Solutions Manual

Engineering Mechanics: Dynamics

1st Edition

Gary L. Gray
The Pennsylvania State University

Francesco Costanzo
The Pennsylvania State University

Michael E. Plesha
University of Wisconsin–Madison

With the assistance of:
Chris Punshon
Andrew J. Miller
Justin High
Chris O'Brien
Chandan Kumar
Joseph Wyne

Version: August 10, 2009

The McGraw-Hill Companies, Inc.

Copyright © 2002–2010
Gary L. Gray, Francesco Costanzo, and Michael E. Plesha

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Important Information about this Solutions Manual

Even though this solutions manual is nearly complete, we encourage you to visit

`http://www.mhhe.com/pgc`

often to obtain the most up-to-date version. In particular, as of July 30, 2009, please note the following:

- The solutions for Chapters 1 and 2 have been accuracy checked and have been edited by us. They are in their final form.
- The solutions for Chapters 4 and 7 have been accuracy checked and should be error free. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapters 3, 6, 8, and 9 are being accuracy checked and the accuracy checked versions should be available by the end of August 2009. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapter 10 should be available in their entirety by the end of August 2009.

All of the figures in Chapters 6–10 are in color. Color will be added to the figures in Chapters 1–5 over the coming weeks.

Contact the Authors

If you find any errors and/or have questions concerning a solution, please do not hesitate to contact the authors and editors via email at:

`dyn_solns@email.esm.psu.edu`

We welcome your input.

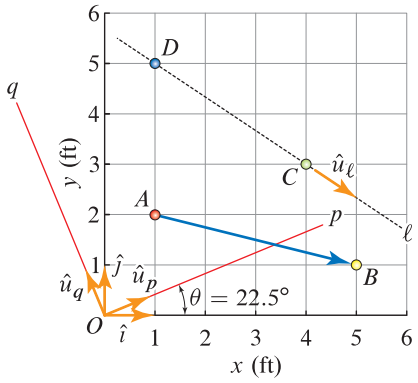
Accuracy of Numbers in Calculations

Throughout this solutions manual, we will generally assume that the data given for problems is accurate to 3 significant digits. When calculations are performed, all intermediate numerical results are reported to 4 significant digits. Final answers are usually reported with 3 significant digits. If you verify the calculations in this solutions manual using the rounded intermediate numerical results that are reported, you should obtain the final answers that are reported to 3 significant digits.

Chapter 1 Solutions

Problem 1.1

Determine $(r_{B/A})_x$ and $(r_{B/A})_y$, the x and y components of the vector $\vec{r}_{B/A}$, so as to be able to write $\vec{r}_{B/A} = (r_{B/A})_x\,\hat{\imath} + (r_{B/A})_y\,\hat{\jmath}$.



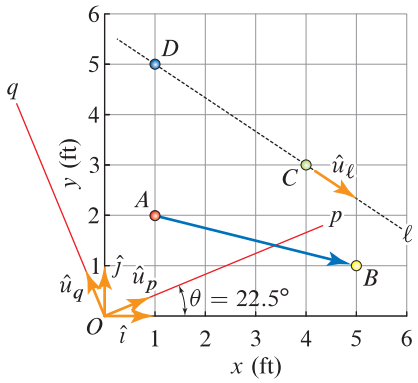
Solution

Using the component system formed by the unit vectors $\hat{\imath}$ and $\hat{\jmath}$, the vector $\vec{r}_{B/A}$ can be written as $\vec{r}_{B/A} = (x_B - x_A)\,\hat{\imath} + (y_B - y_A)\,\hat{\jmath}$, where (x_A, y_A) and (x_B, y_B) are the Cartesian coordinates of points A and B relative to the xy frame, respectively. By inspection we have $(x_A, y_A) = (1.000, 2.000)$ ft and $(x_B, y_B) = (5.000, 1.000)$ ft. Using these relations one obtains

$$\vec{r}_{B/A} = (4.00\,\hat{\imath} - 1.00\,\hat{\jmath})\ \text{ft}.$$

Problem 1.2

If the positive direction of line ℓ is from D to C, find the component of the vector $\vec{r}_{B/A}$ along ℓ.



Solution

Using the component system formed by the unit vectors $\hat{\imath}$ and $\hat{\jmath}$, the vector $\vec{r}_{B/A}$ can be written as $\vec{r}_{B/A} = (x_B - x_A)\,\hat{\imath} + (y_B - y_A)\,\hat{\jmath}$, where (x_A, y_A) and (x_B, y_B) are the Cartesian coordinates of points A and B relative to the xy frame, respectively. By inspection we have $(x_A, y_A) = (1.000, 2.000)$ ft and $(x_B, y_B) = (5.000, 1.000)$ ft. Therefore, we have

$$\vec{r}_{B/A} = (4.000\,\hat{\imath} - 1.000\,\hat{\jmath})\ \text{ft}. \tag{1}$$

Next we find the unit vector parallel to ℓ and pointing from D to C. This is accomplished by finding $\vec{r}_{C/D}$ and scaling it by its own magnitude.

$$\vec{r}_{C/D} = (x_C - x_D)\,\hat{\imath} + (y_C - y_D)\,\hat{\jmath} = (3.000\,\hat{\imath} - 2.000\,\hat{\jmath})\ \text{ft}. \tag{2}$$

Next, we derive the magnitude of $\vec{r}_{C/D}$:

$$|\vec{r}_{C/D}| = \sqrt{(3.000\ \text{ft})^2 + (2.000\ \text{ft})^2} = 3.606\ \text{ft}. \tag{3}$$

Hence we have

$$\hat{u}_\ell = \frac{\vec{r}_{C/D}}{|\vec{r}_{C/D}|} = \left(\frac{3.000\ \text{ft}}{3.606\ \text{ft}}\,\hat{\imath} - \frac{2.000\ \text{ft}}{3.606\ \text{ft}}\,\hat{\jmath}\right) = 0.8319\,\hat{\imath} - 0.5546\,\hat{\jmath}. \tag{4}$$

The component of $\vec{r}_{C/D}$ along ℓ is then obtained by taking the dot product of $\vec{r}_{B/A}$ and $\hat{u}_\ell$, i.e.,

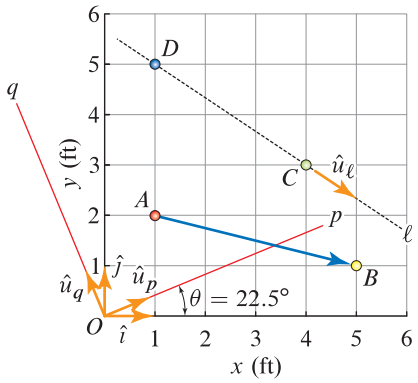
$$(r_{B/A})_\ell = [(4.000\,\hat{\imath} - 1.000\,\hat{\jmath})\ \text{ft}] \cdot (0.8319\,\hat{\imath} - 0.5546\,\hat{\jmath}) = 3.882\ \text{ft}, \tag{5}$$

which, when expressed to three significant figures gives

$$\boxed{(r_{C/D})_\ell = 3.88\ \text{ft}.}$$

Problem 1.3

Find the components of $\vec{r}_{B/A}$ along the p and q axes.



Solution

Using the component system formed by the unit vectors $\hat{\imath}$ and $\hat{\jmath}$, the vector $\vec{r}_{B/A}$ can be written as $\vec{r}_{B/A} = (x_B - x_A)\,\hat{\imath} + (y_B - y_A)\,\hat{\jmath}$, where (x_A, y_A) and (x_B, y_B) are the Cartesian coordinates of points A and B relative to the xy frame, respectively. By inspection we have $(x_A, y_A) = (1.000, 2.000)$ ft and $(x_B, y_B) = (5.000, 1.000)$ ft. Therefore, we have

$$\vec{r}_{B/A} = (4.000\,\hat{\imath} - 1.000\,\hat{\jmath})\ \text{ft}. \tag{1}$$

Now we find the unit vectors $\hat{u}_p$ and $\hat{u}_q$, which are the unit vectors orienting the p and q axes, respectively,

$$\hat{u}_p = \cos\theta_p\,\hat{\imath} + \sin\theta_p\,\hat{\jmath}, \tag{2}$$
$$\hat{u}_q = \cos\theta_q\,\hat{\imath} + \sin\theta_q\,\hat{\jmath}, \tag{3}$$

where θ_p and θ_q are the angles formed by the p and q axes with the positive x axis, respectively, and therefore are given by $\theta_p = \theta = 22.5°$, $\theta_q = (\theta + 90°) = 112.5°$. Consequently, Eqs. (2) and (3) become

$$\hat{u}_p = 0.9239\,\hat{\imath} + 0.3827\,\hat{\jmath} \quad \text{and} \quad \hat{u}_q = -0.3827\,\hat{\imath} + 0.9239\,\hat{\jmath}. \tag{4}$$

Then desired components are obtained by computing the dot product of the vector $\vec{r}_{B/A}$ with the unit vectors in Eqs. (4), i.e.,

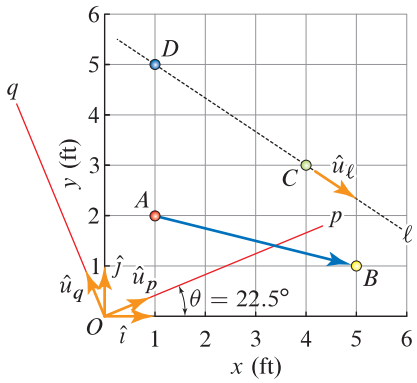
$$(r_{B/A})_p = \hat{u}_p \cdot \vec{r}_{B/A} = \left[(4.000\,\hat{\imath} - 1.000\,\hat{\jmath})\ \text{ft}\right] \cdot (0.9239\,\hat{\imath} + 0.3827\,\hat{\jmath}) = 3.313\,\text{ft}, \tag{5}$$
$$(r_{B/A})_q = \hat{u}_q \cdot \vec{r}_{B/A} = \left[(4.000\,\hat{\imath} - 1.000\,\hat{\jmath})\ \text{ft}\right] \cdot (-0.3827\,\hat{\imath} + 0.9239\,\hat{\jmath}) = -2.455\,\text{ft}, \tag{6}$$

which, when expressed to three significant figures, give

$$\boxed{(r_{B/A})_p = 3.31\,\text{ft} \quad \text{and} \quad (r_{B/A})_q = -2.46\,\text{ft}.}$$

Problem 1.4

Determine expressions for the vector $\vec{r}_{B/A}$ using both the xy and the pq coordinate systems. Next, determine $|\vec{r}_{B/A}|$, the magnitude of $\vec{r}_{B/A}$, using both the xy and the pq representations and establish whether or not the two values for $|\vec{r}_{B/A}|$ are equal to each other.



Solution

Using the component system formed by the unit vectors $\hat{\imath}$ and $\hat{\jmath}$, the vector $\vec{r}_{B/A}$ can be written as $\vec{r}_{B/A} = (x_B - x_A)\,\hat{\imath} + (y_B - y_A)\,\hat{\jmath}$, where (x_A, y_A) and (x_B, y_B) are the Cartesian coordinates of points A and B relative to the xy coordinate system, respectively. By inspection we have $(x_A, y_A) = (1.000, 2.000)$ ft and $(x_B, y_B) = (5.000, 1.000)$ ft. Therefore, we have

$$\vec{r}_{B/A} = (4.000\,\hat{\imath} - 1.000\,\hat{\jmath})\ \text{ft}, \tag{1}$$

which, when expressed to three significant figures, gives

$$\boxed{\vec{r}_{B/A} = (4.00\,\hat{\imath} - 1.00\,\hat{\jmath})\ \text{ft}.}$$

Now we find the unit vectors $\hat{u}_p$ and $\hat{u}_q$, which are the unit vectors orienting the p and q axes, respectively,

$$\hat{u}_p = \cos\theta_p\,\hat{\imath} + \sin\theta_p\,\hat{\jmath} \quad \text{and} \quad \hat{u}_q = \cos\theta_q\,\hat{\imath} + \sin\theta_q\,\hat{\jmath}, \tag{2}$$

where θ_p and θ_q are the angles formed by the p and q axes with the positive x axis, respectively, and therefore are given by $\theta_p = \theta = 22.5°$, $\theta_q = (\theta + 90°) = 112.5°$. Consequently, Eqs. (2) and (3) become

$$\hat{u}_p = 0.9239\,\hat{\imath} + 0.3827\,\hat{\jmath} \quad \text{and} \quad \hat{u}_q = -0.3827\,\hat{\imath} + 0.9239\,\hat{\jmath}. \tag{3}$$

Then, using Eqs. (1) and (3), the expression for $\vec{r}_{B/A}$ in the pq component system is

$$\begin{aligned}
\vec{r}_{B/A} &= (\hat{u}_p \cdot \vec{r}_{B/A})\,\hat{u}_p + (\hat{u}_q \cdot \vec{r}_{B/A})\,\hat{u}_q \\
&= \left\{\left[(4.000\,\hat{\imath} - 1.000\,\hat{\jmath})\ \text{ft}\right] \cdot (0.9239\,\hat{\imath} + 0.3827\,\hat{\jmath})\right\}\hat{u}_p \\
&\quad + \left\{\left[(4.000\,\hat{\imath} - 1.000\,\hat{\jmath})\ \text{ft}\right] \cdot (-0.3827\,\hat{\imath} + 0.9239\,\hat{\jmath})\right\}\hat{u}_q \\
&= (3.313\,\hat{u}_p - 2.455\,\hat{u}_q)\ \text{ft},
\end{aligned} \tag{4}$$

which, when expressed to three significant figures gives

$$\boxed{\vec{r}_{B/A} = (3.31\,\hat{u}_p - 2.46\,\hat{u}_q)\ \text{ft}.}$$

Using Eq. (1), the magnitude of $\vec{r}_{B/A}$ expressed in the xy component system is

$$\left|\vec{r}_{B/A}\right| = \sqrt{(4.000\,\text{ft})^2 + (-1.000\,\text{ft})^2} = 4.123\,\text{ft}. \tag{5}$$

Using the result in Eq. (4), the magnitude of $\vec{r}_{B/A}$ expressed in the pq component system is

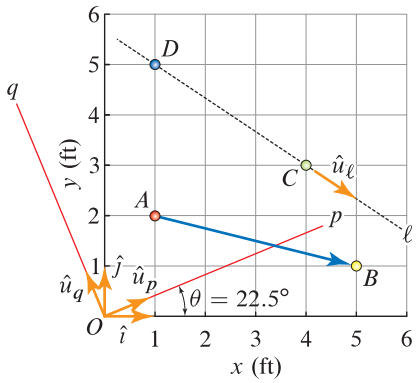
$$\left|\vec{r}_{B/A}\right| = \sqrt{(3.313\,\text{ft})^2 + (-2.455\,\text{ft})^2} = 4.123\,\text{ft}. \tag{6}$$

Expressing the results in the last two equations to three significant figures we have

$$\boxed{\left|\vec{r}_{B/A}\right|_{xy\text{ system}} = \left|\vec{r}_{B/A}\right|_{pq\text{ system}} = 4.12\,\text{ft}.}$$

Problem 1.5

Suppose that you were to compute the quantities $\left|\vec{r}_{B/A}\right|_{xy}$ and $\left|\vec{r}_{B/A}\right|_{pq}$, that is, the magnitude of the vector $\vec{r}_{B/A}$ computed using the xy and pq frames, respectively. Do you expect these two scalar values to be the same or different? Why?

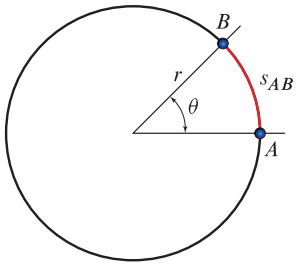


Solution

The magnitude of a vector is a fundamental property of the vector in question and it *must* be independent of how the vector is *represented*. In other words, a choice of frame only affects the values of the components of a vector and *not* its magnitude or direction. For this reason, we expect the value of $\left|\vec{r}_{B/A}\right|_{xy}$ to be same as that of $\left|\vec{r}_{B/A}\right|_{pq}$.

Problem 1.6

The measure of angles in radians is defined according to the following relation: $r\theta = s_{AB}$, where r is the radius of the circle and s_{AB} denotes the length of the circular arc. Determine the dimensions of the angle θ.



Solution

Use the given definition of an angle in radians:

$$[r\theta] = [r]\,[\theta] = [s_{AB}]. \tag{1}$$

The radius r and the arc length s must have dimensions of length, we have that

$$[L]\,[\theta] = [L]. \tag{2}$$

A simplification of the above relation yields that

$$[\theta] = 1, \tag{3}$$

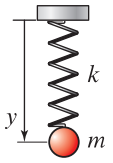
so that we conclude that

The angle θ is dimensionless (or nondimensional).

Observe that since angles can be expressed in a variety of units, such as radian or degree, this problem illustrates the idea that some nondimensional quantities still require proper use of units.

Problem 1.7

A simple oscillator consists of a linear spring fixed at one end and a mass attached at the other end, which is free to move. Suppose that the periodic motion of a simple oscillator is described by the relation $y = Y_0 \sin(2\pi\omega_0 t)$, where y has units of length and denotes the vertical position of the oscillator, Y_0 is called the oscillation amplitude, ω_0 is the oscillation frequency, and t is time. Recalling that the argument of a trigonometric function is an angle, determine the dimensions of Y_0 and ω_0, as well as their units in both the SI and the U.S. Customary systems.



Solution

Because the value of a trigonometric function must be nondimensional and because y has dimensions of length, Y_0 must also have dimensions of length:

$$[Y_0] = [L].$$

A such, the units to express the quantity Y_0 in the SI and the U.S. Customary systems are

Units of Y_0 in the SI system: m.
Units of Y_0 in the US Customary system: ft.

The quantity $(2\pi\omega_0 t)$ is the argument of a trigonometric function and as such it must be nondimensional. Therefore we have

$$[2\pi\omega_0 t] = 1 \quad \Rightarrow \quad [\omega_0]\,[t] = 1,$$

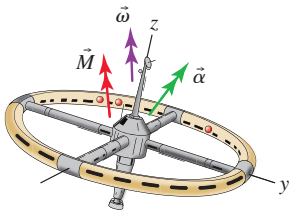
$$[\omega_0] = \frac{1}{[T]}.$$

If we where to infer the units of ω_0 directly from its dimensions, we would conclude that the units of ω_0 are seconds to the power negative one. However, in this particular problem, this would lead to an erroneous conclusion. The reason for this is that, as indicated in the problem statement, the term ω_0 appears as part of the argument $2\pi\omega_0 t$. When this happens, ω_0 is typically seen as the "number of 2π rad per unit time," that is as a quantity that measures the number of oscillation *cycles* per unit time (for additional details, see the discussion about Eqs. (9.4) and (9.8) on p. 674 of th textbook). In both the SI and the U.S. Customary systems, the number of cycles per unit time are measured in hertz, which is a unit corresponding to 1 cycle per second. Hence, we have

In both the SI and the U.S. Customary systems, ω_0 has units of Hz (hertz), i.e., cycles per second.

Problem 1.8

To study the motion of a space station, the station can be modeled as a rigid body and the equations describing its motion can be chosen to be Euler's equations, which read



$$M_x = I_{xx}\,\alpha_x - \left(I_{yy} - I_{zz}\right)\omega_y\,\omega_z,$$
$$M_y = I_{yy}\,\alpha_y - \left(I_{zz} - I_{xx}\right)\omega_x\,\omega_z,$$
$$M_z = I_{zz}\,\alpha_z - \left(I_{xx} - I_{yy}\right)\omega_x\,\omega_y.$$

In the previous equations, M_x, M_y, and M_z denote the x, y, and z components of the moment applied to the body; ω_x, ω_y, and ω_z denote the corresponding components of the angular velocity of the body, where angular velocity is defined as the time rate of change of an angle; α_x, α_y, and α_z denote the corresponding components of the angular acceleration of the body, where angular acceleration is defined as the time rate of change of an angular velocity. The quantities I_{xx}, I_{yy}, and I_{zz} are called the *principal mass moments of inertia* of the body. Determine the dimensions of I_{xx}, I_{yy}, and I_{zz} and determine their units in SI as well as in the U.S. Customary system.

Solution

We consider only one of the three equations as their dimensions are identical:

$$[M_x] = [I_{xx}\alpha_x - (I_{yy} - I_{zz})\omega_y\omega_z] = [I_{xx}][\alpha_x] - [I_{yy} - I_{zz}][\omega_y][\omega_z] \quad \Rightarrow \quad [M_x] = [I_{xx}][\alpha_x], \quad (1)$$

given that the additive terms in above equation must all the same dimensions.

Because α_x is the rate of change of an angular velocity, its units are:

$$[\alpha_x] = \frac{1}{[T]^2}. \quad (2)$$

The term M_x is the component of a moment and therefore it must have the dimensions of force times length:

$$[M_x] = [L]\,[F] = [M]\,[L]^2\,\frac{1}{[T]^2}. \quad (3)$$

Substituting Eqs. (2) and (3) into the last of Eqs. (1), we have

$$[M]\,[L]^2\,\frac{1}{[T]^2} = [I_{xx}]\,\frac{1}{[T]^2}.$$

Therefore, we conclude that the dimensions of the quantities I_{xx}, I_{yy}, and I_{zz} are

$$\boxed{[I_{xx}] = [I_{yy}] = [I_{zz}] = [M][L]^2.}$$

Because the quantities I_{xx}, I_{yy}, and I_{zz} have units of mass time length squared, their units in the SI and U.S. Customary systems are as follows:

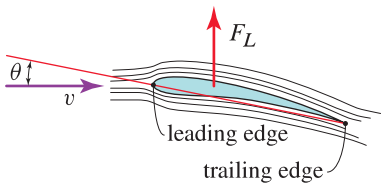
Units of I_{xx}, I_{yy}, and I_{zz} in the SI system: $\mathrm{kg{\cdot}m^2}$,
Units of I_{xx}, I_{yy}, and I_{zz} in the U.S. Customary system: $\mathrm{slug{\cdot}ft^2} = \mathrm{lb{\cdot}s^2{\cdot}ft}$.

Problem 1.9

The lift force F_L generated by the airflow moving over a wing is often expressed as follows:

$$F_L = \tfrac{1}{2}\rho v^2 C_L(\theta) A, \tag{1}$$

where ρ, v, and A denote the mass density of air, the airspeed (relative to the wing), and the wing's nominal surface area, respectively. The quantity C_L is called the *lift coefficient*, and it is a function of the wing's angle of attack θ. Find the dimensions of C_L and determine its units in the SI system.



Solution

The dimensions of each of the quantities in the given equation are

$$[F_L] = \left[\frac{\mathrm{M}\cdot\mathrm{L}}{\mathrm{T}^2}\right], \quad [\rho] = \left[\frac{\mathrm{M}}{\mathrm{L}^3}\right], \quad [A] = \left[\mathrm{L}^2\right], \quad [v] = \left[\frac{\mathrm{L}}{\mathrm{T}}\right]. \tag{2}$$

When the lift force (F_L) equation is written in terms of its dimensions, it becomes

$$\left[\frac{\mathrm{M}\cdot\mathrm{L}}{\mathrm{T}^2}\right] = \left[\frac{\mathrm{M}}{\mathrm{L}^3}\right]\left[\frac{\mathrm{L}}{\mathrm{T}}\right]^2 [C_L(\theta)]\left[\mathrm{L}^2\right]. \tag{3}$$

Simplifying and solving for the dimensions of $C_L(\theta)$ yields

$$\left[\frac{\mathrm{M}\cdot\mathrm{L}}{\mathrm{T}^2}\right] = [C_L(\theta)]\left[\frac{\mathrm{M}}{\mathrm{T}^2}\right][\mathrm{L}] \quad \Rightarrow \quad \left[C_L(\theta)\right] = 1. \tag{4}$$

Therefore we conclude that

> The lift coefficient is dimensionless and has no units associated with it.

Problem 1.10

Are the words *units* and *dimensions* synonyms?

Solution

No, units and dimensions are not synonyms. “Dimensions” refer to a physical quantity which is independent of the specific system of units used to measure it.

Solutions Manual

Engineering Mechanics: Dynamics

1st Edition

Gary L. Gray
The Pennsylvania State University

Francesco Costanzo
The Pennsylvania State University

Michael E. Plesha
University of Wisconsin–Madison

With the assistance of:
Chris Punshon
Andrew J. Miller
Justin High
Chris O'Brien
Chandan Kumar
Joseph Wyne

Version: August 10, 2009

The McGraw-Hill Companies, Inc.

Copyright © 2002–2010
Gary L. Gray, Francesco Costanzo, and Michael E. Plesha

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Important Information about this Solutions Manual

Even though this solutions manual is nearly complete, we encourage you to visit

`http://www.mhhe.com/pgc`

often to obtain the most up-to-date version. In particular, as of July 30, 2009, please note the following:

- The solutions for Chapters 1 and 2 have been accuracy checked and have been edited by us. They are in their final form.
- The solutions for Chapters 4 and 7 have been accuracy checked and should be error free. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapters 3, 6, 8, and 9 are being accuracy checked and the accuracy checked versions should be available by the end of August 2009. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapter 10 should be available in their entirety by the end of August 2009.

All of the figures in Chapters 6–10 are in color. Color will be added to the figures in Chapters 1–5 over the coming weeks.

Contact the Authors

If you find any errors and/or have questions concerning a solution, please do not hesitate to contact the authors and editors via email at:

`dyn_solns@email.esm.psu.edu`

We welcome your input.

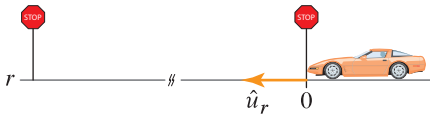
Accuracy of Numbers in Calculations

Throughout this solutions manual, we will generally assume that the data given for problems is accurate to 3 significant digits. When calculations are performed, all intermediate numerical results are reported to 4 significant digits. Final answers are usually reported with 3 significant digits. If you verify the calculations in this solutions manual using the rounded intermediate numerical results that are reported, you should obtain the final answers that are reported to 3 significant digits.

Chapter 2 Solutions

Problem 2.1

The position of a car traveling between two stop signs along a straight city block is given by $r = [9t - (45/2)\sin(2t/5)]$ m, where t denotes time and $0\,\text{s} \le t \le 17.7\,\text{s}$. Compute the displacement of the car between 2.1 and 3.7 s as well as between 11.1 and 12.7 s. For each of these time intervals compute the average velocity.



Solution

We start with observing that the two time intervals considered have the same length. More importantly, we observe that the argument of the sine function in the definition of the function $r(t)$ is understood to be expressed in radians.

The displacement of the car along the street between 2.1 s to 3.7 s is calculated from the function $r(t)$ as

$$\boxed{\Delta r_1 = r(3.7\,\text{s}) - r(2.1\,\text{s}) = 8.75\,\text{m}.} \tag{1}$$

Similarly, the displacement between 11.1 s and 12.7 s is

$$\boxed{\Delta r_2 = r(12.7\,\text{s}) - r(11.1\,\text{s}) = 13.7\,\text{m}.} \tag{2}$$

The average velocity at which the car traveled over the first interval is calculated to be

$$\boxed{(v_{\text{avg}})_1 = \frac{r(3.7\,\text{s}) - r(2.1\,\text{s})}{3.7\,\text{s} - 2.1\,\text{s}} = 5.47\,\text{m/s},} \tag{3}$$

while the average velocity of the car over the second interval is

$$\boxed{(v_{\text{avg}})_2 = \frac{r(12.7\,\text{s}) - r(11.1\,\text{s})}{12.7\,\text{s} - 11.1\,\text{s}} = 8.58\,\text{m/s}.} \tag{4}$$

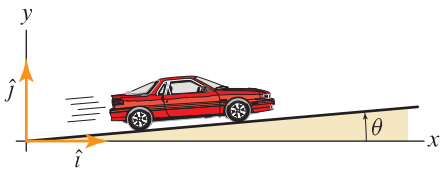
Problems 2.2 and 2.3

The position of the car relative to the coordinate system shown is

$$\vec{r}(t) = [(5.98t^2 + 0.139t^3 - 0.0149t^4)\,\hat{\imath} + (0.523t^2 + 0.0122t^3 - 0.00131t^4)\,\hat{\jmath}]\,\text{ft}.$$

Problem 2.2 Determine the velocity and acceleration of the car at $t = 15\,\text{s}$. In addition, again at $t = 15\,\text{s}$, determine the slope θ of the car's path relative to the coordinate system shown as well as the angle ϕ between velocity and acceleration.

Problem 2.3 Find the difference between the average velocity over the time interval $0\,\text{s} \le t \le 2\,\text{s}$ and the true velocity computed at the midpoint of the interval, i.e., at $t = 1\,\text{s}$. Repeat the calculation for the time interval $8\,\text{s} \le t \le 10\,\text{s}$. What do the results suggest about the approximation of the true velocity by the average velocity over different time intervals?



Solution to 2.2

The velocity is found by taking the derivative with respect to time of the position function:

$$\vec{v}(t) = [(11.96t + 0.4170t^2 - 0.05960t^3)\,\hat{\imath} + (1.046t + 0.03660t^2 - 0.005240t^3)\,\hat{\jmath}]\text{ft/s}. \qquad (1)$$

Hence, evaluating the above expression for $t = 15\,\text{s}$, we have

$$\boxed{\vec{v}(15\,\text{s}) = (72.1\,\hat{\imath} + 6.24\,\hat{\jmath})\,\text{ft/s}.}$$

The acceleration is found by taking the derivative with respect to time of the velocity function:

$$\vec{a}(t) = [(11.96 + 0.8340t - 0.1788t^2)\hat{\imath} + (1.046 + 0.07320t - 0.01572t^2)\,\hat{\jmath}]\,\text{ft/s}^2. \qquad (2)$$

Hence, evaluating the above expression for $t = 15\,\text{s}$, we have

$$\boxed{\vec{a}(15\,\text{s}) = (-15.8\,\hat{\imath} - 1.39\,\hat{\jmath})\,\text{ft/s}^2.}$$

Because the velocity is always tangent to the path, we can compute the angle θ as the angle formed by the velocity vector and the horizontal axis, i.e., $\theta = \tan^{-1}(v_y/v_x)$. Hence, using Eq. (1) for $t = 15\,\text{s}$ we have

$$\boxed{\theta(15\,\text{s}) = 4.95°.}$$

The angle ϕ between the velocity and acceleration at $t = 15\,\text{s}$ is found using the dot product:

$$\vec{v}(15\,\text{s}) \cdot \vec{a}(15\,\text{s}) = \left|\vec{v}(15\,\text{s})\right|\left|\vec{a}(15\,\text{s})\right| \cos\phi(15\,\text{s}) \quad \Rightarrow \quad \phi(15\,\text{s}) = \cos^{-1}\left(\frac{\vec{v}(15\,\text{s}) \cdot \vec{a}(15\,\text{s})}{\left|\vec{v}(15\,\text{s})\right|\left|\vec{a}(15\,\text{s})\right|}\right),$$

which, using Eqs. (1) and (2) can be evaluated to give

$$\boxed{\phi(15\,\text{s}) = 180°.}$$

Solution to 2.3

The average velocity over the interval $0 \leq t \leq 2$ is

$$\vec{v}_{\text{avg}}(t_1, t_2) = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}, \tag{3}$$

which, for $t_1 = 0\,\text{s}$ and $t_2 = 2\,\text{s}$, can be evaluated to obtain

$$\vec{v}_{\text{avg}}(0\,\text{s}, 2\,\text{s}) = (12.40\,\hat{\imath} + 1.084\,\hat{\jmath})\,\text{ft/s}. \tag{4}$$

The exact velocity is found by taking the derivative with respect to time of the position function and evaluating it at $t = 1\,\text{s}$. The velocity is determined as the first time derivative of the position function. This gives

$$\vec{v}(t) = [(11.96t + 0.4170t^2 - 0.05960t^3)\,\hat{\imath} + (1.046t + 0.03660t^2 - 0.005240t^3)\,\hat{\jmath}]\,\text{ft/s}. \tag{5}$$

Evaluating the above expression for $t = 1\,\text{s}$, we have

$$\vec{v}(1\,\text{s}) = (12.32\,\hat{\imath} + 1.077)\,\text{ft/s}. \tag{6}$$

therefore, using the results in Eqs. (4) and (6), we have

$$\boxed{\vec{v}_{\text{avg}}(0\,\text{s}, 2\,\text{s}) - \vec{v}(1\,\text{s}) = (0.0800\,\hat{\imath} + 0.00700\,\hat{\jmath})\,\text{ft/s}}$$

Similarly for $8 \leq t \leq 10$, we have

$$\vec{v}_{\text{avg}}(8\,\text{s}, 10\,\text{s}) = (97.57\,\hat{\imath} + 8.524\,\hat{\jmath})\,\text{ft/s}. \tag{7}$$

For $t = 9\,\text{s}$ we have

$$\vec{v}(9\,\text{s}) = (97.97\,\hat{\imath} + 8.559\,\hat{\jmath})\,\text{ft/s}. \tag{8}$$

Therefore, using Eqs. (7) and (8), we have

$$\boxed{\vec{v}_{\text{avg}}(8\,\text{s}, 10\,\text{s}) - \vec{v}(9\,\text{s}) = (-0.400\,\hat{\imath} - 0.0350\,\hat{\jmath})\,\text{ft/s}.}$$

The quality of the approximation is not uniform, i.e., it depends on the time interval considered.

Problem 2.4

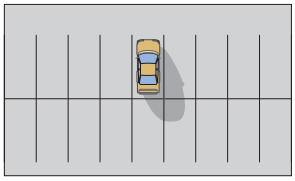
If $\vec{v}_{\text{avg}}$ is the average velocity of a point P over a given time interval, is $|\vec{v}_{\text{avg}}|$, the magnitude of the average velocity, equal to the average speed of P over the time interval in question?

Solution

The two quantities described in the problem statement are not equal in general. The average velocity depends only on the position at the time instants considered as well as the size of the corresponding time interval. Therefore $\left|\vec{v}_{\text{avg}}\right|$ will depend on the same quantities. By contrast the average of the speed depends on the value of the speed at *every* time instant during the time interval considered. Hence the two quantities cannot be expected to be equal. For example, consider the case of a car looping around a circle with constant speed v_0. Suppose also that Δt is the amount of time that the car takes in completing one full circle, i.e., to occupy the same position after one full loop. Then, the average velocity over the the amount of time Δt is equal to zero (since the car occupies the same position at time t as it does at time $t + \Delta t$). Therefore, the magnitude of the average velocity over Δt is also equal to zero. However, since the speed of the car is constant, then the average speed is simply equal to v_0, thus showing that the magnitude of the average velocity is, in general, not equal to the average speed.

Problem 2.5

A car is seen parked in a given parking space at 8:00 A.M. on a Monday morning and is then seen parked in the same spot the next morning at the same time. What is the displacement of the car between the two observations? What is the distance traveled by the car during the two observations?

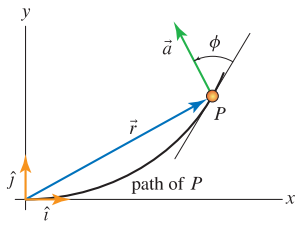


Solution

The displacement is equal to zero, since it only depends on the difference in position at the times indicated. However, there is not enough information to answer the question regarding distance traveled since we would need to know the position as a function of time throughout the entire time interval considered, i.e., between 8 : 00 A.M. Monday and 8 : 00 A.M. the following Tuesday.

Problem 2.6

Let $\vec{r} = [t\,\hat{\imath} + (2 + 3t + 2t^2)\,\hat{\jmath}]$ m describe the motion of the point P relative to the Cartesian frame of reference shown. Determine an analytic expression of the type $y = y(x)$ for the trajectory of P.



Solution

We are given

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}, \tag{1}$$

where

$$x(t) = t\ \text{m} \quad \text{and} \quad y(t) = (2 + 3t + 2t^2)\ \text{m}. \tag{2}$$

Solving the first of Eqs. (2) with respect to time, we have

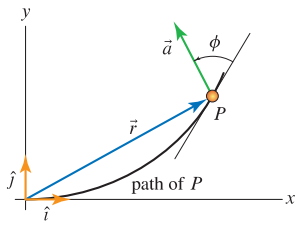
$$t\ \text{m} = x. \tag{3}$$

Substituting Eq. (3) into the second of Eqs. (2), and expressing all known coefficients to three significant figures, we obtain:

$$\boxed{y(x) = \left(2.00 + 3.00x + 2.00x^2\right)\ \text{m}.}$$

Problem 2.7

Let $\vec{r} = [t\,\hat{\imath} + (2 + 3t + 2t^2)\,\hat{\jmath}]$ ft describe the motion of a point P relative to the Cartesian frame of reference shown. Recalling that for any two vectors $\vec{p}$ and $\vec{q}$ we have that $\vec{p} \cdot \vec{q} = |\vec{p}|\,|\vec{q}| \cos\beta$, where β is the angle formed by $\vec{p}$ and $\vec{q}$, and recalling that the velocity vector is *always* tangent to the trajectory, determine the function $\phi(x)$ describing the angle between the acceleration vector and the tangent to the path of P.



Solution

The velocity vector is the time derivative of the position vector:

$$\vec{v} = [1.000\,\hat{\imath} + (3.000 + 4.000t)\,\hat{\jmath}]\ \text{ft/s}. \tag{1}$$

The acceleration vector is the time derivative of the velocity vector:

$$\vec{a} = (4.000\ \text{ft/s}^2)\,\hat{\jmath}. \tag{2}$$

As stated in the problem, ϕ is the angle between the acceleration vector and the tangent to the path of P, that is,

$$\vec{a} \cdot \vec{v} = |\vec{a}||\vec{v}| \cos\phi. \tag{3}$$

From Eqs. (1) and (2) we have that

$$|\vec{v}| = \left(\sqrt{1.000^2 + (3.000 + 4.000t)^2}\right)\text{ft/s} = \left(\sqrt{10.00 + 24.00t + 16.00\,t^2}\right)\text{ft/s and } |\vec{a}| = 4\,\text{ft/s}^2. \tag{4}$$

Substituting the expressions for $|\vec{v}|$ and $|\vec{a}|$ from Eq. (4) in Eq. (3) and solving for ϕ we obtain

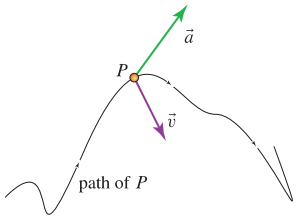
$$\cos\phi = \frac{\vec{a} \cdot \vec{v}}{|\vec{a}||\vec{v}|} \quad \Rightarrow \quad \phi = \cos^{-1}\left(\frac{3.000 + 4.000t}{\sqrt{10.00 + 24.00t + 16.00\,t^2}}\right). \tag{5}$$

Since we have that $r_x = x = (t)$ ft, we can replace t with x in the last of Eqs. (5) to obtain:

$$\boxed{\phi(x) = \cos^{-1}\left(\frac{3.00 + 4.00x}{\sqrt{10.0 + 24.0x + 16.0x^2}}\right).}$$

Problem 2.8

Is it possible for the vector $\vec{v}$ shown to represent the velocity of the point P?

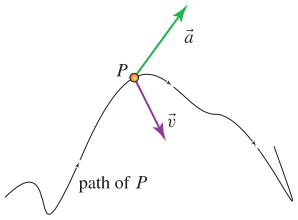


Solution

The vector $\vec{v}$ shown is not tangent to the path at point P. Hence $\vec{v}$ cannot be considered the velocity of P.

Problem 2.9

Is it possible for the vector $\vec{a}$ shown to be the acceleration of the point P?

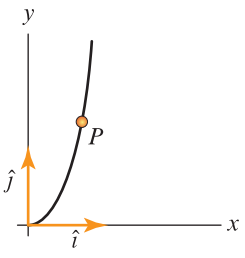


Solution

No, because $\vec{a}$ does not point toward the concave side of the trajectory of P .

Problems 2.10 and 2.11

The motion of a point P with respect to a Cartesian coordinate system is described by $\vec{r} = [2\sqrt{t}\,\hat{\imath} + (4\ln(t+1) + 2t^2)\,\hat{\jmath}]$ ft, where t is time expressed in s.



Problem 2.10 Determine P's displacement between times $t_1 = 4$ s and $t_2 = 6$ s. In addition, determine the average velocity between t_1 and t_2.

Problem 2.11 Determine P's average acceleration between times $t_1 = 4$ s and $t_2 = 6$ s.

Solution to 2.10

The positions of P at $t_1 = 4$ s and $t_2 = 6$ s are given by

$$\vec{r}(t_1) = \{2\sqrt{4}\,\hat{\imath} + [4\ln(4+1) + 2(4)^2]\,\hat{\jmath}\}\text{ ft} = (4.000\,\hat{\imath} + 38.44\,\hat{\jmath})\text{ ft}, \tag{1}$$

$$\vec{r}(t_2) = \{2\sqrt{6}\,\hat{\imath} + [4\ln(6+1) + 2(6)^2]\,\hat{\jmath}\}\text{ ft} = (4.899\,\hat{\imath} + 79.78\,\hat{\jmath})\text{ ft}. \tag{2}$$

The displacement is the change in position of P from t_1 to t_2 and is given by

$$\Delta\vec{r}(t_1,t_2) = \vec{r}(t_2) - \vec{r}(t_1) \quad \Rightarrow \quad \boxed{\Delta\vec{r}(t_1,t_2) = (0.899\,\hat{\imath} + 41.3\,\hat{\jmath})\text{ ft}.}$$

The average velocity of P is the displacement divided by change in time and, recalling that $t_1 = 4$ s, $t_2 = 6$ s, and using the results in Eqs. (1) and (2), is given by

$$\vec{v}_{\text{avg}}(t_1,t_2) = \frac{\Delta\vec{r}(t_1,t_2)}{t_2 - t_1} \quad \Rightarrow \quad \boxed{\vec{v}_{\text{avg}}(t_1,t_2) = (0.450\,\hat{\imath} + 20.7\,\hat{\jmath})\text{ ft/s}.} \tag{3}$$

Solution to 2.11

To find average acceleration we must first find the velocity of P as a function of time by taking the derivative of the position function:

$$\vec{v}(t) = \left[\frac{1}{\sqrt{t}}\,\hat{\imath} + \left(\frac{4.000}{t + 1.000} + 4.000t \right) \hat{\jmath} \right] \text{ft/s}. \tag{4}$$

The velocities of P at $t_1 = 4\,\text{s}$ and $t_2 = 6\,\text{s}$ are given by

$$\vec{v}(t_1) = (0.5000\,\hat{\imath} + 16.80\,\hat{\jmath})\,\text{ft/s}, \tag{5}$$

$$\vec{v}(t_2) = (0.4082\,\hat{\imath} + 24.57\,\hat{\jmath})\,\text{ft/s}. \tag{6}$$

The average acceleration of P is the change in velocity divided by the change in time and, using the results in Eqs. (5) and (6), is given by

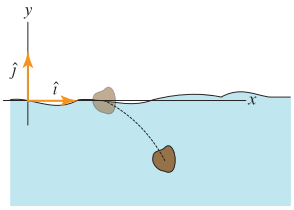
$$\vec{a}_{\text{avg}}(t_1, t_2) = \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1} \quad \Rightarrow \quad \boxed{\vec{a}_{\text{avg}}(t_1, t_2) = (-0.0459\,\hat{\imath} + 3.88\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 2.12

The motion of a stone thrown into a pond is described by

$$\vec{r}(t) = \left[\left(1.5 - 0.3e^{-13.6t}\right)\hat{\imath} + \left(0.094e^{-13.6t} - 0.094 - 0.72t\right)\hat{\jmath}\right] \text{m},$$

where t is time expressed in s, and $t = 0$ s is the time when the stone first hits the water. Determine the stone's velocity and acceleration. In addition, find the initial angle of impact θ of the stone with the water, i.e., the angle formed between the stone's trajectory and the x axis at $t = 0$ s.



Solution

The velocity of the stone is found by differentiating the position vector with respect to time. This gives

$$\vec{v}(t) = \dot{\vec{r}}(t) = [(4.080e^{-13.60t})\,\hat{\imath} + (-0.7200 - 1.278e^{-13.60t})\,\hat{\jmath}]\,\text{m/s}, \tag{1}$$

which, when expressing the equation's coefficients to three significant figures, gives

$$\boxed{\vec{v}(t) = \dot{\vec{r}}(t) = [(4.08e^{-13.6t})\,\hat{\imath} + (-0.720 - 1.28e^{-13.6t})\,\hat{\jmath}]\,\text{m/s}.}$$

The acceleration vector is found by differentiating the velocity vector in Eq. (1) with respect to time. Doing so and expressing the coefficients of the resulting formula to three significant figures, we have

$$\boxed{\vec{a}(t) = \dot{\vec{v}}(t) = [(-55.5e^{-13.6t})\,\hat{\imath} + (17.4e^{-13.6t})\,\hat{\jmath}]\,\text{m/s}^2.} \tag{2}$$

The impact angle θ is the slope of the stone's trajectory at the time that the stone enters the water. Then, recalling that the velocity is always tangent to the trajectory, we can compute θ using the components of $\vec{v}$ at time $t = 0$. Using Eq. (1) to evaluate the velocity components, we have

$$\theta = \tan^{-1}\left(\frac{v_y(0)}{v_x(0)}\right) = \tan^{-1}\left(\frac{-1.998\,\text{m/s}}{4.080\,\text{m/s}}\right) \quad \Rightarrow \quad \boxed{\theta = -26.1°.}$$

Problem 2.13

Two points P and Q happen to go by the same location in space (though at different times).

(a) What must the paths of P and Q have in common if, at the location in question, P and Q have identical speeds?

(b) What must the paths of P and Q have in common if, at the location in question, P and Q have identical velocities?

Solution

Part (a) In the first case, the only thing we can expect the paths to share is that point in space which, at different instants, is occupied by P and Q.

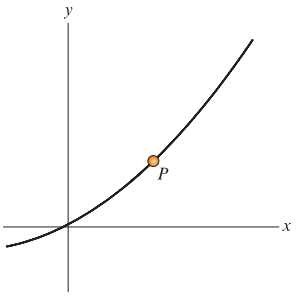
Part (b) In the second case, the paths in question will not only share a point, like in the previous case, but will also have the same tangent line at that point, since the velocity vector is *always* tangent to the path.

Problems 2.14 and 2.15

The position of point P is given by

$$\vec{r}(t) = 2.0\,[0.5 + \sin(\omega t)]\,\hat{\imath} + \left[9.5 + 10.5\sin(\omega t) + 4.0\sin^2(\omega t)\right]\hat{\jmath},$$

with $t \geq 0$, $\omega = 1.3\,\text{s}^{-1}$, and the position is measured in meters.



Problem 2.14 Find the trajectory of P in Cartesian components and then, using the x component of $\vec{r}(t)$, find the maximum and minimum values of x reached by P. The equation for the trajectory is valid for all values of x, yet the maximum and minimum values of x as given by the x component of $\vec{r}(t)$ are finite. What is the origin of this discrepancy?

Problem 2.15

(a) Plot the trajectory of P for $0 \leq t \leq 0.6\,\text{s}$, $0 \leq t \leq 1.4\,\text{s}$, $0 \leq t \leq 2.3\,\text{s}$, and $0 \leq t \leq 5\,\text{s}$.

(b) Plot the $y(x)$ trajectory for $-10 \leq x \leq 10\,\text{s}$.

(c) You will notice that the trajectory found in (b) does not agree with any of those found in (a). Explain this discrepancy by analytically determining the minimum and maximum values of x reached by P. As you look at this sequence of plots, why does the trajectory change between some times and not others?

Solution to 2.14

First write the position of P in component form as

$$x(t) = [2.0(0.5 + \sin\omega t)]\,\text{m} \quad \text{and} \quad y(t) = (9.5 + 10.5\sin\omega t + 4.0\sin^2\omega t)\,\text{m}. \tag{1}$$

Instead of solving the first of Eq. (1) for t, we solve it for $\sin\omega t$ as a function of x. This gives

$$\sin\omega t = \frac{x - 1.000\,\text{m}}{2.000\,\text{m}}, \tag{2}$$

which can then be substituted into the second of Eqs. (1) to obtain the trajectory $y(x)$ as

$$y(x) = \left[9.500 + 10.50\left(\frac{x - 1.000}{2.000}\right) + 4.000\left(\frac{x - 1.000}{2.000}\right)^2\right]\text{m}$$

$$\Rightarrow \quad \boxed{y(x) = \left(5.25 + 3.25x + 1.00x^2\right)\text{m}.} \tag{3}$$

Referring to Eq. (1), for x_{max}, $\sin\omega t = +1 \Rightarrow t = \frac{\pi}{2\omega}\,\text{s}$ and for x_{min}, $\sin\omega t = -1 \Rightarrow t = \frac{3\pi}{2\omega}\,\text{s}$. Thus, the maximum and minimum values are

$$\boxed{x_{\text{max}} = 3.00\,\text{m} \quad \text{and} \quad x_{\text{min}} = -1.00\,\text{m}.}$$

The maximum and minimum values of the x coordinate occur due to the presence of a periodic function in the equation for this component. Therefore, the trajectory, which is valid for all times t, is constrained to oscillate between $-1.000\,\text{m}$ and $3.000\,\text{m}$ on the x axis.

Solution to 2.15

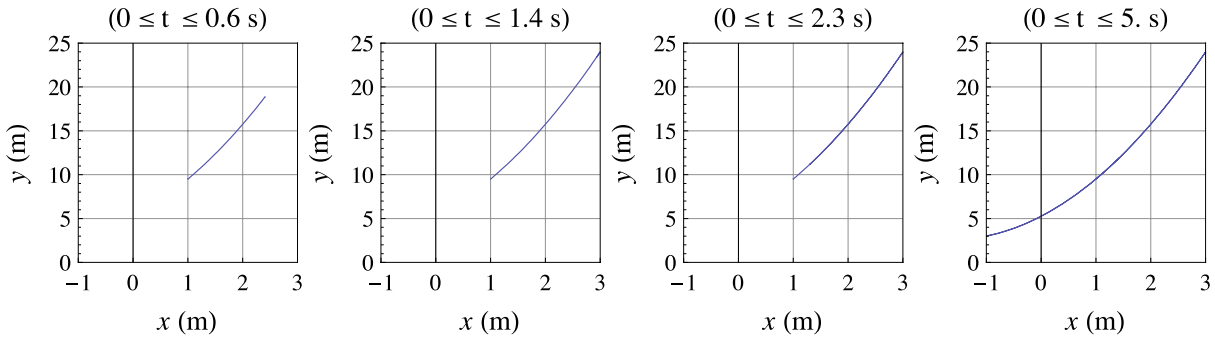
Part (a). Since $\omega = 1.3\,\text{s}^{-1}$, we write the x and y coordinates of point P as

$$x(t) = \{2.0[0.5 + \sin(1.3t)]\}\,\text{m} \quad \text{and} \quad y(t) = [9.5 + 10.5\sin(1.3t) + 4.0\sin^2(1.3t)]\,\text{m}. \tag{4}$$

One strategy to plot the trajectory of point P is to plot the line that connects the points of coordinates $[x(t), y(t)]$ as time t varies within a given time interval. This way of plotting the trajectory *does not* involve finding y as a function of x. Rather, it consists of generating a list of (x, y) values, each of which is computed by first assigning a specific value of time. This procedure is called a *parametric plot*, where the parameter used to generate the plotted points is time and does not appear directly on the plot (i.e., the plot uses x and y axes, but it does not show the time values corresponding to the points on the plot). Parametric plots can be generated using any appropriate numerical software such as MATLAB or *Mathematica*.

The parametric plots shown below of the trajectory of P were generated in *Mathematica* with the following code:

```
x = 2.0 (0.5 + Sin[1.3 t]);
y = 9.5 + 10.5 Sin[1.3 t] + 4 (Sin[1.3 t]) ^ 2;
tf = {0.6, 1.4, 2.3, 5.0};
Table[ParametricPlot[{x, y}, {t, 0, tf[[i]]}, PlotRange → {{-1, 3}, {0, 25}}, Frame → True,
  FrameTicks → {{Automatic, None}, {{-1, 0, 1, 2, 3}, None}}, AspectRatio → 1,
  GridLines → Automatic, FrameLabel → {"x  (m)", "y  (m)"},
  PlotLabel → StringJoin["(0 ≤ t ≤ ", ToString[tf[[i]]], " s)"]], {i, 1, Length[tf]}]
```



Part (b). In this part of the problem we first need to write the trajectory in the form $y = y(x)$. To do so, we start with solving the first of Eq. (4) for $\sin(1.3t)$ as a function of x. This gives

$$\sin(1.3t) = \frac{x - 1.000\,\text{m}}{2.000\,\text{m}}, \tag{5}$$

which can then be substituted into the second of Eqs. (4) to obtain

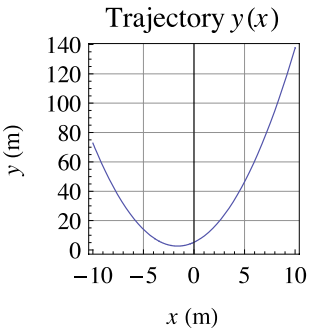
$$y(x) = \left[9.500 + 10.50\left(\frac{x - 1.000}{2.000}\right) + 4.000\left(\frac{x - 1.000}{2.000}\right)^2\right]\text{m}$$

$$\Rightarrow \quad y(x) = \left(5.25 + 3.25x + 1.00x^2\right)\text{m}, \tag{6}$$

where we have expressed all known coefficients to three significant figures.

Now that we have the trajectory in the form $y(x)$, we can plot it over the given interval $-10\,\text{m} \le x \le 10\,\text{m}$ as shown on right. This plot was generated using *Mathematica* with the following code:

```
Plot[5.250 + 3.250 x + 1.00 x^2, {x, -10.0, 10.0}, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, AspectRatio → 1,
 GridLines → Automatic, ImageSize → 170, FrameLabel → {"x (m)", "y (m)"},
 PlotLabel → "Trajectory y(x)"]
```



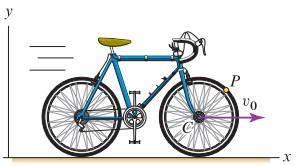
Part (c). Referring to Eq. (4), for $x_{\max}$, $\sin\omega t = +1 \Rightarrow t = \frac{\pi}{2\omega}$ s and for $x_{\min}$, $\sin\omega t = -1 \Rightarrow t = \frac{3\pi}{2\omega}$ s. Thus, the maximum and minimum values are

$$x_{\max} = 3.00\,\text{m} \quad \text{and} \quad x_{\min} = -1.00\,\text{m}.$$

The maximum and minimum values of the x coordinate occur due to the presence of a periodic function in the equation for this component. Therefore, the trajectory, which is valid for all times t, is constrained to oscillate between $-1.00\,\text{m}$ and $3.00\,\text{m}$ on the x axis. This fact explains why in the plot sequence in Part (a) the trajectory seems not to change after a while: point P keeps tracing the same curve segment again and again. The periodicity of the motion of P also explains the discrepancy between the trajectory obtained in Part (a) and that obtained in Part (b). In fact, since the plot generated in Part (a) was based on a direct application of Eqs. (4), we see that the plot in question correctly reflects the periodic time dependence the coordinates of point P. By contrast, the trajectory computed in Part (b) no longer carries any direct relationship with time. Now it is important to realize, that the trajectory plotted in Part (b) does contain the trajectory plotted in Part (a) as a sub-curve. The problem with the trajectory in Part (b) is that we have no direct way of knowing what part of the entire curve actually pertains to the motion of point P.

Problems 2.16 through 2.18

A bicycle is moving to the right at a speed $v_0 = 20\,\text{mph}$ on a horizontal and straight road. The radius of the bicycle's wheels is $R = 1.15\,\text{ft}$. Let P be a point on the periphery of the front wheel. One can show that the x and y coordinates of P are described by the following functions of time:



$$x(t) = v_0 t + R\sin(v_0 t/R) \quad \text{and} \quad y(t) = R\big[1 + \cos(v_0 t/R)\big].$$

Problem 2.16 Determine the expressions for the velocity, speed, and acceleration of P as functions of time.

Problem 2.17 Determine the maximum and minimum speed achieved by P as well as the y coordinate of P when the maximum and minimum speeds are achieved. Finally, compute the acceleration of P when P achieves its maximum and minimum speeds.

Problem 2.18 Plot the trajectory of P for $0\,\text{s} < t < 1\,\text{s}$. For the same time interval, plot the speed as a function of time as well as the components of the velocity and acceleration of P.

Solution to 2.16

The velocity of P is the time derivative of P's position, which, using the coordinate system shown, is given by

$$\vec{r} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} = \big[v_0 t + R\sin(v_0 t/R)\big]\,\hat{\imath} + R\big[1 + \cos(v_0 t/R)\big]\,\hat{\jmath}. \tag{1}$$

Differentiating the above expression with respect to time, we have

$$\vec{v} = v_0\left[1 + \cos\left(\frac{v_0 t}{R}\right)\right]\hat{\imath} - v_0\sin\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{2}$$

Then, recalling that $v_0 = 20\,\text{mph} = 29.33\,\text{ft/s}$ and $R = 1.15\,\text{ft}$, and expressing all known coefficients to three significant figures, we have

$$\boxed{\vec{v} = (29.3\,\text{ft/s})[1 + \cos[(25.5\,\text{rad/s})t]\,\hat{\imath} - (29.3\,\text{ft/s})[1 + \sin[(25.5\,\text{rad/s})t]\,\hat{\jmath}.}$$

The speed is now found by taking the magnitude of the velocity vector. Hence, we have

$$v = \sqrt{v_x^2(t) + v_y^2(t)} = v_0\sqrt{2 + 2\cos\left(\frac{v_0 t}{R}\right)} \quad \Rightarrow \quad \boxed{v = (29.3\,\text{ft/s})\sqrt{2 + 2\cos[(25.5\,\text{rad/s})t].}} \tag{3}$$

The acceleration of P is the time derivative of P's velocity. Hence, from Eq. (2), we have

$$\vec{a} = -\frac{v_0^2}{R}\sin\left(\frac{v_0 t}{R}\right)\hat{\imath} - \frac{v_0^2}{R}\cos\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{4}$$

Then, again recalling that $v_0 = 20\,\text{mph} = 29.33\,\text{ft/s}$ and $R = 1.15\,\text{ft}$, and expressing all known coefficients to three significant figures, we have

$$\boxed{\vec{a} = -\big(748\,\text{ft/s}^2\big)\sin[(25.5\,\text{rad/s})t]\,\hat{\imath} - \big(748\,\text{ft/s}^2\big)\cos[(25.5\,\text{rad/s})t]\,\hat{\jmath}.}$$

Solution to 2.17

The speed of P is the magnitude of the velocity of P. Hence, we first compute the velocity of P, which is the time derivative of P's position. Using the coordinate system shown, the position vector of P is given by

$$\vec{r} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} = \big[v_0 t + R\sin(v_0 t/R)\big]\,\hat{\imath} + R\big[1 + \cos(v_0 t/R)\big]\,\hat{\jmath}. \tag{5}$$

Differentiating the above expression with respect to time, we have

$$\vec{v} = v_0\left[1 + \cos\left(\frac{v_0 t}{R}\right)\right]\hat{\imath} - v_0 \sin\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{6}$$

Hence, the speed is given by

$$v = \sqrt{v_x^2(t) + v_y^2(t)} = v_0\sqrt{2 + 2\cos\left(\frac{v_0 t}{R}\right)}, \tag{7}$$

from which we see that v is maximum when $\cos(v_0 t/R) = 1$ and minimum when $\cos(v_0 t/R) = -1$. In turn, this implies that

$$\boxed{v_{\max} = 2v_0 = 58.7\,\text{ft/s} \quad \text{and} \quad v_{\min} = 0\,\text{ft/s},} \tag{8}$$

where we have used the fact that $v_0 = 20\,\text{mph} = 29.33\,\text{ft/s}$.

Next, recalling that for $v = v_{\min}$ we have $\cos(v_0 t/R) = -1$ and for $v = v_{\max}$ we have $\cos(v_0 t/R) = 1$, using the expression for the y component of the position of P in Eq. (5), we have

$$\boxed{y_{v_{\min}} = 0\,\text{ft} \quad \text{and} \quad y_{v_{\max}} = 2R = 2.30\,\text{ft},}$$

where we have used the fact that $R = 1.15\,\text{ft}$.

To determine the acceleration corresponding to $v_{\min}$ and $v_{\max}$, we first need to determine the acceleration of P by differentiating with respect to time the expression in Eq. (6). This gives

$$\vec{a} = -\frac{v_0^2}{R}\sin\left(\frac{v_0 t}{R}\right)\hat{\imath} - \frac{v_0^2}{R}\cos\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{9}$$

Now, recall that for $v = v_{\min}$ we have $\cos(v_0 t/R) = -1$ and for $v = v_{\max}$ we have $\cos(v_0 t/R) = 1$. In both cases, we have $\sin(v_0 t/R) = 0$. Using this considerations along with Eq. (9), we have

$$\boxed{\vec{a}_{v_{\min}} = \frac{v_0^2}{R}\,\hat{\jmath} = (748\,\text{ft/s}^2)\,\hat{\jmath} \quad \text{and} \quad \vec{a}_{v_{\max}} = -\frac{v_0^2}{R}\,\hat{\jmath} = (-748\,\text{ft/s}^2)\,\hat{\jmath},}$$

where we have used the following numerical data: $v_0 = 20\,\text{mph} = 29.33\,\text{ft/s}$ and $R = 1.15\,\text{ft}$.

Solution to 2.18

The velocity of P is the time derivative of P's position, which, using the coordinate system shown, is given by

$$\vec{r} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} = \big[v_0 t + R\sin(v_0 t/R)\big]\hat{\imath} + R\big[1 + \cos(v_0 t/R)\big]\,\hat{\jmath}. \tag{10}$$

Differentiating the above expression with respect to time, we have

$$\vec{v} = v_0\left[1 + \cos\left(\frac{v_0 t}{R}\right)\right]\hat{\imath} - v_0 \sin\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{11}$$

The speed is now found by taking the magnitude of the velocity vector. Hence, we have

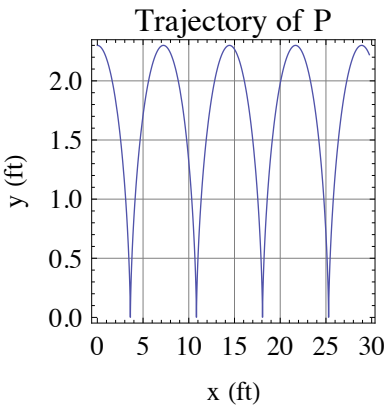
$$v = \sqrt{v_x^2(t) + v_y^2(t)} = v_0\sqrt{2 + 2\cos\left(\frac{v_0 t}{R}\right)}. \tag{12}$$

The acceleration of P is the time derivative of P's velocity. Hence, from Eq. (11), we have

$$\vec{a} = -\frac{v_0^2}{R}\sin\left(\frac{v_0 t}{R}\right)\hat{\imath} - \frac{v_0^2}{R}\cos\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{13}$$

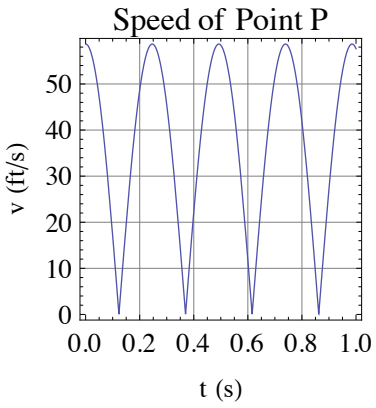
Trajectory of P. We can plot the trajectory of point P for $0 < t < 1$ s by computing the values of the coordinates of the point P at various time values, where the coordinates in question are the component of $\vec{r}$ in Eq. (10). This strategy for plotting the trajectory yields what is typically called a *parametric* plot (since we generate a plot with x and y axis of points whose position is determined by assigning values to the parameter t, as opposed to plotting y as a function of x). This can be done with a variety of pieces of numerical software. Recalling that we have $v_0 = 20$ mph $= 29.33$ ft/s and $R = 1.15$ ft, the plot presented below has been obtained using *Mathematica* with the following code

```
Parameters = {v0 → 29.33, R → 1.15};
x = v0 t + R Sin[v0 t / R]; y = R (1 + Cos[v0 t / R]);
ParametricPlot[{x, y} /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"x (ft)", "y (ft)"}, PlotLabel → "Trajectory of P", AspectRatio → 1]
```



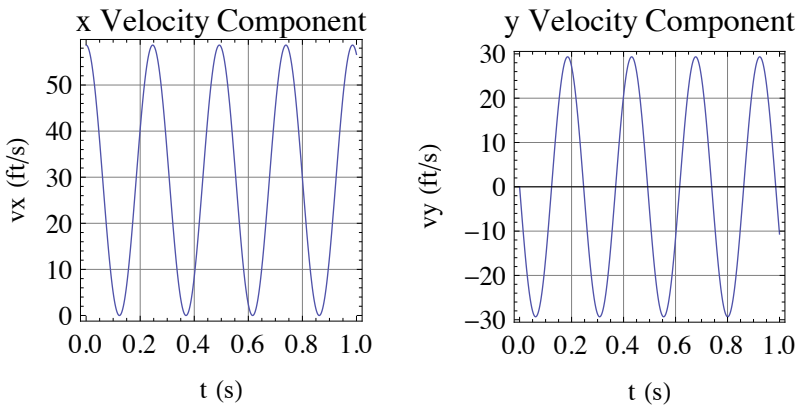
Speed of P. The speed of P can be plotted for $0 < t < 1$ s by providing the known parameters $v_0 = 20$ mph $= 29.33$ ft/s and $R = 1.15$ ft as well as the last expression in Eq. (12) to some appropriate numerical software. The plot shown below was generated using *Mathematica* with the following code:

```
Parameters = {v0 → 29.33, R → 1.15};
v = v0 √(2 + 2 Cos[v0 t / R]) ;
Plot[v /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "v (ft/s)"}, AspectRatio → 1, PlotLabel → "Speed of Point P"]
```



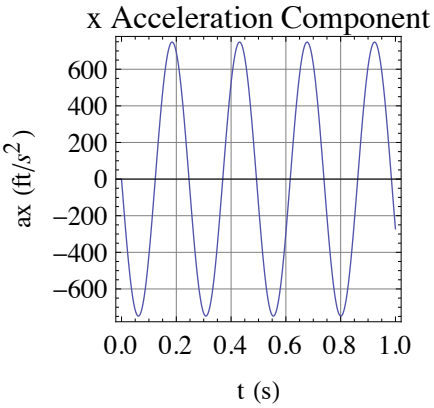
Velocity Components The components of the velocity of P can be plotted for $0 < t < 1$ s by providing the known parameters $v_0 = 20$ mph $= 29.33$ ft/s and $R = 1.15$ ft as well as the expression in Eq. (11) to some appropriate numerical software. The plot shown below was generated using *Mathematica* with the following code:

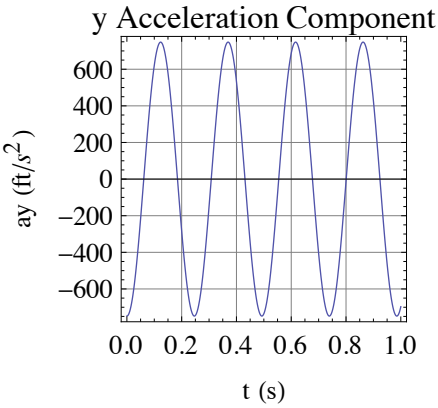
```
Parameters = {v0 → 29.33, R → 1.15};
vx = v0 (1 + Cos[v0 t / R]); vy = -v0 Sin[v0 t / R];
Plot[vx /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "vx (ft/s)"}, PlotLabel → "x Velocity Component", AspectRatio → 1]
Plot[vy /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "vy (ft/s)"}, PlotLabel → "y Velocity Component",
 AspectRatio → 1]
```



Acceleration Components The components of the acceleration of P can be plotted for $0 < t < 1$ s by providing the known parameters $v_0 = 20$ mph $= 29.33$ ft/s and $R = 1.15$ ft as well as the expression in Eq. (13) to some appropriate numerical software. The plot shown below was generated using *Mathematica* with the following code:

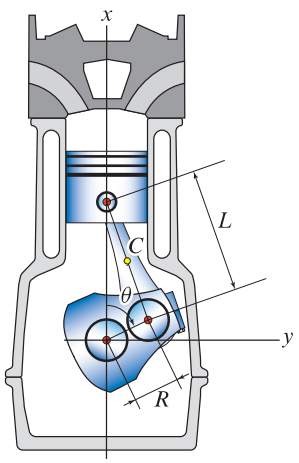
```
Parameters = {v0 → 29.33, R → 1.15};
ax = -(v0^2 / R) Sin[v0 t / R]; ay = -(v0^2 / R) Cos[v0 t / R];
Plot[ax /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "ax (ft/s^2)"}, PlotLabel → "x Acceleration Component",
 AspectRatio → 1]
Plot[ay /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "ay (ft/s^2)"}, PlotLabel → "y Acceleration Component",
 AspectRatio → 1]
```


x Acceleration Component
ax (ft/s^2)
600
400
200
0
−200
−400
−600
0.0 0.2 0.4 0.6 0.8 1.0
t (s)


y Acceleration Component
ay (ft/s^2)
600
400
200
0
−200
−400
−600
0.0 0.2 0.4 0.6 0.8 1.0
t (s)

Problems 2.19 through 2.21

Point C is a point on the connecting rod of a mechanism called a *slider-crank*. The x and y coordinates of C can be expressed as follows: $x_C = R\cos\theta + \frac{1}{2}\sqrt{L^2 - R^2\sin^2\theta}$ and $y_C = (R/2)\sin\theta$, where θ describes the position of the crank. If the crank rotates at a constant rate, then we can express θ as $\theta = \omega t$, where t is time and ω is the crank's angular velocity. Let $R = 0.1\,\text{m}$, $L = 0.25\,\text{m}$, and $\omega = 250\,\text{rad/s}$.



Problem 2.19 Find expressions for the velocity, speed, and acceleration of C.

Problem 2.20 Determine the maximum and minimum speeds of C as well as C's coordinates when the maximum and minimum speeds are achieved. In addition, determine the acceleration of C when the speed is at a minimum.

Problem 2.21 Plot the trajectory of point C for $0\,\text{s} < t < 0.025\,\text{s}$. For the same interval of time, plot the speed as a function of time as well as the components of the velocity and acceleration of C.

Solution to 2.19

The velocity of point C is is the time derivative of the position of C, which, using the coordinate system shown and recalling that $\theta = \omega t$, can be written as

$$\vec{r}_C = x_C\,\hat{\imath} + y_C\,\hat{\jmath} = \left(R\cos\omega t + \tfrac{1}{2}\sqrt{L^2 - R^2\sin^2\omega t}\right)\hat{\imath} + \left(\tfrac{1}{2}R\sin\omega t\right)\hat{\jmath}. \tag{1}$$

Hence, differentiating the above expression with respect to time and simplifying, we have

$$\boxed{\vec{v}_C = \frac{-\omega R}{2}\left(2\sin\omega t + \frac{R\sin\omega t\cos\omega t}{\sqrt{L^2 - R^2\sin^2\omega t}}\right)\hat{\imath} + \frac{\omega R}{2}\cos\omega t\,\hat{\jmath}.} \tag{2}$$

The speed is now found by taking the magnitude of the velocity vector:

$$\boxed{v = \frac{\omega R}{2}\sqrt{4\sin^2\omega t + \frac{4R\sin^2\omega t\cos\omega t}{\sqrt{L^2 - R^2\sin^2\omega t}} + \frac{R^2\sin^2\omega t\cos^2\omega t}{L^2 - R^2\sin^2\omega t} + \cos^2\omega t}.} \tag{3}$$

The acceleration is found by taking the derivative of the velocity. Hence, differentiating Eq. (2) with respect to time, we have

$$\boxed{\vec{a} = \frac{-\omega^2 R}{2}\left[2\cos\omega t + \frac{R(\cos^2\omega t - \sin^2\omega t)}{\sqrt{L^2 - R^2\sin^2\omega t}} + \frac{R^3\cos^2\theta\sin^2\omega t}{(L^2 - R^2\sin^2\omega t)^{3/2}}\right]\hat{\imath} - \frac{\omega^2 R}{2}\sin\omega t\,\hat{\jmath}.} \tag{4}$$

Solution to 2.20

The velocity of point C is is the time derivative of the position of C, which, using the coordinate system shown and recalling that $\theta = \omega t$, can be written as

$$\vec{r}_C = x_C\,\hat{\imath} + y_C\,\hat{\jmath} = \left(R\cos\omega t + \tfrac{1}{2}\sqrt{L^2 - R^2\sin^2\omega t}\right)\hat{\imath} + \left(\tfrac{1}{2}R\sin\omega t\right)\hat{\jmath}. \tag{5}$$

Hence, differentiating the above expression with respect to time and simplifying, we have

$$\vec{v}_C = \frac{-\omega R}{2}\left(2\sin\omega t + \frac{R\sin\omega t\cos\omega t}{\sqrt{L^2 - R^2\sin^2\omega t}}\right)\hat{\imath} + \frac{\omega R}{2}\cos\omega t\,\hat{\jmath} \tag{6}$$

The speed is now found by taking the magnitude of the velocity vector:

$$v = \frac{\omega R}{2}\sqrt{4\sin^2\omega t + \frac{4R\sin^2\omega t\cos\omega t}{\sqrt{L^2 - R^2\sin^2\omega t}} + \frac{R^2\sin^2\omega t\cos^2\omega t}{L^2 - R^2\sin^2\omega t} + \cos^2\omega t}. \tag{7}$$

The acceleration is found by taking the derivative of the velocity. Hence, differentiating Eq. (6) with respect to time, we have

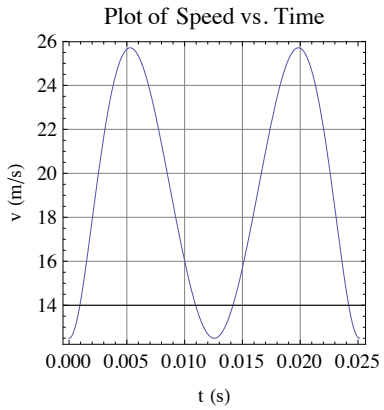
$$\vec{a} = \frac{-\omega^2 R}{2}\left[2\cos\omega t + \frac{R(\cos^2\omega t - \sin^2\omega t)}{\sqrt{L^2 - R^2\sin^2\omega t}} + \frac{R^3\cos^2\theta\sin^2\omega t}{(L^2 - R^2\sin^2\omega t)^{3/2}}\right]\hat{\imath} - \frac{\omega^2 R}{2}\sin\omega t\,\hat{\jmath}. \tag{8}$$

To find the minimum and the maximum of the speed we begin by plotting the speed as a function of time during a full cycle of the crank. Then we can obtain a visual estimate of the minimum and maximum values in question. As indicated in the problem statement, the crank angle θ is related to time as follows: $\theta = \omega t$, where $\omega = 250\,\text{rad/s}$. A full cycle of the crank is such that $0 \le \theta \le 2\pi$ rad. Hence, the corresponding time interval we consider is $0 \le t \le 0.02513\,\text{s}$. Recalling that $R = 0.1\,\text{m}$, $L = 0.25\,\text{m}$, and $\omega = 250\,\text{rad/s}$, the plot of v in Eq. (7) can be obtained with any appropriate piece of numerical software. The plot given below was obtained using *Mathematica* with the following code:

```
Parameters = {ω → 250., L -> 0.25, R → 0.1};

vC = (ω R)/2 Sqrt[4 Sin[ω t]^2 + (4 R Sin[ω t]^2 Cos[ω t])/Sqrt[L^2 - R^2 Sin[ω t]^2] + (R^2 Sin[ω t]^2 Cos[ω t]^2)/(L^2 - R^2 Sin[ω t]^2) + Cos[ω t]^2];

Plot[vC /. Parameters, {t, 0, 0.02513}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "v (m/s)"}, PlotLabel → "Plot of Speed vs. Time",
 AspectRatio → 1]
```



By inspection, the plot indicates that the speed achieves its minimum value of approximately 12.50 m/s for $t = 0$ and for $t = 0.01256$ (midpoint of the time interval considered). In addition, again by inspection, we see that the speed achieves its maximum value of approximately 25.75 m/s for $t = 0.0055$ s and $t = 0.01950$ s. To obtain a more precise estimate of these values, one can take advantage of numerical minimization and maximization techniques that are often available in modern mathematical software packages. For example, in *Mathematica* we can use the following code:

```
Parameters = {ω → 250., L -> 0.25, R → 0.1};
vC = ω R/2 Sqrt[(4 Sin[ω t]^2 + 4 R Sin[ω t]^2 Cos[ω t]/Sqrt[L^2 - R^2 Sin[ω t]^2] + R^2 Sin[ω t]^2 Cos[ω t]^2/(L^2 - R^2 Sin[ω t]^2) + Cos[ω t]^2)];
Minimize[{vC /. Parameters, 0 ≤ t ≤ 0.001}, t]
Minimize[{vC /. Parameters, 0.012 ≤ t ≤ 0.013}, t]
Maximize[{vC /. Parameters, 0 ≤ t ≤ 0.006}, t]
Maximize[{vC /. Parameters, 0.019 ≤ t ≤ 0.021}, t]
```

which gives

$$v_{\min} = 12.50\,\text{m/s} \quad \text{for} \quad t = 1.800\times10^{-14}\,\text{s} \quad \text{and} \quad t = 0.01257\,\text{s}, \tag{9}$$

$$v_{\max} = 25.71\,\text{m/s} \quad \text{for} \quad t = 0.005286\,\text{s} \quad \text{and} \quad t = 0.01985\,\text{s}. \tag{10}$$

Expressing the values of the minimum and maximum speed to three significant figures, we have

$$\boxed{v_{\min} = 12.5\,\text{m/s} \quad \text{and} \quad v_{\max} = 25.7\,\text{m/s}.}$$

Substituting the values of time in Eqs. (9) and (10), into Eq. (5), and recalling that $R = 0.1$ m, $L = 0.25$ m, and $\omega = 250$ rad/s, we have that position of C corresponding to when the speed is at a minimum or at a maximum is described by the following coordinates:

$$\boxed{\begin{aligned} v = v_{\min} \quad &\Rightarrow \quad (x, y) = (0.225, 0.000)\,\text{m} \quad \text{and} \quad (0.0250, -4.54\times10^{5})\,\text{m}, \qquad (11)\\ v = v_{\max} \quad &\Rightarrow \quad (x, y) = (0.140, 0.0485)\,\text{m} \quad \text{and} \quad (0.140, -0.0484)\,\text{m}. \qquad (12)\end{aligned}}$$

Finally, substituting the time values in Eq. (9) into Eq. (8), and again recalling that $R = 0.1$ m, $L = 0.25$ m, and $\omega = 250$ rad/s, we have that when $v = v_{\min}$,

$$\boxed{\vec{a}_C = (-7.50\times10^{3}\,\text{m/s}^2)\,\hat{\imath} + (0\,\text{m/s}^2)\,\hat{\jmath}, \quad \text{and} \quad \vec{a}_C = (5.00\times10^{3}\,\text{m/s}^2)\,\hat{\imath} + (2.84\,\text{m/s}^2)\,\hat{\jmath}.}$$

Solution to 2.21

The velocity of point C is is the time derivative of the position of C, which, using the coordinate system shown and recalling that $\theta = \omega t$, can be written as

$$\vec{r}_C = x_C\,\hat{\imath} + y_C\,\hat{\jmath} = \left(R\cos\omega t + \tfrac{1}{2}\sqrt{L^2 - R^2\sin^2\omega t}\right)\hat{\imath} + \left(\tfrac{1}{2}R\sin\omega t\right)\hat{\jmath}. \tag{13}$$

Hence, differentiating the above expression with respect to time and simplifying, we have

$$\vec{v}_C = \frac{-\omega R}{2}\left(2\sin\omega t + \frac{R\sin\omega t\cos\omega t}{\sqrt{L^2 - R^2\sin^2\omega t}}\right)\hat{\imath} + \frac{\omega R}{2}\cos\omega t\,\hat{\jmath} \tag{14}$$

The speed is now found by taking the magnitude of the velocity vector:

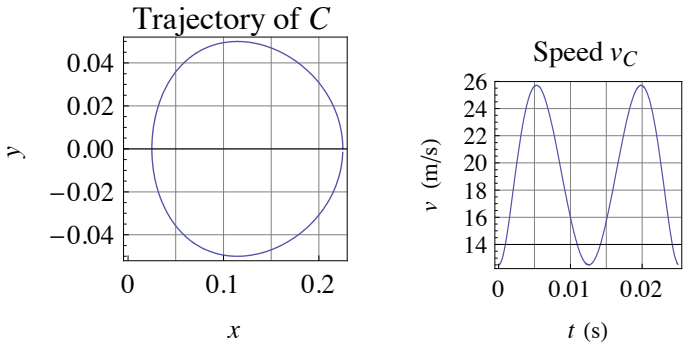
$$v = \frac{\omega R}{2}\sqrt{4\sin^2\omega t + \frac{4R\sin^2\omega t\cos\omega t}{\sqrt{L^2 - R^2\sin^2\omega t}} + \frac{R^2\sin^2\omega t\cos^2\omega t}{L^2 - R^2\sin^2\omega t} + \cos^2\omega t}. \tag{15}$$

The acceleration is found by taking the derivative of the velocity. Hence, differentiating Eq. (14) with respect to time, we have

$$\vec{a} = \frac{-\omega^2 R}{2}\left[2\cos\omega t + \frac{R(\cos^2\omega t - \sin^2\omega t)}{\sqrt{L^2 - R^2\sin^2\omega t}} + \frac{R^3\cos^2\theta\sin^2\omega t}{(L^2 - R^2\sin^2\omega t)^{3/2}}\right]\hat{\imath} - \frac{\omega^2 R}{2}\sin\omega t\,\hat{\jmath}. \tag{16}$$

Plot of the trajectory and speed of C. Plots of the trajectory and speed of C for $0 < t < 0.025$ s can be generated with any appropriate numerical piece of software. The plots presented below were generated using *Mathematica* with the following code:

```
Parameters = {ω → 250., L -> 0.25, R → 0.1};
xC = R Cos[ω t] + 1/2 Sqrt[L^2 - R^2 Sin[ω t]^2] ; yC = R/2 Sin[ω t];
vC = ω R/2 Sqrt[4 Sin[ω t]^2 + 4 R Sin[ω t]^2 Cos[ω t]/Sqrt[L^2 - R^2 Sin[ω t]^2] + R^2 Sin[ω t]^2 Cos[ω t]^2/(L^2 - R^2 Sin[ω t]^2) + Cos[ω t]^2] ;
ParametricPlot[{xC, yC} /. Parameters, {t, 0, 0.025}, Frame → True,
 FrameTicks → {{Automatic, None}, {{0, 0.1, 0.2}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"x ", "y "}, PlotLabel → "Trajectory of C "]
Plot[vC /. Parameters, {t, 0, 0.025}, Frame → True,
 FrameTicks → {{Automatic, None}, {{0, 0.01, 0.02, 0.04}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"t (s)", "v (m/s)"}, PlotLabel → "Speed v_C "]
```



Plots of the components of $\vec{v}_C$ and $\vec{a}_C$. The components of point C's velocity and acceleration are shown below in the plots to the left and right, respectively.

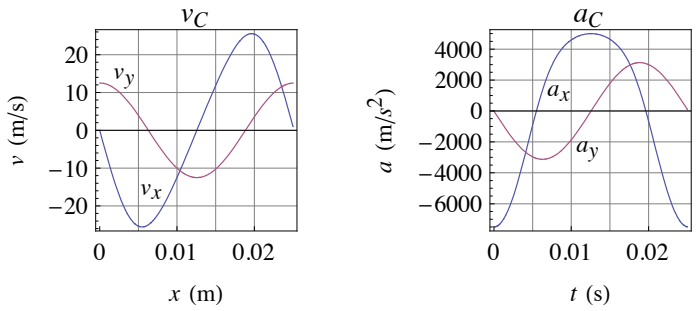
```
Parameters = {ω → 250., L -> 0.25, R → 0.1};
```

$$\texttt{vCx} = \frac{-\omega\,R}{2}\left(2\,\text{Sin}[\omega\,t] + \frac{R\,\text{Sin}[\omega\,t]\,\text{Cos}[\omega\,t]}{\sqrt{L^2 - R^2\,\text{Sin}[\omega\,t]^2}}\right);\quad \texttt{vCy} = \frac{\omega\,R}{2}\,\text{Cos}[\omega\,t];$$

$$\texttt{aCx} = \frac{-\omega^2\,R}{2}\left(2\,\text{Cos}[\omega\,t] + \frac{R\left(\text{Cos}[\omega\,t]^2 - \text{Sin}[\omega\,t]^2\right)}{\sqrt{L^2 - R^2\,\text{Sin}[\omega\,t]^2}} + \frac{R^3\,\text{Cos}[\omega\,t]^2\,\text{Sin}[\omega\,t]^2}{\left(L^2 - R^2\,\text{Sin}[\omega\,t]^2\right)^{3/2}}\right);$$

$$\texttt{aCy} = -\frac{\omega^2\,R}{2}\,\text{Sin}[\omega\,t];$$

```
Plot[{vCx, vCy} /. Parameters, {t, 0, 0.025}, Frame → True,
 FrameTicks → {{Automatic, None}, {{0, 0.01, 0.02, 0.04}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"x  (m)", "v  (m/s)"}, PlotLabel → "v_C "]
Plot[{aCx, aCy} /. Parameters, {t, 0, 0.025}, Frame → True,
 FrameTicks → {{Automatic, None}, {{0, 0.01, 0.02, 0.04}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"t  (s)", "a  (m/s^2)"}, PlotLabel → "a_C "]
```



Problem 2.22

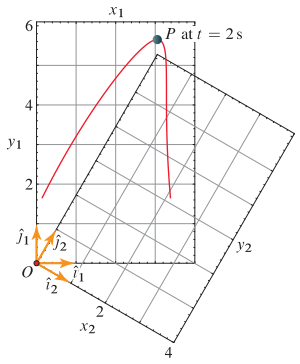
The motion of a point P with respect to Cartesian frames 1 and 2 is described by

$$(\vec{r}_{P/O})_1 = \left[(t + \sin t)\,\hat{\imath}_1 + \left(2 + 4t - t^2\right)\hat{\jmath}_1\right] \text{m}$$

and

$$(\vec{r}_{P/O})_2 = \Big\{\left[(t + \sin t)\cos\theta + \left(2 + 4t - t^2\right)\sin\theta\right]\hat{\imath}_2 + \left[-(t + \sin t)\sin\theta + \left(2 + 4t - t^2\right)\cos\theta\right]\hat{\jmath}_2\Big\} \text{m},$$

respectively, where t is time in seconds. Note that the two frames in this problem share the same origin, and therefore we are writing $(\vec{r}_{P/O})_1$ and $(\vec{r}_{P/O})_2$ to explicitly indicate that $(\vec{r}_{P/O})_1$ is expressed relative to frame 1 and $(\vec{r}_{P/O})_2$ is expressed relative to frame 2. Determine P's velocity and acceleration with respect to the two frames. In addition, determine the speed of P at time $t = 2\,\text{s}$, and verify that the speeds in the two frames are equal.



Solution

The velocity is found by differentiating the position vector:

$$\boxed{\begin{aligned}(\vec{v}_{P/O})_1 &= \left[(1 + \cos t)\,\hat{\imath}_1 + (4 - 2t)\,\hat{\jmath}_1\right] \text{m/s}, \qquad (1)\\ (\vec{v}_{P/O})_2 &= \Big\{\left[(1 + \cos t)\cos\theta - 2(t-2)\sin\theta\right]\hat{\imath}_2 \\ &\qquad - \left[2(t-2)\cos\theta + (1 + \cos t)\sin\theta\right]\hat{\jmath}_2\Big\} \text{m/s}. \qquad (2)\end{aligned}}$$

The acceleration is found by differentiating the velocity vector:

$$\boxed{\begin{aligned}(\vec{a}_{P/O})_1 &= (-\sin t\,\hat{\imath}_1 - 2\,\hat{\jmath}_1)\,\text{m/s}^2, \qquad (3)\\ (\vec{a}_{P/O})_2 &= \left[(-\cos\theta\sin t - 2\sin\theta)\,\hat{\imath}_2 + (\sin\theta\sin t - 2\cos\theta)\,\hat{\jmath}_2\right]\text{m/s}^2. \qquad (4)\end{aligned}}$$

The speed of P in frame 1 is obtained by computing the magnitude of the velocity given in Eq. (1). For $t = 2\,\text{s}$, Eq. (1) gives

$$[\vec{v}_{P/O}((t = 2\,\text{s}))]_1 = [1 + \cos(2\,\text{s})]\,\hat{\imath}_1\,\text{m/s} \quad \Rightarrow \quad \boxed{|\vec{v}_{P/O}(t = 2\,\text{s})|_1 = 0.584\,\text{m/s}.} \qquad (5)$$

The speed of P in frame two is computed by determining the magnitude of the vector in Eq. (2). Now, for $t = 2\,\text{s}$, Eq. (2) reduces to

$$[\vec{v}_{P/O}(t = 2\,\text{s})]_2 = [1 + \cos(2\,\text{s})]\cos\theta\,\hat{\imath}_2 - [1 + \cos(2\,\text{s})]\sin\theta\,\hat{\jmath}_2, \qquad (6)$$

so that we have

$$|\vec{v}_{P/O}(t = 2\,\text{s})\|_2 = \sqrt{\left((1 + \cos 2)\cos\theta\right)^2 + \left((1 + \cos 2)\sin\theta\right)^2}\,\text{m/s} = \sqrt{1 + 2\cos 2 + \cos^2 2}\,\text{m/s},$$

where, in simplifying, we have used the trigonometric identity $\sin^2 x + \cos^2 x = 1$. Evaluating the above expression to three significant figures, we have

$$\boxed{|\vec{v}_{P/O}(t = 2\,\text{s})|_2 = 0.584\,\text{m/s},}$$

which, by comparison with the result in Eq. (5) shows that the speeds are equal for both frames.

Problem 2.23

Let $\vec{r}_{P/A}$, $\vec{v}_{P/A}$, and $\vec{a}_{P/A}$ denote the position, velocity, and acceleration vectors of a point P with respect to the frame with origin at A. Let $\vec{r}_{P/B}$, $\vec{v}_{P/B}$, and $\vec{a}_{P/B}$ be the position, velocity, and acceleration vectors of the same point P with respect to the frame with origin at B. If frame B does not move relative to frame A, and if the frames are distinct, state whether or not each of the following relations is true and why.

(a) $\vec{r}_{P/A} - \vec{r}_{P/B} = \vec{0}$

(b) $\vec{v}_{P/A} - \vec{v}_{P/B} = \vec{0}$

(c) $\vec{v}_{P/A} \cdot \vec{a}_{P/B} = \vec{v}_{P/B} \cdot \vec{a}_{P/B}$

Solution

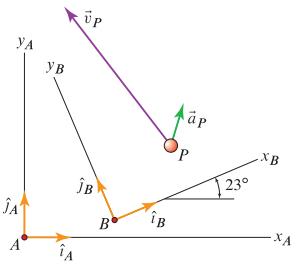
Part (a). Since position vectors are frame dependent, in general we cannot expect them to be identical in different frames. Hence, relation (a) is false.

Part (b). As far as relation (b) is concerned, it is always true. The velocity vector, as a vector, is frame independent and therefore it is always the same whether or not we express it in frame 1 or 2.

Part (c). As far as relation (c) is concerned, again it is always true according to the same argument used in case (b).

Problem 2.24

The velocity of point P relative to frame A is $\vec{v}_{P/A} = (-14.9\,\hat{\imath}_A + 19.4\,\hat{\jmath}_A)$ ft/s, and the acceleration of P relative to frame B is $\vec{a}_{P/B} = (3.97\,\hat{\imath}_B + 4.79\,\hat{\jmath}_B)$ ft/s^2. Knowing that frames A and B do not move relative to one another, determine the expressions for the velocity of P in frame B and the acceleration of P in frame A. Verify that the speed of P and the magnitude of P's acceleration are the same in the two frames.



Solution

The frame A unit vectors can be expressed in terms of frame B as

$$\begin{aligned}\hat{\imath}_A &= \cos 23^\circ\,\hat{\imath}_B - \sin 23^\circ\,\hat{\jmath}_B, & \hat{\jmath}_A &= \sin 23^\circ\,\hat{\imath}_B + \cos 23^\circ\,\hat{\jmath}_B,\\ &= 0.9205\,\hat{\imath}_B - 0.3907\,\hat{\jmath}_B. & &= 0.3907\,\hat{\imath}_B + 0.9205\,\hat{\jmath}_B.\end{aligned}$$

With the frame A unit vectors known in terms of those of frame B, it is found that the velocity in frame B is

$$\vec{v}_{P/B} = \big[-14.9(\cos 23^\circ\,\hat{\imath}_B - \sin 23^\circ\,\hat{\jmath}_B)\big]\text{ ft/s} + \big[19.4(\sin 23^\circ\,\hat{\imath}_B + \cos 23^\circ\,\hat{\jmath}_B)\big]\text{ ft/s},$$

$$\boxed{\vec{v}_{P/B} = (-6.14\,\hat{\imath}_B + 23.7\,\hat{\jmath}_B)\text{ ft/s}.} \tag{1}$$

The speed of the particle in frame A is found by taking the magnitude of the velocity vector $\vec{v}_{P/A}$ where

$$v_{P/A} = \sqrt{(14.9)^2 + (19.4)^2}\text{ ft/s} \quad\Rightarrow\quad \boxed{v_{P/A} = 24.5\text{ ft/s},} \tag{2}$$

Similarly, for frame B,

$$v_{P/B} = \sqrt{(-6.135)^2 + (23.68)^2}\text{ ft/s} \quad\Rightarrow\quad \boxed{v_{P/B} = 24.5\text{ ft/s}.} \tag{3}$$

Therefore, the magnitude of the velocity vector is verified to be independent of the reference frame.
The frame B unit vectors can be expressed in terms of frame A as

$$\begin{aligned}\hat{\imath}_B &= \cos 23^\circ\,\hat{\imath}_A + \sin 23^\circ\,\hat{\jmath}_A, & \hat{\jmath}_B &= -\sin 23^\circ\,\hat{\imath}_A + \cos 23^\circ\,\hat{\jmath}_A,\\ &= 0.9205\,\hat{\imath}_A + 0.3907\,\hat{\jmath}_A. & &= -0.3907\,\hat{\imath}_A + 0.9205\,\hat{\jmath}_A.\end{aligned}$$

With the frame B unit vectors known in terms of those of frame A, we find that the acceleration in frame A is

$$\vec{a}_{P/A} = \big[3.97(\cos 23^\circ\,\hat{\imath}_A + \sin 23^\circ\,\hat{\jmath}_A)\big]\text{ ft/s}^2 + \big[4.79(-\sin 23^\circ\,\hat{\imath}_A + \cos 23^\circ\,\hat{\jmath}_A)\big]\text{ ft/s}^2, \tag{4}$$

$$\boxed{\vec{a}_{P/A} = (1.78\,\hat{\imath}_A + 5.96\,\hat{\jmath}_A)\text{ ft/s}^2.} \tag{5}$$

The magnitue of the acceleration in frame A is the magnitude of the acceleration vector $\vec{a}_{P/A}$:

$$a_{P/A} = \sqrt{(1.783)^2 + (5.960)^2}\text{ ft/s}^2 \quad\Rightarrow\quad \boxed{a_{P/A} = 6.22\text{ ft/s}^2,} \tag{6}$$

Similarly, for frame B,

$$a_{P/B} = \sqrt{(3.97)^2 + (4.79)^2}\text{ ft/s}^2 \quad\Rightarrow\quad \boxed{a_{P/B} = 6.22\text{ ft/s}^2.} \tag{7}$$

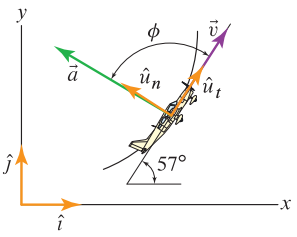
Therefore, the magnitude of the acceleration vector is verified to be independent of the reference frame.

Problem 2.25

At the instant shown, when expressed via the $(\hat{u}_t, \hat{u}_n)$ component system, the airplane's velocity and acceleration are

$$\vec{v} = 135\,\hat{u}_t \text{ m/s} \quad \text{and} \quad \vec{a} = (-7.25\,\hat{u}_t + 182\,\hat{u}_n) \text{ m/s}^2.$$

Treating the $(\hat{u}_t, \hat{u}_n)$ and $(\hat{\imath}, \hat{\jmath})$ component systems as stationary relative to one another, express the airplane's velocity and acceleration in the $(\hat{\imath}, \hat{\jmath})$ component system. Determine the angle ϕ between the velocity and acceleration vectors, and verify that ϕ is the same in the $(\hat{u}_t, \hat{u}_n)$ and $(\hat{\imath}, \hat{\jmath})$ component systems.



Solution

The unit vectors $\hat{u}_n$ and $\hat{u}_t$ need to be expressed in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$:

$$\begin{aligned} \hat{u}_t &= \cos 57^\circ\,\hat{\imath} + \sin 57^\circ\,\hat{\jmath}, & \hat{u}_n &= -\sin 57^\circ\,\hat{\imath} + \cos 57^\circ\,\hat{\jmath}, \\ &= 0.5446\,\hat{\imath} + 0.8387\,\hat{\jmath}. & &= -0.8387\,\hat{\imath} + 0.5446\,\hat{\jmath}. \end{aligned}$$

The velocity in the $(\hat{\imath}, \hat{\jmath})$ component system is

$$\vec{v} = (135 \text{ m/s})(0.5446\,\hat{\imath} + 0.8387\,\hat{\jmath}) = (73.52\,\hat{\imath} + 113.2\,\hat{\jmath}) \text{ m/s}, \tag{1}$$

which, when expressed to three significant figures, gives

$$\boxed{\vec{v} = (73.5\,\hat{\imath} + 113\,\hat{\jmath}) \text{ m/s}.}$$

Similarly, the acceleration vector in the $(\hat{\imath}, \hat{\jmath})$ component system is

$$\vec{a} = (-7.25 \text{ m/s}^2)(0.5446\,\hat{\imath} + 0.8387\,\hat{\jmath}) + (182 \text{ m/s}^2)(-0.8387\,\hat{\imath} + 0.5446\,\hat{\jmath}) = (-156.6\,\hat{\imath} + 93.04\,\hat{\jmath}) \text{ m/s}^2, \tag{2}$$

which, when expressed to three significant figures, gives

$$\boxed{\vec{a} = (-157\,\hat{\imath} + 93.0\,\hat{\jmath}) \text{ m/s}^2.}$$

Using Eqs. (1) and (2), the magnitudes of $\vec{v}$ and $\vec{a}$ in the $(\hat{\imath}, \hat{\jmath})$ component system are

$$|\vec{v}| = \sqrt{(73.52)^2 + (113.2)^2} \text{ m/s} = 135.0 \text{ m/s} \text{ and } |\vec{a}| = \sqrt{(-156.6)^2 + (93.04)^2} \text{ m/s}^2 = 182.2 \text{ m/s}^2.$$

Using the problem's given information, the magnitudes of $\vec{v}$ and $\vec{a}$ in the $(\hat{u}_n, \hat{u}_t)$ component system are

$$|\vec{v}| = 135.0 \text{ m/s}, \quad \text{and} \quad |\vec{a}| = \sqrt{(-7.25)^2 = (182)^2} \text{ m/s}^2 = 182.1 \text{ m/s}^2.$$

One can now determine the angle with the definition of the dot product. Specifically, in the $(\hat{\imath}, \hat{\jmath})$ component system we have

$$\phi_{(\hat{\imath},\hat{\jmath})} = \cos^{-1}\left[\frac{(73.52\,\hat{\imath} + 113.2\,\hat{\jmath}) \cdot (-156.6\,\hat{\imath} + 93.04\,\hat{\jmath})}{(135.0)(182.2)}\right] \quad \Rightarrow \quad \boxed{\phi_{(\hat{\imath},\hat{\jmath})} = 92.3^\circ.}$$

In the $(\hat{u}_t, \hat{u}_n)$ component system we have

$$\phi_{(\hat{u}_t,\hat{u}_n)} = \cos^{-1}\left[\frac{135\,\hat{u}_t \cdot (-7.25\,\hat{u}_t + 182\,\hat{u}_n)}{(135.0)(182.1)}\right] \quad \Rightarrow \quad \boxed{\phi_{(\hat{u}_t,\hat{u}_n)} = 92.3^\circ.}$$

The above two results allow us to say that indeed the two angles computed are identical in the two frames.

