_{iy}$$

It should be noted that the force P_{iz} was actually shown as negative in the original free-body diagram so it does have a negative value for stiffness.

$$P_{iz} = -\frac{6EI_y}{L^2}\theta_{iy} \tag{4.16}$$

Example 4.16 Δ_{iz} stiffness

Derive the Δ_{iz} stiffness using the area moment method for a linear member.

A free-body diagram is shown in Figure 4.29 with an imposed deflection of 1 unit on the *i*-end of the member. The moments are assumed in the negative *y* direction using the right-hand rule and the Cartesian right-hand coordinate system. The forces are shown consistent with the deformation. The moment diagram divided by *EI* is shown for the reaction forces on the *i*-end of the member.

Since both ends of the beam are fixed for rotation, the change in rotation from the *i*-end to the *j*-end is zero. This is the area under the *M*/*EI* diagram between those points.

$$\Delta\theta_{ij} = 0 = \theta_j - \theta_i = -\frac{M_{iy}L}{EI_y} + \frac{P_{iz}L^2}{2EI_y}$$

$$P_{iz} = \frac{2M_{iy}}{L}$$

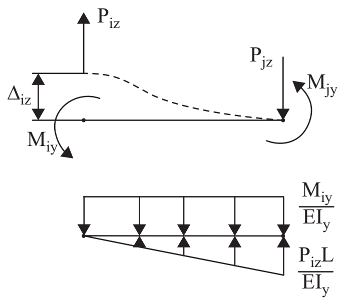

Figure 4.29. Example 4.16 Δ_{iz} stiffness.

The tangential deviation of a point at the *i*-end from a tangent to the curve on the *j*-end is the implied deflection. This is equal to the moment of the area of the *M*/*EI* diagram about the point at the *i*-end.

$$t_{ij} = \Delta_{iz} = \int_i^j \frac{M}{EI}\bar{x}_i dx = -\left(\frac{M_{iy}L}{EI_y}\right)\frac{L}{2} + \left(\frac{P_{iz}L^2}{2EI_y}\right)\frac{2L}{3}$$

Substituting the first equation for P_{iz} into the Δ_{iz} equation results in one of the stiffness terms. The second term is found by substituting the first stiffness term back into the P_{iz} equation. Note that M_{iy} was assumed as negative in the free-body diagram, so the sign must be switched.

$$\Delta_{iz} = \frac{M_{iy}L^2}{6EI_y}$$

$$M_{iy} = -\frac{6EI_y}{L^2}\Delta_{iz} \tag{4.17}$$

$$P_{iz} = \frac{12EI_y}{L^3}\Delta_{iz} \tag{4.18}$$

The four terms given in Equations 4.15 through 4.18 are the flexural stiffness terms for the forces at the *i*-end due to motions at the *i*-end. This is denoted as stiffness matrix $[K_{ii}]$ in Equation 4.19. The stiffness equation and matrix form of this are as follows:

$$[K_{ii}][\delta_i] = [F_i]$$

$$\begin{bmatrix} \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} \\ -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \begin{bmatrix} \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} P_{iz} \\ M_{iy} \end{bmatrix}$$

$$\left[K_{ii}\right] = \begin{bmatrix} \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} \\ -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \tag{4.19}$$

The transmission matrix derived in Section 4.3 for the X-Z system can be used to find the forces at the *j*-end.

$$[T] = \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix}$$

From statics equilibrium, the forces at the *i*-end transmitted to the *j*-end plus the forces at the *j*-end are equal to zero. The forces at the *j*-end due to motions at the *i*-end are denoted as stiffness matrix $[K_{ji}]$.

$$[T][K_{ii}] + \left[K_{ji}\right] = 0$$

$$\left[K_{ji}\right] = -[T][K_{ii}]$$

$$\left[K_{ji}\right] = -\begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} \\ -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -L & -1 \end{bmatrix} \begin{bmatrix} \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} \\ -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix}$$

$$\left[K_{ji}\right] = \begin{bmatrix} -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} \end{bmatrix} \tag{4.20}$$

For a prismatic member, it can be shown that the stiffness matrix is symmetric about the main diagonal. This results in stiffness $[K_{ij}]$ being equal to the transpose of $[K_{ji}]$, where the forces at the *j*-end due to motions at the *i*-end are denoted as stiffness matrix $[K_{ji}]$.

$$\left[K_{ij}\right]=\left[K_{ji}\right]^T=\begin{bmatrix}-\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\\ -\frac{6EI_y}{L^2} & \frac{2EI_y}{L}\end{bmatrix}^T=\begin{bmatrix}-\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}\\ \frac{6EI_y}{L^2} & \frac{2EI_y}{L}\end{bmatrix}$$

$$\left[K_{ij}\right]=\begin{bmatrix}-\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}\\ \frac{6EI_y}{L^2} & \frac{2EI_y}{L}\end{bmatrix} \tag{4.21}$$

From statics equilibrium, the forces at the *i*-end transmitted to the *j*-end plus the forces at the *j*-end are equal to zero. The forces at the *j*-end due to motions at the *j*-end are denoted as stiffness matrix $[K_{jj}]$.

$$[T]\left[K_{ij}\right]+\left[K_{jj}\right]=0$$

$$\left[K_{jj}\right]=-[T]\left[K_{ij}\right]$$

$$\left[K_{jj}\right]=-\begin{bmatrix}1 & 0\\ L & 1\end{bmatrix}\begin{bmatrix}-\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}\\ \frac{6EI_y}{L^2} & \frac{2EI_y}{L}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ -L & -1\end{bmatrix}\begin{bmatrix}-\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}\\ \frac{6EI_y}{L^2} & \frac{2EI_y}{L}\end{bmatrix}$$

$$\left[K_{jj}\right]=\begin{bmatrix}\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\\ \frac{6EI_y}{L^2} & \frac{4EI_y}{L}\end{bmatrix} \tag{4.22}$$

Adding the axial stiffness terms that are derived in most strength of materials textbooks, the entire coplanar X-Z frame stiffness matrix can be found.

$$P_{ix}=\frac{A_xE}{L}\Delta_{ix}$$

$$P_{jx}=-\frac{A_xE}{L}\Delta_{ix}$$

$$P_{ix}=-\frac{A_xE}{L}\Delta_{jx}$$

$$P_{jx}=\frac{A_xE}{L}\Delta_{jx}$$

$$\begin{bmatrix} \frac{A_xE}{L} & 0 & 0 & -\frac{A_xE}{L} & 0 & 0 \\ 0 & \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & 0 & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} \\ 0 & -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} & 0 & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ -\frac{A_xE}{L} & 0 & 0 & \frac{A_xE}{L} & 0 & 0 \\ 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ 0 & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} & 0 & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \\ \Delta_{jx} \\ \Delta_{jz} \\ \theta_{jy} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iz} \\ M_{iy} \\ P_{jx} \\ P_{jz} \\ M_{jy} \end{bmatrix} \tag{4.23}$$

4.11 ELASTIC MEMBER STIFFNESS, X-Y SYSTEM

The X-Y coplanar coordinate system differs from the X-Z system in a few ways. The derivation of elastic member stiffness in the X-Y system will be derived in the two following examples using the area moment method and conjugate beam method.

Example 4.17 θ_{iz} *stiffness*

Derive the θ_{iz} stiffness using the area moment method for a linear member.

A free-body diagram is shown in Figure 4.30 with an imposed rotation of 1 unit on the *i*-end of the member. The moments are assumed in the positive *z* direction using the right-hand rule and the Cartesian right-hand coordinate system. The forces are shown consistent with the deformation. The moment diagram divided by EI is shown for the reaction forces on the *i*-end of the member.

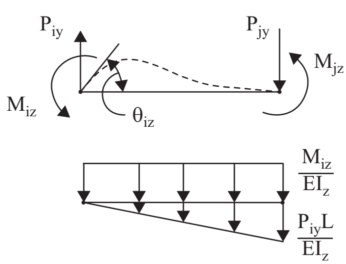

Figure 4.30. Example 4.17 θ_{iz} stiffness.

Since both ends of the beam are fixed for translation, the tangential deviation of a point at the *i*-end from a tangent to the curve on the *j*-end is zero. This is equal to the moment of the area of the *M*/*EI* diagram about the point at the *i*-end.

$$t_{ij} = 0 = \int_i^j \frac{M}{EI}\bar{x}_i dx = -\left(\frac{M_{iz}L}{EI_z}\right)\frac{L}{2} + \left(\frac{P_{iy}L^2}{2EI_z}\right)\frac{2L}{3}$$

$$P_{iy} = \frac{3M_{iz}}{2L}$$

The change in rotation from the *i*-end to the *j*-end is equal to the negative of the implied rotation. This is the area under the *M*/*EI* diagram between those points.

$$\Delta\theta_{ij} = 0 - \theta_{iz} = -\theta_{iz} = -\frac{M_{iz}L}{EI_z} + \frac{P_{iy}L^2}{2EI_z}$$

Substituting the first equation for P_{iy} into the θ_{iz} equation results in one of the stiffness terms. The second term is found by substitution of the first stiffness term back into the P_{iy} equation.

$$-\theta_{iz} = -\frac{M_{iz}L}{4EI_z}$$

$$M_{iz} = \frac{4EI_z}{L}\theta_{iz} \tag{4.24}$$

$$P_{iy} = \frac{6EI_z}{L^2}\theta_{iz} \tag{4.25}$$

Example 4.18 Δ_{iy} *stiffness*

Derive the Δ_{iy} stiffness using the conjugate beam method for a linear member.

A free-body diagram is shown in Figure 4.31 with an imposed deflection of one unit on the *i*-end of the member. The moments are assumed in the positive *z* direction using the right-hand rule and the Cartesian right-hand coordinate system. The forces are shown consistent with the deformation. The moment diagram divided by *EI* is shown for the reaction forces on the *i*-end of the member.

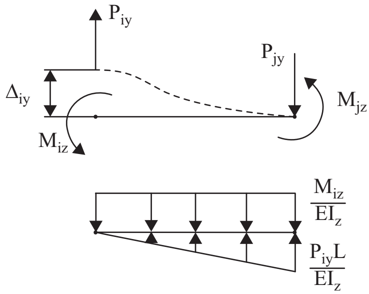

Figure 4.31. Example 4.18 Δ_{iy} stiffness.

The conjugate beam can be constructed for the two basic assumptions. The shear in the conjugate is equal to the slope of the real beam, and the moment of the conjugate is equal to the deflection of the real beam.

$$V_{iconj} = \theta_{ireal} = 0$$
$$M_{iconj} = \Delta_{ireal} = \Delta_{iy}$$

Since moment in the conjugate exists but shear is zero, the conjugate beam is slotted in the *y* direction on the *i*-end. This is a connection that is free to move vertically, but restrained from rotation.

$$V_{jconj} = \theta_{jreal} = 0$$
$$M_{jconj} = \Delta_{jreal} = 0$$

Since both the shear and the moment in the conjugate do not exist, the conjugate beam is free on the *j*-end. The resulting conjugate beam is shown in Figure 4.32.

The conjugate beam method can be applied to find the reactions at the *i*-end of the conjugate beam, which are equal to deformations at the *i*-end of the real beam. The load from Figure 4.31 is applied to the conjugate beam in Figure 4.32.

$$V_{iconj} = \theta_{ireal} = 0 = -\frac{M_{iz}L}{EI_z} + \frac{P_{iy}L^2}{2EI_z}$$
$$M_{iconj} = \Delta_{ireal} = \Delta_{iy} = -\left(\frac{M_{iz}L}{EI_z}\right)\frac{L}{2} + \left(\frac{P_{iy}L^2}{2EI_z}\right)\frac{2L}{3}$$

Figure 4.32. Example 4.18 Δ_{iy} stiffness.

Solving the first equation for P_{iy} in terms of M_{iz} and then substituting into the second equation, the stiffness value can be found.

$$P_{iy} = \frac{2M_{iz}}{L}$$
$$\Delta_{iy} = \frac{M_{iz}L^2}{6EI_z}$$
$$M_{iz} = \frac{6EI_z}{L^2}\Delta_{iy} \tag{4.26}$$

This can be substituted back into the equation for P_{iy} to obtain the last stiffness value.

$$P_{iy} = \frac{12EI_z}{L^3}\Delta_{iy} \tag{4.27}$$

The four terms given in Equations 4.24 through 4.27 are the flexural stiffness terms for the forces at the *i*-end due to motions at the *i*-end. This is denoted as stiffness matrix $[K_{ii}]$ in Equation 4.28. The stiffness equation and matrix form of this are as follows:

$$[K_{ii}][\delta_i] = [F_i]$$
$$\begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \begin{bmatrix} \Delta_{iy} \\ \theta_{iz} \end{bmatrix} = \begin{bmatrix} P_{iy} \\ M_{iz} \end{bmatrix}$$
$$[K_{ii}] = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \tag{4.28}$$

The transmission matrix derived in Section 4.3 for the X-Y system can be used to find the forces at the *j*-end.

$$[T] = \begin{bmatrix} 1 & 0 \\ -L & 1 \end{bmatrix}$$

From statics equilibrium, the forces at the *i*-end transmitted to the *j*-end plus the forces at the j-end are equal to zero. The forces at the *j*-end due to motions at the *i*-end are denoted as stiffness matrix $[K_{ji}]$.

$$[T][K_{ii}]+[K_{ji}]=0$$

$$[K_{ji}]=-[T][K_{ii}]$$

$$[K_{ji}]=-\begin{bmatrix}1 & 0\\ -L & 1\end{bmatrix}\begin{bmatrix}\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2}\\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ L & -1\end{bmatrix}\begin{bmatrix}\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2}\\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L}\end{bmatrix}$$

$$[K_{ji}]=\begin{bmatrix}-\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2}\\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L}\end{bmatrix} \tag{4.29}$$

For a prismatic member it can be shown that the stiffness matrix is symmetrical about the main diagonal. This results in stiffness $[K_{ij}]$ being equal to the transpose of $[K_{ji}]$, where the forces at the *j*-end due to motions at the *i*-end are denoted as stiffness matrix $[K_{ji}]$.

$$[K_{ij}]=[K_{ji}]^T=\begin{bmatrix}-\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2}\\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L}\end{bmatrix}^T=\begin{bmatrix}-\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2}\\ -\frac{6EI_z}{L^2} & \frac{2EI_z}{L}\end{bmatrix}$$

$$[K_{ij}]=\begin{bmatrix}-\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\\ -\frac{6EI_y}{L^2} & \frac{2EI_y}{L}\end{bmatrix} \tag{4.30}$$

From statics equilibrium, the forces at the *i*-end transmitted to the *j*-end plus the forces at the *j*-end are equal to zero. The forces at the *j*-end due to motions at the *j*-end are denoted as stiffness matrix $[K_{jj}]$.

$$[T][K_{ij}]+[K_{jj}]=0$$

$$[K_{jj}]=-[T][K_{ij}]$$

$$\left[K_{jj}\right] = -\begin{bmatrix} 1 & 0 \\ -L & 1 \end{bmatrix}\begin{bmatrix} -\dfrac{12EI_y}{L^3} & \dfrac{6EI_y}{L^2} \\ -\dfrac{6EI_y}{L^2} & \dfrac{2EI_y}{L} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ L & -1 \end{bmatrix}\begin{bmatrix} -\dfrac{12EI_y}{L^3} & \dfrac{6EI_y}{L^2} \\ -\dfrac{6EI_y}{L^2} & \dfrac{2EI_y}{L} \end{bmatrix}$$

$$\left[K_{jj}\right] = \begin{bmatrix} \dfrac{12EI_z}{L^3} & -\dfrac{6EI_z}{L^2} \\ -\dfrac{6EI_z}{L^2} & \dfrac{4EI_z}{L} \end{bmatrix} \tag{4.31}$$

If we add the axial stiffness terms that were shown in the previous section, the entire coplanar X-Y frame stiffness matrix can be found.

$$\begin{bmatrix} \dfrac{A_xE}{L} & 0 & 0 & -\dfrac{A_xE}{L} & 0 & 0 \\ 0 & \dfrac{12EI_z}{L^3} & \dfrac{6EI_z}{L^2} & 0 & -\dfrac{12EI_z}{L^3} & \dfrac{6EI_z}{L^2} \\ 0 & \dfrac{6EI_z}{L^2} & \dfrac{4EI_z}{L} & 0 & -\dfrac{6EI_z}{L^2} & \dfrac{2EI_z}{L} \\ -\dfrac{A_xE}{L} & 0 & 0 & \dfrac{A_xE}{L} & 0 & 0 \\ 0 & -\dfrac{12EI_z}{L^3} & -\dfrac{6EI_z}{L^2} & 0 & \dfrac{12EI_z}{L^3} & -\dfrac{6EI_z}{L^2} \\ 0 & \dfrac{6EI_z}{L^2} & \dfrac{2EI_z}{L} & 0 & -\dfrac{6EI_z}{L^2} & \dfrac{4EI_z}{L} \end{bmatrix}\begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ M_{jz} \end{bmatrix} \tag{4.32}$$

4.12 ELASTIC MEMBER STIFFNESS, 3-D SYSTEM

By combining Equations 4.23 and 4.32 and adding the torsional stiffness terms that are derived in most strength of materials textbooks, we can construct the elastic member stiffness in the three-dimensional (3-D) system Cartesian coordinate system. This will include axial and torsional stiffness, as well as bending about each orthogonal axes of the member cross-section.

$$M_{ix} = \frac{I_xG}{L}\theta_{ix}$$

$$M_{jx} = -\frac{I_x G}{L}\theta_{ix}$$
$$M_{ix} = -\frac{I_x G}{L}\theta_{ix}$$
$$M_{jx} = \frac{I_x G}{L}\theta_{ix}$$

$$\begin{bmatrix}
\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{I_xG}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{I_xG}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{I_xG}{L} & 0 & 0 & 0 & 0 & 0 & \frac{I_xG}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
\begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix}
=
\begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix}
\tag{4.33}$$

4.13 GLOBAL JOINT STIFFNESS

The solution of structures using the global joint stiffness method requires that each component be in the same Cartesian coordinate system. The main components of the system are the members, loads, and supports. The main results desired from the solution are the joint deformations, member forces, and support reactions. The global or joint system is the primary way to organize all the components. The supports and some of the loads will be expressed in this system directly. Some of the loads will have to be transformed into this system. The basic process to set up and solve the structure using this stiffness is summarized as follows:

1. Find the local member stiffness, $[K_m]$, using Equation 4.33 (4.23 or 4.32 for 2-D systems). Rotate the local member stiffness to the global joint stiffness system, $[K_g]$. This is shown in Equation 4.34.

$$\left[K_g\right]=[R]^T[K_m][R] \tag{4.34}$$

2. Assemble all of the members into the global joint stiffness matrix. This is done by using joint labeling to order the matrix.
3. Determine the global joint loading, $[P_g]$, from all direct loads on joints, $[P\ \&\ M_g]$, and loads on members. The member loads are applied as the opposite of the fixed-end forces and moments, $[FEPM_m]$. These must be rotated from the local system to the global system, $[R]^T$.

$$\left[P_g\right]=\left[P\,\&\,M_g\right]-\left[FEPM_m\right][R]^T \tag{4.35}$$

4. Solve the general stiffness equation for global displacements, $[\Delta_g]$. The rows and columns of the matrices corresponding to the deformations restrained by the supports are removed prior to solving the system of equations.

$$\left[K_g\right]\left[\Delta_g\right]=\left[P_g\right] \tag{4.36}$$

5. Determine the reactions at the support, $[P_g]$, from the global deformations. Any fixed-end forces must be subtracted from the results. Only the forces at the supports due to the free deformations need to be found.

$$\left[P_g\right]=\left[K_g\right]\left[\Delta_g\right]+\left[FEPM_m\right][R]^T \tag{4.37}$$

6. Solve for the local member forces and moments, $[P\ \&\ M_m]$, for each member separately from the global joint deformations. The global joint deformations must be rotated into the member system, $[R]$. The fixed-end forces and moments must be added back to get the final member end forces.

$$\left[P\,\&\,M_m\right]=\left[K_m\right][R]\left[\Delta_g\right]+\left[FEPM_m\right] \tag{4.38}$$

The general set-up for the stiffness method of analysis represents a system of linear equations, where the displacement vector is the unknown. Except for those designated as supports, there are six unknown joint displacement components for each joint in the structure in a 3-D structure. Each displacement released at a support is still an unknown displacement component to the system. There is an equation for each degree of freedom of the structure. Each non-related component at a support has a displacement that is set identically equal to zero, and as far as the system of equations is concerned, this particular equation may be omitted, along with any coefficient in the other equations which corresponds to the dropped displacement.

Sometimes the system solution is handled in six by six blocks of coefficients or six rows of equations at a time, where each block representing the accumulated stiffness for a joint in the case of the diagonal, or the carry-over effects from other joints in the case of off-diagonals. In this case, unless the support joint is fully restrained, its corresponding row of six by six blocks is maintained intact and a number of sufficient sizes to simulate "infinite stiffness" in the restrained direction are added to the diagonal of the diagonal block in the master stiffness matrix.

The building of the global joint stiffness matrix consists of various stages of operations. First, the member stiffness matrix is defined in its own system, giving due consideration to member end releases, for each member in the structure. This will be discussed in Chapter 5. The member stiffness matrix can be considered as four separate six by six blocks. These represent the forces at the ends due to the motions at the end and were discussed in Sections 4.10 and 4.11.

$$\left[K_m\right] = \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix}$$

The four separate components of the global member stiffness should be placed in the global joint stiffness matrix. The diagonal terms will be added to other terms representing the stiffness of other members connected to that joint. The off-diagonal terms will simply be placed in their appropriate position. The transformation of the local member system to the global joint system was shown in Equation 4.34 and is derived as follows starting with the local stiffness equation:

$$\left[K_m\right]\left[\Delta_m\right] = \left[P_m\right]$$

This equation represents the force in the local system due to deformations in the local system. To go from local to global we multiply by the rotation transpose, $[R]^T$, on both sides of the equation.

$$[K_m][R]^T[\Delta_m] = [R]^T[P_m]$$

The right side of the equation now represents the forces in the global system, $[P_g]$. The left side represents the force in the global system due to local deformations. The equation needs to be written in terms of the global deformation, $[\Delta_g]$. From Section 4.3, the local deformation is the global deformation multiplied by the rotation matrix, $[R]$.

$$[R]^T[K_m][R][\Delta_g] = [P_g]$$

Example 4.19 Global joint stiffness

Determine the global joint deformations, support reactions, and local member forces for the pin-connected bracing structure loaded as shown in Figure 4.33.

