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V. A. Svetlitsky

# Engineering Vibration Analysis

## Worked Problems 2

Springer

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## Worked Problems 2

Translated by A. S. Lidvansky and R. A. Mukhamedshin

With 138 Figures

Springer

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# Preface

Constantly increasing attention is paid in the course 'Vibration Theory' to vibration of mechanical systems with distributed parameters, since the real elements of machines, devices, and constructions are made of materials that are not perfectly rigid. Therefore, vibrations of the objects including, for example, rod elastic elements excite the vibrations of these elements, which can produce a substantial effect on dynamic characteristics of moving objects and on readings of instruments.

For a mechanical engineer working in the field of design of new technologies the principal thing is his know-how in developing the sophisticated mathematical models in which all specific features of operation of the objects under design in real conditions are meticulously taken into account. So, the main emphasis in this book is made on the methods of derivation of equations and on the algorithms of solving them (exactly or approximately) taking into consideration all features of actual behavior of the forces acting upon elastic rod elements.

The eigen value and eigen vector problems are considered at vibrations of curvilinear rods (including the rods with concentrated masses). Also considered are the problems with forced vibrations. When investigating into these problems an approximate method of numerical solution of the systems of linear differential equations in partial derivatives is described, which uses the principle of virtual displacements.

Some problems are more complicated than others and can be used for practical works of students and their graduation theses.

To facilitate the solution of these problems, the book includes Appendices containing the concise description of the foundations of rod mechanics (static and dynamic). The Appendices are useful not only for solving the problems presented. They can also be used when the problems concerning the dynamics of spatially curvilinear rods are solved. The conditions of these problems can be formulated by an instructor at practical lessons.

Answers to some problems contain short descriptions of solution algorithms without numerical results. Students should continue these solutions by themselves using computer methods. This must promote the deeper understanding of the vibration theory and the better experience in programming while solving nonstandard problems, when no ready library programs are available.

In preparation of the manuscript the author has used the manuals and lecture courses prepared by him at the 'Applied Mechanics' chair in the Bauman Technical University (Moscow, Russia). The monograph is intended for use by students, postgraduates, and lecturers of engineering universities. It can be also useful for mechanical engineers whose practical work is connected with the vibration theory.

Moscow, September, 2003 *Valery A.Svetlitsky*

## BASIC NOTATION

| | |
|---|---|
| $A_{11}$ | torsional stiffness of rod; |
| $A_{22}$ | bending stiffness of rod relative to $y$ (or $x_2$) axis; |
| $A_{33}$ | bending stiffness of rod relative to $x$ (or $x_3$) axis; |
| $c$ | stiffness coefficient; |
| $E$ | modulus of elasticity of the first kind; |
| $F$ | cross section area; |
| $F(t)$ | disturbing force; |
| $G$ | modulus of elasticity of the second kind; |
| $H(\varepsilon)$ | Heaviside function; |
| $J_x, J_y, J_p, J_k$ | geometrical features of rod cross section; |
| $K_{ij}$ | Krylov function; |
| $k$ | stiffness coefficient of elastic base; |
| $M_1, M_2, M_3$ | torsional and two bending momenta; |
| $P_1, P_2, P_3$ | components of concentrated force in a related coordinate system; |
| $P_{x_1}, P_{x_2}, P_{x_3}$ | components of concentrated force in a Cartesian coordinate system; |
| $p_i$ | eigenfrequency (natural frequency) of vibration; |
| $Q_1, Q_2, Q_3$ | axial and two cutting forces; |
| $q_1, q_2, q_3$ | components of a distributed load in; a related coordinate system; |
| $q_{x_1}, q_{x_2}, q_{x_3}$ | components of a distributed load in; a Cartesian coordinate system; |
| $T$ | kinetic energy; |
| $\alpha$ | coefficient of viscous friction; |
| $\delta$ | logarithmic decrement of damping; |
| $\delta(z)$ | Dirac delta function; |
| $\mu$ | dynamic viscosity coefficient; |
| $\Pi$ | potential energy; |
| $\varrho$ | density of material; |
| $\varphi$ | angular displacement; |
| $\Omega$ | angular velocity; |
| $\omega$ | frequency of free vibrations; |
| $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$ | components of concentrated moment in a related coordinate system; |
| $\mathfrak{M}_{x_1}, \mathfrak{M}_{x_2}, \mathfrak{M}_{x_3}$ | components of concentrated moment in a Cartesian coordinate system. |

# Contents

# 1

# Problems and Examples

## 1.1 Vibrations of Perfectly Flexible Rods

**1** Derive the differential equation of small vibrations of a string (Fig. 1.1) subject to the action of a distributed load ($q$ is the load per unit length). The tension of the string is $Q_{10}$, and its mass per unit length is $m_0$ (when deriving the equation, assume that the tension $Q_{10}$ remains constant).

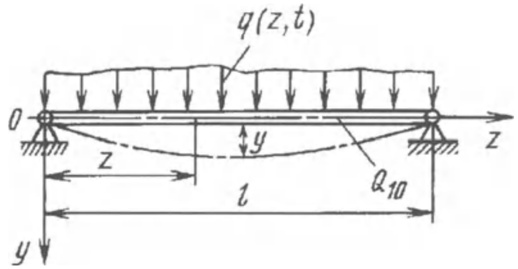


**Fig. 1.1.**

**2** Determine the frequencies of free vibrations of a string (Fig. 1.2) and the velocity of propagation of its transverse displacements (the gravity force of the string should be disregarded). Use the following numerical values: $l = 0.5$ m, $Q_{10} = 30$ N, the string diameter $d = 1$ mm, and the material (steel) density $\varrho = 7800$ kg/m$^3$.

**3** A heavy homogeneous filament of length $l$ is fixed at the point $O$ (Fig. 1.3) and is under the action of the force of gravity in the equilibrium vertical position. The mass of a unit length of the filament is $m_0$.

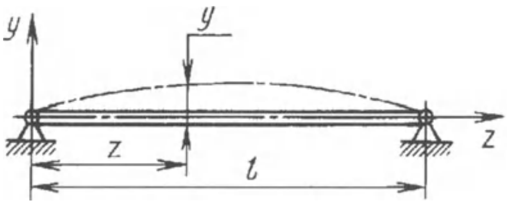


Fig. 1.2.

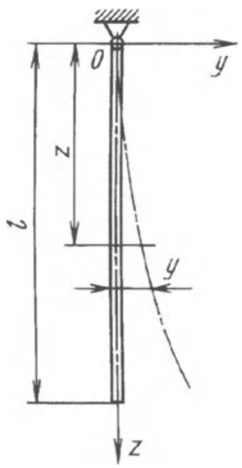


Fig. 1.3.

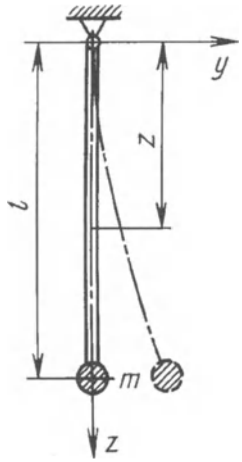


Fig. 1.4.

Derive the differential equation of small free vibrations of the filament with respect to its vertical equilibrium position and determine the frequencies of natural vibrations.

**4** Derive the differential equation of small vibrations and calculate the first frequency of vibrations for a heavy filament with a weight at its end (Fig. 1.4). The weight mass is $m$, and the mass of a unit length of the filament is $m_0$ so that $m = m_0 l$.

**5** A heavy homogeneous filament of length $l$ is fixed at the point $O$ between two vertical planes (Fig. 1.5). Both the filament and the planes rotate about the vertical axis with a constant angular velocity $\omega$.

Derive the differential equation of small free vibrations with respect to the vertical equilibrium position and determine the frequencies of filament vibrations as a function of the angular velocity $\omega$. Establish the least possible value of the critical angular velocity. The mass of a unit length of the filament is $m_0$.

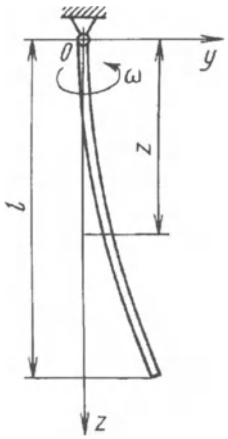


Fig. 1.5.


Fig. 1.6.

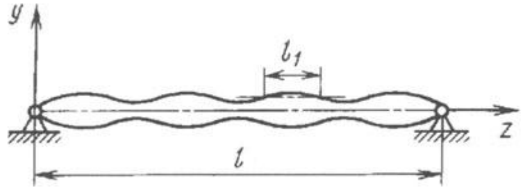


Fig. 1.7.

**6** Determine the first two frequencies of natural transverse vibrations of a string (Fig. 1.6) whose mass (mass of the string unit length) varies according to the law

$$m = m_0 + m_1 \sin \frac{\pi z}{l}.$$

Assume that, under vibrations, the string tension $Q_{10}$ remains practically unchanged.

**7** Determine the lowest frequency of transverse vibrations of a string (Fig. 1.7) whose mass varies (along the string length) as

$$m = m_0 + m_1 \sin \frac{\pi z}{l_1}.$$

**8** A filament fixed at the point $O$ is on a rotating disk (Fig. 1.8). Derive the differential equation of small transverse vibrations of the filament with respect to the equilibrium position, under which the filament has a rectilinear form.

**9** Derive the differential equation of small transverse vibrations of a string lying on a flexible inertialess base (Fig. 1.9) and determine the frequencies of natural vibrations. The string tension is $Q_{10}$ and the mass of a unit length of the filament is $m_0$. When the string is displaced from its equilibrium position, it is under the action of a restoring force proportional to this displacement. The proportionality coefficient is $k$.

**10** At the initial moment, the deviation of a string (Fig. 1.10) has the form $y = y_0 \sin \frac{\pi z}{l}$ , and all velocities are zero. The string tension is $Q_{10}$. Determine the string deviations in subsequent time instants.

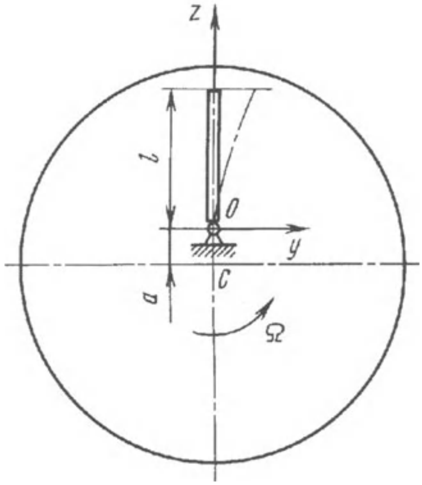


**Fig. 1.8.**

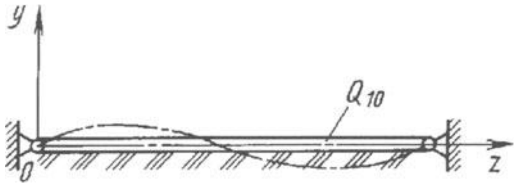


**Fig. 1.9.**

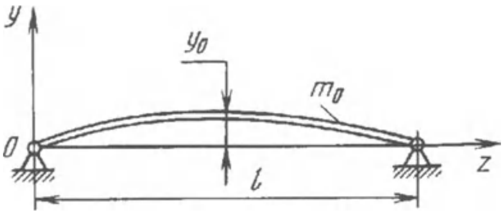


**Fig. 1.10.**

**11** Derive the differential equation of small transverse vibrations of a branch of a flexible gearing (Fig. 1.11). Determine the vibration frequencies, eigen functions, and the critical velocity $\omega$ of motion in the general case and for a particular case when the velocity of motion of the flexible gearing is $\omega = 16$ m/s, the gearing length is $l = 0.6$ m, the mass of a unit length of the flexible gearing is $m_0 = 0.3$ kg/m, the cross section area of the flexible gearing is $F = 2$

cm$^2$, and the initial tensions of the gearing branches are $Q_{10} = F\sigma_{10} = 800$ N and $Q_{20} = 400$ N.

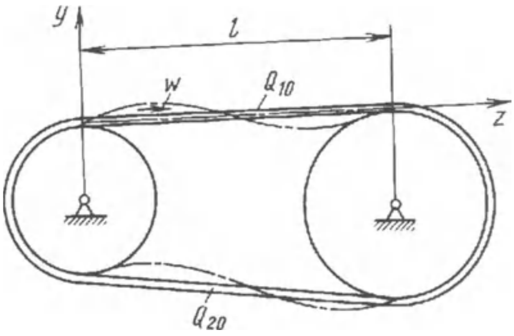


**Fig. 1.11.**

**12** Find the velocities of propagation of perturbation waves along the branches of the flexible gearing (see Problem **11**) and determine the velocity of motion of this gearing, under which the perturbations do not propagate against the gearing motion.

**13** Investigate the stability of transverse vibrations of the branches of an operating flexible gearing under a steady-state regime of vibrations of the gearing blocks (Fig. 1.12). The mass of a unit length of the flexible gearing is $m_0 = 0.3$ kg/m, the gearing length is $l = 0.6$ m, and its cross section area is 2 cm$^2$. Under steady-state vibrations of the blocks, the full tensions in the branches vary according to the following law (see solution to Problem **11**)

$$\sigma_1 = \sigma_{10} + \Delta\sigma_1 \sin \omega t; \quad \sigma_2 = \sigma_{20} + \Delta\sigma_2 \sin \omega t,$$

where $\sigma_{10} = 4$ MPa; $\sigma_{20} = 2$ MPa; $\Delta\sigma_1 = 1.95$ MPa; $\Delta\sigma_2 = 1.86$ MPa; and $\omega = 88$ s$^{-1}$.

The velocity of motion of the flexible gearing is $w = 16$ m/s.

**14** Figure 1.13*a* presents schematically an operating belt conveyer with a nonuniformly distributed weight. The tension of the working branch of the conveyer is $Q_{10}$ (the conveyer branch can be considered as a belt with zero bending stiffness). In a coordinate system fixed to the conveyer belt (in the coordinate system $y_1, z_1$ moving with the velocity $w$) the distribution of the lading mass $m$ is described by the following equation (1.13*b*)

$$m = m_0 + m_1 \sin \frac{2\pi z_1}{l_1} \quad (m \ll m_0).$$

The mass of the belt unit length is $m_2$.

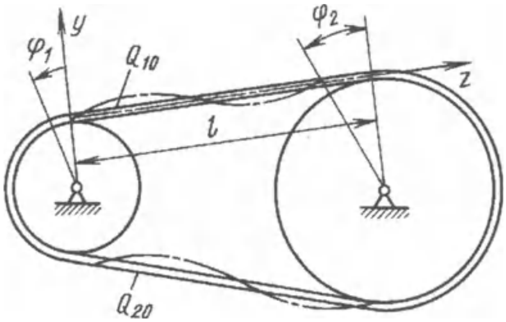


**Fig. 1.12.**

Set up the differential equation of vibrations of a driving branch of the conveyer and examine (approximately) the stability of small vibrations. The numerical values are as follows: $Q_{10} = 2$ kN, $m_0 = 20$ kg/m, $m_1 = 2$ kg/m, $m_2 = 1.2$ kg/m, $l = 2.025$ m, $l_1 = 0.45$ m, and $w = 2$ m/s.

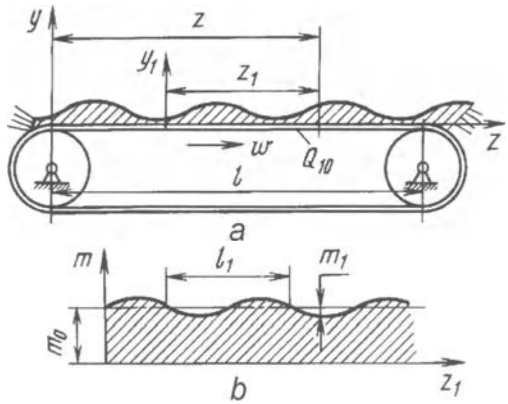


**Fig. 1.13.**

**15** A perfectly incompressible fluid flows along a perfectly flexible vertical pipe (Fig. 1.14). The pipe is fixed at the points $A$ and $B$, and pulled with the tension $Q_{10}$. The fluid velocity $w$ along the height of the pipe and the fluid pressure $p$ on the segment $AB$ can be assumed constant. The mass of a unit length of the pipe is $m_1$ and the mass of fluid per a pipe unit length is $m_2$. The area of the inner cross section of the pipe is $F$. Determine the frequencies of free vibrations of the pipe.

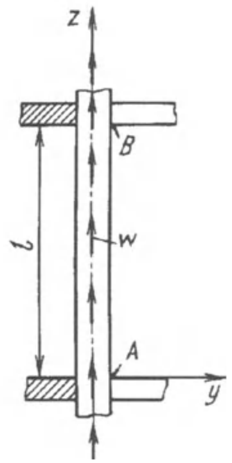


Fig. 1.14.

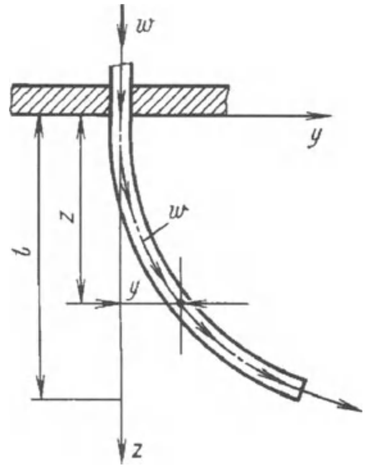


Fig. 1.15.

**16** A perfectly incompressible fluid flows inside a flexible vertical hanging pipe (Fig. 1.15) with the velocity $w$. The mass of a unit length of the pipe is $m_1$; the fluid mass per a pipe unit length is $m_2$. Neglect the fluid pressure.

Set up the differential equation of small transverse vibrations of the pipe with respect to the vertical equilibrium position.

**17** Determine the velocities of propagation of perturbations in the pipe (see Fig. 1.14) and the fluid flow velocity, at which perturbations do not propagate against the fluid flow.

**18** Under transverse vibrations, a string is deformed (extended), which results in a change of its initial tension. It is a usual practice to neglect this additional tension when the equation of string vibrations is derived. However, the error in determining the frequencies of free vibrations remains unclear in this case.

Determine the lowest frequency of free vibrations of a string (Fig. 1.10) taking its extensibility into account and find the error that results from neglecting the string extensibility. The area of the string cross section is $F$, and the Young's modulus of the first kind is $E$. The initial tension of the string is $Q_{10}$.

**19** A steel string is placed between the poles $N$ and $S$ of a permanent magnet (Fig. 1.16). The string tension is $Q_{10}$, the mass of a unit length is $m_0$. The attraction force of the magnet when the string is displaced from its neutral position (the force acting upon a unit length of the string) is equal to

$$q = F_2 - F_1 = \frac{k_1 \Phi_0^2}{(l_1 - y)^2} - \frac{k_1 \Phi_0^2}{(l_1 + y)^2}.$$

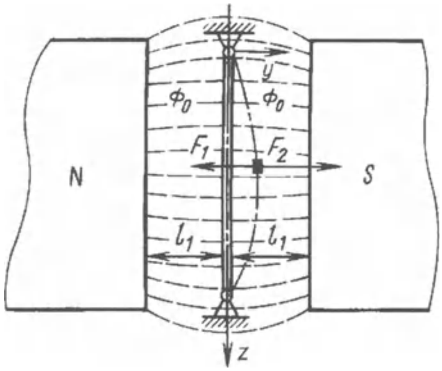


Fig. 1.16.

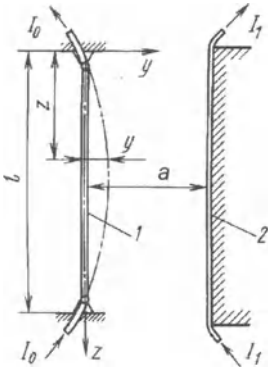


Fig. 1.17.

Assuming the string displacement to be small ($y \ll l_1$), derive the differential equation of small transverse vibrations of the string and determine the frequencies of its free vibrations (see solution to Problem **107** in Part I).

**20** Derive the differential equation of small transverse vibrations of a string in a magnetic field and investigate approximately their stability (Problem **19**), if $k_1\Phi_0^2 = a_{11} + a_{12}\sin\omega t$ (the magnetic field is time-variable). The numerical values of the problem are as follows: $Q_{10} = 160$ N, $m_0 = 6\cdot 10^{-3}$ kg/m, $l = 0.1$ m, $a = 10$ mm, $a_{11} = 0.008$ N/m, $a_{12} = 0.004$ N/m, and $\omega = 5000$ s$^{-1}$.

**21** A tightened string 1 (Fig. 1.17) carries the constant current $I_0$ and is subjected to the action of a magnetic field generated by another, infinitely long stiff cord 2 that carries the current $I_1 = I_{10}\sin\omega t$.

The string remains practically rectilinear due to its large bending stiffness. The force of attraction of the string 1 by the cord 2 (acting upon a unit length of the string) is equal to

$$q = \frac{2I_1 I_0 k}{(a-y)},$$

where $y$ is the string displacement under vibrations. The string tension is $Q_{10}$ and the mass of the string unit length is $m_0$.

Derive the differential equation of small vibrations of the string.

**22** At the instant $t = 0$ a constant force $P_0$ suddenly acts upon a string (Fig. 1.18) at a distance $l_0$ from the left support. The string tension is $Q_{10}$ and the mass of a unit length is $m_0$.

Derive the expression for a transverse displacement of the string at the point of the force $P_0$ application as a function of time.

**23** The point-like load $P_0$ moves with the constant velocity $v$ along a string lying on a linear inertialess flexible base (Fig. 1.19). The stiffness of the base

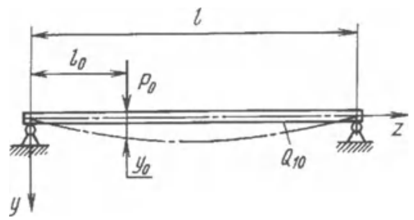


Fig. 1.18.

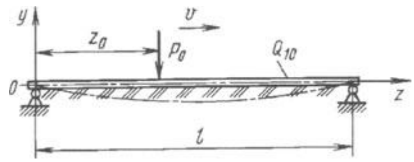


Fig. 1.19.

is $k$, the string tension is $Q_{10}$, and the mass of a unit length is $m_0$. At the initial instant, the load is above the left support.

Determine the string bending deflection as a function of the velocity of motion of the load.

**24** Figure 1.20 presents schematically a moving electric locomotive whose current collector is pressed with a constant force to a tightened wire and slides along the wire with the constant velocity $v$ when the electric locomotive moves. At the initial instant the current collector is at the point $O$ where the wire is fastened.

Investigate the vibrations of the wire (string), assuming that under vibrations the force pressing the current collector to the wire remains practically constant and is equal to $P_0$. The force of the wire tension is $Q_{10}$ and the mass of a unit length is $m_0$.

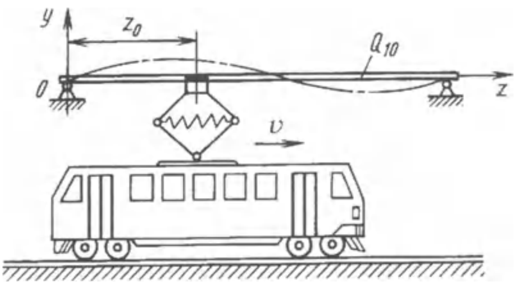


Fig. 1.20.

**25** Figure 1.21 shows schematically a segment of an overhead ropeway with a load of mass $M$ that moves with the constant velocity $v$. The cable tension is $Q_{10}$ and the mass of its unit length is $m_0$. At the initial moment, the load is above the left support.

Derive the differential equation of small transverse vibrations of the cable (string).

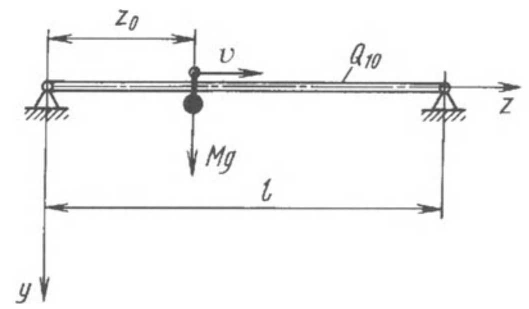


Fig. 1.21.

**26** Determine the vertical displacement of the load with mass $M$ (see Problem **25**) in a particular case when one can neglect the force of inertia $M\ddot{y}_0$ in comparison to the gravity force $Mg$.

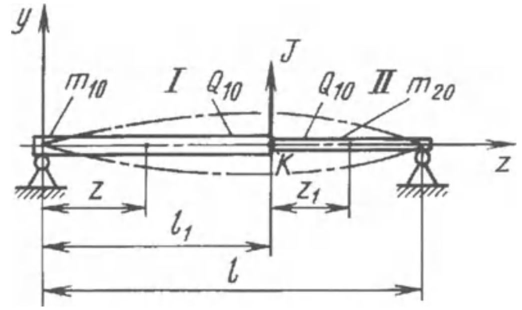


Fig. 1.22.

**27** The string has two segments. The masses of unit lengths of the first and the second segments are $m_{10}$ and $m_{20}$, respectively. The force of the string tension is $Q_{10}$. Set up the equation for determining the frequencies of small vibrations of the string using the method of initial parameters and determine the forms of string vibrations. Solve the problem for the case, when at the instant $t = 0$ a momentum $J(z_K = l_1)$ acts upon the string at the point $K$ (Fig. 1.22).

**28** Derive the equation for determining the frequencies of string vibrations taking the point-like mass $M$ (Fig. 1.23) into account. To this end, take advantage of (1) the method of initial parameters and (2) approximate method with only one term retained (determine the first frequency).

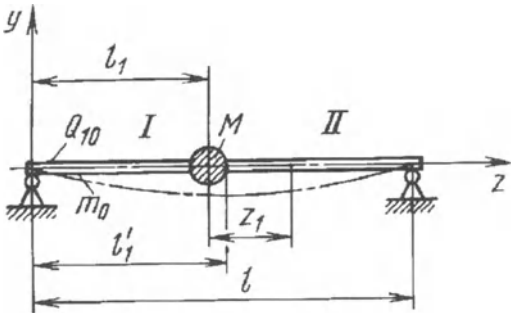


**Fig. 1.23.**

**29** Derive the equation for determining the frequencies of string vibrations taking the concentrated (not point-like) mass (Fig. 1.24) into account. The moment of inertia of the mass $M$ relative to the axis perpendicular to the plane of drawing and passing through the point $K$ (center of mass) is equal to $J_K (J_K = \frac{2}{5} M r^2)$. Neglect the gravity force of mass $M$.

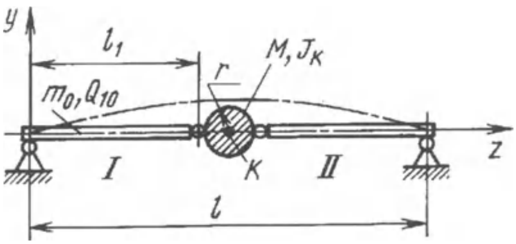


**Fig. 1.24.**

**30** Before applying the concentrated axial 'dead' force $P_0$ (Fig. 1.25) the force of string tension was $Q_{10}$. Determine the frequencies of string vibrations as a function of $P_0$. The area $F$ of string cross-section and the Young modulus $E$ are specified.

**31** The point of the axial line with the coordinate $z_K$ has a forced vertical displacement $y_K = y_{K0} \cos \omega t$ (Fig. 1.26). Determine the amplitudes of steady-state vibrations of the string. The force of string tension is $Q_{10}$.

**32** Figure 1.27 demonstrates a belt moving with the velocity $w$ and a concentrated mass $M$ placed upon it at a distance of $l_1$. The mass is free to move in the vertical direction when the belt vibrates. Determine the frequencies of belt vibrations, restricting to a two-term approximation. The force of belt tension is $Q_{10}$.

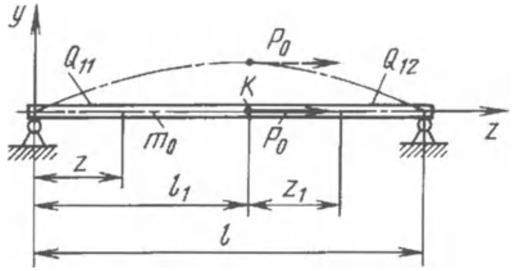


**Fig. 1.25.**

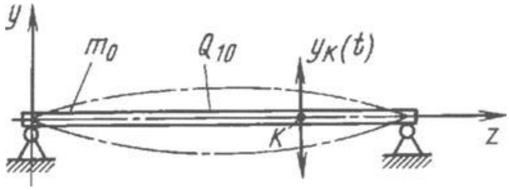


**Fig. 1.26.**

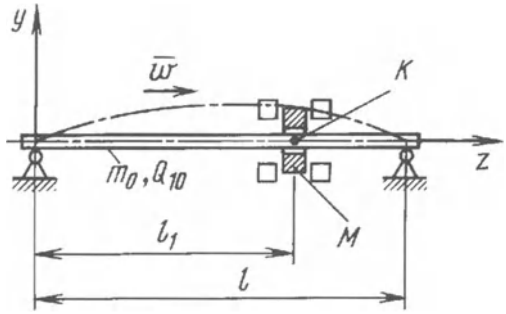


**Fig. 1.27.**

**33** Set up an algorithm for determination of exact values of frequencies for the conditions of Problem **32**.

**34** A string has two segments (Fig. 1.28). The masses of unit lengths are equal to $m_{01}$ and $m_{02}$ for the first and second segments, respectively. The force of string tension is $Q_{10}$. Determine the amplitude of steady-state vibrations of the string under the action of the force $P(t) = P_0 \cos \omega t$ at the point $K$. When solving approximately, take advantage of the Galerkin method, restricting to a single-term approximation.

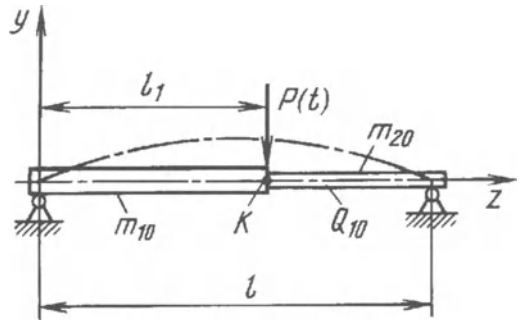


**Fig. 1.28.**

**35** Determine by an approximate method (with restriction to a two-term approximation) the amplitudes of steady-state vibrations of the string under the action of the force $P(t) = P_0 \cos \omega t$ at the points $K_1$ and $K_2$ (Fig. 1.29). The unit length of the string is $m_0$ and the tension force is $Q_{10}$. A spring with force $c$ is installed at the point $K_1$.

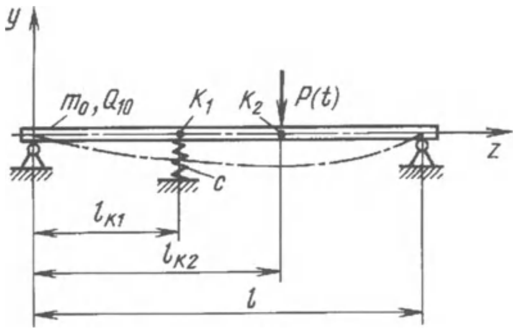


**Fig. 1.29.**

**36** Find the exact solution to the equation of small vibrations of the string and determine the amplitudes of string vibrations at the points $K_1$ and $K_2$ (see Problem **35**) for steady-state vibrations.

**37** Give an algorithm for exact solution of the equation of small steady-state vibrations of the string and determine the amplitudes of string vibrations at the points $K_1$ and $K_2$ (see Problem **35**) taking into account the distributed forces of viscous drag $\alpha \dfrac{\partial y}{\partial t}$, where $\alpha$ is a constant coefficient.

**38** A point-like mass $M$ moves with the constant velocity $v$ along a string lying on a linear flexible base. The stiffness coefficient of the base (the bed coefficient) is equal to $k$ (Fig. 1.30). At the instants $t = 0$ the mass $M$ was at the origin of coordinates. Restrict to the case when the velocity $v$ is small so that (see solution to Problem **25**) the terms $2v\dfrac{\partial^2 y}{\partial z \partial t}$ and $v^2\dfrac{\partial^2 y}{\partial z^2}$ can be neglected.

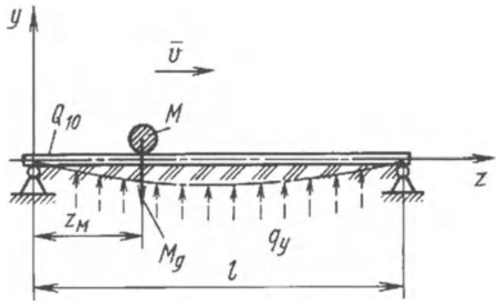


**Fig. 1.30.**

Determine, with restriction to a two-term approximation, the bending deflections at non-steady-state vibrations for the following cases: (1) the mass $M$ is on the string (the deflection under the moving mass is included) and (2) the mass $M$ has left the string. Assume that the relation between the string and the base is bilateral.

**39** Figure 1.31 shows the moving tape of a tape-drive mechanism. The tension of the tape is $Q_{10}$. Solve the equation of small free vibrations of the tape under the steady-state regime (at $w = \text{const}$) assuming that the vibrations are caused by the momentum $J$ applied at the instant $t = 0$ to the tape element located at the distance $z_K$ from the origin of coordinates.

## 1.2 Torsional vibrations of rods

**40** Derive the differential equation of free torsional vibrations of a solid shaft with a round section (Fig. 1.32) and determine the frequencies of free vibrations of the shaft for the cases when it is fastened as shown in Fig. 1.32. The modulus of elasticity in shear of the shaft material is $G$ and its density is $\varrho$.

**41** Determine the velocity of propagation of a torsional wave (shear wave) over the solid shaft if $G = 80$ GPa and $\varrho = 7800$ kg/m$^3$.

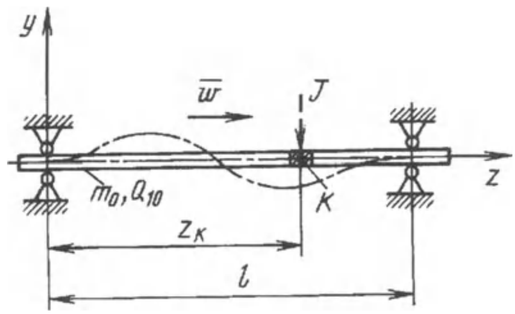


Fig. 1.31.

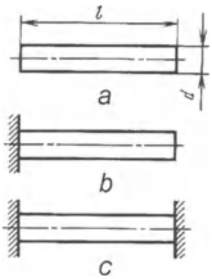


Fig. 1.32.

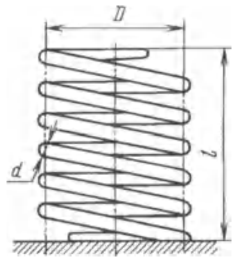


Fig. 1.33.

**42** Determine the velocity of propagation of a torsional wave along a spring (Fig. 1.33) and the frequencies of free torsional vibrations of the spring if $l = 0.2$ m, $D = 0.1$ m; $d = 5$ mm; the number of coils $i = 20$; the first-kind modulus of the material of the wire from which the spring is coiled is $E = 200$ GPa, and its density is $\varrho = 7800$ kg/m$^3$ (the spring has a small angle of lead).

Hint: change the spring for an equivalent rod of round section [5].

**43** Derive the differential equation for determining the frequencies of free vibrations of a shaft with disks at its ends (Fig. 1.34). The moments of inertia of the disks are $J_1$ and $J_2$. The density of the rod material is $\varrho$ and the shear modulus is $G$. Demonstrate that at $\varrho = 0$ (inertialess shaft) the frequency of vibrations for the disks is equal to that obtained in Problem **233** of Part I.

## 1.3 Extensional vibrations of rods

**44** Derive the differential equation of extensional vibrations of a rectilinear rod and determine the frequencies of vibrations for the cases when the rod is fixed as in Fig. 1.32. The Young's modulus of the first kind is $E$ for the rod material, its density is $\varrho$, and the cross section area is $F$; the rod elements

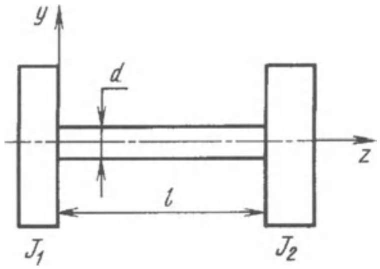


**Fig. 1.34.**

do not execute transverse motions and are displaced only in the longitudinal direction.

**45** Determine the velocity of propagation of compression waves along the rod if $E = 200$ GPa and $\varrho = 7800$ kg/m$^3$.

**46** Derive the differential equation of free extensional vibrations of the rod with a distributed longitudinal load $q(z, t)$ (Fig. 1.35).

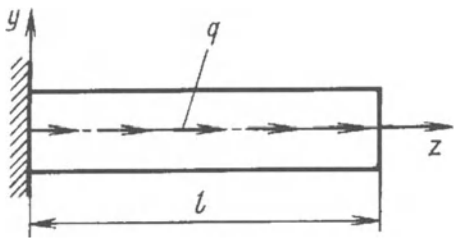


**Fig. 1.35.**

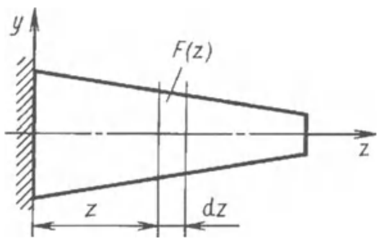


**Fig. 1.36.**

**47** Derive the differential equation of free vibrations of a rod in the case of a variable cross section area (Fig. 1.36).

**48** The left end face of a rod (Fig. 1.37) is linked with a spring of force $c = EF/l$. Derive the differential equation for determination of the frequencies of free vibrations and determine by the graphical method the first three frequencies of small vibrations of the rod. The mass of a unit length of the rod is $m_0$.

**49** For the case or a rod fixed as is shown in Fig. 1.38 derive the equation of frequencies and determine the first two frequencies of free vibrations if $c = EF/l$.

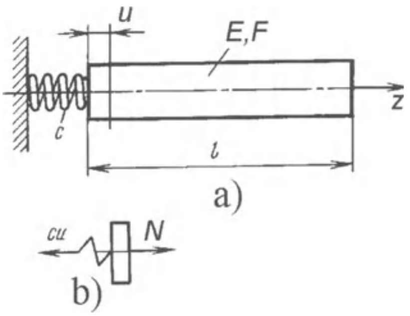


**Fig. 1.37.**

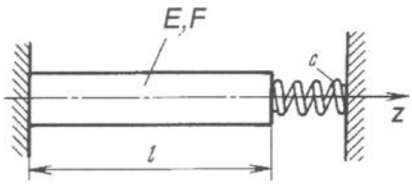


**Fig. 1.38.**

**50** Both end faces of the rod are connected with springs (Fig. 1.39). Derive the equation for determination of the frequencies of rod vibrations and calculate the first two frequencies of free vibrations if $c_1 = 2c_2 = EF/l$.

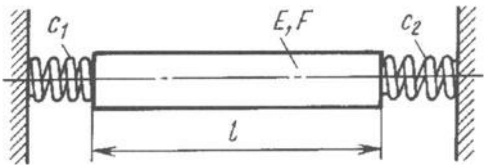


**Fig. 1.39.**

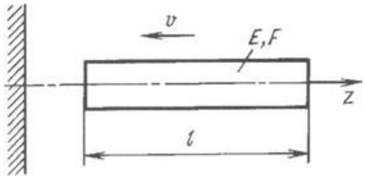


**Fig. 1.40.**

**51** A rod moving with a constant velocity $v$ along the $z$ axis (Fig. 1.40) bumps against a perfectly rigid wall so that the left butt of the rod remains further fixed with the barrier rigidly. Determine the maximum displacement of the right end face and the maximum value of the axial thrust at the left end of the rod.

**52** The rod $II$ flying with a constant velocity $v$ along the $z$ axis hits the rod $I$ at the instant $t = 0$ and, henceforward, they vibrate together (Fig. 1.41). Determine the time variation of the axial force at the place of conjunction of two rods.

**53** Derive the differential equation for determination of the frequencies of free extensional vibrations of a step-shaped rod (Fig. 1.42) of a homogeneous material (with density $\varrho$) for the case when $l_1 = 2l/5$ and $l_2 = 3l/5$. Find the first four frequencies of free rod vibrations.

**54** Derive the equation for determination of the frequencies of free extensional vibrations of the step-shaped rod (Fig. 1.43).

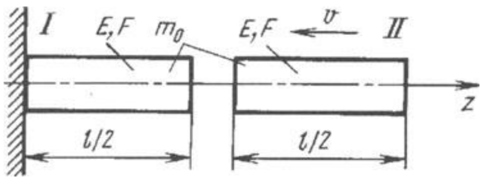


Fig. 1.41.

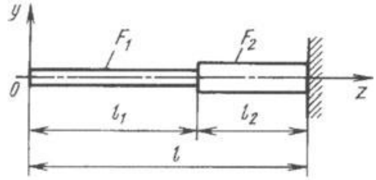


Fig. 1.42.

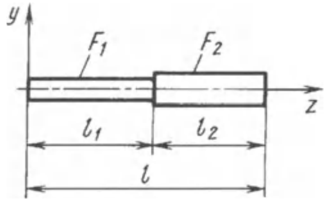


Fig. 1.43.

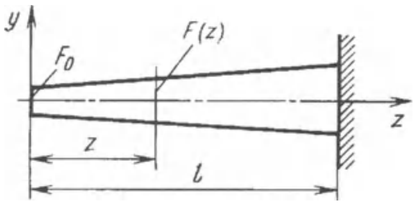


Fig. 1.44.

**55** Determine the first frequency of free extensional vibrations of the rod (Fig. 1.44) whose area of cross section and the mass per unit length vary according to the following formulas

$$F = F_0(1 + z/l); \quad m = m_0(1 + z/l).$$

**56** Find the first two frequencies of free longitudinal vibrations of the rod (see Problem **55**).

**57** A rod pressed by the forces $N$ is suddenly disengaged at the instant $t = 0$ from the action of the forces (unloaded). Establish the law of motion for the rod sections. Figure 1.45 gives a diagram of displacements of the rod sections at the initial moment.

**58** A rod is stretched by the force $P$ (Fig. 1.46) that is suddenly discharged. Establish the law of time variation of the displacement of the right end face of the rod.

**59** Determine approximate values of the amplitudes of forced extensional vibrations of the rod (Fig. 1.47) under the action of harmonic longitudinal force $P = P_0 \sin \omega t$ applied to a free end of the rod. The section area of the rod and its mass per unit length vary as

$$F = F_0(1 + z/l); \quad m = m_0(1 + z/l).$$

**60** A rod begins to move under the action of suddenly applied (at the instant $t = 0$) force $P_0$ the keeps henceforth a constant value (Fig. 1.48). Determine the axial force in the section $z = l/2$ at the moment $t_1 = l/a$ that

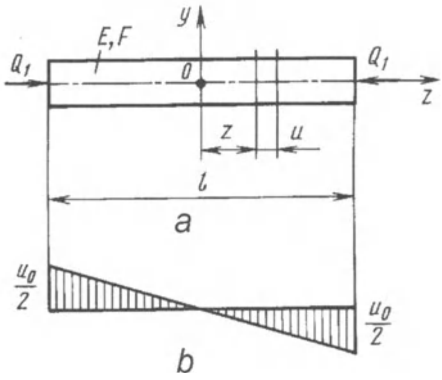


Fig. 1.45.

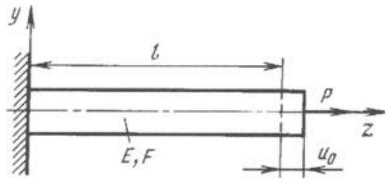


Fig. 1.46.

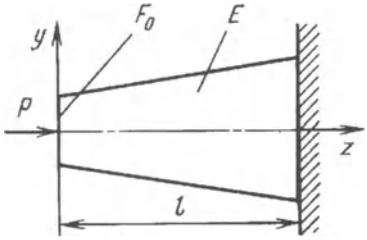


Fig. 1.47.

arises at extensional vibrations of the rod. The mass of the rod unit length is $m_0$ $\left(a = \sqrt{EF/m_0}\right)$.

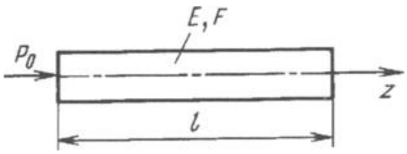


Fig. 1.48.

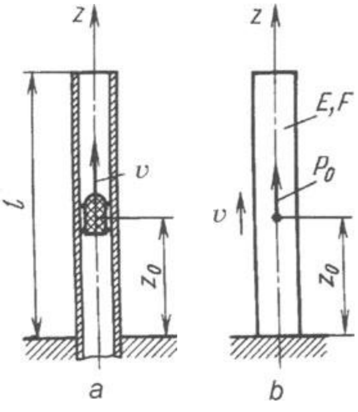


Fig. 1.49.

**61** At the moment $t = 0$ the constant force $P$ is suddenly applied to the right end face of the rod (Fig. 1.46). Determine the maximum value of the displacement of the point of force application and establish the difference with the case when the force $P$ is gradually increases (the rod is loaded with the force $P$ statically).

**62** A bullet moves inside a barrel (Fig. 1.49*a*) with the constant velocity $v$. The force of friction between the barrel and the bullet is constant and equal to $P_0$. The mass of the barrel unit length is $m_0$. Write the expression for axial displacements of the barrel sections as a function of velocity $v$. At $t = 0$ the bullet is at the origin of coordinates. Schematically, the barrel with the bullet can be represented as a rod loaded by the constant stress $P_0$ moving along the axis (Fig. 1.49*b*).

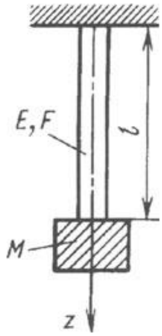


**Fig. 1.50.**

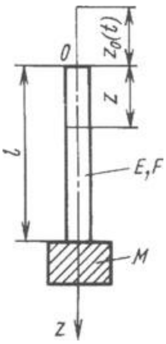


**Fig. 1.51.**

**63** Determine the first three frequencies of free longitudinal vibrations of the rod with mass $M$ at the end (Fig. 1.50) if $M = m_0 l$, where $m_0$ is the mass of the rod unit length.

**64** Determine the first three frequencies of free vibrations of a rod with a concentrated mass at the end (Fig. 1.50) for the case when the upper section is free ($M = m_0 l$, where $m_0$ is the mass of the rod unit length).

**65** The upper section of the rod (Fig. 1.51) is forcibly displaced in the vertical direction according to the law $z = A \sin \omega t$. There is a concentrated mass $M$ on the lower end of the rod. Determine the displacement $u$ of an arbitrary section of the rod under a steady-state regime of vibrations and the amplitude of longitudinal vibrations of mass $M$.

**66** Figure 1.52*a* presents schematically an explosive cartridge *I* of a solid-propellant jet engine. The cartridge is placed inside the motor body. Since in most cases it is required that a jet engine should provide for a constant thrust during the charge burning, the fuel-propellant cartridge is usually manufactured in such a form that keeps its surface constant during combustion. The simplest form of a cartridge providing for a constant combustion surface is a cylindrical tube (Fig. 1.52*b*). In this case, the reduction of the external surface in the process of burning is compensated by equally increasing surface of

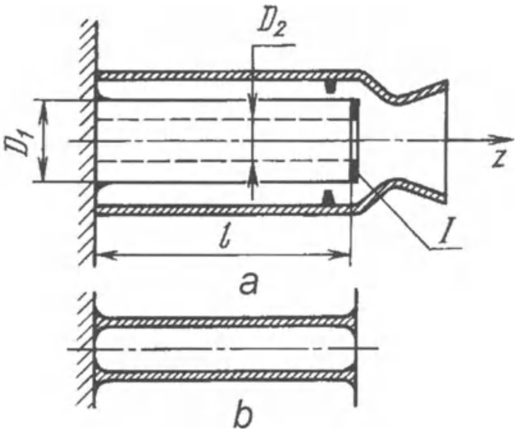


**Fig. 1.52.**

the inner hole. As a rule, the butt section of the cartridge has an inhibitory coating (to avoid the end face combustion).

Derive the differential equation of small free longitudinal vibrations of a burning cartridge, assuming that the pressure in the engine chamber remains constant when the cartridge burns. The full time of combustion is $t_1$, the Young's modulus of the first kind is $E$ for the material of charge, its density is $\varrho$, and the combustion rate (burnt out propellant mass per unit time) is constant. When solving the problem, assume that the modulus $E$ remains constant and does not depend on the propellant temperature.

**67** Determine the velocity of propagation for a compression wave along a cylindrical spring whose lower butt is rigidly fixed to a base (see Fig. 1.33) and find the frequencies of its free vibrations. Take the following numerical data: $l = 0.2$ m, the mean diameter of spring coils is $D = 0.1$ m, the diameter of the spring wire is $d = 5$ mm, the number of sring coils $i = 20$, the elasticity modulus of the second kind is $G = 80$ GPa for the wire material, and the density of this wire material is $\varrho = 7800$ kg/m$^3$. The spring has a small angle of lead.

Hint: change the spring for an equivalent rod [5].

**68** A spring with a small angle of lead of coils is placed into a groove of a disk (Fig. 1.53) and rotates together with the disk with the angular velocity $\Omega$. Derive the differential equation of small extensional vibrations of the spring and determine the lower frequency of its vibrations as a function of the disk angular velocity (neglect the friction between the spring and the disk).

The spring force for extension is $c$, the mean diameter is $D$, the wire diameter is $d$, the number of spring coils is $i$, and the elasticity modulus of the second kind is $G$ for the wire material.

Hint: when deriving the equation of extensional vibrations of the spring, change it for an equivalent rod [5].

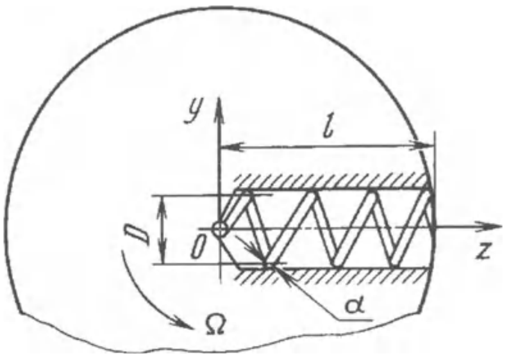


**Fig. 1.53.**

**69** Determine the critical velocity $\Omega_*$ of disk rotation, at which the lower frequency of spring vibrations (see Fig. 1.53) becomes equal to zero.

**70** A spring is on the disk that rotates with a constant angular velocity $\Omega$ (Fig. 1.54). Before disk starts rotating, the spring is stretched with the force $N_0$ and fixed at the points $A$ and $B$. The spring force for stretching is $c$, the mean diameter of spring coils is $D$, the wire diameter is $d$, the elasticity modulus of the second kind for the wire material is $G$, and the number of coils is $i$. The spring has a small angle of lead.

Derive the differential equation of small extensional vibrations of the spring taking the disk angular velocity $\Omega$ into account.

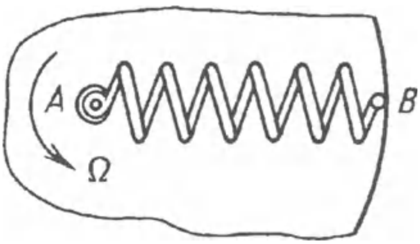


**Fig. 1.54.**

**71** A stretchable filament is fixed on a rotating disk (Fig. 1.55). The mass of the filament unit length is $m_0$, the elasticity modulus of the first kind is $E$

for the filament material, the sectional area is $F$, and the filament tension is $Q_{10}$ at $\Omega = 0$. Derive the differential equations of small free vibrations of the filament (neglecting the gravity force) taking the disk angular velocity into account and determine approximately the lower frequencies of vibrations.

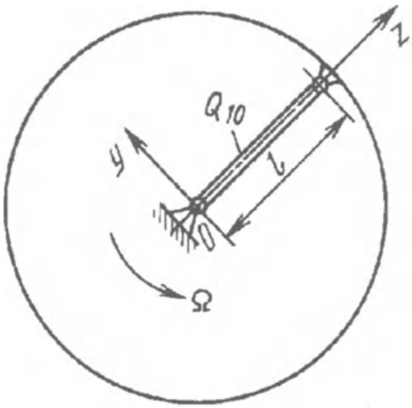


**Fig. 1.55.**

## 1.4 Bending vibrations of rectilinear rods

**72** For the cases of fixing shown in Fig. 1.56 derive the differential equation of small transverse vibrations of the rod and determine the frequencies of free vibrations. The mass of the rod unit length is $m_0$ and its bending stiffness is $EJ_x$.

**73** Determine the frequencies of the rod free vibrations for the cases shown in Fig. 1.57.

**74** Demonstrate that in the case of a variable (over the length) moment of inertia $J_x(z)$ the differential equation of small vibrations of the rod has the following form

$$\frac{\partial^2}{\partial z^2}\left(EJ_x(z)\frac{\partial^2 y}{\partial z^2}\right) = q = -m_0\frac{\partial^2 y}{\partial t^2}$$

**75** Derive the differential equation of small free vibrations of a rod placed into the permanent magnetic field (Fig. 1.58) and determine the frequencies of the free vibrations if, when the rod is deflectefrom its equilibrium position, the force

$$q = F_1 - F_2 = \frac{k_1\Phi_0^2}{(a-y)^2} - \frac{k_1\Phi_0^2}{(a+y)^2}$$

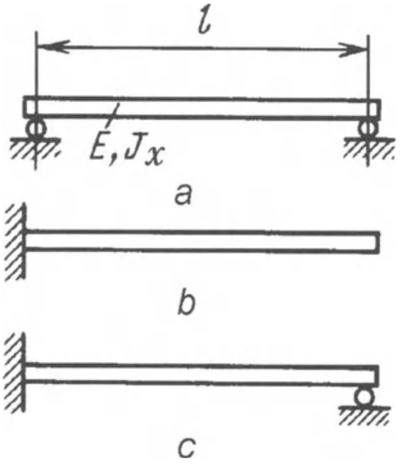


Fig. 1.56.

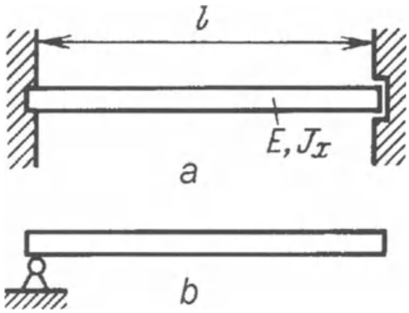


Fig. 1.57.

acts upon its unit length.

Find the critical value of $\Phi_0$.

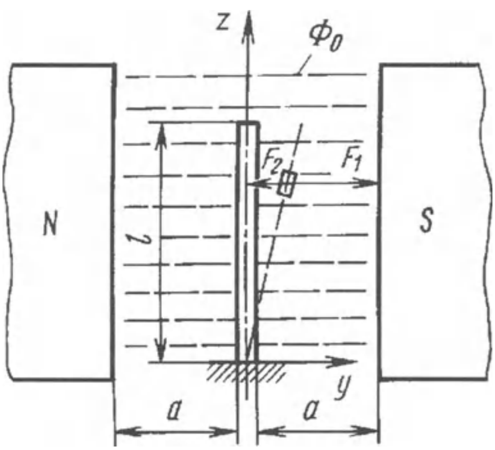


Fig. 1.58.

**76** A hinge-supported rod (Fig. 1.59) has the following parameters variable over length: the bending stiffness

$$EJ_x = EJ_0 \left(1 + \sin \frac{\pi z}{l}\right)^3$$

and the mass per unit length

$$m = m_0\left(1 + \sin\frac{\pi z}{l}\right).$$

Determine using the Galerkin method the fundamental frequency of free vibrations under a single-term approximation.

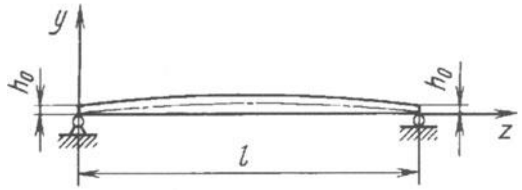


**Fig. 1.59.**

**77** Refine the fundamental frequency obtained under the first approximation (see Problem **76**) by considering the second approximation.

**78** Determine the first frequency of free vibrations of the rod (Fig. 1.60) whose bending stiffness is $EJ_x$, the mass per unit length is $m_0$, the length is $l$, and the distance between supports is $b = l/2$.

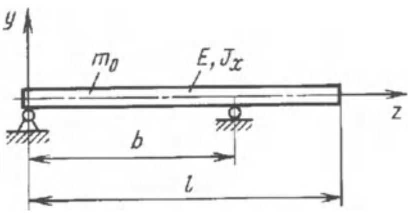


**Fig. 1.60.**

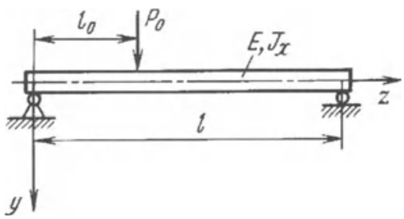


**Fig. 1.61.**

**79** At the moment $t = 0$ the force $P_0$ is suddenly applied to the rod of constant stiffness (Fig. 1.61). The mass of the rod unit length is $m_0$. Investigate the rod vibrations caused by the force $P_0$ applied and determine the time variations of the maximum normal strain in the section $z = l_0$.

**80** Determine the bending deflections of the rod depending on the velocity $v$ of displacement of the force $P_0$ over the rod (Fig. 1.62). The bending stiffness of the rod is $EJ_x$ and the mass of its unit length is $m_0$. At the initial moment the point of application of the force $P_0$ is above the left support. The numerical data are the following: $J_0 = 0.1$ cm$^4$, $E = 200$ GPa, $m_0 = 8$ kg/m, and $l = 15$ m.

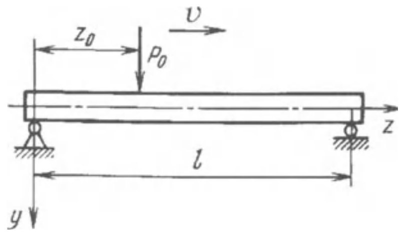


Fig. 1.62.

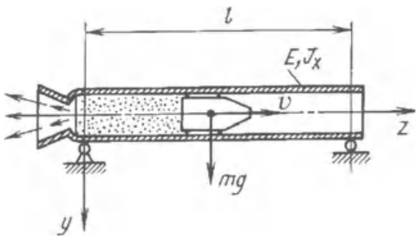


Fig. 1.63.

**81** The dynamical jet gun is designed using the principle of dynamic balancing of the acting force of a shot by the reaction of powder gases effluent from the behind-the-bullet space (Fig. 1.63). The mass of the barrel unit length is $m_0$.

Determine the angular velocity of the bullet attained by it upon flying out of the barrel (assume that the bullet moves with a constant velocity $v$). When solving the problem, assume that the bullet acts upon the barrel with a constant force that is equal to the gravity force of the bullet. Neglect the force of inertia $J_y$.

**82** Derive the differential equation of small free vibrations of a rod lying on an elastic base (Fig. 1.64$a$), if the reactive force acting upon the rod unit length from the side of the elastic base is proportional to its bending deflection $ky$, where $k$ is the stiffness coefficient for the base (the bed coefficient).

The mass of the rod unit length is $m_0$ and the bending stiffness is $EJ_x$. Assume that at small vibrations the rod does not break off the base. Neglect the mass of the base involved into vibrations (i.e., assume that the elastic base is equivalent to a set of uniformly distributed inertialess springs, Fig. 1.64$b$).

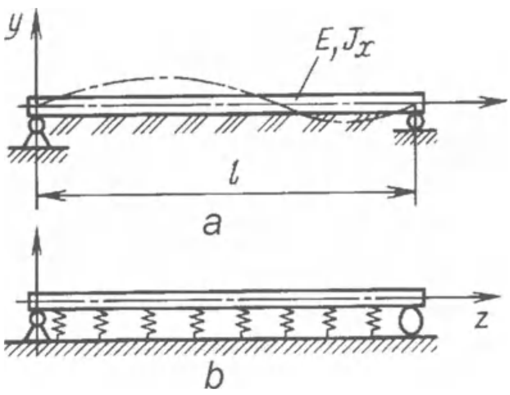


Fig. 1.64.

**83** Using the Rayleigh method, determine the first frequency of free vibrations for the rod lying on an elastic intertialess base (Fig. 1.65), if the stiffness coefficient of the base is $k$, the mass of the rod unit length is $m_0$, and the bending stiffness is $EJ_x$.

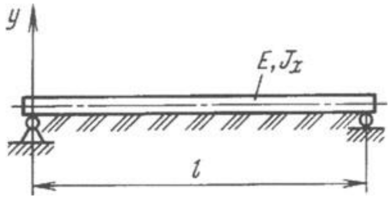


**Fig. 1.65.**

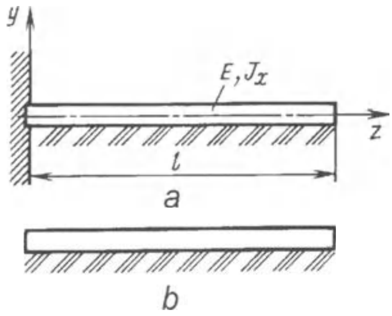


**Fig. 1.66.**

**84** A rod lies on an elastic inertialess base (Fig. 1.66). The elastic restoring force acting upon the rod unit length from the base side is proportional to the bending deflection and is equal to $ky$. Determine the frequencies of free vibrations of the rod for two variants of its fixing.

**85** The force $P_0$ moves with the constant velocity $v$ along the rod lying on an elastic inertialess base (Fig. 1.67). The stiffness coefficient of the elastic base is $k$. Determine the bending deflections of the rod as a function of the velocity $v$ of the force displacement. At $t = 0$ the point of force application is above the left support.

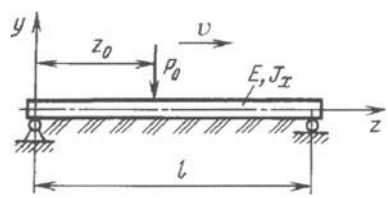


**Fig. 1.67.**

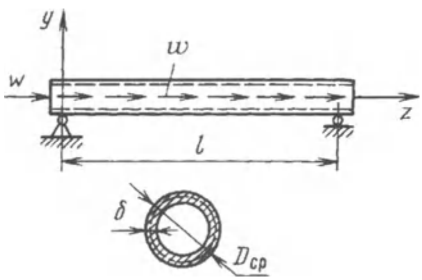


**Fig. 1.68.**

**86** A perfectly incompressible fluid flows with the constant velocity $w$ inside the hinge-supported pipeline (Fig. 1.68). Derive the differential equation of small transverse vibrations of the pipeline taking the moving fluid into account. Determine (by the approximate method) the first two frequencies of vibrations. The mean diameter of the pipeline is $D_m = 0.1$ m, its thickness is $\delta = 20$ mm, the length is $l = 1$ m, the density of the pipeline material is $\varrho_{\mathrm{P}} = 2700$ kg/m$^3$ (duralumin), and the elasticity modulus of the first kind is $E = 70$ GPa. The mass of fluid per unit length of the pipeline is $m_{\mathrm{F}} = 7.68$ kg/m.

Determine the frequencies of vibrations for three values of the velocity of fluid motion: $w_1 = 0$ , $w_2 = 10$, and $w_3 = 20$ m/s. Neglect the gravity force acting upon the pipeline and fluid.

**87** Determine the critical velocity for the fluid flowing inside the hinge-supported pipeline (see Problem **86**).

**88** Derive the differential equation of small transverse vibrations of the hinge-supported pipeline if the fluid jet outflows at an angle $\alpha$ to the pipeline axis (Fig. 1.69). Take the numerical values from Problem **86**.

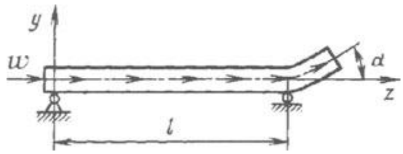


**Fig. 1.69.**

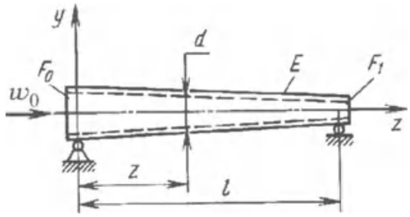


**Fig. 1.70.**

Determine the first two frequencies of pipeline vibrations at $\alpha = 90°$ and for the fluid velocities $w$ equal to 0, 10, and 20 m/s. Determine the critical velocity of the fluid stream.

Hint: When the jet leaves the pipeline at the angle $\alpha$ to the axial line, the force of jet reaction $N = mw^2(1 - \cos\alpha)$ acts upon it in the direction of the $z$ axis.

**89** A perfectly incompressible fluid flows inside a pipeline with variable profile (Fig. 1.70). The inner diameter of the pipe changes according to the formula $d = d_0 - z\,(d_0 - d_1)\,/l$ where $d_0$ and $d_1$ are the diameters of the input and output sections of the pipe. The pipe has the walls of a constant thickness $\delta$ ($\delta \ll d$), the density of the pipe material is $\varrho_{\mathrm{P}}$ , and the fluid density is $\varrho_{\mathrm{F}}$.

Derive the equation of small vibrations of the pipe assuming the constant flow rate.

**90** The ideal incompressible fluid flows with the constant velocity $w$ inside a hinge-supported pipeline having the bending stiffness $EJ_x$ and lying on an elastic base (Fig. 1.71). The mass of the pipeline unit length is $m_1$, the fluid mass per the pipeline unit length is $m_2$, and the stiffness coefficient of the elastic base is $k$ (assume that the elastic base is inertialess).

Determine approximately the first two frequencies of small transverse vibrations of the pipeline.

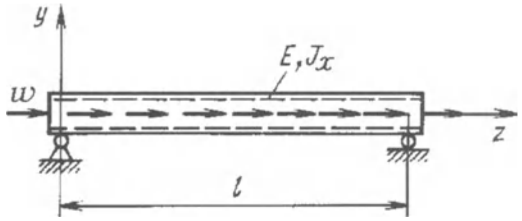


**Fig. 1.71.**

**91** Derive the differential equation of small transverse vibrations of a beam (Fig. 1.72) taking into account a constant compressing force $N$ acting upon it. Determine the frequencies of beam vibrations.

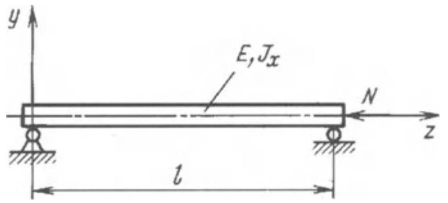


**Fig. 1.72.**

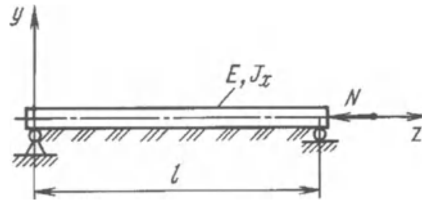


**Fig. 1.73.**

**92** Investigate the stability of the first four modes of the transverse vibrations of the beam (Fig. 1.72) loaded with the applied compressing force $N$ variable in time ($N = N_0 + N_1 \sin \omega t$). Use in the problem the following numerical data: $J_x = 0.1$ cm$^4$, $N_0 = 1000$ N, $N_1 = 200$ N, $m_0 = 0.8$ kg/m, $\omega = 300$ s$^{-1}$, and $l = 1$ m.

**93** How will the solution to Problem **92** be changed, if the constant component $N_0$ of the force $N$ changes its direction into the opposite one? Take the numerical values from Problem **92**.

**94** When deriving the differential equation of small transverse vibrations of a string, its bending stiffness is assumed to vanish. Find the error in determining the frequencies of vibrations of the string whose length is $l = 1$ m, if $J_x = \pi d^4/64 = 5 \cdot 10^{-6}$ cm$^4$, the mass of the string unit length is $m_0 = 6 \cdot 10^{-3}$

kg/m, the tension is $Q_{10} = 100$ N, and the elasticity modulus of the first kind is $E = 200$ GPa for the string material (see Fig. 1.2). The string can be considered as a hinged rod.

**95** A rod lying on an elastic inertialess base is compressed by a constant force $N$. The mass of the rod unit length is $m_0$, the bending stiffness is $EJ_x$, and the stiffness coefficient of the elastic base is $k$. Derive the differential equation of free transverse vibrations of the rod (Fig. 1.73) and determine the vibration frequencies.

## 1.5 Vibrations of rectilinear and curvilinear rods

More complicated problems are formulated in this paragraph. The methods of solving them, as well as the necessary equations, are described in the Appendices A-F. When solving particular problems, one should obtain partial equations from the general equations presented in these Appendices. The equations are given in the dimensionless form, which simplifies their numerical integration. The main emphasis is focused on the development of algorithms for solving the problems that can be implemented in computer calculations.

**96** A rod is loaded with a periodic axial force (Fig. 1.74). Derive (approximately) the equations for the boundaries of the principal region of parametric vibrations. When solving the equations of rod vibrations, take advantage of the virtual displacement principle. One should restrict oneself to a single-term approximation.

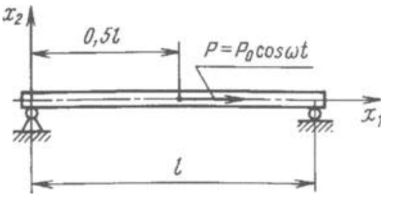


**Fig. 1.74.**

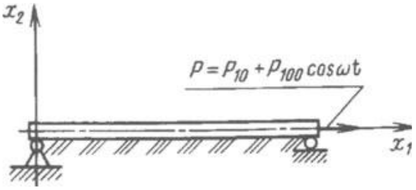


**Fig. 1.75.**

**97** Figure 1.75 demonstrates a rod lying on an elastic base with a linear characteristic. The rod is loaded with the axial periodic force $P(t)$. Using the Rayleigh method, determine the region of the main parametric resonance. Restrict consideration to a single-term approximation.

**98** The force $P$ applied to a point-like mass $m$ (Fig. 1.76) suddenly disappears at the instant $t = 0$. Determine the dynamic reaction in the hinge. The rod section is constant. Neglect the inertia of rotation and the forces of viscous drag. Consider the case when the rod has a variable section.

**99** The momentum $J$ has acted upon a concentrated mass (Fig. 1.77). Determine the angle of rotation of the mass $m$ in the plane of drawing under the free vibrations that arise after termination of the momentum action.

**100** Figure 1.78 demonstrates a rod moving with the velocity $w_0$ in a viscous medium. The follow-up uniformly distributed force $q_1 = -q_{10}\bar{e}_1 \quad (q_{10} = \beta w_0^2)$ acts upon the rod. Derive the equation of small vibrations and determine the first two eigen values, taking advantage of the approximate method of solution.

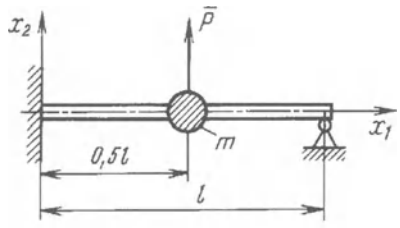


Fig. 1.76.

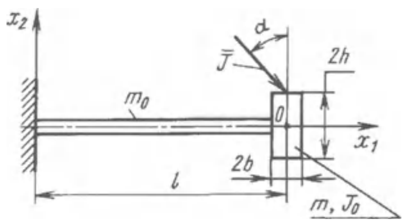


Fig. 1.77.

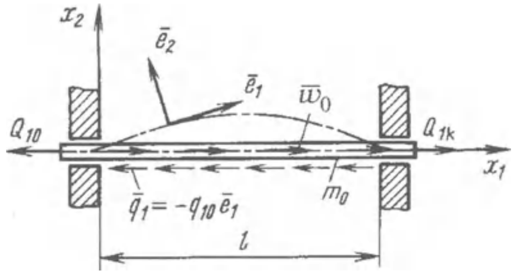


Fig. 1.78.

**101** Determine the amplitude of steady-state vibrations of the rod in section $K$ (Fig. 1.79), taking advantage of the approximate method of solution (see Appendix E).

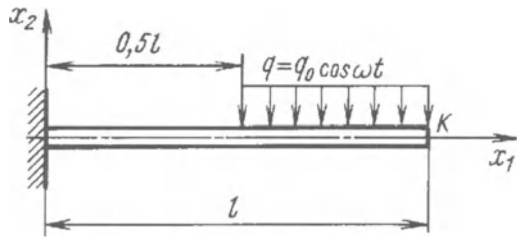


Fig. 1.79.

**102** A forced angular displacement (kinematic perturbation) is specified in the $K$ section of the rod (Fig. 1.80). Determine the amplitude of the moment at the embedment ($x_1 = 0$) under steady-state vibrations.

**103** The periodic force $P(t)$ (Fig. 1.82$b$) is applied to the rod of constant section (Fig. 1.82$a$). Using the Duffing method, obtain the approximate so-

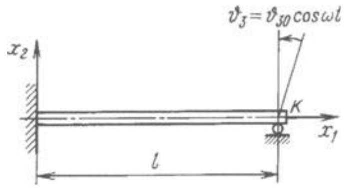


Fig. 1.80.

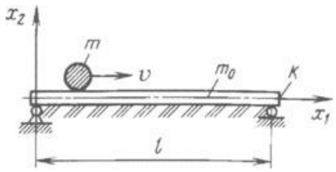


Fig. 1.81.

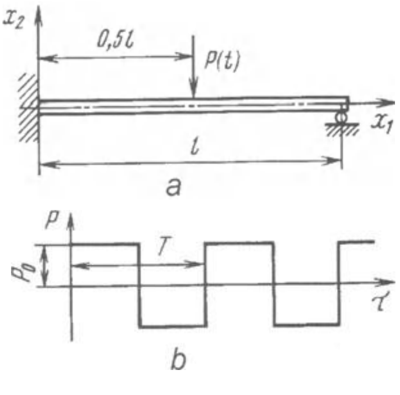


Fig. 1.82.

lution for forced steady-state vibrations. When solving the problem, one can use the virtual displacement principle taking the two-term approximation.

**104** A point-like mass $m$ moves with the velocity $v$ along a rod lying on an elastic base (Fig. 1.81). Determine approximately the angle of rotation of the rod in the $K$ section at the moment when the mass rolls off from it, restricting to a two-term approximation. Consider the specific case when $m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} \approx \frac{\partial^2 y}{\partial t^2}$ (see data of Problem **38**).

**105** Figure 1.83 depicts a segment of the railway that can be considered as a rod lying on the elastic base whose stiffness coefficient is equal to $k$. The train with a length much longer than that of the railway segment moves along the rod. The train can be considered as the one-dimensional medium (since the distance between railcar wheels $l_1$ is much less than $l$) with a zero bending stiffness.

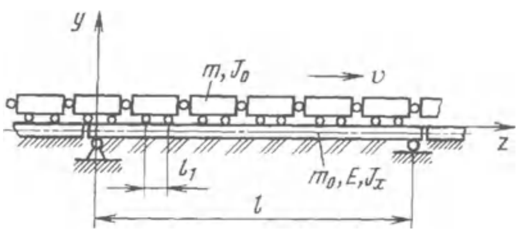


Fig. 1.83.

Derive the equation of free vibrations of the rod loaded with a distributed inertial load and determine approximately the first two frequencies, restricting to a two-term approximation.

Hint: The railcars are not point-like masses, therefore, one should take into account their moments of inertia $J_0$ relative to the car center of mass, i.e., during vibrations, the inertial force and the moment of inertia will both act upon the rails from the cars. In the limit, one can consider that the rod (rails) is loaded with moving distributed inertial load.

**106** A rod is hinged on a disk rotating with the angular velocity $\Omega$ (Fig. 1.84). The bending stiffness of the rod is $EJ_x$ and the mass of the unit length is $m_0$.

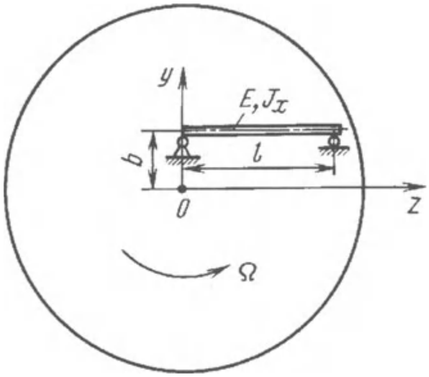


**Fig. 1.84.**

Derive the differential equation of small bending vibrations of the rod and determine approximately the first two frequencies of vibrations assuming that the rod is nonstretchable. Construct the plot of variation of the first frequency $p_1$ versus $\Omega$ at the following values of parameters: $EJ_x = 0.5$ N·m$^2$; $m_0 = 23.4 \cdot 10^{-3}$ kg/m; and $l = 0.2$ m.