Problem 2.26

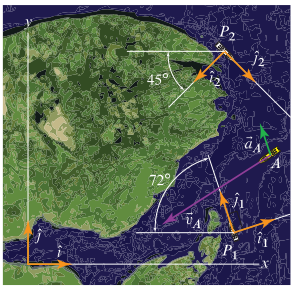
Two Coast Guard patrol boats P_1 and P_2 are stationary while monitoring the motion of a surface vessel A. The velocity of A with respect to P_1 is expressed by

$$\vec{v}_A = (-23\,\hat{\imath}_1 - 6\,\hat{\jmath}_1)\ \text{ft/s},$$

whereas the acceleration of A, expressed relative to P_2, is given by

$$\vec{a}_A = (-2\,\hat{\imath}_2 - 4\hat{\jmath}_2)\ \text{ft/s}^2.$$

Determine the velocity and the acceleration of A expressed with respect to the land-based component system $(\hat{\imath}, \hat{\jmath})$.



Solution

Define $\theta = 72°$. The unit vectors of P_1 can be expressed in terms of the $(\hat{\imath}, \hat{\jmath})$ component system as

$$\begin{aligned}\hat{\imath}_1 &= \sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}, & \hat{\jmath}_1 &= -\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath},\\ &= 0.9511\,\hat{\imath} + 0.3090\,\hat{\jmath}. & &= -0.3090\,\hat{\imath} + 0.9511\,\hat{\jmath}.\end{aligned}$$

The velocity of the surface vessel A in the land-based frame is

$$\vec{v}_A = \big[-23(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) - 6(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\big]\ \text{ft/s},$$

$$\boxed{\vec{v}_A = (-20.0\,\hat{\imath} - 12.8\,\hat{\jmath})\ \text{ft/s}.}$$

Define $\beta = 45°$. The unit vectors of P_2 can be expressed in terms of the $(\hat{\imath}, \hat{\jmath})$ component system as

$$\begin{aligned}\hat{\imath}_2 &= -\cos\beta\,\hat{\imath} - \sin\beta\,\hat{\jmath}, & \hat{\jmath}_2 &= \sin\beta\,\hat{\imath} - \cos\beta\,\hat{\jmath},\\ &= -\frac{1}{\sqrt{2}}\,\hat{\imath} - \frac{1}{\sqrt{2}}\,\hat{\jmath}, & &= \frac{1}{\sqrt{2}}\,\hat{\imath} - \frac{1}{\sqrt{2}}\,\hat{\jmath},\\ &= -0.7071\,\hat{\imath} - 0.7071\,\hat{\jmath}. & &= 0.7071\,\hat{\imath} - 0.7071\,\hat{\jmath}.\end{aligned}$$

As done for the velocity vector, the acceleration vector in the land-based frame is found to be

$$\vec{a}_A = \big[-2\,(-\cos\beta\,\hat{\imath} - \sin\beta\,\hat{\jmath}) - 4(\sin\beta\,\hat{\imath} - \cos\beta\,\hat{\jmath})\big]\ \text{ft/s}^2,$$

$$\boxed{\vec{a}_A = \left(-\sqrt{2}\,\hat{\imath} + 3\sqrt{2}\,\hat{\jmath}\right)\text{ft/s}^2 = (-1.41\,\hat{\imath} + 4.24\,\hat{\jmath})\ \text{ft/s}^2.} \qquad (1)$$

Problem 2.27

For a particle moving along a straight line, the table reports the particle's position x as a function of time. Determine the average velocity between every pair of consecutive time values for this motion, using Eq. (2.6). Provide a plot of the average velocity as a function of time.

t (s)	x (m)	t (s)	x (m)	t (s)	x (m)
0.00	0.000	1.00	1.344	2.00	1.193
0.20	0.331	1.20	1.458	2.20	0.963
0.40	0.645	1.40	1.500	2.40	0.686
0.60	0.928	1.60	1.468	2.60	0.375
0.80	1.165	1.80	1.364	2.80	0.046

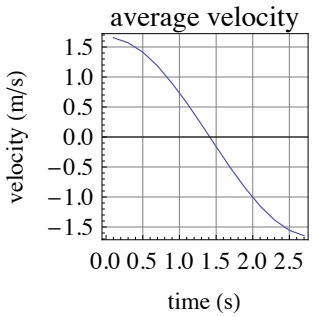
Solution

The following *Mathematica* code will find and plot the average velocity:

```
data = {{0.0, 0.0}, {0.2, 0.331}, {0.4, 0.645}, {0.6, 0.928}, {0.8, 1.165},
   {1.0, 1.344}, {1.2, 1.458}, {1.4, 1.5}, {1.6, 1.468}, {1.8, 1.364}, {2.0, 1.193},
   {2.2, 0.963}, {2.4, 0.686}, {2.6, 0.375}, {2.8, 0.046}};
data2 = Drop[data, 1];
data1 = Drop[data, -1];

Δr = Transpose[data2 - data1][[2]]
Δt = Transpose[data2 - data1][[1]]
tavg = Transpose[(data2 + data1) / 2][[1]]
vavg = Δr / Δt

averagev = Table[0, {Length[vavg]}];
Do[averagev[[C]] = ({ tavg[[C]], vavg[[C]] }), {C, Length[vavg]}]
ListPlot[{averagev}, Joined → True, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"time (s)", "velocity (m/s)"},
 PlotLabel → "average velocity"]
```



Problem 2.28

Continue Prob. 2.27 by treating the average velocities as if they the were the true velocities, and compute the average accelerations corresponding to every pair of consecutive time values as was done in Example 2.6 on p. 43. Provide a plot of the average acceleration as a function of time.

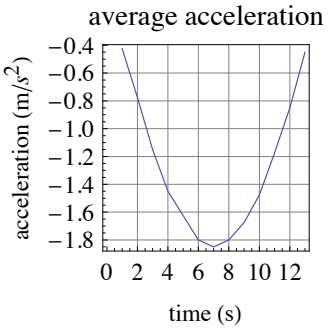
Solution

The following *Mathematica* code must be added onto the notebook created for problem 2.27. The average acceleration is found and plotted with:

```
averagev = Table[0, {Length[vavg]}];
Do[averagev[[C]] = ({ tavg[[C]], vavg[[C]] }), {C, Length[vavg]}]
velocity2 = Drop[averagev, 1];
velocity1 = Drop[averagev, -1];

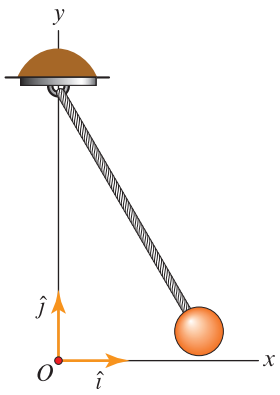
Δv = Transpose[velocity2 - velocity1][[2]];
Δt = Transpose[velocity2 - velocity1][[1]];
tavg2 = Transpose[(velocity2 + velocity1) / 2][[1]]
acelavg = Δv / Δt

ListPlot[{acelavg}, Joined → True, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, AspectRatio → 1,
 GridLines → Automatic, FrameLabel → {"time (s)", "acceleration (m/s^2)"},
 PlotLabel → "average acceleration"]
```



Problem 2.29

The table gives the position vs. time data for a pendulum swinging in the xy plane. Compute the displacement between $t = 0.0\,\text{s}$ and $t = 0.539\,\text{s}$ and between $t = 0.0\,\text{s}$ and $t = 2.023\,\text{s}$. Furthermore, compute the average velocity over the given time intervals. Knowing that the data in the table below concerns a swinging pendulum, interpret the result you obtain for the average velocity between $t = 0.0\,\text{s}$ and $t = 2.023\,\text{s}$.



Time (s)	x (ft)	y (ft)	Time (s)	x (ft)	y (ft)
0.000	1.693	0.530	1.079	-1.679	0.491
0.135	1.641	0.439	1.214	-1.485	0.361
0.270	1.164	0.218	1.348	-0.898	0.129
0.405	0.627	0.052	1.483	-0.219	0.013
0.539	-0.222	0.005	1.618	0.586	0.038
0.674	-0.962	0.124	1.753	1.148	0.240
0.809	-1.489	0.343	1.888	1.542	0.435
0.944	-1.768	0.514	2.023	1.809	0.528

Solution

The displacement for the two time intervals can be calculated as

$$\Delta\vec{r}(0\,\text{s},\ 0.539\,\text{s}) = \vec{r}(0.539\,\text{s}) - \vec{r}(0\,\text{s}) = [x(0.539\,\text{s}) - x(0\,\text{s})]\,\hat{\imath} + [y(0.539\,\text{s}) - y(0\,\text{s})]\,\hat{\jmath},$$

$$\Rightarrow \quad \boxed{\Delta\vec{r}(0\,\text{s},\ 0.539\,\text{s}) = (-1.92\,\hat{\imath} - 0.525\,\hat{\jmath})\,\text{ft}.} \tag{1}$$

$$\Delta\vec{r}(0\,\text{s},\ 2.023\,\text{s}) = \vec{r}(2.023\,\text{s}) - \vec{r}(0\,\text{s}) = [x(2.023\,\text{s}) - x(0\,\text{s})]\,\hat{\imath} + [y(2.023\,\text{s}) - y(0\,\text{s})]\,\hat{\jmath},$$

$$\Rightarrow \quad \boxed{\Delta\vec{r}(0\,\text{s},\ 2.023\,\text{s}) = (0.116\,\hat{\imath} - 0.002\,\hat{\jmath})\,\text{ft}.} \tag{2}$$

The average velocity can then be calculated as

$$\vec{v}_{\text{avg}}(0\,\text{s},\ 0.539\,\text{s}) = \Delta\vec{r}(0\,\text{s},\ 0.539\,\text{s})/(0.539 - 0), \tag{3}$$

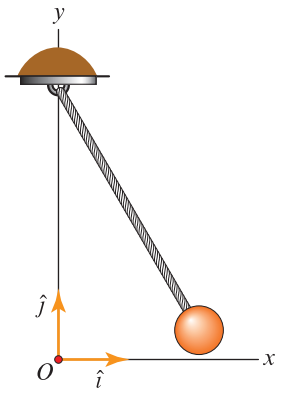
$$\Rightarrow \quad \boxed{\vec{v}_{\text{avg}}(0\,\text{s},\ 0.539\,\text{s}) = (-3.55\,\hat{\imath} - 0.974\,\hat{\jmath})\,\text{ft/s}.} \tag{4}$$

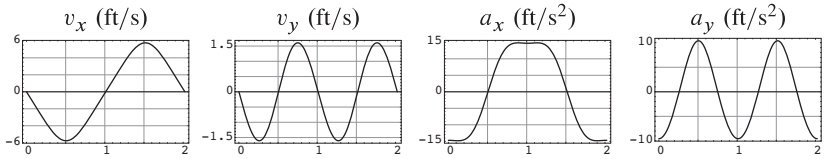
$$\vec{v}_{\text{avg}}(0\,\text{s},\ 2.023\,\text{s}) = \Delta\vec{r}(0\,\text{s},\ 2.023\,\text{s})/(2.023 - 0), \tag{5}$$

$$\Rightarrow \quad \boxed{\vec{v}_{\text{avg}}(0\,\text{s},\ 2.023\,\text{s}) = (0.0573\,\hat{\imath} - 9.89 \times 10^{-4}\,\hat{\jmath})\,\text{ft/s}.} \tag{6}$$

Problem 2.30

The table in Prob. 2.29 gives the position vs. time data for a pendulum swinging in the xy plane. Compute the components of the average velocity as well as the magnitude of the average velocity over each time step and plot these quantities vs. time. Furthermore, compute the components of the approximate acceleration as was done in Example 2.6 on p. 43 by using the average velocity data generated and plot the results vs. time. Finally compare the results to the plots of the components of the exact velocity and acceleration vs. time shown below. In these plots the vertical axes represent the quantity labeling each plot whereas the horizontal axes represent time expressed in seconds.





Solution

The magnitude of the velocity vector is computed using the equation,

$$v_{\text{ave}}(t_i) = \sqrt{v_x^2(t_i) + v_y^2(t_i)}. \tag{1}$$

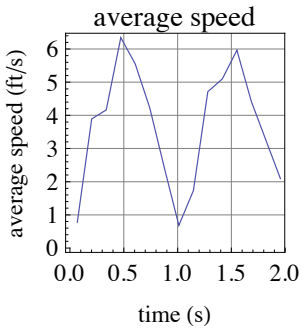
Using *Mathematica* we generate the plot for the magnitude with

```
data = {{0.0, 1.693, 0.530}, {0.135, 1.641, 0.439}, {0.27, 1.164, 0.218},
   {0.405, 0.627, 0.052}, {0.539, -0.222, 0.005}, {0.674, -0.962, 0.124},
   {0.809, -1.489, 0.343}, {0.944, -1.768, 0.514}, {1.079, -1.679, 0.491},
   {1.214, -1.485, 0.361}, {1.348, -0.898, 0.129 }, {1.483, -0.219, 0.013},
   {1.618, 0.586, 0.038}, {1.753, 1.148, 0.240}, {1.888, 1.542, 0.435 },
   {2.023, 1.809, 0.528}};

data2 = Drop[data, 1];
data1 = Drop[data, -1];
Δt = Transpose[data2 - data1][[1]];
Δx = Transpose[data2 - data1][[2]];
Δy = Transpose[data2 - data1][[3]];

vx = Δx / Δt;
vy = Δy / Δt;
tavg = Transpose[(data1 + data2) / 2][[1]];
speed = Sqrt[vx^2 + vy^2];

speedplot = Table[0, {Length[tavg]}];
Do[speedplot[[c]] = {tavg[[c]], speed[[c]]}, {c, 1, Length[tavg]}]
ListPlot[{plotvelocityy}, Joined → True, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, ImageSize → 130, FrameLabel → {"time (s)", "average speed (ft/s)"},
 PlotLabel → "average speed"]
```



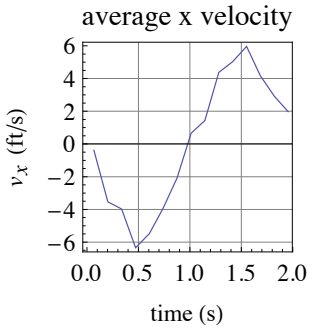
The average of each component of the velocity vector is calculated for each interval i, between time-steps $t = i$ and $t = (i + 1)$, through the equations

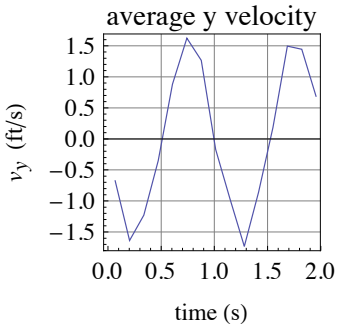
$$(v_x)_i = \frac{[x(t_{i+1}) - x(t_i)]}{(t_{i+1} - t_i)}, \qquad (v_y)_i = \frac{[y(t_{i+1}) - y(t_i)]}{(t_{i+1} - t_i)}.$$

Using the same *Mathematica* notebook we generate the plots for the x and y components of velocity with

```
plotvelocityx = Table[0, {Length[tavg]}];
Do[plotvelocityx[[C]] = {tavg[[C]], vx[[C]]}, {C, 1, Length[tavg]}]
ListPlot[{plotvelocityx}, Joined → True, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, ImageSize → 130, FrameLabel → {"time (s)", "vx (ft/s)"},
 PlotLabel → "average x velocity"]

plotvelocityy = Table[0, {Length[tavg]}];
Do[plotvelocityy[[C]] = {tavg[[C]], vy[[C]]}, {C, 1, Length[tavg]}]
ListPlot[{plotvelocityy}, Joined → True, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, ImageSize → 130, FrameLabel → {"time (s)", "vy (ft/s)"},
 PlotLabel → "average y velocity"]
```





The average acceleration components in each direction can be calculated, using the velocity components, from the equations

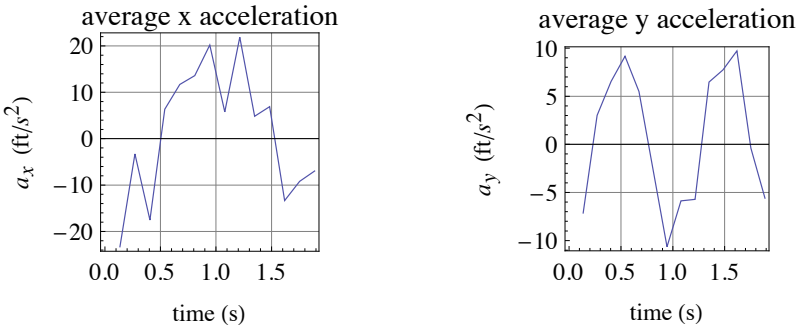
$$(a_x)_i = \frac{[v_x(t_{i+1}) - v_x(t_i)]}{t_{i+1} - t_i}, \qquad (a_y)_i = \frac{[v_y(t_{i+1}) - v_y(t_i)]}{t_{i+1} - t_i}, \tag{2}$$

Using the same *Mathematica* notebook we generate the plot for the x and y components of acceleration with the following code

```
Δt2 = Table[0, {Length[tavg] - 1}];
Do[Δt2[[C]] = tavg[[C + 1]] - tavg[[C]], {C, 1, Length[tavg] - 1}]
t2avg = Table[0, {Length[tavg] - 1}];
Do[t2avg[[C]] = (tavg[[C + 1]] + tavg[[C]]) / 2, {C, 1, Length[tavg] - 1}]

ax = Table[0, {Length[tavg] - 1}];
Do[ax[[C]] = {t2avg[[C]], (vx[[C + 1]] - vx[[C]]) / Δt2[[C]]}, {C, 1, Length[tavg] - 1}]
ListPlot[{ax}, Joined → True, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, ImageSize → 130, FrameLabel → {"time (s)", "a_x (ft/s^2)"},
 PlotLabel → "average x acceleration"]

ay = Table[0, {Length[tavg] - 1}];
Do[ay[[C]] = {t2avg[[C]], (vy[[C + 1]] - vy[[C]]) / Δt2[[C]]}, {C, 1, Length[tavg] - 1}]
ListPlot[{ay}, Joined → True, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, ImageSize → 130, FrameLabel → {"time (s)", "a_y (ft/s^2)"},
 PlotLabel → "average y acceleration"]
```



The plots generated via numerical differentiation are typically not smooth because numerical differentiation is an intrinsically “noise producing” process. By contrast, integration is a smoothing process.

Problem 2.31

Let $f(t)$ be a function of time, and suppose that a table of values of $f(t)$ is provided for a sequence of equally spaced time instants. Then, for any three consecutive values of $f(t)$, i.e., $f(t_i)$, $f(t_{i+1})$, and $f(t_{i+2})$, you can approximate the value of the derivative of $f(t)$ with respect to time at $t = t_i$ by using the formula

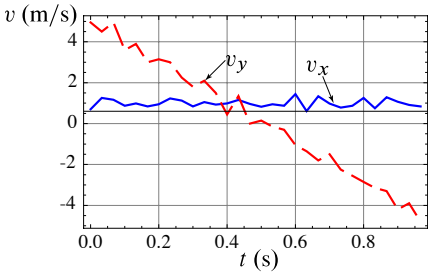
$$\frac{df}{dt}(t_i) \approx \frac{-f(t_{i+2}) + 4f(t_{i+1}) - 3f(t_i)}{2\,\Delta t}, \tag{1}$$

where $\Delta t = t_{i+1} - t_i = t_{i+2} - t_{i+1}$. Use this formula to compute derivatives, and rework Example 2.6 to obtain new plots for the velocity and the acceleration. Does the formula given above allow you to obtain smoother plots for the velocity and acceleration with respect to those in Example 2.6?

Solution

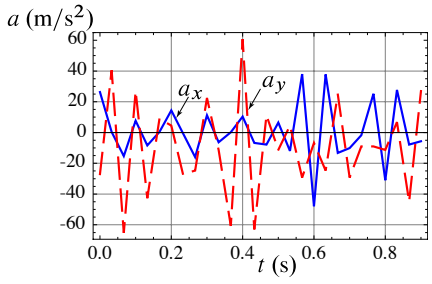
Because of the formula given in Eq. (1), to calculate the velocity at the current time t_i we need data at the next two future time points t_{i+1} and t_{i+2}. This means we can only calculate velocities for the first 30 data points. As an example of the application of the given formula, the v_x component at $t_0 = 0.000\,\text{s}$ can be calculated as:

$$\begin{aligned} v_x(t_0) &\approx \frac{-x(t_2) + 4x(t_1) - 3x(t_0)}{\Delta t} \\ &= \frac{-0.069 + 4(0.029) - 3(0.000)}{2(1/30)} = 0.705\,\text{m/s}. \end{aligned}$$



Finding the acceleration is similar to calculating velocities. Again as an example of the application of the given formula, the a_x component at $t_0 = 0.000\,\text{s}$ can be calculated as:

$$\begin{aligned} a_x(t_0) &\approx \frac{-v_x(t_2) + 4v_x(t_1) - 3v_x(t_0)}{2\Delta t} \\ &= \frac{-1.17 + 4(1.26) - 3(0.705)}{2(1/30)} = 26.3\,\text{m/s}^2. \end{aligned}$$



The results we generated produced plots similar to Figs. 3 and 4 given in Example 2.6 in the text. Our acceleration plot appears to be worse than the one in the example because the peaks are higher. This is an example of the difficulty of numerical differentiation.

Problem 2.32

Find the x and y components of the acceleration in Example 2.5 (except for the plots) by simply differentiating Eqs. (4) and (5) with respect to time. Verify that you get the results given in Example 2.5.

Solution

The x component of acceleration is found with:

$$\dot{x} = \frac{v_0 y}{\sqrt{y^2 + 4a^2}} \quad \Rightarrow \quad \ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dy}\frac{dy}{dt} = \dot{y}\frac{d\dot{x}}{dy}$$

$$\ddot{x} = -\frac{v_0 y^2 \dot{y}}{\left(4a^2 + y^2\right)^{3/2}} + \frac{v_0 \dot{y}}{\sqrt{4a^2 + y^2}},$$

$$= \frac{4a^2 v_0 \dot{y}}{\left(y^2 + 4a^2\right)^{3/2}}$$

Substituting the equation for $\dot{y}$ gives us

$$\dot{y} = \frac{2v_0 a}{\sqrt{y^2 + 4a^2}} \quad \Rightarrow \quad \boxed{\ddot{x} = \frac{8a^3 v_0^2}{\left(4a^2 + y^2\right)^2}.}$$

The y component of acceleration is found with:

$$\dot{y} = \frac{2v_0 a}{\sqrt{y^2 + 4a^2}} \quad \Rightarrow \quad \ddot{y} = \frac{d\dot{y}}{dt} = \frac{d\dot{y}}{dy}\frac{dy}{dt} = \dot{y}\frac{d\dot{y}}{dy}$$

$$\ddot{y} = -\frac{2a v_0 y \dot{y}}{\left(4a^2 + y^2\right)^{3/2}},$$

Substituting the equation for $\dot{y}$ gives us

$$\dot{y} = \frac{2v_0 a}{\sqrt{y^2 + 4a^2}} \quad \Rightarrow \quad \boxed{\ddot{y} = -\frac{4a^2 v_0^2 y}{\left(4a^2 + y^2\right)^2}.}$$

Problem 2.33

Find the x and y components of the acceleration in Example 2.5 (except for the plots) by differentiating the first of Eqs. (3) and the last of Eqs. (1) with respect to time and then solving the resulting two equations for $\ddot{x}$ and $\ddot{y}$. Verify that you get the results given in Example 2.5.

Solution

$$\text{First of Eq. (3) from example:} \quad v_0^2 = \dot{x}^2 + \dot{y}^2 \quad \Rightarrow \quad 0 = 2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}, \tag{1}$$

$$\text{Last of Eq. (1) from example:} \quad y\dot{y} = 2a\dot{x} \quad \Rightarrow \quad 2a\ddot{x} = \dot{y}^2 + y\ddot{y}, \tag{2}$$

Solve the first of Eqs. (1) and (2) to find $\dot{x}$ and $\dot{y}$:

$$\dot{x} = \frac{y\dot{y}}{2a} \quad \Rightarrow \quad v_0^2 = \left(\frac{y^2}{4a^2}\right)\dot{y}^2 + \dot{y}^2 \quad \Rightarrow \quad \dot{y} = \frac{2av_0}{\sqrt{y^2 + 4a^2}}. \tag{3}$$

$$\dot{x} = \frac{y\dot{y}}{2a} \quad \Rightarrow \quad \dot{x} = \frac{yv_0}{\sqrt{y^2 + 4a^2}}. \tag{4}$$

Solving the second of Eqs. (1) and (2) simultaneously for the acceleration components yields

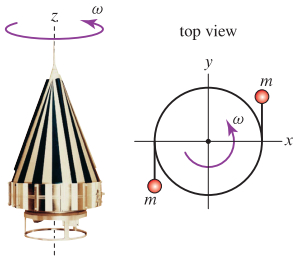
$$\ddot{x} = \frac{\dot{y}^3}{y\dot{x} + 2a\dot{y}}, \qquad \ddot{y} = -\frac{\dot{x}\dot{y}^2}{y\dot{x} + 2a\dot{y}}. \tag{5}$$

Substituting Eqs. (3) and (4) and simplifying yields the same results as Example 2.5.

$$\boxed{\ddot{x} = \frac{8a^3v_0^2}{\left(4a^2 + y^2\right)^2}, \qquad \ddot{y} = -\frac{4a^2v_0^2y}{\left(4a^2 + y^2\right)^2}.} \tag{6}$$

Problem 2.34

Pioneer 3 was a spin-stabilized spacecraft launched on 6 December 1958 by the U.S. Army Ballistic Missile agency in conjunction with NASA. It was a cone-shaped probe 58 cm high and 25 cm in diameter at its base. It was designed with a despin mechanism consisting of two 7 g masses (m in the figure) that could be spooled out to the end of two 150 cm wires when triggered by a hydraulic timer 10 h after launch. As they are deployed, the masses slow the spacecraft spin rate from its initial value to a desired one. The table below reports discrete acceleration vs. time data of one of the masses as it is deployed in a test run in which the spacecraft is kept with its z axis vertical and the masses deploy in the xy plane. Follow the steps described in Example 2.7, and reconstruct the velocity as well as the position of the deployed mass as a function of time. Finally plot the trajectory of the mass. Use the initial conditions $x(0) = 0.125\,\text{m}$, $y(0) = 0\,\text{m}$, $\dot{x}(0) = 0\,\text{m/s}$, and $\dot{y}(0) = 1.25\,\text{m/s}$.



Time (s)	a_x (m/s^2)	a_y (m/s^2)	Time (s)	a_x (m/s^2)	a_y (m/s^2)
0.00	0.0	0.0	0.16	4.1	-78.7
0.02	-3.9	9.2	0.18	38.3	-79.6
0.04	-14.3	13.9	0.20	73.1	-64.9
0.06	-27.8	10.8	0.22	101.0	-34.7
0.08	-39.8	-1.1	0.24	116.0	7.5
0.10	-45.2	-20.6	0.26	112.0	55.5
0.12	-40.3	-43.7	0.28	88.5	101.0
0.14	-23.5	-65.0	0.30	46.1	135.0

Solution

Following the methodology of Example 2.7, the velocity of one of the masses is calculated by *Mathematica* (we could have used other mathematical software) with the following code for the x component:

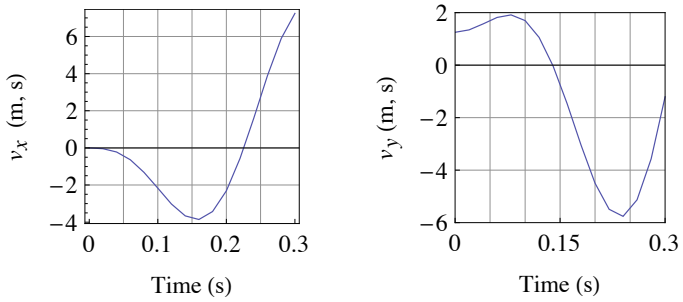
```
ProblemData = {{0, 0, 0}, {0.02, -3.9, 9.2}, {0.04, -3.9, 9.2}, {0.06, -27.8, 10.8},
   {0.08, -39.8, -1.1}, {0.10, -45.2, -20.6}, {0.12, -40.3, -43.7}, {0.14, -23.5, -65.0},
   {0.16, 4.1, -78.7}, {0.18, 38.3, -79.6}, {0.20, 73.1, -64.9}, {0.22, 101.0, -34.7},
   {0.24, 116.0, 7.5}, {0.26, 112.0, 55.5}, {0.28, 88.5, 101.0}, {0.30, 46.1, 135.0}};

SampleTime = 0.3 / (NumberofSamples - 1);

v0x = 0;

velx =
  Table[SampleTime/2 (ProblemData[[1, 2]] + ProblemData[[n, 2]]) +
     SampleTime Sum[ProblemData[[k, 2]], {k, 2, n-1}], {n, 2, Length[ProblemData]}] + v0x;
```

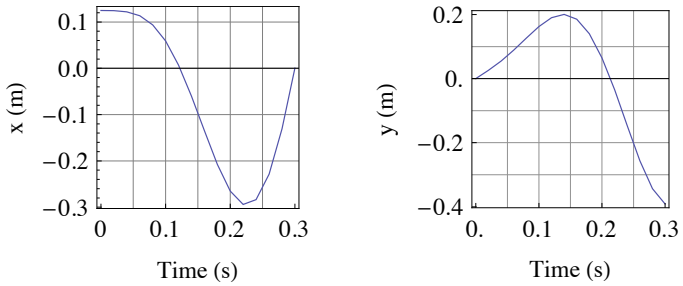
```
velxComplete = Prepend[velx, v0x];
ListPlot[Transpose[{Transpose[ProblemData][[1]], velxComplete}], PlotRange → All,
 GridLines → {Automatic, Automatic},
 FrameTicks → {{Automatic, None}, {{0, 0.15, 0.3}, None}}, Frame → True,
 Joined → True, AspectRatio → 1]
```



In order to determine the position of the spinning mass, the CTR is employed to integrate the velocity data. The x component of position and the trajectory are calculated by *Mathematica* with the following code:

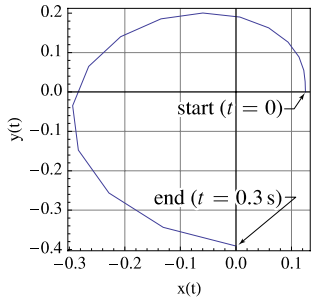
```
posxComplete = Prepend[posx, x0];
posyComplete = Prepend[posy, y0];

ListPlot[Transpose[{Transpose[ProblemData][[1]], posxComplete}], Joined → True,
 PlotRange → All, GridLines → {Automatic, Automatic}, ImageSize → 130,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, FrameLabel → {"Time (s)", "x(m)"},
 Frame → True, AspectRatio → 1]
```



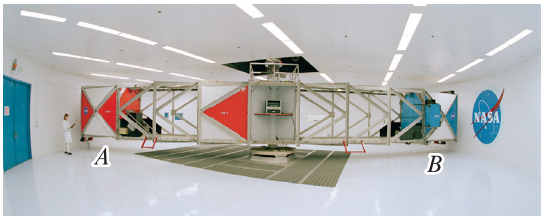
An alternative way to view the position of the mass is by plotting its trajectory:

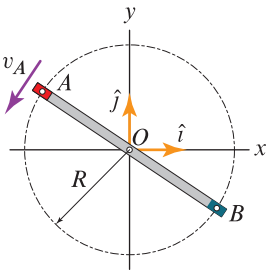
```
ListPlot[Transpose[{posxComplete, posyComplete}], Joined → True, PlotRange → All,
 GridLines → {Automatic, Automatic}, ImageSize → 130,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, FrameLabel → {"x(t)", "y(t)"},
 Frame → True, AspectRatio → 1]
```



Problem 2.35

The Center for Gravitational Biology Research at NASA's Ames Research Center runs a large centrifuge capable of $20g$ of acceleration ($12.5g$ is the maximum for human subjects). The distance from the axis of rotation to the cab at either A or B is $R = 25\,\text{ft}$. The trajectory of A is described by $y_A = \sqrt{R^2 - x_A^2}$ for $y_A \geq 0$ and by $y_A = -\sqrt{R^2 - x_A^2}$ for $y_A < 0$. If A moves at the constant speed $v_A = 120\,\text{ft/s}$, determine the velocity and acceleration of A when $x_A = -20\,\text{ft}$ and $y_A > 0$.





Solution

Starting with the equation of the trajectory for $y > 0$, and differentiating it with respect to time, we have

$$y_A = \sqrt{R^2 - x_A^2} \quad \Rightarrow \quad \dot{y}_A = \frac{-x_A \dot{x}_A}{\sqrt{R^2 - x_A^2}}. \tag{1}$$

Recalling that the speed can be computed as $v = \sqrt{\dot{x}_A^2 + \dot{y}_A^2}$, using Eq. (1), we can express the quantity v^2 as

$$\dot{x}_A^2 + \dot{y}_A^2 = v_A^2 \quad \Rightarrow \quad \dot{x}_A = \frac{-v_A}{R}\sqrt{R^2 - x_A^2} \quad \Rightarrow \quad \dot{y}_A = \frac{v_A x_A}{R}, \tag{2}$$

where we have substituted the second of Eqs. (2) into the last of Eqs. (1) to obtain the last of Eqs. (2). Then, recalling that $v_A = 120\,\text{ft/s}$, $R = 25\,\text{ft}$, and $x_A = -25\,\text{ft}$, evaluating the last two of Eqs. (2), we have

$$\boxed{\vec{v}_A = -(72.0\,\hat{\imath} + 96.0\,\hat{\jmath})\,\text{ft/s}.}$$

Next, differentiating the last of Eqs. (2) with respect to time, we have

$$\ddot{y}_A = \frac{\dot{x}_A v_A}{R} \quad \Rightarrow \quad \ddot{y}_A = \frac{-v_A^2}{R^2}\sqrt{R^2 - x_A^2}. \tag{3}$$

Because the speed is constant, differentiating the first of Eqs. (2) with respect to time, we have

$$2\dot{x}_A \ddot{x}_A + 2\dot{y}_A \ddot{y}_A = 0 \quad \Rightarrow \quad \ddot{x}_A = \frac{-\dot{y}_A \ddot{y}_A}{x_A}. \tag{4}$$

Substituting the last of Eqs. (2) and the last of Eqs. (3) into the last of Eqs. (4), we have

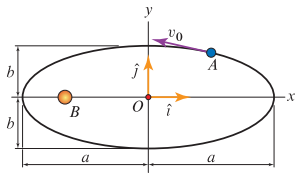
$$\ddot{x}_A = \frac{-x_A v_A^2}{R^2}. \tag{5}$$

Finally, recalling that $v_A = 120\,\text{ft/s}$, $R = 25\,\text{ft}$, and $x_A = -25\,\text{ft}$, evaluating Eq. (5) and the last of Eqs. (3), we have

$$\boxed{\vec{a}_A = (461\,\hat{\imath} - 346\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 2.36

The orbit of a satellite A around planet B is the ellipse shown and is described by the equation $(x/a)^2 + (y/b)^2 = 1$, where a and b are the semimajor and semiminor axes of the ellipse, repectively. When $x = a/2$ and $y > 0$, the satellite is moving with a speed v_0 as shown. Determine the expression for the satellite's velocity $\vec{v}$ in terms of v_0, a, and b for $x = a/2$ and $y > 0$.



Solution

Take the derivative with respect to time of the given equation:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \Rightarrow \quad \frac{x\dot{x}}{a^2} + \frac{y\dot{y}}{b^2} = 0. \tag{1}$$

Substitute $x = a/2$ into $(x/a)^2 + (y/b)^2 = 1$ to find $y = (b\sqrt{3})/2$. Then,substituting the expressions for x and y into the second of Eqs. (1), we have

$$\dot{x} + \frac{a\sqrt{3}\dot{y}}{b} = 0. \tag{2}$$

Now recall that the speed v_0 is given by $v_0 = \sqrt{\dot{x}^2 + \dot{y}^2}$. Therefore, solving Eq. (2) for $\dot{x}$ and substituting the result in the expression for the speed squared, i.e., $\dot{x}^2 + \dot{y}^2 = v_0^2$, we have

$$\frac{3a^2\dot{y}^2}{b^2} + \dot{y}^2 = v_0^2 \quad \Rightarrow \quad \dot{y} = \frac{v_0 b}{\sqrt{3a^2 + b^2}}. \tag{3}$$

Substituting the last of Eqs. (3) into (2), we find

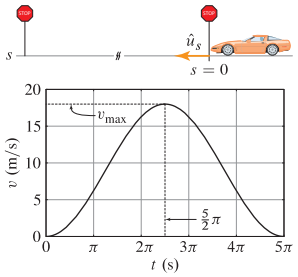
$$\dot{x} = \frac{-\sqrt{3}a v_0}{\sqrt{3a^2 + b^2}}. \tag{4}$$

Then, recalling that the velocity is given by $\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$, using Eq. (4) and the last of Eqs. (3), we can express the velocity as

$$\boxed{\vec{v} = \frac{v_0}{\sqrt{3a^2 + b^2}}\left(-\sqrt{3}a\,\hat{\imath} + b\,\hat{\jmath}\right).}$$

Problems 2.37 through 2.40

The following four problems refer to the car traveling between two stop signs presented at the beginning of this section on p. 55, in which the car's velocity is assumed to be given by $v = [9 - 9\cos(2t/5)]$ m/s for $0 \le t \le 5\pi$ s.



Problem 2.37 Determine $v_{\max}$, the maximum velocity reached by the car. Furthermore, determine the position $s_{v_{\max}}$ and the time $t_{v_{\max}}$ at which $v_{\max}$ occurs.

Problem 2.38 Determine the time at which the brakes are applied and the car starts to slow down.

Problem 2.39 Determine the average velocity of the car between the two stop signs.

Problem 2.40 Determine $|a|_{\max}$, the maximum of the magnitude of the acceleration reached by the car, and determine the position(s) at which $|a|_{\max}$ occurs.

Solution to 2.37

To find $v_{\max}$, we first differentiate v with respect to time and set it equal to zero to find the extrema.

$$a = \frac{dv}{dt} = 0 \quad \Rightarrow \quad \frac{18}{5}\sin\left(\frac{2t}{5}\right) \text{m/s}^2 = 0 \quad \Rightarrow \quad \sin\left(\frac{2t}{5}\right) = 0 \quad \Rightarrow \quad t = 0, \frac{5\pi}{2}\text{ s}, 5\pi\text{ s}, \ldots \tag{1}$$

Find v at the only meaningful solution $t = (5\pi)/2$ s:

$$v\left(\frac{5\pi}{2}\text{ s}\right) = \left[9 - 9\cos\left(\frac{2}{5}\frac{5\pi}{2}\right)\right]\text{m/s} = 18.0\,\text{m/s}. \tag{2}$$

$$\boxed{t_{v_{\max}} = \frac{5\pi}{2}\text{ s} = 7.85\,\text{s} \quad \Rightarrow \quad v_{\max} = 18.0\,\text{m/s}.} \tag{3}$$

To find the position as a function of time, $s(t)$, we integrate

$$s = \int_0^t v(t)\,dt \quad \Rightarrow \quad s = \left[9t - \frac{45}{2}\sin\left(\frac{2t}{5}\right) + C\right]\text{m}, \tag{4}$$

where C is a constant of integration. To determine C, we apply the initial condition according to which $s(0) = 0$ m/s. This implies that $C = 0$. We then find the position at the maximum velocity to be

$$\boxed{s_{v_{\max}} = \frac{45\pi}{2}\text{ m} = 70.7\,\text{m}.} \tag{5}$$

Solution to 2.38

The brakes are applied when the velocity is maximum (when the acceleration changes sign).

$$\frac{dv}{dt} = 0 \quad \Rightarrow \quad \frac{18}{5}\sin\left(\frac{2t}{5}\right)\text{m/s}^2 = 0. \tag{6}$$

The only meaningful solution is:

$$\boxed{t_{\text{braking}} = \frac{5\pi}{2}\,\text{s} = 7.85\,\text{s}.} \tag{7}$$

Solution to 2.39

To find the average velocity, we need to determine the total change in position and the time it took for that change in position to occur. We start by finding the time over which the car was in motion. Since we know that the car started moving at time $t = 0$, we need to determine when the car first came to a stop. The car stops when $v = [9 - 9\cos(2t/5)]\,\text{m/s} = 0$. This occurs when $t = 5\pi$ s.

To compute the change in position, we first determine $s(t)$, the position as a function of time by integrating the expression for the velocity, i.e.,

$$s = \int_0^t v(t)\,dt \quad \Rightarrow \quad s = \left[9t - \frac{45}{2}\sin\left(\frac{2t}{5}\right) + C\right]\text{m} = \left[9t - \frac{45}{2}\sin\left(\frac{2t}{5}\right)\right]\text{m}, \tag{8}$$

where the constant of integration C was found to be equal to zero by applying the initial condition according to which $s(0) = 0\,\text{m/s}$. Next, using the above expression for $s(t)$, for $t = 5\pi$ s, we have

$$s(5\pi\text{ s}) = \left[9(5\pi) - \frac{5}{2}9\sin\left(\frac{2(5\pi)}{5}\right)\right]\text{m/s} = 45\pi\text{ m} = 141.4\,\text{m}. \tag{9}$$

The average velocity is then given by

$$v_{\text{avg}} = \frac{s_{\text{final}} - s_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}} = \frac{45\pi\text{ m} - 0\,\text{m}}{5\pi\text{ s} - 0\,\text{s}} \quad \Rightarrow \quad \boxed{v_{\text{avg}} = 9.00\,\text{m/s}.} \tag{10}$$

Solution to 2.40

Since the acceleration is the time derivative of the velocity, we have

$$a = \frac{dv}{dt} = \frac{18}{5}\sin\left(\frac{2t}{5}\right)\text{m/s}^2. \tag{11}$$

To find $a_{\max}$ we differentiate the above expression with respect to time and set it equal to zero to find the extrema.

$$\dot{a} = \frac{da}{dt} = \frac{36}{25}\cos\left(\frac{2t}{5}\right)\text{m/s}^3 = 0 \quad \Rightarrow \quad \cos\left(\frac{2t}{5}\right) = 0 \quad \Rightarrow \quad t = \frac{5\pi}{4}\,\text{s}, \frac{15\pi}{4}\text{s}, \ldots \tag{12}$$

We then find a at these extremal points in our time range of interest and find the maximum magnitude. This gives

$$\left|a\left(\frac{5\pi}{4}\,\text{s}\right)\right| = \left|\frac{18}{5}\sin\left(\frac{\pi}{2}\right)\right|\text{m/s} = \frac{18}{5}\,\text{m/s}^2, \tag{13}$$

$$\left|a\left(\frac{15\pi}{4}\,\text{s}\right)\right| = \left|\frac{18}{5}\sin\left(\frac{3\pi}{2}\right)\right|\text{m/s} = \frac{18}{5}\,\text{m/s}^2. \tag{14}$$

Therefore, expressing our answers to three significant figures, we have

$$\boxed{t_{|a|_{\max}} = 3.93\,\text{s} \quad \text{and} \quad t_{|a|_{\max}} = 11.8\,\text{s} \quad \text{where} \quad |a|_{\max} = 3.60\,\text{m/s}^2.}$$

To determine the position of the car corresponding to the occurrence of the maximum acceleration, we first determine the position as a function of time by integrating the velocity.

$$s = \int_0^t v(t)\,dt \quad \Rightarrow \quad s = \left[9t - \frac{45}{2}\sin\left(\frac{2t}{5}\right) + C\right]\text{m} = \left[9t - \frac{45}{2}\sin\left(\frac{2t}{5}\right)\right]\text{m}, \tag{15}$$

where the constant of integration C was found to be equal to zero by applying the initial condition according to which $s(0) = 0\,\text{m/s}$. Then evaluating $s(t)$ for the $t = (5\pi/4)\,\text{s}$ and $t = (15\pi/4)\,\text{s}$, we have

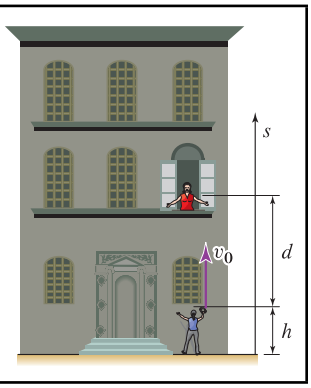
$$s\left(\frac{5\pi}{4}\,\text{s}\right) = \left[9\left(\frac{5\pi}{4}\right) - \frac{45}{2}\sin\frac{2\left(\frac{5}{4}\pi\right)}{5}\right]\text{m} \quad \text{and} \quad s\left(\frac{15\pi}{4}\,\text{s}\right) = \left[9\left(\frac{15\pi}{4}\right) - \frac{45}{2}\sin\frac{2\left(\frac{15}{4}\pi\right)}{5}\right]\text{m}. \tag{16}$$

Therefore, evaluating our result numerically and expressing them to three significant figures, we have

$$\boxed{s_{a_{\max}} = 12.8\,\text{m} \quad \text{and} \quad s_{a_{\max}} = 129\,\text{m}.} \tag{17}$$

Problem 2.41

A ring is thrown straight upward from a height $h = 2.5\,\text{m}$ off the ground and with an initial velocity $v_0 = 3.45\,\text{m/s}$. Gravity causes the ring to have a constant downward acceleration $g = 9.81\,\text{m/s}^2$. Determine $h_{\max}$, the maximum height reached by the ring.



Solution

The release height of the rose is given by $s_0 = 2.5\,\text{m}$. Using the constant acceleration equation, we have

$$v^2 = v_0^2 - 2g(s - s_0), \tag{1}$$

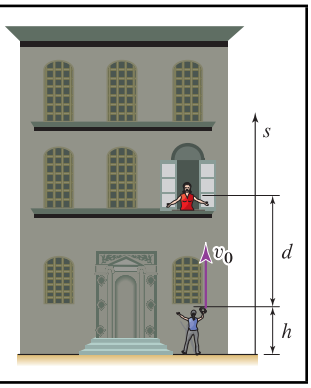
where v_0 is the value of the speed corresponding to the position value s_0. The maximum height is the value of s corresponding to $v = 0$. Hence, setting $s = h_{\max}$ and $v = 0$ in the above equation, and solving for $h_{\max}$, we have

$$h_{\max} = \frac{v_0^2}{2g} + s_0 \quad \Rightarrow \quad \boxed{h_{\max} = 3.11\,\text{m},} \tag{2}$$

where we have used the following numerical data: $v_0 = 3.45\,\text{m/s}$ and $s_0 = 2.5\,\text{m}$.

Problem 2.42

A ring is thrown straight upward from a height $h = 2.5\,\text{m}$ off the ground. Gravity causes the ring to have a constant downward acceleration $g = 9.81\,\text{m/s}^2$. Letting $d = 5.2\,\text{m}$, if the person at the window is to receive the ring in the gentlest possible manner, determine the initial velocity v_0 the ring must be given when first released.



Solution

Substitute $s_0 = h$, $s = (h + d)$, and $v = 0\,\text{m/s}$ into the constant acceleration equation $v^2 = v_0^2 - 2g(s - s_0)$ and solve for v_0:

$$v_0 = \sqrt{2gd} \quad \Rightarrow \quad \boxed{v_0 = 10.1\,\text{m/s},} \tag{1}$$

where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$ and $d = 5.2\,\text{m}$.

Problem 2.43

A car stops 4 s after the application of the brakes while covering a rectilinear stretch 337 ft long. If the motion occurred with a constant acceleration a_c, determine the initial speed v_0 of the car and the acceleration a_c. Express v_0 in mph and a_c in terms of g, the acceleration of gravity.



Solution

Substituting $v = 0$ and $t_0 = 0$ into the constant acceleration equation $v = v_0 + a_c(t - t_0)$, we have

$$0 = v_0 + a_c t \quad \Rightarrow \quad v_0 = -a_c t. \tag{1}$$

Then, we substitute the expression for v_0 in Eq. (1), $s_0 = 0$, and $t_0 = 0$ into the constant acceleration equation $s = s_0 + v_0(t - t_0) + \frac{1}{2}a_c(t - t_0)^2$ to obtain

$$a_c = -\frac{2s}{t^2} = -42.12\,\text{ft/s}^2, \tag{2}$$

where we have used the following numerical data: $s = 337$ ft and $t = 4$ s. Expressing the above result in terms of g, the acceleration due to gravity, and to 3 significant figures, we have

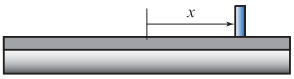
$$a_c = -42.12\,\text{ft/s}^2\left(\frac{g}{32.2\,\text{ft/s}^2}\right) \quad \Rightarrow \quad \boxed{a_c = -1.31g,}$$

Finally, substituting the value of a_c in Eq. (2) into the last of Eqs. (1) and evaluating v_0, we have

$$v_0 = -a_c t = (42.12\,\text{ft/s}^2)(4\,\text{s})\left(\frac{1\,\text{mi}}{5280\,\text{ft}}\right)\left(\frac{3600\,\text{s}}{1\,\text{h}}\right) \quad \Rightarrow \quad \boxed{v_0 = 115\,\text{mph},} \tag{3}$$

where we have used the fact that $t = 4$ s.

Problems 2.44 and 2.45



The motion of a peg sliding within a rectilinear guide is controlled by an actuator in such a way that the peg's acceleration takes on the form $\ddot{x} = a_0(2\cos 2\omega t - \beta \sin \omega t)$, where t is time, $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$.

Problem 2.44 Determine the expressions for the velocity and the position of the peg as functions of time if $\dot{x}(0) = 0\,\text{m/s}$ and $x(0) = 0\,\text{m}$.

Problem 2.45 Determine the total distance traveled by the peg during the time interval $0\,\text{s} \le t \le 5\,\text{s}$ if $\dot{x}(0) = a_0\beta/\omega$.

Solution to 2.44

Integrating the acceleration equation we have

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \ddot{x}(t)\,dt \quad \Rightarrow \quad \dot{x}(t) = \frac{a_0}{\omega}(\sin 2\omega t + \beta \cos \omega t)_0^t$$

$$\Rightarrow \quad \dot{x}(t) = \frac{a_0}{\omega}(\sin 2\omega t + \beta \cos \omega t - \beta),$$

where we have accounted for the fact that $\dot{x}(0) = 0$. Hence, evaluating with above expression, we have,

$$v(t) = 7.00\big[\sin t + 1.50\cos(0.500t) - 1.50\big]\,\text{m/s},$$

where we have expressed known coefficients to three significant figures, and where we have used the following numerical data: $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$.

To determine position as a function of time, we integrate the velocity function obtained above:

$$s(t) = x(t) = x(0) + \int_0^t \dot{x}(t)\,dt \quad \Rightarrow \quad x(t) = \frac{a_0}{\omega^2}\left(-\tfrac{1}{2}\cos 2\omega t + \beta \sin \omega t - \beta\omega t\right)_0^t$$

$$\Rightarrow \quad x(t) = \frac{a_0}{2\omega^2}(1 - \cos 2\omega t + 2\beta \sin \omega t - 2\beta\omega t),$$

which can be evaluated to obtain

$$x(t) = 7.00[1.00 - 1.50\cos t + 3.00\sin(0.500t)]\,\text{m},$$

where we have expressed known coefficients to three significant figures, and where we have used the following numerical data: $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$.

Solution to 2.45

Integrating the acceleration equation and accounting for the condition that $\dot{x}(0) = a_0\beta/\omega$, we have

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \ddot{x}(t)\,dt \quad \Rightarrow \quad \dot{x}(t) = \frac{a_0}{\omega}\big(\sin 2\omega t + \beta\cos\omega t\big). \tag{1}$$

To find the total distance traveled we must first find the sign of $\dot{x}(t)$. To do so, we first rewrite Eq. (1) as follows:

$$\dot{x}(t) = \frac{a_0}{\omega}\cos\omega t\big(2\sin\omega t + \beta\big), \tag{2}$$

where we have used the trigonometric identity $\sin 2\omega t = 2\sin\omega t\cos\omega t$ and then factored the $\cos\omega t$ term. Then, referring to Eq. (2), and keeping in mind that we are only interested in the peg's motion for $0 \le t \le 5$ s, we see that

$$\cos\omega t > 0 \quad \text{for} \quad 0 < \omega t < \frac{\pi}{2}\,\text{rad} \quad \Rightarrow \quad 0 < t < \frac{\pi\,\text{rad}}{2\omega} = \pi\,\text{s} \tag{3}$$

where we have used the fact that $\omega = 0.5\,\text{rad/s}$. In addition, we have that

$$2\sin\omega t + \beta > 0 \quad \text{for all} \quad 0 < t < 5\,\text{s}. \tag{4}$$

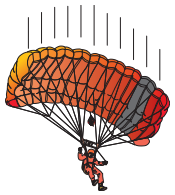
Therefore, we see that $\dot{x}$ changes sign for $t = \pi$ s. This implies that the distance traveled must be computed by integrating the velocity on two separate time intervals as follows:

$$\begin{aligned} d &= \int_{0\,\text{s}}^{\pi\,\text{s}} \frac{a_0}{\omega}\big[\cos\omega t(2\sin\omega t + \beta)\big]\,dt - \int_{\pi\,\text{s}}^{5\,\text{s}} \frac{a_0}{\omega}\big[\cos\omega t(2\sin\omega t + \beta)\big]\,dt, \\ &= \frac{a_0}{2\omega^2}\Big[-\cos 2\omega t + 2\beta\sin\omega t\Big]_{0\,\text{s}}^{\pi\,\text{s}} - \frac{a_0}{2\omega^2}\Big[-\cos 2\omega t + 2\beta\sin\omega t\Big]_{\pi\,\text{s}}^{5\,\text{s}}, \\ &= \frac{a_0}{2\omega^2}\big(3 + 4\beta + \cos 5 - 2\beta\sin 2.5\big)\,\text{m} \quad \Rightarrow \quad \boxed{d = 52.4\,\text{m},} \end{aligned}$$

where we have used the following numerical data: $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$

Problem 2.46

Referring to Example 2.9 on p. 63, and defining *terminal velocity* as the velocity at which a falling object stops accelerating, determine the skydiver's terminal velocity without performing any integrations.



Solution

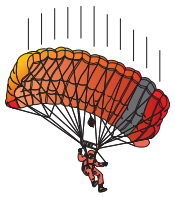
We use the equation $a = g - C_d v^2/m$, given in Example 2.9 on p. 63 and set $a = 0\,\text{m/s}^2$. Then, we can solve ofr the terminal velocity to obtain

$$v_{\text{term}} = \sqrt{\frac{mg}{C_d}} = 5.00\,\text{m/s}, \tag{1}$$

where we have used the following numerical data: $C_d = 43.2\,\text{kg/m}$, $m = 110\,\text{kg}$, and $g = 9.81\,\text{m/s}^2$.

Problem 2.47

Referring to Example 2.9 on p. 63, determine the distance d traveled by the skydiver from the instant the parachute is deployed until the difference between the velocity and the terminal velocity is 10% of the terminal velocity.



Solution

Recall that the acceleration can be related to the velocity and position as follows:

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}. \tag{1}$$

From Example 2.9 on p. 63 we have that $a = g - C_d v^2/m$. Hence, substituting this expression into Eq. (1), rearranging terms, and integrating from v_0 to $v_{\text{qt}} = 1.10\, v_{\text{term}}$ (where the subscript qt stands for quasi-terminal), we have

$$\int_0^d ds = \int_{v_0}^{v_{\text{qt}}} \frac{v}{g - (C_d/m)v^2}\, dv, \tag{2}$$

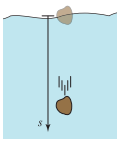
where v_0 is the velocity of the skydiver at the time that the parachute is deployed. Carrying out the above integrations, we have

$$d = \frac{-m}{2C_d} \ln\left[\frac{g - (C_d/m)v_{\text{qt}}^2}{g - (C_d/m)v_0^2}\right]. \tag{3}$$

Now, going back to Example 2.9 on p. 63, we have $m = 110\,\text{kg}$, $v_0 = 44.5\,\text{m/s}$, $C_d = 43.2\,\text{kg/m}$. In addition, recall that $g = 9.81\,\text{m/s}^2$ and that, again from Example 2.9 on p. 63, $v_{\text{term}} = \sqrt{mg/C_d} = 4.998\,\text{m/s}$, so that $v_{\text{qt}} = 5.498\,\text{m/s}$. Hence, using these data to evaluate the right-hand side of Eq. (3), we have

$$\boxed{d = 7.54\,\text{m}.}$$

Problems 2.48 and 2.49



The acceleration of an object in rectilinear free fall while immersed in a linear viscous fluid is $a = g - C_d v/m$, where g is the acceleration of gravity, C_d is a constant drag coefficient, v is the object's velocity, and m is the object's mass.

Problem 2.48 Letting $t_0 = 0$ and $v_0 = 0$, determine the velocity as a function of time and find the terminal velocity.

Problem 2.49 Letting $s_0 = 0$ and $v_0 = 0$, determine the position as a function of velocity.

Solution to 2.48

We observe that in this problem the acceleration is provided as a function of velocity. Hence, we first proceed to find time as a function of velocity and then inverting the result to determine the velocity as a function of time. Specifically, recalling that we have $a = dv/dt \Rightarrow dt = dv/a$, letting $t_0 = 0$, we have

$$t(v) = \int_0^v \frac{dv}{g - (C_d/m)v} = -\frac{m}{C_d} \ln\left[g - (C_d/m)v\right]\Big|_0^v = \frac{-m}{C_d} \ln\left(1 - \frac{C_d}{mg} v\right). \tag{1}$$

We then invert this to find $v(t)$

$$\boxed{v(t) = \frac{mg}{C_d}\left(1 - e^{-\frac{C_d t}{m}}\right).}$$

To find the terminal velocity, we can set $a = g - (C_d/m)v$ equal to 0:

$$0 = g - \frac{C_d v_{\text{term}}}{m} \quad \Rightarrow \quad \boxed{v_{\text{term}} = \frac{mg}{C_d}.} \tag{2}$$

Solution to 2.49

Recall that acceleration velocity and positions can be related as follows:

$$a = v\frac{dv}{ds} \quad \Rightarrow \quad ds = \frac{v\,dv}{a}. \tag{3}$$

Hence, since the acceleration is given as a function of the velocity, we can determine the position as a function of the velocity as follows:

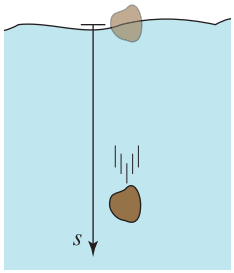
$$s(v) = \int_0^v \frac{v}{g - (C_d/m)v}\,dv = \int_0^v \frac{(C_d/m)v + g - g}{(C_d/m)[g - (C_d/m)v]}\,dv = \int_0^v \left[\frac{mg/C_d}{g - (C_d/m)v} - \frac{m}{C_d}\right]dv, \tag{4}$$

which can be evaluated to obtain

$$s(v) = \frac{-mg}{C_d}\frac{m}{C_d}\ln\left(\frac{g - (C_d/m)v}{g}\right) - \frac{m}{C_d}v \quad \Rightarrow \quad \boxed{s(v) = -\frac{m}{C_d}\left[v + \frac{mg}{C_d}\ln\left(1 - \frac{C_d v}{mg}\right)\right].} \tag{5}$$

Problem 2.50

A 1.5 kg rock is released from rest at the surface of a calm lake. If the resistance offered by the water as the rock falls is directly proportional to the rock's velocity, the rock's acceleration is $a = g - C_d v/m$, where g is the acceleration of gravity, C_d is a constant drag coefficient, v is the rock's velocity, and m is the rock's mass. Letting $C_d = 4.1\,\text{kg/s}$, determine the rock's velocity after 1.8 s.



Solution

We begin by observing that

$$a = g - \frac{C_d}{m}v = \frac{dv}{dt} \quad \Rightarrow \quad \int_0^{t_f} dt = \int_0^{v_f} \frac{dv}{g - (C_d/m)v}$$

$$\Rightarrow \quad t_f = \frac{-m}{C_d}\ln\left(\frac{g - (C_d/m)v_f}{g}\right) \quad \Rightarrow \quad t_f = \frac{-m}{C_d}\ln\left(1 - \frac{C_d}{mg}v_f\right), \quad (1)$$

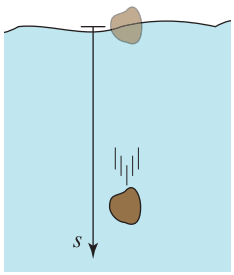
where the subscript f stands for final. Solving the above equation for v_f, we have

$$v_f = \frac{mg}{C_d}\left(1 - e^{-C_d t_f/m}\right) \quad \Rightarrow \quad \boxed{v_f = 3.56\,\text{m/s}}$$

where we have used the following numerical data: $m = 1.5\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $C_d = 4.1\,\text{kg/s}$, and $t_f = 1.8\,\text{s}$.

Problems 2.51 and 2.52

A 3.1 lb rock is released from rest at the surface of a calm lake, and its acceleration is $a = g - C_d v/m$, where g is the acceleration of gravity, $C_d = 0.27\,\text{lb}\cdot\text{s/ft}$ is a constant drag coefficient, v is the rock's velocity, and m is the rock's mass.



Problem 2.51 Determine the depth to which the rock will have sunk when the rock achieves 99% of its terminal velocity.

Problem 2.52 Determine the rock's velocity after it drops 5 ft.

Solution to 2.51

Recalling that we can relate the acceleration to velocity and position as $a = v dv/ds$, we can write

$$ds = \frac{v\,dv}{a} \quad \Rightarrow \quad \int_0^s ds = \int_0^v \frac{v}{g - (C_d/m)v}\,dv = \int_0^v \frac{(C_d/m)v + g - g}{(C_d/m)[g - (C_d/m)v]}\,dv$$

$$\Rightarrow \quad s = \int_0^v \left[\frac{mg/C_d}{g - (C_d/m)v} - \frac{m}{C_d}\right] dv, \quad (1)$$

which can be evaluated to obtain

$$s = \frac{-m^2 g}{C_d^2} \ln\left(1 - \frac{C_d}{mg} v\right) - \frac{m}{C_d} v. \quad (2)$$

We now recall that the terminal velocity can be computed as the value of velocity corresponding to the acceleration to become equal to zero, i.e., $g - (C_d/m)v_{\text{term}} = 0$. Hence, we have

$$v_{\text{term}} = mg/C_d = 11.48\,\text{ft/s}, \quad (3)$$

where we have used the following numerical data: $mg = 3.1\,\text{lb}$ and $C_d = 0.27\,\text{lb}\cdot\text{s/ft}$.

Now, letting $v = 0.99 v_{\text{term}} = 11.37\,\text{ft/s}$, we can evaluate the right-hand side of Eq. (2) to obtain

$$\boxed{s = 14.9\,\text{ft},}$$

where we have used the following numerical data: $m = (3.1\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.09627\,\text{slug}$, $g = 32.2\,\text{ft/s}^2$, and $C_d = 0.27\,\text{lb}\cdot\text{s/ft}$.

Solution to 2.52

Recalling that we can relate the acceleration to velocity and position as $a = vdv/ds$, we can write

$$ds = \frac{vdv}{a} \quad \Rightarrow \quad \int_0^s ds = \int_0^v \frac{v}{g-(C_d/m)v}\,dv = \int_0^v \frac{(C_d/m)v+g-g}{(C_d/m)[g-(C_d/m)v]}\,dv$$

$$\Rightarrow \quad s = \int_0^v \left[\frac{mg/C_d}{g-(C_d/m)v} - \frac{m}{C_d}\right] dv, \quad (4)$$

which can be evaluated to obtain

$$s = \frac{-m^2 g}{C_d^2}\ln\left(1-\frac{C_d}{mg}v\right) - \frac{m}{C_d}v. \quad (5)$$

We now need to solve the above equation for v after setting $s = 5\,\text{ft}$. Since this cannot be done analytically, we will need to do it numerically. As a first step, we recall that we are given the following numerical data: $m = (3.1\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.09627\,\text{slug}$, $g = 32.2\,\text{ft/s}^2$, and $C_d = 0.27\,\text{lb·s/ft}$. Therefore, letting $s = 5\,\text{ft}$ and using the data just listed, we evaluate Eq. (5) (expressing all known coefficients to 4 significant figures) to obtain

$$5.000\,\text{ft} = -(0.3566\,\text{s})v - (4.094\,\text{ft})\ln[1.000-(0.08710\,\text{s/ft})v]. \quad (6)$$

Next we need to use some numerical root finding program to determine the desired value of v. This can de done using several pieces of mathematical software. The solution presented below was obtained using *Mathematica* via the following code:

```
FindRoot[-0.3566 v - 4.094 Log[1 - 0.0871 v] == 5.000, {v, 1.}]
```

where we note that, as required by most root finding algorithms, one needs to specify some initial guess for the solution (we have used $v = 1\,\text{ft/s}$). The execution of the code above, gives the following result (once expressed to 3 significant figures)

$$\boxed{v = 10.1\,\text{ft/s}.}$$

Problem 2.53

Suppose that the acceleration of an object of mass m along a straight line is $a = g - C_d v/m$, where the constants g and C_d are given and v is the object's velocity. If $v(t)$ is unknown and $v(0)$ is given, can you determine the object's velocity via the following integral?

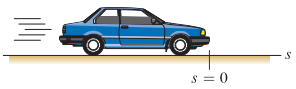
$$v(t) = v(0) + \int_0^t \left(g - \frac{C_d}{m} v \right) dt$$

Solution

No, because the integrand is not an explicit function of the variable of integration. Clearly, if $v(t)$ is provided as an explicit function of time then one could integrate. However in this case there would be no need whatsoever to integrate since $v(t)$ would be given to begin with.

Problem 2.54

A car travels on a rectilinear stretch of road at a constant speed $v_0 = 65\,\text{mph}$. At $s = 0$ the driver applies the brakes hard enough to cause the car to skid. Assume that the car keeps sliding until it stops, and assume that throughout this process the car's acceleration is given by $\ddot{s} = -\mu_k g$, where $\mu_k = 0.76$ is the kinetic friction coefficient and g is the acceleration of gravity. Compute the car's stopping distance and time.



Solution

Using the constant acceleration equation $v^2 = v_0^2 + 2a_c(s - s_0)$ with $v_0 = 65\,\text{mph} = 95.33\,\text{ft/s}$, $v = 0$, and $s_0 = 0$, we have

$$v^2 = v_0^2 + 2\ddot{s}(s - s_0) \quad \Rightarrow \quad s = \frac{v_0^2}{2\mu_k g} \quad \Rightarrow \quad \boxed{s = 186\,\text{ft},} \tag{1}$$

where in addition to the data already listed we have used the following numerical values: $g = 32.2\,\text{ft/s}^2$ and $\mu_k = 0.76$.

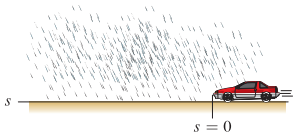
To determine the car's stopping time, we can use the constant acceleration equation $v = v_0 + a_c t$ to find

$$t = \frac{v - v_0}{\ddot{s}} = \frac{v - v_0}{-\mu_k g} \quad \Rightarrow \quad \boxed{t = 3.90\,\text{s},} \tag{2}$$

where we have used the following numerical data: $v = 0$, $v_0 = 65\,\text{mph} = 95.33\,\text{ft/s}$, $\mu_k = 0.76$, and $g = 32.2\,\text{ft/s}^2$.

Problem 2.55

Heavy rains cause a particular stretch of road to have a coefficient of friction that changes as a function of location. Specifically, measurements indicate that the friction coefficient has a 3% decrease per meter. Under these conditions the acceleration of a car skidding while trying to stop can be approximated by $\ddot{s} = -(\mu_k - cs)g$ (the 3% decrease in friction was used in deriving this equation for acceleration), where μ_k is the friction coefficient under dry conditions, g is the acceleration of gravity, and c, with units of m^{-1}, describes the rate of friction decrement. Let $\mu_k = 0.5$, $c = 0.015\,\text{m}^{-1}$, and $v_0 = 45\,\text{km/h}$, where v_0 is the initial velocity of the car. Determine the distance it will take the car to stop and the percentage of increase in stopping distance with respect to dry conditions, i.e., when $c = 0$.



Solution

Recalling that the acceleration can be related to the position and velocity as $a = v\,dv/ds$, we can write

$$\ddot{s} = v\frac{dv}{ds} \quad \Rightarrow \quad \int_{v_0}^{0} v\,dv = -\int_{0}^{s_{\text{wet}}} (\mu_k - cs)g\,ds, \tag{1}$$

where the upper limits of integration for integral on the left-hand side (i.e., $v = 0$) corresponds to the car stopping, and the upper limit of integration for the integral on the right-hand side denotes the stopping distance over wet ground. Carrying out the integration, we have

$$-\tfrac{1}{2}v_0^2 = -\frac{g}{2c}\Big[(\mu_k - cs)^2\Big]_0^{s_{\text{wet}}} \quad \Rightarrow \quad -\tfrac{1}{2}v_0^2 = -\frac{g}{2c}\big[\mu_k^2 - 2\mu_k c s_{\text{wet}} + c^2 s_{\text{wet}}^2 - \mu_k^2\big], \tag{2}$$

which can be simplified to yield

$$cs_{\text{wet}}^2 - 2\mu_k s_{\text{wet}} + v_0^2/g = 0 \quad \Rightarrow \quad s_{\text{wet}} = \frac{1}{c}\Big[\mu_k \pm \sqrt{\mu_k^2 - (cv_0^2)/g}\Big]. \tag{3}$$

Recalling that $c = 0.015\,\text{m}^{-1}$, $\mu_k = 0.5$, $v_0 = 45\,\text{km/h} = 12/50\,\text{m/s}$, and $g = 9.81\,\text{m/s}^2$, we can evaluate the last of Eqs. (3) to obtain $s_{\text{wet}} = 26.31\,\text{m}$ or $40.35\,\text{m}$. Only the first solution is meaningful since the car will not keep moving after it has come to a stop for the first time. Hence, expressing our result to 3 significant figures, we have

$$\boxed{s_{\text{wet}} = 26.3\,\text{m}.}$$

For dry conditions, $c = 0$ so that $\ddot{s} = -\mu_k g$, which implies that the acceleration is constant. Consequently, we use the equation $v^2 = v_0^2 + 2a_c(s - s_0)$ to determine the stopping distance under dry conditions. Hence, denoting by s_{dry} the position s at which $v = 0$ under dry conditions, we have

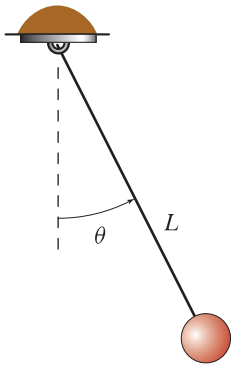
$$0 = v_0^2 - 2\mu_k g(s_{\text{dry}} - s_0) \quad \Rightarrow \quad s_{\text{dry}} = \frac{v_0^2}{2\mu_k g} \quad \Rightarrow \quad s_{\text{dry}} = 15.93\,\text{m},$$

where we have set $s_0 = 0$. Hence, the percentage increase in stopping distance, expressed to 3 significant figures, is

$$\boxed{\frac{(s_{\text{wet}} - s_{\text{dry}})}{s_{\text{dry}}} \times 100 = 65.2\%.}$$

Problems 2.56 through 2.59

As you will learn in Chapter 3, the angular acceleration of a simple pendulum is given by $\ddot{\theta} = -(g/L)\sin\theta$, where g is the acceleration of gravity and L is the length of the pendulum cord.



Problem 2.56 Derive the expression of the angular velocity $\dot{\theta}$ as a function of the angular coordinate θ. The initial conditions are $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$.

Problem 2.57 Let the length of the pendulum cord be $L = 1.5$ m. If $\dot{\theta} = 3.7$ rad/s when $\theta = 14°$, determine the maximum value of θ achieved by the pendulum.

Problem 2.58 The given angular acceleration remains valid even if the pendulum cord is replaced by a massless rigid bar. For this case, let $L = 5.3$ ft and assume that the pendulum is placed in motion at $\theta = 0°$. What is the minimum angular velocity at this position for the pendulum to swing through a full circle?

Problem 2.59 Let $L = 3.5$ ft and suppose that at $t = 0$ s the pendulum's position is $\theta(0) = 32°$ with $\dot{\theta}(0) = 0$ rad/s. Determine the pendulum's period of oscillation, i.e., from its initial position back to this position.

Solution to 2.56

Recalling that we can write $\ddot{\theta} = \dot{\theta}\,d\dot{\theta}/d\theta$, and that this implies $d\dot{\theta} = \ddot{\theta}\,d\theta$, we can then integrate each side to obtain

$$\int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = \int_{\theta_0}^{\theta} \ddot{\theta}(\theta)\,d\theta \quad \Rightarrow \quad \dot{\theta}^2 - \dot{\theta}_0^2 = \frac{-2g}{L}\int_{\theta_0}^{\theta} \sin\theta\,d\theta \quad \Rightarrow \quad \dot{\theta}^2 = \dot{\theta}_0^2 + \frac{2g}{L}(\cos\theta - \cos\theta_0).$$

Solving for $\dot{\theta}$, we then have

$$\dot{\theta}(\theta) = \pm\sqrt{\dot{\theta}_0^2 + \frac{2g}{L}\big(\cos\theta - \cos\theta_0\big)}. \tag{1}$$

Solution to 2.57

Recalling that we can write $\ddot{\theta} = \dot{\theta} d\dot{\theta}/d\theta$, and that this implies $d\dot{\theta} = \ddot{\theta} d\theta$, we can then integrate each side to obtain

$$\int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} = \int_{\theta_0}^{\theta} \ddot{\theta}(\theta)\, d\theta \quad \Rightarrow \quad \dot{\theta}^2 - \dot{\theta}_0^2 = \frac{-2g}{L} \int_{\theta_0}^{\theta} \sin\theta\, d\theta \quad \Rightarrow \quad \dot{\theta}^2 - \dot{\theta}_0^2 = \frac{2g}{L}(\cos\theta - \cos\theta_0).$$

Hence, observing that the maximum angle achieved by the pendulum corresponds to when the pendulum inverts its swinging motion and therefore for $\dot{\theta} = 0$, denoting by $\theta_{\max}$ the angle in question, we can rewrite the last of the above equations as

$$-\dot{\theta}_0^2 = \frac{2g}{L}(\cos\theta_{\max} - \cos\theta_0),$$

which can be solved for $\theta_{\max}$ to obtain

$$\theta_{\max} = \cos^{-1}\left[\frac{-L\dot{\theta}_0^2}{2g} + \cos\theta_0\right] \quad \Rightarrow \quad \boxed{\theta_{\max} = 94.4^\circ,}$$

where we have used the following numerical data: $\dot{\theta}_0 = 3.7\,\text{rad/s}$, $\theta_0 = 14^\circ$, $L = 1.5\,\text{m}$, and $g = 9.81\,\text{m/s}^2$.

Solution to 2.58

Recalling that we can write $\ddot{\theta} = \dot{\theta} d\dot{\theta}/d\theta$, and that this implies $d\dot{\theta} = \ddot{\theta} d\theta$, we can then integrate each side to obtain

$$\int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} = \int_{\theta_0}^{\theta} \ddot{\theta}(\theta)\, d\theta \quad \Rightarrow \quad \dot{\theta}^2 - \dot{\theta}_0^2 = \frac{-2g}{L} \int_{\theta_0}^{\theta} \sin\theta\, d\theta \quad \Rightarrow \quad \dot{\theta}^2 - \dot{\theta}_0^2 = \frac{2g}{L}(\cos\theta - \cos\theta_0).$$

Now observe that the minimum angular speed of the pendulum at $\theta = 0$ is the speed that allows the pendulum to reach the angle $\theta = \pi$ rad with $\dot{\theta} = 0$. Hence, letting $\dot{\theta}_0$ denote the the angular speed of the pendulum for $\theta_0 = 0$, substituting these conditions in the above equation, we have

$$-\dot{\theta}_0^2 = \frac{2g}{L}\big[\cos(\pi \text{ rad}) - \cos 0\big] \quad \Rightarrow \quad \dot{\theta}_0^2 = \frac{4g}{L} \quad \Rightarrow \quad \dot{\theta}_0 = \pm 2\sqrt{\frac{g}{L}},$$

which can be evaluated to obtain

$$\boxed{\left(\dot{\theta}_0\right)_{\text{min}} = \pm 4.93 \text{ rad/s},}$$

where we have used the following numerical data: $g = 32.2\,\text{ft/s}^2$ and $L = 5.3$ ft.

Solution to 2.59

We need to establish a relationship between the angle θ and time. To do so, we begin by establishing a relation between the angular velocity $\dot{\theta}$ and and swing angle, and then we will proceed to establish the desired relation. Recalling that we can write $\ddot{\theta} = \dot{\theta} d\dot{\theta}/d\theta$, and that this implies $d\dot{\theta} = \ddot{\theta} d\theta$, we can then integrate each side to obtain

$$\int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} = \int_{\theta_0}^{\theta} \ddot{\theta}(\theta)\, d\theta \quad \Rightarrow \quad \dot{\theta}^2 - \dot{\theta}_0^2 = \frac{-2g}{L} \int_{\theta_0}^{\theta} \sin\theta\, d\theta \quad \Rightarrow \quad \dot{\theta}^2 = \dot{\theta}_0^2 + \frac{2g}{L}(\cos\theta - \cos\theta_0).$$

Hence, solving for $\dot{\theta}$ as a function of θ, we have

$$\dot{\theta} = \pm\sqrt{\dot{\theta}_0^2 + \frac{2g}{L}\big(\cos\theta - \cos\theta_0\big)}. \tag{2}$$

One the pendulum is released from $\theta_0 = 32°$, the angle θ will decrease until the angle $-32°$ is achieved and it will then swing back tot he original angle. The time taken to go from $32°$ to $-32°$ is equal to the time taken to swing back from $-32°$ to $32°$. Hence, the period of oscillation, which we will denote by p, is twice the time that the pendulum takes to go from $32°$ to $-32°$. With this in mind, and referring to Eq. (2), the relationship between $\dot{\theta}$ and θ for after release and until $\theta = -32°$ is achieved, is

$$\dot{\theta} = -\sqrt{\dot{\theta}_0^2 + \frac{2g}{L}\big(\cos\theta - \cos\theta_0\big)}, \tag{3}$$

given that θ will initially decrease. Now, recalling that $\dot{\theta} = d\theta/dt$, we can write $dt = d\theta/\dot{\theta}$. Hence, using Eq. (3), we can write

$$dt = -\frac{d\theta}{\sqrt{\dot{\theta}_0^2 + \frac{2g}{L}\big(\cos\theta - \cos\theta_0\big)}}, \tag{4}$$

Hence, we can integrating over one half of a full swing to obtain

$$\int_0^{p/2} dt = -\int_{32°}^{-32°} \frac{d\theta}{\sqrt{\dot{\theta}_0^2 + \frac{2g}{L}\big(\cos\theta - \cos\theta_0\big)}} \quad \Rightarrow \quad p = -2\int_{32°}^{-32°} \frac{d\theta}{\sqrt{\frac{2g}{L}\big[\cos\theta - \cos(32°)\big]}}, \tag{5}$$

where we have enforced the fact that $\theta_0 = 32°$ and $\dot{\theta}_0 = 0$. Now recalling that $L = 3.5\,\text{m}$ and $g = 9.81\,\text{m/s}^2$, we have that $2/\sqrt{2g/L} = 0.4663\,\text{s}$. Hence, we can rewrite the last of Eqs. (5) as

$$p = \int_{32°}^{-32°} \frac{(-0.4663\,\text{s})}{\sqrt{\cos\theta - \cos 32°}}\, d\theta.$$

The above integral can be evaluated with some appropriate mathematical software. We have used the following code using *Mathematica*:

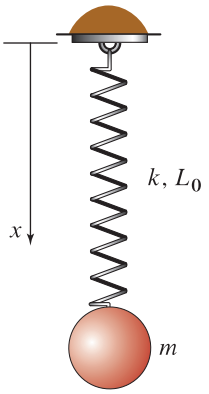
```
NIntegrate[-0.4663/Sqrt[Cos[θ] - Cos[32. Degree]], {θ, 32. Degree, -32. Degree}]
```

which yields the following result:

$$\boxed{p = 2.11\,\text{s}.}$$

Problems 2.60 through 2.62

As we will see in Chapter 3, the acceleration of a particle of mass m suspended by a linear spring with spring constant k and unstretched length L_0 (when the spring length is equal to L_0, the spring exerts no force on the particle) is given by $\ddot{x} = g - (k/m)(x - L_0)$.



Problem 2.60 Derive the expression for the particle's velocity $\dot{x}$ as a function of position x. Assume that at $t = 0$, the particle's velocity is v_0 and its position is x_0.

Problem 2.61 Let $k = 100\,\text{N/m}$, $m = 0.7\,\text{kg}$, and $L_0 = 0.75\,\text{m}$. If the particle is released from rest at $x = 0\,\text{m}$, determine the maximum length achieved by the spring.

Problem 2.62 Let $k = 8\,\text{lb/ft}$, $m = 0.048\,\text{slug}$, and $L_0 = 2.5\,\text{ft}$. If the particle is released from rest at $x = 0\,\text{ft}$, determine how long it takes for the spring to achieve its maximum length. *Hint:* A good table of integrals will come in handy.

Solution to 2.60

The acceleration can be related to the position and the velocity as follows: $\ddot{x} = \dot{x}d\dot{x}/dx$. This implies that we can write $\dot{x}d\dot{x} = \ddot{x}dx$ and we can therefore proceed to integrate this expression as follows:

$$\int_{v_0}^{\dot{x}} \dot{x}\,d\dot{x} = \int_{x_0}^{x} \left[g - \left(\frac{k}{m}\right)(x - L_0) \right] dx, \tag{1}$$

where as indicated in the problem statement, v_0 is the value of $\dot{x}$ for $x = x_0$. Evaluating the integral gives us the velocity as a function of x.

$$\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}v_0^2 = g(x - x_0) - \frac{k}{2m}(x^2 - x_0^2) + \frac{kL_0}{m}(x - x_0). \tag{2}$$

Then multiplying the above equation through by 2 and solving for $\dot{x}$, we have

$$\dot{x} = \pm\sqrt{v_0^2 + 2\left(g + \frac{kL_0}{m}\right)(x - x_0) - \frac{k}{m}(x^2 - x_0^2)}. \tag{3}$$

Solution to 2.61

The acceleration can be related to the position and the velocity as follows: $\ddot{x} = \dot{x}d\dot{x}/dx$. This implies that we can write $\dot{x}d\dot{x} = \ddot{x}dx$ and we can therefore proceed to integrate this expression as follows:

$$\int_{v_0}^{\dot{x}} \dot{x}\, d\dot{x} = \int_{x_0}^{x} \left[g - \left(\frac{k}{m}\right)(x - L_0) \right] dx, \tag{4}$$

where as indicated in the problem statement, v_0 is the value of $\dot{x}$ for $x = x_0$. Now, in this particular problem, since the particle starts from rest at $x = 0$, we set $v_0 = 0$ and $x_0 = 0$. We now observe that the coordinate x (when positive) measures the length of the spring. In addition, denoting the maximum length by $x_{\max}$, we observe that $x_{\max}$ is achieved when $\dot{x} = 0$, i.e., when the spring has stretched to the point that it will start recoiling back. Using these considerations, we can rewrite Eq. (4) as

$$\int_{0}^{0} \dot{x}\, d\dot{x} = \int_{0}^{x_{\max}} \left[g - \left(\frac{k}{m}\right)(x - L_0) \right] dx \quad \Rightarrow \quad 0 = g x_{\max} - \frac{k}{2m} x_{\max}^2 + \frac{k L_0}{m} x_{\max}. \tag{5}$$

The last of the above equations can be solved for $x_{\max}$ to obtain

$$x_{\max} = \frac{2(mg + kL_0)}{k} \quad \Rightarrow \quad \boxed{x_{\max} = 1.64\,\text{m},} \tag{6}$$

where we have used the following numerical data: $m = 0.7\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $k = 100\,\text{N/m}$, and $L_0 = 0.75\,\text{m}$.

Solution to 2.62

The acceleration can be related to the position and the velocity as follows: $\ddot{x} = \dot{x}d\dot{x}/dx$. This implies that we can write $\dot{x}d\dot{x} = \ddot{x}dx$ and we can therefore proceed to integrate this expression as follows:

$$\int_{v_0}^{\dot{x}} \dot{x}\, d\dot{x} = \int_{x_0}^{x} \left[g - \left(\frac{k}{m}\right)(x - L_0) \right] dx, \tag{7}$$

where as indicated in the problem statement, v_0 is the value of $\dot{x}$ for $x = x_0$. Evaluating the integral gives us the velocity as a function of x.

$$\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}v_0^2 = g(x - x_0) - \frac{k}{2m}\left(x^2 - x_0^2\right) + \frac{kL_0}{m}(x - x_0). \tag{8}$$

Then multiplying the above equation through by 2, keeping in mind that we are interested in the motion of the spring for $\dot{x} \geq 0$ (i.e., we are not interested in the recoiling motion of the spring after it has stretched to its maximum length), and solving for $\dot{x}$, we have

$$\dot{x} = \sqrt{v_0^2 + 2\left(g + \frac{kL_0}{m}\right)(x - x_0) - \frac{k}{m}\left(x^2 - x_0^2\right)}. \tag{9}$$

Now, since the particle is released from rest at $x = 0$, we have $v_0 = 0$. Consequently, the above equation can be simplified to obtain

$$\dot{x} = \sqrt{\frac{2(mg + kL_0)}{m}x - \frac{k}{m}x^2}. \tag{10}$$

Next, we determine the maximum length of the spring, which is achieved when $\dot{x} = 0$. Hence, setting $\dot{x} = 0$ and $x = x_{\max}$ in Eq. (10) and solving for $x_{\max}$, we have

$$\frac{2(mg + kL_0)}{m}x_{\max} - \frac{k}{m}x_{\max}^2 = 0 \quad \Rightarrow \quad x_{\max} = \frac{2(mg + kL_0)}{k} \quad \Rightarrow \quad x_{\max}k = 2(mg + kL_0). \tag{11}$$

Substituting Eq. (11) into Eq. (10), we have

$$\dot{x} = \sqrt{\frac{k}{m}xx_{\max} - \frac{k}{m}x^2} \quad \Rightarrow \quad \dot{x} = \sqrt{\frac{k}{m}}\sqrt{x\left(x_{\max} - x\right)}. \tag{12}$$

Now we recall that $\dot{x} = dx/dt$. Therefore, we can rearrange the terms in the last of Eqs. (10) to integrate as follows:

$$\dot{x} = \frac{dx}{dt} \quad \Rightarrow \quad \int_0^{t_{x_{\max}}} dt = \int_0^{x_{\max}} \frac{dx}{\dot{x}} \quad \Rightarrow \quad t_{x_{\max}} = \sqrt{\frac{m}{k}} \int_0^{x_{\max}} \frac{dx}{\sqrt{x\,(x_{\max} - x)}}. \tag{13}$$

The integral on the right-hand side of the last of Eqs. (13) can be carried out by substitution, or by consulting a table of integrals, or by using some symbolic mathematical software. Regarless of the method, we have

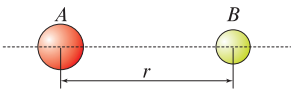
$$\int \frac{dx}{\sqrt{x\,(x_{\max} - x)}} = 2\sin^{-1}\left(\sqrt{\frac{x}{x_{\max}}}\right) + C, \tag{14}$$

where C is a constant of integration. Then using the above result we have that the last of Eqs. (13) gives

$$t_{x_{\max}} = \sqrt{\frac{m}{k}}2[\sin^{-1}(1) - \sin^{-1}(0)] = \pi\sqrt{\frac{m}{k}} \quad \Rightarrow \quad \boxed{t_{x_{\max}} = 0.243\,\text{s},} \tag{15}$$

where we have used the following numerical data: $m = 0.048$ slug and $k = 8$ lb/ft.

Problems 2.63 and 2.64



Two masses m_A and m_B are placed at a distance r_0 from one another. Because of their mutual gravitational attraction, the acceleration of sphere B as seen from sphere A is given by

$$\ddot{r} = -G\left(\frac{m_A + m_B}{r^2}\right),$$

where G is the universal gravitational constant.

Problem 2.63 If the spheres are released from rest, determine

(a) The velocity of B (as seen by A) as a function of the distance r.

(b) The velocity of B (as seen by A) at impact if $r_0 = 7$ ft, the weight of A is 2.1 lb, the weight of B is 0.7 lb, and

(i) The diameters of A and B are $d_A = 1.5$ ft and $d_B = 1.2$ ft, respectively.

(ii) The diameters of A and B are infinitesimally small.

Problem 2.64 Assume that the particles are released from rest at $r = r_0$.

(a) Determine the expression relating their relative position r and time. *Hint:*

$$\int \sqrt{x/(1-x)}\, dx = \sin^{-1}\left(\sqrt{x}\right) - \sqrt{x(1-x)}.$$

(b) Determine the time it takes for the objects to come into contact if $r_0 = 3$ m, A and B have masses of 1.1 and 2.3 kg, respectively, and

(i) The diameters of A and B are $d_A = 22$ cm and $d_B = 15$ cm, respectively.

(ii) The diameters of A and B are infinitesimally small.

Solution to 2.63

Part (a). Recalling that we can relate the acceleration to the velocity and position as $\ddot{r} = \dot{r}\, d\dot{r}/dt$, we can then write

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} \quad \Rightarrow \quad \int_0^{\dot{r}} \dot{r}\, dr = -G\,(m_A + m_B)\int_{r_0}^{r} \frac{1}{r^2}\, dr,$$

which can be evaluated to obtain

$$\tfrac{1}{2}\dot{r}^2 = G(m_A + m_B)\left(\frac{1}{r} - \frac{1}{r_0}\right) \quad \Rightarrow \quad \boxed{\dot{r} = -\sqrt{2G\,(m_A + m_B)}\left(\frac{r_0 - r}{r r_0}\right)^{1/2},} \qquad (1)$$

where we have chosen the negative root because the masses are moving toward each other.

Part (b). Now that we have the velocity as a function of position, we can proceed to answer the questions posed in Part (b) of the problem. For question (i), when the masses touch $r = r_A + r_B = (d_A + d_B)/2$. Using this fact and recalling that $G = 3.439 \times 10^{-8}\,\text{ft}^3/\text{slug}\cdot\text{s}^2$, Eq. (1) yields

$$\boxed{\dot{r} = -\sqrt{2G\,(m_A + m_B)}\left(\frac{2}{d_A + d_B} - \frac{1}{r_0}\right)^{1/2} = -5.98 \times 10^{-5}\,\text{ft/s}.} \tag{2}$$

For part (ii), we take the limit of Eq. (2) as $r \to 0$ to obtain

$$\boxed{\dot{r} \to -\infty.}$$

Solution to 2.64

We begin by finding the relation between position and velocity. To do so observe that we can relate the acceleration to the velocity and position as $\ddot{r} = \dot{r}\,d\dot{r}/dt$. Hence, we can then write

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} \quad \Rightarrow \quad \int_0^{\dot{r}} \dot{r}\,dr = -G\,(m_A + m_B)\int_{r_0}^{r} \frac{1}{r^2}\,dr,$$

which can be evaluated to obtain

$$\tfrac{1}{2}\dot{r}^2 = G\big(m_A + m_B\big)\left(\frac{1}{r} - \frac{1}{r_0}\right) \quad \Rightarrow \quad \dot{r} = -\sqrt{2G\,(m_A + m_B)}\sqrt{\frac{r_0 - r}{r\,r_0}}, \tag{3}$$

where we have chosen the negative root because the masses are moving toward each other.

Part (a) Next, we observe that $\dot{r} = dr/dt$ and we write $dt = dr/\dot{r}$. Using this expression and the expression for $\dot{r}$ in the last of Eqs. (3), we can then write

$$-\frac{1}{\sqrt{2G\,(m_A + m_B)}}\int_{r_0}^{r}\sqrt{\frac{r}{1 - r/r_0}}\,dr = \int_0^t dt,$$

where we have divided both the numerator and the denominator of the fraction under the square root by r_0. Next, making the substitution $x = r/r_0$, we obtain

$$-\frac{r_0^{3/2}}{\sqrt{2G\,(m_A + m_B)}}\int_1^{r/r_0}\sqrt{\frac{x}{1-x}}\,dx = t.$$

Making use of the information given in the problem statement about the integral, we get

$$-\frac{r_0^{3/2}}{\sqrt{2G\,(m_A + m_B)}}\Big[\sin^{-1}\sqrt{x} - \sqrt{x\,(1-x)}\Big]_1^{r/r_0} = t,$$

$$\Rightarrow \quad \boxed{t = \frac{-r_0^{3/2}}{\sqrt{2G\,(m_A + m_B)}}\left[\sin^{-1}\sqrt{\frac{r}{r_0}} - \sqrt{\frac{r}{r_0}\left(1 - \frac{r}{r_0}\right)} - \frac{\pi}{2}\right].} \tag{4}$$

Part (b)

(i) Substituting the values $r_0 = 3\,\text{m}$, $m_A = 1.1\,\text{kg}$, $m_B = 7.5\,\text{kg}$, $G = 6.674\times10^{-11}\,\text{m}^3/\text{kg}\cdot\text{s}^2$, and $r = \frac{d_A + d_B}{2} = 0.1850\,\text{m}$ (where the $d_A = 22\,\text{cm} = 0.2200\,\text{m}$ and $d_B = 15\,\text{cm} = 0.1500\,\text{m}$) in Eq. (4), we obtain

$$t = 380{,}600\,\text{s} \quad \Rightarrow \quad \boxed{t = 106\,\text{h}.}$$

(ii) If the diameters are infinitesimally small, $r \to 0$. Hence, from Eq. (4) we obtain

$$t = \frac{(\pi/2)\,r_0^{3/2}}{\sqrt{2G(m_A + m_B)}} \quad \Rightarrow \quad t = 383{,}100\,\text{s} \quad \Rightarrow \quad \boxed{t = 106\,\text{h},}$$

where again we have used the following numerical data: $r_0 = 3\,\text{m}$, $m_A = 1.1\,\text{kg}$, $m_B = 7.5\,\text{kg}$, $G = 6.674\times10^{-11}\,\text{m}^3/\text{kg}\cdot\text{s}^2$.

Problem 2.65

Suppose that the acceleration $\ddot{r}$ of an object moving along a straight line takes on the form

$$\ddot{r} = -G\left(\frac{m_A + m_B}{r^2}\right),$$

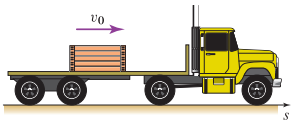
where the constants G, m_A, and m_B are known. If $\dot{r}(0)$ is given, under what conditions can you determine $\dot{r}(t)$ via the following integral?

$$\dot{r}(t) = \dot{r}(0) - \int_0^t G\,\frac{m_A + m_B}{r^2}\,dt$$

Solution

$\dot{r}(t)$ can be determined if the position r of the object is known as a function of time and $r(t) \neq 0$ during the time interval of interest.

Problems 2.66 and 2.67



If the truck brakes and the crate slides to the right relative to the truck, the horizontal acceleration of the crate is given by $\ddot{s} = -g\mu_k$, where g is the acceleration of gravity, $\mu_k = 0.87$ is the kinetic friction coefficient, and s is the position of the crate relative to a coordinate system attached to the ground (rather than the truck).

Problem 2.66 Assuming that the crate slides without hitting the right end of the truck bed, determine the time it takes to stop if its velocity at the start of the sliding motion is $v_0 = 55\,\text{mph}$.

Problem 2.67 Assuming that the crate slides without hitting the right end of the truck bed, determine the distance it takes to stop if its velocity at the start of the sliding motion is $v_0 = 75\,\text{km/h}$.

Solution to 2.66

Since the acceleration is constant, we can use the (constant acceleration) equation $v = v_0 + at$ to find the stopping time. Specifically, letting t_{stop} denote the stopping time and observing that at $t = t_{\text{stop}}$ we have $v = 0$, we can then write

$$0 = v_0 - g\mu_k t_{\text{stop}} \quad \Rightarrow \quad \boxed{t_{\text{stop}} = \frac{v_0}{g\mu_k} = 2.88\,\text{s},} \tag{1}$$

where $v_0 = 55\,\text{mph} = 80.67\,\text{ft/s}$ and $g = 32.2\,\text{ft/s}^2$.

Solution to 2.67

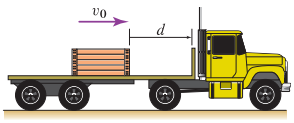
Since the acceleration is constant, we can use the (constant acceleration) equation $v^2 = v_0^2 + 2a_c(s - s_0)$ with $s_0 = 0$, $s = s_{\text{stop}}$, and $v = 0$ to obtain

$$0 = v_0^2 - 2(g\mu_k)s_{\text{stop}} \quad \Rightarrow \quad \boxed{s = \frac{v_0^2}{2g\mu_k} = 25.4\,\text{m},} \tag{2}$$

where $v_0 = 75\,\text{km/h} = 20.83\,\text{m/s}$ and $g = 9.81\,\text{m/s}^2$.

Problem 2.68

If the truck brakes hard enough that the crate slides to the right relative to the truck, the distance d between the crate and the front of the trailer changes according to the relation

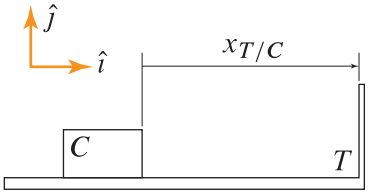


$$\ddot{d} = \begin{cases} \mu_k g + a_T & \text{for } t < t_s, \\ \mu_k g & \text{for } t > t_s, \end{cases}$$

where t_s is the time it takes the truck to stop, a_T is the acceleration of the truck, g is the acceleration of gravity, and μ_k is the kinetic friction coefficient between the truck and the crate. Suppose that the truck and the crate are initially traveling to the right at $v_0 = 60$ mph and the brakes are applied so that $a_T = -10.0\,\text{ft/s}^2$. Determine the minimum value of μ_k so that the crate does not hit the right end of the truck bed if the initial distance d is 12 ft. *Hint:* The truck stops *before* the crate stops.

Solution

Referring to figure on the right, the acceleration of the truck relative to the crate is given by



$$a_{T/C} = \begin{cases} (a_{T/C})_1 = \mu_k g + a_T & \text{for } t < t_s, \\ (a_{T/C})_2 = \mu_k g & \text{for } t > t_s, \end{cases} \tag{1}$$

where t_s is the time at which the truck comes to a stop. Using the constant acceleration equation of the type $v = v_0 + a_c(t - t_0)$, the time at which the truck stops found as

$$0 = (v_T)_0 + a_T t_s \quad \Rightarrow \quad t_s = -\frac{(v_T)_0}{a_T}, \tag{2}$$

where $(v_T)_0$ is the initial speed of the truck and crate. Now, the distance the crate moves relative to the truck during the time the truck is coming to a stop, $(x_{T/C})_1$, is found using the constant acceleration equation of the type $v^2 = v_0^2 + 2a_c(s - s_0)$. Adapting this relation to the problem at hand, we have

$$\begin{aligned} (x_{T/C})_1 &= (x_{T/C})_0 + (v_{T/C})_0 t_s + \tfrac{1}{2}(a_{T/C})_1 t_s^2 \\ &= d + \tfrac{1}{2}(\mu_k g + a_T) t_s^2, \end{aligned} \tag{3}$$

where $(x_{T/C})_0$ is the initial position of the truck relative to the crate and $(v_{T/C})_0$ is the initial velocity of the truck relative to the crate.