The area of each member, A_x, is 10 in^2 and the modulus of elasticity, E, is 10,000 ksi. Note that the structure is in the XZ coordinate system.

Since this is a pin connected structure loaded only at the joint, it will act as a true truss with only axial forces in the members. The stiffness model will be simplified to only include the axial stiffness components, AE/L. Furthermore, rotation at the joints will be excluded since there is no rotational stiffness imparted by the members. Rotation could be included

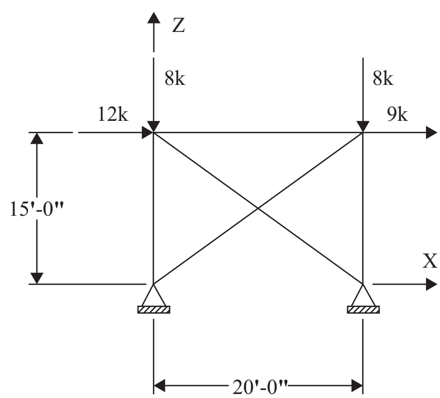

Figure 4.33. Example 4.19 Global joint stiffness.

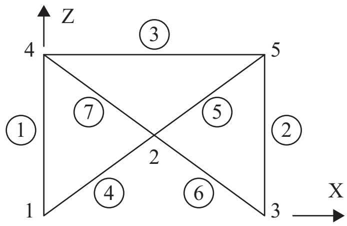

Figure 4.34. Example 4.19 Global joint stiffness.

if it is desired to find the rotation of the member ends, but this can easily be found from the final deformed position of the structure.

A numbering system must be assigned to the joint and members for easy bookkeeping. Figure 4.34 shows a numbering system for the five joints and the seven members of the structure. The member numbers are circled for clarity.

The member stiffness for each member is first found from Equation 4.34 (step 1).

$$[K_g] = [R]^T [K_m][R]$$
$$[K_g] = [\beta]^T [K_m][\beta]$$

Expand the general local member stiffness to include just the x and z forces and motions. Also note that rotation transformation of members will be about the y-axis or β.

$$[K_g] = \begin{bmatrix} cos\beta & sin\beta & 0 & 0 \\ -sin\beta & cos\beta & 0 & 0 \\ 0 & 0 & cos\beta & sin\beta \\ 0 & 0 & -sin\beta & cos\beta \end{bmatrix} \begin{bmatrix} \frac{AE}{L} & 0 & -\frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & \frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\times \begin{bmatrix} cos\beta & sin\beta & 0 & 0 \\ -sin\beta & cos\beta & 0 & 0 \\ 0 & 0 & cos\beta & sin\beta \\ 0 & 0 & -sin\beta & cos\beta \end{bmatrix}$$

To simplify the process, we can multiply the expanded global stiffness equation and factor out the axial stiffness term.

$$\left[K_g\right] = \frac{AE}{L}\begin{bmatrix} cos^2\beta & -sin\beta cos\beta & cos^2\beta & sin\beta cos\beta \\ -sin\beta cos\beta & sin^2\beta & sin\beta cos\beta & -sin^2\beta \\ cos^2\beta & sin\beta cos\beta & cos^2\beta & -sin\beta cos\beta \\ sin\beta cos\beta & -sin^2\beta & -sin\beta cos\beta & sin^2\beta \end{bmatrix}$$

Members 1 and 2 have the same orientation. Selecting the bottom end as the *i*-end, the rotation is –90° or 270°. The axial stiffness is 555.6 k/in.

$$\left[K_{14}\right] = \left[K_{35}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 555.6 & 0 & -555.6 \\ 0 & 0 & 0 & 0 \\ 0 & -555.6 & 0 & 555.6 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{14} \\ K_{41} & K_{44} \end{bmatrix} = \begin{bmatrix} K_{33} & K_{35} \\ K_{53} & K_{55} \end{bmatrix}$$

Member 3 does not need rotations since it is already in the global system orientation. The global stiffness will be the same as the member stiffness. Selecting the left end as the *i*-end, the rotation is 0°. The axial stiffness is 416.7 k/in.

$$\left[K_{45}\right] = \begin{bmatrix} 416.7 & 0 & -416.7 & 0 \\ 0 & 0 & 0 & 0 \\ -416.7 & 0 & 416.7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} K_{44} & K_{45} \\ K_{54} & K_{55} \end{bmatrix}$$

Members 4 and 5 have the same orientation. Selecting the bottom-left end as the *i*-end, the rotation is –36.87° or 323.13°. Instead of using angles, in this case it is easier to use trigonometry directly. The cosine is 0.8 and the sine is –0.6. The axial stiffness is 666.7 k/in.

$$\left[K_{12}\right] = \left[K_{25}\right] = \begin{bmatrix} 426.7 & 320.0 & -426.7 & -320.0 \\ 320.0 & 240.0 & -320.0 & -240.0 \\ -426.7 & -320.0 & 426.7 & 320.0 \\ -320.0 & -240.0 & 320.0 & 240.0 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$
$$= \begin{bmatrix} K_{22} & K_{25} \\ K_{52} & K_{55} \end{bmatrix}$$

Members 6 and 7 have the same orientation. Selecting the top-left end as the *i*-end, the rotation is 36.87° or –323.13°. Instead of using angles, in this case it is easier to use trigonometry directly. The cosine is 0.8 and the sine is 0.6. The axial stiffness is 666.7 k/in.

$$[K_{23}]=[K_{42}]=\begin{bmatrix}426.7 & -320.0 & -426.7 & 320.0\\ -320.0 & 240.0 & 320.0 & -240.0\\ -426.7 & 320.0 & 426.7 & -320.0\\ 320.0 & -240.0 & -320.0 & 240.0\end{bmatrix}=\begin{bmatrix}K_{22} & K_{23}\\ K_{32} & K_{33}\end{bmatrix}$$
$$=\begin{bmatrix}K_{44} & K_{42}\\ K_{24} & K_{22}\end{bmatrix}$$

The global joint stiffness matrix can be assembled using each of the member's contributions (step 2).

$$[K_g]=\begin{bmatrix}K_{11} & K_{12} & 0 & K_{14} & 0\\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25}\\ 0 & K_{32} & K_{33} & 0 & K_{35}\\ K_{41} & K_{42} & 0 & K_{44} & K_{45}\\ 0 & K_{52} & K_{53} & K_{54} & K_{55}\end{bmatrix}$$

$$[K_g]=\begin{bmatrix}427 & 320 & -427 & -320 & 0 & 0 & 0 & 0 & 0 & 0\\ 320 & 796 & -320 & -240 & 0 & 0 & 0 & -566 & 0 & 0\\ -427 & -320 & 1707 & 0 & -427 & 320 & -427 & 320 & -427 & -320\\ -320 & -240 & 0 & 960 & 320 & -240 & 320 & -240 & -320 & -240\\ 0 & 0 & -427 & 320 & 427 & -320 & 0 & 0 & 0 & 0\\ 0 & 0 & 320 & -240 & -320 & 796 & 0 & 0 & 0 & -566\\ 0 & 0 & -427 & 320 & 0 & 0 & 843 & -320 & -417 & 0\\ 0 & -566 & 320 & -240 & 0 & 0 & -320 & 796 & 0 & 0\\ 0 & 0 & -427 & -320 & 0 & 0 & -417 & 0 & 843 & 320\\ 0 & 0 & -320 & -240 & 0 & -566 & 0 & 0 & 320 & 795\end{bmatrix}$$

The global joint loading can be determined from Equation 4.35 directly since all of the applied loads are at the joints and in the global system (step 3). Note that there are no loads applied directly to the members, so there are no fixed-end forces and moments. The load matrix is in units of kips (k).

$$\left[P_g\right]=\left[P\,\&\,M_g\right]-\left[FEPM_m\right]\left[R\right]^T=\begin{bmatrix}P_{1x}\\P_{1y}\\P_{2x}\\P_{2y}\\P_{3x}\\P_{3y}\\P_{4x}\\P_{4y}\\P_{5x}\\P_{6y}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\12\\-8\\9\\-8\end{bmatrix}$$

Using any of the methods for solving non-homogenous linear algebraic equations, the global deformations can be found from the global stiffness Equation 4.36 (step 4). The rows and columns corresponding to the support constraint degrees of freedom must be deleted prior to the solution. This would be both x and y at joints 1 and 3. The solution for the deformations will be in inches.

$$\left[\Delta_g\right]=\left[K_g\right]^{-1}\left[P_g\right]$$

$$\begin{bmatrix}\Delta_{2x}\\\Delta_{2y}\\\Delta_{4x}\\\Delta_{4y}\\\Delta_{5x}\\\Delta_{6y}\end{bmatrix}=\begin{bmatrix}1707&0&-427&320&-427&-320\\0&960&320&-240&-320&-240\\-427&320&843&-320&-417&0\\320&-240&-320&796&0&0\\-427&-320&-417&0&843&320\\-320&-240&0&0&320&795\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\12\\-8\\9\\-8\end{bmatrix}=\begin{bmatrix}0.0246\\-0.0057\\0.0595\\0.0022\\0.0602\\-0.0261\end{bmatrix}$$

The reactions at the support can be found using the solution of the global deformation with Equation 4.37 (step 5). Only the terms in the rows corresponding to the restrained degrees of freedom and in the columns of the unrestrained degrees of freedom need to be included. Since there are no applied loads on the members, the fixed-end forces and moments are omitted. In addition, there are no applied loads at the support locations so the global applied forces and moments are omitted. The reaction forces are in kips (k).

$$\left[P\right]=\left[K_g\right]\left[\Delta_g\right]-\left[P\,\&\,M_g\right]-\left[FEPM_m\right]\left[R\right]^T$$

$$[P]=\left[K_g\right]\left[\Delta_g\right]=\begin{bmatrix}-427 & -320 & 0 & 0 & 0 & 0\\ -320 & -240 & 0 & -566 & 0 & 0\\ -427 & 320 & 0 & 0 & 0 & 0\\ 320 & -240 & 0 & 0 & 0 & -566\end{bmatrix}\begin{bmatrix}0.0246\\ -0.0057\\ 0.0595\\ 0.0022\\ 0.0602\\ -0.0261\end{bmatrix}$$

$$=\begin{bmatrix}8.67\\ -7.75\\ 12.33\\ 23.75\end{bmatrix}$$

The final step is finding the member forces for each of the members using Equation 4.35 (step 6). Since there are no applied forces on the members, the fixed-end forces and moments can be omitted. Note that the local member stiffness matrix is used here and not the global matrix. The member force will be in kips. If the *i*-end is positive, the member is in compression and if it is negative the member is in tension.

$$\left[P\,\&\,M_m\right]=\left[K_m\right][R]\left[\Delta_g\right]+\left[FEPM_m\right]$$
$$\left[P\,\&\,M_m\right]=\left[K_m\right][\beta]\left[\Delta_g\right]$$

For member 1, the deformations at joints 1 and 4 are used.

$$\begin{bmatrix}555.6 & 0 & 0 & 0\\ 0 & -555.6 & 0 & 0\\ -555.6 & 0 & 555.6 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0.0595\\ 0.0022\end{bmatrix}=\begin{bmatrix}-1.24\\ 0\\ 1.24\\ 0\end{bmatrix}$$

For member 2, the deformations at joints 3 and 5 are used.

$$\begin{bmatrix}555.6 & 0 & 0 & 0\\ 0 & -555.6 & 0 & 0\\ -555.6 & 0 & 555.6 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0.0602\\ -0.0261\end{bmatrix}=\begin{bmatrix}14.5\\ 0\\ -14.5\\ 0\end{bmatrix}$$

For member 3, the deformations at joints 4 and 5 are used.

$$\begin{bmatrix} 416.7 & 0 & -416.7 & 0 \\ 0 & 0 & 0 & 0 \\ -416.7 & 0 & 416.7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.0595 \\ 0.0022 \\ 0.0602 \\ -0.0261 \end{bmatrix} = \begin{bmatrix} -0.326 \\ 0 \\ 0.326 \\ 0 \end{bmatrix}$$

For member 4, the deformations at joints 1 and 2 are used.

$$\begin{bmatrix} 666.7 & 0 & -666.7 & 0 \\ 0 & 0 & 0 & 0 \\ -666.7 & 0 & 666.7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.0246 \\ -0.0057 \end{bmatrix} = \begin{bmatrix} -10.8 \\ 0 \\ 10.8 \\ 0 \end{bmatrix}$$

For member 5, the deformations at joints 2 and 5 are used.

$$\begin{bmatrix} 666.7 & 0 & -666.7 & 0 \\ 0 & 0 & 0 & 0 \\ -666.7 & 0 & 666.7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.0246 \\ -0.0057 \\ 0.0602 \\ -0.0261 \end{bmatrix} = \begin{bmatrix} -10.8 \\ 0 \\ 10.8 \\ 0 \end{bmatrix}$$

For member 6, the deformations at joints 2 and 3 are used.

$$\begin{bmatrix} 666.7 & 0 & -666.7 & 0 \\ 0 & 0 & 0 & 0 \\ -666.7 & 0 & 666.7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.0246 \\ -0.0057 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15.4 \\ 0 \\ 15.4 \\ 0 \end{bmatrix}$$

For member 7, the deformations at joints 4 and 2 are used.

$$\begin{bmatrix} 666.7 & 0 & -666.7 & 0 \\ 0 & 0 & 0 & 0 \\ -666.7 & 0 & 666.7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.0595 \\ 0.0024 \\ 0.0246 \\ -0.0057 \end{bmatrix} = \begin{bmatrix} 15.4 \\ 0 \\ 15.4 \\ 0 \end{bmatrix}$$

Example 4.20 Global joint stiffness

Determine the global joint deformations, support reactions, and local member forces for the rigidly connected frame structure loaded as shown in Figure 4.35.

The area of each member, A_x, is 10 in^2, the moment of inertia, I_z, is 1000 in^4, and the modulus of elasticity, E, is 10,000 ksi. Note the structure is in the XY coordinate system and the numbering system is similar to Example 4.19.

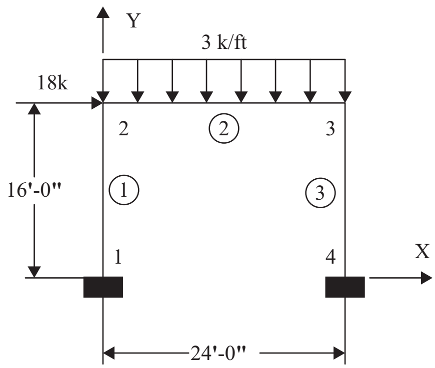

Figure 4.35. Example 4.20 Global joint stiffness.

The member stiffness for each member is first found from Equation 4.34 (step 1).

$$[K_g] = [R]^T [K_m][R]$$
$$[K_g] = [a]^T [K_m][a]$$

Using the local member stiffness for the XY system from Section 4.11 which has translation in x and y direction and rotation about the z direction, the rotation transformation of members will be about the z-axis or α.

$$[a]^T = \begin{bmatrix} cos\alpha & -sin\alpha & 0 & 0 & 0 & 0 \\ sin\alpha & cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & cos\alpha & -sin\alpha & 0 \\ 0 & 0 & 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K_m] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix}$$

$$[a] = \begin{bmatrix} cos a & sin a & 0 & 0 & 0 & 0 \\ -sin a & cos a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & cos a & sin a & 0 \\ 0 & 0 & 0 & -sin a & cos a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Members 1 and 3 have the same orientation. Selecting the bottom end as the *i*-end, the rotation is 90°. Table 4.3 contains the local member stiffness matrix, the rotation matrices, a and a^T, and the global member stiffness matrix.

$$[K_{12}] = [K_{43}] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} K_{44} & K_{43} \\ K_{34} & K_{33} \end{bmatrix}$$

Member 2 is already in the global system and does not need rotation. Selecting the left end as the *i*-end, the rotation is 0°. Table 4.4 contains the local member stiffness matrix, the rotation matrices, a and a^T, and the global member stiffness matrix.

$$[K_{23}] = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix}$$

Table 4.3. Example 4.20 Global joint stiffness

K_L

1510.4	0	0	-1510.4	0	0
0	49.17	4720	0	-49.17	4720
0	4720	604166.67	0	-4720	302083.33
-1510.4	0	0	1510.4	0	0
0	-49.17	-4720	0	49.17	-4720
0	4720	302083.33	0	-4720	604166.67

α

0	1	0	0	0	0
-1	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	1	0
0	0	0	-1	0	0
0	0	0	0	0	1

α_T

0	-1	0	0	0	0
1	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	-1	0
0	0	0	1	0	0
0	0	0	0	0	1

K_G

49.2	0	-4720	-49.2	0	-4720
0	1510	0	0	-1510	0
-4720	0	604166.67	4720	0	302083.33
-49.2	0	4720	49.2	0	4720
0	-1510	0	0	1510	0
-4720	0	302083.33	4720	0	604166.67

The global joint stiffness matrix can be assembled using each of the member's contributions (step 2). Table 4.5 contains the global joint stiffness matrix.

The global joint loading is determined from Equation 4.35. In this case, member 2 is loaded with a uniformly distributed load. The fixed-end forces and moments due to the load must be calculated. Normally, the fixed-end forces and moments are rotated into the global system before they are placed in the global joint loading, but in this case, the member is already in the global system and no rotation is necessary (step 3). The load matrix is in units of kips and inches (k-in).

Table 4.4. Example 4.20 Global joint stiffness

$\mathbf{K_L}$

1006.9	0	0	-1006.9	0	0
0	14.57	2098	0	-14.57	2098
0	2098	402777.78	0	-2098	201388.89
-1006.9	0	0	1006.9	0	0
0	-14.57	-2098	0	14.57	-2098
0	2098	201388.89	0	-2098	402777.78

α

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

α_T

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

K_G

1006.9	0	0	-1006.9	0	0
0	15	2098	0	-15	2098
0	2098	402777.78	0	-2098	201388.89
-1006.9	0	0	1006.9	0	0
0	-15	-2098	0	15	-2098
0	2098	201388.89	0	-2098	402777.78

Table 4.5. Example 4.20 Global joint stiffness

$$\left[K_g\right]=\begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix}$$

$$FEP_{23} = \frac{wL}{2} = \frac{0.25k/in(288in)}{2} = 36k$$

$$FEM_{23} = \frac{wL^2}{12} = \frac{0.25k/in(288in)^2}{12} = 1728k - in$$

$$FEP_{32} = \frac{wL}{2} = \frac{0.25k/in(288in)}{2} = 36k$$

$$FEM_{32} = -\frac{wL^2}{12} = -\frac{0.25k/in(288in)^2}{12} = -1728k - in$$

The 18 k lateral load is placed directly on joint 2 in the *y* direction.

$$\left[P_g\right] = \left[P \,\&\, M_g\right] - \left[FEPM_m\right]\left[R\right]^T = \begin{bmatrix} P_{1x} \\ P_{1y} \\ M_{1z} \\ P_{2x} \\ P_{2y} \\ M_{2z} \\ P_{3x} \\ P_{3y} \\ M_{3z} \\ P_{4x} \\ P_{4y} \\ M_{4z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 18 \\ -36 \\ -1728 \\ 0 \\ -36 \\ 1728 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The global deformations can be found from the global stiffness Equation 4.36 (step 4). The rows and columns corresponding to the support constraint degrees of freedom must be deleted prior to the solution. This would be all three motions at 1 and 4. The resulting matrix is just joints 2 and 3. This is shown in Table 4.6 along with the reduced load. The solution for the deformations will be in inches and radians.

$$\left[\Delta_g\right] = \left[K_g\right]^{-1}\left[P_g\right] = \begin{bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \theta_{2z} \\ \Delta_{3x} \\ \Delta_{3y} \\ \theta_{3z} \end{bmatrix} = \begin{bmatrix} 0.3040 \\ -0.0207 \\ -0.0034 \\ 0.2852 \\ -0.0270 \\ 0.0010 \end{bmatrix}$$

Table 4.6. Example 4.20 Global joint stiffness

K_G						P-FEPM
1056	0	4720	-1007	0	0	18
0	1525	2098	0	-15	2098	-36
4720	2098	1006944	0	-2098	201389	-1728
-1007	0	0	1056	0	4720	0
0	-15	-2098	0	1525	-2098	-36
0	2098	201389	4720	-2098	1006944	1728

The reactions at the supports can be found using the solution of the global deformation with Equation 4.37 (step 5). Only the terms in the rows corresponding to the restrained degrees of freedom and in the columns of the unrestrained degrees of freedom need to be included. Table 4.7 shows the appropriate stiffness terms and deformations needed to find the reactions. Since there were no fixed-end forces and moments at the support joints, the solution is complete. The reaction forces are in kips and inches (k-in).

$$[P]=[K_g][\Delta_g]-[P\,\&\,M_g]-[FEPM_m][R]^T$$
$$[P]=[K_g][\Delta_g]$$

Table 4.7. Example 4.20 Global joint stiffness

K_{G2}						Δ_G	P
-49	0	-4720	0	0	0	0.3040	0.92
0	-1510	0	0	0	0	-0.0207	31.22
4720	0	302083	0	0	0	-0.0034	419.18
0	0	0	-49	0	-4720	0.2852	-18.92
0	0	0	0	-1510	0	-0.0270	40.78
0	0	0	4720	0	302083	0.0010	1659.74

The final step is finding the member forces for each of the members using Equation 4.35 (step 6). The member force will be in kips and inches. The local member stiffness matrix and the rotation matrix were shown in step 1 and are omitted here. The sign convention for the X-Y system applies when interpreting the final-end forces and moments.

$$[P\,\&\,M_m]=[K_m][R][\Delta_g]+[FEPM_m]$$
$$[P\,\&\,M_m]=[K_m][a][\Delta_g]$$

For member 1, the deformations at joints 1 and 2 are used. Table 4.8 contains the final member-end forces in the local system along with global deformations used to find those end forces.