**107** Determine approximately the first two frequencies of transverse vibrations of the rod fixed on a rotating disk (see Fig. 1.84), if one interchanges positions of the hinge and the roller (fixing the rod).

Plot the dependence of the first frequency of transverse vibrations on the disk angular velocity $\Omega$ and compare with the plot of the previous problem. Take the values of parameters from Problem **106**.

**108** Derive the differential equation of bending vibrations of a rod hinged on a rotating disk (the case shown in Fig. 1.85). The mass of the rod unit length is $m_0$ and the bending stiffness is $EJ_x$.

Determine, applying the Galerkin method, the first two frequencies of vibrations.

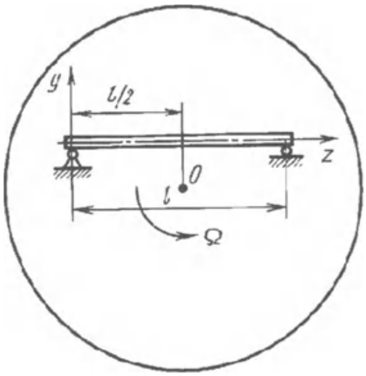


Fig. 1.85.

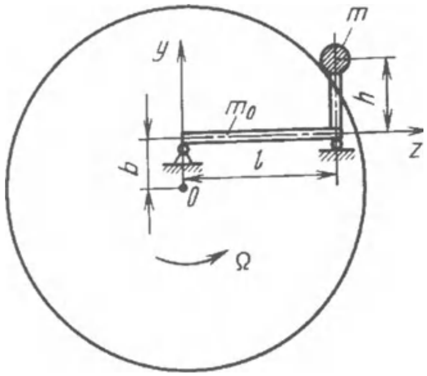


Fig. 1.86.

**109** Figure 1.86 presents a layout of a balancer in the clockwork of a remote fuse. The mass $m$ is located at the end of a perfectly rigid lever that is joined with a flat hinged rod. The balancer is on a disk rotating with the angular velocity $\Omega$. The mass of the rod unit length is $m_0$ and the bending stiffness is $EJ_x$.

Derive the differential equation of small vibrations of the balancer relative to the dynamical equilibrium position taking the rod mass into account; specify the boundary conditions necessary for solving the equation derived. Assume that the rod segment of length $h$ is perfectly rigid.

**110** The mass $m = 0.02$ kg if fixed to a hinged pinned flat rod with the help of a perfectly stiff lever of length $h = 30$ mm. Using the approximate method (Rayleigh method) derive the dependence of the vibration frequency of mass $m$ (see Fig. 86) located on the rotating disk (balancer of a remote fuse clockwork) on the angular velocity $\Omega$. When solving, restrict to the first approximation, approximating the bending deflections by the expression of the form $y = y_1(z)\sin pt$, where $y_1$ is the dynamic bending deflections of the rod with respect to the equilibrium position in the coordinate system rotating with the disk.

The mass of the rod unit length is $m_0 = 23.4 \cdot 10{-}3$ kg/m, the bending stiffness is $EJ_x = 0.5$ N m$^2$; $l = 120$ mm; $b = 30$ mm; and $\Omega = 100$ rad/s (see Fig. 1.86).

**111** Demonstrate that the first frequency of the balancer vibrations (obtained by the Rayleigh method) does not depend on the initial deformed state of the system caused by the field of centrifugal forces (see Problem **110**).

**112** What changes will occur in the exact equation of vibrations of an elastic rod, if one changes its work holding (interchanges positions of the hinge and the roller) in Problem **109**) (see Fig. 1.86).

**113** Using the Rayleigh method, determine the first frequency of vibrations of the rod shown in Fig. 1.84 (see Problem **106**). Take numerical data from Problem **110**.

**114** What condition for parameters of the balancer in Problem **113** should be met in order that the first frequency would be independent of the disk angular velocity.

**115** Determine the first frequency of vibrations of the balancer with mass $m$ placed on a rotating platform (see Fig. 1.86). For solution, use the Rayleigh method and the numerical data of Problem **110**.

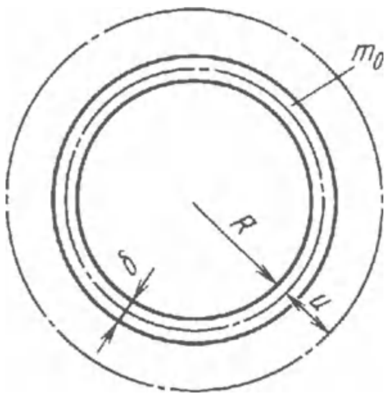


**Fig. 1.87.**

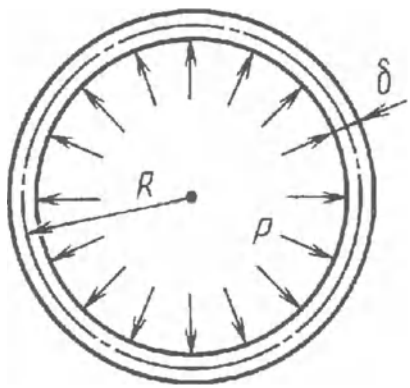


**Fig. 1.88.**

**116** Determine the frequencies of radial vibrations of a thin ring (Fig. 1.87). The mass of the ring unit length is $m_0$, the section area in $F$, the elasticity modulus of the first kind is $E$, and $\delta \ll R$.

**117** A ring (Fig. 1.88) is under the action of the internal pressure $p$, variable in time ($p = p_0 + p_1 \sin \omega t$). The mass of the ring unit length is $m_0$, the cross section area is $F$, the elasticity modulus of the first kind is $E$, the ring width is $h$, and its thickness is $\delta$ ($\delta \ll R$). Determine the amplitude of steady-state radial vibrations of the ring.

**118** A thin ring rotates about the symmetry axis with the angular velocity $\Omega$ (Fig. 1.89). Derive the differential equation of radial vibrations of the ring and determine the frequency of free radial vibrations and the critical velocity of rotation of the ring.

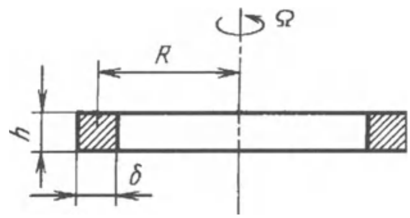


Fig. 1.89.

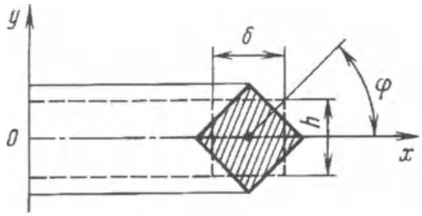


Fig. 1.90.

**119** Determine the frequency of small angular vibrations of the ring with respect to the axial line (see Fig. 1.89) assuming that the axial line of the ring remains strain-free, while its cross sections rotate during vibrations through one and the same angle $\varphi$ (Fig. 1.90). The mass of the ring unit length is $m_0$, the elasticity modulus of the first kind is $E$, and $R \gg \delta$.

**120** A ring from a rod with the constant section (Fig. 1.91) is situated on a rotating disk. The angular velocity $\omega_0$ of disk rotation is constant. Derive the equation of small vibrations of the ring in the plane of drawing taking into account the inertia of rotation of the rod elements.

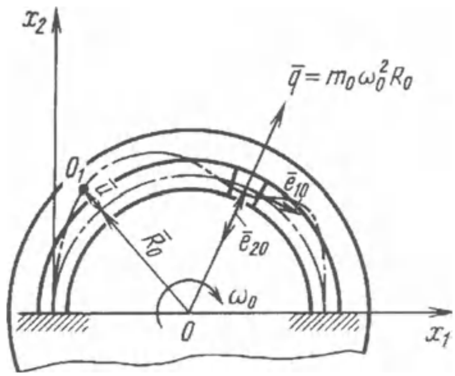


Fig. 1.91.

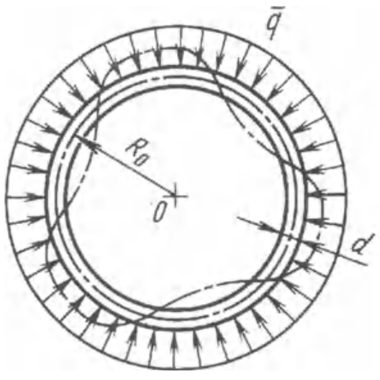


Fig. 1.92.

**121** Figure 1.92 shows a ring of length $l$ made from a rod of constant section. The ring is loaded with a tracking static load $\overline{q}$. Derive the equation of small vibrations of the ring with respect to the plane of drawing taking into account the inertia of rotation of the rod elements.

**122** Derive the equation of small vibrations of the ring (see Fig. 1.92) rotating with the constant angular velocity $\omega_0$ with respect to the plane of drawing taking into account the inertia of the rod elements.

**123** Derive the equation of small vibrations of the rod with concentrated masses $m_1$ and $m_2$ (Fig. 1.93), if the mass $m_1$ is point-like and the mass $m_2$ possesses some inertia of rotation. The tensor of inertia $J_0$ relative to the principal axes of the mass $m_2$ is known. The principal axes of the mass $m_2$ coincide with the principal axes of the rod section (at $\varepsilon = \varepsilon_2$). One can neglect the relative size $a$ in comparison with the rod length ($q \ll l$).

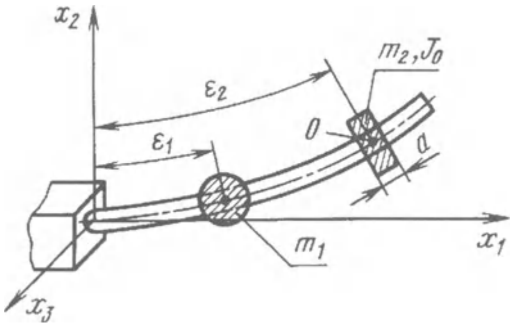


**Fig. 1.93.**

**124** Figure 1.94 shows a rod of variable section with hinge (at $\varepsilon = \varepsilon_1$) and elastic (at $\varepsilon = \varepsilon_2$) intermediate supports. A force directed along the $x_2$ axis arises in the elastic support under vibrations. Derive the equations of small vibrations of the rod in the plane of drawing taking local constraints into account.

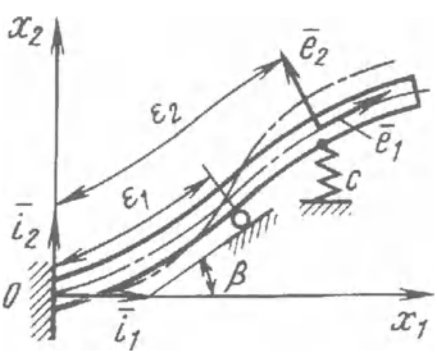


**Fig. 1.94.**

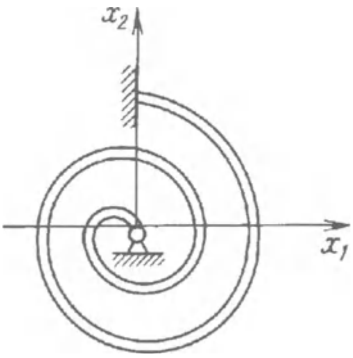


**Fig. 1.95.**

**125** Spiral springs (Fig. 1.95) are used in time-measuring instruments. Derive the equations of small vibrations of the spiral spring of constant section in the plane of drawing, if its axial line is an Archimedean spiral. Determine the spiral curvature as a function of the arc coordinate $\varepsilon$.

**126** At the instant $t = 0$ the force $\overline{P}$ applied to the end face of a constant-section round rod (Fig. 1.96) stops acting. Determine the horizontal displacement (along the $x_1$ axis) of the point $K$ under free vibrations. The vibrations of the rod proceed in the plane of drawing. Neglect the rotation inertia of the rod elements and the drag forces. (Fig. 1.96)

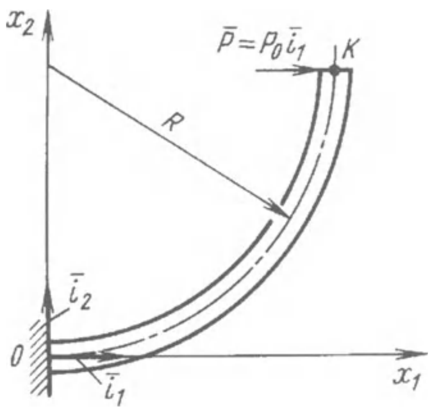


**Fig. 1.96.**

**127** At the instant $t = 0$ a concentrated moment $\overline{\mathfrak{M}}$ is applied to a round rod of constant section (Fig. 1.97*a*). The moment is constant (Fig. 1.97*б*). Determine the moment in the embedment that arises under the rod vibrations. On of principal axes of the rod section is perpendicular to the plane of drawing, therefore, the vibrations of the rod proceed in this plane. When solving, take advantage of the approximate method with restriction to a two-term approximation.

**128** A momentum $\overline{J}$ has acted upon a point-like mass $m$ placed on a rod of constant section (Fig. 1.98). As a result, the rod with mass $m$ begins to execute free vibrations in the plane of drawing. Determine the reaction in the hinge.

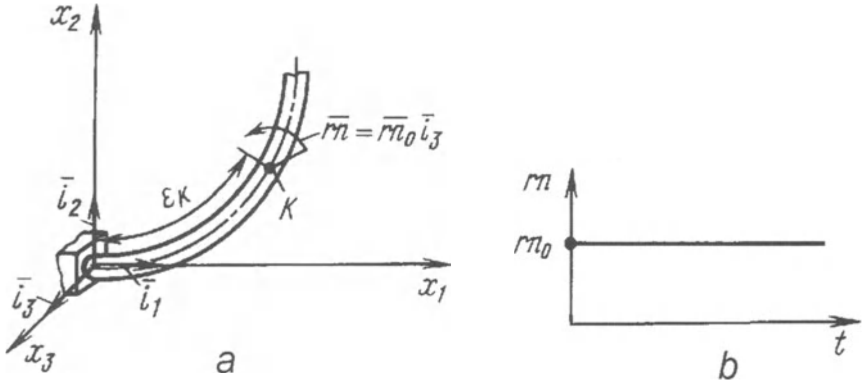


**Fig. 1.97.**

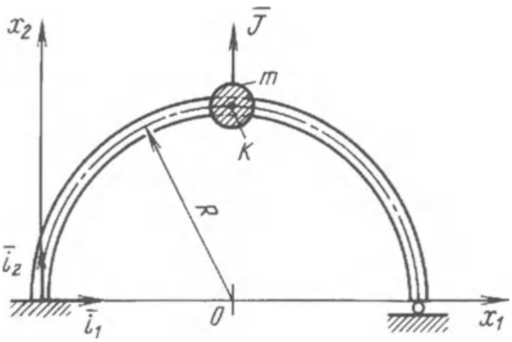


**Fig. 1.98.**

# 2

# Answers and solutions

## 2.1 Vibrations of Perfectly Flexible Rods

**1** Figure 2.1 shows a string element (at an arbitrary instant of time) with forces acting upon it.

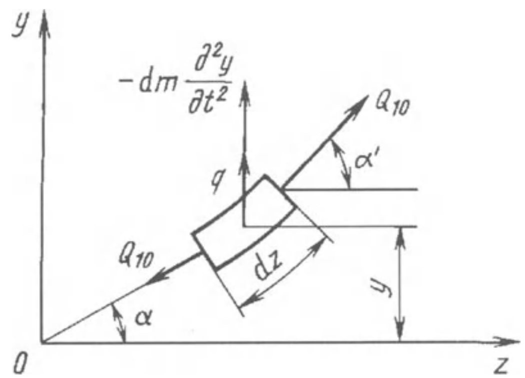


**Fig. 2.1.**

When deriving the formulas we assume the displacements of the string points to be perpendicular to the $Oz$ axis. Under small vibrations, the displacement $y$ and derivatives of $y$ with respect to $z$ are small, therefore, one can neglect the terms with their squares as quantities of the second order of smallness.

Taking advantage of the D'Alambert method, we get the following differential equation (in projections onto the $Oy$ axis)

$$-m_0\frac{\partial^2 y}{\partial t^2} + Q_{10}\frac{\mathrm{d}\alpha}{\mathrm{d}z} + q = 0. \tag{1}$$

When angles are small, to an accuracy of quantities of higher orders of smallness, $\alpha \approx \tan\alpha \approx \frac{\partial y}{\partial z}$, therefore, $\mathrm{d}\alpha/\mathrm{d}z = \frac{\partial^2 y}{\partial z^2}$, and (1) takes on the form

$$\frac{\partial^2 y}{\partial t^2} = \frac{Q_{10}}{m_0}\frac{\partial^2 y}{\partial z^2} + \frac{q}{m_0}.$$

**2** In the case under consideration, $g = 0$, and the differential equation of string vibrations (see solution to Problem **1**) assumes the following form

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial z^2} \quad (a^2 = Q_{10}/m_0) \tag{1}$$

The velocity of propagation of transverse displacements along the string is $a = \sqrt{Q_{10}/m_0}$, and $a \approx 69$ m/s.

We seek the solution to equation (1) in the form $y = y_1(z)\sin pt$.

The function $y_1(z)$ should satisfy the boundary conditions of the problem ($z = 0,\ y_1 = 0;\quad z = l,\ y_1 = 0$). From (1) we have

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}z^2} + \left(\frac{p}{a}\right)^2 y_1 = 0 \tag{2}$$

The solution to (2) has the form

$$y_1 = c_1 \cos\frac{p}{a}z + c_2 \sin\frac{p}{a}z$$

From the boundary conditions it follows that $c_1 = 0$ and $\sin pl/a = 0$, hence $pl/a = \pi n$. Then, the frequencies of vibrations are

$$p_n = (\pi n/l)\sqrt{Q_{10}/m_0}, \quad p_n = 434n \quad (n = 1, 2, ...)$$

**3** In the case considered, the tension in the filament is variable along its length

$$Q_1 = m_0 g(l - z)).$$

Figure 2.2 shows an element of the filament with the forces acting upon it at an arbitrary instant. Let us project the forces onto the $Oy$ axis:

$$-\mathrm{d}z\, m_0\frac{\partial^2 y}{\partial z^2} + (Q_1 + \mathrm{d}Q_1)\sin(\alpha + \mathrm{d}\alpha) - Q_1 \sin\alpha = 0,$$

or

$$m_0\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial z}(Q_1\alpha).$$

Since $\alpha = \partial y/\partial z$, we have finally

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial z}\left[g(l - z)\frac{\partial y}{\partial z})\right]. \tag{1}$$

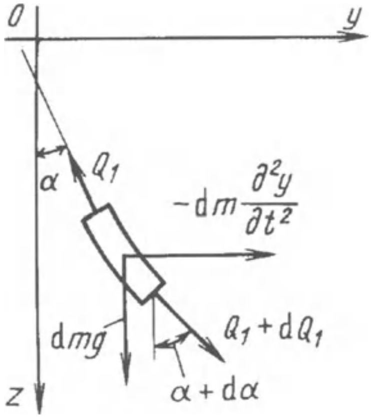


**Fig. 2.2.**

The solution to equation (1) is sought in the form

$$y = y_1(z)\sin pt. \tag{2}$$

After some transformations, we get

$$\frac{d^2 y_1}{dz_1^2} + \frac{1}{z_1}\frac{dy_1}{dz_1} + \frac{p^2}{gz_1}y_1 = 0, \tag{3}$$

where $z_1 = l - z$.

The solution to equation (3) can be expressed through zero-order Bessel functions of the first and second kind

$$y_1 = c_1 I_0\left(2\sqrt{p^2 z_1/g}\right) + c_2 Y_0\left(2\sqrt{p^2 z_1/g}\right) \tag{4}$$

The function $y_1$ should meet the following boundary conditions:

$$z = 0, \quad z_1 = l, \quad y_1 = 0;$$
$$z = l, \quad z_1 = 0, \quad y_1 \neq \infty.$$

The displacement of the lower end of the filament should be finite under small vibrations. Since the zero-order Bessel function of the second kind $Y_0$ goes to infinity when its argument vanishes, one must set $c_2 = 0$ in solution (4). Then $y_1$ is finite at $z = l$.

In order to satisfy the first condition, it is necessary that $I_0(2p\sqrt{l/g}) = 0$.

The first three roots of function $I_0$ are equal to: $K_1 = 2.4$; $K_2 = 5.52$, and $k_3 = 8.65$ [2]. Hence the frequencies of vibrations (the first three frequencies) are as follows

$$p_1 = 1.2\sqrt{g/l}; \quad p_2 = 2.76\sqrt{g/l}; \quad p_3 = 4.325\sqrt{g/l}.$$

**4** In the case under consideration, the filament tension is

$$Q_1 = m_0 g(l - z) + mg \tag{1}$$

The differential equation of the filament vibrations has the form (see solution to Problem **3**)

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial z}\left[\frac{m}{m_0}g + g(l - z)\frac{\mathrm{d}y}{\mathrm{d}z})\right] \tag{2}$$

We seek the solution in the form

$$y = y_1(z)\sin pt \tag{3}$$

Assuming $z_1 = mg + m_0 g(l - z)$, we find from equations (2) and (3)

$$y_1 = c_1 I_0\left(2\frac{p}{g}\sqrt{\frac{z_1}{m_0}}\right) + c_2 Y_0\left(2\frac{p}{g}\sqrt{\frac{z_1}{m_0}}\right) \tag{4}$$

Function $y_1$ should satisfy the following boundary condition: $y_1(0) = 0$ for $z = 0$.

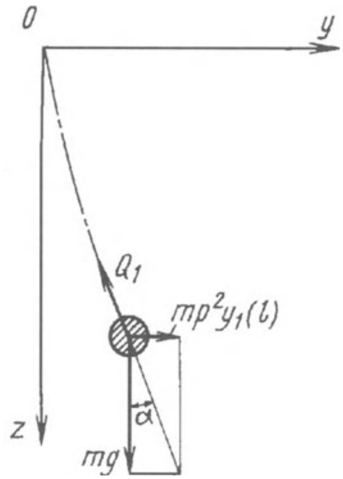


**Fig. 2.3.**

In order to find the second boundary condition, we consider the dynamic equilibrium of the mass $m$ (Fig. 2.3).

The forces of gravity and inertia should yield a net force that balances the force of tension. Hence, the force $Q_1$ is inclined to the vertical at angle $\alpha$.

Since we consider small vibrations, $\alpha = y_1'(l)$. The second condition has the form

$$\left(\frac{\mathrm{d}y_1}{\mathrm{d}z}\right)_{z=l} = \frac{mp^2 y_1(l)}{mg} = \frac{p^2}{g} y_1(l).$$

The first boundary condition allows one to derive the following equation

$$c_1 I_0\left(2p\sqrt{\frac{m+m_0 l}{m_0 g}}\right) + c_2 Y_0\left(2p\sqrt{\frac{m+m_0 l}{m_0 g}}\right) = 0. \qquad (5)$$

Differentiating expression (4) with respect to $z$ and using the relationships linking the derivatives of Bessel functions to each other, we have from the second boundary condition

$$\begin{aligned} c_1\left[\frac{p^2}{g} I_0\left(2p\sqrt{\frac{m}{m_0 g}}\right) - \frac{pm_0}{\sqrt{m_0 mg}} I_1\left(2p\sqrt{\frac{m}{m_0 g}}\right)\right] + \\ + c_2\left[\frac{p^2}{g} Y_0\left(2p\sqrt{\frac{m}{m_0 g}}\right) - \frac{pm_0}{\sqrt{m_0 mg}} Y_1\left(2p\sqrt{\frac{m}{m_0 g}}\right)\right] = 0. \end{aligned} \qquad (6)$$

Since $m = m_0 l$, we find after some algebra ($x = p\sqrt{l/g}$):

$$\begin{aligned} & c_1 I_0\left(\sqrt{8x}\right) + c_2 Y_0\left(\sqrt{8x}\right) = 0; \\ & c_1\left[x^2 I_0(2x) - x I_1(2x)\right] + c_2\left[x^2 Y_0(2x) - x Y_1(2x)\right] = 0. \end{aligned} \qquad (7)$$

Assuming that the determinant of system (7) is equal to zero, we get the equation for vibration frequencies. The first frequency of vibrations correspond to the first root of the equation

$$\begin{aligned} I_0\left(\sqrt{8x}\right)\left[x^2 Y_0(2x) - x Y_1(2x)\right] - \\ - Y_0\left(\sqrt{8x}\right)\left[x^2 I_0(2x) - x I_1(2x)\right] = 0. \end{aligned} \qquad (8)$$

Equation (8) can be solved graphically (to determine the first frequencies). The first root of equation (8) is $x_1 = 1.05$.

Hence, the first frequency is

$$p = 1.05\sqrt{g/l}.$$

**5** When the filament is deflected from the vertical position of equilibrium, in addition to the forces considered above (see solution to Problem **3**), the distributed centrifugal forces $m_0 \overline{y\omega}^2$ act upon it. The differential equation of the filament vibrations has the form

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial z}\left[g(l-z)\frac{\partial y}{\partial z})\right] + \omega^2 y. \tag{1}$$

The solution to equation (1) is sought in the form $y = y_1(z)\sin pt$.

After some transformations (see solution to Problem **3**) we get the Bessel's equation

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}z_1{}^2} + \frac{1}{z_1}\frac{\mathrm{d}y_1}{\mathrm{d}z_1} + \frac{(p^2+\omega^2)}{g z_1} y_1 = 0, \tag{2}$$

where $z_1 = l - z$.

Equation (2) has the following solution

$$y_1 = c_1 I_0\left(2\sqrt{(p^2+\omega^2)\,z_1/g}\right) + c_2 Y_0\left(2\sqrt{(p^2+\omega^2)\,z_1/g}\right)$$

In order to have finite $y_1$ at $z_1 = 0$ (which corresponds to $z = l$), it is necessary to assume that $c_2 = 0$.

For the second boundary condition to be met ($y_1 = 0$ at $z = 0$ ), it is necessary that

$$I_0\left(2\sqrt{(p^2+\omega^2)\,z_1/g}\right) = 0$$

The first three roots of function $I_0$ are as follows

$$k_1 = 2.4; \quad k_2 = 5.52; \quad k_3 = 8.65.$$

Accordingly, the first three frequencies of the filament vibrations are

$$p_1 = \sqrt{1.44g/l - \omega^2}; \quad p_2 = \sqrt{7.6g/l - \omega^2}; \quad p_3 = \sqrt{19g/l - \omega^2}.$$

The least value of the critical angular velocity of the filament is

$$\omega_* = 1.2\sqrt{g/l}.$$

**6** In the case under consideration, the differential equation of string vibrations has the form

$$\left(m_0 + m_1 \sin\frac{\pi z}{l}\right)\frac{\partial^2 y}{\partial t^2} = Q_{10}\frac{\partial^2 y}{\partial z^2}. \tag{1}$$

We seek the solution to equation (1) using the Galerkin method and assuming

$$y = \sin\frac{\pi z}{l} f_1(t) + \sin\frac{2\pi z}{l} f_2(t)$$

Now the frequencies of vibrations are

$$p_1 = \frac{\pi}{l}\sqrt{\frac{Q_{10}}{m_0\left[1 + 8m_1/\left(3\pi m_0\right)\right]}};$$

$$p_2 = \frac{2\pi}{l}\sqrt{\frac{Q_{10}}{m_0\left[1 + 32m_1/\left(15\pi m_0\right)\right]}}.$$

**7** The frequency of vibrations is

$$p = \frac{\pi}{l}\sqrt{\frac{Q_{10}}{m_0\left(1+a_{11}\right)}},$$

where $a_{11} = \dfrac{l}{2\pi}\left(1-\cos\dfrac{\pi l}{l_1}\right)\left(1-\dfrac{\pi l}{l^2-4l_1^2}\right)\dfrac{m_1}{m_0}$.

**8** The force of filament tension in a section at a distance $z$ from the axis of rotation (Fig. 2.4) is equal to

$$Q_1 = \int\limits_z^{l+a} m_0\Omega^2\eta\mathrm{d}\eta = \frac{m_0\Omega^2}{2}\left[(l+a)^2 - z^2\right].$$

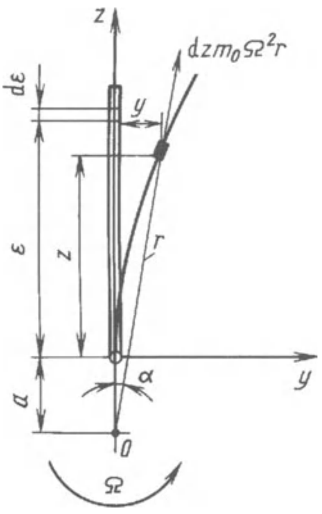


Fig. 2.4.

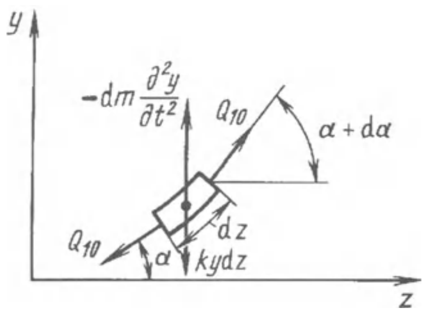


Fig. 2.5.

For small deflections of the filament from the rectilinear form, the force $Q_1$ is practically invariable. The differential equation of small vibrations of the filament has the form

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial z}\left\{\frac{\Omega^2}{2}\left[(l+a)^2 - z^2\right]\frac{\partial y}{\partial z}\right\} + \Omega^2 y.$$

**9** When a string is deflected from its equilibrium position, the forces shown in Fig. 2.5 act upon it.

Projecting the forces onto the $Oy$ axis, we get after transformations

$$m_0 \frac{\partial^2 y}{\partial t^2} = Q_{10} \frac{\partial^2 y}{\partial z^2} - ky. \qquad (1)$$

We seek the solution to equation (1) in the form $y = y_1(z) \sin pt$. For the function $y_1(z)$ we get the following differential equation

$$Q_{10} \frac{\partial^2 y_1}{\partial z^2} + m_0 p^2 y_1 - k y_1 = 0,$$

from whence

$$y_1 = c_1 \cos \sqrt{\frac{m_0 p^2 - k}{Q_{10}}}\, z + c_2 \sin \sqrt{\frac{m_0 p^2 - k}{Q_{10}}}\, z.$$

The function $y_1$ should satisfy the following boundary conditions:

$$z = 0, \quad y_1 = 0;$$
$$z = l, \quad y_1 = 0,$$

hence

$$c_1 = 0; \quad \sin \sqrt{(m_0 p^2 - k)\, Q_{10}} \cdot l = 0,$$

or

$$\frac{m_0 p^2 - k}{Q_{10}} = \frac{\pi^2 n^2}{l^2}.$$

The frequencies of vibrations of the string lying on an elastic base are determined by the equation

$$p_n = \sqrt{\frac{\pi^2 n^2 Q_{10}}{l^2 m_0} + \frac{k}{m_0}} \qquad (n = 1, 2, ...).$$

**10** We seek the solution to the equation of string vibrations (equation (1) in the solution to Problem **2**) using the Fourier method and assuming that

$$y = Y(z) T(t);$$
$$y(z, t) = y_0 \sin \frac{\pi z}{l} \cos \frac{\pi a t}{l}.$$

**11** It is more convenient to investigate small vibrations of the moving branch of the gearing using the Eulerian variables. Therefore, turning from the total time derivatives to local derivatives, we get

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial z} w;$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\partial y^2}{\partial t^2} + 2w \frac{\partial^2 y}{\partial z \partial t} + w^2 \frac{\partial^2 y}{\partial z^2} \qquad (1)$$

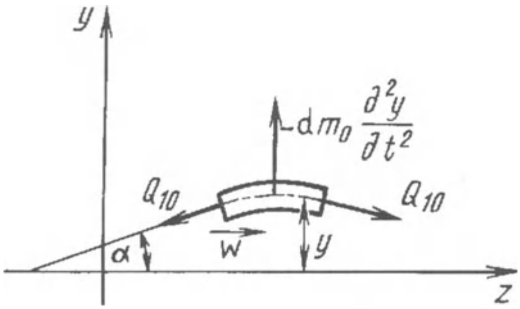


**Fig. 2.6.**

The equations of vibrations of the string (Fig. 2.6) assumes the form

$$\frac{\partial^2 y}{\partial t^2} + 2w\frac{\partial^2 y}{\partial z \partial t} - \left(\frac{Q_{10}}{m_0} - w^2\right)\frac{\partial^2 y}{\partial z^2} = 0. \tag{2}$$

The solution to equation (2) is sought in the form

$$y = y_1(z)e^{ipt} \tag{3}$$

Upon substitution of (3) into equation (2) we have the equations for the function $y_1(z)$:

$$\frac{d^2 y_1}{dz^2} - \frac{2wpi}{\left(\frac{Q_{10}}{m_0} - w^2\right)}\frac{dy_1}{dz} + \frac{p^2}{\left(\frac{Q_{10}}{m_0} - w^2\right)}y_1 = 0. \tag{4}$$

The function $y_1$ should satisfy the following boundary conditions:

$$z = 0,\ y_1 = 0; \quad z = l,\ y_1 = 0.$$

Assuming $y_1 = Ae^{\lambda z}$, the characteristic equation for equation (4) has the form

$$\lambda^2 - a_1 i\lambda + a_2 = 0, \tag{5}$$

where $a_1 = \dfrac{2wp}{\left(\frac{Q_{10}}{m_0} - w^2\right)}$; $\quad a_2 = \dfrac{p^2}{\left(\frac{Q_{10}}{m_0} - w^2\right)}$

The roots of equation (5) are

$$\lambda_{1,2} = i\left(a_1 \pm \sqrt{a_1^2 + 4a_2}\right)/2.$$

The solution to equation (4) can be represented in the form

$$y_1 = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z}. \tag{6}$$

This solution should satisfy homogeneous boundary conditions, and this allows one to write

$$\begin{vmatrix} 1 & 1 \\ e^{\lambda_1 l} & e^{\lambda_2 l} \end{vmatrix} = 0,$$

or

$$e^{(\lambda_2 - \lambda_1)} = 1. \tag{7}$$

Condition (7) is satisfied at $(\lambda_2 - \lambda_1)\, l = 2\pi n \mathrm{i}$ or $l\sqrt{a_1^2 + 4a_2} = 2\pi n$, whence, after some transformations we get the following values of the frequencies of vibrations

$$p_n = \frac{\pi n}{l}\sqrt{\frac{Q_{10}}{m_0}}\left(1 - \frac{m_0 w^2}{Q_{10}}\right) \quad (n = 1, 2, ...).$$

After calculations we find $p_n = 230\ \mathrm{ns}^{-1}$.

The critical velocity at which the frequencies of vibrations are zero is equal to

$$w_* = \sqrt{\frac{Q_{10}}{m_0}}.$$

After calculations we get $w_* = 16.43$ m/s.

Now let us determine the eigen functions. We find the roots of the characteristic equation, $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$, for every frequency $p_n$:

$$\begin{aligned} &\lambda_{1,2}^{(n)} = i\left(\gamma_{1n} \mp \gamma_{2n}\right), \\ &\gamma_{1n} = w p_n; \quad \gamma_{2n} = \pi n; \\ &p_n = \frac{\pi n \left(Q_{10} - m_0 w^2\right)}{\sqrt{Q_{10}}} \end{aligned} \tag{8}$$

For each pair of roots $\lambda_{1,2}^{(n)}$ we write the particular solution

$$\tilde{y}_1^{(n)} = c_1^{(n)} e^{\lambda_1^{(n)} z} + c_2^{(n)} e^{\lambda_2^{(n)} z}.$$

Since $y^{(n)}(0) = 0$ for $z = 0$, then $c_2^{(n)} = -c_1^{(n)}$. Assuming $c_1^{(n)} = 1$, we obtain the eigen function

$$y_1^{(n)}(z) = e^{\lambda_1^{(n)} z} - e^{\lambda_2^{(n)} z} \quad (n = 1, 2).$$

After some transformations we have

$$y_1^{(n)}(z) = 2\sin\gamma_{1n} z \cdot \sin\gamma_{2n} z - 2\mathrm{i}\cos\gamma_{1n} z \cdot \sin\gamma_{2n} z, \tag{9}$$

i.e., the eigen functions are complex functions of the form

$$y_1^{(n)}(z) = y_{11}^{(n)}(z) + \mathrm{i} y_{12}^{(n)}(z),$$

where $y_{11}^{(n)} = 2\sin\gamma_{1n}z \cdot \sin\gamma_{2n}z$; $y_{12}^{(n)} = -2\cos\gamma_{1n}z \cdot \sin\gamma_{2n}z$. Thus, we get the particular solutions

$$y^{(n)} = y_1^{(n)} e^{ip_n t}$$

and the general solution

$$y(z,t) = \sum_{n=1}^{\infty} c_n y_1^{(n)} e^{ip_n t},$$

where $c_n = c_{n1} + ic_{n2}$.

Now represent the general solution in the form

$$y(z,t) = y_{11}(z,t) + iy_{12}(z,t), \tag{10}$$

where

$$\begin{aligned} y_{11} = &\sum_{n=1}^{\infty} c_{1n}\left(y_{11}^{(n)}\cos p_n t + y_{12}^{(n)}\sin p_n t\right) + \\ &\sum_{n=1}^{\infty} c_{2n}\left(y_{12}^{(n)}\cos p_n t - y_{11}^{(n)}\sin p_n t\right); \\ y_{12} = &\sum_{n=1}^{\infty} c_{1n}\left(y_{12}^{(n)}\cos p_n t + y_{11}^{(n)}\sin p_n t\right) + \\ &\sum_{n=1}^{\infty} c_{2n}\left(y_{11}^{(n)}\cos p_n t - y_{12}^{(n)}\sin p_n t\right). \end{aligned}$$

Each of the functions $y_{11}$ and $y_{12}$ satisfies equation of vibrations (2). The arbitrary constants $c_{1n}$ and $c_{2n}$ can be found from the initial conditions. In the general case, at $t = 0$ the bending deflections of the belt and the velocities are known, i.e.,

$$y_{11}(z,0) = \alpha_1(z) = \sum_{n=1}^{\infty}\left(c_{1n}y_{11}^{(n)} + c_{2n}y_{12}^{(n)}\right); \tag{11}$$

$$\dot{y}_{11}(z,0) = \alpha_2(z) = \sum_{n=1}^{\infty}\left(c_{1n}p_n y_{12}^{(n)} + c_{2n}p_n y_{11}^{(n)}\right). \tag{12}$$

Multiplying equation (11) by $p_n y_{12}^{(k)}$ and equation (12) by $y_{11}^{(k)}$, and summing the expressions obtained we have

$$\begin{aligned} \alpha_1 p_n y_2^{(k)} + \alpha_2 y_1^{(k)} = &\sum_{n=1}^{\infty} c_{1n}p_n\left(y_1^{(n)}y_2^{(k)} + y_2^{(k)}y_1^{(n)}\right) \\ &+\sum_{n=1}^{\infty} c_{2n}p_n\left(y_2^{(n)}y_2^{(k)} - y_1^{(n)}y_1^{(k)}\right). \end{aligned} \tag{13}$$

Now we integrate equation (13) between 0 and $l$:

$$\int_0^l \left(\alpha_1 p_n y_2^{(n)} + \alpha_2 y_1^{(n)}\right) \mathrm{d}z = c_{1n} p_n J_{nn}^{(2)} + c_{2n} p_n J_{nn}^{(1)}. \tag{14}$$

Here

$$J_{nn}^{(1)} = -\frac{\sin a_{1n}}{2a_{1n}} + \frac{\sin(2\pi n - a_{1n})}{4(2\pi n - a_{1n})} + \frac{\sin(2\pi n + a_{1n})}{4(2\pi n + a_{1n})};$$

$$J_{nn}^{(2)} = -\frac{\cos a_{1n}}{2a_{1n}} - \frac{1 - \cos(2\pi n - a_{1n})}{4(2\pi n - a_{1n})} + \frac{1 - \cos(2\pi n + a_{1n})}{4(2\pi n + a_{1n})};$$

$$a_{1n} = \frac{2\omega_0 p_n}{Q_{10} - m_0\omega^2};$$

Multiplying expression (12) by $p_n y^{(k)}$ and (11) by $y_2^{(k)}$, we find the difference of the expression obtained and integrate it between 0 and $l$:

$$\int_0^l \left(\alpha_1 p_n y_1^{(n)} - \alpha_2 y_2^{(n)}\right) \mathrm{d}z = -c_{1n} p_n J_{nn}^{(1)} + c_{2n} p_n J_{nn}^{(2)}. \tag{15}$$

When integrating, we have used the conditions of orthogonality of functions $y_1^{(n)}$ and $y_2^{(n)}$:

$$J_{kn}^{(1)} = \int_0^l \left(y_2^{(k)} y_2^{(n)} - y_1^{(k)} y_1^{(n)}\right) \mathrm{d}z = 0;$$

$$J_{kn}^{(2)} = \int_0^l \left(y_1^{(k)} y_2^{(n)} + y_2^{(k)} y_2^{(n)}\right) \mathrm{d}z = 0.$$

From system of equations (14) and (15) we determine $c_{1n}$ and $c_{2n}$ ($n = 1, 2, \ldots$) and get the solution $y_{11}(z, t)$ to the equation of free vibrations of the branch of the flexible gearing.

**12** Write the equation of small vibrations of a gearing branch

$$a_{11}\frac{\partial^2 y}{\partial t^2} + 2a_{12}\frac{\partial^2 y}{\partial z \partial t} + a_{22}\frac{\partial^2 y}{\partial z^2} = 0, \tag{1}$$

where $a_{11} = 1$; $a_{12} = w$; $a_{22} = -\left(\dfrac{Q_{10}}{m_0} - w^2\right)$;

The characteristic equation for (1) has the following form [6]:

$$a_{22}\,\mathrm{d}t^2 - 2a_{12}\,\mathrm{d}z\,\mathrm{d}t + a_{11}\,\mathrm{d}z^2 = 0. \tag{2}$$

Equation (2) has two roots:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = w_1; \quad \frac{\mathrm{d}z}{\mathrm{d}t} = w_2, \tag{3}$$

where

$$\begin{aligned} w_1 &= \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}; \\ w_2 &= \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}. \end{aligned} \tag{4}$$

The straight lines $c_1 = z - w_1 t$ and $c_2 = z - w_2 t$ are the integrals of equations (3). Parameters $w_1$ and $w_2$ are the velocities of propagation of perturbations.

Substituting the coefficients of equation (1) into expressions (4), we get

$$w_1 = w + \sqrt{\frac{Q_{10}}{m_0}}; \quad w_2 = w - \sqrt{\frac{Q_{10}}{m_0}},$$

where $w_1$ and $w_2$ are the velocities of propagation of perturbations along the direction of motion of the flexible gearing and in the opposite direction, respectively.

The velocity $w_*$, at which perturbations do not propagate against the direction of motion of the flexible gearing, is equal to

$$w_* = \sqrt{\frac{Q_{10}}{m_0}}.$$

This expression coincides with the equation for $w_*$ obtained in Problem **11**.

**13** The differential equation of vibrations of a moving flexible gearing (filament) is derived in Problem **11**.

The distinction of the problem considered from Problem **11** consists in the fact that the tension of branches of the flexible gearing is variable in time. Therefore, the equation of vibrations takes on the following forms for the driving and driven branches, respectively:

$$\frac{\partial^2 y}{\partial t^2} + 2\omega \frac{\partial^2 y}{\partial z \partial t} - \left( \frac{F\sigma_{10}}{m_0} + \frac{F\Delta\sigma_1}{m_0} \sin \omega t - \omega^2 \right) \frac{\partial^2 y}{\partial z^2} = 0; \tag{1}$$

$$\frac{\partial^2 y}{\partial t^2} + 2\omega \frac{\partial^2 y}{\partial z \partial t} - \left( \frac{F\sigma_{20}}{m_0} + \frac{F\Delta\sigma_2}{m_0} \sin \omega t - \omega^2 \right) \frac{\partial^2 y}{\partial z^2} = 0; \tag{2}$$

We can solve equations (1) and (2) approximately taking advantage of the principle of virtual displacements (Appendix E), restricting ourselves to a single-term approximation $y = f_1(t) \sin \frac{\pi z}{l}$.

For the $f_1(t)$ function we get the Mathieu's equation. For example, for equation (1) we have

$$\frac{d^2 f_1}{d\tau^2} + (a + 2q\cos 2\tau) f_1 = 0,$$

where $a = 4\left(\frac{\pi}{l}\right)^2 \frac{F\sigma_{10}/m_0 - \omega^2}{\omega^2}$; $q = 4\left(\frac{\pi}{l}\right)^2 \frac{\Delta\sigma_1}{m_0\omega^2}$; $\tau = \frac{\omega t}{2}$. Upon substitution of numerical values, we find $a = 33$ and $q = 6$.

**14** In the fixed system of coordinates ($z_1 = z - wt$) the mass varies according to the law

$$m = m_0 + m_1 \sin\left(\frac{2\pi z}{l_1} - \frac{2\pi wt}{l_1}\right)$$

The differential equation of vibrations in the Eulerian variables is derived for a moving belt in Problem **11**:

$$m_3\left(\frac{\partial^2 y}{\partial t^2} + 2w\frac{\partial^2 y}{\partial z \partial t} + w^2\frac{\partial^2 y}{\partial z^2}\right) = Q_{10}\frac{\partial^2 y}{\partial z^2}. \quad (1)$$

In the case under consideration the mass of a unit length of the belt is

$$\begin{aligned} m_3 &= m_2 + m_0 + m_1 \sin\left(\frac{2\pi z}{l_1} - \frac{2\pi wt}{l_1}\right) = \\ &= (m_2 + m_0)\left[1 + \frac{m_1}{m_2 + m_0}\sin\left(\frac{2\pi z}{l_1} - \frac{2\pi wt}{l_1}\right)\right]. \end{aligned} \quad (2)$$

Dividing both sides of equation (1) by $m_3$ and expanding $\frac{1}{m_3}$ into a series of powers of $m_1$, we get (retaining only a linear part of the expansion)

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2} + 2w\frac{\partial^2 y}{\partial z \partial t} + w^2\frac{\partial^2 y}{\partial z^2} - \\ &-\frac{Q_{10}}{(m_0 + m_2)}\left[1 + \frac{m_1}{m_2 + m_0}\sin\left(\frac{2\pi z}{l_1} - \frac{2\pi wt}{l_1}\right)\right]\frac{\partial^2 y}{\partial z^2} = 0. \end{aligned} \quad (3)$$

To investigate the stability of small vibrations of the belt we apply the Galerkin method (principle of virtual displacements). In this case, we restrict ourselves to a single-term approximation, assuming $y = f(t) sin\frac{\pi z}{l}$ .

Then we have the following equation for unknown function $f(t)$

$$\begin{aligned} &\ddot{f} + \left\{\frac{Q_{10}}{m_0 + m_2}\left(\frac{\pi}{l}\right)^2 - w^2\left(\frac{\pi}{l}\right)^2 - \right. \\ &\left. -\frac{Q_{10}m_1}{m_0 + m_2}\left(\frac{\pi}{l}\right)^2\left(a_{11}\cos\frac{2\pi wt}{l_1} - a_{12}\sin\frac{2\pi wt}{l_1}\right)\right\} f = 0. \end{aligned} \quad (4)$$

Here

$$a_{11}=\int\limits_0^1 \sin\frac{2\pi z}{l_1}\sin^2\frac{\pi z}{l}\mathrm{d}z=\frac{l_1}{4\pi}\left(1-\cos\frac{2\pi l}{l_1}\right)\left(-\frac{l_1^2}{l^2-l_1^2}\right);$$

$$a_{12}=\int\limits_0^1 \cos\frac{2\pi z}{l_1}\sin^2\frac{\pi z}{l}\mathrm{d}z=-\frac{l_1^3}{4\pi}\sin\frac{2\pi l}{l_1}\cdot\frac{1}{(l^2-l_1^2)};$$

Equation (4) can be transformed to the form

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}+\left[a_1-b\sin\left(\frac{2\pi wt}{l_1}+\beta\right)\right]f=0; \tag{5}$$

were $b=\dfrac{Q_{10}m_1}{(m_0+m_2)^2}\left(\dfrac{\pi}{l}\right)^2\sqrt{a_{11}^2+a_{12}^2};\quad \tan\beta=\dfrac{a_{11}}{a_{12}}.$

Now make a conversion to a new independent variable, assuming

$$\frac{2\pi wt}{l_1}+\beta=2\tau-\frac{\pi}{2}.$$

Equation (5) takes on the form of a Mathieu's equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2}+(a+2q\cos 2\tau)\,f=0;$$

Calculate the coefficients:

$$a=\frac{l_1^2}{(\pi\omega)^2}\left(\frac{\pi}{l}\right)^2\left(\frac{Q_{10}}{m_0+m}-w^2\right),\quad a=1.15;$$

$$2q=\frac{l_1^2}{(\pi\omega)^2}b;\quad 2q=0.43,$$

which corresponds to a point on the plane of the Ains-Strett diagram (see Fig. 305 of Part 1) with coordinates (1.15; 0.125) that lies in the unstable region.

**15** Consider an element of the pipe with fluid at an arbitrary instant (Fig. 2.7). Let us project the forces onto the $Oy$ axis:

$$-m_1\mathrm{d}z\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}-m_2\mathrm{d}z\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}+Q_{10}\mathrm{d}\alpha-pF\mathrm{d}\alpha=0, \tag{1}$$

where $\mathrm{d}\alpha/\mathrm{d}z=\partial^2 y/\partial z^2$.

Changing over to the Eulerian variables, we get

$$m_1\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}=m_1\frac{\partial^2 y}{\partial t^2};\quad m_2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}=m_2\left(\frac{\partial^2 y}{\partial t^2}+2w\frac{\partial^2 y}{\partial z\partial t}+w^2\frac{\partial^2 y}{\partial z^2}\right).$$

Equation (1) assumes the form

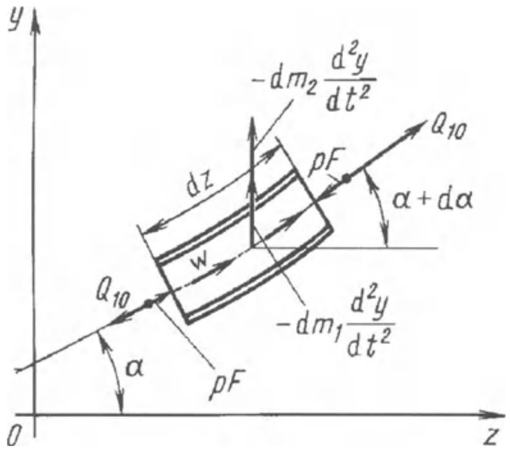


**Fig. 2.7.**

$$\frac{\partial^2 y}{\partial t^2} + \frac{2wm_2}{(m_1+m_2)}\frac{\partial^2 y}{\partial z \partial t} - \frac{Q_*}{(m_1+m_2)}\frac{\partial^2 y}{\partial z^2} = 0, \tag{2}$$

where $Q_* = Q_{10} - pF - m_2 w^2$

We seek the solution of equation (2) in the form (see solution to Problem **11**)

$$y = y_1(z)e^{ipt}.$$

After transformations similar to those made in Problem **11**, we find the frequencies of vibrations of the pipe with flowing fluid:

$$p_n = \frac{\pi n}{l}\sqrt{\frac{Q_{10}-pF}{(m_1+m_2)}\left(1-\frac{m_2 w^2}{Q_{10}-pF}\right)} \quad (n = 1, 2, \ldots).$$

**16** Unlike Problem **15**, in the case considered the force of tension in the pipe is variable with length $Q_1 = m_1 g(l - z)$.

The differential equation of small vibrations of the pipe has the form

$$(m_1+m_2)\frac{\partial^2 y}{\partial t^2} + 2m_2 w\frac{\partial^2 y}{\partial z \partial t} + m_2 w^2 \frac{\partial^2 y}{\partial z^2} = \frac{\partial}{\partial z}\left[m_1 g(l-z)\frac{\partial y}{\partial z}\right].$$

**17** The velocities of propagation of a wave of perturbations (see solution to Problem **12**) are equal to

$$w_1 = \frac{m_2 w}{m_1+m_2} + \sqrt{\frac{Q_{10}-pF}{m_1+m_2}}; \quad w_2 = \frac{m_2 w}{m_1+m_2} - \sqrt{\frac{Q_{10}-pF}{m_1+m_2}}.$$

The velocity of fluid flow, at which perturbations do not propagate along the pipe against the fluid flow, is

$$w_* = \frac{m_1 + m_2}{m_2} \sqrt{\frac{Q_{10} - pF}{m_1 + m_2}}.$$

**18** Figure 2.8 shows the string in a deflected position. Let us find a variation of the string length under vibrations. From the figure it follows that

$$\triangle \mathrm{d}z = \mathrm{d}z - \mathrm{d}z \cos\alpha \approx \mathrm{d}z - \mathrm{d}z\left(1 - \alpha^2/2\right). \tag{1}$$

Integrate equation (1) between 0 and $l$:

$$\triangle l = \frac{1}{2}\int_0^l y'^2 \mathrm{d}z,$$

where $y' = \mathrm{d}y/\mathrm{d}z \approx \alpha$.

Assuming that the relative deformation $\triangle\mathrm{d}z/\mathrm{d}z$ is constant over the string length, we obtain for the additional tension of the string

$$\triangle Q_1 = \frac{EF}{2l}\int_0^l y'^2 \mathrm{d}z,$$

because $\triangle l = \triangle Q_1 l/(EF)$

The equation of the string vibrations with allowance made for the additional tension takes on the form

$$m_0 \frac{\partial^2 y}{\partial t^2} = \left(Q_{10} + \triangle Q_1\right)\frac{\partial^2 y}{\partial z^2}. \tag{2}$$

We solve equations (2) by the Galerkin method assuming that

$$y = y_0 \sin\frac{\pi z}{l} \cdot f(t)$$

After some transformations we get a nonlinear equation for $f(z)$:

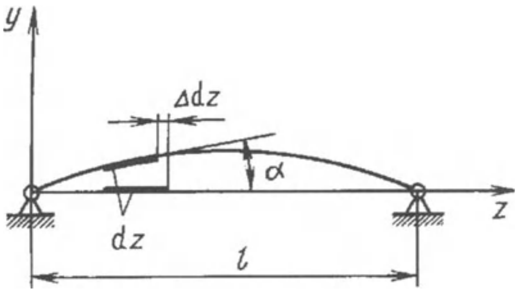


**Fig. 2.8.**

$$\frac{d^2 f}{dt^2} + \frac{Q_{10}}{m_0}\left(\frac{\pi}{l}\right)^2 f + \left(\frac{\pi}{l}\right)^4 \frac{EF}{4m_0} y_0^2 f^3 = 0.$$

The frequency of vibrations with a correction for the nonlinear term of the equation (see solution of Problem **165** in Part 1) is equal to

$$p = \sqrt{\frac{Q_{10}}{m_0}\left(\frac{\pi}{l}\right)^2 + \frac{3}{4}\left(\frac{\pi}{l}\right)^4 \frac{EF}{4m_0} y_0^2}.$$

The error in determination of the frequency in the case, when the changed tension is not taken into account, is equal to

$$\triangle p = \frac{3}{32}\left(\frac{\pi}{l}\right)^2 \frac{EF}{Q_{10}} y_0^2.$$

**19** The differential equation of the string vibrations at small displacements has the following form

$$m_0 \frac{\partial^2 y}{\partial t^2} = Q_{10}\frac{\partial^2 y}{\partial z^2} + q = Q_{10}\frac{\partial^2 y}{\partial z^2} + \frac{4\Phi_0^2 k_1 y}{l_1^3}. \tag{1}$$

We seek the solution to equation (1) in the form $y = y_1(z)\sin pt$. Upon substitution of $y$ into equation (1) we get

$$Q_{10}\frac{\partial^2 y_1}{\partial z^2} + \left(\frac{4\Phi_0^2 k_1}{l_1^3} + m_0 p^2\right) y_1 = 0.$$

The frequencies of vibrations are

$$p_n = \sqrt{\left(\frac{\pi n}{l}\right)^2 \frac{Q_{10}}{m_0} - \frac{4\Phi_0^2 k_1}{m_0 l_1^3}} \quad (n = 1, 2, \ldots).$$

**20** The differential equation of the string motion has the form

$$m_0 \frac{\partial^2 y}{\partial t^2} = Q_{10}\frac{\partial^2 y}{\partial z^2} + \frac{4k_1}{l_1^3}\left(a_1 + a_2 \sin\omega t\right) y. \tag{1}$$

We seek the solution to equation (1) in the form $y = \sum\limits_{n=1}^{\infty} f_n(t)\sin\frac{\pi n z}{l}$.

Using the Galerkin method, we get the following equation for unknown functions of time

$$m_0 \frac{d^2 f_n}{dt^2} + Q_{10}\left(\frac{\pi n}{l}\right)^2 f_n - \frac{4k_1}{l_1^3}\left(a_1 + a_2 \sin\omega t\right) f_n = 0. \tag{2}$$

Equations (2) can be reduced to the form

$$\frac{\mathrm{d}^2 f_n}{\mathrm{d}t^2} + (a_n + 2q_n \cos 2\tau) f_n = 0,$$

where $a_n = \dfrac{4}{w^2}\left[\dfrac{Q_{10}}{m_0}\left(\dfrac{\pi n}{l}\right)^2 - \dfrac{4a_1 k_1}{m_0 l_1^3}\right]$; $\quad 2q_n = \dfrac{4}{w^2}\dfrac{4a_2 k_1}{m_0 l_1^3}$.

Having calculated the values of coefficients, we get (restricting ourselves to the case $n = 1$) $a_1 = 3.1$; and $q_1 = 0.050$, which corresponds to a stable mode of the string vibrations.

**21** Since small vibrations are considered, one can assume that

$$q = \frac{2I_1 I_0 k}{a - y} \approx \frac{2I_1 I_0 k}{a}\left(1 + \frac{y}{a}\right).$$

The equation of the string vibration has the form

$$m_0 \frac{\partial^2 y}{\partial t^2} = Q_{10}\frac{\partial^2 y}{\partial z^2} + q,$$

or

$$\frac{\partial^2 y}{\partial t^2} = \frac{Q_{10}}{m_0}\frac{\partial^2 y}{\partial z^2} + \frac{2I_0 I_1 0 \sin \omega t}{m_0 a^2} y + \frac{2I_0 I_1 0 \sin \omega t}{m_0 a}.$$

**22** In the case of action of a concentrated force $P_0$, the differential equation of the string vibration can be represented in the form

$$m_0 \frac{\partial^2 y}{\partial t^2} = Q_{10}\frac{\partial^2 y}{\partial z^2} + P_0 \delta (z - l_0). \tag{1}$$

where $\delta (z - l_0)$ is the Dirac delta function.

Let us expand the delta function into a Fourier series in terms of the functions that satisfy the boundary conditions of the problem (eigen functions):

$$\delta (z - l_0) = \sum_{n=1}^{\infty} c_n \sin \frac{\pi n z}{l}$$

The expansion coefficients are

$$c_n = \frac{2}{l}\int_0^l \delta (z - l_0) \sin \frac{\pi n z}{l} \mathrm{d}z = \frac{2}{l} \sin \frac{\pi n l_0}{l}$$

Taking these coefficients into account, equation (1) takes the form

$$m_0 \frac{\partial^2 y}{\partial t^2} = Q_{10}\frac{\partial^2 y}{\partial z^2} + \sum_{n=1}^{\infty} \frac{2P_0}{l} \sin \frac{\pi n l_0}{l} \sin \frac{\pi n z}{l}. \tag{2}$$

The solution to equation (2) is sought in the form

$$y = \sum_{n=1}^{\infty} y_n(t) \sin \frac{\pi n z}{l} \tag{3}$$

Substituting (3) into (2), we obtain equations in order to determine functions $y_n(t)$

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} + a^2 \left( \frac{\pi n}{l} \right)^2 y_n = \frac{2P_0}{l m_0} \sin \frac{\pi n l_0}{l} \quad (n = 1, 2, \ldots), \tag{4}$$

where $a^2 = Q_{10}/m_0$

Solving equation (4) at zero initial conditions ($y_n = \dot{y}_n = 0$), we get

$$y_n = \frac{2P_0 l}{Q_{10} m_0 \pi^2 n^2} \sin \frac{\pi n l_0}{l} \cdot \left( 1 - \cos \frac{a \pi n}{l} t \right).$$

Finally, the solution to equation (1) assumes the form

$$y(z, t) = \frac{2P_0 l}{Q_{10} \pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{\pi n l_0}{l} \sin \frac{\pi n z}{l} \cdot \left( 1 - \cos \frac{a \pi n}{l} t \right).$$

The displacement of the point of application for the force $P_0$ is a function of time:

$$y(l_0, t) = \frac{2P_0 l}{Q_{10} \pi^2} \sum_{n=1}^{\infty} \frac{\sin \dfrac{\pi n l_0}{l}}{n^2} \cdot \left( 1 - \cos \frac{a \pi n}{l} t \right).$$

**23** The differential equation of vibrations of the string lying on an elastic inertialess base (see solution to Problem **9**), with allowance made for a concentrated force, has the following form

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial z^2} + \frac{P_0 \delta (z - z_0)}{m_0} - \frac{k y}{m_0}, \tag{1}$$

where $z_0 = vt$

Expanding $\delta (z - z_0)$ into a series and assuming that

$$y = \sum_{n=1}^{\infty} y_n(t) \sin \frac{\pi n z}{l} \quad (n = 1, 2, \ldots),$$

we get the following equation for $y_n(t)$ (see solution to Problem **22**)

$$\ddot{y}_n + \left[ a^2 \left( \frac{\pi n}{l} \right)^2 + \frac{k}{m_0} \right] y_n = \frac{2P_0}{l m_0} \sin \frac{\pi n v t}{l}.$$

Since $y_n = \dot{y}_n = 0$ at $t = 0$, we find after some transformations

$$y(t,z) = \frac{2P_0 l}{m_0 \pi^2} \sum_{n=1}^{\infty} \frac{\sin\dfrac{\pi n v t}{l} - \dfrac{\left(\dfrac{\pi n}{l}\right) v \sin\sqrt{a^2\left(\dfrac{\pi n}{l}\right)^2 + \dfrac{k}{m}t}}{\sqrt{a^2\left(\dfrac{\pi n}{l}\right)^2 + \dfrac{k}{m_0}}}}{n^2\left[a^2\left(\dfrac{\pi n}{l}\right)^2 + \dfrac{k}{m_0} - \left(\dfrac{\pi n}{l}v\right)^2\right]} \sin\frac{\pi n z}{l}.$$

**24** Figure 1.20 to the problem statement shows a string with the moving load $P_0$ $(z_0 = vt)$. The differential equation of vibrations of the wire at moving load is similar to equation (1) in Problem **23** for $k = 0$):

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial z^2} + \frac{2P_0}{l m_0} \delta\left(z - z_0\right). \tag{1}$$

Let us seek the solution to equation (1) in the form

$$y = \sum_{n=1}^{\infty} y_n(t) \sin\frac{\pi n z}{l}.$$

Using the principle of virtual displacements we get after transformations

$$y(t,z) = \frac{2P_0 l}{m_0\left(a^2 - v^2\right)\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{\pi n z}{l} \times \\ \times \left(\sin\frac{\pi n v t}{l} - \frac{v}{a}\sin\frac{a \pi n t}{l}\right).$$

**25** In the case considered, the transverse force $N$ with which the load acts on the cable does not remain constant under vibrations. The force of interaction between the cable and the load is

$$N = Mg - M\ddot{y}_0 \qquad \left(\ddot{y}_0 = \left.\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right|_{z\,=\,z_0}\right).$$

where $y_0$ is the displacement of the string at the point where the load is located, $y_0 = y\big|_{z\,=\,z_0}$; and $z_0 = vt$.

The differential equation of the string vibration (given a traveling force) has the form (see solution to Problem **24**)

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial z^2} + N\,\delta\frac{(z - vt)}{m_0}.$$

Changing for the Eulerian variables we get

$$N = Mg - M\left.\left(\frac{\partial^2 y}{\partial t^2} + 2v\frac{\partial^2 y}{\partial z \partial t} + v^2\frac{\partial^2 y}{\partial z^2}\right)\right|_{z\,=\,z_0}$$

and, finally, we have the equation of small vibrations of the string with a moving concentrated mass $M$

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial z^2} + \frac{Mg}{m_0}\delta(z-vt) - \frac{M}{m_0}\left(\frac{\partial^2 y}{\partial t^2} + 2v\frac{\partial^2 y}{\partial z\partial t} + v^2\frac{\partial^2 y}{\partial z^2}\right)\delta(z-vt).$$

**26** In this case the solution is similar to that presented above for Problem **24**, therefore, changing $P_0$ for $Mg$, we get

$$y = \frac{2Mgl}{m_0\pi^2\left(a^2-v^2\right)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{\pi nz}{l}\left(\sin\frac{\pi nvt}{l} - \frac{v}{a}\sin\frac{a\pi nt}{l}\right).$$

The vertical displacement of the load $Mg$ is

$$y\Big|_{z=vt} = \frac{2Mgl}{m_0\pi^2\left(a^2-v^2\right)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{\pi nvt}{l}\left(\sin\frac{\pi nvt}{l} - \frac{v}{a}\sin\frac{a\pi nt}{l}\right).$$

**27** At each of the segments the equations of small vibrations of the string look like

$$\frac{\partial^2 Y^{\mathrm{I}}}{\partial t^2} - a_1^2\frac{\partial^2 Y^{\mathrm{I}}}{\partial z^2} = 0; \qquad \frac{\partial^2 Y^{\mathrm{II}}}{\partial t^2} - a_2^2\frac{\partial^2 Y^{\mathrm{II}}}{\partial z^2} = 0. \tag{1}$$

where $a_1^2 = Q_{10}/m_{10}$; $a_2^2 = Q_{10}/m_{20}$

Assuming

$$Y^{\mathrm{I}} = Y_0^{\mathrm{I}}(z)\mathrm{e}^{\mathrm{i}pt}; \quad Y^{\mathrm{II}} = Y_0^{\mathrm{II}}(z)\mathrm{e}^{\mathrm{i}pt}, \tag{2}$$

we obtain

$$Y_0''^{\mathrm{II}} + k_1^2 Y_0^{\mathrm{I}} = 0, \quad Y_0''^{\mathrm{II}} + k_2^2 Y_0^{\mathrm{II}} = 0 \quad \left(k_j^2 = \frac{p^2 m_{j0}}{Q_{10}}\right). \tag{3}$$

Further solution is more compact if one takes for each segment its own origin of coordinates (see Fig. 1.22). Then we have for equations (3)

$$\begin{aligned} Y_0^{\mathrm{I}}(z) &= c_1^{\mathrm{I}}\cos k_1 z + c_2^{\mathrm{I}}\sin k_1 z, \\ Y_0'^{\mathrm{I}}(z) &= -c_1^{\mathrm{I}} k_1 \sin k_1 z + c_2^{\mathrm{II}} k_1\cos k_1 z; \end{aligned} \tag{4}$$

$$\begin{aligned} Y_0^{\mathrm{II}}(z) &= c_1^{\mathrm{II}}\cos k_2 z_1 + c_2^{\mathrm{II}}\sin k_2 z_1, \\ Y_0'^{\mathrm{II}}(z) &= -c_1^{\mathrm{II}} k_2 \sin k_2 z_1 + c_2^{\mathrm{II}} k_2\cos k_2 z_1; \end{aligned} \tag{5}$$

Having determined $c_j^{\mathrm{I}}$ and $c_j^{\mathrm{II}}$ $(j = 1, 2)$ from the boundary conditions, we represent relations (4) and (5) in vector form convenient for further transformations:

$$\overline{\mathbf{Y}}_0^{\mathrm{I}} = \mathbf{K}^{\mathrm{I}}(z, k_1)\overline{\mathbf{Y}}_{00}^{\mathrm{I}}, \quad \overline{\mathbf{Y}}_0^{\mathrm{II}} = \mathbf{K}^{\mathrm{II}}(z, k_2)\overline{\mathbf{Y}}_{00}^{\mathrm{II}}, \tag{6}$$

where $\mathbf{K}^{\mathrm{I}}(z, k_1)$ and $\mathbf{K}^{\mathrm{II}}(z, k_2)$ are the matrices of form

$$\mathbf{K}^{\mathrm{I}}(z, k_1) = \begin{bmatrix} \cos k_1 z & \dfrac{\sin k_1 z}{k_1} \\ -k_1 \sin k_1 z & \cos k_1 z \end{bmatrix};$$

$$\mathbf{K}^{\mathrm{II}}(z_1, k_2) = \begin{bmatrix} \cos k_2 z_1 & \dfrac{\sin k_1 z_1}{k_2} \\ -k_2 \sin k_2 z_1 & \cos k_2 z_1 \end{bmatrix};$$

$$\overline{\mathbf{Y}}_{00}^{\mathrm{I}} = \overline{\mathbf{Y}}_{0}^{\mathrm{I}}\Big|_{z=0}; \quad \overline{\mathbf{Y}}_{00}^{\mathrm{II}} = \overline{\mathbf{Y}}_{0}^{\mathrm{II}}\Big|_{z_1=0}.$$

Since at the point of conjunction of two segments

$$Y_0^{\mathrm{I}}(l_1, k_1) = Y_0^{\mathrm{II}}(0, k_2) \quad Y_0'^{\mathrm{I}}(l_1, k_1) = Y_0'^{\mathrm{II}}(0, k_2)$$

or, in the vector form

$$\overline{\mathbf{Y}}_{0}^{\mathrm{I}}(l_1, k_1) = \overline{\mathbf{Y}}_{00}^{\mathrm{II}}(0, k_2), \tag{7}$$

then, from (6) we have

$$\overline{\mathbf{Y}}_{00}^{\mathrm{II}} = \mathbf{K}^{\mathrm{I}}(l_1, k_1)\,\overline{\mathbf{Y}}_{00}^{\mathrm{I}}, \tag{8}$$

where $\mathbf{K}^{\mathrm{I}}(l_1, k_1)$ is the transition matrix.

At the second segment we have for an arbitrary $z_1$ $(0 \le z_1 \le l - l_1)$

$$\overline{\mathbf{Y}}_{0}^{\mathrm{II}} = \mathbf{K}^{\mathrm{II}}(z_1, k_2)\,\mathbf{K}^{\mathrm{I}}\overline{\mathbf{Y}}_{00}^{\mathrm{I}}. \tag{9}$$

For $z = 0$ and $z_1 = l - l_1$ the boundary conditions

$$\overline{\mathbf{Y}}_{0}^{\mathrm{I}}(0, k_1) = \begin{bmatrix} 0 \\ \overline{\mathbf{Y}}_{00}^{\prime\mathrm{I}} \end{bmatrix}; \quad \overline{\mathbf{Y}}_{0}^{\mathrm{II}}(l - l_1, k_2) = \begin{bmatrix} 0 \\ \overline{\mathbf{Y}}_{0}^{\prime\mathrm{II}} \end{bmatrix}.$$

should be satisfied.

At $z_1 = l - l_1$ $\left(\overline{\mathbf{Y}}_{0}^{\mathrm{II}}(l - l_1) = 0\right)$ it follows from condition (7) that

$$k_{12}\overline{\mathbf{Y}}_{00}^{\prime\mathrm{I}} = 0, \tag{10}$$

where $K_{12}$ is the element of the matrix $\mathbf{K} = \mathbf{K}^{\mathrm{II}}(l - l_1, k_2)\,\mathbf{K}^{\mathrm{I}}(l_1, k_1)$,

From (10), we obtain the equation for determination of the frequencies $p_j$ $(j = 1, 2)$:

$$a_1 \cos\left[\frac{p}{a_2}(l - l_1)\right] \sin\left(\frac{p}{a_1} l_1\right) + \\ + a_2 \sin\left[\frac{p}{a_2}(l - l_1)\right] \cos\left(\frac{p}{a_1} l_1\right) = 0 \tag{11}$$

In the specific case, when $m_{10} = m_{20}$, we have (see solution to Problem **2**)

$$\sin\left(\sqrt{\frac{m_{10}}{Q_{10}}}\, p\, l_1\right) = 0.$$

Equation (11) provides for a possibility to determine the frequencies $p_j$ numerically. For every frequency we find the partial solutions (solutions to equations (3)):

$$\begin{aligned} Y_{0j}^{\mathrm{I}} &= c_{1j}^{\mathrm{I}} \cos k_{1j} z + c_{2j}^{\mathrm{I}} \sin k_{1j} z \qquad (k_{1j} = p_i/a_1)\,; \\ Y_{0j}^{\mathrm{II}} &= c_{1j}^{\mathrm{II}} \cos k_{2j} z_1 + c_{2j}^{\mathrm{II}} \sin k_{2j} z_1 \quad (k_{2j} = p_i/a_2)\,. \end{aligned} \tag{12}$$

The partial solutions should satisfy the boundary conditions and condition (7):

$$\begin{aligned} &z = 0, \quad Y_{0j}^{\mathrm{I}} = 0; \quad z_1 = l - l_1, \quad Y_{0j}^{\mathrm{II}} = 0; \\ &z = l_1, \quad Y_{0j}^{\mathrm{I}}(l_1) = Y_{0j}^{\mathrm{II}}(0)\,. \end{aligned}$$

After transformations, we get

$$\begin{aligned} c_{1j}^{\mathrm{I}} &= 0; \quad c_{2j}^{\mathrm{I}} = c_{1j}^{\mathrm{II}} \sin k_{ij} l_1; \\ c_{2j}^{\mathrm{II}} &= -\frac{\cos k_{2j}(l - l_1)}{\sin k_{2j}(l - l_1)} c_{1j}^{\mathrm{II}}. \end{aligned}$$

Then we find from solutions 12

$$\begin{aligned} Y_{0j}^{\mathrm{I}}(z) &= \sin k_{1j} z c_{1j}^{\mathrm{II}}; \\ Y_{0j}^{\mathrm{II}}(z_1) &= \Big[\sin k_{1j} l_1 \cdot \cos k_{2j} z_1 - \\ &\quad - \frac{\cos k_{2j}(l - l_1)}{\sin k_{2j}(l - l_1)} \sin k_{1j} l_1 \cdot \sin k_{2j} z_1\Big] c_{1j}^{\mathrm{II}}. \end{aligned} \tag{13}$$

Assuming $c_{1j}^{\mathrm{II}} = 1$ and passing to $z$ ($z_1 = z - l_1$), we obtain the eigen functions of the boundary value problem (modes of vibrations) for the string as a whole:

$$\varphi_j(z) = \begin{cases} \sin k_{1j} z, & 0 \le z \le l_1; \\ \Big[\cos k_{2j}(z - l_1) - \dfrac{\cos k_{2j}(l - l_1)}{\sin k_{2j}(l - l_1)} \times \\ \quad \times \sin k_{2j}(z - l_1)\Big] \sin k_{1j} l_1, & l_1 < z < l. \end{cases} \tag{14}$$

The functions $\varphi_j(z)$ satisfy the orthogonality condition

$$\int_0^l \varphi_j \varphi_i \mathrm{d}z = 0 \quad (j \ne i).$$

Upon determining the eigen functions, we find the solution to equations (1) under free vibrations

$$y(z,t) = \sum_{j=l}^{\infty} (A_j \cos p_j t + B_j \sin p_j t)\, \varphi_j(z). \tag{15}$$

When the action of momentum $J$ is stopped, the points of the string gain the velocities $v = \dfrac{J}{m_{10}}\,\delta(z - z_k)$, while their displacements are equal to zero. Therefore,

$$y(z,0) \equiv 0; \quad \left.\frac{\partial y}{\partial z}\right|_{t=0} = \frac{J}{m_{10}}\,\delta(z - z_k).$$

It follows from the first initial condition that $A_j = 0$, therefore,

$$y(z,t) = \sum_{j=l}^{\infty} B_j \sin p_j t\, \varphi_j(z). \tag{16}$$

At $t = 0$ we have for the second condition

$$\frac{J}{m_{10}}\,\delta(z - z_k) = \sum_{j=l}^{\infty} B_j p_j \varphi_j. \tag{17}$$

Multiplying equation (17) both from left and from right by $\varphi_k(z)$ and integrating between 0 and $l$, we have, on the strength of orthogonality of functions $\varphi_j(z)$,

$$B_j = \frac{J\varphi_j(z_k)}{m_{10} p_j \int\limits_0^l \varphi_j^2 \mathrm{d}z}.$$

Thus, under vibrations caused by a momentum applied, the bending deflections of the string are equal to

$$y(z,t) = \sum_{j=l}^{\infty} \frac{J\varphi_j(z_k)}{p_j \int\limits_0^l \varphi_j^2 \mathrm{d}z \cdot m_{10}} \sin p_j t \cdot \varphi_j(z).$$

**28** Unlike Problem **27**, in this case it is required to get the matrix of transition from segment I to segment II accounting for the point-like concentrated mass $M$. Figure 2.9 shows the forces acting upon the mass $M$ at an arbitrary instant $t$ ($J_i$ is the force of inertia). It is clear from the figure that the following conditions should be met under vibrations:

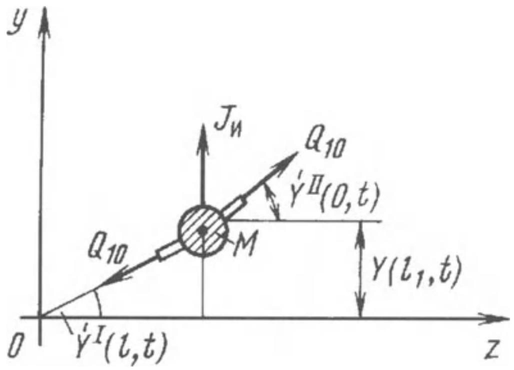


**Fig. 2.9.**

$$J_i + Q_{10}\frac{\partial Y^{\mathrm{II}}}{\partial z}\bigg|_{z_1=0} - Q_{10}\frac{\partial Y^{\mathrm{II}}}{\partial z}\bigg|_{z=l_1} = 0 \quad \left(J_i = -M\frac{\partial^2 Y_0^{\mathrm{I}}}{\partial t^2}\right); \qquad (1)$$

$$Y^{\mathrm{I}}(l_1, t) = Y^{\mathrm{II}}(0, t)$$

Assuming (see solution to Problem **27**) that

$$Y^{\mathrm{I}} = Y_0^{\mathrm{I}} \mathrm{e}^{\mathrm{i}p\,t}, \quad Y^{\mathrm{II}} = Y_0^{\mathrm{II}} \mathrm{e}^{\mathrm{i}p\,t}, \qquad (2)$$

we find from conditions (1)

$$\begin{gathered} Mp^2 Y_0^{\mathrm{I}}(l_1) + Q_{10} Y'^{\mathrm{II}}(0) - Q_{10} Y_0'^{\mathrm{I}}(0) = 0; \\ Y_0^{\mathrm{II}}(0) = Y_0^{\mathrm{I}}(l_1), \end{gathered} \qquad (3)$$

or, in the vector form,

$$\overline{\mathbf{Y}}_0^{\mathrm{II}}(0) = \mathbf{A}\overline{\mathbf{Y}}_0^{\mathrm{I}}(l_1), \qquad (4)$$

where $\overline{\mathbf{Y}}_0^{\mathrm{II}}(0) = \left[\mathbf{Y}_0^{\mathrm{II}}(0), \mathbf{Y}_0'^{\mathrm{II}}(0)\right]^{\mathrm{T}}$ and $\mathbf{A}$ is the matrix of transition though the mass $M$,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -\dfrac{Mp^2}{Q_{10}} & 1 \end{bmatrix}. \qquad (5)$$

The matrix of transition from the section at $z = 0$ to the section at $z_1 = l_1'$ $(l_1' \approx l_1)$ is

$$\mathbf{K}_1 = \mathbf{A} \cdot \mathbf{K}^{\mathrm{I}}(l_1). \qquad (6)$$

The general matrix of transition from $z = 0$ to $z = l$ $(z_1 = l - l_1)$ looks like

$$\mathbf{K} = \mathbf{K}^{\mathrm{II}}(l - l_1)\,\mathbf{A}\mathbf{K}^{\mathrm{I}}(l_1). \qquad (7)$$

Therefore,

$$\overline{\mathbf{Y}}_0^{\mathrm{II}}(l - l_1) = \mathbf{K}\overline{\mathbf{Y}}_{00}^{\mathrm{I}}. \qquad (8)$$

Since the conditions

$$z = 0, \quad Y_{00}^{\mathrm{I}} = 0 \quad \text{and} \quad z_1 = l - l_1, \quad Y_0^{\mathrm{II}} = 0,$$

should hold true, then we get from (8) the following equation for determination of frequencies

$$k_{12} = 0.$$

After some transformations, we have

$$\sin\lambda - \frac{M}{m_{10}l}\lambda \sin\lambda\left(1 - \frac{l_1}{l}\right)\sin\lambda\frac{l_1}{l} = 0, \tag{9}$$

where $\lambda = pl\sqrt{\dfrac{m_{10}}{Q_{10}}}$. Solving equation (9) numerically, we find the roots $\lambda_j$, and then the frequencies

$$p_j = \frac{\lambda_j}{l}\sqrt{\frac{Q_{10}}{m_{10}}}.$$

**29** Unlike Problem **28**, the mass $M$ is not point-like in this problem, therefore, under vibrations, in addition to the force of inertia $J_i$, one should take into account the moment of inertia $M_i$ (Fig. 2.10).