Now, after the truck comes to a stop, the crate continues to slide relative to the truck with an acceleration given by $(a_{T/C})_2$ in Eq. (1). Using this acceleration, the distance the crate moves relative to the truck after the truck comes to a stop can be found using

$$(v_{T/C})_F^2 = (v_{T/C})_1^2 + 2(a_{T/C})_2 \left[(x_{T/C})_F - (x_{T/C})_1\right], \tag{4}$$

where $(v_{T/C})_F = 0$ is the final velocity of the truck relative to the crate, $(v_{T/C})_1$ is the velocity of the truck relative to the crate at time t_s, $(x_{T/C})_F = 0$ is final position of the truck relative to the crate, and $(x_{T/C})_1$ is the position of the truck relative to the crate at time t_s and is given by Eq. (3). Before finishing, we need to find $(v_{T/C})_1$, which can readily be done as

$$(v_{T/C})_1 = (v_{T/C})_0 + (a_{T/C})_1 t_s \quad \Rightarrow \quad (v_{T/C})_1 = (a_{T/C})_1 t_s. \tag{5}$$

Substituting Eqs. (1), (3) and (5) into Eq. (4), we obtain

$$0 = [(\mu_k g + a_T)t_s]^2 + 2(\mu_k g)\left[-d - \tfrac{1}{2}(\mu_k g + a_T)t_s^2\right].$$

Finally, substituting in t_s from Eq. (2), we get the final equation for μ_k

$$0 = \left[(\mu_k g + a_T)\left(-\frac{(v_T)_0}{a_T}\right)\right]^2 + 2(\mu_k g)\left[-d - \tfrac{1}{2}(\mu_k g + a_T)\left(-\frac{(v_T)_0}{a_T}\right)^2\right].$$

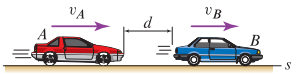
Solving for μ_k we get

$$\boxed{\mu_k = -\frac{(v_T)_0^2 a_T}{g\left[(v_T)_0^2 - 2da_T\right]} = 0.301,}$$

where we have used the following numerical data: $(v_T)_0 = v_0 = 60\,\text{mph} = 88.00\,\text{ft/s}$, $a_T = -10\,\text{ft/s}^2$, $g = 32.2\,\text{ft/s}^2$, and $d = 12\,\text{ft}$.

Problem 2.69

Cars A and B are traveling at $v_A = 72$ mph and $v_B = 67$ mph, respectively, when the driver of car B applies the brakes abruptly, causing the car to slide to a stop. The driver of car A takes 1.5 s to react to the situation and applies the brakes in turn, causing car A to slide as well. If A and B slide with equal accelerations, i.e., $\ddot{s}_A = \ddot{s}_B = -\mu_k g$, where $\mu_k = 0.83$ is the kinetic friction coefficient and g is the acceleration of gravity, compute the minimum distance d between A and B at the time B starts sliding to avoid a collision.



Solution

For both cars we can use the constant acceleration equation $v^2 = v_0^2 + 2a_c(s - s_0)$. Setting to zero the final velocities of both cars, we have

$$d_{B_s}: \qquad 0 = v_{B_i}^2 - 2\mu_k g d_{B_s} \quad \Rightarrow \quad d_{B_s} = \frac{v_{B_i}^2}{2\mu_k g}, \tag{1}$$

$$d_{A_s}: \qquad 0 = v_{A_i}^2 - 2\mu_k g d_{A_s} \quad \Rightarrow \quad d_{A_s} = \frac{v_{A_i}^2}{2\mu_k g} + v_{A_i} t_r, \tag{2}$$

where the $v_{A_i} t_r$ term in Eq. (2) represents the distance traveled during the reaction time, t_r. Realizing that $d_{A_s} = d_{B_s} + d$ gives the minimum distance d to avoid collision, we then have

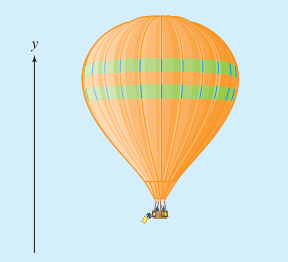
$$d = d_{A_s} - d_{B_s} = \frac{1}{2\mu_k g}\left(v_{A_i}^2 - v_{B_i}^2\right) + v_{A_i} t_r \quad \Rightarrow \quad \boxed{d = 187\,\text{ft},} \tag{3}$$

where we have used the following numerical data: $\mu_k = 0.83$, $g = 32.2\,\text{ft/s}^2$, $v_{Ai} = 72\,\text{mph} = 105.6\,\text{ft/s}$, $v_{Bi} = 67\,\text{mph} = 98.27\,\text{ft/s}$, and $t_r = 1.5\,\text{s}$.

Note: The the value of the quantity d given by the first of Eqs. (3) after substituting the available numerical data and expressing it using 4 significant figures is $d = 186.4\,\text{ft}$. We have rounded d from 186.4 ft to 187 ft because 186 ft will not avoid collision.

Problem 2.70

A hot air balloon is climbing with a velocity of 7 m/s when a sandbag (used as ballast) is released at an altitude of 305 m. Assuming that the sandbag is subject only to gravity and that therefore its acceleration is given by $\ddot{y} = -g$, g being the acceleration due to gravity, determine how long the sandbag takes to hit the ground and its impact velocity.



Solution

Since the acceleration of the sandbag is constant, we can relate position and time via the (constant acceleration) equation $y = y_0 + \dot{y}_0 t + \frac{1}{2} a t^2$, where $a = -g$. Solving this equation for t, we have

$$t = \frac{1}{g}\left(\dot{y}_0 + \sqrt{\dot{y}^2 + 2g y_0}\right) = 8.631\,\text{s}, \tag{1}$$

where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$, $\dot{y}_0 = 7\,\text{m/s}$, and $y_0 = 305\,\text{m}$. Expressing the above result to three significant figures, we have

$$\boxed{t = 8.63\,\text{s}.}$$

The impact velocity can be found by plugging the t value from Eq. (1) into the (constant acceleration) equation $\dot{y} = \dot{y}_0 + at$. Recalling that $a = -g$, this gives

$$\dot{y} = \dot{y}_0 - gt \quad \Rightarrow \quad \boxed{\dot{y} = -77.7\,\text{m/s}.}$$

Problem 2.71

Approximately 1 h 15 min into the movie "King Kong" (the one directed by Peter Jackson), there is a scene in which Kong is holding Ann Darrow (played by the actress Naomi Watts) in his hand while swinging his arm in anger. A quick analysis of the movie indicates that at a particular moment Kong displaces Ann from rest by roughly 10 ft in a span of four frames. Knowing that the DVD plays at 24 frames per second and assuming that Kong subjects Ann to a constant acceleration, determine the acceleration Ann experiences in the scene in question. Express your answer in terms of the acceleration due to gravity g. Comment on what would happen to a person *really* subjected to this acceleration.



Solution

The time it takes for King Kong to displace Ann is $\frac{4}{24}$ s $= 0.1667$ s. Using the constant acceleration equation $s = s_0 + \dot{s}_0 t + 0.5 a_c t^2$, we can the estimate an acceleration a_c given by

$$a_c = \frac{2s}{t^2} \quad \Rightarrow \quad a_c = 719.7\,\text{ft/s}^2 \frac{g}{32.2\,\text{ft/s}^2} = 22.35g \quad \Rightarrow \quad \boxed{a_c = 22.4g,} \tag{1}$$

where the last result has been expressed to 3 significant figures, and where we have used the following numerical data: $s = 10$ ft, $t = 0.1667$ s, $g = 32.2\,\text{ft/s}^2$. The human body can withstand around 10–15g of acceleration. Hence, at such a high acceleration the person would most certainly die.

Problem 2.72

Derive the constant acceleration relation in Eq. (2.41), starting from Eq. (2.33). State what assumption you need to make about the acceleration a to complete the derivation. Finally, use Eq. (2.36), along with the result of your derivation, to derive Eq. (2.42). Be careful to do the integral in Eq. (2.36) before substituting your result for $v(t)$ (try it without doing so, to see what happens). After completing this problem, notice that Eqs. (2.41) and (2.42) are *not* subject to the same assumption you needed to make to solve both parts of this problem.

Solution

Assuming that the acceleration is not equal to 0 and integrating Eq. (2.33), we have

$$t(v) = t_0 + \frac{1}{a_c}\int_{v_0}^{v} dv \quad \Rightarrow \quad t(v) = t_0 + \frac{1}{a_c}(v - v_0) \quad \Rightarrow \quad \boxed{v = v_0 + a_c(t - t_0).} \tag{1}$$

Integrating Eq. (2.36) we have

$$s = s_0 + \frac{1}{a_c}\int_{v_0}^{v} v\, dv \quad \Rightarrow \quad s = s_0 + \frac{1}{2a_c}\left(v^2 - v_0^2\right). \tag{2}$$

Substituting for v from Eq. (1), we obtain

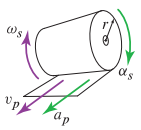
$$s = s_0 + \frac{1}{2a_c}\left[a_c^2(t - t_0)^2 + 2v_0 a_c\,(t - t_0)\right], \tag{3}$$

which can be simplified to obtain

$$\boxed{s = s_0 + v_0(t - t_0) + \tfrac{1}{2}a_c(t - t_0)^2.} \tag{4}$$

Problems 2.73 through 2.75





The spool of paper used in a printing process is unrolled with velocity v_p and acceleration a_p. The thickness of the paper is h, and the outer radius of the spool at any instant is r.

Problem 2.73 If the velocity at which the paper is unrolled is *constant*, determine the angular acceleration α_s of the spool as a function of r, h, and v_p. Evaluate your answer for $h = 0.0048$ in., for $v_p = 1000$ ft/min, and two values of r, that is, $r_1 = 25$ in. and $r_2 = 10$ in.

Problem 2.74 If the velocity at which the paper is unrolled is *not constant*, determine the angular acceleration α_s of the spool as a function of r, h, v_p, and a_p. Evaluate your answer for $h = 0.0048$ in., $v_p = 1000$ ft/min, $a_p = 3$ ft/s^2, and two values of r, that is, $r_1 = 25$ in. and $r_2 = 10$ in.

Problem 2.75 If the velocity at which the paper is unrolled is *constant*, determine the angular acceleration α_s of the spool as a function of r, h, and v_p. Plot your answer for $h = 0.0048$ in. and $v_p = 1000$ ft/min as a function of r for 1 in. $\leq r \leq 25$ in. Over what range does α_s vary?

Solution to 2.73

The radius decreases by the paper thickness h for every one revolution. Hence, letting θ be the angle measuring the angular position of a fixed radial line on the spool (θ increases when the spool turns clockwise), then we have

$$\frac{\Delta r}{\Delta \theta} = \frac{-h}{2\pi}. \tag{1}$$

Assuming that the decrease in radius can be viewed as occurring continuously, we can turn the above relation into a relation in terms of differentials. That is we can write

$$\frac{dr}{d\theta} = \frac{-h}{2\pi}. \tag{2}$$

Next, observing that the angular velocity of the spool is $\omega_s = d\theta/dt$, we can then use the above equation to relate the time rate of change of r to ω_s by applying the chain rule as follows:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad \Rightarrow \quad \dot{r} = \frac{-h}{2\pi}\omega_s \quad \Rightarrow \quad \omega_s = \frac{-2\pi}{h}\dot{r}. \tag{3}$$

Recalling that the linear velocity of the paper is related to the angular velocity of the spool as $v_p = r\omega_s$, we can use Eq. (3) to relate v_p to $\dot{r}$ as follows:

$$v_p = r\omega_s \quad \Rightarrow \quad v_p = \frac{-2\pi}{h} r\dot{r} \quad \Rightarrow \quad \dot{r} = \frac{-h\, v_p}{2\pi r}. \tag{4}$$

Taking the time derivative of v_p in the second of Eqs. (4), accounting for the fact that v_p is constant (i.e., $a_p = 0$), and using the last of Eqs. (4), we have

$$a_p = 0 = \frac{-2\pi}{h}\left(\dot{r}^2 + r\ddot{r}\right) \quad \Rightarrow \quad 0 = \frac{-2\pi}{h}\left(\frac{h^2 v_p^2}{4\pi^2 r^2} + r\ddot{r}\right). \tag{5}$$

Taking the time derivative of the last of Eqs. (3) and the last of Eqs. (4) to find expressions for α_s and $\ddot{r}$, we can write

$$\alpha_s = \frac{-2\pi}{h}\ddot{r} \quad \text{and} \quad \ddot{r} = \frac{-h^2 v_p^2}{4\pi^2 r^3}, \tag{6}$$

which, when taken combined, imply that

$$\boxed{\alpha_s = \frac{h v_p^2}{2\pi r^3}.}$$

Evaluating the expression above for $h = 0.0048\,\text{in.} = 0.0004000\,\text{ft}$, $v_p = 1000\,\text{ft/min} = 16.67\,\text{ft/s}$, $r_1 = 25\,\text{in.} = 2.083\,\text{ft}$, and $r_2 = 10\,\text{in.} = 0.8333\,\text{ft}$, we have

$$\boxed{\alpha_s\big|_{r=r_1} = 0.00196\,\text{rad/s}^2 \quad \text{and} \quad \alpha_s\big|_{r=r_2} = 0.0306\,\text{rad/s}^2.}$$

Solution to 2.74

The radius decreases by the paper thickness h for every one revolution. Hence, letting θ be the angle measuring the angular position of a fixed radial line on the spool (θ increases when the spool turns clockwise), then we have

$$\frac{\Delta r}{\Delta \theta} = \frac{-h}{2\pi}. \tag{7}$$

Assuming that the decrease in radius can be viewed as occurring continuously, we can turn the above relation into a relation in terms of differentials. That is we can write

$$\frac{dr}{d\theta} = \frac{-h}{2\pi}. \tag{8}$$

Next, observing that the angular velocity of the spool is $\omega_s = d\theta/dt$, we can then use the above equation to relate the time rate of change of r to ω_s by applying the chain rule as follows:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad \Rightarrow \quad \dot{r} = \frac{-h}{2\pi}\omega_s \quad \Rightarrow \quad \omega_s = \frac{-2\pi}{h}\dot{r}. \tag{9}$$

Recalling that the linear velocity of the paper is related to the angular velocity of the spool as $v_p = r\omega_s$, we can use Eq. (9) to relate v_p to $\dot{r}$ as follows:

$$v_p = r\omega_s \quad \Rightarrow \quad v_p = \frac{-2\pi}{h} r\dot{r} \quad \Rightarrow \quad \dot{r} = \frac{-h v_p}{2\pi r}. \tag{10}$$

Taking the time derivative of the second of Eqs. (10) and then using the last of Eqs. (10) to substitute for the term $\dot{r}$, we have

$$\dot{v}_p = a_p = \frac{-2\pi}{h}\left(\frac{h^2 v_p^2}{4\pi^2 r^2} + r\ddot{r}\right). \tag{11}$$

To find an expression for the term $\ddot{r}$, we take the time derivative of $\dot{r}$ in Eq. (9) and obtain

$$\ddot{r} = \frac{-h}{2\pi}\alpha_s. \tag{12}$$

Substituting the result from Eq. (12) into Eq. (11), we have

$$a_p = \frac{-2\pi}{h}\left(\frac{h^2 v_p^2}{4\pi^2 r^2} - \frac{hr}{2\pi}\alpha_s\right). \tag{13}$$

Then, solve for α_s and simplifying, we obtain

$$\boxed{\alpha_s = \frac{a_p}{r} + \frac{h v_p^2}{2\pi r^3}.}$$

Evaluating the expression above for $h = 0.0048\,\text{in.} = 0.0004000\,\text{ft}$, $v_p = 1000\,\text{ft/min} = 16.67\,\text{ft/s}$, $a_p = 3\,\text{ft/s}^2$, $r_1 = 25\,\text{in.} = 2.083\,\text{ft}$, and $r_2 = 10\,\text{in.} = 0.8333\,\text{ft}$, we have

$$\boxed{\alpha_s\big|_{r=r_1} = 1.44\,\text{rad/s}^2 \quad \text{and} \quad \alpha_s\big|_{r=r_2} = 3.63\,\text{rad/s}^2.}$$

Solution to 2.75

The radius decreases by the paper thickness h for every one revolution. Hence, letting θ be the angle measuring the angular position of a fixed radial line on the spool (θ increases when the spool turns clockwise), then we have

$$\frac{\Delta r}{\Delta \theta} = \frac{-h}{2\pi}. \tag{14}$$

Assuming that the decrease in radius can be viewed as occurring continuously, we can turn the above relation into a relation in terms of differentials. That is we can write

$$\frac{dr}{d\theta} = \frac{-h}{2\pi}. \tag{15}$$

Next, observing that the angular velocity of the spool is $\omega_s = d\theta/dt$, we can then use the above equation to relate the time rate of change of r to ω_s by applying the chain rule as follows:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad \Rightarrow \quad \dot{r} = \frac{-h}{2\pi}\omega_s \quad \Rightarrow \quad \omega_s = \frac{-2\pi}{h}\dot{r}. \tag{16}$$

Recalling that the linear velocity of the paper is related to the angular velocity of the spool as $v_p = r\omega_s$, we can use Eq. (16) to relate v_p to $\dot{r}$ as follows:

$$v_p = r\omega_s \quad \Rightarrow \quad v_p = \frac{-2\pi}{h} r\dot{r} \quad \Rightarrow \quad \dot{r} = \frac{-h v_p}{2\pi r}. \tag{17}$$

Taking the time derivative of v_p in the second of Eqs. (17), accounting for the fact that v_p is constant (i.e., $a_p = 0$), and using the last of Eqs. (17), we have

$$a_p = 0 = \frac{-2\pi}{h}\left(\dot{r}^2 + r\ddot{r}\right) \quad \Rightarrow \quad 0 = \frac{-2\pi}{h}\left(\frac{h^2 v_p^2}{4\pi^2 r^2} + r\ddot{r}\right). \tag{18}$$

Taking the time derivative of the last of Eqs. (16) and the last of Eqs. (17) to find expressions for α_s and $\ddot{r}$, we can write

$$\alpha_s = \frac{-2\pi}{h}\ddot{r} \quad \text{and} \quad \ddot{r} = \frac{-h^2 v_p^2}{4\pi^2 r^3}, \tag{19}$$

which, when taken combined, imply that

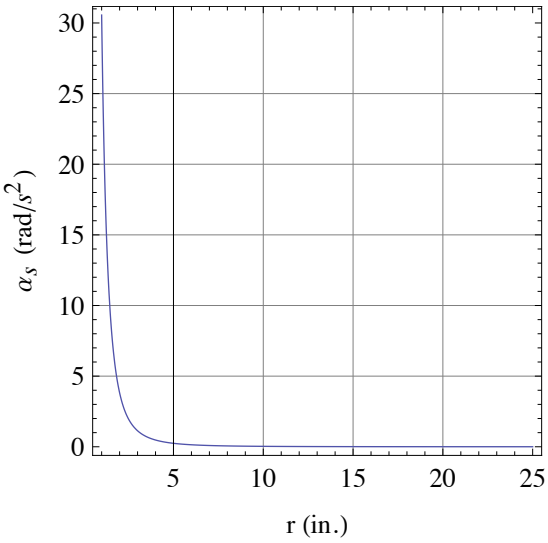
$$\boxed{\alpha_s = \frac{h v_p^2}{2\pi r^3}.}$$

To plot the above function, we first substitute the values of the known coefficients, i.e., we recall that $h = 0.0048\,\text{in.}$, and $v_p = 1000\,\text{ft/min} = 200.0\,\text{in./s}$ and rewrite α_s as

$$\alpha_s = \frac{30.56\,\text{in.}^3/\text{s}^2}{r^3}. \tag{20}$$

The above function can now be plotted with a variety of pieces of mathematical software. The plot presented below was generated using*Mathematica* with the following code:

```
Plot[30.56/r^3, {r, 1, 25}, PlotRange → All, Frame → True, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"r (in.)", "αs (rad/s²)"}]
```



By inspection, α_s appears to vary from 30.5 rad/s^2 to close to zero as r varies from 1 in. to 4 in.

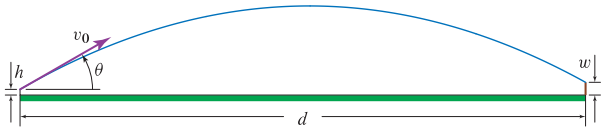
Problem 2.76

The discussion in Example 2.12 revealed that the angle θ had to be greater than $\theta_{\text{min}} = 0.716°$. Find an analytical expressions for θ_{min} in terms of h, w, and d.

Solution

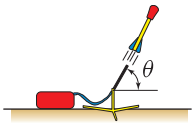
With infinite speed, there is no effect of gravity. Hence

$$\theta = \tan^{-1}\left(\frac{w-h}{d}\right).$$



Problem 2.77

A stomp rocket is a toy consisting of a hose connected to a "blast pad" (i.e., an air bladder) at one end and to a short pipe mounted on a tripod at the other end. A rocket with a hollow body is mounted onto the pipe and is propelled into the air by "stomping" on the blast pad. Some manufactures claim that one can shoot a rocket over 200 ft in the air. Neglecting air resistance, determine the rocket's minimum initial speed such that it reaches a maximum flight height of 200 ft.



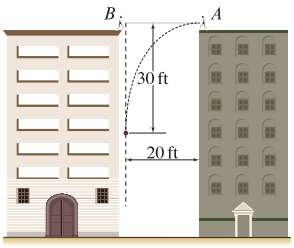
Solution

The rocket must be fired vertically to reach 200 ft with minimum initial speed. Observing that $\ddot{y} = -g =$ constant, recalling that $\dot{y} = 0$, using constant acceleration equations, we have

$$\dot{y}^2 - \dot{y}_0^2 = -2g(h_{\max}) \quad \Rightarrow \quad \dot{y}_0 = \sqrt{2gh_{\max}} \quad \Rightarrow \quad \boxed{\dot{y}_0 = 113\,\text{ft/s}.} \tag{1}$$

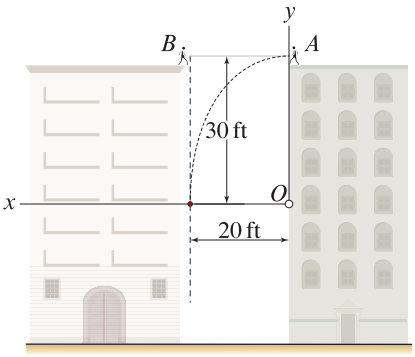
Problem 2.78

Stuntmen A and B are shooting a movie scene in which A needs to pass a gun to B. Stuntman B is supposed to start falling vertically precisely when A throws the gun to B. Treating the gun and the stuntman B as particles, find the velocity of the gun as it leaves A's hand so that B will catch it after falling 30 ft.



Solution

Since the gun and B drop at the same time, from the same height, and are assumed to be both subject to the same acceleration, i.e., gravity, then in order for the gun and B to the same vertical position at the time that B grasps the gun, it is necessary for the gun and B to fall with equal vertical velocities. This can only be achieved if the initial vertical velocity of the gun is equal to that of B, namely, zero. Hence, the rest of the problem is devoted to finding the horizontal component of the velocity of the gun at the time the gun is thrown. To do so, we start by finding the time B takes to fall the distance $h = 30\,\text{ft}$. Using the coordinate system shown on the right, and using constant acceleration equations, we have



$$y_B = y_{B_0} + \dot{y}_{B_0}\,(t - t_0) - \tfrac{1}{2} g\,(t - t_0)^2\,, \tag{1}$$

where y_{B_0} and $\dot{y}_{B_0}$ are the position and the vertical velocity of B at time t_0, respectively. Setting $t_0 = 0$ and recalling that B drops from rest a distance h above the origin of the y axis, we can rewrite the above equation in the following form:

$$y_B = h - \tfrac{1}{2} g t^2. \tag{2}$$

Due to our choice of origin, B will grasp the gun at $y_B = 0$. Letting t_f (the subscript f stands for final) denote the time at which B grasps the gun, from Eq. (2) we then have

$$t_f = \sqrt{2h/g}. \tag{3}$$

We now observe that the motion of the gun in the horizontal direction is also a constant acceleration motion with acceleration equal to zero. Hence, using the subscript A to refer to the gun (as opposed to the stuntman who initially threw it) we have

$$x_A = x_{A_0} + \dot{x}_{A_0}(t - t_0), \tag{4}$$

where x_{A_0} and $\dot{x}_{A_0}$ are the gun's position and velocity in the x direction at time t_0. Having already set $t_0 = 0$ and observing that $x_{A_0} = 0$, and that for $t = t_f$ we must have $x_A = d$, from the above equation we have

$$d = \dot{x}_{A_0}\sqrt{\frac{2h}{g}} \quad \Rightarrow \quad \dot{x}_{A_0} = d\sqrt{\frac{g}{2h}} = 14.65\,\text{ft/s}, \tag{5}$$

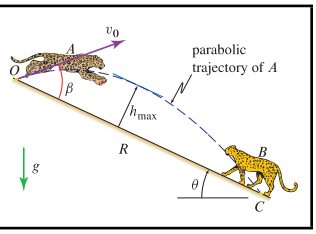
where we have used the following numerical data: $d = 20\,\text{ft}$, $h = 30\,\text{ft}$, and $g = 32.2\,\text{ft/s}^2$.

In summary, expressing our answer in vector form and to 3 significant figures, we have

$$\vec{v}_{\text{gun}} = (14.7\,\text{ft/s})\,\hat{\imath}\,.$$

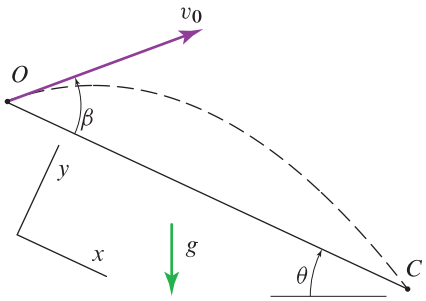
Problem 2.79

The jaguar A leaps from O at speed $v_0 = 6\,\text{m/s}$ and angle $\beta = 35°$ relative to the incline to try to intercept the panther B at C. Determine the distance R that the jaguar jumps from O to C (i.e., R is the distance between the two points of the trajectory that intersect the incline), given that the angle of the incline is $\theta = 25°$.



Solution

The acceleration of A is completely in the vertical direction. Hence, referring to the figure on the right, the components of the acceleration of A in the x and y directions are $\ddot{x} = g\sin\theta$ and $\ddot{y} = -g\cos\theta$. Again referring to the figure on the right, the components of the initial velocity of A are $v_{x0} = v_0\cos\beta$ and $v_{y0} = v_0\sin\beta$. Observing that the x and y component of acceleration of A are both constants, we can use constant acceleration equations to write



$$x = v_0(\cos\beta)t + \tfrac{1}{2}g(\sin\theta)t^2 = t\left[v_0\cos\beta + \tfrac{1}{2}g(\sin\theta)t\right], \quad (1)$$

$$y = v_0(\sin\beta)t - \tfrac{1}{2}g(\cos\theta)t^2 = t\left[v_0\sin\beta - \tfrac{1}{2}g(\cos\theta)t\right]. \quad (2)$$

For $t = t_C$, $y = 0$. Therefore

$$0 = v_0\sin\beta - \tfrac{1}{2}g(\cos\theta)t_C \quad \Rightarrow \quad t_C = \frac{2v_0\sin\beta}{g\cos\theta}. \quad (3)$$

Observing that, for $t = t_C$, $x(t_C) = R$, substituting this value of t into Eq. (1), we have

$$R = \frac{2v_0\sin\beta}{g\cos\theta}\left[v_0\cos\beta + \tfrac{1}{2}g(\sin\theta)\frac{2v_0\sin\beta}{g\cos\theta}\right], \quad (4)$$

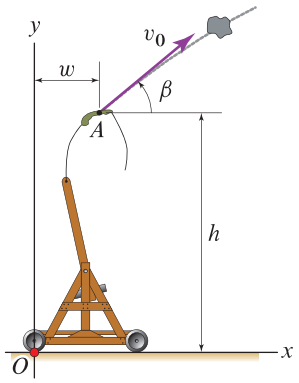
which can be simplified to

$$\boxed{R = \frac{2v_0^2\sin\beta}{g\cos\theta}\left[\cos\beta + \tan\theta\sin\beta\right] = 5.05\,\text{m}.} \quad (5)$$

where we have used the following numerical data: $v_0 = 6\,\text{m/s}$, $\beta = 35°$, $g = 9.81\,\text{m/s}^2$, and $\theta = 25°$.

Problem 2.80

If the projectile is released at A with initial speed v_0 and angle β, derive the projectile's trajectory, using the coordinate system shown. Neglect air resistance.



Solution

Using the coordinate system indicated in the problem statement, we set up the following constant acceleration equations for both the x and y coordinates of the projectile:

$$x = x_A + (v_0 \cos\beta)t, \tag{1}$$

$$y = y_A + (v_0 \sin\beta)t - \tfrac{1}{2}gt^2, \tag{2}$$

where x_A and y_A are the coordinates of the fixed point A at which the projectile is released. Eliminating t between Eqs. (1) and (2), we obtain

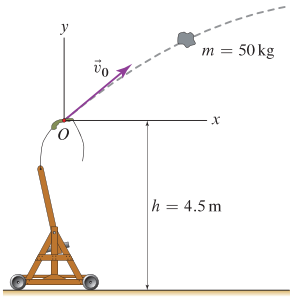
$$y - y_A = \tan\beta(x - x_A) - \frac{g}{2v_0^2\cos^2\beta}(x - x_A)^2. \tag{3}$$

Observing that $x_A = w$ and $y_A = h$, the above equation becomes

$$\boxed{y = h + \tan\beta(x - w) - \frac{g}{2v_0^2\cos^2\beta}(x - w)^2.}$$

Problem 2.81

A trebuchet releases a rock with mass $m = 50\,\text{kg}$ at point O. The initial velocity of the projectile is $\vec{v}_0 = (45\,\hat{\imath} + 30\,\hat{\jmath})\,\text{m/s}$. Neglecting aerodynamic effects, determine where the rock will land and its time of flight.



Solution

Referring to the coordinate system defined in the problem statement, we see that y_{land}, the y coordinate of the rock when it lands on the ground, is $-h$. With this in mind, we can write the following constant acceleration equation for the y displacement:

$$y = v_{y0}t - \tfrac{1}{2}gt^2,$$

where it is understood that $t = 0$ is the time of release and $y = 0$ and v_{y0} are the vertical position and the vertical component of velocity of the rock at time $t = 0$, respectively. Denoting by t_{flight} the time at which the rock impacts the ground, we must have

$$-h = v_{y0}t_{\text{flight}} - \tfrac{1}{2}gt_{\text{flight}}^2 \quad \Rightarrow \quad gt_{\text{flight}}^2 - 2v_{y0} - 2h = 0 \quad \Rightarrow \quad t_{\text{flight}} = \frac{v_{y0} \pm \sqrt{v_{y0}^2 + 2gh}}{g}. \qquad (1)$$

The only physically meaningful solution for t_{flight} is that corresponding to the $+$ sign in front of the square root, that is,

$$t_{\text{flight}} = \frac{1}{g}\left[v_{y0} + \sqrt{v_{y0}^2 + 2gh}\right] = 6.263\,\text{s}, \qquad (2)$$

where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$, $v_{y0} = 30\,\text{m/s}$, and $h = 4.5\,\text{m}$. Expressing the time of flight to three significant figures, we have

$$\boxed{t_{\text{flight}} = 6.26\,\text{s}.}$$

Next observing that the motion is in the x direction is a constant acceleration motion with acceleration equal to zero, we have that the x coordinate of the rock is described by the following (constant acceleration) equation:

$$x = v_{x0}t,$$

where we have accounted for the fact that at $t = 0$ we must have $x = 0$ and where v_{x0} is the x component of the velocity of the rock for $t = 0$. Hence, recalling that $v_{x0} = 45\,\text{m/s}$, for $t = t_{\text{flight}} = 6.263\,\text{s}$ (see Eq. (2)), we have

$$x_{\text{land}} = 45t_{\text{flight}} = 281.8\,\text{m}.$$

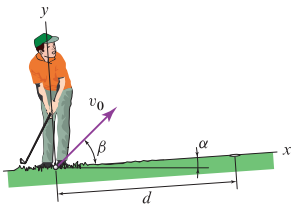
Hence, summarizing, the position vector where the rock will land, expressed to 3 significant figures is

$$\boxed{\vec{r}_{\text{land}} = (282\,\hat{\imath} - 4.50\,\hat{\jmath})\,\text{m},}$$

where we have used the fact that $y_{\text{land}} = -h = 4.5\,\text{m}$.

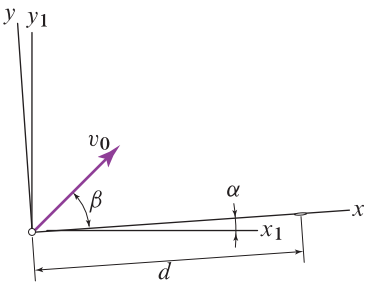
Problem 2.82

A golfer chips the ball into the hole on the fly from the rough at the edge of the green. Letting $\alpha = 4°$ and $d = 2.4\,\text{m}$, verify that the golfer will place the ball within 10 mm of the hole if the ball leaves the rough with a speed $v_0 = 5.03\,\text{m/s}$ and an angle $\beta = 41°$.



Solution

Referring to the figure on the right, we can write the ball's trajectory with respect to the $x_1 y_1$ coordinate system with origin at the golfer's feet, by using Eq. (2.52) on p. 74 of the textbook. This gives



$$y_1 = \tan(\alpha+\beta)x_1 - \left[\frac{g\sec^2(\alpha+\beta)}{2v_0^2}\right]x_1^2. \tag{1}$$

Recalling that the problem statement gives $\alpha = 4°$ and $\beta = 41°$, so that $\alpha + \beta = 45°$. This implies that $\tan(\alpha+\beta) = 1$ and $\sec^2(\alpha+\beta) = 2$, which then implies that Eq. (1) can be given the following simpler form:

$$y_1 = x_1 - \frac{g}{v_0^2}x_1^2. \tag{2}$$

The x_1 and y_1 coordinates of the point at which the ball lands must satisfy the condition $x_1 \tan\alpha = y_1$. Combining this requirement with Eq. (2) we have

$$x_1\tan\alpha = x_1 - \frac{g}{v_0^2}x_1^2 \quad \Rightarrow \quad x_1 = \frac{v_0^2}{g}\big(1-\tan\alpha\big) = 2.399\,\text{m}, \tag{3}$$

where we have used the following numerical values: $v_0 = 5.03\,\text{m}$, $g = 9.81\,\text{m/s}^2$, and $\alpha = 4°$. The value of x_1 in Eq. (3) is the x_1 coordinate of the ball's landing spot. With this information, letting d_L denote the distance between the ball's landing spot and the the golfer, we can determine d_L using trigonometry as follows:

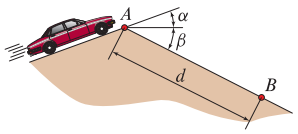
$$d_L = \frac{x_1}{\cos\alpha} = 2.405\,\text{m} \quad \Rightarrow \quad d - d_L = 0.005\,\text{m} < 10\,\text{mm}, \tag{4}$$

where we have used the fact that $\alpha = 4°$ and $d = 2.4\,\text{m}$.

The golfer's chip shot is successful.

Problems 2.83 and 2.84

In a movie scene involving a car chase, a car goes over the top of a ramp at A and lands at B below.

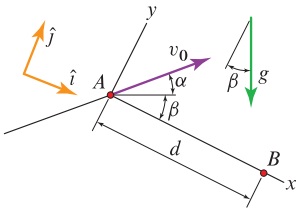


Problem 2.83 If $\alpha = 20°$ and $\beta = 23°$, determine the distance d covered by the car if the car's speed at A is 45 km/h. Neglect aerodynamic effects.

Problem 2.84 Determine the speed of the car at A if the car is to cover distance $d = 150$ ft for $\alpha = 20°$ and $\beta = 27°$. Neglect aerodynamic effects.

Solution to 2.83

Expressing the acceleration vector of the car in the xy coordinate system, we have



$$\vec{a} = g\sin\beta\,\hat{\imath} - g\cos\beta\,\hat{\jmath} \quad \Rightarrow \quad \ddot{x} = g\sin\beta \quad \text{and} \quad \ddot{y} = -g\cos\beta. \tag{1}$$

Now let $t = 0$ be the instant at which the car jumps off at A. Letting A be the origin of the chosen xy coordinate system, the y coordinate of the car at $t = 0$ is equal to zero. Next we observe that, again at time $t = 0$, the velocity components of the cars are as follows:

$$v_x(0) = v_0\cos(\alpha+\beta) \quad \text{and} \quad v_y(0) = v_0\sin(\alpha+\beta). \tag{2}$$

Hence, using the above considerations along with constant acceleration equations, the y coordinate of the car as a function of time is given by

$$y = v_0\sin(\alpha+\beta)t - \tfrac{1}{2}g(\cos\beta)t^2. \tag{3}$$

Point B denotes the point at which the car lands. Observing that $y_B = 0$, the time taken by the car to reach B is obtained by setting Eq. (3) equal to zero. This gives

$$v_0\sin(\alpha+\beta)t_B - \tfrac{1}{2}(g\cos\beta)t_B^2 = 0 \quad \Rightarrow \quad t_B = \frac{2v_0\sin(\alpha+\beta)}{g\cos\beta} = 1.888\,\text{s}, \tag{4}$$

where we have used the following numerical data: $v_0 = 45\,\text{km/h} = 12.50\,\text{ft/s}$, $\alpha = 20°$, $\beta = 23°$, and $g = 9.81\,\text{m/s}^2$.

Next observing that the x component of the acceleration in the second of Eqs. (1) is also constant, using constant acceleration equations, the x coordinate of the car as a function of time is given by

$$x = v_0\cos(\alpha+\beta)t + \tfrac{1}{2}(g\sin\beta)t^2. \tag{5}$$

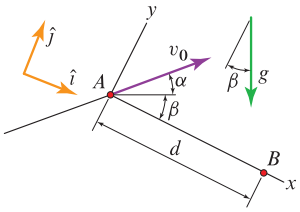
Hence, for $t = t_B$ we have

$$x_B = d = v_0\cos(\alpha+\beta)]t_B + \tfrac{1}{2}(g\sin\beta)t_B^2 = 24.09\,\text{m}, \tag{6}$$

where we have used the following numerical values: $v_0 = 45\,\text{km/h} = 12.50\,\text{ft/s}$, $\alpha = 20°$, $\beta = 23°$, $t_B = 1.888\,\text{s}$ (see Eq. (4)), and $g = 9.81\,\text{m/s}^2$. Hence, expressing our answer to 3 significant digits, we have

$$\boxed{d = 24.1\,\text{m}.} \tag{7}$$

Solution to 2.84



Expressing the acceleration vector of the car in the xy coordinate system, we have

$$\vec{a} = g\sin\beta\,\hat{\imath} - g\cos\beta\,\hat{\jmath} \quad \Rightarrow \quad \ddot{x} = g\sin\beta \quad \text{and} \quad \ddot{y} = -g\cos\beta. \tag{8}$$

Now let $t = 0$ be the instant at which the car jumps off at A. Letting A be the origin of the chosen xy coordinate system, the y coordinate of the car at $t = 0$ is equal to zero. Next we observe that, again at time $t = 0$, the velocity components of the cars are as follows:

$$v_x(0) = v_0\cos(\alpha+\beta) \quad \text{and} \quad v_y(0) = v_0\sin(\alpha+\beta). \tag{9}$$

Hence, using the above considerations along with constant acceleration equations, the y coordinate of the car as a function of time is given by

$$y = v_0\sin(\alpha+\beta)t - \tfrac{1}{2}g(\cos\beta)t^2. \tag{10}$$

Point B denotes the point at which the car lands. Observing that $y_B = 0$, the time taken by the car to reach B is obtained by setting Eq. (10) equal to zero. This gives

$$v_0\sin(\alpha+\beta)t_B - \tfrac{1}{2}(g\cos\beta)t_B^2 = 0 \quad \Rightarrow \quad t_B = \frac{2v_0\sin(\alpha+\beta)}{g\cos\beta}. \tag{11}$$

Next observing that the x component of the acceleration in the second of Eqs. (8) is also constant, using constant acceleration equations, the x coordinate of the car as a function of time is given by

$$x = v_0\cos(\alpha+\beta)t + \tfrac{1}{2}(g\sin\beta)t^2. \tag{12}$$

Hence, for $t = t_B$ we have

$$x_B = d = v_0\cos(\alpha+\beta)\frac{2v_0\sin(\alpha+\beta)}{g\cos\beta} + \tfrac{1}{2}(g\sin\beta)\left(\frac{2v_0\sin(\alpha+\beta)}{g\cos\beta}\right)^2$$
$$\Rightarrow \quad \frac{2v_0^2\sin(\alpha+\beta)}{g\cos\beta}[\cos(\alpha+\beta) + \tan\beta\sin(\alpha+\beta)] = d, \tag{13}$$

which can be solved for v_0 to obtain

$$v_0 = \sqrt{\frac{dg\cos\beta}{2\sin(\alpha+\beta)[\cos(\alpha+\beta) + \tan\beta\sin(\alpha+\beta)]}}. \tag{14}$$

Recalling that $d = 150\,\text{ft}$, $\alpha = 20°$, $\beta = 27°$, and $g = 32.2\,\text{ft/s}^2$, Eq. (14) can be evaluated to obtain

$$\boxed{v_0 = 52.8\,\text{ft/s}.}$$

Problem 2.85

The M777 lightweight 155 mm howitzer is a piece of artillery whose rounds are ejected from the gun with a speed of 829 m/s. Assuming that the gun is fired over a flat battlefield and ignoring aerodynamic effects, determine (*a*) the elevation angle needed to achieve the maximum range, (*b*) the maximum possible range of the gun, and (*c*) the time it would take a projectile to cover the maximum range. Express the result for the range as a percentage of the actual maximum range of this weapon, which is 30 km for unassisted ammunition.



Solution

Let θ be the elevation angle and $v_0 = 829\,\text{m/s}$ be the intial speed. Use the constant acceleration equation $s = v_0 t + 0.5 a_c t^2$, the x and y displacements for constant acceleration are given by

$$x = (v_0 \cos\theta)t, \tag{1}$$

$$y = (v_0 \sin\theta)t - \tfrac{1}{2} g t^2. \tag{2}$$

The time of flight can be obtained by equating Eq. (2) to zero. Hence, we have

$$t = \frac{2 v_0 \sin\theta}{g}. \tag{3}$$

Substituting Eq. (3) into Eq. (1), and using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$, we get

$$x = R = \frac{v_0^2 \sin 2\theta}{g}. \tag{4}$$

(i) From Eq. (4), $R_{\max}$ occurs when $\sin 2\theta = 1$. The meaningful solution yields:

$$2\theta = \frac{\pi}{2}\,\text{rad} \quad \Rightarrow \quad \boxed{\theta_{\max} = \frac{\pi}{4}\,\text{rad} = 45^\circ.} \tag{5}$$

(ii) From Eq. (4), the maximum possible value of x or the range R is given by

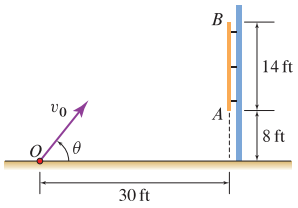
$$R_{\max} = \frac{v_0^2}{g} = 70.06\,\text{km} \quad \Rightarrow \quad \boxed{R_{\max} = 234\%\text{ of the actual maximum range.}}$$

(iii) Substituting the value of $\theta_{\max}$ from Eq. (5) into Eq. (3), we get

$$\boxed{t = \frac{\sqrt{2} v_0}{g} = 120\,\text{s}.}$$

Problem 2.86

You want to throw a rock from point O to hit the vertical advertising sign AB, which is $R = 30\,\text{ft}$ away. You are able to throw a rock with the speed $v_0 = 45\,\text{ft/s}$. The bottom of the sign is 8 ft off the ground and the sign is 14 ft tall. Determine the range of angles at which the projectile can be thrown in order to hit the target, and compare this with the angle subtended by the target as seen from an observer at point O. Compare your results with those found in Example 2.11.



Solution

This problem can be solved by following the same steps illustrated in Example 2.11 on p. 76 of the textbook. Specifically, we recall Eq. (7) in Example 2.11 on p. 76 of the textbook:

$$\tan\theta = \frac{v_0^2 \pm \sqrt{v_0^4 - g(gR^2 + 2yv_0^2)}}{gR}. \tag{1}$$

Observe that we are given all of the data needed to use the above equation. Namely, we have $v_0 = 45\,\text{ft/s}$, $g = 32.2\,\text{ft/s}^2$, $R = 30\,\text{ft}$, so that substituting in the above equation $y_A = 8\,\text{ft}$ and $y_B = 22\,\text{ft}$ we have

$$y = y_A = 8\,\text{ft} \quad \Rightarrow \quad \begin{cases} \theta_1 = 30.44°, \\ \theta_2 = 74.50°, \end{cases} \tag{2}$$

$$y = y_B = 22\,\text{ft} \quad \Rightarrow \quad \begin{cases} \theta_1 = 56.84°, \\ \theta_2 = 69.41°. \end{cases} \tag{3}$$

Following the same logic as in Example 2.11 we obtain the two ranges of firing angles as

$$\boxed{\qquad\qquad} \tag{4}$$

Calculating the sizes of these ranges

$$\Delta\theta_1 = 56.84° - 30.44° = 26.40°, \tag{5}$$

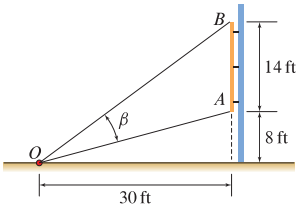
$$\Delta\theta_2 = 74.50° - 69.41° = 5.090°, \tag{6}$$

which, when expressed to 3 significant figures gives

$$\boxed{\Delta\theta_1 = 26.4° \quad \text{and} \quad \Delta\theta_2 = 5.09°.}$$

The angle subtended by the target as seen from an observer at point O is

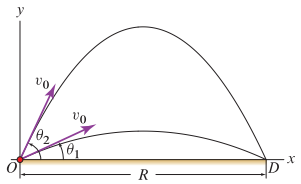
$$\beta = \tan^{-1}\left(\frac{22}{30}\right) - \tan^{-1}\left(\frac{8}{30}\right) \quad \Rightarrow \quad \boxed{\beta = 21.3°.} \tag{7}$$



Unlike Example 2.11, the difference between the angle subtended by the target and $\Delta\theta_1$ or $\Delta\theta_2$ is significant. In addition, we see that the value of $\Delta\theta_1$ is much closer to β than $\Delta\theta_2$.

Problem 2.87

Suppose that you can throw a projectile at a large enough v_0 so that it can hit a target a distance R downrange. Given that you know v_0 and R, determine the general expressions for the *two* distinct launch angles θ_1 and θ_2 that will allow the projectile to hit D. For $v_0 = 30\,\text{m/s}$ and $R = 70\,\text{m}$, determine numerical values for θ_1 and θ_2.



Solution

Applying the constant acceleration equation (Eq. (2.42) on p. 59 of the textbook) for both x and y displacements, we get

$$x = x_0 + (v_0 \cos\theta)t,$$
$$y = y_0 + (v_0 \sin\theta)t - \tfrac{1}{2}gt^2,$$

where $(x_0, y_0) = (0, 0)$, $x = R$ and $y = 0$. Eliminating t, we get

$$gR - 2v_0^2 \sin\theta \cos\theta = 0.$$

By simplifying, we get

$$\sin 2\theta = \sin(180^\circ - 2\theta) = \frac{gR}{v_0^2}.$$

Therefore,

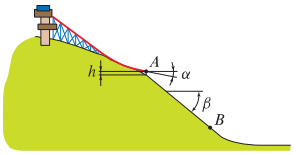
$$\theta_1 = \tfrac{1}{2}\sin^{-1}\left(\frac{gR}{v_0^2}\right) \quad \text{and} \quad \theta_2 = 90^\circ - \tfrac{1}{2}\sin^{-1}\left(\frac{gR}{v_0^2}\right).$$

Plugging in the given values, we get

$$\boxed{\theta_1 = 24.9^\circ \quad \text{and} \quad \theta_2 = 65.1^\circ.}$$

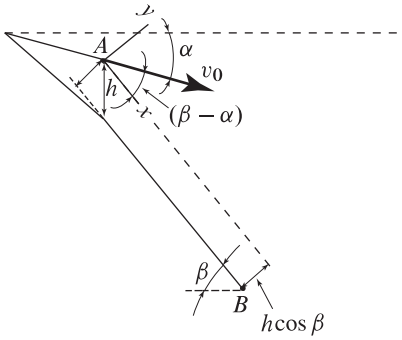
Problem 2.88

An alpine ski jumper can fly distances in excess of 100 m by using his or her body and skis as a "wing" and therefore taking advantage of aerodynamic effects. With this in mind and assuming that a ski jumper could survive the jump, determine the distance the jumper could "fly" without aerodynamic effects, i.e., if the jumper were in free fall after clearing the ramp. For the purpose of your calculation, use the following typical data: $\alpha = 11°$ (slope of ramp at takeoff point A), $\beta = 36°$ (average slope of the hill), $v_0 = 86\,\text{km/h}$ (speed at A), $h = 3\,\text{m}$ (height of takeoff point with respect to the hill). Finally, for simplicity, let the jump distance be the distance between the takeoff point A and the landing point B.



Solution

Referring to the figure on the right, the velocity and acceleration vectors relative to the xy coordinate system is given by



$$\vec{v} = v_0\cos(\beta-\alpha)\,\hat{\imath} + v_0\sin(\beta-\alpha)\,\hat{\jmath} \tag{1}$$

$$\vec{a} = g\sin\beta\,\hat{\imath} - g\cos\beta\,\hat{\jmath} \tag{2}$$

Letting $t = 0$ be the time at which the jumper takes off at A, we can express the coordinates of the jumper as a function of time using constant acceleration equations. Specifically, we have

$$x = v_0\cos(\beta-\alpha)t + \tfrac{1}{2}(g\sin\beta)t^2, \tag{3}$$

$$y = v_0\sin(\beta-\alpha)t - \tfrac{1}{2}(g\cos\beta)t^2. \tag{4}$$

Therefore, letting t_B denote the time at which the jumper lands at B, we have

$$x_B = v_0\cos(\beta-\alpha)t_B + \tfrac{1}{2}(g\sin\beta)t_B^2, \tag{5}$$

$$y_B = v_0\sin(\beta-\alpha)t_B - \tfrac{1}{2}(g\cos\beta)t_B^2. \tag{6}$$

Now, note that we must have $y_B = -h\cos\beta$. So, from Eq. (6) we have

$$-h\cos\beta = v_0\sin(\beta-\alpha)t_B - \tfrac{1}{2}(g\cos\beta)t_B^2 \quad\Rightarrow\quad (g\cos\beta)t_B^2 - 2v_0\sin(\beta-\alpha)t_B - 2h\cos\beta = 0$$

$$\Rightarrow\quad t_B = \frac{v_0\sin(\beta-\alpha) \pm \sqrt{v_0^2\sin^2(\beta-\alpha) + 2hg\cos^2\beta}}{g\cos\beta}. \tag{7}$$

The only acceptable value for t_B is the one obtained using the $+$ sign in front of the square root. With this in mind, and recalling that $v_0 = 86\,\text{km/h} = 23.89\,\text{m/s}$, $\beta = 36°$, $\alpha = 11°$, $h = 3\,\text{m}$, and $g = 9.81\,\text{m/s}^2$, we then have

$$t_B = 2.766\,\text{s}. \tag{8}$$

Using the data listed right above Eq. (8) and substituting the value of t_B from Eq. (8) into Eq. (5), we have $x_B = 81.93\,\text{m}$. Then, recalling that $y_B = -h\cos\beta = -2.427\,\text{m}$, we have that the distance between points A and B can be calculated using the Pythagorean theorem, i.e., $d_{AB} = \sqrt{x_B^2 + y_B^2}$ which gives

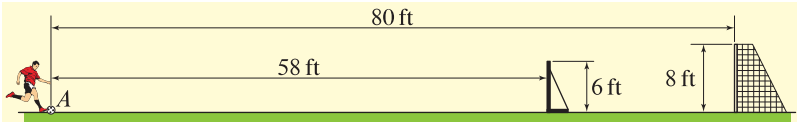
$$\boxed{d_{AB} = 82.0\,\text{m}.} \tag{9}$$

Problems 2.89 and 2.90

A soccer player practices kicking a ball from A directly into the goal (i.e., the ball does not bounce first) while clearing a 6 ft tall fixed barrier.

Problem 2.89 Determine the minimum speed that the player needs to give the ball to accomplish the task. *Hint:* To find $(v_0)_{\min}$, consider the equation for the projectile's trajectory (see, e.g., Eq. (2.52)) for the case in which the ball reaches the goal at its base. Then solve this equation for the initial speed v_0 as a function of the initial angle θ, and finally find $(v_0)_{\min}$ as you learned in calculus. Don't forget to check whether or not the ball clears the barrier.

Problem 2.90 Find the initial speed and angle that allow the ball to barely clear the barrier while barely reaching the goal at its base. *Hint:* As shown in Eq. (2.52), a projectile's trajectory can be given the form $y = C_1 x - C_2 x^2$ where the coefficients C_1 and C_2 can be found by forcing the parabola to go through two given points.



Solution to 2.89

The ball must clear a barrier and it must reach the goal post without bouncing. We will satisfying one of the conditions and see if our solution meets the other required condition. Let v_0 be the intial velocity, β be the elevation angle, and G denote the point at the base of the goal. For $(v_0)_{\min}$, $y = y_G = 0$ ft at $x = x_G = 80$ ft. Using Eq. (2.52) on p. 74 of the textbook, we have that the trajectory of the ball is described by the following equation:

$$y = (\tan\beta)x - \left(\frac{g\sec^2\beta}{2v_0^2}x^2\right). \tag{1}$$

Thus for the ball to land at the base of the goal, we have

$$0 = (\tan\beta)x_G - \left(\frac{g\sec^2\beta}{2v_0^2}x_G^2\right) \quad \Rightarrow \quad v_0 = \sqrt{\frac{gx_G}{\sin 2\beta}}. \tag{2}$$

Minimizing v_0 with respect to β requires making the denominator of the fraction under the square root of the last of Eqs. (2) as large as possible. The maximum value of the sine function is 1, which is achieved when the argument of the sine function is equal to $\pi/2$. Hence, we have

$$\sin 2\beta = 1 \quad \Rightarrow \quad \beta = \frac{\pi}{4}\,\text{rad} \quad \text{and} \quad (v_0)_{\min} = \sqrt{gx_G} = 50.75\,\text{ft/s}, \tag{3}$$

which, when expressed to 3 significant figures, gives

$$\boxed{(v_0)_{\min} = 50.8\,\text{ft/s}.}$$

Substituting $\beta = (\pi/4)$ rad, $(v_0)_{\min}$ from Eq. (3), and the value $x = 58$ ft into Eq. (1), we have

$$y(58\,\text{ft}) = 15.94\,\text{ft} > 6\,\text{ft},$$

that is, the ball clears the obstacle in front of the goal.

Solution to 2.90

As explained in the hint, the trajectory can be given the form

$$y = C_1 x - C_2 x^2, \tag{4}$$

where it is understood that the origin of the coordinate system used is at point A where the ball is kicked. The ball must barely clear the barrier, so we have

$$6\,\text{ft} = C_1(58\,\text{ft}) - C_2(58\,\text{ft})^2. \tag{5}$$

The ball must also barely reach the goal post, so we have

$$0 = C_1(80\,\text{ft}) - C_2(80\,\text{ft})^2. \tag{6}$$

Solving Eqs. (5) and (6), we get $C_1 = 0.3762$ and $C_2 = 4.702\times10^{-3}\,\text{ft}^{-1}$. Using Eq. (2.52) on p. 74 of the textbook, we have that the trajectory of the ball is described by the following equation:

$$y = (\tan\beta)x - \left(\frac{g\sec^2\beta}{2v_0^2}x^2\right). \tag{7}$$

Comparing Eq. (4) with the Eq. (7), we can conclude that

$$C_1 = \tan\beta \quad \text{and} \quad C_2 = \left(\frac{g\sec^2\beta}{2v_0^2}\right). \tag{8}$$

Hence we have

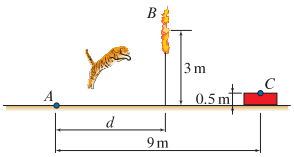
$$\beta = \tan^{-1}(0.3762) = 20.62^\circ \quad \text{and} \quad v_0 = \sqrt{\frac{g\sec^2\beta}{2C_2}} = 62.52\,\text{ft/s}, \tag{9}$$

which, expressed to 3 significant figures, give

$$\boxed{v_0 = 62.5\,\text{ft/s} \quad \text{and} \quad \beta = 20.6^\circ.}$$

Problems 2.91 and 2.92

In a circus act a tiger is required to jump from point A to point C so that it goes through the ring of fire at B. *Hint:* As shown in Eq. (2.52), a projectile's trajectory can be given the form $y = C_1 x - C_2 x^2$ where the coefficients C_1 and C_2 can be found by forcing the parabola to go through two given points.

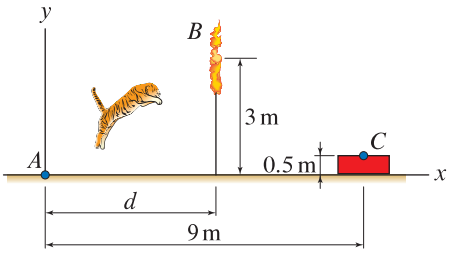


Problem 2.91 Determine the tiger's initial velocity if the ring of fire is placed at a distance $d = 5.5\,\text{m}$ from A. Furthermore, determine the slope of the tiger's trajectory as the tiger goes through the ring of fire.

Problem 2.92 Determine the tiger's initial velocity as well as the distance d so that the slope of the tiger's trajectory as the tiger goes through the ring of fire is completely horizontal.

Solution to 2.91

Referring to the figure on the right, we will be using a Cartesian coordinate system with origin at A. As indicated in the hint given in the problem statement, and due to the choice of coordinate system, i.e., due to the fact that the origin is the point from which the tiger jumps, the trajectory of the tiger can be described by the following equation (of a parabola):



$$y = C_1 x - C_2 x^2, \tag{1}$$

where C_1 and C_2 are constants to be determined by making sure that the tiger passes through point B of coordinates $(x_B, y_B) = (5.5\,\text{m}, 3\,\text{m})$, and then lands on point C of coordinates $(x_C, y_C) = (9\,\text{m}, 0.5\,\text{m})$. Using Eq. (1) to enforce these conditions, we have

$$3\,\text{m} = C_1(5.5\,\text{m}) - C_2(5.5\,\text{m})^2 = (5.5\,\text{m})C_1 - (30.25\,\text{m}^2)C_2, \tag{2}$$

$$0.5\,\text{m} = C_1(9\,\text{m}) - C_2(9\,\text{m})^2 = (9\,\text{m})C_1 - (81.00\,\text{m}^2)C_2, \tag{3}$$

which is a system of two equations in the two unknowns C_1 and C_2 whose solution is

$$C_1 = 1.315 \quad \text{and} \quad C_2 = 0.1400\,\text{m}^{-1}. \tag{4}$$

Recall that the in the textbook we had expressed the trajectory of a projectile in Eq. (2.52) on p. 74, which has the form

$$y = (\tan\beta)x - \left(\frac{g\sec^2\beta}{2v_0^2}x^2\right), \tag{5}$$

where β is the angle formed by the velocity vector with the horizontal at the origin, g is the acceleration due to gravity, and where v_0 is the initial speed of the projectile. Comparing Eq. (1) with Eq. (5), we conclude that

$$\beta = \tan^{-1} C_1 = 52.75^\circ \quad \text{and} \quad v_0 = \sqrt{\frac{g\sec^2\beta}{2C_2}} = 9.779\,\text{m/s}, \tag{6}$$

where we have used the values of C_1 and C_2 in Eqs. (4) in addition to the fact that $g = 9.81\,\text{m/s}^2$. Now that we have the values of β and v_0, the initial velocity of the tiger, expressed to 3 significant digits, is

$$\vec{v}_{\text{initial}} = v_0\cos\beta\,\hat{\imath} + v_0\sin\beta\,\hat{\jmath} = (5.92\,\hat{\imath} + 7.78\,\hat{\jmath})\,\text{m/s}$$

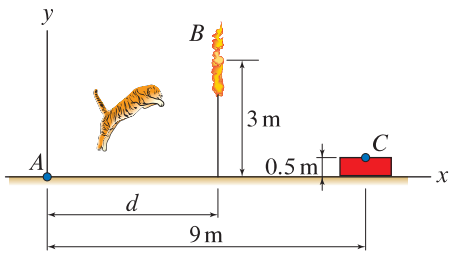
To find the slope of the trajectory at B, we take the derivative of the trajectory equation with respect to x and then substitute $x = x_B = 5.5\,\text{m}$ to get

$$\boxed{\left.\frac{dy}{dx}\right|_{x=x_B} = C_1 - 2C_2 x_B = -0.225,} \tag{7}$$

where we have used the values of C_1 and C_2 in Eqs. (4).

Solution to 2.92

Referring to the figure on the right, we will be using a Cartesian coordinate system with origin at A. As indicated in the hint given in the problem statement, and due to the choice of coordinate system, i.e., due to the fact that the origin is the point from which the tiger jumps, the trajectory of the tiger can be described by the following equation (of a parabola):



$$y = C_1 x - C_2 x^2, \tag{8}$$

where C_1 and C_2 are constants to be determined by making sure that the tiger passes through point B of coordinates $(x_B, y_B) = (d, 3\,\text{m})$ with zero slope, and then lands on point C of coordinates $(x_C, y_C) = (9\,\text{m}, 0.5\,\text{m})$. Using Eq. (8) to enforce these conditions, we have

$$y_B = C_1 x_B - C_2 x_B^2 \quad \Rightarrow \quad 3\,\text{m} = C_1 d - C_2 d^2, \tag{9}$$

$$\left.\frac{dy}{dx}\right|_{x=d} = 0 \quad \Rightarrow \quad 0 = C_1 - 2dC_2 \tag{10}$$

$$y_C = C_1 x_C - C_2 x_C^2 \quad \Rightarrow \quad 0.5\,\text{m} = (9\,\text{m})C_1 - (81.00\,\text{m}^2)C_2. \tag{11}$$

The last of Eqs. (9)–(11) form a system of three equations in the three unknowns d, C_1, and C_2, which can be solved numerically using some appropriate mathematical software. For example we have used *Mathematica* with the following code

```
NSolve[{3. == C1 d - C2 d^2, 0. == C1 - 2 C2 d, 0.5 == C1 9. - C2 81.}, {C1, C2, d}]
```

which yields the following two solutions:

$$C_1 = 0.05809 \qquad C_2 = 0.0002812\,\text{m}^{-1} \qquad d = 103.3\,\text{m}, \tag{12}$$

$$C_1 = 1.275 \qquad C_2 = 0.1355\,\text{m}^{-1} \qquad d = 4.705\,\text{m}. \tag{13}$$

Because the first solution implies that $d > x_C$, the solution in question is not acceptable and therefore we have that the only acceptable solution is

$$C_1 = 1.275, \quad C_2 = 0.1355\,\text{m}^{-1}, \quad d = 4.705\,\text{m}. \tag{14}$$

Expressing the result for d with 3 significant figures, we have

$$\boxed{d = 4.70\,\text{m}.}$$

Now we turn to the determination of the initial velocity of the tiger. To do so, we recall that the in the textbook we had expressed the trajectory of a projectile in Eq. (2.52) on p. 74, which has the form

$$y = (\tan\beta)x - \left(\frac{g\sec^2\beta}{2v_0^2}x^2\right), \tag{15}$$

where β is the angle formed by the velocity vector with the horizontal at the origin, g is the acceleration due to gravity, and where v_0 is the initial speed of the projectile. Comparing Eq. (8) with Eq. (15), we conclude that

$$\beta = \tan^{-1} C_1 = 51.89^\circ \quad \text{and} \quad v_0 = \sqrt{\frac{g\sec^2\beta}{2C_2}} = 9.749\,\text{m/s}, \tag{16}$$

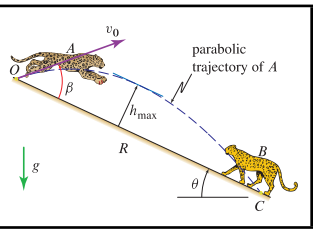
where we have used the values of C_1 and C_2 in Eqs. (14) in addition to the fact that $g = 9.81\,\text{m/s}^2$. Now that we have the values of β and v_0, the initial velocity of the tiger, expressed to 3 significant digits, is

$$\vec{v}_{\text{initial}} = v_0 \cos\beta\,\hat{\imath} + v_0 \sin\beta\,\hat{\jmath} = (6.02\,\hat{\imath} + 7.67\,\hat{\jmath})\,\text{m/s}$$



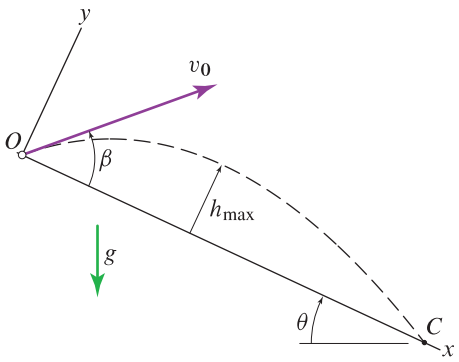
Problem 2.93

A jaguar A leaps from O at speed v_0 and angle β relative to the incline to attack a panther B at C. Determine an expression for the maximum *perpendicular* height $h_{\max}$ above the incline achieved by the leaping jaguar, given that the angle of the incline is θ.



Solution

Align a Cartesian coordinate system with the incline as shown on the right. The acceleration vector is then given by



$$\vec{a} = g\sin\theta\,\hat{\imath} - g\cos\theta\,\hat{\jmath} \quad \Rightarrow \quad a_y = -g\cos\theta.$$

Applying the constant acceleration equation in the y direction, we have

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0). \tag{1}$$

At the $h_{\max}$ position the y component of velocity must be equal to zero. enforcing this condition, we have

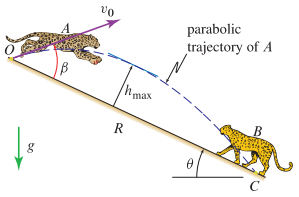
$$0 = (v_0\sin\beta)^2 - 2g\cos\theta(h_{\max} - 0). \tag{2}$$

Solving Eq. (2) for $h_{\max}$, we obtain

$$\boxed{h_{\max} = \frac{v_0^2\sin^2\beta}{2g\cos\theta}.} \tag{3}$$

Problems 2.94 and 2.95

The jaguar A leaps from O at speed v_0 and angle β relative to the incline to intercept the panther B at C. The distance along the incline from O to C is R, and the angle of the incline with respect to the horizontal is θ.

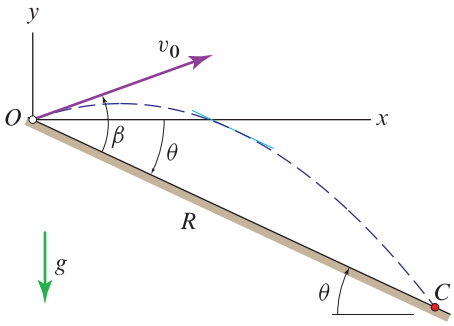


Problem 2.94 Determine an expression for v_0 as a function of β for A to be able to get from O to C.

Problem 2.95 Derive v_0 as a function of β to leap a given distance R along with the optimal value of launch angle β, i.e., the value of β necessary to leap a given distance R with the minimum v_0. Then plot v_0 as a function of β for $g = 9.81\,\text{m/s}^2$, $R = 7\,\text{m}$, and $\theta = 25°$, and find a numerical value of the optimal β and the corresponding value of v_0 for the given set of parameters.

Solution to 2.94

Using the xy coordinate system shown on the right, we write the following two constant acceleration equations describing the x and y coordinates of jaguar A as a function of time:



$$x = x_0 + v_{0x}t, \tag{1}$$

$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2, \tag{2}$$

where it is understood that the jaguar leaps at time $t = 0$, and where (x_0, y_0) are the coordinates of the point from which of the jaguar leaps. Since the jaguar leaps from the origin of the chosen coordinate system, letting t_C denote the time at which jaguar A arrives at C, we have

$$R\cos\theta = [v_0\cos(\beta-\theta)]\,t_C, \tag{3}$$

$$-R\sin\theta = [v_0\sin(\beta-\theta)]\,t_C - \tfrac{1}{2}gt_C^2, \tag{4}$$

where R is distance from O to C. Eliminating t_C from Eqs. (3) and (4) gives

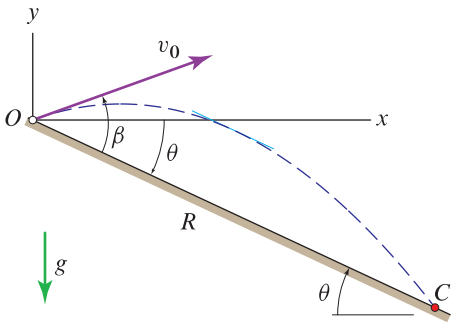
$$-\sin\theta = \cos\theta\tan(\beta-\theta) - \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos^2(\beta-\theta)}. \tag{5}$$

Solving Eq. (5) for v_0 and simplifying with the help of the trigonometric identity: $\sin(A+B) = \sin A\cos B + \cos A\sin B$, we have

$$\boxed{v_0 = \sqrt{\frac{gR}{2}}\frac{\cos\theta}{\sqrt{\sin\beta\cos(\beta-\theta)}}.} \tag{6}$$

Solution to 2.95

Using the xy coordinate system shown on the right, we write the following two constant acceleration equations describing the x and y coordinates of jaguar A as a function of time:



$$x = x_0 + v_{0x}t, \tag{7}$$

$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2, \tag{8}$$

where it is understood that the jaguar leaps at time $t = 0$, and where (x_0, y_0) are the coordinates of the point from which of the jaguar leaps. Since the jaguar leaps from the origin of the chosen coordinate system, letting t_C denote the time at which jaguar A arrives at C, we have

$$R\cos\theta = [v_0\cos(\beta-\theta)]\,t_C, \tag{9}$$

$$-R\sin\theta = [v_0\sin(\beta-\theta)]\,t_C - \tfrac{1}{2}gt_C^2, \tag{10}$$

where R is distance from O to C. Eliminating t_C from Eqs. (9) and (10) gives

$$-\sin\theta = \cos\theta\tan(\beta-\theta) - \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos^2(\beta-\theta)}. \tag{11}$$

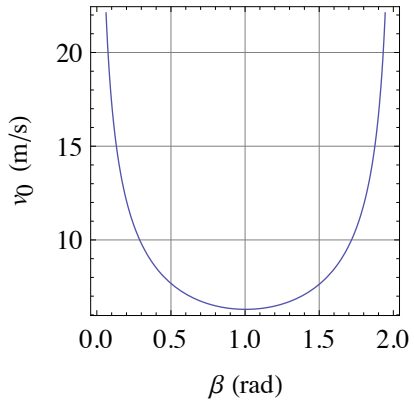
Solving Eq. (11) for v_0 and simplifying with the help of the trigonometric identity: $\sin(A+B) = \sin A\cos B + \cos A\sin B$, we have

$$\boxed{v_0 = \sqrt{\frac{gR}{2}}\frac{\cos\theta}{\sqrt{\sin\beta\cos(\beta-\theta)}}.} \tag{12}$$

Recalling that $g = 9.81\,\text{m/s}^2$, $R = 7\,\text{m}$, and $\theta = 25°$, we can plot the above function with any appropriate mathematical software. The plot shown below was obtained using *Mathematica* with the following code:

```
Parameters = {g → 9.81, θ → 25. Degree, R → 7.};

Plot[Sqrt[g R / 2] Cos[θ] / Sqrt[Sin[β] Cos[β - θ]] /. Parameters, {β, 0, 2}, Frame → True,
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"β (rad)", "v0 (m/s)"}]
```



We can find the optimal value of β to reach a distance of $R = 7\,\text{m}$ by differentiating v_0 with respect to β and setting it equal to zero. Recalling that $\theta = 25^\circ$, this gives

$$\frac{dv_0}{d\beta} = -\frac{1}{2}\sqrt{\frac{gR}{2}}\cos\theta\frac{\cos\beta\cos(\beta-\theta)-\sin\beta\sin(\beta-\theta)}{[\sin\beta\cos(\beta-\theta)]^{3/2}} = 0$$

$$\Rightarrow \quad \cos\beta\cos(\beta-25^\circ)-\sin\beta\sin(\beta-25^\circ) = 0. \quad (13)$$

The above equation is a transcendental equation that we will solve numerically. Again, this can be done with any appropriate mathematical software. We have used *Mathematica* with the following code

```
FindRoot[Cos[β] Cos[β - 25. Degree] - Sin[β] Sin[β - 25. Degree] == 0, {β, 25. Degree}]
```

Note that the use of root finding algorithms generally requires the user to provide a guess of the value of the solution. As can be seen in the above code (see information provided at the end of the code line), we have provided a guess of 25°. The outcome of this calculation gives

$$\beta_{\text{optimal}} = 57.52^\circ. \quad (14)$$

Then, using the above value of β along with $g = 9.81\,\text{m/s}^2$, $R = 7\,\text{m}$, and $\theta = 25^\circ$, from Eq. (12) we have that the corresponding value of v_0 is

$$(v_0)_{\text{optimal}} = 6.297\,\text{m/s}. \quad (15)$$

In summary, expressing our answer to 3 significant figures, we have

$$\boxed{\beta_{\text{optimal}} = 57.5^\circ \quad \text{and} \quad (v_0)_{\text{optimal}} = 6.30\,\text{m/s}.} \quad (16)$$

Problems 2.96 and 2.97



A stomp rocket is a toy consisting of a hose connected to a blast pad (i.e., an air bladder) at one end and to a short pipe mounted on a tripod at the other end. A rocket with a hollow body is mounted onto the pipe and is propelled into the air by stomping on the blast pad.

Problem 2.96 If the rocket can be imparted an initial speed $v_0 = 120\,\text{ft/s}$, and if the rocket's landing spot at B is at the same elevation as the launch point, i.e., $h = 0\,\text{ft}$, neglect air resistance and determine the rocket's launch angle θ such that the rocket achieves the maximum possible range. In addition, compute R, the rocket's maximum range, and t_f, the corresponding flight time.

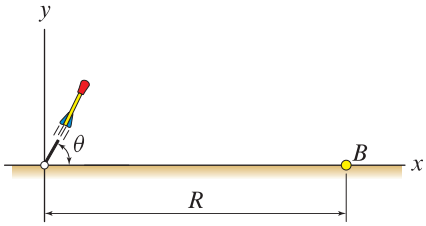
Problem 2.97 Assuming the rocket can be given an initial speed $v_0 = 120\,\text{ft/s}$, the rocket's landing spot at B is 10 ft higher than the launch point, i.e., $h = 10\,\text{ft}$, and neglecting air resistance, find the rocket's launch angle θ such that the rocket achieves the maximum possible range. In addition, as part of the solution, compute the corresponding maximum range and flight time. To do this:

(a) Determine the range R as a function of time.

(b) Take the expression for R found in (a), square it, and then differentiate it with respect to time to find the flight time corresponding to the maximum range and then that maximum range.

(c) Use the time found in (b) to then find the angle required to achieve the maximum range.

Solution to 2.96



Referring to the figure on the right, we will use an xy coordinate system with origin at the launch point of the rocket. Let θ be the elevation angle and $v_0 = 120\,\text{ft/s}$ be the intial speed. The acceleration of the rocket is equal to $-g$ in the y direction and zero in the x direction. Hence, we can use the constant acceleration equation $s = v_0 t + \frac{1}{2} a_c t^2$, to express the x and y coordinates of the rocket as a function of time. This gives

$$x = (v_0 \cos\theta)t, \tag{1}$$

$$y = (v_0 \sin\theta)t - \tfrac{1}{2} g t^2. \tag{2}$$

Since in this problem point B lies on the line $y = 0$, the time taken by the rocket to arrive at B can be obtained by equating Eq. (2) to zero. This gives

$$t_B = \frac{2v_0 \sin\theta}{g}. \tag{3}$$

Substituting Eq. (3) in place of t into Eq. (1), using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$, and observing that $x_B = R$, we get

$$R = \frac{v_0^2 \sin 2\theta}{g}, \tag{4}$$

where R is the range of the rocket. The maximum value of R occurs when $\sin 2\theta = 1$, i.e.,

$$2\theta = \frac{\pi}{2}\,\text{rad} \quad \Rightarrow \quad \boxed{\theta_{\max} = \frac{\pi}{4}\,\text{rad} = 45°.} \tag{5}$$

Substituting $\theta_{\max} = 45°$, into Eq. (4) and recalling that $g = 32.2\,\text{ft/s}^2$, gives

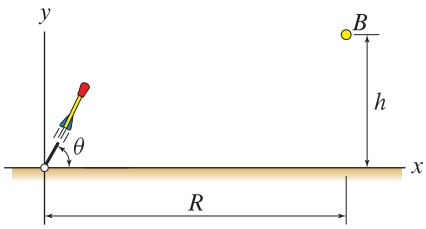
$$\boxed{R = \frac{v_0^2}{g} = 447\,\text{ft}.}$$

Observing that the time of flight is the same as t_B given in Eq. (3), for $\theta = \theta_{\max}$, we have

$$\boxed{t_f = \frac{2v_0 \sin\theta_{\max}}{g} = 5.27\,\text{s}.}$$

Solution to 2.97

Referring to the figure on the right, we will use an xy coordinate system with origin at the launch point of the rocket. Let θ be the elevation angle and $v_0 = 120\,\text{ft/s}$ be the intial speed. The acceleration of the rocket is equal to $-g$ in the y direction and zero in the x direction. Hence, we can use the constant acceleration equation $s = v_0 t + \frac{1}{2}a_c t^2$, to express the x and y coordinates of the rocket as a function of time. This gives



$$x = (v_0 \cos\theta)t, \tag{6}$$

$$y = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2. \tag{7}$$

For $t = t_f$ the rocket is at B, so that we must have

$$h = v_0(\sin\theta)t_f - \tfrac{1}{2}gt_f^2 \quad \Rightarrow \quad \sin\theta = \frac{2h + gt_f^2}{2v_0 t_f}, \tag{8}$$

$$R = v_0(\cos\theta)t_f \quad \Rightarrow \quad \cos\theta = \frac{R}{v_0 t_f}. \tag{9}$$

Next, recalling that $\sin^2\theta + \cos^2\theta = 1$, using the last of Eqs. (8) and (9), we have

$$\left(\frac{2h + gt_f^2}{2v_0 t_f}\right)^2 + \left(\frac{R}{v_0 t_f}\right)^2 = 1, \tag{10}$$

which, for convenience, we view as an equation for R^2 whose solution is

$$R^2 = v_0^2 t_f^2 - \left(h + \tfrac{1}{2}gt_f^2\right)^2. \tag{11}$$

Now we maximize R with respect to t_f. Since the value of t_f for which R is maximum coincides with the value of t_f for which R^2 is maximum, we can find the value of t_f in question by differentiating Eq. (11) with respect to t_f and then setting the results to 0. This gives

$$\frac{dR^2}{dt_f} = 0 = 2v_0 t_f - 2\left(h + \tfrac{1}{2}gt_f^2\right)(gt_f) \quad \Rightarrow \quad t_{f_{\max}} = \sqrt{\frac{2}{g^2}\left(v_0^2 - hg\right)} = 5.211\,\text{s}, \tag{12}$$

where we have used the following numerical values: $g = 32.2\,\text{ft/s}^2$, $v_0 = 120\,\text{ft/s}$, $h = 10\,\text{ft}$. Expressing this result to 3 significant figures, we have

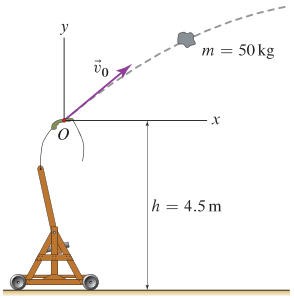
$$\boxed{t_{f_{\max}} = 5.21\,\text{s}.}$$

To find $R_{\max}$, we substitute the value of t_f from the last of Eqs. (12) into Eq. (11) and then we take a square root. To find the corresponding value of θ we substitute the value of t_f from the last of Eqs. (12) into the last of Eqs. (8) and solve for θ. Recalling that we have $g = 32.2\,\text{ft/s}^2$, $v_0 = 120\,\text{ft/s}$, $h = 10\,\text{ft}$, these operations yield the following results:

$$\boxed{R_{\max} = 437\,\text{ft} \quad \text{and} \quad \theta_{R_{\max}} = 45.7°.}$$

Problem 2.98

A trebuchet releases a rock with mass $m = 50\,\text{kg}$ at the point O. The initial velocity of the projectile is $\vec{v}_0 = (45\,\hat{\imath} + 30\,\hat{\jmath})\,\text{m/s}$. If one were to model the effects of air resistance via a drag force directly proportional to the projectile's velocity, the resulting accelerations in the x and y directions would be $\ddot{x} = -(\eta/m)\dot{x}$ and $\ddot{y} = -g - (\eta/m)\dot{y}$, respectively, where g is the acceleration of gravity and $\eta = 0.64\,\text{kg/s}$ is a viscous drag coefficient. Find an expression for the trajectory of the projectile.



Solution

We can integrate the x and the y components of acceleration to get the x and y displacement as a function of time. Starting with the given acceleration components, using $\ddot{x} = \frac{d\dot{x}}{dt}$, we can integrate the expression for the x component of acceleration to get $\dot{x}$.

$$\int_{(v_0)_x}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{\eta}{m}\,dt \quad \Rightarrow \quad \dot{x} = (v_0)_x e^{-\frac{\eta}{m}t}, \tag{1}$$

where $(v_0)_x$ is the x component of the velocity of the projectile at $t = 0$. Integrating Eq. (1) again with respect to time, we obtain

$$\int_0^x dx = (v_0)_x \int_0^t e^{-\frac{\eta}{m}t}\,dt \quad \Rightarrow \quad x = \frac{m(v_0)_x}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right). \tag{2}$$

Using $\ddot{y} = \frac{d\dot{y}}{dt}$, we can integrate the expression for the y component of acceleration to get $\dot{y}$.

$$-\int_{(v_0)_y}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad \Rightarrow \quad \dot{y} = \frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t} \tag{3}$$

where $(v_0)_y$ is the y component of the velocity of the projectile at $t = 0$. Integrating Eq. (3) again with respect to time, we obtain

$$y = \left(\frac{m^2 g}{\eta^2} - \frac{mgt}{\eta} + \frac{m}{\eta}(v_0)_y\right) - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right) e^{-\frac{\eta}{m}t}. \tag{4}$$

From Eq. (2) we find

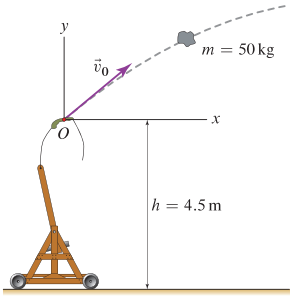
$$e^{-\frac{\eta}{m}t} = \left(1 - \frac{\eta x}{m(v_0)_x}\right) \quad \Rightarrow \quad t = -\frac{m}{\eta}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{5}$$

Substitute the last of Eqs. (5) into Eq. (4) and plugging in $m = 50\,\text{kg}$, $(v_0)_x = 45\,\text{m/s}$, $(v_0)_y = 30\,\text{m/s}$ and $\eta = 0.64\,\text{kg/s}$, we obtain

$$\boxed{y = \left[59.9\times10^3 \ln\left(1 - 2.84\times10^{-4}x\right) + 17.7x\right]\text{ m}.} \tag{6}$$

Problem 2.99

Continue Prob. 2.98 and, for the case where $\eta = 0.64\,\text{kg/s}$, determine the maximum height from the ground reached by the projectile and the time it takes to achieve it. Compare the result with what you would obtain in the absence of air resistance.



Solution

We begin by determining part of the solution to Problem 2.98, namely, the function $y(t)$ describing the vertical position of the projectile as a function of time. To do so, we can integrate the x and the y components of acceleration to get the x and y displacement as a function of time. Starting with the given acceleration components, using $\ddot{x} = \frac{d\dot{x}}{dt}$, we can integrate the expression for the x component of acceleration to get $\dot{x}$.

$$\int_{(v_0)_x}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{\eta}{m}\,dt \quad \Rightarrow \quad \dot{x} = (v_0)_x e^{-\frac{\eta}{m}t}, \tag{1}$$

where $(v_0)_x$ is the x component of the velocity of the projectile at $t = 0$. Integrating Eq. (1) again with respect to time, we obtain

$$\int_0^x dx = (v_0)_x \int_0^t e^{-\frac{\eta}{m}t}\,dt \quad \Rightarrow \quad x = \frac{m(v_0)_x}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right). \tag{2}$$

Using $\ddot{y} = \frac{d\dot{y}}{dt}$, we can integrate the expression for the y component of acceleration to get $\dot{y}$.

$$-\int_{(v_0)_y}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad \Rightarrow \quad \dot{y} = \frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t}. \tag{3}$$

Integrating Eq. (3) again with respect to time, we obtain

$$y = \left(\frac{m^2 g}{\eta^2} - \frac{mgt}{\eta} + \frac{m}{\eta}(v_0)_y\right) - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right) e^{-\frac{\eta}{m}t}, \tag{4}$$

where $(v_0)_y$ is the y component of the velocity of the projectile at $t = 0$.

Now that we have an expression for $y(t)$, we observe that when maximum height is reached $v_y = 0$. Hence, letting $H_{\max}$ denote the maximum height achieved by the projectile and $t_{H_{\max}}$ the time at which this height is achieved, setting Eq. (3) to zero, we obtain

$$\left[\frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t_{H_{\max}}} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t_{H_{\max}}}\right] = 0 \quad \Rightarrow \quad t_{H_{\max}} = \frac{m}{\eta}\ln\left[\frac{mg + (v_0)_y\eta}{mg}\right] = 3.000\,\text{s}, \tag{5}$$

where we have used the following numerical data: $m = 50\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $(v_0)_y = 30\,\text{m/s}$ and $\eta = 0.64\,\text{kg/s}$. Expressing the result in Eq. (5) using 3 significant figures, we have

$$\boxed{t_{H_{\max}} = 3.00\,\text{s}.}$$

Then, the maximum height in the presence of air resistance can be calculated as $H_{\max} = y_{\max} + h$. In turn, we can find $y_{\max}$ by substituting the value of time found in Eq. (5) into Eq. (4) along with the known values $m = 50\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $(v_0)_y = 30\,\text{m/s}$ and $\eta = 0.64\,\text{kg/s}$. Doing so, we obtain $y_{\max} = 44.71\,\text{m}$. Hence, $H_{\max}$ is given by

$$H_{\max} = y_{\max} + h = 49.21\,\text{m}, \tag{6}$$

which, when expressed to 3 significant figures, gives

$$\boxed{H_{\max} = 49.2\,\text{m}.}$$

In the case of absence of air resistance, the maximum height is given by $(H_{\max})_{\text{no air}} = (y_{\max})_{\text{no air}} + h$, where $(y_{\max})_{\text{no air}}$ can be calculated by equating the y component of the velocity to zero. Using a constant acceleration equation we have

$$v_y^2 = (v_0)_y^2 + 2a(y_{\max})_{\text{no air}} = 0 \quad \Rightarrow \quad (y_{\max})_{\text{no air}} = -\frac{(v_0)_y^2}{2a} = 45.87\,\text{m}, \tag{7}$$

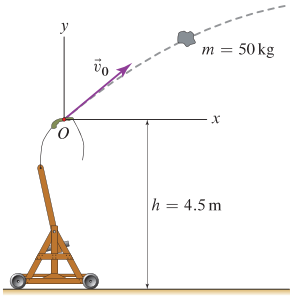
where $a = -g$. Hence, the maximum height $(H_{\max})_{\text{no air}}$ in this case is given by

$$(H_{\max})_{\text{no air}} = (y_{\max})_{\text{no air}} + h = 50.37\,\text{m}. \tag{8}$$

The maximum height reached with no air resistance 2.38% greater than that with air resistance.

Problem 2.100

Continue Prob. 2.98 and, for the case where $\eta = 0.64\,\text{kg/s}$, determine t_I and x_I, the value of t, and the x position corresponding to the projectile's impact with the ground.



Solution

We begin by working part of the solution to Prob. 2.98. Specifically, we can integrate the x and the y components of acceleration to get the x and y displacement as a function of time. Starting with the given acceleration components, using $\ddot{x} = \frac{d\dot{x}}{dt}$, we can integrate the expression for the x component of acceleration to get $\dot{x}$.

$$\int_{(v_0)_x}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{\eta}{m}\,dt \quad \Rightarrow \quad \dot{x} = (v_0)_x e^{-\frac{\eta}{m}t} \tag{1}$$

where $(v_0)_x$ is the velocity component of the projectile Integrating Eq. (1) again with respect to time, we obtain

$$\int_0^x dx = (v_0)_x \int_0^t e^{-\frac{\eta}{m}t}\,dt \quad \Rightarrow \quad x = \frac{m v_{x0}}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right). \tag{2}$$

Using $\ddot{y} = \frac{d\dot{y}}{dt}$, we can integrate the expression for the y component of acceleration to get $\dot{y}$.

$$-\int_{(v_0)_y}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad \Rightarrow \quad \dot{y} = \frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t}. \tag{3}$$

Integrating Eq. (3) again with respect to time, we obtain

$$y = \left(\frac{m^2 g}{\eta^2} - \frac{mgt}{\eta} + \frac{m}{\eta}(v_0)_y\right) - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right) e^{-\frac{\eta}{m}t}. \tag{4}$$

Next, from Eq. (2) we find

$$e^{-\frac{\eta}{m}t} = \left(1 - \frac{\eta x}{m(v_0)_x}\right) \quad \Rightarrow \quad t = -\frac{m}{\eta}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{5}$$

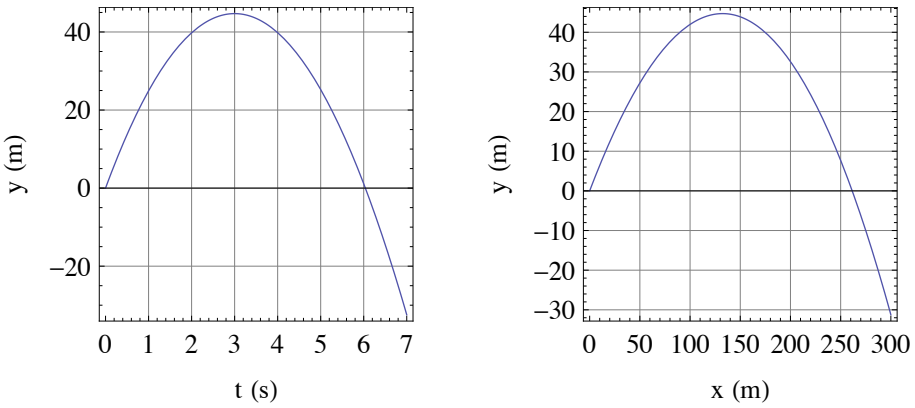
Substituting the last of Eqs. (5) into Eq. (4), we have

$$y = \left[\frac{m^2 g}{\eta^2} + \frac{m^2 g}{\eta^2}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right) + \frac{m}{\eta}(v_0)_y\right] - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right)\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{6}$$

To find the time of impact t_I and the location x_I of the impact we observe that the impact is characterized by the condition $y = -h = -4.5\,\text{m}$. Hence we can use a numerical root finding method to find the the value of t in Eq. (4) for which the condition $h = -h$ is satisfied. Similarly, we can use a numerical root finding method to find the the value of x in Eq. (6) for which the condition $h = -h$ is satisfied. Because the majority of root finding methods require us to provide a guess of the solution, before using any such methods, we proceed to plot $y(t)$ as given in Eq. (4) and of $y(x)$ as given in Eq. (6). These plots can be done with any appropriate mathematical software. Recalling that we are given $m = 50\,\text{kg}$, $(v_0)_x = 45\,\text{m/s}$, $(v_0)_y = 30\,\text{m/s}$ and $\eta = 0.64\,\text{kg/s}$, the plots shown below were obtained using *Mathematica* with the following code:

```
Parameters = {m → 50., g → 9.81, h → 4.5, v0x → 45., v0y → 30., η → 0.64};
yt = (m^2 g/η^2 - m g t/η + m/η v0y) - (m^2 g/η^2 + m/η v0y) e^(-t η/m);
yx = (m^2 g/η^2 + m^2 g/η^2 Log[1 - η x/(m v0x)] + m/η v0y) - (m^2 g/η^2 + m/η v0y) (1 - η x/(m v0x));
Plot[yt /. Parameters, {t, 0, 7}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "y (m)"}, AspectRatio → 1]
Plot[yx /. Parameters, {x, 0, 300}, Frame → True, GridLines → Automatic,
 FrameLabel → {"x (m)", "y (m)"}, AspectRatio → 1]
```

which gives



From the above two plots, we see that t_I is close to 6 s and x_I is close to 250 m. Hence, we will use the values just listed as guesses in an appropriate root finding numerical method to find more precise values to the quantities t_I and x_I. For example, this can be don using *Mathematica* with the following code:

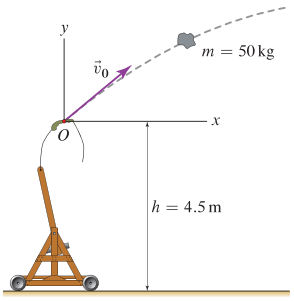
```
Parameters = {m → 50., g → 9.81, h → 4.5, v0x → 45., v0y → 30., η → 0.64};
yt = (m^2 g/η^2 - m g t/η + m/η v0y) - (m^2 g/η^2 + m/η v0y) e^(-t η/m);
yx = (m^2 g/η^2 + m^2 g/η^2 Log[1 - η x/(m v0x)] + m/η v0y) - (m^2 g/η^2 + m/η v0y) (1 - η x/(m v0x));
FindRoot[yt == -h /. Parameters, {t, 6}]
FindRoot[yx == -h /. Parameters, {x, 250}]
```

Evaluating the outcome of the above code to 3 significant figures, we have

$$t_I = 6.19\,\text{s} \quad \text{and} \quad x_I = 268\,\text{m}.$$

Problem 2.101

With reference to Probs. 2.98 and 2.100, assume that an experiment is conducted so that the measured value of x_I is 10% smaller that what is predicted in the absence of viscous drag. Find the value of η that would be required for the theory in Prob. 2.98 to match the experiment.



Solution

For the case of no air drag, the time of impact can be calculated by equating the constant acceleration equation for the y coordinate of the projectile to $-h$. Letting t_I denote the time of impact, we would have

$$y = (v_0)_y t - \tfrac{1}{2} g t^2 \quad \Rightarrow \quad (v_0)_y t_I - \tfrac{1}{2} g t_I^2 = -h \quad \Rightarrow \quad g t_I^2 - 2(v_0)_y t_I - 2h = 0$$

$$\Rightarrow \quad t_I = \frac{(v_0)_y \pm \sqrt{v_{0y}^2 + gh}}{g} \quad \Rightarrow \quad t_I = 6.263\,\text{s}, \quad (1)$$

where we have discarded the solution with the negative square root because it yields a negative time value, and where we have used the following numerical data: $(v_0)_y = 30\,\text{m/s}$, $g = 9.81\,\text{m/s}^2$, and $h = 4.5\,\text{m}$. The impact distance x_I for the case of no air drag can be calculated again by using constant acceleration equations (with $a_x = 0$). This gives

$$x = (v_0)_x t \quad \Rightarrow \quad x_I = (v_0)_x t_I = 281.8\,\text{m}. \quad (2)$$

The problem statement indicates that the x position of the rock in the presence of air drag is: $(x_I)_{\text{air}} = 0.9 x_I$, i.e.,

$$(x_I)_{\text{air}} = 253.6\,\text{m}. \quad (3)$$

To be able to use these results, we first determine the trajectory of the projectile in the presence of air resistance. We begin by working part of the solution to Prob. 2.98. Specifically, we can integrate the x and the y components of acceleration to get the x and y displacement as a function of time. Starting with the given acceleration components, using $\ddot{x} = \frac{d\dot{x}}{dt}$, we can integrate the expression for the x component of acceleration to get $\dot{x}$.

$$\int_{(v_0)_x}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{\eta}{m}\, dt \quad \Rightarrow \quad \dot{x} = (v_0)_x e^{-\frac{\eta}{m}t} \quad (4)$$

where $(v_0)_x$ is the velocity component of the projectile Integrating Eq. (4) again with respect to time, we obtain

$$\int_0^x dx = (v_0)_x \int_0^t e^{-\frac{\eta}{m}t}\, dt \quad \Rightarrow \quad x = \frac{m v_{x0}}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right). \quad (5)$$

Using $\ddot{y} = \frac{d\dot{y}}{dt}$, we can integrate the expression for the y component of acceleration to get $\dot{y}$.

$$-\int_{(v_0)_y}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad \Rightarrow \quad \dot{y} = \frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t}. \quad (6)$$

Integrating Eq. (6) again with respect to time, we obtain

$$y = \left(\frac{m^2 g}{\eta^2} - \frac{mgt}{\eta} + \frac{m}{\eta}(v_0)_y\right) - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right) e^{-\frac{\eta}{m}t}. \tag{7}$$

Next, from Eq. (5) we find

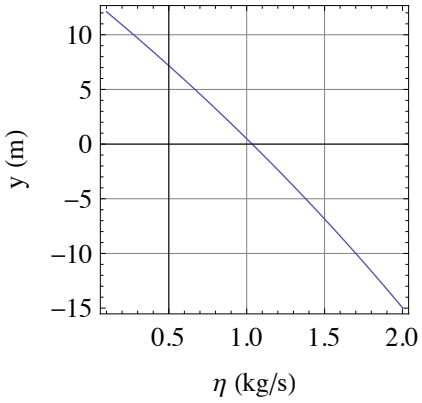
$$e^{-\frac{\eta}{m}t} = \left(1 - \frac{\eta x}{m(v_0)_x}\right) \quad \Rightarrow \quad t = -\frac{m}{\eta} \ln\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{8}$$

Substituting the last of Eqs. (8) into Eq. (7), we have

$$y = \left[\frac{m^2 g}{\eta^2} + \frac{m^2 g}{\eta^2} \ln\left(1 - \frac{\eta x}{m(v_0)_x}\right) + \frac{m}{\eta}(v_0)_y\right] - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right)\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{9}$$

We now observe that at impact the x and y coordinates of the rock are $(x_I)_{\text{air}}$ and $-h$. By enforcing this condition in Eq. (9) we obtain an equation in η that can be solved numerically by using some appropriate root finding algorithm as can be found in common mathematical software. Because most root finding algorithms require the user to supply a guess of the solution, we begin by plotting the value of y for $x = (x_I)_{\text{air}} = 253.6\,\text{m}$ (see Eq. (3)) as a function of η. This again can be done using appropriate numerical software. The plot presented below was obtained in *Mathematica* using the following code:

```
Parameters = {m → 50., g → 9.81, h → 4.5, v0x → 45., v0y → 30., xAir → 253.6};
yx = (m^2 g/η^2 + m^2 g/η^2 Log[1 - η x/(m v0x)] + m/η v0y) - (m^2 g/η^2 + m/η v0y) (1 - η x/(m v0x));
Plot[yx /. x → xAir /. Parameters, {η, 0.1, 2}, Frame → True, GridLines → Automatic,
 FrameLabel → {"η (kg/s)", "y (m)"}, AspectRatio → 1]
```



From the plot above, we see that $y = 0$ for $x = (x_I)_{\text{air}}$ when η is a bit greater then 1 kg/s. Hence, we will use the value $\eta = 1\,\text{kg/s}$ as the guess for a root finding algorithm in order to find a more accurate value of the value of η. When using Mathematica, this can be done with the following code

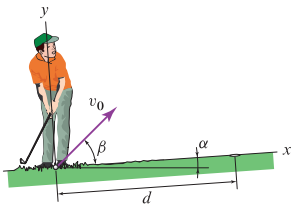
```
Parameters = {m → 50., g → 9.81, h → 4.5, v0x → 45., v0y → 30., xAir → 253.6};
yx = (m^2 g/η^2 + m^2 g/η^2 Log[1 - η x/(m v0x)] + m/η v0y) - (m^2 g/η^2 + m/η v0y) (1 - η x/(m v0x));
FindRoot[yx == -h /. x → xAir /. Parameters, {η, 1}]
```

which yields the following solution

$$\eta = 1.35\,\text{kg/s}.$$

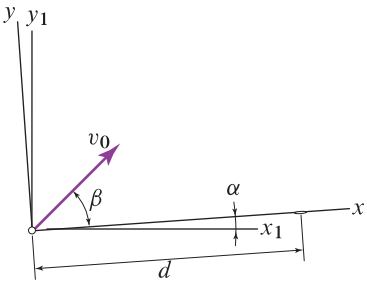
Problem 2.102

Express the trajectory of the golf ball using the axes shown and in terms of initial speed v_0, initial angle β, slope α, and the acceleration of gravity g.