Table 4.8. Example 4.20 Global joint stiffness

Δ_G		P_L	
0	Δ_{x1}	31.22	kips
0	Δ_{y1}	-0.92	kips
0	θ_{z1}	419.18	kip-in
0.3040	Δ_{x2}	-31.22	kips
-0.0207	Δ_{y2}	0.92	kips
-0.0034	θ_{z2}	-596.37	kip-in

For member 2, the deformations at joints 2 and 3 are used. Table 4.9 contains the final member-end forces in the local system along with global deformations used to find those end forces.

Table 4.9. Example 4.20 Global joint stiffness

Δ_G		P_L	
0.3040	Δ_{x2}	18.92	kips
-0.0207	Δ_{y2}	-4.78	kips
-0.0034	θ_{z2}	-1131.63	kip-in
0.2852	Δ_{x3}	-18.92	kips
-0.0270	Δ_{y3}	4.78	kips
0.0010	θ_{z3}	-245.46	kip-in

For member 3, the deformations at joints 4 and 3 are used. Table 4.10 contains the final member-end forces in the local system along with global deformations used to find those end forces.

Table 4.10. Example 4.20 Global joint stiffness

Δ_G		P_L	
0	Δ_{x4}	40.78	kips
0	Δ_{y4}	18.92	kips
0	θ_{z4}	1659.74	kip-in
0.2852	Δ_{x3}	-40.78	kips
-0.0270	Δ_{y3}	-18.92	kips
0.0010	θ_{z3}	1973.46	kip-in

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CHAPTER 5

Advanced Structural Stiffness

In this chapter, concepts learned in previous chapters are expanded and applied to advanced structural stiffness. This method is applied in computers to solve complex structures that are statically determinate or statically indeterminate. Matrices will be utilized to determine internal member forces and displacements within a structure. Small pieces of the structure are analyzed and then compiled into a larger matrix in order to view the structure as a whole. This procedure is the basis for finite element analyses.

5.1 MEMBER END RELEASES, X-Z SYSTEM

Joint stiffness is expressed in the master matrix for a structure, but two situations exist that may cause them to vary. First, a joint that is being utilized as a support may be released. For example, the support becomes slotted or pinned. When looking at a joint release, the joint is fully designed as a support before the release. This creates the reaction components. A support release is in the global system and is handled in the reduction of the joint stiffness matrix. Second, a member can be physically released from a joint for one or more of the six possible end displacements. When a member release occurs, the member is released in some direction and the stiffness contribution that member was making to the joint changes or goes to zero. When a member is released in the local system, this release changes the member stiffness matrix. Some of these released stiffness matrices will be derived in the following examples. The order of the motions and forces on the member's end are given by the deflections and rotations at the *i*-end followed by the deflections and rotations at the *j*-end. The following is a full list of stiffness values simplified from Equation 4.33:

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{2,2} & 0 & 0 & 0 & k_{2,6} & 0 & k_{2,8} & 0 & 0 & 0 & k_{2,12} \\ 0 & 0 & k_{3,3} & 0 & k_{3,5} & 0 & 0 & 0 & k_{3,9} & 0 & k_{3,11} & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & k_{5,3} & 0 & k_{5,5} & 0 & 0 & 0 & k_{5,9} & 0 & k_{5,11} & 0 \\ 0 & k_{6,2} & 0 & 0 & 0 & k_{6,6} & 0 & k_{6,8} & 0 & 0 & 0 & k_{6,12} \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{8,2} & 0 & 0 & 0 & k_{8,6} & 0 & k_{8,8} & 0 & 0 & 0 & k_{8,12} \\ 0 & 0 & k_{9,3} & 0 & k_{9,5} & 0 & 0 & 0 & k_{9,9} & 0 & k_{9,11} & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & k_{11,3} & 0 & k_{11,5} & 0 & 0 & 0 & k_{11,9} & 0 & k_{11,11} & 0 \\ 0 & k_{12,2} & 0 & 0 & 0 & k_{12,6} & 0 & k_{12,8} & 0 & 0 & 0 & k_{12,12} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.1}$$

Example 5.1 Δ_{iz} end release

Derive the local member stiffness for a Δ_{iz} member end release using the conjugate beam method.

A free-body diagram of the released beam is shown in Figure 5.1. Since the beam is allowed to move at the *i*-end in the *z* direction, the reaction P_{iz} is equal to zero. The loaded conjugate beam is also shown in Figure 5.1. Note that the shear in the conjugate beam is equal to the rotation in the real beam and the moment in the conjugate beam is equal to the deflection in the real beam.

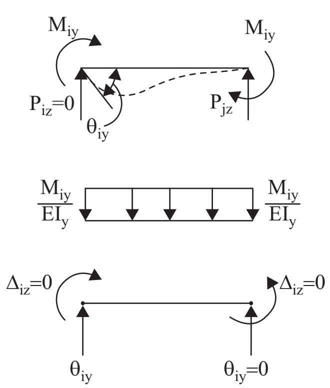

Figure 5.1. Example 5.1 Δ_{iz} end release.

If a motion Δ_{iz} is imposed, there is no resistance and therefore no forces. The resulting forces are the stiffness values due to the motion. The following are forces and stiffness due to Δ_{iz}:

$$P_{iz} = M_{iy} = P_{jz} = M_{jy} = 0$$
$$k_{3,3} = k_{5,3} = k_{9,3} = k_{11,3} = 0$$

If a motion Δ_{jz} is imposed, there is no resistance and therefore no forces. The resulting forces are the stiffness values due to the motion. The following are forces and stiffness due to Δ_{jz}:

$$P_{iz} = M_{iy} = P_{jz} = M_{jy} = 0$$
$$k_{3,9} = k_{5,9} = k_{9,9} = k_{11,9} = 0$$

If a motion θ_{iy} is imposed, there is resistance and therefore forces. The resulting forces are derived using conjugate beam.

$$\Sigma F_z = \theta_{iy} - \frac{M_{iy} L}{EI_y} = 0$$
$$M_{iy} = \frac{EI_y}{L}\theta_{iy}$$
$$k_{5,5} = \frac{EI_y}{L}\theta_{iy}$$

From statics on the real beam:

$$\Sigma M_y = M_{iy} + M_{jy} = 0$$
$$M_{jy} = -M_{iy} = -\frac{EI_y}{L}\theta_{iy}$$
$$k_{11,5} = -\frac{EI_y}{L}\theta_{iy}$$
$$P_{iz} = P_{jz} = 0$$
$$k_{3,5} = k_{9,5} = 0$$

From symmetry of the stiffness matrix, the following terms can be found:

$$k_{5,11} = k_{11,5}$$
$$k_{5,11} = -\frac{EI_y}{L}\theta_{jy}$$

From statics on the real beam considering θ_{jy}:

$$M_{iy} = -M_{jy} = \frac{EI_y}{L}\theta_{jy}$$

$$k_{11,11} = \frac{EI_y}{L}\theta_{jy}$$

$$P_{iz} = P_{jz} = 0$$

$$k_{3,11} = k_{9,11} = 0$$

The resulting stiffness matrix is shown in Equation 5.2 with only the affected terms replaced with the new values.

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{2,2} & 0 & 0 & 0 & k_{2,6} & 0 & k_{2,8} & 0 & 0 & 0 & k_{2,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{EI_y}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EI_y}{L} & 0 \\ 0 & k_{6,2} & 0 & 0 & 0 & k_{6,6} & 0 & k_{6,8} & 0 & 0 & 0 & k_{6,12} \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{8,2} & 0 & 0 & 0 & k_{8,6} & 0 & k_{8,8} & 0 & 0 & 0 & k_{8,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{EI_y}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EI_y}{L} & 0 \\ 0 & k_{12,2} & 0 & 0 & 0 & k_{12,6} & 0 & k_{12,8} & 0 & 0 & 0 & k_{12,12} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.2}$$

Example 5.2 θ_{iy} end release

Derive the local member stiffness for a θ_{iy} member end release using the conjugate beam method.

A free-body diagram of the released beam is shown in Figure 5.2. Since the beam is allowed to rotate at the *i*-end in the *y* direction, the reaction M_{iy} is equal to zero. The loaded conjugate beam is also shown in Figure 5.2. Note that the shear in the conjugate beam is equal to the rotation in the real beam and the moment in the conjugate beam is equal to the deflection in the real beam.

If a motion Δ_{iz} is imposed, there is resistance and therefore forces. The resulting forces are derived using conjugate beam.

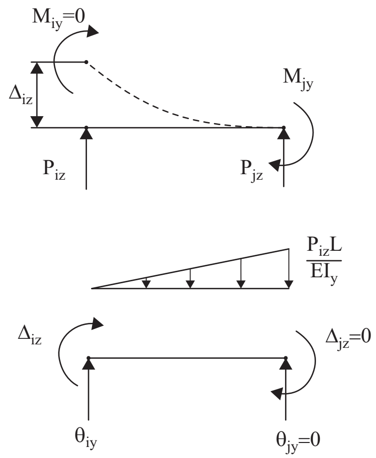

Figure 5.2. Example 5.2 θ_{iy} end release.

$$\Sigma M_{jy} = -\Delta_{iz} + \frac{P_{iz}L}{EI_y}\frac{L}{2}\frac{2L}{3} = 0$$

$$P_{iz} = \frac{3EI_y}{L^3}\Delta_{iz}$$

$$k_{3,3} = \frac{3EI_y}{L^3}\Delta_{iz}$$

From statics on the real beam:

$$\Sigma M_{jy} = P_{iz}L + M_{jy} = 0$$

$$M_{jy} = -P_{iz}L = -\frac{3EI_y}{L^2}\Delta_{iz}$$

$$k_{11,3} = -\frac{3EI_y}{L^2}\Delta_{iz}$$

$$\Sigma F_z = P_{iz} + P_{jz} = 0$$

$$P_{jz} = -P_{iz} = -\frac{3EI_y}{L^3}\Delta_{iz}$$

$$k_{9,3} = -\frac{3EI_y}{L^3}\Delta_{iz}$$

$$M_{iy} = 0$$

$$k_{5,3} = 0$$

If a motion Δ_{jz} is imposed, there is resistance and therefore forces. The free-body diagram of the deflected shape is shown in Figure 5.3. The conjugate beam will be the same as shown in Figure 5.2. The resulting forces are derived using conjugate beam. Note that the direction of P_{iz} in Figure 5.3 will be the opposite of that in Figure 5.2, so all the values will be the opposite of those from Δ_{iz}.

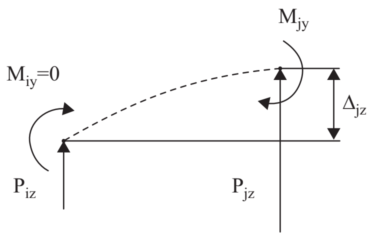

Figure 5.3. Example 5.2 θ_{iy} end release.

$$P_{iz} = -\frac{3EI_y}{L^3}\Delta_{jz}$$

$$k_{3,9} = -\frac{3EI_y}{L^3}\Delta_{jz}$$

$$M_{jy} = -P_{iz}L = \frac{3EI_y}{L^2}\Delta_{jz}$$

$$k_{11,9} = \frac{3EI_y}{L^2}\Delta_{jz}$$

$$P_{jz} = -P_{iz} = \frac{3EI_y}{L^3}\Delta_{jz}$$

$$k_{9,9} = \frac{3EI_y}{L^3}\Delta_{jz}$$

If a motion θ_{iy} is imposed, there is no resistance and therefore no forces. The resulting forces are the stiffness values due to the motion. The following are forces and stiffness due to Δ_{iz}:

$$P_{iz} = M_{iy} = P_{jz} = M_{jy} = 0$$

$$k_{3,5} = k_{5,5} = k_{9,5} = k_{11,5} = 0$$

If a motion θ_{jy} is imposed, there is resistance and therefore forces. The free-body diagram of the deflected shape is shown in Figure 5.4. The

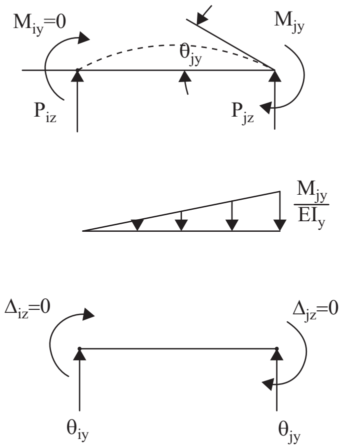

Figure 5.4. Example 5.2 θ_{iy} end release.

conjugate beam is also shown in Figure 5.4. The resulting forces are derived using conjugate beam.

$$\Sigma M_{iy} = \theta_{jy} L - \frac{M_{jy}}{EI_y} \frac{L}{2} \frac{2L}{3} = 0$$

$$M_{jy} = \frac{3EI_y}{L} \theta_{jy}$$

$$k_{11,11} = \frac{3EI_y}{L} \theta_{jy}$$

From statics on the real beam:

$$\Sigma M_{jy} = P_{iz} L + M_{jy} = 0$$

$$P_{iz} = -\frac{M_{jy}}{L} = -\frac{3EI_y}{L^2} \theta_{jy}$$

$$k_{3,11} = -\frac{3EI_y}{L^2} \theta_{jy}$$

$$\Sigma F_z = P_{iz} + P_{jz} = 0$$

$$P_{jz} = -P_{iz} = \frac{3EI_y}{L^2} \theta_{jy}$$

$$k_{9,11} = \frac{3EI_y}{L^2} \theta_{jy}$$

$$M_{iy} = 0$$

$$k_{5,11} = 0$$

The resulting stiffness matrix is shown in Equation 5.3 with only the affected terms replaced with the new values.

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{2,2} & 0 & 0 & 0 & k_{2,6} & 0 & k_{2,8} & 0 & 0 & 0 & k_{2,12} \\ 0 & 0 & \frac{3EI_y}{L^3} & 0 & 0 & 0 & 0 & 0 & -\frac{3EI_y}{L^3} & 0 & \frac{-3EI_y}{L^2} & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{6,2} & 0 & 0 & 0 & k_{6,6} & 0 & k_{6,8} & 0 & 0 & 0 & k_{6,12} \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{8,2} & 0 & 0 & 0 & k_{8,6} & 0 & k_{8,8} & 0 & 0 & 0 & k_{8,12} \\ 0 & 0 & -\frac{3EI_y}{L^3} & 0 & 0 & 0 & 0 & 0 & \frac{3EI_y}{L^3} & 0 & \frac{3EI_y}{L^2} & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & -\frac{3EI_y}{L^2} & 0 & 0 & 0 & 0 & 0 & \frac{3EI_y}{L^2} & 0 & \frac{3EI_y}{L} & 0 \\ 0 & k_{12,2} & 0 & 0 & 0 & k_{12,6} & 0 & k_{12,8} & 0 & 0 & 0 & k_{12,12} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.3}$$

The member stiffness for releasing Δ_{jz} and θ_{jy} can be derived in a similar manner to Δ_{iz} and θ_{iy}. The resulting stiffness matrices are shown in Equations 5.4 and 5.5 with only the affected terms replaced with the new values.

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{2,2} & 0 & 0 & 0 & k_{2,6} & 0 & k_{2,8} & 0 & 0 & 0 & k_{2,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{EI_y}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EI_y}{L} & 0 \\ 0 & k_{6,2} & 0 & 0 & 0 & k_{6,6} & 0 & k_{6,8} & 0 & 0 & 0 & k_{6,12} \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{8,2} & 0 & 0 & 0 & k_{8,6} & 0 & k_{8,8} & 0 & 0 & 0 & k_{8,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{EI_y}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EI_y}{L} & 0 \\ 0 & k_{12,2} & 0 & 0 & 0 & k_{12,6} & 0 & k_{12,8} & 0 & 0 & 0 & k_{12,12} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.4}$$

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{2,2} & 0 & 0 & 0 & k_{2,6} & 0 & k_{2,8} & 0 & 0 & 0 & k_{2,12} \\ 0 & 0 & \frac{3EI_y}{L^3} & 0 & -\frac{3EI_y}{L^2} & 0 & 0 & 0 & -\frac{3EI_y}{L^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & -\frac{3EI_y}{L^2} & 0 & \frac{3EI_y}{L} & 0 & 0 & 0 & \frac{3EI_y}{L^2} & 0 & 0 & 0 \\ 0 & k_{6,2} & 0 & 0 & 0 & k_{6,6} & 0 & k_{6,8} & 0 & 0 & 0 & k_{6,12} \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{8,2} & 0 & 0 & 0 & k_{8,6} & 0 & k_{8,8} & 0 & 0 & 0 & k_{8,12} \\ 0 & 0 & -\frac{3EI_y}{L^3} & 0 & \frac{3EI_y}{L^2} & 0 & 0 & 0 & \frac{3EI_y}{L^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{12,2} & 0 & 0 & 0 & k_{12,6} & 0 & k_{12,8} & 0 & 0 & 0 & k_{12,12} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.5}$$

When more than one of the four degrees of freedom is released on a flexural member, two conditions may exist. The first is that the member will provide no joint stiffness. This occurs when both deflection and rotation at either end are released, when rotation is released at both ends, or when deflection is released at one end and rotation is released at the other end. The stiffness matrix for this condition is shown in Equation 5.6. The resulting beam is either a cantilever beam, a pinned-pinned beam, or a pinned-slotted beam.

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{2,2} & 0 & 0 & 0 & k_{2,6} & 0 & k_{2,8} & 0 & 0 & 0 & k_{2,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{6,2} & 0 & 0 & 0 & k_{6,6} & 0 & k_{6,8} & 0 & 0 & 0 & k_{6,12} \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{8,2} & 0 & 0 & 0 & k_{8,6} & 0 & k_{8,8} & 0 & 0 & 0 & k_{8,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{12,2} & 0 & 0 & 0 & k_{12,6} & 0 & k_{12,8} & 0 & 0 & 0 & k_{12,12} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.6}$$

The second condition is an unstable beam. This occurs when both ends are released for deflection or more than two of the four degrees of freedom are released. Table 5.1 summarizes all the flexural stiffness conditions for the X-Z system. In this table, 1 indicates that the degree of freedom is released and 0 indicates that the degree of freedom is not released.

Table 5.1. Release codes—X-Z system

Δ_{iz}	θ_{iy}	Δ_{jz}	θ_{jy}	Equation
0	0	0	0	4.33
1	0	0	0	5.2
0	1	0	0	5.3
0	0	1	0	5.4
0	0	0	1	5.5
1	1	0	0	5.6
1	0	1	0	Unstable
1	0	0	1	5.6
0	1	1	0	5.6
0	1	0	1	5.6
0	0	1	1	5.6
0	1	1	1	Unstable
1	0	1	1	Unstable
1	1	0	1	Unstable
1	1	1	0	Unstable
1	1	1	1	Unstable

5.2 MEMBER END RELEASES, X-Y SYSTEM

The X-Y member stiffness can be derived in a similar manner to the X-Z system. In general, only the sign of the moments will change. The resulting stiffness matrices for releasing Δ_{iz}, θ_{iy}, Δ_{jz}, and θ_{jy} are shown in Equations 5.7 through 5.10 with only the affected terms replaced with the new values. Equation 5.11 contains the stiffness matrix used when the member has no stiffness contribution in the case of the cantilever beam, a pinned-pinned beam, or a pinned-slotted beam.