Taking advantage of the d'Alembert principle, we get two following equations

$$J_i + Q_{10}Y'^{\mathrm{II}}(0,t) - Q_{10}Y'^{\mathrm{I}}(0,t) = 0; \tag{1}$$

$$M_i + Q_{10}r\left(\left.\frac{\partial Y^{\mathrm{II}}}{\partial z}\right|_{z_1=0} - \vartheta_K\right) + Q_{10}r\left(\left.\frac{\partial Y^{\mathrm{II}}}{\partial z}\right|_{z_1=l_1} - \vartheta_K\right) = 0. \tag{2}$$

Displacements of the points of the string attachment are related by the expression

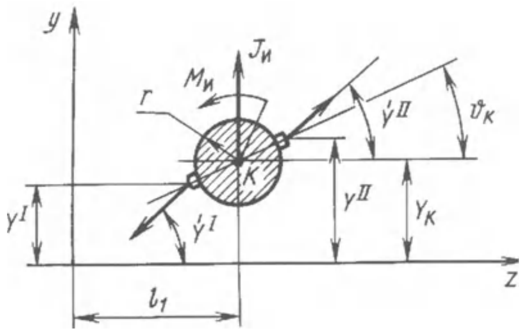


**Fig. 2.10.**

$$Y^{\mathrm{II}}\Big|_{z_1=0} = Y^{\mathrm{I}}\Big|_{z=l_1} + 2r\vartheta_K. \tag{3}$$

The quantities $J_i$ and $M_i$ appearing in equations (1) and (2) are equal to, respectively,

$$J_i = -M\frac{\partial^2 Y_i}{\partial t^2} \quad \left(Y_i = \frac{Y^{\mathrm{II}}\Big|_{z_1=0} + Y^{\mathrm{I}}\Big|_{z=l_1}}{2}\right);$$

$$M_i = -J_K\frac{\partial^2 \vartheta_K}{\partial t^2}.$$

Assuming

$$Y^{\mathrm{I}}(z,t) = Y_0^{\mathrm{I}}\mathrm{e}^{\mathrm{i}p\,t}, \quad Y^{\mathrm{II}}(z,t) = Y_0^{\mathrm{II}}\mathrm{e}^{\mathrm{i}p\,t},$$
$$\vartheta_K = \vartheta_{K_0}\mathrm{e}^{\mathrm{i}p\,t},$$

we get after transformations (exclusion of $\vartheta_{K_0}$ from (1) and (2) ) two equations relating the end of the first segment with the beginning of the second segment of the string:

$$\frac{Mp^2}{2Q_{10}}Y_0^{\mathrm{II}}\Big|_{z_1=0} + Y_0'^{\mathrm{II}}\Big|_{z_1=0} = -\frac{Mp^2}{2Q_{10}}Y_0^{\mathrm{I}}\Big|_{z=l_1} + Y_0'^{\mathrm{I}}\Big|_{z=l_1}, \tag{4}$$

$$\left(\frac{J_ip^2 - 2rQ_{10}}{2rQ_{10}}\right)Y_0^{\mathrm{II}}\Big|_{z_1=0} + rY_0'^{\mathrm{II}}\Big|_{z_1=0} =$$
$$= -\left(\frac{J_ip^2 - 2rQ_{10}}{2rQ_{10}}\right)Y_0^{\mathrm{I}}\Big|_{z=l_1} - rY_0'^{\mathrm{I}}\Big|_{z=l_1}, \tag{5}$$

or, in the vector form

$$\mathbf{A}_1\overline{\mathbf{Y}}_0^{\mathrm{II}}\Big|_{z_1=0} = \mathbf{A}_2\overline{\mathbf{Y}}_0^{\mathrm{I}}\Big|_{z=l_1}. \tag{6}$$

Here

$$\mathbf{A}_1 = \begin{bmatrix} \dfrac{Mp^2}{2Q_{10}} & 1 \\ \left(\dfrac{J_Kp^2 - 2rQ_{10}}{2rQ_{10}}\right) & r \end{bmatrix};$$

$$\mathbf{A}_2 = \begin{bmatrix} -\dfrac{Mp^2}{2Q_{10}} & 1 \\ -\left(\dfrac{J_Kp^2 - 2rQ_{10}}{2rQ_{10}}\right) & -r \end{bmatrix}.$$

From relationship (6) we find

$$\overline{\mathbf{Y}}_0^{\mathrm{II}}\Big|_{z_1=0} = \mathbf{A}\overline{\mathbf{Y}}^{\mathrm{I}}\Big|_{z=l_1},$$

where

$$\mathbf{A}_1 = \mathbf{A}_1^{-1}\mathbf{A}_2 =$$

$$= \begin{bmatrix} -\frac{1}{\triangle}\left[\left(\frac{rM}{2Q_{10}} + \frac{J_K}{2rQ_{10}}\right)p^2 - 1\right] & \frac{2r}{Q_{10}} \\ \frac{Mp^2}{\triangle Q_{10}}\left(\frac{J_K p^2}{2rQ_{10}} - 1\right) & -\frac{1}{\triangle}\left[\left(\frac{rM}{2Q_{10}} + \frac{J_K}{2rQ_{10}}\right)p^2 - 1\right] \end{bmatrix};$$

$$\triangle = \left(\frac{rM}{2Q_{10}} - \frac{J_K}{2rQ_{10}}\right)p^2 + 1. \tag{7}$$

At $r = 0$ we get from (7) a matrix of transition through the concentrated mass $M$ (see Problem **28**). The transition matrices $\mathbf{K}^{\mathrm{I}}\Big|_{z=l_1}$ and $\mathbf{K}^{\mathrm{II}}\Big|_{z_1=z_K}$ $(z_K = l - l_1 - 2r)$ for two segments of the string are similar to the matrices determined in Problem **28**, therefore, the equation relating vectors $\overline{\mathbf{Y}}_{00}^{\mathrm{I}} = \overline{\mathbf{Y}}_0^{\mathrm{II}}\Big|_{z=0}$ and $\overline{\mathbf{Y}}_{0K}^{\mathrm{II}} = \overline{\mathbf{Y}}_0^{\mathrm{II}}\Big|_{z_1=z_K}$ has the form

$$\overline{\mathbf{Y}}_{0K}^{\mathrm{II}} = \mathbf{K}(p)\overline{\mathbf{Y}}_{00}^{\mathrm{I}}, \tag{8}$$

where

$$\mathbf{K}(p) = \mathbf{K}^{\mathrm{II}}(z_1)\mathbf{A}\mathbf{K}^{\mathrm{I}}(l_1).$$

Since $Y_{0K}^{\mathrm{II}} = Y_{00}^{\mathrm{I}} = 0$, from (8) we find the following equation to determine the frequencies $p_j$:

$$k_{12}(p) = 0,$$

where $k_{12}(p)$ is the element of matrix $\mathbf{K}$.

**30** After application of the force $P_0$, with allowance for the initial tension $Q_{10}$, the string tensions $Q_{11}$ and $Q_{12}$ on the segments I and II, respectively, are equal to

$$Q_{11} = Q_{10} + P_0\left(1 - \frac{l_1}{l}\right); \quad Q_{12} = Q_{10} - P_0\frac{l_1}{l}$$

The equation of small vibrations has the following form for every segment of the string

$$\frac{\partial^2 Y^{\mathrm{I}}}{\partial t^2} - a_1^2\frac{\partial^2 Y^{\mathrm{I}}}{\partial z^2} = 0; \quad \frac{\partial^2 Y^{\mathrm{II}}}{\partial t^2} - a_2^2\frac{\partial^2 Y^{\mathrm{II}}}{\partial z^2} = 0. \tag{1}$$

where $a_1^2 = Q_{11}/m_0$; $a_2^2 = Q_{12}/m_0$.

Assuming $Y^{\mathrm{I}} = Y_0^{\mathrm{I}} e^{ipt}$ and $Y^{\mathrm{II}} = Y_0^{\mathrm{II}} e^{ipt*}$, after transformations we get the following solution in the vector for (see Problem **27**)

$$\overline{\mathbf{Y}}_0^{\mathrm{I}}(z) = \mathbf{K}^{\mathrm{I}}(z)\overline{\mathbf{Y}}_{00}^{\mathrm{I}}; \quad \overline{\mathbf{Y}}_0^{\mathrm{II}}(z_1) = \mathbf{K}^{\mathrm{II}}(z_1)\overline{\mathbf{Y}}_{00}^{\mathrm{II}}. \tag{2}$$

Since the conditions

$$Y_0^{\mathrm{II}}\Big|_{z_1=0} = Y_0^{\mathrm{I}}\Big|_{z=l_1}; \quad Y_0'^{\mathrm{II}}\Big|_{z_1=0} = Y_0'^{\mathrm{I}}\Big|_{z=l_1},$$

should be satisfied at $z = l_1$, the matrix of transition to the segment II is equal to

$$\mathbf{K}^{\mathrm{I}}(l,k) = \begin{bmatrix} \cos k_1 l_1 & \dfrac{\sin k_1 l_1}{k_1} \\ -k_1 \sin k_1 l_1 & \cos k_1 l_1 \end{bmatrix}, \tag{3}$$

where $k_1 = p/a_1$. Therefore,

$$\overline{\mathbf{Y}}_{00}^{\mathrm{II}} = \mathbf{K}^{\mathrm{I}}(l_1,k_1)\,\overline{\mathbf{Y}}_{00}^{\mathrm{I}}.$$

On the second segment we have

$$\overline{\mathbf{Y}}_0^{\mathrm{II}}(z_1) = \mathbf{K}^{\mathrm{II}}(z_1,k_2)\,\mathbf{K}^{\mathrm{I}}(l_1,k_1)\,\overline{\mathbf{Y}}_{00}^{\mathrm{I}}.$$

Since the condition $Y_0^{\mathrm{II}} = 0$ must be met at $z_1 = l - l_1$, then

$$k_{12} = 0,$$

where $k_{12}$ is the element of the matrix $\mathbf{K} = \mathbf{K}^{\mathrm{II}}(l-l_1,k_2)\,\mathbf{K}^{\mathrm{I}}(l_1,k_1)$. After some transformations we get the following equation for determination of frequencies:

$$a_1 \cos\left[\frac{P}{a_2}(l-l_1)\right] \sin\left(\frac{P}{a_1}l_1\right) + a_2 \sin\left[\frac{P}{a_2}(l-l_1)\right] \cos\left(\frac{P}{a_1}l_1\right) = 0. \tag{4}$$

**31** In this problem, the 'kinematic' excitation of vibrations takes place, which presents some difficulties for a solver. Therefore, we describe below the general method of solving similar problems that can be used for both exact and approximate solutions.

We assume that an unknown vertical strength $P(t)$ is applied to the string at point $K$, generating vibrations and displacing the point $K$ according to the law $y_K(t)$. This allows us to consider the problem in question as a problem in forced vibrations. The equation of forced vibrations of a string has the form

$$L(y) = m_0\frac{\partial^2 y}{\partial t^2} - Q_{10}\frac{\partial^2 y}{\partial z^2} - P(t)\,\delta(z - z_K) = 0. \tag{1}$$

We seek the approximate solution to equation (1) in the form

$$\tilde{y}(z,t) = f_1 \sin\frac{\pi z}{l} + f_2 \sin\frac{2\pi z}{l}. \tag{2}$$

Making transformation with the use of the Galerkin method we arrive at two equations

$$\begin{aligned} &\ddot{f}_1 + P_1^2 f_1 = P(t)\sin\frac{\pi z_K}{l} \qquad (P = P_0\cos\omega t)\,; \\ &\ddot{f}_2 + P_2^2 f_2 = P(t)\sin\frac{2\pi z_K}{l}. \end{aligned} \tag{3}$$

Under steady-state vibrations we have

$$f_1 = \frac{P_0\sin\frac{\pi z_K}{l}}{(p_1^2-\omega^2)}\cos\omega t; \quad f_2 = \frac{P_0\sin\frac{\pi z_K}{l}}{(p_2^2-\omega^2)}\cos\omega t. \tag{4}$$

As a result, we find

$$\tilde{y}(z,t) = P_0\left(\frac{\sin\frac{\pi z_K}{l}}{(p_1^2-\omega^2)}\sin\frac{\pi z}{l} + \frac{\sin\frac{2\pi z_K}{l}}{(p_2^2-\omega^2)}\sin\frac{2\pi z}{l}\right). \tag{5}$$

Since at $z = z_K$ the amplitude of displacement of the point $K$ should be equal to $y_{K0}$, then we get from relationship (5) the following equation for determination of the unknown amplitude of force $P_0$:

$$y_{K0} = P_0\left[\frac{\left(\sin\frac{\pi z_K}{l}\right)^2}{p_1^2-\omega^2} + \frac{\left(\sin\frac{2\pi z_K}{l}\right)^2}{p_2^2-\omega^2}\right] = P_0 a. \tag{6}$$

Now, having determined $P_0$ from equation (6), we get the solution to the problem stated in the form

$$\tilde{y}(z,t) = \left[\frac{y_{K0}\sin\frac{\pi z_K}{l}}{a\,(p_1^2-\omega^2)}\sin\frac{\pi z}{l} + \frac{y_{K0}\sin\frac{2\pi z_K}{l}}{a\,(p_2^2-\omega^2)}\sin\frac{2\pi z}{l}\right]\cos\omega t.$$

**32** The equation of vibrations of a moving string, taking its interaction with a point-like mass $M$ into account, has the form (see solution to Problem **11**)

$$L(y) = \frac{\partial^2 y}{\partial z^2} + 2w\frac{\partial^2 y}{\partial t\partial z} - \left(\frac{Q_{10}}{m_0} - w^2\right)\frac{\partial^2 y}{\partial t^2} + \frac{M}{m_0}\frac{\partial^2 y}{\partial t^2}\delta\left(z - z_k\right) = 0. \tag{1}$$

The solution to equation (1) we seek in the form (two-term approximation)

$$\tilde{y} = f_1 \sin\frac{\pi z}{l} + f_2 \sin\frac{2\pi z}{l}. \tag{2}$$

Taking advantage of the Galerkin method, we have the following relations

$$\int_0^l L(\widetilde{y}) \sin \frac{\pi z}{l} \mathrm{d}z = 0; \quad \int_0^l L(\widetilde{y}) \sin \frac{2\pi z}{l} \mathrm{d}z = 0,$$

from which two equations for $f_1$ and $f_2$ are derived

$$\begin{aligned} a_{11}\ddot{f}_1 + a_{12}\ddot{f}_2 - b_{12}\dot{f}_2 + c_{11}f_1 &= 0; \\ a_{21}\ddot{f}_1 + a_{22}\ddot{f}_2 + b_{21}\dot{f}_1 + c_{22}f_2 &= 0, \end{aligned} \tag{3}$$

where $a_{11} = 1 + \alpha_1$; $a_{12} = a_{21} = \alpha$; $a_{22} = 1 + \alpha_2$; $b_{12} = -b_{21} = \frac{8}{3}\frac{w}{l}$; $c_{11} = \left(\frac{Q_{10}}{m_0} - w^2\right)\left(\frac{\pi}{l}\right)^2$; $c_{22} = \left(\frac{Q_{10}}{m_0} - w^2\right)\left(\frac{2\pi}{l}\right)^2$; $\alpha_1 = \frac{M}{m_0}\sin^2\frac{\pi z_K}{l}$; $\alpha = \frac{M}{m_0}\sin\frac{\pi z_K}{l}\sin\frac{2\pi z_K}{l}$; $\alpha_2 = \frac{M}{m_0}\sin^2\frac{2\pi z_K}{l}$.

Assuming

$$f_1 = f_{10}\mathrm{e}^{\mathrm{i}p\,t}; \quad f_2 = f_{20}\mathrm{e}^{\mathrm{i}p\,t},$$

let us write based on system (3) the following equation to determine the frequencies $p_j$

$$\begin{vmatrix} \left(-p^2a_{11} + c_{11}\right) & \left(-b_{12}\mathrm{i}p - \alpha p^2\right) \\ \left(b_{22}\mathrm{i}p - \alpha p^2\right) & \left(-p^2a_{22} + c_{22}\right) \end{vmatrix} = 0, \tag{4}$$

or

$$p^4 - \frac{a_{11}c_{22} + a_{22}c_{11} + b_{12}^2}{a_{11}a_{22} - \alpha^2}p^2 + \frac{c_{11}c_{22}}{a_{11}a_{22} - \alpha^2} = 0. \tag{5}$$

From (5) we find the desired frequencies

$$p_1 = \sqrt{\frac{a_1 - \sqrt{a_1^2 - 4a_2}}{2}}; \quad p_2 = \sqrt{\frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2}},$$

where $a_1 = \dfrac{a_{11}c_{22} + a_{22}c_{11} + b_{12}^2}{a_{11}a_{22} - \alpha^2}$; $\quad a_2 = \dfrac{c_{11}c_{22}}{a_{11}a_{22} - \alpha^2}$.

**33** When solving Problem **11** we have derived the equation of small vibrations of the moving branch of a gearing that was considered as a perfectly flexible rod (string). This problem differs in the fact that the moving string interacts in the section $K$ with a point-like mass $M$.

Derive the conversion matrix taking the moving string into account. Figure 2.11 illustrates the forces acting upon the mass $M$. Unlike Problem **28**, here an additional concentrated force $J_w$ appears that is caused by changing momentum of the string per unit time

$$J_w = -mw^2\left(\bar{\mathbf{e}}_{1+} - \bar{\mathbf{e}}_{1-}\right). \tag{1}$$

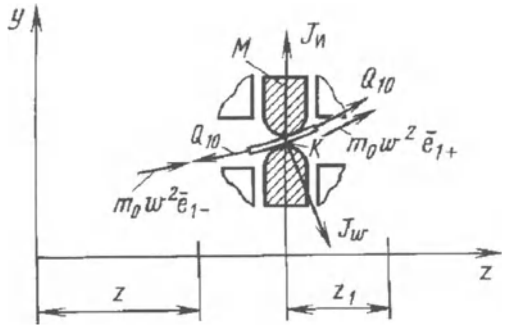

**Fig. 2.11.**

In the projection onto the vertical axis

$$J_w = -m_0 w^2 \left[y'^{\mathrm{II}}(0) - y'^{\mathrm{I}}(l_1)\right]. \tag{2}$$

The equation of dynamic equilibrium of the mass $M$, taking $J_w$ into account, has the following form

$$-M\frac{\partial^2 y_K}{\partial t^2} + \left(Q_{10} - m_0 w^2\right) y'^{\mathrm{II}}(0,t) - \left(Q_{10} - m_0 w^2\right) y'^{\mathrm{I}}(l_1,t) = 0,$$

where $y_K = y^{\mathrm{I}}(l_1,t)$.

For vertical displacements of the end of the segment $I$ and the beginning of the segment $II$ of the string the following relation holds true (the point $K$ is taken as zero point for the segment $II$)

$$y^{\mathrm{II}}(0,t) = y^{\mathrm{I}}(l_1,t). \tag{3}$$

Assuming

$$y^{\mathrm{I}}(l_1,t) = y_0^{\mathrm{I}}(l,1)\mathrm{e}^{\mathrm{i}pt}; \quad y^{\mathrm{II}}(0,t) = y_0^{\mathrm{II}}(0)\mathrm{e}^{\mathrm{i}pt}, \tag{4}$$

we get after some transformations the following transition matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -\dfrac{Mp^2}{Q_{10} - m_0 w^2} \\ 0 & 1 \end{bmatrix}. \tag{5}$$

Now write the transition matrices for segments $I$ and $II$. It was established in solution to Problem **11** that

$$y_1(\eta) = c_1 \mathrm{e}^{\lambda_1 \eta} + c_2 \mathrm{e}^{\lambda_2 \eta}; \quad (\eta = z/l), \tag{6}$$

where $\lambda_{1,2} = \mathrm{i}\dfrac{\left(a_1 \pm \sqrt{a_1^2 + 4a_2}\right)}{2}$; $a_1 = \dfrac{2wpm_0}{(Q_{10} - m_0 w^2)}$;
$a_2 = \dfrac{m_0 p^2}{(Q_{10} - m_0 w^2)}$. Differentiating equation (6) we get

$$y_1'(\eta) = c_1\lambda_1 e^{\lambda_1\eta} + c_2\lambda_2 e^{\lambda_2\eta}. \qquad (7)$$

Now, assuming $\overline{\mathbf{Z}} = (y_1', y_1)^{\mathrm{T}}$, let us write down the matrix relating the string section at the zero point with an arbitrary section on the segment $I$ $(\eta < \eta_K)$ as

$$\overline{\mathbf{Z}}(\eta) = \begin{bmatrix} y_1'(\eta) \\ y_1(\eta) \end{bmatrix} = \begin{bmatrix} \lambda_1 e^{\lambda_1\eta} & \lambda_2 e^{\lambda_2\eta} \\ e^{\lambda_1\eta} & e^{\lambda_2\eta} \end{bmatrix} \begin{bmatrix} y_1'(0) \\ y_1(0) \end{bmatrix}. \qquad (7)$$

The matrix $\mathbf{K}^{\mathrm{I}}$ (from $\eta = 0$ to $\eta = \eta_K$) is equal to

$$\overline{\mathbf{K}}^{\mathrm{I}} = \begin{bmatrix} \lambda_1 e^{\lambda_1\eta_K} & \lambda_2 e^{\lambda_2\eta_K} \\ e^{\lambda_1\eta_K} & e^{\lambda_2\eta_K} \end{bmatrix}. \qquad (8)$$

In a similar way we compose the matrix of transition from $\eta_K$ to $\eta = 1$ as

$$\overline{\mathbf{K}}^{\mathrm{II}} = \begin{bmatrix} \lambda_1 e^{\lambda_1(1-\eta_K)} & \lambda_2 e^{\lambda_2(1-\eta_K)} \\ e^{\lambda_1(1-\eta_K)} & e^{\lambda_2(1-\eta_K)} \end{bmatrix}. \qquad (9)$$

Finally, we have

$$\overline{\mathbf{Z}}(1) = \mathbf{K}^{\mathrm{II}}\mathbf{A}\mathbf{K}^{\mathrm{I}}\overline{\mathbf{Z}}(0) = \mathbf{K}(p,\omega)\overline{\mathbf{Z}}(0). \qquad (10)$$

Since $y_1(0) = 0$ at $\eta = 0$, and $y_1(1) = 0$ at $\eta = 1$, then we get from the matrix $\mathbf{K}(p, w)$

$$k_{22}(p, w) = 0. \qquad (11)$$

Now from condition (11) one determine the frequencies $p_j$ numerically as functions of $w$.

**34** Invoking the mass

$$m_0 = m_{10}\left[H(z) - H\left(z - l_1\right)\right] + m_{20}H\left(z - l_1\right), \qquad (1)$$

we get the equation of small forced vibrations of a string with allowance for a concentrated force $P(t)$

$$L(y) = m_0\frac{\partial^2 y}{\partial t^2} - Q_{10}\frac{\partial^2 y}{\partial z^2} - P(t)\delta\left(z - l_1\right) = 0. \qquad (2)$$

The approximate solution to equation (2) is sought in the form (single-term approximation)

$$y = f_1(t)\varphi_1(z), \qquad (3)$$

where $\varphi_1(z)$ is the function satisfying the boundary conditions, for example,

$$\varphi_1(z) = \sin\frac{\pi z}{l}. \qquad (4)$$

For a string having the segments with masses $m_{10}$ and $m_{20}$ the exact eigen function at $j = 1$ (see solution to Problem **27**) is equal to

$$\varphi_1(z) = \begin{cases} \sin k_{11} z, & 0 \le z \le l_1; \\ \cos k_{21}(z-l) - \dfrac{\cos k_{21}(l-l_1)}{\sin k_{21}(l-l_1)} \times \\ \quad \times \sin k_{21}(z-l_1) \sin k_{11} l_1, & l_1 < z \le l. \end{cases} \tag{5}$$

In accordance with the Galerkin method we obtain

$$\int_0^l L(\tilde{y}) \varphi_1 \mathrm{d}z = 0, \tag{6}$$

or

$$\ddot{f}_1 \left[ m_{10} \int_0^{l_1} \varphi_1^{\mathrm{I}} \varphi_1^{\mathrm{I}} \mathrm{d}z + m_{20} \int_{l_1}^{l} \varphi_1^{\mathrm{II}} \varphi_1^{\mathrm{II}} \mathrm{d}z \right] - \\ - f_1 Q_{10} \left[ \int_0^{l_1} \ddot{\varphi}_1^{\mathrm{I}} \varphi_1^{\mathrm{I}} \mathrm{d}z + \int_{l_1}^{l} \ddot{\varphi}_1^{\mathrm{II}} \varphi_1^{\mathrm{II}} \mathrm{d}z \right] - P(t) \varphi_1^{\mathrm{I}}(l_1) = 0. \tag{7}$$

After transformations at $\varphi_1(z) = \sin \dfrac{\pi z}{l}$ $(z_K = l_1)$ we have

$$\ddot{f}_1 + p_1^2 f_1 = \varphi_1^{\mathrm{I}}(l_1) P_0 \cos \omega t_0, \tag{8}$$

where $p_1$ is the approximate value of the first frequency. (If for the function $\varphi_1(z)$ we take eigen function (5), then we have the exact value of the first frequency.)

Under a steady-state regime

$$f_1 = \frac{\varphi_1(l_1) P_0 \cos \omega t}{(p_1^2 - \omega^2)}. \tag{9}$$

Expression (9) holds true provided that $\omega \ne p_1$, therefore,

$$\tilde{y} = \frac{\varphi_1(l_1) \varphi_1(z) P_0}{p_1^2 - \omega^2} \cos \omega t.$$

The amplitude $y_K$ of the string vertical displacement at point $K$ is equal to

$$y_K = \frac{[\varphi_1(l_1)]^2 P_0}{p_1^2 - \omega^2}. \tag{10}$$

Now we find the problem's exact solution, assuming that

$$y(z,t) = \sum_{j=1}^{\infty} f_j(t)\varphi(z), \tag{11}$$

where $\varphi_j(z)$ are the eigen functions (see solution to Problem **27**). Substituting (11) into equation (2) and taking advantage of the Galerkin method, we have

$$\int_0^l L(\tilde{y})\,\varphi_j(z)\mathrm{d}z = 0 \qquad (j = 1, 2, \ldots n). \tag{12}$$

From (12) we derive the following system of equations for functions $f_j(t)$

$$\ddot{f}_j + p_j^2 f_j = P(t)\,\varphi_j^{\mathrm{I}}(l_1) \quad (j = 1, 2, \ldots n). \tag{13}$$

From (13) we determine $f_j$ under steady-state vibrations

$$f_j(t) = \frac{\varphi_1^{\mathrm{I}}(l_1)\,P_0 \cos\omega t}{\left(p_j^2 - \omega^2\right)}. \tag{14}$$

The exact solution has the form

$$y(z,t) = \sum_{j=1}^{\infty} \frac{\varphi_j^{\mathrm{I}}(l_1)\,P_0}{\left(p_j^2 - \omega^2\right)} \cos\omega t\,\varphi_j(z).$$

**35** Changing the action of an elastic constraint on the rod by a concentrated force $N$ (Fig. 2.12) we arrive at the equation of small forced vibrations

$$L(y) = m_0 \frac{\partial^2 y}{\partial t^2} - Q_{10} \frac{\partial^2 y}{\partial z^2} - N\delta\left(z - l_{K_1}\right) - P\delta\left(z - l_{K_2}\right) = 0, \tag{1}$$

where $N = -cy\big|_{z=l_{K_1}}$.

The approximate solution is sought in the form

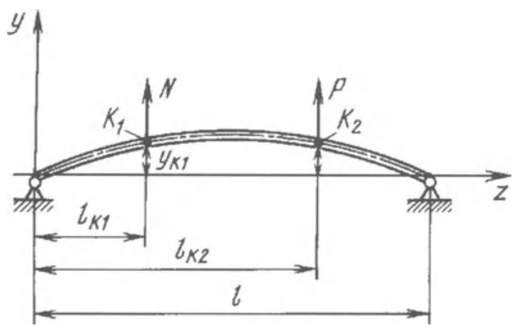


**Fig. 2.12.**

$$\tilde{y} = f_1(t)\varphi_1(z) + f_2(t)\varphi_2(z). \tag{2}$$

As functions $\varphi_1$ and $\varphi_2$ we choose the functions $\sin\dfrac{\pi z}{l}$ and $\sin\dfrac{2\pi z}{l}$, respectively. Substituting $\tilde{y}$ into (1) and making use of an algorithm of the Galerkin method we write down the following relations

$$\int\limits_0^l L(\tilde{y})\sin\frac{\pi z}{l}\mathrm{d}z = 0, \quad \int\limits_0^l L(\tilde{y})\sin\frac{2\pi z}{l}\mathrm{d}z = 0, \tag{3}$$

from which we derive the differential equations for $f_1$ and $f_2$:

$$\begin{aligned} \ddot{f}_1 + a_{11}f_1 + a_{12}f_2 &= b_1; \\ \ddot{f}_2 + a_{21}f_1 + a_{22}f_2 &= b_2. \end{aligned} \tag{4}$$

Here

$$\begin{aligned} a_{11} &= \frac{Q_{10}}{m_0}\left(\frac{\pi}{l}\right)^2 + \frac{2c}{lm_0}\left(\sin\frac{\pi l_{K_1}}{l}\right)^2; \\ a_{12} = a_{21} &= \frac{2c}{lm_0}\sin\frac{\pi l_{K_1}}{l}\sin\frac{2\pi l_{K_1}}{l}; \\ a_{22} &= \frac{Q_{10}}{m_0}\left(\frac{2\pi}{l}\right)^2 + \frac{2c}{lm_0}\left(\sin\frac{2\pi l_{K_1}}{l}\right)^2; \\ b_1 &= P_0\sin\frac{\pi l_{K_2}}{l}\cos\omega t; \\ b_2 &= P_0\sin\frac{2\pi l_{K_2}}{l}\cos\omega t. \end{aligned}$$

Let us represent system of equations (4) in the vector form

$$\ddot{\bar{\mathbf{f}}} + \mathbf{A}\bar{\mathbf{f}} = \bar{\mathbf{b}}, \tag{5}$$

where $\bar{\mathbf{b}} = \bar{\mathbf{b}}_0\cos\omega t$; $\bar{\mathbf{b}}_0 = [b_{01}, b_{02}]^{\mathrm{T}}$. Under a steady-state regime, we have

$$\bar{\mathbf{f}} = \bar{\mathbf{f}}_0\cos\omega t. \tag{6}$$

Substituting (6) into equation (5), we find

$$\bar{\mathbf{f}}_0 = \left[\mathbf{A} - \omega^2 E\right]^{-1} = \mathbf{A}_1\bar{b}_0, \tag{7}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} \dfrac{a_{22}-\omega^2}{\triangle} & -\dfrac{a_{12}}{\triangle} \\ -\dfrac{a_{21}}{\triangle} & \dfrac{a_{11}-\omega^2}{\triangle} \end{bmatrix};$$

$$\triangle = \left(a_{11} - \omega^2\right)\left(a_{22} - \omega^2\right) - a_{12}a_{21}.$$

From (7), we determine the components of vector $\overline{f}_0$:

$$f_{01} = \frac{(a_{11} - \omega^2)}{\triangle} b_{01} - \frac{a_{12}}{\triangle} b_{02};$$

$$f_{02} = -\frac{a_{21}}{\triangle} b_{01} + \frac{(a_{11} - \omega^2)}{\triangle} b_{02},$$

where $b_{01} = P_0 \sin \frac{\pi l_{K_2}}{l}$; $b_{02} = P_0 \sin \frac{2\pi l_{K_2}}{l}$.

The approximate solution to equation (1) under steady-state vibrations, accounting for a local elastic (bilateral) constraint, has the form

$$y(z,t) = f_{01} \sin \frac{\pi z}{l} \cos \omega t + f_{02} \sin \frac{2\pi z}{l} \cos \omega t.$$

The amplitude values of the rod displacements at points $K_1$ and $K_2$ are equal, respectively, to

$$y_{K_1} = f_{01} \sin \frac{\pi l_{K_1}}{l} + f_{02} \sin \frac{2\pi l_{K_1}}{l};$$

$$y_{K_2} = f_{01} \sin \frac{\pi l_{K_2}}{l} + f_{02} \sin \frac{2\pi l_{K_2}}{l}.$$

**36** The equation of forced vibrations of a string (see solution to Problem **35**) has the form

$$L(y) = m_0 \frac{\partial^2 y}{\partial t^2} - Q_{10} \frac{\partial^2 y}{\partial z^2} + cy\,\delta\,(z - z_{K_1}) - P_0 \cos \omega t\,\delta\,(z - z_{K_2}) = 0. \quad (1)$$

Under steady-state vibrations one can assume that

$$y = y_0(z) \cos \omega t. \quad (2)$$

As a result, we have the following equation

$$Q_{10} \frac{\partial^2 y_0}{\partial z^2} + m_0 \omega^2 y_0(z) = c y_0\,\delta\,(z - z_{K_1}) - P_0\,\delta\,(z - z_{K_2}), \quad (3)$$

whose general solution is

$$y_0 = c_1 \cos kz + c_2 \sin kz + \frac{1}{k} \int_0^z \sin k(z - h) \cdot b(h) \mathrm{d}h, \quad (4)$$

where $k = \omega \sqrt{\frac{m_0}{Q_{10}}}$ and $b = c y_0\,\delta\,(h - z_{K_1}) - P_0\,\delta\,(h - z_{K_2})$.

Taking the integral in equation (4) we obtain

$$y_0 = c_1 \cos kz + c_2 \sin kz + \\ + \frac{1}{k} c y_0 (z_{K_1}) \sin k (z - z_{K_1}) \cdot H (z - z_{K_1}) - \\ - P_0 \frac{1}{k} \sin k (z - z_{K_2}) \cdot H (z - z_{K_2}) .$$

In order to determine three unknowns $c_1$, $c_2$, and $y_0(z_{K_1})$, we have three conditions

$$\begin{aligned} &1)\ z = 0, \quad && y_0 = 0, \quad && c_1 = 0; \\ &2)\ z = l, \quad && y_0 = 0; \\ &3)\ z = z_{K_1}, \quad && y_0 = y_0(z_{K_1}), \quad && c_2 \sin k z_{K_1} - y_0(z_{K_1}) = 0. \end{aligned}$$

As a result, we get the following system of inhomogeneous equations

$$c_2 \sin k z_{K_1} - y_0 (z_{K_1}) = 0;$$
$$c_2 \sin kl + \frac{c}{k} \sin (l - z_{K_1}) \cdot y_0 (z_{K_1}) = \frac{P_0}{k} \sin k (l - z_{K_2}) ,$$

from which we find $c_2$ and $y_0(z_{K_1})$. The exact solution has the form

$$y(z,t) = \left[ c_2 \sin kz + \frac{c}{k} y_0 (z_{K_1}) \sin k (z - z_{K_1}) \cdot H (z - z_{K_1}) - \right. \\ \left. - \frac{P_0}{k} \sin k (z - z_{K_2}) \cdot H (z - z_{K_2}) \right] \cos \omega t.$$

**37** Accounting for the forces of viscous drag, the equation of small vibrations has the form

$$m_0 \frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - Q_{10} \frac{\partial^2 y}{\partial z^2} + c y \, \delta (z - z_{K_1}) - P_0 \cos \omega t \, \delta (z - z_{K_2}) = 0. \quad (1)$$

The solution to equation (1) in the case of steady-state vibrations we seek in the form

$$y = y_{01}(z) \cos \omega t + y_{02}(z) \sin \omega t. \quad (2)$$

Substituting (2) into equation (1) let us write two equations for $y_{01}$ and $y_{02}$:

$$y_{01}'' + \frac{m_0 \omega^2}{Q_{10}} y_{01} - \frac{\alpha \omega}{Q_{10}} y_{02} = \frac{c}{Q_{10}} y_{01} \, \delta (z - z_{K_1}) - \\ - \frac{P_0}{Q_{10}} \delta (z - z_{K_2}) ; \quad (3)$$
$$y_{02}'' + \frac{\alpha \omega}{Q_{10}} y_{01} + \frac{m_0 \omega^2}{Q_{10}} y_{02} = \frac{c}{Q_{10}} y_{02} \, \delta (z - z_{K_1}) .$$

Assuming $y_{01}' = x_1$, $y_{02}' = x_2$, $y_{01} = x_3$, and $y_{02} = x_4$ we obtain the equation

$$\overline{\mathbf{X}}' + \mathbf{A}\overline{\mathbf{X}} = \overline{\mathbf{b}}, \tag{4}$$

in which $\overline{\mathbf{X}} = [X_1, X_2, X_3, X_4]^{\mathrm{T}} = [y'_{01}, y'_{02}, y_{01}, y_{02}]^{\mathrm{T}}$;

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \dfrac{m_0\omega^2}{Q_{10}} & -\dfrac{\alpha\omega}{Q_{10}} \\ 0 & 0 & \dfrac{\alpha\omega}{Q_{10}} & \dfrac{m_0\omega^2}{Q_{10}} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \ \overline{\mathbf{b}} = \begin{bmatrix} \dfrac{c}{Q_{10}} y_{01}\delta_1 - \dfrac{P_0}{Q_{10}}\delta_2 \\ \dfrac{c}{Q_{10}} y_{01}\delta_1 \\ 0 \\ 0 \end{bmatrix};$$

where $\delta_1 = \delta(z - z_{K_1})$ and $\delta_2 = \delta(z - z_{K_2})$.
The solution to equation (4) has the form

$$\begin{aligned} \overline{\mathbf{X}} &= \mathbf{K}(z)\overline{\mathbf{C}} + \int_0^z \mathbf{K}(z-h)\overline{\mathbf{b}}(h)\mathrm{d}h; \\ \overline{\mathbf{C}} &= [c_1, c_2, c_3, c_4]^{\mathrm{T}}, \end{aligned} \tag{5}$$

or

$$\begin{aligned} \overline{\mathbf{X}} = \mathbf{K}(z)\overline{\mathbf{C}} + \mathbf{K}(z - z_{K_1})\overline{\mathbf{b}}_{01} H(z - z_{K_1}) + \\ + \mathbf{K}(z - z_{K_2})\overline{\mathbf{b}}_{02} H(z - z_{K_2}), \end{aligned} \tag{6}$$

where

$$\overline{\mathbf{b}}_{01} = \begin{bmatrix} \dfrac{c}{Q_{10}} y_{01}(z_{K_1}) \\ \dfrac{c}{Q_{10}} y_{02}(z_{K_1}) \\ 0 \\ 0 \end{bmatrix}; \quad \overline{\mathbf{b}}_{02} = \begin{bmatrix} -\dfrac{P_0}{Q_{10}} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The components of vector $\overline{\mathbf{X}}$ should satisfy the following boundary conditions

$$\begin{aligned} z = 0; \quad x_3 = x_4 = 0; \\ z = l; \quad x_3 = x_4 = 0; \end{aligned}$$

From the first condition $(z = 0)$ we find $c_3 = c_4 = 0$. From the second conditions $(z = l)$ we derive two equations for four unknowns $(c_1, c_2, y_{01}(z_{K_1}), y_{02}(z_{K_1}))$:

$$\begin{aligned} & k_{31}(l)c_1 + k_{32}(l)c_2 + \frac{c}{Q_{10}} y_{01}(z_{K_1}) k_{31}(l - z_{K_1}) + \\ & \quad + \frac{c}{Q_{10}} y_{02}(z_{K_1}) k_{32}(l - z_{K_1}) - \frac{P_0}{Q_{10}} k_{31}(l - z_{K_2}) = 0; \\ & k_{41}(l)c_1 + k_{42}(l)c_2 + \frac{c}{Q_{10}} y_{01}(z_{K_1}) k_{41}(l - z_{K_1}) + \\ & \quad + \frac{c}{Q_{10}} y_{02}(z_{K_1}) k_{42}(l - z_{K_1}) - \frac{P_0}{Q_{10}} k_{41}(l - z_{K_2}) = 0. \end{aligned} \tag{7}$$

Two more equations we write from the conditions $z = z_K$, $x_3 = y_{01}(z_{K_1})$ and $x_4 = y_{02}(z_{K_1})$:

$$\begin{aligned} y_{01}(z_{K_1}) &= k_{31}(z_{K_1})\,c_1 + k_{32}(z_{K_1})\,c_2; \\ y_{02}(z_{K_1}) &= k_{41}(z_{K_1})\,c_1 + k_{42}(z_{K_1})\,c_2. \end{aligned} \tag{8}$$

From systems of equations (7) and (8) we find $c_1, c_2, y_{01}(z_{K_1})$ and $y_{02}(z_{K_2})$, as well as $y_{01}(z)$ and $y_{02}(z)$:

$$\begin{aligned} y_{01}(z) = {} & k_{31}(z)\,c_1 + k_{32}(z)\,c_2 + k_{31}(z - z_{K_1})\frac{c}{Q_{10}} y_{01}(z_{K_1}) \times \\ & \times H(z - z_{K_1}) + k_{32}(z - z_{K_1})\frac{c}{Q_{10}} y_{02}(z_{K_1})\,H(z - z_{K_1}) - \\ & - \frac{P_0}{Q_{10}} k_{31}(z - z_{K_2})\,H(z - z_{K_2}); \\ y_{02}(z) = {} & k_{41}(z)\,c_1 + k_{42}(z)\,c_2 + k_{41}(z - z_{K_1})\frac{c}{Q_{10}} y_{01}(z_{K_1}) \times \\ & \times H(z - z_{K_1}) + k_{42}(z - z_{K_1})\frac{c}{Q_{10}} y_{02}(z_{K_1})\,H(z - z_{K_1}) - \\ & - \frac{P_0}{Q_{10}} k_{41}(z - z_{K_2})\,H(z - z_{K_2}). \end{aligned}$$

The exact solution to equation (1) has the form

$$y(z,t) = y_{01}(z)\cos\omega t + y_{02}(z)\sin\omega t$$

**38** The equation of small vibrations of a string lying on an elastic base with linear characteristic, taking into account the moving mass $M$ and its gravity force $Mg$, has the form

$$m_0\frac{\partial^2 y}{\partial t^2} = Q_{10}\frac{\partial^2 y}{\partial z^2} - ky + \left(-M\frac{\partial^2 y}{\partial t^2}\right)\delta(z - z_M) - Mg\,\delta(z - z_M), \tag{1}$$

where $z_M = vt$.

The approximate solution to equation (1) we seek (restricting ourselves to two-term approximation) in the form

$$\tilde{y} = \sum_{j=1}^{2} f_j(t)\sin\frac{\pi j z}{l}. \tag{2}$$

Substituting (2) into (1) we get

$$\begin{aligned} \sum_{j=1}^{2}\left[m_0\ddot{f}_j\sin\frac{\pi j z}{l} + Q_{10}\left(\frac{\pi j}{l}\right)^2\sin\frac{\pi j z}{l} f_j + k f_j\sin\frac{\pi j z}{l}\right] = \\ = -M\sum_{j=1}^{2}\ddot{f}_j\sin\frac{\pi j z}{l}\delta - Mg\delta. \end{aligned} \tag{3}$$

Using the principle of virtual displacements we obtain the equations for unknown functions $f_j(t)$:

$$\begin{aligned}
&\ddot{f}_1+Q_{10}\left(\frac{\pi}{l}\right)^2\frac{1}{m_0}f_1+\frac{k}{m_0}f_1=-\frac{M}{m_0}\frac{2}{l}\ddot{f}_1\sin\frac{\pi vt}{l}-\\
&\quad -M\frac{2}{lm_0}\ddot{f}_2\sin\frac{\pi vt}{l}\sin\frac{2\pi vt}{l}-\frac{2}{lm_0}Mg\sin\frac{\pi vt}{l};\\
&\ddot{f}_2+Q_{10}\left(\frac{2\pi}{l}\right)^2\frac{1}{m_0}f_2+\frac{k}{m_0}f_2=-\frac{M}{m_0}\frac{2}{l}\ddot{f}_1\sin\frac{\pi vt}{l}\sin\frac{2\pi vt}{l}-\\
&\quad -M\frac{2}{lm_0}\ddot{f}_2\sin\frac{2\pi vt^2}{l}-\frac{2}{lm_0}Mg\sin\frac{2\pi vt}{l},
\end{aligned}\tag{4}$$

and then find

$$\begin{aligned}
a_{11}(t)\ddot{f}_1+a_{12}(t)\ddot{f}_2+c_{11}(t)f_1&=-\frac{2Mg}{m_0 l}\sin\frac{\pi vt}{l};\\
a_{21}(t)\ddot{f}_1+a_{22}(t)\ddot{f}_2+c_{22}(t)f_2&=-\frac{2Mg}{m_0 l}\sin\frac{2\pi vt}{l}.
\end{aligned}\tag{5}$$

In the vector form equations (5) look like

$$\ddot{\overline{\mathbf{f}}}+\mathbf{A}^{-1}\mathbf{C}\overline{\mathbf{f}}=\mathbf{A}^{-1}\overline{\mathbf{f}},\tag{6}$$

where $\mathbf{A}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$; $\mathbf{C}=\begin{bmatrix}c_{11} & 0\\ 0 & c_{22}\end{bmatrix}$.

Now solve the inhomogeneous equation numerically at zero initial conditions: $t=0$, $\overline{\mathbf{f}}(0)=0$, $\dot{\overline{\mathbf{f}}}(0)=0$ on the time interval $0\le t\le l/v$. Having determined $f_j(t)$, we find the string bending deflections at the point $z(0<z<l)$ at any instant $(0<t<t_K)$, as well as the bending deflection under the moving point-like mass

$$\tilde{y}(z_M)=f_1(t)\sin\frac{\pi vt}{l}+f_2(t)\sin\frac{2\pi vt}{l}.\tag{7}$$

At the time $t_k$ the mass $M$ leaves the string, and the string begins to execute free vibrations under the following initial conditions:

$$\begin{aligned}
y(t_K,z)&=f_1(t_K)\sin\frac{\pi z}{l}+f_2(t_K)\sin\frac{2\pi z}{l};\\
\dot{y}(t_K,z)&=\dot{f}_1(t_K)\sin\frac{\pi z}{l}+\dot{f}_2(t_K)\sin\frac{2\pi z}{l}.
\end{aligned}\tag{8}$$

The equation of free vibrations of the string is

$$m_0\frac{\partial^2 y}{\partial t^2}=Q_{10}\frac{\partial^2 y}{\partial z^2}-ky.\tag{9}$$

Taking advantage of the Fourier method and assuming $y=T(t)Y(z)$, we get two equations

$$\ddot{T} + \frac{\lambda^2}{m_0} T = 0; \tag{10}$$

$$Y'' + \beta^2 Y = 0 \quad \left(\beta^2 = \frac{\lambda^2 - k}{Q_{10}}\right). \tag{11}$$

Then determine $T$ and $Y$:

$$\begin{aligned} T &= c_1 \cos \frac{\lambda}{\sqrt{m_0}} t + c_2 \sin \frac{\lambda}{\sqrt{m_0}} t; \\ Y &= c_3 \cos \beta z + c_4 \sin \beta z. \end{aligned} \tag{12}$$

The function $Y$ should satisfy the boundary conditions $z = 0$, $Y = 0$ and $z = l$, $Y = 0$, which are fulfilled at $c_3 = 0$ and $\beta l = \pi n$. This allows us to determine $\lambda$:

$$\lambda_n = \sqrt{(\pi n)^2 Q_{10} + k} \quad (n = 1, 2, \ldots).$$

As a result, we have the partial solution

$$Y^{(n)} = \left(c_1^{(n)} \cos p_n t + c_2^{(n)} \sin p_n t\right) \sin \pi n z, \tag{13}$$

where $p_n = \sqrt{(\pi n)^2 \frac{Q_{10}}{m_0} + \frac{k}{m_0}}$.
The general solution has the following form

$$Y = \sum_{n=1}^{\infty} Y^{(n)} = \sum_{n=1}^{\infty} \left(c_1^{(n)} \cos p_n t + c_2^{(n)} \sin p_n t\right) \sin \pi n z. \tag{14}$$

At $t = 0$ (we take the instant when the mass $M$ leaves the string as the zero time moment) the bending deflections and velocities of the string are known (see equations (8)):

$$Y(0, z) = Y(t_K, z); \quad \dot{Y}(0, z) = \dot{Y}(t_K, z).$$

Therefore, from equation (14) we find

$$\begin{aligned} Y(0, z) &= \sum_{n=1}^{\infty} c_1^{(n)} \sin \pi n z; \\ \dot{Y}(0, z) &= \sum_{n=1}^{\infty} p_n c_2^{(n)} \sin \pi n z. \end{aligned} \tag{15}$$

The arbitrary constants are nonzero only for $n = 1$ and $n = 2$. Taking equations (8) and (14) into account, these constants are equal to

$$\begin{aligned} c_1^{(1)} &= f_1(t_K), \quad c_1^{(2)} = f_2(t_K), \\ c_2^{(1)} &= \dot{f}_1(t_K), \quad c_2^{(2)} = \dot{f}_2(t_K). \end{aligned}$$

Finally, we get the following solution to equation (9) of free vibrations of the string after the mass $M$ has left it:

$$y = \sum_{j=1}^{2} \left( c_1^{(j)} \cos p_j t + c_2^{(j)} \sin p_j t \right) \sin \pi j z .$$

**39** The algorithm of solving the equations of free vibrations of a tape was described in Problem **11**. In this case, at $t = 0$ the bending deflections vanish, i.e., $\alpha_1(z) \equiv 0$, while vertical velocities of the tape axial points satisfy the following condition

$$\alpha_2(z) = \frac{J}{m_0} \delta \left( z - z_K \right) .$$

From system of equations (14) and (15) (see solution to Problem **11**) we have

$$\begin{aligned} +\frac{J}{m_0} y_1^{(n)} \left( z_K \right) &= \quad c_{1n} p_n J_{nn}^{(2)} + c_{2n} p_n J_{nn}^{(1)} ; \\ -\frac{J}{m_0} y_2^{(n)} \left( z_K \right) &= - c_{1n} p_n J_{nn}^{(1)} + c_{2n} p_n J_{nn}^{(2)} . \end{aligned}$$

Taking the real part of solution (10) from Problem **11**, we obtain

$$\begin{aligned} y_n = y_1(z,t) = \sum_{j=1}^{\infty} \Big[ c_{1n} \left( y_1^{(n)} \cos p_n t + y_2^{(n)} \sin p_n t \right) + \\ + c_{2n} \left( y_1^{(n)} \cos p_n t - y_2^{(n)} \sin p_n t \right) \Big] . \end{aligned}$$

## 2.2 Torsional vibrations of rods

**40** Figure 2.13 shows the shaft element of length $dz$ with the moments $M_1$ and $M_1 + dM_1$ acting upon it.

The differential equation of rotation for the shaft element has the form

$$\frac{\partial^2 \varphi}{\partial t^2} = a^2 \frac{\partial^2 \varphi}{\partial z^2} \qquad \left(a^2 = G/\varrho\right). \tag{1}$$

In order to determine the frequencies of torsional vibrations of the shaft, we seek the solution to equation (1) in the following form

$$\varphi = \varphi_1(z) \sin p\, t,$$

where $p$ is the unknown frequency of vibrations.

From (1) we have

$$\varphi_1 = c_1 \cos \frac{pz}{a} + c_2 \sin \frac{pz}{a}.$$

The function $\varphi_1$ should satisfy certain boundary conditions depending on the way of holding the end butts.

In the case shown in Fig. 1.32*a* (free end butts) the intrinsic moment in the end sections is equal to zero:

$$M_1 \Big|_{z=0} = \frac{\partial \varphi}{\partial z} G J_\varrho = \sin p\, t \cdot G J_\varrho \frac{\partial \varphi_1}{\partial z} = 0.$$

This allows us to derive the conditions $\left.\dfrac{\partial \varphi}{\partial z}\right|_{z=0;\, l} = 0$. Hence,

$$c_2 \frac{p}{a} = 0, \tag{2}$$

$$c_1 \frac{p}{a} \sin \frac{p}{a} l = c_2 \frac{p}{a} \cos \frac{p}{a} l = 0. \tag{3}$$

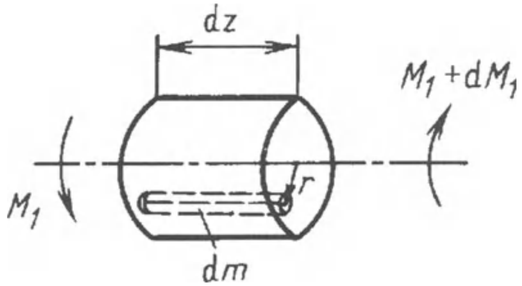


**Fig. 2.13.**

It follows from relation (2) that $c_2 = 0$ , while from equation (3) $\sin \frac{p}{a} l = 0$ (since $c_1$ should be different from zero). This is possible if $pl/a = \pi n \quad (n = 1, 2, \ldots)$.

The frequencies of torsional vibrations of a free shaft (see Fig. 1.32*a*) are

$$p_n = \frac{\pi n}{l}\sqrt{\frac{G}{\varrho}} \quad (n = 1, 2, \ldots).$$

If one of shaft sections is fixed (see Fig. 1.32*b*), the boundary conditions have the form

$$z = 0, \quad \varphi_1 = 0;$$
$$z = l, \quad \frac{\partial \varphi}{\partial z} = 0.$$

After similar calculations, we have frequencies of torsional vibrations of the shaft

$$p_n = \frac{2n-1}{2l}\sqrt{\frac{G}{\varrho}} \qquad (n = 1, 2, \ldots).$$

For the case of holding the both end butts, the boundary conditions have the form (see Fig.1.32*c*)

$$z = 0, \quad \varphi_1 = 0;$$
$$z = l, \quad \varphi_1 = 0,$$

and the frequencies of vibrations

$$p_n = \frac{\pi n}{l}\sqrt{\frac{G}{\varrho}} \qquad (n = 1, 2, \ldots).$$

**41** The velocity of propagation of a shear wave is

$$a = \sqrt{G/\varrho}.$$

Upon substituting the numerical values we have $a = 3.2$ m/s.

**42** Let us change the spring for an equivalent rod of the round cross section, equating their torsional rigidities

$$\frac{GJ_\rho}{l} = \frac{E\pi d^4/64}{\pi D i},$$

where $J_\rho$ is the geometric moment of inertia for the equivalent rod section.

The moment of inertia of the equivalent rod is equal to the moment inertia of the spring mass:

$$\varrho J_\rho l = \varrho \frac{\pi d^2}{4} \pi D i \frac{D^2}{4}.$$

(The spring has a small angle of lead, therefore, it can be schematically represented as consisting of $i$ coils. The moment of inertia of a coil is $J = mD^2/4$, where $m = \varrho\left(\pi d^2/4\right)\pi D$ is the coil mass, and $Ji$ is the moment of inertia of the entire spring.)

The velocity of propagation of a shear wave

$$a = \frac{ld}{2\pi i D^2}\sqrt{\frac{E}{\varrho}}.$$

After substitution of numerical values we find $a \approx 4.1$ m/s.

Now the frequencies of torsional vibrations of the spring (the spring is equivalent to a rod, one end butt of which is rigidly fixed)

$$p_n = \frac{2n-1}{2l}\frac{dl}{2\pi i D^2}\sqrt{\frac{E}{\varrho}} \qquad (n = 1, 2, \ldots).$$

**43** The differential equation of small torsional vibrations of a shaft (see Problem **40**) is written as

$$\frac{\partial^2\varphi}{\partial t^2} = a^2\frac{\partial^2\varphi}{\partial z^2} \qquad \left(a^2 = G/\varrho\right). \tag{1}$$

Solution of (1) is searched in the form $\varphi = \varphi_1(z)\sin pt$ that permits, after substituting the solution into (1), to get an equation for $\varphi_1$:

$$\frac{\partial^2\varphi_1}{\partial z^2} + \frac{p^2\varphi_1}{a^2} = 0. \tag{2}$$

The solution to this equation has the form

$$\varphi_1 = c_1\cos\frac{p}{a}z + c_2\sin\frac{p}{a}z.$$

The moments of inertia of the disks applied to the end butts of the shaft, and this allows us to derive two boundary conditions

$$GJ_\rho\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = J_1\left(\frac{\partial^2\varphi}{\partial t^2}\right)_{z=0}; \tag{3}$$

$$GJ_\rho\left(\frac{\partial\varphi}{\partial z}\right)_{z=l} = -J_2\left(\frac{\partial^2\varphi}{\partial t^2}\right)_{z=l}, \tag{4}$$

where $J_\rho$ is the polar moment of inertia for a shaft section.

The fulfilled boundary conditions (3) and (4) allow one to get the following system of homogeneous equations for $c_1$ and $c_2$:

$$GJ_\rho \frac{p}{a} c_2 + J_1 p^2 c_1 = 0;$$

$$\left(GJ_\rho \frac{p}{a} \cos \frac{p}{a} l + J_2 p^2 \sin \frac{p}{a} l\right) c_2 -$$
$$- \left(GJ_\rho \frac{p}{a} \sin \frac{p}{a} l + J_2 p^2 \cos \frac{p}{a} l\right) c_1 = 0.$$

Equating the determinant of this system to zero, we have the following equation for determination of the frequencies of free vibrations:

$$\tan \beta = \frac{\beta(m+n)}{mn\beta^2 - 1}, \tag{5}$$

where $\beta = p\,l/a$; $m = J_1/J_0$; $n = J_2/J_0$, and $J_0 = \varrho l J_\rho$. The formula for the frequencies of the disks' vibrations derived in solution to Problem **233** in Part I follows from (5) as a specific case. To this end we represent (5) in the form

$$\frac{\tan \beta}{\beta} = \frac{m+n}{mn\beta^2 - 1}$$

and pass to the limit at $\varrho \to 0$.

Then we express $m, n,$, and $\beta$ through the system parameters:

$$\frac{\tan \dfrac{p\,l\sqrt{\varrho}}{\sqrt{G}}}{\sqrt{\varrho}} = \frac{(J_1 + J_2)}{J_\rho \sqrt{G} \left(\dfrac{J_1 J_2 p^2}{{J_\rho}^2 G} - \varrho\right)}.$$

In the limit (at $\varrho \to 0$) we have

$$p^2 = (J_1 + J_2) \frac{GJ_\rho}{J_1 J_2 l}.$$

## 2.3 Extensional vibrations of rods

**44** Figure 2.14 shows a rod element taken at an arbitrary distance $z$. The inertia force $\mathrm{d}J = -\mathrm{d}z \cdot F\varrho \dfrac{\mathrm{d}^2 u}{\mathrm{d}t^2}$ and the forces $N$ and $N + \mathrm{d}N$ act upon the rod element.

Taking advantage of the d'Alembert's principle (assuming $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} = \dfrac{\partial^2}{\partial t^2}$), we have

$$\mathrm{d}z \cdot m_0 \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \mathrm{d}N \qquad (m_0 = F\varrho)\,. \tag{1}$$

Substituting $N = EF\dfrac{\partial u}{\partial z}$ into (1), we get

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial z^2} \qquad \left(a^2 = EF/m_0\right). \tag{2}$$

We seek the solution to equation (2) in the form

$$u = u(z) \sin p\,t, \tag{3}$$

where $p$ is the frequency of vibrations of the rod.

After substitution of (3) into equation (2) we find

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + \frac{p^2 u_1}{a^2} = 0. \tag{4}$$

The solution to equation (4) has the form

$$u_1 = c_1 \cos \frac{pz}{a} + c_2 \sin \frac{pz}{a}.$$

Consider the case of a free rod (see Fig. 1.32$a$). The solution to the equation should satisfy the boundary conditions (the force $N$ at the rod ends is equal to zero at $z = 0$):

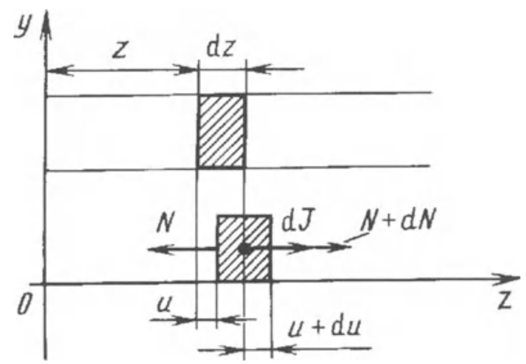


**Fig. 2.14.**

$$N = EF\frac{\partial u}{\partial z} = EF\sin pt\frac{\mathrm{d}u_1}{\mathrm{d}z} = 0,$$

i.e., $\left.\frac{\mathrm{d}u_1}{\mathrm{d}z}\right|_{z=0} = 0$. Similarly, for the second end section we have $\left.\frac{\mathrm{d}u_1}{\mathrm{d}z}\right|_{z=l} = 0$. The arbitrary constants $c_1$ and $c_2$ cannot be equal to zero simultaneously (then $u = 0$ and no motion takes place). Therefore, in order that the boundary conditions would be met, it is necessary that $c_2 = 0$ and $\sin\frac{pl}{a} = 0$, i.e., $\frac{pl}{a} = \pi n$ $(n = 1, 2, \ldots)$.

For the case shown in Fig. 1.32*a* the frequency of rod vibrations is equal to

$$p_n = \frac{\pi n}{l}\sqrt{\frac{EF}{m_0}} \qquad (n = 1, 2, \ldots).$$

The boundary conditions for the case presented in Fig. 1.32*b* have the form

$$z = 0, \quad u_1 = 0;$$
$$z = l, \quad \frac{\mathrm{d}u_1}{\mathrm{d}z} = 0,$$

and the frequencies of vibrations are determined by the expression

$$p_n = \frac{2n-1}{2l}\sqrt{\frac{EF}{m_0}} \qquad (n = 1, 2, \ldots).$$

Accordingly, for the case shown in Fig. 1.32*c*

$$z = 0, \quad u_1 = 0;$$
$$z = l, \quad u_1 = 0$$

and

$$p_n = \frac{\pi n}{l}\sqrt{\frac{EF}{m_0}} \qquad (n = 1, 2, \ldots).$$

**45** The velocity of propagation of perturbations

$$a = \sqrt{EF/m_0} = \sqrt{E/\varrho}.$$

After substitution of numerical values we obtain $a = 5 \cdot 10^3$ m/s.

**46** In the case considered, under free vibrations (see solution to Problem **44**) the force equal to $\mathrm{d}z \cdot q(z, t)$ is added to the forces acting upon the rod element, and the differential equation of rod vibrations takes on the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial z^2} + \frac{q}{m_0}.$$

**47** In the case under consideration (see solution to Problem **44**)

$$N = EF(z)\frac{\partial u}{\partial z} \qquad \Big(m_0 = \varrho F(z)\Big).$$

The equation of extensional vibrations assumes the form

$$\varrho F(z)\frac{\partial^2 u}{\partial t^2} = \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}\left[EF(z)\frac{\partial u}{\partial z}\right].$$

**48** The boundary conditions have the form

$$z = 0, \quad \frac{\partial u}{\partial z}EF = cu;$$
$$z = l, \quad \frac{\partial u}{\partial z}EF = 0.$$

The sign of the elastic force can be taken from Fig. 1.37*b*, where a rod element is shown close to the elastic fixation with a positive direction of the inner strength $N$ (assumed at the equation derivation) and a positive displacement of the end butt.

After transformations, we get the equation for determination of the frequency of rod vibrations

$$\tan k = \frac{c}{kEF/l} \qquad (k = pl/a).$$

or, for the case considered, at $c = EF/l$

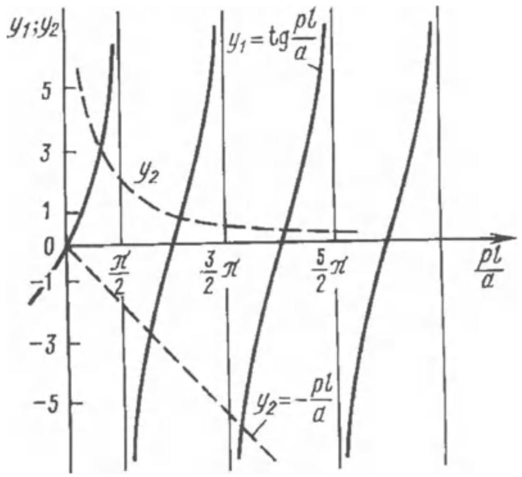


**Fig. 2.15.**

$$\tan k = \frac{1}{k}. \tag{1}$$

Equation (1) has the following roots: $k_1 \approx 3\pi/8$; $k_2 \approx 9\pi/8$; and $k_3 \approx 33\pi/16$ (Fig. 2.15). Hence, the frequencies of vibrations are equal to

$$p_1 = \frac{3}{8}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}; \quad p_2 = \frac{9}{8}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}; \quad p_3 = \frac{33}{16}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}.$$

**49** In the general case (for an arbitrary stiffness $c$), the equation of frequencies has the form

$$\tan k = -\frac{EFk}{lc}.$$

Since $c = EF/l$, then

$$\tan k = -k. \tag{1}$$

Equation (1), when solved graphically (see Fig. 2.15), has the following roots: $k_1 \approx 2\pi/3$ and $k_2 \approx 3\pi/2$. Hence, the frequencies are

$$p_1 \approx \frac{2}{3}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}; \qquad p_2 \approx \frac{3}{2}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}.$$

**50** In the case considered, we have the following boundary conditions:

$$z = 0, \quad \frac{\partial u}{\partial z}EF = c_1 u;$$
$$z = l, \quad \frac{\partial u}{\partial z}EF = -c_2 u.$$

The equation of frequencies has the following form

$$\tan k = -\frac{kEF\left(c_2 + c_1\right)/l}{c_1 c_2 - \left(EF/l\right)^2 k^2} \qquad \left(k = \frac{pl}{a}\right).$$

Hence, the equation has the roots $k_1 \approx 1.25$ and $k_2 \approx 3.5$, and the frequencies of vibrations

$$p_1 \approx \frac{1.25}{l}\sqrt{\frac{EF}{m_0}}; \qquad p_2 \approx \frac{3.5}{l}\sqrt{\frac{EF}{m_0}}.$$

**51** At the initial instant $u \equiv 0$ and $\dfrac{\partial u}{\partial t} = -v$, therefore, after some transformations we find

$$u = -\frac{8vl}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{\pi nz}{2l}\sin\frac{\pi nat}{2l} \qquad (n = 1, 3, 5, \ldots).$$

The displacement of the right end face $(z = l)$:

$$u(l,t) = \frac{8v\,l}{\pi^2 a}\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin\frac{\pi nat}{l} \qquad \left(a = \sqrt{\frac{EF}{m_0}}\right).$$

The displacement of the right end face reaches its maximum value at the time $t_1 = l/a$ $\left(\text{at this moment } \sin\frac{\pi nat_1}{2l} = (-1)^{\frac{n-1}{2}}\right)$.

The displacement is

$$u_{max} = \frac{8v\,l}{\pi^2 a}\left(1 + \frac{1}{9} + \frac{1}{25} + \ldots\right) = -\frac{v\,l}{a}.$$

The axial strength is

$$N = EF\frac{\partial u}{\partial z} = -EF\frac{8v\,l}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{\pi n}{2l}\right)\cos\frac{\pi nz}{2l}\sin\frac{\pi nat}{2l}.$$

And, finally, the maximum value of strength

$$N_{max} = -\frac{4vEF}{\pi a}\sum_{n=1}^{\infty}\frac{1}{2n-1}(-1)^{n+1} = -\frac{vEF}{a} = -v\sqrt{EFm_0}.$$

**52** At the initial instant

$$u \equiv 0; \quad \dot{u} = \begin{cases} 0 & \text{for } 0 \le z \le l/2; \\ -v & \text{for } l/2 < z \le l. \end{cases}$$

The expression for the displacement of an arbitrary section of the rod assumes the form

$$u = \sum_{n=1}^{\infty} c_n \sin\frac{\pi nz}{2\,l}\sin\frac{\pi nat}{2\,l} \qquad (n = 1, 3, 5, \ldots).$$

Since $\dot{u} = \dot{u}_0$ at $t = 0$, then the coefficients of the series are equal to

$$c_n = -\frac{8v\,l}{\pi^2 an^2}\cos\frac{\pi n}{4} \qquad (n = 1, 3, 5, \ldots).$$

The axial strength under rod vibrations

$$N = EF\frac{\partial u}{\partial z} =$$
$$= -\frac{4EFv}{a\pi}\sum_{n}\frac{1}{n}\cos\frac{\pi n}{4}\cos\frac{\pi nz}{2\,l}\sin\frac{\pi nat}{2\,l} \qquad (n = 1, 3, 5, \ldots).$$

Assuming $z = l/2$, we get the axial strength at the place of conjunction of two rods:

$$N = -\frac{4EFv}{a\pi}\sum_n \frac{1}{n}\cos^2\frac{\pi n}{4}\sin\frac{\pi nat}{2l} \quad (n = 1, 3, 5, \ldots).$$

**53** For every part of the rod the following equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial z^2} \tag{1}$$

is valid.

Seeking the solution to equation (1) in the form $u = u_1(z)\sin pt$ we get

$$u_1 = c_1\cos\frac{pz}{a} + c_2\sin\frac{pz}{a}. \tag{2}$$

The amplitude of the longitudinal force

$$N = EF\frac{\mathrm{d}u_1}{\mathrm{d}z}.$$

Invoke the following designations: $u_{11}$ and $N_1$ are the displacement and the longitudinal force at the first segment, $u_{12}$ and $N_2$ are the same quantities at the second segment.