Solution

The components of gravity and initial velocities in the xy coordinate system are:



$$\ddot{x} = -g\sin\alpha, \quad \text{and} \quad \ddot{y} = -g\cos\alpha. \tag{1}$$

$$\dot{x} = v_0\cos\beta, \quad \text{and} \quad \dot{y} = v_0\sin\beta. \tag{2}$$

Therefore, keeping in mind that the ball start at the origin to the chosen coordinate system, the x and y of the ball are given by

$$x(t) = v_0\cos\beta t - \tfrac{1}{2}g\sin\alpha t^2, \tag{3}$$

$$y(t) = v_0\sin\beta t - \tfrac{1}{2}g\cos\alpha t^2. \tag{4}$$

The trajectory is now found by eliminating t from the two equations above. One way to accomplish this is to essentially start over and consider the motion of the ball in the x_1y_1 coordinate system. Specifically, let consider the expression for the horizontal component of velocity in the x_1y_1 system. Observing that $\ddot{x}_1 = 0$, and that the ball starts at the origin of the x_1y_1 coordinate system, using constant acceleration equations, we have

$$\ddot{x}_1 = 0 \quad \Rightarrow \quad x_1(t) = v_0\cos(\alpha+\beta)t \quad \Rightarrow \quad t = \frac{x_1}{v_0\cos(\alpha+\beta)}. \tag{5}$$

To go back to the xy coordinate system, we now express x_1 in terms of x and y as follows

$$x_1 = x\cos\alpha - y\sin\alpha \quad \Rightarrow \quad t = \frac{x\cos\alpha - y\sin\alpha}{v_0\cos(\alpha+\beta)}. \tag{6}$$

Substituting Eq. (6) into Eq. (4), we then obtain

$$\boxed{y = v_0\sin\beta\left[\frac{x\cos\alpha - y\sin\alpha}{v_0\cos(\alpha+\beta)}\right] - \tfrac{1}{2}g\cos\alpha\left[\frac{x\cos\alpha - y\sin\alpha}{v_0\cos(\alpha+\beta)}\right]^2.}$$

Problem 2.103

Consider the vectors $\vec{a} = 2\,\hat{\imath} + 1\,\hat{\jmath} + 7\,\hat{k}$ and $\vec{b} = 1\,\hat{\jmath} + 2\,\hat{\jmath} + 3\,\hat{k}$. Compute the following quantities.

(a) $\vec{a} \times \vec{b}$

(b) $\vec{b} \times \vec{a}$

(c) $\vec{a} \times \vec{b} + \vec{b} \times \vec{a}$

(d) $\vec{a} \times \vec{a}$

(e) $(\vec{a} \times \vec{a}) \times \vec{b}$

(f) $\vec{a} \times (\vec{a} \times \vec{b})$

Parts (a)–(d) of this problem are meant to be a reminder that the cross product is an *anticommutative* operation while Parts (e) and (f) are meant to be a reminder that the cross product is an operation that is *not associative*.

Solution

Using the vectors given in the problem statement, various properties of the cross-product are illustrated through a few simple exercises.

Part (a) The commutative relationship for the cross-product is demonstrated by first evaluating

$$\vec{a} \times \vec{b} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 7 \\ 1 & 2 & 3 \end{vmatrix} \quad \Rightarrow \quad \boxed{\vec{a} \times \vec{b} = \left(-11\,\hat{\imath} + 1\,\hat{\jmath} + 3\,\hat{k}\right).} \tag{1}$$

Part (b) The cross-product is again evaluated, but this time in the opposite order such that

$$\vec{b} \times \vec{a} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 1 & 7 \end{vmatrix} \quad \Rightarrow \quad \boxed{\vec{b} \times \vec{a} = \left(11\,\hat{\imath} - 1\,\hat{\jmath} - 3\,\hat{k}\right).} \tag{2}$$

Thus the cross-product is *anti-commutative* because the results are equal in magnitude, but opposite in direction (sign).

Part (c) The fact that the cross-product relation between two vector is anti-commutative is also demonstrated through the equation below, where

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{a} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 7 \\ 1 & 2 & 3 \end{vmatrix} + \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 1 & 7 \end{vmatrix} = \left(-11\,\hat{\imath} + 1\,\hat{\jmath} + 3\,\hat{k}\right) + \left(11\,\hat{\imath} - 1\,\hat{\jmath} - 3\,\hat{k}\right), \tag{3}$$

$$\Rightarrow \quad \boxed{\vec{a} \times \vec{b} + \vec{b} \times \vec{a} = \vec{0}.} \tag{4}$$

Part (d) The cross-product of a vector with itself, such as

$$\boxed{\vec{a}\times\vec{a} = \det\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 1 & 7\\ 2 & 1 & 7\end{vmatrix} = \vec{0},} \tag{5}$$

will always yield the zero vector.

Part (e) Demonstrating the non-associative nature of the cross-product, the example shows one possible way to calculate the product of three vectors, where

$$\boxed{\left(\vec{a}\times\vec{a}\right)\times\vec{b} = \vec{0}\times\vec{b} = \vec{0}.} \tag{6}$$

Part (f) The associative property does not hold for cross-products since the result of part (e) is not equal to the result of

$$\vec{a}\times\left(\vec{a}\times\vec{b}\right) = \vec{a}\times\det\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 1 & 7\\ 1 & 2 & 3\end{vmatrix} = \det\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 1 & 7\\ -11 & 1 & 3\end{vmatrix},$$

$$\Rightarrow \boxed{\vec{a}\times\left(\vec{a}\times\vec{b}\right) = \left(-4\,\hat{\imath} - 83\,\hat{\jmath} + 13\,\hat{k}\right).} \tag{7}$$

Problem 2.104

Consider two vectors $\vec{a} = 1\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$ and $\vec{b} = -6\,\hat{\imath} + 3\,\hat{\jmath}$.

(a) Verify that $\vec{a}$ and $\vec{b}$ are perpendicular to one another.

(b) Compute the vector triple product $\vec{a} \times (\vec{a} \times \vec{b})$.

(c) Compare the result from calculating $\vec{a} \times (\vec{a} \times \vec{b})$ with the vector $-|\vec{a}|^2\,\vec{b}$.

The purpose of this exercise is to show that as long as $\vec{a}$ and $\vec{b}$ are perpendicular to one another, you can always write $\vec{a} \times (\vec{a} \times \vec{b}) = -|\vec{a}|^2\,\vec{b}$. This identity turns out to be very useful in the study of the planar motion of rigid bodies.

Solution

Part (a) Two vectors are perpendicular if their dot product is zero. Thus,

$$\boxed{\vec{a} \cdot \vec{b} = 1 \cdot -6 + 2 \cdot 3 + 3 \cdot 0 = 0.} \tag{1}$$

Part (b) The triple product is evaluated by first calculating the cross-product of $\vec{a}$ and $\vec{b}$ and then taking the cross-product of $\vec{a}$ with the cross-product of $\vec{a}$ and $\vec{b}$. The calculation proceeds by

$$\vec{a} \times \left(\vec{a} \times \vec{b}\right) = \vec{a} \times \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -6 & 3 & 0 \end{vmatrix} = \vec{a} \times \left(-9\,\hat{\imath} - 18\,\hat{\jmath} + 15\,\hat{k}\right) = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -9 & -18 & 15 \end{vmatrix}, \tag{2}$$

$$\Rightarrow \quad \boxed{\vec{a} \times \left(\vec{a} \times \vec{b}\right) = \left(84\,\hat{\imath} - 42\,\hat{\jmath} + 0\,\hat{k}\right).} \tag{3}$$

Part (c) The vector expression is evaluated as

$$-|\vec{a}|^2\vec{b} = -\left(\sqrt{1^2 + 2^2 + 3^2}\right)^2 \left(-6\,\hat{\imath} + 3\,\hat{\jmath} + 0\,\hat{k}\right), \tag{4}$$

$$= \left(84\,\hat{\imath} - 42\,\hat{\jmath} + 0\,\hat{k}\right), \tag{5}$$

which is the same as $\vec{a} \times \left(\vec{a} \times \vec{b}\right)$.

Problem 2.105

Let $\vec{r}$ be the position vector of a point P with respect to a Cartesian coordinate system with axes x, y, and z. Let the motion of P be confined to the xy plane, so that $\vec{r} = r_x\,\hat{\imath} + r_y\,\hat{\jmath}$ (i.e., $\vec{r}\cdot\hat{k} = 0$). Also, let $\vec{\omega}_r = \omega_r\,\hat{k}$ be the angular velocity vector of the vector $\vec{r}$. Compute the outcome of the products $\vec{\omega}_r \times (\vec{\omega}_r \times \vec{r})$ and $\vec{\omega}_r \times (\vec{r} \times \vec{\omega}_r)$.

Solution

Use the property verified in part (c) of the solution to Problem 2.104 :

$$\vec{\omega}_r \times \left(\vec{\omega}_r \times \vec{r}\right) = -|\vec{\omega}_r|^2\vec{r} = -\omega_r^2\left(r_x\,\hat{\imath} + r_y\,\hat{\jmath}\right). \tag{1}$$

The cross-product is anti-commutative, as verified in part (b) of the solution to Problem 2.103. Therefore the triple cross-product $\vec{\omega}_r \times \left(\vec{r} \times \vec{\omega}_r\right)$ is Eq. (1) multiplied by -1.

$$\vec{\omega}_r \times \left(\vec{r} \times \vec{\omega}_r\right) = \vec{\omega}_r \times -\left(\vec{\omega}_r \times \vec{r}\right) = \omega_r^2\left(r_x\,\hat{\imath} + r_y\,\hat{\jmath}\right). \tag{2}$$

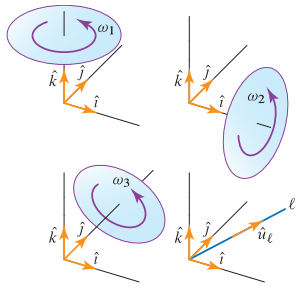
Problem 2.106

The three propellers shown are all rotating with the same *angular speed* of 1000 rpm about different coordinate axes.

(a) Provide the proper vector expressions for the *angular velocity* of each of the three propellers.

(b) Suppose that an identical propeller rotates at 1000 rpm about the axis ℓ oriented by the unit vector $\hat{u}_\ell$. Let any point P on ℓ have coordinates such that $x_P = y_P = z_P$. Find the vector representation of the angular velocity of this fourth propeller.

Express the answers using units of radians per second.



Solution

Part (a) 1000 rpm $= (100\pi/\sqrt{3})$ rad/s $= 104.7$ rad/s. The angular velocity vectors can be written as

$$\boxed{\vec{\omega}_1 = \left(105\,\hat{k}\right)\text{rad/s}, \quad \vec{\omega}_2 = \left(105\,\hat{\imath}\right)\text{rad/s} \quad \text{and} \quad \vec{\omega}_3 = \left(-105\,\hat{\jmath}\right)\text{rad/s}.} \tag{1}$$

Part (b) The unit vector in the ℓ direction is

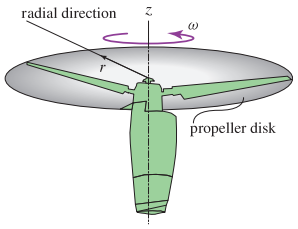
$$\hat{u}_\ell = \frac{1}{\sqrt{3}}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right). \tag{2}$$

Therefore, the angular velocity is expressed as

$$\boxed{\vec{\omega}_\ell = \frac{100\pi}{3\sqrt{3}}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)\text{rad/s} \quad \Rightarrow \quad 60.5\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)\text{rad/s}.} \tag{3}$$

Problem 2.107

The propeller shown has a diameter of 38 ft and is rotating with a constant angular speed of 400 rpm. At a given instant, a point P on the propeller is at $\vec{r}_P = (12.5\,\hat{\imath} + 14.3\,\hat{\jmath})$ ft. Use Eqs. (2.62) and (2.64) to compute the velocity and acceleration of P, respectively.



Solution

400 rpm = $(80\pi/3)$ rad/s = 41.89 rad/s. We note that $\dot{r} = 0$, $\vec{\omega} = \omega\,\hat{k}$, and the magnitude of ω is constant. Therefore Eq.(2.62) becomes

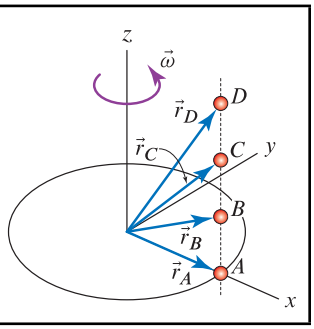
$$\dot{\vec{r}}_P = (\vec{\omega} \times \vec{r}_P) \quad \Rightarrow \quad \boxed{\dot{\vec{r}}_P = (-599\,\hat{\imath} + 524\,\hat{\jmath})\ \text{ft/s}.}$$

The acceleration of point P is calculated using Eq. (2.64). The only non-zero term in the acceleration equation is evaluated to find the acceleration as

$$\ddot{\vec{r}}_P = [\vec{\omega} \times (\vec{\omega} \times \vec{r}_P)] \quad \Rightarrow \quad \boxed{\ddot{\vec{r}}_P = (-21{,}900\,\hat{\imath} - 25{,}100\,\hat{\jmath})\ \text{ft/s}^2.}$$

Problem 2.108

Consider the four points whose positions are given by the vectors $\vec{r}_A = \left(2\,\hat{\imath} + 0\,\hat{k}\right)$ m, $\vec{r}_B = \left(2\,\hat{\imath} + 1\,\hat{k}\right)$ m, $\vec{r}_C = \left(2\,\hat{\imath} + 2\,\hat{k}\right)$ m, and $\vec{r}_D = \left(2\,\hat{\imath} + 3\,\hat{k}\right)$ m. Knowing that the magnitude of these vectors is constant and that the angular velocity of these vectors at a given instant is $\vec{\omega} = 5\,\hat{k}$ rad/s, apply Eq. (2.62) to find the velocities $\vec{v}_A$, $\vec{v}_B$, $\vec{v}_C$, and $\vec{v}_D$. Explain why all the velocity vectors are the same even though the position vectors are not.



Solution

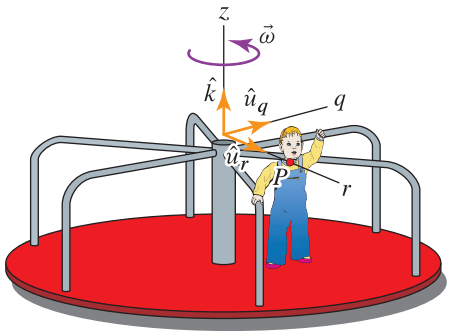
The position vectors do not change magnitude, $\dot{\vec{r}} = 0$. The equation reduces to $\vec{v} = \vec{\omega} \times \vec{r}$.

$$\vec{v}_A = 5\,\hat{k} \times 2\,\hat{\imath} = 10\,\hat{\jmath}\ \text{m/s}, \qquad \vec{v}_B = 5\,\hat{k} \times \left(2\,\hat{\imath} + 1\,\hat{k}\right) = 10\,\hat{\jmath}\ \text{m/s},$$
$$\vec{v}_C = 5\,\hat{k} \times \left(2\,\hat{\imath} + 2\,\hat{k}\right) = 10\,\hat{\jmath}\ \text{m/s}, \qquad \vec{v}_D = 5\,\hat{k} \times \left(2\,\hat{\imath} + 3\,\hat{k}\right) = 10\,\hat{\jmath}\ \text{m/s}.$$

The velocities are the same because all the points have the same distance from the axis of rotation.

Problem 2.109

A child on a merry-go-round is moving radially outward at a constant rate of 4 ft/s. If the merry-go-round is spinning at 30 rpm, determine the velocity and acceleration of point P on the child when the child is 0.5 and 2.3 ft from the spin axis. Express the answers using the component system shown.



Solution

$\dot{r}$ is 4 ft/s and the angular velocity expressed in rad/s is $\pi\,\hat{k}$ rad/s. The velocity is expressed as

$$\dot{\vec{r}} = \dot{r}\,\hat{u}_r + \vec{\omega}_r \times \vec{r}\,\hat{u}_r \quad \Rightarrow \quad \dot{\vec{r}} = 4\,\hat{u}_r + \pi r\,\hat{u}_q.$$

$$\begin{aligned} &\text{For } r = 0.5\,\text{ft}: \quad \vec{v} = (4\,\hat{u}_r + 1.57\,\hat{u}_q)\,\text{ft/s}. \\ &\text{For } r = 2.3\,\text{ft}: \quad \vec{v} = (4\,\hat{u}_r + 7.23\,\hat{u}_q)\,\text{ft/s}. \end{aligned}$$

The acceleration is expressed as

$$\vec{a} = \ddot{\vec{r}} = \ddot{r}\,\hat{u}_r + 2\vec{\omega}_r \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}}_r \times \vec{r} + \vec{\omega}_r \times \left(\vec{\omega}_r \times \vec{r}\right). \tag{1}$$

Now because $\dot{r}$ and ω are both constant we plug $\ddot{r} = 0$ and $\dot{\omega} = 0$ into Eq. (1) to get

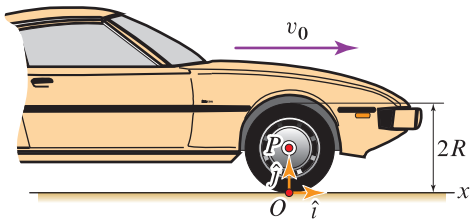
$$\vec{a} = 2\vec{\omega}_r \times \dot{r}\,\hat{u}_r + \vec{\omega}_r \times \left(\vec{\omega}_r \times \vec{r}\,\hat{u}_r\right) \quad \Rightarrow \quad \vec{a} = -\pi^2 r\,\hat{u}_r + 8\pi\,\hat{u}_q. \tag{2}$$

Substituting the given values for r into Eq. (2) we get

$$\begin{aligned} &\text{for } r = 0.5\,\text{ft}, \quad \vec{a} = (-4.93\,\hat{u}_r + 25.1\,\hat{u}_q)\,\text{ft/s}^2, \\ &\text{for } r = 2.3\,\text{ft}, \quad \vec{a} = (-22.7\,\hat{u}_r + 25.1\,\hat{u}_q)\,\text{ft/s}^2. \end{aligned}$$

Problem 2.110

When a wheel rolls without slipping on a stationary surface, the point on the wheel that is in contact with the rolling surface has zero velocity. With this in mind, consider a nondeformable wheel rolling without slip on a flat stationary surface. The center of the wheel P is traveling to the right with a constant speed of 23 m/s. Letting $R = 0.35$ m, determine the angular velocity of the wheel, using the stationary component system shown.



Solution

We can express $\vec{v}_P$ in two ways:

$$\vec{v}_P = v_0\,\hat{\imath} \quad \text{and} \quad \vec{v}_P = \dot{\vec{r}}_P = \dot{r}_P\,\hat{\imath} + \omega\,\hat{k} \times r_P\,\hat{\jmath}. \tag{1}$$

Combine Eqs. (1) and substitute $v_0 = 23$ m/s, $r_P = R$, $\dot{r}_P = 0$, and solve for ω.

$$v_0\,\hat{\imath} = -\omega R\,\hat{\imath} \quad \Rightarrow \quad \boxed{\vec{\omega} = \frac{-v_0}{R}\,\hat{k} = -(65.7\,\text{rad/s})\,\hat{k}.} \tag{2}$$

Problem 2.111

Starting with Eq. (2.62), show that the second derivative with respect to time of an arbitrary vector $\vec{A}$ is given by Eq. (2.64). Keep the answer in pure vector form, and do not resort to using components in any component system.

Solution

Beginning with Eq. (2.62),

$$\dot{\vec{A}} = \dot{A}\,\hat{u}_A + \vec{\omega}_A \times \vec{A}. \tag{1}$$

and taking the derivative with respect to time, gives

$$\ddot{\vec{A}} = \ddot{A}\,\hat{u}_A + \dot{A}\,\dot{\hat{u}}_A + \dot{\vec{\omega}} \times \vec{A} + \vec{\omega} \times \dot{\vec{A}}, \tag{2}$$

but, $\dot{\hat{u}}_A = \vec{\omega} \times \hat{u}_A$ so

$$\ddot{\vec{A}} = \ddot{A}\,\hat{u}_A + \dot{A}\left(\vec{\omega} \times \hat{u}_A\right) + \left(\dot{\vec{\omega}} \times \vec{A}\right) + \vec{\omega} \times \left(\dot{A}\,\hat{u}_A + \vec{\omega} \times \vec{A}\right), \tag{3}$$

distributing the terms of the products, yields

$$\ddot{\vec{A}} = \ddot{A}\,\hat{u}_A + \dot{A}\left(\vec{\omega} \times \hat{u}_A\right) + \left(\dot{\vec{\omega}} \times \vec{A}\right) + \left(\vec{\omega} \times \dot{A}\,\hat{u}_A\right) + \vec{\omega} \times \left(\vec{\omega} \times \vec{A}\right). \tag{4}$$

The above equation is very similar to the form of Eq. 2.60 in the book. Utilizing a property of cross-products where a is a scalar and $\vec{c}$, $\vec{d}$ are vectors it can be stated that

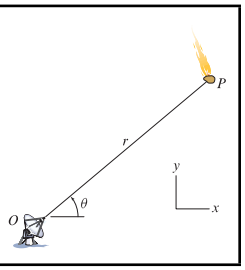
$$a\left(\vec{c} \times \vec{d}\right) = \left(a\,\vec{c} \times \vec{d}\right) = \left(\vec{c} \times a\,\vec{d}\right). \tag{5}$$

Therefore, applying the property of cross products to Eq. (4), it is concluded that

$$\begin{aligned}\ddot{\vec{A}} &= \ddot{A}\,\hat{u}_A + \left(\vec{\omega} \times \dot{A}\,\hat{u}_A\right) + \left(\vec{\omega} \times \dot{A}\,\hat{u}_A\right) + \left(\dot{\vec{\omega}} \times \vec{A}\right) + \vec{\omega} \times \left(\vec{\omega} \times \vec{A}\right), \\ &= \ddot{A}\,\hat{u}_A + \left(2\vec{\omega} \times \dot{A}\,\hat{u}_A\right) + \left(\dot{\vec{\omega}} \times \vec{A}\right) + \vec{\omega} \times \left(\vec{\omega} \times \vec{A}\right).\end{aligned}$$

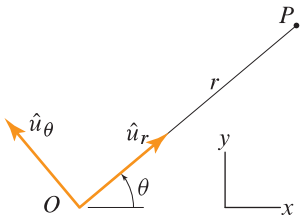
Problem 2.112

The radar station at O is tracking the meteor P as it moves through the atmosphere. At the instant shown, the station measures the following data for the motion of the meteor: $r = 21{,}000\,\text{ft}$, $\theta = 40°$, $\dot{r} = -22{,}440\,\text{ft/s}$, and $\dot{\theta} = -2.935\,\text{rad/s}$. Use Eq. (2.62) to determine the magnitude and direction (relative to the xy coordinate system shown) of the velocity vector at this instant.



Solution

We define a coordinate system as shown in the figure. Then Eq. (2.62) becomes



$$\begin{aligned}\dot{\vec{r}} &= \dot{r}\,\hat{u}_r + \vec{\omega}_r \times \vec{r},\\ &= \dot{r}\,\hat{u}_r + r\omega_r\,\hat{u}_\theta.\end{aligned} \tag{1}$$

The $(\hat{u}_r, \hat{u}_\theta)$ coordinate system is expressed in the (x, y) coordinate system as

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \qquad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{2}$$

Now we replace ω_r with $\dot{\theta}$, $\dot{\vec{r}}$ with $\vec{v}$, and plug Eq. (2) into Eq. (1) to get

$$\begin{aligned}\vec{v} &= \dot{r}\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + r\dot{\theta}\,(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}),\\ &= \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\hat{\imath} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\hat{\jmath}.\end{aligned} \tag{3}$$

Recalling that $r = 21{,}000\,\text{ft}$, $\theta = 40°$, $\dot{r} = -22{,}440\,\text{ft/s}$, and $\dot{\theta} = -2.935\,\text{rad/s}$, we can evaluate the terms in Eq. (3) to obtain

$$\vec{v} = (22{,}430\,\hat{\imath} - 61{,}640\,\hat{\jmath})\,\text{ft/s} \tag{4}$$

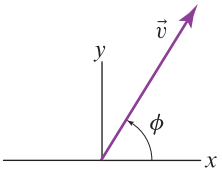
The magnitude of the velocity is given by

$$|\vec{v}| = \sqrt{v_x^2 + x_y^2}. \tag{5}$$

Hence, since from Eq. (4) we have that $v_x = 22{,}430\,\text{ft/s}$ and $v_y = -61{,}640\,\text{ft/s}$, evaluating Eq. (5), we have

$$\boxed{|\vec{v}| = 65{,}600\,\text{ft/s}.}$$

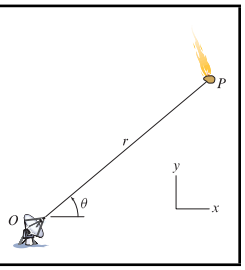
Referring to the figure on the right, let ϕ denote the angle formed by the velocity vector and the x axis such that $\tan\phi = v_y/v_x$. Then the we can obtain ϕ as $\tan^{-1}(v_y/v_x)$. Hence, recalling again that from Eq. (4) we have $v_x = 22{,}430\,\text{ft/s}$ and $v_y = -61{,}640\,\text{ft/s}$, we obtain



$$\boxed{\phi = \tan^{-1}\left(\frac{v_y}{v_x}\right) = -70.0°.} \tag{6}$$

Problem 2.113

The radar station at O is tracking the meteor P as it moves through the atmosphere. At the instant shown, the station measures the following data for the motion of the meteor: $r = 21{,}000\,\text{ft}$, $\theta = 40°$, $\dot{r} = -22{,}440\,\text{ft/s}$, $\dot{\theta} = -2.935\,\text{rad/s}$, $\ddot{r} = 187{,}500\,\text{ft/s}^2$, and $\ddot{\theta} = -5.409\,\text{rad/s}^2$. Use Eq. (2.64) to determine the magnitude and direction (relative to the xy coordinate system shown) of the acceleration vector at this instant.



Solution

Applying Eq. (2.64) on p. 92 of the textbook, we have

$$\ddot{\vec{r}}_{P/O} = \ddot{r}_{P/O}\,\hat{u}_{P/O} + 2\vec{\omega}_r \times \dot{r}_{P/O}\,\hat{u}_{P/O} + \dot{\vec{\omega}}_r \times r_{P/O} + \vec{\omega}_r \times \left(\vec{\omega}_r \times \vec{r}_{P/O}\right). \tag{1}$$

Using the Cartesian coordinate system shown and its companion component system, we can describe $\vec{r}_{P/O}$ and $\vec{\omega}_r$ as follows:

$$\vec{r}_{P/O} = r\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \quad \text{and} \quad \vec{\omega}_r = \dot{\theta}\,\hat{k}. \tag{2}$$

Rewriting each term in Eq. (1) using the expressions in Eqs. (2), we obtain

$$\begin{aligned}
\ddot{r}_{P/O}\,\hat{u}_{P/O} &= \ddot{r}\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}),\\
2\vec{\omega}_r \times \dot{r}_{P/O}\,\hat{u}_{P/O} &= 2\dot{r}\vec{\omega}_r \times \hat{u}_{P/O},\\
&= 2\dot{r}\dot{\theta}\,(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath}),\\
\dot{\vec{\omega}}_r \times \vec{r}_{P/O} &= r\ddot{\theta}\,(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath}),\\
\vec{\omega}_r \times \left(\vec{\omega}_r \times \vec{r}_{P/O}\right) &= r\dot{\theta}^2\,(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}).
\end{aligned} \tag{3}$$

Now we substitute Eqs. (3) into Eq. (1) and separate components to obtain:

$$\begin{aligned}
\vec{a}_{P/O} = &\left(\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta\right)\hat{\imath}\\
&+ \left(\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta\right)\hat{\jmath}.
\end{aligned} \tag{4}$$

Recalling that we have $r = 21{,}000\,\text{ft}$, $\theta = 40°$, $\dot{r} = -22{,}440\,\text{ft/s}$, $\dot{\theta} = -2.935\,\text{rad/s}$, $\ddot{r} = 187{,}500\,\text{ft/s}^2$, and $\ddot{\theta} = -5.409\,\text{rad/s}^2$, we can evaluate the right-hand side of Eq. (4) to obtain

$$\vec{a}_{P/O} = (-6599\,\hat{\imath} + 18130\,\hat{\jmath})\,\text{ft/s}^2. \tag{5}$$

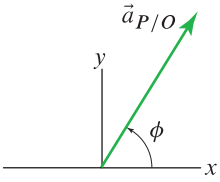
The magnitude of $\vec{a}_{P/O}$ is given by

$$|\vec{a}_{P/O}| = \sqrt{(a_{P/O})_x^2 + (a_{P/O})_y^2}. \tag{6}$$

Observing that from Eq. (5) we have $(a_{P/O})_x = -6599\,\text{ft/s}^2$ and $(a_{P/O})_y = 18{,}130\,\text{ft/s}^2$, we can evaluate $|\vec{a}_{P/O}|$ in Eq. (6) to obtain

$$\boxed{|\vec{a}_{P/O}| = 19{,}300\,\text{ft/s}^2.}$$

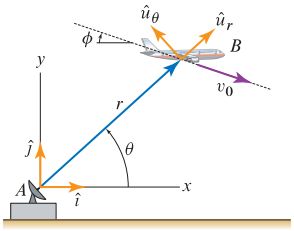
Referring to the figure on the right, let the angle ϕ denote the direction of the vector $\vec{a}_{P/O}$ relative to the x axis such that $\tan\phi = (a_{P/O})_y/(a_{P/O})_x$. Then the we can obtain ϕ as $\tan^{-1}[(a_{P/O})_y/(a_{P/O})_x]$. Hence, recalling again that from Eq. (5) we have $(a_{P/O})_x = -6599\,\text{ft/s}^2$ and $(a_{P/O})_y = 18{,}130\,\text{ft/s}^2$ and paying attention to the fact that $(a_{P/O})_x < 0$ and $(a_{P/O})_y > 0$, we obtain



$$\phi = \tan^{-1}\left(\frac{(a_{P/O})_y}{(a_{P/O})_x}\right) = 110^\circ$$

Problem 2.114

A plane B is approaching a runway along the trajectory shown while the radar antenna A is monitoring the distance r between A and B as well as the angle θ. If the plane has a constant approach speed v_0 as shown, use Eq. (2.62) to determine the expressions for $\dot{r}$ and $\dot{\theta}$ in terms of r, θ, v_0, and ϕ.



Solution

Express the velocity in the following two ways:

$$\vec{v} = v_0 \cos\phi\, \hat{\imath} - v_0 \sin\phi\, \hat{\jmath} \quad \text{and} \quad \vec{v} = \dot{r}\, \hat{u}_r + \dot{\theta}\, \hat{k} \times \hat{u}_r = \dot{r}\, \hat{u}_r + r\dot{\theta}\, \hat{u}_\theta. \tag{1}$$

Express $(\hat{u}_r, \hat{u}_\theta)$ in terms of $(\hat{\imath}, \hat{\jmath})$.

$$\hat{u}_r = \cos\theta\, \hat{\imath} + \sin\theta\, \hat{\jmath}; \quad \hat{u}_\theta = -\sin\theta\, \hat{\imath} + \cos\theta\, \hat{\jmath}. \tag{2}$$

Plug Eqs. (2) into the last of Eqs. (1) and collect the $\hat{\imath}$ and $\hat{\jmath}$ terms.

$$\vec{v} = \Big(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\Big)\hat{\imath} + \Big(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\Big)\hat{\jmath}. \tag{3}$$

Equate the $\hat{\imath}$ and $\hat{\jmath}$ components of velocity given by the first of Eqs. (1) and Eq. (3)

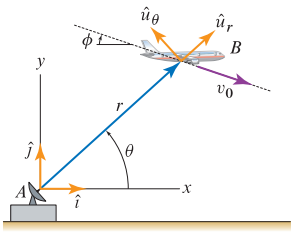
$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = v_0\cos\phi,$$
$$\dot{r}\sin\theta + r\dot{\theta}\cos\theta = -v_0\sin\phi.$$

Solve these equations simultaneously to obtain

$$\dot{r} = v_0\cos(\theta+\phi) \quad \text{and} \quad \dot{\theta} = \frac{-v_0}{r}\sin(\theta+\phi).$$

Problem 2.115

A plane B is approaching a runway along the trajectory shown with $\phi = 15°$, while the radar antenna A is monitoring the distance r between A and B as well as the angle θ. The plane has a constant approach speed v_0. In addition, when $\theta = 20°$, it is known that $\dot{r} = 216\,\text{ft/s}$ and $\dot{\theta} = -0.022\,\text{rad/s}$. Use Eq. (2.62) to determine the corresponding values of v_0 and of the distance between the plane and the radar antenna.



Solution

We begin by express the velocity in the following two ways:

$$\vec{v} = v_0 \cos\phi\,\hat{\imath} - v_0 \sin\phi\,\hat{\jmath} \quad \text{and} \quad \vec{v} = \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times \hat{u}_r = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{1}$$

Expressing $(\hat{u}_r, \hat{u}_\theta)$ in terms of $(\hat{\imath}, \hat{\jmath})$, we have

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}; \quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{2}$$

Substituting Eqs. (2) into the last of Eqs. (1) and collecting the $\hat{\imath}$ and $\hat{\jmath}$ terms, we have

$$\vec{v} = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\hat{\imath} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\hat{\jmath}. \tag{3}$$

Equating the $\hat{\imath}$ and $\hat{\jmath}$ components of velocity given by the first of Eqs. (1) and Eq. (3), we have

$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = v_0\cos\phi,$$
$$\dot{r}\sin\theta + r\dot{\theta}\cos\theta = -v_0\sin\phi.$$

Solving these equations simultaneously, we obtain

$$\dot{r} = v_0\cos(\theta+\phi) \quad \text{and} \quad \dot{\theta} = \frac{-v_0}{r}\sin(\theta+\phi). \tag{4}$$

Now, solving the first of Eqs. (4) for v_0, we have

$$v_0 = \frac{\dot{r}}{\cos(\theta+\phi)}, \tag{5}$$

which, recalling that we have $\dot{r} = 216\,\text{ft/s}$, $\theta = 20°$ and $\phi = 15°$, can be evaluated to obtain

$$\boxed{v_0 = 264\,\text{ft/s}.}$$

Substituting Eq. (5) into the second of Eqs. (4) and solving for r, we obtain

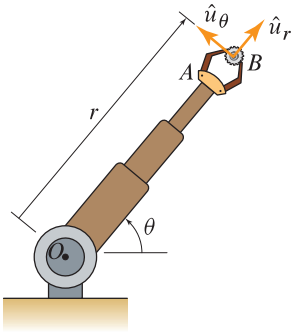
$$r = \frac{-\dot{r}}{\cos(\theta+\phi)}\frac{\sin(\theta+\phi)}{\dot{\theta}} = -\frac{\dot{r}}{\dot{\theta}}\tan(\theta+\phi), \tag{6}$$

which, recalling that $\dot{r} = 216\,\text{ft/s}$, $\dot{\theta} = -0.022\,\text{rad/s}$, $\theta = 20°$ and $\phi = 15°$, can be evaluated to obtain

$$\boxed{r = 6870\,\text{ft}.}$$

Problem 2.116

The end B of a robot arm is being extended with the constant rate $\dot{r} = 4\,\text{ft/s}$. Knowing that $\dot{\theta} = 0.4\,\text{rad/s}$ and is constant, use Eqs. (2.62) and (2.64) to determine the velocity and acceleration of B when $r = 2\,\text{ft}$. Express your answer using the component system shown.



Solution

The velocity is expressed using Eq. (2.62) on p. 92 of the textbook. This gives

$$\vec{v}_B = \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times r\,\hat{u}_r \quad \Rightarrow \quad \vec{v}_B = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta, \tag{1}$$

which, recalling that we have $\dot{r} = 4\,\text{ft/s}$, $\dot{\theta} = 0.4\,\text{rad/s}$, and $r = 2\,\text{ft}$, can be evaluated to obtain

$$\boxed{\vec{v}_B = (4.00\,\hat{u}_r + 0.800\,\hat{u}_\theta)\,\text{ft/s}.}$$

The acceleration is expressed using Eq. (2.64) on p. 92 of the textbook while keeping in mind that $\ddot{r} = 0$ and $\ddot{\theta} = 0$. This gives

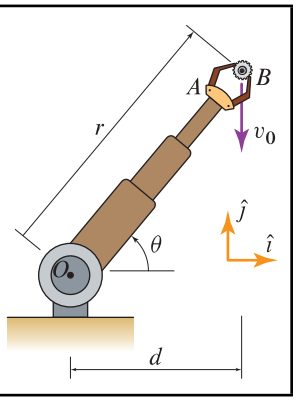
$$\vec{a}_B = 2\dot{\theta}\,\hat{k} \times \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times \left(\dot{\theta}\,\hat{k} \times r\,\hat{u}_r\right) \quad \Rightarrow \quad \vec{a}_B = -\dot{\theta}^2 r\,\hat{u}_r + 2\dot{\theta}\dot{r}\,\hat{u}_\theta$$

which, recalling that we have $\dot{r} = 4\,\text{ft/s}$, $\dot{\theta} = 0.4\,\text{rad/s}$, and $r = 2\,\text{ft}$, can be evaluated to obtain

$$\boxed{\vec{a} = (-0.320\,\hat{u}_r + 3.20\,\hat{u}_\theta)\,\text{ft/s}.}$$

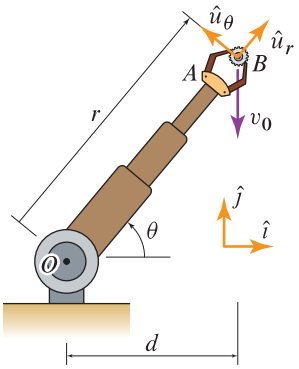
Problem 2.117

The end B of a robot arm is moving vertically down with a constant speed $v_0 = 2\,\text{m/s}$. Letting $d = 1.5\,\text{m}$, apply Eq. (2.62) to determine the rate at which r and θ are changing when $\theta = 37°$.



Solution

Referring to the figure on the right, the velocity is expressed both in terms of the $\hat{\imath}$ $\hat{\jmath}$ component system and using the component system $\hat{u}_r$ $\hat{u}_\theta$ along with Eq. (2.62) on p. 92 of the textbook. This gives



$$\vec{v} = -v_0\,\hat{\jmath} \quad \text{and} \quad \vec{v} = \dot{\vec{r}} = \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times r\,\hat{u}_r = \dot{r}\,\hat{u}_r + \dot{\theta} r \hat{u}_\theta. \tag{1}$$

Expressing the $(\hat{u}_r, \hat{u}_\theta)$ coordinate system in terms of $(\hat{\imath}, \hat{\jmath})$, we have

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \qquad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{2}$$

Substituting Eqs. (2) into the last of Eqs. (1) and collecting the $\hat{\imath}$ and $\hat{\jmath}$ terms, we have

$$\vec{v} = \Big(\dot{r}\cos\theta - \dot{\theta} r \sin\theta\Big)\,\hat{\imath} + \Big(\dot{r}\sin\theta + \dot{\theta} r \cos\theta\Big)\,\hat{\jmath} \tag{3}$$

Equating the $\hat{\imath}$ and $\hat{\jmath}$ components of velocity given by the first of Eqs. (1) and Eq. (3), and keeping in mind that $r = d/\cos\theta$, we have

$$\dot{r}\cos\theta - \dot{\theta} r \sin\theta = 0 \quad \Rightarrow \quad \dot{r}\cos\theta - \dot{\theta} d \tan\theta = 0 \tag{4}$$

$$\dot{r}\sin\theta + \dot{\theta} r \cos\theta = -v_0 \quad \Rightarrow \quad \dot{r}\sin\theta + \dot{\theta} d = -v_0. \tag{5}$$

Solving these equations simultaneously for $\dot{r}$ and $\dot{\theta}$, gives

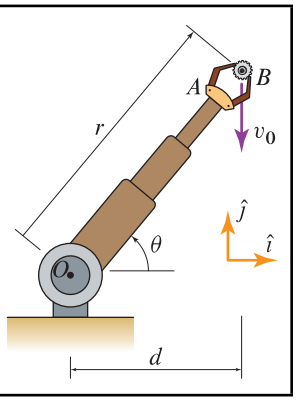
$$\dot{r} = -v_0 \sin\theta \quad \text{and} \quad \dot{\theta} = \frac{-v_0}{d(1+\tan^2\theta)} = \frac{-v_0}{d}\cos^2\theta, \tag{6}$$

which, recalling that we have $v_0 = 2\,\text{m/s}$, $\theta = 37°$, and $d = 1.5\,\text{m}$, can be evaluated to obtain

$$\boxed{\dot{r} = 1.20\,\text{m/s} \quad \text{and} \quad \dot{\theta} = -0.850\,\text{rad/s}.}$$

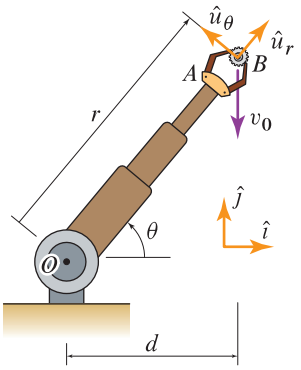
Problem 2.118

The end B of a robot arm is moving vertically down with a constant speed $v_0 = 6\,\text{ft/s}$. Letting $d = 4\,\text{ft}$, use Eqs. (2.62) and (2.64) to determine $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ when $\theta = 0°$.



Solution

Referring to the figure on the right, the velocity is expressed both in terms of the $\hat{\imath}$ $\hat{\jmath}$ component system and using the component system $\hat{u}_r$ $\hat{u}_\theta$ along with Eq. (2.62) on p. 92 of the textbook. This gives



$$\vec{v} = -v_0\,\hat{\jmath} \quad \text{and} \quad \vec{v} = \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times r\,\hat{u}_r = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta. \tag{1}$$

Expressing $(\hat{u}_r, \hat{u}_\theta)$ in terms of $(\hat{\imath}, \hat{\jmath})$, we have

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{2}$$

Substituting Eqs. (2) into the last of Eqs. (1) and collecting the $\hat{\imath}$ and $\hat{\jmath}$ terms, we have

$$\vec{v} = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\hat{\imath} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\hat{\jmath}. \tag{3}$$

Equating the $\hat{\imath}$ and $\hat{\jmath}$ components of velocity given by the first of Eqs. (1) and Eq. (3), and keeping in mind that $r = d/\cos\theta$, we have

$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = 0 \quad \Rightarrow \quad \dot{r}\cos\theta - \dot{\theta} d\tan\theta = 0 \tag{4}$$

$$\dot{r}\sin\theta + r\dot{\theta}\cos\theta = -v_0 \quad \Rightarrow \quad \dot{r}\sin\theta + \dot{\theta} d = -v_0. \tag{5}$$

Substituting $\theta = 0$ into Eqs. (4) and (5), we have

$$\boxed{\dot{r} = 0 \quad \text{and} \quad \dot{\theta} = \frac{-v_0}{d} = -1.50\,\text{rad/s},} \tag{6}$$

where we have used the following numerical data: $v_0 = 6\,\text{ft/s}$ and $d = 4\,\text{ft}$.

Using Eq. (2.64) on p. 92, the acceleration expressed in the $(\hat{u}_r, \hat{u}_\theta)$ component system is:

$$\vec{a} = \ddot{r}\,\hat{u}_r + 2\dot{\theta}\,\hat{k} \times \dot{r}\,\hat{u}_r + \ddot{\theta}\,\hat{k} \times r\,\hat{u}_r - \dot{\theta}^2 r\,\hat{u}_r.$$

On the other hand, the acceleration expressed in the $(\hat{\imath}, \hat{\jmath})$ component system is:

$$\vec{a} = \ddot{r}\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + 2\dot{\theta}\dot{r}\,(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) + \ddot{\theta} r\,(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) - \dot{\theta}^2 r\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}).$$

Collecting $\hat{\imath}$ and $\hat{\jmath}$ terms then substituting $r = d$, $\theta = 0$, and the expressions in Eqs. (6), we have

$$\vec{a}\big|_{\theta=0°} = \left(\ddot{r} - \frac{v_0^2}{d}\right)\hat{\imath} + \left(\ddot{\theta} d\right)\hat{\jmath}.$$

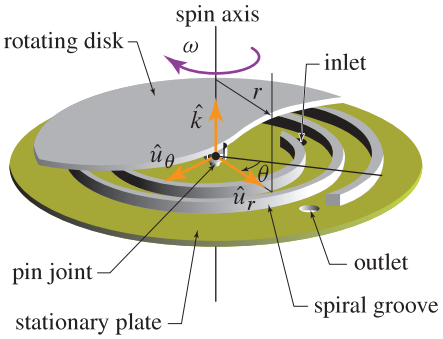
Since $\vec{v}$ is constant $\vec{a} = 0$

$$\ddot{r} = \frac{v_0^2}{d} = 9.00\,\text{ft/s}^2 \quad \text{and} \quad \ddot{\theta} = 0\,\text{rad/s}^2,$$

where, again we have used the fact that $v_0 = 6\,\text{ft/s}$ and $d = 4\,\text{ft}$.

Problem 2.119

A micro spiral pump consists of a spiral channel attached to a stationary plate. This plate has two ports, one for fluid inlet and the other for outlet, the outlet being farther from the center of the plate than the inlet. The system is capped by a rotating disk. The fluid trapped between the rotating disk and stationary plate is put in motion by the rotation of the top disk, which pulls the fluid through the spiral channel. With this in mind, consider a channel with geometry given by the equation $r = \eta\theta + r_0$, where $\eta = 12\,\mu$m is called the polar slope, $r_0 = 146\,\mu$m is the radius at the inlet, r is the distance from the spin axis, and θ is the angular position of a point in the spiral channel. If the top disk rotates with a constant angular speed $\omega = 30{,}000$ rpm, and assuming that the fluid particles in contact with the rotating disk are essentially stuck to it, determine the velocity and acceleration of one of such fluid particles when it is at $r = 170\,\mu$m. Express the answer using the component system shown (which rotates with the top disk).



Solution

Using Eq. (2.62) on p. 92 of the textbook, we can express the velocity of a particle moving through the pump as

$$\vec{v} = \dot{r}\,\hat{u}_r + \vec{\omega} \times \vec{r}, \tag{1}$$

where $\vec{\omega} = -\dot{\theta}\,\hat{k}$, $\dot{\theta} = 1000\pi$ rad/s, $r = \eta\theta + r_0$, $r_0 = 146\,\mu$m, and $\eta = 12\,\mu$m. Hence, the above expression can be rewritten as

$$\vec{v} = \eta\dot{\theta}\,\hat{u}_r + \dot{\theta}\,(\eta\theta + r_0)\,\hat{u}_\theta. \tag{2}$$

Solving $r = \eta\theta + r_0$ for θ and evaluating the result for $r = 170\,\mu$m, we find $\theta = 2.000$ rad. Using this result along with the known data $r_0 = 146\,\mu$m, $\eta = 12\,\mu$m, and $\dot{\theta} = 1000\pi$ rad/s, we can evaluate $\vec{v}$ in Eq. (1) to obtain

$$\boxed{\vec{v} = (3.77\times10^{-2}\,\hat{u}_r + 0.534\,\hat{u}_\theta)\ \text{m/s}.}$$

We now continue with the acceleration analysis. Using Eq. (2.64) on p. 92 of the textbook, and recalling that $\ddot{r}$ and $\dot{\omega}$ are equal to zero, we have

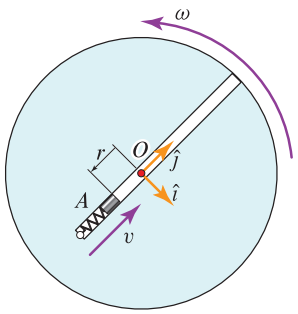
$$\vec{a} = \ddot{r}\,\hat{u}_r + 2\vec{\omega} \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad \Rightarrow \quad \vec{a} = -\dot{\theta}^2(\eta\theta + r_0)\,\hat{u}_r + 2\eta\dot{\theta}^2\,\hat{u}_\theta, \tag{3}$$

where, in obtaining the second of Eqs. (3), we have substituted for $\vec{\omega} = -\dot{\theta}\,\hat{k}$, $\vec{r} = r\,\hat{u}_r = (\eta\theta + r_0)\,\hat{u}_r$, and $\dot{r} = \eta\dot{\theta}$. Recalling again that we have $r_0 = 146\,\mu$m, $\eta = 12\,\mu$m, and $\dot{\theta} = 1000\pi$ rad/s, we can evaluate the last of Eqs. (3) to obtain

$$\boxed{\vec{a} = \left(-1680\,\hat{u}_r + 237\,\hat{u}_\theta\right)\text{m/s}^2.}$$

Problem 2.120

A disk rotates about its center, which is the fixed point O. The disk has a straight channel whose centerline passes by O and within which a collar A is allowed to slide. If, when A passes by O, the speed of A relative to the channel is $v = 14\,\text{m/s}$ and is increasing in the direction shown with a rate of $5\,\text{m/s}^2$, determine the acceleration of A given that $\omega = 4\,\text{rad/s}$ and is constant. Express the answer using the component system shown, which rotates with the disk. *Hint:* Apply Eq. (2.64) to the vector describing the position of A relative to O and then let $r = 0$.



Solution

Let $\vec{r}$ be the position of A relative to O. Then $\vec{r} = r\,\hat{u}_r$ where $\hat{u}_r = -\hat{\jmath}$. For $r = 0$ the acceleration of $\vec{r}$ is

$$\vec{a} = \ddot{r}\,\hat{u}_r + 2\vec{\omega} \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\,\right),$$

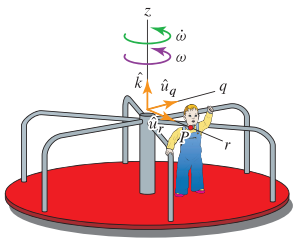
$$\vec{a} = \ddot{r}\,\hat{u}_r + 2\vec{\omega} \times \dot{r}\,\hat{u}_r,$$

$$\vec{a} = (-5\,\text{m/s}^2)(-\hat{\jmath}) + 2(4\,\text{rad/s})(-14\,\text{m/s})\,\hat{k} \times (-\hat{\jmath}).$$

$$\boxed{\vec{a} = (-112\,\hat{\imath} + 5\,\hat{\jmath})\,\text{m/s}^2}$$

Problem 2.121

At the instant shown, the angular velocity and acceleration of the merry-go-round are as indicated in the figure. Assuming that the child is walking along a radial line, should the child walk outward or inward to make sure that he does not experience any sideways acceleration (i.e., in the direction of $\hat{u}_q$)?

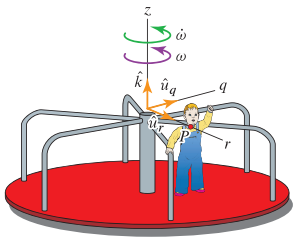


Solution

The acceleration of the child is $\vec{a} = \ddot{r}\,\hat{u}_r + 2\vec{\omega} \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right)$. The terms that contribute to the acceleration in the direction of $\hat{u}_q$ are $2\vec{\omega} \times \dot{r}\,\hat{u}_r$ and $\dot{\vec{\omega}} \times \vec{r}$. Since $\vec{\omega}$ and $\dot{\vec{\omega}}$ are in the same direction then we must have $\dot{r} < 0$ if we hope that the terms will cancel. Hence, the child should move inward.

Problem 2.122

Assuming that the child shown is moving on the merry-go-round along a radial line, use Eq. (2.64) to determine the relation that ω, $\dot{\omega}$, r, and $\dot{r}$ must satisfy so that the child will not experience any sideways acceleration.



Solution

The acceleration of the child takes on the form: $\vec{a} = \ddot{r}\,\hat{u}_r + 2\vec{\omega} \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}} \times \vec{r} - \omega^2 r\,\hat{u}_r$. The acceleration in the direction of $\hat{u}_q$ is

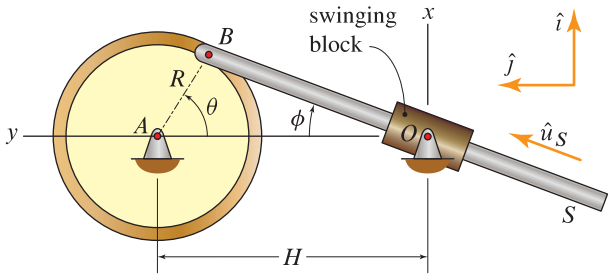
$$a_q = (2\omega\dot{r} + \dot{\omega}r)\,\hat{u}_q. \tag{1}$$

Hence we must have

$$\boxed{2\omega\dot{r} + \dot{\omega}r = 0.}$$

Problem 2.123

The mechanism shown is called a *swinging block* slider crank. First used in various steam locomotive engines in the 1800s, this mechanism is often found in door-closing systems. If the disk is rotating with a constant angular velocity $\dot{\theta} = 60\,\text{rpm}$, $H = 4\,\text{ft}$, $R = 1.5\,\text{ft}$, and r is the distance between B and O, compute $\dot{r}$ and $\dot{\phi}$ when $\theta = 90°$. *Hint:* Apply Eq. (2.62) to the vector describing the position of B relative to O.



Solution

We begin by observing that we can express the velocity of B in two ways. First as the time derivative of the position vector $\vec{r}_{B/A}$ and second as the time derivative of the position vector $\vec{r}_{B/O}$. Referring to the figure in the problem statement, we can express these two position vectors as follows:

$$\vec{r}_{B/A} = R\,\hat{u}_{B/A} \quad \text{and} \quad \vec{r}_{B/O} = r\,\hat{u}_S, \tag{1}$$

where, as given the problem statement, r is the distance between B and O, and where we observe that the angular velocities of the unit vector in the above equations are

$$\vec{\omega}_{\hat{u}_{B/A}} = \dot{\theta}\,\hat{k} \quad \text{and} \quad \vec{\omega}_{\hat{u}_S} = -\dot{\phi}\,\hat{k}. \tag{2}$$

Hence, observing that $\dot{R} = 0$ since R is a constant, using Eq. (2.62) on p. 92 of the textbook, we have

$$\vec{v}_B = \dot{\vec{r}}_{B/A} = \dot{\theta}\,\hat{k} \times R\,\hat{u}_{B/A} \quad \text{and} \quad \vec{v}_B = \dot{\vec{r}}_{B/O} = \dot{r}\,\hat{u}_S - \dot{\phi}\,\hat{k} \times r\,\hat{u}_S. \tag{3}$$

Next, we observe that, for $\theta = 90°$, we have

$$\hat{u}_{B/A} = \hat{\imath}, \quad r = \sqrt{H^2 + R^2}, \quad \text{and} \quad \hat{u}_S = \frac{1}{\sqrt{R^2 + H^2}}(R\,\hat{\imath} + H\,\hat{\jmath}). \tag{4}$$

Substituting Eqs. (4) into Eqs. (3), for $\theta = 90°$, we have

$$\vec{v}_B\big|_{\theta=90°} = R\dot{\theta}\,\hat{\jmath} \quad \text{and} \quad \vec{v}_B\big|_{\theta=90°} = \frac{\dot{r}}{\sqrt{R^2 + H^2}}(R\,\hat{\imath} + H\,\hat{\jmath}) + H\dot{\phi}\,\hat{\imath} - R\dot{\phi}\,\hat{\jmath}. \tag{5}$$

Equating the two above expressions for $\vec{v}_B$ component by component, we have

$$\hat{\imath}: \quad \frac{\dot{r}R}{\sqrt{R^2 + H^2}} + H\dot{\phi} = 0, \tag{6}$$

$$\hat{\jmath}: \quad \frac{\dot{r}H}{\sqrt{R^2 + H^2}} - R\dot{\phi} = R\dot{\theta}. \tag{7}$$

Equations (6) and (7) form a system of two equations in the two unknowns $\dot{r}$ and $\dot{\phi}$ (at $\theta = 90°$) whose solution is

$$\dot{r}\big|_{\theta=90°} = \frac{RH\dot{\theta}}{\sqrt{R^2+H^2}} \quad \text{and} \quad \dot{\phi}\big|_{\theta=90°} = -\frac{R^2\dot{\theta}}{R^2+H^2}. \tag{8}$$

Recalling that we have $\dot{\theta} = 60\,\text{rpm} = 6.283\,\text{rad/s}$, $H = 4\,\text{ft}$, $R = 1.5\,\text{ft}$, we can evaluate the quantities in Eqs. (8) to obtain

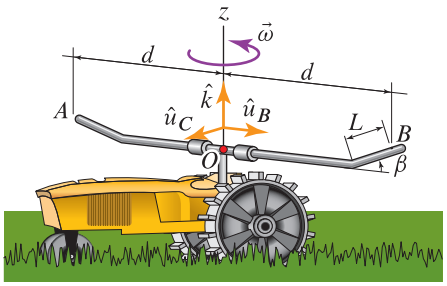
$$\boxed{\dot{r}\big|_{\theta=90°} = 8.82\,\text{ft/s} \quad \text{and} \quad \dot{\phi}\big|_{\theta=90°} = -0.775\,\text{rad/s}.}$$

Problems 2.124 and 2.125

A sprinkler essentially consists of a pipe AB mounted on a hollow shaft. The water comes in the pipe at O and goes out the nozzles at A and B, causing the pipe to rotate. Assume that the particles of water move through the pipe at a constant rate *relative to the pipe* of 5 ft/s and that the pipe AB is rotating at a constant angular velocity of 250 rpm. In all cases, express the answers using the right-handed and orthonormal component system shown.

Problem 2.124 Determine the acceleration of the water particles when they are at $d/2$ from O (still within the horizontal portion of the pipe). Let $d = 7$ in.

Problem 2.125 Determine the acceleration of the water particles right before they are expelled at B. Let $d = 7$ in., $\beta = 15°$, and $L = 2$ in. *Hint:* In this case the vector describing the position of a water particle at B goes from O to B and is best written as $\vec{r} = r_B\,\hat{u}_B + r_z\,\hat{k}$.



Solution to 2.124

Using Eq. (2.64) on p. 92, we can express the acceleration of water particle as

$$\vec{a} = \ddot{r}\,\hat{u}_r + 2\vec{\omega} \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}} \times \vec{r} - \omega^2 r\,\hat{u}_r. \tag{1}$$

next we observe that in our problem we have $\dot{r} = 5$ ft/s, $\ddot{r} = 0$, $\hat{u}_r = \hat{u}_B$, $\vec{\omega} = \omega\,\hat{k}$, $\omega = 250$ rpm $= 26.18$ rad/s, $\dot{\vec{\omega}} = \vec{0}$, and $\vec{r} = (d/2)\,\hat{u}_b$, where $d = 7$ in. $= 0.5833$ ft. Thus we have

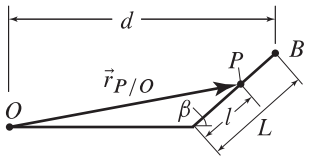
$$\ddot{\vec{r}} = 2\omega\,\hat{k} \times \dot{r}\,\hat{u}_B - \frac{\omega^2 d}{2}\,\hat{u}_B \quad \Rightarrow \quad \ddot{\vec{r}} = -2\omega\dot{r}\,\hat{u}_C - \frac{\omega^2 d}{2}\,\hat{u}_B, \tag{2}$$

which can be evaluated to obtain

$$\boxed{\vec{a} = (-262\,\hat{u}_C - 200\,\hat{u}_B)\ \text{ft/s}.}$$

Solution to 2.125

The expression for the position vector of a general point P *after* the horizontal section is



$$\vec{r}_{P/O} = (d - L\cos\beta + l\cos\beta)\,\hat{u}_B + (l\sin\beta)\,\hat{k}. \qquad (3)$$

Now we take the derivative of Eq. (3), noting that d, β, and L are constants, to get

$$\dot{\vec{r}}_{P/O} = \dot{l}\cos\beta\,\hat{u}_B + (d - L\cos\beta + l\cos\beta)\left(\vec{\omega}\times\hat{u}_B\right) + \dot{l}\sin\beta\,\hat{k}, \qquad (4)$$

$$\dot{\vec{r}}_{P/O} = -\omega\,(d - L\cos\beta + l\cos\beta)\,\hat{u}_C + \dot{l}\cos\beta\,\hat{u}_B + \dot{l}\sin\beta\,\hat{k}. \qquad (5)$$

Now we take the second derivative of $\vec{r}_{P/O}$.

$$\ddot{\vec{r}}_{P/O} = -\omega\dot{l}\cos\beta\,\hat{u}_C - \omega\,(d - L\cos\beta + l\cos\beta)\left(\vec{\omega}\times\hat{u}_C\right) + \dot{l}\cos\beta\left(\vec{\omega}\times\hat{u}_B\right) \qquad (6)$$

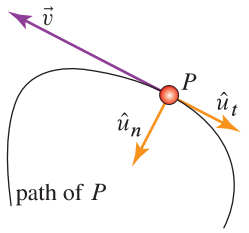
$$\ddot{\vec{r}}_{P/O} = -2\dot{l}\omega\cos\beta\,\hat{u}_C - \omega^2\,(d - L\cos\beta + l\cos\beta)\,\hat{u}_B. \qquad (7)$$

For P at B: $l = L$, $\beta = 15°$, $\omega = 250\,\text{rpm} = 26.18\,\text{rad/s}$, and $\dot{l} = 5\,\text{ft/s}$ so we have:

$$\boxed{\vec{a}_{P/O} = -2\dot{l}\omega\cos\beta\,\hat{u}_C - \omega^2 d\,\hat{u}_B \quad \Rightarrow \quad \vec{a}_{P/O} = (-253\,\hat{u}_C - 400\,\hat{u}_B)\,\text{ft/s}^2}$$

Problem 2.126

A particle P is moving along a path with the velocity shown. Is the sketch of the normal-tangential component system at P correct?

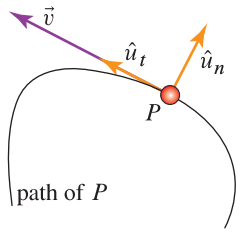


Solution

No, the unit vector $\hat{u}_t$ must point in the direction of $\vec{v}$.

Problem 2.127

A particle P is moving along a path with the velocity shown. Is the sketch of the normal-tangential component system at P correct?

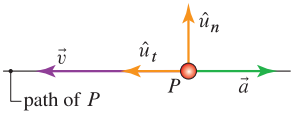


Solution

No, the unit vector $\hat{u}_n$ must point toward the concave side of the curve.

Problem 2.128

A particle P is moving along a straight line with the velocity and acceleration shown. What is wrong with the unit vectors shown in the figure?

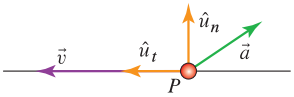


Solution

The unit vector $\hat{u}_n$ is not defined for a straight line.

Problem 2.129

A particle P is moving along some path with the velocity and acceleration shown. Can the path of P be the straight line shown?



Solution

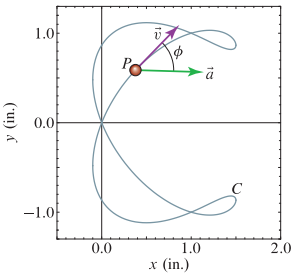
No, because the path is straight. It would be compatible if the path were curved with a tangent at P coincident with $\hat{u}_t$ and concavity on the side of $\hat{u}_n$.

Problem 2.130

A particle P is moving along the curve C, whose equation is given by

$$\left(y^2 - x^2\right)(x-1)(2x-3) = 4\left(x^2 + y^2 - 2x\right)^2,$$

at a *constant* speed v_c. For any position on the curve C for which the radius of curvature is defined (i.e., *not* equal to infinity), what *must* be the angle ϕ between the velocity vector $\vec{v}$ and the acceleration vector $\vec{a}$?

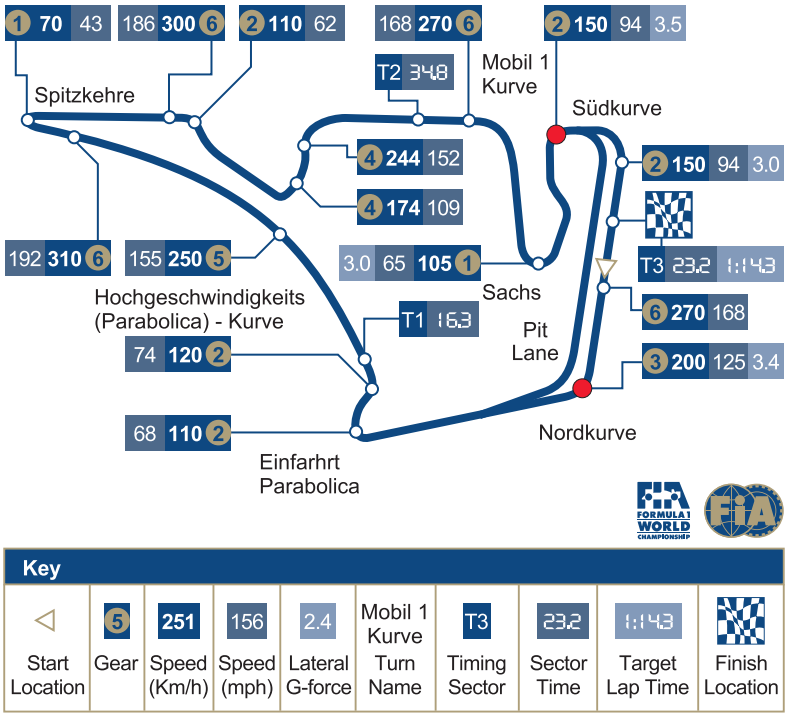


Solution

The speed is constant. This tells us that there is no component of acceleration in the direction of velocity. Therefore the angle ϕ between $\vec{v}$ and $\vec{a}$ must be 90°.

Problem 2.131

Making the same assumptions stated in Example 2.17, consider the map of the Formula 1 circuit at Hockenheim in Germany and estimate the radius of curvature of the curves Südkurve and Nordkurve (at the locations indicated in red).



Solution

At Südkurve the car is traveling at a speed of $150\,\text{km/hr} = 41.67\,\text{m/s}$ with an acceleration of $3.5g = 34.34\,\text{m/s}^2$. Therefore the curvature is

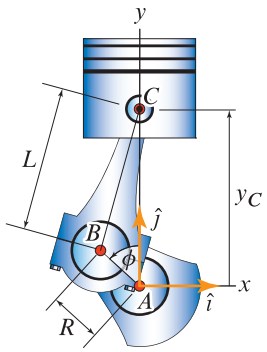
$$a = \frac{v^2}{\rho} \quad \Rightarrow \quad \rho = \left(\frac{41.67^2}{34.34}\right)\text{m} \quad \Rightarrow \quad \boxed{\rho = 50.6\,\text{m}.}$$

At Nordkurve the car is traveling at a speed of $200\,\text{km/hr} = 55.56\,\text{m/s}$ with an acceleration of $3.4g = 33.35\,\text{m/s}^2$. Therefore the curvature is

$$a = \frac{v^2}{\rho} \quad \Rightarrow \quad \rho = \left(\frac{55.56^2}{33.35}\right)\text{m} \quad \Rightarrow \quad \boxed{\rho = 92.5\,\text{m}.}$$

Problem 2.132

The motion of the piston C as a function of the crank angle ϕ and the lengths of the crank AB and connecting rod BC is given by $y_C = R\cos\phi + L\sqrt{1-(R\sin\phi/L)^2}$ and $x_C = 0$. Using the component system shown, express $\hat{u}_t$, the unit vector tangent to the trajectory of C, as a function of the crank angle ϕ for $0 \leq \phi \leq 2\pi$ rad.

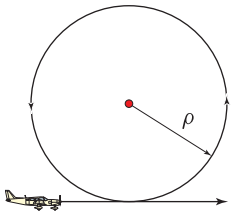


Solution

$\hat{u}_t = -\hat{j}$ for $0 < \phi < \pi$ rad. $\hat{u}_t = \hat{j}$ for $\pi < \phi < 2\pi$ rad. $\hat{u}_t$ is undefined at $\phi = 0$ and $\phi = 2\pi$ rad because the path of C folds back on itself at these two positions.

Problem 2.133

An aerobatics plane initiates the basic loop maneuver such that, at the bottom of the loop, the plane is going 140 mph, while subjecting the plane to approximately $4g$ of acceleration. Estimate the corresponding radius of the loop.



Solution

$4g = 128.8\,\text{ft/s}^2$. $140\,\text{mi/hr} = 205.3\,\text{ft/s}$. Thus, the radius of the loop is found with

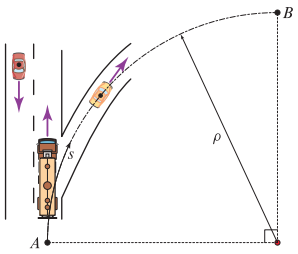
$$a_n = \frac{v^2}{\rho} \quad \Rightarrow \quad \rho = \left(\frac{205.3^2}{128.8}\right)\text{ft}, \tag{1}$$

which gives

$$\boxed{\rho = 327\,\text{ft}.}$$

Problem 2.134

Suppose that a highway exit ramp is designed to be a circular segment of radius $\rho = 130\,\text{ft}$. A car begins to exit the highway at A while traveling at a speed of 65 mph and goes by point B with a speed of 25 mph. Compute the acceleration vector of the car as a function of the arc length s, assuming that the tangential component of the acceleration is constant between points A and B.



Solution

The change in speed is completely due to the tangential acceleration. Hence, since this component of acceleration is assumed to be constant, the speeds at A and B are related via the following constant acceleration equation:

$$v_B^2 - v_A^2 = 2a_t(s_B - s_A) \quad \Rightarrow \quad a_t = \frac{v_B^2 - v_A^2}{2(s_B - s_A)}. \tag{1}$$

Next, recalling that we have $s_A = 0$, $v_A = 65\,\text{mph} = 95.33\,\text{ft/s}$, $v_B = 25\,\text{mph} = 36.67\,\text{ft/s}$, and $s_B = \rho\frac{\pi}{2} = (65\pi)\,\text{ft} = 204.2\,\text{ft}$, the above equation can be evaluated to obtain

$$a_t = -18.96\,\text{ft/s}^2. \tag{2}$$

The expression for the speed in terms of arc length, s, can be found again using constant acceleration equations, i.e.,

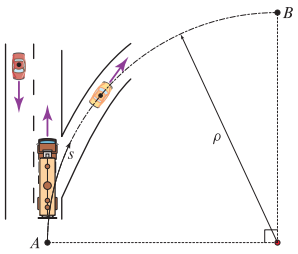
$$v^2(s) = v_A^2 + 2a_t(s - s_A) \quad \Rightarrow \quad v^2(s) = [9088 - (37.92)s]\ \text{ft}^2/\text{s}^2, \tag{3}$$

where we have used the fact that $v_A = 65\,\text{mph} = 95.33\,\text{ft/s}$ and that $a_t = -18.96\,\text{ft/s}^2$ (see Eq. (2)). Recalling that $a_n = v^2/\rho$, we have that the acceleration vector is

$$\vec{a}(s) = a_t\,\hat{u}_t + \frac{v^2(s)}{\rho}\,\hat{u}_n \quad \Rightarrow \quad \boxed{\vec{a}(s) = \{-19.0\,\hat{u}_t + [69.9 - (0.292)s]\ \hat{u}_n\}\,\text{ft/s}^2.} \tag{4}$$

Problem 2.135

Suppose that a highway exit ramp is designed to be a circular segment of radius $\rho = 130\,\text{ft}$. A car begins to exit the highway at A while traveling at a speed of 65 mph and goes by point B with a speed of 25 mph. Compute the acceleration vector of the car along the car's path as a function of the arc length s, assuming that between A and B the speed was controlled so as to maintain constant the rate dv/ds.



Solution

Let K be the constant value of the quantity dv/ds. Then, the appropriate expression for the speed can be found by integrating the equation $\frac{dv}{ds} = K$ as follows:

$$dv = K\,ds \quad \Rightarrow \quad \int_{v_A}^{v_B} dv = \int_{s_A}^{s_B} K\,ds \quad \Rightarrow \quad v_B = v_A + Ks. \tag{1}$$

Applying the conditions that at $s_A = 0$, $v_A = 65\,\text{mph} = 95.33\,\text{ft/s}$, and at $s_B = \rho\frac{\pi}{2} = (65\pi)\,\text{ft} = 204.2\,\text{ft}$, $v_B = 25\,\text{mph} = 36.67\,\text{ft/s}$, it is found that

$$K = \frac{v_B - v_A}{s_B} = -0.2873\,\text{s}^{-1}. \tag{2}$$

Hence the speed as a function of the arc length s is

$$v(s) = v_A + Ks. \tag{3}$$

Next we recall that the quantity $\dot{v}$ can be written as

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} \quad \Rightarrow \quad \dot{v} = v(s)\frac{dv}{ds} = (v_A + Ks)\,K. \tag{4}$$

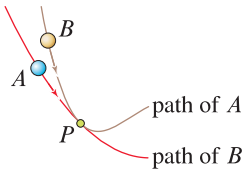
Finally, recalling that the acceleration vector is given by $\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n$, we can write

$$\begin{aligned}\vec{a} &= \left(v_A K + K^2 s\right)\hat{u}_t + \frac{(v_A + Ks)^2}{\rho}\hat{u}_n \\ &= \left\{\left[-27.4 + (825\times10^{-4})s\right]\hat{u}_t + \left[69.9 - (0.421)s + (635\times10^{-6})s^2\right]\hat{u}_n\right\}\text{ft/s}^2,\end{aligned}$$

where we have used the value of K in Eq. (2) and the following numerical data: $v_A = 65\,\text{mph} = 95.33\,\text{ft/s}$ and $\rho = 130\,\text{ft}$.

Problem 2.136

Particles A and B are moving in the plane with the same constant speed v, and their paths are tangent at P. Do these particles have zero acceleration at P? If not, do these particles have the same acceleration at P?

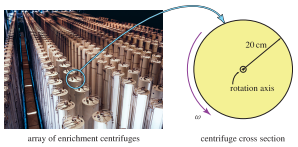


Solution

Without knowledge of the radii of curvature for the two curves it is not possible to answer the question. If the curvature of the two curves is equal to zero at P, then the acceleration of the particles at P would be equal to zero since the particles are moving with constant speed. If the radii of curvature for the two curves at P are finite and equal but not equal to zero, then the acceleration of the two particles at P would be equal. If the radii of curvature are different, then the accelerations of the two particles would be different.

Problem 2.137

Uranium is used in light water reactors to produce a controlled nuclear reaction for the generation of power. When first mined, uranium comes out as the oxide U_3O_8, which is 0.7% of the isotope U-235 and 99.3% of the isotope U-238. To be used in a nuclear reactor, the concentration of U-235 must be in the 3–5% range. The process of increasing the percentage of U-235 is called *enrichment*, and it is done in a number of ways. One method uses centrifuges, which spin at very high rates to create artificial gravity. In these centrifuges, the heavy U-238 atoms concentrate on the outside of the cylinder (where the acceleration is largest), and the lighter U-235 atoms concentrate near the spin axis. Before centrifuging, the uranium is processed into gaseous uranium hexafluoride or UF_6, which is then injected into the centrifuge. Assuming that the radius of the centrifuge is 20 cm and that it spins at 70,000 rpm, determine



(a) The velocity of the outer surface of the centrifuge.

(b) The acceleration in g experienced by an atom of uranium that is on the inside of the outer wall of the centrifuge.

Solution

We denote by ω the angular speed of the centrifuge. Hence, we have that $\omega = 70{,}000\,\text{rpm} = 7330\,\text{rad/s}$.

Part (a). Using the fact that the speed v and the angular speed ω are related as $v = \omega\rho$, we have that the speed of points on the outer surface of the centrifuge is

$$v = \rho\omega = 1466\,\text{m/s}, \tag{1}$$

where we have used the fact that $\omega = 7330\,\text{rad/s}$ and $\rho = 20\,\text{cm} = 0.2000\,\text{m}$. Expressing the above result to 3 significant figures, we have

$$\boxed{v = \rho\omega = 1470\,\text{m/s}.}$$

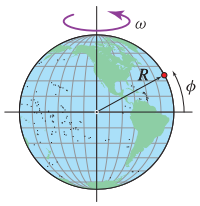
Part (b). Under the assumption that the centrifuge is spinning with a constant angular speed, the only component of acceleration of a point on the wall of the centrifuge will be the normal component. Hence, the acceleration experienced by an atom at the inside surface of the centrifuge is

$$a_n = \frac{v^2}{\rho} = \frac{(1466\,\text{m/s})^2}{0.2\,\text{m}} = \frac{g\,(10.75\times10^6\,\text{m/s}^2)}{9.81\,\text{m/s}^2} \quad \Rightarrow \quad \boxed{\vec{a} = 1.10\times10^6 g\,\hat{u}_n,}$$

where the unit vector v always points from a point on the periphery of the centrifuge toward the center of the centrifuge,and where we have used the the value of v in Eq. (1) along with the fact that $\rho = 20\,\text{cm} = 0.2000\,\text{m}$ and $g = 9.81\,\text{m/s}^2$.

Problem 2.138

Treating the center of the Earth as a fixed point, determine the magnitude of the acceleration of points on the surface of the earth as a function of the angle ϕ shown. Use $R = 6371$ km as the radius of the Earth.



Solution

Let ω denote the angular speed of the Earth. Since the Earth undergoes one full revolution per day, we have $\frac{2\pi}{24(3600)}$ rad/s.

$$\omega = \frac{1\text{ rev}}{1\text{ day}} = \frac{2\pi\text{ rad}}{(24\text{ h})(3600\text{ s/h})} = 72.72\times10^{-6}\text{ rad/s}. \tag{1}$$

Next, let ρ denote the distance between the point indicated on the figure and the axis of rotation of the Earth, i.e., a point on the surface of the Earth characterized by the angle ϕ off the equator and the axis of rotation of the Earth. Then, we have we

$$\rho = R\cos\phi. \tag{2}$$

Next we observe that under the assumption that the angular speed of the Earth is constant, the only component of the acceleration of the point in question is the normal component. Hence, we must have

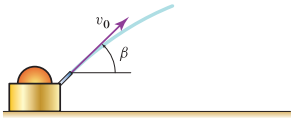
$$|\vec{a}| = a_n = \frac{v^2}{\rho} = \rho\omega^2 = R\omega^2\cos\phi,$$

which, recalling that $R = 6371$ km $= 6{,}371{,}000$ m, using the value of ω in Eq. (1), and expressing all known coefficients to 3 significant figures, can be evaluated to obtain

$$\boxed{|\vec{a}| = (33.7\times10^{-3}\cos\phi)\text{ m/s}^2.}$$

Problem 2.139

A water jet is ejected from the nozzle of a fountain with a speed $v_0 = 12\,\text{m/s}$. Letting $\beta = 33°$, determine the rate of change of the speed of the water particles as soon as these are ejected as well as the corresponding radius of curvature of the water path.



Solution

The normal and tangential components of acceleration are:

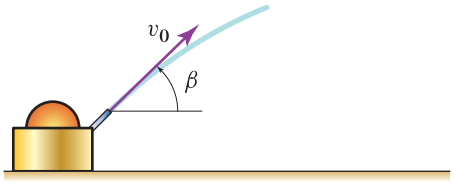
$$a_n = g\cos\beta = \frac{v_0^2}{\rho}; \quad \text{and} \quad a_t = -g\sin\beta = \dot{v}.$$

Hence, recalling that $g = 9.81\,\text{m/s}^2$, $v_0 = 12\,\text{m/s}$, and $\beta = 33°$, we have

$$\rho = \frac{v_0^2}{g\cos\beta} = 17.5\,\text{m} \quad \text{and} \quad \dot{v} = -5.34\,\text{m/s}.$$

Problem 2.140

A water jet is ejected from the nozzle of a fountain with a speed v_0. Letting $\beta = 21°$, determine v_0 so that the radius of curvature at the highest point on the water arch is 10 ft.



Solution

We model the water particles in the jet as projectiles subject to the constant acceleration due to gravity. Hence, at the highest point on the water arch the vertical component of the velocity of the water particles is equal to zero and the only nonzero component of velocity is equal to the initial component of velocity in the horizontal direction. This component of velocity is equal to $v_0 \cos\beta$. Therefore, at the highest point on the water arch we have

$$\vec{v} = v_0 \cos\beta\, \hat{u}_t \quad \Rightarrow \quad v = v_0 \cos\beta. \tag{1}$$

Recalling that speed and radius of curvature are related through the relation $a_n = v^2/\rho$ and observing that at the highest point on the water arch the acceleration due to gravity is completely in the normal direction, i.e., $a_n = g$, then we must have

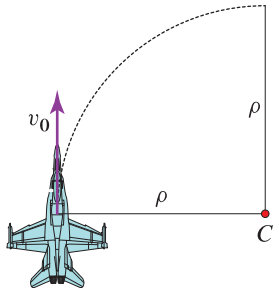
$$a_n = g = \frac{v^2}{\rho} \quad \Rightarrow \quad g = \frac{v_0^2 \cos^2\beta}{\rho} \quad \Rightarrow \quad v_0 = \sqrt{\frac{\rho g}{\cos^2\beta}}. \tag{2}$$

Recalling that $\beta = 21°$, $g = 32.2\,\text{ft/s}^2$, and $\rho = 10\,\text{ft}$, we can evaluate the above expression to obtain

$$\boxed{v_0 = 19.2\,\text{ft/s}.}$$

Problem 2.141

A jet is flying at a constant speed $v_0 = 750$ mph while performing a constant speed circular turn. If the magnitude of the acceleration needs to remain constant and equal to $9g$, where g is the acceleration due to gravity, determine the radius of curvature of the turn.



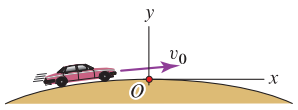
Solution

Using the given data, we have $v_0 = 750$ mph $= 1100$ ft/s. Since the speed is constant, the only component of the acceleration is the normal component, namely, a_n. Relating this component of acceleration to the speed and the radius of curvature, we have

$$a_n = \frac{v^2}{\rho} = 9g \quad \Rightarrow \quad \boxed{\rho = \frac{v^2}{9g} = 4180 \text{ ft.}} \tag{1}$$

Problem 2.142

A car traveling with a speed $v_0 = 65$ mph almost loses contact with the ground when it reaches the top of the hill. Determine the radius of curvature of the hill at its top.



Solution

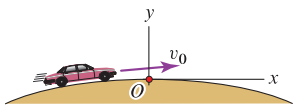
For the car to almost loose contact with the ground we have

$$a_n = \frac{v_0^2}{\rho} = g \quad \Rightarrow \quad \boxed{\rho = \frac{v_0^2}{g} = 282\,\text{ft},} \tag{1}$$

where we have used the following numerical data: $v_0 = 65$ mph $= 95.33$ ft/s and $g = 32.2\,\text{ft/s}^2$.

Problem 2.143

A car is traveling over a hill. If, using a Cartesian coordinate system with origin O at the top of the hill, the hill's profile is described by the function $y = -(0.003\,\text{m}^{-1})x^2$, where x and y are in meters, determine the minimum speed at which the car would lose contact with the ground at the top of the hill. Express the answer in km/h.



Solution

We need to calculate the curvature at the top of the hill ($x = 0$). TO do so, we will apply the formula giving the radius of curvature in terms of derivatives of the curve $y = -(0.003\,\text{m}^{-1})x^2$ with respect to x. Specifically, we have

$$\rho(x) = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|}. \tag{1}$$

With this in mind, we have

$$\frac{dy}{dx} = -(0.006000\,\text{m}^{-1})x \quad \text{and} \quad \frac{d^2y}{dx^2} = -0.006000\,\text{m}^{-1}. \tag{2}$$

Substituting the values from Eqs. (2) into Eq. (1) for $x = 0$, we have

$$\rho(0) = \frac{1}{0.006000\,\text{m}^{-1}} = 166.7\,\text{m}. \tag{3}$$

Now we observe that for the car to almost loose contact with the ground we have

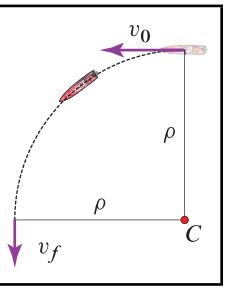
$$a_n = \frac{v_{\text{min}}^2}{\rho} = g, \tag{4}$$

where v_{min} is the value of the speed that we are trying to determine. Since g is known, i.e., $g = 9.81\,\text{m/s}^2$ and we have just determined that $\rho = 166.7\,\text{m}$, we can solve the above equation to obtain

$$v_{\text{min}} = \sqrt{g\rho(0)} \quad \Rightarrow \quad v_{\text{min}} = 40.44\,\text{m/s} \quad \Rightarrow \quad \boxed{v_{\text{min}} = 146\,\text{km/h}.} \tag{5}$$

Problem 2.144

A race boat is traveling at a constant speed $v_0 = 130\,\text{mph}$ when it performs a turn with constant radius ρ to change its course by 90° as shown. The turn is performed while losing speed uniformly in time so that the boat's speed at the end of the turn is $v_f = 125\,\text{mph}$. If the maximum allowed normal acceleration is equal to $2g$, where g is the acceleration due to gravity, determine the tightest radius of curvature possible and the time needed to complete the turn.



Solution

Since the normal acceleration is given by $a_n = v^2/\rho$, under the assumption that $\rho =$ constant, the maximum normal acceleration occurs where the speed is maximum. In turn, this occurs at the beginning of the turn given that the speed is monotonically decreasing. Specifically, we have

$$v_{\max} = v_0 = 130\,\text{mph} = 190.7\,\text{ft/s}. \tag{1}$$

then, going back to the formula $a_n = v^2/\rho$, we have

$$a_n = 2g = \frac{v_{\max}^2}{\rho_{\min}} \quad \Rightarrow \quad \boxed{\rho_{\min} = \frac{v_0^2}{2g} = 565\,\text{ft},} \tag{2}$$

where we have used the value of v_0 in Eq. (1) and the fact that $g = 32.2\,\text{ft/s}^2$.

Next, recalling that the speed is decreasing uniformly in time, i.e., that $\dot{v}$ is constant, using the constant acceleration equation $v_f^2 - v_0^2 = 2a_t\,(s - s_0)$ to find a_t, we have

$$s = \frac{\pi\rho}{2} \quad \text{and} \quad s_0 = 0 \quad \Rightarrow \quad a_t = \frac{v_f^2 - v_0^2}{\pi\rho_{\min}} \tag{3}$$

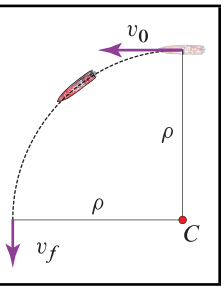
Now use the constant acceleration equation $v_f = v_0 + a_t t_f$ to find the time to complete the turn, where $v_f = 125\,\text{mph} = 183.3\,\text{ft/s}$.

$$t_f = \frac{\pi\rho_{\min}}{v_f + v_0} \quad \Rightarrow \quad \boxed{t_f = \frac{\pi v_0^2}{2g\,(v_f + v_0)} = 4.74\,\text{s},} \tag{4}$$

where we have used the following numerical values: $v_0 = 190.7\,\text{ft/s}$ (see Eq. (1)), $v_f = 125\,\text{mph} = 183.3\,\text{ft/s}$, and $g = 32.2\,\text{ft/s}^2$.

Problem 2.145

A race boat is traveling at a constant speed $v_0 = 130\,\text{mph}$ when it performs a turn with constant radius ρ to change its course by 90° as shown. The turn is performed while losing speed uniformly in time so that the boat's speed at the end of the turn is $v_f = 116\,\text{mph}$. If the magnitude of the acceleration is not allowed to exceed $2g$, where g is the acceleration due to gravity, determine the tightest radius of curvature possible and the time needed to complete the turn.



Solution

The given values of speed at the beginning and end of the turn are

$$v_0 = 130\,\text{mph} = 190.7\,\text{ft/s} \quad \text{and} \quad v_f = 116\,\text{mph} = 170.1\,\text{ft/s}. \tag{1}$$

Treating the value of ρ as known and equal to the minimum value that we need to determine, then the normal component of acceleration is given by

$$a_n = \frac{v^2}{\rho_{\text{min}}}, \tag{2}$$

where v is the speed of the boat. Next, recalling that the tangential acceleration is constant, applying constant acceleration equations, we have

$$v_f^2 - v_0^2 = 2a_t(s_f - s_0) = 2a_t\frac{\pi}{2}\rho_{\text{min}}, \tag{3}$$

where $(\pi/2)\rho_{\text{min}}$ is the distance covered by the boat along its path while performing the turn. Hence, we can solve the above equation for a_t to obtain

$$a_t = \frac{v_f^2 - v_0^2}{\pi\rho_{\text{min}}}. \tag{4}$$

Now, using Eqs. (2) and (4) we can compute the magnitude of the acceleration of the boat. In addition, recalling that we want the maximum value of the acceleration not to exceed $2g$, we can solve for ρ_{min} by writing

$$|\vec{a}|^2 = a_n^2 + a_t^2 = \frac{v_0^4}{\rho_{\text{min}}^2} + \frac{\left(v_f^2 - v_0^2\right)^2}{\pi^2\rho_{\text{min}}^2} = 4g^2 \quad \Rightarrow \quad \rho_{\text{min}} = \sqrt{\frac{v_0^4}{4g^2} + \frac{\left(v_f^2 - v_0^2\right)^2}{4\pi^2 g^2}} = 565.9\,\text{ft}, \tag{5}$$

where we have the speed values in Eqs. (1) and the fact that $g = 32.2\,\text{ft/s}^2$.

Now we solve for t_f the time needed to perform the turn by observing that we must have $v_f = v_0 + a_t t_f$. Hence, using the expression for a_t in Eq. (4), we can write

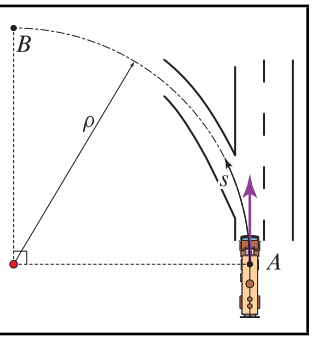
$$t_f = \frac{v_f - v_0}{a_t} = \frac{v_f - v_0}{v_f^2 - v_0^2}\pi\rho_{\text{min}} \quad \Rightarrow \quad t_f = \frac{\pi\rho_{\text{min}}}{v_f + v_0} = 4.927\,\text{s},$$

where we have used the speed values in Eq. (1) and the value of ρ_{min} in Eq. (5). Summarizing and expressing our final answers to 3 significant figures, we have

$$\boxed{\rho_{\text{min}} = 566\,\text{ft} \quad \text{and} \quad t_f = 4.93\,\text{s}.}$$

Problem 2.146

A truck takes an exit ramp with a speed $v_0 = 55\,\text{mph}$. The ramp is a circular arc with radius $\rho = 150\,\text{ft}$. Determine the constant rate of change of the truck speed that will allow the truck to stop at B.



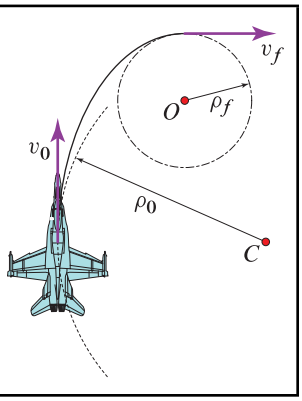
Solution

Since the tangential component of the acceleration of the truck is assumed to be constant, we can solve the problem by using the constant acceleration equation $v^2 - v_0^2 = 2\dot{v}(s - s_0)$. Doing so and recalling that we have letting $s_0 = 0$, $s = (\pi\rho)/2 = 235.6\,\text{ft}$, $v_f = 0$, and $v_0 = 55\,\text{mph} = 80.67\,\text{ft/s}$, we can write

$$\dot{v} = \frac{-v_0^2}{2s} \quad \Rightarrow \quad \boxed{\dot{v} = -13.8\,\text{ft/s}^2.} \tag{1}$$

Problem 2.147

A jet is flying straight and level at a speed $v_0 = 1100\,\text{km/h}$ when it turns to change its course by 90° as shown. In an attempt to progressively tighten the turn, the speed of the plane is uniformly decreased in time while keeping the normal acceleration constant and equal to $8g$, where g is the acceleration due to gravity. At the end of the turn, the speed of the plane is $v_f = 800\,\text{km/h}$. Determine the radius of curvature ρ_f at the end of the turn and the time t_f that the plane takes to complete its change in course.



Solution

Keeping in mind that we have $v_f = 800\,\text{km/h} = 222.2\,\text{m/s}$ and that $g = 9.81\,\text{m/s}^2$, the radius of curvature at the end of the turn is

$$a_n = 8g = \frac{v_f^2}{\rho_f} \quad \Rightarrow \quad \boxed{\rho_f = \frac{v_f^2}{8g} = 629\,\text{m}} \tag{1}$$

Now, we need to relate the speed to the fact that the plane undergoes a change of course of 90°. To do so, let ω denote the anguar speed of the plane. Therefore ω is also the angular speed of the radial line connecting the plane to center of curvature on an instant by instant basis. With this in mind, we have $v = \omega\rho$ where $\omega = \frac{d\theta}{dt}$, where θ is any angle that one can conveniently define to characterize the orientation of the radial line connecting the plane to center of curvature on an instant by instant basis. So we have

$$\omega = \frac{d\theta}{dt} = \frac{v}{\rho} = \frac{8g}{v}, \tag{2}$$

where, to obtain the last expression we have used the fact that the normal acceleration must remain constant and equal to $8g$, i.e., $a_n = v^2/rho = 8g$ so that $v/\rho = 8g/v$. Now, we enforce the condition that the speed is decrased uniformily in time. This means that we can used constant acceleration equations and write

$$v = v_0 + a_t t, \tag{3}$$

where a_t is the (constant) tangential acceleration. Therefore, substituting Eq. (3) into Eq. (2) we have

$$\frac{d\theta}{dt} = \frac{8g}{v_0 + a_t t} \quad \Rightarrow \quad d\theta = \frac{8g}{v_0 + a_t t}\,dt \quad \Rightarrow \quad \int_0^{t_f} \frac{8g}{v_0 + a_c t}\,dt = \int_0^{\pi/2} d\theta, \tag{4}$$

where in the above integral we have finally managed to relate the speed of the airplane to the fact that the airplane undergoes a change of course of 90°. Carrying out the above integration, and recalling that we must have $a_t = (v_f - v_0)/t_f$, we have

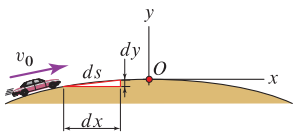
$$\frac{8g}{a_t}\ln\left(1 + \frac{a_t t_f}{v_0}\right) = \frac{\pi}{2} \quad \Rightarrow \quad \frac{8gt_f}{v_f - v_0}\ln\left(1 + \frac{v_f - v_0}{v_0}\right) = \frac{\pi}{2}.$$

Hence, recalling that we have $v_f = 800\,\text{km/h} = 222.2\,\text{m/s}$, $v_0 = 1100\,\text{km/h} = 305.6\,\text{ft/s}$, and $g = 9.81\,\text{m/s}^2$, we can solve the above equation for t_f to obtain

$$\boxed{t_f = \frac{\pi(v_f - v_0)}{16g\ln(v_f/v_0)} = 5.24\,\text{s}.}$$

Problem 2.148

A car is traveling over a hill with a constant speed $v_0 = 70\,\text{mph}$. Using the Cartesian coordinate system shown, the hill's profile is described by the function $y = -(0.0005\,\text{ft}^{-1})x^2$, where x and y are measured in feet. At $x = -300\,\text{ft}$, the driver applies the brakes, causing a constant time rate of change of speed $\dot{v} = -3\,\text{ft/s}^2$ until the car arrives at O. Determine the distance traveled while applying the brakes along with the time to cover this distance.



Hint: To compute the distance traveled by the car along the car's path, observe that $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\,dx$, and that

$$\int \sqrt{1 + C^2x^2}\,dx = \frac{x}{2}\sqrt{1 + C^2x^2} + \frac{1}{2C}\ln\left(Cx + \sqrt{1 + C^2x^2}\right).$$

Solution

Taking the derivative of $y(x)$ with respect to x, we have

$$y = -(0.0005\,\text{ft}^{-1})x^2 \quad \Rightarrow \quad \frac{dy}{dx} = -(0.001000\,\text{ft}^{1})x. \tag{1}$$

Making use of the equation for ds given in the problem statement. For convenience, let $C = -0.001000\,\text{ft}^{-1}$. Then we can write

$$ds = \sqrt{1 + (dy/dx)^2}\,dx \quad \Rightarrow \quad ds = \sqrt{1 + C^2x^2}\,dx. \tag{2}$$

Integrating Eq. (2) from $x = -300\,\text{ft}$ to $x = 0$, we have

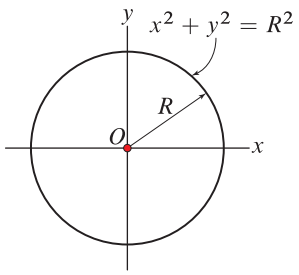
$$s = \int_{-300\,\text{ft}}^{0} \sqrt{1 + C^2x^2}\,dx \quad \Rightarrow \quad s = \left[\frac{x}{2}\sqrt{1 + C^2x^2} + \frac{1}{2C}\ln\left(Cx + \sqrt{1 + C^2x^2}\right)\right]_{-300\,\text{ft}}^{0}. \tag{3}$$

Evaluating the last of Eqs. (3) and expressing the final result to 3 significant figures, we find

$$\boxed{s = 304\,\text{ft}.}$$

Problem 2.149

Recalling that a circle of radius R and center at the origin O of a Cartesian coordinate system with axes x and y can be expressed via the formula $x^2+y^2 = R^2$, use Eq. (2.72) to verify that the radius of curvature of this circle is equal to R.



Solution

Taking the differential of $x^2 + y^2 = R^2$ with respect to x to find $\frac{dy}{dx}$, we have

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}. \tag{1}$$

Takinf the derivative of Eq. (1) with respect to x and substituting Eq. (1) back into the results for $\frac{dy}{dx}$, we have

$$\frac{d^2y}{dx^2} = \frac{-1}{y} + \frac{x}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-(x^2+y^2)}{y^3}. \tag{2}$$

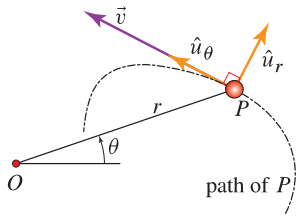
Then

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad \Rightarrow \quad \rho = \frac{\left[1 + x^2/y^2\right]^{3/2}}{\left|x^2+y^2\right|}\left|y^3\right| \quad \Rightarrow \tag{3}$$

$$\rho = \frac{\left(x^2+y^2\right)^{3/2}}{\left|x^2+y^2\right|} = \frac{R^3}{R^2} \quad \Rightarrow \quad \boxed{\rho = R.} \tag{4}$$

Problem 2.150

A particle P is moving along a path with the velocity shown. Discuss in detail whether or not there are incorrect elements in the sketch of the polar component system at P.