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{3,3} & 0 & k_{3,5} & 0 & 0 & 0 & k_{3,9} & 0 & k_{3,11} & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & k_{5,3} & 0 & k_{5,5} & 0 & 0 & 0 & k_{5,9} & 0 & k_{5,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{EI_z}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-EI_z}{L} \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{9,3} & 0 & k_{9,5} & 0 & 0 & 0 & k_{9,9} & 0 & k_{9,11} & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & k_{11,3} & 0 & k_{11,5} & 0 & 0 & 0 & k_{11,9} & 0 & k_{11,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{EI_z}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EI_z}{L} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.7}$$

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3EI_z}{L^3} & 0 & 0 & 0 & 0 & 0 & -\frac{3EI_z}{L^3} & 0 & 0 & 0 & \frac{3EI_z}{L^2} \\ 0 & 0 & k_{3,3} & 0 & k_{3,5} & 0 & 0 & 0 & k_{3,9} & 0 & k_{3,11} & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & k_{5,3} & 0 & k_{5,5} & 0 & 0 & 0 & k_{5,9} & 0 & k_{5,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3EI_z}{L^3} & 0 & 0 & 0 & 0 & 0 & \frac{3EI_z}{L^3} & 0 & 0 & 0 & -\frac{3EI_z}{L^2} \\ 0 & 0 & k_{9,3} & 0 & k_{9,5} & 0 & 0 & 0 & k_{9,9} & 0 & k_{9,11} & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & k_{11,3} & 0 & k_{11,5} & 0 & 0 & 0 & k_{11,9} & 0 & k_{11,11} & 0 \\ 0 & \frac{3EI_z}{L^2} & 0 & 0 & 0 & 0 & 0 & -\frac{3EI_z}{L^2} & 0 & 0 & 0 & \frac{3EI_z}{L} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.8}$$

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{3,3} & 0 & k_{3,5} & 0 & 0 & 0 & k_{3,9} & 0 & k_{3,11} & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & k_{5,3} & 0 & k_{5,5} & 0 & 0 & 0 & k_{5,9} & 0 & k_{5,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{EI_z}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-EI_z}{L} \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{9,3} & 0 & k_{9,5} & 0 & 0 & 0 & k_{9,9} & 0 & k_{9,11} & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & k_{11,3} & 0 & k_{11,5} & 0 & 0 & 0 & k_{11,9} & 0 & k_{11,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{EI_z}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EI_z}{L} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.9}$$

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3EI_z}{L^3} & 0 & 0 & 0 & \frac{3EI_z}{L^2} & 0 & -\frac{3EI_z}{L^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{3,3} & 0 & k_{3,5} & 0 & 0 & 0 & k_{3,9} & 0 & k_{3,11} & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & k_{5,3} & 0 & k_{5,5} & 0 & 0 & 0 & k_{5,9} & 0 & k_{5,11} & 0 \\ 0 & \frac{3EI_z}{L^2} & 0 & 0 & 0 & \frac{3EI_z}{L} & 0 & -\frac{3EI_z}{L^2} & 0 & 0 & 0 & 0 \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3EI_z}{L^3} & 0 & 0 & 0 & -\frac{3EI_z}{L^2} & 0 & \frac{3EI_z}{L^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{9,3} & 0 & k_{9,5} & 0 & 0 & 0 & k_{9,9} & 0 & k_{9,11} & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & k_{11,3} & 0 & k_{11,5} & 0 & 0 & 0 & k_{11,9} & 0 & k_{11,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.10}$$

$$\begin{bmatrix} k_{1,1} & 0 & 0 & 0 & 0 & 0 & k_{1,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{3,3} & 0 & k_{3,5} & 0 & 0 & 0 & k_{3,9} & 0 & k_{3,11} & 0 \\ 0 & 0 & 0 & k_{4,4} & 0 & 0 & 0 & 0 & 0 & k_{4,10} & 0 & 0 \\ 0 & 0 & k_{5,3} & 0 & k_{5,5} & 0 & 0 & 0 & k_{5,9} & 0 & k_{5,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{7,1} & 0 & 0 & 0 & 0 & 0 & k_{7,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{9,3} & 0 & k_{9,5} & 0 & 0 & 0 & k_{9,9} & 0 & k_{9,11} & 0 \\ 0 & 0 & 0 & k_{10,4} & 0 & 0 & 0 & 0 & 0 & k_{10,10} & 0 & 0 \\ 0 & 0 & k_{11,3} & 0 & k_{11,5} & 0 & 0 & 0 & k_{11,9} & 0 & k_{11,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.11}$$

Table 5.2. Release codes—X-Y system

Δ_{jy}	θ_{iz}	Δ_{jy}	θ_{jz}	Equation
0	0	0	0	4.33
1	0	0	0	5.1
0	1	0	0	5.2
0	0	1	0	5.3
0	0	0	1	5.4
1	1	0	0	5.5
1	0	1	0	Unstable
1	0	0	1	5.5
0	1	1	0	5.5
0	1	0	1	5.5
0	0	1	1	5.5
0	1	1	1	Unstable
1	0	1	1	Unstable
1	1	0	1	Unstable
1	1	1	0	Unstable
1	1	1	1	Unstable

Table 5.2 summarizes all the flexural stiffness conditions for the X-Y system. In this table, 1 indicates that the degree of freedom is released and 0 indicates that the degree of freedom is not released.

5.3 MEMBER END RELEASES, 3-D SYSTEM

When member releases occur in both coplanar systems, X-Z and X-Y, the member stiffness matrix should be created by combining the appropriate conditions from each system. The process for creating the global joint stiffness is the same as outlined in Section 4.13 with four modifications. First, the released member stiffness matrix should be used for the appropriate members in step one. Second, if a released member has a member load the fixed-end forces and moments must be modified. These modified forces and moments can be found in most structural analysis textbooks. Third, the released member stiffness matrix should be used to solve for the local member forces in step six. Lastly, the modified fixed-end forces and moments must be added to the local member end forces and moments in step six. The following example uses a released member to model a structure.

Example 5.3 Member stiffness

Determine the global joint deformations, support reactions, and local member forces for the pin-connected bracing structure loaded as shown in Figure 5.5.

The area of each member, A_x, is 20 in^2, the moment of inertia, I_z, is 1,000 in^4, and the modulus of elasticity, E, is 10,000 ksi. Note the structure is in the X-Y coordinated system and the numbering system is shown.

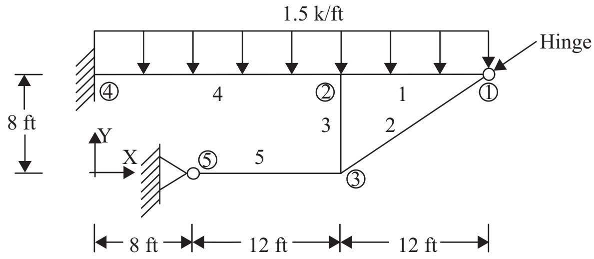

Figure 5.5. Example 5.3 Member stiffness.

The member stiffness for each member is first found from Equation 4.34 (step 1). Member 2 will have a released stiffness matrix and will be created last. Selecting member 2 instead of member 1 will result in not having to use modified fixed-end forces and moments.

$$[K_g] = [R]^T [K_m][R]$$
$$[K_g] = [a]^T [K_m][a]$$

Using the local member stiffness for the X-Y system from Section 4.11, which has translation in the *x* and *y* directions and rotation about the *z* direction, the rotation transformation of members will be about the *z*-axis or *a*.

$$[a]^T = \begin{bmatrix} cos\alpha & -sin\alpha & 0 & 0 & 0 & 0 \\ sin\alpha & cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & cos\alpha & -sin\alpha & 0 \\ 0 & 0 & 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K_m]=\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix}$$

$$[a]=\begin{bmatrix} cos\alpha & sin\alpha & 0 & 0 & 0 & 0 \\ -sin\alpha & cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & cos\alpha & sin\alpha & 0 \\ 0 & 0 & 0 & -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Members 1, 4, and 5 are already in the global system and do not need rotation. Selecting the left end as the *i*-end, the rotation is 0°. Tables 5.3 through 5.5 contain the local member stiffness matrices, the rotations matrices, a and a^T, and the global member stiffness matrices.

$$[K_{21}]=\begin{bmatrix} K_{22} & K_{21} \\ K_{12} & K_{11} \end{bmatrix}$$

$$[K_{42}]=\begin{bmatrix} K_{44} & K_{42} \\ K_{24} & K_{22} \end{bmatrix}$$

$$[K_{53}]=\begin{bmatrix} K_{55} & K_{53} \\ K_{35} & K_{33} \end{bmatrix}$$

Member 3 is vertical and selecting the bottom end as the *i*-end, the rotation is 90°. Table 5.6 contains the local member stiffness matrix, the rotations matrices, a and a^T, and the global member stiffness matrix.

$$[K_{32}]=\begin{bmatrix} K_{33} & K_{32} \\ K_{23} & K_{22} \end{bmatrix}$$

Table 5.3. Example 5.3 Member stiffness, member 1

$\mathbf{K_L}$

1388.9	0.0	0.0	−1388.9	0.0	0.0
0.0	40.2	2893.5	0.0	−40.2	2893.5
0.0	2893.5	277777.8	0.0	−2893.5	138888.9
−1388.9	0.0	0.0	1388.9	0.0	0.0
0.0	−40.2	−2893.5	0.0	40.2	−2893.5
0.0	2893.5	138888.9	0.0	−2893.5	277777.8

$\boldsymbol{\alpha}$

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

$\boldsymbol{\alpha}^T$

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

$\mathbf{K_G}$

1388.9	0.0	0.0	−1388.9	0.0	0.0
0.0	40.2	2893.5	0.0	−40.2	2893.5
0.0	2893.5	277777.8	0.0	−2893.5	138888.9
−1388.9	0.0	0.0	1388.9	0.0	0.0
0.0	−40.2	−2893.5	0.0	40.2	−2893.5
0.0	2893.5	138888.9	0.0	−2893.5	277777.8

For member 2, we will release the *i*-end at joint 1 for rotation. Selecting the *i*-end at joint 1 the rotation is 213.7°. Table 5.7 contains the local member stiffness matrix, the rotation matrices, α and α^T, and the global member stiffness matrix. The general released stiffness matrix is also shown.

$$[K_{13}] = \begin{bmatrix} K_{11} & K_{13} \\ K_{31} & K_{33} \end{bmatrix}$$

Table 5.4. Example 5.3 Member stiffness, member 4

$\mathbf{K_L}$

833.3	0.0	0.0	−833.3	0.0	0.0
0.0	8.7	1041.7	0.0	−8.7	1041.7
0.0	1041.7	166666.7	0.0	−1041.7	83333.3
−833.3	0.0	0.0	833.3	0.0	0.0
0.0	−8.7	−1041.7	0.0	8.7	−1041.7
0.0	1041.7	83333.3	0.0	−1041.7	166666.7

$\boldsymbol{\alpha}$

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

$\boldsymbol{\alpha}^T$

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

$\mathbf{K_G}$

833.3	0.0	0.0	−833.3	0.0	0.0
0.0	8.7	1041.7	0.0	−8.7	1041.7
0.0	1041.7	166666.7	0.0	−1041.7	83333.3
−833.3	0.0	0.0	833.3	0.0	0.0
0.0	−8.7	−1041.7	0.0	8.7	−1041.7
0.0	1041.7	83333.3	0.0	−1041.7	166666.7

$$[K_m] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{3EI_z}{L^3} & 0 & 0 & -\frac{3EI_z}{L^3} & \frac{3EI_z}{L^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{3EI_z}{L^3} & 0 & 0 & \frac{3EI_z}{L^3} & -\frac{3EI_z}{L^2} \\ 0 & \frac{3EI_z}{L^2} & 0 & 0 & -\frac{3EI_z}{L^2} & \frac{3EI_z}{L} \end{bmatrix}$$

Table 5.5. Example 5.3 Member stiffness, member 5

K$_L$

1388.9	0.0	0.0	−1388.9	0.0	0.0
0.0	40.2	2893.5	0.0	−40.2	2893.5
0.0	2893.5	277778	0.0	−2893.5	138888.9
−1388.9	0.0	0.0	1388.9	0.0	0.0
0.0	−40.2	−2893.5	0.0	40.2	−2893.5
0.0	2893.5	138888.9	0.0	−2893.5	277778

α

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

α^T

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

K$_G$

1388.9	0.0	0.0	−1388.9	0.0	0.0
0.0	40.2	2893.5	0.0	−40.2	2893.5
0.0	2893.5	277778	0.0	−2893.5	138888.9
−1388.9	0.0	0.0	1388.9	0.0	0.0
0.0	−40.2	−2893.5	0.0	40.2	−2893.5
0.0	2893.5	138888.9	0.0	−2893.5	277778

The global joint stiffness matrix can be assembled using each of the member's contributions (step 2). Table 5.8 contains the global joint stiffness matrix.

$$\left[K_g\right]=\begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & 0 \\ K_{31} & K_{32} & K_{33} & 0 & K_{35} \\ 0 & K_{42} & 0 & K_{44} & 0 \\ 0 & 0 & K_{53} & 0 & K_{55} \end{bmatrix}$$

Table 5.6. Example 5.3 Member stiffness, member 3

K_L

2083.3	0.0	0.0	−2083.3	0.0	0.0
0.0	135.6	6510.4	0.0	−135.6	6510.4
0.0	6510.4	416666.7	0.0	−6510.4	208333.3
−2083.3	0.0	0.0	2083.3	0.0	0.0
0.0	−135.6	−6510.4	0.0	135.6	−6510.4
0.0	6510.4	208333.3	0.0	−6510.4	416666.7

α

6.12574E-17	1	0	0	0	0
−1	6.12574E-17	0	0	0	0
0	0	1	0	0	0
0	0	0	6.12574E-17	1	0
0	0	0	−1	6.12574E-17	0
0	0	0	0	0	1

α^T

6.12574E-17	−1	0	0	0	0
1	6.12574E-17	0	0	0	0
0	0	1	0	0	0
0	0	0	6.12574E-17	−1	0
0	0	0	1	6.12574E-17	0
0	0	0	0	0	1

K_G

135.6	0.0	−6510.4	−135.6	0.0	−6510.4
0.0	2083.3	0.0	0.0	−2083.3	0.0
−6510.4	0.0	416666.7	6510.4	0.0	208333.3
−135.6	0.0	6510.4	135.6	0.0	6510.4
0.0	−2083.3	0.0	0.0	2083.3	0.0
−6510.4	0.0	208333.3	6510.4	0.0	416666.7

The global joint loading is determined from Equation 4.35. In this case, members 1 and 4 are loaded with a uniformly distributed load. The fixed-end forces and moments due to the load must be calculated. Normally, the fixed-end forces and moments are rotated into the global system before they are placed in the global joint loading, but in this case the member is already in the global system and no rotation is necessary (step 3). The load matrix is in units of kips and inches (k-in).

Table 5.7. Example 5.3 Member stiffness, member 2

K_L

1155.6	0.0	0.0	−1155.6	0.0	0.0
0.0	5.8	0.0	0.0	−5.8	1001.6
0.0	0.0	0.0	0.0	0.0	0.0
−1155.6	0.0	0.0	1155.6	0.0	0.0
0.0	−5.8	0.0	0.0	5.8	−1001.6
0.0	1001.6	0.0	0.0	−1001.6	173343.8

α

−0.832050294	−0.554700196	0	0	0	0
0.554700196	−0.832050294	0	0	0	0
0	0	1	0	0	0
0	0	0	−0.832050294	−0.554700196	0
0	0	0	0.554700196	−0.832050294	0
0	0	0	0	0	1

α^T

−0.832050294	0.554700196	0	0	0	0
−0.554700196	−0.832050294	0	0	0	0
0	0	1	0	0	0
0	0	0	−0.832050294	0.554700196	0
0	0	0	−0.554700196	−0.832050294	0
0	0	0	0	0	1

K_G

801.8	530.7	0.0	-801.8	-530.7	555.6
530.7	359.6	0.0	-530.7	-359.6	-833.4
0.0	0.0	0.0	0.0	0.0	0.0
-801.8	-530.7	0.0	801.8	530.7	-555.6
-530.7	-359.6	0.0	530.7	359.6	833.4
555.6	-833.4	0.0	-555.6	833.4	173343.8

$$FEP_{42} = \frac{wL}{2} = \frac{0.125\text{k/in}(240\text{in})}{2} = 15\text{k}$$

$$FEM_{42} = \frac{wL^2}{12} = \frac{0.125\text{k/in}(240\text{in})^2}{12} = 600\text{k-in}$$

$$FEP_{24} = \frac{wL}{2} = \frac{0.125\text{k/in}(240\text{in})}{2} = 15\text{k}$$

$$FEM_{24} = -\frac{wL^2}{12} = -\frac{0.125\text{k/in}(240\text{in})^2}{12} = -600\text{k-in}$$

$$FEP_{21} = \frac{wL}{2} = \frac{0.125\text{k/in}(144\text{in})}{2} = 9\text{k}$$

$$FEM_{21} = \frac{wL^2}{12} = \frac{0.125\text{k/in}(144\text{in})^2}{12} = 216\text{k-in}$$

Table 5.8. Example 5.3 Member stiffness

K_G

2191	531	0	−1389	0	0	−802	−531	556						
531	400	−2894	0	−40	−2894	−531	−360	−833						
0	−2894	277778	0	2894	138889	0	0	0						
−1389	0	0	2358	0	6510	−136	0	6510	−833	0	0			
0	−40	2894	0	2132	1852	0	−2083	0	0	−9	−1042			
0	−2894	138889	6510	1852	861111	−6510	0	208333	0	1042	83333			
−802	−531	0	−136	0	−6510	2326	531	−7066				−1389	0	0
−531	−360	0	0	−2083	0	531	2483	−2060				0	−40	−2894
556	−833	0	6510	0	208333	−7066	−2060	867788				0	2894	138889
			−833	0	0				833	0	0			
			0	−9	1042				0	9	1042			
			0	−1042	83333				0	1042	166667			
						−1389	0	0				1389	0	0
						0	−40	2894				0	40	2894
						0	−2894	138889				0	2894	277778

$$\mathrm{FEP}_{12} = \frac{wL}{2} = \frac{0.125\text{k/in}(144\text{in})}{2} = 9\text{k}$$

$$\mathrm{FEM}_{12} = -\frac{wL^2}{12} = -\frac{0.125\text{k/in}(144\text{in})^2}{12} = -216\text{k-in}$$

$$[P_g] = [P \,\&\, M_g] - [FEPM_m][R]^T = \begin{bmatrix} P_{1x} \\ P_{1y} \\ M_{1z} \\ P_{2x} \\ P_{2y} \\ M_{2z} \\ P_{3x} \\ P_{3y} \\ M_{3z} \\ P_{4x} \\ P_{4y} \\ M_{4z} \\ P_{5x} \\ P_{5y} \\ M_{5z} \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ -216 \\ 0 \\ 24 \\ -384 \\ 0 \\ 0 \\ 0 \\ 0 \\ 15 \\ 600 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The global deformations can be found from the global stiffness Equation 4.36 (step 4). The rows and columns corresponding to the support constraint degrees of freedom must be deleted prior to the solution. This would be all three motions at 4 and the X and Z motions at joint 5. The resulting matrix is shown in Table 5.9 along with the reduced load. The solution for the deformations will be in inches and radians.

$$\left[\Delta_g\right]=\left[K_g\right]^{-1}\left[P_g\right]=\begin{bmatrix}\Delta_{1x}\\ \Delta_{1y}\\ \theta_{1z}\\ \Delta_{2x}\\ \Delta_{2y}\\ \theta_{2z}\\ \Delta_{3x}\\ \Delta_{3y}\\ \theta_{3z}\\ \theta_{5z}\end{bmatrix}=\begin{bmatrix}0.07505\\ -2.45212\\ 0.00004\\ 0.06168\\ -2.25618\\ -0.00261\\ -0.03701\\ -2.25254\\ -0.00449\\ -0.02122\end{bmatrix}$$

The reactions at the supports can be found using the solution of the global deformation with Equation 4.37 (step 5). Only the terms in the rows corresponding to the restrained degrees of freedom and in the columns of the unrestrained degrees of freedom need to be included. Table 5.10 shows the appropriate stiffness terms and deformations needed to find the reactions. Since there are fixed-end forces and moments at support joint 4, they must

Table 5.9. Example 5.3 Member stiffness

K_G										-FEPM
2191	531	0	−1389	0	0	−802	−531	556	0	0
531	400	−2894	0	−40	−2894	−531	−360	−833	0	−9
0	−2894	277778	0	2894	138889	0	0	0	0	216
−1389	0	0	2358	0	6510	−136	0	6510	0	0
0	−40	2894	0	2132	1852	0	−2083	0	0	−24
0	−2894	138889	6510	1852	861111	−6510	0	208333	0	384
−802	−531	0	−136	0	−6510	2326	531	−7066	0	0
−531	−360	0	0	−2083	0	531	2483	−2060	−2894	0
556	−833	0	6510	0	208333	−7066	−2060	867788	138889	0
0	0	0	0	0	0	0	−2894	138889	277778	0

Table 5.10 Example 5.3 Member stiffness

0	0	0	–833	0	0	0	0	0	0	0.0750	–51.4	0	–51.4
0	0	0	0	–9	1042	0	0	0	0	–2.4521	16.9	15	31.9
0	0	0	0	–1042	83333	0	0	0	0	0.0000	2132.9	600	2732.9
0	0	0	0	0	0	–1389	0	0	0	0.0617	51.4	0	51.4
0	0	0	0	0	0	0	–40	2894	2894	–2.2562	16.1	0	16.1
										–0.0026			
										–0.0370			
										–2.2525			
										–0.0045			
										–0.0212			

be added back to get the final reactions. The reaction forces are in kips and inches (k-in).

$$[P]=\left[K_g\right]\left[\Delta_g\right]-\left[P\,\&\,M_g\right]-\left[FEPM_m\right][R]^T$$
$$[P]=\left[K_g\right]\left[\Delta_g\right]$$

The final step is finding the member forces for each of the members using Equation 4.35 (step 6). The member force will be in kips and inches. The local member stiffness matrix and the rotation matrix were shown in step 1 and are omitted here. The sign convention for the X-Y system applies when interpreting the final end forces and moments.

$$\left[P\,\&\,M_m\right]=\left[K_m\right][R]\left[\Delta_g\right]+\left[FEPM_m\right]$$
$$\left[P\,\&\,M_m\right]=\left[K_m\right][a]\left[\Delta_g\right]$$

For member 1, the deformations at joints 2 and 1 are used. Since this member had a load, the fixed-end forces and moments must be added to the results. Table 5.11 contains the final member end forces in the local system along with global deformations used to find those end forces.

For member 2, the deformations at joints 1 and 3 are used. Table 5.12 contains the final member end forces in the local system along with global deformations used to find those end forces.

For member 3, the deformations at joints 3 and 2 are used. Table 5.13 contains the final member end forces in the local system along with global deformations used to find those end forces.

Table 5.11. Example 5.3 Member stiffness

Δ_G		P_L	Final P	
0.06	Δ_{x2}	–18.56	–18.56	kips
–2.26	Δ_{y2}	0.45	9.45	kips
0.00	θ_{z2}	–151.73	64.27	kip-in
0.08	Δ_{x1}	18.56	18.56	kips
–2.45	Δ_{y1}	–0.45	8.55	kips
0.00	θ_{z1}	216.00	0.00	kip-in

Table 5.12. Example 5.3 Member stiffness

Δ_G		P_L	Final P	
0.08	Δ_{x1}	20.19	20.19	kips
–2.45	Δ_{y1}	–3.18	–3.18	kips
0.00	θ_{z1}	0.00	0.00	kip-in
–0.04	Δ_{x3}	–20.19	–20.19	kips
–2.25	Δ_{y3}	3.18	3.18	kips
0.00	θ_{z3}	–550.21	–550.21	kip-in

Table 5.13. Example 5.3 Member stiffness

Δ_G		P_L	Final P	
–0.04	Δ_{x3}	7.58	7.58	kips
–2.25	Δ_{y3}	–32.84	–32.84	kips
0.00	θ_{z3}	–1772.68	–1772.68	kip-in
0.06	Δ_{x2}	–7.58	–7.58	kips
–2.26	Δ_{y2}	32.84	32.84	kips
0.00	θ_{z2}	–1379.89	–1379.89	kip-in

For member 4, the deformations at joints 4 and 2 are used. Since this member had a load, the fixed-end forces and moments must be added to the results. Table 5.14 contains the final member end forces in the local system along with global deformations used to find those end forces.