At the first segment $N_1 = 0$ and, therefore, $c_2 = 0$, i.e., at $z = 0$

$$u_{11} = c_1\cos\frac{pz}{a}; \qquad N_1 = c_1.$$

At the end of the first segment

$$u_{11} = c_1\cos\frac{pl_1}{a}; \qquad N_1 = -\frac{p}{a}EFc_1\sin\frac{pl_1}{a}.$$

One can choose any value for the amplitude value of the displacement of the free end butt, for example, $u_{11}(0) = 1$, then $c_1 = 1$. At the second segment

$$u_{12}(l_1) = u_{11}(l_1) = \cos\frac{pl_1}{a};$$

$$N_1(l_1) = N_2(l_1) = -EF_1\frac{p}{a}\sin\frac{pl_1}{a}.$$

Having determined $c_1$ and $c_2$ in solution (2) for the second segment, where $z$ changes from 0 to $l_2$, we obtain for $u_{12}(z)$ and $N_2(z)$ the following equations:

$$u_{12}(z) = -\frac{F_1}{F_2}\sin\frac{pl_1}{a}\sin\frac{pz}{a} + \cos\frac{pl_1}{a}\cos\frac{pz}{a};$$

$$N_2(z) = -\frac{p}{a}EF_1\sin\frac{pl_1}{a}\cos\frac{pz}{a} - \frac{p}{a}EF_2\cos\frac{pl_1}{a}\sin\frac{pz}{a}.$$

The condition $u_{12}(l_2) = 0$ entails the following transcendent equation of frequencies

$$\frac{F_1}{F_2}\tan\frac{l_1 k}{l}\cot\frac{l_2 k}{l} \qquad \left(k=\frac{pl}{a}\right). \tag{3}$$

Solving equation (3) (for example, graphically), one can obtain the following values of the first four roots: $k_1 = 1.89; k_2 = 4.53; k_3 = 7.85$; and $k_4 = 11.2$. Thus, the first four frequencies of rod vibrations are equal to

$$p_1 = \frac{1.89}{l}\sqrt{\frac{E}{\varrho}}; \qquad p_2 = \frac{4.53}{l}\sqrt{\frac{E}{\varrho}};$$
$$p_3 = \frac{7.85}{l}\sqrt{\frac{E}{\varrho}}; \qquad p_4 = \frac{11.2}{l}\sqrt{\frac{E}{\varrho}}.$$

**54** The equation has the following form

$$\frac{F_1}{F_2}\tan\frac{l_1 k}{l} = -\tan\frac{l_2 k}{l} \qquad \left(k=\frac{pl}{a}\right).$$

**55** We seek the solution to the equation of extensional vibrations of the rod using the Galerkin method and assuming

$$u = f(t)\,u_1(z) = f(t)\,u_0\left(1 - z^2/l^2\right). \tag{1}$$

The function $u_1(z)$ satisfies the boundary conditions of the problem:

$$z = 0, \qquad \frac{\mathrm{d}u_1}{\mathrm{d}z} = 0;$$
$$z = l, \qquad u_1 = 0.$$

Substituting solution (1) into the equation of rod vibrations, we get (according to the Galerkin method)

$$m_0\ddot{f}\int_0^l u_2 u_1 \mathrm{d}z - f\int_0^l \frac{\mathrm{d}}{\mathrm{d}z}\left[EF(z)\frac{\mathrm{d}u_1}{\mathrm{d}z}\right]u_1\mathrm{d}z = 0.$$

After calculations we have

$$\ddot{f} + \frac{10}{3}\frac{EF_0}{l^2 m_0}f = 0.$$

The first frequency of rod vibrations (in the first approximation)

$$p_1 = \frac{1.826}{l}\sqrt{\frac{EF_0}{m_0}}.$$

**56** In order to determine the first two frequencies of the rod vibrations we seek the solution to the differential equation of extensional vibrations of the rod in the form

$$u = \left[u_{10}\left(1 - (z/l)^2\right) + u_{20}\left(1 - (z/l)^3\right)\right] \sin p\,t$$

After calculations we have

$$p_1 = \frac{1.794}{l}\sqrt{\frac{EF_0}{m_0}}; \qquad p_2 = \frac{5.033}{l}\sqrt{\frac{EF_0}{m_0}}.$$

**57** We seek the solution to the equation of the extensional vibrations of the rod using the Fourier method. Assuming $u = Z(z)T(t)$, we get two equations

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} + p^2T = 0; \tag{1}$$

$$\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} + \frac{p^2Z}{a} = 0 \quad \left(a^2 = \frac{EF_0}{m_0}\right), \tag{2}$$

where $p$ is the frequency of vibrations.

The solution to equations (1) and (2) has the form

$$u = \left(c_1 \cos\frac{pz}{a} + c_2 \sin\frac{pz}{a}\right)(c_3 \cos p\,t + c_4 \sin p\,t)$$

and should satisfy the following boundary conditions:

$$z = 0, \qquad \frac{\partial u}{\partial z} = 0;$$

$$z = l, \qquad \frac{\partial u}{\partial z} = 0.$$

Then, we find $c_2 = 0$ , $p\,l/a = 0$, and the frequencies of vibrations are $p_n = \dfrac{\pi na}{l}$.

Since different modes of rod vibrations with frequencies $p_n$ are possible, the general solution to the equation of extensional vibrations of the rod is equal to the sum of partial solutions:

$$u = \sum_{n=1}^{\infty} \cos\frac{\pi nz}{l}\left(c_{3n}\cos\frac{\pi nat}{l} + c_{4n}\sin\frac{\pi nat}{l}\right). \tag{3}$$

At the initial instant

$$u(0, z) = u_0\left(\frac{1}{2} - \frac{z}{l}\right); \tag{4}$$

$$\frac{\partial u}{\partial t} \equiv 0. \tag{5}$$

From condition (5) we get $c_{4n} = 0$, and from expression (3) we find

$$u = \sum_{n=1}^{\infty} \cos\frac{\pi nz}{l} \cdot c_{3n} \cos\frac{\pi nat}{l}. \tag{6}$$

At $t = 0$

$$f(z) = u_0\left(\frac{1}{2} - \frac{z}{l}\right) = \sum_{n=1}^{\infty} \cos\frac{\pi nz}{l}. \tag{7}$$

Relation (7) is an expansion of the function $f(z)$ into a Fourier series. The above coefficients follow from the theory of the Fourier analysis

$$c_{3n} = \frac{2}{l}\int_0^l f(z)\cos\frac{\pi nz}{l}\mathrm{d}z,$$

or

$$c_{3n} = \frac{2u_0}{l}\int_0^l \left(\frac{1}{2} - \frac{z}{l}\right)\cos\frac{\pi nz}{l}\mathrm{d}z.$$

After calculations, we have

$$c_{3n} = \begin{cases} 0 & \text{for even } n; \\ 4u_0/\left(\pi^2 n^2\right) & \text{for odd } n. \end{cases}$$

Expression (6) takes on the form

$$u = \frac{4u_0}{\pi^2}\sum_n \frac{1}{n^2}\cos\frac{\pi nz}{l}\cos\frac{\pi nat}{l} \quad (n = 1, 3, 5, \ldots).$$

**58** It follows from the boundary conditions of the problem that $c_1 = 0$, $\cos p\,l/a = 0$, hence, the frequencies

$$p_n = \frac{\pi na}{2l} \qquad \left(a = \sqrt{\frac{EF_0}{m_0}}, \qquad n = 1, 3, 5, \ldots\right).$$

Since at the initial time $\dot{u} = 0$, then (see solution to equation **57**) the displacement of an arbitrary section of the rod has the form

$$u = \sum_n c_n \sin\frac{\pi nz}{2l}\cos\frac{\pi nat}{2l} \qquad (n = 1, 3, 5, \ldots).$$

The displacement at initial instant is specified in the following form

$$u = u_0\frac{z}{l}.$$

The coefficients of the expansion

$$c_n = \frac{2u_0}{l}\int\limits_0^l z\sin\frac{\pi nz}{2\,l}\mathrm{d}z = \frac{8u_0}{\pi^2 n^2}(-1)^{\frac{n-1}{2}}$$

The expression for the displacement of an arbitrary section of the rod assumes the form

$$u = \frac{8u_0}{\pi^2}\sum_n^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin\frac{\pi nz}{2\,l}\cos\frac{\pi nat}{2\,l}\qquad (n = 1,3,5,\ldots).$$

The displacement of the right end face of the rod

$$u(l,t) = \frac{8u_0}{\pi^2}\sum_n^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\cos\frac{\pi nat}{2\,l}\qquad (n = 1,3,5,\ldots).$$

**59** Let us write the differential equation of vibrations of the rod with a concentrated force $P$:

$$m\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\left[EF(z)\frac{\partial u}{\partial z}\right] + P\delta(z), \tag{1}$$

where $\delta(z)$ is the Dirac delta.

The solution to equation (1) under steady-state forced vibrations of the rod we seek in the form $u = u_1(z)\sin\omega t$. After its substitution into equation (1) we get

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[EF(z)\frac{\mathrm{d}u_1}{\mathrm{d}z}\right] + mu_1\omega^2 = -P_0\delta(z).$$

The function $u_1$ can be chosen in the form $u_1 = a_1\cos\frac{\pi z}{2\,l}$, where $a_1$ is the amplitude of steady-state vibrations that corresponds to the first mode of natural vibrations of a homogeneous rod (the function used in solution to Problem **35** can be also used as function $u_1$).

According to the Galerkin method,

$$\int\limits_0^l m_0\omega^2 a_1\left(1+\frac{z}{l}\right)\cos\frac{\pi z}{2\,l} - \frac{EF_0 a_1\pi}{2\,l}\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(1+\frac{z}{l}\right)\sin\frac{\pi z}{2\,l}\right] - P_0\delta(z)\cos\frac{\pi z}{2\,l}\mathrm{d}z = 0.$$

After calculations we obtain

$$a_1 = \frac{16P_0 l\pi^2}{4m_0\omega^2 l^2\left(3\pi^2-4\right) - EF_0\pi^2\left(3\pi^2+8\right)}.$$

**60** The differential equation of extensional vibrations of the rod has the following form (see solution to Problem **59**):

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial z^2} + \frac{P_0 \delta(z-l)}{m_0}. \tag{1}$$

We seek the solution to equation (1) in the form

$$u = \sum_{n=1}^{\infty} u_n(t) \cos \frac{\pi n z}{l}.$$

At the initial instant $u(0,z) = \dot{u}(0,z) = 0$ , therefore, finally (see solution to Problem **51**)

$$u = \frac{2P_0 l}{EF\pi^2} \sum_{n=1}^{\infty} \frac{\cos(\pi n z/l)}{n^2} \left(1 - \cos \frac{\pi n z t}{l}\right) (-1)^n.$$

The rod also moves as a rigid body. Let us write the equation of motion of the rod as a rigid body:

$$m_0 l \ddot{u}_0 = P_0,$$

from where we have

$$u_0 = \frac{P_0 t^2}{2m_0 l}.$$

The total displacement of the rod sections is

$$u = \frac{P_0 t^2}{2m_0 l} + \frac{2P_0 l}{EF\pi^2} \sum_{n=1}^{\infty} \frac{\cos(\pi n z/l)}{n^2} \left(1 - \cos \frac{\pi n a t}{l}\right) (-1)^n.$$

The axial strength in the rod equals

$$N = EF\frac{\partial u}{\partial z} = -\frac{2P_0 l}{\pi} \sum_{n=1}^{\infty} \frac{\sin(\pi n z/l)}{n} \left(1 - \cos \frac{\pi n a t}{l}\right) (-1)^n.$$

This strength in the section $z = l/2$ at the moment $t_1 = l/a$ is equal to

$$N = -\frac{4P_0}{\pi} \left(-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \ldots\right) = P_0.$$

**61** The differential equation of extensional vibrations of the rod can be presented (taking the force $P$ into account) in the following form

$$m_0 \frac{\partial^2 u}{\partial t^2} = EF \frac{\partial^2 u}{\partial z^2} + P_0 \delta(z-l), \tag{1}$$

where $\delta(z-l)$ is the Dirac delta.

We seek the solution to equation (1) in the form

$$u = \sum_n u_n(t) \sin \frac{\pi n z}{l} \qquad (n = 1, 3, 5, \ldots). \tag{2}$$

Now substitute solution (2) into equation (1), multiply this equation by $\sin \frac{\pi n z}{l}$, and integrate between 0 and $l$. As a result, we have

$$\ddot{u}_n + a^2 \left(\frac{\pi n}{2l}\right)^2 u_n = \frac{2P}{l m_0}(-1)^{\frac{n-1}{2}} \quad (n = 1, 3, 5, \ldots). \tag{3}$$

Since at the initial instant $\dot{u}_0 = u(0) = 0$, the solution to equation (3) has the form

$$u_n = \frac{8P l^2}{l m_0 \pi^2 a^2 n^2} \left(1 - \cos \frac{\pi n a t}{2l}\right) (-1)^{\frac{n-1}{2}}.$$

Finally, we find

$$u(z,t) = \frac{8Pl}{m_0 \pi^2 a^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin \frac{\pi n z}{l} \left(1 - \cos \frac{\pi n a t}{2l}\right).$$

The quantity $u$ reaches its maximum value at the time $t_1 = 2l/a$ for the point where the force $P$ is applied ($z = l$):

$$u_{\max} = \frac{16Pl}{EF\pi^2} \sum_n \frac{1}{n^2} \qquad (n = 1, 3, 5, \ldots). \tag{4}$$

The sum of the series appearing in expression (4) is $\sum_n (1/n^2) = \pi^2/8$, therefore, $u_{\max} = 2Pl/(EF)$, i.e., at a sudden application of the load the displacement of the rod butt is twice as large as at a static loading.

**62** When a concentrated force acts, the differential equation of extensional vibrations of the rod has the following form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial z^2} + \frac{P_0 \delta(z - z_0)}{m_0}, \tag{1}$$

where $z_0 = vt$.

We seek the solution to equation (1) in the form

$$u = \sum_n u_n(t) \sin \frac{\pi n z}{2l}.$$

For the functions $u_n(t)$ we get

$$\ddot{u}_n + a^2 \left(\frac{\pi n}{2l}\right)^2 u_n = \frac{2P_0}{l m_0} \sin \frac{\pi n v t}{2l}. \tag{2}$$

Since at the initial instant $\dot{u}(0) = u(0) = 0$ or $u_n(0) = \dot{u}_n(0) = 0$, we find from equation (2)

$$u_n = \frac{8P_0 l}{m_0\pi^2 (a^2 - v^2) n^2} \left( \sin \frac{\pi n v t}{2\,l} - \sin \frac{\pi n a t}{2\,l} \right).$$

The solution to equation (1) takes on the form

$$u(t,z) = \frac{8P_0 l}{m_0\pi^2 (a^2 - v^2)} \sum_n \frac{\sin \dfrac{\pi n z}{2\,l}}{n^2} \times \times \left( \sin \frac{\pi n v t}{2\,l} - \sin \frac{\pi n a t}{2\,l} \right) \qquad (n = 1, 3, 5, \ldots).$$

**63** In this case, the following boundary conditions take place

$$z = 0, \quad u = 0;$$
$$z = l, \quad M\frac{\partial^2 u}{\partial t^2} = -EF\frac{\partial u}{\partial z}.$$

The solution to the equation of extensional vibrations of the rod has the following form

$$u = \left( c_1 \cos \frac{pz}{a} + c_2 \sin \frac{pz}{a} \right) \sin p\,t.$$

In order that the first boundary condition would be satisfied, it is necessary to assume $c_1 = 0$. The fulfillment of the second boundary condition is provided by the equation

$$m_0 \cos \frac{p\,l}{a} = \frac{Mp}{a} \sin \frac{p\,l}{a}, \tag{1}$$

or

$$\tan \frac{p\,l}{a} = \frac{m_0 l a}{M p\,l}. \tag{2}$$

Equation (2) can be solved graphically (to obtain the series of first frequencies). Figure 2.15 presents for $M = m_0 l$ the plots of the functions

$$y_1 = \tan \frac{p\,l}{a}, \quad y_2 = \frac{m_0 l a}{M p\,l}.$$

From the roots (points of intersection of the plots for $y_1$ and $y_2$)

$$p_1 l/a = 3\pi/8; \quad p_2 l/a = 9\pi/8; \quad p_3 l/a = 33\pi/16,$$

we can deduce the frequencies

$$p_1 = \frac{3}{8}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}; \qquad p_2 = \frac{9}{8}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}; \qquad p_3 = \frac{33}{16}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}.$$

**64** The boundary conditions have the following form

$$z = 0, \qquad \frac{\partial u}{\partial z} = 0;$$

$$z = l, \qquad M\frac{\partial^2 u}{\partial t^2} = -EF\frac{\partial u}{\partial z}.$$

For determination of frequencies we have the equation

$$\tan\frac{p\,l}{a} = \frac{Mp\,l}{m_0 l a}.$$

Figure 2.15 shows the points of intersection of the plots $y_1 = \tan\frac{p\,l}{a}$ and $y_1 = -\frac{p\,l}{a}$ . Consequently, the desired frequencies are

$$p_1 = \frac{5}{8}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}; \qquad p_2 = \frac{3}{2}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}; \qquad p_3 = \frac{5}{2}\frac{\pi}{l}\sqrt{\frac{EF}{m_0}}.$$

**65** The boundary conditions of the problem:

$$z = 0, \qquad u = z_0(t);$$

$$z = l, \qquad M\frac{\partial^2 u}{\partial t^2} = -EF\frac{\partial u}{\partial z}.$$

We seek the solution to the equation of steady-state longitudinal vibrations of the rod in the following form

$$u = u_1(z)\sin\omega t \qquad \left(u_1 = c_1\cos\frac{\omega z}{a} + c_2\sin\frac{\omega z}{a}\right).$$

The arbitrary constants are

$$c_1 = A; \qquad c_2 = A\frac{\dfrac{M}{m_0 l}\dfrac{\omega\,l}{a}\cos\dfrac{\omega\,l}{a} + \sin\dfrac{\omega\,l}{a}}{\cos\dfrac{\omega\,l}{a} - \dfrac{M}{m_0 l}\dfrac{\omega\,l}{a}\sin\dfrac{\omega\,l}{a}}.$$

The displacement of an arbitrary section has the form

$$u(z,t) = A\frac{\cos\dfrac{\omega(l-z)}{a} - \dfrac{M}{m_0 l}\dfrac{\omega\,l}{a}\sin\dfrac{\omega(l-z)}{a}}{\cos\dfrac{\omega\,l}{a} - \dfrac{M}{m_0 l}\dfrac{\omega\,l}{a}\sin\dfrac{\omega\,l}{a}}\sin\omega t.$$

The amplitude of longitudinal vibrations of the mass $M$ is

$$u_0(l,t) = \frac{A}{\cos\dfrac{\omega\,l}{a} - \dfrac{M}{m_0 l}\dfrac{\omega\,l}{a}\sin\dfrac{\omega\,l}{a}}. \qquad (1)$$

It follows from formula (1) that, when the denominator vanishes, the amplitude of vibrations of the mass $M$ becomes infinitely large. The values of $\omega$, for which this takes place, we find from the equation

$$\tan\frac{\omega l}{a}=\frac{lm_0 a}{M\omega l}. \tag{2}$$

From the comparison of equation (2) of Problem **65** with equation (2) of Problem **63** it follows that the values of $\omega$, at which the denominator of formula (1) in Problem **65** vanishes, coincide with the frequencies of natural vibrations of the system.

**66** In the case under consideration the mass per unit length of the rod is variable. Therefore, when deriving the equations of motion, we take advantage of the theorem of momentum variation for an element of the rod mass ($\mathrm{d}m=\varrho E\mathrm{d}z$) and obtain

$$\varrho\frac{\partial}{\partial t}\left(F\frac{\partial u}{\partial t}\right)=EF\frac{\partial^2 u}{\partial z^2}, \tag{1}$$

where $F$ is the current area of the charge section.

At an arbitrary instant the charge mass $M=M_0-Mt$, where $M_0$ is the initial charge mass and $M$ is the constant consumption per second (the charge mass burning out per unit time).

At the instant $t$ of termination of the engine operation the mass $M$ is zero, therefore, $M=M_0/t_1$. Since the charge mass $M=\varrho lF$, and at the initial instant $M_1=\varrho lF_0$, the law of time behavior of the area of the charge section has the form

$$F=F_0\left(1-t/t_1\right).$$

Equation (1) can be transformed to the form

$$\frac{\partial^2 u}{\partial t^2}-\frac{1}{t_1\left(1-t/t_1\right)}\frac{\partial u}{\partial t}=\frac{E}{\varrho}\frac{\partial^2 u}{\partial z^2}.$$

**67** The extension force of the spring (with a small angle of lead) is equal to $c=Gd^4/\left(8D^3 i\right)$. The stiffness of an equivalent rod of the same length is $EF_1/l=Gd^4/\left(8D^3 i\right)$, where $E_1$ and $F_1$ are the Young's modulus and the section area of the equivalent rod, respectively.

The equivalent rod mass is equal to the spring mass:

$$\varrho F_1 l=\pi D i\varrho\pi d^2/4.$$

The velocity of propagation of a longitudinal wave in the rod is

$$a=\sqrt{\frac{E_1F_1}{\varrho F_1}}=\frac{ld}{\pi i D^2}\sqrt{\frac{G}{2\varrho}},\qquad a=3.7\ \mathrm{m/s}.$$

The frequencies of spring vibrations (see solution to Problem **42**) are

$$p_n = \frac{n-1}{2l}\frac{ld}{iD^2}\sqrt{\frac{G}{2\varrho}} \qquad (n = 1, 2, \ldots).$$

**68** Let us change the spring for a continuous equivalent rod (of the same length $l$) whose extension stiffness should be equal to the spring force:

$$c = E_1 F_1 / l, \tag{1}$$

where $E_1$ and $F_1$ are, respectively, the elasticity modulus of the first kind and the section area of the equivalent rod.

The mass per unit length for the equivalent rod should be equal to the mass per unit length for the spring:

$$\varrho F_1 = \frac{\pi^2 i \varrho D d^2}{14}. \tag{2}$$

From relationship (2) and from equation (1) we determine the area $F_1$ and the modulus $E_1$, respectively. Figure 2.16 shows an element of the equivalent rod with the forces acting upon it ($u$ is the longitudinal displacement of the rod element under vibrations). The distributed forces act upon the rod in the equilibrium state (the inertia forces are $q = (\varrho F_1)\,\Omega^2 z$). Due to the displacement of the element under vibrations, an additional force $\triangle q = \varrho F_1 \Omega^2 u$ acts upon it.

The differential equation of the extensional vibrations has the following form

$$\mathrm{d}z \cdot \varrho F_1 \frac{\partial^2 u}{\partial t^2} = \frac{\partial N}{\partial z}\mathrm{d}z + \triangle q \mathrm{d}z.$$

After transformations, we find

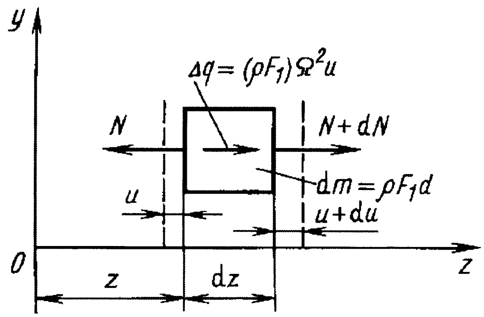


**Fig. 2.16.**

$$\frac{\partial^2 u}{\partial t^2} = \frac{E_1}{\varrho}\frac{\partial^2 u}{\partial z^2} + \Omega^2 u. \tag{3}$$

Now we seek the solution to equation (3) in the form $u = u_1(z)\sin pt$ and get the following equation for the function $u_1$

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}z^2} + \frac{\left(\Omega^2 + p^2\right) u_1}{a^2} = 0 \qquad \left(a^2 = E_1/\varrho\right). \tag{4}$$

The solution to equation (4) has the form

$$u_1 = c_1 \cos\lambda z + c_2 \sin\lambda z \qquad \left(\lambda = \sqrt{\frac{\Omega^2 + p^2}{a^2}}\right).$$

The function $u_1$ should satisfy the following boundary conditions:

$$z = 0, \qquad u_1 = 0;$$
$$z = l, \qquad \frac{\mathrm{d}u_1}{\mathrm{d}z} = 0.$$

which holds true at $c_1 = 0$ and $\sin\lambda l = 0$ or $\lambda l = \pi n$ $(n = 1, 2, \ldots)$. Thus we obtain the frequencies of vibrations of the spring in a field of centrifugal forces:

$$p_n = \frac{\pi n}{l}\sqrt{\frac{E_1}{\varrho}}\sqrt{\left(1 - \frac{\Omega^2 l^2}{\pi^2 n^2 a^2}\right)} \qquad (n = 1, 2, \ldots).$$

**69** The first frequency of vibrations (see solution to Problem **68**) is equal to zero at $\Omega_*^2 l^2/\left(\pi^2 a^2\right) = 1$, so that we get the critical angular velocity of the disk as

$$\Omega_* = \frac{\pi}{l}\sqrt{\frac{F_1}{\varrho}}.$$

**70** The spring can changed for an equivalent rod assuming (see solution to Problem **68**)

$$c = \frac{Gd^4}{8D^3 i} = \frac{E_1 F_1}{l}; \quad \varrho F_1 = \frac{1}{l}\pi D i \varrho \frac{\pi d^2}{4},$$

where $E_1$ and $F_1$ are the elasticity modulus of the first kind and the section area of the equivalent rod, respectively.

The mass per unit length of the equivalent rod is $m_0 = \varrho F_1$.

When the disk rotates, the distributed forces act on the spring (per unit length), $q = m_0 \Omega^2 z$, which causes a change in the initial level of $N_0$. Let us find the strength $N$ in the equivalent rod when it rotates. The additional reaction forces $R_1$ and $R_2$ arising at the points of rod fixing (caused by the forces $q$) satisfy the equilibrium condition

$$R_1 + R_2 = \int_0^l q\mathrm{d}z = \frac{m_0\Omega^2 l^2}{2}.$$

Considering the deformations of the rod, we derive one more equation. The total change of the rod length is equal to zero, therefore, taking advantage of the principle of independent action of forces, we get (neglecting the embedment and changing it for the reaction force $R_2$)

$$\frac{R_2 l}{E_1 F_1} = \int_0^l \frac{N_q(z)\mathrm{d}z}{E_1 F_1}.$$

The longitudinal strength $N_q(z)$ caused only by the forces $q$ is

$$N_q(z) = \int_0^z m_0\Omega^2 \xi \mathrm{d}\xi = \frac{m_0\Omega^2 z^2}{2}.$$

Therefore,

$$R_2 = \frac{m_0\Omega^2 l^2}{6}.$$

The total strength in the rod during rotation changes according to the law

$$N = N_0 + R_2 - \frac{m_0\Omega^2 z^2}{2}.$$

The differential equation of longitudinal vibrations of the equivalent rod is similar to the following equation given above in Problem **68**

$$\frac{\partial^2 u}{\partial t^2} = \frac{E_1}{\varrho}\frac{\partial^2 u}{\partial z^2} + \Omega^2 u.$$

**71** As the disk rotates the tension in the filament changes and becomes (see solution to Problem **8**) variable in length. In order to determine the filament tension at $\Omega \neq 0$ let us consider the filament loaded with distributed forces (Fig. 2.17*a*).

We find the tension in the filament caused only by the forces $q$. The sum of reaction forces at the points of fixing the filament is

$$R_1 + R_2 = \int_0^l m_0\Omega^2 z\mathrm{d}z = \frac{m_0\Omega^2}{2} l^2.$$

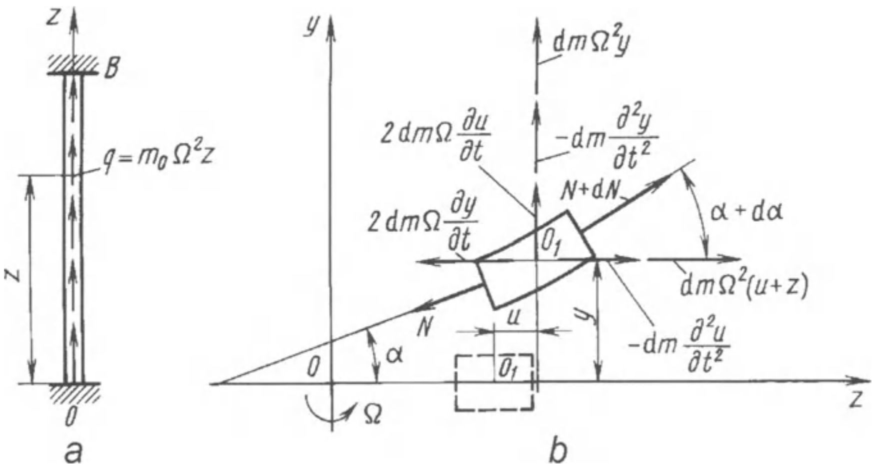


**Fig. 2.17.**

Since the variation of the filament length is zero, we get the following relation

$$\frac{R_A l}{EF} = \int_0^l \frac{N_1(z)\mathrm{d}z}{EF} \qquad \left(N_1(z) = \frac{m_0\Omega^2 z^2}{2}\right).$$

After calculations, we find

$$R_A = \frac{m_0 l^2 \Omega^2}{6}.$$

The tension force in an arbitrary section of the filament (taking initial $Q_{10}$ into account) is

$$N = Q_{10} + R_A - N_1 = Q_{10} + 0.5 m_0 l^2 \Omega \left(\frac{1}{3} - \frac{z^2}{l^2}\right). \tag{1}$$

Figure 2.17*b* shows an element of the filament with the forces acting upon it. Under vibrations, the filament element is displaced along both the $z$ and $y$ axes. In Fig. 2.17*b* $-\mathrm{d}m\dfrac{\partial^2 u}{\partial t^2}$ and $-\mathrm{d}m\dfrac{\partial^2 y}{\partial t^2}$ are the forces of inertia; $2\mathrm{d}m\,\Omega\dfrac{\partial u}{\partial t}$ and $2\mathrm{d}m\,\Omega\dfrac{\partial y}{\partial t}$ are the Coriolis forces; and $\mathrm{d}m\,\Omega^2 u$ and $\mathrm{d}m\,\Omega^2 y$ are the additional centrifugal forces that arise when the element is displaced from its initial position.

The longitudinal strength $N$ (under vibrations) is

$$N' = N + EF\frac{\partial u}{\partial z},$$

where $EF\dfrac{\partial u}{\partial z}$ is the strength in the filament that appear under extensional vibrations.

Projecting the forces onto the axes $y$ and $z$, we get after some transformations two differential equations of the form

$$\frac{\partial^2 y}{\partial t^2} - 2\Omega\frac{\partial u}{\partial t} - \frac{1}{m_0}\frac{\partial}{\partial z}\left(N'\frac{\partial y}{\partial z}\right) - \Omega^2 y = 0; \tag{2}$$

$$\frac{\partial^2 u}{\partial t^2} + 2\Omega\frac{\partial y}{\partial t} - \frac{1}{m_0}\frac{\partial N'}{\partial z} - \Omega^2(z+u) = 0 \tag{3}$$

Since $y$ and $u$, as well as their derivatives, are small, one can assume in equations (2) and (3) $N' \approx N_0$. In equation (3)

$$-\Omega^2 z - \frac{1}{m_0}\frac{\partial N_0}{\partial z} = 0.$$

Then differential equations (2) and (3) assume the form

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2} - \Omega^2 y - 2\Omega\frac{\partial u}{\partial t} = \frac{1}{m_0}\frac{\partial}{\partial z}\left(N_0\frac{\partial y}{\partial z}\right); \\ &\frac{\partial^2 u}{\partial t^2} - \Omega^2 u + 2\Omega\frac{\partial y}{\partial t} = \frac{1}{m_0}EF\frac{\partial^2 u}{\partial z^2}. \end{aligned} \tag{4}$$

We seek the approximate solution to system of equations (4)) using the Galerkin method in the form

$$y = y_1(t)\sin\frac{\pi z}{l}, \qquad u = u_1(t)\sin\frac{\pi z}{l}.$$

After transformations we obtain the following system of differential equations for $y_1(t)$ and $u_1(t)$:

$$\begin{aligned} &\ddot{y}_1 + \left[\frac{Q_{10}}{m_0}\left(\frac{\pi}{l}\right)^2 - \frac{5}{4}\Omega^2\right] y_1 - 2\Omega\dot{u}_1 = 0; \\ &\ddot{u}_1 + \left[\frac{EF}{m_0}\left(\frac{\pi}{l}\right)^2 - \Omega^2\right] u_1 + 2\Omega\dot{y}_1 = 0. \end{aligned} \tag{5}$$

In order to solve system of equations (5) we assume

$$y_1 = A\sin pt, \qquad u_1 = B\sin pt. \tag{6}$$

Substituting (6) into system (5) we get the system of two algebraic homogeneous equations for $A$ and $B$. Equating the determinant of this system to zero

$$\begin{vmatrix} -p^2 + \dfrac{Q_{10}}{m_0}\left(\dfrac{\pi}{l}\right)^2 - \dfrac{5}{4}\Omega^2 & -2\Omega \\ 2\Omega p & -p^2 + \dfrac{EF}{m_0}\left(\dfrac{\pi}{l}\right)^2 - \Omega^2 \end{vmatrix} = 0,$$

we find the approximate values of the first frequencies of filament vibrations

$$p_{1,2} = \sqrt{a \pm \sqrt{a^2 - 4b/2}},$$

where

$$a = \left(\frac{\pi}{l}\right)^2 \left(\frac{EF}{m_0} + \frac{Q_{10}}{m_0}\right) - \frac{25}{4}\Omega^2;$$

$$b = \left[\left(\frac{\pi}{l}\right)^2 \frac{EF}{m_0} - \Omega^2\right] \left[\left(\frac{\pi}{l}\right)^2 \frac{Q_{10}}{m_0} - \frac{5}{4}\Omega^2\right].$$

## 2.4 Bending vibrations of rectilinear rods

**72** The distributed forces of inertia $q$ act upon the rod (Fig. 2.18$a$) under vibrations. As is known, the equation of bending deflections of the rod can be represented in the form

$$EJ_x \frac{\partial^4 y}{\partial z^4} = q$$

The distributed forces of inertia acting upon the rod unit length are

$$q = -m_0 \frac{\partial^2 y}{\partial t^2}.$$

The differential equation of the bending deflections of the rod takes on the form

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial z^4} = 0 \qquad \left(a^2 = \frac{EJ_x}{m_0}\right). \tag{1}$$

The solution to equation (1) is sought in the form $y = y_1(z) \sin pt$. For the functions $y_1(z)$ we have

$$\frac{\partial^4 y_1}{\partial z^4} - \lambda^4 y_1 = 0 \qquad \left(\lambda^4 = \frac{p^2}{a^2}\right). \tag{2}$$

The solution to equation (2) has the form

$$y_1 = c_1 \mathrm{K}_1(\lambda z) + c_2 \mathrm{K}_2(\lambda z) + c_3 \mathrm{K}_3(\lambda z) + c_4 \mathrm{K}_4(\lambda z),$$

where $\mathrm{K}_i(\lambda z)$ are the following Krylov functions [4]

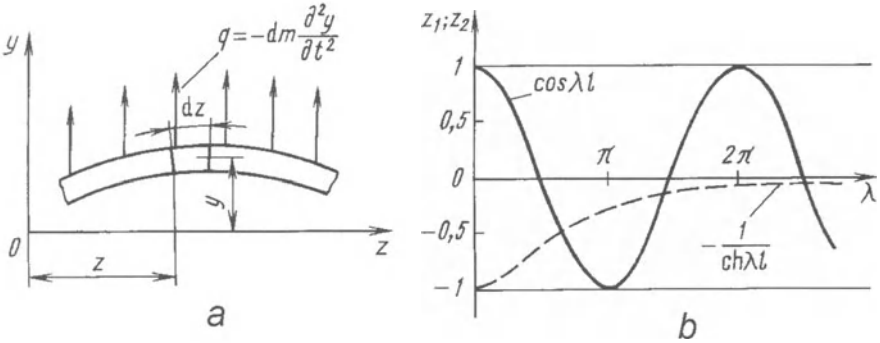


**Fig. 2.18.**

$$\mathrm{K}_1(\lambda z) = \frac{1}{2}(\cosh\lambda z + \cos\lambda z); \qquad \mathrm{K}_2(\lambda z) = \frac{1}{2}(\sinh\lambda z + \sin\lambda z);$$
$$\mathrm{K}_3(\lambda z) = \frac{1}{2}(\cosh\lambda z - \cos\lambda z); \qquad \mathrm{K}_4(\lambda z) = \frac{1}{2}(\sinh\lambda z - \sin\lambda z).$$

At $z = 0$ $\mathrm{K}_1 = 1$ and $\mathrm{K}_2 = \mathrm{K}_3 = \mathrm{K}_4 = 0$.

In the case of hinge fixing (see Fig. 1.56$a$) the function $y_1$ should satisfy the following boundary conditions:

$$z = 0; \qquad y_1' = y = 0;$$
$$z = l; \qquad y_1 = y_1'' = 0.$$

In the places of hinging of the rod ends the bending deflection and bending moment are equal to zero. The bending moment is proportional to the second derivative: $M = EJ_x y''$.

In order to satisfy the boundary conditions at the left end of the rod one needs to set $c_1 = c_2 = 0$.

From the boundary conditions at the right end of the rod we get

$$\begin{aligned} &c_2\mathrm{K}_2(\lambda l) + c_4\mathrm{K}_4(\lambda l) = 0; \\ &c_2\lambda^2\mathrm{K}_4(\lambda l) + c_4\lambda^2\mathrm{K}_2(\lambda l) = 0. \end{aligned} \tag{3}$$

Now equate the determinant of system (3) to zero:

$$\mathrm{K}_2^2(\lambda l) - \mathrm{K}_4^2(\lambda l) = 0,$$

or

$$\sinh\lambda l \,\sin\lambda l = 0, \tag{4}$$

Since $\sinh\lambda l \neq 0$, it follows from equation (4) that $\lambda l = 0$ or $\lambda l = \pi n$. The frequencies of vibrations are

$$p_n = \frac{\pi^2 n^2}{l^2}\sqrt{\frac{EJ_x}{m_0}} \qquad (n = 1, 2, 3, \ldots).$$

The following boundary conditions take place in the case of a cantilever rod (see Fig. 1.56$b$):

$$z = 0; \qquad y_1 = y_1' = 0;$$
$$z = l; \qquad y_1'' = y_1''' = 0.$$

At the free end of the rod the moment and the cutting force $Q = EJ_x y'''$ are equal to zero.

The equation of frequencies has the form

$$\mathrm{K}_1^2(\lambda l) - \mathrm{K}_2(\lambda l)\,\mathrm{K}_4(\lambda l) = 0,$$

so that

$$\cos \lambda l = -1/(\cosh \lambda l). \tag{5}$$

Equation (5) is solved graphically by plotting the functions $z_1 = \cos \lambda l$ and $z_2 = -1/(\cosh \lambda l)$ (Fig. 2.18*b*).

The first two roots of equation ((5) are

$$(\lambda l)_1 = 1.875; \qquad (\lambda l)_2 = 4.694.$$

The remaining roots, as it follows from the plots, can be represented in the form

$$(\lambda l)_n \approx (2n-1)\pi/2 \qquad (n > 2).$$

The desired frequencies are

$$p_1 = 3.52\sqrt{\frac{EJ_x}{m_0 l^4}}; \qquad p_1 = 22\sqrt{\frac{EJ_x}{m_0 l^4}}; \qquad p_n = \frac{(2n-1)^2\pi^2}{4}\sqrt{\frac{EJ_x}{m_0 l^4}}.$$

Finally, in the case shown in Fig. 1.56*c* the equation of frequencies has the following form

$$\tan \lambda l = \tanh \lambda l,$$

and the frequencies are

$$p_1 = 16.4\sqrt{\frac{EJ_x}{m_0 l^4}}; \qquad p_1 = 49\sqrt{\frac{EJ_x}{m_0 l^4}}; \qquad p_n = \frac{(4n+1)^2\pi^2}{16}\sqrt{\frac{EJ_x}{m_0 l^4}}.$$

**73** For the case presented in Fig. 1.57*a* the equation of frequencies has the form

$$\cosh \lambda l \cos \lambda l - 1 = 0.$$

The roots of this equation are

$$(\lambda l)_1 = 4.73; \qquad (\lambda l)_2 = 7.85; \qquad (\lambda l)_n = \frac{2n+1}{2}\pi \qquad (n > 1)$$

Then the frequencies are

$$p_1 = 22.5\sqrt{\frac{EJ_x}{m_0 l^4}}; \quad p_2 = 62\sqrt{\frac{EJ_x}{m_0 l^4}};$$

$$p_n = \left[\frac{2(n+1)+1}{2}\right]^2 \pi^2\sqrt{\frac{EJ_x}{m_0 l^4}}.$$

For the case of Fig. 1.57*b* the equation of frequencies looks like

$$\cosh \lambda l \cos \lambda l + 1 = 0.$$

The roots of this equation are

$$(\lambda l)_1 = 1.875; \qquad (\lambda l)_2 = 4.694; \qquad (\lambda l)_n = (2n-1)\,\pi/2$$

Consequently, the desired frequencies are

$$p_1 = 2.5\sqrt{\frac{EJ_x}{m_0 l^4}}; \qquad p_2 = 22\sqrt{\frac{EJ_x}{m_0 l^4}};$$
$$p_n = \left(\frac{2n-1}{2}\right)^2 \pi^2 \sqrt{\frac{EJ_x}{m_0 l^4}}.$$

**74** As is known, the differential equation of the elastic line of a rod with variable section has the form

$$EJ_x(z)\frac{\partial^2 y}{\partial z^2} = M(z).$$

Differentiating this equation twice with respect to $z$ we find

$$\frac{\partial^2}{\partial z^2}\left(EJ_x\frac{\partial^2 y}{\partial z^2}\right) = q = -m_0(z)\frac{\partial^2 y}{\partial t^2}.$$

**75** The differential equation of transverse vibrations of a rod under the action of a distributed load has the form

$$EJ_x\frac{\partial^4 y}{\partial z^4} + m_0\frac{\partial^2 y}{\partial t^2} = q,$$

or, since we consider small vibrations,

$$EJ_x\frac{\partial^4 y}{\partial z^4} + m_0\frac{\partial^2 y}{\partial t^2} = 4k\Phi y/a^3. \qquad (1)$$

We seek the solution to equation (1) in the form

$$y = y_1(z)\sin pt$$

Then we obtain the following equation for the function $y_1(z)$

$$\frac{\partial^4 y_1}{\partial z^4} - \lambda^4 y_1 = 0, \qquad (2)$$

where $\lambda^4 = \dfrac{p^2 m_0 + 4k\Phi_0^2/a^3}{EJ_x}$

Equation (2) is similar to equation (2) in Problem **72**, where the following values of the two first roots were obtained for a cantilever rod (see Fig. 2.18*b*): $(\lambda l)_1 = 1.875$ and $(\lambda l)_2 = 4.694$.

The vibration frequencies corresponding to them are

$$p_1 = \sqrt{(1.875)^4 \frac{EJ_x}{m_0 l^4} - \frac{4k\Phi_0^2}{a^3 m_0}}; \qquad p_2 = \sqrt{(4.694)^4 \frac{EJ_x}{m_0 l^4} - \frac{4k\Phi_0^2}{a^3 m_0}}.$$

The critical (minimum) value of the magnetomotive force $\Phi_{0*}$ is found by the formula

$$\Phi_{0*} = \sqrt{(1.875)^4 \frac{EJ_x a^3}{l^4 4k}}.$$

**76** The differential equation of vibrations of a rod with a variable moment of inertia has the form

$$\frac{\partial^2}{\partial z^2}\left(EJ_x \frac{\partial^2 y}{\partial z^2}\right) + m\frac{\partial^2 y}{\partial t^2} = 0. \tag{1}$$

We seek the solution to equation (1) in the form

$$y = y_0 \sin\frac{\pi z}{l} f(t)$$

The function $\sin\frac{\pi z}{l}$ satisfies the problem's boundary conditions:

$$z = 0; \qquad y = y'' = 0;$$
$$z = l; \qquad y'' = 0.$$

Using the Galerkin method we get

$$\ddot{f}\int_0^l m\sin\frac{\pi z}{l}\mathrm{d}z - f\left(\frac{\pi}{l}\right)^2 E\int_0^l \frac{\partial^2}{\partial z^2}\left(J_x \sin\frac{\pi z}{l}\right)\sin\frac{\pi z}{l}\mathrm{d}z = 0.$$

After calculations we have

$$\ddot{f} + \frac{\pi^2}{l^4}\frac{EJ_0}{m_0} 4.7 f = 0,$$

and in the first approximation the fundamental frequency is

$$p_1 = 2.16\frac{\pi^2}{l^2}\sqrt{\frac{EJ_0}{m_0}}.$$

**77** The symmetric mode of vibrations corresponds to the first frequency, therefore, we seek the solution to the equation of vibrations in the form

$$y = \left(y_1 \sin\frac{\pi z}{l} + y_2 \sin\frac{3\pi z}{l}\right)\sin pt$$

(the function $\sin\frac{3\pi z}{l}$ corresponds to the symmetric mode of vibrations).

After some transformations we get the system of equations

$$\left(0.356m_0lp^2 - 1.62\frac{EJ_0}{l^3}\pi^4\right)y_1 + \\ + \left(-0.085m_0lp + 6.98\frac{EJ_0}{l^3}\pi^4\right)y_2 = 0;$$

$$\left(-0.085m_0lp^2 + 6.98\frac{EJ_0}{l^3}\pi^4\right)y_1 + \\ + \left(0.414m_0lp - 98.92\frac{EJ_0}{l^3}\pi^4\right)y_2 = 0,$$

so that we find the refined value of the first frequency

$$p_1 = 1.781\frac{\pi^2}{l^2}\sqrt{\frac{EJ_0}{m_0}}.$$

and the frequency $p_3$ corresponding to the second mode of rod vibrations

$$p_3 = 14\frac{\pi^2}{l^2}\sqrt{\frac{EJ_0}{m_0}}.$$

**78** In the case considered the rod consist of two segments, therefore, for each of the segments we have

$$\begin{aligned} y_1 &= c_1\mathrm{K}_1(\lambda z) + c_2\mathrm{K}_2(\lambda z) + c_3\mathrm{K}_3(\lambda z) + c_4\mathrm{K}_4(\lambda z) \quad (0 \le z \le b); \\ y_2 &= c_1'\mathrm{K}_1(\lambda z) + c_2'\mathrm{K}_2(\lambda z) + c_3'\mathrm{K}_3(\lambda z) + c_4'\mathrm{K}_4(\lambda z) \quad (b \le z \le l). \end{aligned} \tag{1}$$

At the first segment $y_1(0) = y_0''(0) = 0$, therefore, $c_1 = c_3 = 0$.
At the point of conjunction of two segments $y_1(b) = y_2(b)$ and

$$\left.\frac{\partial y_1}{\partial z}\right|_{z=b} = \left.\frac{\partial y_2}{\partial z}\right|_{z=b}.$$

The transverse forces at the end of the first segment and at the beginning of the second segment differ by a value of the support reaction, i.e.,

$$EJ_x\left.\frac{\partial^3 y_1}{\partial z^3}\right|_{z=b} = EJ_x\left.\frac{\partial^3 y_2}{\partial z^3}\right|_{z=b} - R,$$

where $R$ is the support reaction.

The conditions of connection of the segments can be satisfied if the bending deflections at the second segment are represented as

$$y_2 = y_1 + \frac{R}{\lambda^3 EJ_x}\mathrm{K}_4[\lambda(z-b)], \tag{2}$$

so that the function $K_4[\lambda(z-b)]$ is identically equal to zero at $z \le b$, and it is nonzero at $z > b$.

Expression (2) is valid for the entire rod. Substituting the relation for $y_1$ into (2), we have at $c_1 = c_3 = 0$

$$y = c_2 \mathrm{K}_2(\lambda z) + c_4 \mathrm{K}_4(\lambda z) + \frac{R}{\lambda^3 E J_x} \mathrm{K}_4[\lambda(z-b)]. \tag{3}$$

Equation (3) involves three constants ($c_2$, $c_4$, and $R$) that can be determined from the conditions

$$z = b; \qquad y = 0;$$
$$z = l; \qquad y'' = y''' = 0.$$

We get the following system of three equations for finding the constants $c_2$, $c_4$, and $R$:

$$\begin{aligned}
&c_2 \mathrm{K}_2(\lambda b) + c_4 \mathrm{K}_4(\lambda b) = 0;\\
&c_2 \mathrm{K}_4(\lambda l) + c_4 \mathrm{K}_1(\lambda l) + \frac{R}{\lambda^2 E J_x} K_2[\lambda(l-b)] = 0;\\
&c_2 \mathrm{K}_3(\lambda l) + c_4 \mathrm{K}_1(\lambda l) + \frac{R}{\lambda^3 E J_x} \mathrm{K}_1[\lambda(l-b)] = 0.
\end{aligned} \tag{4}$$

Equating to zero the determinant of system (4), we have the equation of frequencies

$$\begin{vmatrix} \mathrm{K}_2(\lambda l) & \mathrm{K}_4(\lambda b) & 0 \\ \mathrm{K}_2(\lambda l) & \mathrm{K}_2(\lambda l) & \mathrm{K}_2[\lambda(l-b)] \\ \mathrm{K}_3(\lambda l) & \mathrm{K}_1(\lambda l) & \mathrm{K}_1[\lambda(l-b)] \end{vmatrix} = 0,$$

or

$$\begin{aligned}
&\mathrm{K}_2(\lambda b)\,\mathrm{K}_2(\lambda b)\,\mathrm{K}_1[\lambda(l-b)] + \mathrm{K}_3(\lambda b)\,\mathrm{K}_4(\lambda b)\,\mathrm{K}_2[\lambda(l-b)] = \\
&= \mathrm{K}_4(\lambda b)\,\mathrm{K}_4(\lambda b)\,\mathrm{K}_1[\lambda(l-b)] + \mathrm{K}_1(\lambda b)\,\mathrm{K}_2(\lambda b)\,\mathrm{K}_2[\lambda(l-b)].
\end{aligned} \tag{5}$$

The roots of equation (5) can be determined graphically. For the case under consideration ($b = l/2$) the first root is $(\lambda l) = 0.311$, and the frequency corresponding to this root is

$$p_1 = 9.066\sqrt{\frac{EJ_x}{m_0 l^4}}.$$

**79** The differential equation of vibrations of the rod has the following form

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial z^4} = \frac{P_0}{m_0}\,\delta(z - l_0). \tag{1}$$

We seek the solution to equation (1)) in the form

$$y = \sum_{n=1}^{\infty} y_n(t) \sin\frac{\pi n z}{l}.$$

For the function $y_n(t)$ we get

$$\ddot{y}_n + a^2\left(\frac{\pi n}{l}\right)^4 y_n = \frac{2P_0}{lm_0}\sin\frac{\pi n\, l_0}{l} \qquad (n = 1, 2, \ldots). \tag{2}$$

At the initial moment $y_n = \dot{y}_n = 0$, therefore, the solutions to equation (2) have the following form

$$y_n = \frac{2P_0\, l^3 \sin\dfrac{\pi n\, l_0}{l}}{m_0 a^2 \pi^4 n^4}\left[1 - \cos\frac{a(\pi n)^2 t}{l^2}\right].$$

The solution to equation (1) can be represented as:

$$y = \frac{2P_0\, l^3}{a^2\pi^4}\sum_{n=1}^{\infty}\frac{\sin\dfrac{\pi n\, l_0}{l}}{n^4}\sin\frac{\pi n z}{l}\left[1 - \cos\frac{a(\pi n)^2 t}{l^2}\right].$$

The bending moment at an arbitrary section of the rod is

$$M = \frac{\partial^2 y}{\partial z^2}EJ_x = -\frac{EJ_x 2P_0\, l^3}{a^2\pi^4}\times$$
$$\times\sum_{n=1}^{\infty}\frac{\pi^2}{n^2 l^2}\sin\frac{\pi n\, l_0}{l}\sin\frac{\pi n z}{l}\left[1 - \cos\frac{a(\pi n)^2 t}{l^2}\right].$$

The maximum normal strength in the section, where the force is applied, is

$$|\sigma_{\max}| = \left.\frac{M}{W_x}\right|_{z=l_0} = \frac{2EJ_x}{a^2\pi^2 W_x}\sum_{n=1}^{\infty}\frac{\sin\dfrac{\pi n\, l_0}{l}}{n^2}\left[1 - \cos\frac{a(\pi n)^2 t}{l^2}\right].$$

**80** Let us write down the equation of rod vibrations:

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial z^4} = \frac{P_0}{m_0}\delta(z - z_0), \tag{1}$$

where $z_0 = vt$.

The solution to equation (1) is sought in the form

$$y = \sum_{n=1}^{\infty} y_n(t)\sin\frac{\pi n z}{l}. \tag{2}$$

Upon substitution of solution (2) into equation (1) and some transformations (see solution to Problem **22**) we have for the functions $y_n(t)$ the following equations

$$\ddot{y}_n + a^2\left(\frac{\pi n}{l}\right)^4 y_n = \frac{2P_0}{m_0 l}\sin\frac{\pi n v t}{l}. \tag{3}$$

The solution to equation (3) has the form

$$y_n = c_1 \cos\frac{a(\pi n)^2 t}{l^2} + c_2 \sin\frac{a(\pi n)^2 t}{l^2} + \frac{\dfrac{2P_0}{m_0\, l}\sin\dfrac{\pi n v t}{l}}{a^2(\pi n/l)^4 - (\pi n v/l)^2}$$

Since $y_n(0) = \dot{y}_n(0) = 0$ at $t = 0$, then

$$y_n = \frac{2P_0}{m_0\, l}\,\frac{\sin\dfrac{\pi n v t}{l} - \dfrac{\pi n v/l}{a^2(\pi n/l)^2}\sin\dfrac{a(\pi n)^2 t}{l^2}}{a^2(\pi n/l)^4 - (\pi n/b)^2 v^2}.$$

The solution to equation (1) has the form

$$y(t,z) = \frac{2P_0}{m_0\, l}\sum_{n=1}^{\infty}\sin\frac{\pi n z}{l}\,\frac{\sin\dfrac{\pi n v t}{l} - \dfrac{\pi n v/l}{a^2(\pi n/l)^2}\sin\dfrac{a(\pi n)^2 t}{l^2}}{a^2(\pi n/l)^4 - (\pi n/b)^2 v^2}.$$

It follows from the solution derived that there are such values of the velocity $v$, at which the denominator in the serial terms is equal to zero, however, their numerator at these values of $v$ is also zero. If one evaluates this indeterminate form, a finite number is obtained, i.e., no critical velocities for the moving force exist.

**81** The angular velocity of the bullet at the moment of its exit out of the barrel is

$$\omega = \left.\frac{\partial^2 y}{\partial z \partial t}\right|_{\substack{z = l \\ t = l/v}}.$$

Using the solution to Problem **80**, in which one should take $mg$ instead of $P_0$, we get after appropriate transformations

$$\omega = \frac{2mg}{m_0\, l}\sum_{n=1}^{\infty}\frac{1 - (-1)^n \cos\dfrac{a\pi^2 n^2}{lv}}{a^2(\pi n/l)^2 - v^2}.$$

**82** At vibrations of the rod on the elastic base, an additional force $\mathrm{d}q_1 = \mathrm{d}z\, ky$ acts upon its element. This force is directed against the displacement $y$. Therefore, the equation of vibrations has the form (see solution to Problem **75**)

$$\frac{\partial^2 y}{\partial t^2} + \frac{EJ_x}{m_0}\frac{\partial^4 y}{\partial z^4} + \frac{k}{m_0} y = 0. \qquad (1)$$

**83** According to the Rayleigh method, $T_{\max} = \Pi_1 + \Pi_2$ , where $\Pi_1$ and $\Pi_2$ are, respectively, the potential energies of the rod bending and of the base deformation. They are equal

$$\Pi_1 = \frac{1}{2}\int_0^l EJ_x y_1'' \mathrm{d}z; \quad \Pi_2 = \frac{1}{2}\int_0^l k y_1^2 \mathrm{d}z.$$

The maximum kinetic energy is

$$T_{\max} = \frac{p^2}{2}\int_0^l m_0 y_1^2 \mathrm{d}z.$$

The squared frequency of vibrations is

$$p^2 = \frac{EJ_x \int_0^l y_1''^2 \mathrm{d}z + k\int_0^l y_1^2 \mathrm{d}z}{m_0 \int_0^l y_1^2 \mathrm{d}z}.$$

Instead of $y_1(z)$ one can take the function $\sin\dfrac{\pi z}{l}$, which satisfies all boundary conditions of the problem. After transformations we have

$$p = \sqrt{\left(\frac{\pi}{l}\right)^4 \frac{EJ_x}{m_0} + km_0}.$$

**84** The differential equation of rod vibrations is similar to equation (1) of Problem **82**. We seek the solution in the form $y = y_1(z)\sin pt$.

For the function $y_1(z)$ we have

$$y_1^{IV} - \lambda^4 y_1 = 0 \qquad \left(\lambda^4 = \frac{p^2}{a^2} - \frac{k}{a^2 m_0}\right). \tag{1}$$

For the case of fixation shown in Fig. 1.66*a* the roots of the equation of frequencies (see solution to Problem **72**) are equal to

$$(\lambda l)_1 = 1.875; \qquad (\lambda l)_2 = 4.694; \qquad (\lambda l)_n = (2n-1)\pi/2 \qquad (n > 2).$$

Hence,

$$p_1 = \sqrt{\frac{(1.875)^4}{l^4}\frac{EJ_x}{m_0} + \frac{k}{m_0}}; \qquad p_2 = \sqrt{\frac{(4.69)^4}{l^4}\frac{EJ_x}{m_0} + \frac{k}{m_0}}.$$

For the case shown in Fig. 1.66*b* the equation of frequencies has the form

$$\cosh\lambda l \cdot \cos\lambda l - 1 = 0. \tag{2}$$

The first three roots of equation (2) can be determined graphically and are equal to

$$(\lambda l)_1 = 0; \qquad (\lambda l)_2 = 4.73; \qquad (\lambda l)_3 = 7.85.$$

The frequencies of vibration in this case are

$$p_1 = \sqrt{\frac{k}{m_0}}; \qquad p_2 = \sqrt{\frac{(4.73)^4}{l^4}\frac{EJ_x}{m_0} + \frac{k}{m_0}};$$
$$p_3 = \sqrt{\frac{(7.85)^4}{l^4}\frac{EJ_x}{m_0} + \frac{k}{m_0}}.$$

**85** The differential equation of rod vibrations on an elastic base has the form

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial z^4} + \frac{k}{m_0}y = \frac{P_0}{m_0}\,\delta\,(z - z_0)\,.$$

After calculations similar to those made in Problem **80**, we get

$$y(t,z) = \frac{2P_0}{m_0 l}\sum_{n=1}^{\infty} \sin\frac{\pi n z}{l}\,\frac{\left(\sin\frac{\pi n v t}{l} - \frac{v(\pi n/l)}{pn}\sin p_n t\right)}{\left[p_n^2 - \left(\frac{\pi n}{l}\right)^2 v^2\right]},$$

where $p_n = \sqrt{a^2\left(\frac{\pi n}{l}\right)^4 + \frac{k}{m_0}}$.

**86** The differential equation of rod bending under the action of a distributed load has the form

$$EJ_x\frac{\partial^4 y}{\partial z^4} = q_1(z,t).$$

In the case under consideration the distributed load $q_1(z,t)$ is represented by the force of inertia of both fluid and pipeline (Fig. 2.19). Using the Eulerian variables we can write

$$q_1(z,t) = -m_{\mathrm{P}}\frac{\partial^2 y}{\partial t^2} - m_{\mathrm{F}}\left(\frac{\partial^2 y}{\partial t^2} + 2w\frac{\partial^2 y}{\partial z \partial t} + w^2\frac{\partial^2 y}{\partial z^2}\right),$$

where $m_{\mathrm{P}}$ is the mass of a pipeline unit length, $m_{\mathrm{P}} = \varrho_{\mathrm{P}} F_{\mathrm{P}}$ $(F_{\mathrm{P}} = \pi D_m)$ and $m_{\mathrm{F}}$ is the mass of fluid per unit length of the pipeline.

We derive the following differential equation of transverse vibrations of the pipeline

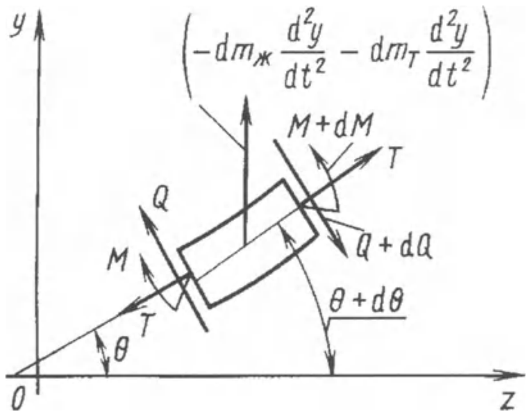


**Fig. 2.19.**

$$EJ_x\frac{\partial^4 y}{\partial z^4} + (m_P + m_F)\frac{\partial^2 y}{\partial t^2} + 2m_F w\frac{\partial^2 y}{\partial z \partial t} + m_F w^2\frac{\partial^2 y}{\partial z^2} = 0. \qquad (1)$$

Now we seek the solution to equation (1) in the form

$$y = y_1(z)e^{ipt}$$

For the function $y_1(z)$ we have the following differential equation:

$$\frac{d^4 y_1}{dz^4} - ap^2 y_1 + ibp\frac{dy_1}{dz} + c\frac{d^2 y_1}{dz^2} = 0, \qquad (2)$$

where $a = (m_P + m_F)/(EJ_x)$; $b = 2wm_F/(EJ_x)$; and $c = w^2 m_F/(EJ_x)$.

Let us seek the solution to equation (2) by the Galerkin method

$$y_1 = A\sin\frac{\pi z}{l} + B\sin\frac{2\pi z}{l}. \qquad (3)$$

The solution to equation (2) should satisfy the boundary conditions of the problem:

$$z = 0, \quad y_1 = 0, \quad y_1'' = 0;$$
$$z = l, \quad y_1 = l, \quad y_1'' = 0.$$

Substituting (3) into equation (2), multiplying sequentially the resulting expression by $\sin\frac{\pi z}{l}$ and by $\sin\frac{2\pi z}{l}$, and integrating it between 0 and $l$, we obtain the following system of two linear homogeneous equations for unknown constants $A$ and $B$:

$$\begin{aligned} &A\left[\left(\frac{\pi}{l}\right)^4 - ap^2 - c\left(\frac{\pi}{l}\right)^2\right] - B\frac{8ib}{3l}p = 0; \\ &A\frac{8ib}{3l}p + B\left[16\left(\frac{\pi}{l}\right)^4 - ap^2 - 4c\left(\frac{\pi}{l}\right)^2\right] = 0. \end{aligned} \qquad (4)$$

After calculations, we have the following values of the first two frequencies for a series of the velocities of fluid motion:

**Table 2.1.**

| $w$, m/s . . . . . . . . . . | 0 | 10 | 20 |
|---|---|---|---|
| $p_1$, s$^{-1}$ . . . . . . . . . . | 24.9 | 24.2 | 21.7 |
| $p_2$, s$^{-1}$ . . . . . . . . . . | 98.8 | 101.7 | 101 |

**87** The first frequency of vibrations of the pipeline becomes zero at the critical velocity of fluid flow. Calculating the determinant of systems of equations (4) in Problem **86**, we can obtain the equation of frequencies. This equation has a zero root if its free term is equal to zero:

$$\left[\left(\frac{\pi}{l}\right)^2 - c\right]\left[4\left(\frac{\pi}{l}\right)^2 - c\right] = 0 \qquad \left(c = \frac{m_{\mathrm{F}} w^2}{EJ_x}\right). \tag{1}$$

The lowest velocity (critical velocity) at which condition (1) is satisfied is

$$w_* = \frac{\pi}{l}\sqrt{\frac{EJ_*}{m_{\mathrm{F}}}}.$$

After substitution of numerical values we get $w = 47.8$ m/s.

**88** Figure 2.19 shows a pipeline element with forces acting upon it. The equation of small vibrations of the pipeline has the form

$$EJ_x \frac{\partial^4 y}{\partial z^4} + m_{\mathrm{P}} \frac{\partial^2 y}{\partial t^2} + +m_{\mathrm{F}} \left(\frac{\partial^2 y}{\partial t^2} + 2w \frac{\partial^2 y}{\partial z \partial t} + w^2 \frac{\partial^2 y}{\partial z^2}\right) - N \frac{\partial^2 y}{\partial z^2} = 0. \tag{1}$$

After transformations we have, taking the formula for $N$ into account (see statement of the problem),

$$\frac{\partial^4 y}{\partial z^4} + a \frac{\partial^2 y}{\partial t^2} + b \frac{\partial^2 y}{\partial z \partial t} + c \cos\alpha \frac{\partial^2 y}{\partial z^2} = 0, \tag{2}$$

where $a = (m_{\mathrm{P}} + m_{\mathrm{F}}) / (EJ_x)$; $b = 2w m_{\mathrm{F}} / (EJ_x)$; and $c = w^2 m_{\mathrm{F}} / (EJ_x)$

The characteristic equation has the form

$$\begin{aligned} &a^2 p^4 - p^2 \left[17a\left(\frac{\pi}{l}\right)^4 - 5ac\left(\frac{\pi}{l}\right)^2 \cos\alpha + \left(\frac{8}{3}\frac{b}{l}\right)^2\right] + \\ &+4\left(\frac{\pi}{l}\right)^4 \left[\left(\frac{\pi}{l}\right)^2 - y\cos\alpha\right]\left[4\left(\frac{\pi}{l}\right)^2 - c\cos\alpha\right] = 0. \end{aligned} \tag{3}$$

At $\alpha = 90°$ equation (3) has no zero roots, i.e., no critical velocity exists; at an arbitrary angle $\alpha$ the critical velocity is

$$w_* = \frac{\pi}{l}\sqrt{\frac{EJ_x}{m_F \cos\alpha}}.$$

and for $\alpha \to 90°$ $\quad w_* \to \infty$.

After calculations (at $\alpha = 90°$) we have the following values of frequencies

**Table 2.2.**

| $w$, m/s . . . . . . . . . . | 0 | 10 | 20 |
|---|---|---|---|
| $p_1$, $s^{-1}$ . . . . . . . . . . | 24.9 | 24.5 | 24.3 |
| $p_2$, $s^{-1}$ . . . . . . . . . . | 98.8 | 99.5 | 102.5 |

**89** In the case considered the flow velocity is variable along the length of the pipe. Since the fluid is incompressible, then

$$\varrho_P w_0 F_0 = \varrho_F w(z) F(z) \qquad \left(F(z) = \pi d^2(z)/4\right).$$

Hence, the velocity in the arbitrary section of the pipe is

$$w(z) = w_0 F_0 / F(z).$$

As a result, we have the following differential equation (see solution to Problem **86**):

$$\frac{\partial^2}{\partial z^2}\left(EJ_x\frac{\partial^2 y}{\partial z^2}\right) + \left(\varrho_P \pi d\delta + \varrho_F\frac{\pi d^2}{4}\right)\frac{\partial^2 y}{\partial t^2} + \\ + 2\varrho_F w_0 F_0 \frac{\partial^2 y}{\partial z \partial t} + \frac{\varrho_F w_0^2 F_0^2}{F}\frac{\partial^2 y}{\partial z^2} = 0.$$

**90** The differential equation of vibrations of a pipeline lying on an elastic base can be derived from equation (1) of the solution to Problem **88** by invoking an additional elastic force $ky$ that acts to the pipeline from the side of the elastic base. By this means, we can write

$$\frac{\partial^2 y}{\partial t^2} + \frac{EJ_x}{(m_1+m_2)}\frac{\partial^4 y}{\partial z^4} + \frac{2m_2 w}{(m_1+m_2)}\frac{\partial^2 y}{\partial z \partial t} + \\ + \frac{m_2 w^2}{(m_1+m_2)}\frac{\partial^2 y}{\partial z^2} + \frac{k}{(m_1+m_2)}y = 0. \qquad (1)$$

Now we seek the solution to equation (1) using the Galerkin method and assuming sequentially

$$y_1 = A \sin \frac{\pi z}{l} \sin p t; \tag{2}$$

$$y_2 = A \sin \frac{2\pi z}{l} \sin p t; \tag{3}$$

Solutions (2) and (3) allow one to determine in the first approximation the first and the second frequencies of vibrations, respectively.

Substituting (2) and (3) into equation (1), after some transformations, we have

$$p_1 = \sqrt{\frac{EJ_x}{(m_1 + m_2)} \left(\frac{\pi}{l}\right)^4 + \frac{k}{(m_1 + m_2)} - \frac{m_2 w^2}{(m_1 + m_2)} \left(\frac{\pi}{l}\right)^2};$$

$$p_2 = \sqrt{\frac{EJ_x}{(m_1 + m_2)} \left(\frac{2\pi}{l}\right)^4 + \frac{k}{(m_1 + m_2)} - \frac{m_2 w^2}{(m_1 + m_2)} \left(\frac{2\pi}{l}\right)^2}.$$

**91** One can assume that the axial compressing strength is constant over the rod length under small vibrations. The differential equation of transverse vibrations of the rod with allowance for the longitudinal force is a particular case of equation (1) in the solution to Problem **88**:

$$EJ_x \frac{\partial^4 y}{\partial z^4} + m_0 \frac{\partial^2 y}{\partial t^2} + N \frac{\partial^2 y}{\partial z^2} = 0. \tag{1}$$

We seek the solution to equation (1) in the form

$$y = \sum_n f_n(t) \sin \frac{\pi n z}{l}.$$

For the functions $f_n(t)$ we get

$$\ddot{f}_n + \left[\left(\frac{\pi n}{l}\right)^4 \frac{EJ_x}{m_0} - \frac{N}{m_0} \frac{\pi n}{l^2}\right] f_n = 0.$$

The desired frequencies of vibrations of the rod are

$$p_n = \sqrt{\left(\frac{\pi n}{l}\right)^4 \frac{EJ_x}{m_0} - \frac{N}{m_0} \left(\frac{\pi n}{l}\right)^2}.$$

**92** The differential equation of transverse vibrations of the rod (see solution to Problem **91**) has the form

$$EJ_x \frac{\partial^4 y}{\partial z^4} + m_0 \frac{\partial^2 y}{\partial t^2} + (N_0 + N_1 \sin \omega t) \frac{\partial^2 y}{\partial z^2} = 0. \tag{1}$$

We seek the solution to equation (1) in the form

$$y = \sum_n f_n(t) \sin \frac{\pi n z}{l} \qquad (n = 1, 2, 3, \ldots).$$

After transformations, we obtain

$$\ddot{f}_n + \left[ \frac{EJ_x}{m_0} \left( \frac{\pi n}{l} \right)^4 - \frac{N}{m_0} \left( \frac{\pi n}{l} \right)^2 - \frac{N_1}{m_0} \left( \frac{\pi n}{l} \right)^2 \sin \omega t \right] f_n = 0.$$

Now let us make a conversion to a new independent variable (assuming $\omega t = 2\tau - \pi/2$):

$$\ddot{f}_n + [a_n + 2q_n \cos 2\tau] f_n = 0,$$

where

$$a_n = \frac{4}{\omega^2} \left[ \frac{EJ_x}{m_0} \left( \frac{\pi n}{l} \right)^4 - \frac{N}{m_0} \left( \frac{\pi n}{l} \right)^2 \right]; \qquad 2q_n = \frac{4}{\omega^2} \frac{N_1}{m_0} \left( \frac{\pi n}{l} \right)^2$$

The numerical coefficients for several $n$ are as follows:

**Table 2.3.**

| $n$ . . . . . . . . . . | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $a_n$ . . . . . . . . . . | 0.535 | 15.2 | 83 | 269 |
| $q_2$ . . . . . . . . . . | 0.05 | 0.22 | 0.49 | 0.88 |

The corresponding points $(a_n, q_n)$ are inside the stable regions on the diagram (see Appendix A of Part I).

**93** In this case the coefficients have the values given in Table 2.4.

**Table 2.4.**

| $n$ . . . . . . . . . . | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $a_n$ . . . . . . . . . . | 1.66 | 20 | 93.95 | 292.81 |
| $q_2$ . . . . . . . . . . | 0.05 | 0.22 | 0.49 | 0.88 |

The corresponding points $(a_n, q_n)$ are inside the stable regions on the diagram (see Appendix A of Part I).

**94** In order to estimate the error in determining the frequencies, let us consider the differential equation of rod vibrations under the action of the stretching force $Q_{10}$ constant over the rod length ($Q_{10} = N$), taking into account the bending stiffness (a special case of the equation derived in Problem **88**):

$$EJ_x \frac{\partial^4 y}{\partial z^4} + m_0 \frac{\partial^2 y}{\partial t^2} - Q_{10} \frac{\partial^2 y}{\partial z^2} = 0. \tag{1}$$

The solution to equation (1) is sought in the form

$$y = \sum_n^{\infty} f_n(t) \sin \frac{\pi n z}{l}. \tag{1}$$

Substituting (1) into equation (1) and taking advantage of the principle of virtual displacements, after appropriate transformations we arrive at the following equations

$$m_0 \ddot{f}_n + \left[ Q_{10} \left( \frac{\pi n}{l} \right)^2 + EJ_x \left( \frac{\pi n}{l} \right)^4 \right] f_n = 0 \qquad (n = 1, 2, \ldots).$$

The frequencies of vibrations are

$$p_n = \frac{\pi n}{l} \sqrt{\frac{Q_{10}}{m_0} \left[ 1 + \frac{EJ_x}{Q_{10}} \left( \frac{\pi n}{l} \right)^2 \right]} \qquad (n = 1, 2, \ldots).$$

If the value of $EJ_x$ is small (as is usually the case in real strings), then

$$p_n = \frac{\pi n}{l} \sqrt{\frac{Q_{10}}{m_0}} \left[ 1 + \frac{1}{2} \frac{EJ_x}{Q_{10}} \left( \frac{\pi n}{l} \right)^2 \right].$$

Substituting numerical data we obtain

$$p_n = \frac{\pi n}{l} \sqrt{\frac{Q_{10}}{m_0}} \cdot (1 + 50 \cdot 10^{-4} n^2).$$

At small values of $n (n < 10)$ the error does not exceed 5%. For larger values of $n$, determination of the frequencies of vibrations of a real string, using the formula for a perfectly flexible string, yields a fairly large error.

**95** The equation of transverse vibrations of a rod lying on an elastic base is derived in Problem **82**. We add to it a term dependent on the longitudinal force:

$$EJ_x \frac{\partial^4 y}{\partial z^4} + m_0 \frac{\partial^2 y}{\partial t^2} + ky + N \frac{\partial^2 y}{\partial z^2} = 0. \tag{1}$$

Then we seek the solution to equation (1) in the form

$$y = \sum_n f_n(t) \sin \frac{\pi n z}{l} \qquad (n = 1, 2, 3, \ldots).$$

For the functions $f_n(t)$ we get the equations of the form

$$\ddot{f}_n + \left[ \left( \frac{\pi n}{l} \right)^4 \frac{EJ_x}{m_0} + \frac{k}{m_0} - \frac{N}{m_0} \left( \frac{\pi n}{l} \right)^2 \right] f_n = 0. \tag{2}$$

The frequencies of vibrations of the rod are

$$p_n = \sqrt{\left( \frac{\pi n}{l} \right)^4 \frac{EJ_x}{m_0} + \frac{k}{m_0} - \frac{N}{m_0} \left( \frac{\pi n}{l} \right)^2}.$$

## 2.5 Vibrations of rectilinear and curvilinear rods

**96** The equation of parametric vibrations of the rod has the following form in the dimensionless notation (see paragraph C.2 in Appendix C)

$$L = \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^4 u}{\partial \eta^4} + P_0 \cos\omega\tau \left[H(\eta) - H(\eta - 0.5)\right] \frac{\partial^2 u}{\partial \eta^2} = 0. \tag{1}$$

The solution to equation (1) we seek in the form

$$u = f^{(1)}(\tau) \sin \pi\eta. \tag{2}$$

Taking advantage of the principle of virtual displacements, we have after transformations

$$\int_0^1 L \sin \pi\eta \, d\eta = 0.$$

or

$$f^{(1)} + \left(\pi^4 - \pi^2 0.5 \cos\omega\tau\right) f^{(1)} = 0. \tag{3}$$

When solving equation (3) by the Rayleigh method we assume

$$f^{(1)} = a_1 \cos\frac{\omega}{2}\tau + b_1 \sin\frac{\omega}{2}\tau. \tag{4}$$

Substituting (4) into equation (3) we have two relations

$$\left(\pi^4 + \frac{\pi^2}{4} - \frac{\omega^2}{4}\right) b_1 = 0; \tag{5}$$

$$\left(\pi^4 - \frac{\pi^2}{4} - \frac{\omega^2}{4}\right) a_1 = 0. \tag{6}$$

Equating the expressions in brackets to zero, we find the boundaries of the main region of instability.