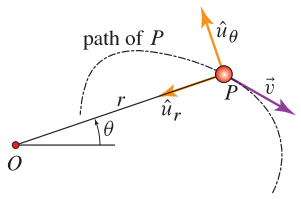


Solution

The unit vector $\hat{u}_r$ must be oriented along the radial line r and point away from the origin. The unit vector $\hat{u}_\theta$ is oriented perpendicular to $\hat{u}_r$ and pointing in the direction of increasing θ.

Problem 2.151

A particle P is moving along a path with the velocity shown. Discuss in detail whether or not there are incorrect elements in the sketch of the polar component system at P.

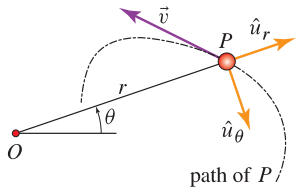


Solution

The unit vector $\hat{u}_\theta$ is oriented correctly but $\hat{u}_r$ must point in the direction opposite to the direction shown.

Problem 2.152

A particle P is moving along a path with the velocity shown. Discuss in detail whether or not there are incorrect elements in the sketch of the polar component system at P.

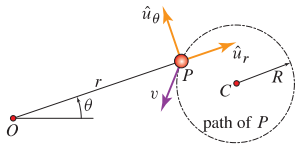


Solution

The unit vector $\hat{u}_r$ is oriented correctly. $\hat{u}_\theta$ must be oriented opposite to the direction shown.

Problem 2.153

A particle P is moving along a circle with center C and radius R in the direction shown. Letting O be the origin of a polar coordinate system with the coordinates r and θ shown, discuss in detail whether or not there are incorrect elements in the sketch of the polar component system at P.

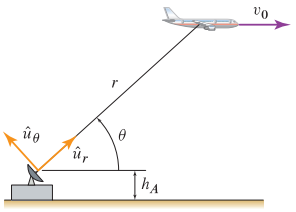


Solution

Both $\hat{u}_r$ and $\hat{u}_\theta$ are oriented as they should be.

Problem 2.154

A radar station is tracking a plane flying at a constant altitude with a speed $v_0 = 550\,\text{mph}$. If at a given instant $r = 7\,\text{mi}$ and $\theta = 32°$, determine the corresponding values of $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$.



Solution

The velocity is $\vec{v} = v_0\,\hat{\imath}$, where $\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta$. The velocity in polar coordinates is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$. Therefore

$$v_0\cos\theta\,\hat{u}_r - v_0\sin\theta\,\hat{u}_\theta = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta.$$

This vector equation gives us two scalar equations we use to solve for the desired quantites.

$$v_0\cos\theta = \dot{r} \quad \Rightarrow \quad \boxed{\dot{r} = 466\,\text{mph}}$$

$$-v_0\sin\theta = r\dot{\theta} \quad \Rightarrow \quad \boxed{\dot{\theta} = -41.6\,\text{rad/h}.}$$

The problem statement implies that the velocity of the airplane is constant and that therefore the acceleration of the airplane is equal to zero. Use the following equation to solve for the desired quantities. This implies that we must have

$$\vec{0} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta,$$

This vector equation gives us two scalar equations we use to solve for the desired quantites.

$$\ddot{r} = r\dot{\theta}^2 \quad \Rightarrow \quad \boxed{\ddot{r} = 12{,}100\,\text{mi/h}^2,}$$

and

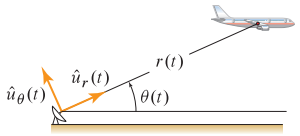
$$\ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r} \quad \Rightarrow \quad \boxed{\ddot{\theta} = 5550\,\text{rad/h}^2.}$$

Problem 2.155

During a given time interval, a radar station tracking an airplane records the readings

$$\dot{r}(t) = [449.8\cos\theta(t) + 11.78\sin\theta(t)]\,\text{mph},$$
$$r(t)\dot{\theta}(t) = [11.78\cos\theta(t) - 449.8\sin\theta(t)]\,\text{mph},$$

where t denotes time. Determine the speed of the plane. Furthermore, determine whether the plane being tracked is ascending or descending and the corresponding climbing rate (i.e., the rate of change of the plane's altitude) expressed in ft/s.



Solution

Using the expressions for $\dot{r}(t)$ and $r(t)\dot{\theta}(t)$ provided by the problem statement, and realizing that $\dot{r}(t) = v_r$ and $r(t)\dot{\theta}(t) = v_\theta$, we have

$$\begin{aligned} v^2 &= v_r^2 + v_\theta^2 \\ &= \dot{r}^2 + (r\dot{\theta})^2 \\ &= \left[(449.8)^2\cos^2\theta + (11.78)^2\sin^2\theta + 2\sin\theta\cos\theta(449.8)(11.78)\right]\text{mph} \\ &\quad + \left[(11.78)^2\cos^2\theta + (449.8)^2\sin^2\theta - 2\sin\theta\cos\theta(449.8)(11.78)\right]\text{mph}. \end{aligned} \tag{1}$$

Using the trigonometric identity $\sin^2\omega + \cos^2\omega = 1$, we can simplify the above expression to

$$v^2 = \left[(449.8)^2 + (11.78)^2\right]\text{mph} \quad \Rightarrow \quad \boxed{v = 450\,\text{mph}.} \tag{2}$$

Letting y denote the elevation of the airplane, we have that $y = r\sin\theta$. Taking the derivative of y with respect to time we have

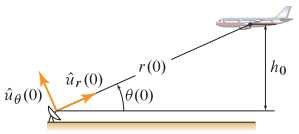
$$\begin{aligned} \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ &= [(449.8)\sin\theta\cos\theta + (11.78)\sin^2\theta + (11.78)\cos^2\theta - (449.8)\sin\theta\cos\theta]\,\text{mph} \\ &= 11.78\,\text{mph}. \end{aligned}$$

Converting the above result and expressing to 3 significant figures, we have

The airplane is ascending at a rate of 17.3 ft/s.

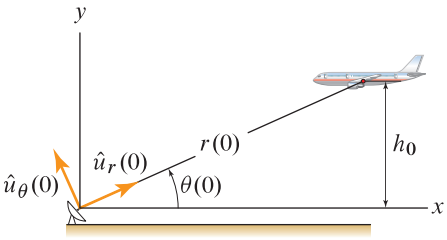
Problem 2.156

At a given instant, an airplane flying at an altitude $h_0 = 10{,}000$ ft begins its descent in preparation for landing when it is $r(0) = 20$ mi from the radar station at the destination's airport. At that instant, the aiplane's speed is $v_0 = 300$ mph, the climb rate is -5 ft/s, and the horizontal component of velocity is decreasing steadily at a rate of 15 ft/s^2. Determine the $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ that would be observed by the radar station.



Solution

Referring to the figure on the right, at time $t = 0$, we have $\theta = \sin^{-1}(h/r) = 5.434°$, $\dot{x}(0) = -\sqrt{v_0^2 - \dot{y}(0)^2} = -440.0$ ft/s, and $\dot{y}(0) = -5$ ft/s. In addition, we can relate the polar coordinates (r, θ) to the Cartesian coordinates (x, y) as follows:



$$y = r\sin\theta \quad \Rightarrow \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta, \tag{1}$$

$$x = r\cos\theta \quad \Rightarrow \quad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta. \tag{2}$$

Viewing the above equations as a system of two equations in the two unknowns $\dot{r}$ and $\dot{\theta}$, solving the system in question and evaluating the solution for $t = 0$, we have

$$\dot{r}(0) = -438 \text{ ft/s} \quad \text{and} \quad \dot{\theta}(0) = 347\times10^{-6} \text{ rad/s},$$

where we have used the data listed earlier.

Differentiating the last of Eqs. (1) and (2) yields the following system of two equations in the two unknowns $\ddot{r}$ and $\ddot{\theta}$:

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta$$

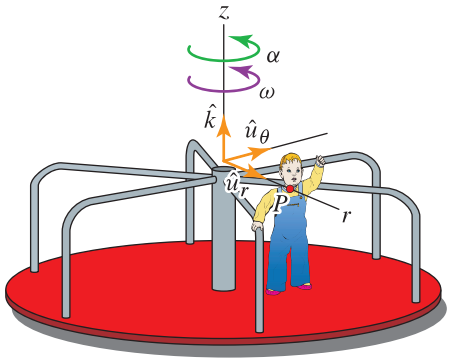
$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta.$$

Recalling that we have $\ddot{x}(0) = 15$ ft/s^2 and $\ddot{y}(0) = 0$, solving the above system and evaluating the solution at $t = 0$, we have

$$\ddot{r}(0) = 14.9 \text{ ft/s}^2 \quad \text{and} \quad \ddot{\theta}(0) = -10.6\times10^{-6} \text{ rad/s}^2.$$

Problem 2.157

At a given instant, the merry-go-round is rotating with an angular velocity $\omega = 20\,\text{rpm}$ while the child is moving radially outward at a constant rate of $0.7\,\text{m/s}$. Assuming that the angular velocity of the merry-go-round remains constant, i.e., $\alpha = 0$, determine the magnitudes of the speed and of the acceleration of the child when he is $0.8\,\text{m}$ away from the spin axis.



Solution

The child's velocity is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, where $\dot{r} = 0.7\,\text{m/s}$ and $\dot{\theta} = \omega = 20\,\text{rpm} = 2.094\,\text{rad/s}$. Hence, for $r = 8\,\text{m}$, we have

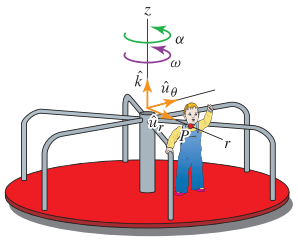
$$v = |\vec{v}| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \quad \Rightarrow \quad \boxed{v = 1.82\,\text{m/s}.} \tag{1}$$

The child's acceleration is $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$, where $\ddot{\theta} = \alpha = 0$ and $\ddot{r} = 0$.

$$|\vec{a}| = \sqrt{(-r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta})^2} \quad \Rightarrow \quad \boxed{|\vec{a}| = 4.57\,\text{m/s}^2.} \tag{2}$$

Problem 2.158

At a given instant, the merry-go-round is rotating with an angular velocity $\omega = 18\,\text{rpm}$, and it is slowing down at a rate of $0.4\,\text{rad/s}^2$. When the child is 2.5 ft away from the spin axis, determine the time rate of change of the child's distance from the spin axis so that the child experiences no transverse acceleration while moving along a radial line.



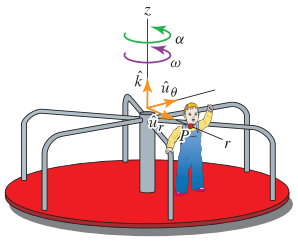
Solution

The transverse component of acceleration is $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, where $\dot{\theta} = \omega = 18\,\text{rpm} = 1.885\,\text{rad/s}$ and $\ddot{\theta} = \alpha = -0.4\,\text{rad/s}^2$. Hence, for $r = 2.5\,\text{ft}$, we have

$$\dot{r} = -\frac{r\ddot{\theta}}{2\dot{\theta}} \quad \Rightarrow \quad \boxed{\dot{r} = 0.265\,\text{ft/s}.}$$

Problem 2.159

At a given instant, the merry-go-round is rotating with an angular velocity $\omega = 18$ rpm. When the child is 0.45 m away from the spin axis, determine the second derivative with respect to time of the child's distance from the spin axis so that the child experiences no radial acceleration.



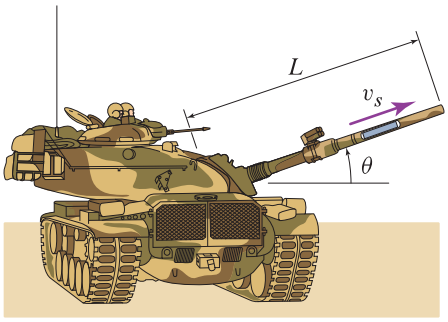
Solution

The radial component of acceleration is $a_r = \ddot{r} - r\dot{\theta}^2$, where $\dot{\theta} = \omega = 18$ rpm $= 1.885$ rad/s. Hence, for $r = 0.45$ m, we have

$$\ddot{r} = r\omega^2 \quad \Rightarrow \quad \boxed{\ddot{r} = 1.60\,\text{m/s}^2.}$$

Problem 2.160

The cutaway of the gun barrel shows a projectile that, upon exit, moves with a speed $v_s = 5490\,\text{ft/s}$ relative to the gun barrel. The length of the gun barrel is $L = 15\,\text{ft}$. Assuming that the angle θ is increasing at a constant rate of $0.15\,\text{rad/s}$, determine the speed of the projectile right when it leaves the barrel. In addition, assuming that the projectile acceleration along the barrel is constant and that the projectile starts from rest, determine the magnitude of the acceleration upon exit.



Solution

We will use a coordinate system with origin at the base of the gun, the coordinate r define as the distance of a point along the gun from its base, and the coordinate θ as shown in the figure. In this coordinate system, we have that the rate of change of the radial position of the projectile along the gun is $\dot{r}$. In addition, we are told that $\dot{\theta} = 0.15\,\text{rad/s}$. Recall that in polar coordinates the velocity of the projectile is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$. Therefore, upon exit, i.e., for $r = L$ and $\dot{r} = v_s = 5490\,\text{ft/s}$, the speed of the projectile is

$$\boxed{|\vec{v}| = \sqrt{\dot{r}^2 + (L\dot{\theta})^2} = 5490\,\text{ft/s}.}$$

The assumption that the acceleration of the projectile along the gun is constant implies that the quantity $\ddot{r}$ is assumed constant. Hence, we can compute $\ddot{r}$ using constant acceleration equations, i.e., $\dot{r}^2 = \dot{r}_0^2 + 2\ddot{r}(r - r_0)$. letting $r_0 = 0$, since $\dot{r}_0 = 0$ (the projectile start from rest), and since we know that for $r = L$ we have $\dot{r} = v_s$, we can write

$$\ddot{r} = \frac{v_s^2}{2L}.$$

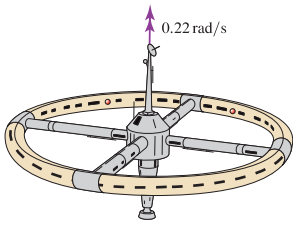
Next, recalling that in polar coordinate the acceleration of the projectile is $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$, and that we have $\ddot{\theta} = 0$, upon exit, we can write

$$\boxed{|\vec{a}| = \sqrt{\left(\frac{v_s^2}{2L} - L\dot{\theta}^2\right)^2 + \left(2v_s\dot{\theta}\right)^2} = 1.00\times10^6\,\text{ft/s}^2,}$$

where we have used the fact that $v_s = 5490\,\text{ft/s}$, $L = 15\,\text{ft}$, and $\dot{\theta} = 0.15\,\text{rad/s}$.

Problem 2.161

A space station is rotating in the direction shown at a constant rate of 0.22 rad/s. A crew member travels from the periphery to the center of the station through one of the radial shafts at a constant rate of 1.3 m/s (relative to the shaft) while holding onto a handrail in the shaft. Taking $t = 0$ to be the instant at which travel through the shaft begins and knowing that the radius of the station is 200 m, determine the velocity and acceleration of the crew member as a function of *time*. Express your answer using a polar coordinate system with origin at the center of the station.



Solution

We will use a polar coordinate system with origin at the center of the station and radial coordinate going from the center to the person moving along the shaft. The velocity of the astronaut is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, where $\dot{r} = -1.3$ m/s and $\dot{\theta} = 0.22$ rad/s. Let r_0 represent the radius of the station. The position is $r = r_0 - 1.3\,t$. So we have

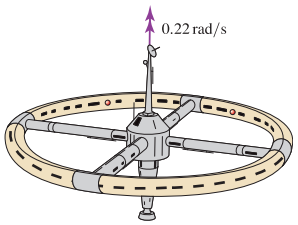
$$\vec{v} = [-1.30\,\hat{u}_r + (44.0 - 0.286\,t)\,\hat{u}_\theta]\ \text{m/s}.$$

The acceleration is $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$, where $\ddot{r} = 0$ and $\ddot{\theta} = 0$.

$$\vec{a} = [(-9.68 + 0.0629\,t)\,\hat{u}_r - 0.572\,\hat{u}_\theta]\ \text{m/s}^2.$$

Problem 2.162

Solve Prob. 2.161 and express your answers as a function of *position* along the shaft traveled by the astronaut.



Solution

The velocity of the astronaut is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$. Therefore

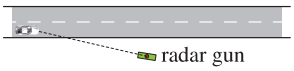
$$\boxed{\vec{v} = (-1.3\,\hat{u}_r + 0.22\,r\,\hat{u}_\theta)\ \text{m/s}.}$$

The acceleration is $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$, where $\ddot{r}$ and $\ddot{\theta} = 0$.

$$\boxed{(-0.0484\,r\,\hat{u}_r - 0.572\,\hat{u}_\theta)\ \text{m/s}^2.}$$

Problem 2.163

A person driving along a rectilinear stretch of road is fined for speeding, having been clocked at 75 mph when the radar gun was pointing as shown. The driver claims that, because the radar gun is off to the side of the road instead of directly in front of his car, the radar gun overestimates his speed. Is he right or wrong and why?

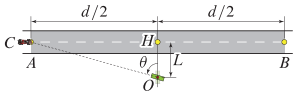


Solution

The driver is wrong. The speed recorded is the component of the actual speed along the radial direction. As such, i.e. being a component, it can only be smaller than the true speed.

Problem 2.164

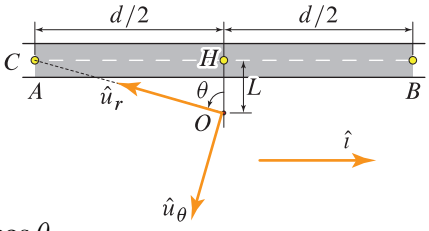
A motion tracking camera is placed along a rectilinear stretch of a racetrack (the figure is not to scale). A car C enters the stretch at A with a speed $v_A = 110$ mph and accelerates uniformly in time so that at B it has a speed $v_B = 175$ mph, where $d = 1$ mi. Letting the distance $L = 50$ ft, if the camera is to track the motion of C, determine the camera's angular velocity as well as the time rate of change of the angular velocity when the car is at A and at H.



Solution

At A the velocity of the car is $\vec{v} = v_A\,\hat{\imath}$. Expressing the unit vector $\hat{\imath}$ via the $(\hat{u}_r, \hat{u}_\theta)$ component system, we have

$$\hat{\imath} = -\sin\theta\hat{u}_r - \cos\theta\hat{u}_\theta.$$



To find $\dot{\theta}$, we consider the transverse component of velocity v_θ, i.e.,

$$v_\theta = r\dot{\theta} = -v\cos\theta \quad \Rightarrow \quad \dot{\theta} = \frac{-v\cos\theta}{r}. \tag{1}$$

For the point A we have $r = \sqrt{(d/2)^2 + L^2}$, $\cos\theta = \frac{L}{r}$, and $v = v_A$. Eq. (1) becomes

$$\boxed{\dot{\theta}_A = \frac{-v_A L}{(d/2)^2 + L^2} = -1.16\times10^{-3}\ \text{rad/s}.} \tag{2}$$

where we have used the following numerical data: $v_A = 110$ mph $= 161.3$ ft/s, $L = 50$ ft, and $d = 1$ mi $= 5280$ ft.

Next, using the constant acceleration equation $v^2 = v_0^2 + 2a_c(s - s_0)$ the acceleration of the car is $\vec{a} = \left(\frac{v_B^2 - v_A^2}{2d}\right)\hat{\imath}$. Using the same constant acceleration equation the velocity at H is $\vec{v}_H = \sqrt{\frac{v_A^2 + v_B^2}{2}}\,\hat{\imath}$. For the point H we have $r = L = 50$ ft and $\cos\theta = 1$ so that Eq. (1) gives

$$\boxed{\dot{\theta}_H = \frac{-1}{L}\sqrt{\frac{v_A^2 + v_B^2}{2}} = -4.29\,\text{rad/s},} \tag{3}$$

where we have used the following numerical data: $v_A = 110$ mph $= 161.3$ ft/s, $v_B = 175$ mph $= 256.7$ ft/s, $L = 50$ ft, and $d = 1$ mi $= 5280$ ft.

To find $\ddot{\theta}$ we need to find $\dot{r}$ at A and H by using the radial component of velocity $v_r = \dot{r} = -v\sin\theta$. At A, we have that $\sin\theta = \frac{d}{2r} = \frac{d}{2\sqrt{(d/2)^2 + L^2}}$. At H, we have that $\sin\theta = 0$. Therefore, we can write

$$v_r = \dot{r} = -v\sin\theta \quad \Rightarrow \quad \dot{r}_A = \frac{-v_A d}{2\sqrt{(d/2)^2 + L^2}}, \tag{4}$$

$$\Rightarrow \quad \dot{r}_H = 0. \tag{5}$$

Now we use the $\hat{u}_\theta$ component of acceleration to find $\ddot{\theta}$. Keeping in mind that $\vec{a} = \left(\frac{v_B^2 - v_A^2}{2d}\right)\hat{\imath}$, we have

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{(v_B^2 - v_A^2)}{2d}\cos\theta \quad \Rightarrow \quad \ddot{\theta} = \frac{1}{r}\left[\frac{(v_A^2 - v_B^2)}{2d}\cos\theta - 2\dot{r}\dot{\theta}\right]. \tag{6}$$

To find $\ddot{\theta}$ at A we start with Eq. (6) and replace $\dot{r}$ with Eq. (4), $\dot{\theta}$ with Eq. (2), and $\cos\theta$ with $\frac{L}{r}$, i.e.,

$$\boxed{\ddot{\theta}_A = \frac{1}{r^2}\left[\frac{\left(v_A^2 - v_B^2\right)L}{2d} - \frac{v_A^2 dL}{r^2}\right] = -1.68\times10^{-4}\,\text{rad/s}^2,}$$

where $r = \sqrt{(d/2)^2 + L^2}$, and where we have used the following numerical data: $v_A = 110\,\text{mph} = 161.3\,\text{ft/s}$, $v_B = 175\,\text{mph} = 256.7\,\text{ft/s}$, $L = 50\,\text{ft}$, and $d = 1\,\text{mi} = 5280\,\text{ft}$.

To find $\ddot{\theta}$ at H we start with Eq. (6) and replace $\dot{r}$ with Eq. (5), r with L, and $\cos\theta$ with 1.

$$\boxed{\ddot{\theta} = \frac{\left(v_A^2 - v_B^2\right)}{2dL} = -0.0755\,\text{rad/s}^2,}$$

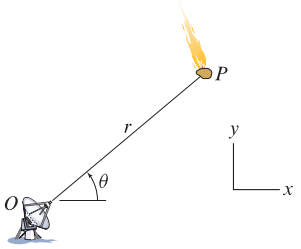
where we have used the following numerical data: $v_A = 110\,\text{mph} = 161.3\,\text{ft/s}$, $v_B = 175\,\text{mph} = 256.7\,\text{ft/s}$, $L = 50\,\text{ft}$, and $d = 1\,\text{mi} = 5280\,\text{ft}$.

Problem 2.165

The radar station at O is tracking a meteor P as it moves through the atmosphere. At the instant shown, the station measures the following data for the motion of the meteor: $r = 21{,}000\,\text{ft}$, $\theta = 40°$, $\dot{r} = -22{,}440\,\text{ft/s}$, $\dot{\theta} = -2.935\,\text{rad/s}$, $\ddot{r} = 187{,}500\,\text{ft/s}^2$, and $\ddot{\theta} = -5.409\,\text{rad/s}^2$.

(a) Determine the magnitude and direction (relative to the xy coordinate system shown) of the velocity vector at this instant.

(b) Determine the magnitude and direction (relative to the xy coordinate system shown) of the acceleration vector at this instant.



Solution

Part (a). Using the polar coordinate system defined in the figure of the problem statement, we have that the velocity of the meteor is

$$\vec{v} = v_r\,\hat{u}_r + v_\theta\,\hat{u}_\theta = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta = -(22{,}440\,\text{ft/s})\,\hat{u}_r - (6.164\times10^4\,\text{ft/s})\,\hat{u}_\theta, \tag{1}$$

where we have used the following numerical data: $\dot{r} = -22{,}440\,\text{ft/s}$, $r = 21{,}000\,\text{ft}$, and $\dot{\theta} = -2.935\,\text{rad/s}$. Then the magnitude of $\vec{v}$ is

$$|\vec{v}| = \sqrt{v_r^2 + v_\theta^2} = 6.56\times10^4\,\text{ft/s},$$

where, from Eq. (1), we have used the fact that $v_r = -22{,}440\,\text{ft/s}$ and $v_\theta = -6.164\times10^4\,\text{ft/s}$. Let β denote the angle by which the unit vector $\hat{u}_r$ must be rotated counterclockwise to point in the same direction as the vector $\vec{v}$. Then, because both v_r and v_θ are negative, the angle β is given by

$$\beta = 180° + \tan^{-1}\left(\frac{v_\theta}{v_r}\right) = 250.0°. \tag{2}$$

Since the radial direction is oriented at 40° counterclockwise with respect to the x axis, then the angle formed by the velocity vector and the x axis is $\beta + 40°$, where it is understood that this angle is measured counterclockwise from the x axis. Hence, we have

Orientation of $\vec{v}$ from x axis $= 290°$ (ccw).

Part (b). Using the polar coordinate system defined in the figure of the problem statement, we have that the acceleration of the meteor is

$$\vec{v} = a_r\,\hat{u}_r + a_\theta\,\hat{u}_\theta = (\ddot{r} - r\dot{\theta}^{2)}\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta = (6.601\times10^3\,\text{ft/s}^2)\,\hat{u}_r + (18.13\times10^3\,\text{ft/s}^2)\,\hat{u}_\theta, \tag{3}$$

where we have used the following numerical data: $\ddot{r} = 187{,}500\,\text{ft/s}^2$, $\dot{r} = -22{,}440\,\text{ft/s}$, $\dot{\theta} = -2.935\,\text{rad/s}$, $r = 21{,}000\,\text{ft}$, and $\ddot{\theta} = -5.409\,\text{rad/s}^2$. Then the magnitude of $\vec{a}$ is

$$|\vec{a}| = \sqrt{a_r^2 + a_\theta^2} = 19.3\times10^3\,\text{ft/s}^2,$$

where, from Eq. (3), we have used the fact that $a_r = 6.601 \times 10^3$ ft/s^2 and $a_\theta = 18.13 \times 10^3$ ft/s^2. Let γ denote the angle by which the unit vector $\hat{u}_r$ must be rotated counterclockwise to point in the same direction as the vector $\vec{a}$. Then, because both a_r and a_θ are positive, the angle γ is given by

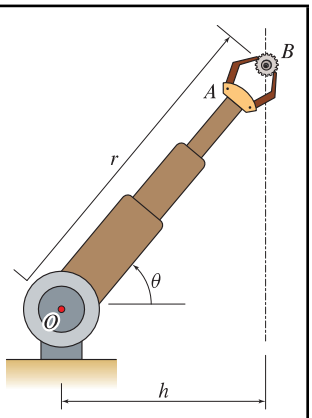
$$\gamma = \tan^{-1}\left(\frac{a_\theta}{a_r}\right) = 69.99^\circ. \tag{4}$$

Since the radial direction is oriented at 40° counterclockwise with respect to the x axis, then the angle formed by the acceleration vector and the x axis is $\gamma + 40^\circ$, where it is understood that this angle is measured counterclockwise from the x axis. Hence, we have

Orientation of $\vec{a}$ from x axis $= 110^\circ$ (ccw).

Problems 2.166 and 2.167

As a part of an assembly process, the end effector at A on the robotic arm needs to move the gear at B along the vertical line shown with some known velocity v_0 and acceleration a_0. Arm OA can vary its length by telescoping via internal actuators, and a motor at O allows it to pivot in the vertical plane.

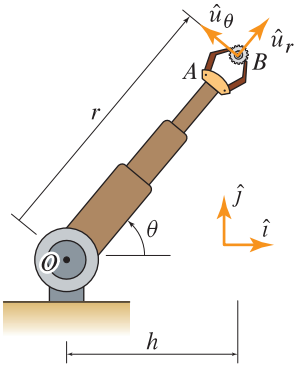


Problem 2.166 When $\theta = 50°$, it is required that $v_0 = 8\,\text{ft/s}$ (down) and that it be slowing down at $a_0 = 2\,\text{ft/s}^2$. Using $h = 4\,\text{ft}$, determine, at this instant, the values for $\ddot{r}$ (the extensional acceleration) and $\ddot{\theta}$ (the angular acceleration).

Problem 2.167 Letting v_0 and a_0 be positive if the gear moves and accelerates upward, determine expressions for r, $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$ that are valid for any value of θ.

Solution to 2.166

Referring to the figure on the right, The $(\hat{\imath}, \hat{\jmath})$ component system is converted to the $(\hat{u}_r, \hat{u}_\theta)$ component system as



$$\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta \quad \text{and} \quad \hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta. \tag{1}$$

The velocity $\vec{v} = -v_0\,\hat{\jmath}$ is expressed in the $(\hat{u}_r, \hat{u}_\theta)$ coordinate system as

$$\vec{v} = -v_0\,(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta)\,. \tag{2}$$

The general expression for velocity in polar coordinates is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$. Thus $\dot{r}$ and $\dot{\theta}$ are

$$\dot{r} = -v_0 \sin\theta \quad \text{and} \quad \dot{\theta} = \frac{-v_0\cos\theta}{r}. \tag{3}$$

The acceleration $\vec{a} = a_0\,\hat{\jmath}$ is converted to the $(\hat{u}_r, \hat{u}_\theta)$ coordinate system as

$$\vec{a} = a_0\,(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta)\,. \tag{4}$$

The general expression for acceleration in polar coordinates is $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta$. Using the results of Eq. (3) and noting that $r = h/\cos\theta$ we obtain

$$\ddot{r} - r\dot{\theta}^2 = a_0 \sin\theta, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_0\cos\theta, \tag{5}$$

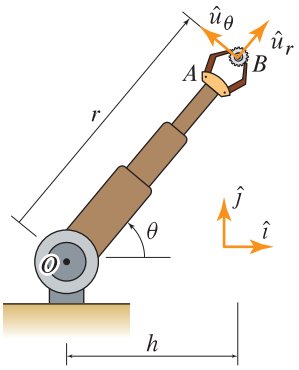
$$\ddot{r} = a_0\sin\theta + \frac{v_0^2\cos^3\theta}{h}, \qquad \ddot{\theta} = \frac{\cos\theta}{h}\left(a_0\cos\theta - \frac{2v_0^2\cos^2\theta\sin\theta}{h}\right). \tag{6}$$

Recalling that we have $\theta = 50°$, $v_0 = 8\,\text{ft/s}$, $a_0 = 2\,\text{ft/s}^2$, and $h = 4\,\text{ft}$, we can evaluated the expressions for $\ddot{r}$ and $\ddot{\theta}$ to obtain

$$\boxed{\ddot{r} = 5.78\,\text{ft/s}^2 \quad \text{and} \quad \ddot{\theta} = -1.42\,\text{rad/s}^2.}$$

Solution to 2.167

Referring to the figure on the right, the length of the arm is



$$\boxed{r = \frac{h}{\cos\theta},} \tag{7}$$

The $(\hat{\imath}, \hat{\jmath})$ component system is converted to the $(\hat{u}_r, \hat{u}_\theta)$ componenet system as

$$\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta \quad \text{and} \quad \hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta. \tag{8}$$

The velocity $\vec{v} = -v_0\,\hat{\jmath}$ is expressed in the $(\hat{u}_r, \hat{u}_\theta)$ coordinate system as

$$\vec{v} = -v_0\,(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta)\,. \tag{9}$$

The general expression for velocity in polar coordinates is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$. Thus $\dot{r}$ and $\dot{\theta}$ are

$$\boxed{\dot{r} = v_0 \sin\theta \quad \text{and} \quad \dot{\theta} = \frac{v_0\cos^2\theta}{h}.} \tag{10}$$

The acceleration $\vec{a} = a_0\,\hat{\jmath}$ is converted to the $(\hat{u}_r, \hat{u}_\theta)$ coordinate system as

$$\vec{a} = a_0\,(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta)\,. \tag{11}$$

The general expression for acceleration in polar coordinates is $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta$. Using the results of Eq. (10) and noting that $r = h/\cos\theta$ we obtain

$$\ddot{r} - r\dot{\theta}^2 = a_0\sin\theta \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_0\cos\theta, \tag{12}$$

which can be solved for $\ddot{r}$ and $\ddot{\theta}$ to obtain

$$\boxed{\ddot{r} = a_0\sin\theta + \frac{v_0^2\cos^3\theta}{h} \quad \text{and} \quad \ddot{\theta} = \frac{a_0\cos^2\theta}{h} - \frac{2v_0^2\cos^3\theta\sin\theta}{h^2}.}$$

Problem 2.168

The time derivative of the acceleration, i.e., $\dot{\vec{a}}$, is usually referred to as the *jerk* because "jerky" motion is generally associated with quickly changing acceleration. Starting from Eq. (2.89), compute the jerk in polar coordinates.

Solution

Equation (2.89) on p. 119 of the textbook states

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta = a_r\,\hat{u}_r + a_\theta\,\hat{u}_\theta. \tag{1}$$

Therefore, we have

$$\dot{\vec{a}} = \dot{a}_r\,\hat{u}_r + a_r\dot{\hat{u}}_r + \dot{a}_\theta\,\hat{u}_\theta + a_\theta\,\dot{\hat{u}}_\theta. \tag{2}$$

Recall that

$$\dot{\hat{u}}_r = \dot{\theta}\,\hat{k} \times \hat{u}_r = \dot{\theta}\hat{u}_\theta \quad \text{and} \quad \dot{\hat{u}}_\theta = \dot{\theta}\,\hat{k} \times \hat{u}_\theta = -\dot{\theta}\hat{u}_r. \tag{3}$$

Substituting Eq. (3) into Eq. (2) and rearranging terms, we have Differentiate $\vec{a} = a_r\,\hat{u}_r + a_\theta\,\hat{u}_\theta$. Remember that $\dot{\hat{u}}_r = \dot{\theta}\,\hat{k} \times \hat{u}_r$ and $\dot{\hat{u}}_\theta = \dot{\theta}\,\hat{k} \times \hat{u}_\theta$.

$$\dot{\vec{a}} = \left(\dot{a}_r - \dot{\theta}a_\theta\right)\hat{u}_r + \left(\dot{a}_\theta + \dot{\theta}a_r\right)\hat{u}_\theta. \tag{4}$$

Then, time differentiating the expressions $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, gives

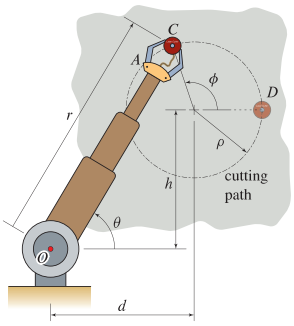
$$\dot{a}_r = \dddot{r} - \dot{r}\dot{\theta}^2 - 2r\dot{\theta}\ddot{\theta} \quad \text{and} \quad \dot{a}_\theta = \dot{r}\ddot{\theta} + r\dddot{\theta} + 2\ddot{r}\dot{\theta} + 2\dot{r}\ddot{\theta}. \tag{5}$$

Substituting Eqs. (5) into Eq. (4) and simplifying, we have

$$\boxed{\dot{\vec{a}} = \left(\dddot{r} - 3r\dot{\theta}\ddot{\theta} - 3\dot{r}\dot{\theta}^2\right)\hat{u}_r + \left[r\left(\dddot{\theta} - \dot{\theta}^3\right) + 3\ddot{r}\dot{\theta} + 3\dot{r}\ddot{\theta}\right]\hat{u}_\theta.}$$

Problems 2.169 and 2.170

In the cutting of sheet metal, the robotic arm OA needs to move the cutting tool at C counterclockwise at a constant speed v_0 along a circular path of radius ρ. The center of the circle is located in the position shown relative to the base of the robotic arm at O.



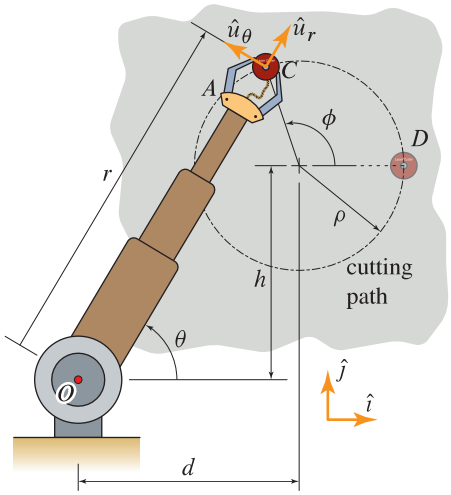
Problem 2.169 When the cutting tool is at D ($\phi = 0$), determine r, $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ as functions of the given quantities (i.e., d, h, ρ, v_0).

Problem 2.170 For all positions along the circular cut (i.e., for any value of ϕ), determine r, $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ as functions of the given quantities (i.e., d, h, ρ, v_0). These quantities can be found "by hand," but it is tedious, so you might consider using symbolic algebra software such as Mathematica or Maple.

Solution to 2.169

Referring to the figure on the right, the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are related to the unit vectors of the polar coordinate system as follows:

$$\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta \quad \text{and} \quad \hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta. \tag{1}$$



The kinematic quantities describing the motion of the arm, in polar coordinates, are now found for the case where $\phi = 0$. The velocity vector of the cutting tool at C (moving with constant speed along a circular path) is expressed in terms of our xy coordinate system as

$$\vec{v} = v_0\,\hat{\jmath}. \tag{2}$$

By substituting the second of Eqs. (1) into Eq. (2), we have

$$\vec{v} = (v_0 \sin\theta)\,\hat{u}_r + (v_0\cos\theta)\,\hat{u}_\theta. \tag{3}$$

Since in polar coordinates we have $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, then from Eq. (3) we deduce that

$$\dot{r} = v_0 \sin\theta \quad \text{and} \quad \dot{\theta} = \frac{v_0\cos\theta}{r}. \tag{4}$$

Now, using geometry we see that, for $\phi = 0$,

$$\boxed{r = \sqrt{h^2 + (d+\rho)^2}.} \tag{5}$$

In addition, again for $\phi = 0$, the sines and cosines of angle θ are

$$\sin\theta = \frac{h}{r}, \quad \cos\theta = \frac{d+\rho}{r}. \tag{6}$$

Substituting Eqs. (5) and (6) into Eqs. (4) . we have

$$\boxed{\dot{r} = \frac{v_0 h}{\sqrt{h^2 + (d+\rho)^2}} \quad \text{and} \quad \dot{\theta} = \frac{v_0\,(d+\rho)}{h^2 + (\rho+d)^2}.} \tag{7}$$

Since the speed of point C is constant, the acceleration of point C is only the direction normal to the path of C, i.e., is directed toward the center of the cutting path. Therefore, for $\phi = 0$, the acceleration of C can be written as

$$\vec{a} = -\left(\frac{v_0^2}{\rho}\right)\hat{\imath}, \tag{8}$$

Substituting the first of Eqs. (1) into Eq. (8), we have

$$\vec{a} = -\frac{v_0^2 \cos\theta}{\rho}\,\hat{u}_r + \frac{v_0^2 \sin\theta}{\rho}\,\hat{u}_\theta. \tag{9}$$

Now recall that in polar coordinates, we have $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$. Hence, we can use the above expression to conclude that

$$\ddot{r} = \frac{r\dot{\theta}^2\rho - v_0^2\cos\theta}{\rho}, \qquad \ddot{\theta} = \frac{v_0^2\sin\theta - 2\dot{r}\dot{\theta}\rho}{r\rho}. \tag{10}$$

Employing the expressions for r, $\sin\theta$, and $\cos\theta$ of Eq. (5) and Eq. (6) respectively, and simplifying, we have

$$\ddot{r} = \frac{1}{\rho}\left(\dot{\theta}^2\rho\sqrt{h^2 + (d+\rho)^2} - \frac{v_0^2(d+\rho)}{\sqrt{h^2 + (d+\rho)^2}}\right) \quad \text{and} \quad \ddot{\theta} = \frac{\frac{hv_0^2}{\sqrt{h^2+(d+\rho)^2}} - 2\dot{r}\dot{\theta}\rho}{\rho\sqrt{h^2 + (d+\rho)^2}}. \tag{11}$$

Finally, using the expressions in Eqs. (7) and simplifying, we have

$$\boxed{\ddot{r} = -\frac{v_0^2(d+\rho)\left[h^2 + d(d+\rho)\right]}{\rho\left[h^2 + (d+\rho)^2\right]^{3/2}} \quad \text{and} \quad \ddot{\theta} = \frac{hv_0^2\left(d^2 + h^2 - \rho^2\right)}{\rho\left[h^2 + (d+\rho)^2\right]^2}.} \tag{12}$$

Solution to 2.170

Referring to the figure on the right, the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are related to the unit vectors of the polar coordinate system as follows:

$$\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta \quad \text{and} \quad \hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta. \tag{13}$$

For a generic value of ϕ, the velocity vector of point C can be expressed as

$$\vec{v} = (-v_0 \sin\phi)\,\hat{\imath} + (v_0 \cos\phi)\,\hat{\jmath}. \tag{14}$$

Substituting Eqs. (13) in Eq. (14) and rearranging terms, we have

$$\vec{v} = (v_0 \cos\phi \sin\theta - v_0 \cos\theta \sin\phi)\,\hat{u}_r + (v_0 \cos\theta \cos\phi + v_0 \sin\theta \sin\phi)\,\hat{u}_\theta. \tag{15}$$

Since in polar coordinates we have $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, then from Eq. (15) we deduce that

$$\dot{r} = v_0 \sin\theta - v_0 \cos\theta \sin\phi \quad \text{and} \quad \dot{\theta} = \frac{v_0 \cos\theta \cos\phi + v_0 \sin\theta \sin\phi}{r}. \tag{16}$$

Referring to the figure and using geometry, we have

$$\boxed{r = \sqrt{(h + \rho\sin\phi)^2 + (d + \rho\cos\phi)^2}.} \tag{17}$$

In addition, the sines and cosines of angle θ are given by

$$\sin\theta = \frac{h + \rho\sin\phi}{r} \quad \text{and} \quad \cos\theta = \frac{d + \rho\cos\phi}{r}. \tag{18}$$

Substituting Eqs. (17) and (18) into Eqs. (16) and simplifying, we have

$$\boxed{\dot{r} = \frac{v_0\,(h\cos\phi - d\sin\phi)}{\sqrt{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2}} \quad \text{and} \quad \dot{\theta} = \frac{v_0\,(\rho + d\cos\phi + h\sin\phi)}{d^2 + h^2 + \rho^2 + 2d\rho\cos\phi + 2h\rho\sin\phi}.} \tag{19}$$

Since the speed of point C is constant, the acceleration of point C is only the direction normal to the path of C, i.e., is directed toward the center of the cutting path. Therefore, the acceleration of C can be written as

$$\vec{a} = -\frac{v_0^2}{\rho}\cos\phi\,\hat{\imath} + \frac{v_0^2}{\rho}\sin\phi\,\hat{\jmath}, \tag{20}$$

which, along with Eq. (13), imply

$$\vec{a} = \left(-\frac{v_0^2\cos\theta\cos\phi}{\rho} - \frac{v_0^2\sin\theta\sin\phi}{\rho}\right)\hat{u}_r + \left(\frac{v_0^2\cos\phi\sin\theta}{\rho} - \frac{v_0^2\cos\theta\sin\phi}{\rho}\right)\hat{u}_\theta. \tag{21}$$

Now recall that in polar coordinates, we have $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$, and therefore, by comparison with the above expression, we can conclude that

$$\ddot{r} = \frac{r\dot{\theta}^2\rho - v_0^2\cos\theta\cos\phi - v_0^2\sin\theta\sin\phi}{\rho} \quad \text{and} \quad \ddot{\theta} = \frac{-2\dot{r}\dot{\theta}\rho + v_0^2\cos\phi\sin\theta - v_0^2\cos\theta\sin\phi}{r\rho}. \tag{22}$$

Employing the expressions for r and θ of Eq. (17) and, after simplification, it is found that

$$\ddot{r} = \frac{\rho\left(\dot{\theta}^2\left(d^2+h^2+\rho^2\right)-v_0^2\right)-\left(v_0^2-2\dot{\theta}^2\rho^2\right)(d\cos\phi+h\sin\phi)}{\rho\sqrt{(d+\rho\cos\phi)^2+(h+\rho\sin(\phi))^2}}, \tag{23}$$

$$\ddot{\theta} = \frac{h\cos(\phi)\mathrm{v}0^2-d\sin\phi v_0^2-2\dot{r}\dot{\theta}\rho\sqrt{(d+\rho\cos\phi)^2+(h+\rho\sin\phi)^2}}{\rho\left((d+\rho\cos\phi)^2+(h+\rho\sin\phi)^2\right)}. \tag{24}$$

Substituting Eqs. (19) into the above expression and simplifying, we have

$$\ddot{r} = -\frac{v_0^2(\rho+d\cos\phi+h\sin\phi)\left(d^2+\rho d\cos\phi+h^2+h\rho\sin\phi\right)}{\rho\left(d^2+2\rho d\cos\phi+h^2+\rho^2+2h\rho\sin\phi\right)^{3/2}}, \tag{25}$$

$$\ddot{\theta} = -\frac{v_0^2\left(d^2+h^2-\rho^2\right)(d\sin\phi-h\cos\phi)}{\rho\left(d^2+2\rho d\cos\phi+h^2+\rho^2+2h\rho\sin\phi\right)^2}. \tag{26}$$

Problem 2.171

Considering the system analyzed in Example 2.23, let $h = 15\,\text{ft}$, $v_0 = 55\,\text{mph}$, and $\phi = 25°$. Plot the trajectory of the projectile in two different ways: (1) by solving the projectile motion problem using Cartesian coordinates and plotting y versus x and (2) by using a computer to solve Eqs. (3), (4), (9), and (10) in Example 2.23. You should, of course, get the same trajectory regardless of the coordinate system used.

Solution

To describe the trajectory of a projectile subject only to gravity and under the conditions stated in Example (2.23) on p. 126 of the textbook, we can use Eq. (2.52) on p. 74 of the textbook. Using this equation and observing that the projectile is released from a height h above the origin of the chosen coordinate system, we have

$$y = (\tan\phi)x - \left(\frac{g\sec^2\phi}{2v_0^2}\right)x^2 + h, \tag{1}$$

For the purpose of providing a meaningful comparison between the plot of the above function and that obtained from the equations in Example (2.23) on p. 126 of the textbook, we will compute the value of x for, say $t = 3\,\text{s}$ and then produce plots that have the same range for the x coordinate. To do so, recall that the acceleration equation in the x direction is equal to zero. hence, with reference to Example (2.23) on p. 126, we have that the x coordinate of the projectile as a function of time is given by

$$x = v_0\cos\phi t. \tag{2}$$

Hence, recalling that $v_0 = 55\,\text{mph} = 80.67\,\text{ft/s}$ and $\phi = 25°$, for $t = s\,3$ we have

$$x\big|_{t=3\,\text{s}} = 219.3\,\text{ft}. \tag{3}$$

Again, the above value of x will simply we used to make sure that the plots we generate in this solution have a range of x value going from 0 to 219.3 ft, which corresponds to a time range going from 0 to 3 s.

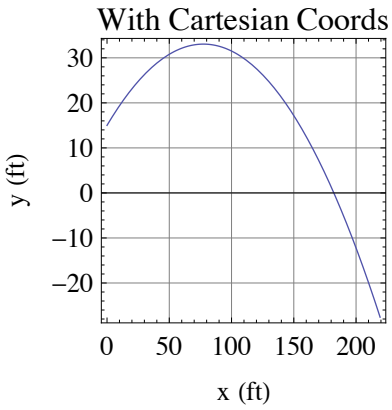
With the bove in mind, recalling that we have $h = 15\,\text{ft}$, $v_0 = 55\,\text{mph} = 80.67\,\text{ft/s}$, $\phi = 25°$, and $g = 32.2\,\text{ft/s}^2$, we can plot the above formula using any appropriate mathematical software. The plots shown below was obtained using *Mathematica* with the following code:.

```
Parameters = {h → 15., v0 → 80.67, ϕ → 25. Degree, g → 32.2};

y = Tan[ϕ] x - g Sec[ϕ]^2 / (2 v0^2) x^2 + h;

Plot[y /. Parameters, {x, 0, 219.3}, Frame → True, GridLines → Automatic,
 FrameLabel → {"x (ft)", "y (ft)"}, PlotLabel → "With Cartesian Coords",
 AspectRatio → 1]
```

Solution continued on next page . . .



We now will determine the trajectory of the projectile starting from Eqs. (3) and (4) derived in Example 2.23 on p. 126 of the textbook, and subject to the initial conditions in Eqs. (9) and (10) of the example in question. For convenience, we repeat the equations we need here below:

$$\ddot{r} - r\dot{\theta}^2 = -g\sin\theta, \tag{4}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -g\cos\theta, \tag{5}$$

with the initial conditions

$$r(0) = h, \qquad \theta(0) = \frac{\pi}{2}\,\text{rad}. \tag{6}$$

$$\dot{r}(0) = v_0\sin\phi, \qquad \dot{\theta}(0) = -\frac{v_0}{h}\cos\phi. \tag{7}$$

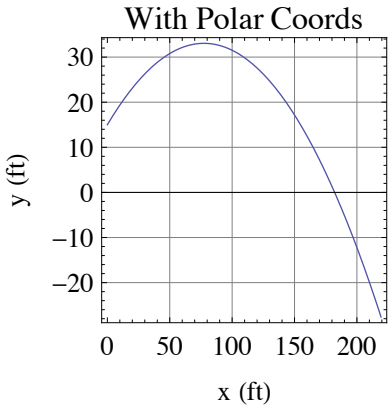
The above system of differential equations and initial conditions can be integrated with any appropriate mathematical software. However, before proceeding to the solution of the equations in question, we must keep in mind that the mathematical software will yield r and θ as a function of time. Then to get the plot of the trajectory, we must resort to parametric plot, i.e., a plot of the coordinates of the projectile corresponding to a certain range of the time parameter. Furthermore, to produce a plot that can be compared to the one shown above, we must plots values of x and y corresponding to the values of r and θ given by the numerical solution. This is done by simply noticing that the (x, y) coordinates of a point are expressed via the (r, θ) components using the following relations:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{8}$$

With the above considerations in mind, we have used *Mathematica* with the following code to obtain a solution with time t going from 0 to 3 s (see discussion above concerning the choice of the range of x for the plots shown earlier):

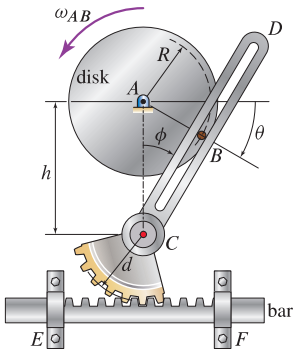
```
Parameters = {h → 15., v0 → 80.67, ϕ → 25. Degree, g → 32.2};
Equations = {r''[t] - r[t] (θ'[t])^2 == -g Sin[θ[t]],
   r[t] θ''[t] + 2 r'[t] θ'[t] == -g Cos[θ[t]], r[0] == h, θ[0] == π/2, r'[0] == v0 Sin[ϕ],
   θ'[0] == -(v0 Cos[ϕ])/h};
Motion = NDSolve[Equations /. Parameters, {r, θ}, {t, 0, 3}]
ParametricPlot[{r[t] Cos[θ[t]], r[t] Sin[θ[t]]} /. Motion[[1]], {t, 0, 3},
 Frame → True, GridLines → Automatic, FrameLabel → {"x (ft)", "y (ft)"},
 PlotLabel → "With Polar Coords", AspectRatio → 1]
```

The code above yields the following trajectory, which can be seen to be identical to the one obtained earlier (as expected).



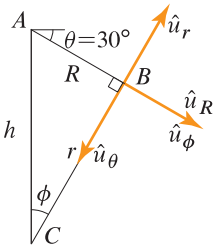
Problem 2.172

The *reciprocating rectilinear motion* mechanism shown consists of a disk pinned at its center at A that rotates with a constant angular velocity ω_{AB}, a slotted arm CD that is pinned at C, and a bar that can oscillate within the guides at E and F. As the disk rotates, the peg at B moves within the slotted arm, causing it to rock back and forth. As the arm rocks, it provides a slow advance and a quick return to the reciprocating bar due to the change in distance between C and B. Letting $\theta = 30°$, $\omega_{AB} = 50\,\text{rpm} = \text{constant}$, $R = 0.3\,\text{ft}$, and $h = 0.6\,\text{ft}$, determine $\dot{\phi}$ and $\ddot{\phi}$, i.e., the angular velocity and angular acceleration of the slotted arm CD, respectively.



Solution

Let $R = \overline{AB}$ and $r = \overline{BC}$. The velocity of B in the $(\hat{u}_R, \hat{u}_\theta)$ and $(\hat{u}_r, \hat{u}_\phi)$ component systems are



$$\vec{v}_B = -R\omega_{AB}\,\hat{u}_\theta \quad \text{and} \quad \vec{v}_B = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi. \tag{1}$$

To find $\dot{\phi}$ take advantage of the fact that for $\theta = 30°$ and $R = 0.5h$ we have a right triangle ABC. Then $\hat{u}_r = -\hat{u}_\theta$ and $\hat{u}_R = \hat{u}_\phi$. Converting the $\hat{u}_\phi$ component of $\vec{v}_B$ to $\hat{u}_R$ and equating components we find

$$r\dot{\phi}\,\hat{u}_R = 0 \quad \Rightarrow \quad \boxed{\dot{\phi} = 0.} \tag{2}$$

The acceleration of B in the $(\hat{u}_R, \hat{u}_\theta)$ and $(\hat{u}_r, \hat{u}_\phi)$ component systems are

$$\vec{a}_B = -R\omega_{AB}^2\,\hat{u}_R \quad \text{and} \quad \vec{a}_B = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{u}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{u}_\phi. \tag{3}$$

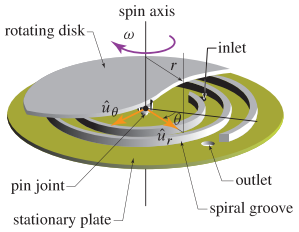
Convert the $\hat{u}_\phi$ component of $\vec{a}_B$ to $\hat{u}_R$, equate components of Eq. (3), plug in Eq. (2), and substitute $r = \sqrt{h^2 - R^2}$ to find

$$r\ddot{\phi} = -R\omega_{AB}^2 \quad \Rightarrow \quad \boxed{\ddot{\phi} = \frac{-R\omega_{AB}^2}{\sqrt{h^2 - R^2}} = -15.8\,\text{rad/s}^2,}$$

where we have used the following numerical data: $R = 0.3\,\text{ft}$, $\omega_{AB} = 50\,\text{rpm} = 5.236\,\text{rad/s}$, and $h = 0.6\,\text{ft}$.

Problems 2.173 and 2.174

A micro spiral pump consists of a spiral channel attached to a stationary plate. This plate has two ports, one for fluid inlet and another for outlet, the outlet being farther from the center of the plate than the inlet. The system is capped by a rotating disk. The fluid trapped between the rotating disk and the stationary plate is put in motion by the rotation of the top disk, which pulls the fluid through the spiral channel.



Problem 2.173 Consider a spiral channel with the geometry given by the equation $r = \eta\theta + r_0$, where $r_0 = 146\,\mu$m is the starting radius, r is the distance from the spin axis, and θ is the angular position of a point in the spiral channel. Assume that the radius at the outlet is $r_{\text{out}} = 190\,\mu$m, that the top disk rotates with a constant angular speed ω, and that the fluid particles in contact with the rotating disk are essentially stuck to it. Determine the constant η and the value of ω (in rpm) such that after 1.25 rev of the top disk, the speed of the particles in contact with this disk is $v = 0.5$ m/s at the outlet.

Problem 2.174 Consider a spiral channel with the geometry given by the equation $r = \eta\theta + r_0$, where $\eta = 12\,\mu$m is called the polar slope, $r_0 = 146\,\mu$m is the starting radius, r is the distance from the spin axis, and θ is the angular position of a point in the spiral channel. If the top disk rotates with a constant angular speed $\omega = 30{,}000$ rpm, and assuming that the fluid particles in contact with the rotating disk are essentially stuck to it, use the polar coordinate system shown and determine the velocity and acceleration of one fluid particle when it is at $r = 170\,\mu$m.

Solution to 2.173

To find η we start with $r = \eta\theta + r_0$ and substitute $\theta = 1.25\,\text{rav}(2\pi\ \text{rad/rev}) = 7.854\,\text{rad}$ and $r = r_{\text{out}}$.

$$r = \eta\theta + r_0 \quad \Rightarrow \quad \eta = \frac{r_{\text{out}} - r_0}{7.854\,\text{rad}} = 5.602\,\mu\text{m} \quad \Rightarrow \quad \boxed{\eta = 5.60\,\mu\text{m},} \tag{1}$$

where we have used the fact that $r_{\text{out}} = 190\,\mu$m and $r_0 = 146\,\mu$m. Taking the derivative with respect to time of $r = \eta\theta + r_0$, we have

$$r = \eta\theta + r_0 \quad \Rightarrow \quad \dot{r} = \eta\dot{\theta}. \tag{2}$$

Next, recalling that the speed in polar coordinates is given by $v^2 = \dot{r}^2 + (r\dot{\theta})^2$ and that $\dot{\theta} = \omega$, we have

$$v = \sqrt{(\eta\omega)^2 + (r\omega)^2} \quad \Rightarrow \quad \omega = \frac{v}{\sqrt{\eta^2 + r^2}} \quad \Rightarrow \quad \boxed{\omega = 2630\,\text{rad/s} = 25{,}100\,\text{rpm},} \tag{3}$$

where we have used the fact that $v = 0.5$ m/s, $\eta = 5.602\,\mu$m (see Eq. (1)), and $r = r_{\text{out}} = 190\,\mu$m.

Solution to 2.174

The velocity of a particle is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$ where $\dot{r} = \eta\dot{\theta}$ and $\dot{\theta} = \omega$. Hence, we can write

$$\vec{v} = \eta\omega\,\hat{u}_r + r\omega\,\hat{u}_\theta \quad \Rightarrow \quad \boxed{\vec{v} = (37.7\times10^{-3}\,\hat{u}_r + 0.534\,\hat{u}_\theta)\,\text{m/s},} \tag{4}$$

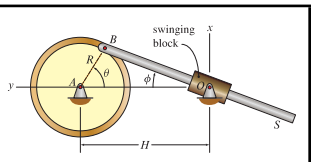
where we have used the fact that $\eta = 12\,\mu$m, $\omega = 30{,}000\,$rpm $= 3142\,$rad/s, and $r = 170\,\mu$m. The general expression for the acceleration of a particle is $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta$. Note that $\ddot{r} = 0$ and $\ddot{\theta} = 0$. Hence, recalling again that $\dot{r} = \eta\dot{\theta}$ and $\dot{\theta} = \omega$, we have

$$\vec{a} = -r\omega^2\,\hat{u}_r + 2\eta\omega^2\,\hat{u}_\theta \quad \Rightarrow \quad \boxed{\vec{a} = (-1680\,\hat{u}_r + 237\,\hat{u}_\theta)\,\text{m/s}^2,} \tag{5}$$

where, again, we have used the fact that $\eta = 12\,\mu$m, $\omega = 30{,}000\,$rpm $= 3142\,$rad/s, and $r = 170\,\mu$m.

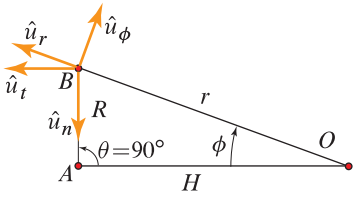
Problem 2.175

The mechanism shown is called a *swinging block* slider crank. First used in various steam locomotive engines in the 1800s, this mechanism is often found in door-closing systems. If the disk is rotating with a constant angular velocity $\dot{\theta} = 60\,\text{rpm}$, $H = 4\,\text{ft}$, $R = 1.5\,\text{ft}$, and r denotes the distance between B and O, compute $\dot{r}$, $\dot{\phi}$, $\ddot{r}$, and $\ddot{\phi}$ when $\theta = 90°$.



Solution

The motion of B is uniform circular motion. The velocity of B at $\theta = 90°$ in the $(\hat{u}_n, \hat{u}_t)$ and $(\hat{u}_r, \hat{u}_\phi)$ component systems is



$$\vec{v} = R\dot{\theta}\,\hat{u}_t \quad \text{and} \quad \vec{v} = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi. \tag{1}$$

The $(\hat{u}_n, \hat{u}_t)$ unit vectors are express in the $(\hat{u}_r, \hat{u}_\phi)$ component system as

$$\hat{u}_t = \cos\phi\,\hat{u}_r - \sin\phi\,\hat{u}_\phi \qquad \hat{u}_n = -\sin\phi\,\hat{u}_r - \cos\phi\,\hat{u}_\phi. \tag{2}$$

Convert $\vec{v} = R\dot{\theta}\,\hat{u}_t$ to the $(\hat{u}_r, \hat{u}_\phi)$ component system and equate components of Eqs. (1). Note that $\sin\phi = \frac{R}{\sqrt{R^2+H^2}}$, $\cos\phi = \frac{H}{\sqrt{R^2+H^2}}$, $r = \sqrt{R^2+H^2}$, and $\dot{\theta} = 2\pi\,\text{rad/s}$.

$$\vec{v} = R\dot{\theta}\left(\cos\phi\,\hat{u}_r - \sin\phi\,\hat{u}_\phi\right). \tag{3}$$

$$\dot{r} = R\dot{\theta}\cos\phi \quad \Rightarrow \quad \boxed{\dot{r} = \frac{HR\dot{\theta}}{\sqrt{R^2+H^2}} = 8.82\,\text{ft/s}.} \tag{4}$$

$$-R\dot{\theta}\sin\phi = r\dot{\phi} \quad \Rightarrow \quad \boxed{\dot{\phi} = \frac{-R^2\dot{\theta}}{R^2+H^2} = -0.775\,\text{rad/s}.} \tag{5}$$

The acceleration of B at $\theta = 90°$ in the $(\hat{u}_n, \hat{u}_t)$ and $(\hat{u}_r, \hat{u}_\phi)$ component systems is

$$\vec{a} = R\dot{\theta}^2\,\hat{u}_n \quad \text{and} \quad \vec{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{u}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{u}_\phi \tag{6}$$

Convert $\vec{a} = R\dot{\theta}^2\,\hat{u}_n$ to the $(\hat{u}_r, \hat{u}_\phi)$ component system and equate components of Eqs. (6). Plug in Eqs. (4) and (5) and simplify.

$$\vec{a} = R\dot{\theta}^2\left(-\sin\phi\,\hat{u}_r - \cos\phi\,\hat{u}_\phi\right).$$

Recalling that $\dot{\theta} = 60\,\text{rpm} = 6.283\,\text{rad/s}$, $H = 4\,\text{ft}$, $R = 1.5\,\text{ft}$, we have

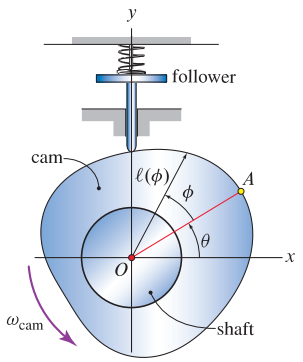
$$\ddot{r} - r\dot{\phi}^2 = -R\dot{\theta}^2\sin\phi \quad \Rightarrow \quad \boxed{\ddot{r} = \frac{-\dot{\theta}^2H^2R^2}{\left(R^2+H^2\right)^{3/2}} = -18.2\,\text{ft/s}^2,}$$

and

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = -R\dot{\theta}^2\cos\phi \quad \Rightarrow \quad \boxed{\ddot{\phi} = \frac{\dot{\theta}^2HR\left(R^2-H^2\right)}{\left(R^2+H^2\right)^2} = -9.78\,\text{rad/s}^2.}$$

Problem 2.176

The cam is mounted on a shaft that rotates about O with constant angular velocity ω_{cam}. The profile of the cam is described by the function $\ell(\phi) = R_0(1 + 0.25\cos^3\phi)$, where the angle ϕ is measured relative to the segment OA, which rotates with the cam. Letting $\omega_{\text{cam}} = 3000\,\text{rpm}$ and $R_0 = 3\,\text{cm}$, determine the velocity and acceleration of the follower when $\theta = 33°$. Express the acceleration of the follower in terms of g, the acceleration due to gravity.



Solution

Let r denote the distance between the point on the follower in contact with the cam and point O. Then, the velocity of the follower is the time rate of change of the radius of the cam, i.e., $\vec{v}_{\text{follower}} = \dot{r}\,\hat{\jmath}$. Then, using the chain rule, we have

$$\frac{dr}{dt} = \frac{d\ell}{d\phi_f}\frac{d\phi_f}{d\theta}\frac{d\theta}{dt},$$

where ϕ_f is the value of the angle ϕ identifying the radial line going from O to the follower (the subscript f stands for follower). The radial line in question lies always along the y axis. and therefore we have that ϕ_f is such that $\phi_f = 90° - \theta$. Consequently, we have

$$\frac{d\ell}{d\phi_f} = -0.75R_0\cos^2\phi_f\sin\phi_f, \quad \frac{d\phi_f}{d\theta} = -1, \quad \text{and} \quad \frac{d\theta}{dt} = \omega_{\text{cam}}.$$

Hence, recalling that $R_0 = 3\,\text{cm} = 0.03000\,\text{m}$, $\omega_{\text{cam}} = 3000\,\text{rpm} = 314.2\,\text{rad/s}$, and $\theta = 33°$, we have

$$\vec{v}_{\text{follower}} = \dot{r}\,\hat{\jmath} = 0.75R_0\omega_{\text{cam}}\cos^2(90-\theta)\sin(90-\theta)\,\hat{\jmath} \quad \Rightarrow \quad \boxed{\vec{v}_{\text{follower}} = (1.76\,\text{m/s})\,\hat{\jmath}.} \qquad (1)$$

The acceleration of the follower is given by $\vec{a}_{\text{follower}} = \ddot{r}\,\hat{\jmath}$. Hence, referring to Eq. (1), we can write

$$\frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\phi_f}\frac{d\phi_f}{d\theta}\frac{d\theta}{dt},$$

where

$$\frac{d\dot{r}}{d\phi_f} = 0.75R_0\omega_{\text{cam}}\cos\phi_f\left(\cos^2\phi_f - 2\sin^2\phi_f\right), \quad \frac{d\phi_f}{d\theta} = -1, \quad \text{and} \quad \frac{d\theta}{dt} = \omega_{\text{cam}}.$$

Hence, recalling again that recalling that $R_0 = 3\,\text{cm} = 0.03000\,\text{m}$, $\omega_{\text{cam}} = 3000\,\text{rpm} = 314.2\,\text{rad/s}$, and $\theta = 33°$, we can write

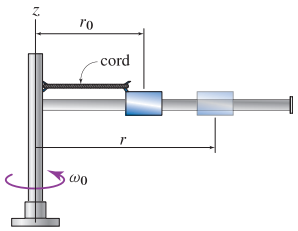
$$\vec{a}_{\text{follower}} = \ddot{r}\,\hat{\jmath} = -0.75R_0\omega_{\text{cam}}^2\cos\left(90°-\theta\right)\left[\cos^2\left(90°-\theta\right) - 2\sin^2\left(90°-\theta\right)\right]\hat{\jmath} = (1343\,\text{m/s}^2)\,\hat{\jmath},$$

which, when expressed to three siginificant figures and in terms of the acceleration due to gravity $g = 9.81\,\text{m/s}^2$, gives

$$\boxed{\vec{a}_{\text{follower}} = (137g)\,\hat{\jmath}.}$$

Problem 2.177

The collar is mounted on the horizontal arm shown, which is originally rotating with the angular velocity ω_0. Assume that after the cord is cut, the collar slides along the arm in such a way that the collar's total acceleration is equal to zero. Determine an expression of the radial component of the collar's velocity as a function of r, the distance from the spin axis. *Hint:* Using polar coordinates, observe that $d(r^2\dot{\theta})/dt = ra_\theta$.



Solution

The radial component of acceleration is

$$a_r = 0 \quad \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = 0. \tag{1}$$

The transverse component of acceleration is

$$a_\theta = 0 \quad \Rightarrow \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad ra_\theta = 0. \tag{2}$$

Because $d(r^2\dot{\theta})/dt = ra_\theta = 0$, $r^2\dot{\theta} = K$ where K is a constant. Initially $r = r_0$ and $\dot{\theta} = \omega_0$. So

$$r^2\dot{\theta} = r_0^2\omega_0 \quad \Rightarrow \quad \dot{\theta} = \frac{r_0^2\omega_0}{r^2}. \tag{3}$$

Take the results of Eq. (1) and substitute Eq. (3) for $\dot{\theta}$.

$$\ddot{r} = r\dot{\theta}^2 \quad \Rightarrow \quad \ddot{r} = \frac{r_0^4\omega_0^2}{r^3}. \tag{4}$$

Rewrite $\ddot{r}$ using the chain rule. Eq. (4) becomes

$$\ddot{r} = \frac{d\dot{r}}{dr}\frac{dr}{dt} \quad \Rightarrow \quad \dot{r}\frac{d\dot{r}}{dr} = \frac{r_0^4\omega_0^2}{r^3}. \tag{5}$$

Integrate Eq. (5):

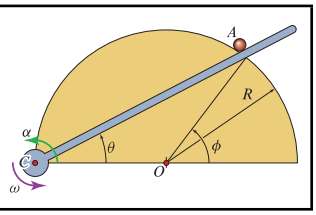
$$\int_0^{\dot{r}} \dot{r}\, d\dot{r} = \int_{r_0}^{r} \frac{r_0^4\omega_0^2}{r^3}\, dr \quad \Rightarrow \quad \tfrac{1}{2}\dot{r}^2 = \frac{-r_0^4\omega_0^2}{2r^2}\bigg|_{r_0}^{r} \quad \Rightarrow \quad \dot{r} = r_0^2\omega_0\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right)^{1/2}, \tag{6}$$

where, in taking the square root of the second of Eqs. (6) we have paid attention to make sure that $\dot{r}$ is positive, given that the collar is sliding outward and therefore causing r to increase. Simplifying the last of Eqs. (6), we have

$$\boxed{\dot{r} = \frac{r_0\omega_0}{r}\sqrt{r^2 - r_0^2}.}$$

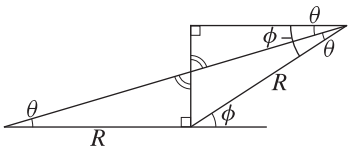
Problem 2.178

Particle A slides over the semicylinder while pushed by the arm pinned at C. The motion of the arm is controlled such that it starts from rest at $\theta = 0$, ω increases uniformly as a function of θ, and $\omega = 0.5\,\text{rad/s}$ for $\theta = 45°$. Letting $R = 4\,\text{in.}$, determine the speed and the magnitude of the acceleration of A when $\phi = 32°$.



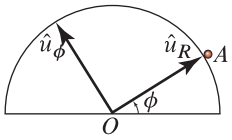
Solution

We need to find $\omega(\theta)$ such that $\frac{d\omega}{d\theta} = K = \text{constant}$. Therefore, we must have $\omega(\theta) = K\theta + C$ where K and C are constants. To evaluate K and C, we recall that we must have



$$0 = K(0) + C = 0 \quad \Rightarrow \quad C = 0, \tag{1}$$

$$0.5\,\text{rad/s} = K\frac{\pi}{4}\,\text{rad} \quad \Rightarrow \quad K = \frac{2}{\pi}\,\text{s}^{-1}. \tag{2}$$



Therefore $\omega(\theta) = K\theta$. It is demonstrated in the top figure that $\phi = 2\theta$. Therefore $\dot{\phi} = 2\omega$ and $\ddot{\phi} = 2\alpha$. Now we can determine $\omega(\phi)$.

$$\omega(\theta) = K\theta \quad \Rightarrow \quad \omega(\phi) = \frac{K\phi}{2}. \tag{3}$$

To determine the velocity and acceleration of A we now adopt a polar coordinate system with O as the origin. Observing that $\dot{R} = 0$, the velocity of A is

$$\vec{v}_A = \dot{R}\,\hat{u}_R + R\dot{\phi}\,\hat{u}_\phi \quad \Rightarrow \quad \vec{v}_A = R\dot{\phi}\,\hat{u}_\phi. \tag{4}$$

Starting with (4), replacing $\dot{\phi}$ with 2ω, and using Eqs. (2) and (3), we have that the speed of A at $\phi = 32°$ is

$$v = \left|R\dot{\phi}\right| \quad \Rightarrow \quad v = 2R\omega \quad \Rightarrow \quad v = KR\phi \quad \Rightarrow \quad \boxed{v = 0.119\,\text{ft/s},} \tag{5}$$

where we have used the fact that $R = 4\,\text{in.} = 0.3333\,\text{ft}$ and $K = (2/\pi)\,\text{s}^{-1}$. Next, observing that $\ddot{R} = 0$ and $\dot{R} = 0$, the acceleration of A is

$$\vec{a} = \left(\ddot{R} - R\dot{\phi}^2\right)\hat{u}_R + \left(R\ddot{\phi} + 2\dot{R}\dot{\phi}\right)\hat{u}_\phi \quad \Rightarrow \quad \vec{a} = -R\dot{\phi}^2\,\hat{u}_R + R\ddot{\phi}\,\hat{u}_\phi. \tag{6}$$

Converting ϕ to θ, the magnitude of acceleration can be obtained as follows:

$$\vec{a} = -4R\omega^2\,\hat{u}_R + 2R\ddot{\theta}\,\hat{u}_\phi \quad \Rightarrow \quad |\vec{a}| = 2R\sqrt{4\omega^4 + \ddot{\theta}^2}. \tag{7}$$

Using the chain rule we rewrite $\ddot{\theta}$ as

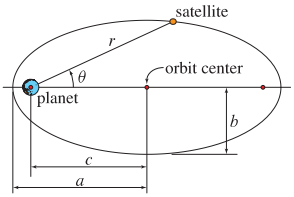
$$\ddot{\theta} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = K^2\theta \quad \Rightarrow \quad \ddot{\theta} = \frac{4\theta}{\pi^2\,\text{s}^2} = \frac{2\phi}{\pi^2\,\text{s}^2}, \tag{8}$$

where, from Eqs. (2), we have used the fact that $K = (2/\pi)\,\text{s}^{-1}$. Therefore, recalling that $R = 4\,\text{in.} = 0.3333\,\text{ft}$ and $\phi = 32°$, the magnitude of the acceleration of A becomes

$$|\vec{a}| = 2R\sqrt{4\omega^4 + \ddot{\theta}^2} \quad \Rightarrow \quad |\vec{a}| = \frac{4R\phi}{\pi^2\,\text{s}^2}\sqrt{\phi^2 + 1} \quad \Rightarrow \quad \boxed{|\vec{a}| = 0.0864\,\text{ft/s}^2.} \tag{9}$$

Problem 2.179

A satellite is moving along the elliptical orbit shown. Using the polar coordinate system in the figure, the satellite's orbit is described by the equation



$$r(\theta) = 2b^2 \frac{a + \sqrt{a^2 - b^2}\cos\theta}{a^2 + b^2 - (a^2 - b^2)\cos(2\theta)},$$

which implies the following identity

$$\frac{rr'' - 2(r')^2 - r^2}{r^3} = -\frac{a}{b^2},$$

where the prime indicates differentiation with respect to θ. Using this identity and knowing that the satellite moves so that $K = r^2\dot{\theta}$ with K constant (i.e., according to Kepler's laws), show that the radial component of acceleration is proportional to $-1/r^2$, which is in agreement with Newton's universal law of gravitation.

Solution

We need to show that $a_r = (\text{constant}) \times \left(-\frac{1}{r^2}\right)$. First we will rewrite Kepler's law as

$$K = r^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{K}{r^2}. \tag{1}$$

We are given $r = r(\theta)$ so we use the chain rule to write its derivative with respect to time as

$$\dot{r} = r'\dot{\theta} \quad \Rightarrow \quad \dot{r} = \frac{Kr'}{r^2}, \tag{2}$$

where we have substituted Eq. (1) for $\dot{\theta}$. Next we take the second derivative of r with respect to time to get

$$\ddot{r} = \frac{-2K}{r^3}\left(r'\dot{\theta}\right)r' + \frac{K}{r^2}\left(r''\dot{\theta}\right) \quad \Rightarrow \quad \frac{K^2}{r^2}\left[\frac{r''}{r^2} - \frac{2(r')^2}{r^3}\right], \tag{3}$$

where we have factored $K\dot{\theta}$ out of the first of Eq. (3) and substituted Eq. (1) for $\dot{\theta}$. Now recall that $a_r = \ddot{r} - r\dot{\theta}^2$ and we have expressions for r, $\ddot{r}$, and $\dot{\theta}$ so we can write

$$a_r = \frac{K^2}{r^2}\left[\frac{r''}{r^2} - \frac{2(r')^2}{r^3}\right] - r\frac{K^2}{r^4} \quad \Rightarrow \quad \frac{K^2}{r^2}\left[\frac{rr'' - 2(r')^2 - r^2}{r^3}\right]. \tag{4}$$

We see that the expression in brackets is our identity so we will replace it with $-a/b^2$. Recalling that K, a, and b are constants, we prove the radial component of acceleration is proportional to $-1/r^2$.

$$\boxed{a_r = \frac{K^2 a}{b^2}\left(-\frac{1}{r^2}\right).} \tag{5}$$

Problem 2.180

Reference frame A is translating relative to reference frame B. Both frames track the motion of a particle C. If at one instant the velocity of particle C is the same in the two frames, what can you infer about the motion of frames A and B at that instant?

Solution

Frame A has zero velocity relative to frame B at the instant considered. Explanation:

$$\vec{v}_{C/B} = \vec{v}_{A/B} + \vec{v}_{C/A}. \quad \text{If} \quad \vec{v}_{C/B} = \vec{v}_{C/A} \quad \Rightarrow \quad \vec{v}_{A/B} = 0. \tag{1}$$

Problem 2.181

Reference frame A is translating relative to reference frame B with velocity $\vec{v}_{A/B}$ and acceleration $\vec{a}_{A/B}$. A particle C appears to be stationary relative to frame A. What can you say about the velocity and acceleration of particle C relative to frame B?

Solution

The velocity and acceleration of particle C relative to frame B are $\vec{v}_{A/B}$ and $\vec{a}_{A/B}$. Explanation:

$$\vec{v}_{C/B} = \vec{v}_{A/B} + \vec{v}_{C/A}. \quad \text{If} \quad \vec{v}_{C/A} = \vec{0} \quad \Rightarrow \quad \vec{v}_{C/B} = \vec{v}_{A/B}. \tag{1}$$

Similarly,

$$\vec{a}_{C/B} = \vec{a}_{A/B} + \vec{a}_{C/A}. \quad \text{If} \quad \vec{a}_{C/A} = \vec{0} \quad \Rightarrow \quad \vec{a}_{C/B} = \vec{a}_{A/B}. \tag{2}$$

Problem 2.182

Reference frame A is translating relative to reference frame B with constant velocity $\vec{v}_{A/B}$. A particle C appears to be in uniform rectilinear motion relative to frame A. What can you say about the motion of particle C relative to frame B?

Solution

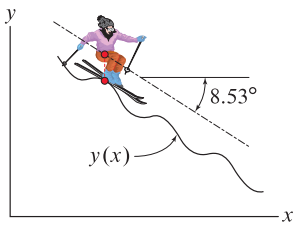
Particle C will appear to be in uniform rectilinear motion relative to frame B. Explanation:

$$\vec{v}_{C/B} = \vec{v}_{A/B} + \vec{v}_{C/A}.$$

If $\vec{v}_{C/A}$ is constant uniform rectilinear motion relative to A and $\vec{v}_{A/B}$ is constant then $\vec{v}_{C/B}$ is uniform rectilinear motion.

Problem 2.183

A skier is going down an undulating *slope* with moguls. Let the skis be short enough for us to assume that the skier's feet are tracking the moguls' profile. Then if the skier is skilled enough to maintain her hips on a straight line trajectory and vertically aligned over her feet, determine the velocity and acceleration of her hips relative to her feet when her speed is equal to 15 km/h. For the profile of the moguls, use the formula $y(x) = h_I - 0.15x + 0.125\sin(\pi x/2)$ m, where h_I is the elevation at which the skier starts the descent.



Solution

The position equations for the skier's hips and feet are denoted by y_H and y_F respectively.

$$y_H = (h_I - x\tan 8.53°)\,\text{m}, \tag{1}$$

$$y_F = \left[h_I - 0.15x + 0.125\sin\left(\frac{\pi x}{2}\right)\right]\text{m}. \tag{2}$$

Now, calculating the relative position of her hips with respect to her feet,

$$y_{H/F} = y_H - y_F = \left[(0.15 - \tan 8.53°)x - 0.125\sin\left(\frac{\pi x}{2}\right)\right]\text{m}, \tag{3}$$

and taking the derivative of Eq. (3) with respect to time yields

$$v_{H/F} = \dot{x}\left[0.15 - \tan 8.53° - \frac{0.125\pi}{2}\cos\left(\frac{\pi x}{2}\right)\right]\text{m/s}. \tag{4}$$

However, $\dot{x} = v_0\cos 8.53°$ and $v_0 = 15\,\text{km/hr} = 4.167\,\text{m/s}$. With this substitution Eq. (4) becomes

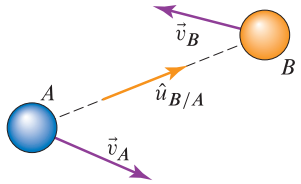
$$\boxed{v_{H/F} = \left[5.63\times10^{-5} - 0.809\cos\left(\frac{\pi x}{2}\right)\right]\text{m/s}.} \tag{5}$$

The relative acceleration of the skier's hips with respect to her feet can now be determined by differentiating Eq. (5) and substituting $\dot{x} = v_0\cos 8.53°$. This yields

$$a_{H/F} = 1.271\dot{x}\,\sin\left(\frac{\pi x}{2}\right)\text{m/s}^2 \quad \Rightarrow \quad \boxed{a_{H/F} = 5.24\sin\left(\frac{\pi x}{2}\right)\text{m/s}^2.}$$

Problem 2.184

Two particles A and B are moving in a plane with arbitrary velocity vectors $\vec{v}_A$ and $\vec{v}_B$, respectively. Letting the *rate of separation* (ROS) be defined as the component of the relative velocity vector along the line connecting particles A and B, determine a general expression for ROS. Express your result in terms of $\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$, where $\vec{r}_A$ and $\vec{r}_B$ are the position vectors of A and B, respectively, relative to some chosen fixed point in the plane of motion.



Solution

We begin by writing $\hat{u}_{B/A}$ in terms of $\vec{r}_{B/A}$

$$\hat{u}_{B/A} = \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}. \tag{1}$$

The velocity of B relative to A is:

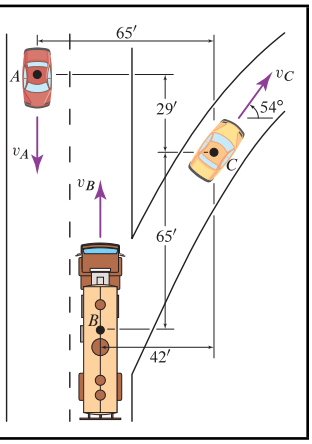
$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A. \tag{2}$$

The component of Eq. (2) in the direction of Eq. (1) is the ROS.

$$\text{ROS} = \vec{v}_{B/A} \cdot \hat{u}_{B/A} \quad \Rightarrow \quad \boxed{\text{ROS} = (\vec{v}_B - \vec{v}_A) \cdot \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}.}$$

Problem 2.185

Three vehicles A, B, and C are in the positions shown and are moving with the indicated directions. We define the *rate of separation* (ROS) of two particles P_1 and P_2 as the component of the relative velocity of, say, P_2 with respect to P_1 in the direction of the relative position vector of P_2 with respect to P_1, which is along the line that connects the two particles. At the given instant, determine the rates of separation ROS_{AB} and ROS_{CB}, that is, the rate of separation between A and B and between C and B. Let $v_A = 60$ mph, $v_B = 55$ mph, and $v_C = 35$ mph. Furthermore, treat the vehicles as particles and use the dimensions shown in the figure.



Solution

We need to derive a convenient expression for the rate of separation. We begin by writing $\hat{u}_{B/A}$ in terms of $\vec{r}_{B/A}$

$$\hat{u}_{B/A} = \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}. \tag{1}$$

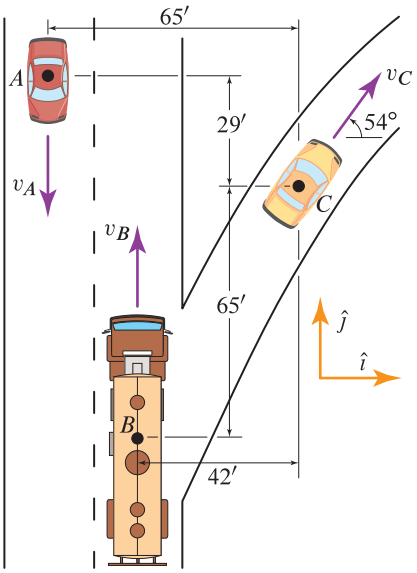
The velocity of B relative to A is:

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A. \tag{2}$$

The component of Eq. (2) in the direction of Eq. (1) is the ROS, i.e.,

$$\text{ROS} = \vec{v}_{B/A} \cdot \hat{u}_{B/A} \quad \Rightarrow \quad \text{ROS} = (\vec{v}_B - \vec{v}_A) \cdot \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}. \tag{3}$$

Now that we have a formula for the ROS, consider the $(\hat{\imath}, \hat{\jmath})$ component system shown in the figure on the right.



$$\frac{\vec{r}_{B/A}}{|r_{B/A}|} = \frac{(23\,\hat{\imath} - 94\,\hat{\jmath})\ \text{ft}}{\sqrt{(23)^2 + (-94)^2}\ \text{ft}} = (0.2377\,\hat{\imath} - 0.9713\,\hat{\jmath}), \tag{4}$$

$$\frac{\vec{r}_{C/B}}{|r_{C/B}|} = \frac{(42\,\hat{\imath} + 65\,\hat{\jmath})\ \text{ft}}{\sqrt{(42)^2 + (65)^2}\ \text{ft}} = (0.5427\,\hat{\imath} + 0.8399\,\hat{\jmath}). \tag{5}$$

Recalling that $v_A = 60\,\text{mph} = 88.00\,\text{ft/s}$, $v_B = 55\,\text{mph} = 80.67\,\text{ft/s}$, and $35\,\text{mph} = 51.33\,\text{ft/s}$. In vector form, we have

$$\vec{v}_A = (-88.00\,\hat{\jmath})\ \text{ft/s}, \quad \vec{v}_B = (80.67\,\hat{\jmath})\ \text{ft/s}, \quad \vec{v}_C = (30.17\,\hat{\imath} + 41.53\,\hat{\jmath})\ \text{ft/s}. \tag{6}$$

The rate of separation between vehicles A and B is found as

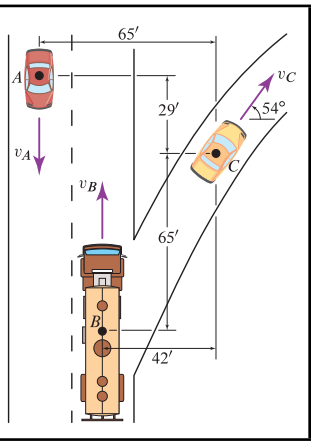
$$\text{ROS}_{AB} = (\vec{v}_B - \vec{v}_A) \cdot \frac{\vec{r}_{B/A}}{|r_{B/A}|} \quad \Rightarrow \quad \boxed{\text{ROS}_{AB} = -164\,\text{ft/s}.}$$

Similarly, the rate of separation between vehicles C and B is found as

$$\text{ROS}_{CB} = (\vec{v}_B - \vec{v}_C) \cdot \frac{\vec{r}_{C/B}}{|r_{C/B}|} \quad \Rightarrow \quad \boxed{\text{ROS}_{CB} = -16.5\,\text{ft/s}.} \tag{7}$$

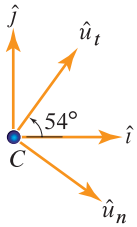
Problem 2.186

Car A is moving at a constant speed $v_A = 75\,\text{km/h}$, while car C is moving at a constant speed $v_C = 42\,\text{km/h}$ on a circular exit ramp with radius $\rho = 80\,\text{m}$. Determine the velocity and acceleration of C relative to A.



Solution

Orient an $(\hat{\imath}, \hat{\jmath})$ frame with $\hat{\jmath}$ in the direction of $\vec{v}_B$. The velocities of A and C are



$$\vec{v}_A = (-75\,\hat{\jmath})\,\text{km/hr} = (-20.83\,\hat{\jmath})\,\text{m/s}, \quad \text{and} \tag{1}$$

$$\vec{v}_C = \left(42\cos 54^\circ\,\hat{\imath} + 42\sin 54^\circ\,\hat{\jmath}\right)\text{km/hr} = (6.857\,\hat{\imath} + 9.469\,\hat{\jmath})\,\text{m/s}. \tag{2}$$

The relative velocity of C with respect to A is

$$\vec{v}_{C/A} = \vec{v}_C - \vec{v}_A \quad \Rightarrow \quad \boxed{\vec{v}_{C/A} = (6.86\,\hat{\imath} + 30.3\,\hat{\jmath})\,\text{m/s}.} \tag{3}$$

The accelerations of A and C are

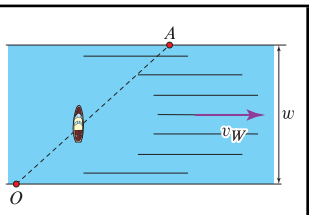
$$\vec{a}_A = \vec{0} \quad \text{and} \quad \vec{a}_C = \frac{v_C^2}{\rho}\,\hat{u}_n = \frac{v_C^2}{\rho}\left(\sin 54^\circ\,\hat{\imath} - \cos 54^\circ\,\hat{\jmath}\right). \tag{4}$$

The relative acceleration of C with respect to A is

$$\vec{a}_{C/A} = \vec{a}_C - \vec{a}_A \quad \Rightarrow \quad \boxed{\vec{a}_{C/A} = (1.38\,\hat{\imath} + 1.00\,\hat{\jmath})\,\text{m/s}^2.} \tag{5}$$

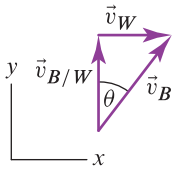
Problem 2.187

A remote control boat, capable of a maximum speed of 10 ft/s in still water, is made to cross a stream with a width $w = 35$ ft that is flowing with a speed $v_W = 7$ ft/s. If the boat starts from point O and keeps its orientation parallel to the cross-stream direction, find the location of point A at which the boat reaches the other bank while moving at its maximum speed. Furthermore, determine how long the crossing requires.



Solution

We will use a Cartesian coordinate system with its origin centered at O. The velocity of the boat with reference to the stream bank is



$$\vec{v}_B = \vec{v}_W + \vec{v}_{B/W},$$
$$\vec{v}_B = (7\,\hat{\imath} + 10\,\hat{\jmath})\ \text{ft/s}. \qquad (1)$$

The y component of velocity is constant so the time of crossing is

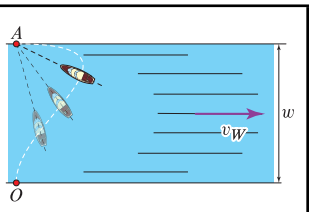
$$w = v_y t \quad \Rightarrow \quad \boxed{t = 3.50\ \text{s}.} \qquad (2)$$

The x component of velocity is constant. Use the time given in Eq. (2) to calculate the position of A as

$$x = v_x t \quad \Rightarrow \quad \boxed{x = 24.5\ \text{ft}.} \qquad (3)$$

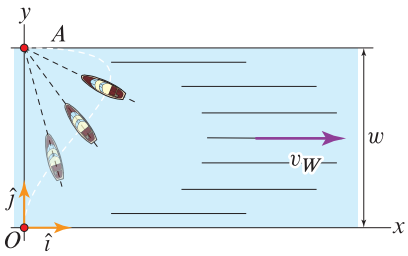
Problem 2.188

A remote control boat, capable of a maximum speed of 10 ft/s in still water, is made to cross a stream of width $w = 35$ ft that is flowing with a speed $v_W = 7$ ft/s. The boat is placed in the water at O, and it is *intended* to arrive at A by using a homing device that makes the boat always point toward A. Determine the time the boat takes to get to A and the path it follows. Also, consider a case in which the maximum speed of the boat is equal to the speed of the current. In such a case, does the boat ever make it to point A? *Hint:* To solve the problem, write $\vec{v}_{B/W} = v_{B/W}\,\hat{u}_{A/B}$, where the unit vector $\hat{u}_{A/B}$ always points from the boat to point A and is therefore a function of time.



Solution

Referring to the figure on the right, we will use a Cartesian coordinate system as shown. As it is central to the description of the motion of the boat, which we denote by B, we begin with finding an expression for the unit vector that points from the boat to points A. Denoting the coordinates of the boat as by (x, y), the position of points A and B are



$$\vec{r}_A = w\,\hat{j} \quad \text{and} \quad \vec{r}_B = x\,\hat{\imath} + y\,\hat{j}. \tag{1}$$

Therefore the position of A relative to B is the vector

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = -x\,\hat{\imath} + (w - y)\,\hat{j}. \tag{2}$$

Dividing $\vec{r}_{A/B}$ by its magnitude we then find the unit vector mentioned earlier, i.e.,

$$\hat{u}_{A/B} = \frac{-x}{\sqrt{x^2 + (w-y)^2}}\,\hat{\imath} + \frac{w - y}{\sqrt{x^2 + (w-y)^2}}\,\hat{j}. \tag{3}$$

Next, we write in vector form what we know about the velocity of the water, which we denote by $\vec{v}_W$ and the velocity of the boat relative to the water, which we denote by $\vec{v}_{B/W}$. Specifically, we have

$$\vec{v}_W = v_w\,\hat{\imath} \quad \text{and} \quad \vec{v}_{B/W} = v_{B/W}\,\hat{u}_{B/W}, \tag{4}$$

where $v_W = 7$ ft/s, $v_{B/W} = 10$ ft/s when the boat is moving at its maximum speed (relative to the water), and where $\hat{u}_{B/W}$ is given in Eq. (3).

Now, to determine the time and path of the boat, we need to write the expression of the velocity of the boat $\vec{v}_B$. Using relative kinematics, we have $\vec{v}_B = \vec{v}_W + \vec{v}_{B/W} = \vec{v}_W + v_{B/W} \cdot \hat{u}_{B/A}$, which gives

$$\vec{v}_B = \left(v_W - \frac{v_{B/W}\,x}{\sqrt{x^2 + (w-y)^2}}\right)\hat{\imath} + \frac{v_{B/W}\,(w-y)}{\sqrt{x^2 + (w-y)^2}}\,\hat{j}. \tag{5}$$

Now recall that the velocity of the boat can be written as

$$\vec{v}_B = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{j}. \tag{6}$$

Hence, equating Eqs. (5) and (6) component by component we find that the motion of the boat is govered by the following differential equations:

$$\dot{x} = v_W - \frac{v_{B/W}\,x}{\sqrt{x^2 + (w-y)^2}} \quad \text{and} \quad \dot{y} = \frac{v_{B/W}\,(w-y)}{\sqrt{x^2 + (w-y)^2}}, \tag{7}$$

which, given that the boat starts at O at time $t = 0$, are subject to the following initial conditions:

$$x(0) = 0 \quad \text{and} \quad y(0) = 0. \tag{8}$$

These equations can be integrated using some appropriate mathematical software. We have used *Mathematica*. Before presenting the code used to solve the problem we must observe something about Eqs. (7) which is important to properly understand the response of the mathematical software used to solve them. Let's suppose that the boat does make it to point A. In this case, the coordinates of the boat would be $(x = 0, y = w)$. When this happens, notice that the argument of the square root in Eqs. (7) becomes equal to zero. In turn, because the square roots appear at the denominator of fractions, when the boat makes it to A, the equations suffer a division by zero and will cause the numerical software to fail. Now, when using a numerical method to solve Eqs. (7) we must specify the time interval we want the solution to cover. As just discussed, as soon as the boat makes it to A the numerical integration fails. Therefore, we need to use trial and error to find the maximum amount of time for which the equations can be integrated. However, this maximum amount of time will correspond to the time taken by the boat to reach A, which is one of the quantities we need to determine in this problem. With this all the above in mind, we have integrated the equations in question for the boat traveling at the maximum speed relative to water, i.e., $w_{B/W} = 10\,\text{ft/s}$, using *Mathematica* with the following code:

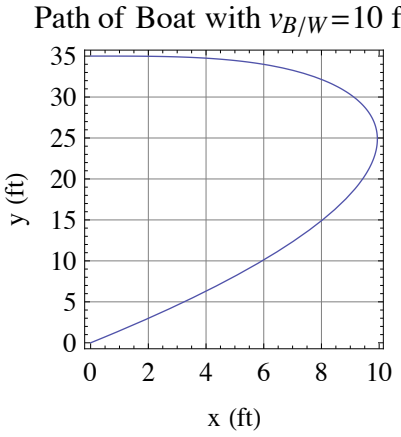
```
Parameters = {vW → 7., vBrelW → 10., w → 35.};
Equations = {x'[t] == vW - vBrelW x[t] / Sqrt[x[t]^2 + (w - y[t])^2], y'[t] == vBrelW (w - y[t]) / Sqrt[x[t]^2 + (w - y[t])^2],
    x[0] == 0, y[0] == 0};
Motion = NDSolve[Equations /. Parameters, {x, y}, {t, 0, 6.8627}]
```

Notice that, using trial and error, we were able to integrate our equation only up to $t = 6.8627\,\text{s}$. Hence, expressing this result to three significant figures, we will say that

> For $v_{B/W} = 10\,\text{s}$, the boat reaches A in 6.86 s.

The solution obtained using *Mathematica* with the above code, can be plotted to depict the path followed by the boat. This path was plotted with the following code:

```
ParametricPlot[{x[t], y[t]} /. Motion[[1]], {t, 0, 6.8627}, Frame → True,
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"x (ft)", "y (ft)"},
 PlotLabel → "Path of Boat with v_B/W=10 ft/s"]
```

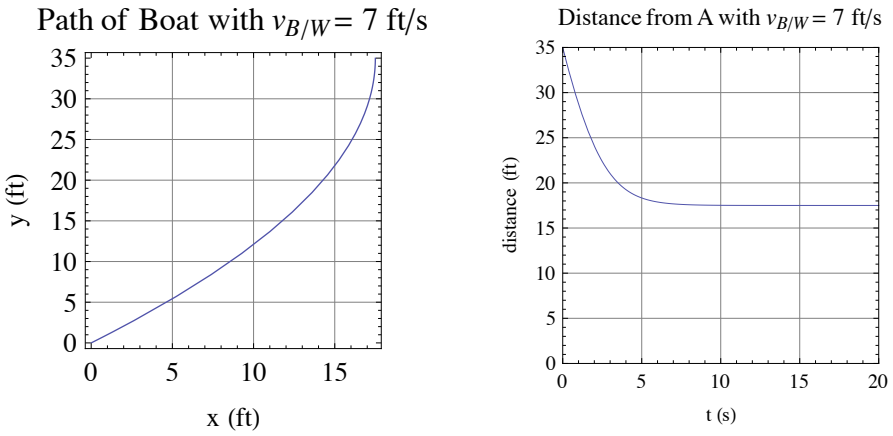


Now we consider the case for which the speed of the current is equal to the speed of the boat. We repeat the same calculations given above but with $v_{B/W} = 7$ ft/s. This time we find that the mathematical software does not fail for any amount of time, indicating that

> When $v_{B/W} = 7$ s, the boat does not reach A.