For member 5, the deformations at joints 5 and 3 are used. Table 5.15 contains the final member end forces in the local system along with global deformations used to find those end forces.

Releases can also occur for the axial and torsional components of the member stiffness. If either end of the member is released for Δ_x or θ_x, then all the corresponding stiffness components are zero. If both ends of the

Table 5.14. Example 5.3 Member stiffness

Δ_G		P_L	Final P	
0.00	Δ_{x4}	–51.40	–51.40	kips
0.00	Δ_{y4}	16.87	31.87	kips
0.00	θ_{z4}	2132.90	2732.90	kip-in
0.06	Δ_{x2}	51.40	51.40	kips
–2.26	Δ_{y2}	–16.87	–1.87	kips
0.00	θ_{z2}	1915.62	1315.62	kip-in

Table 5.15. Example 5.3 Member stiffness

Δ_G		P_L	Final P	
0.00	Δ_{x5}	51.40	51.40	kips
0.00	Δ_{y5}	16.13	16.13	kips
–0.02	θ_{z5}	0.00	0.00	kip-in
–0.04	Δ_{x3}	–51.40	–51.40	kips
–2.25	Δ_{y3}	–16.13	–16.13	kips
0.00	θ_{z3}	2322.89	2322.89	kip-in

member are released, then the member is unstable. The axial components of the member stiffness matrix are $k_{1,1}$, $k_{1,7}$, $k_{7,1}$, and $k_{7,7}$. The torsional components of the member stiffness matrix are $k_{4,4}$, $k_{4,10}$, $k_{10,4}$, and $k_{10,10}$.

5.4 NON-PRISMATIC MEMBERS

Non-prismatic members have cross-sectional properties that vary along the length of the member. The stiffness of these members can be handled in two ways. First, the member could be divided into prismatic sections and modeled with several different members of constant cross-section. Second, the member can be modeled with stiffness derived from the mathematical model of the cross-sectional variation. The following derivation is for the stiffness of a non-prismatic member in the X-Z system.

Example 5.4 Non-prismatic member stiffness

Derive the local member stiffness in the X-Z system for a non-prismatic cross-section using Castigliano's theorems.

Figure 5.6 shows the beam with an applied deflection and an applied rotation at the *i*-end. These can be used simultaneously to derive the stiffness of the member.

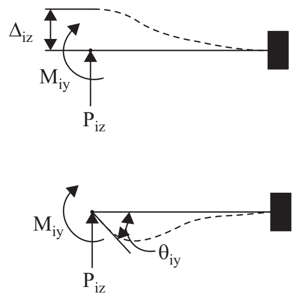

Figure 5.6. Example 5.4 Non-prismatic member stiffness.

The internal moment, M_x, at any point, x, can be found from statics and the partial derivatives of that moment can be found with respect to the applied force and moment at the i-end.

$$M_x = -P_{iz}x - M_{iy}$$
$$\frac{\delta M_x}{\delta P_{iz}} = -x$$
$$\frac{\delta M_x}{\delta M_{iy}} = -1$$

Castigliano's second theorem states that the first partial derivative of strain energy with respect to a particular force is equal to the displacement of the point of application of that force in the direction of its line of action. This can be applied for both Δ_{iz} and θ_{iy}.

$$\Delta_{iz} = \int M_x \frac{\delta M_x}{\delta P_{iz}} \frac{dx}{EI_y} = \int P_{iz}x^2 \frac{dx}{EI_y} + \int M_{iy}x \frac{dx}{EI_y}$$
$$\theta_{iy} = \int M_x \frac{\delta M_x}{\delta M_{iy}} \frac{dx}{EI_y} = \int P_{iz}x \frac{dx}{EI_y} + \int M_{iy} \frac{dx}{EI_y}$$

Since the cross-sectional properties vary, the moment of inertia, I_y, varies. Let the values S_1, S_2, and S_3 be used and substituted into the previous two equations. These values can be pre-derived for the cross-sectional variation.

$$S_1 = \int \frac{dx}{EI_y}$$
$$S_2 = \int x \frac{dx}{EI_y}$$

$$S_3 = \int x^2 \frac{dx}{EI_y}$$
$$\Delta_{iz} = P_{iz} S_3 + M_{iy} S_2$$
$$\theta_{iy} = P_{iz} S_2 + M_{iy} S_1$$

Written in matrix form this is the flexibility matrix:

$$\begin{bmatrix} \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} S_3 & S_2 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} \mathrm{P}_{iz} \\ \mathrm{M}_{iy} \end{bmatrix}$$
$$[\delta_i] = [f_{ii}][F_i]$$

We can solve this matrix equation for the forces P_{iz} and M_{iy} by the cofactor method. Substituting the determinant of the flexibility matrix, $D = S_3 S_1 - S_2^2$, to simplify the equation. This is the stiffness form of the equation.

$$\begin{bmatrix} P_{iz} \\ M_{iy} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} S_1 & -S_2 \\ -S_2 & S_3 \end{bmatrix} \begin{bmatrix} \Delta_{iz} \\ \theta_{iy} \end{bmatrix}$$
$$[F_i] = [K_{ii}][\delta_i]$$

We can use the transmission matrix Equation 4.6 to find the forces at the j-end of the member, where the values of x are cause minus effect or $x = x_i - x_j = 0 - L = -L$.

$$[T] = \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix}$$
$$\left[K_{ji}\right] = -[T][K_{ii}]$$
$$\left[K_{ji}\right] = \begin{bmatrix} -1 & 0 \\ -L & -1 \end{bmatrix} \begin{bmatrix} S_1 & -S_2 \\ -S_2 & S_3 \end{bmatrix} \frac{1}{D} = \begin{bmatrix} -S_1 & S_2 \\ S_2 - S_1 L & S_2 L - S_3 \end{bmatrix} \frac{1}{D}$$
$$\begin{bmatrix} P_{jz} \\ M_{jy} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -S_1 & S_2 \\ S_2 - S_1 L & S_2 L - S_3 \end{bmatrix} \begin{bmatrix} \Delta_{iz} \\ \theta_{iy} \end{bmatrix}$$
$$\left[F_j\right] = \left[K_{ji}\right][\delta_i]$$

The carry-over factor (COF) used in the moment distribution method can be found by observing the ratio of the moment at the j-end to the moment at the i-end.

$$COF_{i\to j} = \frac{M_{jy}}{M_{iy}} = \frac{\left[(S_2 - S_1 L)\Delta_{iz} + (S_2 L - S_3)\theta_{iy}\right]\frac{1}{D}}{\left[-S_2\Delta_{iz} + S_3\theta_{iy}\right]\frac{1}{D}}$$

When only the rotational deformation is considered, the following is the COF:

$$COF_{i\to j} = \frac{S_2 L - S_3}{S_3}$$

The distribution factor (DF) used in the moment distribution method is the ratio of the rotational stiffness of a member to the sum of the rotational stiffness of all members at the joint. The rotational stiffness is the moment at a joint due to the rotation at a joint. This is the term K_{ii} for rotation and moment only.

$$K_{Miy,\theta iy} = \frac{S_3}{D}$$

The DF for a member at a joint can be written as follows:

$$DF = \frac{-\frac{S_3}{D}}{\sum \frac{S_3}{D}}$$

We could find the deflection and rotation at the j-end using the same method. Alternatively, since the stiffness matrix is symmetric we can find the forces at the i-end due to motions at the j-end directly.

$$\left[K_{ij}\right] = \left[K_{ji}\right]^T = \frac{1}{D}\begin{bmatrix} -S_1 & S_2 \\ S_2 - S_1 L & S_2 L - S_3 \end{bmatrix}^T = \frac{1}{D}\begin{bmatrix} -S_1 & S_2 - S_1 L \\ S_2 & S_2 L - S_3 \end{bmatrix}$$

$$\begin{bmatrix} P_{iz} \\ M_{iy} \end{bmatrix} = \frac{1}{D}\begin{bmatrix} -S_1 & S_2 - S_1 L \\ S_2 & S_2 L - S_3 \end{bmatrix}\begin{bmatrix} \Delta_{jz} \\ \theta_{jy} \end{bmatrix}$$

$$\left\lceil F_i \right\rceil = \left\lceil K_{ii} \right\rceil \left\lceil \delta_i \right\rceil$$

We can use the transmission matrix equation again to find the forces at the j-end of the member.

$$\left[K_{jj}\right] = -[T]\left[K_{ij}\right]$$

$$\left[K_{jj}\right] = \begin{bmatrix} -1 & 0 \\ -L & -1 \end{bmatrix} \begin{bmatrix} -S_1 & S_2 - S_1L \\ S_2 & S_2L - S_3 \end{bmatrix} \frac{1}{D} = \begin{bmatrix} S_1 & S_1L - S_2 \\ S_1L - S_2 & S_1L^2 - 2S_2L + S_3 \end{bmatrix} \frac{1}{D}$$

$$\begin{bmatrix} \mathrm{P}_{jz} \\ \mathrm{M}_{jy} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} S_1 & S_1L - S_2 \\ S_1L - S_2 & S_1L^2 - 2S_2L + S_3 \end{bmatrix} \begin{bmatrix} \Delta_{jz} \\ \theta_{jy} \end{bmatrix}$$

$$\left[\mathrm{F}_j\right] = \left[K_{jj}\right]\left[\delta_j\right]$$

The COF from the j-end to the i-end is as follows considering the deformations at the j-end:

$$COF_{j\to i} = \frac{\mathrm{M}_{iy}}{\mathrm{M}_{jy}} = \frac{\left[S_2\Delta_{jz} + \left(S_2L - S_3\right)\theta_{jy}\right]\frac{1}{D}}{\left[\left(S_1L - S_2\right)\Delta_{jz} + \left(S_1L^2 - 2S_2L + S_3\right)\theta_{jy}\right]\frac{1}{D}}$$

When only the rotational deformation is considered the following is the COF:

$$COF_{j\to i} = \frac{S_2L - S_3}{S_1L^2 - 2S_2L + S_3}$$

The DF is a ratio of the rotational stiffness of a member to the sum of the rotational stiffness of all members at the joint.

Rotational stiffness, the stiffness due to rotation at point A:

$$K_{\theta B} = \frac{S_1L^2 - 2S_2L + S_3}{D}$$

The DF for a member at a joint can be written as follows:

$$DF = \frac{-\frac{S_1L^2 - 2S_2L + S_{33}}{D}}{\sum \frac{S_1L^2 - 2S_2L + S_3}{D}}$$

The axial stiffness terms can be derived directly from strength of materials. The axial stiffness is the inverse of the flexibility, which can be written as follows. The torsional stiffness would look the same as the axial stiffness with GI_x substituted for EA_x:

$$f = \int \frac{dx}{EA_x}$$

The coplanar X-Z nonprismatic stiffness matrix is shown in Equation 5.12. The X-Y system will have the same values with the sign of the moments due to deflection reversed.

$$[K_m] = \begin{bmatrix} \frac{1}{f} & 0 & 0 & -\frac{1}{f} & 0 & 0 \\ 0 & \frac{S_1}{D} & -\frac{S_2}{D} & 0 & -\frac{S_1}{D} & \frac{S_2 - S_1 L}{D} \\ 0 & -\frac{S_2}{D} & \frac{S_3}{D} & 0 & \frac{S_2}{D} & \frac{S_2 L - S_3}{D} \\ -\frac{1}{f} & 0 & 0 & \frac{1}{f} & 0 & 0 \\ 0 & -\frac{S_1}{D} & \frac{S_2}{D} & 0 & \frac{S_1}{D} & \frac{S_1 L - S_2}{D} \\ 0 & \frac{S_2 - S_1 L}{D} & \frac{S_2 L - S_3}{D} & 0 & \frac{S_1 L - S_2}{D} & \frac{S_1 L^2 - 2S_2 L + S_3}{D} \end{bmatrix} \tag{5.12}$$

FIXED-END MOMENTS

The fixed-end forces and moments must be derived for a non-prismatic member. The changes in stiffness along the length of the member will change how the forces and moments are distributed by the member. The following examples derive two of the most common member loads.

Example 5.5 Non-prismatic member stiffness

Derive the fixed-end forces and moments due to a uniformly distributed load in the X-Z system for a non-prismatic cross-section using Castigliano's theorems.

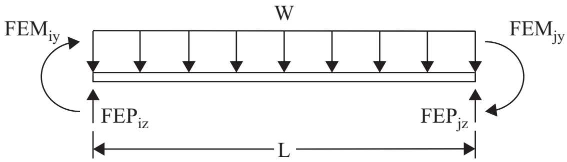

Figure 5.7. Example 5.5 Non-prismatic member stiffness.

The free-body diagram of the beam is shown in Figure 5.7. The procedure for Castigliano's second theorem used in Example 5.4 will be repeated here.

The internal moment, M_x, at any point, x, can be found from statics and the partial derivatives of that moment can be found with respect to the applied force and moment at the i-end.

$$M_x = \frac{wx^2}{2} - FEP_{iz}x - FEM_{iy}$$

$$\frac{\delta M_x}{\delta FEP_{iz}} = -x$$

$$\frac{\delta M_x}{\delta FEM_{iy}} = -1$$

$$\Delta_{iz} = 0 = \int M_x \frac{\delta M_x}{\delta FEP_{iz}} \frac{dx}{EI_y} = \int FEP_{iz} x^2 \frac{dx}{EI_y} + \int FEM_{iy} x \frac{dx}{EI_y} - \int \frac{wx^3}{2} \frac{dx}{EI_y}$$

$$\theta_{iy} = 0 = \int M_x \frac{\delta M_x}{\delta FEM_{iy}} \frac{dx}{EI_y} = \int FEP_{iz} x \frac{dx}{EI_y} + \int FEM_{iy} \frac{dx}{EI_y} - \int \frac{wx^2}{2} \frac{dx}{EI_y}$$

Observing that there is a new term that varies with x, we will substitute S_4 as follows:

$$S_4 = \int x^3 \frac{dx}{EI_y}$$

$$0 = \mathrm{FEP}_{iz} S_3 + \mathrm{FEM}_{iy} S_2 - \frac{w}{2} S_4$$

$$0 = \mathrm{FEP}_{iz} S_2 + \mathrm{FEM}_{iy} S_1 - \frac{w}{2} S_3$$

$$\begin{bmatrix} S_3 & S_2 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} \mathrm{FEP}_{iz} \\ \mathrm{FEM}_{iy} \end{bmatrix} = \begin{bmatrix} S_4 \\ S_3 \end{bmatrix} \frac{w}{2}$$

$$\begin{bmatrix} \mathrm{FEP}_{iz} \\ \mathrm{FEM}_{iy} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} S_1 & -S_2 \\ -S_2 & S_3 \end{bmatrix} \begin{bmatrix} S_4 \\ S_3 \end{bmatrix} \frac{w}{2} = \frac{w}{2D} \begin{bmatrix} S_1 S_4 - S_2 S_3 \\ S_3{}^2 - S_2 S_4 \end{bmatrix}$$

Apply equilibrium on Figure 5.7 to find the *j*-end forces and moments.

$$FEP_{jz} = wL - FEP_{iz}$$

$$FEM_{jy} = \frac{wL^2}{2} - FEP_{iz}L - FEM_{iy}$$

Example 5.6 Non-prismatic member stiffness

Derive the fixed-end forces and moments due to a concentrated load in the X-Z system for a non-prismatic cross-section using Castigliano's theorems.

The free-body diagram of the beam is shown in Figure 5.8. The procedure used in Example 5.5 will be repeated here.

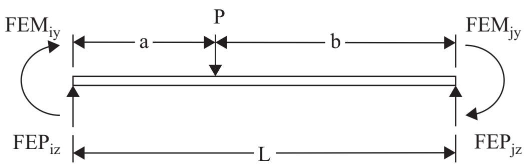

Figure 5.8. Example 5.6 Non-prismatic member stiffness.

The internal moment, M_x, at any point, x, can be found from statics and the partial derivatives of that moment can be found with respect to the applied force and moment at the *i*-end. In this case, two moment equations must be written. The first is M_{x1}, with x from the *i*-end to the point load ($0 \le x \le a$) and the second is M_{x2}, from the point load to the *j*-end ($a \le x \le L$).

$$M_{x1} = -FEP_{iz}x - FEM_{iy}$$

$$M_{x2} = -P(x-a) - FEP_{iz}x - FEM_{iy}$$

The partial derivatives are the same for either of the two moment equations.

$$\frac{\delta M_x}{\delta FEP_{iz}} = -x$$

$$\frac{\delta M_x}{\delta FEM_{iy}} = -1$$

$$\Delta_{iz} = 0 = \int M_x \frac{\delta M_x}{\delta FEP_{iz}} \frac{dx}{EI_y} = \int FEP_{iz}x^2 \frac{dx}{EI_y} + \int FEM_{iy}x \frac{dx}{EI_y} - \int_a^L Px(x-a)\frac{dx}{EI_y}$$

$$\theta_{iy} = 0 = \int M_x \frac{\delta M_x}{\delta FEM_{iy}} \frac{dx}{EI_y} = \int FEP_{iz}x \frac{dx}{EI_y} + \int FEM_{iy} \frac{dx}{EI_y} - \int_a^L P(x-a)\frac{dx}{EI_y}$$

Observing that there are two new terms that vary with x, we will substitute S_5 and S_6 as follows:

$$S_5 = \int (x-a)\frac{dx}{EI_y}$$

$$S_6 = \int (x^2-ax)\frac{dx}{EI_y}$$

$$0 = \text{FEP}_{iz}S_3 + \text{FEM}_{iy}S_2 - PS_6$$

$$0 = \text{FEP}_{iz}S_2 + \text{FEM}_{iy}S_1 - PS_5$$

$$\begin{bmatrix} S_3 & S_2 \\ S_2 & S_1 \end{bmatrix}\begin{bmatrix} \text{FEP}_{iz} \\ \text{FEM}_{iy} \end{bmatrix} = \begin{bmatrix} S_6 \\ S_5 \end{bmatrix} P$$

$$\begin{bmatrix} \text{FEP}_{iz} \\ \text{FEM}_{iy} \end{bmatrix} = \frac{1}{D}\begin{bmatrix} S_1 & -S_2 \\ -S_2 & S_3 \end{bmatrix}\begin{bmatrix} S_6 \\ S_5 \end{bmatrix} P = \frac{P}{D}\begin{bmatrix} S_1S_6 - S_2S_5 \\ S_3S_5 - S_2S_6 \end{bmatrix}$$

Apply equilibrium on Figure 5.8 to find the j-end forces and moments.

$$\text{FEP}_{jz} = P - \text{FEP}_{iz}$$

$$\text{FEM}_{jy} = Pb - \text{FEP}_{iz}L - \text{FEM}_{iy}$$

The process for solving stiffness problems involving non-prismatic members is the same as with prismatic members with the same four modifications for members with end releases. The following example shows this process.

Example 5.7 Non-prismatic member stiffness

Determine the deformations at the free end of the non-prismatic beam using only three degrees of freedom in the stiffness solution. Also determine the final end forces and the support reactions. The beam is shown in Figure 5.9.