**97** The equation of bending parametric vibrations of the rod has the form

$$\frac{\partial^4 u}{\partial \eta^4} - (P_{10} + P_{100} \cos\omega_0\tau) \frac{\partial^2 u}{\partial \eta^2} + \alpha \frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial \tau^2} + ku = 0. \tag{1}$$

Setting $u = f(\tau) \sin \pi\eta$ we write the equation for $f$ as

$$\ddot{f} + \alpha \dot{f} + [k + \pi^4 + \pi^2 (P_{10} + P_{100} \cos\omega_0\tau)] f = 0,$$

or

$$\ddot{f} + \alpha \dot{f} + (a_1 + a_2 \cos\omega_0\tau) f = 0, \tag{2}$$

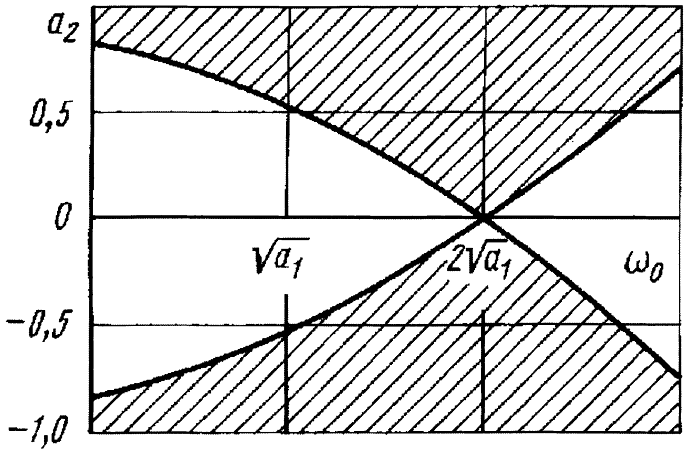


**Fig. 2.20.**

where $a_1 = k + \pi^4 + \pi^2 P_{10}$ and $a_2 = \pi^2 P_{100}$. In accordance with the Rayleigh method we assume

$$f = A_1 \cos \frac{\omega_0}{2}\tau + B_1 \sin \frac{\omega_0}{2}\tau. \tag{3}$$

Substituting (3) into equation (2) we have

$$\begin{aligned} &\left[a_1 - \left(\frac{\omega_0}{2}\right)^2 + \frac{a_2}{2}\right] A_1 + \frac{\alpha\omega_0}{2} B_1 = 0; \\ &-\frac{\alpha\omega_0}{2} A_1 + \left[a_1 - \left(\frac{\omega_0}{2}\right)^2 - \frac{a_2}{2}\right] B_1 = 0. \end{aligned} \tag{4}$$

Equating the determinant of system (4) to zero, we get the equation

$$\left[a_1 - \left(\frac{\omega_0}{2}\right)^2 + \frac{a_2}{2}\right]\left[a_1 - \left(\frac{\omega_0}{2}\right)^2 - \frac{a_2}{2}\right] + \frac{\alpha^2\omega_0^2}{4} = 0, \tag{5}$$

from which we determine the boundaries of the main region of the parametric resonance. At $\alpha = 0$ we find

$$a_2 = 2\left[-a_1 + \left(\frac{\omega_0}{2}\right)^2\right]; \quad a_2 = 2\left[a_1 - \left(\frac{\omega_0}{2}\right)^2\right].$$

The region of instability is hatched in Fig. 2.20.

**98** In order to solve the equation of free vibrations, one should know the bending deflection $u$ of the rod loaded with the force $P$, i.e., it is necessary to solve the equation

$$\frac{\partial^4 u_{x_2}^{(0)}}{\partial \eta^4} = P\,\delta(\eta - 0.5). \tag{1}$$

Taking the boundary condition into account, we obtain from expression (1)

$$u_{x_2}^{(0)} = \frac{(\eta - 0.5)^3}{6} PH(\eta - 0.5) - \frac{1.375}{12} P\eta^3 + \frac{0.375}{4} P\eta^2.$$

Let us consider two variants of solving this problem.

1. The variant of solution using the Krylov functions (it can be realized only for the rod of constant section) allows one to get the answer in the analytical form.

Considering the force of inertia of the mass as a concentrated force applied to the rod, we have the following equation:

$$\frac{\partial^4 u_{x_2}}{\partial \eta^4} + \frac{\partial^2 u_{x_2}}{\partial \tau^2} = -n_2 \frac{\partial^2 u_{x_2}}{\partial \tau^2}\,\delta(\eta - 0.5) \qquad \left(n_2 = \frac{m}{m_0\,l}\right). \tag{2}$$

Assuming $u_{x_2} = u_{x_{20}} \mathrm{e}^{\mathrm{i}\lambda\tau}$, the solution to equation (2) expressed through the Krylov functions has the form

$$\begin{aligned} u_{x_{20}} = {} & c_1 \mathrm{K}_1 + c_2 \mathrm{K}_2 + c_3 \mathrm{K}_3 + c_4 \mathrm{K}_4 - \\ & - \frac{\mathrm{K}_4(\eta - 0.5)}{\lambda_0^3} n_2 \lambda_0 u_{x_2}(0.5) H(\eta - 0.5), \end{aligned} \tag{3}$$

where $K_j(\lambda_0 \eta)$ $(\lambda_0 = \sqrt{\lambda})$.

Since $u_{x_{20}} = u'_{x_{20}} = 0$ for $\eta = 0$, $c_1 = c_2 = 0$, and, therefore,

$$u_{x_{20}}(0.5) = c_3 \mathrm{K}_3(0.5) + c_4 \mathrm{K}_4(0.5). \tag{4}$$

Excluding $u_{x_2}(0.5)$ from equation (3) we write

$$\begin{aligned} u_{x_{20}} = {} & [\mathrm{K}_3(\eta) - \lambda_0 n_2 \mathrm{K}_4(\eta - 0.5) \mathrm{K}_3(0.5) H(\eta - 0.5)]\, c_3 + \\ & + [\mathrm{K}_4(\eta) - \lambda_0 n_2 \mathrm{K}_4(\eta - 0.5) \mathrm{K}_4(0.5) H(\eta - 0.5)]\, c_4. \end{aligned} \tag{5}$$

Since the boundary conditions should be satisfied, i.e., $u(1) = 0$ for $\eta = 1$, we obtain from (5) the following system of two homogeneous equations

$$\begin{aligned} & [\mathrm{K}_3(\lambda_0 1) - \lambda_0 n_2 \mathrm{K}_4(\lambda_0 0.5) \mathrm{K}_3(\lambda_0 0.5)]\, c_3 + \\ & \qquad + \left[\mathrm{K}_4(\lambda_0 1) - \lambda_0 n_2 \mathrm{K}_4^2(\lambda_0 0.5)\right] c_4 = 0; \\ & [\mathrm{K}_1(\lambda_0 1) - \lambda_0 n_2 \mathrm{K}_2(\lambda_0 0.5) \mathrm{K}_3(\lambda_0 0.5)]\, c_3 + \\ & \qquad + [\mathrm{K}_2(\lambda_0 1) - \lambda_0 n_2 \mathrm{K}_2(\lambda_0 0.5) \mathrm{K}_4(\lambda_0 0.5)]\, c_4 = 0. \end{aligned} \tag{6}$$

From the condition $D = 0$, where $D$ is the determinant of system (6), we find $\lambda_{0j}$. The dimensionless frequencies are $\lambda_j = \sqrt{\lambda_{0j}}$. Then, for every $\lambda_j$ we determine $c_3^{(j)}$, assuming $c_4^{(j)} = 1$:

$$c_3^{(j)} = -\frac{\mathrm{K}_4(\lambda_{0j}1) - \lambda_{0j}n_2\mathrm{K}_2^2(\lambda_{0j}0.5)}{\mathrm{K}_3(\lambda_{0j}1) - \lambda_{0j}n_2\mathrm{K}_4(\lambda_{0j}0.5)\mathrm{K}_3(\lambda_{0j}0.5)}.$$

As a result we get the eigen functions

$$\begin{aligned}\varphi^{(j)}(\eta) = u_{x_2}^{(j)} = \{&\mathrm{K}_3(\lambda_{0j}\eta)- \\ &- \lambda_{0j}n_2\mathrm{K}_4\left[\lambda_{0j}(\eta-0.5)\right]\mathrm{K}_3(\lambda_{0j}0.5)H(\eta-0.5)\}c_3^{(j)}+ \\ &+ \{\mathrm{K}_4(\lambda_{0j}\eta) - \lambda_{0j}n_2\mathrm{K}_4\left[\lambda_{0j}(\eta-0.5)\right]\mathrm{K}_4(\lambda_{0j}0.5)H(\eta-0.5)\}.\end{aligned}$$

When solving equation (2) we assumed that $u_{x_2} = u_{x_2 0}\mathrm{e}^{\mathrm{i}\lambda_j\tau}$, therefore, the general solution to equation (1) has the form

$$u_{x_2} = \sum_{j=1}^{n}\left(C^{(j)}\cos\lambda_j\tau + B^{(j)}\sin\lambda_j\tau\right)\varphi^{(j)}(\eta). \tag{7}$$

For $\tau = 0$ we have the following initial conditions:

$$u_{x_2}(0,\eta) = u_{x_2 0}(\eta) = \sum_{j=0}^{n} c^{(j)}\varphi^{(j)}(\eta);$$

$$\dot{u}_{x_2}(0,\eta) = 0.$$

From the second initial condition it follows that $B^{(j)} = 0$. Then we find the arbitrary constants $C^{(j)}$ from the equations

$$\int_0^1 u_{x_2 0}\varphi^{(k)}\mathrm{d}\eta = \sum_{j=2}^{n} C^{(j)}\int_0^1 \varphi^{(l)}\varphi^{(k)}\mathrm{d}\eta \qquad (k = 1,2,\ldots,n).$$

Having determined $C^{(j)}$ and $B^{(j)}$, we obtain the solution to equation (1) that satisfies the initial and boundary conditions:

$$u_{x_2}(\tau,\eta) = \sum_{j=1}^{n} C^{(j)}\varphi^{(j)}(\eta)\cos\lambda_j\tau. \tag{8}$$

The reaction force in the hinge is

$$R = Q_{x_2}(\tau,1) = -u_{x_2}'''(\tau,1) = -\sum_{j=1}^{n} C^{(j)}\varphi'''^{(j)}(1)\cos\lambda\tau, \tag{9}$$

where $\varphi'''^{(j)}(1) = \left[\lambda_{0j}^3 K_4(\lambda_{0j}1) - \lambda_{0j}^4 n_2 K_1(\lambda_{0j}0.5)K_3(\lambda_{0j}0.5)\right]c_3^{(j)} + \lambda_{0j}^3 K_1(\lambda_{0j}1) - \lambda_{0j}^4 n_2 K_1(\lambda_{0j}0.5)K_4(\lambda_{0j}0.5)$.

As a consequence, we have the solution to the problem in the analytical form (except for $\lambda_{0j}$ that were determined numerically).

2. The variant of numerical solution with the use of computers (can be realized for a rod of any section). In solving the problem numerically we take advantage of the system of equations (a special case of equations (C.19) from Appendix C) that, for example, for a rectilinear rod of variable section has the form (taking the inertia of rotation into account)

$$\begin{aligned}
&n_1(\eta)\frac{\partial^2 u_{x2}}{\partial\tau^2} - \frac{\partial Q_{x2}}{\partial\eta} = -n_2\frac{\partial^2 u_{x2}}{\partial\tau^2}\,\delta(\eta-0.5);\\
&J_{33}(\eta)\frac{\partial^2\vartheta_3}{\partial\tau^2} - \frac{\partial M_{x3}}{\partial\eta} - Q_{x_2} = 0;\\
&\frac{\partial\vartheta_3}{\partial\eta} - \frac{1}{A_{33}(\eta)}M_{x_3} = 0;\\
&\frac{\partial u_{x_2}}{\partial\eta} - \vartheta_3 = 0,
\end{aligned} \tag{10}$$

or, in the vector form

$$\mathbf{A}^{(1)}\frac{\partial^2\overline{\mathbf{Z}}}{\partial\tau^2} + \frac{\partial\overline{\mathbf{Z}}}{\partial\eta} + \mathbf{A}^{(2)}\overline{\mathbf{Z}} = \triangle\overline{\boldsymbol{\Phi}}. \tag{11}$$

Here,

$$\mathbf{A}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & -n_1\\ 0 & 0 & -J_{33} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix};\ \mathbf{A}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & -\dfrac{1}{A_{33}} & 0 & 0\\ 0 & 0 & -1 & 0\end{bmatrix};$$

$$\overline{\mathbf{Z}} = \begin{bmatrix} Q_{x2}\\ M_{x3}\\ \vartheta_3\\ u_{x2}\end{bmatrix};\ \triangle\overline{\boldsymbol{\Phi}} = \begin{bmatrix} -n_2\dfrac{\partial^2 u_{x2}}{\partial\tau^2}\\ 0\\ 0\\ 0\end{bmatrix}\delta(\eta-0.5);$$

$$n_1 = \frac{m(s)}{m_0} = \frac{F(l\eta)}{F(0)};\quad J_{33} = \frac{J_3(l\eta)}{F(0)\,l^2};\quad A_{33} = \frac{J_3(l\eta)}{J_3(0)};$$

and $F(0)$ and $J_3(0)$ are, respectively, the section area of the rod and its moment of inertia with respect to the axis $x_3$ at $\eta = 0$ (see Appendix C). Assuming $\overline{Z} = \overline{Z}_0\mathrm{e}^{\mathrm{i}\lambda\tau}$ we get from equation (11)

$$\overline{\mathbf{Z}}_0' + \mathbf{B}(\lambda;\eta)\overline{\mathbf{Z}}_0 = \triangle\overline{\boldsymbol{\Phi}}_0 \qquad \left(\mathbf{B}(\lambda;\eta) = \mathbf{A}^{(2)} - \lambda^2\mathbf{A}^{(1)}\right), \tag{12}$$

where $\triangle\overline{\boldsymbol{\Phi}}_0 = \triangle\overline{\boldsymbol{\Phi}}_0\left[n_2\lambda^2 u_{x_2}\,\delta(\eta-0.5),0,0,0\right]^{\mathrm{T}}$.

The algorithm of numerical determination of eigen values $\lambda_j$ and eigen functions $\overline{\mathbf{Z}}_0^{(j)}$ is presented in Appendix D.

Having determined $\lambda_j$ and $\overline{\mathbf{Z}}_0^{(j)}$ we find the following solution to equation (10) (see Appendix E)

$$\overline{\mathbf{Z}}(\tau,\eta) = \sum_{j=1}^{n} \left( C^{(j)} \overline{\mathbf{Z}}_0^{(j)} \cos\lambda_j \tau + B^{(j)} \overline{\mathbf{Z}}_0^{(j)} \sin\lambda_j \tau \right). \tag{13}$$

From equation (12) we obtain the expression for $u_{x_2}$:

$$u_{x_2} = \sum_{j=1}^{n} \left( C^{(j)} \mathbf{Z}_{04}^{(j)} \cos\lambda_j \tau + B^{(j)} \mathbf{Z}_{04}^{(j)} \sin\lambda_j \tau \right). \tag{14}$$

It follows from the initial conditions that

$$B^{(j)} = 0;$$
$$u_{x_2}(0,\eta) = u_{x_2}^{(0)} = \sum_{j=1}^{n} C^{(j)} Z_{04}^{(j)}.$$

We determine the arbitrary constants $C^{(j)}$ numerically from the system of equations

$$\int_0^1 u_{x_2} Z_{04}^{(k)} \mathrm{d}\eta = \sum_{j=1}^{n} C^{(j)} Z_{04}^{(j)} Z_{04}^{(k)} \mathrm{d}\eta \qquad (k = 1, 2, \ldots, n).$$

Since $Q_{x_2} = \sum_{j=1}^{n} C^{(j)} Z_{01}^{(j)} \cos\lambda_j\tau$, the reaction force in the hinge is

$$R = Q_{x_2}\Big|_{\eta=1} = \sum_{j=1}^{n} C^{(j)} Z_{01}^{(j)}(1) \cos\lambda_j \tau.$$

**99** The impulsive force $P_{x_2}$ and moment $M_{x_3}$ act relative to the center of mass (point $O$). They are equal to

$$P_{x_2} = -J\cos\alpha \cdot \bar{\mathrm{i}}_2; \qquad M_{x_3} = -Jh\sin\alpha \cdot \bar{\mathrm{i}}_2.$$

We do not take into account the projection $\overline{\mathbf{J}}$ onto the axis $x_1$, because the rod is assumed to be nonstretchable. In what follows we presume that the momentum $|\overline{\mathbf{J}}|$, mass $m$, and the moment of inertia $J_0$ are reduced to the dimensionless form. After termination of action of $\overline{\mathbf{J}}$, the mass $m$ attains the linear and angular velocities that are, respectively, equal to

$$\dot{u}_{x_2}(0,1) = -n_2\cos\alpha; \qquad \dot{\vartheta}_3(0,1) = -n_3 \sin\alpha, \tag{1}$$

where $n_2 = \dfrac{J}{mp_0\, l}$ and $n_3 = \dfrac{Jh}{J_0 p_0}$.

From equation (13) we can write (see solution to Problem **98**)

$$\begin{aligned}\vartheta_3(\tau,\eta) &= \sum_{j=1}^{n}\left(C^{(j)}\cos\lambda_j\tau + B^{(j)}\sin\lambda_j\tau\right) Z_{03}^{(j)};\\ u_{x_2}(\tau,\eta) &= \sum_{j=1}^{n}\left(C^{(j)}\cos\lambda_j\tau + B^{(j)}\sin\lambda_j\tau\right) Z_{04}^{(j)}.\end{aligned} \tag{2}$$

This problem has the following initial conditions:

$$\begin{aligned}&\tau=0, \quad u_{x_2}(0,\eta)=0;\\ &\tau=0, \quad \dot{\vartheta}_3(0,\eta) = -n_3\sin\alpha\cdot\delta(\eta-1),\\ &\qquad\quad\ \dot{u}_{x_2}(0,\eta) = -n_2\cos\alpha\cdot\delta(\eta-1).\end{aligned}$$

From the first condition we get $C^{(j)}=0$. The second condition results in the relations

$$\sum_{j=1}^{n} B^{(j)}\lambda_j Z_{03}^{(j)} = -n_3\sin\alpha\cdot\delta(\eta-1); \tag{3}$$

$$\sum_{j=1}^{n} B^{(j)}\lambda_j Z_{04}^{(j)} = -n_2\cos\alpha\cdot\delta(\eta-1). \tag{4}$$

Let us determine the values of $B^{(j)}$ at which conditions (3) and (4) are satisfied most precisely. Consider the integral of the sum of squared errors:

$$I = \int_0^1 \left\{\left[\frac{J\sin\alpha\cdot h}{J_0}\delta(\eta-1) + \sum_{j=1}^{n} B^{(j)}\lambda_j Z_{03}^{(j)}\right]^2 + \right.$$
$$\left. + \left[\frac{J\cos\alpha}{n}\delta(\eta-1) + \sum_{j=1}^{n} B^{(j)}\lambda_j Z_{04}^{(j)}\right]^2\right\} \mathrm{d}\eta.$$

From the condition of minimum $I$ we have the following system of equations:

$$\frac{\partial I}{\partial B^{(k)}} = 0; \qquad (k=1,2,\ldots,n)$$

or

$$\sum_{j=1}^{n} B^{(j)}\lambda_j \int_0^1 \left(Z_{03}^{(j)}Z_{03}^{(k)} + Z_{04}^{(j)}Z_{04}^{(k)}\right)\mathrm{d}\eta + n_3\sin\alpha\cdot Z_{0k}^{(k)}(1) +$$
$$+ n_2\cos\alpha\cdot Z_{04}^{(k)}(1) = 0.$$

After determination of the arbitrary constants $B^{(j)}$, we find the angle of rotation of the mass $m$ from equations (2):

$$\vartheta_3(\tau, 1) = \sum_{j=1}^{n} B^{(j)} Z_{03}^{(j)}(1) \sin \gamma_j \tau.$$

**100** In addition to the term $\dfrac{\partial [Q_1(\eta) u']}{\partial \eta}$ the term $-q_1 \dfrac{\partial u}{\partial \eta}$ also appears in the equation of transverse vibrations of the rod in case of follow-up axial distributed $q_1$ Taking advantage of the algorithm for solving Problem **86** with allowance for the axial forces $Q_1$ and $q_1$ we have the following equation (in the dimensionless form)

$$\frac{\partial^4 u}{\partial \eta^4} - \frac{\partial}{\partial \eta}\left(Q_1(\eta)\frac{\partial u}{\partial \eta}\right) + q_1 \frac{\partial u}{\partial \eta} + 2w_0 \frac{\partial^2 u}{\partial \tau \partial \eta} + w_0^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \tau^2} = 0. \quad (1)$$

Let us consider the approximate solution to equation (1) taking the two-term approximation (since, under the conditions of the problem, it is required to determine the first two frequencies)

$$u = f^{(1)}(\tau)\varphi^{(1)}(\eta) + f^{(2)}(\tau)\varphi^{(2)}(\eta), \quad (2)$$

where $\varphi^{(1)}$ and $\varphi^{(2)}$ are any independent functions satisfying the boundary conditions of the given problem: $\eta = 0$, $u = u' = 0$; $\eta = 1$, $u = u' = 0$. One can take as such functions the eigen functions satisfying the given boundary conditions. If the Krylov functions are used, we obtain

$$\varphi^{(j)}(\eta) = \mathrm{K}_4(\lambda_{0j}\eta) - \frac{\mathrm{K}_4(\lambda_{0j}1)}{\mathrm{K}_3(\lambda_{0j}1)} \mathrm{K}_3(\lambda_{0j}\eta).$$

Substituting solution (2) into equation (1), we have

$$\begin{aligned} L = {} & \ddot{f}^{(1)}\varphi^{(1)} + \ddot{f}^{(2)}\varphi^{(2)} + 2w_0\left(\dot{f}^{(1)}\varphi'^{(1)} + \dot{f}^{(2)}\varphi'^{(2)}\right) + \\ & + w_0^2\left(f^{(1)}\varphi''^{(1)} + f^{(2)}\varphi''^{(2)}\right) - f^{(1)}\left(Q_1\varphi'^{(1)}\right) - f^{(2)}\left(Q_1\varphi'^{(2)}\right) + \\ & + f^{(1)}q_1\varphi'^{(1)} + f^{(2)}q_1\varphi'^{(2)} + f^{(1)}\left(\varphi^{(1)}\right)^{\mathrm{IV}} + f^{(2)}\left(\varphi^{(2)}\right)^{\mathrm{IV}}. \end{aligned} \quad (3)$$

Taking advantage of the principle of virtual displacements we can write two equations

$$\int_0^1 L\varphi^{(1)} \mathrm{d}\eta = 0; \qquad \int_0^1 L\varphi^{(2)} \mathrm{d}\eta = 0,$$

or

$$\begin{aligned} h_{11}\ddot{f}^{(1)} + h_{12}\ddot{f}^{(2)} + b_{11}\dot{f}^{(1)} + b_{12}\dot{f}^{(2)} + c_{11}f^{(1)} + c_{12}f^{(2)} = 0; \\ h_{21}\ddot{f}^{(1)} + h_{22}\ddot{f}^{(2)} + b_{21}\dot{f}^{(1)} + b_{22}\dot{f}^{(2)} + c_{21}f^{(1)} + c_{22}f^{(2)} = 0. \end{aligned} \quad (4)$$

Here

$$h_{ij} = \int_0^1 \varphi^{(i)}\varphi^{(j)}\mathrm{d}\eta; \qquad b_{ij} = 2w_0 \int_0^1 \left(\varphi'^{(j)}\varphi^{(i)}\right)\mathrm{d}\eta;$$

$$c_{ij} = \int_0^1 \left[w_0^2\varphi'^{(j)} + \left(\varphi^{(j)}\right)^{\mathrm{IV}} - \left(Q_1\varphi'^{(j)}\right)' + q_1\varphi'^{(j)}\right]\varphi^{(i)}\mathrm{d}\eta.$$

The coefficients $b_{ij}$ satisfy the condition $b_{ij} = -b_{ji}$, therefore, $b_{11} = b_{22} = 0$. Consider in more detail the coefficients $c_{ij}$. Integrating every term by parts, with allowance made for homogeneous boundary conditions to which the functions $\varphi^{(j)}(\eta)$ and their first derivatives satisfy, we get

$$c_{ij} = c_{ij}^{(0)} + c_{ij}^{(1)},$$

where $c_{ij}^{(0)} = \int_0^1 \left(\varphi''^{(j)}\varphi''^{(k)} - w_0^2\varphi'^{(j)}\varphi'^{(k)} + Q_1\varphi'^{(j)}\varphi'^{(k)}\right)\mathrm{d}\eta;$

$$c_{ij}^{(1)} = -c_{ij}^{(1)} = q_1 \int_0^1 \varphi'^{(j)}\varphi^{(k)}\mathrm{d}\eta; \qquad c_{11}^{(1)} = c_{22}^{(1)} = 0.$$

We seek the solution to system of equations (4) in the form

$$f^{(1)} = f_{10}\mathrm{e}^{\lambda\tau}; \qquad f^{(2)} = f_{20}\mathrm{e}^{\lambda\tau}. \quad (5)$$

Substituting (5) into (4), after transformations we have the following characteristic equation

$$\det \begin{bmatrix} h_{11}\lambda^2 + c_{11}^{(0)} & h_{12}\lambda^2 + c_{12}^{(0)} + c_{12}^{(1)} + b_{12}\lambda \\ h_{12}\lambda^2 + c_{12}^{(0)} - c_{12}^{(1)} - b_{12}\lambda & h_{22}\lambda^2 + c_{22}^{(0)} \end{bmatrix} = 0; \quad (6)$$

When composing the above determinant, we have used the following properties of the coefficients: $h_{12} = h_{21}$, $c_{12}^{(0)} = c_{21}^{(0)}$, $b_{12} = -b_{21}$ and $c_{12}^{(1)} = -c_{21}^{(1)}$. From (6) we derive the characteristic equation

$$a_0\lambda^4 + a_2\lambda^2 + a_3\lambda + a_4 = 0. \quad (7)$$

where $a_0 = h_{11}h_{22} - h_{12}^2$; $a_1 = 0$; $a_2 = h_{11}c_{22}^{(0)} + h_{22}c_{11}^{(0)} - 2h_{12}c_{12}^{(0)} + b_{12}^2$; $a_3 = 2c_{12}^{(1)}b_{12}$ and $a_4 = c_{11}^{(0)}c_{22}^{(0)} - \left(c_{12}^{(0)}\right)^2 + \left(c_{12}^{(1)}\right)^2$. From equation (7) we find the complex eigen values

$$\lambda_{1,2} = a_1 \pm \mathrm{i}\beta_1; \qquad \lambda_{3,4} = a_2 \pm \mathrm{i}\beta_2,$$

i.e., the problem is dissipative.

**101** The equation of forced vibrations in the dimensionless form looks like

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^4 u}{\partial \eta^4} = q_0 \cos \omega_0 \tau H(\eta - 0.5), \tag{1}$$

where $(\omega_0 = \omega_/ p_0)$. Assuming $u = u_0 \cos \omega_0 \tau$, we get

$$u_0^{\mathrm{IV}} - \omega_0^2 u_0 = q_0 H(\eta - 0.5). \tag{2}$$

Now let us consider a more general case when the right-hand side of equation (2) is an arbitrary function. For a rod with constant section

$$u_0' = \vartheta_{30}; \qquad u_0'' = M_{x30}; \qquad u_0''' = Q_{x20},$$

therefore, equation (2) can be represented as the following system of four first-order equations:

$$\begin{aligned} &Q_{x20}' + \omega_0^2 u_0 = b_1(\eta); \\ &M_{x30} + Q_{x20} = 0; \\ &\vartheta_{30} - M_{x30} = 0; \\ &u_0 - \vartheta_{30} = 0, \end{aligned} \tag{3}$$

or, in the vector form

$$\overline{\mathbf{Z}}_0 + \mathbf{A}\overline{\mathbf{Z}}_0 = \overline{\mathbf{b}} \qquad \left(\overline{\mathbf{b}} = [b_1, 0, 0, 0]^{\mathrm{T}}\right). \tag{4}$$

The general solution to equation (4) has the following form (for equations with constant coefficients)

$$\overline{\mathbf{Z}}_0 = \mathbf{K}(\eta)\overline{\mathbf{c}} + \int_0^\eta \mathbf{K}(\eta - h_1)\overline{\mathbf{b}}(h)\mathrm{d}h_1. \tag{5}$$

The fundamental matrix $\mathbf{K}(\eta)$ depends on $\omega_0$. From equation (5) we get the expression for the displacements of points of the axial line of the rod:

$$u_0 = \sum_{j=1}^{4} k_{4j} c_j + \int_0^\eta \widetilde{k}_{41}(\eta - h_1) b_1(h)\mathrm{d}h_1. \tag{6}$$

For the problem under consideration

$$b_1(h) = g_0 H(h - 0.5); \qquad \widetilde{k}_{41} = \mathrm{e}^{\omega_0(\eta - h_1)},$$

therefore,

$$\int_0^\eta k_{41} b_1 \mathrm{d}h_1 = -\frac{q_0}{\omega_0}\left(1 - \mathrm{e}^{\omega_0(\eta-0.5)}\right) H(\eta - 0.5).$$

From the boundary conditions at $\eta = 0$ it follows that $c_1 = c_2 = 0$. We calculate the arbitrary constants $c_3$ and $c_4$ from the boundary conditions at $\eta = 1$:

$$\begin{aligned} M_{x30}(1) &= u_0''(1) = k_{23}(1)c_3 + k_{24}(1)c_4 - \omega_0 q_0\left(1 - \mathrm{e}^{0.5\omega_0}\right); \\ Q_{x20}(1) &= u_0'''(1) = k_{13}(1)c_3 + k_{14}(1)c_4 - \omega_0^2 q_0\left(1 - \mathrm{e}^{0.5\omega_0}\right). \end{aligned} \tag{7}$$

Upon determining $c_3$ and $c_4$ from system (7) we find the amplitude of steady-state vibrations of the rod at the point $K$

$$u_{0K}(1) = k_{43}(1)c_3 + k_{44}(1)c_4 - \frac{q_0}{\omega_0}\left(1 - \mathrm{e}^{0.5\omega_0}\right).$$

**102** One can assume that the unknown moment $\mathfrak{M}_0 \cos\omega\tau$ acts in the section $K$, so that the equation of forced vibrations in the dimensionless form is

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^4 u}{\partial \eta^4} = \mathfrak{M}_0 \cos\omega\tau\, \delta'(\eta - 1). \tag{1}$$

Under steady-state vibrations we seek the solution to equation (1) in the form

$$u = u_0(\eta)\cos\omega\tau.$$

Then, from (1) we obtain

$$u_0(\eta) = \sum_{i=1}^{4} c_i \mathrm{K}_i\left(\sqrt{\omega}\eta\right) + \frac{\mathfrak{M}_0}{\omega}\mathrm{K}_3\left[\sqrt{\omega}(\eta - 1)\right] H(\eta - 1). \tag{2}$$

It follows from the boundary conditions at $\eta = 0$ that $c_1 = c_2 = 0$. At $\eta = 1$ solution (2) should satisfy the following three conditions: $u_0 = 0$; $u_0' = \vartheta_{30}$; and $u_0'' = \mathfrak{M}_0$, or

$$\begin{aligned} &c_3 K_3(1) + c_4 K_4(1) = 0; \\ &c_3\sqrt{\omega}K_2(1) + c_4\sqrt{\omega}K_3(1) = \vartheta_{30}; \\ &c_3\omega K_1(1) + c_4\omega K_2(1) - M_0 = 0. \end{aligned} \tag{3}$$

From system (3) we determine $c_3, c_4$, and $\mathfrak{M}_0$ as functions of $\theta_{30}$:

$$c_3 = \alpha_1\vartheta_{30}; \qquad c_4 = \alpha_2\vartheta_{30}; \qquad \mathfrak{M}_0 = \alpha_3\vartheta_{30}, \tag{4}$$

and derive the expression for $u_0(\eta)$:

$$u_0(\eta) = \left[\alpha_1 K_3(\sqrt{\omega}\eta) + \alpha_2 K_4(\sqrt{\omega}\eta)\right] \vartheta_{30} + \frac{\alpha_3 \vartheta_{30}}{\omega} K_3 \left[\sqrt{\omega}(\eta - 1)\right]. \qquad (5)$$

The amplitude of the moment in the embedment is

$$M_0 = u_0''(0) = \omega \alpha_1 \vartheta_{30}.$$

**103** The equation of forced vibrations of the rod in the dimensionless form looks like

$$L(u) = \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^4 u}{\partial \eta^4} - P(\tau)\,\delta(\eta - 0.5) = 0. \qquad (1)$$

We seek the solution to equation (1) by the approximate method (principle of virtual displacements), restricting ourselves in accordance with the conditions of the problem to the two-term approximation:

$$u = f^{(1)}(\tau)\varphi^{(1)}(\eta) + f^{(2)}(\tau)\varphi^{(2)}(\eta). \qquad (2)$$

One can choose the first two modes of free vibrations of the rod as the functions $\varphi^{(i)}(\eta)$. For the boundary conditions of the problem we have the following eigen functions:

$$\varphi^{(j)} = \mathrm{K}_4(\lambda_{0j}\eta) - \frac{\mathrm{K}_4(\lambda_{0j}1)}{\mathrm{K}_3(\lambda_{0j}1)} \mathrm{K}_3(\lambda_{0j}\eta), \qquad (3)$$

where $\mathrm{K}_i$ are the Krylov functions.

We represent the virtual displacements in the form

$$\delta u^{(j)} = \delta b_j \varphi^{(j)}.$$

Substituting solution (2) into equation (1), in accordance with the principle of virtual displacements we get after some transformations

$$\ddot{f}^{(1)} + \lambda_{01}^4 f^{(1)} = \frac{P(\tau)}{h_{11}} \varphi^{(1)}(0.5); \qquad (4)$$

$$\ddot{f}^{(2)} + \lambda_{02}^4 f^{(2)} = \frac{P(\tau)}{h_{22}} \varphi^{(2)}(0.5), \qquad (5)$$

where $h_{11} = \int\limits_0^1 \left(\varphi^{(1)}\right)^2 \mathrm{d}\eta$ and $h_{22} = \int\limits_0^1 \left(\varphi^{(2)}\right)^2 \mathrm{d}\eta$.

For the problem under consideration the equations for determination of $f^{(i)}$ turned out to be independent (by virtue of orthogonality of the functions $\varphi^{(j)}$, therefore, we choose only one of them, for example, equation (4). Its solution for an arbitrary right-hand side has the form

$$\begin{aligned} f^{(1)} &= c_1 \cos \lambda_1 \tau + c_2 \sin \lambda_1 \tau + \frac{\varphi^{(1)}(0.5)}{h_{11}\lambda_1} \int\limits_0^\tau \sin \lambda_1(\tau - \tau_1) P \mathrm{d}\tau_1; \\ \dot{f}^{(1)} &= -\lambda_1 c_1 \sin \lambda_1 \tau + \lambda_1 c_2 \cos \lambda_1 \tau + \frac{\varphi^{(1)}}{h_{11}} \int\limits_0^\tau \cos \lambda_1(\tau - \tau_1) P \mathrm{d}\tau_1. \end{aligned} \qquad (6)$$

According to the Duffing method we find the functions $f^{(1)}$ and $\dot{f}^{(1)}$ at $\tau = 0$ and $\tau = T$:

$$\begin{aligned}
&f^{(1)}(0) = c_1; \qquad \dot{f}^{(1)}(0) = c_2\lambda_1 \qquad (\lambda_1 = \lambda_{01}^2);\\
&f^{(1)}(T) = f^{(1)}(0)\cos\lambda_1 T + \frac{\dot{f}^{(1)}(0)}{\lambda_1}\sin\lambda_1 T +\\
&\qquad\qquad + \frac{a_1}{\lambda_1}\int_0^T \sin\lambda_1(T-\tau_1)P\,\mathrm{d}\tau_1;\\
&\dot{f}^{(1)}(T) = -\lambda_1 f^{(1)}(0)\sin\lambda_1 T + \dot{f}^{(1)}(0)\cos\lambda_1 T +\\
&\qquad\qquad + a_1\int_0^T \cos\lambda_1(T-\tau_1)P\,\mathrm{d}\tau_1,
\end{aligned} \tag{7}$$

where $a_1 = \varphi^{(1)}(0.5)/h_{11}$.

Under steady-state vibrations the conditions $f^{(1)}(0) = f^{(1)}(T)$ and $\dot{f}^{(1)}(0) = \dot{f}^{(1)}(T)$ must be satisfied. Therefore, we obtain from system (7) two inhomogeneous equations to determine $f^{(1)}(0)$ and $\dot{f}^{(1)}(0)$:

$$\begin{aligned}
&(1-\cos\lambda_1 T)f^{(1)}(0) - \frac{\sin\lambda_1 T}{\lambda_1}\dot{f}^{(1)}(0) = b_1;\\
&\sin\lambda_1 f^{(1)}(0) + (1-\cos\lambda_1 T)\dot{f}^{(1)}(0) = b_2,
\end{aligned} \tag{8}$$

where

$$b_1 = \frac{a_1 P_0}{\lambda_1}\left(\int_0^{T/2}\sin\lambda_1(T-\tau_1)\mathrm{d}\tau_1 - \int_{T/2}^{T}\sin\lambda_1(T-\tau_1)\mathrm{d}\tau_1\right);$$

$$b_2 = a_1 P_0\left(\int_0^{T/2}\cos\lambda_1(T-\tau_1)\mathrm{d}\tau_1 - \int_{T/2}^{T}\cos\lambda_1(T-\tau_1)\mathrm{d}\tau_1\right).$$

Having determined $f^{(1)}(0)$ and $\dot{f}^{(1)}(0)$, we find the solution to equation (4) on the interval $0 \le \tau \le T$:

$$f^{(1)}(\tau) = f^{(1)}(0)\cos\lambda_1\tau + \frac{\dot{f}^{(1)}(0)}{\lambda_1}\sin\lambda_1\tau +$$
$$+\begin{cases}
\dfrac{a_1P_0}{\lambda_1^2}(1-\cos\lambda_1\tau) & \left(0\le\tau\le\dfrac{T}{2}\right);\\
\dfrac{a_1P_0}{\lambda_1^2}\left\{\left(1-\cos\lambda_1\dfrac{T}{2}\right)-\right. & \\
\qquad\left.-\left[1-\cos\lambda_1\left(\tau-\dfrac{T}{2}\right)\right]\right\} & \left(\dfrac{T}{2}\le\tau\le T\right).
\end{cases}$$

Similar expression we have for the function $f^{(2)}(\tau)$:

$$f^{(2)}(\tau) = f^{(2)}(0)\cos\lambda_2\tau + \frac{\dot f^{(2)}(0)}{\lambda_2}\sin\lambda_2\tau +$$
$$+\begin{cases} \dfrac{a_2P_0}{\lambda_2^2}(1-\cos\lambda_2\tau) & \left(0 \le \tau \le \dfrac{T}{2}\right); \\ \dfrac{a_2P_0}{\lambda_2^2}\left\{\left(1-\cos\lambda_2\dfrac{T}{2}\right) - \right. & \\ \left. \quad -\left[1-\cos\lambda_2\left(\tau-\dfrac{T}{2}\right)\right]\right\} & \left(\dfrac{T}{2}\le\tau\le T\right). \end{cases}$$

The solutions derived are valid under the condition that the determinants of system (8) and similar system for $f^{(2)}(0)$ and $\dot f^{(2)}(0)$) of the form

$$D = \begin{vmatrix} (1-\cos\lambda_i T) & -\sin\lambda_i T/\lambda_i \\ \sin\lambda_i T & (1-\cos\lambda_i T) \end{vmatrix}$$

are not equal to zero.

**104** The equation of small vibrations of the rod has the following form

$$m_0\frac{\partial^2 u_{x_2}}{\partial t^2} + A_{33}\frac{\partial^4 u_{x_2}}{\partial x_1{}^4} + k_1 u_{x_2} - \left(mg - m\frac{\partial^2 u_x}{\partial t^2}\right)\delta(x_1 - vt) = 0, \qquad (1)$$

where $A_{33} = EJ_{x_3}$ is the bending stiffness of the rod, and $k_1$ is the stiffness coefficient for the base. Assuming $u_{x_2} = ul$, $x_1 = \eta l$, $\tau = p_0 t$, and $p_0 = \left(A_{33}/m_0 l^4\right)^{1/2}$, we reduce equation (1) to the dimensionless form:

$$L = \frac{\partial^2 u}{\partial\tau^2} + \frac{\partial^4 u}{\partial\eta^4} + \frac{\partial^2 u}{\partial\tau^2}\delta(\eta - v_0\tau) + ku - n_2\,\delta(\eta - v_0\tau) = 0, \qquad (2)$$

where $n_1 = m/(m_0 l)$; $k = k_1 l^4/A_{33}$; $n_2 = mg\,l^2/A_{33}$, and $v_0 = v/(lp_0)$.

Under the two-term approximation we seek the solution to equation (2)) in the form

$$u = f^{(1)}(\tau)\sin\pi\eta + f^{(2)}(\tau)\sin 2\pi\eta. \qquad (3)$$

Taking advantage of the principle of virtual displacements, we can write

$$\int_0^1 L\sin\pi\eta\,\mathrm{d}\eta = 0; \qquad \int_0^1 L\sin 2\pi\eta\,\mathrm{d}\eta = 0,$$

so that after transformations we have

$$\begin{aligned} h_{11}\ddot f^{(1)} + h_{12}\ddot f^{(2)} + \left(\pi^4 + k\right)f^{(1)} &= 2n_2\sin\pi v_0\tau; \\ h_{21}\ddot f^{(1)} + h_{22}\ddot f^{(2)} + \left(16\pi^4 + k\right)f^{(2)} &= 2n_2\sin 2\pi v_0\tau, \end{aligned} \qquad (4)$$

where $h_{11} = 1 + 2n_1 \sin^2 \pi v_0 \tau$; $h_{22} = 1 + 2n_1 \sin^2 2\pi v_0 \tau$; and $h_{12} = h_{21} = 2n_1 \sin \pi v_0 \tau \sin 2\pi v_0 \tau$.

As a result, we obtained a system of equations with periodic coefficients. The main feature of the given problem is the fact that the time of the process (the time of motion of the mass along the rod) is limited. Therefore, the vibrations of the rod are unsteady. The time of motion of the mass $m$ along the rod is $t_K = l/v$, and the dimensionless time is $\tau_K = 1/v_0$. Let us write system of equations (4) in the form

$$\ddot{\overline{\mathbf{f}}} + \mathbf{H}^{-1}\mathbf{B}\overline{\mathbf{f}} = \mathbf{H}^{-1}\overline{\mathbf{b}}. \tag{5}$$

Here

$$\mathbf{H} = \begin{bmatrix} 1 + 2n_1 \sin^2 \pi v_0 \tau & 2n_1 \sin \pi v_0 \tau \sin 2\pi v_0 \tau \\ 2n_1 \sin \pi v_0 \tau \sin 2\pi v_0 \tau & 1 + 2n_1 \sin^2 2\pi v_0 \tau \end{bmatrix};$$

$$\mathbf{B} = \begin{bmatrix} \pi^4 + k & 0 \\ 0 & 16\pi^4 + k \end{bmatrix}; \quad \overline{\mathbf{b}} = \begin{bmatrix} 2n_2 \sin \pi v_0 \tau \\ 2n_2 \sin 2\pi v_0 \tau \end{bmatrix}.$$

The determinant of the matrix $\mathbf{H}$

$$D = 1 + 2n_1 \sin^2 \pi v_0 \tau + 2n_1 \sin^2 2\pi v_0 \tau$$

is always greater than zero, i.e., the matrix $\mathbf{H}$ is not degenerate.

Equation (5) can be solved numerically at zero initial data. As a result, at $\tau = \tau_K$ we get $f^{(1)}(\tau_K)$ and $f^{(2)}(\tau_K)$. Since for the chosen approximate solution the rotation angle of the rod in any section is equal to

$$\vartheta = \frac{\partial u}{\partial \eta} = \pi \cos \pi\eta \cdot f^{(1)}(\tau) + 2\pi \cos 2\pi\eta \cdot f^{(2)}(\tau),$$

then the angle of rotation in the section $K$ at the moment when the mass $m$ rolls off the rod (at $\eta = 1$) is equal to

$$\vartheta_{3K} = -\pi f^{(1)}(\tau_K) + 2\pi f^{(2)}(\tau_K).$$

**105** Under vibrations, an inertial load from the side of railcars acts upon the rod (rails). This load can be considered (in the limit) as distributed. Two contact forces are applied to every railcar, and they can be reduced to the resultant force $J_i$ and the moment $\mu_i$ (Fig. 2.21*a*):

$$J_i = -m_i \frac{\partial^2 u_2}{\partial t^2}; \qquad \mu_i = -J_0 \frac{\partial^2 \vartheta}{\partial t^2}, \tag{1}$$

where $m$ and $J_0$ are the mass of a railcar and its moment of inertia with respect to the axis perpendicular to the plane of drawing and passing through the railcar center, and $u_2$ is the displacements of points of the rod axial line.

Figure 2.21*b* demonstrates an element of the rod with all forces applied to it. Taking advantage of the d'Alembert's principle we obtain the equations of

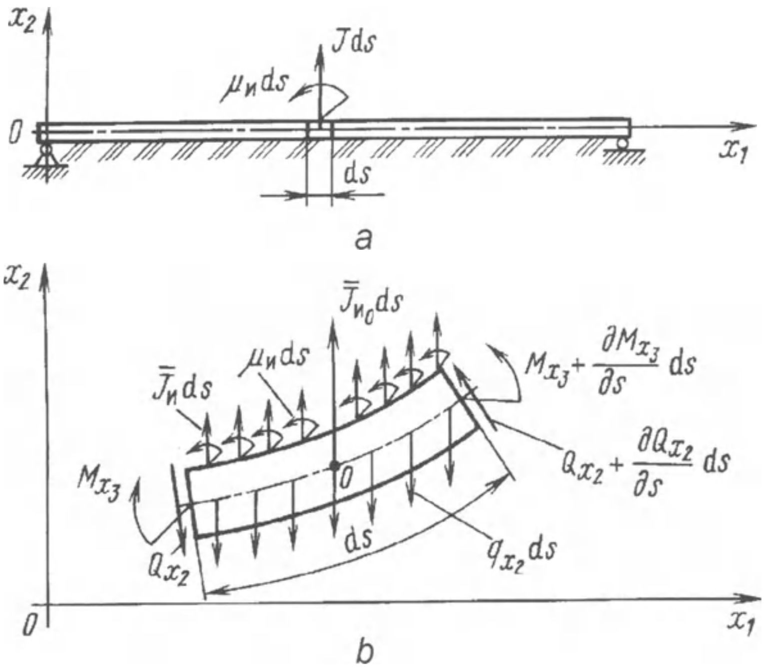


**Fig. 2.21.**

motion (translational motion along the axis $x2$ and rotational about the axis perpendicular to the plane of drawing) in the dimensionless form:

$$J_{i0} + \frac{\partial Q_{x_2}}{\partial \eta} + J_i + q_{x_2} = 0; \tag{2}$$

$$\frac{\partial M_3}{\partial \eta} + Q_{x_2} + \mu_i = 0, \tag{3}$$

where $J_{i0} = -m_0 \dfrac{\partial^2 u_2}{\partial \tau^2}$ and $q_{x_2} = -ku_2$ (equation (3) does not involves the term accounting for the inertia of rotation of the rod element).

Let us write the equations relating the moment $M_{x_3}$ to the rod curvature (see Appendix A; for the rod of constant section $A_{33} = 1$) and the displacement $u_2$ of the axial line points to the angle of section rotation (for small deflections of the axial line of the rod from the straight line):

$$\frac{\partial \vartheta_3}{\partial \eta} - M_{x_3} = 0; \tag{4}$$

$$\frac{\partial u_2}{\partial \eta} - \vartheta_3 = 0. \tag{5}$$

Since the railcars move with the velocity $v$, we make use of the Eulerian coordinates. As a result, converting to the dimensionless notation, we have

$$\begin{aligned} J_i &= -n_{11}\left(\frac{\partial^2 u_2}{\partial\tau^2} + 2v_0\frac{\partial^2 u_2}{\partial\tau\partial\eta} + v_0^2\frac{\partial^2 u_2}{\partial\eta^2}\right); \\ \mu_i &= -J_0^{(0)}\left(\frac{\partial^3 u_2}{\partial\tau^2\partial\eta} + 2v_0\frac{\partial^3 u_2}{\partial\tau\partial\eta^2} + v_0^2\frac{\partial^3 u_2}{\partial\eta^3}\right) \qquad \left(\vartheta_3 = \frac{\partial u_2}{\partial\eta}\right), \end{aligned} \tag{6}$$

where $n_{11} = m/m_0 l$; $J_0^{(0)} = J_0/m_0 l^3$; and $v_0 = v/lp_0$ are dimensionless coefficients.

Excluding $\vartheta_3$, $M_{x_3}$, and $Q_{x_2}$ from equations (2)-(5), we obtain the equation of free vibrations of the rod with an account of the moving load:

$$\begin{aligned} L = (1+n_{11})\frac{\partial^2 u_2}{\partial\tau^2} + \left(1 - J_0^{(0)}v_0^2\right)\frac{\partial^4 u_2}{\partial\eta^4} + 2v_0 n_{11}\frac{\partial^2 u_2}{\partial\tau\partial\eta} - \\ -2v_0 J_0^{(0)}\frac{\partial^4 u_2}{\partial\tau\partial\eta^3} - J_0^{(0)}\frac{\partial^4 u_2}{\partial\tau^2\partial\eta^2} + n_{11}v_0^2\frac{\partial^2 u_2}{\partial\eta^2} + ku_2 = 0. \end{aligned} \tag{7}$$

Assuming that the virtual displacements

$$\delta u_2 = \sum_{i=1}^{n} \delta a_i\, u_2^{(i)},$$

where $u_2^{(i)}$ is the functions satisfying the boundary conditions of the problem (for hinged fixity of the rod ends $u_2^{(i)} = \sin\pi i\eta$), and restricting ourselves to a two-term approximation, we find the approximate solution to equation (7) as

$$u_2 = f^{(1)}{u_2}^{(1)} + f^{(2)}{u_2}^{(2)}.$$

In accordance with the principle of virtual displacements we have

$$\int_0^1 L\left({u_2}^{(1)}, {u_2}^{(2)}\right){u_2}^{(1)}\mathrm{d}\eta = 0;$$

$$\int_0^1 L\left({u_2}^{(1)}, {u_2}^{(2)}\right){u_2}^{(2)}\mathrm{d}\eta = 0,$$

or, after transformations,

$$\begin{aligned} a_{11}\ddot{f}_1 + a_{12}\ddot{f}_2 + b_{11}\dot{f}_1 + b_{12}\dot{f}_2 + c_{11}f_1 + c_{12}f_2 = 0; \\ a_{21}\ddot{f}_1 + a_{22}\ddot{f}_2 + b_{21}\dot{f}_1 + b_{22}\dot{f}_2 + c_{21}f_1 + c_{22}f_2 = 0. \end{aligned} \tag{8}$$

Here

$$a_{11} = \frac{1}{2}\left[(1+n_{11}) + \pi^2 J_0^{(0)}\right]; \qquad a_{12} = 0;$$
$$a_{21} = 0; \qquad a_{22} = \frac{1}{2}\left[(1+n_{11}) + 4\pi^2 J_0^{(0)}\right];$$
$$b_{11} = 0; \qquad b_{12} = -\frac{8}{3}\left(n_{11} + 4\pi^2 J_0^{(0)}\right);$$
$$b_{21} = \frac{8}{3}\left(n_{11} + \pi^2 J_0^{(0)}\right); \qquad b_{22} = 0;$$
$$c_{11} = \frac{1}{2}\left[\left(1 - J_0^{(0)}\right)\pi^4 - n_{11}\pi^2 + k\right]; \qquad c_{12} = 0;$$
$$c_{21} = 0; \qquad c_{22} = \frac{1}{2}\left[16\pi^2\left(1 - J_0^{(0)}\right) - n_{11}4\pi^2 + k\right].$$

Assuming that $f_1 = f_{10}\mathrm{e}^{\lambda\tau}$ and $f_2 = f_{20}\mathrm{e}^{\lambda\tau}$, after some transformations we get the characteristic equation

$$a_0\lambda^4 + a_2\lambda^2 + a_4 = 0, \tag{9}$$

where $a_0 = a_{11}a_{22}$; $a_2 = a_{11}c_{22} + a_{22}c_{11} + b_{12}^2$; and $a_4 = c_{11}c_{22}$. The roots of equation (9) are

$$\lambda_{1,2} = \pm\mathrm{i}\beta_1, \ \lambda_{3,4} = \pm\mathrm{i}\beta_2,$$

where the frequencies are equal to

$$\beta_1 = \sqrt{\frac{a_2 - \sqrt{a_2^2 - 4a_4a_0}}{2a_0}}, \text{ and } \beta_2 = \sqrt{\frac{a_2 + \sqrt{a_2^2 - 4a_4a_0}}{2a_0}}.$$

**106** Figure 2.22*a* demonstrates a position of the rod at an arbitrary instant. Projecting the forces onto the axis $y$ (Fig. 2.22*b*) we obtain

$$-m_0\mathrm{d}z\frac{\partial^2 y}{\partial t^2} - \mathrm{d}Q + \mathrm{d}z\Delta q_y + (N + \mathrm{d}N)\sin\alpha' - N\sin\alpha = 0,$$

or $\left(\text{since } \dfrac{\partial Q}{\partial z} = EJ_x\dfrac{\partial^4 y}{\partial z^4}\right)$

$$EJ_x\frac{\partial^4 y}{\partial z^4} + m_0\frac{\partial^2 y}{\partial t^2} - \Delta q_y - \frac{\partial}{\partial z}\left(N\frac{\partial y}{\partial z}\right) = 0, \tag{1}$$

where $\alpha' = \alpha + \mathrm{d}\alpha$. The distributed load $\Delta q_y$ acting upon the rod under vibrations is equal to $\Delta q_y = m_0\Omega^2 y$. The distributed load $q_z$ remains invariable at small vibrations of the rod. The longitudinal strength $N$ depends on $q_z = m_0\Omega^2 z$:

$$N = \int_z^l q_z\mathrm{d}z = \frac{m_0\Omega^2}{2}\left(l^2 - z^2\right).$$

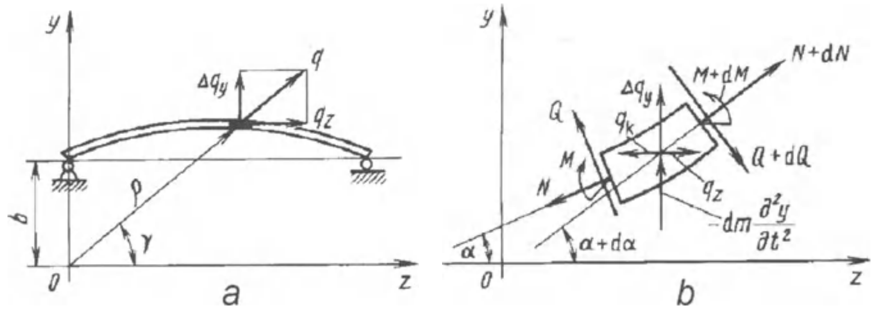


**Fig. 2.22.**

Let us substitute the expressions for $N$ and $\Delta q_y$ into equation (1):

$$L(y) = EJ_x \frac{\partial^4 y}{\partial z^4} + m_0 \frac{\partial^2 y}{\partial t^2} - \Omega^2 m_0 y + \\ + m_0 \Omega^2 z \frac{\partial y}{\partial z} - \frac{m_0 \Omega^2}{2} \left(l^2 - z^2\right) \frac{\partial^2 y}{\partial z^2} = 0. \tag{2}$$

Additionally, the distributed Coriolis force $q_C = 2m_0 \Omega \dot{y}$ directed along the $z$ axis acts on the rod during its vibrations. Therefore, more exact formula for the longitudinal force looks like

$$N = \int_z^l \left(q_z - 2m_0 \Omega \dot{y}\right) \mathrm{d}z.$$

Since in equation (1) the force $N$ is multiplied by $\frac{\partial y}{\partial z}$, one can neglect the term $\int_z^l 2m_0 \Omega \frac{\partial y}{\partial t} \mathrm{d}z \frac{\partial y}{\partial z}$ as a value of the second order of smallness in comparison to $\int_z^l q_z \mathrm{d}z \frac{\partial y}{\partial z}$. Consequently, the influence of the Coriolis force can also be neglected.

To determine the approximate values of vibration frequencies we seek the solution to equation (2) in the form

$$y = y_{11} \sin pt + y_{12} \sin pt = \left(A_1 \sin \frac{\pi z}{l} + A_2 \sin \frac{2\pi z}{l}\right) \sin pt. \tag{3}$$

The solution should satisfy the boundary conditions of the problem

$$z = 0,\ y = y'' = 0;$$
$$z = l,\ y = y'' = 0.$$

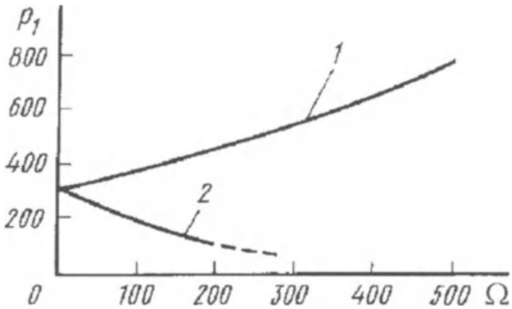


**Fig. 2.23.**

Using the Galerkin method we arrive at two relationships:

$$\int_0^l L(y_{11}, y_{12})y_{11}\,\mathrm{d}z = 0; \qquad \int_0^l L(y_{11}, y_{12})y_{12}\,\mathrm{d}z = 0.$$

After integration we have two homogeneous equations for unknowns $A_1$ and $A_2$:

$$A_1\left[EJ_x\left(\frac{\pi}{l}\right)^4 - m_0p^2 + \frac{25}{12}m_0\Omega^2\right] - \frac{20}{9}m_0\Omega^2A_2 = 0;$$

$$\frac{20}{9}m_0\Omega^2A_1 + A_2\left[EJ_x\left(\frac{2\pi}{l}\right)^4 - m_0p^2 + \frac{145}{12}m_0\Omega^2\right] = 0.$$

The first two frequencies of rod vibrations are equal to

$$p_{1,2} = \sqrt{\left(a \pm \sqrt{a^2 - 4b}\right)/2},$$

where

$$a = 17\frac{EJ_x}{m_0}\left(\frac{\pi}{l}\right)^4 + 14\Omega^2;$$

$$b = 16\left(\frac{EJ_x}{m_0}\right)^2\left(\frac{\pi}{l}\right)^8 + 44\frac{EJ_x}{m_0}\left(\frac{\pi}{l}\right)^4\Omega^2 + 19\Omega^4.$$

The variation of the frequency of vibrations $p_1$ is plotted in Fig. 2.23 (curve 1) as a function of $\Omega$.

**107** The differential equation of rod vibrations (see solution to Problem **106**) has the following form for the considered case of fixing

$$EJ_x\frac{\partial^4 y}{\partial z^4} + m_0\frac{\partial^2 y}{\partial t^2} - m_0\Omega^2 y + m_0\Omega^2 z\frac{\partial y}{\partial z} + \frac{m_0\Omega^2}{2}z^2\frac{\partial^2 y}{\partial z^2} = 0.$$

The frequencies of vibrations are

$$p_{1,2} = \sqrt{\left(a \pm \sqrt{a^2 - 4b}\right)/2},$$

where

$$a = 17\frac{EJ_x}{m_0}\left(\frac{\pi}{l}\right)^4 + 11\Omega^2;$$

$$b = 16\left(\frac{EJ_x}{m_0}\right)^2\left(\frac{\pi}{l}\right)^8 - 54.3\frac{EJ_x}{m_0}\left(\frac{\pi}{l}\right)^4\Omega^2 + 18\Omega^4.$$

The variation of $p_1$ versus the disk angular velocity $\Omega$ is plotted in Fig. 2.23 (curve 2).

**108** The differential equation of bending vibrations of the rod is similar to the equation in Problem **106**, except for the fact that now the longitudinal strength is

$$N = m_0\Omega^2\left(lz - z^2\right)/2,$$

The equation of vibrations of the rod takes on the form (see solution to Problem **106**)

$$EJ_x\frac{\partial^4 y}{\partial z^4} + m_0\frac{\partial^2 y}{\partial t^2} - \Omega^2 m_0 y - \\ -m_0\Omega^2\left(\frac{l}{2} - z\right)\frac{\partial y}{\partial z} - \frac{m_0\Omega^2}{2}\left(lz - z^2\right)\frac{\partial^2 y}{\partial z^2} = 0.$$

The frequencies of vibrations are

$$p_{1,2} = \sqrt{\left(a \pm \sqrt{a^2 + 4b}\right)/2},$$

where

$$a = 17\frac{EJ_x}{m_0}\left(\frac{\pi}{l}\right)^4 + 15.76\Omega^2;$$

$$b = \left[\frac{EJ_x}{m_0}\left(\frac{\pi}{l}\right)^4 - 0.426\Omega^2\right]\left[16\frac{EJ_x}{m_0}\left(\frac{\pi}{l}\right)^4 + 2\Omega^2\right] + \frac{16}{9}\Omega^4.$$

**109** Figure 2.24 shows a position of the system at an arbitrary instant. The bending deflection $y$ and the angle $\varphi$ characterize the deviation of the system from the dynamical equilibrium position in the field of centrifugal forces of inertia. Projecting the forces onto the $y$ axis we obtain (see solution to Problem **106**)

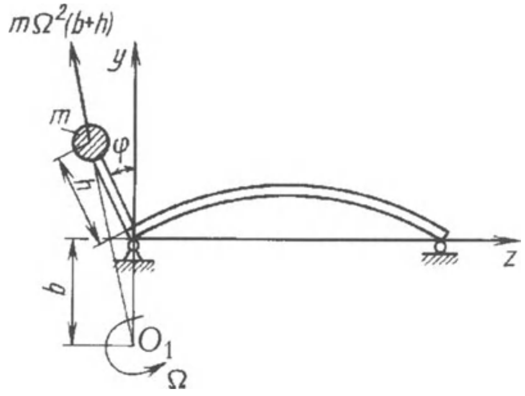


**Fig. 2.24.**

$$EJ_x \frac{\partial^4 y}{\partial z^4} + m_0 \frac{\partial^2 y}{\partial t^2} + m_0 \Omega^2 y - \frac{\partial}{\partial z}\left(N \frac{\partial y}{\partial z}\right) = 0, \tag{1}$$

where $N = m_0 \Omega^2 \left(l^2 - z^2\right)/2$.

The equation of vibrations of the mass $m$ has the form

$$J_0 \ddot{\varphi} = -m_0 \Omega^2 (h+b)d - EJ_x \left.\frac{\partial^2 y}{\partial z^2}\right|_{z=0},$$

where $J_0$ is the moment of inertia of the mass relative to hinge $O_1$ $(J_0 = mh^2)$; $EJ_x(\partial^2 y/\partial z^2)_{z=0}$ is the elastic moment acting upon the mass from the side of the spring; $d$ is the arm of force $m_0\Omega^2(h+b)d$ relative to the hinge (at small vibrations the Coriolis force moment $\left[m \frac{\partial z}{\partial t}(\Omega + \dot{\varphi})\right]$ with respect to the point $O_1$ is equal to zero).

Since $d = hb\varphi/(h+b)$ and $\varphi = \partial y/\partial z|_{z=0}$, we obtain the following boundary condition, to which the solution to equation (1) should satisfy

$$J_0 \left.\frac{\partial^3 y}{\partial t^2 \partial z}\right|_{z=0} + m_0 \Omega^2 hb \left.\frac{\partial y}{\partial z}\right|_{z=0} + EJ_x \left.\frac{\partial^2 y}{\partial z^2}\right|_{z=0}.$$

The remaining boundary conditions have the form

$$z = 0,\ y = 0; \qquad z = l,\ y = y'' = 0.$$

**110** Using the Rayleigh method we determine the maximum values of kinetic and potential energies of the system in its relative motion:

$$T_{\max} = \frac{p^2}{2} J_0 \left(\frac{\partial y_1}{\partial z}\right)^2_{z=0} + \frac{p^2}{2} \int_0^l m_0 {y_1}^2 \mathrm{d}z; \tag{1}$$

$$\Pi_{\max} = \Pi_1 + \Pi_{q_z} + \Pi_{q_y} + \Pi_2.$$

Here, $\Pi_1$ is the potential energy of spring bending; $\Pi_{q_z}$ and $\Pi_{q_y}$ are the variations of the potential of distributed forces $q_z$ and $q_y$, respectively. In this case, $\Pi_{q_z} = -A_{q_z}$ and $\Pi_{q_y} = -A_{q_y}$ ($A_{q_z}$ and $A_{q_y}$ are the works done by the forces $q_z$ and $q_y$ in moving through the displacements due to vibrations); while $\Pi_2 = A$ ($A$ is the work of the forces of inertia acting upon the concentrated mass $m$).

In expanded form, the expression for the potential energy looks like

$$\Pi_{\max} = \frac{1}{2}\int_0^l EJ_x\left(\frac{\partial^2 y_1}{\partial z^2}\right)dz + \int_0^l q_z\triangle\,dz - \frac{1}{2}\int_0^l q_y y_1\,dz + \frac{1}{2}m\Omega^2 bh\left(\frac{\partial y_1}{\partial z}\right)^2, \tag{2}$$

where $\triangle = \frac{1}{2}\int_0^z \left(\frac{\partial y_1}{\partial z}\right)^2 dz$.

The last term in equation (2) represents the work of centrifugal forces that act on the mass $m$ when it is deflected by the angle $\varphi$.

Figure 2.24 demonstrates the forces acting upon the mass $m$. The work done by the inertia forces is

$$A = \int_0^z F_z\,dz - \int_y^h F_y\,dy,$$

where $F_z = m\Omega^2 z$ and $F_y = m\Omega^2(b+y)$.

After integration we have

$$A = m\Omega^2\frac{z^2}{2} - m\Omega^2\left[b(h-y) + \frac{1}{2}(h^2-y^2)\right].$$

Since $z = h\sin\varphi$ and $y = h\cos\varphi$, we get after substitution and transformations

$$A = m\Omega^2\left[\frac{h^2}{2}\sin^2\varphi - hb(1-\cos\varphi) - \frac{h^3}{2}(1-\cos^2\varphi)\right] = \\ = hbm\Omega^2(1-\cos\varphi) \approx hbm\Omega^2\frac{\varphi^2}{2}.$$

Equating $T_{\max}$ and $\Pi_{\max}$, we find the frequency as

$$p^2 = \frac{\int\limits_0^l EJ_x y_1''\mathrm{d}z + m_0 \int\limits_0^l \Omega^2 z \int\limits_0^z (y_1')^2 \,\mathrm{d}z\mathrm{d}z}{J_0 (y_1')^2 \Big|_{z=0} + \int\limits_0^l m_0 y_1^2 \mathrm{d}z} -$$

$$- \frac{\int\limits_0^l m\Omega^2 y_1^2 \mathrm{d}z + m\Omega^2 bh (y_1')^2}{J_0 (y_1')^2 \Big|_{z=0} + \int\limits_0^l m_0 y_1^2 \mathrm{d}z}. \qquad (3)$$

Since in the first approximation $y_1 = A \sin \dfrac{\pi z}{l}$, the we have after calculations

$$p^2 = \frac{EJ_x \dfrac{\pi^4}{2\,l^3} + \Omega^2 \left[ l m_0 \left( \dfrac{\pi^2}{6} - \dfrac{1}{2} \right) + mbh \dfrac{\pi^2}{l^2} \right]}{m_0 \dfrac{l}{2} + mh^2 \dfrac{\pi^2}{l^2}}. \qquad (4)$$

Now we substitute the numerical data of the problem into equation (4):

$$p^2 = 1.13\Omega^2 + 1.04 \cdot 10^5.$$

For the angular velocity $\Omega = 100$ rad/s, the lowest frequency of system vibrations is $p = 332$ s$^{-1}$.

**111** The solid line in Fig. 2.25 shows the position of the system at an arbitrary moment under vibrations, while the dashed line marks the position in the state of dynamical equilibrium in the field of centrifugal forces.

According to the Rayleigh method, the maximum kinetic energy value is equal to the maximum gain of the potential energy:

$$\triangle \Pi_{\max} = T_{\max}.$$

Let us write the potential energy gain in the general form (see solution to Problem **110**):

$$\triangle \Pi_{\max} = \triangle \Pi_1 + \triangle \Pi_2 + \triangle \Pi_{q_z} + \triangle \Pi_{q_y}. \qquad (1)$$

We can find the increment of the terms in the right-hand side of equality (1):

$$\triangle\Pi_1 = \Pi_1 - \Pi_{10} = \frac{1}{2}\int\limits_0^l J_x(y_0'' + y_1'')^2 \mathrm{d}z -$$

$$-\frac{1}{2}\int\limits_0^l EJ_x(y_0'')^2\,\mathrm{d}z = EJ_x\int\limits_0^l \left(y_0''y_1'' + \frac{1}{2}(y_1'')^2\right)\mathrm{d}z;$$

$$\triangle\Pi_2 = hbm\Omega^2\left[\frac{1}{2}(\varphi_0 + \varphi_1)^2 - \frac{1}{2}y_0^2\right] = hbm\Omega^2\left[\varphi_0\varphi_1 + \frac{1}{2}\varphi_1^2\right];$$

$$\triangle\Pi_{q_z} = m_0\Omega^3\left[\frac{1}{2}\int\limits_0^l z\int\limits_0^z (y_0' + y_1')^2 \mathrm{d}\eta\,\mathrm{d}z - \frac{1}{2}\int\limits_0^l z\int\limits_0^z (y_0'')^2 \mathrm{d}\eta\,\mathrm{d}z = \right.$$

$$= m_0\Omega^2\left[\int\limits_0^l z\int\limits_0^z y_0'y_1'\mathrm{d}\eta\,\mathrm{d}z + \frac{1}{2}\int\limits_0^l (y_0')^2\mathrm{d}\eta\,\mathrm{d}z\right];$$

$$\triangle\Pi_{q_y} = -m_0\Omega^2\left[\frac{1}{2}\int\limits_0^l (y_0 + y_1)^2\,\mathrm{d}z - \frac{1}{2}\int\limits_0^l y_0^2\,\mathrm{d}z = \right.$$

$$= -m_0\Omega^2\left[\int\limits_0^l y_0y_1\,\mathrm{d}z + \frac{1}{2}\int\limits_0^l y_1^2\,\mathrm{d}z\right].$$

Consider the terms (appearing in $\triangle\Pi_1$, $\triangle\Pi_{q_z}$, and $\triangle\Pi_{q_y}$) depending on the initial state. After integration by parts they can be represented in the following form

$$\begin{aligned} J_1 &= EJ_x\int\limits_0^l y_0''y_1''\,\mathrm{d}z = EJ_x\Big|_0^l - y_0'''y_1\Big|_0^l + \int\limits_0^l y_0^{\mathrm{IV}}y_1\,\mathrm{d}z; \\ J_2 &= \int\limits_0^l z\int\limits_0^z y_0'y_1'\,\mathrm{d}\eta\,\mathrm{d}z = \int\limits_0^l z\left(y_0'y_1\Big|_0^z - \int\limits_0^z y_0'y_1\mathrm{d}\eta\right)\mathrm{d}z. \end{aligned} \tag{2}$$

Since $y_1 = 0$ at $z = 0$; and $y_1 = 0$, $y_0^2 = 0$, and $y_1\big|_{z=0} = \varphi$ at $z = l$, then

$$J_1 = -EJ_xy_0''\Big|_{z=0}\varphi_1 + EJ_x\int\limits_0^l y_0^{\mathrm{IV}}y_1\,\mathrm{d}z;$$

$$J_2 = \int\limits_0^l z\left[y_0'(z)y_1(z) - \int\limits_0^z y_0''(\eta)y_1(\eta)\mathrm{d}\eta\right]\mathrm{d}z -$$

$$-\int\limits_0^l zy_0'(z)y_1(z)\,\mathrm{d}z - \int\limits_0^l z\int\limits_0^z y_0''(\eta)y_1(\eta)\,\mathrm{d}\eta\,\mathrm{d}z.$$

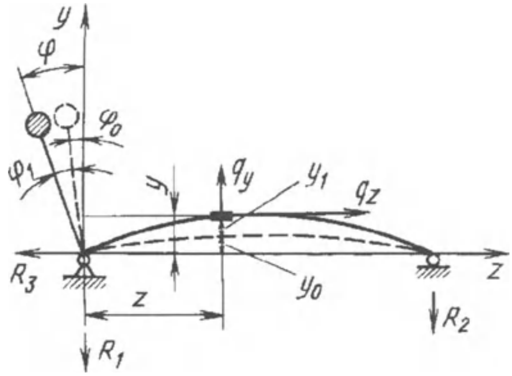


**Fig. 2.25.**

Taking advantage of the Dirichlet's formula, the second term in the expression for $J_2$ can be converted to the form

$$\int_0^l z \int_0^z y_0''(\eta) y_1(\eta)\,\mathrm{d}\eta\,\mathrm{d}z = \int_0^l y_1(z) \int_0^l z \left[ y_0'(z) y_1(z) - \int_z^l \eta y_0''(z)\,\mathrm{d}\eta\,\mathrm{d}z. \right.$$

Combining in expression (1) all terms depending on the initial deformed state, we have:

$$A = \int_0^l \left[ -EJ_x y_0'' \Big|_{z=0} + m\Omega^2 hb\varphi_0 \right] \varphi_1\,\mathrm{d}z + \\ + \int_0^l \left[ EJ_x y_0^{\mathrm{IV}} + z y_1' m_0 \Omega^2 - m_0 \Omega^2 \int_z^l \eta y_0''(z)\,\mathrm{d}\eta - m_1 \Omega^2 y_0 \right] y_1\,\mathrm{d}z. \tag{3}$$

In equilibrium, there is a balance between the elastic moment (in the section $z = 0$) and the moment of inertia force that acts on the mass:

$$EJ_x y_0'' \Big|_{z=0} = mhb\Omega^2 \varphi_0^2,$$

therefore, the first term in the right-hand side of equation (3) is equal to zero.

In order to demonstrate that the second term in this equation is also equal to zero, we consider the equation of the rod deflection curve under bending by the moment $M_1 = m\Omega^2 hby_0$:

$$EJ_x y_0'' = M_1 - R_1 z + \int_0^z q_y (z - z_1)\,\mathrm{d}z_1 + R_3 y_0 - \\ - \int_0^z q_z \Big[ y_0(z) - y_0(z_1) \Big]\,\mathrm{d}z_1. \qquad (4)$$

Since $R_3 = \int_0^l q_z \mathrm{d}z$, expression (2.5) can be converted to the form

$$EJ_x y_0'' = M_1 - R_1 z + \int_0^z m_0 \Omega^2 y_0(z_1)(z - z_1)\,\mathrm{d}z_1 + \\ + \int_z^l y_0(z) m_0 \Omega^2 z_1\,\mathrm{d}z_1 + \int_0^l m_0 \Omega^2 z_1 y_0(z_1)\,\mathrm{d}z_1. \qquad (5)$$

Differentiating equation (2.5) twice with respect to $z$ we get

$$EJ_x y^{\mathrm{IV}} - m_0 \Omega^2 y_0(z) - \int_z^l y_0''(z) m_0 \Omega^2 z_1\,\mathrm{d}z_1 + m_0 \Omega^2 z y_0'(z) = 0.$$

Thus, $A = 0$ (see formula (3)), i.e., the frequency of vibrations of the balancing lever does not depend on the initial strained state.

**112** The equation of vibrations of the flexible rod (spring) is as follows

$$EJ_x \frac{\partial^4 y}{\partial z^4} + m_0 \frac{\partial^2 y}{\partial t^2} + m_0 \Omega^2 y - \frac{\partial}{\partial z}\left( m_0 \frac{\Omega^2}{2} z^2 \frac{\partial y}{\partial z} \right) = 0.$$

**113** In this case (as opposed to Problem **110**), the work of the forces $q_z$ is positive and equal to (see solution to Problem **110**)

$$A_{q_z} = \int_0^l m_0 \Omega^2 z \int_0^l (y_1')^2\,\mathrm{d}\eta\,\mathrm{d}z.$$

The frequency of vibrations is

$$p^2 = \frac{EJ_x \dfrac{\pi^4}{2\,l^3} - \Omega^2 l m_0 \left( \dfrac{\pi^2}{12} + \dfrac{1}{2} \right) + mbh \dfrac{\pi^2}{l^2} \Omega^2}{m_0 \dfrac{l}{2} + m\pi^2 \left( \dfrac{\pi}{l} \right)^2}, \qquad (1)$$

or

$$p^2 = 0.62\Omega^2 + 10^5.$$

At the disk angular velocity $\Omega = 100$ rad/s we obtain

$$p = 324 \text{ s}^{-1}.$$

**114** When solving Problem **113** we have received expression (1) for the frequency of vibrations. It follows from this expression that, when the condition

$$lm_0\left(\frac{\pi^2}{12} + \frac{1}{2}\right) = \frac{mbh\pi^2}{l^2}$$

is met, the frequency of vibrations does not depend on the disk angular velocity.

**115** The expression for the frequency of vibrations of the system is the same as in Problem **110** considered above.

**116** The equation of radial vibrations of the ring has the form

$$\ddot{u} + \frac{EF}{m_0R^2}u = 0.$$

The frequency of vibrations is

$$p = R^{-1}\sqrt{EF/m_0}.$$

**117** Consider an element of the ring (Fig. 2.26). Projecting all forces onto the radius ($dm_0 = m_0 ds$) we obtain

$$\ddot{u} + \frac{EF}{m_0R^2}u = \frac{p_0}{m_0} + \frac{p_1}{m_1}\sin\omega t.$$

The constant pressure $p_0$ makes up a static component of the radial displacement of the ring. The amplitude of steady-state vibrations of the ring is

$$u_1 = \frac{p_1}{m_1\left(\dfrac{EF}{m_0R^2} - \omega^2\right)}.$$

**118** Under radial vibrations of the rotating ring an additional radial force acts upon it. This force (per unit length) is equal to $m_0\Omega^2u$.