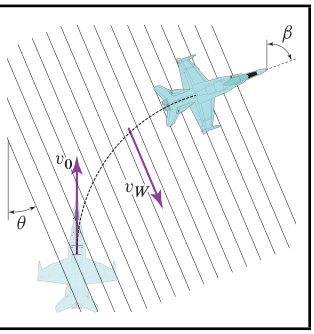
We present our solution for t going from zero to 20 s and we plot both the path of the boat and the distance $d_{AB} = \sqrt{x^2 + (w - y)^2}$ of the boat from A as a function of time. As can be seen from the plot of the path of the boat, the boat does reach the other side of the stream but when it does it points toward A while moving relative to the water with speed equal to the (absolute) speed of the water. As a result, an observer stationary with the banks of the stream sees the boat become stationary. The fact that the boat no longer moves relative to the banks can be seen from the plot of the distance d_{AB}, which, after about 9 s becomes constant (and remains different from zero).

```
Parameters = {vW → 7., vBrelW → 7., w → 35.};
Equations = {x'[t] == vW - vBrelW x[t] / Sqrt[x[t]^2 + (w - y[t])^2], y'[t] == vBrelW (w - y[t]) / Sqrt[x[t]^2 + (w - y[t])^2],
   x[0] == 0, y[0] == 0};
Motion = NDSolve[Equations /. Parameters, {x, y}, {t, 0, 20}];
ParametricPlot[{x[t], y[t]} /. Motion[[1]], {t, 0, 20}, PlotRange → All, Frame → True,
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"x (ft)", "y (ft)"},
 PlotLabel → "Path of Boat with v_B/W = 7 ft/s"]
Plot[Sqrt[x[t]^2 + (w - y[t])^2] /. Parameters /. Motion[[1]], {t, 0, 20},
 PlotRange → {{0, 20}, {0, 35}}, Frame → True, GridLines → Automatic, AspectRatio → 1,
 FrameLabel → {"t (s)", "distance (ft)"}, PlotLabel → "Distance from A with v_B/W = 7 ft/s"]
```



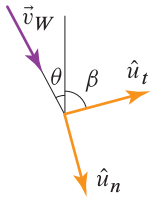
Problem 2.189

A plane is initially flying north with a speed $v_0 = 430\,\text{mph}$ relative to the ground while the wind has a constant speed $v_W = 12\,\text{mph}$ forming an angle $\theta = 23°$ with the north-south direction. The plane performs a course change of $\beta = 75°$ eastward while maintaining a constant reading of the airspeed indicator. Letting $\vec{v}_{P/A}$ be the velocity of the airplane relative to the air and assuming that the airspeed indicator measures the magnitude of the component of $\vec{v}_{P/A}$ in the direction of motion of the airplane, determine the speed of the plane relative to the ground after the course correction.



Solution

We will express the velocity of the airplane using a normal tangential component system. In this manner, the tangent direction will always be the direction of motion of the airplane. In addition, we will use subscripts 1 and 2 to denote quantities before and after the change in course, respectively. Before turning, the velocity of the airplane and wind relative to the ground are



$$\vec{v}_{P_1} = v_0\,\hat{u}_{t_1} \quad \text{and} \quad \vec{v}_{A_1} = v_W(-\cos\theta\hat{u}_{t_1} + \sin\theta\,\hat{u}_{n_1}).$$

The airspeed of the plane before the turn is

$$v_1 = \left(\vec{v}_{P_1} - \vec{v}_{W_1}\right)\cdot\hat{u}_{t_1} \quad \Rightarrow \quad v_1 = v_0 + v_W\cos\theta.$$

After turning, the velocity of the wind is

$$\vec{v}_{A_2} = v_W\left[\cos(\beta+\theta)\;\hat{u}_{t_2} + \sin(\beta+\theta)\,\hat{u}_{n_2}\right].$$

The airspeed of the plane after the turn is

$$v_2 = \left(\vec{v}_{P_2} - \vec{v}_{W_2}\right)\cdot\hat{u}_{t_2} \quad \Rightarrow \quad v_2 = v_{P_2} - v_w\cos(\beta+\theta).$$

Enforcing the condition that $v_2 = v_1$, we have

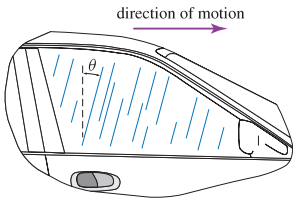
$$v_0 + v_W\cos\theta = v_{P_2} - v_w\cos(\beta+\theta) \quad \Rightarrow \quad v_{P_2} = v_0 + v_W[\cos\theta + \cos(\beta+\theta)].$$

Recalling that $v_0 = 430\,\text{mph}$, $v_W = 12\,\text{mph}$, $\theta = 23°$, and $\beta = 75°$, we can evaluate v_{P_2} to obtain

$$\boxed{v_{P_2} = 439\,\text{mph}.}$$

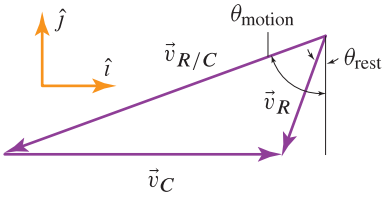
Problem 2.190

An interesting application of the relative motion equations is the experimental determination of the speed at which rain falls. Say you perform an experiment in your car in which you park your car in the rain and measure the angle the falling rain makes on your side window. Let this angle be $\theta_{\text{rest}} = 20°$. Next you drive forward at 25 mph and measure the new angle $\theta_{\text{motion}} = 70°$ that the rain makes with the vertical. Determine the speed of the falling rain.



Solution

We will use a Cartesian Coordinate system where x and y represent the horizontal and vertical directions, respectively. The orientation of the angle θ allows us to write the velocity of the rain with respect to the stationary car as



$$\vec{v}_R = -v_R\,(\sin\theta_{\text{rest}}\,\hat{\imath} + \cos\theta_{\text{rest}}\,\hat{\jmath})\,. \tag{1}$$

The second piece of information allows us to write the velocity of the rain with respect to the moving car as

$$\vec{v}_{R/C} = -v_{R/C}\,(\sin\theta_{\text{motion}}\,\hat{\imath} + \cos\theta_{\text{motion}}\,\hat{\jmath}). \tag{2}$$

Relative kinematics tells us that we must have

$$\vec{v}_R = \vec{v}_{R/C} + \vec{v}_C\,. \tag{3}$$

Substituting Eq. (1) and Eq. (2) into Eq. (3),

$$-v_R\sin\theta_{\text{rest}}\,\hat{\imath} - v_R\cos\theta_{\text{rest}}\,\hat{\jmath} = v_C\,\hat{\imath} - v_{R/C}\sin\theta_{\text{motion}}\,\hat{\imath} - v_{R/C}\cos\theta_{\text{motion}}\,\hat{\jmath}. \tag{4}$$

This vector equation is a system of two equations in two unknowns v_R and $v_{R/C}$

$$-v_R\sin\theta_{\text{rest}} = v_C - v_{R/C}\sin\theta_{\text{motion}}, \tag{5}$$
$$-v_R\cos\theta_{\text{rest}} = -v_{R/C}\cos\theta_{\text{motion}}, \tag{6}$$

The solution to the system of equations yields

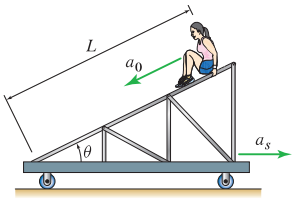
$$v_R = \frac{v_C}{\cos\theta_{\text{rest}}\tan\theta_{\text{motion}} - \sin\theta_{\text{rest}}}.$$

Recalling that we have $\theta_{\text{rest}} = 20°$, $\theta_{\text{motion}} = 70°$, $v_C = 25\,\text{mph} = 36.67\,\text{ft/s}$, we can evaluate the above expression to obatin

$$\boxed{v_{\text{rain}} = 16.4\,\text{ft/s}.}$$

Problem 2.191

A woman is sliding down an incline with a constant acceleration of $a_0 = 2.3\,\text{m/s}^2$ relative to the incline. At the same time the incline is accelerating to the right at $1.2\,\text{m/s}^2$ relative to the ground. Letting $\theta = 34°$ and $L = 4\,\text{m}$ and assuming that both the woman and the incline start from rest, determine the horizontal distance traveled by the woman with respect to the ground when she reaches the bottom of the slide.



Solution

The time it takes the person to slide down the incline is found using the constant acceleration equation

$$s = s_0 + v_0^2 + \tfrac{1}{2}a_c t^2 \quad \Rightarrow \quad L = \tfrac{1}{2}a_0 t^2 \quad \Rightarrow \quad t = \sqrt{\frac{2L}{a_0}}. \tag{1}$$

Adopting a Cartesian coordinate system with $\hat{\imath}$ in the direction of a_s and $\hat{\jmath}$ opposite to the direction of gravity, we can describe the acceleration of the incline and the acceleration of the person relative to the incline as follows:

$$\vec{a}_s = a_s\,\hat{\imath} \quad \text{and} \quad \vec{a}_{p/s} = -a_0(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \tag{2}$$

Therefore, using relative kinematics, we have

$$\vec{a}_p = \vec{a}_s + \vec{a}_{p/s} = (a_s - a_0\cos\theta)\hat{\imath} - a_0\sin\theta\,\hat{\jmath}. \tag{3}$$

The horizontal distance covered by the person is found with the same constant acceleration equation used earlier but with the acceleration set to the horizontal component of $\vec{a}_p$. This yields

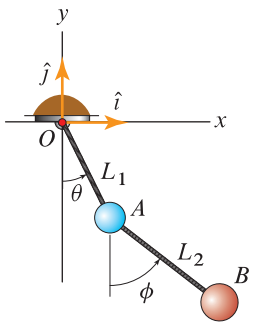
$$x_p = \tfrac{1}{2}\,(a_s - a_0\cos\theta)\,t^2 \quad \Rightarrow \quad x_p = \frac{L}{a_0}\,(a_s - a_0\cos\theta) = -1.23\,\text{m}.$$

where t was replaced by its expression given in the last of Eqs. (1) and where we have used the following numerical data: $L = 4\,\text{m}$, $a_0 = 2.3\,\text{m/s}^2$, $a_s = 1.2\,\text{m/s}^2$, and $\theta = 34°$. In summary, we can say

The person travels 1.23 m to the left of where she started.

Problem 2.192

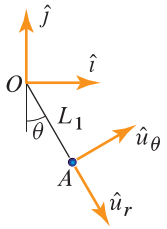
The pendulum bob A swings about O, which is a fixed point, while bob B swings about A. Express the components of the acceleration of B relative to the component system shown with origin at the fixed point O in terms of L_1, L_2, θ, ϕ, and the necessary time derivatives of ϕ and θ.



Solution

The acceleration of A is

$$\vec{a}_A = a_{A_r}\,\hat{u}_r + a_{A_\theta}\,\hat{u}_\theta, \quad \text{where} \quad a_{A_r} = \ddot{L}_1 - L_1\dot{\theta}^2 \quad \text{and} \quad a_{A_\theta} = L_1\ddot{\theta} + 2\dot{L}_1\dot{\theta}.$$
$$\vec{a}_A = -L_1\dot{\theta}^2\,\hat{u}_r + L_1\ddot{\theta}\,\hat{u}_\theta, \quad \text{because } \dot{L}_1 \text{ and } \ddot{L}_1 = 0.$$

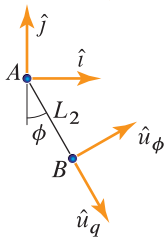


The $(\hat{u}_r, \hat{u}_\theta)$ component system is expressed in the $(\hat{\imath}, \hat{\jmath})$ component system as

$$\hat{u}_r = \sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\theta = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}.$$

Therefore the acceleration of A in the $(\hat{\imath}, \hat{\jmath})$ component system is

$$\vec{a}_A = \left(L_1\ddot{\theta}\cos\theta - L_1\dot{\theta}^2\sin\theta\right)\hat{\imath} + \left(L_1\dot{\theta}^2\cos\theta + L_1\ddot{\theta}\sin\theta\right)\hat{\jmath}.$$



Similarly the acceleration of B with respect to A is

$$\vec{a}_{B/A} = -L_2\dot{\phi}^2\,\hat{u}_q + L_2\ddot{\phi}\,\hat{u}_\phi.$$

The $(\hat{u}_q, \hat{u}_\phi)$ component system is expressed in the $(\hat{\imath}, \hat{\jmath})$ component system as

$$\hat{u}_q = \sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\phi = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}.$$

Therefore $\vec{a}_{B/A}$ in the $(\hat{\imath}, \hat{\jmath})$ component system is

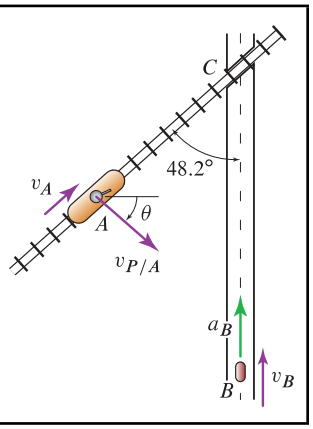
$$\vec{a}_{B/A} = \left(L_2\ddot{\phi}\cos\phi - L_2\dot{\phi}^2\sin\phi\right)\hat{\imath} + \left(L_2\dot{\phi}^2\cos\phi + L_2\ddot{\phi}\sin\phi\right)\hat{\jmath}.$$

$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$. Therefore

$$\vec{a}_B = \left(L_1\ddot{\theta}\cos\theta - L_1\dot{\theta}^2\sin\theta + L_2\ddot{\phi}\cos\phi - L_2\dot{\phi}^2\sin\phi\right)\hat{\imath} + \left(L_1\dot{\theta}^2\cos\theta + L_1\ddot{\theta}\sin\theta + L_2\dot{\phi}^2\cos\phi + L_2\ddot{\phi}\sin\phi\right)\hat{\jmath}.$$

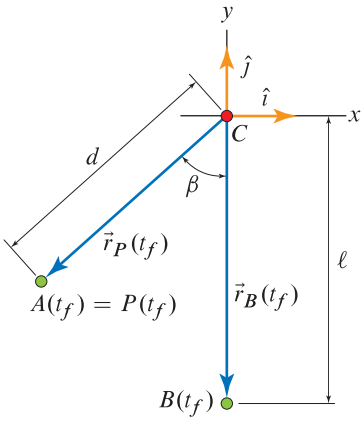
Problem 2.193

Revisit Example 2.26 in which the movie's hero is traveling on train car A with constant speed $v_A = 18\,\text{m/s}$ while the target B is moving at a constant speed $v_B = 40\,\text{m/s}$ (so that $a_B = 0$). Recall that 4 s before an otherwise inevitable collision between A and B, a projectile P traveling at a speed of 300 m/s relative to A is shot toward B. Take advantage of the solution in Example 2.26, and determine the time it takes the projectile P to reach B and the projectile's distance traveled.



Solution

Referring to Example 2.26 on p. 141 of the textbook and to the figure on the right, which is Fig. 3 in Example 2.26, we denote by β the angle $A\hat{C}B$, so that $\beta = 48.2°$. In addition, we denote by t_f and t_h the time at which the projectile is fired and at which the projectile hits the target, respectively. Finally, again as was done in Example 2.26, we denote by d the distance between A and C at $t = t_f$, so that $d = v_A(4\,\text{s}) = 72.00\,\text{m}$. The time taken by the projectile to hit its target is



$$\Delta t = t_h - t_f. \tag{1}$$

To find this time, we observe that the distance traveled by the projectile in the x direction, is given by

$$d_x = d\sin\beta. \tag{2}$$

In Example 2.26 on p. 141 we had determined that the (absolute) velocity of the projectile was

$$\vec{v}_P = (v_A\sin\beta + v_{P/A}\cos\theta)\,\hat{\imath} + (v_A\cos\beta - v_{P/A}\sin\theta)\,\hat{\jmath}. \tag{3}$$

Observing that v_P is constant, then we have that Δt in Eq. (1) is simply equal to the distance in Eq. (2) divided by the x component of $\vec{v}_P$, i.e.,

$$\Delta t = \frac{d\sin\beta}{v_A\sin\beta + v_{P/A}\cos\theta}, \tag{4}$$

which, recalling that we have $d = 72.00\,\text{m}$, $\beta = 48.2°$, $v_{P/A} = 300\,\text{m/s}$, and $\theta = 64.4°$ (see final result in Example 2.26), can be evaluated to obtain

$$\boxed{\Delta t = 0.375\,\text{s}.}$$

To find the distance traveled, we observe again that $\vec{v}_P$ is constant, which implies that the absolute path of the projectile is a straight line segment. Letting $\vec{r}_P(t)$ denote the position vector of the projectile as a

function of time, the two endpoints of the segment in question are identified by $\vec{r}(t_f)$ and $\vec{r}(t_h)$. Hence, again because the velocity of P is constant, the distance traveled by the projectile is given by

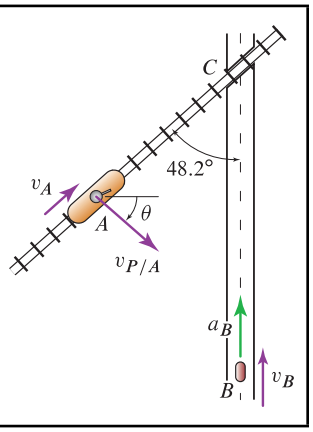
$$|\vec{r}_P(t_h) - \vec{r}_P(t_f)| = |\vec{v}_P \Delta t| = \frac{\sqrt{(v_A \sin\beta + v_{P/A}\cos\theta)^2 + (v_A \cos\beta - v_{P/A}\sin\theta)^2}(d \sin\beta)}{v_A \sin\beta + v_{P/A}\cos\theta}, \quad (5)$$

where we have used the expressions for v_P in Eq. (3) and Δt in Eq. (4). Recalling that we have $d = 72.00\,\text{m}$, $\beta = 48.2°$, $v_{P/A} = 300\,\text{m/s}$, and $\theta = 64.4°$ (see final result in Example 2.26), we can evaluate the right-hand side of Eq. (5) to obtain

Distance traveled by $P = 111\,\text{m}$.

Problem 2.194

Consider the following variation of the problem in Example 2.26 in which a movie hero needs to destroy a mobile robot B, except this time they are not going to collide at C. Assume that the hero is traveling on the train car A with constant speed $v_A = 18\,\text{m/s}$ while the robot B travels at a constant speed $v_B = 50\,\text{m/s}$. In addition, assume that at time $t = 0\,\text{s}$ the train car A and the robot B are 72 and 160 m away from C, respectively. To prevent B from reaching its intended target, at $t = 0\,\text{s}$ the hero fires a projectile P at B. If P can travel at a constant speed of 300 m/s relative to the gun, determine the orientation θ that must be given to the gun to hit B. *Hint:* An equation of the type $\sin\beta \pm A\cos\beta = C$ has the solution $\beta = \mp\gamma + \sin^{-1}(C\cos\gamma)$, if $|C\cos\gamma| \le 1$, where $\gamma = \tan^{-1} A$.



Solution

We base the solution of this problem on the solution of Example 2.26 on p. 141 of the textbook. This means that the definition of all the quantities used in this solution are understood as having been defined in Example 2.26. In addition, we report here Eq. (14) from Example 2.26, which remains valid under the conditions stated in this problem and which determines the value of the angle θ that we want to determine. The equation in question reads

$$(\ell - d\cos\beta)\cos\theta - d\sin\beta\sin\theta = \frac{\sin\beta}{v_{P/A}}(v_B d - v_A \ell)\,. \tag{1}$$

We then proceed to divide this equation through by the term $-d\sin\beta$, so that we have

$$\sin\theta - \frac{\ell - d\cos\beta}{d\sin\beta}\cos\theta = \frac{v_A\ell - v_B d}{v_{P/A}d}. \tag{2}$$

The above equation is a transcendental equation in θ whose solution can be obtained using the following technique. We consider the term multiplying the $\cos\theta$ on the left-hand side of the equation and we define an angle ϕ such that

$$\tan\phi = \frac{\ell - d\cos\beta}{d\sin\beta} \quad\Rightarrow\quad \phi = \tan^{-1}\left(\frac{\ell - d\cos\beta}{d\sin\beta}\right) = 64.40°, \tag{3}$$

where, from Example 2.26, we have used the following numerical data: $\ell = 160\,\text{m}$, $d = 72\,\text{m}$, and $\beta = 48.2°$. Then, using the definition in the first of Eqs. (3) and recalling that $\tan\phi = \sin\phi/\cos\phi$, we can rewrite Eq. (2) as

$$\sin\theta - \frac{\sin\phi}{\cos\phi}\cos\theta = \frac{v_A\ell - v_B d}{v_{P/A}d} \quad\Rightarrow\quad \sin\theta\cos\phi - \sin\phi\cos\theta = \frac{(v_A\ell - v_B d)\cos\phi}{v_{P/A}d}$$

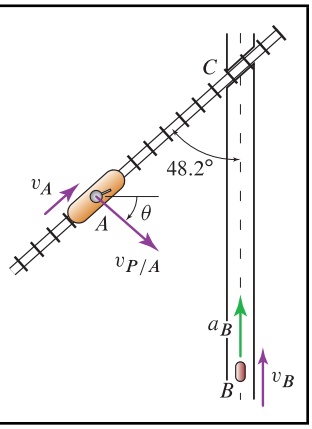
$$\Rightarrow\quad \sin(\theta - \phi) = \frac{(v_A\ell - v_B d)\cos\phi}{v_{P/A}d}, \tag{4}$$

where we have used the trigonometric identity $\sin\theta\cos\phi - \sin\phi\cos\theta = \sin(\theta - \phi)$. Then, again recalling that (from Example 2.26) we have $\ell = 160\,\text{m}$, $d = 72\,\text{m}$, and $\beta = 48.2°$, using the result in the last of Eqs. (3), the last of Eq. (4) can now be solved for θ to obtain

$$\theta = \phi + \sin^{-1}\left[\frac{(v_A\ell - v_B d)\cos\phi}{v_{P/A}d}\right] = 63.6°.$$

Problem 2.195

Consider the following variation of the problem in Example 2.26 in which a movie hero needs to destroy a mobile robot B. As was done in that problem, assume that the movie hero is traveling on the train car A with constant speed $v_A = 18\,\text{m/s}$ and that, 4 s before an otherwise inevitable collision at C, the hero fires a projectile P traveling at 300 m/s relative to A. Differently from the problem in Example 2.26, assume that the robot B travels with a constant acceleration $a_B = 10\,\text{m/s}^2$ and that $v_B(0) = 20\,\text{m/s}$, where $t = 0$ is the time of firing. Determine the orientation θ of the gun fired by the hero so that B can be destroyed before the collision at C.



Solution

The general strategy for the solution of any moving target problem has been discussed in the Road Map of Example 2.26 on p. 141. The main idea underlying the strategy in question is that, letting $t = 0$ be the time at which the projectile is fired, there exists a time $t_I > 0$ such that

$$\vec{r}_{P/B}(t_I) = \vec{r}_P(t_I) - \vec{r}_B(t_I) = \vec{0}, \tag{1}$$

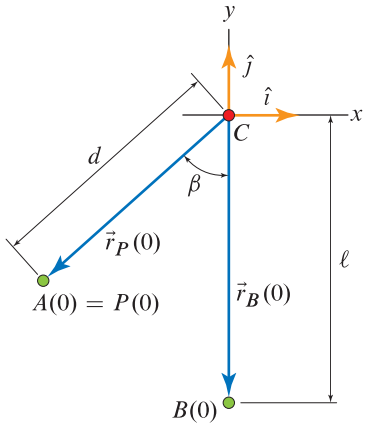
that is, there must exist a time at which the projectile and the target meet. To solve this problem we need find the position of the projectile and of the target as functions of time and then set them equal to each other as required by the above equation.

Because A is moves at a constant speed along a straight line, then $\vec{v}_A$ is constant. Once P is fired, its velocity is also a constant given by

$$\vec{v}_P = \vec{v}_A + v_{P/A}\,\hat{u}_{P/A}(0), \tag{2}$$

where $v_{P/A} = 300\,\text{m/s}$. Hence,

$$\vec{r}_P(t) = \vec{r}_P(0) + \left[\vec{v}_A(0) + v_{P/A}\,\hat{u}_{P/A}(0)\right]t. \tag{3}$$



Because B is has a constant acceleration, constant acceleration equations can be used to express $\vec{r}_B(t)$. Specifically, we have

$$\vec{r}_B(t) = \vec{r}_B(0) + \vec{v}_B(0)t + \tfrac{1}{2}\vec{a}_B t^2. \tag{4}$$

Using Eqs. (3), and (4) to enforce Eq. (1), we have

$$\vec{r}_P(0) + \left(\vec{v}_A(0) + v_{P/A}\,\hat{u}_{P/A}(0)\right)t_I - \vec{r}_B(0) - \vec{v}_B(0)t_I - \tfrac{1}{2}\vec{a}_B t_I^2 = \vec{0}, \tag{5}$$

Referring to the figure on the right (which is essentially identical to Fig. 3 in Example 2.26 and represents the geometry system at the time of firing) we have that $\vec{r}_A(0) = \vec{r}_P(0)$, so that Eq. (5) can be rewritten as

$$\vec{r}_{A/B}(0) + \vec{v}_{A/B}(0)t_I + v_{P/A}\,\hat{u}_{P/A}(0)t_I - \tfrac{1}{2}\vec{a}_B t_I^2 = \vec{0}. \tag{6}$$

The problem is solved when we are able to express all of the terms in Eq. (6) in terms of known quantities and the only two unknowns of the problem, which are t_I and the firing angle θ. We therefore proceed to determine convenient expressions for each of the vectors in Eq. (6).

Since the position at $t = 0$ in this problem is the same as that in the figure shown, we have

$$\vec{r}_{A/B}(0) = -d \sin\beta\, \hat{\imath} + (\ell - d\cos\beta)\, \hat{\jmath} = [(-72.0 \sin 48.2^\circ)\, \hat{\imath} + (160 - 72.0 \cos 48.2^\circ)\, \hat{\jmath}]\,\text{m}, \quad (7)$$

where, using the results in Example 2.26, we have $d = 72.0\,\text{m}$ and $\ell = 160\,\text{m}$. Equation (7) can be evaluated to obtain

$$\vec{r}_{A/B}(0) = (-53.67\, \hat{\imath} + 112.0\, \hat{\jmath})\,\text{m}. \quad (8)$$

Also, again referring to the geometry at $t = 0$, we have

$$\vec{v}_A(0) = (18\,\text{m/s})\, \hat{u}_{C/A} = (18\,\text{m/s})[(\sin 48.2^\circ)\, \hat{\imath} + (\cos 48.2^\circ)]\, \hat{\jmath} = (13.41\, \hat{\imath} + 12.00\, \hat{\jmath})\,\text{m/s} \quad (9)$$

and

$$\vec{v}_B(0) = (20\, \hat{\jmath})\,\text{m/s}, \quad (10)$$

so that

$$\vec{v}_{A/B}(0) = (13.42\, \hat{\imath} - 8\, \hat{\jmath})\,\text{m/s}. \quad (11)$$

Since $t = 0$ is the time of firing, we must have

$$\hat{u}_{P/A}(0) = \cos\theta\, \hat{\imath} - \sin\theta\, \hat{\jmath}. \quad (12)$$

Finally, the problem statement along with the geometry of the problem tells that

$$\vec{a}_B = a_B\, \hat{\jmath} = (10\,\text{m/s}^2)\, \hat{\jmath}. \quad (13)$$

Substituting the expressions in Eqs. (8), (11)–(13) into Eq. (6) and expressing the result on a component by component basis, we have

$$(-53.67 + 13.42 t_I + 300 t_I \cos\theta)\,\text{m} = 0 \quad \text{and} \quad (112 - 8 t_I - 300 t_I \sin\theta - 5 t_I^2)\,\text{m} = 0. \quad (14)$$

The equations above form a system of two equations in the two unknowns θ and t_I that can be solved numerically with some appropriate mathematical software. We have used *Mathematica* with the following code:

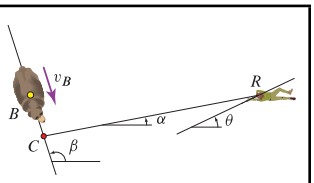
```
FindRoot[{-53.67 + 13.42 tI + 300. tI Cos[θ] == 0, 112. - 8. tI - 300. tI Sin[θ] - 5. tI^2 == 0},
 {{θ, 64.4 Degree}, {tI, 0.375}}]
```

Notice that we have provided the root finding algorithm an initial guess for the solution consisting of the values $\theta = 64.4^\circ$ and $t_I = 0.1\,\text{s}$, which coincide with the solution to the case discussed in Example 2.26. The above code yields the following solution (expressed to 3 significant figures):

$$\theta = 65.9^\circ \quad \text{and} \quad t_I = 0.39\,\text{s}.$$

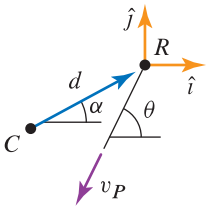
Problem 2.196

A park ranger R is aiming a rifle armed with a tranquilizer dart at a bear (the figure is not to scale). The bear is moving in the direction shown at a constant speed $v_B = 25$ mph. The ranger fires the rifle when the bear is at C at a distance of 150 ft. Knowing that $\alpha = 10°$, $\beta = 108°$, the dart travels with a constant speed of 425 ft/s, and the dart and the bear are moving in a horizontal plane, determine the orientation θ of the rifle so that the ranger can hit the bear. *Hint:* An equation of the type $\sin\beta \pm A\cos\beta = C$ has the solution $\beta = \mp\gamma + \sin^{-1}(C\cos\gamma)$, if $|C\cos\gamma| \le 1$, where $\gamma = \tan^{-1} A$.



Solution

Place a Cartesian coordinate system with origin at R. Let P be the projectile. Thus, we have the situation depicted by the figure at the right. The velocity vector of the bear and the dart are then written in the xy plane as



$$\vec{v}_B = -v_B(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}), \qquad \vec{v}_P = -v_P(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}), \tag{1}$$

where $v_B = 25$ mph $= 36.66$ ft/s, and $v_P = 425$ ft/s. Let $t = 0$ be the time of firing, and t_I be the time at which P hits B. Then it must be the case that

$$\vec{r}_{P/B}(t_I) = \vec{0}, \tag{2}$$

where

$$\vec{r}_P = \vec{v}_P t = -v_P t(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}), \tag{3}$$

and

$$\vec{r}_B = \vec{r}_B(0) + \vec{v}_B t, \text{ with } \vec{r}_B(0) = -d(\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}). \tag{4}$$

Enforcing Eq. (2) requires that

$$(-v_P\cos\theta t_I + d\cos\alpha + v_B\cos\beta t_I)\,\hat{\imath} + (-v_P\sin\theta t_I + d\sin\alpha + v_B\sin\beta t_I)\,\hat{\jmath} = \vec{0}, \tag{5}$$

whereupon, considering the vector components individually, it is determined that

$$\hat{\jmath}: \qquad d\sin\alpha = v_P t_I\left(\sin\theta - \frac{v_B}{v_P}\sin\beta\right), \tag{6}$$

$$\hat{\imath}: \qquad d\cos\alpha = v_P t_I\left(\cos\theta - \frac{v_B}{v_P}\cos\beta\right). \tag{7}$$

Dividing Eq. (6) by Eq. (7) (since $\cos\theta - \frac{v_B}{v_P}\cos\beta \neq 0$), we have

$$\tan\alpha = \frac{\sin\theta - \left(\frac{v_B}{v_P}\right)\sin\beta}{\cos\theta - \left(\frac{v_B}{v_P}\right)\cos\beta}. \tag{8}$$

Rearranging and multiplying by $\cos\alpha$ yields

$$\sin\theta\cos\alpha - \cos\theta\sin\alpha = \frac{v_B}{v_P}(\sin\beta\cos\alpha - \cos\beta\sin\alpha), \tag{9}$$

$$\sin(\theta - \alpha) = \frac{v_B}{v_P} \sin(\beta - \alpha), \tag{10}$$

$$\theta = \alpha + \sin^{-1}\left[\frac{v_B}{v_P} \sin(\beta - \alpha)\right]. \tag{11}$$

Hence, satisfying the values for α, β, v_B, and v_P it is determined that

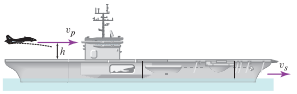
$$\theta = 10^\circ + \sin^{-1}\left[\frac{36.66}{425} \sin(108^\circ - 10^\circ)\right], \tag{12}$$

which can be evaluated to obtain

$$\boxed{\theta = 14.9^\circ.}$$

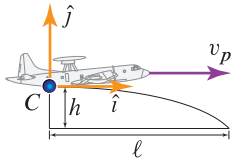
Problem 2.197

An airplane flying horizontally with a speed $v_p = 110\,\text{km/h}$ relative to the water, drops a crate onto a carrier when vertically over the back end of the ship, which is traveling at a speed $v_s = 26\,\text{km/h}$ relative to the water. If the plane drops the crate from a height $h = 20\,\text{m}$, at what distance from the back of the ship will the crate first land on the deck of the ship?



Solution

Let C represent the crate which undergoes projectile motion after being dropped. The acceleration of C is $\vec{a}_c = -g\,\hat{\jmath}$. The velocity of the crate is



$$\vec{v}_c = v_p\,\hat{\imath} - gt\,\hat{\jmath}. \tag{1}$$

Note that the $(\hat{\imath}, \hat{\jmath})$ component system is fixed relative to the water and the origin is at the point where C is released. The position of the crate is

$$\vec{r}_c(t) = v_p t\,\hat{\imath} - \tfrac{1}{2}gt^2\,\hat{\jmath}. \tag{2}$$

The position of the back of the ship is

$$\vec{r}_s(t) = v_s t\,\hat{\imath}. \tag{3}$$

Let t_d be the time at which the crate hits the deck.

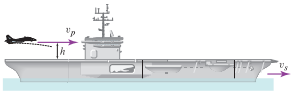
$$r_{cy}(t_d) = r_{cy}(0) - h \quad \Rightarrow \quad -\tfrac{1}{2}gt_d^2 = -h \quad \Rightarrow \quad t_d = \sqrt{\frac{2h}{g}} \tag{4}$$

$\vec{r}_{c/s} = \vec{r}_c - \vec{r}_s$, where for $t = t_d$ $r_{cy} = r_{sy}$.

$$\left(\vec{r}_{c/s}\right)_x = \ell = (v_p - v_s)\sqrt{\frac{2h}{g}} = 47.1\,\text{m}.$$

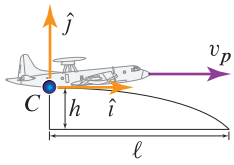
Problem 2.198

An airplane flying horizontally with a speed v_p relative to the water drops a crate onto a carrier when vertically over the back end of the ship, which is traveling at a speed $v_s = 32$ mph relative to the water. The length of the carrier's deck is $\ell = 1000$ ft, and the drop height is $h = 50$ ft. Determine the maximum value of v_p so that the crate will first impact within the rear half of the deck.



Solution

Let C be the crate. Use a component system fixed relative to the water and with its origin at the point where C is released. C undergoes projectile motion. The acceleration of C is $\vec{a} = -g\,\hat{\jmath}$. Therefore the velocity of the crate is



$$\vec{v}_c = v_p\,\hat{\imath} - gt\,\hat{\jmath}. \tag{1}$$

The position of the crate is

$$\vec{r}_c(t) = v_p t\,\hat{\imath} - \tfrac{1}{2}gt^2\,\hat{\jmath}. \tag{2}$$

The position of the back of the ship is

$$\vec{r}_s(t) = v_s t\,\hat{\imath} - h\,\hat{\jmath}. \tag{3}$$

Let t_d be the time when the crate will hit the deck. Considering the vertical component of $\vec{r}_c$, we have

$$-h = -\tfrac{1}{2}gt_d^2 \quad \Rightarrow \quad t_d = \sqrt{\frac{2h}{g}}. \tag{4}$$

Now we want to solve the equation

$$\vec{r}_{c/s}(t_d)\cdot\hat{\imath} = \tfrac{1}{2}\ell \tag{5}$$

for $v_{p_{\max}}$, where

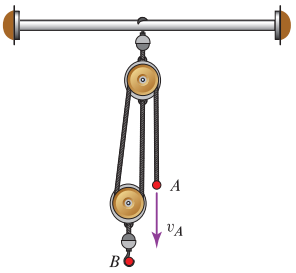
$$\vec{r}_{c/s}(t_d)\cdot\hat{\imath} = [\vec{r}_c(t_d) - \vec{r}_s(t_d)]\cdot\hat{\imath}. \tag{6}$$

Using Eqs. (2) and (3) along with the result in Eq. (4) we have

$$v_{p_{\max}}t_d - v_s t_d = \frac{\ell}{2} \quad \Rightarrow \quad \boxed{v_{p_{\max}} = v_s + \frac{\ell}{2}\sqrt{\frac{g}{2h}} = 331\ \text{ft/s} = 225\ \text{mph}.} \tag{7}$$

Problem 2.199

The object in the figure is called a *gun tackle*, and it used to be very common on sailboats to help in the operation of front-loaded guns. If the end at A is pulled down at a speed of 1.5 m/s, determine the velocity of B. Neglect the fact that some portions of the rope are not vertically aligned.



Solution

By neglecting the fact that some portions of the rope are not vertically aligned we are able to write the length of the cord as

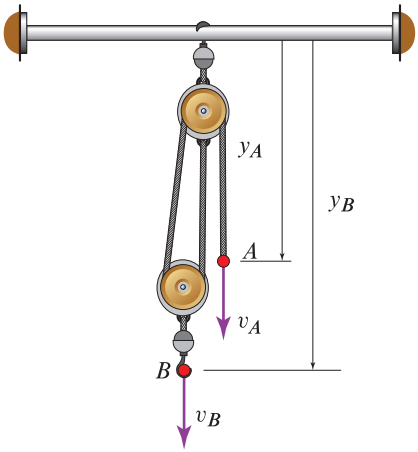
$$L = y_A + 2y_B.$$

The length of the cord is not changing so we calculate its derivative with respect to time as

$$0 = \dot{y}_A + 2\dot{y}_B.$$

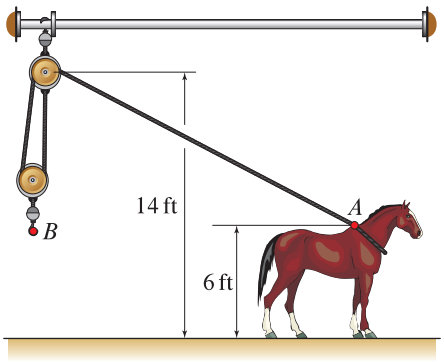
Solving for $\dot{y}_B$, substituting the given data $\dot{y}_A = 1.5\text{m/s}$, expressing our answer in vector for and to 3 significant digits, we have

$$\vec{v}_B = (-0.75\,\text{m/s})\,\hat{j}\ \ \hat{\imath}\ \ \hat{\jmath}\ .$$



Problem 2.200

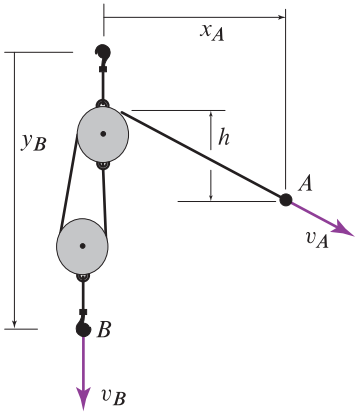
The gun tackle shown in the figure is operated with the help of a horse. If the horse moves to the right at a speed of 7 ft/s, determine the velocity and acceleration of B when the horizontal distance from B to A is 15 ft. Except for the part of the rope attached to the horse, neglect the fact that some portions are not vertically aligned.



Solution

We can write the rope length L as

$$L = 2y_B + \sqrt{x_A^2 + h^2}. \tag{1}$$



The length of the cord is not changing so we calculate its derivative with respect to time as

$$0 = 2\dot{y}_B + \frac{x_A \dot{x}_A}{\sqrt{x_A^2 + h^2}}. \tag{2}$$

Solving the above equation for $\dot{y_B} = v_B$, substituting the given data $h = 8$ ft, $\dot{x}_A = 7$ ft/s, and $x_A = 15$ ft, we have

$$\dot{y}_B = \frac{-x_A \dot{x}_A}{2\sqrt{x_A^2 + h^2}} = -3.088 \text{ ft/s}. \tag{3}$$

Expressing the above answer in vector form and using 3 significant digits, we have

$$\boxed{\vec{v}_B = -3.09 \text{ ft/s} \downarrow_{\hat{j}}^{\hat{i}}.} \tag{4}$$

To obtain the acceleration of B we differentiate with respect to time the expression for $\dot{y}_B$ in Eq. (3) and, observing that $\ddot{x} = 0$, we obtain

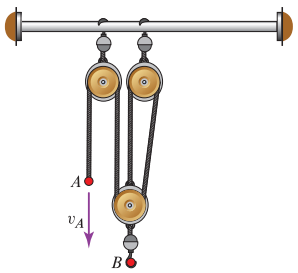
$$\ddot{y}_B = \frac{-h^2 \dot{x}_A^2}{2(x_A^2 + h^2)^{3/2}}. \tag{5}$$

Again recalling that we have $h = 8$ ft, $\dot{x}_A = 7$ ft/s, and $x_A = 15$ ft, expressing our answer in vector form and to 3 significant figures, we have

$$\boxed{\vec{a}_B = -0.319 \text{ ft/s}^2 \downarrow_{\hat{j}}^{\hat{i}}.}$$

Problem 2.201

The figure shows an inverted gun tackle with snatch block, which used to be common on sailboats. If the end at A is pulled at a speed of 1.5 m/s, determine the velocity of B. Neglect the fact that some portions of the rope are not vertically aligned.



Solution

By neglecting the fact that some portions of the rope are not vertically aligned we are able to write the length of the cord as

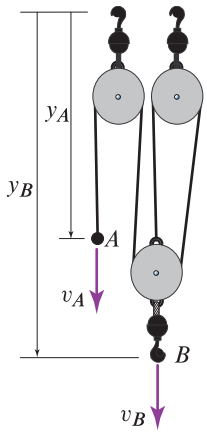
$$L = y_A + 3y_B.$$

The length of the cord is not changing so its derivative with respect to time is

$$0 = \dot{y}_A + 3\dot{y}_B.$$

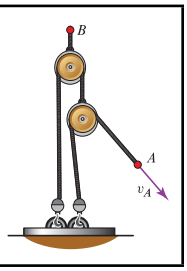
Solving for $\dot{y}_B$, substituting the given data $\dot{y}_A = 1.5\,\text{m/s}$, expressing our answer in vector form and to 3 significant figures, we have

$$\vec{v}_B = (-0.5\,\text{m/s})\,\hat{j} \quad \text{(1)}$$



Problem 2.202

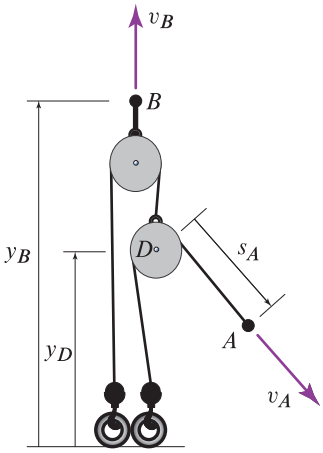
In maritime speak, the system in the figure is often called a *whip-upon-whip purchase* and is used for controlling certain types of sails on small cutters (by attaching point B to the sail to be unfurled). If the end of the rope at A is pulled with a speed of 4 m/s, determine the velocity of B. Neglect the fact that some portions of the rope are not vertically aligned.



Solution

We will refer to the length of the rope attached to A as L_2 and the length of the other rope is referred to as L_1. We write the lengths of the rope as

$$L_1 = 2y_B - y_D,$$
$$L_2 = y_D + s_A.$$



The length of the ropes are not changing so we calculate their derivatives with respect to time as

$$0 = 2\dot{y}_B - \dot{y}_D, \quad (1)$$
$$0 = \dot{y}_D + \dot{s}_A. \quad (2)$$

Substituting $\dot{s}_A = v_A$ into Eq. (2) and eliminating $\dot{y}_D$ from Eq. (1) and Eq. (2) we get

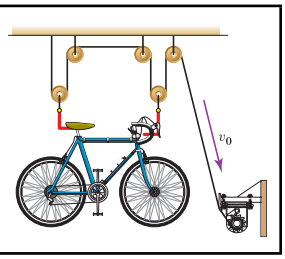
$$2\dot{y}_B = -\frac{v_A}{2}.$$

Substituting the given data $v_A = 4$ m/s and expressing our answer in vector form, we obtain

$$\vec{v}_B = (-2.00\,\text{m/s})\,\hat{j}$$

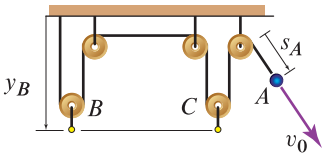
Problem 2.203

The pulley system shown is used to store a bicycle in a garage. If the bicycle is hoisted via a winch that winds the rope at a rate $v_0 = 5\,\text{in./s}$, determine the vertical speed of the bicycle.



Solution

Assume that pulleys B and C are always at the same height. Let A represent an arbitrary point on the rope segment approaching the motor. The length of the rope is



$$L = 4y_B + s_A.$$

The length of the rope does not change as it is winched in. Note that $\dot{s}_A = v_0$.

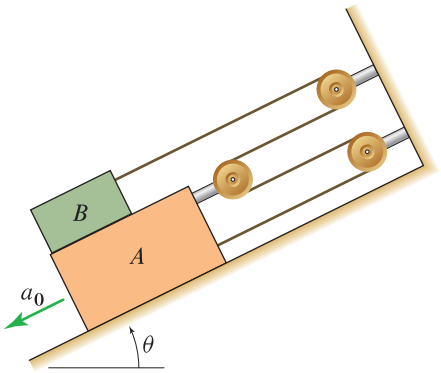
$$0 = 4\dot{y}_B + \dot{s}_A \quad \Rightarrow \quad \dot{y}_B = \frac{-\dot{s}_A}{4} = \frac{-v_0}{4}.$$

Recalling that we have $v_0 = 5\,\text{in.} = 0.4167\,\text{ft/s}$, and observing that the speed of the bicycle is equal to $|\dot{y}_B|$, we have

$$\boxed{v_{\text{bicycle}} = 0.104\,\text{ft/s}.}$$

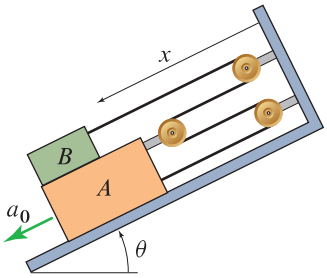
Problem 2.204

Block A is released from rest and starts sliding down the incline with an acceleration $a_0 = 3.7\,\text{m/s}^2$. Determine the acceleration of block B relative to the incline. Also, determine the time needed for B to move a distance $d = 0.2\,\text{m}$ relative to A.



Solution

Expressing the length of the rope L in terms of the coordinates of A and B and then differentiating L with respect to time, we have



$$L = 3x_A + x_B \quad \Rightarrow \quad 0 = 3\dot{x}_A + \dot{x}_B \quad \Rightarrow \quad 0 = 3\ddot{x}_A + \ddot{x}_B. \qquad (1)$$

Therefore

$$\ddot{x}_B = -3\ddot{x}_A = -3a_0. \qquad (2)$$

Hence, recalling that $a_0 = 3.7\,\text{m/s}^2$ and expressing our answer in vector form and to 3 significant figures, we have

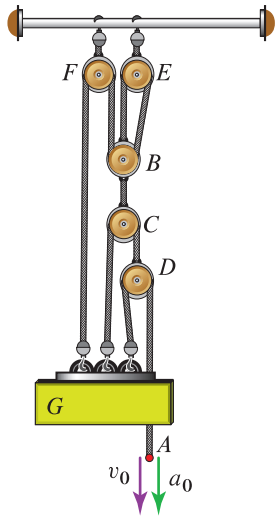
$$\vec{a}_B = (-11.1\,\text{m/s}^2)\,\hat{\imath} \;\angle\, @\theta.$$

Now we use the constant acceleration equation $s = s_0 + \dot{s}_0 t + \frac{1}{2}a_c t^2$ where $s = -d$, $s_0 = 0$, $\dot{s}_0 = 0$, and $a_c = \ddot{x}_{B/A} = \ddot{x}_B - \ddot{x}_A = -4a_0$.

$$-d = -2a_0 t^2 \quad \Rightarrow \quad \boxed{t = \sqrt{\frac{d}{2a_0}} = 0.164\,\text{s}.} \qquad (3)$$

Problem 2.205

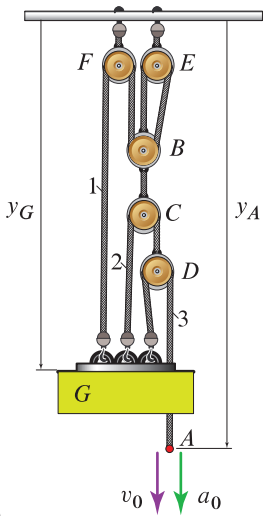
Assuming that all ropes are vertically aligned, determine the velocity and acceleration of the load G if $v_0 = 3\,\text{ft/s}$ and $a_0 = 1\,\text{ft/s}^2$.



Solution

Referring to the figure on the right, the length of ropes 1, 2, and 3 are

$$L_1 = y_G + 3y_B,$$
$$L_2 = y_G + y_D - 2y_B,$$
$$L_3 = y_G + y_A - 2y_D.$$



The derivatives with respect to time of the rope lengths are

$$0 = \dot{y}_G + 3\dot{y}_B, \tag{1}$$
$$0 = \dot{y}_G + \dot{y}_D - 2\dot{y}_B, \tag{2}$$
$$0 = \dot{y}_G + \dot{y}_A - 2\dot{y}_D. \tag{3}$$

Eliminating $\dot{y}_B$ from Eqs. (1) and (2), we obtain

$$5\dot{y}_G + 3\dot{y}_D = 0 \quad \Rightarrow \quad \dot{y}_D = -\tfrac{5}{3}\dot{y}_G. \tag{4}$$

Recalling that $\dot{y}_A = v_0$, substituting Eq. (4) into Eq. (3) and solving for $\dot{y}_G$, we have

$$\dot{y}_G = \frac{-3\dot{y}_A}{13} \quad \Rightarrow \quad \dot{y}_G = \frac{-3v_0}{13}. \tag{5}$$

Recalling that $v_0 = 3\,\text{ft/s}$, expressing our answer in vector form and to three significant digits, we have

$$\vec{v}_G = (-0.692\,\text{ft/s})\,\hat{\jmath}$$

Taking the derivative with respect to time of the first of Eqs. (5) and recalling that $\ddot{y}_A = a_0$, we have

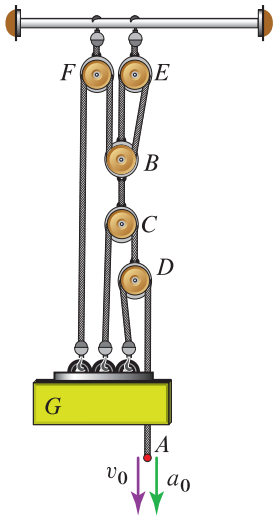
$$\ddot{y}_G = \frac{-3\ddot{y}_A}{13} \quad \Rightarrow \quad \ddot{y}_G = \frac{-3a_0}{13}. \tag{6}$$

Recalling that $a_0 = 1\,\text{ft/s}^2$, expressing our answer in vector form and to three significant digits, we have

$$\vec{a}_G = (-0.231\,\text{ft/s}^2)\,\hat{\jmath}$$

Problem 2.206

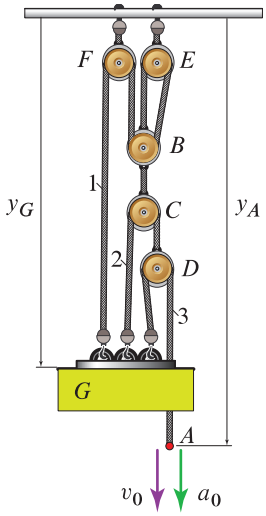
The load G is initially at rest when the end A of the rope is pulled with the constant acceleration a_0. Determine a_0 so that G is lifted 2 ft in 4.3 s.



Solution

Referring to the figure on the right, the length of ropes 1, 2, and 3 are

$$L_1 = y_G + 3y_B,$$
$$L_2 = y_G + y_D - 2y_B,$$
$$L_3 = y_G + y_A - 2y_D.$$



The derivatives with respect to time of the rope lengths are

$$0 = \dot{y}_G + 3\dot{y}_B, \tag{1}$$
$$0 = \dot{y}_G + \dot{y}_D - 2\dot{y}_B, \tag{2}$$
$$0 = \dot{y}_G + \dot{y}_A - 2\dot{y}_D. \tag{3}$$

Eliminating $\dot{y}_B$ from Eqs. (1) and (2), we obtain

$$5\dot{y}_G + 3\dot{y}_D = 0 \quad \Rightarrow \quad \dot{y}_D = -\tfrac{5}{3}\dot{y}_G. \tag{4}$$

Recalling that $\dot{y}_A = v_0$, substituting Eq. (4) into Eq. (3) and solving for $\dot{y}_G$, we have

$$\dot{y}_G = \frac{-3\dot{y}_A}{13} \quad \Rightarrow \quad \ddot{y}_G = \frac{-3\ddot{y}_A}{13} \quad \Rightarrow \quad \ddot{y}_G = \frac{-3a_0}{13}, \tag{5}$$

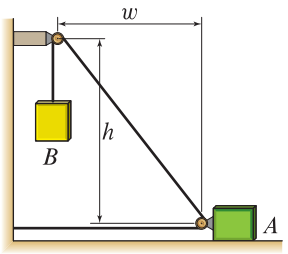
where we have used the fact that $\ddot{y}_A = a_0$. Observe that the last of Eqs. (5) implies that the acceleration of G is constant. Hence, using the constant acceleration equation $s = s_0 + \dot{s}_0 t + \frac{1}{2}a_c t^2$ with $s - s_0 = -d = -2$ ft, $\dot{s}_0 = 0$, and $a_c = \ddot{y}_G$, we have

$$-d = -\frac{3a_0}{26}t^2 \quad \Rightarrow \quad \boxed{a_0 = \frac{26d}{3t^2} = 0.937\ \text{ft/s}^2,}$$

where we have set $t = 4.3$ s.

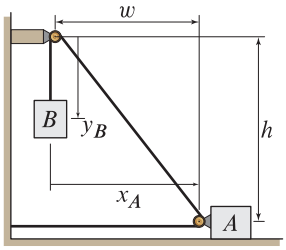
Problem 2.207

At the instant shown, block A is moving at a constant speed $v_0 = 3\,\text{m/s}$ to the left and $w = 2.3\,\text{m}$. Using $h = 2.7\,\text{m}$, determine how much time is needed to lower B 0.75 m from this position.



Solution

Referring to the figure on the right, the length of the rope can be described as



$$L = x_A + y_B + \sqrt{x_A^2 + h^2}.$$

We will denote quantities evaluated at the initial and final positions of the system by the subscripts 1 and 2, respectively. Because the length of the rope is constant, we set the expressions of the length of the rope corresponding to the initial and final positions of the system equal to each other. Initially we have $x_{A1} = w$. In addition, we have $y_{B2} - y_{B1} = d = 0.75\,\text{m}$. Hence, we have

$$\begin{aligned} w + y_{B1} + \sqrt{w^2 + h^2} &= x_{A2} + y_{B2} + \sqrt{x_{A2}^2 + h^2} \\ \Rightarrow \quad x_{A2} + \left(d - w - \sqrt{w^2 + h^2}\right) &= -\sqrt{x_{A2}^2 + h^2} \\ \Rightarrow \quad x_{A2}^2 + \left(d - w - \sqrt{w^2 + h^2}\right)^2 + 2x_{A2}\left(d - w - \sqrt{w^2 + h^2}\right) &= x_{A2}^2 + h^2. \end{aligned}$$

The last of the above equations can be solved for x_{A2} to obtain

$$x_{A2} = \frac{h^2 - \left(d - w - \sqrt{w^2 + h^2}\right)^2}{2\left(d - w - \sqrt{w^2 + h^2}\right)}. \tag{1}$$

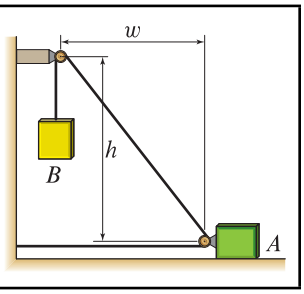
The expression above gives the position of A corresponding to the final position achieved by B as given by the problem statement. Because A and B are connected by in inextensible rope that is assumed not to go slack, the time taken by B to achieve its final position will be equal to the time taken by A to achieve its final position. With this in mind, letting t denote the time to be determined, and recalling that the velocity of A is constant, we then have

$$x_{A2} = x_{A1} - v_0 t \quad \Rightarrow \quad \boxed{t = \frac{1}{v_0}\left[w - \frac{h^2 - \left(d - w - \sqrt{w^2 + h^2}\right)^2}{2\left(w - d + \sqrt{w^2 + h^2}\right)}\right] = 0.156\,\text{s},}$$

where we have used the fact that $x_{A1} = w$ and where we have used the following numerical data: $v_0 = 3\,\text{m/s}$, $h = 2.7\,\text{m}$, $d = 0.75\,\text{m}$, and $w = 2.3\,\text{m}$.

Problem 2.208

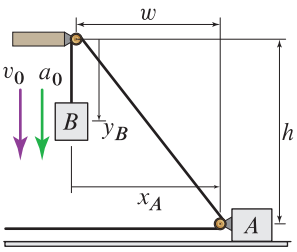
At the instant shown, $h = 10\,\text{ft}$, $w = 8\,\text{ft}$, and block B is moving with a speed $v_0 = 5\,\text{ft/s}$ and an acceleration $a_0 = 1\,\text{ft/s}^2$, both downward. Determine the velocity and acceleration of block A.



Solution

Referring to the figure on the right, the length of the rope can be described as

$$L = x_A + y_B + \sqrt{x_A^2 + h^2}.$$



Since the length of the rope is constant, differentiating the above equation with respect to time, we have

$$\dot{x}_A + \frac{x_A \dot{x}_A}{\sqrt{x_A^2 + h^2}} = -\dot{y}_B \quad \Rightarrow \quad \dot{x}_A = \frac{-\dot{y}_B \sqrt{x_A^2 + h^2}}{x_A + \sqrt{x_A^2 + h^2}}. \qquad (1)$$

Recalling that $\dot{y}_B = v_0 = 5\,\text{ft/s}$ and $h = 10\,\text{ft}$, for $x = w = 8\,\text{ft}$, expressing our answer in vector form, we have

$$\vec{v}_A = \frac{-v_0 \sqrt{w^2 + h^2}}{w + \sqrt{w^2 + h^2}}\,\hat{\imath} = (-3.08\,\text{ft/s})\,\hat{\imath}\ \ \hat{\imath}\ \hat{\jmath}.$$

Taking the derivative with respect to time of the last of Eqs. (1), we have

$$\ddot{x}_A = \frac{-\ddot{y}_B \sqrt{x_A^2 + h^2} - \frac{\dot{y}_B x_A \dot{x}_A}{\sqrt{h^2 + x_A^2}}}{x_A + \sqrt{h^2 + x_A^2}} + \frac{\dot{y}_B \sqrt{h^2 + x_A^2}\left(\dot{x}_A + \frac{\dot{x}_A \dot{x}_A}{\sqrt{h^2 + x_A^2}}\right)}{\left(x_A + \sqrt{h^2 + x_A^2}\right)^2}.$$

Replace $\dot{x}_A$ with its expression in the last of Eqs. (1), we obtain

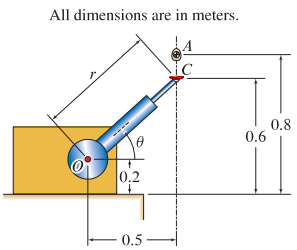
$$\ddot{x}_A = \frac{\frac{x_A \dot{y}_B^2}{x_A + \sqrt{h^2 + x_A^2}} - \sqrt{h^2 + x_A^2}\,\ddot{y}_B}{x_A + \sqrt{h^2 + x_A^2}} - \frac{\dot{y}_B \sqrt{h^2 + x_A^2}\left(\frac{x_A \dot{y}_B}{x_A + \sqrt{h^2 + x_A^2}} + \frac{\sqrt{h^2 + x_A^2}\,\dot{y}_B}{x_A + \sqrt{h^2 + x_A^2}}\right)}{\left(x_A + \sqrt{h^2 + x_A^2}\right)^2}.$$

Recalling that $\dot{y}_B = v_0 = 5\,\text{ft/s}$, $\ddot{y}_B = a_0 = 1\,\text{ft/s}^2$, $h = 10\,\text{ft}$, for $x_A = w = 8\,\text{ft}$, after simplification, the above expression, written in vector form, becomes

$$\vec{a}_A = \frac{v_0^2\left(w - \sqrt{h^2 + w^2}\right) - a_0\left[h^2 + w\left(w + \sqrt{h^2 + w^2}\right)\right]}{\left(w + \sqrt{h^2 + w^2}\right)^2}\,\hat{\imath} = (-0.893\,\text{ft/s}^2)\,\hat{\imath}\ \ \hat{\imath}\ \hat{\jmath}.$$

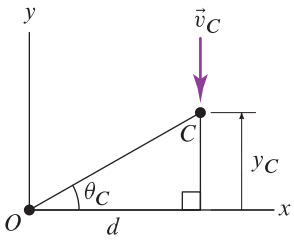
Problem 2.209

As a part of a robotics competition, a robotic arm is to be designed so as to catch an egg without breaking it. The egg is released from rest at $t = 0$ from point A while the arm C is initially also at rest in the position shown. The arm starts moving when the egg is released, and it is to catch the egg at some point in such a way as to avoid any impact between the egg and the robot hand. The arm catches the egg without any impact by specifying that the arm and the egg have to be at the same position at the same time with identical velocities. A student proposes that this can be done by specifying a constant value of $\ddot{\theta}$ for which (after a fair bit of work) it is found that the arm catches the egg at $t = 0.4391$ s for $\ddot{\theta} = -13.27\,\text{rad/s}^2$. Using these values of t and $\ddot{\theta}$, determine the acceleration of both the arm and the egg at the time of catch. Once you have done this, explain whether or not using a constant value of $\ddot{\theta}$, as has been proposed, is an acceptable strategy.



Solution

Referring to the figure on the right, we will use a Cartesian coordinate system with its origin at O. Since $\ddot{\theta}_C$ is constant, we can use the constant acceleration relations to obtain



$$\theta_C = \theta_{C0} + \dot{\theta}_{C0}t + \tfrac{1}{2}\ddot{\theta}_C t^2 = \theta_{C0} + \tfrac{1}{2}\ddot{\theta}_C t^2, \tag{1}$$

where θ_C is related to y_C as follows:

$$y_C = d\tan\theta_C. \tag{2}$$

Since the egg is initially falling under the action of gravity, using constant acceleration equations, at the time of catch, i.e, for $t = 0.4391$ s, the y coordinate of the egg is given by

$$y_e\big|_{t=0.4391\,\text{s}} = 0.6000\,\text{m} - \tfrac{1}{2}(9.81\,\text{m/s}^2)(0.4391\,\text{s})^2 = -0.3457\,\text{m}. \tag{3}$$

At the time of catch we mush have $y_e = y_C$. Therefore, recalling that $d = 0.5$ m, from Eq. (2),

$$\theta_C\big|_{t=0.4391\,\text{s}} = -34.66°. \tag{4}$$

Combining this result with the general expression for θ_C in Eq. (1) we then deduce that

$$\theta_{C0} = 38.66°. \tag{5}$$

Now, we turn to the computation of the acceleration of the egg and of point C at the time of catch. As already stated, up to the time of catch the acceleration of the egg is

$$\boxed{\ddot{y}_e = -9.81\,\text{m/s}^2.}$$

For the arm, we differentiate Eq. (2) twice with respect to t to obtain

$$\ddot{y}_C = d\ddot{\theta}_C\sec^2\theta_C + 2d\dot{\theta}_C^2\sec^2\theta_C\tan\theta_C. \tag{6}$$

To evaluate the expression above we need the value of $\dot{\theta}_C$ at the time of catch. To obtain this value, we differentiate Eq. (1) and evaluate it for $t = 0.4391$ s. Recalling that $\ddot{\theta}_C = -13.27\,\text{rad/s}^2$, this gives

$$\dot{\theta}_C\big|_{t=0.4391\,\text{s}} = \ddot{\theta}_C\,(0.4391\,\text{s}) = -5.827\,\text{rad/s}. \tag{7}$$

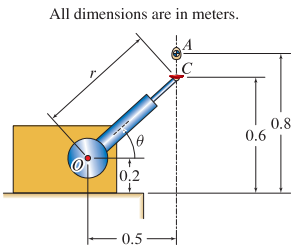
Using this result along with the fact that $d = 0.5$ m and that $\theta_C|_{t=0.4391\,\text{s}} = -34.66°$ (see eq. (4)), we have

$$\boxed{\ddot{y}_C = -44.5\,\text{m/s}^2.}$$

Since the acceleration $|\ddot{y}_C| > |\ddot{y}_e|$, the arm and egg will only be in contact for an instant and will then separate again. Consequently, the proposed strategy is not acceptable for catching the egg.

Problem 2.210

Referring to the problem of a robot arm catching an egg (Prob. 2.209), the strategy is that the arm and the egg must have the same velocity and the same position at the same time for the arm to gently catch the egg. In addition, what should be true about the accelerations of the arm and the egg for the catch to be successful *after* they rendezvous with the same velocity at the same position and time? Describe what happens if the accelerations of the arm and egg do not match.

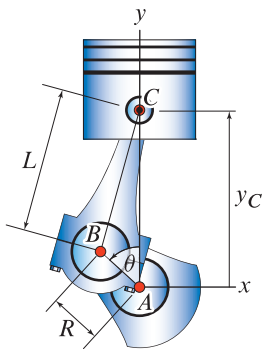


Solution

The relative acceleration of the arm with respect to the egg must be zero. If $\ddot{y}_e < \ddot{y}_C$ then the arm and egg will separate right after the catch. If $\ddot{y}_e > \ddot{y}_C$ the egg will experience a jerk.

Problem 2.211

The piston head at C is constrained to move along the y axis. Let the crank AB be rotating counterclockwise at a constant angular speed $\dot{\theta} = 2000\,\text{rpm}$, $R = 3.5\,\text{in.}$, and $L = 5.3\,\text{in.}$ Determine the velocity of C when $\theta = 35°$.



Solution

Since the piston is constrained to move along the y axis, the law of cosines gives the relation between the distances,

$$L^2 = R^2 + y_C^2 - 2Ry_C\cos\theta, \tag{1}$$

from which y_C is found to be

$$y_C = R\cos\theta \pm \sqrt{L^2 - R^2 + R^2\cos^2\theta} = R\cos\theta \pm \sqrt{L^2 - R^2\sin^2\theta}. \tag{2}$$

To determine the appropriate solution, observe that for $\theta = 0$ we expect $y_C = R + L$. For this reason we select the solution with the $+$ sign in front of the square root and we therefore state that

$$y_C = R\cos\theta + \sqrt{L^2 - R^2 + R^2\cos^2\theta}. \tag{3}$$

Next taking the derivative of Eq. (3) with respect to time and simplifying yields

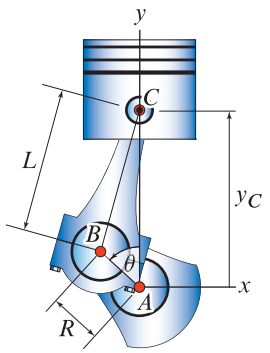
$$\dot{y}_C = -R\dot{\theta}\sin\theta - \frac{R^2\dot{\theta}\sin\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}. \tag{4}$$

Recalling that we need to find the velocity of C for $\theta = 35°$, and recalling that $R = 3.5\,\text{in.} = 0.2917\,\text{ft}$, $L = 5.3\,\text{in.} = 0.4417\,\text{ft}$, and $\dot{\theta} = 2000\,\text{rpm} = 209.4\,\text{rad/s}$, using Eq. (4) and expressing our result in vector form, we have

$$\vec{v}_C = \dot{y}_C\,\hat{\jmath} = (-55.5\,\text{ft/s})\,\hat{\jmath}.$$

Problem 2.212

The piston head at C is constrained to move along the y axis. Let the crank AB be rotating counterclockwise at a constant angular speed $\dot{\theta} = 2000$ rpm, $R = 3.5$ in., and $L = 5.3$ in. Determine expressions for the velocity and acceleration of C as a function of θ and the given parameters.



Solution

Since the piston is constrained to move along the y axis, the law of cosines gives the relation between the distances,

$$L^2 = R^2 + y_C^2 - 2Ry_C \cos\theta, \tag{1}$$

from which y_C is found to be

$$y_C = R\cos\theta \pm \sqrt{L^2 - R^2 + R^2\cos^2\theta} = R\cos\theta \pm \sqrt{L^2 - R^2\sin^2\theta}. \tag{2}$$

To determine the appropriate solution, observe that for $\theta = 0$ we expect $y_C = R + L$. For this reason we select the solution with the $+$ sign in front of the square root and we therefore state that

$$y_C = R\cos\theta + \sqrt{L^2 - R^2 + R^2\cos^2\theta}. \tag{3}$$

Next taking the derivative of Eq. (3) with respect to time and simplifying yields

$$\dot{y}_C = -R\dot{\theta}\sin\theta - \frac{R^2\dot{\theta}\sin\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}} = -R\dot{\theta}\sin\theta\left(1 + \frac{R\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right). \tag{4}$$

Recalling that $R = 3.5$ in. $= 0.2917$ ft, $L = 5.3$ in. $= 0.4417$ ft, and $\dot{\theta} = 2000$ rpm $= 209.4$ rad/s, Eq. (4) can be expressed numerically and in vector form, as follows:

$$\vec{v}_C = (-6.61\ \text{ft/s})\sin\theta\left(1 + \frac{0.292\cos\theta}{\sqrt{0.195 - 0.0851\sin^2\theta}}\right)\hat{\jmath},$$

where we have expressed all known coefficients to 3 significant figures.

Next, recalling that $\ddot{\theta} =$ constant, differentiating with respect to time $\dot{y}_C$ in Eq. (4), we have

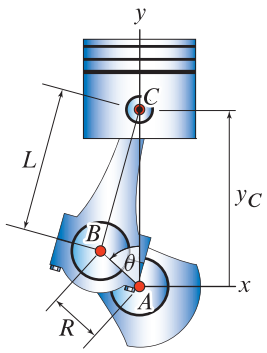
$$\ddot{y}_C = -R\dot{\theta}^2\left[\cos\theta\left(1 + \frac{R\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right) + \sin\theta\left(\frac{R^3\cos^2\theta\sin\theta}{(L^2 - R^2\sin^2\theta)^{3/2}} - \frac{R\sin\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right)\right]. \tag{5}$$

Then, again recalling that $R = 3.5$ in. $= 0.2917$ ft, $L = 5.3$ in. $= 0.4417$ ft, and $\dot{\theta} = 2000$ rpm $= 209.4$ rad/s, Eq. (5) can be expressed numerically and in vector form, as follows:

$$\vec{a}_C = (-1.28\times10^4\ \text{ft/s}^2)\left[\cos\theta\left(1 + \frac{0.292\cos\theta}{\sqrt{0.195 - 0.0851\sin^2\theta}}\right) + \sin\theta\left(\frac{0.0248\cos^2\theta\sin\theta}{(0.195 - 0.0851\sin^2)^{3/2}} - \frac{0.292\sin\theta}{\sqrt{0.195 - 0.0851\sin^2\theta}}\right)\right]\hat{\jmath}.$$

Problem 2.213

Let $\vec{\omega}_{BC}$ denote the angular velocity of the relative position vector $\vec{r}_{C/B}$. As such, $\vec{\omega}_{BC}$ is also the angular velocity of the connecting rod BC. Using the concept of time derivative of a vector given in Section 2.4 on p. 88, determine the component of the relative velocity of C with respect to B along the direction of the connecting rod BC.

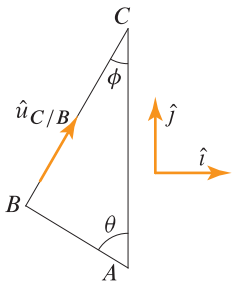


Solution

We being by observing that $\vec{v}_{C/B} = \dot{\vec{r}}_{C/B}$, where $\vec{r}_{C/B} = L\,\hat{u}_{C/B}$. Since L is constant, we must have $\dot{\vec{r}}_{C/B} = L\,\dot{\hat{u}}_{C/B}$. Recalling that $\dot{\hat{u}}_{C/B} = \vec{\omega} \times \hat{u}_{C/B}$

$$\vec{v}_{C/B} = L\vec{\omega} \times \hat{u}_{C/B}.$$

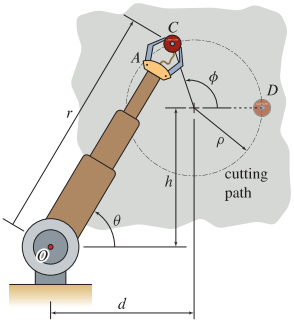
We now observe that $\vec{\omega} \times \hat{u}_{C/B}$ is orthogonal to $\hat{u}_{C/B}$. Therefore the component of $\vec{v}_{C/B}$ along the direction $\hat{u}_{C/B}$ is



$$\text{Component of } \vec{v}_{C/B} \text{ along } \overline{BC} = \left(L\vec{\omega} \times \hat{u}_{C/B}\right) \cdot \hat{u}_{C/B} = 0.$$

Problems 2.214 and 2.215

In the cutting of sheet metal, the robotic arm OA needs to move the cutting tool at C counterclockwise at a constant speed v_0 along a circular path of radius ρ. The center of the circle is located in the position shown relative to the base of the robotic arm at O.

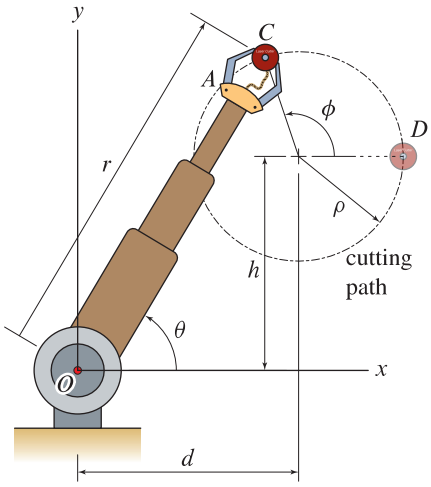


Problem 2.214 For all positions along the circular cut (i.e., for any value of ϕ), determine r, $\dot{r}$, and $\dot{\theta}$ as functions of the given quantities (i.e., d, h, ρ, v_0). Use one or more geometric constraints and their derivatives to do this. These quantities can be found "by hand," but it is tedious, so you might consider using symbolic algebra software such as Mathematica or Maple.

Problem 2.215 For all positions along the circular cut (i.e., for any value of ϕ), determine $\ddot{r}$ and $\ddot{\theta}$ as functions of the given quantities (i.e., d, h, ρ, v_0). These quantities can be found by hand, but it is very tedious, so you might consider using symbolic algebra software such as Mathematica or Maple.

Solution to 2.214

Referring to the figure on the right, we define a Cartesian coordinate system with origin at O. Then the coordinates of point C are such that



$$r^2 = x_C^2 + y_C^2, \tag{1}$$

where

$$x_C = d + \rho\cos\phi = r\cos\theta, \tag{2}$$

$$y_C = h + \rho\sin\phi = r\sin\theta, \tag{3}$$

so that r can be expressed as

$$\boxed{r = \sqrt{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2}.} \tag{4}$$

Differentiating Eqs. (2) and (3) with respect to time, we have

$$-\rho\dot{\phi}\sin\phi = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \tag{5}$$

$$\rho\dot{\phi}\cos\phi = \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \tag{6}$$

Next, observe that we have

$$\sin\theta = \frac{h + \rho\sin\phi}{r}, \quad \cos\theta = \frac{d + \rho\cos\phi}{r}, \quad \text{and} \quad \dot{\phi} = v_0/\rho. \tag{7}$$

Hence, substituting Eqs. (4) and (7) into Eqs.(5) and (6), we obtain a system of two equation in the two unknowns $\dot{r}$ and $\dot{\theta}$ whose solution is

$$\boxed{\dot{\theta} = \frac{v_0(\rho + d\cos\phi + h\sin\phi)}{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2} \quad \text{and} \quad \dot{r} = \frac{v_0(h\cos\phi - d\sin\phi)}{\sqrt{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2}}.}$$

Solution to 2.215

Referring to the figure on the right, we define a Cartesian coordinate system with origin at O. Then the coordinates of point C are such that

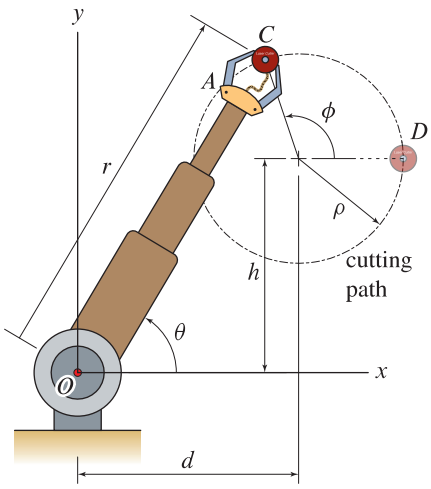
$$r^2 = x_C^2 + y_C^2, \tag{8}$$

where

$$x_C = d + \rho\cos\phi = r\cos\theta, \tag{9}$$
$$y_C = h + \rho\sin\phi = r\sin\theta, \tag{10}$$

so that r can be expressed as

$$r = \sqrt{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2}. \tag{11}$$



Differentiating Eqs. (9) and (10) with respect to time, we have

$$-\rho\dot{\phi}\sin\phi = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \tag{12}$$
$$\rho\dot{\phi}\cos\phi = \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \tag{13}$$

Next, observe that we have

$$\sin\theta = \frac{h + \rho\sin\phi}{r}, \quad \cos\theta = \frac{d + \rho\cos\phi}{r}, \quad \text{and} \quad \dot{\phi} = v_0/\rho. \tag{14}$$

Hence, substituting Eqs. (11) and (14) into Eqs.(12) and (13), we obtain a system of two equation in the two unknowns $\dot{r}$ and $\dot{\theta}$ whose solution is

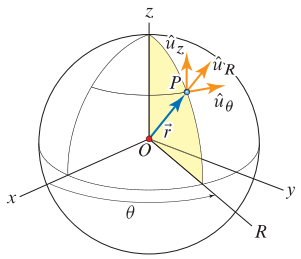
$$\dot{\theta} = \frac{v_0(\rho + d\cos\phi + h\sin\phi)}{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2} \quad \text{and} \quad \dot{r} = \frac{v_0(h\cos\phi - d\sin\phi)}{\sqrt{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2}}. \tag{15}$$

To find expressions for $\ddot{r}$ and $\ddot{\theta}$ we must take the second derivatives of Eqs. (15) and then replace $\dot{\phi}$ with its expression in terms of v_0 and ρ, i.e. $\dot{\phi} = v_0/\rho$. Doing so, after simplification, yields the following expressions:

$$\ddot{r} = -\frac{v_0^2(\rho + d\cos\phi + h\sin\phi)(d^2 + h^2 + \rho d\cos\phi + h\rho\sin\phi)}{\rho\left[(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2\right]^{3/2}},$$
$$\ddot{\theta} = -\frac{v_0^2\left(d^2 + h^2 - \rho^2\right)(d\sin\phi - h\cos\phi)}{\rho\left[(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2\right]^2}.$$

Problem 2.216

Although point P is moving on a sphere, its motion is being studied with the *cylindrical* coordinate system shown. Discuss in detail whether or not there are incorrect elements in the sketch of the cylindrical component system at P.

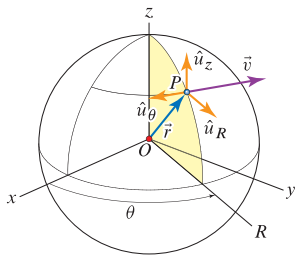


Solution

The unit vector $\hat{u}_R$ points in the direction of $\vec{r}$. This is incorrect. For a cylindrical coordinate system, the unit vector $\hat{u}_R$ must be parallel to the θ-R plane and point in the R direction.

Problem 2.217

Although point P is moving on a sphere, its motion is being studied with the *cylindrical* coordinate system shown. Discuss in detail whether or not there are incorrect elements in the sketch of the cylindrical component system at P.

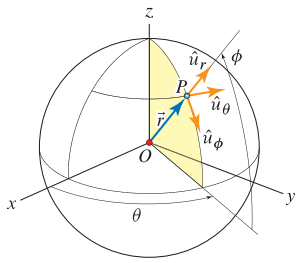


Solution

The unit vector $\hat{u}_\theta$ points in the direction of decreasing θ. This is incorrect, as $\hat{u}_\theta$ must point in the direction of increasing θ.

Problem 2.218

Discuss in detail whether or not (a) there are incorrect elements in the sketch of the cylindrical component system at P and (b) the formulas for the velocity and acceleration components derived in the section can be used with the coordinate system shown.



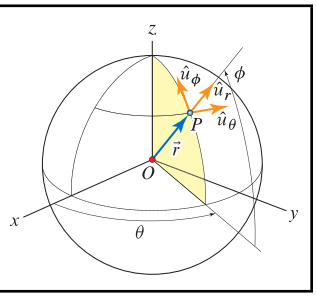
Solution

(i) The unit vector $\hat{u}_\phi$ is pointing in the direction of decreasing ϕ. This is incorrect. It must point in the direction of increasing ϕ.

(ii) No, the formulas derived in the section cannot be used since the angle ϕ in this figure is defined from the xy plane to the line OP. The formulas of this section require that ϕ be defined from the z axis to the line OP.

Problem 2.219

Discuss in detail whether or not (a) there are incorrect elements in the sketch of the cylindrical component system at P and (b) the formulas for the velocity and acceleration components derived in the section can be used with the coordinate system shown.



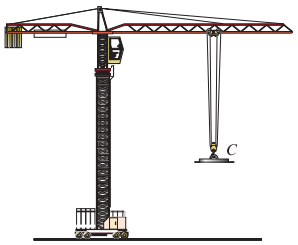
Solution

(i) The orientations of the unit vectors in relation to the positive directions of r, ϕ, and θ are correct.

(ii) No, the formulas derived in the section cannot be used since the angle ϕ in this figure is defined from the xy plane to the line OP. The formulas of this section require that ϕ be defined from the z axis to the line OP. Also the $(\hat{u}_r, \hat{u}_\phi, \hat{u}_\theta)$, triad is *NOT* right-handed.

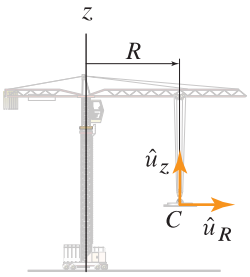
Problem 2.220

A top-slewing crane is lifting an object C at a constant rate of 5.3 ft/s while rotating at a constant rate of 0.12 rad/s about the vertical axis. If the distance between the object and the axis of rotation of the crane's boom is currently 46 ft and it is being reduced at a constant rate of 6.5 ft/s, find the velocity and acceleration of C, assuming that the swinging motion of C can be neglected.



Solution

Referring to the figure on the right, we define a cylindrical coordinate system with the R direction parallel to the crane's boom going from the vertical axis of the crane toward point C; with the z axis coinciding with the vertical axis of the crane and pointing in the direction opposite to gravity; with the θ direction defined in such a way the the triad $(\hat{u}_R, \hat{u}_\theta, \hat{u}_z$ is right-handed. Interpreting the data given in the problem statement, in addition to having $R = 46$ ft, we can write:



$$\dot{R} = -6.5\,\text{ft/s}, \qquad \dot{\theta} = 0.12\,\text{rad/s}, \qquad \dot{z} = 5.3\,\text{ft/s}, \tag{1}$$

$$\ddot{R} = 0, \qquad \ddot{\theta} = 0, \qquad \ddot{z} = 0. \tag{2}$$

Substituting the values into the equation for the velocity, namely, $\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z$, we have

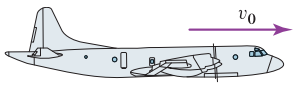
$$\boxed{\vec{v}_C = (-6.5\,\hat{u}_r + 5.52\,\hat{u}_\theta + 5.3\,\hat{u}_z)\,\text{ft/s}.}$$

For the acceleration, substituting the given values into the equation $\vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z$, gives

$$\boxed{\vec{a}_C = (-0.662\,\hat{u}_r - 1.56\,\hat{u}_\theta)\,\text{ft/s}^2.}$$

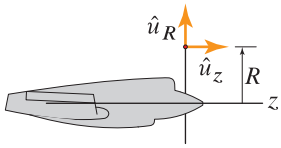
Problem 2.221

An airplane is flying horizontally at a speed $v_0 = 320\,\text{mph}$ while its propellers rotate at an angular speed $\omega = 1500\,\text{rpm}$. If the propellers have a diameter $d = 14\,\text{ft}$, determine the magnitude of the acceleration of a point on the periphery of the propeller blades.



Solution

Referring to the figure on the right, we define a cylindrical coordinate system with the z axis coinciding with the shaft on which the propeller is mounted but remaining fixed (i.e, not moving with the plane), R direction perpendicular to the shaft and going from the z axis to the point whose acceleration we want to measure, an such that the triad $(\hat{u}_R, \hat{u}_\theta, \hat{u}_z$ is right-handed. Next, we recall that in cylindrical coordinates the acceleration is given by the following formula:



$$\vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z \tag{1}$$

Using the problem's given information we that we have

$$R = \tfrac{1}{2}d = 7.000\,\text{ft}, \quad \dot{R} = 0, \quad \ddot{R} = 0, \quad \dot{\theta} = \omega = 1500\,\text{rpm} = 157.1\,\text{rad/s}, \quad \ddot{z} = 0. \tag{2}$$

Substituting the above information in the formula for the acceleration we have

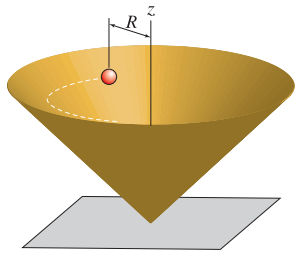
$$\vec{a} = -R\dot{\theta}^2\,\hat{u}_R = (-1.728\times 10^5\,\text{ft/s}^2)\,\hat{u}_R. \tag{3}$$

The magnitude of the above vector, expressed to 3 significant figures, is

$$\boxed{|\vec{a}| = 17{,}300\,\text{ft/s}^2.}$$

Problem 2.222

A particle is moving over the surface of a right cone with angle β and under the constraint that $R^2\dot{\theta} = K$, where K is a constant. The equation describing the cone is $R = z\tan\beta$. Determine the expressions for the velocity and the acceleration of the particle in terms of K, β, z, and the time derivatives of z.



Solution

We use the the cylindrical coordinate system implied by the problem's figure. Next we recall that the general expressions for the velocity and acceleration in cylindrical coordinates are as follows:

$$\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z \quad \text{and} \quad \vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z. \tag{1}$$

The problem is solved by determining all of the terms in the expressions for $\vec{v}$ and $\vec{a}$ and then substituting the terms in question into the above equations.

We begin with computing the terms related to the coordinate R. We are told that $R = z\tan\beta$. Hence, recalling that β is constant, we have

$$R = z\tan\beta \quad \Rightarrow \quad \dot{R} = \dot{z}\tan\beta \quad \Rightarrow \quad \ddot{R} = \ddot{z}\tan\beta. \tag{2}$$

Next we consider the terms related to θ. Specifically, we start with the constraint equation $R^2\dot{\theta} = K$, and obtain

$$K = R^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{K}{R^2} = \frac{K}{z^2\tan^2\beta} \quad \Rightarrow \quad \ddot{\theta} = -\frac{2K\dot{z}}{z^3\tan^2\beta}. \tag{3}$$

Substituting the first two of Eqs. (2) and the second of Eqs. (3) into the first of Eqs. (1), we have

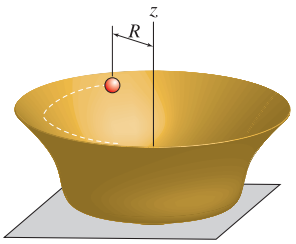
$$\boxed{\vec{v} = \dot{z}\tan\beta\,\hat{u}_R + \frac{K}{z\tan\beta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z.}$$

Substituting Eqs. (2) and the last two of Eqs. (3) into the second of Eqs. (1), we have

$$\boxed{\vec{a} = \left(\ddot{z}\tan\beta - \frac{K^2}{z^3\tan^3\beta}\right)\hat{u}_R + \ddot{z}\,\hat{u}_z.}$$

Problem 2.223

Solve Prob. 2.222 for general surfaces of revolution; that is, R is no longer equal to $z\tan\beta$, but is now an arbitrary function of z, that is, $R = f(z)$. The expressions you are required to find will contain K, $f(z)$, derivatives of $f(z)$ with respect to z, as well as derivatives of z with respect to time.



Solution

We use the the cylindrical coordinate system implied by the problem's figure. Next we recall that the general expressions for the velocity and acceleration in cylindrical coordinates are as follows:

$$\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z \quad \text{and} \quad \vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z. \tag{1}$$

The problem is solved by determining all of the terms in the expressions for $\vec{v}$ and $\vec{a}$ and then substituting the terms in question into the above equations.

We begin with computing the terms related to the coordinate R. Since we are told that $R = f(z)$, and keeping in mind that $z = z(t)$, using the chain rule, we have

$$\dot{R} = \frac{dR}{dz}\frac{dz}{dt} \quad \Rightarrow \quad \dot{R} = \dot{z}\frac{df}{dz} \quad \text{and} \quad \ddot{R} = \frac{d}{dt}\left(\dot{z}\frac{df}{dz}\right) \quad \Rightarrow \quad \ddot{R} = \ddot{z}\frac{df}{dz} + \dot{z}^2\frac{d^2f}{dz^2}. \tag{2}$$

Next we consider the terms concerning the coordinate θ. Recalling that we have the constraint equation $K = R^2\dot{\theta}$. Hence, we can write

$$K = R^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{K}{f^2(z)} \quad \Rightarrow \quad \ddot{\theta} = \frac{d\dot{\theta}}{dz}\frac{dz}{dt} = \frac{-2K}{f^3(z)}\frac{df}{dz}\dot{z}. \tag{3}$$

Recalling that $R = f(z)$, substituting the expression for $\dot{R}$ from Eqs.(2) along with the expression for $\dot{\theta}$ from Eqs. (3) into the first of Eqs. (1) we have

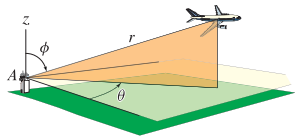
$$\boxed{\vec{v} = \frac{df}{dz}\dot{z}\,\hat{u}_r + \frac{K}{f(z)}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z.}$$

Again recalling that $R = f(z)$, substituting the expressions for $\dot{R}$ and $\ddot{R}$ from Eqs.(2) along with the expressions for $\dot{\theta}$ and $\ddot{\theta}$ from Eqs. (3) into the second of Eqs. (1) we have

$$\boxed{\vec{a} = \left(\ddot{z} + \dot{z}^2\frac{d^2f}{dz^2} - \frac{K^2}{f^3(z)}\right)\hat{u}_R + \ddot{z}\,\hat{u}_z.}$$

Problem 2.224

Revisit Example 2.31, and assuming that the plane is accelerating, determine the relation that the radar readings obtained by the station at A need to satisfy for you to conclude that the jet is flying along a straight line whether at constant altitude or not.



Solution

For the plane to fly along a straight line, the airplane's velocity and acceleration vectors must be parallel. This condition is expressed by the following vector equation:

$$\vec{v} \times \vec{a} = \vec{0}. \tag{1}$$

Recalling that we can write

$$\vec{v} = v_r\,\hat{u}_r + v_\phi\,\hat{u}_\phi + v_\theta\,\hat{u}_\theta \quad \text{and} \quad \vec{a} = a_r\,\hat{u}_r + a_\phi\,\hat{u}_\phi + a_\theta\,\hat{u}_\theta, \tag{2}$$

the condition in Eq. (1) takes on the form

$$\vec{v} \times \vec{a} = (v_\phi a_\theta - v_\theta a_\phi)\,\hat{u}_r + (v_\theta a_r - v_r a_\theta)\,\hat{u}_\phi + (v_r a_\phi - v_\phi a_r)\,\hat{u}_\theta = \vec{0}. \tag{3}$$

The above equation is satisfies if and only if

$$v_\phi a_\theta - v_\theta a_\phi = 0, \quad v_\theta a_r - v_r a_\theta = 0, \quad \text{and} \quad v_r a_\phi - v_\phi a_r = 0. \tag{4}$$

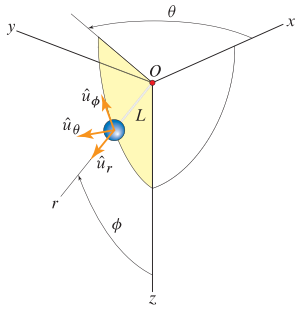
Next, using Eq. (2.138) on p. 157 of the textbook and Eq. (2.140) on p. 157 of the textbook, we can rewrite Eqs. (4) as follows:

$$\begin{aligned} r\dot{\phi}(r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi) - r\dot{\theta}\sin\phi(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi) &= 0, \\ r\dot{\theta}\sin\phi(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi) - \dot{r}(r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi) &= 0, \\ \dot{r}(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi) - r\dot{\phi}(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi) &= 0. \end{aligned}$$

The above equations are those that need to be satisfied by the radar reading to conclude that the plane is flying along a straight line.

Problem 2.225

The system depicted in the figure is called a *spherical pendulum*. The fixed end of the pendulum is at O. Point O behaves as a spherical joint; i.e., the location of O is fixed while the pendulum's cord can swing in any direction in the three-dimensional space. Assume that the pendulum's cord has a constant length L, and use the coordinate system depicted in the figure to derive the expression for the acceleration of the pendulum.



Solution

Keeping in mind that the length of the pendulum is constant, we have that the position vector of the pendulum bob is described as $\vec{r} = L\,\hat{u}_r$, where the radial coordinate r is such that

$$r = L = \text{constant}. \tag{1}$$

Therefore the time derivatives of the radial coordinate must be equal to zero, i.e.,

$$\dot{r} = 0 \quad \text{and} \quad \ddot{r} = 0. \tag{2}$$

Now recall that the acceleration in spherical coordinates is given by

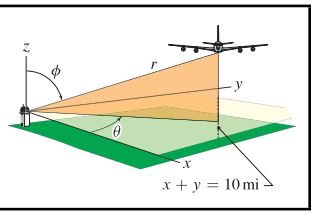
$$\vec{a} = \left(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi\right)\hat{u}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi\right)\hat{u}_\phi + \left(r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi\right)\hat{u}_\theta.$$

Therefore, using Eqs. (1) and (2) the acceleration becomes

$$\boxed{\vec{a} = -L\left(\dot{\phi}^2 + \dot{\theta}^2\sin^2\phi\right)\hat{u}_r + L\left(\ddot{\phi} - \dot{\theta}^2\sin\phi\cos\phi\right)\hat{u}_\phi + L\left(\ddot{\theta}\sin\phi + 2\dot{\phi}\dot{\theta}\cos\phi\right)\hat{u}_\theta.}$$

Problem 2.226

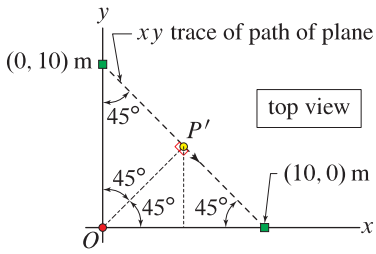
An airplane is traveling at a constant altitude of 10,000 ft, with a constant speed of 450 mph, within the plane whose equation is given by $x + y = 10$ mi and in the direction of increasing x. Find the expressions for $\dot{r}$, $\dot{\theta}$, $\dot{\phi}$, $\ddot{r}$, $\ddot{\theta}$, and $\ddot{\phi}$ that would be measured when the airplane is closest to the radar station.



Solution

The figure on the right shows the trace of the path of the airplane on the xy plane. Using the coordinate system shown, then velocity and acceleration of the plane are

$$\vec{v} = \frac{v_0}{\sqrt{2}}(\hat{\imath} - \hat{\jmath}) \quad \text{and} \quad \vec{a} = \vec{0}$$



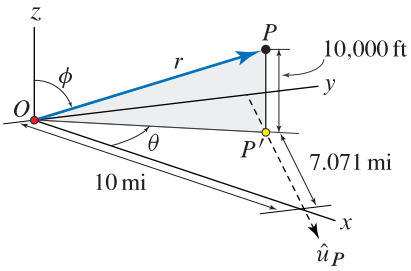
where $v_0 = 450\,\text{mph} = 660.0\,\text{ft}$. Point P' in the figure is the projection on the xy plane of the point on the airplane's path that is closest to the origin O. Based on the geometry of the path, we have that $x_{P'} = y_{P'} = 5\,\text{mi} = 2.640\times10^4$ ft. Therefore, referring to the bottom figure on the right, the coordinates of the point P (the point on the path closest to the origin) are

$$P : (x_{P'}, y_{P'}, h) = (26{,}400, 2.6400, 10{,}000)\,\text{ft}, \tag{1}$$

where $h = 10{,}000$ ft is the altitude at which the airplane is flying.

Referring to the figure on the right (bottom), the distance between P and O is

$$r = \sqrt{h^2 + \frac{x_P^2}{4} + \frac{y_P^2}{4}} = 38{,}650\,\text{ft}.$$



The angles θ and ϕ when the plane is closest to the radar are

$$\theta = 45^\circ \quad \text{and} \quad \phi = 90^\circ - \tan^{-1}\left(\frac{h\sqrt{2}}{x_0}\right) = 75.01^\circ. \tag{2}$$

Then, the unit vectors $\hat{u}_r$, $\hat{u}_\phi$, and $\hat{u}_\theta$ of a spherical coordinate system with origin at O are expressed as

$$\begin{aligned}
\hat{u}_r &= \sin\phi\cos\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\phi\,\hat{k},\\
\hat{u}_\phi &= \cos\phi\cos\theta\,\hat{\imath} + \cos\phi\sin\theta\,\hat{\jmath} - \sin\phi\,\hat{k},\\
\hat{u}_\theta &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}.
\end{aligned}$$

The components of the velocity in the $\hat{u}_r$, $\hat{u}_\phi$, and $\hat{u}_\theta$ directions for $\theta = 45^\circ$ are

$$v_r = \dot{r} = \vec{v}\cdot\hat{u}_r = \frac{v_0}{\sqrt{2}}(\sin\phi\cos\theta - \sin\phi\sin\theta) \quad \Rightarrow \quad \boxed{\dot{r} = 0.}$$

$$v_\phi = r\dot{\phi} = \vec{v}\cdot\hat{u}_\phi = \frac{v_0}{\sqrt{2}}(\cos\phi\cos\theta - \cos\phi\sin\theta) \quad \Rightarrow \quad \boxed{\dot{\phi} = 0.}$$

$$v_\theta = r\dot{\theta}\sin\phi = \vec{v}\cdot\hat{u}_\theta = \frac{v_0}{\sqrt{2}}(-\sin\theta - \cos\theta) \quad \Rightarrow \quad \boxed{\dot{\theta} = \frac{-v_0}{r\sin\phi} = -0.0177\,\text{rad/s},}$$

where we have used the fact that $v_0 = 450\,\text{mph} = 660.0\,\text{ft}$, $\theta = 45°$, and $\phi = 75.01°$ (see Eqs. (2)).