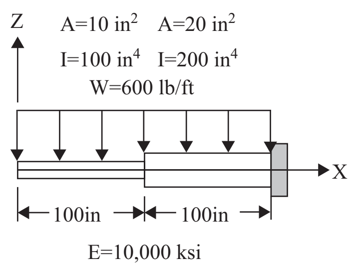

Figure 5.9. Example 5.7 Non-prismatic member stiffness.

$$S_1 = \int \frac{dx}{EI_y} = \int_0^{100} \frac{dx}{100E} + \int_{100}^{200} \frac{dx}{200E} = \frac{1}{10000}\left(\frac{100-0}{100} + \frac{200-100}{200}\right) = 0.00015$$

$$S_2 = \int x \frac{dx}{EI_y} = \int_0^{100} \frac{xdx}{100E} + \int_{100}^{200} \frac{xdx}{200E} = \frac{1}{10000}\left[\frac{100^2-0^2}{2(100)} + \frac{200^2-100^2}{2(200)}\right]$$
$$= 0.0125$$

$$S_3 = \int x^2 \frac{dx}{EI_y} + \int_0^{100} \frac{x^2dx}{100E} + \int_{100}^{200} \frac{x^2dx}{200E} = \frac{1}{10000}\left[\frac{100^3-0^3}{3(100)} + \frac{200^3-100^3}{3(200)}\right] = 1.5$$

$$D = S_1S_3 - S_2{}^2 = 0.00015(1.5) - (0.0125)^2 = 0.00006875$$

$$f = \int \frac{dx}{EA_x} = \int_0^{100} \frac{dx}{10E} + \int_{100}^{200} \frac{dx}{20E} = \frac{1}{10000}\left(\frac{100-0}{10} + \frac{200-100}{20}\right) = 0.0015$$

$$K = \frac{1}{f} = \frac{1}{0.0015} = 666.67$$

Since the *j*-end of the member is fixed, there is no need to build the entire member stiffness matrix. The *j*-end motions will be eliminated and only the *i*-end of the member stiffness needs to be developed.

$$[K_{ii}] = \begin{bmatrix} \frac{1}{f} & 0 & 0 \\ 0 & \frac{S_1}{D} & -\frac{S_2}{D} \\ 0 & -\frac{S_2}{D} & \frac{S_3}{D} \end{bmatrix} = \begin{bmatrix} 666.67 & 0 & 0 \\ 0 & 2.1818 & -181.81 \\ 0 & -181.81 & 21818 \end{bmatrix}$$

The fixed-end forces and moments must be derived and applied to the system.

$$S_4 = \int x^3 \frac{dx}{EI_y} \int_0^{100} \frac{x^3dx}{100E} + \int_{100}^{200} \frac{x^3dx}{200E} = \frac{1}{10000}\left[\frac{100^4-0^4}{4(100)} + \frac{200^4-100^4}{4(200)}\right] = 212.5$$

$$[P_g] = [P] - [FEPM] = \begin{bmatrix} 0 \\ -\text{FEP}_{iz} \\ -\text{FEM}_{iy} \end{bmatrix} = \frac{w}{2D}\begin{bmatrix} 0 \\ -(S_1S_4 - S_2S_3) \\ -(S_3{}^2 - S_2S_4) \end{bmatrix}$$

$$[P_g] = \frac{0.05}{2(0.00006875)}\begin{bmatrix} 0 \\ -[0.00015(212.5) - 0.0125(1.5)] \\ -[(1.5)^2 - 0.0125(212.5)] \end{bmatrix} = \begin{bmatrix} 0 \\ -4.7727 \\ 147.73 \end{bmatrix}$$

The general stiffness equation can be set-up and solved as follows. The units are in inches and radians:

$$[P_i] = [K_{ii}][\delta_i]$$

$$[\delta_i] = [K_{ii}]^{-1}[P_i]$$

$$[\delta_i] = \begin{bmatrix} 666.67 & 0 & 0 \\ 0 & 2.1818 & -181.81 \\ 0 & -181.81 & 21818 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -4.7727 \\ 147.73 \end{bmatrix} = \begin{bmatrix} 0 \\ -5.3123 \\ -0.0375 \end{bmatrix}$$

5.5 SHEAR STIFFNESS, X-Z SYSTEM

The shear stiffness of a member should be included when it is significant. This effect was developed by Timoshenko (1921) in 1921. For normal frame structures, the stiffness contributions due to shear are minor and are sometimes ignored. For frames and structures with larger or deeper members, the shear stiffness contribution is appreciable and should be included. The following two sections and corresponding examples derive the combined flexural and shear stiffness of members.

Example 5.8 Shear stiffness

Derive the local member shear stiffness for θ_{iy} using Castigliano's theorems.

The free-body diagram of the beam with an imposed rotation of θ_{iy} is shown in Figure 5.10. Also shown is a left-hand free-body of the beam cut at any distance x from the i-end.

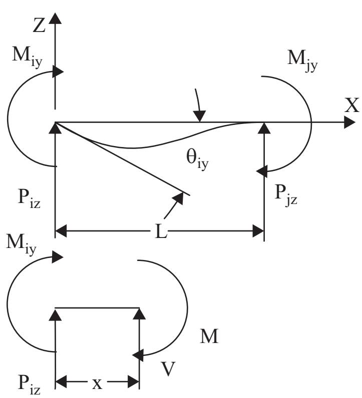

Figure 5.10. Example 5.8 Shear stiffness.

The internal shear, V_x, and moment, M_x, at any point, x, can be found from statics and the partial derivatives of that shear and moment can be found with respect to the applied force and moment at the i-end.

$$V_x = -P_{iz}$$
$$\frac{\delta V_x}{\delta P_{iz}} = -1$$
$$\frac{\delta V_x}{\delta M_{iy}} = 0$$
$$M_x = -P_{iz}x - M_{iy}$$
$$\frac{\delta M_x}{\delta P_{iz}} = -x$$
$$\frac{\delta M_x}{\delta M_{iy}} = -1$$

Castigliano's second theorem can be applied noting that at the i-end the deflection is zero and the rotation is θ_{iy}.

$$\Delta_{iz} = 0 = \int M_x \frac{\delta M_x}{\delta P_{iz}} \frac{dx}{EI_y} + \int V_x \frac{\delta V_x}{\delta P_{iz}} \frac{dx}{GA_z} = \int \left(P_{iz}x^2 + M_{iy}x\right) \frac{dx}{EI_y} + \int P_{iz} \frac{dx}{GA_z}$$
$$0 = \frac{P_{iz}L^3}{3EI_y} + \frac{M_{iy}L^2}{2EI_y} + \frac{P_{iz}L}{GA_z}$$
$$\frac{M_{iy}}{EI_y} = -\frac{2P_{iz}L}{3EI_y} - \frac{2P_{iz}}{LGA_z}$$
$$\theta_{iy} = \int M_x \frac{\delta M_x}{\delta M_{iy}} \frac{dx}{EI_y} + \int V_x \frac{\delta V_x}{\delta M_{iy}} \frac{dx}{GA_z} = \int \left(P_{iz}x + M_{iy}\right) \frac{dx}{EI_y} + \int 0 \frac{dx}{GA_z}$$
$$\theta_{iy} = \frac{P_{iz}L^2}{2EI_y} + \frac{M_{iy}L}{EI_y}$$

Substituting the first equation into the second equation results in the following:

$$\theta_{iy} = \frac{P_{iz}L^2}{2EI_y} - \frac{2P_{iz}L^2}{3EI_y} - \frac{2P_{iz}}{GA_z} = -\frac{P_{iz}L^2}{6EI_y} - \frac{2P_{iz}}{GA_z}$$
$$\theta_{iy} = \left(-\frac{L^2}{6EI_y} - \frac{2}{GA_z}\right)P_{iz}$$
$$P_{iz} = -\left(\frac{6EI_y A_z G}{L^2 A_z G + 12EI_y}\right)\theta_{iy} \qquad (5.13)$$

Substituting Equation 5.13 into the following equation repeated from earlier, results in Equation 5.14 for the second stiffness value.

$$\frac{M_{iy}}{EI_y} = -\frac{2P_{iz}L}{3EI_y} - \frac{P_{iz}}{LGA_z}$$

$$M_{iy} = \left(\frac{4EI_y \left(L^2 A_z G + 3EI_y \right)}{L^3 A_z G + 12LEI_y} \right) \theta_{iy} \tag{5.14}$$

Example 5.9 Shear stiffness

Derive the local member shear stiffness for Δ_{iz} using Castigliano's theorems.

The free-body diagram of the beam with an imposed deflection of Δ_{iz} is shown in Figure 5.11. Also shown is a left-hand free-body of the beam cut at any distance x from the i-end.

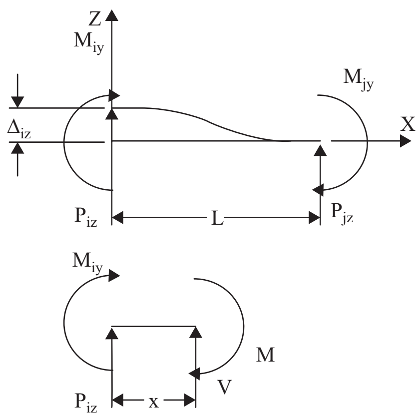

Figure 5.11. Example 5.9 Shear stiffness.

The internal shear, V_x, and moment, M_x, are exactly the same as in Example 5.8. The partial derivatives are also the same.

$$V_x = -P_{iz}$$

$$\frac{\delta V_x}{\delta P_{iz}} = -1$$

$$\frac{\delta V_x}{\delta M_{iy}} = 0$$

$$M_x = -P_{iz}x - M_{iy}$$

$$\frac{\delta M_x}{\delta P_{iz}} = -x$$
$$\frac{\delta M_x}{\delta M_{iy}} = -1$$

Castigliano's second theorem can be applied noting that at the i-end the rotation is zero and the deflection is Δ_{iz}.

$$\theta_{iy} = 0 = \int M_x \frac{\delta M_x}{\delta M_{iy}} \frac{dx}{EI_y} + \int V_x \frac{\delta V_x}{\delta M_{iy}} \frac{dx}{GA_z} = \int \left(P_{iz} x + M_{iy}\right) \frac{dx}{EI_y} + \int 0 \frac{dx}{GA_z}$$
$$0 = \frac{P_{iz} L^2}{2EI_y} + \frac{M_{iy} L}{EI_y}$$
$$M_{iy} = -\frac{P_{iz} L}{2}$$
$$\Delta_{iz} = \int M_x \frac{\delta M_x}{\delta P_{iz}} \frac{dx}{EI_y} + \int V_x \frac{\delta V_x}{\delta P_{iz}} \frac{dx}{GA_z} = \int \left(P_{iz} x^2 + M_{iy} x\right) \frac{dx}{EI_y} + \int P_{iz} \frac{dx}{GA_z}$$
$$\Delta_{iz} = \frac{P_{iz} L^3}{3EI_y} + \frac{M_{iy} L^2}{2EI_y} + \frac{P_{iz} L}{GA_z}$$

Substituting the first equation into the second equation results in the following:

$$\Delta_{iz} = \frac{P_{iz} L^3}{3EI_y} - \frac{P_{iz} L^3}{4EI_y} + \frac{P_{iz} L}{GA_z} = \frac{P_{iz} L^3}{12EI_y} + \frac{P_{iz} L}{GA_z}$$
$$\Delta_{iz} = \left(\frac{L^3}{12EI_y} + \frac{L}{GA_z}\right) P_{iz}$$
$$P_{iz} = \left(\frac{12EI_y A_z G}{L^3 A_z G + 12LEI_y}\right) \Delta_{iz} \tag{5.15}$$

Substituting Equation 5.15 into the following equation repeated from earlier, results in Equation 5.16 for the second stiffness value.

$$M_{iy} = -\frac{P_{iz} L}{2}$$
$$M_{iy} = -\left(\frac{6EI_y A_z G}{L^2 A_z G + 12EI_y}\right) \Delta_{iz} \tag{5.16}$$

The forces at the *j*-end of the member due to the motions at the *i*-end can be found using the transmission matrix. Then, the forces at the *i*-end due to the motions at the *j*-end can be found by symmetry of the stiffness matrix. Finally, the forces at the *j*-end due to motions at the *j*-end can be found using the transmission matrix. This process was illustrated previously in Chapter 4. The resulting terms are shown in the matrices given as Equations 5.21 through 5.24.

5.6 SHEAR STIFFNESS, X-Y SYSTEM

The following contains the combined flexural and shear stiffness of members in the X-Y system.

Example 5.10 Shear stiffness

Derive the local member shear stiffness for θ_{iz} using Castigliano's theorems.

The free-body diagram of the beam with an imposed rotation of θ_{iz} is shown in Figure 5.12. Also shown is a left-hand free-body of the beam cut at any distance x from the *i*-end.

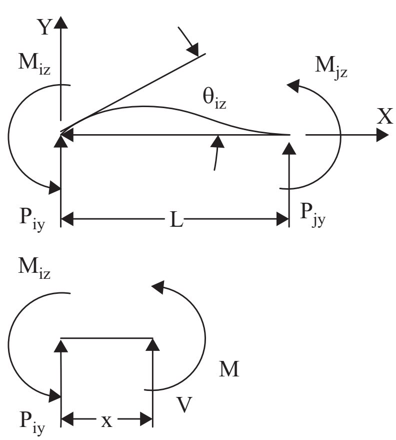

Figure 5.12. Example 5.10 Shear stiffness.

The internal shear, V_x, and moment, M_x, at any point, x, can be found from statics and the partial derivatives of that shear and moment can be found with respect to the applied force and moment at the *i*-end.

$$V_x = -P_{iy}$$
$$\frac{\delta V_x}{\delta P_{iy}} = -1$$
$$\frac{\delta V_x}{\delta M_{iz}} = 0$$
$$M_x = P_{iy}x - M_{iz}$$
$$\frac{\delta M_x}{\delta P_{iy}} = x$$
$$\frac{\delta M_x}{\delta M_{iz}} = -1$$

Castigliano's second theorem can be applied noting that at the i-end the deflection is zero and the rotation is θ_{iz}.

$$\Delta_{iy} = 0 = \int M_x \frac{\delta M_x}{\delta P_{iy}} \frac{dx}{EI_z} + \int V_x \frac{\delta V_x}{\delta P_{iy}} \frac{dx}{GA_y} = \int \left(P_{iy}x^2 - M_{iz}x\right)\frac{dx}{EI_z} + \int P_{iy} \frac{dx}{GA_y}$$
$$0 = \frac{P_{iy}L^3}{3EI_z} - \frac{M_{iz}L^2}{2EI_z} + \frac{P_{iy}L}{GA_y}$$
$$\frac{M_{iz}}{EI_z} = \frac{2P_{iy}L}{3EI_z} - \frac{2P_{iy}}{LGA_y}$$
$$\theta_{iz} = \int M_x \frac{\delta M_x}{\delta M_{iz}} \frac{dx}{EI_z} + \int V_x \frac{\delta V_x}{\delta M_{iz}} \frac{dx}{GA_y} = \int \left(-P_{iy}x + M_{iz}\right)\frac{dx}{EI_z} + \int 0 \frac{dx}{GA_y}$$
$$\theta_{iz} = -\frac{P_{iy}L^2}{2EI_z} + \frac{M_{iz}L}{EI_z}$$

Substituting the first equation into the second equation results in the following:

$$\theta_{iz} = -\frac{P_{iy}L^2}{2EI_z} + \frac{2P_{iy}L^2}{3EI_z} + \frac{2P_{iy}}{GA_y} = \frac{P_{iy}L^2}{6EI_z} + \frac{2P_{iy}}{GA_y}$$
$$\theta_{iz} = \left(\frac{L^2}{6EI_z} + \frac{2}{GA_y}\right)P_{iy}$$
$$P_{iy} = \left(\frac{6EI_zA_yG}{L^2A_yG + 12EI_z}\right)\theta_{iz} \tag{5.17}$$

Substituting Equation 5.17 into the following equation repeated from earlier, results in Equation 5.18 for the second stiffness value.

$$\frac{M_{iz}}{EI_z} = \frac{2P_{iy}L}{3EI_z} - \frac{2P_{iy}}{LGA_y}$$

$$M_{iz} = \left(\frac{4EI_z\left(L^2 A_y G + 3EI_z\right)}{L^3 A_y G + 12LEI_z} \right)\theta_{iz} \tag{5.18}$$

Example 5.11 Shear stiffness

Derive the local member shear stiffness for Δ_{iy} using the Castigliano's theorems.

The free-body diagram of the beam with an imposed deflection of Δ_{iy} is shown in Figure 5.13. Also shown is a left-hand free-body of the beam cut at any distance x from the i-end.

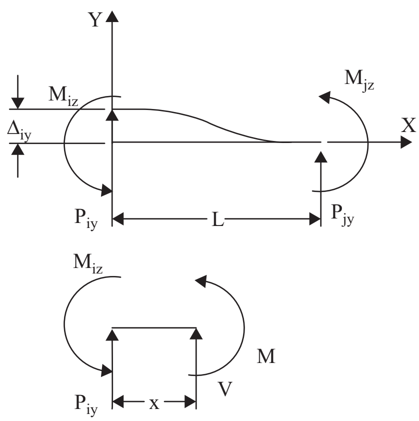

Figure 5.13. Example 5.11 Shear stiffness.

The internal shear, V_x, and moment, M_x, are exactly the same as in Example 5.10. The partial derivatives are also the same.

$$V_x = -P_{iy}$$

$$\frac{\delta V_x}{\delta P_{iy}} = -1$$

$$\frac{\delta V_x}{\delta M_{iz}} = 0$$

$$M_x = P_{iy}x - M_{iz}$$

$$\frac{\delta M_x}{\delta P_{iy}} = x$$
$$\frac{\delta M_x}{\delta M_{iz}} = -1$$

Castigliano's second theorem can be applied noting that at the i-end the rotation is zero and the deflection is Δ_{iy}.

$$\theta_{iz} = 0 = \int M_x \frac{\delta M_x}{\delta M_{iz}} \frac{dx}{EI_z} + \int V_x \frac{\delta V_x}{\delta M_{iz}} \frac{dx}{GA_y} = \int \left(-P_{iy}x + M_{iz}\right) \frac{dx}{EI_z} + \int 0 \frac{dx}{GA_y}$$
$$0 = -\frac{P_{iy}L^2}{2EI_z} + \frac{M_{iz}L}{EI_z}$$
$$M_{iz} = \frac{P_{iy}L}{2}$$
$$\Delta_{iy} = \int M_x \frac{\delta M_x}{\delta P_{iy}} \frac{dx}{EI_z} + \int V_x \frac{\delta V_x}{\delta P_{iy}} \frac{dx}{GA_y} = \int \left(P_{iy}x^2 - M_{iz}x\right) \frac{dx}{EI_z} + \int P_{iy} \frac{dx}{GA_y}$$
$$\Delta_{iy} = \frac{P_{iy}L^3}{3EI_z} - \frac{M_{iz}L^2}{2EI_z} + \frac{P_{iy}L}{GA_y}$$

Substituting the first equation into the second equation results in the following:

$$\Delta_{iy} = \frac{P_{iy}L^3}{3EI_z} - \frac{P_{iy}L^3}{4EI_z} + \frac{P_{iy}L}{GA_y} = \frac{P_{iy}L^3}{12EI_z} + \frac{P_{iy}L}{GA_y}$$
$$\Delta_{iy} = \left(\frac{L^3}{12EI_z} + \frac{L}{GA_y}\right) P_{iy}$$
$$P_{iy} = \left(\frac{12EI_z A_y G}{L^3 A_y G + 12LEI_z}\right) \Delta_{iy} \quad . \tag{5.19}$$

Substituting Equation 5.19 into the following equation repeated from earlier, results in Equation 5.20 for the second stiffness value.

$$M_{iz} = \frac{P_{iy}L}{2}$$
$$M_{iz} = \left(\frac{6EI_z A_y G}{L^2 A_y G + 12EI_z}\right) \Delta_{iy} \tag{5.20}$$

The resulting terms are shown in the matrices given as Equations 5.21 through 5.24.

5.7 SHEAR STIFFNESS, 3-D SYSTEM

The total three-dimensional (3-D) shear stiffness matrices are shown in four parts in the following text as Equations 5.21 through 5.24. These parts represent the force at the *i*-end or *j*-end due to the motions at the *i*-end or *j*-end. This is shown in the total stiffness matrix as follows:

$$[K]=\begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix}$$

$$[K_{ii}]=\begin{bmatrix} \frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_zA_yG}{L^3A_yG+12LEI_z} & 0 & 0 & 0 & \frac{6EI_zA_yG}{L^2A_yG+12EI_z} \\ 0 & 0 & \frac{12EI_yA_zG}{L^3A_zG+12LEI_y} & 0 & -\frac{6EI_yA_zG}{L^2A_zG+12EI_y} & 0 \\ 0 & 0 & 0 & \frac{I_xG}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_yA_zG}{L^2A_zG+12EI_y} & 0 & \frac{4EI_y\left(L^2A_zG+3EI_y\right)}{L^3A_zG+12LEI_y} & 0 \\ 0 & \frac{6EI_zA_yG}{L^2A_yG+12EI_z} & 0 & 0 & 0 & \frac{4EI_z\left(L^2A_yG+3EI_z\right)}{L^3A_yG+12LEI_z} \end{bmatrix} \tag{5.21}$$

$$[K_{ji}]=\begin{bmatrix} -\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_zA_yG}{L^3A_yG+12LEI_z} & 0 & 0 & 0 & -\frac{6EI_zA_yG}{L^2A_yG+12EI_z} \\ 0 & 0 & -\frac{12EI_yA_zG}{L^3A_zG+12LEI_y} & 0 & -\frac{6EI_yA_zG}{L^2A_zG+12EI_y} & 0 \\ 0 & 0 & 0 & -\frac{I_xG}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_yA_zG}{L^2A_zG+12EI_y} & 0 & \frac{2EI_y\left(L^2A_zG+6EI_y\right)}{L^3A_zG+12LEI_y} & 0 \\ 0 & \frac{6EI_zA_yG}{L^2A_yG+12EI_z} & 0 & 0 & 0 & \frac{2EI_z\left(L^2A_yG+6EI_z\right)}{L^3A_yG+12LEI_z} \end{bmatrix} \tag{5.22}$$

$$[K_{ij}] = \begin{bmatrix} -\frac{A_x E}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z A_y G}{L^3 A_y G + 12LEI_z} & 0 & 0 & 0 & \frac{6EI_z A_y G}{L^2 A_y G + 12EI_z} \\ 0 & 0 & -\frac{12EI_y A_z G}{L^3 A_z G + 12LEI_y} & 0 & -\frac{6EI_y A_z G}{L^2 A_z G + 12EI_y} & 0 \\ 0 & 0 & 0 & -\frac{I_x G}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y A_z G}{L^2 A_z G + 12EI_y} & 0 & \frac{2EI_y\left(L^2 A_z G - 6EI_y\right)}{L^3 A_z G + 12LEI_y} & 0 \\ 0 & -\frac{6EI_z A_y G}{L^2 A_y G + 12EI_z} & 0 & 0 & 0 & \frac{2EI_z\left(L^2 A_y G - 6EI_z\right)}{L^3 A_y G + 12LEI_z} \end{bmatrix} \tag{5.23}$$

$$[K_{jj}] = \begin{bmatrix} \frac{A_x E}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z A_y G}{L^3 A_y G + 12LEI_z} & 0 & 0 & 0 & -\frac{6EI_z A_y G}{L^2 A_y G + 12EI_z} \\ 0 & 0 & \frac{12EI_y A_z G}{L^3 A_z G + 12LEI_y} & 0 & \frac{6EI_y A_z G}{L^2 A_z G + 12EI_y} & 0 \\ 0 & 0 & 0 & \frac{I_x G}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y A_z G}{L^2 A_z G + 12EI_y} & 0 & \frac{4EI_y\left(L^2 A_z G + 3EI_y\right)}{L^3 A_z G + 12LEI_y} & 0 \\ 0 & -\frac{6EI_z A_y G}{L^2 A_y G + 12EI_z} & 0 & 0 & 0 & \frac{4EI_z\left(L^2 A_y G + 3EI_z\right)}{L^3 A_y G + 12LEI_z} \end{bmatrix} \tag{5.24}$$

5.7.1 SHEAR AREA

The shear area is the cross-sectional property that is used for shear energy resistance. It can be found for a cross-section using the shear stress equation for a beam derived in most strength of materials textbooks. The basic equations for shear in the Y and Z direction are given as follows:

$$\tau_{xy} = \frac{V_y Q_y}{I_z t_z} = \frac{V_y A' \bar{y}'}{I_z t_z}$$

$$\tau_{xz} = \frac{V_z Q_z}{I_y t_y} = \frac{V_z A' \bar{z}'}{I_y t_y}$$

If these equations are written using a single term to represent all the cross-sectional properties, it results in the following:

$$\tau_{xy} = \frac{V_y Q_y}{I_z t_z} = \frac{V_y A'\overline{y}'}{I_z t_z} = \frac{V_y}{A_y}$$

$$\tau_{xz} = \frac{V_z Q_z}{I_y t_y} = \frac{V_z A'\overline{z}'}{I_y t_y} = \frac{V_z}{A_z}$$

The equations for the shear areas A_y and A_z can then be found.

$$A_y = \frac{I_z t_z}{A'\overline{y}'}$$

$$A_z = \frac{I_y t_y}{A'\overline{z}'}$$

There are three terms in these equations. The first is the centroidal moment of inertia. The second is the moment of the area, $A'\overline{y}'$ or $A'\overline{z}'$, between the centroid and the extreme fiber taken about the centroid. The third is the value t, which is the width at the centroid.