The differential equation of radial vibrations of the ring has the form

$$\ddot{u} + \left(\frac{EF}{m_0R^2} - \Omega^2\right)u = 0.$$

The frequency of ring vibrations is

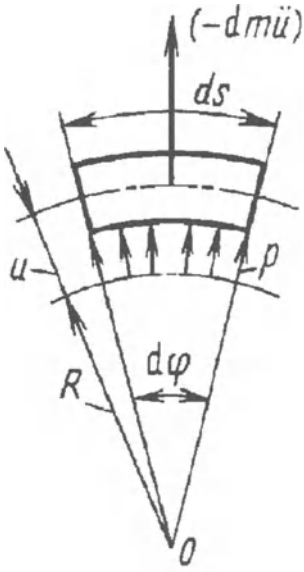


**Fig. 2.26.**

$$p = \sqrt{\frac{EF}{m_0 R^2} - \Omega^2}.$$

Thus, the critical angular velocity of the ring is

$$\Omega_* = \sqrt{EF/(m_0 R^2)}.$$

**119** At the section rotation by an angle $\varphi$ every point of the ring section is displaced along an arc equal to $\varrho\varphi$, where $\varrho$ is the polar radius of the point (Fig. 2.27). The projection of this displacement onto the radial direction is $u = \varrho\varphi \sin\alpha = y\varphi$, which corresponds to an elongation over the circumference (of this filament) by $\Delta l = 2\pi y\varphi$.

Since the ring filaments are in tension, the potential energy is

$$\Pi = \int\limits_F \frac{E}{2}\left(\frac{\Delta l}{l}\right)^2 \mathrm{d}F = \frac{\pi E J_x \varphi^2}{R}. \tag{1}$$

The kinetic energy of the ring rotation about the axial line is

$$T = \int\limits_0^{2\pi R} \frac{J_0 \dot{\varphi}^2 \,\mathrm{d}s}{2} = \frac{J_0 \dot{\varphi}^2}{2} 2\pi R. \tag{2}$$

The moment of inertia of the ring's unit of length with respect to the axial line is

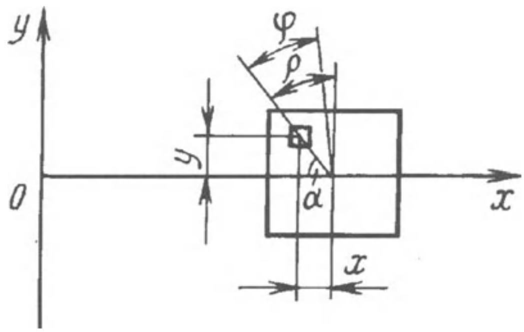


**Fig. 2.27.**

$$J_0 = \varrho_1 J_\varrho = F\varrho_1 J_\varrho / F = m_0 J_\varrho / F,$$

where $J_\varrho$ is the polar moment of inertia of the ring cross section; and $F$ is the cross section area.

From relations (1) and (2) we get the differential equation of the torsional vibrations of the ring

$$\ddot{\varphi} + \frac{EF}{m_0 R^2} \frac{J_x}{J_\varrho} \varphi = 0.$$

The frequency of vibrations is

$$p = \sqrt{\frac{EF}{m_0 R^2} \frac{J_x}{J_\varrho}}.$$

**120** The vibrations of the rod take place around its equilibrium state, therefore, one needs first to determine the static mode of deformation of the rod. The rod is unstretchable and located symmetrically about the axis of rotation, therefore,

$$M_0 = 0; \quad q_{10} = q_{30} = 0; \quad q_{20} = -\omega_0^2 R_0.$$

From the equations of equilibrium we get

$$Q_{20} = 0; \quad Q_{10} = -\frac{q_{20}}{æ_{30}} = \omega_0^2 R_0^2.$$

The equations of small vibrations of the rod whose axial line is a plane curve are given in Appendix C. System of equations (C.24) describes small vibrations of the rod with a 'runaway' of the axial line out of the plane, i.e., the most general case of vibrations of a 'plane' curvilinear rod.

If the 'plane' curvilinear rod executes free vibrations in the plane of drawing (Fig. 2.28), then, in order to derive a system of equations describing these vibrations, one should put $Q_3 = M_1 = M_2 = \Delta æ_1 = \Delta æ_2 = 0$. As a result, we have for a rod of constant section ($A_{33} = n_1 = 1$) the following system of equations describing these vibrations, with allowance made for rotational inertia of rod elements and in dimensionless notation (C.24):

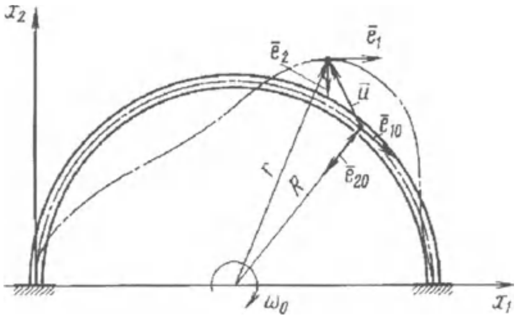


**Fig. 2.28.**

$$\frac{\partial^2 u_1}{\partial \tau^2} - \frac{\partial Q_1}{\partial \eta} + æ_{30} Q_2 = P_1; \tag{1}$$

$$\frac{\partial^2 u_2}{\partial \tau^2} - \frac{\partial Q_2}{\partial \eta} - Q_{10}\Delta æ_3 + æ_{30} Q_2 = P_2; \tag{2}$$

$$J_{33}\frac{\partial^2 \vartheta_3}{\partial \tau^2} - \frac{\partial M_3}{\partial \eta} - Q_2 = 0; \tag{3}$$

$$\frac{\partial \vartheta_3}{\partial \eta} - \Delta æ_3 = 0; \tag{4}$$

$$\frac{\partial u_1}{\partial \eta} - æ_{30} u_2 = 0; \tag{5}$$

$$\frac{\partial u_2}{\partial \eta} - æ_{30} u_1 - \vartheta_3 = 0, \tag{6}$$

where $M_3 = \Delta æ_3$.

Consider the right-hand sides of equations (1)-(3). Since no concentrated forces act upon the rod, one should put in equations (C.24)

$$P_1 = q_1, \quad P_2 = q_2.$$

Let us find expressions for the dynamic loads $q_1$ and $q_2$ that appear under rod vibrations.

The dot-and-dash curve in Fig. 1.91 shows the position of the rod axial line under vibrations. It follows from the figure that the absolute velocity of the point $O_1$ is

$$\overline{\mathbf{v}}_{O_1} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\overline{\mathbf{R}}_0 + \overline{\mathbf{u}}\right). \tag{7}$$

The increment of the absolute velocity of the point $O_1$ is

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}_{O_1} - \overline{\mathbf{v}}_0 = \frac{\mathrm{d}\overline{\mathbf{u}}}{\mathrm{d}\tau} \qquad \left(\overline{\mathbf{v}}_0 = \frac{\overline{\mathbf{R}}_0}{\mathrm{d}\tau}\right).$$

Passing to local derivatives (see Appendix B) we have

$$\overline{\mathbf{v}} = \frac{\partial\overline{\mathbf{u}}}{\partial\tau} + \overline{\omega}_0 \times \overline{\mathbf{u}} \qquad (|\overline{\omega}_0| = \mathrm{const}). \tag{8}$$

It is worth noting that the angular velocity of rotation of coupled axes is

$$\overline{\omega}' = \overline{\omega}_0 + \overline{\omega}$$

where $\overline{\omega}$ is the additional angular velocity of a rod element that arises under vibrations. However, as under small vibrations one can consider the components of vectors $\overline{\omega}$ and $\overline{\mathbf{u}}$ as small quantities, the product $\overline{\omega}_0 \times \overline{\mathbf{u}}$ can be neglected.

The increment of the absolute acceleration of point $O_1$ is (the tilde symbol in the local derivative notation is omitted)

$$\frac{\mathrm{d}\overline{\mathbf{v}}}{\mathrm{d}\tau} = \frac{\partial}{\partial\tau}\left(\frac{\partial\overline{\mathbf{u}}}{\partial\tau} + \overline{\omega}_0 \times \overline{\mathbf{u}}\right) + \overline{\omega}_0 \times \left(\frac{\partial\overline{\mathbf{u}}}{\partial\tau} + \overline{\omega}_0 \times \overline{\mathbf{u}}\right) \tag{9}$$

or, assuming that for plane vibrations $\overline{\mathbf{u}} = u_1\overline{\mathbf{e}}_1 + u_2\overline{\mathbf{e}}_2$,

$$\frac{\mathrm{d}\overline{\mathbf{v}}}{\mathrm{d}\tau} = \frac{\partial^2\overline{\mathbf{u}}}{\partial\tau^2} + 2\overline{\omega}_0 \times \dot{\overline{\mathbf{u}}} - \omega_0^2 u_1\overline{\mathbf{e}}_1 + \omega_0^2 u_2\overline{\mathbf{e}}_2. \tag{10}$$

It follows from equation (10) that the additional dynamic load $q$ acting upon the rod that is placed on a rotating disk is equal to

$$\overline{\mathbf{q}} = -2\left(\overline{\omega}_0 \times \dot{\overline{\mathbf{u}}}\right) + \omega_0^2 u_1\overline{\mathbf{e}}_1 - \omega_0^2 u_2\overline{\mathbf{e}}_2$$

or (since $\overline{\omega}_0 = \omega_0\overline{\mathbf{e}}_3$)

$$\begin{aligned} q_1 &= 2\omega_0\frac{\partial u_2}{\partial\tau} + \omega_0^2 u_1; \\ q_2 &= -2\omega_0\frac{\partial u_1}{\partial\tau} - \omega_0^2 u_2. \end{aligned} \tag{11}$$

Finally, we have the following system of equations for small vibrations of a round rod in the plane of drawing:

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial\tau^2} - \frac{\partial Q_1}{\partial\eta} + æ_{30}Q_2 - 2\omega_0\frac{\partial u_2}{\partial\tau} - \omega_0^2 u_1 = 0;\\
&\frac{\partial^2 u_2}{\partial\tau^2} - \frac{\partial Q_2}{\partial\eta} - \omega_0^2 R_0^2 M_3 - æ_{30}Q_1 + 2\omega_0\frac{\partial u_1}{\partial\tau} + \omega_0^2 u_2 = 0;\\
&J_{33}\frac{\partial^2 \vartheta_3}{\partial\tau^2} - \frac{\partial M_3}{\partial\eta} - Q_2 = 0;\\
&\frac{\partial\vartheta_3}{\partial\eta} - M_3 = 0;\\
&\frac{\partial u_1}{\partial\eta} - æ_{30}u_2 = 0;\\
&\frac{\partial u_2}{\partial\eta} + æ_{30}u_1 - \vartheta_3 = 0; \qquad \left(M_3 = \Delta æ_{33};\ æ_{30} = \frac{1}{R_0}\right),
\end{aligned} \tag{12}$$

or, in the vector form of notation (see Appendix C)

$$\mathbf{A}^{(1)}\frac{\partial^2\overline{\mathbf{Z}}}{\partial\tau^2} + \mathbf{A}^{(3)}\frac{\partial\overline{\mathbf{Z}}}{\partial\tau} + \frac{\partial\overline{\mathbf{Z}}}{\partial\eta} + \mathbf{A}^{(2)}\overline{\mathbf{Z}} = 0. \tag{13}$$

Here

$$\mathbf{A}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & -J_{33} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{A}^{(2)} = \begin{bmatrix} 0 & \frac{1}{R_0} & 0 & 0 & -\omega_0^2 & 0\\ \frac{1}{R_0} & 0 & -\omega_0^2 R_0^2 & 0 & 0 & \omega_0^2\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{R_0}\\ 0 & 0 & 0 & -1 & \frac{1}{R_0} & 0 \end{bmatrix};$$

$$\mathbf{A}^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -2\omega_0\\ 0 & 0 & 0 & 0 & 2\omega_0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\overline{\mathbf{Z}} = [Q_1, Q_2, M_3, \vartheta_3, u_1, u_2]^{\mathrm{T}}.$$

**121** The ring is loaded with a static distributed force, therefore, $Q_{10} = -q_0 R_0$, $Q_{20} = Q_{30} = 0$, and $M_{10} = M_{20} = M_{30} = 0$. The equations of free vibrations of the rod whose axial line is a plane curve are given in Appendix C (see equations (C.24)). In the case under consideration they fall apart into two independent systems that describe rod vibrations in the plane of drawing and with respect to the plane of drawing. For the rod of constant section we derive from system (C.24) the following system of equations of rod vibrations with respect to the plane of drawing:

$$
\begin{aligned}
&\frac{\partial^2 u_3}{\partial \tau^2} - \frac{\partial Q_3}{\partial \eta} - q_0 R_0 M_2 = 0;\\
&J_{11}\frac{\partial^2 \vartheta_1}{\partial \tau^2} - \frac{\partial M_1}{\partial \eta} + \frac{M_2}{R_0} = 0;\\
&J_{22}\frac{\partial^2 \vartheta_2}{\partial \tau^2} - \frac{\partial M_2}{\partial \eta} - \frac{M_1}{R_0} + Q_3 = 0;\\
&\frac{\partial \vartheta_1}{\partial \eta} - \frac{\vartheta_2}{R_0} - \frac{M_1}{A_{11}} = 0;\\
&\frac{\partial \vartheta_2}{\partial \eta} + \frac{\vartheta_1}{R_0} - M_2 = 0;\\
&\frac{\partial u_3}{\partial \eta} + \vartheta_2 = 0.
\end{aligned}
\qquad (1)
$$

For the round section the dimensionless moments of inertia $J_{11}$ and $J_{22}2$ and stiffnesses $A_{11}$ and $A_{22}$ are

$$J_{11} = \frac{1}{8}\left(\frac{d}{l}\right)^2, \; J_{22} = \frac{1}{16}\left(\frac{d}{l}\right)^2; \qquad A_{11} = 2\frac{G}{E}, \; A_{22} = 1.$$

In the vector form of notation we have

$$\mathbf{A}^{(1)}\frac{\partial^2 \overline{\mathbf{Z}}}{\partial \tau^2} + \frac{\partial \overline{\mathbf{Z}}}{\partial \eta} + \mathbf{A}^{(2)}\overline{\mathbf{Z}} = 0.$$

Here

$$\mathbf{A}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -J_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -J_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{A}^{(2)} = \begin{bmatrix} 0 & 0 & q_0 R_0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{R_0} & 0 & 0 & 0 \\ 0 & -\dfrac{1}{R_0} & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{1}{A_{11}} & 0 & 0 & -\dfrac{1}{R_0} & 0 \\ 0 & 0 & -1 & \dfrac{1}{R_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

$$\overline{\mathbf{Z}} = [Q_3, M_1, M_2, \vartheta_1, \vartheta_2, u_3]^{\mathrm{T}}.$$

**122** One can derive the equations of small vibrations of the rotating ring as a special case of equations (C.24) (see Appendix C).

It follows from relation (9) (see solution to Problem **120**) that at $\overline{\omega}_0 = \omega_0 \overline{\mathbf{e}}_3$ the absolute acceleration does not depend on $u_3$, so $q_3 = 0$. Thus, we obtain the following equations (at $Q_{10} = 0$ and $Q_{20} = \omega_0^2 R_0^2$):

$$\begin{aligned} &\frac{\partial^2 u_3}{\partial \tau^2} - \frac{\partial Q_3}{\partial \eta} + \omega_0^2 \frac{R_0^2}{A_{22}} M_2 = 0; \\ &J_{11} \frac{\partial^2 \vartheta_1}{\partial \tau^2} - \frac{\partial M_1}{\partial \eta} + \frac{æ_{30}}{A_{22}} M_2 = 0; \\ &J_{22} \frac{\partial^2 \vartheta_2}{\partial \tau^2} - \frac{\partial M_2}{\partial \eta} - \frac{æ_{30}}{A_{11}} M_1 + Q_3 = 0; \\ &\frac{\partial \vartheta_1}{\partial \eta} - æ_{30} \vartheta_2 - \frac{M_1}{A_{11}} = 0; \\ &\frac{\partial \vartheta_2}{\partial \eta} + æ_{30} \vartheta_1 - \frac{M_2}{A_{22}} = 0; \\ &\frac{\partial u_3}{\partial \eta} + \vartheta_2 = 0 \qquad (M_1 = A_{11} \Delta æ_1,\ M_2 = A_{22} \Delta æ_2). \end{aligned} \tag{1}$$

System of equations (1) can be represented in the form

$$\mathbf{A}^{(1)} \frac{\partial^2 \overline{\mathbf{Z}}}{\partial \tau^2} + \frac{\partial \overline{\mathbf{Z}}}{\partial \eta} + \mathbf{A}^{(2)} \overline{\mathbf{Z}} = 0,$$

where

$$\mathbf{A}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -J_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -J_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{A}^{(2)} = \begin{bmatrix} 0 & 0 & \omega_0^2 R_0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{æ_{30}}{A_{22}} & 0 & 0 & 0 \\ 1 & -\dfrac{æ_{30}}{A_{11}} & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{1}{A_{11}} & 0 & 0 & -æ_{30} & 0 \\ 0 & 0 & -\dfrac{1}{A_{11}} & æ_{30} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

$$\overline{\mathbf{Z}} = [Q_3, M_1, M_3, \vartheta_1, \vartheta_2, u_3]^{\mathrm{T}}.$$

**123** The presence of concentrated masses results in the appearance of concentrated forces of inertia that arise under vibrations. Therefore, we take advantage of equations (C.13) of small vibrations (see Appendix C) putting in the first two equations

$$\begin{aligned} \overline{\mathbf{P}}^{(1)} &= \overline{\mathbf{J}}_{\mathrm{i}}^{(1)}\delta(\eta-\eta_1) + \overline{\mathbf{J}}_{\mathrm{i}}^{(2)}\delta(\eta-\eta_2); \\ \overline{\mathfrak{M}}^{(1)} &= \overline{\mathfrak{M}}_{\mathrm{i}}^{(2)}\delta(\eta-\eta_2), \end{aligned} \tag{1}$$

where $\overline{\mathbf{J}}_{\mathrm{i}}^{(1)}$, $\overline{\mathbf{J}}_{\mathrm{i}}^{(2)}$, and $\overline{\mathfrak{M}}_{\mathrm{i}}^{(2)}$ are dimensionless forces and the moment of inertia, respectively,

$$\begin{aligned} \overline{\mathbf{J}}_{\mathrm{i}}^{(1)} &= -\left(\sum_{j=1}^{n} \frac{\partial^2 u_j}{\partial \tau^2}\overline{\mathbf{e}}_j\right); \\ \overline{\mathbf{J}}_{\mathrm{i}}^{(2)} &= -\left(\sum_{j=1}^{n} \frac{\partial^2 u_j}{\partial \tau^2}\overline{\mathbf{e}}_j\right); \\ \overline{\mathfrak{M}}_{\mathrm{i}}^{(2)} &= -\left(\sum_{j=1}^{n} \frac{\partial^2 \vartheta_j}{\partial \tau^2}\overline{\mathbf{e}}_j\right). \end{aligned} \tag{2}$$

The remaining equations of system (C.13) are kept unchanged.

**124** We consider vibrations of the rod about its natural state (i.e., at $Q_{10} = Q_{20} = M_{30} = 0$) in the plane $x_1Ox_2$ (see Fig. 1.94). In this case, two concentrated forces $\overline{\mathbf{R}}_1$ and $\overline{\mathbf{R}}_2$ act upon the rod in the sections $\eta_1$ and $\eta_2$. They are equal, respectively, to

$$\overline{\mathbf{R}}_1 = R_2\overline{\mathbf{e}}_2, \qquad \overline{\mathbf{R}}_2 = R_{x_2}\overline{\mathbf{i}}_2$$

where $R_{x_2} = -c\left(\overline{\mathbf{u}}\,\overline{\mathbf{i}}_2\right)$, therefore, similarly to Problem **123** we have (see Appendix C)

$$\begin{aligned} P_1 &= -c\left(\overline{\mathbf{u}}\,\overline{\mathbf{i}}_2\right)\left(\overline{\mathbf{i}}_2\overline{\mathbf{e}}_1\right)\delta(\eta-\eta_2); \\ P_2 &= R_2\delta(\eta-\eta_1) - c\left(\overline{\mathbf{u}}\,\overline{\mathbf{i}}_2\right)\left(\overline{\mathbf{i}}_2\overline{\mathbf{e}}_2\right)\delta(\eta-\eta_2); \\ \overline{\mathbf{i}}_2 &= \sin(\vartheta_{30}+\triangle\vartheta_3)\overline{\mathbf{e}}_1 + \cos(\vartheta_{30}+\triangle\vartheta_3)\overline{\mathbf{e}}_2, \end{aligned} \tag{1}$$

where $\vartheta_{30}$ is the angle between vectors $\overline{\mathbf{e}}_1$ and $\overline{\mathbf{i}}_1$ .

Since small vibrations are considered, we obtain after some transformations (assuming that $\overline{\mathbf{e}}_1 \approx \overline{\mathbf{e}}_{10}$)

$$\begin{aligned} P_1 &= -c\left(u_1\sin^2\vartheta_{30} + u_2\cos\vartheta_{30}\sin\vartheta_{30}\right)\delta(\eta-\eta_2); \\ P_2 &= R_2\delta(\eta-\eta_1) - \\ &\quad -c\left(u_1\cos\vartheta_{30}\sin\vartheta_{30} + u_2\cos^2\vartheta_{30}\right)\delta(\eta-\eta_2). \end{aligned} \tag{2}$$

The first two equations of system (C.25) (see Appendix C) after transformation of their right-hand sides take on the form

$$\begin{aligned} n_1\frac{\partial^2 u_1}{\partial\tau^2} - \frac{\partial Q_1}{\partial\eta} + æ_{30}Q_2 &= -c\sin^2\vartheta_{30}u_1\delta(\eta-\eta_2) - \\ &\quad -c\cos\vartheta_{30}\sin\vartheta_{30}u_2\delta(\eta-\eta_2); \\ n_1\frac{\partial^2 u_2}{\partial\tau^2} - \frac{\partial Q_2}{\partial\eta} + æ_{30}Q_1 &= R_2\delta(\eta-\eta_1) - \\ -cu_1\cos\vartheta_{30}\sin\vartheta_{30}\delta(\eta-\eta_2) &- c\cos^2\vartheta_{30}u_2\delta(\eta-\eta_2). \end{aligned} \tag{3}$$

The remaining equations of system (C.25) are kept unchanged.

**125** Small free vibrations of the spiral in the plane of drawing are described by system of equations (C.25) (see Appendix C). However, in order to use them in numerical calculations one needs to know the dependence of $æ_{30}$ on the coordinate $\eta$. The equation of the Archimedean spiral in the polar coordinate system (Fig. 2.29) has the form

$$r = a\varphi; \quad x_1 = r\cos\varphi; \quad x_2 = r\sin\varphi, \tag{1}$$

where $a$ is the spiral parameter. The length of the spiral is $l$.

From the conditions of fixing the spiral ($\varphi = 0$, $r = 0$, $\varphi = a\pi/2$, and $r = r_0$) we derive the range of variation for the angle $\varphi$ ($0 \le \varphi \le a\pi/2$). The differential of the arc in the polar coordinate system is

$$\mathrm{d}\eta^2 = \widetilde{a}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\eta}\right)^2 + \widetilde{a}^2\varphi^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\eta}\right)^2, \tag{2}$$

therefore,

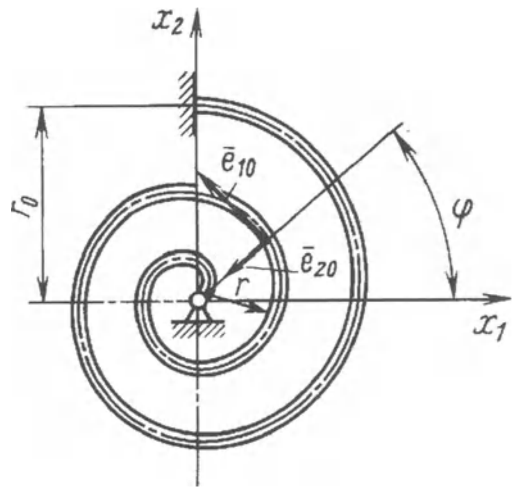


**Fig. 2.29.**

$$d\eta = \tilde{a}\sqrt{1+\varphi^2}\,d\varphi \qquad (\tilde{a} = a/l). \tag{3}$$

Integrating equation (3) between 0 and 1 we obtain

$$1 = \tilde{a}\int_0^{a\pi/2}\sqrt{1+\varphi^2}\,d\varphi. \tag{4}$$

From expression (4) we determine $\tilde{a}$. Dividing (1) by $l$ we pass to the dimensionless form of notation:

$$\tilde{r} = \tilde{a}\varphi; \qquad \tilde{x}_1 = \tilde{r}\cos\varphi; \qquad \tilde{x}_r = \tilde{r}\sin\varphi.$$

Calculating $\tilde{a}$, we can find the dimensionless value of $\tilde{r}_0 = 9\tilde{a}\pi/2$. From this point on we omit the symbol of tilde above dimensionless quantities. From equation (2) we get

$$\frac{d\varphi}{d\eta} = \frac{1}{a\sqrt{1+\varphi^2}}. \tag{5}$$

Integrating equation (5) we find the function $\varphi(\eta)$ ($\varphi = 0$ at $\eta = 0$). Taking relation (5) into account, we determine the derivatives $x_1'(\eta)$ and $x_2'(\eta)$:

$$\begin{aligned} x_1'(\eta) &= a\varphi'\cos\varphi - a\varphi\varphi'\sin\varphi = \frac{\cos\varphi - \varphi\sin\varphi}{\sqrt{1+\varphi^2}}; \\ x_2'(\eta) &= a\varphi'\sin\varphi + a\varphi\varphi'\cos\varphi = \frac{\sin\varphi + \varphi\cos\varphi}{\sqrt{1+\varphi^2}}. \end{aligned} \tag{6}$$

The formula for the curvature $æ_{30}(\eta)$ of a plane curve has the form

$$æ_{30}(\eta) = \sqrt{x_1''^2 + x_2''^2}. \tag{7}$$

Differentiating (6) with respect to $\eta$ and substituting the resulting relations into equation (7) we have

$$æ_{30}(\eta) = \frac{1}{a}\frac{(2+\varphi^2)}{(1+\varphi^2)^{3/2}}. \tag{8}$$

Solving equation (5) we find the dependence of $\varphi$ on $\eta$, and then determine from (8) $æ_{30}(\eta)$ as a function of $\eta$. Taking advantage of equation (C.25) (see Appendix C) and knowing the function $æ_{30}(\eta)$, one can find the equations of small vibrations of an Archimedean spiral.

**126** In order to solve the problem it is necessary first to determine static mode of deformation (see Appendix A) and the displacement vector of points of the rod axial line. The rod is loaded with a dead force $\overline{\mathbf{P}}$, and the displacements are assumed to be small. The solution of equations of the rod equilibrium is considered in detail in [4].

When the force $\overline{\mathbf{P}}$ stops acting, we have $\overline{\mathbf{Q}}_0 = \overline{\mathbf{M}}_0 = 0$. As a result of solution of equilibrium equation (A.92) (see Appendix A) we find the components $u_{01}$ and $u_{02}$ of the vector of displacements and the angle $\vartheta_{03}$. The we take advantage of equations (C.25) of small vibrations of the rod in the plane of drawing (see Appendix C), putting in them $Q_{10} = Q_{20} = M_{30} = 0$. In addition, for the rod of constant round section $æ_{30} = \pi/2$; $n_1 = 1$; $A_{33} = 1$, and $J_{33} = d/(16\,l)$, where $d$ and $l$ are the rod diameter and length, respectively. As a result, we have the following system of equations

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial \tau^2} - \frac{\partial Q_1}{\partial \eta} + \frac{\pi}{2} Q_2 = 0;\\
&\frac{\partial^2 u_2}{\partial \tau^2} - \frac{\partial Q_2}{\partial \eta} - \frac{\pi}{2} Q_1 = 0;\\
&J_{33}\frac{\partial^2 \vartheta_3}{\partial \tau^2} - \frac{\partial M_3}{\partial \eta} - Q_2 = 0;\\
&\frac{\partial \vartheta_3}{\partial \eta} - M_3 = 0;\\
&\frac{\partial u_1}{\partial \eta} - \frac{\pi}{2} u_2 = 0;\\
&\frac{\partial u_2}{\partial \eta} + \frac{\pi}{2} u_1 - \vartheta_3 = 0.
\end{aligned} \tag{1}$$

Putting $u_1 = u_{10}(\eta)e^{i\lambda\tau}$, $u_2 = u_{20}(\eta)e^{i\lambda\tau}$, and so on, we obtain the equation

$$\overline{\mathbf{Z}}_0' + \mathbf{A}\overline{\mathbf{Z}}_0 = 0, \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & \pi/2 & 0 & 0 & \lambda^2 & 0 \\ \pi/2 & 0 & 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & J_{33}\lambda^2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\pi/2 \\ 0 & 0 & 0 & -1 & \pi/2 & 0 \end{bmatrix}; \ \overline{\mathbf{Z}}_0 = \begin{bmatrix} Q_{10} \\ Q_{20} \\ M_{30} \\ \vartheta_{30} \\ u_{10} \\ u_{20} \end{bmatrix}.$$

For the boundary conditions of the problem

$$\eta = 0, \ u_{10} = u_{20} = \vartheta_{30} = 0;$$
$$\eta = 1, \ Q_1 = Q_2 = M_3 = 0$$

we determine $\lambda_j$ $(j = 1, 2, 3)$. For every $\lambda_j$ we find the eigen vector $\overline{\mathbf{Z}}_0^j$ (see Appendix D). In order to solve this problem we need to know only $u_{k0}^{(j)}(\eta)$ $(k = 1, 2;\ j = 1, 2, 3)$.

The initial conditions have the form

$$\tau = 0, \qquad \overline{\mathbf{u}} = \overline{\mathbf{u}}_0, \qquad \dot{\overline{\mathbf{u}}} = 0.$$

Let us write the expressions for the components of vector $\overline{\mathbf{u}}$u:

$$u_1 = c^{(1)}u_{10}^{(1)}\cos\lambda_1\tau + c^{(2)}u_{10}^{(2)}\cos\lambda_2\tau + c^{(3)}u_{10}^{(3)}\cos\lambda_3\tau;$$
$$u_2 = c^{(1)}u_{20}^{(1)}\cos\lambda_1\tau + c^{(2)}u_{20}^{(2)}\cos\lambda_2\tau + c^{(3)}u_{20}^{(3)}\cos\lambda_3\tau, \tag{3}$$

or, in the vector form of notation,

$$\overline{\mathbf{u}} = \sum_{j=1}^{n} c^{(j)}\overline{\mathbf{u}}^{(j)}\cos\lambda_j\tau.$$

At $\tau = 0$ $\left(\overline{\mathbf{u}}_j = \overline{\mathbf{Z}}_u^{(j)}\right)$

$$\overline{\mathbf{u}}_0 = \sum_{j=1}^{3} c^{(j)}\overline{\mathbf{Z}}_u^{(j)},$$

where $\overline{\mathbf{Z}}_u^{(j)} = \begin{bmatrix} u_{10}^{(j)} \\ u_{20}^{(j)} \end{bmatrix}.$

We determine the coefficients $c^{(j)}$ from the equations

$$\int_0^1 \left(\overline{\mathbf{u}}_0 \overline{\mathbf{Z}}_u^{(1)}\right) \mathrm{d}\eta = c^{(1)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(1)}\right)^2 \mathrm{d}\eta +$$

$$+ c^{(2)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(2)} \overline{\mathbf{Z}}_u^{(1)}\right) \mathrm{d}\eta + c^{(3)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(3)} \overline{\mathbf{Z}}_u^{(1)}\right) \mathrm{d}\eta;$$

$$\int_0^1 \left(\overline{\mathbf{u}}_0 \overline{\mathbf{Z}}_u^{(2)}\right) \mathrm{d}\eta = c^{(1)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(1)} \overline{\mathbf{Z}}_u^{(2)}\right) \mathrm{d}\eta +$$

$$+ c^{(2)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(2)}\right)^2 \mathrm{d}\eta + c^{(3)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(3)} \overline{\mathbf{Z}}_u^{(2)}\right) \mathrm{d}\eta;$$

$$\int_0^1 \left(\overline{\mathbf{u}}_0 \overline{\mathbf{Z}}_u^{(3)}\right) \mathrm{d}\eta = c^{(1)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(1)} \overline{\mathbf{Z}}_u^{(3)}\right) \mathrm{d}\eta +$$

$$+ c^{(2)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(2)} \overline{\mathbf{Z}}_u^{(3)}\right) \mathrm{d}\eta + c^{(3)} \int_0^1 \left(\overline{\mathbf{Z}}_u^{(3)}\right)^2 \mathrm{d}\eta.$$

Provided that the rotation angles of the end section are small, the horizontal displacement of the point $K$ along the axis $x_1$ is

$$u_{x_1} = (\overline{\mathbf{u}}_k \overline{\mathbf{i}}_1) = (u_{1k}\overline{\mathbf{e}}_1 + u_{2k}\overline{\mathbf{e}}_2)\overline{\mathbf{i}}_1 = -u_2,$$

or

$$u_{x_1} = -\left(c^{(1)} u_{20}^{(1)}(1) \cos \lambda_1 \tau + c^{(2)} u_{20}^{(2)}(1) \cos \lambda_2 \tau + c^{(3)} u_{20}^{(3)}(1) \cos \lambda_3 \tau\right).$$

**127** We take advantage of equations (C.25) (see Appendix C) putting in them $Q_{10} = Q_{20} = 0$, $n_1 = 1$, and $A_{33} = 1$. Let us represent system (C.25) in the vector form

$$L_1 = \mathbf{A}^{(1)} \frac{\partial^2 \overline{\mathbf{Z}}}{\partial \tau^2} + \frac{\partial \overline{\mathbf{Z}}}{\partial \eta} + \mathbf{A}^{(2)} \overline{\mathbf{Z}} - \triangle \overline{\mathbf{\Phi}} = 0, \qquad (1)$$

where

$$\mathbf{A}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & J_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{A}^{(2)} = \begin{bmatrix} 0 & æ_{30} & 0 & 0 & 1 & 0 \\ -æ_{30} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -æ_{30} \\ 0 & 0 & 0 & -1 & æ_{30} & 0 \end{bmatrix};$$

$$\overline{\mathbf{Z}} = [Q_1, Q_2, \mathfrak{M}_3, \vartheta_3, u_1, u_2]^{\mathrm{T}}; \quad \triangle\overline{\mathbf{\Phi}} = [0, 0, \mathfrak{M}_3, 0, 0, 0]^{\mathrm{T}} \delta(\eta - \eta_1)$$

For approximate solution of equation (1) it is required to determine first (for two-term approximation) the eigen functions $\overline{\mathbf{Z}}_0^{(1)}$ and $\overline{\mathbf{Z}}_0^{(2)}$ (see equations (1) of Problem **126**). Further on, we seek the solution to equation (1) in the form

$$\overline{\mathbf{Z}} = f^{(1)}(\tau)\overline{\mathbf{Z}}_0^{(1)}(\eta) + f^{(2)}(\tau)\overline{\mathbf{Z}}_0^{(2)}(\eta). \tag{2}$$

Let us employ the principle of virtual displacements (see Appendix E) assuming

$$\delta\overline{\mathbf{Z}}_0^{(1)} = \delta b_1 \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)}; \qquad \delta\overline{\mathbf{Z}}_0^{(2)} = \delta b_2 \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)},$$

where $\mathbf{E}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{E}_0\overline{\mathbf{Z}}_0^{(i)} = \begin{bmatrix} u_{10}^{(i)} \\ u_{20}^{(i)} \\ \vartheta_{30}^{(i)} \\ M_{30}^{(i)} \\ Q_{10}^{(i)} \\ Q_{20}^{(i)} \end{bmatrix}.$

The matrix $E_0$ is a special case of matrix (E.18) (see Appendix E). In accordance with the algorithm of deriving equations for $f^{(1)}$ and $f^{(2)}$) we write two equations

$$\int_0^1 \left(\overline{\mathbf{L}}_1 \cdot \mathbf{E}_0\overline{\mathbf{Z}}_0^{(1)}\right) \mathrm{d}\eta = 0; \qquad \int_0^1 \left(\overline{\mathbf{L}}_1 \cdot \mathbf{E}_0\overline{\mathbf{Z}}_0^{(2)}\right) \mathrm{d}\eta = 0,$$

or

$$\begin{aligned} h_{11}\ddot{f}^{(1)} + h_{12}\ddot{f}^{(2)} + b_{11}f^{(1)} + b_{12}f^{(2)} = b_1; \\ h_{21}\ddot{f}^{(1)} + h_{22}\ddot{f}^{(2)} + b_{21}f^{(1)} + b_{22}f^{(2)} = b_2. \end{aligned} \tag{3}$$

Here

$$h_{11} = \int_0^1 \left( \mathbf{A}^{(1)} \overline{\mathbf{Z}}_0^{(1)} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \right) \mathrm{d}\eta; \quad h_{12} = \int_0^1 \left( \mathbf{A}^{(1)} \overline{\mathbf{Z}}_0^{(2)} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \right) \mathrm{d}\eta;$$

$$h_{21} = \int_0^1 \left( \mathbf{A}^{(1)} \overline{\mathbf{Z}}_0^{(1)} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \right) \mathrm{d}\eta; \quad h_{22} = \int_0^1 \left( \mathbf{A}^{(1)} \overline{\mathbf{Z}}_0^{(2)} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \right) \mathrm{d}\eta;$$

$$b_{11} = \int_0^1 \left( \overline{\mathbf{Z}}_0^{\prime(1)} + \mathbf{A}^{(2)} \overline{\mathbf{Z}}_0^{(1)} \right) \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \mathrm{d}\eta;$$

$$b_{12} = \int_0^1 \left( \overline{\mathbf{Z}}_0^{\prime(2)} + \mathbf{A}^{(2)} \overline{\mathbf{Z}}_0^{(2)} \right) \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \mathrm{d}\eta;$$

$$b_{21} = \int_0^1 \left( \overline{\mathbf{Z}}_0^{\prime(1)} + \mathbf{A}^{(2)} \overline{\mathbf{Z}}_0^{(1)} \right) \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \mathrm{d}\eta;$$

$$b_{22} = \int_0^1 \left( \overline{\mathbf{Z}}_0^{\prime(2)} + \mathbf{A}^{(2)} \overline{\mathbf{Z}}_0^{(2)} \right) \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \mathrm{d}\eta.$$

In a more detailed notation the coefficients $h_{11}$ and $b_{11}$ are determined by the following expressions:

$$h_{11} = \int_0^1 \left[ \left( u_{10}^{(1)} \right)^2 + \left( u_{20}^{(1)} \right)^2 + J_{33} \left( \vartheta_{30}^{(1)} \right)^2 \right] \mathrm{d}\eta;$$

$$b_{11} = \int_0^1 \Big[ Q_{10}^{\prime(1)} u_{10}^{(1)} + Q_{20}^{\prime(1)} u_{20}^{(1)} + M_{30}^{\prime(1)} \vartheta_{30}^{(1)} + \vartheta_{30}^{\prime(1)} M_{30}^{(1)} + u_{10}^{\prime(1)} \triangle Q_{10}^{(1)} +$$
$$+ u_{20}^{\prime(1)} Q_{20}^{(1)} + æ_{30} Q_{20}^{(1)} u_{10}^{(1)} - æ_{30} Q_{10}^{(1)} u_{20}^{(1)} - Q_{20}^{(1)} \vartheta_{30}^{(1)} - \left( M_{30}^{(1)} \right)^2 -$$
$$- æ_{30} u_{20}^{(1)} Q_{10}^{(1)} + æ_{30} u_{10}^{(1)} - \vartheta_{30}^{(1)} Q_{20}^{(1)} \Big] \mathrm{d}\eta.$$

The right-hand sides of the equations of system (3) are equal to

$$b_1 = \int_0^1 \left( \triangle \overline{\mathbf{\Phi}} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \right) \mathrm{d}\eta = \mathfrak{M}_3 \vartheta_{30}^{(1)}(\eta_1);$$

$$b_2 = \int_0^1 \left( \triangle \overline{\mathbf{\Phi}} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \right) \mathrm{d}\eta = \mathfrak{M}_3 \vartheta_{30}^{(2)}(\eta_1).$$

Let us write system (3) in the vector form

$$\mathbf{H}\ddot{\overline{\mathbf{f}}} + \mathbf{H}^{-1}\mathbf{B}\overline{\mathbf{f}} = \mathbf{H}^{-1}\overline{\mathbf{b}}. \tag{4}$$

Assuming $\dot{\overline{\mathbf{f}}} = \overline{\mathbf{F}}_1$ and $\overline{\mathbf{f}} = \overline{\mathbf{F}}_2$, we represent equation (4) in the form of a system of first-order equations

$$\dot{\overline{\mathbf{F}}}_1 + \mathbf{H}^{-1}\mathbf{B}\overline{\mathbf{F}}_2 = \mathbf{H}^{-1}\overline{\mathbf{b}};$$
$$\dot{\overline{\mathbf{F}}}_2 - \overline{\mathbf{F}}_1 = 0,$$

or

$$\dot{\overline{\mathbf{F}}} + \mathbf{A}\overline{\mathbf{F}} = \overline{\mathbf{b}}^{(1)}, \tag{5}$$

where $\overline{\mathbf{F}} = \begin{bmatrix} \dot{\overline{\mathbf{f}}} \\ \overline{\mathbf{f}} \end{bmatrix}$; $\mathbf{A} = \begin{bmatrix} 0 & \mathbf{H}^{-1}\mathbf{B} \\ -\mathbf{E} & 0 \end{bmatrix}$; $\overline{\mathbf{b}}^{(1)} = \begin{bmatrix} \mathbf{H}^{-1}\overline{\mathbf{b}} \\ 0 \end{bmatrix}$; $\overline{\mathbf{b}} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$.
The solution to equation (5) has the form

$$\overline{\mathbf{F}} = \mathbf{K}(\tau)\overline{\mathbf{C}} + \int_0^\tau \mathbf{G}(\tau,\tau_1)\overline{\mathbf{b}}^{(1)}d\tau, \tag{6}$$

where $\mathbf{K}(\tau)$ is the fundamental matrix of solutions to homogeneous equation (5). This matrix satisfies the condition $\mathbf{K}(0) = \mathbf{E}$.

The problem under consideration has zero initial conditions $\left(\overline{\mathbf{f}}(0) = 0, \dot{\overline{\mathbf{f}}} = 0\right)$, therefore,

$$\overline{\mathbf{F}}(\tau) = \int_0^\tau \mathbf{G}(\tau,\tau_1)\overline{\mathbf{b}}^{(1)}\,d\tau_1 \qquad \left(\mathbf{G}(\tau,\tau_1) = \mathbf{K}(\tau)\mathbf{K}^{-1}(\tau_1)\right), \tag{7}$$

or, since the vector $\overline{\mathbf{b}}^{(1)}$ does not depend on $\tau$,

$$\overline{\mathbf{F}}(\tau) = \int_0^\tau \mathbf{G}(\tau,\tau_1)d\tau_1 \cdot \overline{\mathbf{b}}^{(1)}. \tag{8}$$

Let us derive the solution to equation (5) for the case when its right-hand side does not depend on $\tau$, i.e.,

$$\overline{\mathbf{F}} = \mathbf{K}(\tau)\overline{\mathbf{C}} + \overline{\mathbf{F}}_0. \tag{9}$$

We seek the partial solution $\overline{\mathbf{F}}_0$ to inhomogeneous equation (5) in the form

$$\overline{\mathbf{F}}_0 = \overline{\mathbf{C}}_1, \tag{10}$$

where the components of vector $\overline{\mathbf{C}}_1$ are constant quantities.

Substituting (10) into equation (5) we get $\mathbf{A}\overline{\mathbf{C}}_1 = \overline{\mathbf{b}}^{(1)}$, so that $\overline{\mathbf{C}}_1 = \mathbf{A}^{-1}\overline{\mathbf{b}}^{(1)}$. Therefore,

$$\overline{\mathbf{F}} = \mathbf{K}(\tau)\overline{\mathbf{C}} + \mathbf{A}^{-1}\overline{\mathbf{b}}^{(1)}.$$

Since $\overline{\mathbf{F}} = 0$ and $\mathbf{K}(0) = \mathbf{E}$ at $\tau = 0$, then $\mathbf{C} = -\mathbf{A}^{-1}\overline{\mathbf{b}}^{(1)}$. Finally, we have

$$\overline{\mathbf{F}} = \Big(\mathbf{E} - \mathbf{K}(\tau)\Big)\mathbf{A}^{-1}\overline{\mathbf{b}}^{(1)} = \mathbf{P}\overline{\mathbf{b}}^{(1)},$$

or

$$\overline{\mathbf{F}} = \begin{bmatrix} \dot{\overline{\mathbf{f}}} \\ \overline{\mathbf{f}} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \mathbf{H}^{-1}\overline{\mathbf{b}} \\ 0 \end{bmatrix}. \tag{11}$$

Now determine from (11) the vector $\overline{\mathbf{f}}$ whose components $f^{(1)}$ and $f^{(2)}$ appear in accepted approximate solution (2)

$$\overline{\mathbf{f}} = \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix} = P_{11}(\tau)\mathbf{H}^{-1}\overline{\mathbf{b}}. \tag{12}$$

From (12) we find

$$f^{(1)} = d_{11}(\tau)b_1 + d_{12}(\tau)b_2; \qquad f^{(2)} = d_{21}(\tau)b_1 + d_{22}(\tau)b_2,$$

where $d_{ij}$ are the elements of the matrix $P_{11}(\tau)\mathbf{H}^{-1}$.

As a result, we get the following solution

$$\overline{\mathbf{Z}} = (d_{11}b_1 + d_{12}b_2)\overline{\mathbf{Z}}_0^{(1)} + (d_{21}b_1 + d_{22}b_2)\overline{\mathbf{Z}}_0^{(2)}. \tag{13}$$

Under the condition of the problem it is required to determine the moment $M_3$ in the embedment (at $\eta = 0$). According to (13),

$$Z_3(0,\tau) = M_3 = (d_{11}b_1 + d_{12}b_2)M_{30}^{(1)}(0) + (d_{21}b_1 + d_{22}b_2)M_{30}^{(2)}(0).$$

**128** After termination of the action of momentum the mass $m$ attains the velocity $\dot{\overline{u}} = (J/n)\overline{\mathbf{i}}_2$, therefore, at $\tau = 0$ we have the following initial conditions:

$$\overline{\mathbf{u}}(0,\eta) = 0; \qquad \dot{\overline{\mathbf{u}}}(0,\eta) = -\frac{J}{n}\overline{\mathbf{e}}_{2k}\delta(\eta - 0.5) \qquad (n = m\, l p_0). \tag{1}$$

Consider vibrations of the rod about its natural state in the plane of drawing (Fig. 1.98). One needs to put $Q_{10} = Q_{20} = 0$, $n_1 = 1$, and $A_{33} = 1$ in the equations of small vibrations, for example, in equations (C.25) (see Appendix C). The concentrated mass leads to the appearance of $P_1$ and $P_2$ in the first two equations of system (C.25):

$$P_1 = -n_2 \frac{\partial^2 u_1}{\partial \tau^2} \delta(\eta - 0.5);$$
$$P_2 = -n_2 \frac{\partial^2 u_2}{\partial \tau^2} \delta(\eta - 0.5) \qquad \left(n_2 = \frac{m}{m_0\, l}\right). \tag{2}$$

As a result, we obtain the following system of equations

$$\begin{gathered}
\frac{\partial^2 u_1}{\partial \tau^2} - \frac{\partial Q_1}{\partial \eta} + æ_{30} Q_2 = -n_2 \frac{\partial^2 u_1}{\partial \tau^2} \delta(\eta - 0.5);\\
\frac{\partial^2 u_2}{\partial \tau^2} - \frac{\partial Q_2}{\partial \eta} - æ_{30} Q_1 = -n_2 \frac{\partial^2 u_2}{\partial \tau^2} \delta(\eta - 0.5);\\
J_{33} \frac{\partial^2 \vartheta_3}{\partial \tau^2} - \frac{\partial M_3}{\partial \eta} - Q_2 = 0; \qquad \frac{\partial \vartheta_3}{\partial \eta} - \triangle M_3 = 0;\\
\frac{\partial u_1}{\partial \eta} - æ_{30} u_2 = 0; \qquad \frac{\partial u_2}{\partial \eta} + æ_{30} u_1 - \vartheta_3 = 0,
\end{gathered} \tag{3}$$

where $æ_{30} = l/R = \pi$.

Next we determine the eigen values and eigen functions (vectors). Assuming that

$$Q_j = Q_{j0} e^{i\lambda\tau}; \quad u_j = u_{j0} e^{i\lambda\tau}; \quad \vartheta_3 = \vartheta_{30} e^{i\lambda\tau}; \quad M_3 = M_{30} e^{i\lambda\tau},$$

we arrive at the equation

$$\dot{\overline{\mathbf{Z}}}_0 + \mathbf{A}\overline{\mathbf{Z}}_0 = \triangle \overline{\mathbf{\Phi}}, \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -\pi & 0 & 0 & \lambda^2 & 0 \\ \pi & 0 & 0 & 0 & 0 & \lambda^2 \\ 0 & 1 & 0 & J_{33}\lambda^2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\pi \\ 0 & 0 & 0 & -1 & \pi & 0 \end{bmatrix};$$

$$\triangle \overline{\mathbf{\Phi}} = \begin{bmatrix} n_2\lambda^2 u_{10} \\ n_2\lambda^2 u_{20} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \delta(\eta - 0.5) = \triangle \overline{\mathbf{\Phi}}_0 \delta(\eta - 0.5).$$

In the problem considered the boundary conditions have the form

$$\eta = 0, \quad u_{10} = u_{20} = \vartheta_{30} = 0;$$
$$\eta = 1, \quad u_{10} = Q_{20} = M_{30} = 0.$$

The solution to equation (4) looks like

$$\overline{\mathbf{Z}}_0 = \mathbf{K}(\eta)\overline{\mathbf{C}} + \int\limits_0^\eta \mathbf{G}(\eta, h_1)\triangle\overline{\mathbf{\Phi}}(\eta)\,\mathrm{d}h_1, \tag{5}$$

where $\mathbf{K}(\eta)$ is the fundamental matrix of solutions; $\mathbf{G}(\eta, h_1)$ is the Green's matrix.

At $\eta = 0$ from the boundary conditions it follows that $c_4 = c_5 = c_6 = 0$. From equation (5) we get for $\eta = 1$

$$\overline{\mathbf{Z}}_0(1) = \mathbf{K}(1)\overline{\mathbf{C}} + \mathbf{G}(1, 0.5)\triangle\overline{\mathbf{\Phi}}_0.$$

From the boundary conditions at $\eta = 1$ follows

$$\begin{aligned} &k_{21}(1)c_1 + k_{22}(1)c_2 + k_{23}(1)c_3 + g_{21}n_2\lambda^2 u_{10}(0.5) + \\ &\qquad + g_{22}n_2\lambda^2 u_{20}(0.5) = 0; \\ &k_{31}(1)c_1 + k_{32}(1)c_2 + k_{33}(1)c_3 + g_{21}n_2\lambda^2 u_{10}(0.5) + \\ &\qquad + g_{32}n_2\lambda^2 u_{20}(0.5) = 0; \\ &k_{51}(1)c_1 + k_{52}(1)c_2 + k_{53}(1)c_3 + g_{51}n_2\lambda^2 u_{10}(0.5) + \\ &\qquad + g_{52}n_2\lambda^2 u_{20}(0.5) = 0; \end{aligned} \tag{6}$$

In turn, since the integral in (5) is equal to zero at $\eta = 0.5$,

$$\begin{aligned} u_{10}(0.5) &= k_{51}(0.5)c_1 + k_{52}(0.5)c_2 + k_{53}(0.5)c_3; \\ u_{20}(0.5) &= k_{61}(0.5)c_1 + k_{62}(0.5)c_2 + k_{63}(0.5)c_3. \end{aligned} \tag{7}$$

Substituting expressions for $u_{10}(0.5)$ and $u_{20}(0.5)$ into system (6), we obtain after transformations the following system of homogeneous equations for unknowns $c_1$, $c_2$, and $c_3$:

$$\mathbf{e}_{j1}c_1 + \mathbf{e}_{j2}c_2 + \mathbf{e}_{j3}c_3 = 0 \qquad (j = 1, 2, 3). \tag{8}$$

Equating the determinant of system (8) to zero, we write the equation for determination of frequencies. Determining, for example, three frequencies $\lambda_j$ $(j = 1, 2, 3)$, we find three eigen vectors $\overline{\mathbf{Z}}_0^{(j)}$ corresponding to them. Assuming $c_3^{(j)} = 1$, for every $\lambda_j$ we find $c_1^{(j)}$ and $c_2^{(j)}$ from (8). As a consequence, the vector $\overline{\mathbf{C}}^{(j)}$ for each $\lambda_j$ is equal to

$$\overline{\mathbf{C}}^{(j)} = \left(c_1^{(j)}, c_2^{(j)}, 1, 0, 0, 0\right)^{\mathrm{T}}.$$

The eigen vectors are

$$\overline{\mathbf{Z}}_0^{(j)} = \mathbf{K}(\eta, \lambda_j)\overline{\mathbf{C}}^{(j)} + \mathbf{G}(\eta, 0.5)\triangle\overline{\mathbf{\Phi}}_0 H(\eta - 0.5).$$

The vector $\triangle\overline{\mathbf{\Phi}}_0$ depending on $u_{10}(0.5)$ and $u_{20}(0.5)$ can be represented in the form

$$\triangle\overline{\mathbf{\Phi}}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & n_2\lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_2\lambda^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \overline{\mathbf{Z}}_0^{(j)}(0.5) = \overline{\mathbf{D}}\,\overline{\mathbf{Z}}^{(j)}(0.5).$$

Since $\overline{\mathbf{Z}}^{(j)}(0.5) = \mathbf{K}(0.5, \lambda_j)\overline{\mathbf{C}}^{(j)}$, the expressions for eigen vectors have the form

$$\overline{\mathbf{Z}}_0^{(j)}(\eta) = \mathbf{K}(\eta, \lambda_j) + \mathbf{GDK}(0.5, \lambda_j)H(\eta - 0.5)\overline{\mathbf{C}}^{(j)}.$$

Partial solutions to system of equations (3) are

$$\overline{\mathbf{Z}}_{(1)}^{(j)} = \overline{\mathbf{Z}}_{(0)}^{(j)} \cos\lambda_j\tau; \quad \overline{\mathbf{Z}}_{(2)}^{(j)} = \overline{\mathbf{Z}}_{(0)}^{(j)} \sin\lambda_j\tau.$$

At final number of partial solutions the solution is as follows

$$\overline{\mathbf{Z}} = \sum_{j=1}^{n} C^{(j)}\overline{\mathbf{Z}}_{(1)}^{(j)} + \sum_{j=1}^{n} B^{(j)}\overline{\mathbf{Z}}_{(2)}^{(j)} = \sum_{j=1}^{n}\Big(C^{(j)}\overline{\mathbf{Z}}_{(0)}^{(j)}\cos\lambda_j\tau + \\ + B^{(j)}\overline{\mathbf{Z}}_{(0)}^{(j)}\sin\lambda_j\tau\Big). \tag{9}$$

From (9) we have the following expression for vector $\overline{\mathbf{u}}$:

$$\overline{\mathbf{u}} = \sum_{j=1}^{n}\Big(C^{(j)}\overline{\mathbf{Z}}_u^{(j)}\cos\lambda_j\tau + B^{(j)}\overline{\mathbf{Z}}_u^{(j)}\sin\lambda_j\tau\Big),$$

where $\overline{\mathbf{Z}}_u^{(j)} = \begin{bmatrix} u_{10}^{(j)} \\ u_{20}^{(j)} \end{bmatrix}$.

Now we determine the arbitrary constants $C^{(j)}$ and $B^{(j)}$ using the initial conditions. Since $\overline{\mathbf{u}} = 0$ for $\tau = 0$, hence $C^{(j)} = 0$. The second initial condition gives the relation

$$-\frac{J}{n}\delta(\eta - 0.5)\overline{\mathbf{e}}_{2k} = \sum_{j=1}^{n} B^{(j)}\lambda_j\overline{\mathbf{Z}}_u^{(j)}. \tag{10}$$

Multiplying equation (10) by $\overline{\mathbf{Z}}_u(\nu)$ and integrating between 0 and 1 we obtain the system of equations of the form

$$-\frac{J}{n}u_{20}(\nu) = \sum_{j=1}^{n} B^{(j)}\lambda_j \int_0^1 \Big(\overline{\mathbf{Z}}_u^{(j)}, \overline{\mathbf{Z}}_u(\nu)\Big)\, d\eta.$$

Having determined $C^{(j)}$ and $B^{(j)}$ we have finally

$$\overline{\mathbf{Z}} = \sum_{j=1}^{n} B^{(j)} \overline{\mathbf{Z}}_0^{(j)} \sin \lambda_j \tau.$$

In the conditions of the problem it is required to determine the reaction force $\overline{R}$ in the hinge, which is equal to

$$\overline{\mathbf{R}} = Q_1(1, \tau)\overline{\mathbf{e}}_1,$$

where

$$Q_1(1, \tau) = \mathbf{Z}_1 = \sum_{j=1}^{n} B^{(j)} Q_{10}^{(j)}(1) \sin \lambda_j \tau.$$

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# A

# Statics of rods: basic equations

Solving the problems of dynamics of curvilinear rods when the vibrations of the rod takes place about the loaded state, it necessary to know the rod's static mode of deformation: the equations of small vibrations of the rod depend on it (for more details see Appendix C). Therefore, in order to investigate small vibrations of a rod (either free or forced), one needs to solve first the equations of vibrations of the rod loaded with static forces.

When deriving the equations of rod equilibrium, two orthogonal systems of coordinates are used. One is a Cartesian system with unit vectors $\bar{\mathbf{i}}_j$, relative to which the rod position is determined. The second system with unit vectors $\bar{\mathbf{e}}_j$ is movable (Fig. A.1), and it is rigidly fixed to the axial line of the rod. The fixed (movable) axes can be directed arbitrarily. In order that the equations of equilibrium and the equations of motion of a rod element would be simpler,

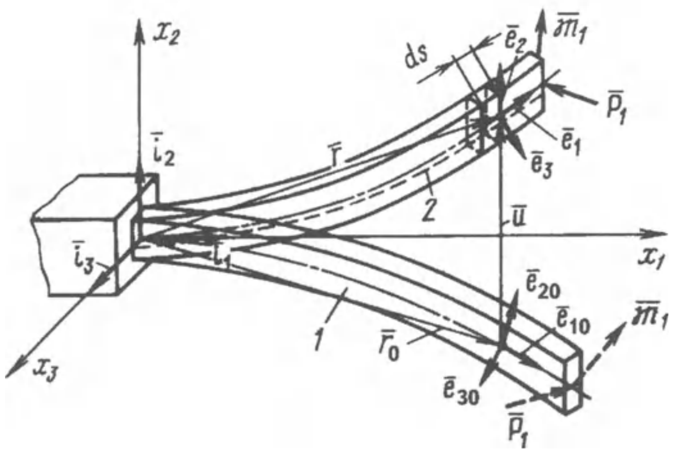


**Fig. A.1.**

it is worthwhile, when the orientation of axes is chosen, to take into account the following. The origin of coordinates should coincide with the centroid of the rod cross section. Next, one of the axes, for example, the axis determined by the unit vector $\overline{\mathbf{e}}_1$ (see Fig. A.1), should be oriented along the tangent to the axial line of the rod in the direction of increasing coordinate $s$. It is reasonable to align two other axes with the principal central axes of cross section. The axes fixed to the principal section axes will be referred to as principal axes. Figure A.1 shows two positions of the rod: in the unloaded (natural) and loaded states. Under the action of slowly increasing forces $\overline{\mathbf{P}}_{(i)}$ and moments $\overline{\mathfrak{M}}_{(\nu)}$ the rod is strained changing from state 1 into state 2. Ii is seen from Fig. A.1 that elastic displacements can be so large that the axial line of the loaded rod differs strongly from its original form (before loading). The external forces in the process of strain can change their directions. The directions of vectors $\overline{\mathbf{P}}_i$ and $\overline{\mathfrak{M}}_i$ at the moment of their application to the rod are shown in Fig. A.1 by dashed lines.

## A.1 Derivation of nonlinear equations of rod equilibrium

When deriving the equations of rod equilibrium we use the following basic assumptions:

1) normal sections that are plane before the strain remain plane after the strain too;

2) the axial line of the rod is unstretchable;

3) dimensions of cross section are small as compared to the rod length and the curvature radius the rod axial line;

4) different but statistically equivalent local loads produce in the rod (if local tensions near the point of load application are not taken into account) one and the same stressed state (principle of Saint-Venent).

Let us consider an element of the rod of length d$s$ and draw all forces applied to it (Fig. A.2). The following notation is used in Fig. A.2: $\overline{\mathbf{Q}}$ is the vector of internal strength, $\overline{\mathbf{M}}$ is the intrinsic moment, $\overline{\mathbf{q}}$ is the vector of distributed load ($\overline{\mathbf{q}} = q_1\overline{\mathbf{e}}_1 + q_2\overline{\mathbf{e}}_2 + q_3\overline{\mathbf{e}}_3$), and $\overline{\mu}$ is the vector of distributed moment ($\overline{\mu} = \mu_1\overline{\mathbf{e}}_1 + \mu_2\overline{\mathbf{e}}_2 + \mu_3\overline{\mathbf{e}}_3$). The directions of the axes of a connected trihedral that are determined by unit vectors $\overline{\mathbf{e}}_2$ and $\overline{\mathbf{e}}_3$ coincide with the directions of principal axes of the rod section. The rod element is in equilibrium, hence, the sums of all forces and all moments are zero, and this results in two vector equations:

$$\mathrm{d}\overline{\mathbf{Q}} + \overline{\mathbf{q}}\mathrm{d}s = 0; \tag{A.1}$$

$$\mathrm{d}\overline{\mathbf{M}} + (\overline{\mathbf{e}}_1 \times \overline{\mathbf{Q}})\mathrm{d}s + \overline{\mu}\mathrm{d}s = 0, \tag{A.2}$$

or

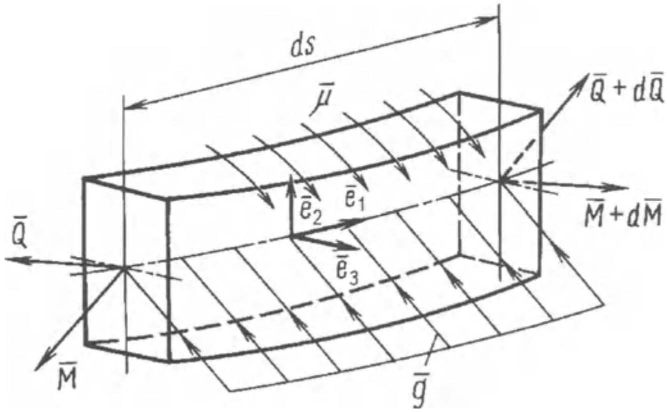


**Fig. A.2.**

$$\frac{\mathrm{d}\overline{\mathbf{Q}}}{\mathrm{d}s} + \overline{\mathbf{q}} = 0; \tag{A.3}$$

$$\frac{\mathrm{d}\overline{\mathbf{M}}}{\mathrm{d}s} + \overline{\mathbf{e}}_1 \times \overline{\mathbf{Q}} + \overline{\mu} = 0. \tag{A.4}$$

In order to convert from equations (A.3) and (A.4) to the equations written through the components of vectors in the bases $\{\overline{\mathbf{i}}_j\}$ or $\{\overline{\mathbf{e}}_j\}$, these vectors should be represented in the form of decompositions in base vectors. In equation systems (A.3) and (A.4) the vectors $\overline{\mathbf{Q}}$, $\overline{\mathbf{M}}$, and $\overline{\mathbf{e}}_1$ are unknown quantities. The acting distributed loads $\overline{\mathbf{q}}$ and $\overline{\mu}$, concentrated forces and moments applied to the rod (see Fig. A.1), and the conditions of rod fixing are known quantities in these systems. The concentrated forces $\overline{\mathbf{P}}^{(i)}$ and moments $\overline{\mathfrak{M}}^{(\nu)}$ can be included into equations (A.3) and (A.4) taking advantage of the Dirac $\delta$. As a result, we have the following equations of equilibrium:

$$\frac{\mathrm{d}\overline{\mathbf{Q}}}{\mathrm{d}s} + \overline{\mathbf{q}} + \sum_{j=1}^{n} \overline{\mathbf{P}}^{(i)} \delta(s - s_i) = 0; \tag{A.5}$$

$$\frac{\mathrm{d}\overline{\mathbf{M}}}{\mathrm{d}s} + \overline{\mathbf{e}}_1 \times \overline{\mathbf{Q}} + \overline{\mu} + \sum_{\nu=1}^{\rho} \overline{\mathfrak{M}}^{(\nu)} \delta(s - s_\nu) = 0, \tag{A.6}$$

where $s_i$ and $s_\nu$ are the coordinates of points of application of concentrated forces and moments, respectively.

The distributed force $\overline{\mathbf{q}}$ appears in equations (A.5) and (A.6). It can act on a part and not over the full length of the rod. In this case, one can write the equations using the Heaviside function, i.e.,

$$\overline{\mathbf{q}} = \overline{\mathbf{q}}(s)[H(s) - H(s - s_i)];$$
$$\overline{\mu} = \overline{\mu}(s)[H(s) - H(s - s_\nu)],$$

where $H(s)$ is the Heaviside function.

Equations (A.5) and (A.6) hold true for the case when the rod is unloaded in the initial state. The system of these equations is not complete, since in the general case it is impossible to determine $\overline{\mathbf{Q}}$ and $\overline{\mathbf{M}}$ from it. The point is that equation (A.6) involves the unit vector $\overline{\mathbf{e}}_1$ whose position in space is unknown, since it depends on the rod deformation. Therefore, we need to derive the relations that would allow us to determine the spatial positions of the base vectors $\overline{\mathbf{e}}_j$ under deformation of the rod.

## A.2 Transformations of base vectors

When a rod changes from its natural state 1 (see Fig. A.1) into state 2 with loading by external forces, the base vectors $\overline{\mathbf{e}}_{i0}$ bound to the axial line of the rod are displaced to another point of space. This transition of the base into another point is characterized by the vector $\overline{\mathbf{u}}$ of displacement of the base origin (see Fig. A.1) and by the rotation of bound coordinate axes (vectors $\overline{\mathbf{e}}_j$).

Let us derive the relations that allow one to change from one orthogonal base to another. Let $\{\overline{\mathbf{e}}_i\}$ $(i = 1, 2, 3)$ be a certain base (determining the directions of bound coordinate axes) in the three-dimensional space, and $\{\overline{\mathbf{e}}_{10}\}$ be the base fixed to the same rod section before this section is loaded by external forces (Fig. A.3). Each vector of the base $\{\overline{\mathbf{e}}_i\}$ can be decomposed in vectors of the original base $\{\overline{\mathbf{e}}_{10}\}$:

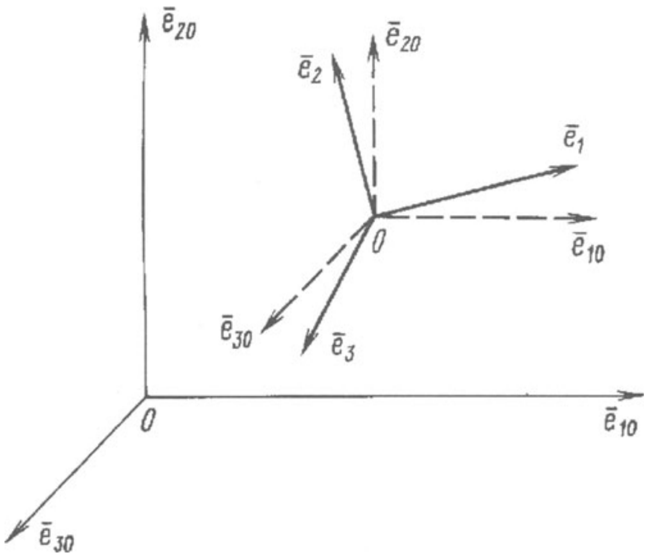


**Fig. A.3.**

$$\begin{aligned}\bar{\mathbf{e}}_1 &= l_{11}\bar{\mathbf{e}}_{10} + l_{12}\bar{\mathbf{e}}_{20} + l_{13}\bar{\mathbf{e}}_{30};\\ \bar{\mathbf{e}}_2 &= l_{21}\bar{\mathbf{e}}_{10} + l_{22}\bar{\mathbf{e}}_{20} + l_{23}\bar{\mathbf{e}}_{30};\\ \bar{\mathbf{e}}_3 &= l_{31}\bar{\mathbf{e}}_{10} + l_{32}\bar{\mathbf{e}}_{20} + l_{33}\bar{\mathbf{e}}_{30},\end{aligned} \tag{A.7}$$

where $l_{ij}$ are the projections of base vectors $\bar{\mathbf{e}}_i$ onto the directions determined by vectors $\bar{\mathbf{e}}_{j0}$. In system of equations (A.7) the coefficients $l_{ij}$ form the matrix **L**:

$$\mathbf{L} = [l_{ij}] = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \tag{A.8}$$

that is referred to as the transition matrix from the base $\{\bar{\mathbf{e}}_{j0}\}$ to the base $\{\bar{\mathbf{e}}_j\}$. Relations (A.7) can written in a more compact form:

$$\bar{\mathbf{e}}_i = \mathbf{L}^{\mathrm{T}}\bar{\mathbf{e}}_{i0}, \tag{A.9}$$

where $\mathbf{L}^{\mathrm{T}}$ is the transposed matrix. At a reverse conversion of the base $\{\bar{\mathbf{e}}_i\}$ to the base $\{\bar{\mathbf{e}}_{10}\}$, we have

$$\bar{\mathbf{e}}_{i0} = \mathbf{L}\bar{\mathbf{e}}_i, \tag{A.10}$$

Let us find the transformation matrix under an arbitrary displacement and rotation of the triple of base vectors (Fig. A.4). Since the base vectors at a translational displacement of coordinate axes coincide with original base vectors, one can restrict oneself to considering only the transformation due to rotation of base vectors. An arbitrary rotation of coordinate axes can be represented as three independent rotations, therefore, we determine the matrix **L** in the following way. Consider a rotation of initial coordinate axes about the axis coinciding with the direction of vector $\bar{\mathbf{e}}_{10}$ through a positive angle $\vartheta_1$ (see Fig. A.4*a*). As a result, we have

$$\begin{aligned}\bar{\mathbf{i}}_1' &= \bar{\mathbf{e}}_{10};\\ \bar{\mathbf{i}}_2' &= \cos\vartheta_1 \cdot \bar{\mathbf{e}}_{20} + \sin\vartheta_1 \cdot \bar{\mathbf{e}}_{30} \quad ;\\ \bar{\mathbf{i}}_3' &= -\sin\vartheta_1 \cdot \bar{\mathbf{e}}_{20} + \cos\vartheta_1 \cdot \bar{\mathbf{e}}_{30};\end{aligned}$$

The corresponding transition matrix has the form

$$\mathbf{L}_{\vartheta_1} = \begin{matrix} \\ \bar{\mathbf{i}}_1' \\ \bar{\mathbf{i}}_2' \\ \bar{\mathbf{i}}_3' \end{matrix} \begin{matrix} \begin{matrix} \bar{\mathbf{e}}_{10} & \quad \bar{\mathbf{e}}_{20} & \quad \bar{\mathbf{e}}_{30} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta_1 & \sin\vartheta_1 \\ 0 & -\sin\vartheta_1 & \cos\vartheta_1 \end{bmatrix} \end{matrix},$$

As for any matrix of rotation of coordinate axes, the elements of matrix $\mathbf{L}_{\vartheta_1}$ can be considered as direction cosines between the base vectors $\{\bar{\mathbf{e}}_i\}$ and $\{\bar{\mathbf{i}}_i\}$. We make the second rotation through a positive angle $\vartheta_3$ about the axis coinciding with the direction of vector $\bar{\mathbf{i}}_3$ (see Fig. A.4*b*). As this takes place,

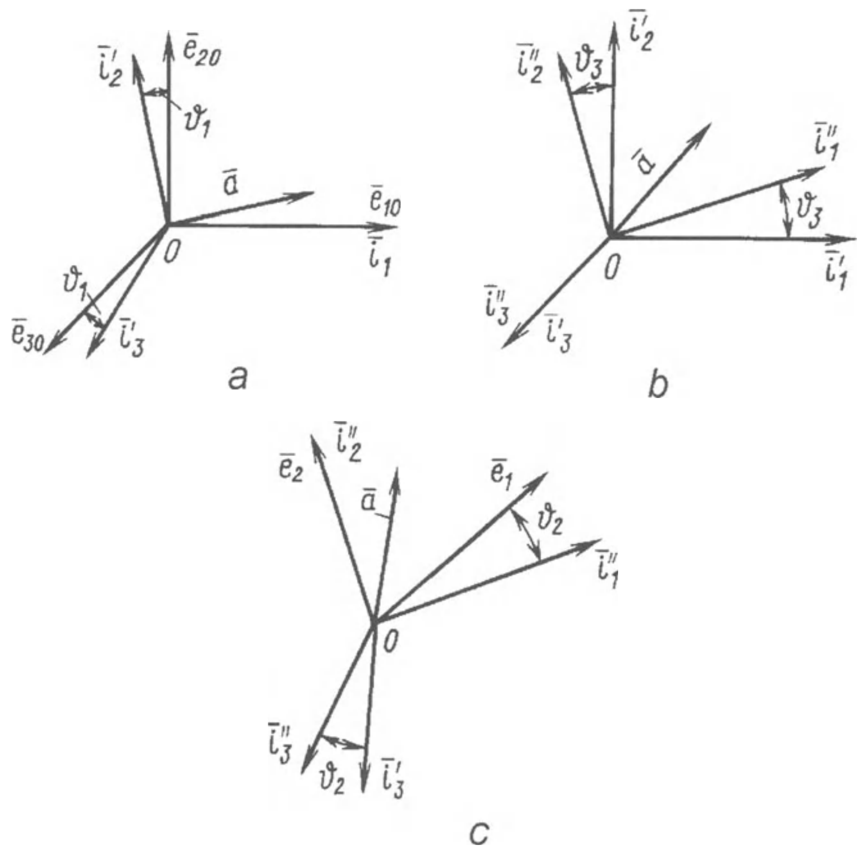


**Fig. A.4.**

$$
\begin{aligned}
\bar{\mathbf{i}}_1'' &= \cos\vartheta_3 \cdot \bar{\mathbf{i}}_1' + \sin\vartheta_3 \cdot \bar{\mathbf{i}}_2';\\
\bar{\mathbf{i}}_2'' &= -\sin\vartheta_3 \cdot \bar{\mathbf{i}}_1' + \cos\vartheta_3 \cdot \bar{\mathbf{i}}_2';\\
\bar{\mathbf{i}}_3'' &= \bar{\mathbf{i}}_3';
\end{aligned}
$$

In this case, the transition matrix looks like

$$
\mathbf{L}_{\vartheta_3} = \begin{matrix} \\ \bar{\mathbf{i}}_1''' \\ \bar{\mathbf{i}}_2''' \\ \bar{\mathbf{i}}_3''' \end{matrix}
\begin{matrix} \begin{matrix} \bar{\mathbf{i}}_1 & \quad\bar{\mathbf{i}}_2 & \quad\bar{\mathbf{i}}_3 \end{matrix} \\
\begin{bmatrix} \cos\vartheta_3 & \sin\vartheta_3 & 0 \\ -\sin\vartheta_3 & \cos\vartheta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}.
$$

Finally, we make the last rotation of coordinate axes through a positive angle $\vartheta_2$ about the axis aligned with the vector $\bar{\mathbf{i}}_2'' = \bar{\mathbf{e}}_2$ (see Fig. A.4*c*). After that, the base vectors $\bar{\mathbf{i}}_j''$ coincide with vectors $\bar{\mathbf{e}}_i$. The corresponding transition matrix has the form

$$\mathbf{L}_{\vartheta_2} = \begin{matrix} & \bar{\mathbf{i}}_1' & \bar{\mathbf{i}}_2' & \bar{\mathbf{i}}_3' \\ \bar{\mathbf{i}}_1'' & \cos\vartheta_2 & 0 & \sin\vartheta_2 \\ \bar{\mathbf{i}}_2'' & 0 & 1 & 0 \\ \bar{\mathbf{i}}_3'' & -\sin\vartheta_2 & 0 & \cos\vartheta_2 \end{matrix} .$$

The general matrix $\mathbf{L}$ of conversion from the base $\{\bar{\mathbf{e}}_{10}\}$ to base $\{\bar{\mathbf{e}}_i\}$ (the transition matrix for rotation of coordinate axes) is equal to the product of matrices $\mathbf{L}_{\vartheta_2}, \mathbf{L}_{\vartheta_3}$, and $\mathbf{L}_{\vartheta_1}$:

$$\mathbf{L} = [l_{ij}] = \mathbf{L}_{\vartheta_2}\mathbf{L}_{\vartheta_3}\mathbf{L}_{\vartheta_1} \tag{A.11}$$

or

$$\mathbf{L} = \begin{matrix} & \bar{\mathbf{e}}_{10} & \bar{\mathbf{e}}_{20} & \bar{\mathbf{e}}_{30} \\ \bar{\mathbf{e}}_1 & \cos\vartheta_2\cos\vartheta_3 & \begin{matrix}\cos\vartheta_2\sin\vartheta_3\cos\vartheta_1+ \\ +\sin\vartheta_2\sin\vartheta_1\end{matrix} & \begin{matrix}\cos\vartheta_2\sin\vartheta_3\sin\vartheta_1- \\ -\sin\vartheta_2\cos\vartheta_1\end{matrix} \\ \bar{\mathbf{e}}_2 & -\sin\vartheta_3 & \cos\vartheta_1\cos\vartheta_3 & \cos\vartheta_3\sin\vartheta_1 \\ \bar{\mathbf{e}}_3 & \sin\vartheta_2\cos\vartheta_3 & \begin{matrix}\sin\vartheta_2\sin\vartheta_3\cos\vartheta_1- \\ -\cos\vartheta_2\sin\vartheta_1\end{matrix} & \begin{matrix}\sin\vartheta_2\sin\vartheta_3\sin\vartheta_1- \\ -\cos\vartheta_2\cos\vartheta_1\end{matrix} \end{matrix} . \tag{A.12}$$

Other sequences of rotations of coordinate axes are also possible.