The components of the acceleration in the $\hat{u}_r$, $\hat{u}_\phi$, and $\hat{u}_\theta$ directions for $\theta = 45°$ are

$$a_r = \ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin^2\phi = 0 \quad \Rightarrow \quad \boxed{\ddot{r} = \frac{v_0^2}{r} = 11.3\,\text{ft/s}^2.}$$

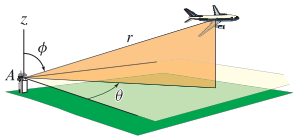
$$a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin\phi\cos\phi = 0 \quad \Rightarrow \quad \boxed{\ddot{\phi} = \frac{v_0^2 \cos\phi}{r^2 \sin\phi} = 78.1\times10^{-6}\,\text{rad/s}^2.}$$

$$a_\theta = r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi = 0 \quad \Rightarrow \quad \boxed{\ddot{\theta} = 0,}$$

where, again, we have used the fact that $v_0 = 450\,\text{mph} = 660.0\,\text{ft}$, $\theta = 45°$, and $\phi = 75.01°$ (see Eqs. (2)).

Problem 2.227

An airplane is being tracked by a radar station at A. At the instant $t = 0$, the following data is recorded: $r = 15\,\text{km}$, $\phi = 80°$, $\theta = 15°$, $\dot{r} = 350\,\text{km/h}$, $\dot{\phi} = -0.002\,\text{rad/s}$, $\dot{\theta} = 0.003\,\text{rad/s}$. If the airplane were flying so as to keep each of the velocity components constant for a few minutes, determine the components of the airplane's acceleration when $t = 30\,\text{s}$.



Solution

Referring to Eq. (2.140) on p. 157 of the textbook, recall that, in spherical coordinates, the components of acceleration are given by

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin^2\phi, \\ a_\phi &= r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin\phi\cos\phi, \\ a_\theta &= r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi. \end{aligned} \tag{1}$$

Therefore, to solve this problem, we must provide the values of r, $\dot{r}$, $\ddot{r}$, ϕ, $\dot{\phi}$, $\ddot{\phi}$, $\dot{\theta}$, and $\ddot{\theta}$ for $t = 30\,\text{s}$.

We will now use the assumption that the velocity components are constant to determine the value of the quantities just listed. With this in mind, it is useful to recall that the expression of the velocity in spherical coordinates is (see Eq. (2.137) on p. 157 of the textbook)

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi + r\dot{\theta}\sin\phi\,\hat{u}_\theta = v_r\,\hat{u}_r + v_\phi\,\hat{u}_\phi + v_\theta\,\hat{u}_\theta. \tag{2}$$

Then, under the assumption that v_r is constant, for the radial coordinate r we have

$$v_r(0) = \dot{r}(0) = \text{constant} \quad \Rightarrow \quad \ddot{r}(t) = 0 \quad \text{and} \quad r(t) = r(0) + \dot{r}(0)t. \tag{3}$$

Recalling that $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$ and that $\dot{r}(0) = 350\,\text{km/h} = 97.22\,\text{m/s}$, then, using Eqs. (3), at $t = 30\,\text{s}$ we have

$$r(30\,\text{s}) = 17920, \quad \dot{r}(30\,\text{s}) = 97.22\,\text{m/s}, \quad \text{and} \quad \ddot{r}(30\,\text{s}) = 0. \tag{4}$$

Next we proceed to determine the values of ϕ, $\dot{\phi}$, and $\ddot{\phi}$ at $t = 30\,\text{s}$. To do so, referring to Eq. (2), we start from the consideration that $v_\phi = r\dot{\phi}$. Hence, using the expression for r in the last of Eqs. (3) we have

$$v_\phi = r\dot{\phi} = \text{constant} = r(0)\dot{\phi}(0), \quad \Rightarrow \quad \dot{\phi} = \frac{r(0)\dot{\phi}(0)}{r(0) + \dot{r}(0)t} \quad \Rightarrow \quad \ddot{\phi} = -\frac{r(0)\dot{r}(0)\dot{\phi}(0)}{[r(0) + \dot{r}(0)t]^2}. \tag{5}$$

The last two of Eqs. (5) will allow us to compute $\dot{\phi}$ and $\ddot{\phi}$ at $t = 30\,\text{s}$. However, we also need the value of ϕ at $t = 30\,\text{s}$. To compute such a value we now proceed to integrate the second of Eqs. (5) with respect to time. This gives

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{r(0)\dot{\phi}(0)}{r(t)} \quad \Rightarrow \quad \phi(t) - \phi(0) = \int_0^t \frac{r(0)\dot{\phi}(0)}{r(0) + \dot{r}(0)t}\,dt$$

$$\Rightarrow \quad \phi(t) = \phi(0) + \frac{r(0)\dot{\phi}(0)}{\dot{r}(0)} \ln\left(1 + \frac{\dot{r}(0)}{r(0)}t\right). \tag{6}$$

Using the last of Eqs. (6) and the last two of Eqs. (5) we then have

$$\phi(30\,\text{s}) = 76.86°, \quad \dot{\phi}(30\,\text{s}) = -0.001674\,\text{rad/s}, \quad \text{and} \quad \ddot{\phi}(30\,\text{s}) = 9.086\times10^{-6}\,\text{rad/s}^2, \tag{7}$$

where we have used the following numerical data: $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$, $\dot{r}(0) = 350\,\text{km/h} = 97.22\,\text{m/s}$, $\phi(0) = 80°$, and $\dot{\phi}(0) = -0.002\,\text{rad/s}$.

Next we proceed to determine the values of $\dot{\theta}$, and $\ddot{\theta}$ at $t = 30\,\text{s}$. We start from the fact that $v_\theta = r\dot{\theta}\sin\phi$ is assumed to be constant. Hence, we have

$$r(t)\dot{\theta}(t)\sin\phi(t) = r(0)\dot{\theta}(0)\sin\phi(0) \quad \Rightarrow \quad \dot{\theta}(t) = \frac{r(0)\dot{\theta}(0)\sin\phi(0)}{r(t)\sin\phi(t)}$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{r(0)\dot{\theta}(0)\sin\phi(0)\{\dot{r}(t)\sin\phi(t) + r(t)[\cos\phi(t)]\dot{\phi}(t)\}}{r^2(t)\sin^2\phi(t)}. \tag{8}$$

Hence, recalling that $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$, $\dot{r}(0) = 350\,\text{km/h} = 97.22\,\text{m/s}$, $\phi(0) = 80°$, $\dot{\theta}(0) = 0.003\,\text{rad/s}$, and using Eqs. (4) and (7), for $t = 30\,\text{s}$, we have

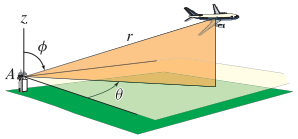
$$\dot{\theta}(30\,\text{s}) = 0.002540\,\text{rad/s} \quad \text{and} \quad \ddot{\theta}(30\,\text{s}) = -0.00001278\,\text{rad/s}^2. \tag{9}$$

In conclusion, substituting Eqs. (4), (7), and (9) into Eqs. (1), for $t = 30\,\text{s}$ we have

$$\boxed{a_r = -0.160\,\text{m/s}^2, \quad a_\phi = -0.188\,\text{m/s}^2, \quad \text{and} \quad a_\theta = 0.223\,\text{m/s}^2.}$$

Problem 2.228

An airplane is being tracked by a radar station at A. At the instant $t = 0$, the following data is recorded: $r = 15\,\text{km}$, $\phi = 80°$, $\theta = 15°$, $\dot{r} = 350\,\text{km/h}$, $\dot{\phi} = -0.002\,\text{rad/s}$, $\dot{\theta} = 0.003\,\text{rad/s}$. If the airplane were flying so as to keep each of the velocity components constant, plot the trajectory of the airplane for $0 < t < 150\,\text{s}$.



Solution

To plot the trajectory of the airplane we need to find expressions for the coordinates of the airplane as a function of time. Clearly, this must be done while enforcing the condition that the velocity components remain constant during the time interval of interest. For this purpose, we consider the general expression for the velocity in spherical coordinates, namely (see Eq. (2.137) on p. 157 of the textbook),

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi + r\dot{\theta}\sin\phi\,\hat{u}_\theta = v_r\,\hat{u}_r + v_\phi\,\hat{u}_\phi + v_\theta\,\hat{u}_\theta. \tag{1}$$

Then, under the assumption that v_r is constant, for the radial coordinate r we have

$$v_r(0) = \dot{r}(0) = \text{constant} \quad \Rightarrow \quad r(t) = r(0) + \dot{r}(0)t, \tag{2}$$

where $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$ and that $\dot{r}(0) = 350\,\text{km/h} = 97.22\,\text{m/s}$.

Next we proceed to determine an expression for $\phi(t)$. To do so, referring to Eq. (1), we start from the consideration that $v_\phi = r\dot{\phi}$. Hence, using the expression for r in the last of Eqs. (2) we have

$$v_\phi = r\dot{\phi} = \text{constant} = r(0)\dot{\phi}(0), \quad \Rightarrow \quad \dot{\phi} = \frac{r(0)\dot{\phi}(0)}{r(0) + \dot{r}(0)t}. \tag{3}$$

Then, to determine $\phi(t)$ we now proceed to integrate the last of Eqs. (3) with respect to time. This gives

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{r(0)\dot{\phi}(0)}{r(t)} \quad \Rightarrow \quad \phi(t) - \phi(0) = \int_0^t \frac{r(0)\dot{\phi}(0)}{r(0) + \dot{r}(0)t}\,dt$$

$$\Rightarrow \quad \phi(t) = \phi(0) + \frac{r(0)\dot{\phi}(0)}{\dot{r}(0)} \ln\left(1 + \frac{\dot{r}(0)}{r(0)}t\right), \tag{4}$$

where $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$, $\dot{r}(0) = 350\,\text{km/h} = 97.22\,\text{m/s}$, $\phi(0) = 80°$, and $\dot{\phi}(0) = -0.002\,\text{rad/s}$.

Next we try and provide an expression for $\theta(t)$. We start from the fact that $v_\theta = r\dot{\theta}\sin\phi$ is assumed to be constant. Hence, we have

$$r(t)\dot{\theta}(t)\sin\phi(t) = r(0)\dot{\theta}(0)\sin\phi(0) \quad \Rightarrow \quad \dot{\theta}(t) = \frac{r(0)\dot{\theta}(0)\sin\phi(0)}{r(t)\sin\phi(t)}. \tag{5}$$

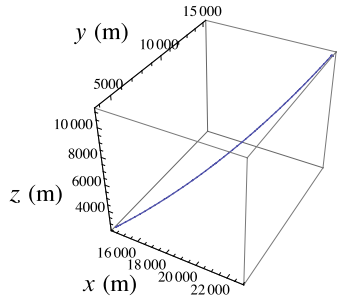
Although the expressions for $r(t)$ and $\phi(t)$ are currently known, even if one were to substitute these expressions into the above equation, we would obtain an expression for $\dot{\theta}$ that cannot be integrated with respect to time in closed-form. Hence, we must proceed to integrate with respect to time numerically. This can be done with a variety of pieces of mathematical software. We have used *Mathematica* as described below.

```
vr = 350 * 1000/3600; vϕ = -30; vϴ = 44.32; r0 = 15 000;

r[t_] := r0 + vr * t;

ϕ[t_] := vϕ/vr Log[1 + vr/r0 t] + 80 °;
```

```
θSolution = NDSolve[{θ'[t] == vθ / (r[t] Sin[ϕ[t]]), θ[0] == 15. Degree}, θ[t], {t, 0, 150}];
x[t_] := Evaluate[r[t] Sin[ϕ[t]] Cos[θ[t]] /. θSolution[[1]]]
y[t_] := Evaluate[r[t] Sin[ϕ[t]] Sin[θ[t]] /. θSolution[[1]]]
z[t_] := r[t] Cos[ϕ[t]]
ParametricPlot3D[{x[t], y[t], z[t]}, {t, 0, 150}]
```

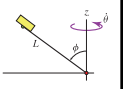


Referring to the above trajectory, the airplane is moving from the upper right to the lower left.

Problem 2.229

A carnival ride called the *octopus* consists of eight arms that rotate about the z axis at the constant angular velocity $\dot{\theta} = 6\,\text{rpm}$. The arms have a length $L = 22\,\text{ft}$ and form an angle ϕ with the z axis. Assuming that ϕ varies with time as $\phi(t) = \phi_0 + \phi_1 \sin\omega t$ with $\phi_0 = 70.5°$, $\phi_1 = 25.5°$, and $\omega = 1\,\text{rad/s}$, determine the magnitude of the acceleration of the outer end of an arm when ϕ achieves its maximum value.





Solution

Since both ϕ_0 and ϕ_1 are positive, the function $\phi(t) = \phi_0 + \phi_1 \sin\omega t$ is maximum when $\sin\omega t = 1$. Therefore, we have

$$\phi = \phi_{\max} = \phi_0 + \phi_1 = 96.00° \quad \text{for} \quad \omega t = (\pi/2)\,\text{rad}. \tag{1}$$

Now that we have determined the maximum value of the angle ϕ, we observe that to compute the magnitude of the acceleration we need to compute the component of the acceleration. In turn, given that we are using a spherical coordinate system, we recall that, referring to Eq. (2.140) on p. 157 of the textbook, the components in question are

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin^2\phi, \\ a_\phi &= r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin\phi\cos\phi, \\ a_\theta &= r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi. \end{aligned} \tag{2}$$

Therefore, to solve this problem, we must provide the values of r, $\dot{r}$, $\ddot{r}$, ϕ, $\dot{\phi}$, $\ddot{\phi}$, $\dot{\theta}$, and $\ddot{\theta}$ corresponding to $\phi = \phi_{\max} = 96.00°$. Clearly, the value of ϕ is already known, since $\phi = \phi_{\max} = 96.00°$. In addition, we have

$$r = L = 22\,\text{ft} = \text{costant} \quad \Rightarrow \quad \dot{r} = 0 \quad \text{and} \quad \ddot{r} = 0. \tag{3}$$

Also, from Eq. (1), we have that for $\phi = \phi_{\max} = 96.00°$

$$\dot{\phi} = 0 \quad \text{and} \quad \ddot{\phi} = -\phi_1\omega^2 \sin\omega t\big|_{\omega t = (\pi/2)\,\text{rad}} = 0.4451\,\text{rad/s}^2, \tag{4}$$

where we have used the fact that $\phi_1 = 25.5°$ and $\omega = 1\,\text{rad/s}$. Finally, for $\dot{\theta}$ and $\ddot{\theta}$ we have

$$\dot{\theta} = 6\,\text{rpm} = 0.6283\,\text{rad/s} = \text{constant} \quad \Rightarrow \quad \ddot{\theta} = 0. \tag{5}$$

Using the results in Eqs. (3)–(4), the a_r component of acceleration for $\phi == \phi_{\max} = 96.00°$ is

$$a_r = \ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin^2\phi \quad \Rightarrow \quad a_r = -L\dot{\theta}^2 \sin^2\phi = -8.590\,\text{ft/s}^2. \tag{6}$$

Similarly, the a_ϕ component of acceleration for $\phi = \phi_{\max} = 96.00°$ is

$$a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin\phi\cos\phi \quad \Rightarrow \quad a_\phi = -L\phi_1\omega^2 - L\dot{\theta}^2 \sin\phi\cos\phi = -8.888\,\text{ft/s}^2. \tag{7}$$

Similarly, the a_θ component of acceleration for $\phi = \phi_{\max} = 96.00°$ is

$$a_\theta = r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi \quad \Rightarrow \quad a_\theta = 0\,\text{ft/s}^2. \tag{8}$$

Now we recall that the magnitude of the acceleration for $\phi = \phi_{\max} = 96.00°$ is

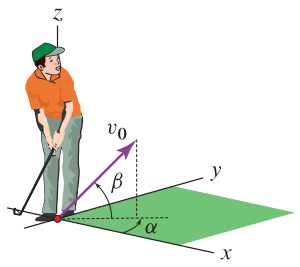
$$|\vec{a}_{\phi_{\max}}| = \sqrt{a_r^2 + a_\phi^2 + a_\theta^2} \quad \Rightarrow \quad |\vec{a}_{\phi_{\max}}| = \sqrt{\left(-L\dot{\theta}^2 \sin^2\phi\right)^2 + \left(-L\phi_1\omega^2 - L\dot{\theta}^2 \sin\phi\cos\phi\right)^2},$$

which, using the results in Eqs. (6)–(8), can be evaluated to obtain

$$\boxed{|\vec{a}_{\phi_{\max}}| = 12.4\,\text{ft/s}^2.}$$

Problem 2.230

A golfer chips the ball as shown. Letting $\alpha = 23°$, $\beta = 41°$, and the initial speed be $v_0 = 6\,\text{m/s}$, determine the x and y coordinates of the place where the ball will land.



Solution

For projectile motion $a_x = 0$, $a_y = 0$, and $a_z = -g$. Hence

$$x = x_0 + \dot{x}_0 t, \quad y = y_0 + \dot{y}_0 t, \quad \text{and} \quad z = z_0 + \dot{z}_0 t - \tfrac{1}{2} g t^2.$$

At time $t = 0$ the ball is at the origin of our coordinate system so $x_0 = y_0 = z_0 = 0$. The initial components of the velocity are

$$\dot{x}_0 = v_0 \cos\beta \cos\alpha, \quad \dot{y}_0 = v_0 \cos\beta \sin\alpha, \quad \text{and} \quad \dot{z}_0 = v_0 \sin\beta.$$

Using the equations written so far, we have that the motion of the ball is described by the following equations:

$$x = (v_0 \cos\beta \cos\alpha)t, \quad y = (v_0 \cos\beta \sin\alpha)t, \quad \text{and} \quad z = (v_0 \sin\beta)t - \tfrac{1}{2} g t^2.$$

To determine the location of landing, we observe that the z coordinate of the landing spot must be $z = 0$. Next we find the time corresponding to $z = 0$, i.e.,

$$z = (v_0 \sin\beta)t - \tfrac{1}{2} g t^2 = 0 \quad \Rightarrow \quad t = \frac{2 v_0 \sin\beta}{g}.$$

The x and y components of the position corresponding to this time are

$$x_{\text{land}} = (v_0 \cos\beta \cos\alpha)\left(\frac{2 v_0 \sin\beta}{g}\right) \quad \Rightarrow \quad x_{\text{land}} = \frac{v_0^2 \sin 2\beta \cos\alpha}{g} = 3.35\,\text{m},$$

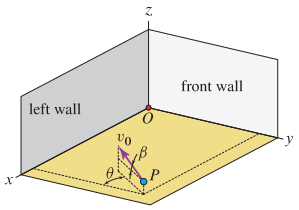
$$y_{\text{land}} = (v_0 \cos\beta \sin\alpha)\left(\frac{2 v_0 \sin\beta}{g}\right) \quad \Rightarrow \quad y_{\text{land}} = \frac{v_0^2 \sin 2\beta \sin\alpha}{g} = 1.42\,\text{m},$$

where we have used the following numerical data: $v_0 = 6\,\text{m/s}$, $\alpha = 23°$, $\beta = 41°$, and $g = 9.81\,\text{m/s}^2$. In summary, the location of the ball's landing spot is identified by the following coordinates

$$x_{\text{land}} = 3.35\,\text{m}, \quad y_{\text{land}} = 1.42\,\text{m}, \quad \text{and} \quad z_{\text{land}} = 0.$$

Problem 2.231

In a racquetball court, at point P with coordinates $x_P = 35\,\text{ft}$, $y_P = 16\,\text{ft}$, and $z_P = 1\,\text{ft}$, a ball is imparted a speed $v_0 = 90\,\text{mph}$ and a direction defined by the angles $\theta = 63°$ and $\beta = 8°$ (β is the angle formed by the initial velocity vector and the xy plane). The ball bounces off the left vertical wall to then then hit the front wall of the court. Assume that the rebound off the left vertical wall occurs such that (1) the component of the ball's velocity tangent to the wall before and after rebound is the same and (2) the component of velocity normal to the wall right after impact is equal in magnitude and opposite in direction to the same component of velocity right before impact. Accounting for the effect of gravity, determine the coordinates of the point on the front wall that will be hit by the ball after rebounding off the left wall.



Solution

From P to the left wall the racquetball undergoes projectile motion and therefore the components of the ball's acceleration in the given coordinate system are $a_x = 0$, $a_y = 0$, and $a_z = -g$. In turn, the coordinates of the ball as a function of time are

$$x = x_P + \dot{x}_0 t, \quad y = y_P + \dot{y}_0 t, \quad \text{and} \quad z = z_P + \dot{z}_0 t - \tfrac{1}{2} g t^2,$$

where $x_P = 35\,\text{ft}$, $y_P = 16\,\text{ft}$, and $z_P = 1\,\text{ft}$. Next, the initial components of velocity are

$$\dot{x}_0 = -v_0 \cos\beta \sin\theta, \quad \dot{y}_0 = -v_0 \cos\beta \cos\theta, \quad \text{and} \quad \dot{z}_0 = v_0 \sin\beta.$$

Hence, first part of the motion of the ball is described by

$$x = x_P - (v_0 \cos\beta \sin\theta)t, \quad y = y_P - (v_0 \cos\beta \cos\theta)t, \quad \text{and} \quad z = z_P + (v_0 \sin\beta)t - \tfrac{1}{2} g t^2.$$

The racquetball will impact the left wall when $y(t) = 0$. Recalling that that $v_0 = 90\,\text{mph} = 132.0\,\text{ft/s}$, $\theta = 63°$, and $\beta = 8°$, and letting t_{lw} denote the time at which the ball hits the left wall, we have

$$y\big|_{\text{left wall}} = y_P - (v_0 \cos\beta \cos\theta)\, t_{\text{lw}} = 0 \quad \Rightarrow \quad t_{\text{lw}} = \frac{y_P}{v_0 \cos\beta \cos\theta} = 0.2696\,\text{s}.$$

The corresponding x and z coordinates at $t = t_{\text{lw}}$ are

$$x_{\text{lw}} = x_P - (v_0 \cos\beta \sin\theta)\, t_{\text{lw}} \quad \Rightarrow \quad x_{\text{lw}} = 3.598\,\text{ft},$$
$$z_{\text{lw}} = z_P + (v_0 \sin\beta)\, t_{\text{lw}} - \tfrac{1}{2} g t_{\text{lw}}^2 \quad \Rightarrow \quad z_{\text{lw}} = 4.783\,\text{ft}.$$

After impact the x and y components of velocity are

$$\dot{x}_{\text{lw}} = -v_0 \cos\beta \sin\theta \quad \text{and} \quad \dot{y}_{\text{lw}} = v_0 \cos\beta \cos\theta.$$

The z component of velocity after impact is calculated with the constant acceleration equation $v = v_0 + at$.

$$\dot{z}_{\text{lw}} = v_0 \sin\beta - g t_{\text{lw}}.$$

After the impact with the left wall, the ball is projectile motion. Hence, the equations describing the motion of the ball after the impact are

$$x = x_{\text{lw}} - (v_0 \cos\beta \sin\theta)t, \quad y = (v_0 \cos\beta \cos\theta)t, \quad \text{and} \quad z = z_{\text{lw}} + (v_0 \sin\beta - g t_{\text{lw}})t - \tfrac{1}{2} g t^2,$$

where we have "reset" the time variable so that $t = 0$ now corresponds to the when the ball bounces off the the left wall. We are now ready to consider the impact of the ball with the front wall, which is described by the condition $x(t) = 0$. Letting t_{fw} denote the time at which the ball impacts the front wall, we have

$$x\big|_{\text{front wall}} = x_{\text{lw}} - (v_0 \cos\beta \sin\theta) t_{\text{fw}} = 0 \quad \Rightarrow \quad t_{\text{fw}} = \frac{x_{\text{lw}}}{v_0 \cos\beta \sin\theta} = 0.03089\,\text{s}.$$

The corresponding values of the y and z coordinates at $t = t_{\text{fw}}$ are

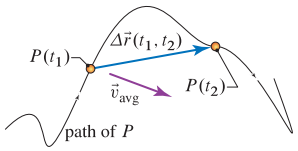
$$y_{\text{fw}} = (v_0 \cos\beta \cos\theta)\, t_{\text{fw}} \quad \Rightarrow \quad y_{\text{fw}} = 1.833\,\text{ft},$$
$$z_{\text{fw}} = z_{\text{lw}} + (v_0 \sin\beta - g t_{\text{lw}})\, t_{\text{fw}} - \tfrac{1}{2} g t_{\text{fw}}^2 \quad \Rightarrow \quad z_{\text{fw}} = 5.067\,\text{ft}.$$

In summary, the coordinates of the point on the front wall that is impacted by the ball, we have

$$\boxed{x_{\text{fw}} = 0, \quad y_{\text{fw}} = 1.83\,\text{ft}, \quad \text{and} \quad z_{\text{fw}} = 5.07\,\text{ft}.}$$

Problem 2.232

The figure shows the displacement vector of a point P between two time instants t_1 and t_2. Is it possible for the vector $\vec{v}_{\text{avg}}$ shown to be the average velocity of P over the time interval $[t_1, t_2]$?

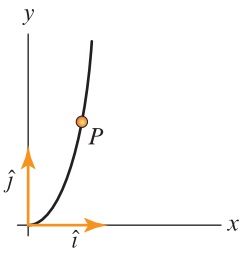


Solution

No. $\vec{v}_{\text{avg}}$ needs to have the same direction as $\Delta\vec{r}\,(t_1, t_2)$.

Problem 2.233

The motion of a point P with respect to a Cartesian coordinate system is described by $\vec{r} = \{2\sqrt{t}\,\hat{\imath} + [4\ln(t+1) + 2t^2]\,\hat{\jmath}\}$ ft, where t is time expressed in seconds. Determine the average velocity between $t_1 = 4$ s and $t_2 = 6$ s. Then find the time $\bar{t}$ for which the x component of P's velocity is *exactly* equal to the x component of P's average velocity between times t_1 and t_2. Is it possible to find a time at which P's velocity and P's average velocity are exactly equal? Explain why. *Hint:* Velocity is a vector.



Solution

Recall that the average velocity over the time interval $[t_1, t_2]$ is

$$\vec{v}_{\text{avg}}(t_1, t_2) = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}. \tag{1}$$

Using the given function for $\vec{r}$ and letting $t_1 = 4$ s and $t_2 = 6$ s, we have

$$\vec{v}_{\text{avg}}(t_1, t_2) = (0.4495\,\hat{\imath} + 20.67\,\hat{\jmath})\ \text{ft/s}, \tag{2}$$

which, expressed to 3 significant figures, gives

$$\boxed{\vec{v}_{\text{avg}}(t_1, t_2) = (0.449\,\hat{\imath} + 20.7\,\hat{\jmath})\ \text{ft/s}.}$$

Next, we compute the velocity of P by differentiating the position vector $\vec{r}(t)$ with respect to time. This gives

$$\vec{v} = \left[\frac{1}{\sqrt{t}}\,\hat{\imath} + \left(4t + \frac{4}{1+t}\right)\hat{\jmath}\right]\text{ft/s}. \tag{3}$$

To determine $\bar{t}$ we then need to solve the equation

$$v_x(\bar{t}) = \left(\frac{1}{\sqrt{\bar{t}}}\right)\text{ft/s} = 0.4495\,\text{ft/s} \quad \Rightarrow \quad \bar{t} = 4.949\,\text{s}, \tag{4}$$

which, expressed to 3 significant figures is

$$\boxed{\bar{t} = 4.95\,\text{s}.}$$

Next, we use the result in the last of Eqs. (4) to compute the y component of the velocity vector in Eq. (3). This gives

$$v_y(\bar{t}) = 20.47\,\text{ft/s} \neq (v_{\text{avg}})_y. \tag{5}$$

This result implies that

> In general, it is not possible to find a time instant in an interval $[t_1, t_2]$ for which the velocity and the average velocity are equal.

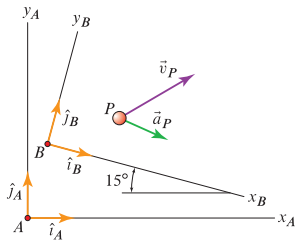
While it is always possible for a scalar function to find a value of time $\bar{t}$ at which the function is equal to its average over a time interval containing $\bar{t}$, when it comes to a vector function, finding a common time that works for every scalar component is, in general, not possible.

Problem 2.234

The velocity and acceleration of point P expressed relative to frame A at some time t are

$$\vec{v}_{P/A} = (12.5\,\hat{\imath}_A + 7.34\,\hat{\jmath}_A)\ \text{m/s} \quad \text{and} \quad \vec{a}_{P/A} = (7.23\,\hat{\imath}_A - 3.24\,\hat{\jmath}_A)\ \text{m/s}^2.$$

Knowing that frame B does not move relative to frame A, determine the expressions for the velocity and acceleration of P with respect to frame B. Verify that the speed of P and the magnitude of P's acceleration are the same in the two frames.



Solution

In order to express the velocity and acceleration vectors in terms of reference frame B, the unit vectors of frame A need to be expressed in terms of the unit vectors of frame B. Letting $\alpha = 15^\circ$ the unit vectors of frame A can be expressed as

$$\begin{aligned} \hat{\imath}_A &= \cos\alpha\,\hat{\imath}_B + \sin\alpha\,\hat{\jmath}_B, & \hat{\jmath}_1 &= -\sin\alpha\,\hat{\imath}_B + \cos\alpha\,\hat{\jmath}_B, \\ &= 0.9659\,\hat{\imath}_B + 0.2588\,\hat{\jmath}_B. & &= -0.2588\,\hat{\imath}_B + 0.9659\,\hat{\jmath}_B. \end{aligned}$$

With the frame A unit vectors known in terms of those of frame B, it is found that the velocity in frame B is

$$\vec{v}_{P/B} = \Big[12.5(0.9659\,\hat{\imath}_2 + 0.2588\,\hat{\jmath}_2)\Big], \text{m/s} + \Big[7.34(-0.2588\,\hat{\imath}_2 + 0.9659\,\hat{\jmath}_2)\Big]\,\text{m/s},$$

$$\boxed{\vec{v}_{P/B}\,(10.2\,\hat{\imath}_2 + 10.3\,\hat{\jmath}_2)\ \text{m/s}.}$$

Similarly, the acceleration vector in frame 2 is found to be

$$\vec{a}_{P/B} = \Big[7.23(0.9659\,\hat{\imath}_2 + 0.2588\,\hat{\jmath}_2)\Big]\,\text{m/s}^2 - \Big[3.24(-0.2588\,\hat{\imath}_2 + 0.9659\,\hat{\jmath}_2)\Big]\,\text{m/s}^2,$$

$$\boxed{\vec{a}_{P/B} = (7.82\,\hat{\imath}_2 - 1.26\,\hat{\jmath}_2)\ \text{m/s}^2.}$$

The speed of the particle in frame A is found by taking the magnitude of the velocity vector $\vec{v}_{P/A}$ where

$$v_{P/A} = \sqrt{(12.5)^2 + (7.34)^2}\ \text{m/s} = 14.5\,\text{m/s},$$

Similarly, for frame B,

$$v_{P/B} = \sqrt{(10.2)^2 + (10.3)^2}\ \text{m/s} = 14.5\,\text{m/s}.$$

The magnitue of the acceleration in frame A is found by taking the magnitude of the acceleration vector $\vec{a}_{P/A}$ where

$$a_{P/A} = \sqrt{(7.23)^2 + (-3.24)^2}\ \text{m/s}^2 = 7.92\,\text{m/s}^2,$$

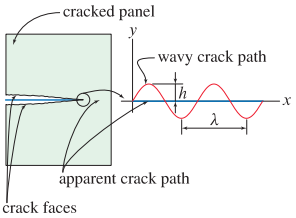
Similarly, for frame B,

$$a_{P/B} = \sqrt{(7.82)^2 + (-1.26)^2}\ \text{m/s}^2 = 7.92\,\text{m/s}^2.$$

Therefore, the magnitude of the acceleration and velocity vectors are verified to be independent of the reference frame.

Problems 2.235 and 2.236

A dynamic fracture model proposed to explain the behavior of cracks propagating at high velocity views the crack path as a *wavy path*. In this model, a crack tip appearing to travel along a straight path actually travels at roughly the speed of sound along a wavy path. Let the wavy path of the crack tip be described by the function $y = h\,\sin(2\pi x/\lambda)$, where h is the amplitude of the crack tip fluctuations in the direction perpendicular to the crack plane and λ is the corresponding period. Assume that the crack tip travels along the wavy path at a constant speed v_s (e.g., the speed of sound).



Problem 2.235 Find the expression for the x component of the crack tip velocity as a function of v_s, λ, h, and x.

Problem 2.236 Denote the *apparent* crack tip velocity by v_a, and define it as the average value of the x component of the crack velocity, that is,

$$v_a = \frac{1}{\lambda}\int_0^{\lambda} v_x\,dx.$$

In dynamic fracture experiments on polymeric materials, $v_a = 2v_s/3$, v_s is found to be close to 800 m/s, and λ is of the order of 100 μm. What value of h would you expect to find in the experiments if the wavy crack theory were confirmed to be accurate?

Solution to 2.235

Using the coordinate system shown, the velocity of the crack tip has the form $\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$. Hence, letting v_s be the speed of the crack, we must have

$$v_s = \sqrt{\dot{x}^2 + \dot{y}^2}. \tag{1}$$

Now recalling that $y = h\,\sin(2\pi x/\lambda)$, we have

$$\dot{y} = \frac{dy}{dx}\frac{dx}{dt} \quad \Rightarrow \quad \dot{y} = \frac{2h\pi\dot{x}}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right). \tag{2}$$

Substituting Eq. (2) into Eq. (1) we have

$$v_s = \sqrt{\dot{x}^2 + \left[\frac{2h\pi\dot{x}}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2} = \pm\dot{x}\sqrt{1 + \left[\frac{2h\pi}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2}, \tag{3}$$

which can be solved to find that the x component of the velocity is

$$\boxed{\dot{x} = \frac{v_s\lambda}{\sqrt{\lambda^2 + 4h^2\pi^2\cos^2\left(\frac{2\pi x}{\lambda}\right)}},} \tag{4}$$

where we have chosen the positive solution will be used since the crack is assumed to propagate in the positive x direction.

Solution to 2.236

To solve this problem we need to first determine the x component of the velocity of the crack tip. Once, we determine an expression for v_x, then we will solve the equation stated in the problem numerically for h.

We begin by observing that, using the coordinate system shown, the velocity of the crack tip has the form $\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$. Hence, letting v_s be the speed of the crack, we must have

$$v_s = \sqrt{\dot{x}^2 + \dot{y}^2}. \tag{5}$$

Now recalling that $y = h\sin(2\pi x/\lambda)$, we have

$$\dot{y} = \frac{dy}{dx}\frac{dx}{dt} \quad \Rightarrow \quad \dot{y} = \frac{2h\pi\dot{x}}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right). \tag{6}$$

Substituting Eq. (6) into Eq. (5) we have

$$v_s = \sqrt{\dot{x}^2 + \left[\frac{2h\pi\dot{x}}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2} = \pm\dot{x}\sqrt{1 + \left[\frac{2h\pi}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2}, \tag{7}$$

which can be solved to find that the x component of the velocity is

$$v_x = \dot{x} = \frac{v_s\lambda}{\sqrt{\lambda^2 + 4h^2\pi^2\cos^2\left(\frac{2\pi x}{\lambda}\right)}}, \tag{8}$$

where we have chosen the positive solution will be used since the crack is assumed to propagate in the positive x direction.

$$v_a = \frac{1}{\lambda}\int_0^\lambda \frac{v_s\lambda}{\sqrt{\lambda^2 + 4h^2\pi^2\cos^2\left(\frac{2\pi x}{\lambda}\right)}}\,dx = \int_0^\lambda \frac{v_s}{\sqrt{\lambda^2 + 4h^2\pi^2\cos^2\left(\frac{2\pi x}{\lambda}\right)}}\,dx. \tag{9}$$

Recall that we have $v_s = 800\,\text{m/s}$, $v_a = \frac{2}{3}v_s = 533.3\,\text{m/s}$, and $\lambda = 100\,\mu\text{m}$. Hence, since the variable x in Eq. (9) has the role of dummy variable of integration, Eq. (9) is an equation in the single unknown h. The equation in question is an integral equation and can be solved numerically using appropriate mathematical software. We have used *Mathematica* with the code given below. As is often the case with the numerical solution of equations, we had to supply *Mathematica* with an initial guess for the solution. Since the quantity h is expected to be of the same order of magnitude as λ, our guess for h was set to the same value as the given value for λ.

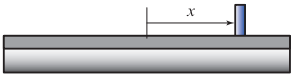
```
EqIntegral[h_] := NIntegrate[ 800. / Sqrt[(100. * 10^-6)^2 + 4 h^2 π^2 Cos[2 π x / (100. * 10^-6)]^2],
  {x, 0, (100. * 10^-6)}]
FindRoot[533.3 == EqIntegral[h], {h, 10.^-4}]
```

The execution of the code above yields the following result (expressed to 3 significant figures):

$$h = 29.4\times10^{-6}\,\text{m}.$$

Problem 2.237

The motion of a peg sliding within a rectilinear guide is controlled by an actuator in such a way that the peg's acceleration takes on the form $\ddot{x} = a_0(2\cos 2\omega t - \beta \sin \omega t)$, where t is time, $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$. Determine the total distance traveled by the peg during the time interval $0\,\text{s} \le t \le 5\,\text{s}$ if $\dot{x}(0) = a_0\beta/\omega + 0.3\,\text{m/s}$. When compared with Prob. 2.45, why does the addition of 0.3 m/s in the initial velocity turn this into a problem that requires a computer for the solution?



Solution

We have acceleration as a function of time so we can integrate it as

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \ddot{x}(t)\,dt = a_0\frac{\beta}{\omega} + 0.3\,\text{m/s} + a_0\left(\frac{2}{2\omega}\sin 2\omega t + \frac{\beta}{\omega}\cos\omega t\right) - a_0\frac{\beta}{\omega}.$$

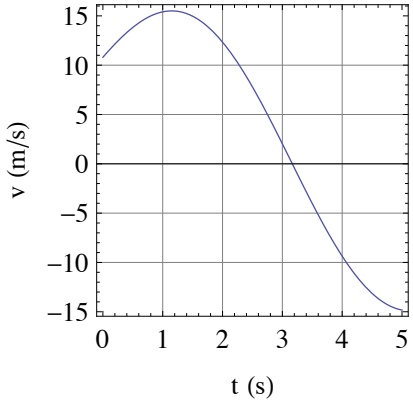
We can use the double angle formula, $\sin 2x = 2\sin x\cos x$, to obtain

$$v(t) = \frac{a_0}{\omega}\left[\sin 2\omega t + \beta\cos\omega t\right] + 0.3\,\text{m/s} \quad \Rightarrow \quad v(t) = \frac{a_0}{\omega}\cos\omega t(2\sin\omega t + \beta) + 0.3\,\text{m/s}.$$

To find the distance traveled, we need to established if and when the peg switches direction during the time interval considered. This can be easily done by using any appropriate mathematical software that can plot the function $v(t)$ over the time interval considered. Recalling that $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$, we have used *Mathematica* with the following code:

```
Parameters = {a0 → 3.5, ω → 0.5, β → 1.5};
Plot[a0/ω Cos[ω t] (2 Sin[ω t] + β) + 0.3 /. Parameters, {t, 0, 5}, Frame → True,
 FrameLabel → {"t (s)", "v (m/s)"}, GridLines → Automatic, AspectRatio → 1]
```

The above code yields the following plot:



As can be seen from the above plot, the velocity changes sign near $t - 3\,\text{s}$. Hence, we need to solve the equation $v(t) = 0$ to determine where exactly the sign switch occurs. Because the expression we have for $v(t)$ includes the constant term 0.3 m, the equation $v(t) = 0$ cannot be solved analytically. Hence, we must use a computer to find the times when velocity goes to zero. This can be done with any appropriate

mathematical software. As it is common with root finding algorithms, we need to provide a guess for the solution. Based on the graph presented above, we set our guess to $t = 3\,\text{s}$ and then use the following *Mathematica* code:

```
Parameters = {a0 → 3.5, ω → 0.5, β → 1.5};
FindRoot[a0/ω Cos[ω t] (2 Sin[ω t] + β) + 0.3 == 0 /. Parameters, {t, 3}]
```

which yields the following solution (expressed to 4 significant figures):

$$t = 3.166\,\text{s}. \tag{1}$$

to find the root numerically at 3.166 s. Using this result, and letting d denote the distance traveled, we have

$$d = \int_0^{3.166\,\text{s}} \left[\frac{a_0}{\omega}\cos\omega t\,(2\sin\omega t+\beta)+0.3\,\text{m/s}\right] dt - \int_{3.166\,\text{s}}^{5\,\text{s}} \left[\frac{a_0}{\omega}\cos\omega t\,(2\sin\omega t+\beta)+0.3\,\text{m/s}\right] dt. \tag{2}$$

The above integrals can be computed either numerically or analytically. Since this is a computer problem, we have chosen to carry out the integration numerically. As usual, this can be done with any appropriate numerical software. we have used *Mathematica* with the following code:

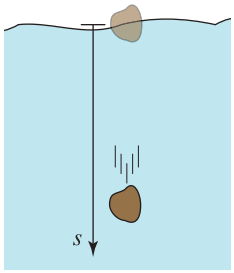
```
Parameters = {a0 → 3.5, ω → 0.5, β → 1.5};
d = NIntegrate[a0/ω Cos[ω t] (2 Sin[ω t] + β) + 0.3 /. Parameters, {t, 0, 3.166}] -
  NIntegrate[a0/ω Cos[ω t] (2 Sin[ω t] + β) + 0.3 /. Parameters, {t, 3.166, 5}]
```

The execution of the above code, yields the following result (expressed to 3 significant figures):

$$\boxed{d = 52.8\,\text{m}.}$$

Problem 2.238

The acceleration of an object in rectilinear free fall while immersed in a linear viscous fluid is $a = g - C_d v/m$, where g is the acceleration of gravity, C_d is a constant drag coefficient, v is the object's velocity, and m is the object's mass. Letting $v = 0$ and $s = 0$ for $t = 0$, where s is position and t is time, determine the position as a function of time.



Solution

We must integrate the acceleration which is a function of velocity. If we let $t_0 = 0$ and $s_0 = 0$, it follows that

$$a = \frac{dv}{dt} = g - C_d v/m \quad \Rightarrow \quad t = \int_0^v \frac{dv}{g - C_d v/m} \quad \Rightarrow \quad t = \frac{-m}{C_d} \ln\left(1 - \frac{C_d}{mg} v\right).$$

We invert $t(v)$ to find velocity as a function of time.

$$1 - \frac{C_d}{mg} v = e^{\frac{-C_d}{m} t} \quad \Rightarrow \quad v(t) = \frac{mg}{C_d}\left(1 - e^{\frac{-C_d}{m} t}\right).$$

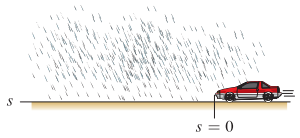
Now that we have velocity as a function of time we can integrate it to obtain position as a function of time.

$$v = \frac{ds}{dt} \quad \Rightarrow \quad s = \frac{mg}{C_d} \int_0^t 1 - e^{\frac{-C_d}{m} t}\, dt \quad \Rightarrow$$

$$\boxed{s(t) = \frac{mg}{C_d^2}\left[C_d t + m\left(e^{\frac{-C_d}{m} t} - 1\right)\right].}$$

Problem 2.239

Heavy rains cause a particular stretch of road to have a coefficient of friction that changes as a function of location. Under these conditions the acceleration of a car skidding while trying to stop can be approximated by $\ddot{s} = -(\mu_k - cs)g$, where μ_k is the friction coefficient under dry conditions, g is the acceleration of gravity, and c, with units of m^{-1}, describes the rate of friction decrement. Let $\mu_k = 0.5$, $c = 0.015\,\text{m}^{-1}$, and $v_0 = 45\,\text{km/h}$, where v_0 is the initial velocity of the car. Determine the time it will take the car to stop and the percentage of increase in stopping time with respect to dry conditions, i.e., when $c = 0$.



Solution

Since we are given an acceleration as a function of position, we will first relate the acceleration to the velocity using the chain rule, and then relate the velocity to time. We therefore proceed as follows:

$$a(s) = \frac{d\dot{s}}{ds}\frac{ds}{dt} \quad \Rightarrow \quad a(s) = \frac{d\dot{s}}{ds}\dot{s} \quad \Rightarrow \quad \int_{v_0}^{v} \dot{s}\, d\dot{s} = \int_0^s a\, ds.$$

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \frac{g}{2c}(cs - \mu_k)^2\Big|_0^s \quad \Rightarrow \quad v = \sqrt{\frac{g}{c}}\sqrt{\frac{c}{g}v_0^2 + (cs - \mu_k)^2 - \mu_k^2}. \tag{1}$$

We need this expression later. For now set $v = 0$ and solve for s.

$$s_f = \frac{\mu_k}{c} \pm \frac{1}{c}\sqrt{\mu_k^2 - \frac{cv_0^2}{g}} \quad \Rightarrow \quad s_f = 26.31\,\text{m} \quad \text{or} \quad 40.35\,\text{m}.$$

$s_f = 26.31$ m is the meaningful solution because it occurs first, where the subscript f stands for final.

We now go back to the velocity, given by Eq. (1), as we relate it to time as follows:

$$v(s) = \frac{ds}{dt} \quad \Rightarrow \quad \int_0^t dt = \int_0^s \frac{ds}{v} \quad \Rightarrow \quad t = \sqrt{\frac{c}{g}}\int_0^s \frac{ds}{\sqrt{\frac{c}{g}v_0^2 + (cs - \mu_k)^2 - \mu_k^2}}. \tag{2}$$

We now perform the following change of variables of integration:

$$A = \frac{c}{g}v_0^2 - \mu_k^2, \qquad x = cs - \mu_k \quad \Rightarrow \quad dx = c\, ds \quad \Rightarrow \quad ds = \frac{1}{c}dx. \tag{3}$$

For $s = 0$ we have $x = -\mu_k$ and for $s = s_f$ we have $x = cs_f - \mu_k$. Hence, the integral in the last of Eqs. (2) becomes

$$t_f = \frac{1}{\sqrt{cg}}\int_{-\mu_k}^{cs_f - \mu_k} \frac{dx}{\sqrt{A + x^2}}. \tag{4}$$

Using a table of integrals or some appropriate mathematical software we obtain

$$t_f = \frac{1}{\sqrt{cg}}\ln\left(x + \sqrt{A + x^2}\right)\Big|_{-\mu_k}^{cs_f - \mu_k} \quad \Rightarrow \quad t_f = \frac{1}{\sqrt{cg}}\ln\frac{\left(x + \sqrt{A + x^2}\right)\Big|_{x = cs_f - \mu_k}}{\left(x + \sqrt{A + x^2}\right)\Big|_{x = -\mu_k}},$$

which, recalling the definition of A in the first of Eqs. (3), after after simplification yields

$$t_f = \frac{1}{\sqrt{cg}}\ln\frac{cs_f - \mu_k}{v_0\sqrt{c/g} - \mu_k} = 5.840\,\text{s}. \tag{5}$$

where we have used the following numerical data: $c = 0.015\,\text{m}^{-1}$, $g = 9.81\,\text{m/s}^2$, $\mu_k = 0.5$, and $v_0 = 45\,\text{km/s} = 12.50\,\text{m/s}$. Expressing t_f to 3 significant figures, we have

$$\boxed{t_f = 5.84\,\text{s}.}$$

With dry conditions, i.e., for $c = 0$, the acceleration of the car would be $\ddot{s} = -\mu_k g$. Hence, we can use the constant acceleration equation $\dot{s} = \dot{s}_0 + a_c t$ to find the time. This gives

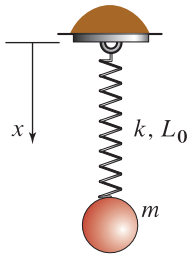
$$0 = v_0 - \mu_k g t_f \quad \Rightarrow \quad t_f = 2.548\,\text{s}.$$

Comparing this result to that in Eq. (5), we find the percent increase to be

$$\frac{t_{f_{\text{wet}}} - t_{f_{\text{dry}}}}{t_{f_{\text{dry}}}} \times 100 \quad \Rightarrow \quad \boxed{129\%}$$

Problem 2.240

The acceleration of a particle of mass m suspended by a linear spring with spring constant k and unstretched length L_0, (when the spring length is equal to L_0, the spring exerts no force on the particle) is given by $\ddot{x} = g - (k/m)(x - L_0)$. Assuming that at $t = 0$ the particle is at rest and its position is $x = 0$ m, derive the expression of the particle's position x as a function of time. *Hint:* A good table of integrals will come in handy.



Solution

We have acceleration as a function of position and we integrate it as follows:

$$a = \frac{d\dot{x}}{dx}\frac{dx}{dt} \quad \Rightarrow \quad \int_0^{\dot{x}} \dot{x}\, d\dot{x} = \int_0^x \left[g - \frac{k}{m}(x - L_0)\right] dx.$$

$$\tfrac{1}{2}\dot{x}^2 = gx - \frac{k}{2m}x^2 + \frac{kL_0}{m}x \quad \Rightarrow \quad \dot{x} = \sqrt{2gx - \frac{k}{m}x^2 + \frac{2kL_0}{m}x}.$$

Now we have $\dot{x}$ as a function of x and we therefore relate it to time as follows:

$$\dot{x} = \frac{dx}{dt} \quad \Rightarrow \quad \int_0^t dt = \int_0^x \frac{dx}{\dot{x}} \quad \Rightarrow \quad t = \int_0^x \frac{dx}{\sqrt{x\left(\frac{2kL_0}{m} + 2g\right) - \frac{k}{m}x^2}}.$$

Now we let $A = \frac{m}{k}\left(\frac{2kL_0}{m} + 2g\right) = 2\left(L_0 + \frac{gm}{k}\right)$ and we rewrite the expression for t as follows:

$$t = \sqrt{\frac{m}{k}}\int_0^x \frac{dx}{\sqrt{Ax - x^2}} \quad \Rightarrow \quad t = 2\sqrt{\frac{m}{k}}\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{A - x}}\right)\Bigg|_0^x.$$

Making use of the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$, we have that

$$\tan^2\left(\frac{t}{2}\sqrt{\frac{k}{m}}\right) = \frac{x}{A - x} \quad \Rightarrow \quad x\left[\sec^2\left(\frac{t}{2}\sqrt{\frac{k}{m}}\right)\right] = A\tan^2\left(\frac{t}{2}\sqrt{\frac{k}{m}}\right).$$

Now making use of the trigonometric identity $\sin^2(\theta/2) = \frac{1}{2}(1 - \cos\theta)$, we can express x as

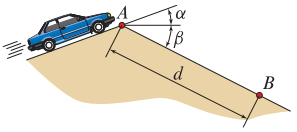
$$x = A\sin^2\left(\frac{t}{2}\sqrt{\frac{k}{m}}\right) \quad \Rightarrow \quad x = \frac{A}{2}\left[1 - \cos\left(\sqrt{\frac{k}{m}}\,t\right)\right].$$

Finally, recalling that A is the quantity $2\left(L_0 + \frac{gm}{k}\right)$, the expression for x can be written as

$$\boxed{x = \left(L_0 + \frac{gm}{k}\right)\left[1 - \cos\left(\sqrt{\frac{k}{m}}\,t\right)\right].}$$

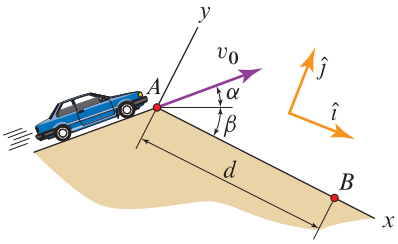
Problem 2.241

In a movie scene involving a car chase, a car goes over the top of a ramp at A and lands at B below. Letting $\alpha = 18°$ and $\beta = 25°$, determine the speed of the car at A if the car is to be airborne for a full 3 s. Furthermore, determine the distance d covered by the car during the stunt as well as the impact speed and angle at B. Neglect aerodynamic effects. Express your answer using the U.S. Customary system of units.



Solution

This problem is a projectile problem and we begin by relating the jump speed v_0 to the velocity of the car and then to the car's trajectory. Referring tot he figure on the right, the initial velocity and acceleration in the (x, y) coordinate system are



$$\dot{x}_0 = v_0 \cos(\alpha + \beta), \qquad \dot{y}_0 = v_0 \sin(\alpha + \beta),$$
$$\ddot{x} = g \sin\beta, \qquad \ddot{y} = -g \cos\beta.$$

Using the constant acceleration equation $s = \dot{s}t + \frac{1}{2}a_c t^2$, we have that the coordinates of the (airborne) car as a function of time are

$$x = v_0 \cos(\alpha + \beta)t + \tfrac{1}{2} g \sin\beta\, t^2, \tag{1}$$
$$y = v_0 \sin(\alpha + \beta)t - \tfrac{1}{2} g \cos\beta\, t^2. \tag{2}$$

We want to satisfy the condition that $y = 0$ at $t = t_f = 3$ s, where the subscript f stands for flight.

$$0 = v_0 \sin(\alpha + \beta)t_f - \tfrac{1}{2} g \cos\beta\, t_f^2 \quad \Rightarrow \quad v_0 = \frac{g t_f \cos\beta}{2\sin(\alpha + \beta)} = 64.19\,\text{ft/s}, \tag{3}$$

where we have used the following numerical data: $g = 32.2\,\text{ft/s}^2$, $t_f = 3$ s, $\beta = 25°$, and $\alpha = 18°$. Expressing the result for v_0 to 3 significant figures, we have

$$\boxed{v_0 = 64.2\,\text{ft/s}.}$$

We now observe that the distance d is equal to the value of x at time $t = t_f$. Hence, using Eq. (1) and the value of v_0 we computed in the last of Eqs. (3), we have that the distance d is given by

$$\boxed{d = \frac{g t_f \cos\beta}{2\sin(\alpha + \beta)} \cos(\alpha + \beta)t_f + \tfrac{1}{2} g \sin\beta\, t_f^2 = 202\,\text{ft}.}$$

To find the impact speed and the impact angle, we need to determine the velocity at impact. To do so, we use the constant acceleration equation $v = v_0 + a_c t$ and we obtain

$$\vec{v}_i = [v_0 \cos(\alpha + \beta) + g \sin\beta\, t_f]\,\hat{\imath} + [v_0 \sin(\alpha + \beta) - g \cos\beta\, t_f]\,\hat{\jmath} = (87.77\,\hat{\imath} - 43.77\,\hat{\jmath})\,\text{ft/s}, \tag{4}$$

where we have used the value of v_0 in the last of Eqs. (3) along with the following numerical data: $g = 32.2\,\text{ft/s}^2$, $t_f = 3$ s, $\beta = 25°$, and $\alpha = 18°$. Therefore the speed at impact is given by

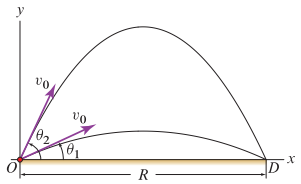
$$\boxed{v_i = \sqrt{v_{xi}^2 + v_{yi}^2} = 98.1\,\text{ft/s}.}$$

We now let ϕ be the impact angle of the car with the ground at B, i.e., with respect to the slope. Hence, this angle is given by

$$\phi = \tan^{-1} \frac{-v_y}{v_x} = 26.5^\circ \text{ measured clockwise from the slope.}$$

Problem 2.242

Consider the problem of launching a projectile a distance R from O to D with a known launch speed v_0. It is probably clear to you that you also need to know the launch angle θ if you want the projectile to land exactly at R. But it turns out that the condition determining whether or not v_0 is large enough to get to R does not depend on θ. Determine this condition on v_0. *Hint:* Find v_0 as a function of R and θ, and then remember that the sine function is bounded by 1.



Solution

For projectile motion we have the initial velocity and acceleration in the (x, y) coordinate system, i.e.,

$$\dot{x}_0 = v_0 \cos\theta, \qquad \dot{y}_0 = v_0 \sin\theta,$$
$$\ddot{x} = 0, \qquad \ddot{y} = -g.$$

Using the constant acceleration equation $s = \dot{s}t + 0.5a_c t^2$ we obtain

$$x = v_0 \cos\theta\, t, \tag{1}$$
$$y = v_0 \sin\theta\, t - \tfrac{1}{2} g t^2. \tag{2}$$

For $t = t_D$ we have $y(t_D) = 0$. Using this fact in Eq. (2), yields the following expression for t_D:

$$t_D = \frac{2v_0 \sin\theta}{g}.$$

The substituting the above expression for t_D into Eq. (1), setting $x = R$ and using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$, we obtain

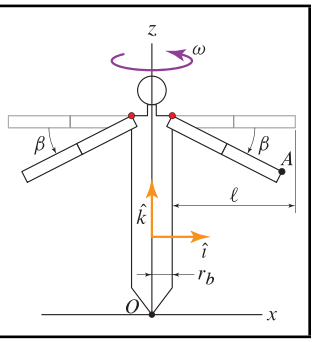
$$R = \frac{v_0^2 \sin 2\theta}{g} \quad \Rightarrow \quad v_0^2 = \frac{gR}{\sin 2\theta}.$$

The smallest value of v_0 corresponds to $\sin 2\theta = 1$. Therefore the condition on v_0 is

$$\boxed{v_0 \geq \sqrt{gR}.}$$

Problem 2.243

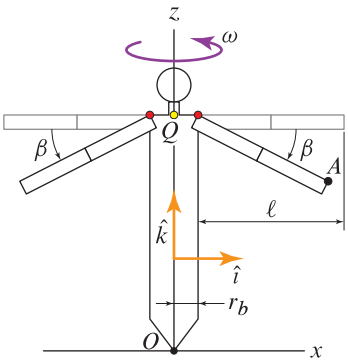
A skater is spinning with her arms completely stretched out and with an angular velocity $\omega = 60\,\text{rpm}$. Letting $r_b = 0.55\,\text{ft}$, and $\ell = 2.2\,\text{ft}$ and neglecting the change in ω as the skater lowers her arms, determine the velocity and acceleration of the hand A right when $\beta = 0°$ if the skater lowers her arms at the constant rate $\dot{\beta} = 0.2\,\text{rad/s}$. Express the answers using the component system shown, which rotates with the skater and for which the unit vector $\hat{\jmath}$ (not shown) is such that $\hat{\jmath} = \hat{k} \times \hat{\imath}$.



Solution

We begin by providing a convenient description of the position of point A. Referring to the figure on the right, let Q be the *fixed* point on the spin axis that is at the same height as the shoulders. Then the position of A relative to Q is given by

$$\vec{r}\,A/Q = (r_b + \ell\cos\beta)\,\hat{\imath} - \ell\sin\beta\,\hat{k}.$$



Since Q is a fixed point, the time derivative of $\vec{r}_{A/Q}$ coincides with the velocity of point A. Hence, we have

$$\vec{v}_A = -\ell\dot{\beta}\sin\beta\,\hat{\imath} + (r_b + \ell\cos\beta)\,\dot{\hat{\imath}} - \ell\dot{\beta}\cos\beta\,\hat{k} - \ell\sin\beta\,\dot{\hat{k}}. \qquad (1)$$

Now we observe that since the unit vector $\hat{\imath}$ always points towards A, then is changes its orientation with the angular velocity $\vec{\omega} = \omega\,\hat{k}$. In addition, since $\hat{k}$ does not change its orientation, then $\hat{k}$ is a constant unit vector. Hence, using the concept of time derivative of a unit vector, we have

$$\dot{\hat{\imath}} = \omega\,\hat{k} \times \hat{\imath} = \omega\,\hat{\jmath} \quad \text{and} \quad \dot{\hat{k}} = \vec{0}. \qquad (2)$$

Substituting Eqs. (2) into Eq. (1), we the have that the velocity of A can be expressed as follows:

$$\vec{v}_A = -\ell\dot{\beta}\sin\beta\,\hat{\imath} + (r_b + \ell\cos\beta)\omega\,\hat{\jmath} - \ell\dot{\beta}\cos\beta\,\hat{k}. \qquad (3)$$

For $\beta = 0$, we then have

$$\boxed{\vec{v}_A\big|_{\beta=0} = (r_b + \ell)\omega\,\hat{\jmath} - \ell\dot{\beta}\,\hat{k} = \big(17.3\,\hat{\jmath} - 0.44\,\hat{k}\big)\,\text{ft/s}.}$$

TO obtain the acceleration, we now differentiate $\vec{v}_A$ in Eq. (3) with respect to time and, recalling that $\dot{\beta}$ and $\hat{k}$ are constant, we obtain

$$\vec{a}_A = -\ell\dot{\beta}^2\cos\beta\,\hat{\imath} - \ell\dot{\beta}\sin\beta\,\dot{\hat{\imath}} - \ell\dot{\beta}\omega\sin\beta\,\hat{\jmath} + (r_b + \ell\cos\beta)\omega\,\dot{\hat{\jmath}} + \ell\dot{\beta}^2\sin\beta\,\hat{k}. \qquad (4)$$

Using the first of Eqs. (2) and observing that the same idea that allowed us to derive the first of Eqs. (2) tells us that $\dot{\hat{\jmath}} = \omega\,\hat{k} \times \hat{\jmath} = -\omega\,\hat{\imath}$, Eq. (4) can be simplified to

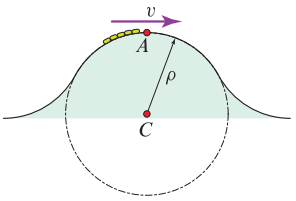
$$\vec{a}_A = -\big[\ell\dot{\beta}^2\cos\beta + \omega^2(r_b + \ell\cos\beta)\big]\,\hat{\imath} - 2\omega\ell\dot{\beta}\sin\beta\,\hat{\jmath} + \ell\dot{\beta}^2\sin\beta\,\hat{k}. \qquad (5)$$

Hence, for $\beta = 0$ the acceleration of A is

$$\boxed{\vec{a}_A\big|_{\beta=0} = \big[-\ell\dot{\beta}^2 - \omega^2(r_b + \ell)\big]\hat{\imath} = (-109\,\hat{\imath})\,\text{ft/s}^2.}$$

Problem 2.244

A roller coaster travels over the top A of the track section shown with a speed $v = 60\,\text{mph}$. Compute the largest radius of curvature ρ at A such that the passengers on the roller coaster will experience weightlessness at A.



Solution

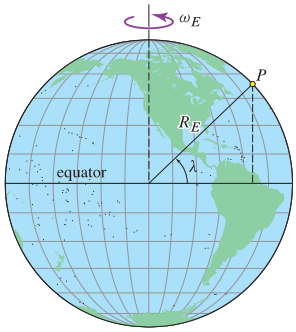
For this condition the centripetal acceleration must be entirely from gravity.

$$\frac{v^2}{\rho} = g \quad \Rightarrow \quad \rho = \frac{v^2}{g} \quad \Rightarrow \quad \boxed{\rho = 240\,\text{ft}}$$

where we have used the fact that $v = 60\,\text{mph} = 88.00\,\text{ft/s}$ and $g = 32.2\,\text{ft/s}^2$.

Problem 2.245

Determine, as a function of the latitude λ, the normal acceleration of the point P on the surface of the Earth due to the spin ω_E of Earth about its axis. In addition, determine the normal acceleration of the Earth due to its rotation about the Sun. Using these results, determine the latitude above which the acceleration due to the orbital motion of the Earth is more significant than the acceleration due to the spin of the Earth about its axis. Use $R_E = 6371$ km for the mean radius of the Earth, and assume the Earth's orbit about the Sun is circular with radius $R_O = 1.497\times10^8$ km.



Solution

The tangential velocity of P due to the rotation of Earth is $v_P = R_E\omega_E\cos\lambda$. The radius of curvature as a function of λ is $\rho_P = R_E\cos\lambda$. The normal acceleration of P due to rotation is

$$a_P = \frac{v_P^2}{\rho_P} = \frac{\omega_E^2 R_E^2\cos^2\lambda}{R_E\cos\lambda} \quad \Rightarrow \quad \boxed{\omega_E^2 R_E\cos\lambda,} \tag{1}$$

where $R_E = 6371\,\text{km} = 6371\times10^3$ m and

$$\omega_E = (1\,\text{rev/day})(2\pi\,\text{rad/rev})(24\,\text{h/day})^{-1}(3600\,\text{s/h})^{-1} = 72.72\times10^{-6}\,\text{rad/s}. \tag{2}$$

The tangential velocity of P due to the motion of the Earth arounf the Sun is $v_E = R_O\omega_O$, where

$$\omega_O = \frac{(1\,\text{rev/year})(2\pi\,\text{rad/rev})}{(365\,\text{day/year})(24\,\text{h/day})(3600\,\text{s/h})} = 1.992\times10^{-7}\,\text{rad/s}. \tag{3}$$

Hence, the normal acceleration of P due to to the motion of the Earth around the Sun is

$$a_O = \frac{v_E^2}{\rho_O} = \frac{\omega_O^2 R_O^2}{R_O} \quad \Rightarrow \quad \boxed{a_O = \omega_O^2 R_O = 5.94\times10^{-3}\,\text{m/s}^2,}$$

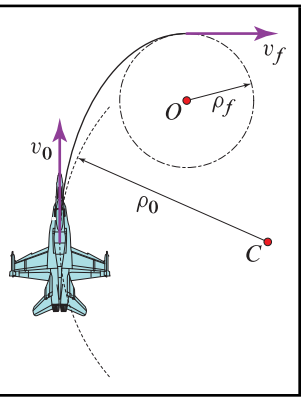
where we have used the fact that $R_O = 1.497\times10^8$ km $= 1497\times10^8$ m. Consequently, the latitude λ above which the acceleration due to the motion of the Earth around the Sun is greater than the acceleration of due to the Earth's spin about its own axis is

$$\boxed{\lambda = \cos^{-1}\left(\frac{\omega_O^2 R_O}{\omega_E^2 R_E}\right) = 79.8°,}$$

where we have used the fact that $R_O = 1.497\times10^8$ km $= 1497\times10^8$ m, $R_E = 6371$ km $= 6{,}371{,}000$ m, and the values of ω_E and ω_O in Eqs. (2) and (3), respectively.

Problem 2.246

A jet is flying straight and level at a speed $v_0 = 1100\,\text{km/h}$ when it turns to change its course by 90° as shown. The turn is performed by decreasing the path's radius of curvature uniformly as a function of the position s along the path while keeping the normal acceleration constant and equal to $8g$, where g is the acceleration due to gravity. At the end of the turn, the speed of the plane is $v_f = 800\,\text{km/h}$. Determine the radius of curvature ρ_f at the end of the turn and the time t_f that the plane takes to complete its change in course.



Solution

Recalling that we are given $a_n = 8g = 78.48\,\text{m/s}^2$ (since $g = 9.81\,\text{m/s}^2$), $v_0 = 1100\,\text{km/h} = 305.6\,\text{m/s}$, and $v_f = 800\,\text{km/s} = 222.2\,\text{m/s}$, the radii of curvature at the beginning and end of the turn are

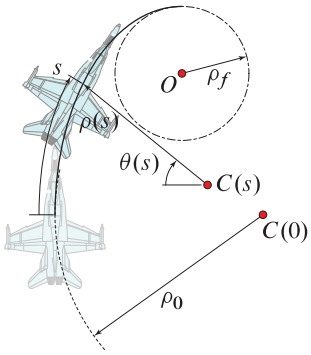
$$\rho_0 = \frac{v_0^2}{a_n} = 1190\,\text{m} \quad \text{and} \quad \rho_f = \frac{v_f^2}{a_n} = 629.1\,\text{m}. \tag{1}$$

Expressing the value of ρ_f to 3 significant figures, we have

$$\boxed{\rho_f = 629\,\text{m}.}$$

Next, recalling that we are told that ρ decreases uniformly with the position s along the airplane's path, i.e., that $d\rho/ds = \text{constant}$, denoting the constant in question by γ, we have that the radius of curvature, expressed as a function of s must have the following form

$$\rho(s) = \rho_0 + \gamma s, \tag{2}$$



where, referring to the figure on the right, we have set $s = 0$ to correspond to the location of the airplane along the path at the beginning of the turn, and where the nondimensional constant γ will be determined later. Now, we recall that that, at every point along the airplane's trajectory, we can define an osculating circle. Let $C(s)$ denote the center of the osculating circle corresponding to the position s along the path. Then, consider the radial segment of length $\rho(s)$ going from $C(s)$ to the airplane. In addition, let the orientation of the segment in question be described by the angle $\theta(s)$ that the segment forms with the line connecting $C(0)$ and the airplane when $s = 0$. Based on how we have defined $\theta(s)$ we have $\theta(0) = 0$, $\theta(s_f) = (\pi/2)\,\text{rad}$, and we have that θ changes with time with an angular velocity we will denote by ω, i.e., $\dot{\theta} = \omega$. Because of the properties of the osculating circle and because of how we have defined θ, we must have

$$\omega(s) = \frac{v(s)}{\rho(s)}, \tag{3}$$

where v is the speed of the plane. Then manipulating the above equation, we have

$$\omega = \frac{v}{\rho} = \frac{d\theta}{dt} = \frac{d\theta}{ds}\frac{ds}{dt} = v\frac{d\theta}{ds} \quad \Rightarrow \quad \frac{d\theta}{ds} = \frac{1}{\rho}. \tag{4}$$

Separating variables in the last of Eqs. (4) and integrating, we have

$$\int_0^{(\pi/2)\,\text{rad}} d\theta = \int_0^{s_f} \frac{ds}{\rho_0 + \gamma s} \quad \Rightarrow \quad \frac{\pi}{2}\,\text{rad} = \frac{1}{\gamma}\ln\left(\frac{\rho_0 + \gamma s_f}{\rho_0}\right) \quad \Rightarrow \quad \frac{\pi}{2}\,\text{rad} = \frac{1}{\gamma}\ln\left(\frac{\rho_f}{\rho_0}\right), \tag{5}$$

which can be solved for γ to obtain

$$\gamma = \frac{2}{\pi}\ln\left(\frac{\rho_f}{\rho_0}\right) = -0.4058. \tag{6}$$

Knowing that γ represents the rate of change of curvature with respect to position ($\gamma = \frac{d\rho}{ds}$) we can take the derivative with respect to s of the equation $a_n = \frac{v^2}{\rho} = 8g$ to obtain

$$v^2 = 8g\rho \quad \Rightarrow \quad 2v\frac{dv}{ds} = 8g\frac{d\rho}{ds} \quad \Rightarrow \quad v\frac{dv}{ds} = 4g\gamma. \tag{7}$$

However, $v\frac{dv}{ds}$ represents the quantity $\dot{v}$. Therefore, we have

$$\frac{dv}{dt} = 4g\gamma \quad \Rightarrow \quad \int_0^{t_f} dt = \int_{v_0}^{v_f} \frac{dv}{4g\gamma} \quad \Rightarrow \quad t_f = \frac{1}{4g\gamma}(v_f - v_0). \tag{8}$$

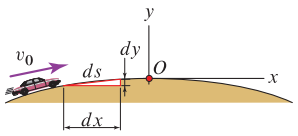
Using the value of γ computed in Eq. (6), and recalling that we have $g = 9.81\,\text{m/s}^2$, $v_0 = 1100\,\text{km/h} = 305.6\,\text{m/s}$, and $v_f = 800\,\text{km/s} = 222.2\,\text{m/s}$, we can evaluate the last of Eqs. (8) to obtain

$$\boxed{t_f = 5.24\,\text{s}.}$$

Problem 2.247

A car is traveling over a hill with a speed $v_0 = 160\,\text{km/h}$. Using the Cartesian coordinate system shown, the hill's profile is described by the function $y = -(0.003\,\text{m}^{-1})x^2$, where x and y are measured in meters. At $x = -100\,\text{m}$, the driver realizes that her speed will cause her to lose contact with the ground once she reaches the top of the hill at O. Verify that the driver's intuition is correct, and determine the minimum constant time rate of change of the speed such that the car will not lose contact with the ground at O. *Hint:* To compute the distance traveled by the car along the car's path, observe that $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\,dx$ and that

$$\int \sqrt{1 + C^2x^2}\,dx = \frac{x}{2}\sqrt{1 + C^2x^2} + \frac{1}{2C}\ln\left(Cx + \sqrt{1 + C^2x^2}\right).$$



Solution

The minimum speed to loose contact with the ground is such that

$$\frac{v_{\text{min}}^2}{\rho} = g \quad \Rightarrow \quad v_{\text{min}}^2 = g\rho(0). \tag{1}$$

To calculate the radius of curvature at the origin of the coordinate system indicated in the problem's figure we use the following equation:

$$\rho(x) = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad \Rightarrow \quad \rho(0) = \frac{\left[1 + (0.006x)^2\right]^{3/2}}{0.006} = 166.7\,\text{m}. \tag{2}$$

Therefore

$$v_{\text{min}} = \sqrt{\rho g} = 40.44\,\text{m/s}. \tag{3}$$

Since $v_0 = 160\,\text{km/h} = 44.44\,\text{m/s}$, we conclude that

the car will loose contact with the ground.

Now we have to find the minimum constant value of $\dot{v} = a_c$ such that the car does not loose contact. Since we need to relate a change in speed to a change in position, we can use the constant acceleration equation $v^2 - v_0^2 = 2a_c\,(s - s_0)$ with $s_0 = 0$ and $v_f = v_{\text{min}}$, where s is the path coordinate along the profile of the hill. This gives

$$v_{\text{min}}^2 - v_0^2 = 2a_c s \quad \Rightarrow \quad a_c = \frac{v_{\text{min}}^2 - v_0^2}{2s_f}, \tag{4}$$

where the subscript f stands for final. To evaluate this equation we need to express the path coordinate s in terms of the Cartesian coordinates x and y. Taking advantage of the hint given in the problem, we can write

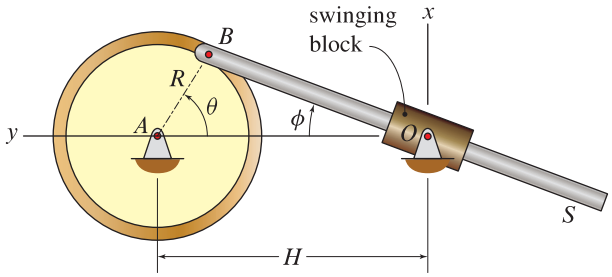
$$s_f = \int_{-100}^{0} \sqrt{1 + (-0.006x)^2}\,dx = \left[\frac{x}{2}\sqrt{1 + C^2x^2} + \frac{1}{2C}\ln\left(Cx + \sqrt{1 + C^2x^2}\right)\right]_{-100}^{0} = 105.7\,\text{m}. \tag{5}$$

Using this result in Eq. (4), along with the fact that $v_0 = 160\,\text{km/h} = 44.44\,\text{m/s}$ and $v_{\text{min}} = 40.44\,\text{m/s}$ (see Eq. (3)), we have

$$a_c = -1.61\,\text{m/s}^2.$$

Problem 2.248

The mechanism shown is called a *swinging block* slider crank. First used in various steam locomotive engines in the 1800s, this mechanism is often found in door-closing systems. Let $H = 1.25\,\text{m}$, $R = 0.45\,\text{m}$, and r denote the distance between B and O. Assuming that the speed of B is constant and equal to $5\,\text{m/s}$, determine $\dot{r}$, $\dot{\phi}$, $\ddot{r}$, and $\ddot{\phi}$ when $\theta = 180°$.



Solution

We start by defining a polar coordinate system with origin at O and radial direction along the segment OB so that the coordinate r measures the distance from O to B. We take as transverse coordinate, the angle ϕ. Observe that for $0 < \theta < 180°$, r is growing. For $\theta = 180°$ r achieves its maximum value $r_{\max} = R + H$, and for $180° < \theta < 360°$, r decreases. Therefore, the rate of change of r for $\theta = 180°$ must be equal to zero, i.e.,

$$\boxed{\dot{r}\big|_{\theta=180°} = 0.} \tag{1}$$

In polar coordinates, the velocity of B is expressed as

$$\vec{v}_B = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi. \tag{2}$$

Therefore, in view of Eq. (1), for $\theta = 180°$, recalling that B is moving counterclockwise with the constant speed $v_0 = 5\,\text{m/s}$, we have

$$\vec{v}_B\big|_{\theta=180°} = r_{\max}\big(\dot{\phi}\big|_{\theta=180°}\big)\,\hat{u}_\phi = -v_0\,\hat{u}_\phi \quad \Rightarrow \quad \dot{\phi}\big|_{\theta=180°} = -\frac{v_0}{R+H} = -2.941\,\text{rad/s}, \tag{3}$$

where we have used the fact that $R = 0.45\,\text{m}$ and $H = 1.25\,\text{m}$. Expressing the above result to 3 significant figures, we have

$$\boxed{\dot{\phi}\big|_{\theta=180°} = -2.94\,\text{rad/s}.}$$

Next we recall that the general expression of the acceleration in polar coordinates is

$$\vec{a}_B = (\ddot{r} - r\dot{\phi}^2)\,\hat{u}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{u}_\phi. \tag{4}$$

Recalling that for $\theta = 180°$, $r = R + H$, $\dot{r} = 0$ and $\dot{\phi}$ takes on the expression in the second of Eqs. (3), we have that, for $\theta = 180°$, Eq. (4) reduces to

$$\vec{a}_B\big|_{\theta=180°} = \left(\ddot{r}\big|_{\theta=180°} - \frac{v_0^2}{R+H}\right)\hat{u}_r + [(R+H)\ddot{\phi}\big|_{\theta=180°}]\,\hat{u}_\phi. \tag{5}$$

We now observe that, due to the fact that B is in uniform circular motion along a circle with center at A, the acceleration of B is always directed toward A and, for $\theta = 180°$, we must have

$$\vec{a}_B\big|_{\theta=180°} = -\frac{v_0^2}{R}\,\hat{u}_r. \tag{6}$$

Setting Eqs. (5) and (6) equal to each other component by component, we have

$$\ddot{r}\big|_{\theta=180^\circ} - \frac{v_0^2}{R+H} = \frac{-v_0^2}{R} \quad \Rightarrow \quad \ddot{r}\big|_{\theta=180^\circ} = v_0^2\left(\frac{1}{R+H} - \frac{1}{R}\right),$$

and

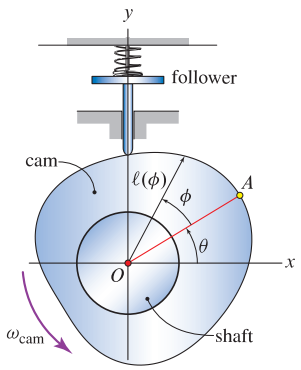
$$(R+H)\ddot{\phi}\big|_{\theta=180^\circ} = 0. \tag{7}$$

Recalling that $v_0 = 5\,\text{m/s}$, $R = 0.45\,\text{m}$, and $H = 1.25\,\text{m}$, we can then express our results in numerical form as follows:

$$\boxed{\ddot{r}\big|_{\theta=180^\circ} = -40.8\,\text{m/s}^2 \quad \text{and} \quad \ddot{\phi}\big|_{\theta=180^\circ} = 0.}$$

Problem 2.249

The cam is mounted on a shaft that rotates about O with constant angular velocity ω_{cam}. The profile of the cam is described by the function $\ell(\phi) = R_0(1 + 0.25\cos^3\phi)$, where the angle ϕ is measured relative to the segment OA, which rotates with the cam. Letting $R_0 = 3$ cm, determine the maximum value of angular velocity $\omega_{\max}$ such that the maximum speed of the follower is limited to 2 m/s. In addition, compute the smallest angle $\theta_{\min}$ for which the follower achieves it maximum speed.



Solution

Let y denote the position of the follower when in contact with the cam. In addition, let ϕ_f (where the subscript f stands for follower) denote the value of ϕ that identifies the radial line on the cam that goes from point O to the follower. Hence, we have that the relation between ϕ_f and θ is $\phi_f = 90° - \theta$. Keeping in mind that $\dot{y}$ describes the velocity of the follower, we have

$$y = \ell(\phi_f) = R_0(1 + 0.25\cos^3\phi_f) \quad \Rightarrow \quad \dot{y} = \frac{d\ell}{d\phi_f}\frac{d\phi_f}{dt}. \tag{1}$$

Since $\dot{\phi_f} = -\dot{\theta}$ and $\dot{\theta} = \omega_{\text{cam}}$, we have

$$\dot{y} = -\dot{\theta}\frac{d\ell}{d\phi_f} \quad \Rightarrow \quad \dot{y} = -\omega_{\text{cam}}\frac{d\ell}{d\phi_f}. \tag{2}$$

Since ω_{cam} is constant, $\dot{y}$ is maximum when $\frac{d\ell}{d\phi_f}$ is maximum. Hence, taking the derivative of $\ell(\phi_f)$ with respect to ϕ_f, we have

$$\frac{d\ell}{d\phi_f} = -0.75R_0\cos^2\phi_f\sin\phi_f. \tag{3}$$

To maximize $\frac{d\ell}{d\phi_f}$, we differential the above quantity again with respect to ϕ_f and set the result equal to 0. This gives

$$\frac{d^2\ell}{d\phi_f^2} = 1.5R_0\cos\phi_f\sin^2\phi_f - 0.75R_0\cos^3\phi_f = 0 \quad \Rightarrow \quad \cos^2\phi_f - 2\sin^2\phi_f = 0 \tag{4}$$

$$\Rightarrow \quad \sin\phi_f\big|_{\dot{y}_{\max}} = \sqrt{1/3} \quad \text{and} \quad \cos^2\phi_f\big|_{\dot{y}_{\max}} = 2/3. \tag{5}$$

Thus, the maximum magnitude of Eq. (3) and the maximum magnitude of Eq. (2) are, respectively,

$$\left|\frac{d\ell}{d\phi_f}\right|_{\max} = \frac{R_0}{2\sqrt{3}} \quad \text{and} \quad |\dot{y}_{\max}| = \frac{\omega_{\text{cam}}R_0}{2\sqrt{3}}. \tag{6}$$

Setting $|\dot{y}_{\max}| = v_{\max}$ and solving for ω_{cam}, we have

$$\boxed{\omega_{\max} = \frac{v_{\max}2\sqrt{3}}{R_0} = 231\ \text{rad/s}.}$$

The minimum angle θ for which the follower achieves its maximum speed is obtained by recalling that $\phi_f = 90^\circ - \theta$ and that, by the first of Eqs. (5), we have

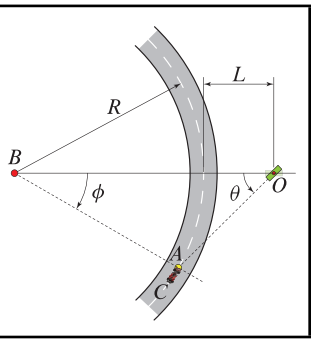
$$\sin\phi_f = \sqrt{1/3} \quad \Rightarrow \quad \phi = \sin^{-1}\sqrt{1/3}\Big|_{\dot{y}_{\max}} = 90^\circ - \theta \quad \Rightarrow \quad \theta_{\min} = 90^\circ - \sin^{-1}\sqrt{1/3}, \qquad (7)$$

where the value of $\sin^{-1}\sqrt{1/3}$ is understood to be limited to the range $0 < \sin^{-1}\sqrt{1/3} < 90^\circ$. Therefore, the numerical value of $\theta_{\min}$ is

$$\boxed{\theta_{\min} = 54.7^\circ.}$$

Problem 2.250

A car is traveling at a constant speed $v_0 = 210\,\text{km/h}$ along a circular turn with radius $R = 137\,\text{m}$ (the figure is not to scale). The camera at O is tracking the motion of the car. Letting $L = 15\,\text{m}$, determine the camera's rotation rate as well as the corresponding time rate of change of the rotation rate when $\phi = 30°$.



Solution

Let r represent the distance from O to the car. From the geometry of the figure we have

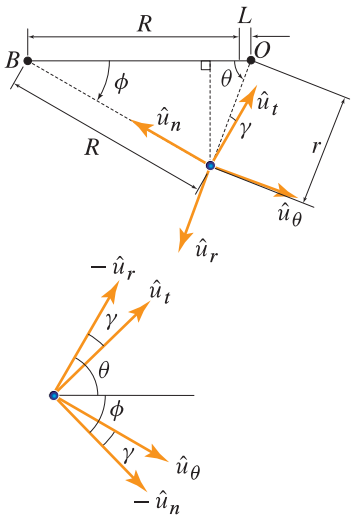
$$r\cos\theta + R\cos\phi = R + L, \quad r\sin\theta = R\sin\phi, \tag{1}$$

and

$$\tan\theta = \frac{R\sin\phi}{R + L - R\cos\phi}. \tag{2}$$

Recalling that $R = 137\,\text{m}$, $L = 15\,\text{m}$, when $\phi = 30°$ from the above equations we have

$$\theta = 64.04° \quad \text{and} \quad r = 76.19\,\text{m}. \tag{3}$$



Next. referring to the figure on the right, we see that we have defined both a polar component system with unit vectors $\hat{u}_r$ and $\hat{u}_\theta$, as well as a normal-tangential component system with unit vectors $\hat{u}_t$ and $\hat{u}_n$. For later use we now write the equations relating these two component systems. To facilitate this task, we define γ to be the angle between $\hat{u}_t$ and $-\hat{u}_r$, using the first of Eqs. (3), for $\phi = 30°$, we have

$$90° = \phi - \gamma + \theta \quad \Rightarrow \quad \gamma = 4.037°. \tag{4}$$

Hence, going back to writing the relationships between the unit vectors of the polar and normal-tangential component systems defined, we have

$$\hat{u}_t = -\cos\gamma\,\hat{u}_r + \sin\gamma\,\hat{u}_\theta \qquad \hat{u}_n = -\sin\gamma\,\hat{u}_r - \cos\gamma\,\hat{u}_\theta.$$

Now we observe that the velocity of the car in the $(\hat{u}_t, \hat{u}_n)$ component system is converted to the $(\hat{u}_r, \hat{u}_\theta)$ component system as

$$\vec{v} = v_0\,\hat{u}_t \quad \Rightarrow \quad \vec{v} = v_0\left(-\cos\gamma\,\hat{u}_r + \sin\gamma\,\hat{u}_\theta\right). \tag{5}$$

Then, matching Eq. (5) with the expression of the velocity in polar components, namely, $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, and recalling that $v_0 = 210\,\text{km/h} = 58.33\,\text{m/s}$ and using the value of γ in Eq. (4), for $\phi = 30°$, we have

$$\dot{r}\big|_{\phi=30°} = -v_0\cos\gamma = -58.19\,\text{m/s} \quad \text{and} \quad r\dot{\theta} = v_0\sin\gamma \quad \Rightarrow \quad \dot{\theta}\big|_{\phi=30°} = \frac{v_0\sin\gamma}{r} = 0.05390\,\text{rad/s}. \tag{6}$$

Expressing $\dot{\theta}$ at $\phi = 30°$ to 3 significant figures, we have

$$\boxed{\dot{\theta}\big|_{\phi=30°} = 0.0539\,\text{rad/s}.}$$

Next, the acceleration of the car in the $(\hat{u}_t, \hat{u}_n)$ component system is converted to the $(\hat{u}_r, \hat{u}_\theta)$ component system as

$$\vec{a} = \frac{v_0^2}{R}\,\hat{u}_n \quad \Rightarrow \quad \vec{a} = \frac{v_0^2}{R}\,(-\sin\gamma\,\hat{u}_r - \cos\gamma\,\hat{u}_\theta)\,. \tag{7}$$

Matching the expression in the radial component of acceleration in the second of Eqs. (7) with the general expression for the radial component of acceleration in polar coordinates, namely, $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, we have

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{-v_0^2}{R}\cos\gamma \quad \Rightarrow \quad \ddot{\theta} = \frac{-1}{r}\left(\frac{v_0^2}{R}\cos\gamma + 2\dot{r}\dot{\theta}\right).$$

Hence, recalling that $v_0 = 210\,\text{km/h} = 58.33\,\text{m/s}$ and $R = 137\,\text{m}$, and that for $\phi = 30°$ we have r is given by the second of Eqs. (3), γ is given by the second of Eqs. (4), and using Eqs. (6), for $\phi = 30°$, we have

$$\boxed{\ddot{\theta}\big|_{\phi=30°} = -0.243\,\text{rad/s}^2.}$$

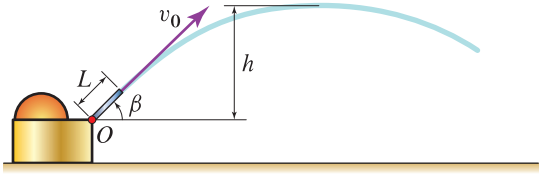
Problems 2.251 through 2.253

A fountain has a spout that can rotate about O and whose angle β is controlled so as to vary with time according to $\beta = \beta_0[1 + \sin^2(\omega t)]$, with $\beta_0 = 15°$ and $\omega = 0.4\pi$ rad/s. The length of the spout is $L = 1.5$ ft, the water flow through the spout is constant, and the water is ejected at a speed $v_0 = 6$ ft/s, measured relative to the spout.

Problem 2.251 Determine the largest speed with which the water particles are released from the spout.

Problem 2.252 Determine the magnitude of the acceleration immediately before release when $\beta = 15°$.

Problem 2.253 Determine the highest position reached by the resulting water arc.



Solution to 2.251

We define a polar coordinate system with origin at O and transverse coordinate coinciding with β. The expression of the velocity of the water particles as they leave the spout is given by

$$\vec{v} = \dot{r}\,\hat{u}_r + L\dot{\beta}\,\hat{u}_\beta \quad \Rightarrow \quad v = \sqrt{\dot{r}^2 + (L\dot{\beta})^2}. \tag{1}$$

where, based on the problem statement

$$\dot{r} = v_0 = 6\,\text{ft/s} \quad \text{and} \quad \dot{\beta} = 2\beta_0\omega \sin\omega t \cos\omega t = \beta_0\omega \sin(2\omega t), \tag{2}$$

where we have used the trigonometric identity $2\sin x \cos x = \sin(2x)$. Substituting Eqs. (2) into the last of Eqs. (1) we obtain the following expression for the speed:

$$v = \sqrt{v_0^2 + [L\beta_0\omega \sin(2\omega t)]^2}. \tag{3}$$

From the above expression, recalling that v_0, β_0, and ω are constant, we se that v is maximum when $\sin(2\omega t)$ is maximum. Since the sine function has maximum value equal to 1, we have

$$\boxed{v_{\max} = \sqrt{v_0^2 + L^2\beta_0^2\omega^2} = 6.02\,\text{ft/s},}$$

where we have used the following numerical data: $v_0 = 6$ ft/s, $L = 1.5$ ft, $\beta_0 = 15°$, and $\omega = 0.4\pi$ rad/s.

Solution to 2.252

We define a polar coordinate system with origin at O and transverse coordinate coinciding with β. The expression of the acceleration of in polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\beta}^2)\,\hat{u}_r + (r\ddot{\beta} + 2\dot{r}\dot{\beta})\,\hat{u}_\beta. \tag{4}$$

Now, recalling that v_0 is constant, when the water particles leave the spout we have

$$r = L,\ \dot{r} = v_0,\ \ddot{r} = 0,\ \dot{\beta} = 2\beta_0\omega\sin\omega t\cos\omega t = \beta_0\omega\sin(2\omega t),\ \text{and}\ \ddot{\beta} = 2\omega^2\beta_0\cos(2\omega t), \tag{5}$$

where we have used the trigonometric identity $2\sin x\cos x = \sin(2x)$. Now, we observe that $\beta_0 = 15^\circ$ and therefore for $\beta = 15^\circ$, we must have $\beta = \beta_0$ and $\sin(\omega t) = 0$, which implies $t = 0$. Hence, for $\beta = 15^\circ$, we have

$$r = L, \quad \dot{r} = v_0, \quad \ddot{r} = 0, \quad \dot{\beta} = 0, \quad \text{and} \quad \ddot{\beta} = 2\omega^2\beta_0, \tag{6}$$

so that, for $\beta = 15^\circ$, the acceleration takes on the form

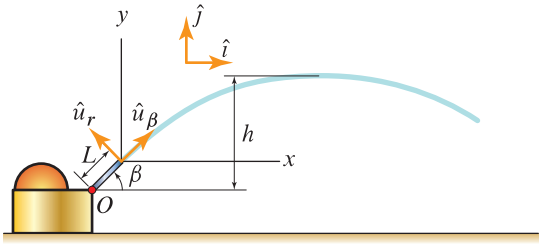
$$\vec{a} = 2L\omega^2\beta_0\,\hat{u}_\beta. \tag{7}$$

Consequently, for $\beta = 15^\circ$, we have

$$\boxed{|\vec{a}| = 2L\omega^2\beta_0 = 1.24\,\text{ft/s}^2,}$$

where we have used the following numerical data: $L = 1.5\,\text{ft}$, $\beta_0 = 15^\circ$, and $\omega = 0.4\pi\ \text{rad/s}$.

Solution to 2.253



Referring to the figure on the right, in the solution of this problem we will use two coordinate systems. The first is a polar coordinate system with origin at O and transverse coordinate β. Let $\bar{\beta}$ be the value of β corresponding to the trajectory of water jet achieving the maximum height. Then the second is a fixed Cartesian coordinate system with origin coinciding with the position of the mouth of the spout corresponding to $\bar{\beta}$. Letting v denote the speed of the water particles at the spout's mouth, then the velocity of the water particles at the spout's mouth is

$$\vec{v} = v(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}). \tag{8}$$

Once the water particles leave the spout their motion is a projectile motion and the trajectory with the maximum height is determined both by the vertical component of the speed, i.e,

$$v_y = v\sin\beta, \tag{9}$$

and the vertical position of mouth of the water spout.

To derive an expression for the speed v of the water particles as they leave the spout, we use the chosen polar coordinate system, for which the velocity at the mouth of the spout is expressed as

$$\vec{v} = \dot{r}\,\hat{u}_r + L\dot{\beta}\,\hat{u}_\beta \quad \Rightarrow \quad v = \sqrt{\dot{r}^2 + (L\dot{\beta})^2}. \tag{10}$$

where, based on the problem statement

$$\dot{r} = v_0 = 6\,\text{ft/s} \quad \text{and} \quad \dot{\beta} = 2\beta_0\omega\sin\omega t\cos\omega t = \beta_0\omega\sin(2\omega t), \tag{11}$$

where we have used the trigonometric identity $2\sin x\cos x = \sin(2x)$. Substituting Eqs. (2) into the last of Eqs. (10) we obtain the following expression for the speed:

$$v = \sqrt{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2}. \tag{12}$$

Consequently, the vertical component of velocity at the spout is

$$v_y = \sqrt{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2}\,\sin\left[\beta_0(1+\sin^2\omega t)\right]. \tag{13}$$

Now let $h = h_1 + h_2$ where h_1 is height of the mouth of the spout with respect to the base of the spout, and where h_2 is the maximum elevation of the water jet measured from the horizontal line going through the mouth of the spout. Hence, for h_1 we have

$$h_1 = L\sin\beta = L\sin\left[\beta_0(1+\sin^2\omega t)\right]. \tag{14}$$

By contrast, h_2 is found using the constant acceleration equation $v^2 = v_0^2 + 2a(s - s_0)$ in the vertical direction. Recalling that for $y = h_2$ $v_y = 0$, we have

$$0 = \left\{\sqrt{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2}\,\sin\left[\beta_0(1+\sin^2\omega t)\right]\right\}^2 - 2gh_2,$$

which can be solved for h_2 to obtain

$$h_2 = \frac{1}{2g}\left\{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2\right\}\sin^2\left[\beta_0(1+\sin^2\omega t)\right]. \tag{15}$$

Consequently, the expression for the height of the water jet as a function of time is

$$h = L\sin\left[\beta_0(1+\sin^2\omega t)\right] + \frac{1}{2g}\left\{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2\right\}\sin^2\left[\beta_0(1+\sin^2\omega t)\right]. \tag{16}$$

This function needs to be maximized. This can be done by differentiating h with respect to time and setting the result equal to zero. Doing so, after simplification, yields the following equation:

$$\frac{\beta_0\omega}{2g}\sin(2\omega t)\Big\{2Lg\cos\left[\beta_0(1+\sin^2\omega t)\right] + 2L\beta_0\cos(2\omega t)\sin^2\left[\beta_0(1+\sin^2\omega t)\right] + [v_0^2 + L^2\beta_0^2 + \omega^2\sin^2(2\omega t)]\sin\left[2\beta_0(1+\sin^2\omega t)\right]\Big\} = 0 \tag{17}$$

Recalling that we have $L = 1.5\,\text{ft}$, $\beta_0 = 15°$, $\omega = 0.4\pi\,\text{rad/s}$, and $g = 32.2\,\text{ft/s}^2$, and although this may require plotting the terms within braces as a function of time, it turns out that the term within braces can never be equal to zero. Hence, the solution of the above equation reduces to the solution of the equation

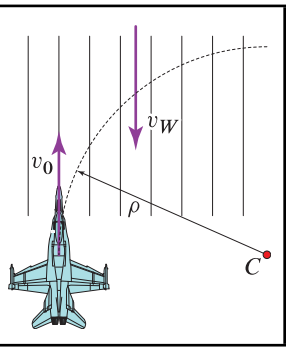
$$\sin(2\omega t) = 0 \quad \Rightarrow \quad t = 0 \pm \frac{n\pi}{\omega} \quad \text{and} \quad t = \frac{\pi}{2\omega} \pm \frac{n\pi}{\omega}, \quad \text{with} \quad n = 0, 1, 2, \ldots \tag{18}$$

Since the function $\beta = \beta_0(1+\sin^2\omega t)$ is at a maximum for $t = \frac{\pi}{2\omega} \pm \frac{n\pi}{\omega}$ the function h will also achieve its maxima for $t = \frac{\pi}{2\omega} \pm \frac{n\pi}{\omega}$. In addition, since the function h is a periodic function with period $\frac{\pi}{\omega}$, the values of h for $t = \frac{\pi}{2\omega} \pm \frac{n\pi}{\omega}$ are all identical to one another and we can therefore evaluate $h_{\max}$ by simply letting $t = \frac{\pi}{2\omega}$, i.e., for $n = 0$. Hence, recalling that $L = 1.5\,\text{ft}$, $\beta_0 = 15°$, $\omega = 0.4\pi\,\text{rad/s}$, and $g = 32.2\,\text{ft/s}^2$, for $t = \frac{\pi}{2\omega} = 1.250\,\text{s}$, we can evaluate h in Eq. (16) to obtain

$$\boxed{h_{\max} = 0.890\,\text{ft}.}$$

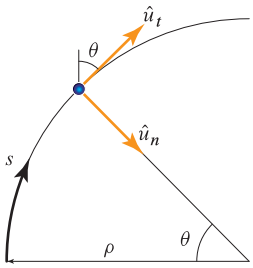
Problem 2.254

A plane is initially flying north with a speed $v_0 = 430$ mph relative to the ground while the wind has a constant speed $v_W = 12$ mph in the north-south direction. The plane performs a circular turn with radius of $\rho = 0.45$ mi. Assume that the airspeed indicator on the plane measures the absolute value of the component of the relative velocity of the plane with respect to the air in the direction of motion. Then determine the value of the tangential component of the airplane's acceleration when the airplane is halfway through the turn, assuming that the airplane maintains constant the reading of the airspeed indicator.