Example 5.12 Shear area

Determine the shear areas A_y and A_z for the a rectangular section.

Figure 5.14 shows the rectangle in the orientation to calculate A_z that corresponds to bending about the Y axis. To find A_y the area to the left of the Z axis will be used.

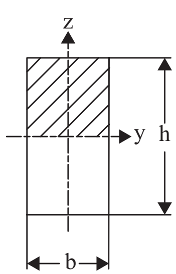

Figure 5.14. Example 5.12 Shear area.

$$A_y = \frac{I_z t_z}{A'\overline{y}'} = \frac{\frac{hb^3}{12}h}{\frac{bh}{2}\frac{b}{4}} = \frac{2bh}{3}$$

$$A_z = \frac{I_y t_y}{A'\overline{z}'} = \frac{\frac{bh^3}{12}b}{\frac{bh}{2}\frac{h}{4}} = \frac{2bh}{3}$$

For a rectangle, the shear areas are both two-thirds the cross-sectional area.

Example 5.13 Shear area

Determine the shear area A_z for the T-shaped section shown in Figure 5.15.

The centroid and the centroidal moment of inertia are found using the moment of area principles.

$$\overline{z} = \frac{\sum A\overline{z}}{\sum A} = \frac{2(12)6 + 12(2)13}{2(12)2} = 9.50 \text{ in}$$

$$I_y = \sum I'_y + Ad_z^2 = \frac{2(12)^3}{12} + 2(12)(9.5-6)^2 + \frac{12(2)^3}{12} + 12(2)(9.5-13)^2$$
$$= 884.0 \text{ in}^4$$

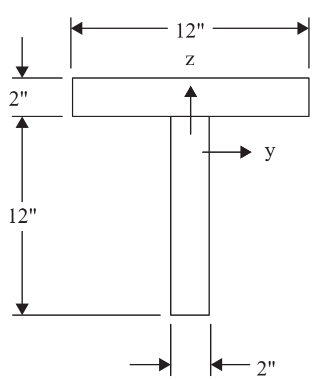

Figure 5.15. Example 5.13 Shear area.

The area of the web below the centroid will be used to find A_z.

$$A_z = \frac{I_y t_y}{A'\overline{z}'} = \frac{884.0(2)}{2(9.5)\frac{9.5}{2}} = 19.59 \text{ in}^2$$

Table 5.16 shows the calculations in tabular form.

Table 5.16. Example 5.13 Shear area

Element	b	h	z	A	Az	I	d	Ad²
Chord	12	2	13.0	24	312	8	3.5	294
Web	2	12	6	24	144	288	–3.5	294
Σ		z = 9.5		48	456	296		588
		Iy = 884.00		A'z' = 90.25			A_z = 19.590	

5.8 GEOMETRIC STIFFNESS, X-Y SYSTEM

An ordinary stiffness analysis, whether it includes shear deformations or not, makes no adjustments for the changing geometry of a loaded structure. Forces and moments are calculated from the original positions of the joints, not from their deformed positions. Elastic buckling, which is a function of joint deformations, is therefore impossible to predict using ordinary stiffness analysis. A procedure to include member and joint deformations in force and moment calculations can be developed by assuming a deformed shape and calculating the additional moment such as deformation would cause. Figure 5.16 shows a member subjected to bending and axial force in an un-deformed and deformed shape. An alternate, yet similar, derivation is published by Ketter, Lee, and Prawel (1979).

The first is a beam in Figure 5.16 that was used to derive the ordinary elastic stiffness matrix in Section 4.11 for the X-Y system. In that case, the

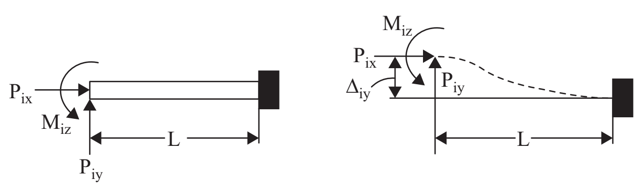

Figure 5.16. Geometric stiffness.

moment in the beam is a function of only the end shears and moments, as given by the following equation:

$$M_x = P_{iy}x - M_{iz}$$

The second beam in Figure 5.16 shows the bending deformations. In this case, the internal moment is a function of not only the end shears and moments, but also a function of the axial force multiplied by the beam's lateral deflection, y.

$$M_x = P_{iy}x - M_{iz} + P_{ix}y$$

This additional moment, the product of axial force, P_{ix}, and lateral deflection, y, is usually called the "P-delta effect." To derive a stiffness matrix that includes the P-delta effect, equilibrium of the deformed beam must be considered.

Example 5.14 Geometric stiffness

Derive the Δ_{iy} stiffness using Castigliano's theorems for a linear member including the geometric effects.

Using the principle of superposition, consider a beam with an applied deflection while the rotation is held to zero. Figure 5.17 shows the deformed beam with applied end forces. Also shown is a left-hand free-body of the beam cut at any distance x from the i-end.

The internal bending moment in the beam is found from equilibrium.

$$M_x = P_{iy}x - M_{iz} + P_{ix}\left(\Delta_{iy} - y\right)$$

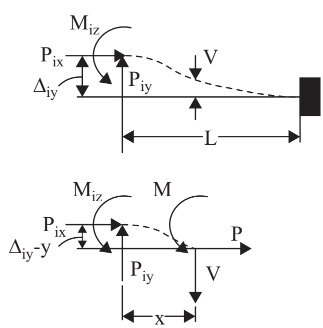

Figure 5.17. Example 5.14 Geometric stiffness.

If the lateral deformation, y, is assumed to be a general cubic function, the four known boundary conditions can be used to find the particular solution.

$$y = ax^3 + bx^2 + cx + d$$
$$y(0) = \Delta_{iy}$$
$$y(L) = 0$$
$$y'(0) = 0$$
$$y'(L) = 0$$
$$y = \Delta_{iy}\left(\frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1\right)$$

Substituting this equation into the internal moment equation yields the following:

$$M_x = P_{iy}x - M_{iz} + P_{ix}\Delta_{iy}\left(-\frac{2x^3}{L^3} + \frac{3x^2}{L^2}\right)$$

It is assumed that axial shortening is caused only by the axial force, P_{ix}. This is the same assumption used for the ordinary elastic stiffness derivation. The geometric stiffness derivation considers the lateral and rotational deformations, Δ_{iy} and θ_{iz}. From Castigliano's theorem, the general deflection and rotation of the free end are as follows:

$$\Delta_{iy} = \int M_x \frac{\delta M_x}{\delta P_{iy}} \frac{dx}{EI_z}$$
$$\theta_{iz} = \int M_x \frac{\delta M_x}{\delta M_{iz}} \frac{dx}{EI_z}$$

The partial derivatives of the internal moment equation with respect to applied shear force and moment are the following:

$$\frac{\delta M_x}{\delta P_{iy}} = x$$
$$\frac{\delta M_x}{\delta M_{iz}} = -1$$

It should be noted that rotation and deflection are functions of moment only, since shear deformations are ignored in this derivation. Setting θ_{iz} to zero and solving for P_{iy} and M_{iz} in terms of the deflection will result in two terms of the stiffness matrix.

$$\theta_{iz} = 0 = \int \left[-P_{iy}x + M_{iz} + P_{ix}\Delta_{iy}\left(\frac{2x^3}{L^3} - \frac{3x^2}{L^2}\right)\right]\frac{dx}{EI_z}$$

$$0 = -P_{iy}\frac{L^2}{2EI_z} + M_{iz}\frac{L}{EI_z} - P_{ix}\Delta_{iy}\frac{L}{2EI_z}$$

$$M_{iz} = P_{iy}\frac{L}{2} + P_{ix}\Delta_{iy}\frac{1}{2}$$

$$\Delta_{iy} = \int \left[P_{iy}x^2 - M_{iz}x + P_{ix}\Delta_{iy}\left(-\frac{2x^4}{L^3} + \frac{3x^3}{L^2}\right)\right]\frac{dx}{EI_z}$$

$$\Delta_{iy} = P_{iy}\frac{L^3}{3EI_z} - M_{iz}\frac{L^2}{2EI_z} + P_{ix}\Delta_{iy}\frac{7L^2}{10EI_z}$$

Substituting the first equation into the second equations yields the following:

$$\Delta_{iy} = P_{iy}\frac{L^3}{12EI_z} + P_{ix}\Delta_{iy}\frac{L^2}{10EI_z}$$

$$P_{iy} = \Delta_{iy}\frac{12EI_z}{L^3} + P_{ix}\Delta_{iy}\frac{-6}{5L} \tag{5.25}$$

Substituting Equation 5.25 into the following equation repeated from earlier results in Equation 5.25 for the second stiffness value.

$$M_{iz} = P_{iy}\frac{L}{2} + P_{ix}\Delta_{iy}\frac{1}{2}$$

$$M_{iz} = \Delta_{iy}\frac{6EI_z}{L^2} + P_{ix}\Delta_{iy}\frac{-1}{10} \tag{5.26}$$

Take note that the first terms in each of these stiffness equations are the same as the elastic stiffness values derived in Equations 4.26 and 4.27. The second term is the geometric component due to the deflected shape and the axial thrust.

Example 5.15 Geometric stiffness

Derive the θ_{iz} stiffness using Castigliano's theorems for a linear member including the geometric effects.

In this case, consider a beam with a known rotation while the deflection is held to zero. Figure 5.18 shows the deformed beam with applied end forces. Also shown is a left-hand free-body of the beam cut at any distance x from the i-end.

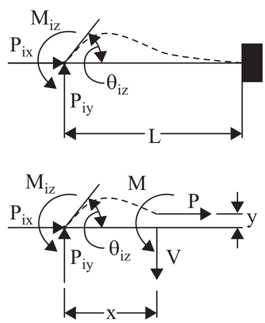

Figure 5.18. Example 5.15 Geometric stiffness.

The internal bending moment in the beam is found from equilibrium.

$$M_x = P_{iy}x - M_{iz} - P_{ix}y$$

If the lateral deformation, y, is assumed to be a general cubic function, the four known boundary conditions can be used to find the particular solution.

$$y = ax^3 + bx^2 + cx + d$$
$$y(0) = 0$$
$$y(L) = 0$$
$$y'(0) = \theta_{iz}$$
$$y'(L) = 0$$
$$y = \theta_{iz}\left(\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right)$$
$$M_x = P_{iy}x - M_{iz} + P_{ix}\theta_{iz}\left(-\frac{x^3}{L^2} + \frac{2x^2}{L} - x\right)$$

The general deflection and rotation of the free end are the same as Example 5.14.

$$\Delta_{iy} = \int M_x \frac{\delta M_x}{\delta P_{iy}} \frac{dx}{EI_z}$$

$$\theta_{iz} = \int M_x \frac{\delta M_x}{\delta M_{iz}} \frac{dx}{EI_z}$$

The partial derivatives of the internal moment equation with respect to applied shear force and moment are the same as in Example 5.14.

$$\frac{\delta M_x}{\delta P_{iy}} = \mathrm{x}$$

$$\frac{\delta M_x}{\delta M_{iz}} = -1$$

Setting Δ_{iy} to zero and solving for P_{iy} and M_{iz} in terms of the deflection will result in two terms of the stiffness matrix.

$$\theta_{iz} = \int \left[-P_{iy}x + M_{iz} + P_{ix}\theta_{iz}\left(\frac{x^3}{L^2} - \frac{2x^2}{L} + x \right) \right] \frac{dx}{EI_z}$$

$$\theta_{iz} = -P_{iy}\frac{L^2}{2EI_z} + M_{iz}\frac{L}{EI_z} + P_{ix}\theta_{iz}\frac{L^2}{12EI_z}$$

$$M_{iz} = P_{iy}\frac{L}{2} - P_{ix}\theta_{iz}\frac{L}{12} - \theta_{iz}\frac{EI_z}{L}$$

$$\Delta_{iy} = 0 = \int \left[P_{iy}x^2 - M_{iz}x + P_{ix}\theta_{iz}\left(-\frac{x^4}{L^2} + \frac{2x^3}{L} - x^2 \right) \right] \frac{dx}{EI_z}$$

$$0 = P_{iy}\frac{L^3}{3EI_z} - M_{iz}\frac{L^2}{2EI_z} - P_{ix}\theta_{iz}\frac{L^3}{30EI_z}$$

Substituting the first equation into the second equation yields the following:

$$P_{iy} = \theta_{iz}\frac{6EI_z}{L^2} + P_{ix}\theta_{iz}\frac{-1}{10} \tag{5.27}$$

Substituting Equation 5.27 into the following equation repeated from earlier results in Equation 5.28 for the second stiffness value.

$$M_{iz} = P_{iy}\frac{L}{2} - P_{ix}\theta_{iz}\frac{L}{12} - \theta_{iz}\frac{EI_z}{L}$$

$$M_{iz} = \theta_{iz}\frac{4EI_z}{L} + P_{ix}\theta_{iz}\frac{-2L}{15} \tag{5.28}$$

Take note that the first terms in each of these stiffness equations are the same as the elastic stiffness values derived in Equations 4.24 and 4.25. The second term is the geometric component due to the deflected shape and the axial thrust. All four terms can be written in matrix form.

$$\left(\begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} + P_{ix}\begin{bmatrix} \frac{-6}{5L} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{-2L}{15} \end{bmatrix}\right)\begin{bmatrix} \Delta_{iy} \\ \theta_{iz} \end{bmatrix} = \begin{bmatrix} P_{iy} \\ M_{iz} \end{bmatrix}$$

The first matrix on the left side of the equation in the basic elastic stiffness matrix will be called [K]. The second matrix on the left side of the equation in the geometric stiffness matrix will be called [G]. In general terms, the equation may be written as follows:

$$\left([K] + \mathrm{P}_{ix}[G]\right)[\Delta] = [F]$$

The same transformation used for previous stiffness matrix derivations can be applied to find the rest of the geometric stiffness matrix. The geometric stiffness matrix for the coplanar X-Y system is given as Equation 5.29. The sign convention on P_{ix} is positive for tension.

$$[K_m] = P_{ix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5L} & \frac{1}{10} & 0 & -\frac{6}{5L} & \frac{1}{10} \\ 0 & \frac{1}{10} & \frac{2L}{15} & 0 & -\frac{1}{10} & -\frac{L}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5L} & -\frac{1}{10} & 0 & \frac{6}{5L} & -\frac{1}{10} \\ 0 & \frac{1}{10} & -\frac{L}{30} & 0 & -\frac{1}{10} & \frac{2L}{15} \end{bmatrix} \tag{5.29}$$

5.9 GEOMETRIC STIFFNESS, X-Z SYSTEM

The geometric stiffness matrix for the coplanar X-Z system can be derived in a similar manner to the X-Y system performed in Section 5.8. The primary difference is that the signs of the moments due to translation and the forces due to rotation will be the opposite of Equation 5.29. The geometric stiffness matrix for the coplanar X-Z system is given as Equation 5.30.

$$[K_m] = P_{ix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5L} & -\frac{1}{10} & 0 & -\frac{6}{5L} & -\frac{1}{10} \\ 0 & -\frac{1}{10} & \frac{2L}{15} & 0 & \frac{1}{10} & -\frac{L}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5L} & \frac{1}{10} & 0 & \frac{6}{5L} & \frac{1}{10} \\ 0 & -\frac{1}{10} & -\frac{L}{30} & 0 & \frac{1}{10} & \frac{2L}{15} \end{bmatrix} \tag{5.30}$$

Example 5.16 Geometric stiffness

Determine the deformations at the free end of the beam by including both the elastic and geometric stiffness contributions to the stiffness solution. The beam is shown in Figure 5.19.

Since the *j*-end of the member is fixed, there is no need to build the entire member stiffness matrix. The *j*-end motions will be eliminated and

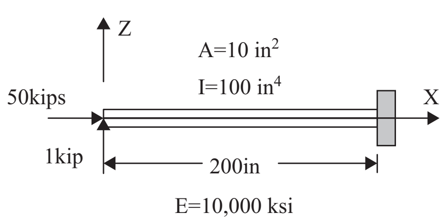

Figure 5.19. Example 5.16 Geometric stiffness.

only the *i*-end of the member stiffness needs to be developed. The value of P_{ix} in this case is known to be −50 kips.

$$\left(\begin{bmatrix} \frac{EA_x}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} + \mathrm{P}_{ix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{6}{5L} & \frac{-1}{10} \\ 0 & \frac{-1}{10} & \frac{2L}{15} \end{bmatrix}\right)\begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iz} \\ M_{iy} \end{bmatrix}$$

$$\left(\begin{bmatrix} 500 & 0 & 0 \\ 0 & 1.5 & -150 \\ 0 & -150 & 20{,}000 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -.3 & 5 \\ 0 & 5 & -1333.3 \end{bmatrix}\right)\begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} 50 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 500 & 0 & 0 \\ 0 & 1.2 & -145 \\ 0 & -145 & 18{,}666.7 \end{bmatrix}\begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} 50 \\ 1 \\ 0 \end{bmatrix}$$

Solving the general solution as shown in the following equations, the resulting deformations can be found. The solution is in inches and radians.

$$([K] + \mathrm{P}_{ix}[G])[\Delta] = [F]$$

$$[\Delta] = ([K] + \mathrm{P}_{ix}[G])^{-1}[F]$$

$$\begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} 0.1000 \\ 13.5757 \\ 0.10545 \end{bmatrix}$$

If the solution was performed with the geometric stiffness omitted, the result would be as follows:

$$\begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} 0.1000 \\ 2.6667 \\ 0.02000 \end{bmatrix}$$

Example 5.17 Geometric stiffness

Determine the deformations at the free end of the beam by including both the elastic and geometric stiffness contributions to the stiffness solution. The beam is shown in Figure 5.20.

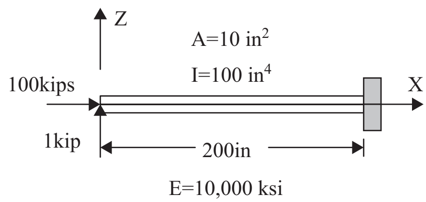

Figure 5.20. Example 5.17 Geometric stiffness.

This is the same as Example 5.16, except the axial force has been increased to 100 kips.

$$\left(\begin{bmatrix} 500 & 0 & 0 \\ 0 & 1.5 & -150 \\ 0 & -150 & 20{,}000 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -.6 & 10 \\ 0 & 10 & -2666.7 \end{bmatrix}\right)\begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} 100 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 500 & 0 & 0 \\ 0 & 0.9 & -140 \\ 0 & -140 & 17{,}333.3 \end{bmatrix}\begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} 100 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_{ix} \\ \Delta_{iz} \\ \theta_{iy} \end{bmatrix} = \begin{bmatrix} 0.2000 \\ -4.33333 \\ -0.00350 \end{bmatrix}$$

Observe that the *i*-end of the beam moved in the negative Z direction. This does not make logical sense. The reason of the backward motion is that the member has buckled elastically. The actual elastic buckling load of this column is 61.685 kips.