At small angles of rotation the matrix $\mathbf{L}$ (see expression (A.13)), which we denote as $\triangle\mathbf{L}$, takes on the form

$$\mathbf{L}_{\vartheta_2} = \begin{matrix} & \bar{\mathbf{e}}_{10} & \bar{\mathbf{e}}_{20} & \bar{\mathbf{e}}_{30} \\ \bar{\mathbf{e}}_1 & 1 & \vartheta_3 & -\vartheta_2 \\ \bar{\mathbf{e}}_2 & -\vartheta_3 & 1 & \vartheta_1 \\ \bar{\mathbf{e}}_3 & \vartheta_2 & -\vartheta_1 & 1 \end{matrix} , \tag{A.13}$$

or

$$\triangle\mathbf{L} = \mathbf{E} + \triangle\mathbf{L}_1,$$

where

$$\triangle\mathbf{L}_1 = \begin{bmatrix} 0 & \vartheta_3 & -\vartheta_2 \\ -\vartheta_3 & 0 & \vartheta_1 \\ \vartheta_2 & -\vartheta_1 & 0 \end{bmatrix} . \tag{A.14}$$

The matrix $\mathbf{L}$ of transformation of base vectors allows one to establish the relation between components of vector $\bar{\mathbf{a}}$ in different bases:

$$a_i = a_{k0} l_{ik},$$

where $a_i$ are the components of the vector $\bar{\mathbf{a}}$ in the base $\{\bar{\mathbf{e}}_i\}$, and $a_{k0}$ are the components of the vector $\bar{\mathbf{a}}$ in the base $\{\bar{\mathbf{e}}_{10}\}$.

In the vector form of notation we have:

$$\bar{\mathbf{a}} = \mathbf{L}\bar{\mathbf{a}}_0. \tag{A.15}$$

If it is required to express components of the vector $\bar{\mathbf{a}}$ in the base $\{\bar{\mathbf{e}}_{i0}\}$ through its components in the base $\{\bar{\mathbf{e}}_i\}$, then from (A.15) we get

$$\bar{\mathbf{a}}_0 = \mathbf{L}^{\mathrm{T}}\bar{\mathbf{a}}.$$

Similarly, one can determine the transition matrix between the bases $\{\bar{\mathbf{i}}_j\}$ and $\{\bar{\mathbf{e}}_{j0}\}$. The latter characterizes the natural state of the rod (before loading). Let us denote the angles of rotation as $\vartheta_j^0$. The matrix $\mathbf{L}^0 0$ obtained similar to the matrix $\mathbf{L}$ has the form

$$\mathbf{L}^0 = \begin{bmatrix} \cos\vartheta_2^0\cos\vartheta_3^0 & \begin{matrix}\cos\vartheta_2^0\sin\vartheta_3^0\cos\vartheta_1^0 + \\ +\sin\vartheta_2^0\sin\vartheta_1^0\end{matrix} & \begin{matrix}\cos\vartheta_2^0\sin\vartheta_3^0\sin\vartheta_1^0 - \\ -\sin\vartheta_2^0\cos\vartheta_1^0\end{matrix} \\ -\sin\vartheta_3^0 & \cos\vartheta_1^0\cos\vartheta_3^0 & \cos\vartheta_3^0\sin\vartheta_1^0 \\ \sin\vartheta_2^0\cos\vartheta_3^0 & \begin{matrix}\sin\vartheta_2^0\sin\vartheta_3^0\cos\vartheta_1^0 - \\ -\cos\vartheta_2^0\sin\vartheta_1^0\end{matrix} & \begin{matrix}\sin\vartheta_2^0\sin\vartheta_3^0\sin\vartheta_1^0 - \\ -\cos\vartheta_2^0\cos\vartheta_1^0\end{matrix} \end{bmatrix}. \tag{A.16}$$

Let us determine the transition matrix between the bases $\{\bar{\mathbf{i}}_j\}$ and $\{\bar{\mathbf{e}}_j\}$ (see Fig. A.1). Since

$$\begin{aligned} \bar{\mathbf{e}}_{j0} &= l_{j1}^0\bar{\mathbf{i}}_1 + l_{j2}^0\bar{\mathbf{i}}_2 + l_{j3}^0\bar{\mathbf{i}}_3; \\ \bar{\mathbf{e}}_k &= l_{k1}\bar{\mathbf{e}}_{10} + l_{k2}\bar{\mathbf{e}}_{20} + l_{k3}\bar{\mathbf{e}}_{30}, \end{aligned} \tag{A.17}$$

then excluding $\bar{\mathbf{e}}_{j0}$ from (A.17) we get

$$\bar{\mathbf{e}}_k = \sum_{j=1}^{3}\sum_{\nu=1}^{3} l_{kj}l_{j\nu}^0\bar{\mathbf{i}}_\nu = \sum_{\nu=1}^{3} l_{k\nu}^{(1)}\bar{\mathbf{i}}_\nu, \tag{A.18}$$

where $l_{k\nu}^{(1)}$ are the elements of the matrix

$$\mathbf{L}^{(1)} = \mathbf{L}\mathbf{L}^0. \tag{A.19}$$

Let us recall that $\mathbf{L}^0$ is a matrix with known elements that characterize the spatial configuration of the axial line in the unloaded (natural) state; while $\mathbf{L}$ is the matrix that characterizes the rotation of vectors of the base $\{\bar{\mathbf{e}}_j\}$ with respect to their natural state. If a rod is rectilinear in its natural state, then

$$\bar{\mathbf{L}}^0 = \mathbf{E}.$$

Knowing the matrix $\overline{\mathbf{L}}^{(1)}$, one can derive the relations linking the vectors of the bases $\{\overline{\mathbf{e}}_j\}$ and $\{\overline{\mathbf{i}}_j\}$:

$$\overline{\mathbf{e}}_j = \left[\mathbf{L}^{(1)}\right]^{\mathrm{T}} \overline{\mathbf{i}}_j = \left[\mathbf{L}^0\right]^{\mathrm{T}} \mathbf{L}^{\mathrm{T}} \overline{\mathbf{i}}_j \qquad (j = 1, 2, 3).$$

Matrix (A.15) allows one to determine the components $a_j$ of vector $\overline{\mathbf{a}}$ in the base $\{\overline{\mathbf{e}}_j\}$ for known components $a_{xj}$ of vector $\overline{\mathbf{a}}$ in the base $\overline{\mathbf{i}}_j$, i.e.,

$$\begin{aligned} &\overline{\mathbf{a}} = \overline{\mathbf{L}}^{(1)} \overline{\mathbf{a}}_x \qquad \left(\overline{\mathbf{a}}_x = \sum_{j=1}^{3} a_{xj} \overline{\mathbf{i}}_j\right); \\ &a_i = \sum_{j=1}^{3} a_{xj} l_{ij}^{(1)}. \end{aligned} \tag{A.20}$$

The transition matrices of base vectors are necessary when deriving the equations of equilibrium and motion for rods, and when accounting for real behaviour of force vectors in the process of loading the rod.

## A.3 Derivatives of base vectors with respect to arc coordinate $s$

The derivative of a vector with respect to a scalar argument is a vector, therefore, it can be represented in the form

$$\frac{\mathrm{d}\overline{\mathbf{e}}_i}{\mathrm{d}s} = \sum_{j=1}^{3} æ_{ij} \overline{\mathbf{e}}_j, \tag{A.21}$$

where $æ_{ij}$ are the elements of a certain matrix. Multiplying expression (A.21) scalarly by $\mathbf{e}_k$ we have

$$\frac{\mathrm{d}\overline{\mathbf{e}}_i}{\mathrm{d}s} \cdot \overline{\mathbf{e}}_k = æ_{ik}. \tag{A.22}$$

Since

$$\overline{\mathbf{e}}_i \cdot \overline{\mathbf{e}}_k = \begin{cases} 1, & \text{if } i = k; \\ 0, & \text{if } i \neq k, \end{cases} \tag{A.23}$$

we have after differentiation of equation (A.22)

$$\frac{\mathrm{d}\overline{\mathbf{e}}^i}{\mathrm{d}s} \cdot \overline{\mathbf{e}}_k + \frac{\mathrm{d}\overline{\mathbf{e}}^k}{\mathrm{d}s} \cdot \overline{\mathbf{e}}_i = 0.$$

Hence, it follows, taking (A.23) into account,

$$æ_{ik} = -æ_{ki},$$

therefore, the matrix $[æ_{ij}]$ has only three independent elements:

$$[æ_{ij}] = \begin{bmatrix} 0 & -æ_3 & æ_2 \\ æ_3 & 0 & -æ_1 \\ -æ_2 & -æ_1 & 0 \end{bmatrix}.$$

Let us invoke the vector

$$\bar{æ} = \sum_{j=1}^{3} æ_j \bar{\mathbf{e}}_j = æ_1\bar{\mathbf{e}}_1 + æ_2\bar{\mathbf{e}}_2 + æ_3\bar{\mathbf{e}}_3.$$

The derivatives of unit vectors with respect to coordinate $s$ (see relations (A.17)) can be represented in the form

$$\frac{\mathrm{d}\bar{\mathbf{e}}_{\nu 0}}{\mathrm{d}s} = \bar{æ}_0 \times \bar{\mathbf{e}}_{\nu 0} = \varepsilon_{n\rho\nu} æ_{\rho 0} \bar{\mathbf{e}}_{n0};$$
$$\frac{\mathrm{d}\bar{\mathbf{e}}_i}{\mathrm{d}s} = \bar{æ} \times \bar{\mathbf{e}}_i = \varepsilon_{kji} æ_j \bar{\mathbf{e}}_k,$$

where $\varepsilon_{kji}$ and $\varepsilon_{n\rho\nu}$ are the Levi-Civita symbols.

Then the following relations

$$\frac{\mathrm{d}\bar{\mathbf{e}}_1}{\mathrm{d}s} = æ_3\bar{\mathbf{e}}_2 - æ_2\bar{\mathbf{e}}_3;$$
$$\frac{\mathrm{d}\bar{\mathbf{e}}_2}{\mathrm{d}s} = æ_1\bar{\mathbf{e}}_3 + æ_3\bar{\mathbf{e}}_1; \tag{A.24}$$
$$\frac{\mathrm{d}\bar{\mathbf{e}}_3}{\mathrm{d}s} = æ_2\bar{\mathbf{e}}_1 - æ_1\bar{\mathbf{e}}_2,$$

hold true.

Taking relations (A.24) into account, the derivative of an arbitrary vector $\bar{\mathbf{a}}$ in the bound coordinate system is equal to

$$\frac{\mathrm{d}\bar{\mathbf{a}}}{\mathrm{d}s} = \frac{\mathrm{d}'\bar{\mathbf{a}}}{\mathrm{d}s} + \bar{æ} \times \bar{\mathbf{a}}, \tag{A.25}$$

where $\frac{\mathrm{d}'}{\mathrm{d}s}$ is the local derivative.

Let us demonstrate the geometrical meaning of the components $æ_j$j of the vector $\bar{æ}$. When relations (A.24) were established, no constraints were imposed on the directions of vectors $\bar{\mathbf{e}}_2$ and $\bar{\mathbf{e}}_3$. Therefore, one can consider these relations as general. For natural axes, which are a special case of bound axes, we have

$$\frac{\mathrm{d}\bar{\mathbf{e}}_1}{\mathrm{d}s} = \frac{1}{\varrho}\bar{\mathbf{e}}_2, \tag{A.26}$$

where $\varrho$ is the curvature radius of the curve.

For the case under consideration we introduce a new designation for the vector $\bar{æ}$: the vector $\overline{\Omega}$ that is known in differential geometry as the Darboux vector. Then, according to the first equation of (A.24), one can write

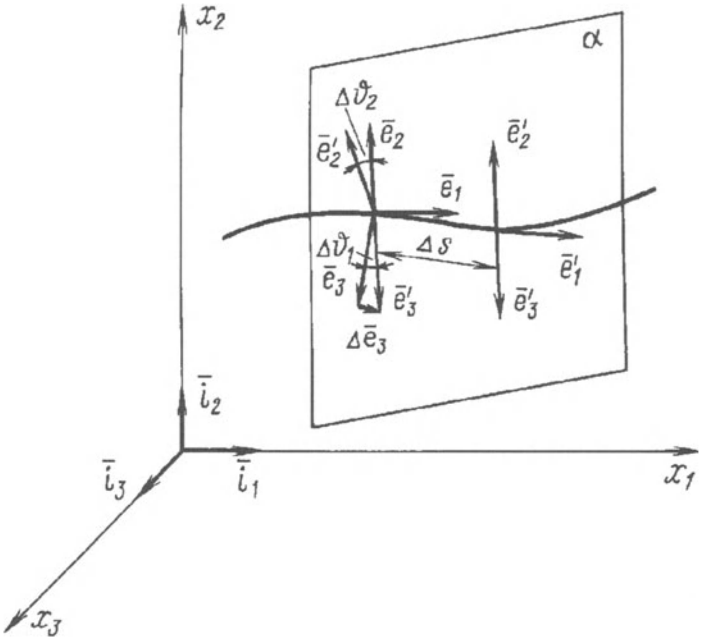


**Fig. A.5.**

$$\frac{d\overline{\mathbf{e}}_1}{ds} = \frac{1}{\varrho}\overline{\mathbf{e}}_2 = \Omega_3\overline{\mathbf{e}}_2 + \Omega_2\overline{\mathbf{e}}_3 = æ_3\overline{\mathbf{e}}_2 - æ_2\overline{\mathbf{e}}_3.$$

Hence, it follows that

$$\Omega_2 = æ_2 = 0; \quad \Omega_3 = æ_3 = \frac{1}{\varrho},$$

i.e., $æ_3$ is the curvature of the curve.

For natural axes, we derive from relations (A.24) the following formulas of Serret-Frenet

$$\frac{d\overline{\mathbf{e}}_1}{ds} = \Omega_3\overline{\mathbf{e}}_2; \quad \frac{d\overline{\mathbf{e}}_2}{ds} = -\Omega_3\overline{\mathbf{e}}_1 + \Omega_1\overline{\mathbf{e}}_3; \quad \frac{d\overline{\mathbf{e}}_3}{ds} = -\Omega_1\overline{\mathbf{e}}_2. \tag{A.27}$$

The vector $\overline{\mathbf{e}}_3'$ is orthogonal to the vector $\overline{\mathbf{e}}_3$. In addition, it lies in the plane $\alpha$ (Fig. A.5) that is orthogonal to the osculating plane, therefore,

$$\frac{d\overline{\mathbf{e}}_3}{ds} = \lim_{\triangle s\to 0}\left(\frac{\triangle\overline{\mathbf{e}}_3}{\triangle s}\right)\overline{\mathbf{e}}_2 = -\lim_{\triangle s\to 0}\left|\frac{\triangle\vartheta_1}{\triangle s}\right|\overline{\mathbf{e}}_2 = -\frac{d\vartheta_1}{ds}\overline{\mathbf{e}}_2,$$

or

$$\frac{d\overline{\mathbf{e}}_3}{ds} = -\Omega_1\overline{\mathbf{e}}_2 = -\frac{d\vartheta_1}{ds}\overline{\mathbf{e}}_2, \tag{A.28}$$

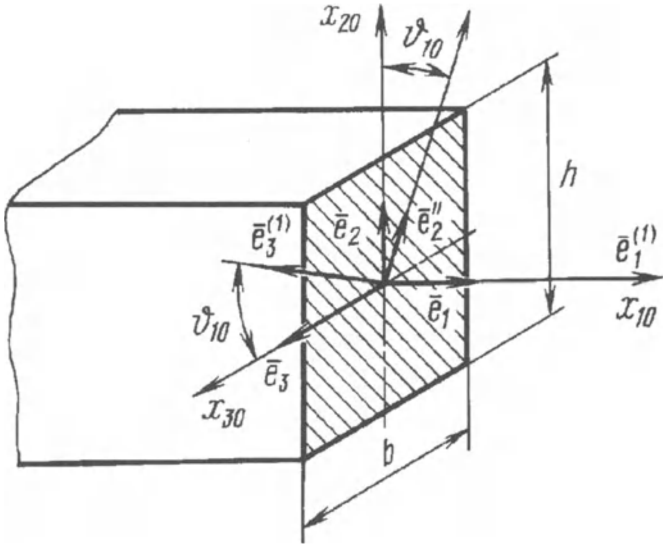


**Fig. A.6.**

hence,

$$\Omega_1 = \mathrm{d}\vartheta_1/\mathrm{d}s.$$

The component $\Omega_1$ of the Darboux vector characterizes the torsion of the curve.

When changing from natural bound $\left(\bar{\mathbf{e}}_j^{(1)}\right)$ axes (Fig. A.6) to any other bound axes ($\bar{\mathbf{e}}_j$) rotated by a known angle $\vartheta$ with respect to the vector $\bar{\mathbf{e}}$, the components of vector $\bar{\text{æ}}$ are expressed through the components of vector $\overline{\Omega}$ in the following way:

$$\bar{\text{æ}} = \sum_{j=1}^{3} \text{æ}_j \bar{\mathbf{e}}_j = \text{æ}_1 \bar{\mathbf{e}}_1 + \Omega_3 \sin\vartheta_{10} \bar{\mathbf{e}}_2 + \Omega_3 \cos\vartheta_{10} \bar{\mathbf{e}}_3, \tag{A.29}$$

where $\text{æ}_1 = \Omega_1 + \dfrac{\mathrm{d}\vartheta_{10}}{\mathrm{d}s}$.

## A.4 Equations relating $\text{æ}_j$ to angles $\vartheta_k$

Let us consider the relations

$$\bar{\mathbf{e}}_i = \sum_{\nu=1}^{3} l_{i\nu} \bar{\mathbf{e}}_{\nu 0}; \qquad \bar{\mathbf{e}}_{\nu 0} = \sum_{\nu=1}^{3} l_{k\nu} \bar{\mathbf{e}}_k,$$

where $l_{ij}$ is the element of matrix (A.13). Differentiating $\overline{\mathbf{e}}_i$, we have

$$\frac{\mathrm{d}\overline{\mathbf{e}}_i}{\mathrm{d}s} = \sum_{\nu=1}^{3}\left(\frac{\mathrm{d}l_{i\nu}}{\mathrm{d}s}\overline{\mathbf{e}}_{\nu0} + l_{i\nu}\frac{\mathrm{d}\overline{\mathbf{e}}_{\nu0}}{\mathrm{d}s}\right) = \overline{æ}\times\overline{\mathbf{e}}_i, \tag{A.30}$$

since

$$\frac{\mathrm{d}\overline{\mathbf{e}}_{\nu0}}{\mathrm{d}s} = \overline{æ}_0\times\overline{\mathbf{e}}_{\nu0} = \overline{æ}_0\times\left(\sum_{k=1}^{3} l_{k\nu}\overline{\mathbf{e}}_k\right).$$

Excluding $\overline{\mathbf{e}}_{\nu0}$ from equation (A.30) we obtain

$$\sum_{\nu=1}^{3}\left\{\frac{\mathrm{d}l_{i\nu}}{\mathrm{d}s}\left(\sum_{k=1}^{3} l_{k\nu}\overline{\mathbf{e}}_k\right) + l_{i\nu}\left[\overline{æ}_0\times\left(\sum_{k=1}^{3} l_{k\nu}\overline{\mathbf{e}}_k\right)\right]\right\} = \overline{æ}\times\overline{\mathbf{e}}_i. \tag{A.31}$$

After some transformations we find from equation (A.31)

$$\begin{aligned}
æ_1 &= \frac{\mathrm{d}l_{21}}{\mathrm{d}s}l_{31} + \frac{\mathrm{d}l_{22}}{\mathrm{d}s}l_{32} + \frac{\mathrm{d}l_{23}}{\mathrm{d}s}l_{33} + (l_{22}l_{33} - l_{23}l_{32})æ_{10} + \\
&\quad + (l_{23}l_{31} - l_{21}l_{33})æ_{20} + (l_{21}l_{32} - l_{22}l_{31})æ_{30};\\
æ_2 &= \frac{\mathrm{d}l_{31}}{\mathrm{d}s}l_{11} + \frac{\mathrm{d}l_{32}}{\mathrm{d}s}l_{12} + \frac{\mathrm{d}l_{33}}{\mathrm{d}s}l_{13} + (l_{32}l_{13} - l_{33}l_{12})æ_{10} + \\
&\quad + (l_{33}l_{11} - l_{31}l_{13})æ_{20} + (l_{31}l_{12} - l_{32}l_{11})æ_{30};\\
æ_3 &= \frac{\mathrm{d}l_{11}}{\mathrm{d}s}l_{21} + \frac{\mathrm{d}l_{12}}{\mathrm{d}s}l_{22} + \frac{\mathrm{d}l_{13}}{\mathrm{d}s}l_{23} + (l_{12}l_{23} - l_{13}l_{22})æ_{10} + \\
&\quad + (l_{13}l_{21} - l_{11}l_{23})æ_{20} + (l_{11}l_{22} - l_{12}l_{21})æ_{30}.
\end{aligned} \tag{A.32}$$

Let us express $l_{ij}$ in relations (A.32) through the angles $\vartheta_1$, $\vartheta_2$ and $\vartheta_3$. As a result, we have

$$\begin{aligned}
æ_1 &= \left(\frac{\mathrm{d}\vartheta_1}{\mathrm{d}s} + æ_{10}\right)\cos\vartheta_2\cos\vartheta_3 - \frac{\mathrm{d}\vartheta_3}{\mathrm{d}s}\sin\vartheta_2 + \\
&\quad + (\sin\vartheta_2\sin\vartheta_1 + \cos\vartheta_2\sin\vartheta_3\cos\vartheta_1)\,æ_{20} + \\
&\quad + (\cos\vartheta_2\sin\vartheta_3\sin\vartheta_1 - \sin\vartheta_2\cos\vartheta_1)\,æ_{30};\\
æ_2 &= \frac{\mathrm{d}\vartheta_2}{\mathrm{d}s} - \left(\frac{\mathrm{d}\vartheta_1}{\mathrm{d}s} + æ_{10}\right)\sin\vartheta_3 + \cos\vartheta_3\cos\vartheta_1æ_{20} + \\
&\quad + \cos\vartheta_3\sin\vartheta_1æ_{30};\\
æ_3 &= \frac{\mathrm{d}\vartheta_3}{\mathrm{d}s}\cos\vartheta_2 + \left(\frac{\mathrm{d}\vartheta_1}{\mathrm{d}s} + æ_{10}\right)\sin\vartheta_2\cos\vartheta_3 + \\
&\quad + (\sin\vartheta_2\sin\vartheta_3\cos\vartheta_1 - \cos\vartheta_2\sin\vartheta_1)\,æ_{20} + \\
&\quad + (\cos\vartheta_2\cos\vartheta_1 + \sin\vartheta_2\sin\vartheta_3\sin\vartheta_1)\,æ_{30}.
\end{aligned} \tag{A.33}$$

When writing expressions (A.33) the angles $\vartheta_1$, $\vartheta_2$ and $\vartheta_3$ were reckoned from the base axes $\{\overline{\mathbf{e}}_{i0}\}$ whose position characterizes the natural state of the rod and is assumed to be the initial state. For the sake of convenience of transformations, system of relations (A.33) can be written in the form of a single vector relationship:

$$\bar{\boldsymbol{æ}} = \mathbf{L}_1 \frac{\mathrm{d}\bar{\vartheta}}{\mathrm{d}s} + \mathbf{L}\bar{\boldsymbol{æ}}_0^{(1)} \qquad (\bar{\boldsymbol{æ}}_0^{(1)} = æ_{i0}\bar{\mathbf{e}}_i), \tag{A.34}$$

where

$$\bar{\vartheta} = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix}; \qquad \frac{\mathrm{d}\bar{\vartheta}}{\mathrm{d}s} = \begin{bmatrix} \vartheta_1' \\ \vartheta_2' \\ \vartheta_3' \end{bmatrix};$$
$$\mathbf{L}_1 = \begin{bmatrix} \cos\vartheta_2\cos\vartheta_3 & 0 & -\sin\vartheta_2 \\ -\sin\vartheta_3 & 1 & 0 \\ \sin\vartheta_2\sin\vartheta_3 & 0 & \cos\vartheta_2 \end{bmatrix}. \tag{A.35}$$

Vector $\bar{\boldsymbol{æ}}_0^{(1)}$ is not equal to vector $\bar{\boldsymbol{æ}}_0$ that characterizes the curve geometry in the initial state. The vector $\bar{\boldsymbol{æ}}_0^{(1)}$ has the components in the base $\{\bar{\mathbf{e}}_i\}$ that are equal to the vector $\bar{\boldsymbol{æ}}_0$ components in the base $\{\bar{\mathbf{e}}_{i0}\}$. Expressions (A.33) give a possibility to determine variations of the components of vector $\bar{\boldsymbol{æ}}$ that characterize the geometry of the axial line in the loaded state, if the geometry of the axial line in the initial state ($æ_{i0}$) is known.

## A.5 Vector equation of displacements of points of the rod axial line

Since (see Fig. A.1)

$$\bar{\mathbf{u}} = \bar{\mathbf{r}} - \bar{\mathbf{r}}_0, \tag{A.36}$$

we can derive, differentiating this equation with respect to $s$,

$$\frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}s} = \bar{\mathbf{e}}_1 - \bar{\mathbf{e}}_{10}. \tag{A.37}$$

Using the matrix $\mathbf{L}$ of form (A.13) we have

$$\bar{\mathbf{e}}_{10} = l_{11}\bar{\mathbf{e}}_1 + l_{21}\bar{\mathbf{e}}_2 1 + l_{31}\bar{\mathbf{e}}_3,$$

and, consequently,

$$\frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}s} = (1 - l_{11})\bar{\mathbf{e}}_1 - l_{21}\bar{\mathbf{e}}_2 - l_{31}\bar{\mathbf{e}}_3. \tag{A.38}$$

Converting to the local derivative (see relation (A.25)), we get

$$\frac{\widetilde{\mathrm{d}\overline{\mathbf{u}}}}{\mathrm{d}s} + \overline{\text{æ}} \times \overline{\mathbf{u}} = (1 - l_{11})\overline{\mathbf{e}}_1 - l_{21}\overline{\mathbf{e}}_2 + l_{31}\overline{\mathbf{e}}_3, \tag{A.39}$$

or, in the scalar form of notation,

$$\begin{aligned} &\frac{\mathrm{d}u_1}{\mathrm{d}s} + \text{æ}_2 u_3 - \text{æ}_3 u_2 + l_{11} - 1 = 0; \\ &\frac{\mathrm{d}u_2}{\mathrm{d}s} + \text{æ}_3 u_1 - \text{æ}_1 u_3 + l_{21} = 0; \\ &\frac{\mathrm{d}u_3}{\mathrm{d}s} + \text{æ}_1 u_2 - \text{æ}_2 u_1 + l_{31} = 0. \end{aligned} \tag{A.40}$$

## A.6 Equation connecting the vectors $\overline{\mathbf{M}}$ and $\overline{\text{æ}}$

Consider in the bound coordinate system ( Fig. A.7) a rod element in the strained state. In the planes passing through the principal axes of the section, the axial line projections have curvatures $\text{æ}_2$ and $\text{æ}_3$ and are projections of the curvature of the spatial axial line. Since the radius of curvature $\varrho$ is directed along a binormal to natural axes that is rotated by the angle $\vartheta_{10}$ with respect to the section principal axes (see Fig. A.7), then

$$\text{æ}_2 = \frac{\sin\vartheta_{10}}{\varrho} = \Omega_3 \sin\vartheta_{10}; \qquad \text{æ}_3 = \frac{\cos\vartheta_{10}}{\varrho} = \Omega_3 \cos\vartheta_{10}. \tag{A.41}$$

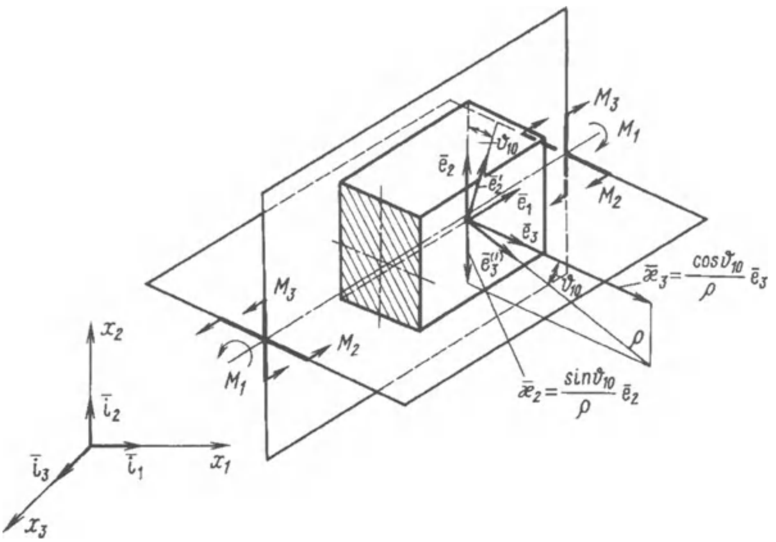


Fig. A.7.

In addition to bending moments $M_2$ and $M_3$, the moment $M_1$ acts upon the rod element, which results in torsion of the rod axial line. This torsion is characterized by the component $æ_1$ of vector $\overline{\boldsymbol{æ}}$. Assuming that the moments $M_1$, $M_2$, and $M_3$ are proportional to variations of torsion and curvature, we obtain three equations:

$$\begin{aligned} M_1 &= A_{11}(æ_1 - æ_{10}) \qquad \left(æ_1 = \Omega_1 + \frac{\mathrm{d}\vartheta_{10}}{\mathrm{d}\eta}\right); \\ M_2 &= A_{22}(æ_2 - æ_{20}); \\ M_3 &= A_{33}(æ_3 - æ_{30}); \\ æ_j - æ_{j0} &= \triangle æ_j, \end{aligned} \tag{A.42}$$

where $A_{ii}$ are the torsional and bending stiffnesses that, for a rod of variable section, depend on $s$ $(A_{11} = GJ_\varrho;\ A_{22} = EJ_y;\ A_{33} = EJ_z)$; while $æ_{i0}$ is the torsion and curvature in the strain-free (natural) state.

System of equations (A.42) can be written in the form of a single vector equation:

$$\overline{\mathbf{M}} = \mathbf{A}(\overline{\boldsymbol{æ}} - \overline{\boldsymbol{æ}}_0^{(1)}), \tag{A.43}$$

where $\mathbf{A} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$.

It should be emphasized that vector $\overline{æ}_0^{(1)}$ is not equal to the vector $\overline{æ}_0$ that characterizes the initial state of the rod.

The vector $\overline{\boldsymbol{æ}}_0$ is known in the base $\{\overline{\mathbf{e}}_{i0}\}$:

$$\overline{\boldsymbol{æ}}_0 = æ_{10}\overline{\mathbf{e}}_{10} + æ_{20}\overline{\mathbf{e}}_{20} + æ_{30}\overline{\mathbf{e}}_{30}. \tag{A.44}$$

In order to find the increments of curvature vectors $\triangle æ_i$ appearing in equations (A.42), one should assume that the vector $\overline{\boldsymbol{æ}}_0^{(1)}$ remains unchanged in the bound system of coordinates: this takes place if its projections in this coordinate system are invariable. In this case, in the base $\{\overline{\mathbf{e}}_i\}$

$$\overline{\boldsymbol{æ}}_0^{(1)} = æ_{10}\overline{\mathbf{e}}_1 + æ_{20}\overline{\mathbf{e}}_2 + æ_{30}\overline{\mathbf{e}}_3. \tag{A.45}$$

## A.7 System of nonlinear equations of rod equilibrium

The system of vector equations of equilibrium of a three-dimensional curvilinear rod has the form

$$\begin{aligned}
&\frac{d\overline{\mathbf{Q}}}{ds} + \overline{\mathbf{P}} = 0;\\
&\frac{d\overline{\mathbf{M}}}{ds} + \overline{\mathbf{e}}_1\overline{\mathbf{Q}} + \overline{\mathfrak{M}} = 0;\\
&\overline{\mathbf{L}}_1\frac{d\overline{\vartheta}}{ds} - æ + \mathbf{L}\overline{æ}_0^{(1)} = 0;\\
&\frac{d\overline{\mathbf{u}}}{ds} - (1 - l_{11})\overline{\mathbf{e}}_1 + l_{21}\overline{\mathbf{e}}_2 + l_{31}\overline{\mathbf{e}}_3 = 0;\\
&\overline{\mathbf{M}} = \mathbf{A}\left(\overline{æ} - \overline{æ}_0^{(1)}\right).
\end{aligned} \tag{A.46}$$

Here

$$\begin{aligned}
\overline{\mathbf{P}} &= \overline{\mathbf{q}} + \sum_{i=1}^{n} \overline{\mathbf{P}}^{(i)}\delta(s - s_i);\\
\overline{\mathfrak{M}} &= \overline{\mu} + \sum_{\nu=1}^{\rho} \overline{\mathfrak{M}}^{(\nu)}\delta(s - s_\nu).
\end{aligned} \tag{A.47}$$

System (A.46) involving five vector equations contains five unknown vectors: $\overline{\mathbf{Q}}, \overline{\mathbf{M}}, \overline{\vartheta}, \overline{æ}$, and $\overline{\mathbf{u}}$.

## A.8 Reduction of equations to dimensionless notation

Let us introduce the following new quantities:

$$\begin{aligned}
&s = \eta\, l, \qquad \tilde{\overline{\mathbf{q}}} = \overline{\mathbf{q}}\Big/\frac{A_{33}(0)}{l^3},\\
&\tilde{\overline{\mathbf{M}}} = \overline{\mathbf{M}}\Big/\frac{A_{33}(0)}{l}, \qquad \tilde{\overline{\mu}} = \overline{\mu}\Big/\frac{A_{33}(0)}{l^2},\\
&\tilde{\overline{\mathbf{Q}}} = \overline{\mathbf{Q}}\Big/\frac{A_{33}(0)}{l^2}, \qquad \tilde{\overline{æ}} = \overline{æ}\, l, \qquad \tilde{A}_{ii}(\eta) = A_{ii}(\eta)\Big/A_{33}(0),\\
&\tilde{\overline{\mathbf{P}}}^{(i)} = \overline{\mathbf{P}}^{(i)}\Big/\frac{A_{33}(0)}{l^2}, \qquad \tilde{\overline{\mathfrak{M}}}^{(\nu)} = \overline{\mathfrak{M}}^{(\nu)}\Big/\frac{A_{33}(0)}{l},
\end{aligned} \tag{A.48}$$

where $\sim$ is the superscript denoting a dimensionless quantity; $A_{33}(0)$ is the bending stiffness at the origin of coordinates. Since centimeter powered to minus unity ($cm^{-1}$) is the unit of $\delta$-function in equations (A.47), we have upon passing to a dimensionless coordinate

$$\delta\left[l(\eta - \eta_i)\right] = \frac{1}{l}\tilde{\delta}(\eta - \eta_i), \tag{A.49}$$

where $\tilde{\delta}$ is the dimensionless function. Substituting relations (A.48) into equations (A.46) and (A.47), we have after transformations the system of nonlinear equations of rod equilibrium in the dimensionless form of notation (the symbol $\sim$ is omitted in dimensionless quantities):

$$\frac{\mathrm{d}\overline{\mathbf{Q}}}{\mathrm{d}\eta} + \overline{\mathbf{q}} + \sum_{i=1}^{n} \overline{\mathbf{P}}^{(i)} \delta(\eta - \eta_i) = 0; \tag{A.50}$$

$$\frac{\mathrm{d}\overline{\mathbf{M}}}{\mathrm{d}\eta} + \overline{\mathbf{e}}_1 \times \overline{\mathbf{Q}} + \overline{\mu} + \sum_{\nu=1}^{\rho} \overline{\mathfrak{M}}^{(\nu)} \delta(\eta - \eta_i) = 0; \tag{A.51}$$

$$\widetilde{\mathbf{L}}_1 \frac{\mathrm{d}\overline{\vartheta}}{\mathrm{d}\eta} + \mathbf{L}_2 \overline{\text{æ}}_0^{(1)} - \mathbf{A}^{-1}\overline{\mathbf{M}} = 0 \qquad (\mathbf{L}_2 = \mathbf{L} - \mathbf{E}); \tag{A.52}$$

$$\frac{\mathrm{d}\overline{\mathbf{u}}}{\mathrm{d}\eta} + (l_{11} - 1)\overline{\mathbf{e}}_1 + l_{21}\overline{\mathbf{e}}_2 + l_{31}\overline{\mathbf{e}}_3 = 0; \tag{A.53}$$

$$\overline{\mathbf{M}} = \mathbf{A}\left(\overline{\text{æ}} - \overline{\text{æ}}_0^{(1)}\right). \tag{A.54}$$

Let us consider in more detail the derived system of nonlinear vector equations of equilibrium of a three-dimensional curvilinear rod. Equations (A.50) and (A.51) are valid for any base, i.e., they are invariant with respect to coordinate systems. From them, for example, one can express the vectors both in the fixed coordinate system:

$$\overline{\mathbf{Q}} = \sum_{j=1}^{3} Q_{x_j}\bar{\mathbf{i}}_j; \qquad \overline{\mathbf{q}} = \sum_{j=1}^{3} q_{x_j}\bar{\mathbf{i}}_j; \qquad \overline{\mathbf{P}}^{(i)} = \sum_{j=1}^{3} P_{x_j}^{(i)}\bar{\mathbf{i}}_j,$$

and in the moving system:

$$\overline{\mathbf{Q}} = \sum_{j=1}^{3} Q_j\overline{\mathbf{e}}_j; \qquad \overline{\mathbf{q}} = \sum_{j=1}^{3} q_j\overline{\mathbf{e}}_j; \qquad \overline{\mathbf{P}}^{(i)} = \sum_{j=1}^{3} P_j^{(i)}\overline{\mathbf{e}}_j.$$

As for equations (A.52) and (A.53), the vectors $\overline{\mathbf{M}}$, $\overline{\vartheta}$, $\overline{\text{æ}}$, $\overline{\text{æ}}_0^{(1)}$, and $\overline{\mathbf{u}}$ in them are related only to the base $\{\overline{\mathbf{e}}_j\}$, i.e.,

$$\overline{\mathbf{M}} = \sum_{j=1}^{3} M_j\overline{\mathbf{e}}_j; \qquad \overline{\text{æ}} = \sum_{j=1}^{3} \text{æ}_j\overline{\mathbf{e}}_j; \qquad \overline{\text{æ}}_0^{(1)} = \sum_{j=1}^{3} \text{æ}_{j0}\overline{\mathbf{e}}_j;$$

$$\overline{\vartheta} = \sum_{j=1}^{3} \vartheta_j\overline{\mathbf{e}}_j; \qquad \overline{\mathbf{u}} = \sum_{j=1}^{3} u_j\overline{\mathbf{e}}_j.$$

## A.9 Boundary conditions

When solving the equations of rod equilibrium, both homogeneous and inhomogeneous boundary conditions are possible. For a three-dimensional curvilinear rod the total number of boundary conditions is 12 (6 conditions at the left end of the rod for $\eta = 0$ and 6 conditions at its right end for $\eta = 1$). For a cantilever rod (see Fig. A.1) we have the following boundary conditions: $\eta = 0$, $\overline{\mathbf{u}} = 0$, $\overline{\vartheta} = 0$ and $\eta = 1$, $\overline{\mathbf{Q}} = 0$, $\overline{\mathbf{M}} = 0$. For hinged rod for $\eta = 1$, if the hinge allows to the rod's front section to rotate with regard to three axes, the boundary conditionsfollowing are as follows: $\overline{\mathbf{u}} = 0$ and $\overline{\mathbf{M}} = 0$.

Other variants of fixing the rod ends are possible, for example, those forbidding the displacement of a butt section of the rod in some direction that is determined by the unit vector $\overline{\mathbf{e}}_\alpha$; in this case the following condition

$$(\overline{\mathbf{u}} \cdot \overline{\mathbf{e}}_\alpha) = 0 \tag{A.55}$$

should be met.

Similar boundary conditions are possible for a butt section rotated by some angle (for example, $\vartheta_1 \neq 0$, $\vartheta_2 = \vartheta_3 = 0$).

## A.10 External load and its behaviour under rod loading process

Earlier, we have derived the general vector equations of equilibrium of the rod loaded with external forces and moments (see expressions (A.50)-(A.54)). The equations of equilibrium or motion can be solved only in the case when the external load is known. Therefore, it is presumed that all necessary data concerning external forces and moments are available. Let us consider in more detail possible behaviour of the external load (distributed and concentrated forces and moments) appearing in vector equations (A.50) and (A.51). Equations (A.50)-(A.54) are valid for large displacements of the rod under the action of external forces, therefore, one needs to know first the behaviour of the external forces in the process of rod loading and, second, whether these forces remain constant in direction and magnitude under rod deformation. If, in the process of rod strain, external forces and moments keep their direction with respect to a fixed coordinate system (as, for example, does the moment $\overline{\mathfrak{M}}$ in Fig. A.8), they are called 'dead' forces. If they keep their direction with respect to bound axes (for example, the force $\overline{\mathbf{P}}$ in Fig. A.8), then we refer to them as the 'tracking' forces. When general vector equations of equilibrium and motion are derived, the 'behaviour' of the external load is immaterial. It plays important part when the equations are written in particular bases (for example, $\{\overline{\mathbf{e}}_i\}$ or $\{\overline{\mathbf{i}}_j\}$) and, especially, when these equations are written in the scalar form (for numerical methods of solution). If the external load is dead, and the equations of rod equilibrium are written in projections onto fixed axes in the base $\{\overline{\mathbf{i}}_j\}$, then projections of the forces $P_{x_j}^{(i)}$, $\mathfrak{M}_{x_j}^{(\nu)}$, $q_{x_j}$, and $\mu_{x_j}$ do not depend on the strained state of the rod. If the load is tracking (the force $\mathbf{P}$ in Fig. A.8), then the projections of external forces in these equations depend on the strained state of the rod.

Let us consider the tracking force

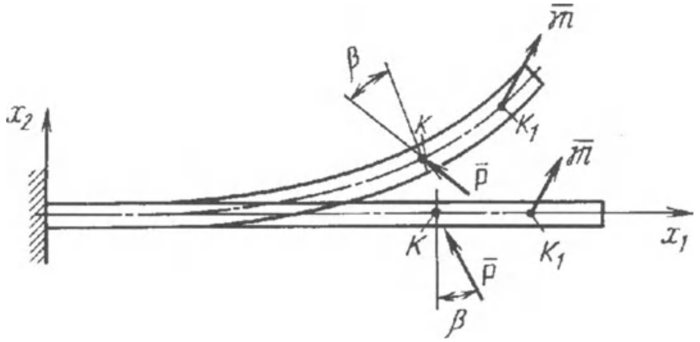


**Fig. A.8.**

$$\overline{\mathbf{P}} = \sum_{i=1}^{3} P_i \overline{\mathbf{e}}_i,$$

where $P_i$ are constant in the base $\{\overline{\mathbf{e}}_i\}$. Making conversion to the fixed coordinate system we get

$$\overline{\mathbf{P}} = \sum_{i=1}^{3} P_i \overline{\mathbf{e}}_i = \sum_{j=1}^{3} P_{x_j} \overline{\mathbf{i}}_j. \tag{A.56}$$

Here

$$P_{x_j} = \sum_{i=1}^{3} P_i l_{ij}^{(1)}(\eta) \qquad (j = 1, 2, 3), \tag{A.57}$$

And $l_{ij}^{(1)}$ are the elements of matrix $\mathbf{L}^{(1)}$ (see expression (A.19)).

Relation (A.57) can be represented in the vector form of notation

$$\overline{\mathbf{P}}_x = \left(L^{(1)}\right)^{\mathrm{T}} \overline{\mathbf{P}}. \tag{A.58}$$

The projections $P_{x_j}$ depend on $l_{ij}^{(1)}(\eta)$, therefore, they are not constant quantities. Similarly, one can write down the distributed loads too

$$\overline{\mathbf{q}} = \sum_{j=1}^{3} q_{x_j} \overline{\mathbf{i}}_j \qquad \left(q_{x_j} = \sum_{i=1}^{3} q_i l_{ij}^{(1)}\right);$$

$$\overline{\mu} = \sum_{j=1}^{3} \mu_{x_j} \overline{\mathbf{i}}_j \qquad \left(\mu_{x_j} = \sum_{i=1}^{3} \mu_i l_{ij}^{(1)}\right),$$

or, in the vector form,

$$\overline{\mathbf{q}}_x = \left(L^{(1)}\right)^{\mathrm{T}} \overline{\mathbf{q}}, \qquad \overline{\mu}_x = \left(L^{(1)}\right)^{\mathrm{T}} \overline{\mu}. \tag{A.59}$$

For the dead force $\overline{\mathbf{P}}$ we have

$$\overline{\mathbf{P}} = \sum_{j=1}^{3} P_{x_j} \bar{\mathbf{i}}_j,$$

where $P_{x_j}$ are constant in the base $\{\bar{\mathbf{i}}_j\}$.

In the bound coordinate system the projections of dead distributed loads are determined by the expressions

$$\overline{\mathbf{q}}_i = \sum_{j=1}^{3} q_{x_j} l_{ij}^{(1)}; \qquad \overline{\mu}_i = \sum_{i=1}^{3} \mu_{x_j} l_{ij}, \tag{A.60}$$

where $q_{x_j}$ and $\mu_{x_j}$ are specified functions.

In the vector notation, the dead forces in the base $\{\overline{\mathbf{e}}_j\}$ have the form

$$\overline{\mathbf{q}} = \mathbf{L}^{(1)}\overline{\mathbf{q}}_x; \quad \overline{\mu} = \mathbf{L}^{(1)}\overline{\mu}_x; \quad \overline{\mathbf{P}} = \mathbf{L}^{(1)}\overline{\mathbf{P}}_x; \quad \overline{\mathfrak{M}} = \mathbf{L}^{(1)}\overline{\mathfrak{M}}_x. \tag{A.61}$$

## A.11 Vector nonlinear equations of rod equilibrium in the bound coordinate system

In order to derive the equations of equilibrium in the projections onto coordinate axes, it is necessary to represent vectors in an appropriate base (for example, the base $\{\overline{\mathbf{e}}_i\}$) bound to the section's principal axes. In this case, it is well to bear in mind that not only projections of vectors depend on the coordinate $\eta$, but the vectors of the base $\{\overline{\mathbf{e}}_i\}$ too. Taking advantage of formula (A.25), let us change over in equations (A.50)-(A.53) to local derivatives:

$$\frac{\tilde{\mathrm{d}}\overline{\mathbf{Q}}}{\mathrm{d}\eta} + \overline{æ} \times \overline{\mathbf{Q}} + \overline{\mathbf{P}} = 0; \tag{A.62}$$

$$\frac{\tilde{\mathrm{d}}\overline{\mathbf{M}}}{\mathrm{d}\eta} + \overline{æ} \times \overline{\mathbf{M}} + \overline{\mathbf{e}}_1 \times \overline{\mathbf{Q}} + \overline{\mathfrak{M}} = 0; \tag{A.63}$$

$$\frac{\tilde{\mathrm{d}}\overline{\vartheta}}{\mathrm{d}\eta} + \mathbf{L}_2\overline{æ}_0^{(1)} - \mathbf{A}^{-1}\overline{\mathbf{M}} = 0; \tag{A.64}$$

$$\frac{\tilde{\mathrm{d}}\overline{\mathbf{u}}}{\mathrm{d}\eta} + \overline{æ} \times \overline{\mathbf{u}} + (l_{11} - 1)\overline{\mathbf{e}}_1 + l_{21}\overline{\mathbf{e}}_2 + l_{31}\overline{\mathbf{e}}_3 = 0; \tag{A.65}$$

$$\overline{\mathbf{M}} = \mathbf{A}\left(\overline{æ} - \overline{æ}_0^{(1)}\right), \tag{A.66}$$

where

$$\overline{\mathbf{P}} = \overline{\mathbf{q}} + \sum_{i=1}^{3} \overline{\mathbf{P}}^{(i)} \delta(\eta - \eta_i); \quad \overline{\mathfrak{M}} = \overline{\mu} + \sum_{\nu=1}^{\rho} \overline{\mathfrak{M}}^{(\nu)} \delta(\eta - \eta_\nu); \tag{A.67}$$

$$\overline{\mathbf{L}}_2 = \begin{bmatrix} \cos\vartheta_2 \cos\vartheta_3 - 1 & \begin{matrix}\cos\vartheta_2 \sin\vartheta_3 \cos\vartheta_1 + \\ + \sin\vartheta_2 \sin\vartheta_1\end{matrix} & \begin{matrix}\cos\vartheta_2 \sin\vartheta_3 \sin\vartheta_1 - \\ - \sin\vartheta_2 \cos\vartheta_1\end{matrix} \\ -\sin\vartheta_3 & \cos\vartheta_1 \cos\vartheta_3 - 1 & \cos\vartheta_3 \sin\vartheta_1 \\ \sin\vartheta_2 \cos\vartheta_3 & \begin{matrix}\sin\vartheta_2 \sin\vartheta_3 \cos\vartheta_1 - \\ - \cos\vartheta_2 \sin\vartheta_1\end{matrix} & \begin{matrix}\sin\vartheta_2 \sin\vartheta_3 \sin\vartheta_1 - \\ - \cos\vartheta_2 \cos\vartheta_1 - 1\end{matrix} \end{bmatrix}. \tag{A.68}$$

All vectors appearing in expressions (A.67) must be represented in the base $\{\overline{\mathbf{e}}_j\}$. If the vectors $\overline{\mathbf{q}}$, $\overline{\mathbf{P}}^{(i)}$, $\overline{\mu}$, and $\overline{\mathfrak{M}}^{(\nu)}$ are known in bound axes, then no additional transformations are required. If these vectors or some of them are specified (known) in Cartesian axes, then one should write them in the bound axes using the matrix of conversion from the base $\{\overline{\mathbf{i}}_j\}$ to the base $\{\overline{\mathbf{e}}_j\}$. How to determine the vector components when making conversion from one coordinate system to another is described in the paragraph A.2 of this Appendix.

## A.12 Equations of rod equilibrium in projections onto bound axes

In applied problems it is more convenient to use equations in projections onto bound axes. In addition, the components $Q_{i0}$ and $M_{i0}$ of vectors $\overline{\mathbf{Q}}_0$ and $\overline{\mathbf{M}}_0$ have clear physical meaning in the bound axes: $Q_{10}$ is the axial force; $Q_2$ and $Q_3$ are the crosscutting forces; $M_1$ is the torsion moment; and $M_2$ and $M_3$ are the bending moments.

For better understanding we restrict ourselves to the case when one concentrate force $\overline{\mathbf{P}}_0$ and one concentrated moment $\overline{\mathfrak{M}}_0$ are applied to the rod. The equations of equilibrium have the following form

$$\left.\begin{aligned} &\frac{dQ_{10}}{d\eta} + Q_{30}æ_{20} - Q_{20}æ_{30} + q_{10} + P_{10}\delta_P = 0; \\ &\frac{dQ_{20}}{d\eta} + Q_{10}æ_{30} - Q_{30}æ_{10} + q_{20} + P_{20}\delta_P = 0; \\ &\frac{dQ_{30}}{d\eta} + Q_{20}æ_{10} - Q_{10}æ_{20} + q_{30} + P_{30}\delta_P = 0; \end{aligned}\right\} \tag{A.69}$$

$$\left.\begin{aligned} &\frac{dM_{10}}{d\eta} + M_{30}æ_{20} - M_{20}æ_{30} + \mu_{10} + \mathfrak{M}_{10}\delta_{\mathfrak{M}} = 0; \\ &\frac{dM_{20}}{d\eta} + M_{10}æ_{30} - M_{30}æ_{10} - Q_{30} + \mu_{20} + \mathfrak{M}_{20}\delta_{\mathfrak{M}} = 0; \\ &\frac{dM_{30}}{d\eta} + M_{20}æ_{10} - M_{10}æ_{20} + Q_{20} + \mu_{30} + \mathfrak{M}_{30}\delta_{\mathfrak{M}} = 0; \end{aligned}\right\} \tag{A.70}$$

$$\left.\begin{aligned} M_{10} &= A_{11}(æ_{10} - æ_{100}); \\ M_{20} &= A_{22}(æ_{20} - æ_{200}); \\ M_{30} &= A_{33}(æ_{30} - æ_{300}); \end{aligned}\right\} \tag{A.71}$$

$$\left.\begin{aligned} æ_{10} = &\left(\frac{d\vartheta_{10}}{d\eta} + æ_{100}\right)\cos\vartheta_{20}\cos\vartheta_{30} - \frac{d\vartheta_{30}}{d\eta}\sin\vartheta_{20} + \\ &+(\sin\vartheta_{20}\sin\vartheta_{10} + \cos\vartheta_{10}\cos\vartheta_{20}\sin\vartheta_{30})æ_{200} + \\ &+(\cos\vartheta_{20}\sin\vartheta_{30}\sin\vartheta_{10} - \sin\vartheta_{20}\sin\vartheta_{10})æ_{300}; \\ æ_{20} = &\frac{d\vartheta_{20}}{d\eta} - \left(\frac{d\vartheta_{10}}{d\eta} + æ_{100}\right)\sin\vartheta_{30} + \\ &+\cos\vartheta_{30}\cos\vartheta_{10}æ_{200} + \cos\vartheta_{30}\sin\vartheta_{10}æ_{300}; \\ æ_{30} = &\frac{d\vartheta_{30}}{d\eta}\cos\vartheta_{20} + \left(\frac{d\vartheta_{10}}{d\eta} + æ_{100}\right)\sin\vartheta_{20}\cos\vartheta_{30} + \\ &+(\sin\vartheta_{20}\sin\vartheta_{30}\cos\vartheta_{10} - \cos\vartheta_{20}\sin\vartheta_{10})æ_{200} + \\ &+(\cos\vartheta_{20}\cos\vartheta_{10} + \sin\vartheta_{10}\sin\vartheta_{20}\sin\vartheta_{30})æ_{300}; \end{aligned}\right\} \tag{A.72}$$

$$\left.\begin{aligned} &\frac{du_{10}}{d\eta} + u_{30}æ_{20} - u_{20}æ_{30} + l_{11}^0 - 1 = 0; \\ &\frac{du_{20}}{d\eta} + u_{10}æ_{30} - u_{30}æ_{10} + l_{21}^0 = 0; \\ &\frac{du_{30}}{d\eta} + u_{20}æ_{10} - u_{10}æ_{20} + l_{31}^0 = 0. \end{aligned}\right\} \tag{A.73}$$

The quantities $æ_{i00}$ appearing in equations (A.71) and (A.72) are assumed to be known.

## A.13 Special cases of equilibrium equations

### Nonlinear equations of equilibrium for a rod whose axial line is a plane curve both before and after loading

The equations of equilibrium for the case under considerations have the following form

$$\frac{dQ_{10}}{d\eta} - æ_{20}Q_{20} + q_{10} + P_{10}\delta_P = 0; \tag{A.74}$$

$$\frac{dQ_{20}}{d\eta} + æ_{30}Q_{10} + q_{20} + P_{20}\delta_P = 0; \tag{A.75}$$

$$\frac{dM_{30}}{d\eta} + Q_{20} + M_{30} + \mathfrak{M}_{30}\delta_{\mathfrak{M}} = 0; \tag{A.76}$$

$$\frac{d\vartheta_{30}^{(1)}}{d\eta} - \frac{1}{A_{33}}M_{30} = æ_{300}; \tag{A.77}$$

$$\frac{du_{10}}{d\eta} - u_{20}æ_{30} + l_{11} - 1 = 0; \tag{A.78}$$

$$\frac{du_{20}}{d\eta} + u_{10}æ_{30} + l_{21} = 0; \tag{A.79}$$

$$M_{30} = A_{33}(æ_{30} - æ_{300}), \tag{A.80}$$

where $l_{11} = \cos\vartheta_{30}$; $l_{21} = -\sin\vartheta_{30}$; $æ_{30} = \frac{d\vartheta_{30}^{(1)}}{d\eta}$, and $æ_{300} = \frac{d\vartheta_{300}}{d\eta}$. Since the angle between the tangents to the axial line of the rod before and after loading (between the vectors $\bar{e}_{10}$ and $\bar{e}_{100}$) is $\vartheta_{30} = \vartheta_{30}^{(1)} - \vartheta_{300}$, equation (A.77) can be represented in the form

$$\frac{d\vartheta_{30}}{d\eta} - \frac{M_{30}}{A_{33}} = 0. \tag{A.81}$$

System of equations (A.74)-(A.79) allows one to determine the static mode of deformation of a plane curvilinear rod at large displacements of points of the rod axial line ($u_{10}$, $u_{20}$) and at large angle of rotation $\vartheta_{30}$. The equations of small vibrations depend on $Q_{01}$, $Q_{02}$, and $M_{03}$ (see Appendix C).

### Equations of equilibrium at small displacements of points of the rod axial line and small rotation angles of the vectors $\bar{e}_j$ of bound axes

We assume that displacements $u_{j0}$ and angles $\vartheta_j$ are small, and the vector $\bar{æ}_0$ is

$$\bar{æ}_0 = \bar{æ}_{00} + \Delta\bar{æ}_0,$$

where $\bar{æ}_{00}$ is the vector characterizing the axial line geometry for the rod in its natural (unloaded) state; and $\Delta\bar{æ}_0$ is the small increment of the vector $\bar{æ}_0$,

$$\Delta\bar{æ}_0 = \sum_{j=1}^{3} \Delta æ_{j0}\bar{e}_{j0} \qquad (\bar{e}_{j0} \approx \bar{e}_j).$$

The intrinsic moments $M_{10}$, $M_{20}$, and $M_{30}$ are proportional to the components of vector $\triangle\overline{\text{æ}}_0$:

$$M_{j0} = A_{ii}\triangle\text{æ}_{j0}.$$

From nonlinear equations (A.62)-(A.65) we obtain after transformations the linear equations of equilibrium:

$$\frac{d\overline{\mathbf{Q}}_0}{d\eta} + \overline{\text{æ}}_{00} \times \overline{\mathbf{Q}}_0 + \overline{\mathbf{q}}_0 + \triangle\overline{\mathbf{q}}_0 + (\overline{\mathbf{P}}_0 + \triangle\overline{\mathbf{P}}_0)\delta_P = 0; \qquad \text{(A.82)}$$

$$\frac{d\overline{\mathbf{M}}_0}{d\eta} + \overline{\text{æ}}_{00} \times \overline{\mathbf{M}}_0 + \overline{\mathbf{e}}_{10} \times \overline{\mathbf{Q}}_0 + \overline{\mu}_0 + \triangle\overline{\mu}_0 + (\overline{\mathfrak{M}} + \triangle\overline{\mathfrak{M}})\delta_{\mathfrak{M}} = 0; \qquad \text{(A.83)}$$

$$\frac{d\overline{\vartheta}_0}{d\eta} + \overline{\text{æ}}_{00} \times \overline{\vartheta}_0 - \mathbf{A}^{-1}\overline{\mathbf{M}}_0 = 0; \qquad \text{(A.84)}$$

$$\frac{d\overline{\mathbf{u}}_0}{d\eta} + \overline{\text{æ}}_{00} \times \overline{\mathbf{u}}_0 + \vartheta_{30}\overline{\mathbf{e}}_{20} + \vartheta_{20}\overline{\mathbf{e}}_{30} = 0, \qquad \text{(A.85)}$$

where $\overline{\text{æ}}_{00}$ and $\overline{\mathbf{e}}_{10}$ are known vectors.

If the loads are tracking, then in the bound coordinate system $\triangle\overline{\mathbf{q}}_0 = \triangle\overline{\mu}_0 = \triangle\overline{\mathbf{P}}_0 = \triangle\overline{\mathfrak{M}} = 0$. If the loads are dead, then the increments of vectors $\triangle\overline{\mathbf{q}}_0$ etc. are not equal to zero. In this case, they depend on $\vartheta_{j0}$ and $u_{j0}$ linearly [4].

The vector products can be represented as

$$\begin{aligned} &\overline{\text{æ}}_{00} \times \overline{\mathbf{Q}}_0 = \mathbf{A}_{\text{æ}}\overline{\mathbf{Q}}_0; \qquad \overline{\text{æ}}_{00} \times \overline{\mathbf{M}}_0 = \mathbf{A}_{\text{æ}}\overline{\mathbf{M}}_0;\\ &\overline{\text{æ}}_{00} \times \overline{\vartheta}_0 = \mathbf{A}_{\text{æ}}\overline{\vartheta}_0; \qquad \overline{\text{æ}}_{00} \times \overline{\mathbf{u}}_0 = \mathbf{A}_{\text{æ}}\overline{\mathbf{u}}_0, \end{aligned} \qquad \text{(A.86)}$$

where $\mathbf{A}_{\text{æ}} = \begin{bmatrix} 0 & -\text{æ}_{300} & \text{æ}_{200} \\ \text{æ}_{300} & 0 & -\text{æ}_{100} \\ -\text{æ}_{200} & \text{æ}_{100} & 0 \end{bmatrix}$. In addition,

$$\overline{\mathbf{e}}_{10} \times \overline{\mathbf{Q}}_0 = \mathbf{A}_1 \times \overline{\mathbf{Q}}_0; \quad \vartheta_{30}\overline{\mathbf{e}}_{20} + \vartheta_{20}\overline{\mathbf{e}}_{30} = \mathbf{A}_1\overline{\vartheta}_0,$$

where $\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

As a result, we have the system of linear vector equations that allow one to determine the static mode of deformation of a three-dimensional curvilinear rod:

$$\frac{\mathrm{d}\overline{\mathbf{Q}}_0}{\mathrm{d}\eta} + \mathbf{A}_{æ}\overline{\mathbf{Q}}_0 + \overline{\mathbf{q}}_0 + \triangle\overline{\mathbf{q}}_0 + (\overline{\mathbf{P}}_0 + \triangle\overline{\mathbf{P}}_0)\delta_P = 0; \tag{A.87}$$

$$\frac{\mathrm{d}\overline{\mathbf{M}}_0}{\mathrm{d}\eta} + \mathbf{A}_{æ}\overline{\mathbf{M}}_0 + \mathbf{A}_1\overline{\mathbf{Q}}_0 + \overline{\mu}_0 + \triangle\overline{\mu}_0 + (\overline{\mathfrak{M}} + \triangle\overline{\mathfrak{M}})\delta_{\mathfrak{M}} = 0; \tag{A.88}$$

$$\frac{\mathrm{d}\overline{\vartheta}_0}{\mathrm{d}\eta} + \mathbf{A}_{æ}\overline{\vartheta}_0 - \mathbf{A}^{-1}\overline{\mathbf{M}}_0 = 0 \qquad (\overline{\mathbf{M}}_0 = \mathbf{A}\triangle\overline{æ}_0); \tag{A.89}$$

$$\frac{\mathrm{d}\overline{\mathbf{u}}_0}{\mathrm{d}\eta} + \mathbf{A}_{æ}\overline{\mathbf{u}}_0 + \mathbf{A}_1\overline{\vartheta}_0 = 0. \tag{A.90}$$

**Linear equations of equilibrium for the specific case when the rod axial line was a plane curve before loading**

For the case when the rod axial line becomes a spatial curve after loading the equations in projections onto bound axes can be derived from system (A.87)-(A.90):

$$\begin{aligned}
&\frac{\mathrm{d}Q_{10}}{\mathrm{d}\eta} - æ_{30}Q_{20} + P_{10} = 0;\\
&\frac{\mathrm{d}Q_{20}}{\mathrm{d}\eta} + æ_{30}Q_{10} + P_{20} = 0;\\
&\frac{\mathrm{d}Q_{30}}{\mathrm{d}\eta} + P_{30} = 0;\\
&\frac{\mathrm{d}M_{10}}{\mathrm{d}\eta} - æ_{30}M_{20} + \mathfrak{M}_{10} = 0;\\
&\frac{\mathrm{d}M_{20}}{\mathrm{d}\eta} + æ_{30}M_{10} - Q_{30} + \mathfrak{M}_{20} = 0;\\
&\frac{\mathrm{d}M_{30}}{\mathrm{d}\eta} + Q_{20} + \mathfrak{M}_{30} = 0;\\
&M_{10} = A_{11}\triangle æ_{10}, \quad M_{20} = A_{22}\triangle æ_{20}, \quad M_{30} = A_{33}\triangle æ_{30}, \qquad\qquad (A.91)\\
&\frac{\mathrm{d}\vartheta_{10}}{\mathrm{d}\eta} - æ_{30}\vartheta_{20} - \triangle æ_{10} = 0;\\
&\frac{\mathrm{d}\vartheta_{20}}{\mathrm{d}\eta} + æ_{30}\vartheta_{10} - \triangle æ_{20} = 0;\\
&\frac{\mathrm{d}\vartheta_{30}}{\mathrm{d}\eta} - æ_{30} = 0;\\
&\frac{\mathrm{d}u_{10}}{\mathrm{d}\eta} - æ_{30}u_{20} = 0;\\
&\frac{\mathrm{d}u_{20}}{\mathrm{d}\eta} + æ_{30}u_{10} - \vartheta_{30} = 0;\\
&\frac{\mathrm{d}u_{30}}{\mathrm{d}\eta} + \vartheta_{20} = 0.
\end{aligned}$$

If the rod axial line after loading persists to be a plane curve (which is possible if $P_{30} = \mathfrak{M}_{10} = \mathfrak{M}_{20} = 0$), then one should set in system of equations (A.91): $\vartheta_{30} = 0$; $M_{10} = M_{20} = 0$; $\Delta æ_{10} = \Delta æ_{20} = 0$; $\vartheta_{10} = \vartheta_{20} = 0$, and $u_{30} = 0$. As a result, we get the following system of equations:

$$\begin{aligned}
&\frac{\mathrm{d}Q_{10}}{\mathrm{d}\eta} - æ_{30}Q_{20} + P_{10} = 0;\\
&\frac{\mathrm{d}Q_{20}}{\mathrm{d}\eta} + æ_{30}Q_{10} + P_{20} = 0;\\
&\frac{\mathrm{d}M_{30}}{\mathrm{d}\eta} + Q_{20} + \mathfrak{M}_{30} = 0;\\
&\frac{\mathrm{d}\vartheta_{30}}{\mathrm{d}\eta} - \frac{M_{30}}{A_{33}} = 0 \qquad (M_{30} = A_{33}\Delta æ_3);\\
&\frac{\mathrm{d}u_{10}}{\mathrm{d}\eta} - æ_{30}u_{20} = 0;\\
&\frac{\mathrm{d}u_{20}}{\mathrm{d}\eta} + æ_{30}u_{10} + \vartheta_{30} = 0.
\end{aligned} \tag{A.92}$$

# B

# Basic equations of rod kinematics

When deriving the equations of motion for rods, it is necessary to know kinematic relations that establish a link between generalized displacements and their first derivatives with respect to time.

## B.1 Time derivatives of vectors of the base $\{\overline{\mathbf{e}}_i\}$

Figure B.1 shows the positions of movable coordinate axes at different instants $t_0$ and $t_1$. The relations that set up a correspondence between the base vectors $\overline{\mathbf{i}}_j$, $\overline{\mathbf{e}}_{j0}$, and $\overline{\mathbf{e}}_j$ at a variation of their position in space were derived in Appendix A. The position of bound axes can be changed either due to translation of the axes when the rod moves (coordinate $s$ being fixed) (see Fig. B.1), or due to a displacement of the axes at a fixed instant $t_0$ when axes drift along the $s$ coordinate. Thus, in the general case the base vectors $\overline{\mathbf{e}}_i$ depend on two independent variables $t$ and $s$.

In the former case the variation of axes' position depends on the variation of variable $t$ at a fixed value of the $s$ variable, while in the latter case it depends on variation of $s$ at a fixed value of $t$. As the rod moves, the position of its axial line in space changes continuously. In order to describe the rod motion and to determine the form of its axial line at any time, one needs to know the derivatives of the vectors $\overline{\mathbf{e}}_j$ of a fixed base with respect to arguments $t$ and $s$. The appropriate relations for derivatives of vectors $\overline{\mathbf{e}}_j$ with respect to $s$ are given in Appendix A.

The derivative of vector $\overline{\mathbf{e}}_j$ with respect to $t$ is a vector that can be decomposed in vectors of the base $\{\overline{\mathbf{e}}_i\}$, i.e., we can present it in the form

$$\frac{\partial \overline{\mathbf{e}}_i}{\partial t} = \sum_{j=1}^{3} \omega_{ij}\overline{\mathbf{e}}_j = \omega_{ij}\overline{\mathbf{e}}_j \qquad (i, j = 1, 2, 3), \tag{B.1}$$

where $\omega_{ij}$ are the elements of a certain matrix $[\omega_{ij}]$ similar to the matrix $[æ_{ij}]$. The matrix $[\omega_{ij}]$ is skew symmetric and has only three independent elements:

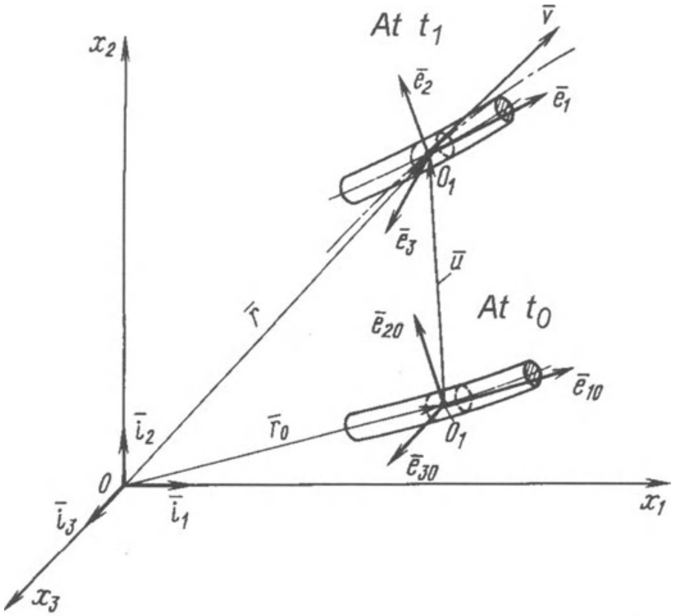


**Fig. B.1.**

$$\mathbf{A}_\omega = [\omega_{ij}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{B.2}$$

As a result, we get from (B.2)

$$\frac{\partial \overline{\mathbf{e}}_1}{\partial t} = \omega_3 \overline{\mathbf{e}}_2 - \omega_2 \overline{\mathbf{e}}_3; \quad \frac{\partial \overline{\mathbf{e}}_2}{\partial t} = -\omega_3 \overline{\mathbf{e}}_1 + \omega_1 \overline{\mathbf{e}}_3; \quad \frac{\partial \overline{\mathbf{e}}_3}{\partial t} = \omega_2 \overline{\mathbf{e}}_1 - \omega_1 \overline{\mathbf{e}}_2.$$

The following equation

$$\mathbf{A}_\omega \overline{\mathbf{e}}_i = \overline{\omega} \times \overline{\mathbf{e}}_i \qquad (i = 1, 2, 3), \tag{B.3}$$

holds true, where $\overline{\omega}$ is the angular velocity vector for rotation of the bound coordinate system ($\overline{\omega} = \omega_1 \overline{\mathbf{e}}_1 + \omega_2 \overline{\mathbf{e}}_2 + \omega_3 \overline{\mathbf{e}}_3$).

For time derivatives of the vectors of a movable base we have the following expressions

$$\frac{\partial \overline{\mathbf{e}}_1}{\partial t} = \overline{\omega} \times \overline{\mathbf{e}}_1; \qquad \frac{\partial \overline{\mathbf{e}}_2}{\partial t} = \overline{\omega} \times \overline{\mathbf{e}}_2; \qquad \frac{\partial \overline{\mathbf{e}}_3}{\partial t} = \overline{\omega} \times \overline{\mathbf{e}}_3. \tag{B.4}$$

## B.2 Absolute and local derivatives of a vector with respect to time

Let us consider the vector $\overline{\mathbf{a}}(t)$ in a bound (moving) coordinate system (see Fig. B.1):

$$\overline{\mathbf{a}}(t) = a_1(t)\overline{\mathbf{e}}_1 + a_2(t)\overline{\mathbf{e}}_2 + a_3(t)\overline{\mathbf{e}}_3.$$

In the moving coordinate system the components $a_i$ of the vector $\overline{\mathbf{a}}(t)$ and the base vectors $\overline{\mathbf{e}}_i$ depend on time, therefore, taking relations (B.4) into account, the derivative of vector $\overline{\mathbf{a}}(s,t)$ has the form

$$\frac{\partial \overline{\mathbf{a}}}{\partial t} = \frac{\partial \overline{\mathbf{a}}_1}{\partial t}\overline{\mathbf{e}}_1 + \frac{\partial \overline{\mathbf{a}}_2}{\partial t}\overline{\mathbf{e}}_2 + \frac{\partial \overline{\mathbf{a}}_3}{\partial t}\overline{\mathbf{e}}_3 + a_1(\overline{\omega} \times \overline{\mathbf{e}}_1) + \tag{B.5}$$

$$+ a_2(\overline{\omega} \times \overline{\mathbf{e}}_2) + a_3(\overline{\omega} \times \overline{\mathbf{e}}_3) = \frac{\widetilde{\partial} \overline{\mathbf{a}}}{\partial t} + \overline{\omega} \times \overline{\mathbf{a}}, \tag{B.6}$$

where $\dfrac{\widetilde{\partial}}{\partial t}$ is the local partial derivative of vector $\overline{\mathbf{a}}$ that characterizes the time variation of vector $\overline{\mathbf{a}}$ relative to the moving coordinate system; $\overline{\omega} \times \overline{\mathbf{a}}$ is the vector characterizing the time variation of vector $\overline{\mathbf{a}}$ due to rotation of the coordinate axes.