5.10 GEOMETRIC STIFFNESS, 3-D SYSTEM

By combining Equations 5.29 and 5.30 and adding the torsional stiffness terms, the geometric stiffness matrix in the 3-D Cartesian coordinate system can be found.

$$P_{ix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5L} & 0 & 0 & 0 & \frac{1}{10} & 0 & -\frac{6}{5L} & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & \frac{6}{5L} & 0 & -\frac{1}{10} & 0 & 0 & 0 & -\frac{6}{5L} & 0 & -\frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{10} & 0 & \frac{2L}{15} & 0 & 0 & 0 & \frac{1}{10} & 0 & -\frac{L}{30} & 0 \\ 0 & \frac{1}{10} & 0 & 0 & 0 & \frac{2L}{15} & 0 & -\frac{1}{10} & 0 & 0 & 0 & -\frac{L}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5L} & 0 & 0 & 0 & -\frac{1}{10} & 0 & \frac{6}{5L} & 0 & 0 & 0 & -\frac{1}{10} \\ 0 & 0 & -\frac{6}{5L} & 0 & \frac{1}{10} & 0 & 0 & 0 & \frac{6}{5L} & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{10} & 0 & -\frac{L}{30} & 0 & 0 & 0 & \frac{1}{10} & 0 & \frac{2L}{15} & 0 \\ 0 & \frac{1}{10} & 0 & 0 & 0 & -\frac{L}{30} & 0 & -\frac{1}{10} & 0 & 0 & 0 & \frac{2L}{15} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix} = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \tag{5.31}$$

The geometric stiffness of a member can also be derived based on a general transcendental equation. The full derivation is published by Blette (1985).

$$y = asin\left(\frac{n\pi}{2L}\right) + bcos\left(\frac{n\pi}{2L}\right) + cx + d$$

The particular solution for Δ_{iy} is as follows:

$$y = \Delta_{iy}\left(\frac{2}{4-\pi}\right)\left(-sin\frac{\pi x}{2L} + cos\frac{\pi x}{2L} + \frac{\pi x}{2L} - \frac{\pi}{2}\right)$$

The particular solution for θ_{iz} is as follows:

$$y = \theta_{iz}\left(\frac{4L}{4\pi-\pi^2}\right)\left[\left(1-\frac{\pi}{2}\right)sin\frac{\pi x}{2L} + cos\frac{\pi x}{2L} + \frac{\pi x}{2L} - 1\right]$$

These two relationships can be used to develop the geometric stiffness matrix.

$$P_{ix}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1.2036}{L} & 0 & 0 & 0 & 0.1018 & 0 & -\frac{1.2036}{L} & 0 & 0 & 0 & 0.1018 \\
0 & 0 & \frac{1.2036}{L} & 0 & -0.1018 & 0 & 0 & 0 & -\frac{1.2036}{L} & 0 & -0.1018 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.1018 & 0 & 0.1379L & 0 & 0 & 0 & 0.1018 & 0 & -0.0361L & 0 \\
0 & 0.1018 & 0 & 0 & 0 & 0.1379L & 0 & -0.1018 & 0 & 0 & 0 & -0.0361L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1.2036}{L} & 0 & 0 & 0 & -0.1018 & 0 & \frac{1.2036}{L} & 0 & 0 & 0 & -0.1018 \\
0 & 0 & -\frac{1.2036}{L} & 0 & 0.1018 & 0 & 0 & 0 & \frac{1.2036}{L} & 0 & 0.1018 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.1018 & 0 & -0.0361L & 0 & 0 & 0 & 0.1018 & 0 & 0.1379L & 0 \\
0 & 0.1018 & 0 & 0 & 0 & -0.0361L & 0 & -0.1018 & 0 & 0 & 0 & 0.1379L
\end{bmatrix}
\begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \Delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \Delta_{jx} \\ \Delta_{jy} \\ \Delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix}
=
\begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix}
\tag{5.32}$$

If more digits are desired for accuracy, the following substitutions can be made:

$$1.2036 = 1.20362445$$
$$0.1018 = 0.1018122226$$
$$0.1379 = 0.1378809597$$
$$0.0361 = 0.03606873710$$

5.11 GEOMETRIC AND SHEAR STIFFNESS

The effect of both the geometric and shear stiffness could be included in the flexural stiffness derivations. This matrix was derived by Karl J. Blette (Blette 1985). The procedure to derive the stiffness matrix would be to include the shear stiffness contribution used in Sections 5.5 and 5.6 in the geometric stiffness of Sections 5.8 and 5.9. Equation 5.33 shows the elastic and shear member stiffness in the 3-D Cartesian coordinate system. Equation 5.34 shows the elastic geometric and shear member stiffness in the 3-D Cartesian coordinate system.

The terms α and β are defined as follows:

$$\alpha_y = \frac{L^3}{12EI_y}$$

$$\alpha_z = \frac{L^3}{12EI_z}$$

$$\beta_y = \frac{L}{GA_z}$$

$$\beta_z = \frac{L}{GA_y}$$

$$\begin{bmatrix}
\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\alpha_z+\beta_z} & 0 & 0 & 0 & \frac{L}{2(\alpha_z+\beta_z)} & 0 & -\frac{1}{\alpha_z+\beta_z} & 0 & 0 & 0 & \frac{L}{2(\alpha_z+\beta_z)} \\
0 & 0 & \frac{1}{\alpha_y+\beta_y} & 0 & -\frac{L}{2(\alpha_y+\beta_y)} & 0 & 0 & 0 & -\frac{1}{\alpha_y+\beta_y} & 0 & -\frac{L}{2(\alpha_y+\beta_y)} & 0 \\
0 & 0 & 0 & \frac{I_xG}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{I_xG}{L} & 0 & 0 \\
0 & 0 & -\frac{L}{2(\alpha_y+\beta_y)} & 0 & \frac{EI_y(4\alpha_y+\beta_y)}{L(\alpha_y+\beta_y)} & 0 & 0 & 0 & \frac{L}{2(\alpha_y+\beta_y)} & 0 & \frac{EI_y(2\alpha_y-\beta_y)}{L(\alpha_y+\beta_y)} & 0 \\
0 & \frac{L}{2(\alpha_z+\beta_z)} & 0 & 0 & 0 & \frac{EI_z(4\alpha_z+\beta_z)}{L(\alpha_z+\beta_z)} & 0 & -\frac{L}{2(\alpha_z+\beta_z)} & 0 & 0 & 0 & \frac{EI_z(2\alpha_z-\beta_z)}{L(\alpha_z+\beta_z)} \\
-\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\alpha_z+\beta_z} & 0 & 0 & 0 & -\frac{L}{2(\alpha_z+\beta_z)} & 0 & \frac{1}{\alpha_z+\beta_z} & 0 & 0 & 0 & -\frac{L}{2(\alpha_z+\beta_z)} \\
0 & 0 & -\frac{1}{\alpha_y+\beta_y} & 0 & \frac{L}{2(\alpha_y+\beta_y)} & 0 & 0 & 0 & \frac{1}{\alpha_y+\beta_y} & 0 & \frac{L}{2(\alpha_y+\beta_y)} & 0 \\
0 & 0 & 0 & -\frac{I_xG}{L} & 0 & 0 & 0 & 0 & 0 & \frac{I_xG}{L} & 0 & 0 \\
0 & 0 & -\frac{L}{2(\alpha_y+\beta_y)} & 0 & \frac{EI_y(2\alpha_y-\beta_y)}{L(\alpha_y+\beta_y)} & 0 & 0 & 0 & \frac{L}{2(\alpha_y+\beta_y)} & 0 & \frac{EI_y(4\alpha_y+\beta_y)}{L(\alpha_y+\beta_y)} & 0 \\
0 & \frac{L}{2(\alpha_z+\beta_z)} & 0 & 0 & 0 & \frac{EI_z(2\alpha_z-\beta_z)}{L(\alpha_z+\beta_z)} & 0 & -\frac{L}{2(\alpha_z+\beta_z)} & 0 & 0 & 0 & \frac{EI_z(4\alpha_z+\beta_z)}{L(\alpha_z+\beta_z)}
\end{bmatrix}
=
\begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix}
\tag{5.33}$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{(6\alpha_z+5\beta_z)}{5L(\alpha_z+\beta_z)} & 0 & 0 & 0 & -\frac{\alpha_z}{10(\alpha_z+\beta_z)} & 0 & \frac{(6\alpha_z+5\beta_z)}{5L(\alpha_z+\beta_z)} & 0 & 0 & 0 & -\frac{\alpha_z}{10(\alpha_z+\beta_z)} \\
0 & 0 & -\frac{(6\alpha_y+5\beta_y)}{5L(\alpha_y+\beta_y)} & 0 & \frac{\alpha_y}{10(\alpha_y+\beta_y)} & 0 & 0 & 0 & \frac{(6\alpha_y+5\beta_y)}{5L(\alpha_y+\beta_y)} & 0 & \frac{\alpha_y}{10(\alpha_y+\beta_y)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\alpha_y}{10(\alpha_y+\beta_y)} & 0 & -\frac{L(8\alpha_y+5\beta_y)}{60(\alpha_y+\beta_y)} & 0 & 0 & 0 & -\frac{\alpha_y}{10(\alpha_y+\beta_y)} & 0 & \frac{L(2\alpha_y+5\beta_y)}{60(\alpha_y+\beta_y)} & 0 \\
0 & \frac{L}{2(\alpha_z+\beta_z)} & 0 & 0 & 0 & -\frac{L(8\alpha_z+5\beta_z)}{60(\alpha_z+\beta_z)} & 0 & \frac{\alpha_z}{10(\alpha_z+\beta_z)} & 0 & 0 & 0 & \frac{L(2\alpha_z+5\beta_z)}{60(\alpha_z+\beta_z)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(6\alpha_z+5\beta_z)}{5L(\alpha_z+\beta_z)} & 0 & 0 & 0 & \frac{\alpha_z}{10(\alpha_z+\beta_z)} & 0 & -\frac{(6\alpha_z+5\beta_z)}{5L(\alpha_z+\beta_z)} & 0 & 0 & 0 & \frac{\alpha_z}{10(\alpha_z+\beta_z)} \\
0 & 0 & \frac{(6\alpha_y+5\beta_y)}{5L(\alpha_y+\beta_y)} & 0 & -\frac{\alpha_y}{10(\alpha_y+\beta_y)} & 0 & 0 & 0 & -\frac{(6\alpha_y+5\beta_y)}{5L(\alpha_y+\beta_y)} & 0 & -\frac{\alpha_y}{10(\alpha_y+\beta_y)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\alpha_y}{10(\alpha_y+\beta_y)} & 0 & \frac{L(2\alpha_y+5\beta_y)}{60(\alpha_y+\beta_y)} & 0 & 0 & 0 & -\frac{\alpha_y}{10(\alpha_y+\beta_y)} & 0 & -\frac{L(8\alpha_y+5\beta_y)}{60(\alpha_y+\beta_y)} & 0 \\
0 & -\frac{\alpha_z}{10(\alpha_z+\beta_z)} & 0 & 0 & 0 & \frac{L(2\alpha_z+5\beta_z)}{60(\alpha_z+\beta_z)} & 0 & \frac{\alpha_z}{10(\alpha_z+\beta_z)} & 0 & 0 & 0 & -\frac{L(8\alpha_z+5\beta_z)}{60(\alpha_z+\beta_z)}
\end{bmatrix} \tag{5.34}$$

5.12 TORSION

The torsional stiffness of slender linear members is composed of two parts. The first is known as St. Venant's torsion, which is uniform on a member at any distance, ρ, from the longitudinal axis. This is the torsional stiffness that is included in the elastic member stiffness Equation 4.33. This is the primary resistance to torsion for circular crosses-sections that have area distributed uniformly about the longitudinal axis. The second type of torsional stiffness is known as warping torsion. Warping torsion is the primary stiffness in thin-walled open cross-sections such as angles shapes and wide flange shapes. The warping torsion will cause longitudinal deformations in the cross-section that will cause certain portions to elongate and other portions to shorten. This warping effect can be included in the derivation of the stiffness. "Structural Analysis and Design," by Ketter, Lee, and Prawel, Jr., covers this derivation (Ketter, Lee, and Prawel 1979). The torsional stiffness at each degree of freedom is represented as two components instead of the single values used in the normal elastic stiffness. Equation 5.35 shows the elastic torsional member stiffness in the 3-D Cartesian coordinate system.

$$\begin{bmatrix}
\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -T_1 & -T_2 & 0 & 0 & 0 & 0 & 0 & T_1 & -T_2 & 0 & 0 \\
0 & 0 & 0 & -T_2 & -T_3 & 0 & 0 & 0 & 0 & 0 & T_2 & T_4 & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{A_xE}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_z}{L^3} & 0 & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & T_1 & T_2 & 0 & 0 & 0 & 0 & 0 & -T_1 & -T_2 & 0 & 0 \\
0 & 0 & 0 & -T_2 & T_4 & 0 & 0 & 0 & 0 & 0 & T_2 & T_3 & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & 0 & \frac{2EI_y}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix} \tag{5.35}$$

The resulting member stiffness matrix is 14×14 in size with the terms T_1, T_2, T_3, and T_4 defined as follows:

$$T_1 = G\kappa_T \frac{\lambda sinh(\lambda L)}{2[cosh(\lambda L)-1]-\lambda L sinh(\lambda L)}$$

$$T_2 = G\kappa_T \frac{cosh(\lambda L)-1}{2[cosh(\lambda L)-1]-\lambda L sinh(\lambda L)}$$

$$T_3 = \frac{G\kappa_T}{\lambda} \frac{sinh(\lambda L)-\lambda L cosh(\lambda L)}{2[cosh(\lambda L)-1]-\lambda L sinh(\lambda L)}$$

$$T_4 = \frac{G\kappa_T}{\lambda} \frac{\lambda L-\lambda sinh(\lambda L)}{2[cosh(\lambda L)-1]-\lambda L sinh(\lambda L)}$$

$$\lambda = \sqrt{\frac{G\kappa_T}{EI_\omega}}$$

In the equation for λ, κ_T is the St. Venant torsion constant, which is typically the polar moment of inertia, I_x. The value of I_ω is known as the warping constant. Both of these values are normally tabulated in handbooks or specifications.

5.13 SUB-STRUCTURING

When a structure is of large enough size that the contents for the global joint stiffness matrix cannot be contained in the RAM of a computer, the matrix can be transformed into segments by reduction or decomposition. The resulting transformed matrix can take many forms depending on the process used. One of the common transformations is the N-matrix, due to the configuration of resulting values. The following is a general description of the operation used to solve large systems using the N-matrix. This method can be used by operating on the individual degrees of freedom or on the entire joint as matrix operations.

The original equation set is normally a sparse matrix with most of the values near the main diagonal. Equation 4.34 represents the original stiffness solution set. For clarity, the zero values are left out of the matrices and X indicates where values exist.

$$[K_g][\Delta_g]=[P_g]$$

$$\begin{bmatrix} X & X & & & & \\ X & X & X & X & & \\ & X & X & X & & \\ & X & X & X & X & \\ & & & X & X & X \\ & & & & X & X \end{bmatrix} \begin{bmatrix} \\ \Delta \\ \\ \end{bmatrix} = \begin{bmatrix} \mathrm{P} \\ - \\ \mathrm{FEPM} \end{bmatrix} \tag{5.36}$$

The matrix is reduced by the appropriate reduction method. The degrees of freedom or joints used to represent the stiffness should be placed at the outer ends of the matrix. These will be the degrees of freedom in the outer columns of Equation 5.37. All of the values above and below the main diagonal of the matrix will be reduced to zero except those retained in the outer columns. The resulting matrix is shown in Equation 5.37 and is the N-matrix.

$$\begin{bmatrix} K_g & & & & & K_g \\ X & X & & & & X \\ X & & X & & & X \\ X & & & X & & X \\ X & & & & X & X \\ K_g & & & & & K_g \end{bmatrix} \begin{bmatrix} \Delta_g \\ \delta \\ \delta \\ \delta \\ \delta \\ \Delta_g \end{bmatrix} = \begin{bmatrix} P_g \\ S \\ A \\ \mathrm{V} \\ \mathrm{E} \\ P_g \end{bmatrix} \tag{5.37}$$

The information contained in the N-matrix is used in two ways. The values in the extreme boundaries of the matrices represent the stiffness, deformation, or force (K_g, Δ_g, and P_g) of the structure related to the retained degrees of freedom or joints. These values can be combined with other members in a stiffness analysis. The interior portions of the matrices represent the values of the stiffness, deformation, or force of the degrees of freedom or joints eliminated in the reduction. These values are used to find the deformations and forces of these degrees of freedom or joints once the main analysis results are known. The N-matrix can be compressed as shown in Equation 5.38. The interior stiffness values can be inverted for later processing.

$$\begin{bmatrix} K_g & & K_g \\ X & X^{-1} & X \\ X & X^{-1} & X \\ X & X^{-1} & X \\ X & X^{-1} & X \\ K_g & & K_g \end{bmatrix} \begin{bmatrix} \Delta_g \\ \Delta \\ \Delta \\ \Delta \\ \Delta \\ \Delta_g \end{bmatrix} = \begin{bmatrix} P_g \\ S \\ A \\ \mathrm{V} \\ \mathrm{E} \\ P_g \end{bmatrix} \tag{5.38}$$

The main analysis is performed and the global deformation of the retained degrees of freedom of joint is found.

$$\left[\Delta_g\right] = \left[K_g\right]^{-1}\left[P_g\right]$$

The interior loads to be used to find the remainder of the deformations are shown in Equation 5.39. This represents the loads due to the interior loading that was saved and the load due to the motion of the main analysis motions on the interior joints.

$$\begin{bmatrix} \\ L \\ O \\ \mathrm{A} \\ \mathrm{D} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ S \\ A \\ \mathrm{V} \\ \mathrm{E} \\ \\ \end{bmatrix} + \begin{bmatrix} & \\ X & X \\ X & X \\ X & X \\ X & X \\ & \end{bmatrix} \begin{bmatrix} \Delta_g \\ \\ \\ \\ \\ \Delta_g \end{bmatrix} \tag{5.39}$$

The deformations of the interior degrees of freedom or joints can be found from Equation 5.40.

$$\begin{bmatrix} \\ \Delta \\ \Delta \\ \Delta \\ \Delta \\ \\ \end{bmatrix} = \begin{bmatrix} \\ X^{-1} \\ X^{-1} \\ X^{-1} \\ X^{-1} \\ \\ \end{bmatrix} \begin{bmatrix} \\ L \\ O \\ \mathrm{A} \\ \mathrm{D} \\ \\ \end{bmatrix} \tag{5.40}$$

The local member forces are found from the deformation in Equation 5.40, the same as in step 6 of the general stiffness procedure given in Section 4.13. The values of Δ are used for Δ_g in Equation 4.38 and repeated here.

$$\left[P\,\&\,M_m\right] = \left[K_m\right]\left[R\right]\left[\Delta_g\right] + \left[FEPM_m\right]$$

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About the Authors

Steven O'Hara is a Professor of Architectural Engineering and Licensed Engineer who has taught architectural engineering at the Oklahoma State University School of Architecture since 1988. His primary areas of interest include the design and analysis of masonry, steel and timber structures, with special interest in classical numerical structural analysis and the design of concrete structures. Professor O'Hara is one of four faculty members in the Architectural Engineering program at OSU, and as teaches courses in the AE program at all levels; he enjoys his close mentoring relationship with the students in the AE program at OSU, as he also performs the role of their academic advisor. Outside the OSU classroom, Professor O'Hara is an Affiliate Professor of Civil Engineering and Architecture in "Project Lead the Way," a nationwide program for high-school students. He trains the high school teachers responsible for introducing engineering principles into the secondary curriculum, through project-based learning and has coauthored the workbook for the curriculum Civil Engineering & Architecture Workbook. He is the coauthor of *ARE Review Manual* used by thousands of architects to prepare for their licensing exams.

As a faculty member, Professor O'Hara has received numerous awards: In 2013, he was elected to the first class of Faculty Teaching Fellows at Oklahoma State University and was elevated to Chapter Honor Member by Chi Epsilon. In 2001, he was the honored recipient of the first Melvin R. Lohmann Professorship. In 1999, he received the Chi Epsilon Excellence in Teaching Award. In 1995, he was recognized as the Halliburton Excellent Young Teacher for the College of Engineering, Architecture, and Technology at OSU.

Carisa H. Ramming joined the faculty at Oklahoma State University in January 2009. Professor Ramming is a graduate of Oklahoma State University with a bachelor of architectural engineering degree and dual master's degrees: master of science in civil engineering with an emphasis in Construction Engineering and master of architectural engineering. Professor Ramming teaches engineering mechanics: statics, steel design, and building foundations.

Before returning to academics, Professor Ramming worked as a consulting engineer primarily designing large structural steel retail facilities, and obtained her license as a professional engineer in the state of Oklahoma.

Professor Ramming is a member of the American Society of Engineering Educators and the Oklahoma Structural Engineers Association. She has served the engineering community as Director, Vice-President, President, and Delegate of Oklahoma's Chapter of the National Council of Structural Engineering. Professor Ramming is the advisor for OSU's student chapter of the Architectural Engineering Institute; the College of Engineering, Architecture and Technology's Student Council; and Tau Beta Pi.

Professor Ramming recently coauthored *Civil Engineering & Architecture Workbook* for Project Lead the Way, which provides curriculum for science, technology, engineering, and mathematics education used in middle and high schools. She was also named the Outstanding Faculty Member from the College of Engineering, Architecture, and Technology by The Panhellenic Council and Interfraternity Council in 2010 and 2012. In 2013, she was awarded the Halliburton Young Teaching Award and in 2014, she received the Excellent Teacher Award for the College of Engineering, Architecture and Technology.

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Numerical Structural Analysis

Steven O'Hara and Carisa H. Ramming

As structural engineers move further into the age of digital computation and rely more heavily on computers to solve problems, it remains paramount that they understand the basic mathematics and engineering principles used to design and analyze building structures. The link between the basic concepts and application to real world problems is one of the most challenging learning endeavors that structural engineers face.

The primary purpose of *Numerical Structural Analysis* is to assist structural engineering students with developing the ability to solve complex structural analysis problems. This book will cover numerical techniques to solve mathematical formulations, which are necessary in developing the analysis procedures for structural engineering. Once the numerical formulations are understood, engineers can then develop structural analysis methods that use these techniques. This will be done primarily with matrix structural stiffness procedures. Finally, advanced stiffness topics will be developed and presented to solve unique structural problems, including member end releases, non-prismatic, shear, geometric, and torsional stiffness.

Steven O'Hara is a professor of architectural engineering and licensed engineer at the Oklahoma State University School of Architecture since 1988. Professor O'Hara is an affiliate professor of civil engineering and architecture in "Project Lead the Way," a nationwide program for high-school students. He trains the high school teachers responsible for introducing engineering principles into the secondary curriculum, through project-based learning and coauthored the workbook for the curriculum, *Civil Engineering & Architecture Workbook*. He is also coauthor of the *ARE Review Manual.*

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