Let us derive expressions relating the projections $\omega_j$ of the angular velocity vector $\omega$ to the angles $\vartheta_1$, $\vartheta_2$, and $\vartheta_3$. We take advantage of the relations

$$\overline{\mathbf{e}}_i = l_{i\rho}\overline{\mathbf{e}}_{\rho 0}, \tag{B.7}$$

where $l_{i\rho}$ are the elements of matrix [A.12]; $\overline{\mathbf{e}}_{\rho 0}$ are unit vectors of the base at $t = t_0$. Differentiating with respect to $t$, we have

$$\frac{\partial l_{i\rho}}{\partial t}\overline{\mathbf{e}}_{\rho 0} = \varepsilon_{kji}\omega_j\overline{\mathbf{e}}_k. \tag{B.8}$$

Since

$$\overline{\mathbf{e}}_{\rho 0} = l_{k\rho}\overline{\mathbf{e}}_k, \tag{B.9}$$

after substitution of (B.9) into (B.8) we get

$$\varepsilon_{kji}\omega_j = \frac{\partial l_{i\rho}}{\partial t} l_{k\rho}. \tag{B.10}$$

Let us find, for example, the expressions for $\omega_1$ making all operations of summation. Setting $j = 1$, $k = 3$, $i = 2\,(\varepsilon_{312} = 1)$ and summing the right-hand side of (B.10) over $\varrho$, we have

$$\omega_1 = \frac{\partial l_{21}}{\partial t} l_{31} + \frac{\partial l_{22}}{\partial t} l_{32} + \frac{\partial l_{23}}{\partial t} l_{33}. \tag{B.11}$$

Setting $k = 2$, $i = 3\,(\varepsilon_{213} = -1)$, one can write one more expression for $\omega_1$:

$$-\omega_1 = \frac{\partial l_{31}}{\partial t} l_{21} + \frac{\partial l_{32}}{\partial t} l_{22} + \frac{\partial l_{33}}{\partial t} l_{23}, \tag{B.12}$$

which can be used to check the validity of relation (B.11) when a conversion is made from angles $\vartheta_j$ to explicit expressions for $l_{ij}$.

For example, for the airplane angles final expressions for projections $\omega_j$ of the angular velocity look like:

$$\begin{aligned}
\omega_1 &= \frac{\partial \vartheta_1}{\partial t} \cos\vartheta_2 \cos\vartheta_3 - \frac{\partial \vartheta_3}{\partial t} \sin\vartheta_2; \\
\omega_2 &= \frac{\partial \vartheta_2}{\partial t} - \frac{\partial \vartheta_1}{\partial t} \sin\vartheta_3; \\
\omega_3 &= \frac{\partial \vartheta_3}{\partial t} \cos\vartheta_2 + \frac{\partial \vartheta_1}{\partial t} \sin\vartheta_2 \cos\vartheta_3.
\end{aligned} \tag{B.13}$$

Relations (B.13) can be represented in the vector form

$$\overline{\omega} = \mathbf{L}_1 \frac{\partial \overline{\vartheta}}{\partial t},$$

where $\mathbf{L}_1 = \begin{bmatrix} \cos\vartheta_2 \cos\vartheta_3 & 0 & -\sin\vartheta_2 \\ -\sin\vartheta_3 & 1 & 0 \\ \sin\vartheta_2 \cos\vartheta_3 & 0 & \cos\vartheta_2 \end{bmatrix}$; $\overline{\vartheta} = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix}$. At small angles $\vartheta_j$ the components of vector $\overline{\omega}$ are equal to

$$\omega_1 = \partial\vartheta_1/\partial t, \quad \omega_2 = \partial\vartheta_2/\partial t, \quad \omega_3 = \partial\vartheta_3/\partial t.$$

To an accuracy of quantities of the second order of smallness one can assume

$$\mathbf{L}_1 \frac{\partial \overline{\vartheta}}{\partial t} = \mathbf{E} \frac{\partial \overline{\vartheta}}{\partial t}$$

where $\mathbf{E}$ is the unit matrix.

## B.3 Velocity and acceleration of a point of the rod axial line

Let us consider basic statements of point kinematics as applied to the problems of rod dynamics. When moving, every point of the rod axial line has a certain velocity $\overline{\mathbf{v}}$ that is related to the time derivative of radius vector $\overline{\mathbf{r}}$ (see Fig. B.1) as

$$\overline{\mathbf{v}} = \frac{\mathrm{d}\overline{\mathbf{r}}(s,t)}{\mathrm{d}t}. \tag{B.14}$$

Vector $\overline{\mathbf{v}}$ is directed along the tangent to the trajectory of motion of the axial line point that is shown by the dashed line. The distinction of the velocity of the rod element from the material point velocity lies in the fact $\overline{\mathbf{r}}$ and $\overline{\mathbf{v}}$

are the functions of two independent variables $s$ and $t$. For example, if the coordinate $s$ of point $O_1$ of the axial line remains invariable as the rod moves (does not depend on $t$), then we have from expression (B.14)

$$\overline{\mathbf{v}} = \frac{\partial \overline{\mathbf{r}}(s,t)}{\partial t},$$

where $\partial/\partial t$ is the complete partial derivative.

In the Cartesian system of coordinates

$$\overline{\mathbf{v}} = \sum_{j=1}^{3} \dot{x}_j \overline{\mathbf{i}}_j.$$

In the bound coordinate system (in the base $\{\overline{\mathbf{e}}_j\}$), converting to the local derivative, we obtain

$$\overline{\mathbf{v}} = \frac{\tilde{\partial} \overline{\mathbf{r}}}{\partial t} + \overline{\omega} \times \overline{\mathbf{r}}, \tag{B.15}$$

where $\overline{\omega}(s,t)$ is the angular velocity vector of rotation of the base $\{\overline{\mathbf{e}}_j\}$. Since (see Fig. B.1)

$$\overline{\mathbf{r}} = \overline{\mathbf{r}}_0 + \overline{\mathbf{u}} \qquad \left( \overline{\mathbf{r}}_0 = \sum_{j=1}^{3} r_{0_j} \overline{\mathbf{i}}_j + \sum_{j=1}^{3} u_j \overline{\mathbf{e}}_j \right),$$

then we find from (B.15)

$$\overline{\mathbf{v}} = \frac{\tilde{\partial} \overline{\mathbf{u}}}{\partial t} + \overline{\omega} \times \overline{\mathbf{u}}.$$

The accelerations of points of the rod axial line are

$$\frac{\mathrm{d}^2 \overline{\mathbf{r}}}{\mathrm{d}t^2} = \frac{\mathrm{d}\overline{\mathbf{v}}}{\mathrm{d}t} = \frac{\partial^2 \overline{\mathbf{r}}}{\partial t^2}.$$

Accordingly, in the Cartesian and bound axes

$$\begin{gathered} \frac{\partial^2 \overline{\mathbf{r}}}{\partial t^2} = \sum_{j=1}^{3} \ddot{x}_j \overline{\mathbf{i}}_j; \\ \frac{\mathrm{d}\overline{\mathbf{v}}}{\mathrm{d}t} = \frac{\partial^2 \overline{\mathbf{r}}}{\partial t^2} = \frac{\tilde{\partial} \overline{\mathbf{v}}}{\partial t} + \overline{\omega} \times \overline{\mathbf{v}}. \end{gathered} \tag{B.16}$$

The right-hand side of expression (B.16) in the base $\{\overline{\mathbf{e}}_j\}$ can be represented as:

$$\frac{\mathrm{d}\overline{\mathbf{v}}}{\mathrm{d}t} = \left( \frac{\tilde{\partial} v_1}{\partial t} + \omega_2 v_3 - \omega_3 v_2 \right) \overline{\mathbf{e}}_1 + \left( \frac{\tilde{\partial} v_2}{\partial t} + \omega_3 v_1 - \omega_1 v_3 \right) \overline{\mathbf{e}}_2 + \\ + \left( \frac{\tilde{\partial} v_3}{\partial t} + \omega_1 v_2 - \omega_2 v_1 \right) \overline{\mathbf{e}}_3.$$

At small vibrations the terms $v_j\omega_k$ can be neglected as quantities of the second order of smallness, therefore, we get the following expressions for the components of the acceleration vector of a point of the rod axial line:

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \frac{\tilde{\partial} v_1}{\partial t}; \quad \frac{\mathrm{d}v_2}{\mathrm{d}t} = \frac{\tilde{\partial} v_2}{\partial t}; \quad \frac{\mathrm{d}v_3}{\mathrm{d}t} = \frac{\tilde{\partial} v_3}{\partial t}.$$

At small vibrations one can set $\overline{\omega} \times \overline{\mathbf{u}} = 0$, therefore, the components of the velocity vector $\overline{\mathbf{v}}$ are equal to

$$v_1 = \frac{\tilde{\partial} u_1}{\partial t}; \quad v_2 = \frac{\tilde{\partial} u_2}{\partial t}; \quad v_3 = \frac{\tilde{\partial} u_3}{\partial t};$$

The components of the acceleration vector when expressed through the components of the displacement vector have the form

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \frac{\tilde{\partial}^2 u_1}{\partial t^2}; \quad \frac{\mathrm{d}v_2}{\mathrm{d}t} = \frac{\tilde{\partial}^2 u_2}{\partial t^2}; \quad \frac{\mathrm{d}v_3}{\mathrm{d}t} = \frac{\tilde{\partial}^2 u_3}{\partial t^2}.$$

# C

# Basic equations of rod dynamics

## C.1 Nonlinear vector equations of motion of three-dimensional curvilinear rods

Let us consider the rod element (Fig. C.1*a*) whose translational velocity is $\overline{\mathbf{v}}$ and the angular velocity of rotation is $\overline{\omega}$. We restrict ourselves to the case when the axial line of the rod can be considered as unstretchable. In the general case both constant and variable distributed forces and moments can act upon the element.

It should be emphasized that such a separation of loads requires additional explanation, because the loads depend on the chosen coordinate system. For example, the tracking load (invariable in magnitude) in the bound coordinate system is constant in time, since its projections do not depend on $t$, while in the Cartesian coordinate system the load continuously changes its direction so that its projections depend on $t$.

When studying the rod motion, the internal force factors (vectors $\overline{\mathbf{Q}}^{(1)}$ and $\overline{\mathbf{M}}^{(1)}$), as well as vectors $\overline{\text{æ}}$, $\overline{\mathbf{u}}$, $\overline{\mathbf{v}}$, $\overline{\vartheta}_j$, and $\overline{\omega}$ are the functions of two variables, $s$ and $t$.

The vectors $\overline{\mathbf{Q}}^{(1)}$ and $\overline{\mathbf{M}}^{(1)}$ are, respectively, equal to

$$\overline{\mathbf{Q}}^{(1)} = \overline{\mathbf{Q}}_0 + \overline{\mathbf{Q}}, \qquad \overline{\mathbf{M}}^{(1)} = \overline{\mathbf{M}}_0 + \overline{\mathbf{M}},$$

where $\overline{\mathbf{Q}}_0$ and $\overline{\mathbf{M}}_0$ are static components of the vector of internal forces and of the vector of moments, respectively; $\overline{\mathbf{Q}}$ and $\overline{\mathbf{M}}$ are dynamic components of the same vectors.

If the vibrations of an unloaded rod is considered, then one should assume $\overline{\mathbf{Q}}_0 = \overline{\mathbf{M}}_0 = 0$. The static mode of deformation of the rod can be determined from equilibrium equations whose derivation is presented in Appendix A, where all explanations to the adopted notation are also given.

In order to derive the equations of rod motion we tale advantage of the d'Alembert's principle. Consider the rod element on which the following force and moment of inertia act (Fig. C.1*b*):

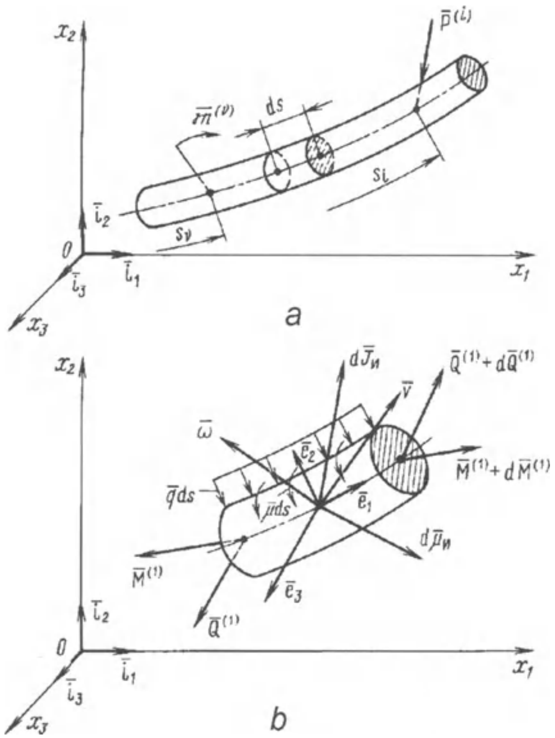


**Fig. C.1.**

$$\mathrm{d}\overline{\mathbf{J}}_i = -\mathrm{d}m\frac{\mathrm{d}\overline{\mathbf{v}}}{\mathrm{d}t}, \quad \mathrm{d}\overline{\mu}_i = -\frac{\mathrm{d}}{\mathrm{d}t}\left(J_0\overline{\omega}\right)\mathrm{d}s, \tag{C.1}$$

where $\mathrm{d}m = m_0(s)\mathrm{d}s$; $m_0(s) = F\varrho$ is the mass of unit length of the rod; $F$ is the area of the rod cross section; $\varrho$ is the rod material density; and

$$\mathbf{J}_0(s) = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} \qquad (J_{ii} = \varrho J_i)$$

($J_i$ are geometrical moments of inertia of the section relative to principal axes).

The velocity vector is

$$\overline{\mathbf{v}} = \frac{\mathrm{d}\overline{\mathbf{u}}}{\mathrm{d}t}, \tag{C.2}$$

where $\overline{\mathbf{u}}$ is the vector of displacement of the points of the rod axial line.

For an unstretchable rod the arc coordinate $s$ of a point of the rod axial line remains unchanged when the rod moves, therefore,

$$\mathrm{d}\overline{\mathbf{J}}_i = -\mathrm{d}m\frac{\partial \overline{\mathbf{v}}}{\partial t}, \qquad \mathrm{d}\overline{\mu}_i = -\frac{\partial}{\partial t}\left(J_0\overline{\omega}\right)\mathrm{d}s, \tag{C.3}$$

where $\overline{\mathbf{v}} = \dfrac{\mathrm{d}\overline{\mathbf{u}}}{\mathrm{d}t}$.

Using the d'Alembert's principle, we get the following vector equations of rod motion (see Fig. C.1), with allowance made for the inertia of rotation and for concentrated forces and moments:

$$m_0\frac{\partial \overline{\mathbf{v}}}{\partial t} = \frac{\partial \overline{\mathbf{Q}}^{(1)}}{\partial s} + \overline{\mathbf{q}}^{(1)} + \sum_{i=1}^{n} \overline{\mathbf{P}}^{(i)(1)}\delta\left(s - s_i\right); \tag{C.4}$$

$$\frac{\partial}{\partial t}\left(\mathbf{J}_0\overline{\omega}\right) = \frac{\partial \overline{\mathbf{M}}^{(1)}}{\partial s} + \overline{\mathbf{e}}_1 \times \overline{\mathbf{Q}}^{(1)} + \overline{\mu}^{(1)} + \sum_{\nu=1}^{\rho} \overline{\mathfrak{M}}^{(\nu)(1)}\delta\left(s - s_\nu\right). \tag{C.5}$$

Changing over to local derivatives in equations (C.4) and (C.5) (see Appendix A) and omitting the tilde symbol in their notation, we have

$$m_0\left(\frac{\partial \overline{\mathbf{v}}}{\partial t} + \overline{\omega} \times \overline{\mathbf{v}}\right) = \frac{\partial \overline{\mathbf{Q}}^{(1)}}{\partial s} + \overline{\text{æ}}^{(1)} \times \overline{\mathbf{Q}}^{(1)} + \overline{\mathbf{P}}^{(1)}; \tag{C.6}$$

$$\frac{\partial}{\partial t}\left(\mathbf{J}_0\overline{\omega}\right) - \frac{\partial \overline{\mathbf{M}}^{(1)}}{\partial s} - \overline{\text{æ}}^{(1)} \times \overline{\mathbf{M}}^{(1)} - \overline{\mathbf{e}}_1 \times \overline{\mathbf{Q}}^{(1)} - \overline{\mathfrak{M}}^{(1)} = 0. \tag{C.7}$$

Here

$$\begin{aligned} \overline{\mathbf{P}}^{(1)} &= \overline{\mathbf{q}}^{(1)} + \sum_{i=1}^{n} \overline{\mathbf{P}}^{(i)(1)}\delta\left(s - s_i\right); \\ \overline{\mathfrak{M}}^{(1)} &= \overline{\mu}^{(1)} + \sum_{\nu=1}^{\rho} \overline{\mathfrak{M}}^{(\nu)(1)}\delta\left(s - s_\nu\right). \end{aligned} \tag{C.8}$$

The force $\overline{\mathbf{P}}^{(1)}$ and the moment $\overline{\mathfrak{M}}^{(1)}$ introduced for more compact notation of the equations consist of distributed ($\overline{\mathbf{q}}$ and $\overline{\mu}$) and concentrated ($\overline{\mathbf{P}}^{(i)}$ and $\overline{\mathfrak{M}}^{(\nu)}$) forces and moments applied to the rod. In turn, they can have static components independent of time, i.e.,

$$\begin{aligned} &\overline{\mathbf{q}}^{(1)} = \overline{\mathbf{q}}_0 + \overline{\mathbf{q}}; \qquad \overline{\mu}^{(1)} = \overline{\mu}_0 + \overline{\mu}; \\ &\overline{\mathbf{P}}^{(i)(1)} = \overline{\mathbf{P}}_0^{(i)} + \overline{\mathbf{P}}^{(i)}; \qquad \overline{\mathfrak{M}}^{(\nu)(1)} = \overline{\mathfrak{M}}_0^{(\nu)} + \overline{\mathfrak{M}}^{(\nu)}, \end{aligned}$$

where $\overline{\mathbf{q}}_0$, $\overline{\mu}_0$, $\overline{\mathbf{P}}_0^{(i)}$, and $\overline{\mathfrak{M}}_0^{(\nu)}$ are static components; and $\overline{\mathbf{q}}$, $\overline{\mu}$, $\overline{\mathbf{P}}^{(i)}$, and $\overline{\mathfrak{M}}^{(\nu)}$ are dynamic components.

Static forces produce the initial static mode of deformation, relative to which vibrations occur (this is the most general case).

Each specific applied problem requires to study very carefully the forces that arise under vibrations and depend on the components of vector $\overline{\mathbf{u}}$ and on the angles $\vartheta_j$. For example, the equation connecting vector $\overline{\mathbf{M}}^{(1)}$ with the increment of vector $\Delta\overline{\text{æ}}$ and the equation for the vector of displacements $\overline{\mathbf{u}}$ (see Appendix A) have the form

$$\overline{\mathbf{M}}^{(1)} = \mathbf{A}\left(\overline{\text{æ}}^{(1)} - \overline{\text{æ}}_0^{(1)}\right) = \mathbf{A}\Delta\overline{\text{æ}}; \tag{C.9}$$

$$\frac{\partial\overline{\mathbf{u}}}{\partial s} + \overline{\text{æ}}^{(1)} \times \overline{\mathbf{u}} = (1 - l_{11})\,\overline{\mathbf{e}}_1 - l_{21}\overline{\mathbf{e}}_2 - l_{31}\overline{\mathbf{e}}_3. \tag{C.10}$$

Let us write two more equations relating the components of vectors $\overline{\text{æ}}$ and $\overline{\omega}$ to the angles $\vartheta_j$ (see Appendices A and B):

$$\overline{\text{æ}}^{(1)} = \mathbf{L}_1\frac{\partial\overline{\vartheta}}{\partial s} + \mathbf{L}\overline{\text{æ}}_0^{(1)}; \tag{C.11}$$

$$\overline{\omega} = \mathbf{L}_1\frac{\partial\overline{\vartheta}}{\partial t}. \tag{C.12}$$

## C.2 Reduction of equations to dimensionless form

Equations (C.6)-(C.11) can be reduced to the dimensionless form:

$$\tau = p_0 t, \quad \overline{\omega} = \tilde{\overline{\omega}} p_0, \quad \overline{\mathbf{v}} = \tilde{\overline{\mathbf{v}}} p_0\, l, \quad \overline{\mathbf{M}}^{(1)} = \tilde{\overline{\mathbf{M}}}^{(1)}\frac{A_{33}(0)}{l},$$
$$\overline{\mu}^{(1)} = \tilde{\overline{\mu}}^{(1)}\frac{A_{33}(0)}{l}, \quad \overline{\mathbf{Q}}^{(1)} = \tilde{\overline{\mathbf{Q}}}^{(1)}\frac{A_{33}(0)}{l^2}, \quad \overline{\mathbf{q}}^{(1)} = \tilde{\overline{\mathbf{q}}}^{(1)}\frac{A_{33}(0)}{l^3},$$
$$\overline{\mathbf{P}}^{(j)} = \tilde{\overline{\mathbf{P}}}^{(j)}\frac{A_{33}(0)}{l^2}, \quad \overline{\text{æ}}^{(1)} = \frac{1}{l}\tilde{\overline{\text{æ}}}^{(1)},$$
$$\tilde{J}_{ii}(\eta) = \frac{J_{ii}(\eta)}{F_0 l^2}, \quad \tilde{A}_{ii}(\eta) = \frac{A_{ii}(\eta)}{A_{33}(0)}, \quad \overline{\mathfrak{M}}^{(\nu)} = \tilde{\overline{\mathfrak{M}}}^{(\nu)}\frac{A_{33}(0)}{l},$$

where $p_0 = \left(\frac{A_{33}(0)}{m_0(0)l^4}\right)^{\frac{1}{2}}$; $m_0(0)$, and $F_0$ are the mass of a unit length of the rod and the section area in the origin of coordinates, respectively; the symbol "$\sim$" here denotes dimensionless quantities.

In an arbitrary section of the rod (changing over to the dimensionless coordinate $\eta = s/l$) one can express the mass of the rod unit length through $m_0(0)$:

$$m_0(\eta) = m_0(0)n_1(\eta) = \varrho F_0 n_1(\eta),$$

where $n_1(\eta)$ is a dimensionless function.

In the dimensionless form we have the following system of differential nonlinear vector equations of the rod motion in the bound coordinate system (omitting the symbol of tilde in designations of local derivatives and dimensionless quantities):

$$
\begin{aligned}
&n_1(\eta)\left(\frac{\partial \overline{\mathbf{v}}}{\partial \tau}+\overline{\omega}\times\overline{\mathbf{v}}\right)-\frac{\partial \overline{\mathbf{Q}}^{(1)}}{\partial \eta}-\overline{\text{æ}}^{(1)}\times\overline{\mathbf{Q}}^{(1)}-\overline{\mathbf{P}}^{(1)}=0;\\
&\frac{\partial}{\partial \tau}\left(\mathbf{J}_0\overline{\omega}\right)-\frac{\partial \overline{\mathbf{M}}^{(1)}}{\partial \eta}-\overline{\text{æ}}^{(1)}\times\overline{\mathbf{M}}^{(1)}-\overline{\mathbf{e}}_1\times\overline{\mathbf{Q}}^{(1)}-\overline{\mathfrak{M}}^{(1)}=0;\\
&\mathbf{L}_1\frac{\partial \overline{\vartheta}}{\partial \eta}+\mathbf{L}\overline{\text{æ}}_0^{(1)}-\overline{\text{æ}}^{(1)}=0;\\
&\frac{\partial \overline{\mathbf{u}}}{\partial \eta}+\overline{\text{æ}}^{(1)}\times\overline{\mathbf{u}}+(l_{11}-1)\,\overline{\mathbf{e}}_1+l_{21}\overline{\mathbf{e}}_2+l_{31}\overline{\mathbf{e}}_3=0;\\
&\mathbf{L}_1\frac{\partial \overline{\vartheta}}{\partial \tau}-\overline{\omega}=0;\\
&\overline{\mathbf{M}}^{(1)}=\mathbf{A}\left(\overline{\text{æ}}^{(1)}-\overline{\text{æ}}_0^{(1)}\right)=\mathbf{A}\triangle\overline{\text{æ}},
\end{aligned}
\tag{C.13}
$$

where $\mathbf{J}_0$ is the matrix with dimensionless elements $J_{ii}$;

$$
\mathbf{L}_1=\begin{bmatrix}\cos\vartheta_2\cos\vartheta_3 & 0 & -\sin\vartheta_2\\ -\sin\vartheta_3 & 1 & 0\\ \sin\vartheta_2\cos\vartheta_3 & 0 & \cos\vartheta_2\end{bmatrix};
$$

$$
\mathbf{L}=\begin{bmatrix}\cos\vartheta_2\cos\vartheta_3 & \begin{matrix}\cos\vartheta_2\sin\vartheta_3\cos\vartheta_1+\\ +\sin\vartheta_2\sin\vartheta_1\end{matrix} & \begin{matrix}\cos\vartheta_2\sin\vartheta_3\cos\vartheta_1-\\ -\sin\vartheta_2\cos\vartheta_1\end{matrix}\\ -\sin\vartheta_3 & \cos\vartheta_1\cos\vartheta_3 & \cos\vartheta_3\sin\vartheta_1\\ \sin\vartheta_2\cos\vartheta_3 & \begin{matrix}\sin\vartheta_2\sin\vartheta_3\cos\vartheta_1-\\ -\cos\vartheta_2\sin\vartheta_1\end{matrix} & \begin{matrix}\sin\vartheta_2\sin\vartheta_3\sin\vartheta_1+\\ +\cos\vartheta_2\cos\vartheta_1\end{matrix}\end{bmatrix}.
$$

Let us recall that angles $\vartheta_j$ (components of the vector $\overline{\vartheta}$) are the angles of rotation of the base $\{\overline{\mathbf{e}}_j\}$ with respect to the base $\{\overline{\mathbf{e}}_{j0}\}$. System of equations (C.13) includes six unknown vectors: $\overline{\mathbf{Q}}^{(1)}$, $\overline{\mathbf{M}}^{(1)}$, $\overline{\text{æ}}^{(1)}$, $\overline{\mathbf{u}}$, $\overline{\omega}$, and $(\vartheta_1,\vartheta_2,\vartheta_3)$. The displacements $u_j$ can be determined (after finding the vector $\overline{\text{æ}}^{(1)}(\eta,\tau)$ and the angles $\vartheta_j(\eta,\tau)$ from the fourts equation of system (A.13).

## C.3 Equations of small vibrations of rods (linear equations)

Let us derive the vector equations of small vibrations of a rod about its equilibrium state, assuming that additional internal forces and moments arising under vibrations are small, as well as displacements $u_j$ and angles $\vartheta_j$. We put

$$
\begin{aligned}
&\overline{\mathbf{Q}}^{(1)}=\overline{\mathbf{Q}}_0+\overline{\mathbf{Q}}; \qquad \overline{\mathbf{M}}^{(1)}=\overline{\mathbf{M}}_0+\overline{\mathbf{M}};\\
&\overline{\text{æ}}^{(1)}=\overline{\text{æ}}_0+\triangle\overline{\text{æ}}; \qquad \overline{\mathbf{q}}^{(1)}=\overline{\mathbf{q}}_0+\overline{\mathbf{q}};\\
&\overline{\mu}^{(1)}=\overline{\mu}_0+\overline{\mu}; \qquad \overline{\mathbf{P}}^{(1)}=\overline{\mathbf{P}}_0+\overline{\mathbf{P}}; \qquad \overline{\mathfrak{M}}^{(1)}=\overline{\mathfrak{M}}_0+\overline{\mathfrak{M}}.
\end{aligned}
\tag{C.14}
$$

At small vibrations one can consider the components of dynamic parts of above vectors to be small, therefore, when deriving equations, we can neglect their products (both vector and scalar):

$$\overline{\omega} \times \overline{\mathbf{v}} = 0, \qquad \Delta\overline{\text{æ}} \times \overline{\mathbf{Q}} = 0, \qquad \Delta\overline{\text{æ}} \times \overline{\mathbf{M}} = 0, \qquad \overline{\mathbf{u}} \times \Delta\overline{\text{æ}} = 0.$$

Upon substituting (C.14) into (C.13) and taking into account only the terms that linearly depend on small quantities, we get the equations of small vibrations of a rod. In the bound coordinate system the vector equations of small forced vibrations of a rod have the following form

$$\begin{aligned}
&n_1(\eta)\frac{\partial^2\overline{\mathbf{u}}}{\partial\tau^2} - \frac{\partial\overline{\mathbf{Q}}}{\partial\eta} - \Delta\overline{\text{æ}} \times \overline{\mathbf{Q}}_0 - \overline{\text{æ}}_0 \times \overline{\mathbf{Q}} = \overline{\mathbf{P}};\\
&\mathbf{J}_0\frac{\partial^2\overline{\vartheta}}{\partial\tau^2} - \frac{\partial\overline{\mathbf{M}}}{\partial\eta} - \Delta\overline{\text{æ}} \times \overline{\mathbf{M}}_0 - \overline{\text{æ}}_0 \times \overline{\mathbf{M}} - \overline{\mathbf{e}}_{10} \times \overline{\mathbf{Q}} - \overline{\mathfrak{M}} = 0;\\
&\frac{\partial\overline{\vartheta}}{\partial\eta} + \overline{\text{æ}}_0 \times \overline{\vartheta} - \Delta\overline{\text{æ}} = 0;\\
&\frac{\partial\overline{\mathbf{u}}}{\partial\eta} + \overline{\text{æ}}_0 \times \overline{\mathbf{u}} - \vartheta_3\overline{\mathbf{e}}_2 + \vartheta_2\overline{\mathbf{e}}_3 = 0;\\
&\overline{\mathbf{M}} = \mathbf{A}\Delta\overline{\text{æ}},
\end{aligned} \tag{C.15}$$

where

$$\begin{aligned}
\overline{\mathbf{P}} &= \overline{\mathbf{q}} + \sum_{i=1}^{n} \overline{\mathbf{P}}^{(i)}\delta(\eta - \eta_i);\\
\overline{\mathfrak{M}} &= \overline{\mu} + \sum_{\nu=1}^{\rho} \overline{\mathfrak{M}}^{(\nu)}\delta(\eta - \eta_\nu).
\end{aligned} \tag{C.16}$$

The quantities $\overline{\mathbf{q}}$, $\overline{\mu}$, $\overline{\mathbf{P}}^{(i)}$, and $\overline{\mathfrak{M}}^{(\nu)}$ appearing in the right-hand sides of expression (C.16) are dynamic loads. When solving equations (C.15), it is more convenient to represent them in the vector-matrix form, since the vectors $\overline{\mathbf{Q}}_0$, $\overline{\mathbf{M}}_0$, and $\overline{\text{æ}}_0$ can be determined form equations of equilibrium (see Appendix A). For example, for the vector products $\Delta\overline{\text{æ}} \times \overline{\mathbf{Q}}_0$ and $\Delta\overline{\text{æ}} \times \overline{\mathbf{M}}_0$ one can write

$$\Delta\overline{\text{æ}} \times \overline{\mathbf{Q}}_0 = \mathbf{A}_Q\Delta\overline{\text{æ}}, \quad \Delta\overline{\text{æ}} \times \overline{\mathbf{M}}_0 = \mathbf{A}_M\Delta\overline{\text{æ}},$$

where $\mathbf{A}_Q = \begin{bmatrix} 0 & Q_{30} & -Q_{20} \\ -Q_{30} & 0 & Q_{10} \\ Q_{20} & -Q_{10} & 0 \end{bmatrix}; \quad \mathbf{A}_M = \begin{bmatrix} 0 & M_{30} & -M_{20} \\ -M_{30} & 0 & M_{10} \\ M_{20} & -M_{10} & 0 \end{bmatrix}.$

In a similar manner we can also derive the expressions for remaining vector products that appear in system of equations (C.15). After some transformations, we have

$$
\begin{aligned}
&n_1(\eta)\frac{\partial^2\overline{\mathbf{u}}}{\partial\tau^2}-\frac{\partial\overline{\mathbf{Q}}}{\partial\eta}-\mathbf{A}_Q\Delta\overline{\text{æ}}-\mathbf{A}_{\text{æ}}\overline{\mathbf{Q}}=\overline{\mathbf{P}};\\
&\mathbf{J}_0\frac{\partial^2\overline{\vartheta}}{\partial\tau^2}-\frac{\partial\overline{\mathbf{M}}}{\partial\eta}-\mathbf{A}_M\Delta\overline{\text{æ}}-\mathbf{A}_{\text{æ}}\overline{\mathbf{M}}-\mathbf{A}_1\overline{\mathbf{Q}}=\overline{\mathfrak{M}};\\
&\frac{\partial\overline{\vartheta}}{\partial\eta}+\mathbf{A}_{\text{æ}}\overline{\vartheta}-\Delta\overline{\text{æ}}=0;\\
&\frac{\partial\overline{\mathbf{u}}}{\partial\eta}+\mathbf{A}_{\text{æ}}\overline{\mathbf{u}}+\mathbf{A}_1\vartheta=0;\\
&\overline{\mathbf{M}}=\mathbf{A}\Delta\overline{\text{æ}},
\end{aligned}
\tag{C.17}
$$

where $\mathbf{A}_{\text{æ}}=\begin{bmatrix}0 & -\text{æ}_{30} & \text{æ}_{20}\\ \text{æ}_{30} & 0 & -\text{æ}_{10}\\ -\text{æ}_{20} & \text{æ}_{10} & 0\end{bmatrix}$; $\quad\mathbf{A}_1=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}$.

We can write system of equations (C.17) in the form of a single vector equation (excluding $\Delta\overline{\text{æ}}$):

$$
\mathbf{A}^{(1)}\frac{\partial^2\overline{\mathbf{Z}}}{\partial\tau^2}+\frac{\partial\overline{\mathbf{Z}}}{\partial\eta}+\mathbf{A}^{(2)}\overline{\mathbf{Z}}=\overline{\mathbf{\Phi}}
\tag{C.18}
$$

where $\overline{\mathbf{Z}}=\begin{bmatrix}\overline{\mathbf{Q}}\\ \overline{\mathbf{M}}\\ \overline{\vartheta}\\ \overline{\mathbf{u}}\end{bmatrix}$; $\quad\overline{\mathbf{\Phi}}=\begin{bmatrix}-\overline{\mathbf{P}}\\ -\overline{\mathfrak{M}}\\ 0\\ 0\end{bmatrix}$; $\quad\mathbf{A}^{(1)}=\begin{bmatrix}0&0&0&-n_1\overline{\mathbf{E}}\\0&0&-J&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$;

and $\quad\mathbf{A}^{(2)}=\begin{bmatrix}\mathbf{A}_{\text{æ}} & \mathbf{A}_Q\mathbf{A}^{-1} & 0 & 0\\ \mathbf{A}_1 & (\mathbf{A}_M\mathbf{A}^{-1}+\mathbf{A}_{\text{æ}}) & -J & 0\\ 0 & -\mathbf{A}^{-1} & \mathbf{A}_{\text{æ}} & 0\\ 0 & 0 & \mathbf{A}_1 & \mathbf{A}_{\text{æ}}\end{bmatrix}$.

## C.4 Equations of small vibrations in projections onto bound axes

From relations (C.17) we derive the following equations in projections onto bound axes:

$$
\left.
\begin{aligned}
&n_1\frac{\partial^2 u_1}{\partial\tau^2}-\frac{\partial Q_1}{\partial\eta}+Q_{20}\Delta\text{æ}_3-Q_{30}\Delta\text{æ}_2+\\
&\qquad\qquad+\text{æ}_{30}Q_2-\text{æ}_{20}Q_3=P_1;\\
&n_1\frac{\partial^2 u_2}{\partial\tau^2}-\frac{\partial Q_2}{\partial\eta}+Q_{30}\Delta\text{æ}_1-Q_{10}\Delta\text{æ}_3+\\
&\qquad\qquad+\text{æ}_{10}Q_3-\text{æ}_{30}Q_1=P_2;\\
&n_1\frac{\partial^2 u_3}{\partial\tau^2}-\frac{\partial Q_3}{\partial\eta}+Q_{10}\Delta\text{æ}_2-Q_{20}\Delta\text{æ}_1+\\
&\qquad\qquad+\text{æ}_{20}Q_1-\text{æ}_{10}Q_2=P_3;
\end{aligned}
\right\}
\tag{C.19}
$$

$$\left.\begin{aligned}
&J_{11}\frac{\partial^2\overline{\vartheta}_1}{\partial\tau^2} - \frac{\partial M_1}{\partial\eta} - æ_{20}M_3 + æ_{30}M_2 - \\
&\qquad\qquad -M_{30}\Delta æ_2 + M_{20}\Delta æ_3 = \mathfrak{M}_1; \\
&J_{22}\frac{\partial^2\overline{\vartheta}_2}{\partial\tau^2} - \frac{\partial M_2}{\partial\eta} - æ_{30}M_1 + æ_{10}M_3 - \\
&\qquad -M_{10}\Delta æ_3 + M_{30}\Delta æ_1 + Q_3 = \mathfrak{M}_2; \\
&J_{33}\frac{\partial^2\overline{\vartheta}_3}{\partial\tau^2} - \frac{\partial M_3}{\partial\eta} - æ_{10}M_2 + æ_{20}M_1 - \\
&\qquad -M_{20}\Delta æ_1 + M_{10}\Delta æ_2 - Q_2 = \mathfrak{M}_3;
\end{aligned}\right\} \tag{C.20}$$

$$\left.\begin{aligned}
&\frac{\partial\vartheta_1}{\partial\eta} + æ_{20}\vartheta_3 - æ_{30}\vartheta_2 - \Delta æ_1 = 0; \\
&\frac{\partial\vartheta_2}{\partial\eta} + æ_{30}\vartheta_1 - æ_{10}\vartheta_3 - \Delta æ_2 = 0; \\
&\frac{\partial\vartheta_3}{\partial\eta} + æ_{10}\vartheta_2 - æ_{20}\vartheta_1 - \Delta æ_3 = 0;
\end{aligned}\right\} \tag{C.21}$$

$$\left.\begin{aligned}
&\frac{\partial u_1}{\partial\eta} + æ_{20}u_3 - æ_{30}u_2 = 0; \\
&\frac{\partial u_2}{\partial\eta} + æ_{30}u_1 - æ_{10}u_3 - \vartheta_3 = 0; \\
&\frac{\partial u_3}{\partial\eta} + æ_{10}u_2 - æ_{20}u_1 + \vartheta_2 = 0;
\end{aligned}\right\} \tag{C.22}$$

$$M_1 = A_{11}\Delta æ_1; \quad M_2 = A_{22}\Delta æ_2; \quad M_3 = A_{33}\Delta æ_3. \tag{C.23}$$

Putting $P_j = \mathfrak{M}_j = 0$ $(j = 1,2,3)$ in expressions (C.19) and (C.20), we obtain the equations of free vibrations of a curvilinear rod relative to its equilibrium state for the case when the rod is loaded with static tracking forces, i.e., $Q_{j0} \neq 0$ and $M_{j0} \neq 0$.

## C.5 Equations of small vibrations of a rod whose axial line in the unloaded state is a plane curve

### A special case of system (C.18) - (C.22)

Figure 1.95 shows a spiral spring whose axial line in its natural state is a plane curve. If the spring were deflected in the plane of drawing $x_1Ox_2$, it would execute small vibrations in the plane of drawing. If it is deflected with respect to this plane, small three-dimensional vibrations take place. For example, let a spiral (the flexible element of an instrument) be on an object moving with acceleration (the object acceleration $\bar{\mathbf{a}}$ is parallel to the plane $x_1Ox_2$). Then, the distributed forces of inertia act upon the spiral in the plane of drawing. If the axial line of the rod remains to be a plane curve in the loaded state, then we should put in equations (C.19) - (C.23)

$$Q_{10} \neq 0, \quad Q_{20} \neq 0, \quad Q_{30} = 0;$$
$$M_{10} = M_{20} = 0, \quad M_{30} \neq 0;$$
$$æ_{10} = æ_{20} = 0, \quad æ_{30} \neq 0.$$

Considering that the rod axial line becomes a three-dimensional curve, from (C.19) - (C.23) we get the following equations of forced vibrations of the plane curvilinear rod

$$
\begin{aligned}
& n_1 \frac{\partial^2 u_1}{\partial \tau^2} - \frac{\partial Q_1}{\partial \eta} + Q_{20} \triangle æ_3 + æ_{30} Q_2 = P_1; \\
& n_1 \frac{\partial^2 u_2}{\partial \tau^2} - \frac{\partial Q_2}{\partial \eta} - Q_{10} \triangle æ_3 + æ_{30} Q_1 = P_2; \\
& n_1 \frac{\partial^2 u_3}{\partial \tau^2} - \frac{\partial Q_3}{\partial \eta} + Q_{10} \triangle æ_2 + Q_2 \triangle æ_1 = P_3; \\
& J_{11} \frac{\partial^2 \vartheta_1}{\partial \tau^2} - \frac{\partial M_1}{\partial \eta} + æ_{30} M_2 - M_{30} \triangle æ_2 = \mathfrak{M}_1; \\
& J_{22} \frac{\partial^2 \vartheta_2}{\partial \tau^2} - \frac{\partial M_2}{\partial \eta} - æ_{30} M_1 + M_{30} \triangle æ_1 + Q_3 = \mathfrak{M}_2; \\
& J_{33} \frac{\partial^2 \vartheta_3}{\partial \tau^2} - \frac{\partial M_3}{\partial \eta} - Q_2 = \mathfrak{M}_3; \\
& \frac{\partial \vartheta_1}{\partial \eta} - æ_{30} \vartheta_2 - \triangle æ_1 = 0; \\
& \frac{\partial \vartheta_2}{\partial \eta} + æ_{30} \vartheta_1 - \triangle æ_2 = 0; \\
& \frac{\partial \vartheta_3}{\partial \eta} - \triangle æ_3 = 0; \\
& \frac{\partial u_1}{\partial \eta} - æ_{30} u_2 = 0; \\
& \frac{\partial u_2}{\partial \eta} + æ_{30} u_1 - \vartheta_3 = 0; \\
& \frac{\partial u_3}{\partial \eta} + \vartheta_2 = 0,
\end{aligned}
\tag{C.24}
$$

where $M_1 = A_{11} \triangle æ_1$; $M_2 = A_{22} \triangle æ_2$, and $M_3 = A_{33} \triangle æ_3$.

If at arising vibrations the line does not leave the plane $x_1 O x_2$, then one should put $Q_j = 0$, $M_1 = M_2 = 0$, $\vartheta_1 = \vartheta_2 = 0$, $\triangle æ_1 = \triangle æ_2 = 0$, $u_j = 0$, $P_j = 0$, and $\mathfrak{M}_1 = \mathfrak{M}_2 = 0$ in equations (C.24). As a result, we have

$$\begin{aligned}
&n_1\frac{\partial^2 u_1}{\partial\tau^2}-\frac{\partial Q_1}{\partial\eta}+Q_{20}\Delta æ_3+æ_{30}Q_2=P_1;\\
&n_1\frac{\partial^2 u_2}{\partial\tau^2}-\frac{\partial Q_2}{\partial\eta}-Q_{10}\Delta æ_3-æ_{30}Q_1=P_2;\\
&J_{33}\frac{\partial^2 \vartheta_3}{\partial\tau^2}-\frac{\partial M_3}{\partial\eta}-Q_2=\mathfrak{M}_3;\\
&\frac{\partial\vartheta_3}{\partial\eta}-\Delta æ_3=0;\\
&\frac{\partial u_1}{\partial\eta}-æ_{30}u_2=0;\\
&\frac{\partial u_2}{\partial\eta}+æ_{30}u_1-\vartheta_3=0.
\end{aligned}\tag{C.25}$$

By analogy with the general case one can present systems of equations (C.24), (C.25) in the form of one vector equation

$$\mathbf{A}^{(1)}\frac{\partial^2\overline{\mathbf{Z}}}{\partial\tau^2}+\mathbf{A}^{(2)}\frac{\partial\overline{\mathbf{Z}}}{\partial\eta}+\mathbf{A}^{(3)}\overline{\mathbf{Z}}=\overline{\mathbf{b}}.\tag{C.26}$$

When free and forced vibrations the plane $x_1Ox_2$ of a rod unloaded with static forces are studied one should set $Q_{10}=Q_{20}=M_{30}=0$ in equations (C.25).

# D

# Exact numerical method of determining the frequencies and modes of rod vibrations

In Appendix C we derived the vector equation of small free vibrations of a three-dimensional curvilinear rod without taking the drag forces into account. At $\overline{\mathbf{\Phi}} = 0$ we have

$$\mathbf{A}^{(1)}\frac{\partial^2 \overline{\mathbf{Z}}}{\partial \tau^2} + \frac{\partial \overline{\mathbf{Z}}}{\partial \eta} + \mathbf{A}^{(2)}\overline{\mathbf{Z}} = 0 \qquad \left(\overline{\mathbf{Z}} = \left[\overline{\mathbf{Q}}, \overline{\mathbf{M}}, \overline{\vartheta}, \overline{\mathbf{u}}\right]^{\mathrm{T}}\right). \tag{D.1}$$

We seek the solution to equation (D.1) in the form

$$\overline{\mathbf{Z}}(\tau, \eta) = \overline{\mathbf{Z}}_0(\eta)\mathrm{e}^{\mathrm{i}\lambda\tau}, \tag{D.2}$$

or

$$\begin{aligned} \overline{\mathbf{Q}}(\tau, \eta) &= \overline{\mathbf{Q}}_0(\eta)\mathrm{e}^{\mathrm{i}\lambda\tau}, \quad & \overline{\mathbf{M}}(\tau, \eta) &= \overline{\mathbf{M}}_0(\eta)\mathrm{e}^{\mathrm{i}\lambda\tau}; \\ \overline{\vartheta}(\tau, \eta) &= \overline{\vartheta}_0(\eta)\mathrm{e}^{\mathrm{i}\lambda\tau}, \quad & \overline{\mathbf{u}}(\tau, \eta) &= \overline{\mathbf{u}}_0(\eta)\mathrm{e}^{\mathrm{i}\lambda\tau}. \end{aligned} \tag{D.3}$$

After substitution of (D.2) into (D.1) we get

$$\frac{\mathrm{d}\overline{\mathbf{Z}}_0}{\mathrm{d}\eta} + \mathbf{B}(\eta, \lambda)\overline{\mathbf{Z}}_0 = 0, \tag{D.4}$$

where $\mathbf{B}(\eta, \lambda) = \begin{bmatrix} \mathbf{A}_{æ} & \mathbf{A}_Q\mathbf{A}^{-1} & 0 & \lambda^2 n_1 \mathbf{E} \\ \mathbf{A}_1 & (\mathbf{A}_m\mathbf{A}^{-1} + \mathbf{A}_{æ}) & \lambda^2\mathbf{J} & 0 \\ 0 & -\mathbf{A}^{-1} & \mathbf{A}_{æ} & 0 \\ 0 & 0 & \mathbf{A}_1 & \mathbf{A}_{æ} \end{bmatrix}.$

## D.1 Determination of eigen values (frequencies)

Specifying the value of $\lambda(1)$, we find (numerically) the solution to equation (D.4):

$$\overline{\mathbf{Z}}_0 = \mathbf{K}[\eta, \lambda(1)]\overline{\mathbf{C}} \qquad \Big(\mathbf{K}(0) = \mathbf{E}\Big), \tag{D.5}$$

where $\mathbf{K}[\eta, \lambda(1)]$ is the fundamental matrix of solutions of a homogeneous equation.

In order to find the fundamental matrix $\mathbf{K}(\eta)$ one should solve equation (D.4) twelve times with the following initial conditions

$$\overline{\mathbf{Z}}_0^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \overline{\mathbf{Z}}_0^{(2)} = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \ \ldots, \overline{\mathbf{Z}}_0^{(12)} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}. \tag{D.6}$$

One can refine matrix $\mathbf{K}(\eta)$ using the Picard method. The matrix $\mathbf{K}(\eta)$ satisfies the following equation

$$\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}\eta} + B\mathbf{K} = 0, \tag{D.7}$$

From (D.7) we find

$$\mathbf{K}^{(2)}(\eta) = -\int_0^{\eta} B\mathbf{K}^{(1)}(h)\mathrm{d}h + \mathbf{E}, \tag{D.8}$$

where $\mathbf{K}^{(1)}(\eta)$ is the fundamental matrix obtained as the result of solving equation (D.7) at initial conditions (D.6) (first approximation).

For homogeneous boundary conditions six component of the vector $\overline{\mathbf{C}}$ are equal to zero, since at $\eta = 0$ six components of the vector $\overline{\mathbf{Z}}_0$ are zero. For example, at $\eta = 0$ (rigid fastening) $\vartheta_0(0) = 0$, and $\overline{\mathbf{u}}_0(0) = 0$. In this case, $c_7 = c_8 = \ldots = c_{12} = 0$. The remaining six components of the vector $\overline{\mathbf{C}}$ can be found from six boundary conditions at $\eta = 1$:

$$\sum_{j=1}^{6} k_{ij}(1)c_j = 0. \tag{D.9}$$

Depending on particular boundary conditions, the indices $i$ and $j$ assume six different values. For example, if the right end of the rod ($\eta = 1$) is free $\Big(\overline{\mathbf{Q}}(1) = \overline{\mathbf{M}}(1) = 0\Big)$, then the indices $i$ and $j$, in accordance with the indices of components of the vector $\overline{\mathbf{Z}}_0$, assume the values $i = 1, 2, \ldots, 6$; $j = 1, 2, \ldots, 6$, i.e., system (D.9) takes on the form

$$\begin{gathered} k_{11}c_1 + k_{12}c_2 + \ldots + k_{16}c_6 = 0; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ k_{61}c_1 + k_{62}c_2 + \ldots + k_{66}c_6 = 0. \end{gathered} \tag{D.10}$$

In order that the solution to system (D.10) would be nonzero, it is necessary to have its determinant zero, i.e.,

$$\mathbf{D}(1) = 0. \tag{D.11}$$

Solving equation (D.11) for a number of values of $\lambda$, we find (numerically) such $\lambda_i$ for which the determinant $\mathbf{D}(1)$ can be considered zero with a preset degree of accuracy. These values of $\lambda_i$ are dimensionless frequencies of rod vibrations. It should be emphasized that for three-dimensional curvilinear rods the type of fastening can be highly variable, i.e., the determinant $\mathbf{D}(1)$ can be obtained for very different combinations of matrix elements $\mathbf{K}(1, \lambda_k)$. However, it is always sufficient to have only six columns of matrix $\mathbf{K}(\eta, \lambda_k)$ in order to find the determinant $\mathbf{D}(1)$, which considerably reduces the computing time when frequencies are determined.

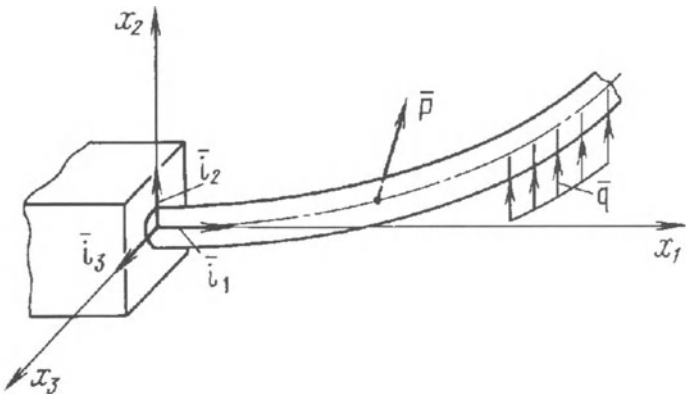


**Fig. D.1.**

Under vibrations of the rod, for example, in the plane $x_1Ox_2$, we derive the equation similar to (D.4):

$$\overline{\mathbf{Z}}_0' + \mathbf{B}\overline{\mathbf{Z}}_0 = 0, \tag{D.12}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & -æ_{30} & -Q_{20}/A_{33} & 0 & n_1\lambda^2 & 0 \\ æ_{30} & 0 & Q_{10}/A_{33} & 0 & 0 & n_1\lambda^2 \\ 0 & 1 & 0 & J_{33}\lambda^2 & 0 & 0 \\ 0 & 0 & -1/A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -æ_{30} \\ 0 & 0 & 0 & -1 & æ_{30} & 0 \end{bmatrix}; \quad \overline{\mathbf{Z}}_0 = \begin{bmatrix} Q_{10} \\ Q_{20} \\ M_{30} \\ \vartheta_{30} \\ U_{10} \\ U_{20} \end{bmatrix}.$$

The solution to equation (D.12)

$$\overline{\mathbf{Z}}_0 = \mathbf{K}(\eta)\overline{\mathbf{C}} \tag{D.13}$$

should meet six boundary conditions (three boundary conditions at each end of the rod). For a rod fixed as it is shown in Fig. D.1, we have

$$\begin{aligned} &\eta = 0, \quad u_{10} = u_{20} = 0, \quad \vartheta_{30} = 0; \\ \text{for} \quad &\eta = 1, \quad M_{30} = 0, \quad Q_{10} = Q_{20} = 0. \end{aligned}$$

Since for $\eta = 0$ matrix $\mathbf{K}(\eta)$ is the unit matrix, three components of the vector $\overline{\mathbf{C}}$ $(c_4, c_5, c_6)$ are equal to zero. From the boundary conditions at $\eta = 1$, we obtain three equations:

$$\begin{aligned} k_{11}c_1 + k_{12}c_2 + k_{13}c_3 &= 0; \\ k_{21}c_1 + k_{22}c_2 + k_{23}c_3 &= 0; \\ k_{31}c_1 + k_{32}c_2 + k_{33}c_3 &= 0. \end{aligned} \tag{D.14}$$

Equating the determinant of system (D.14) to zero,

$$\overline{\mathbf{D}} = \begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} = 0,$$

one can find the equation for calculating the frequencies $\lambda_j$.

## D.2 Determination of eigen functions for conservative problems

Knowing the frequencies $\lambda_j$ and using equations (D.12), one can determine corresponding eigen vectors $\overline{\mathbf{Z}}_0^{(j)}$:

$$\frac{\mathrm{d}\overline{\mathbf{Z}}_0^{(j)}}{\mathrm{d}\eta} + \mathbf{B}(\eta, \lambda_j)\overline{\mathbf{Z}}_\nu^{(j)} = 0. \tag{D.15}$$

From equation (D.15) we have

$$\overline{\mathbf{Z}}_0^{(j)} = \mathbf{K}\,(\eta, \lambda_j)\,\overline{\mathbf{C}}^{(j)}, \tag{D.16}$$

where $\mathbf{K}\,(\eta, \lambda_j)$ is the fundamental matrix of solutions for the eigen value $\lambda_j$.

The numerical method of determination of the fundamental matrix is described in Appendix D.1. For example, for a cantilever rod $c_7^{(j)} = c_8^{(j)} = \ldots = c_{12}^{(j)} = 0$ , while $c_1^{(j)}, c_2^{(j)}, \ldots, c_6^{(j)}$ are not equal to zero and enter as unknowns into the system of homogeneous equations

$$\begin{aligned} &k_{11}(1,\lambda_j)c_1^{(j)} + k_{12}(1,\lambda_j)c_2^{(j)} + \ldots + k_{16}(1,\lambda_j)c_6^{(j)} = 0; \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &k_{61}(1,\lambda_j)c_1^{(j)} + k_{62}(1,\lambda_j)c_2^{(j)} + \ldots + k_{66}(1,\lambda_j)c_6^{(j)} = 0. \end{aligned} \tag{D.17}$$

We express $c_1^{(j)}, c_2^{(j)}, \ldots, c_5^{(j)}$ in system (D.17) through $c_6^{(j)}$:

$$c_k^{(j)} = \beta_k c_6^{(j)} \qquad (k = 1, 2, \ldots, 5).$$

The arbitrary multiplier $c_6^{(j)}$ can be assumed to be equal to unity. Having determined $\overline{\mathbf{C}}^{(j)}$ from equation (D.16), we obtain the eigen vector

$$\overline{\mathbf{Z}}_0^{(j)} = \mathbf{K}(\eta, \lambda_j)\, \overline{\mathbf{C}}_0^{(j)},$$

where $\overline{\mathbf{C}}_0^{(j)} = [\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, 1]^{\mathrm{T}}$.

The components $Z_{07}^{(j)}, \ldots, Z_{012}^{(j)}$ of the vector $\overline{\mathbf{Z}}_0^{(j)}$ are equal to

$$Z_{07}^{(j)} = \vartheta_{01}^{(j)}; \quad Z_{08}^{(j)} = \vartheta_{02}^{(j)}; \quad Z_{09}^{(j)} = \vartheta_{03}^{(j)};$$

$$Z_{010}^{(j)} = u_{01}^{(j)}; \quad Z_{011}^{(j)} = u_{02}^{(j)}; \quad Z_{012}^{(j)} = u_{03}^{(j)},$$

and are called the modes of rod vibrations. The components $Z_{01}^{(j)}, \ldots, Z_{06}^{(j)}$ characterize the variation with $\eta$ of amplitudes of the components of vectors $\overline{\mathbf{Q}}_0^{(j)}$ and $\overline{\mathbf{M}}_0^{(j)}$ under rod vibrations for every frequency $\lambda_j$. For further transformations, it is convenient to represent eigen vectors in the form

$$\overline{\mathbf{Z}}_0^{(j)} = \begin{bmatrix} \overline{\psi}^{(j)} \\ \overline{\varphi}^{(j)} \end{bmatrix},$$

where $\overline{\psi}^{(j)} = \begin{bmatrix} \overline{\mathbf{Q}}_0^{(j)} \\ \overline{\mathbf{M}}_0^{(j)} \end{bmatrix}; \quad \overline{\varphi}^{(j)} = \begin{bmatrix} \overline{\vartheta}_0^{(j)} \\ \overline{\mathbf{u}}_0^{(j)} \end{bmatrix}$

The vectors $\overline{\psi}^{(j)}$ and $\overline{\varphi}^{(j)}$ characterize the stressed and deformed states of the rod, respectively. Such a representation of eigen vectors is convenient at solving the problems of rod dynamics approximately.

# E

# Approximate numerical determination of frequencies at small vibrations of rods

An exact numerical method of determination of the frequencies and of the modes of rod vibrations corresponding to them is described in Appendix D for conservative problems. One of the most efficient approximate methods is based on the fundamental principle of mechanics, the principle of virtual displacements.

Consider a homogeneous system of equations of free vibrations similar to system (C.15) (see Appendix C):

$$\overline{\mathbf{L}}_1 = n_1(\eta)\frac{\partial^2 \overline{\mathbf{u}}}{\partial \tau^2} - \frac{\partial \overline{\mathbf{Q}}}{\partial \eta} - \mathbf{A}_Q \mathbf{A}^{-1}\overline{\mathbf{M}} - \mathbf{A}_{æ}\overline{\mathbf{Q}} = 0; \tag{E.1}$$

$$\overline{\mathbf{L}}_2 = \mathbf{J}(\eta)\frac{\partial^2 \overline{\vartheta}}{\partial \tau^2} + \frac{\partial \overline{\mathbf{M}}}{\partial \eta} - \mathbf{A}_M \mathbf{A}^{-1}\overline{\mathbf{M}} - \mathbf{A}_{æ}\overline{\mathbf{M}} - \mathbf{A}_1\overline{\mathbf{Q}} = 0; \tag{E.2}$$

$$\overline{\mathbf{L}}_3 = \frac{\partial \overline{\vartheta}}{\partial \eta} + \mathbf{A}_{æ}\overline{\vartheta} - \mathbf{A}^{-1}\overline{\mathbf{M}} = 0; \tag{E.3}$$

$$\overline{\mathbf{L}}_4 = \frac{\partial \overline{\mathbf{u}}}{\partial \eta} + \mathbf{A}_{æ}\overline{\mathbf{u}} + \mathbf{A}_1\overline{\vartheta} = 0 \tag{E.4}$$

Let us represent system (E.1) -(E.4) in the form of a single equation (see Appendix C):

$$\overline{\mathbf{L}} = \mathbf{A}^{(1)}\frac{\partial^2 \overline{\mathbf{Z}}}{\partial \tau^2} + \frac{\partial \overline{\mathbf{Z}}}{\partial \eta} + \mathbf{A}^{(2)}\overline{\mathbf{Z}} = 0, \tag{E.5}$$

where $\overline{\mathbf{Z}} = \left(\overline{\mathbf{Q}}, \overline{\mathbf{M}}, \overline{\vartheta}, \overline{\mathbf{u}}\right)^{\mathrm{T}}$

In Appendix D we described a method of determining the eigen vectors $\overline{\mathbf{Z}}_0^{(j)}$. They can be written as

$$\overline{\mathbf{Z}}_0^{(j)} = \begin{bmatrix} \overline{\psi}^{(j)} \\ \overline{\varphi}^{(j)} \end{bmatrix}, \tag{E.6}$$

where vector $\overline{\psi}^{(j)}$ characterizes the coordinate variations of amplitude values of the components of vectors $\overline{\mathbf{Q}}_0^{(j)}$ and $\overline{\mathbf{M}}_0^{(j)}$, and vector $\overline{\varphi}^{(j)}$ characterizes variations of amplitude values of the components of vectors $\overline{\vartheta}_0^{(j)}$ and $\overline{\mathbf{u}}_0^{(j)}$ (modes of vibrations) under vibrations of the rod with the frequency $\lambda_j$. In turn, each vector $\overline{\psi}^{(j)}$ and $\overline{\varphi}^{(j)}$ can be represented in the form

$$\overline{\psi}^{(j)} = \begin{bmatrix} \overline{\psi}_Q^{(j)} \\ \overline{\psi}_M^{(j)} \end{bmatrix}; \qquad \overline{\varphi}^{(j)} = \begin{bmatrix} \overline{\varphi}_\vartheta^{(j)} \\ \overline{\varphi}_u^{(j)} \end{bmatrix}, \tag{E.7}$$

where $\overline{\psi}_Q^{(j)} = \overline{\mathbf{Q}}_0^{(j)}$; $\overline{\psi}_M^{(j)} = \overline{\mathbf{M}}_0^{(j)}$; $\overline{\varphi}_\vartheta^{(j)} = \overline{\vartheta}^{(j)}$; $\overline{\varphi}_u^{(j)} = \overline{\mathbf{u}}^{(j)}$.

For approximate solution of equation (E.5) one should first determine the vector coordinate functions $\overline{\mathbf{Z}}_0^{(j)}(\eta)$ satisfying the boundary conditions of the problem. As such functions one can take eigen vectors of free vibrations of an unloaded rod of constant section without taking the inertia of rotation of rod elements into account. For example, for determination of $\overline{\mathbf{Z}}_0^{(j)}$ one can use the following simpler system

$$\frac{\partial^2 \overline{\mathbf{u}}}{\partial \tau^2} - \frac{\partial \overline{\mathbf{Q}}}{\partial \eta} - \mathbf{A}_æ \overline{\mathbf{Q}} = 0; \tag{E.8}$$

$$\frac{\partial \overline{\mathbf{M}}}{\partial \eta} + \mathbf{A}_æ \overline{\mathbf{M}} - \mathbf{A}_1 \overline{\mathbf{Q}} = 0; \tag{E.9}$$

$$\frac{\partial \overline{\vartheta}}{\partial \eta} + \mathbf{A}_æ \overline{\vartheta} - \mathbf{A}^{-1} \overline{\mathbf{M}} = 0; \tag{E.10}$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial \eta} + \mathbf{A}_æ \overline{\mathbf{u}} - \mathbf{A}_1 \overline{\vartheta} = 0 \tag{E.11}$$

(for a rod of constant section $n_1 = 1$ and $A_{33} = 1$). Solving this system numerically one can determine the vector functions

$$\overline{\mathbf{Z}}_0^{(i)} = \begin{bmatrix} \overline{\psi}^{(i)} \\ \overline{\varphi}^{(i)} \end{bmatrix}; \quad \overline{\psi}^{(i)} = \begin{bmatrix} \overline{\psi}_Q^{(i)} \\ \overline{\psi}_M^{(i)} \end{bmatrix}; \qquad \overline{\varphi}^{(i)} = \begin{bmatrix} \overline{\varphi}_\vartheta^{(i)} \\ \overline{\varphi}_u^{(i)} \end{bmatrix}. \tag{E.12}$$

We seek the approximate solution to equation (E.5) in the form

$$\overline{\mathbf{Z}} = \sum_{i=1}^{n} f_i(\tau) \overline{\mathbf{Z}}_0^{(i)}(\eta), \tag{E.13}$$

where $f_i(\tau)$ is a continuous function of time. We restrict ourselves to two-term approximation

$$\overline{\mathbf{Z}} = f_1 \overline{\mathbf{Z}}_0^{(1)} + f_2 \overline{\mathbf{Z}}_0^{(2)}. \tag{E.14}$$

Substituting (E.14) into equation (E.5) we obtain

$$\overline{\mathbf{L}}(\overline{\mathbf{Z}}) = \overline{\gamma}, \tag{E.15}$$

where vector $\overline{\gamma}$ characterizes the error due to approximate computation of vector $\overline{\mathbf{Z}}$ according to formula (E.14).

Initial system of equations (E.1) -(E.4) has two 'physical' equations $\Big($(E.1) and (E.2)$\Big)$ and two 'geometrical' equations $\Big($(E.3) and (E.4)$\Big)$. Dimensions of terms in these equations are different, therefore, the first six components of the vector $\overline{\mathbf{Z}}(Q_j, M_j)$ $(j = 1, 2, 3)$ have dimensions of distributed forces and distributed moments, while the remaining six components $(\vartheta_j, u_j)$ have dimensions of angular and linear displacements. As a generalized displacement $\delta\overline{\mathbf{Z}}_0$ in the vector form we take the functions proportional to vector functions $\overline{\mathbf{Z}}_0^{(j)}$ and satisfying all boundary conditions of the problem:

$$\delta\overline{\mathbf{Z}}_0^{(1)} = \delta b_1 \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)}; \quad \delta\overline{\mathbf{Z}}_0^{(2)} = \delta b_2 \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)}. \tag{E.16}$$

Then under two-term approximation

$$\delta\overline{\mathbf{Z}}_0 = \delta\overline{\mathbf{Z}}_0^{(1)} + \delta\overline{\mathbf{Z}}_0^{(2)} = \delta b_1 \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} + \delta b_2 \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \tag{E.17}$$

where $\delta b_j$ are independent arbitrary quantities;

$$\mathbf{E}_0 = \begin{bmatrix} 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{bmatrix}; \quad \overline{\mathbf{Z}}_0^{(i)} = \begin{bmatrix} \overline{\mathbf{Q}}_0^{(i)} \\ \overline{\mathbf{M}}_0^{(i)} \\ \overline{\vartheta}_0^{(i)} \\ \overline{\mathbf{u}}_0^{(i)} \end{bmatrix}; \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{E.18}$$

Here, one can use as generalized and virtual displacements not only variations of linear $(\delta\overline{\mathbf{u}})$ and angular $(\delta\overline{\vartheta})$ displacements, but also variations of internal forces $\delta\overline{\mathbf{Q}}$ and moments $\delta\overline{\mathbf{M}}$.

The matrix $\overline{\mathbf{E}}_0$ is introduced in order that all scalar products

$$\overline{\mathbf{Z}} \cdot \overline{\mathbf{E}}_0 \overline{\mathbf{Z}}^{(1)}, \ \overline{\mathbf{Z}} \cdot \overline{\mathbf{E}}_0 \overline{\mathbf{Z}}^{(2)}, \ldots, \overline{\mathbf{Z}} \cdot \overline{\mathbf{E}}_0 \overline{\mathbf{Z}}^{(n)}$$

have the dimension of work (in accordance with the principle of virtual displacements). Since

$$\mathbf{E}_0 \overline{\mathbf{Z}}_0^{(i)} = \begin{bmatrix} \overline{\mathbf{u}}_0^{(i)} \\ \overline{\vartheta}_0^{(i)} \\ \overline{\mathbf{M}}_0^{(i)} \\ \overline{\mathbf{Q}}_0^{(i)} \end{bmatrix},$$

then

$$\overline{\mathbf{Z}} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}^{(i)} = \overline{\mathbf{Q}} \cdot \overline{\mathbf{u}}_0^{(i)} + \overline{\mathbf{M}} \cdot \overline{\vartheta}_0^{(i)} + \overline{\vartheta} \cdot \overline{\mathbf{M}}_0^{(i)} + \overline{\mathbf{u}} \cdot \overline{\mathbf{Q}}_0^{(i)}, \tag{E.19}$$

i.e., all terms in the right-hand side of (E.19) have the dimension of work. Substituting into equation (E.15) expression (E.14) for the vector $\overline{\mathbf{Z}}$, we get

$$\mathbf{A}^{(1)}\overline{\mathbf{Z}}_0^{(1)}\ddot{f}_1 + \mathbf{A}^{(1)}\overline{\mathbf{Z}}_0^{(2)}\ddot{f}_2 + \left(\mathbf{E}_1\overline{\mathbf{Z}}_0^{\prime(1)} + \mathbf{A}^{(2)}\overline{\mathbf{Z}}_0^{(1)}\right) f_1 + \tag{E.20}$$

$$+ \left(\mathbf{E}_1\overline{\mathbf{Z}}_0^{\prime(2)} + \mathbf{A}^{(2)}\overline{\mathbf{Z}}_0^{(2)}\right) f_2 = \overline{\gamma}, \tag{E.21}$$

where $\mathbf{E}_1$ is a $(12 \times 12)$ matrix and $\overline{\gamma} \neq 0$.

We require that the integral of vector $\overline{\gamma}$ on virtual generalized displacements taken along the entire rod length should be equal to zero, i.e.,

$$\int_0^1 \overline{\gamma}\,\delta\mathbf{Z}\,\mathrm{d}\eta = 0, \tag{E.22}$$

or, under a two-term approximation,

$$\delta b_1 \int_0^1 \left(\overline{\gamma}\cdot\mathbf{E}_0\overline{\mathbf{Z}}_0^{(1)}\right) \mathrm{d}\eta + \delta b_2 \int_0^1 \left(\overline{\gamma}\cdot\mathbf{E}_0\overline{\mathbf{Z}}_0^{(2)}\right) \mathrm{d}\eta = 0. \tag{E.23}$$

Since $\delta b_j$ are independent, it follows from equation (E.23) that

$$\int_0^1 \left(\overline{\gamma}\cdot\mathbf{E}_0\overline{\mathbf{Z}}_0^{(1)}\right) \mathrm{d}\eta = 0; \tag{E.24}$$

$$\int_0^1 \left(\overline{\gamma}\cdot\mathbf{E}_0\overline{\mathbf{Z}}_0^{(2)}\right) \mathrm{d}\eta = 0. \tag{E.25}$$

After transformations we obtain two equations for $f_1$ and $f_2 2$:

$$a_{11}\ddot{f}_1 + a_{12}\ddot{f}_2 + c_{11}f_1 + c_{12}f_2 = 0; \tag{E.26}$$

$$a_{21}\ddot{f}_1 + a_{22}\ddot{f}_2 + c_{21}f_1 + c_{22}f_2 = 0, \tag{E.27}$$

where

$$a_{11} = \int\limits_0^1 \left( \mathbf{A}^{(1)} \overline{\mathbf{Z}}_0^{(1)} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \right) \mathrm{d}\eta; \quad a_{12} = \int\limits_0^1 \left( \mathbf{A}^{(1)} \overline{\mathbf{Z}}_0^{(2)} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \right) \mathrm{d}\eta;$$

$$c_{11} = \int\limits_0^1 \left( \overline{\mathbf{Z}}_0'^{(1)} + \mathbf{A}^{(2)} \overline{\mathbf{Z}}_0^{(1)} \right) \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \mathrm{d}\eta;$$

$$c_{12} = \int\limits_0^1 \left( \overline{\mathbf{Z}}_0'^{(2)} + \mathbf{A}^{(2)} \overline{\mathbf{Z}}_0^{(2)} \right) \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(1)} \mathrm{d}\eta;$$

$$a_{21} = \int\limits_0^1 \left( \mathbf{A}^{(1)} \overline{\mathbf{Z}}_0^{(1)} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \right) \mathrm{d}\eta; \quad a_{22} = \int\limits_0^1 \left( \mathbf{A}^{(1)} \overline{\mathbf{Z}}_0^{(2)} \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \right) \mathrm{d}\eta;$$

$$c_{21} = \int\limits_0^1 \left( \overline{\mathbf{Z}}_0'^{(1)} + \mathbf{A}^{(2)} \overline{\mathbf{Z}}_0^{(1)} \right) \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \mathrm{d}\eta;$$

$$c_{22} = \int\limits_0^1 \left( \overline{\mathbf{Z}}_0'^{(2)} + \mathbf{A}^{(2)} \overline{\mathbf{Z}}_0^{(2)} \right) \cdot \mathbf{E}_0 \overline{\mathbf{Z}}_0^{(2)} \mathrm{d}\eta.$$

Putting $f_1 = f_{10} \mathrm{e}^{\mathrm{i}\lambda\tau}$ and $f_2 = f_{20} \mathrm{e}^{\mathrm{i}\lambda\tau}$, one can find from (E.26) the characteristic equation for determination of frequencies $\lambda_j$.

# F

# Approximate solution of equation of rod forced vibrations

Under forced vibrations of the rod we have without taking the forces of viscous drag into account $\left(\text{see equation (C.19)}\right)$

$$\overline{\mathbf{L}} = \mathbf{A}^{(1)} \frac{\partial^2 \overline{\mathbf{Z}}}{\partial \tau^2} + \frac{\partial \overline{\mathbf{Z}}}{\partial \eta} + \mathbf{A}^{(2)} \overline{\mathbf{Z}} - \overline{\mathbf{b}} = 0. \tag{F.1}$$

In the case when the forces of viscous drag are taken into account the equation of rod forced vibrations takes on the following form:

$$\overline{\mathbf{L}} = \mathbf{A}_1^{(1)} \frac{\partial^2 \overline{\mathbf{Z}}}{\partial \tau^2} + \mathbf{A}^{(3)} \frac{\partial \overline{\mathbf{Z}}}{\partial \tau} + \frac{\partial \overline{\mathbf{Z}}}{\partial \eta} + \mathbf{A}^{(2)} \overline{\mathbf{Z}} - \overline{\mathbf{b}} = 0. \tag{F.2}$$

Here

$$\overline{\mathbf{b}} = \begin{bmatrix} \overline{P}(\tau, \eta) \\ \overline{\mathfrak{M}}(\tau, \eta) \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{A}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & B^{(1)} \\ 0 & 0 & B^{(2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Equation (F.2) includes the forces and moments of viscous drag that can be represented in the dimensionless form of notation:

$$\overline{\mathbf{f}}_1 = \mathbf{B}^{(1)} \frac{\partial \overline{\mathbf{u}}}{\partial \tau}; \qquad \overline{\mathbf{f}}_2 = \mathbf{B}^{(2)} \frac{\partial \overline{\vartheta}}{\partial \tau};$$

where

$$\mathbf{B}^{(1)} = \begin{bmatrix} b_{11}^{(1)} & 0 & 0 \\ 0 & b_{22}^{(1)} & 0 \\ 0 & 0 & b_{33}^{(1)} \end{bmatrix}; \quad \mathbf{B}^{(2)} = \begin{bmatrix} b_{11}^{(2)} & 0 & 0 \\ 0 & b_{22}^{(2)} & 0 \\ 0 & 0 & b_{33}^{(2)} \end{bmatrix}.$$

We seek the solution to equations (F.1) and (F.2) in the form

$$\overline{\mathbf{Z}} = \sum_{j=1}^{n} f^{(j)}(\tau)\overline{\mathbf{Z}}_0^{(j)}.$$

Taking advantage of the principle of virtual displacements (see Appendix E), we obtain under a two-term approximation

$$a_{j1}\ddot{f}_1 + a_{j2}\ddot{f}_2 + c_{j1}f_1 + c_{j2}f_2 = b_j \qquad (j = 1, 2), \tag{F.3}$$

where $b_j = \int_0^1 \left(\overline{\mathbf{b}} \cdot \mathbf{E}_0\mathbf{Z}_0^{(j)}\right) \mathrm{d}\eta = \int_0^1 \left(\overline{\mathbf{P}} \cdot \overline{\mathbf{u}}\right) \mathrm{d}\eta + \int_0^1 \left(\overline{\mathfrak{M}} \cdot \overline{\vartheta}\right) \mathrm{d}\eta.$

Solving numerically system of equations (F.3), one can find the approximate solution to the equations of small forced vibrations of form (F.1) and (F.2).

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