Solution

Let the subscripts P and W denote quantities pertaining to the airplane and wind, respectively. Then, referring to the figure on the the right and using a normal tangential component system, the velocity of the airplane and wind are



$$\vec{v}_P = v_P\,\hat{u}_t \quad \text{and} \quad \vec{v}_W = v_W(-\cos\theta\,\hat{u}_t + \sin\theta\,\hat{u}_n). \tag{1}$$

Therefore the velocity of the airplane relative to the wind is

$$\vec{v}_{P/W} = (v_P + v_W\cos\theta)\,\hat{u}_t - v_W\sin\theta\,\hat{u}_n. \tag{2}$$

Now we recall that that the quantity measured by the airspeed indicator is the component of $\vec{v}_{P/W}$ that is in the direction of motion, i.e.,

$$v_{\text{ai}} = |\vec{v}_{P/A}\cdot\hat{u}_t| = v_P + v_W\cos\theta,$$

where the subscript 'ai' stands for airspeed indicator. Now, recalling that the measure of θ in radians is given by $\theta = s/\rho$, we can rewrite v_{ai} as follows:

$$v_{\text{ai}} = v_P + v_W\cos\frac{s}{\rho}. \tag{3}$$

Observing that for $\theta = 0$ we have $v_r = v_0 + v_W$ and recalling that v_{ai} is maintained constant along the turn, we can solve Eq. (3) for v_P as a function of s to obtain

$$v_P(s) = v_0 + v_W\left(1 - \cos\frac{s}{\rho}\right). \tag{4}$$

The tangential component of acceleration is the time derivative of Eq. (4). Hence, we can write

$$a_t = \frac{dv_P}{ds}\frac{ds}{dt} = v_P\frac{dv_P}{ds} \quad \Rightarrow \quad a_t = \left[v_0 + v_W\left(1 - \cos\frac{s}{\rho}\right)\right]\frac{v_W}{\rho}\sin\frac{s}{\rho}. \tag{5}$$

For $s/\rho = (\pi/4)$ rad, we have $\sin(s/\rho) = \cos(s/\rho) = \sqrt{2}/2$. Therefore, midway through the turn, we have

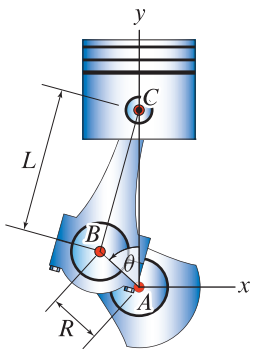
$$a_t = \left[v_0 + v_W\left(1 - \frac{\sqrt{2}}{2}\right)\right]\frac{v_W\sqrt{2}}{2\rho}. \tag{6}$$

Recalling that $v_0 = 430$ mph $= 630.7$ ft/s, $v_W = 12$ mph $= 17.60$ ft/s, and $\rho = 0.45$ mi $= 2376$ ft, we can evaluate the above expression to obtain

$$\boxed{a_t = 3.33\ \text{ft/s}^2.}$$

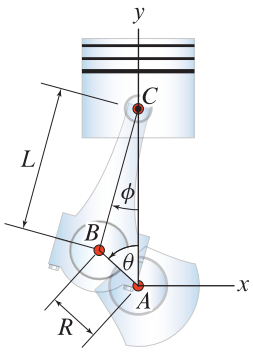
Problem 2.255

The piston head at C is constrained to move along the y axis. Let the crank AB be rotating counterclockwise at a constant angular speed $\dot{\theta} = 2000\,\text{rpm}$, $R = 3.5\,\text{in.}$, and $L = 5.3\,\text{in.}$ Obtain the angular velocity of the connecting rod BC by differentiating the relative position vector of C with respect to B when $\theta = 35°$. *Hint:* You will also need to determine the velocity of B and enforce the constraint that demands that C move only along the y axis.



Solution

Referring to the figure on the right, the can describe the position of C relative to B as follows:



$$\vec{r}_{C/B} = L\,\hat{u}_{C/B}. \tag{1}$$

Then, using the concept of time derivative of a vector, we have that

$$\vec{v}_{C/B} = L\,\dot{\hat{u}}_{C/B} = L\vec{\omega}_{BC} \times \hat{u}_{C/B}, \tag{2}$$

where the angular velocity $\vec{\omega}_{BC}$, which is the angular velocity of the vector $\vec{r}_{C/B}$, is the angular velocity of the connecting rod. Next, again referring to the figure on the right, we must have

$$\vec{\omega}_{BC} = -\dot{\phi}\,\hat{k} \quad \text{and} \quad \hat{u}_{C/B} = \sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}. \tag{3}$$

Substituting Eqs. (3) into Eq. (2) and carrying out the cross-product, we have

$$\vec{v}_{C/B} = L\,\dot{\hat{u}}_{C/B} = L\dot{\phi}(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}). \tag{4}$$

To enforce the condition that C can only move only in the y direction, we must compute the velocity of B and then apply the relative kinematics equation $\vec{v}_C = \vec{v}_B + \vec{v}_{C/B}$. Since B is in uniform circular motion along a circle of radius R and center A, we have

$$\vec{v}_B = -\dot{\theta}R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \tag{5}$$

Therefore, combining Eqs. (5) and Eq. (3), we have

$$\vec{v}_C = \vec{v}_B + \vec{v}_{C/B} = (L\dot{\phi}\cos\phi - \dot{\theta}R\cos\theta)\,\hat{\imath} - (L\dot{\phi}\sin\phi + \dot{\theta}R\sin\theta)\,\hat{\jmath}. \tag{6}$$

Since $v_{Cx} = 0$, we have

$$L\dot{\phi}\cos\phi - \dot{\theta}R\cos\theta = 0 \quad \Rightarrow \quad \dot{\phi} = \frac{R\cos\theta}{L\cos\phi}\dot{\theta} \quad \Rightarrow \quad \vec{\omega}_{BC} = -\frac{R\cos\theta}{L\cos\phi}\dot{\theta}\,\hat{k}, \tag{7}$$

where we have used the definition in the first of Eqs. (3). In order to complete our calculation, we need to determine the angle ϕ. Using trigonometry, we see that

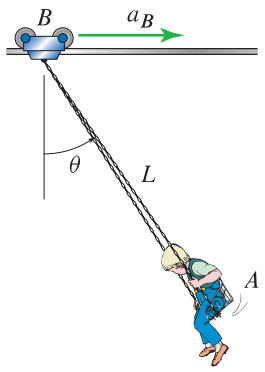
$$L\sin\phi = R\sin\theta \quad \Rightarrow \quad \phi = \sin^{-1}[(R/L)\sin\theta] \quad \Rightarrow \quad \phi\big|_{\theta=35°} = 22.26°, \tag{8}$$

where we have used the fact that $R = 3.5\,\text{in.}$ and $L = 5.3\,\text{in.}$. Hence, evaluating the last of Eqs. (7) using the value of ϕ in Eq. (8) and recalling that $\dot{\theta} = 2000\,\text{rpm}$, for $\theta = 35°$ we have

$$\boxed{\vec{\omega}_{BC} = (-1170\,\text{rpm})\,\hat{k}.}$$

Problem 2.256

A child A is swinging from a swing that is attached to a trolley that is free to move along a fixed rail. Letting $L = 3.2\,\text{m}$, if at a given instant $a_B = 3.4\,\text{m/s}^2$, $\theta = 23°$, $\dot{\theta} = 0.45\,\text{rad/s}$, and $\ddot{\theta} = -0.2\,\text{rad/s}^2$, determine the magnitude of the acceleration of the child relative to the rail at that instant.



Solution

The acceleration of A relative to B is most easily described by setting up a polar coordinate system with origin at B, radial coordinate r going from B to A, and transverse coordinate θ. Recalling that the expression of the angular acceleration in polar coordinates is $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$ and observing that the length of the rope is constant so that $r = L$, we have

$$\vec{a}_{A/B} = -L\dot{\theta}^2\,\hat{u}_r + L\ddot{\theta}\,\hat{u}_\theta, \tag{1}$$

The acceleration of the trolley can be described using the chosen component system as follows:

$$\vec{a}_B = a_B(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta). \tag{2}$$

Hence, using relative kinematics, the acceleration of A relative to the fixed rail is

$$\vec{a}_A = \left(a_B\sin\theta - L\dot{\theta}^2\right)\hat{u}_r + \left(a_B\cos\theta + L\ddot{\theta}\right)\hat{u}_\theta. \tag{3}$$

Consequently, we have

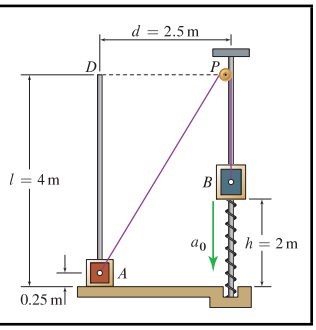
$$|\vec{a}_A| = \sqrt{\left(a_B\sin\theta - L\dot{\theta}^2\right)^2 + \left(a_B\cos\theta + L\ddot{\theta}\right)^2}. \tag{4}$$

Recalling that $L = 3.2\,\text{m}$, $a_B = 3.4\,\text{m/s}^2$, $\theta = 23°$, $\dot{\theta} = 0.45\,\text{rad/s}$, and $\ddot{\theta} = -0.2\,\text{rad/s}^2$, we can evaluate the above expression to obtain

$$\boxed{|\vec{a}_A| = 2.58\,\text{m/s}^2.}$$

Problem 2.257

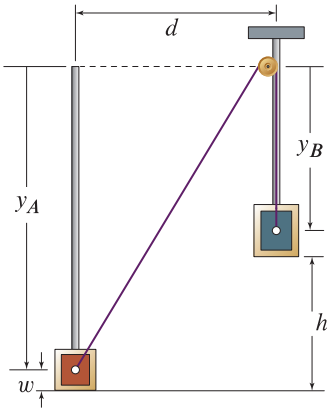
Block B is released from rest at the position shown, and it has a constant acceleration downward $a_0 = 5.7\,\text{ft/s}^2$. Determine the velocity and acceleration of block A at the instant that B touches the floor.



Solution

Note: The units of the given quantity a_0 are incorrect. The given value of a_0 should be $a_0 = 5.7\,\text{m/s}^2$.

The length of the rope is



$$L = \sqrt{d^2 + y_A^2} + y_B. \tag{1}$$

For $t = 0$, we have that $y_A(0) = l - w$. Letting t_f represent the time at the final position, the length of the rope at the initial and final positions is

$$L = \sqrt{d^2 + y_A^2(0)} + y_B(0) \quad \text{and} \quad L = \sqrt{d^2 + y_A^2(t_f)} + y_B(t_f). \tag{2}$$

Subtracting these two equations and rearranging terms, we have

$$\sqrt{d^2 + y_A^2(t_f)} = \sqrt{d^2 + y_A^2(0)} - [y_B(t_f) - y_B(0)] = 0. \tag{3}$$

Since $y_B(t_f) - y_B(0) = h$, squaring both sides of the above equation we obtain

$$d^2 + y_A^2(t_f) = d^2 + y_A^2(0) - 2h\sqrt{d^2 + y_A^2(0)} + h^2. \tag{4}$$

Solving the above equation for y_A, we have

$$y_A(t_f) = \sqrt{y_A^2(0) + h^2 - 2h\sqrt{d^2 + y_A^2(0)}} = 0.1864\,\text{m}, \tag{5}$$

where we have used the following numerical data: $y_A(0) = l - w = 0.25\,\text{m}$, $h = 2\,\text{m}$, and $d = 2.5\,\text{m}$.

Next, differentiating Eq. (1) with respect to time and solving for $\dot{y}_A$, we have

$$0 = \frac{y_A \dot{y}_A}{\sqrt{d^2 + y_A^2}} + \dot{y}_B \quad \Rightarrow \quad \dot{y}_A = \frac{-\dot{y}_B}{y_A}\sqrt{d^2 + y_A^2}. \tag{6}$$

Now, we wse the constant acceleration equation $\dot{s}^2 = \dot{s}_0^2 + 2a_c(s - s_0)$ to find $\dot{y}_B$ after B has traveled a distance h. This gives

$$\dot{y}_B^2 = 2a_B h \quad \Rightarrow \quad \dot{y}_B = \sqrt{2a_B h} = 4.775\,\text{m/s}, \tag{7}$$

where we have used the fact that $a_B = a_0 = 5.7\,\text{m/s}^2$ and $h = 2\,\text{m}$. Then, evaluating the last of Eqs. (6), the velocity of A when B hits the ground is

$$\dot{y}_A(t_f) = -64.22\,\text{m/s}, \tag{8}$$

which, when expressed to 3 significant figures, gives

$$\boxed{\dot{y}_A(t_f) = -64.2\,\text{m/s}.}$$

Now we differentiate Eq. (6) with respect to time to obtain

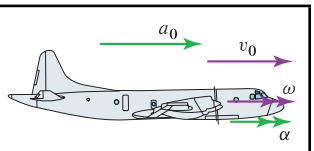
$$\ddot{y}_A = \frac{-\ddot{y}_B}{y_A}\sqrt{d^2 + y_A^2} + \frac{\dot{y}_B \dot{y}_A}{y_A^2}\sqrt{d^2 + y_A^2} - \frac{\dot{y}_B \dot{y}_A}{\sqrt{d^2 + y_A^2}}.$$

Recalling that $d = 2.5\,\text{m}$ and $\ddot{y}_B = a_0 = 5.7\,\text{m/s}^2$, and since we determined that $\dot{y}_A(t_f) = 4.775\,\text{m/s}$ and $\dot{y}_B(t_f) = -64.22\,\text{m/s}$, we can evaluate the above expression for when B hits the ground to obtain

$$\boxed{\ddot{y}_A(t_f) = -22{,}100\,\text{m/s}^2.}$$

Problem 2.258

At a given instant, an airplane is flying horizontally with speed $v_0 = 290\,\text{mph}$ and acceleration $a_0 = 12\,\text{ft/s}^2$. At the same time, the airplane's propellers rotate at an angular speed $\omega = 1500\,\text{rpm}$ while accelerating at a rate $\alpha = 0.3\,\text{rad/s}^2$. Knowing that propeller diameter is $d = 14\,\text{ft}$, determine the magnitude of the acceleration of a point on the periphery of the propellers at the given instant.



Solution

Using a cylindrical coordinate system with origin at the propeller's axis of rotation and z axis in the direction of motion, the general expression for acceleration is

$$\vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z.$$

For the propeller tip we have $R = d/2$, $\dot{R} = \ddot{R} = 0$, $\ddot{z} = a_0$, $\dot{\theta} = \omega$, and $\ddot{\theta} = \alpha$. Thus the acceleration is

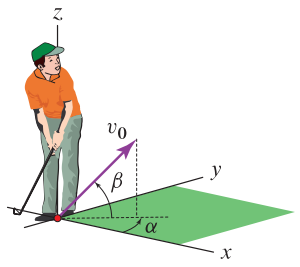
$$\vec{a} = \frac{-d\omega^2}{2}\,\hat{u}_R + \frac{d\alpha}{2}\,\hat{u}_\theta + a_0\,\hat{u}_z,$$

and the magnitude of the acceleration is

$$|\vec{a}| = \sqrt{\left(\frac{-d\omega^2}{2}\right)^2 + \left(\frac{d\alpha}{2}\right)^2 + a_0^2} = 173{,}000\,\text{ft/s}^2.$$

Problem 2.259

A golfer chips the ball as shown. Treating α, β, and the initial speed v_0 as given, find an expression for the radius of curvature of the ball's trajectory as a function of time and the given parameters. *Hint:* Use the Cartesian coordinate system shown to determine the acceleration and the velocity of the ball. Then reexpress these quantities, using normal-tangential components.



Solution

This is 3-D projectile motion. we will follow the hint and develop equations both in Cartersian components as well as in normal-tangetial components. In Cartesian coordinates the components of acceleration are

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \text{and} \quad \ddot{z} = -g.$$

Using constant acceleration equations, the velocity vector expressed in the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ and $(\hat{u}_t, \hat{u}_n, \hat{u}_b)$ component systems are

$$\vec{v} = v_0 \cos\beta \cos\alpha\, \hat{\imath} + v_0 \cos\beta \sin\alpha\, \hat{\jmath} + (v_0 \sin\beta - gt)\, \hat{k},$$
$$\vec{v} = v\, \hat{u}_t,$$

where, v is the speed, which can be given the form

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad \Rightarrow \quad v = \sqrt{v_0^2 + g^2t^2 - 2v_0 gt \sin\beta}, \tag{1}$$

and where the tangent unit vector $\hat{u}_t$ can be related to the base vectors of the Cartesian component system as follows:

$$\hat{u}_t = \frac{v_0 \cos\beta \cos\alpha}{v}\,\hat{\imath} + \frac{v_0 \cos\beta \sin\alpha}{v}\,\hat{\jmath} + \frac{v_0 \sin\beta - gt}{v}\,\hat{k}. \tag{2}$$

To express the acceleration in normal-tangetial component, we will need an expression for the time derivative of the speed. Hence, we proceed to provide differentiate v with respect to time, to obtain

$$\dot{v} = \frac{g^2t - v_0 g \sin\beta}{\sqrt{v_0^2 + g^2t^2 - 2v_0 gt \sin\beta}} = \frac{g}{v}\big(gt - v_0 \sin\beta\big). \tag{3}$$

The acceleration vector expressed in the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ and $(\hat{u}_t, \hat{u}_n, \hat{u}_b)$ component systems are

$$\vec{a} = \dot{v}\, \hat{u}_t + \frac{v^2}{\rho}\, \hat{u}_n \quad \text{and} \quad \vec{a} = -g\, \hat{k}.$$

Equating the above expressions for the acceleration, we obtain an expression for the normal unit vector

$$\hat{u}_n = \frac{-\rho}{v^2}\big(\dot{v}\, \hat{u}_t + g\, \hat{k}\big). \tag{4}$$

Using Eq. (3) we can rewrite Eq. (4) as follows:

$$\hat{u}_n = \frac{-\rho g}{v^2}\left[\frac{1}{v}\,(gt - v_0 \sin\beta)\, \hat{u}_t + \hat{k}\right].$$

Now, recalling that we must have $\hat{u}_n \cdot \hat{u}_n = 1$, we obtain the following equation:

$$1 = \frac{\rho^2 g^2}{v^4}\left[1 + \frac{1}{v^2}\left(gt - v_0 \sin\beta\right)^2 + \frac{2}{v}\left(gt - v_0 \sin\beta\right)\hat{k} \cdot \hat{u}_t\right]. \tag{5}$$

Observe that the last term on the right-hand side of Eq. (5) requires the computation of the product $\hat{k} \cdot \hat{u}_t$. This computation can be done by using Eq. (2), which shows that $\hat{k} \cdot \hat{u}_t = \frac{v_0 \sin\beta - gt}{v}$. Consequently, Eq. (5) becomes

$$1 = \frac{\rho^2 g^2}{v^4}\left[1 + \frac{1}{v^2}(gt - v_0 \sin\beta)^2 - \frac{2}{v^2}(gt - v_0 \sin\beta)^2\right] = \frac{\rho^2 g^2}{v^6}\left[v^2 - (gt - v_0 \sin\beta)^2\right]. \tag{6}$$

Keeping in mind that, in the end, the above expression is to be solved for ρ, the term $v^2 - (gt - v_0 \sin\beta)^2$ can be simplified using Eq. (1) and the trigonometric identity $\cos^2\theta = 1 - \sin^2\theta$ as follows:

$$\begin{aligned} v^2 - (gt - v_0 \sin\beta)^2 = v^2 - g^2t^2 + 2gtv_0 \sin\beta - v_0^2 \sin^2\beta = v_0^2 - v_0^2 \sin^2\beta \\ = v_0^2(1 - \sin^2\beta) = v_0^2 \cos^2\beta. \end{aligned} \tag{7}$$

This result, allows to rewrite Eq. (6) as follows:

$$1 = \frac{\rho^2 g^2}{v^6} v_0^2 \cos^2\beta. \tag{8}$$

Solving Eq. (6) for ρ, we have

$$\rho = \frac{v^3}{g v_0 \cos\beta}. \tag{9}$$

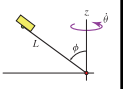
Finally, substituting the expression for v in Eq. (1), we have

$$\boxed{\rho = \frac{(v_0^2 + g^2t^2 - 2v_0gt \sin\beta)^{3/2}}{g v_0 \cos\beta}.}$$

Problem 2.260

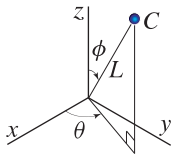
A carnival ride called the octopus consists of eight arms that rotate about the z axis with a constant angular velocity $\dot{\theta} = 6\,\text{rpm}$. The arms have a length $L = 8\,\text{m}$ and form an angle ϕ with the z axis. Assuming that ϕ varies with time as $\phi(t) = \phi_0 + \phi_1 \sin\omega t$ with $\phi_0 = 70.5°$, $\phi_1 = 25.5°$, and $\omega = 1\,\text{rad/s}$, determine the magnitude of the acceleration of the outer end of an arm when ϕ achieves its minimum value.





Solution

Using a spherical coordinate system the components of acceleration of a point C at the end of an arm with constant length L are



$$a_r = -L\phi_1^2\omega^2\cos^2\omega t - L\dot{\theta}^2\sin^2(\phi_0 + \phi_1\sin\omega t), \tag{1}$$

$$a_\phi = -L\omega^2\phi_1\sin\omega t - L\dot{\theta}^2\sin(\phi_0 + \phi_1\sin\omega t)\cos(\phi_0 + \phi_1\sin\omega t), \tag{2}$$

$$a_\theta = 2L\phi_1\omega\dot{\theta}\cos\omega t\cos(\phi_0 + \phi_1\sin\omega t). \tag{3}$$

To determine the minimum value of phi re call that $\phi = \phi_0 + \omega_1\sin\omega t$. Because the iminum of the sine function is equal to -1, then the minimum value of ϕ is

$$\phi_{\text{min}} = \phi_0 - \phi_1 = 45° = \tfrac{1}{4}\pi\ \text{rad}. \tag{4}$$

In addition, we observe that such a minimum value occurs when

$$\omega t\big|_{\phi_{\text{min}}} = 270° = \tfrac{3}{2}\pi\ \text{rad}. \tag{5}$$

Evaluating the acceleration components for $\phi = \phi_{\text{min}} = \phi_0 - \phi_1$ and $\omega t = \frac{3}{2}\pi$ rad, we have

$$a_r = -\tfrac{1}{2}L\dot{\theta}^2, \quad a_\phi = L\big(\omega^2\phi_1 - \tfrac{1}{2}\dot{\theta}^2\big), \quad \text{and} \quad a_\theta = 0. \tag{6}$$

Consequently, the magnitude of the acceleration is for $\phi = \phi_{\text{min}}$, we have $|\vec{a}|_{\phi_{\text{min}}} = \sqrt{a_r^2 + a_\phi^2 + a_\theta^2}$, which gives

$$\boxed{|\vec{a}|_{\phi_{\text{min}}} = L\sqrt{\tfrac{1}{2}\dot{\theta}^4 + \omega^4\phi_1^2 - \omega^2\dot{\theta}^2\phi_1} = 2.53\,\text{m/s}^2,}$$

where we have used the following numerical data: $L = 8\,\text{m}$, $\dot{\theta} = 6\,\text{rpm} = 0.6283\,\text{rad/s}$, $\omega = 1\,\text{rad/s}$, and $\phi_1 = 25.5° = 0.4451\,\text{rad}$.

Solutions Manual

Engineering Mechanics: Dynamics

1st Edition

Gary L. Gray
The Pennsylvania State University

Francesco Costanzo
The Pennsylvania State University

Michael E. Plesha
University of Wisconsin–Madison

With the assistance of:
Chris Punshon
Andrew J. Miller
Justin High
Chris O'Brien
Chandan Kumar
Joseph Wyne

Version: August 10, 2009

The McGraw-Hill Companies, Inc.

Copyright © 2002–2010
Gary L. Gray, Francesco Costanzo, and Michael E. Plesha

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Important Information about this Solutions Manual

Even though this solutions manual is nearly complete, we encourage you to visit

`http://www.mhhe.com/pgc`

often to obtain the most up-to-date version. In particular, as of July 30, 2009, please note the following:

- The solutions for Chapters 1 and 2 have been accuracy checked and have been edited by us. They are in their final form.
- The solutions for Chapters 4 and 7 have been accuracy checked and should be error free. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapters 3, 6, 8, and 9 are being accuracy checked and the accuracy checked versions should be available by the end of August 2009. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapter 10 should be available in their entirety by the end of August 2009.

All of the figures in Chapters 6–10 are in color. Color will be added to the figures in Chapters 1–5 over the coming weeks.

Contact the Authors

If you find any errors and/or have questions concerning a solution, please do not hesitate to contact the authors and editors via email at:

`dyn_solns@email.esm.psu.edu`

We welcome your input.

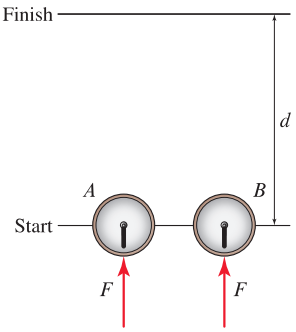
Accuracy of Numbers in Calculations

Throughout this solutions manual, we will generally assume that the data given for problems is accurate to 3 significant digits. When calculations are performed, all intermediate numerical results are reported to 4 significant digits. Final answers are usually reported with 3 significant digits. If you verify the calculations in this solutions manual using the rounded intermediate numerical results that are reported, you should obtain the final answers that are reported to 3 significant digits.

Chapter 3 Solutions

Problem 3.1

Two curling stones A and B, with masses m and $4m$, respectively, are pushed by two identical forces F over the distance d. Which stone arrives first to the finish line?

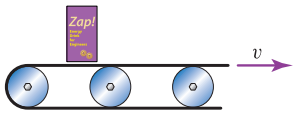


Solution

A will arrive at the finish line first. Since the forces acting on A and B are identical in magnitude and direction and the mass of A is smaller than the mass of B, Newtons second law tells us that A will experience a larger acceleration than B. In turn, A will cross the finish line before B.

Problem 3.2

An object is lowered very slowly onto a conveyer belt. What is the direction of the friction force acting on the object at the instant the object touches the belt?

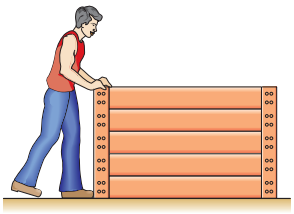


Solution

The friction force will be directed to the right. The horizontal velocity of the object as it is lowered onto the belt is equal to zero. Also, the vertical velocity of the object is negligible. Therefore, the relative velocity of the object with respect to the belt is horizontal and pointing to the left. Since the kinetic friction force opposes relative motion, then this force must point to the right.

Problem 3.3

A person is trying to move a heavy crate by pushing on it. While the person is pushing, what is the net or total force acting on the crate if the crate does not move?

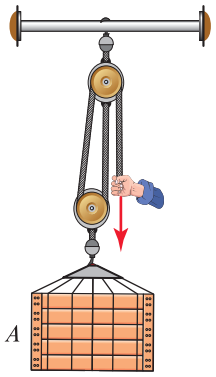


Solution

The net or total force on the crate is equal to zero. If the object is not moving, then its acceleration is equal to zero. Consequently, Newtons second law dictates that the total force acting on the object be equal to zero.

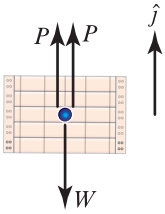
Problem 3.4

A person is lifting a 75 lb crate A by applying a constant force $P = 40$ lb to the pulley system shown. Neglecting friction and the inertia of the pulleys, determine the acceleration of the crate.



Solution

The tension in the rope is constant. Thus the force balance is



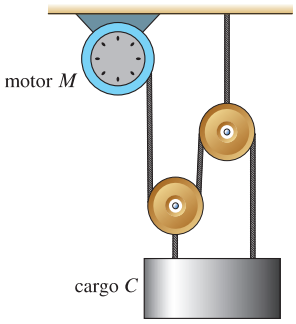
$$\sum F_y : \quad 2P - W = ma_y.$$

The mass of the crate is W/g where g is the acceleration due to gravity. Therefore

$$a_y = \frac{g\,(2P - W)}{W}.$$

Problem 3.5

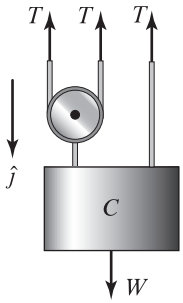
The motor M is at rest when someone flips a switch and it starts pulling in the rope. The acceleration of the rope is uniform and such that it takes 1 s to achieve a retraction rate of 4 ft/s. After 1 s the retraction rate becomes constant. Determine the tension in the rope during and after the initial 1 s interval. The cargo C weighs 130 lb, the weight of the ropes and pulleys is negligible, and friction in the pulleys is negligible.



Solution

Balance Principles:

$$\sum F_y: \quad -3T + W = \frac{W}{g} a_C. \tag{1}$$



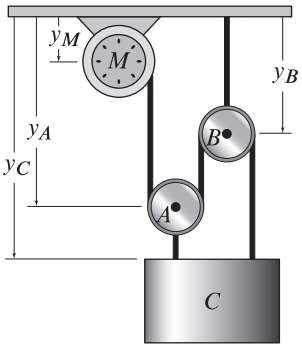
Kinematics

$$L = 3y_C + \text{constants}. \tag{2}$$

Differentiate Eq. (2) with respect to time to obtain relations between v_C and the retraction rate of the rope $\dot{L}$, and a_C and the change in retraction rate $\ddot{L}$ as

$$\dot{L} = 3v_C, \qquad \text{and} \qquad \ddot{L} = 3a_C. \tag{3}$$

The motor accelerates uniformly from rest, which implies that $\ddot{L}$ is constant. Then $\dot{L}(t) = \dot{L}(0) + \ddot{L}t$, where the initial rate $\dot{L}(0) = 0$ ft/s. Since $\dot{L}(t = 1\text{ s}) = -4$ ft/s (which is negative because the rope is being retracted), we have $\ddot{L} = -4\text{ ft/s}^2$. Combining Eqs. (1) and (3), the tension of the rope in terms of $\ddot{L}$ is



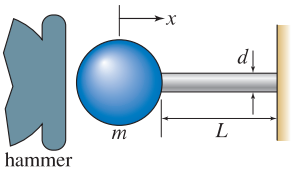
$$\boxed{T = \frac{W}{3}\left(1 - \frac{\ddot{L}}{3g}\right) = 45.1\text{ lb}.} \tag{4}$$

To find the tension in the cable after the motor is finished accelerating we set $\ddot{L}$ equal to 0 in Eq. (4):

$$\boxed{T = \frac{W}{3} = 43.3\text{ lb}.}$$

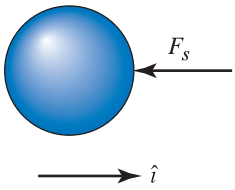
Problem 3.6

A hammer hits a mass m on the end of a metal bar. In Chapter 5 we will see that this imparts an instantaneous initial velocity v_0 at $x = 0$ to the mass. Treating the bar as a massless spring, determine the equation of motion of the mass m. The equivalent spring constant of a bar in compression is given by $k_{\text{eq}} = EA/L$, where E is Young's modulus of the bar, A is the cross-sectional area of the bar, and L the length of the bar.



Solution

First we write the Newton-Euler equations for the mass FBD shown to the right as



$$\sum F_x : \quad -F_s = ma_x.$$

Apply the force law for F_s

$$F_s = k_{eq}x \quad \Rightarrow \quad F_s = \frac{EA}{L}x.$$

Combine these equations to find that the acceleration is

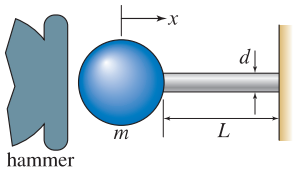
$$\boxed{\ddot{x} + \frac{EA}{mL}x = 0.}$$

Problem 3.7

For the mass described in Prob. 3.6:

(a) Integrate the equation of motion to determine the speed of the mass $v(x)$ as a function of x.

(b) Integrate the result found in Part (a) to obtain the position of the mass as a function of time $x(t)$ from the initial time up until the mass stops for the first time.



Solution

Part (a) The equation of motion is given by

$$\ddot{x} + \frac{EA}{mL}x = 0, \tag{1}$$

which was derived in the solution to problem 3.6. For part (a), to find $v(x)$ we integrate $\ddot{x}$ using the chain rule

$$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \quad \Rightarrow \quad \int_{v_0}^{v} v\,dv = \frac{EA}{mL}\int_0^x x\,dx, \tag{2}$$

where the lower limits correspond to $v = v_0$ when $x = 0$. The solution for $v(x)$ is

$$\boxed{v(x) = \sqrt{v_0^2 - \frac{EA}{mL}x^2}.} \tag{3}$$

Part (b) To find $x(t)$ we first find $t(x)$ by integrating Eq. (3). We replace v by dx/dt and integrate as

$$\int_0^t dt = \int_0^x \frac{dx}{\sqrt{v_0^2 - \frac{EA}{mL}x^2}}, \tag{4}$$

where the lower limits correspond to $x = 0$ when $t = 0$. The integral on the right hand side of Eq. (4) can be solved with the trig substitution $x \to v_0\sqrt{mL/EA}\sin\theta$. We obtain $t(x)$ as

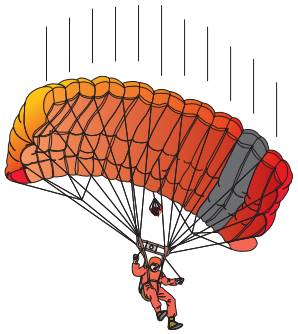
$$t(x) = \sqrt{\frac{mL}{EA}}\sin^{-1}\left(\sqrt{\frac{EA}{mL}}\frac{x}{v_0}\right), \tag{5}$$

which we can solve to find $x(t)$ as

$$\boxed{x(t) = v_0\sqrt{\frac{mL}{EA}}\sin\left(\sqrt{\frac{EA}{mL}}t\right).} \tag{6}$$

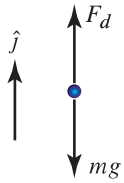
Problem 3.8

As the skydiver moves downward with a speed v, the air drag exerted by the parachute on the skydiver has a magnitude $F_d = C_d v^2$ (C_d is a drag coefficient) and a direction opposite to the direction of motion. Determine the expression of the skydiver's acceleration in terms of C_d, v, the mass of the skydiver m, and the acceleration due to gravity.



Solution

The force balance in the y direction is



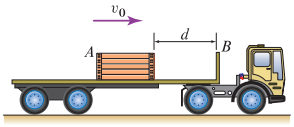
$$\sum F_y : \quad F_d - mg = ma_y.$$

The force law for F_d is given as $F_d = C_d v^2$. Solving for a_y we get

$$C_d v^2 - mg = ma_y \quad \Rightarrow \quad \boxed{a_y = \frac{C_d}{m} v^2 - g.}$$

Problems 3.9 through 3.11

The truck shown is traveling at $v_0 = 60$ mph when the driver applies the brakes to come to a stop. The deceleration of the truck is constant and the truck comes to a complete stop after braking for a distance of 350 ft. Treat the crate as a particle so that tipping can be neglected.



Problem 3.9 Determine the minimum coefficient of static friction between the crate A and the truck so that the crate does *not* slide relative to the truck.

Problem 3.10 If the coefficient of kinetic friction between the crate A and the bed of the truck is 0.3 and static friction is not sufficient to prevent slip, determine the minimum distance d between the crate and the truck B so that the crate never hits the truck at B.

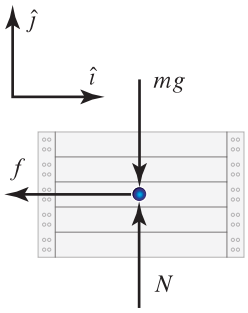
Problem 3.11 If the coefficient of kinetic friction between the crate A and the bed of the truck is 0.3, static friction is not sufficient to prevent slip, and the distance d from the front of the crate to the truck at B is 10 ft, determine the speed *relative to the truck* with which the crate strikes the truck at B.

Solution to 3.9

The Newton-Euler equations for crate A are

$$\sum F_x: \qquad -f = m(a_A)_x, \tag{1}$$

$$\sum F_y: \quad N - mg = m(a_A)_y, \tag{2}$$



where f is the friction force , N is the normal force, and mg is the weight of the crate. The kinematic equations are given by

$$(a_A)_x = (a_B)_x, \quad \text{and} \quad (a_A)_y = 0, \tag{3}$$

where $(a_B)_x$ is the acceleration of truck B. Because the deceleration of B is constant, we can use the constant acceleration equation $v^2 - v_0^2 = 2a_c(s - s_0)$. Let L be the distance the truck brakes.

$$(a_B)_x = \frac{v^2 - v_0^2}{2(x - x_0)} = \frac{-v_0^2}{2L}, \tag{4}$$

Combining Eqs. (1), (3) and (4) we find the friction force f to be

$$f = \frac{mv_0^2}{2L}, \tag{5}$$

and combining Eqs. (2) and (3) the normal force is

$$N = mg. \tag{6}$$

The minimum possible μ_s that prevents slipping would be $\mu_s = f/N$ which we find using solutions for f and N from Eqs. (5) and (6) as

$$\boxed{\mu_s = \frac{v_0^2}{2gL} = 0.344,} \tag{7}$$

where the braking distance is $L = 350$ ft, and g = 32.2 ft/s^2.

Solution to 3.10

The FBD and Eqs. (1) and (2) are identical to that of problem 3.9. The kinematic equation $(a_A)_y = 0$ is also true. For slipping $f = \mu_k N$ where μ_k is the kinetic coefficient of friction. Combining our material and kinematic equations with the Newton-Euler equations, we can solve for $(a_A)_x$ as

$$(a_A)_x = -\mu_k g. \tag{8}$$

Use the same constant acceleration equation as before to find

$$(x - x_0) = \frac{v^2 - v_0^2}{2(a_A)_x}. \tag{9}$$

Let $x_0 = -d$, $x = L$, and $v = 0$. Solve for d.

$$\boxed{d = \frac{v_0^2}{2\mu_k g} - L = 50.8\,\text{ft},} \tag{10}$$

where $\mu_k = 0.3$, $v_0 = 88\,\text{ft/s}$, $L = 350\,\text{ft}$, and g = $32.2\,\text{ft/s}^2$.

Solution to 3.11

To find the velocity of crate A relative to the truck when it strikes point B, we need to determine whether the crate strikes B before or after the truck stops. For this problem the FBD of the crate in the solution to problem 3.9 is still valid, and the Newton-Euler equations are given by Eqs. (1) and (2). As with the solution to problem 3.10, $(a_A)_y = 0$ and $f = \mu_k N$ where $\mu_k = 0.3$. The acceleration of the crate $(a_A)_x$ is given by Eq. (8) and the acceleration of the truck $(a_B)_x$, from the time that the truck begins to accelerate until the time the truck comes to a stop, is given by Eq. (4). After the truck stops, its acceleration is zero. This implies that the acceleration of the crate relative to the truck may not be constant during the entire motion of the crate. Because the acceleration of the truck is constant during braking, we can determine the time it takes for the truck to stop by manipulating a constant acceleration equation to find

$$t = \frac{v - v_0}{(a_B)_x} + t_0 \quad \Rightarrow \quad t_s = \frac{2L}{v_0}, \tag{11}$$

where $(a_B)_x = -v_0^2/2L$ from Eq. (4) and we designate $t = t_s$ as the time when the truck stops. Using a constant acceleration equation we derive an expression for the position of the truck as a function of time

$$x_B = \tfrac{1}{2}(a_B)_x t^2 + v_0 t + x_0 \quad \Rightarrow \quad x_B = -\frac{v_0^2}{4L} t^2 + v_0 t, \tag{12}$$

The initial velocity is v_0, and the acceleration $(a_B)_x$ is again given by Eq. (4). Also we can write the position of the crate as a function of time as

$$x_A = \tfrac{1}{2}(a_A)_x t^2 + v_0 t + x_0 \quad \Rightarrow \quad x_A = -\frac{\mu_k g}{2} t^2 + v_0 t - d, \tag{13}$$

where $x_0 = -d$ because A is initially distance d behind point B, the initial velocity is v_0 which is the same as the truck's initial velocity, and $(a_A)_x = -\mu_k g$ as given by Eq. (8). During the braking time, the position of A relative to B is $x_{A/B} = x_A - x_B$ which we obtain by subtracting Eq. (12) from Eq. (13) to obtain

$$x_{A/B} = \left(\frac{v_0^2}{2L} - \mu_k g \right) \frac{t^2}{2} - d. \tag{14}$$

It is important to understand that Eq. (14) is valid only during braking. After the truck comes to a stop, $x_B = L$ and results in a different formula for $x_{A/B}$. If we suppose that the crate strikes point B before the truck stops, we can determine the time of the collision by letting the relative position $x_{A/B} = 0$ in Eq. (14) and solving for t as

$$t = t_c = \sqrt{\frac{4dL}{v_0^2 - 2\mu_k g L}}, \tag{15}$$

where we designate $t = t_c$ as the collision time. We may now compare the collision time t_c with the truck stopping time t_s. If $t_c \leq t_s$ then we know that the collision occurs before the truck stops. We can either calculate both t_s and t_c by substituting values, or find a simpler relation that gives us a criterion for values of d. If we square both times we have the equivalent comparison $t_c^2 \leq t_s^2$ in which we substitute for t_c from Eq. (15) and for t_s from Eq. (11) and after some algebra we obtain the criterion

$$d \leq L\left(1 - \frac{2\mu_k g L}{v_0^2}\right), \tag{16}$$

and for given values of $L = 350\,\text{ft}$, $v_0 = 88\,\text{ft/s}$, $\mu_k = 0.3$ and $g = 32.2\,\text{ft/s}^2$ the right hand side evalutes to $44.38\,\text{ft}$ which is greater than the given $d = 10\,\text{ft}$. Thus Eq. (14) is valid when the crate hits the truck, and we could find the relative velocity by differentiating Eq. (14) and letting $t = t_c$, or we can use the fact that the relative acceleration is constant *during braking* which we may write as

$$a_{A/B} = a_A - a_B = -\mu_k g + \frac{v_0^2}{2L}, \tag{17}$$

where $a_A = (a_A)_x$ and $a_B = (a_B)_x$ to simplify notation. Since $a_{A/B}$ is constant, we solve Eq. (2.43) from the text to find the relative velocity as a function of relative position as

$$v_{A/B} = \sqrt{\left((v_{A/B})_0\right)^2 + 2a_{A/B}\left(x_{A/B} - (x_{A/B})_0\right)} = \sqrt{\left(-2\mu_k g + \frac{v_0^2}{L}\right)\left(x_{A/B} + d\right)}, \tag{18}$$

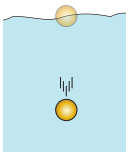
where the initial relative velocity $(v_{A/B})_0 = 0$, the initial relative position is $(x_{A/B})_0 = -d$, and the relative acceleration $a_{A/B}$ is given by Eq. (17). The collision occurs when $x_{A/B} = 0$ and the collision velocity is then

$$\boxed{v_{A/B} = \sqrt{d\left(-2\mu_k g + \frac{v_0^2}{L}\right)} = 5.30\,\text{ft/s},} \tag{19}$$

where $L = 350\,\text{ft}$, $d = 10\,\text{ft}$, $v_0 = 88\,\text{ft/s}$, $\mu_k = 0.3$ and $g = 32.2\,\text{ft/s}^2$.

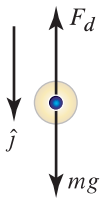
Problem 3.12

A metal ball with mass $m = 0.15\,\text{kg}$ is dropped from rest in a fluid. The magnitude of the resistance due to the fluid is given by $C_d v$, where C_d is a drag coefficient and v is the ball's speed. If $C_d = 2.1\,\text{kg/s}$, determine the ball's speed 4 s after release.



Solution

The Newton-Euler equation in the $\hat{\jmath}$ direction is



$$\sum F_y: \quad mg - F_d = ma_y.$$

The force law is

$$F_d = C_d v.$$

The kinematic equations are

$$a_y = \ddot{y} \quad \text{and} \quad v = \dot{y}.$$

Combine the equations and solve for $\ddot{y}$.

$$mg - C_d \dot{y} = m\ddot{y} \quad \Rightarrow \quad \ddot{y} = g - \frac{C_d \dot{y}}{m}.$$

Integrate $\ddot{y}$ recalling that

$$\ddot{y} = \frac{d\dot{y}}{dt} \quad \Rightarrow \quad \int_0^t dt = \int_0^{\dot{y}} \frac{d\dot{y}}{g - (C_d/m)\dot{y}} \quad \Rightarrow \quad t = \frac{-m}{C_d} \ln\left(g - (C_d/m)\dot{y}\right)\Big|_0^{\dot{y}}$$

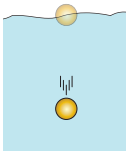
$$\Rightarrow \quad t = \frac{-m}{C_d} \ln\left(1 - \frac{C_d}{mg}\dot{y}\right).$$

Invert this equation to find $\dot{y}(t)$ and evaluate at $t = 4\,\text{s}$.

$$\dot{y} = \frac{mg}{C_d}\left(1 - e^{\frac{-C_d t}{m}}\right) = 0.701\,\text{m/s}.$$

Problem 3.13

A metal ball weighing 0.35 lb is dropped from rest in a fluid. The magnitude of the resistance due to the fluid is given by $C_d v$, where C_d is a drag coefficient and v is the ball's speed. It is observed that 2 s after release the speed of the ball is 25 ft/s. Determine the value of C_d.



Solution

We take advantage of the equation for $\dot{y}$ derived in Problem 3.12.

$$\dot{y} = \frac{mg}{C_d}\left(1 - e^{\frac{-C_d t}{m}}\right).$$

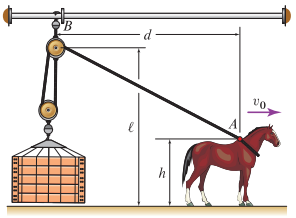
Everything but C_d is fixed so we can find a numeric solution using *Mathematica* with the following code:

```
rules = {m → 0.35 / g, g → 32.2, t → 2};
v[Cd_] := m g / Cd (1 - e^(-Cd t / m))
soln = NSolve[v[Cd] == 25 /. rules, Cd]
```

$$C_d = 0.0126 \text{ slug/s}.$$

Problem 3.14

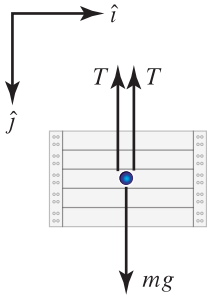
A horse is lifting a 500 lb crate by moving to the right at a constant speed $v_0 = 3\,\text{ft/s}$. Observing that B is fixed and letting $h = 6\,\text{ft}$ and $\ell = 14\,\text{ft}$, determine the tension in the rope when the horizontal distance d between B and point A on the horse is 10 ft.



Solution

The Newton-Euler equations are

$$\sum F_x: \quad 0 = ma_x,$$
$$\sum F_y: \quad mg - 2T = ma_y. \tag{1}$$

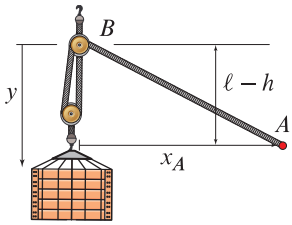


The kinematic equation is $a_y = \ddot{y}$. We must examine the pulley kinematics. The length of the rope is

$$L = 2y + \sqrt{x_A^2 + (\ell - h)^2}.$$

The length of the rope does not change with time. Therefore

$$0 = 2\dot{y} + \frac{x_A \dot{x}_A}{\sqrt{x_A^2 + (\ell - h)^2}}.$$



Solve for $\dot{y}$:

$$\dot{y} = \frac{-x_A \dot{x}_A}{2\sqrt{x_A^2 + (\ell - h)^2}}.$$

Differentiate $\dot{y}$ with respect to time. Recall that $\dot{x}_A = v_0$ and $\ddot{x} = 0$ because v_0 is constant.

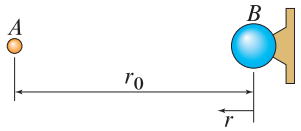
$$\ddot{y} = \frac{-\dot{x}_A^2}{2\sqrt{x_A^2 + (\ell - h)^2}} + \frac{x_A^2 \dot{x}_A^2}{2\left[x_A^2 + (\ell - h)^2\right]^{3/2}} \quad \Rightarrow \quad \ddot{y} = \frac{-v_0^2 (\ell - h)^2}{2\left[x_A^2 + (\ell - h)^2\right]^{3/2}}.$$

Solve Eq. (1) for T:

$$T = \frac{m}{2}(g - \ddot{y}) \quad \Rightarrow \quad \boxed{T = \frac{m}{2}\left\{g + \frac{v_0^2 (\ell - h)^2}{2\left[x_A^2 + (\ell - h)^2\right]^{3/2}}\right\} = 251\,\text{lb}.}$$

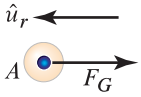
Problem 3.15

The centers of two spheres A and B with masses $m_A = 1\,\text{kg}$ and $m_B = 2\,\text{kg}$ are a distance $r_0 = 1\,\text{m}$ apart. B is fixed in space, and A is initially at rest. Using Eq. (1.6) on p. 5, which is Newton's universal law of gravitation, determine the speed with which A impacts B if the radii of the two spheres are $r_A = 0.05\,\text{m}$ and $r_B = 0.15\,\text{m}$. Assume that the two masses are infinitely far from any other mass so that they are only influenced by their mutual attraction.



Solution

A is the only object that moves, therefore we will focus on it. The Newton-Euler equation is



$$\sum F_r : \quad -F_G = m_A a_r.$$

The force law is

$$F_G = G\frac{m_A m_B}{r^2}.$$

The kinematic equation is $a_r = \ddot{r}$. Therefore we have

$$-G\frac{m_A m_B}{r^2} = m_A \ddot{r} \quad \Rightarrow \quad \ddot{r} = \frac{-Gm_B}{r^2}.$$

Integrate $\ddot{r}$ recalling that

$$\ddot{r} = \frac{d\dot{r}}{dt} = \dot{r}\frac{d\dot{r}}{dr} \quad \Rightarrow \quad -Gm_B \int_{r_0}^{r_A+r_B} \frac{dr}{r^2} = \int_0^{v_c} \dot{r}\, d\dot{r},$$

$$\tfrac{1}{2}v_c^2 = Gm_B\left(\frac{1}{r_A + r_B} - \frac{1}{r_0}\right),$$

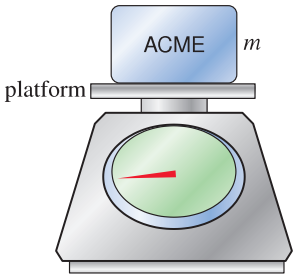
where v_c represents the speed at impact.

$$v_c = \sqrt{2Gm_B\left(\frac{1}{r_A + r_B} - \frac{1}{r_0}\right)} = 32.7\times10^{-6}\,\text{m/s},$$

where $G = 6.6742\times10^{-11}\,\text{m}^3/\text{kg}\cdot\text{s}^2$.

Problems 3.16 and 3.17

Spring scales work by measuring the displacement of a spring that supports both the platform and the object, of mass m, whose weight is being measured. Neglect the mass of the platform on which the mass sits and assume that the spring is uncompressed before the mass is placed on the platform. In addition, assume that the spring is linear elastic with spring constant k.

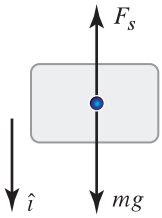


Problem 3.16 If the mass m is gently placed on the spring scale (i.e., it is dropped from zero height above the scale), determine the *maximum reading* on the scale after the mass is released.

Problem 3.17 If the mass m is gently placed on the spring scale (i.e., it is dropped from zero height above the scale), determine the *maximum speed* attained by the mass m as the spring compresses.

Solution to 3.16

The Newton-Euler equations are



$$\sum F_x: \quad mg - F_s = ma_x, \tag{1}$$

The force law is $F_s = -k\delta$ (F_s is negative because it is the force from the spring). Choose a coordinate system such that $x = -\delta$. The kinematic equation is

$$a_x = \ddot{x}, \quad \text{therefore} \quad \ddot{x} = g - \frac{k}{m}x.$$

Use the following equation to obtain the velocity as a function of position

$$v^2(s) = v_0^2 + 2\int_{s_0}^{s} a(s)\,ds \quad \Rightarrow \quad v^2(x) = v_0^2 + 2\int_{x_0}^{x} g - \frac{k}{m}x\,dx \quad \Rightarrow \quad v^2(x) = 2gx - \frac{k}{m}x^2. \tag{2}$$

The maximum reading on the scale occurs when the velocity of the mass is zero, so we set Eq. (2) equal to zero and find $x_{\max}$ (the position where $|\delta|$ is maximum):

$$0 = 2gx - \frac{k}{m}x^2 \quad \Rightarrow \quad x = 2\frac{mg}{k}. \tag{3}$$

The maximum reading on the scale will be the value of the spring force:

$$\boxed{(F_s)_{\max} = kx_{\max} = 2mg.} \tag{4}$$

Solution to 3.17

Take the derivative with respect to x of Eq. (2) and set the results equal to 0:

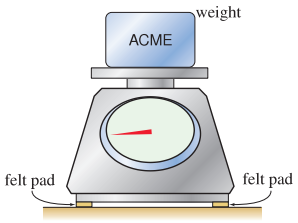
$$\frac{d}{dx}v^2(x) = 0 \quad \Rightarrow \quad 2g - 2\frac{k}{m}x = 0, \tag{5}$$

for x from which obtain $x = mg/k$ for the position that maximizes $v^2(x)$. We substitute this value for x into Eq. (2) to obtain the maximum velocity as

$$\boxed{v_{\max} = g\sqrt{\frac{m}{k}}.} \tag{6}$$

Problem 3.18

A scale is to be used on a wood countertop made from some very nice Brazilian cherry. To protect the countertop, the owner attaches self-sticking felt pads to the feet of the scale. When the weight was placed on the scale before the felt pads were applied, the scale read a certain value. Will the value be higher, lower, or the same when the same weight is placed on the scale but with the felt pads between the scale and the countertop? Ignore the transient dynamic effects that occur immediately after the weight is placed on the scale.



Solution

The reading on the scale is unaffected by the presence of the felt pads. The reading on the scale depends on the deformation of the spring supporting the scale's plate. Based on a simple FBD of the object being weighed, at equilibrium, the spring will have to provide the scale's plate a supporting force equal to the object's weight. Hence, as long as the felt pads do not interfere with the spring's ability to deform, the reading on the scale will be the weight of the object placed on the scales plate.

Problems 3.19 and 3.20



A vehicle is stuck on the railroad tracks as a 430,000 lb locomotive is approaching with a speed of 75 mph. As soon as the problem is detected, the locomotive's emergency brakes are activated, locking the wheels and causing the locomotive to slide.

Problem 3.19 If the coefficient of kinetic friction between the locomotive and the track is 0.45, what is the minimum distance d at which the brakes must be applied to avoid a collision? What would that distance be if instead of a locomotive, there was a 30×10^6 lb train? Treat the locomotive and the train as particles, and assume that the railroad tracks are rectilinear and horizontal.

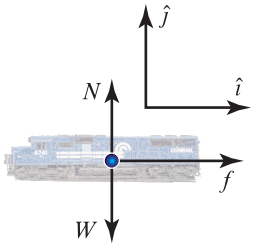
Problem 3.20 Continue Prob. 3.19 and determine the time required to stop the locomotive.

Solution to 3.19

The Newton-Euler equations are

$$\sum F_x: \qquad f = \frac{W}{g}a_x, \tag{1}$$

$$\sum F_y: \quad N - W = \frac{W}{g}a_y. \tag{2}$$



The force law is $f = \mu_k N$. The kinematic equation is $a_y = 0$. Thus the acceleration in the x direction is

$$a_x = \mu_k g. \tag{3}$$

Use the constant acceleration equation $v^2 = v_0^2 + 2a_c(s - s_0)$ to find an expression for the change in position:

$$\boxed{d = \frac{v_0^2}{2\mu_k g} = 418\,\text{ft}.}$$

where we let $x_0 = d$ and $x = 0$.

If an entire train weighing 30×10^6 lb was skidding to a stop, instead of a locomotive, we would have the same stopping distance because the weight does not appear in our solution.

Solution to 3.20

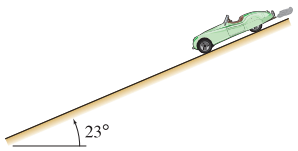
Everything derived in Problem 3.19, is still valid. Use the constant acceleration equation $v = v_0 + a_c t$ along with Eq. (3) to find

$$\boxed{t = \frac{v - v_0}{\mu_k g} = 7.59\,\text{s}} \tag{4}$$

where we let the initial time be $t_0 = 0$, the initial velocity is $v_0 = -75\,\text{mph} = -110\,\text{ft/s}$, and and the final velocity is $v = 0$.

Problem 3.21

A car is driving down a 23° rough incline at 55 km/h when its brakes are applied. Treating the car as a particle and neglecting all forces except gravity and friction, determine the stopping distance if



(a) The tires slide and the coefficient of kinetic friction between the tires and the road is 0.7.

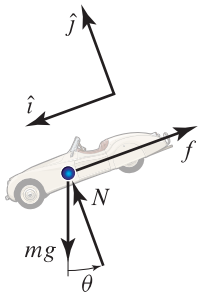
(b) The car is equipped with antilock brakes and the tires do not slide. Use 0.9 for the coefficient of static friction between the tires and the road.

Solution

Let $\theta = 23°$. The Newton-Euler equations are

$$\sum F_x: \quad mg\sin\theta - f = ma_x, \tag{1}$$

$$\sum F_y: \quad N - mg\cos\theta = ma_y, \tag{2}$$



The kinematic equation is

$$a_y = 0. \tag{3}$$

Part (a) When the tires slide the material equation is

$$f = \mu_k N, \tag{4}$$

Solve for the acceleration of the car a_x by combining Eqs. (1)–(3) with Eq. (4) so that

$$a_x = g\left(\sin\theta - \mu_k\cos\theta\right). \tag{5}$$

a_x is constant so use a constant acceleration equation to find the braking distance:

$$d = x - x_0 = \frac{v^2 - v_0^2}{2a_x}, \tag{6}$$

Combining Eqs. (5) and (6), we find the braking distance is

$$\boxed{d = \frac{-v_0^2}{2g\left(\sin\theta - \mu_k\cos\theta\right)} = 46.9\,\text{m}.}$$

Part (b) In this case we have static friction. We assume that sliding is impending and use the material equation

$$f = \mu_s N. \tag{7}$$

We solve for the acceleration of the car a_x by combining Eqs. (1)–(3) with Eq. (7) so that

$$a_x = g\left(\sin\theta - \mu_s\cos\theta\right). \tag{8}$$

a_x is constant so use a constant acceleration equation to find the braking distance:

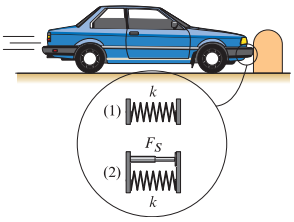
$$d = x - x_0 = \frac{v^2 - v_0^2}{2a_x}, \tag{9}$$

Combine Eqs. (9) and (8)

$$\boxed{d = \frac{-v_0^2}{2g\,(\sin\theta - \mu_s \cos\theta)} = 27.2\,\text{m}.}$$

Problems 3.22 through 3.24

Car bumpers are designed to limit the extent of damage to the car in the case of low-velocity collisions. Consider a 3300 lb passenger car impacting a concrete barrier while traveling at a speed of 4.0 mph. Model the car as a particle, and consider two types of bumper: (1) a simple linear spring with constant k and (2) a linear spring of constant k in parallel with a shock absorbing unit generating a nearly constant force of 700 lb over 0.25 ft.



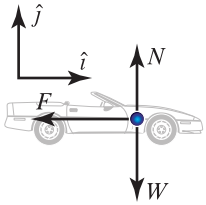
Problem 3.22 If the bumper is of type 1 and if $k = 6500\,\text{lb/ft}$, find the spring compression necessary to stop the car.

Problem 3.23 If the bumper is of type 1, find the value of k necessary to stop the car when the bumper is precompressed 0.25 ft.

Problem 3.24 If the bumper is of type 2, find the value of k necessary to stop the car when the bumper is precompressed 0.25 ft.

Solution to 3.22

Let W represent the weight of the car. The Newton-Euler equations are



$$\sum F_x: \qquad -F = \frac{W}{g}a_x, \tag{1}$$

$$\sum F_y: \quad N - W = \frac{W}{g}a_y, \tag{2}$$

The force law for the bumper is

$$F = kx \tag{3}$$

where x is the *compression* of the bumper. Thus the acceleration of the car is

$$a_x = -\frac{gk}{W}x. \tag{4}$$

We have acceleration as a function of position. Use the equation

$$v^2(s) = v_0^2 + 2\int_0^s a(s)\,ds \quad \Rightarrow \quad v_x^2(x) = v_0^2 - \frac{2gk}{W}\int_0^{x_{\text{stop}}} x\,dx, \tag{5}$$

which we integrate to obtain

$$v_x^2(x) = v_0^2 - \frac{gk}{W}x_{\text{stop}}^2. \tag{6}$$

Let $v_x = 0$ in Eq. (6) and solve for x as

$$\boxed{x_{\text{stop}} = v_0\sqrt{\frac{W}{gk}} = 0.737\,\text{ft},} \tag{7}$$

where the initial velocity is $v_0 = 4.0\,\text{mph} = 5.867\,\text{ft/s}$.

Solution to 3.23

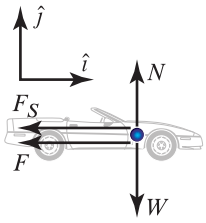
This problem is identical to 3.22 up to Eq. (6). Now instead of solving for x_{stop}, we let $v_x = 0$ and solve for the required spring constant k as

$$\boxed{k = \frac{W}{g}\frac{v_0^2}{x_{\text{stop}}^2} = 5.64\times10^4\,\text{lb/ft},} \tag{8}$$

where $v_0 = 4.0\,\text{mph} = 5.867\,\text{ft/s}$, and $W = 3300\,\text{lb}$.

Solution to 3.24

Let W represent the weight of the car. The Newton-Euler equations are



$$\sum F_x: \quad -F_S - F = \frac{W}{g}a_x, \tag{9}$$

$$\sum F_y: \quad N - W = \frac{W}{g}a_y, \tag{10}$$

The force laws are $F_S = 700\,\text{lb}$ and $F = kx$. The kinematics are $a_y = 0$ and $a_x = \ddot{x}$. Solve Eq. (9) for $a_x = \ddot{x}$ and note that we have acceleration as a function of position. Rewrite $\ddot{x}$ and integrate the results:

$$\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} \quad \Rightarrow \quad -\int_0^{x_{\text{stop}}}(F_S + kx)\,dx = \frac{W}{g}\int_{\dot{x}_0}^{0}\dot{x}\,d\dot{x} \quad \Rightarrow \quad -2F_S x_{\text{stop}} - kx_{\text{stop}}^2 = -\frac{W}{g}\dot{x}_0$$

$$\boxed{k = \frac{Wv_0^2}{gx_{\text{stop}}^2} - \frac{2F_S}{x_{\text{stop}}} = 50{,}800\,\text{lb/ft},}$$

where $v_0 = 4.0\,\text{mph} = 5.867\,\text{ft/s}$, and $W = 3300\,\text{lb}$.

Problem 3.25

What would it mean for the static or kinetic friction coefficients to be negative? Is this possible? Can either the static or kinetic friction coefficients be greater than 1? If yes, explain and give an example.

Solution

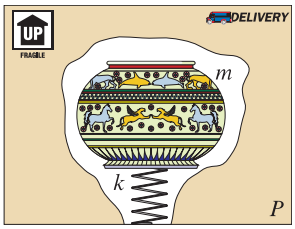
A negative coefficient of static friction is not possible by the definition of μ_s: $|F| \leq \mu_s |N|$. When this inequality is solved for μ_s the results can never be negative.

A negative coefficient of kinetic friction would mean that $\vec{f}_k$ would not be opposite to the direction of relative motion. A negative coefficient of kinetic friction would therefore act in the direction of motion which is not physically meaningful.

It is possible to generate friction forces greater than the normal force between two objects, therefore both the static and kinetic coefficients of friction can exceed one. For example, velcro is lightweight so the normal force between its straps is small, assuming there are no other forces that contribute to the normal force besides the weight of the velcro, yet it will provide large friction forces to oppose motion between the hooked strap and the loop strap. Another example is hot rubber on asphalt. Formula 1 race cars have their tires preheated to increase the adhesive forces between the rubber and the racetrack.

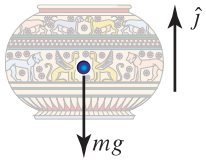
Problem 3.26

Packages for transporting delicate items (e.g., a laptop or glass) are designed to "absorb" some of the energy of the impact in order to protect the contents. These energy absorbers can get pretty complicated to model (e.g., the mechanics of styrofoam peanuts is not easy), but we can begin to understand how they work by modeling them as a linear elastic spring of constant k that is placed between the contents (an expensive vase) of mass m and the package P. Assuming that $m = 3\,\text{kg}$ and that the box is dropped from a height of $1.5\,\text{m}$, determine the maximum displacement of the vase relative to the box and the maximum force on the vase if $k = 3500\,\text{N/m}$. Treat the vase as a particle and neglect all forces except for gravity and the spring force. Assume that the spring relaxes after the box is dropped and that it does not oscillate.



Solution

The Newton-Euler equations for the vase during free fall are



$$\sum F_x: \quad 0 = ma_x, \tag{1}$$

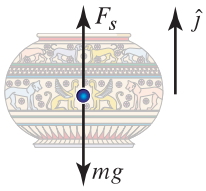
$$\sum F_y: \quad -mg = ma_y. \tag{2}$$

The velocity of the package when it hits the floor is found using the constant acceleration equation $v^2 = v_0^2 + 2a(s - s_0)$. Call v_1 this velocity, then

$$v_1 = -\sqrt{v_0^2 + 2a_y\,(y_1 - y_0)} = -\sqrt{2gh}, \tag{3}$$

We choose a negative sign for the square root because the velocity is downward.

Once the package strikes the floor, the Netwon-Euler equation is



$$\sum F_y: \quad F_s - mg = ma_y, \tag{4}$$

where F_s is the force on the vase from the foam packaging that we model using the material equation for a spring force as

$$F_s = ky, \tag{5}$$

where y is the compression of the spring, and it is the quantity we are trying to find. We combine Eqs. (4) and (5) to solve for the acceleration a_y during the relative motion as a function of y as

$$a_y = -g + \frac{k}{m}y. \tag{6}$$

Since a_y in Eq. (6) is a function of position y, we may use the following equation to find the velocity of the vase relative to the package as

$$\int_{v_1}^{0} \dot{y}\,d\,\dot{y} = \int_0^{\delta_{\max}} \left(-g + \frac{k}{m}y\right) dy, \tag{7}$$

where the initial velocity is $v_1 = -\sqrt{2gh}$ which we found in Eq. (3). We integrate Eq. (7) and substitute for v_1 to obtain

$$v_1^2 = 2g\delta_{\max} - \frac{k}{m}\delta_{\max}^2. \tag{8}$$

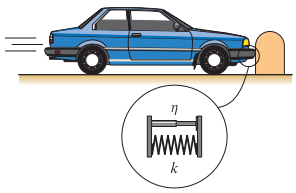
This is a quadratic equation in δ whose solution can be simplified to obtain:

$$\delta_{\max} = \frac{mg}{k}\left(1 \pm \sqrt{1 + \frac{kv_1^2}{mg^2}}\right) \quad \Rightarrow \quad \boxed{y_{\max} = 0.167\,\text{m}.} \tag{9}$$

where we have substituted Eq. (3) for v_1 with $h = 1.5\,\text{m}$ $g = 9.81\,\text{m/s}$. We have chosen a positive sign on the square root because we expect positive values for δ. Similarly, the maximum force will occur when $\delta = \delta_{\max}$, thus the maximum force exerted on the base will be

$$\boxed{\left(F_{\text{sp}}\right)_{\max} = k\delta_{\max} = 586\,\text{N}.}$$

Problems 3.27 and 3.28



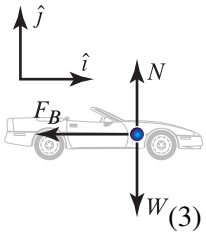
Car bumpers are designed to limit the extent of damage to the car in the case of low-velocity collisions. Consider a passenger car impacting a concrete barrier while traveling at a speed of 4.0 mph. Model the car as a particle of mass m, and assume that the bumper has a spring element in parallel with a shock absorber so that the overall force exerted by the bumper is $F_B = k\delta + \eta\dot{\delta}$, where k, δ, and η denote the spring constant, the spring compression, and the bumper damping coefficient, respectively.

Problem 3.27 Derive the equations of motion for the car during the collision.

Problem 3.28 Let the weight of the car be 3300 lb, $k = 6500$ lb/ft, and $\eta = 300$ lb·s/ft, and let the car be traveling at 4.0 mph at impact. Determine the maximum compression of the bumper necessary to bring the car to a stop. Also determine the time required to stop the car.

Solution to 3.27

The Newton-Euler equations for the car during its collision with the block are



$$\sum F_x: \quad -F_B = ma_x, \tag{1}$$

$$\sum F_y: \quad N - W = ma_y, \tag{2}$$

F_B is defined by the material equation

$$F_B = k\delta + \eta\dot{\delta}. \tag{3}$$

We choose a coordinate system that measures the position x of the front end of the car as it compresses the bumper. Then we let $x = 0$ when the bumper initially contacts the block, and x increases as the car continues to compress the bumper. In these coordinates, the compression of the bumper δ is $\delta = x$. We can substitute x for δ and $\dot{x}$ for $\dot{\delta}$ in Eq. (3) and substitute $\ddot{x}$ for a_x in Eq. (1), and then combine Eqs. (3) and (1) to obtain the equation of motion as

$$\boxed{m\ddot{x} + \eta\dot{x} + kx = 0.} \tag{4}$$

Solution to 3.28

Using a software package such as *Mathematica* we solve Eq. (4) with the following code

```
Parameters = {g → 32.2, w → 3300, k → 6500, η → 300, v0 → 5.8666};
eom = 0 == w/g δ''[t] + η δ'[t] + k δ[t];
initialConds = {δ[0] == 0, δ'[0] == v0};
delta = DSolve[{eom /. Parameters, initialConds /. Parameters}, δ[t], t]
```

$$\delta(t) = \left[0.7494e^{-1.464\,t}\sin(7.828\,t)\right]\text{ft},$$

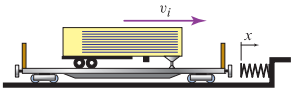
We find the maximum displacement with the following *Mathematica* code:

```
Maximize[δ[t] /. delta, t]
```

$$\boxed{\delta_{\max} = 0.568\,\text{ft}, \quad \text{when} \quad t = 0.177\,\text{s}.}$$

Problems 3.29 through 3.31

A railcar with an overall mass of 75,000 kg traveling with a velocity v_i is approaching a barrier equipped with a bumper constisting of a nonlinear spring whose force vs. compression law is given by $F_s = \beta x^3$, where $\beta = 640 \times 10^6\ \mathrm{N/m^3}$ and x is the compression of the bumper.



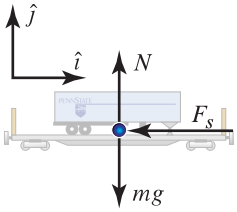
Problem 3.29 Treating the system as a particle and assuming that the contact between the railcar and rails is frictionless, determine the maximum value of v_i so that the compression of the bumper is limited to 20 cm.

Problem 3.30 Treating the system as a particle, assuming that the contact between railcar and rails is frictionless, and letting $v_i = 6\,\mathrm{km/h}$, determine the bumper compression necessary to bring the railcar to a stop.

Problem 3.31 Treating the system as a particle, assuming that the contact between the railcar and rails is frictionless, and letting $v_i = 6\,\mathrm{km/h}$, determine how long it takes for the bumper to bring the railcar to a stop.

Solution to 3.29

Balance Principles:



$$\sum F_x: \qquad -F_s = ma_x, \tag{1}$$

$$\sum F_y: \quad N - mg = ma_y. \tag{2}$$

The force law for F_s is $F_s = \beta x^3$. The kinematic equations are $a_y = 0$ and $a_x = \ddot{x}$. We find the acceleration a_x as a function of x from Eq. (1) as

$$a_x = -\frac{\beta}{m} x^3. \tag{3}$$

To find the velocity as a function of x, we can use the following equation

$$v^2(x) = v_i^2 + 2\int_0^x a_x\, dx \quad \Rightarrow \quad v^2(x) = v_i^2 - \frac{2\beta}{m}\int_0^{x_{\text{stop}}} x^3\, dx \quad \Rightarrow \quad v^2(x) = v_i^2 - \frac{\beta x_{\text{stop}}^4}{2m}. \tag{4}$$

Set $v = 0$ and solve Eq. (4) for the initial velocity v_i in terms of x_{stop} as

$$\boxed{v_i = x_{\text{stop}}^2\sqrt{\frac{\beta}{2m}} = 2.61\,\mathrm{m/s}.} \tag{5}$$

Solution to 3.30

The FBD and Newton-Euler equations for this problem are the same as in problem 3.29, and we can use Eq. (4) to find the maximum deflection for a given $v_i = 6\,\mathrm{km/h} = 1.667\,\mathrm{m/s}$. In Eq. (4) we let $v(x) = 0$ at the maximum deflection x_{stop} and solve for x_{stop} as

$$\boxed{x_{\text{stop}} = \sqrt[4]{\frac{2mv_i^2}{\beta}} = 0.160\,\mathrm{m}.} \tag{6}$$

Solution to 3.31

The FBD and Newton-Euler equations for this problem are the same as in problem 3.29, and we make use of the velocity expression in Eq. (4). To find the time it takes for the railcar to come to a stop, we can use the following equation to get time as a function of x as

$$t(x) = t_i + \int_0^x \frac{dx}{v(x)}. \tag{7}$$

$$t(\delta) = \int_0^\delta \frac{dx}{\sqrt{v_i^2 - \beta x^4/(2m)}}. \tag{8}$$

The integral in Eq. (8) is solved most easily by using a computer. The integral can be simplified by performing the substitution $x \to u\sqrt[4]{2mv_i^2/\beta}$, which given the result from problem 3.30 in Eq. (6), is the same as substituting $x \to u\delta$. The properly transformed integral in terms of u is then

$$t(\delta) = \frac{\delta}{v_i}\int_0^1 \frac{du}{\sqrt{1-u^4}}. \tag{9}$$

To solve Eq. (9) in *Mathematica*, we use the `Integrate` function as shown in the following code:

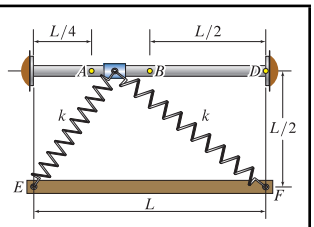
```
m = 75000; beta = 640*10^6; v1 = 6 * (1000/(60*60));
delta = ((2 * m * v1^2)/beta)^(1/4);
t = (delta / v1) * Integrate[1/Sqrt[1-u^4], {u, 0, 1}] // N
```

The above code returns the time t. The stopping time calculated by these numerical codes is

$$\boxed{t(\delta) = 0.126\,\text{s}.} \tag{10}$$

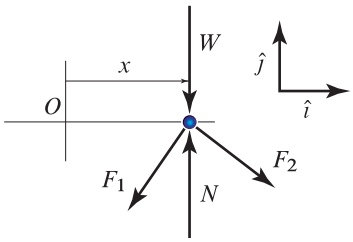
Problem 3.32

A 6 lb collar is constrained to travel along a rectilinear and frictionless bar of length $L = 5$ ft. The springs attached to the collar are identical, and are unstretched when the collar is at B. Treating the collar as a particle, neglecting air resistance, and knowing that at A the collar is moving to the right with a speed of 11 ft/s, determine the linear spring constant k so that the collar reaches D with zero speed. Points E and F are fixed.



Solution

On the FBD to the right, the forces F_1 and F_2 are the forces of the springs acting on the collar. Taking the origin O to be the left end of the bar, the spring forces are written in Cartesian coordinates as



$$\vec{F}_1 = k\delta_1 \left(\frac{-x\,\hat{\imath} - (L/2)\,\hat{\jmath}}{L_1} \right), \tag{1}$$

$$\vec{F}_2 = k\delta_2 \left(\frac{(L-x)\,\hat{\imath} - (L/2)\,\hat{\jmath}}{L_2} \right), \tag{2}$$

where the lengths of the springs, L_1 and L_2, are given by

$$L_1 = \sqrt{x^2 + L^2/4} \quad \text{and} \quad L_2 = \sqrt{(L-x)^2 + L^2/4}, \tag{3}$$

and the stretches of the springs, δ_1 and δ_2, are given by

$$\delta_1 = L_1 - L_0 \quad \text{and} \quad \delta_2 = L_2 - L_0, \tag{4}$$

and the unstretched length of the springs is $L_0 = L/\sqrt{2}$. The Newton-Euler equations for the collar are

$$\sum F_x : \qquad -k\delta_1 \frac{x}{L_1} + k\delta_2 \frac{L-x}{L_2} = \frac{W}{g}\ddot{x}, \tag{5}$$

$$\sum F_y : \quad N - W - k\delta_1 \frac{L}{2L_1} - k\delta_2 \frac{L}{2L_2} = 0. \tag{6}$$

Because the collar is constrained to move along the bar, the kinematic equations are $a_x = \ddot{x}$ and $a_y = 0$, and Eq. (5) is the collar's equation of motion. The acceleration $\ddot{x}$ is purely a function of position x in Eq. (5) and the velocity $\dot{x}$ as a function of x may be found by applying the chain rule $\ddot{x} = \dot{x}\,d\dot{x}/dx$ and integrating the equation of motion over x as

$$\int_{v_0}^{v_f} \frac{W}{g}\dot{x}\,d\dot{x} = \int_{L/4}^{L} \left(-k\delta_1 \frac{x}{L_1} + k\delta_2 \frac{L-x}{L_2} \right) dx. \tag{7}$$

Substitute Eqs. (3) and (4) for L_1, L_2, δ_1, and δ_2, with $L_0 = L/\sqrt{2}$, into the integral on the right hand side of Eq. (7) to obtain

$$\frac{W}{g}\int_{v_0}^{v_f} \dot{x}\,d\dot{x} = k\int_{L/4}^{L} \left(-x + \frac{L}{\sqrt{2}}\frac{x}{\sqrt{x^2 + L^2/4}} + (L-x) - \frac{L}{\sqrt{2}}\frac{(L-x)}{\sqrt{(L-x)^2 + L^2/4}} \right) dx. \tag{8}$$

When evaluated, Eq. (8) becomes

$$\frac{W}{2g}\left(v_f^2 - v_0^2\right) = \frac{kL^2}{4}\left(-\frac{15}{8} + \sqrt{\frac{5}{2}} + \frac{9}{8} + \frac{2-\sqrt{13}}{\sqrt{2}} \right), \tag{9}$$

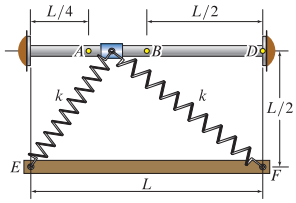
which we solve for k to obtain

$$k = \frac{8W}{gL^2}\left(v_f^2 - v_0^2\right) \Big/ \left[2\sqrt{2}\left(\sqrt{5} - \sqrt{13} + 2\right) - 3\right] = 5.93\,\text{lb/ft}, \tag{10}$$

where $v_0 = 11\,\text{ft/s}$, $v_f = 0$, $W = 6\,\text{lb}$, $L = 5\,\text{ft}$, and $g = 32.2\,\text{ft/s}^2$.

Problem 3.33

A 10 lb collar is constrained to travel along a rectilinear and frictionless bar of length $L = 5$ ft. The springs attached to the collar are identical, they have a spring constant $k = 4$ lb/ft, and they are unstretched when the collar is at B. Treating the collar as a particle, neglecting air resistance, and knowing that at A the collar is moving to the right with a speed of 14 ft/s, determine the speed with which the collar arrives at D. Points E and F are fixed.



Solution

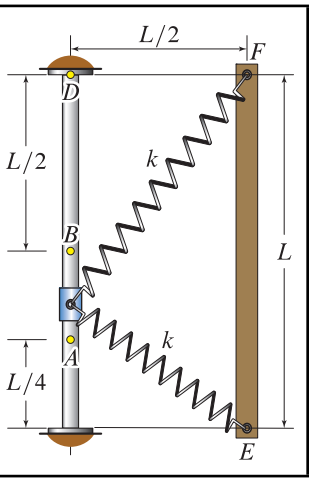
The solution is identical to Problem 3.32. Solve Eq. (10) for v_f.

$$k = \frac{8W}{gL^2}\left(v_f^2 - v_0^2\right) \Big/ \left[2\sqrt{2}\left(\sqrt{5} - \sqrt{13} + 2\right) - 3\right],$$

$$v_f = \sqrt{\frac{kL^2 g}{8W}\left[2\sqrt{2}\left(\sqrt{5} - \sqrt{13} + 2\right) - 3\right] + v_0^2} = 12.1 \text{ ft/s}.$$

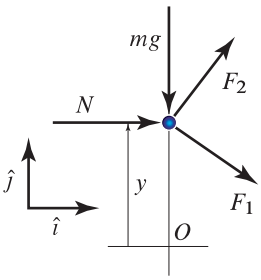
Problem 3.34

An 11 kg collar is constrained to travel along a rectilinear and frictionless bar of length $L = 2$ m. The springs attached to the collar are identical, and they are unstretched when the collar is at B. Treating the collar as a particle, neglecting air resistance, and knowing that at A the collar is moving upward with a speed of 23 m/s, determine the linear spring constant k so that the collar reaches D with zero speed. Points E and F are fixed.



Solution

Taking the origin O to be the bottom end of the bar, the spring forces are written in Cartesian coordinates as



$$\vec{F}_1 = k\delta_1 \left(\frac{(L/2)\,\hat{\imath} - y\,\hat{\jmath}}{L_1} \right), \tag{1}$$

$$\vec{F}_2 = k\delta_2 \left(\frac{(L/2)\,\hat{\imath} + (L-y)\,\hat{\jmath}}{L_2} \right), \tag{2}$$

where the lengths of the springs, L_1 and L_2, are given by

$$L_1 = \sqrt{y^2 + L^2/4} \quad \text{and} \quad L_2 = \sqrt{(L-y)^2 + L^2/4}, \tag{3}$$

and the stretches of the springs, δ_1 and δ_2, are given by

$$\delta_1 = L_1 - L_0 \quad \text{and} \quad \delta_2 = L_2 - L_0, \tag{4}$$

and the unstretched length of each spring is $L_0 = L/\sqrt{2}$. The Newton-Euler equations for the collar are

$$\sum F_x : \qquad N + k\delta_1 \frac{L}{2L_1} + k\delta_2 \frac{L}{2L_2} = 0, \tag{5}$$

$$\sum F_y : \quad -mg - k\delta_1 \frac{y}{L_1} + k\delta_2 \frac{L-y}{L_2} = m\ddot{y}. \tag{6}$$

Because the collar is constrained to move along the bar, the kinematic equations are $a_x = 0$ and $a_y = \ddot{y}$, and Eq. (6) is the collar's equation of motion. The acceleration $\ddot{y}$ is purely a function of position y in Eq. (6) and the velocity $\dot{y}$ as a function of y may be found by applying the chain rule $\ddot{y} = \dot{y}\,d\dot{y}/dy$ and integrating the equation of motion over y as

$$\int_{v_0}^{v_f} m\dot{y}\,d\dot{y} = \int_{L/4}^{L} \left(-mg - k\delta_1 \frac{y}{L_1} + k\delta_2 \frac{L-y}{L_2} \right) dy. \tag{7}$$

Substitute Eqs. (3) and (4) for L_1, L_2, δ_1 and δ_2, with $L_0 = L/\sqrt{2}$, into the integral on the right hand side of Eq. (7) to obtain

$$m\int_{v_0}^{v_f} \dot{y}\,d\dot{y} = \int_{L/4}^{L} \left[-mg + k\left(-y + \frac{L}{\sqrt{2}} \frac{y}{\sqrt{y^2 + L^2/4}} + (L-y) - \frac{L}{\sqrt{2}} \frac{(L-y)}{\sqrt{(L-y)^2 + L^2/4}} \right) \right] dy. \tag{8}$$

When evaluated, Eq. (8) becomes

$$\frac{m}{2}\left(v_f^2 - v_0^2\right) = -\frac{3}{4}mgL + \frac{kL^2}{4}\left(-\frac{15}{8} + \sqrt{\frac{5}{2}} + \frac{9}{8} + \frac{2-\sqrt{13}}{\sqrt{2}}\right), \tag{9}$$

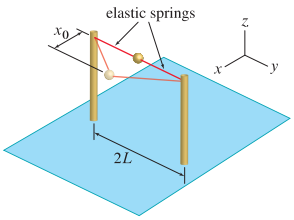
which we solve for k to obtain

$$\boxed{k = \left[\frac{8m}{L^2}\left(v_f^2 - v_0^2\right) + \frac{12mg}{L}\right] \Big/ \left[2\sqrt{2}\left(\sqrt{5} - \sqrt{13} + 2\right) - 3\right] = 9.03\,\text{kN/m},} \tag{10}$$

where $v_0 = 23\,\text{m/s}$, $v_f = 0$, $m = 11\,\text{kg}$, $L = 2\,\text{m}$, and $g = 9.81\,\text{m/s}^2$.

Problem 3.35

Derive the equation of motion of the mass m released from rest at $x = x_0$ from the slingshot. Assume that the cords connecting the mass to the slingshot are linear springs with spring constant k and unstretched length L_0. In addition, assume that the mass is equidistant from the two supports and that the mass and both springs lie in the xy plane. Ignore gravity and assume $L > L_0$.

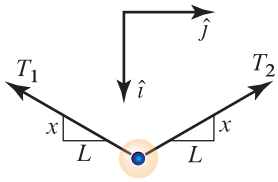


Solution

Balance Principles:

$$\sum F_x: \quad -(T_1 + T_2)\frac{x}{\sqrt{x^2 + L^2}} = ma_x, \tag{1}$$

$$\sum F_y: \quad (-T_1 + T_2)\frac{L}{\sqrt{x^2 + L^2}} = ma_y, \tag{2}$$



Force Laws:

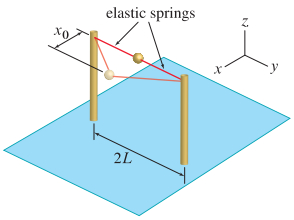
$$T_1 = T_2 = k\left(\sqrt{x^2 + L^2} - L_0\right). \tag{3}$$

Because $T_1 = T_2$, solving Eq. (2) gives us $a_y = 0$, and hence, there is no motion of the mass in the $\hat{j}$ direction. The only kinematics relation we need is $a_x = \ddot{x}$ and then we combine Eqs. (1) and (3) to obtain the equation of motion for the mass m as

$$\ddot{x} + \frac{2kx}{m}\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right) = 0.$$

Problem 3.36

Determine the speed of the mass m when it reaches $x = 0$ if it is released from rest at $x = x_0$ from the slingshot. Assume that the cords connecting the mass to the slingshot are linear springs with spring constant k and unstretched length L_0. Ignore gravity and assume $L > L_0$.



Solution

The FBD and Newton-Euler equations are identical to those determined in the solution to Problem 3.35. The material properties are also the same, and the equation of motion is

$$\ddot{x} + \frac{2kx}{m}\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right) = 0. \tag{1}$$

as derived in the solution to problem 3.35. To obtain the velocity as a function of position, we can write the acceleration $\ddot{x}$ as $v dv/dx$ in Eq. (1) and separate the variables and integrate as

$$\int_0^v v dv = -2\frac{k}{m}\int_{x_0}^{x}\left(x - L_0\frac{x}{\sqrt{x^2 + L^2}}\right)dx, \tag{2}$$

where the lower limit is $v = 0$ at $x = x_0$. Evaluating Eq. (2) gives velocity as a function of position $v(x)$ as

$$v(x) = -2\sqrt{\frac{k}{m}\left[\frac{x_0^2 - x^2}{2} + L_0\left(\sqrt{x^2 + L^2} - \sqrt{x_0^2 + L^2}\right)\right]}, \tag{3}$$

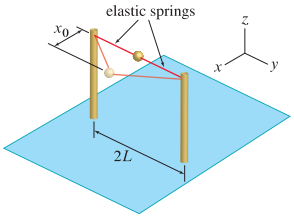
where we choose the sign to be negative because the mass will initially move in the negative $\hat{\imath}$ direction. We evaluate $v(x)$ at $x = 0$ to find $v(0)$ as

$$v(0) = -2\sqrt{\frac{k}{m}\left[\frac{x_0^2}{2} + L_0\left(L - \sqrt{x_0^2 + L^2}\right)\right]}. \tag{4}$$

Problem 3.37

Given the approximation

$$\frac{1}{\sqrt{(x/L)^2+1}} \approx 1 - \frac{(x/L)^2}{2}, \tag{1}$$



show that the equation of motion for the mass m when it is released from rest at $x = x_0$ from the slingshot can be written as

$$\ddot{x} + \omega_0^2 x\left(1 + \Lambda x^2\right) = 0, \tag{2}$$

where

$$\omega_0^2 = \frac{2k}{m}\left(\frac{L - L_0}{L}\right) \quad \text{and} \quad \Lambda = \frac{L_0}{2L^2(L - L_0)}. \tag{3}$$

Assume that the cords connecting the mass to the slingshot are linear springs with spring constant k, length L, and unstretched length L_0. Ignore gravity and assume $L > L_0$. Equation (2) is a famous differential equation in mechanics called *Duffing's equation*.

Solution

The FBD and Newton-Euler equations are identical to those determined in the solution to Problem 3.35. The material properties are also the same, and the equation of motion is

$$\ddot{x} + \frac{2kx}{m}\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right) = 0. \tag{4}$$

as derived in the solution to problem 3.35. Eq. (4) can be rewritten as

$$\ddot{x} + \frac{2kx}{m}\left(1 - \frac{L_0/L}{\sqrt{(x/L)^2+1}}\right) = 0, \tag{5}$$

so that the square root term has the same form as Eq. (1). Using the approximation from Eq. (1) we write the approximate equation of motion as

$$\ddot{x} + \frac{2kx}{m}\left[1 - \frac{L_0}{L}\left(1 - \frac{(x/L)^2}{2}\right)\right] = 0, \tag{6}$$

and with some rearrangement as

$$\ddot{x} + \frac{2kx}{m}\left[\frac{L - L_0}{L} + \frac{L_0}{L}\frac{(x/L)^2}{2}\right] = 0. \tag{7}$$

We factor $(L - L_0)/L$ from both terms in the square brackets to obtain

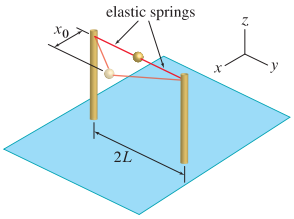
$$\ddot{x} + x\frac{2k}{m}\left(\frac{L - L_0}{L}\right)\left[1 + \frac{L_0}{2L^2(L - L_0)}x^2\right] = 0, \tag{8}$$

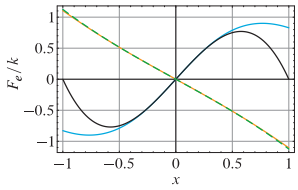
from which it is obvious that we can substitute the parameters ω_0^2 and Λ that are defined in Eq. (2) to arrive at Duffing's Equation as

$$\boxed{\ddot{x} + \omega_0^2 x\left(1 + \Lambda x^2\right) = 0.} \tag{9}$$

Problem 3.38

The force on the mass for the slingshot in Prob. 3.35 and the force on the mass in the Duffing equation obtained from the slingshot equation of motion (and defined by Eqs. (2) and (3)) can be plotted as a function of x. The nature of that force depends on whether or not the springs are initially stretched ($L > L_0$) or initially compressed ($L < L_0$). The figure shows the elastic restoring force on the mass m as a function of the displacement x for four different cases:





- Force for slingshot with $L = 2$ and $L_0 = 1$.
- Force for slingshot with $L = 1$ and $L_0 = 2$.
- Force for the Duffing equation with $L = 2$ and $L_0 = 1$.
- Force for the Duffing equation with $L = 1$ and $L_0 = 2$.

For small x, the force given by the Duffing equation is a good approximation to the force in the slingshot. Explain which of the curves corresponds to a "hardening" spring (a spring that gets stiffer as you pull it) and which corresponds to a "softening" spring (a spring that gets less stiff as you pull it), and explain physically why we see this behavior.

Solution

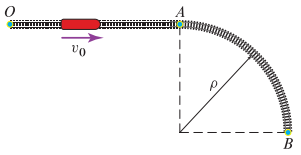
The cords in the slingshot are both modeled as linear springs with force $F = k\delta$, but because the angles of the force vectors depend nonlinearly on x and because the stretch of the cords δ also depends nonlinearly on x, the mass experiences a nonlinear restoring force.

In the figure given we see that the cases where $L > L_0$ always results in a force that behaves like a hardening spring. The slopes of the force-displacment curves become steeper as x moves away from 0. However, at large distances of x, the force curve will have a more constant slope so that the hardening is limited. This is because for large x, change in the angle of the force vectors becomes small, and the stretch of the cords depends more linearly on x.

For the cases where $L < L_0$ the cords are in compression when $x = 0$, and if the mass is slightly moved in either direction from 0, the restoring force tends to push the mass away from 0. The pushing force becomes smaller as x increases until the cords reach their natural length at which point the cords go into tension and pull on the mass as x continues to increase. In other words, the system behaves like a softening spring until the cords reach and pass their unstreched length at which point the restoring force exhibits a more linear dependence on x.

Problem 3.39

A train is traveling with constant speed v_0 along the level track OAB, which lies in the horizontal plane. The section between points O and A is straight, and the section between points A and B is circular with radius of curvature ρ. The train starts at point O at time $t = 0$, reaches point A at time t_A, and reaches point B at time t_B. For this motion, sketch the magnitude of the acceleration vector as a function of time. Would you want to design a train track with this shape?



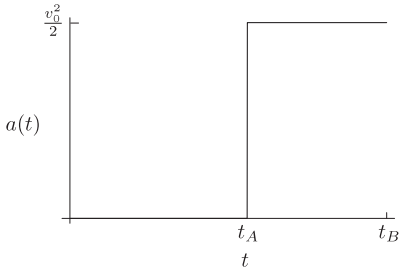
Solution

Recall that in tangential components the acceleration is expressed by

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n. \tag{1}$$

In the given motion, $v =$ constant. Thus, we have

$$\vec{a} = \frac{v^2}{\rho}\,\hat{u}_n. \tag{2}$$



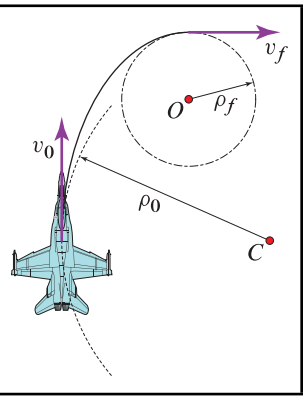
Now, when the train is traveling along the rectilinear stretch, $\vec{a} = \vec{0}$ ($\rho \to \infty$). As soon as the train begins to turn,

$$|\vec{a}| = \frac{v^2}{\rho} \neq 0. \tag{3}$$

Hence, the required plot should look like the one to the right. In the interest of passenger comfort, the turn should be designed with a smooth transition in terms of ρ.

Problem 3.40

A plane is turning along a horizontal path at constant speed. What is the total force acting on the plane in the direction of the path?

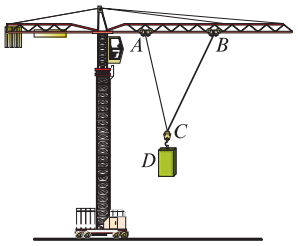


Solution

The total force along the path is equal to zero because the tangential acceleration is equal to zero. The tangential acceleration is equal to zero because the speed of the plane is constant.

Problem 3.41

Body D is in equilibrium when cable AC suddenly breaks. If B is fixed, does the tension in the inextensible cable BC increase or decrease at the instant of release?

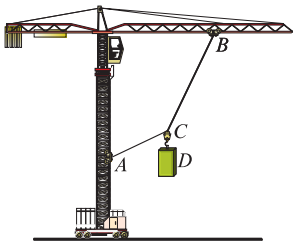


Solution

Because cable BC is inextensible and B is fixed, the path of D after release is a circle centered at B. The tension in the cable BC depends on the component of the weight parallel to the segment BC along with the component of the acceleration of D along BC. Using normal-tangential components, we see that the acceleration is given by v^2/ρ where v is the speed of D and ρ is the distance from B to C. Immediately after release, the position of D is the same as before release. In addition, the speed v is equal to 0. Therefore, the tension in BC will increase since the component of weight along BC is now supported without the help of cable AC.

Problem 3.42

Body D is in equilibrium when cable AC suddenly breaks. If B is fixed, does the tension in the inextensible cable BC increase or decrease at the instant of release?

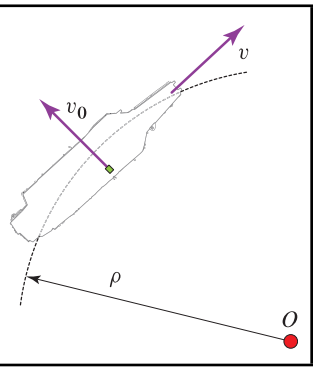


Solution

Because cable BC is inextensible and B is fixed, the path of D is a circle centered at B. The tension in BC depends on the component of the weight parallel to the segment BC along with the acceleration of D along BC. Using normal-tangential components, we see that the acceleration is given by v^2/ρ where v is the speed of D and ρ is the distance from B to C. Immediately after release, the position of D is the same as before release. In addition, the speed v is equal to 0. Therefore, the tension in BC will decrease since BC will support the same component of the weight of D, but without the extra load due to the tension in AC before release.

Problem 3.43

An aircraft carrier is turning with a constant speed v along a circular path with radius ρ and center O. During the maneuver, a forklift is being driven across the deck of the ship at a constant speed v_0, relative to the deck. Does the friction force between the forklift and the deck have a component perpendicular to the relative velocity of the forklift with respect to the deck?

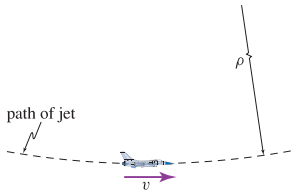


Solution

The situation depicted in the figure is conceptually identical to "Case 3" discussed in Example 3.8: *Coriolis Effect*. Therefore we can say that, due to the Coriolis component of acceleration, the forklift must be subject to a component of force perpendicular to the radial component of velocity. Such a force points to the right of the forklift in the direction of v and is provided by friction between the lift and the deck.

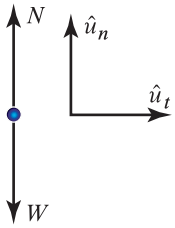
Problem 3.44

A jet is coming out of a dive, and a sensor in the pilot's seat measures a force of 800 lb for a pilot whose weight is 180 lb. If the jet's instruments indicate that the plane is traveling at 850 mph, determine the radius of curvature of the plane's path at this instant.



Solution

The Newton-Euler equations using normal-tangential components as



$$\sum F_n: \quad N - W = \frac{W}{g} a_n, \tag{1}$$

$$\sum F_t: \quad 0 = \frac{W}{g} a_t, \tag{2}$$

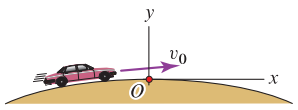
where the normal force on the pilot from the seat is $N = 800$ lb, the weight of the pilot is $W = 180$ lb and the mass of the pilot is W/g . For normal-tangential components we have the kinematic equation $a_n = v^2/\rho$ where v is the speed of the plane and ρ is the radius of curvature. We solve for ρ by combining the kinematic equation with Eq. (1) as

$$\boxed{\rho = \frac{W v^2}{g\,(N - W)} = 1.40 \times 10^4 \text{ ft},} \tag{3}$$

where $v = 850$ mi/h $= 1247$ ft/s and $g = 32.2$ ft/s^2.

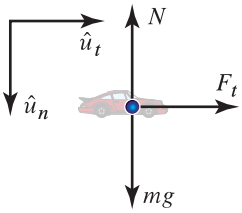
Problem 3.45

A car traveling over a hill at constant speed starts to lose contact with the ground at O. If the radius of curvature of the hill is 282 ft, determine the speed of the car at O.



Solution

Balance Principles:



$$\sum F_t: \quad F_t = ma_t,$$
$$\sum F_n: \quad mg - N = ma_n. \qquad (1)$$

Kinematics:

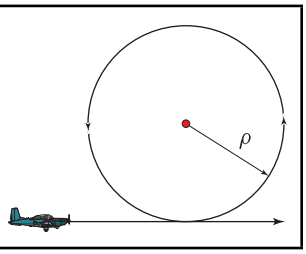
$$a_n = v^2/\rho \quad \text{and} \quad a_t = 0.$$

For the car to almost loose contact we have $N = 0$. Eq. (1) becomes

$$m\frac{v^2}{\rho} = mg \quad \Rightarrow \quad \boxed{v = \sqrt{g\rho} = 95.3\,\text{ft/s}.}$$

Problem 3.46

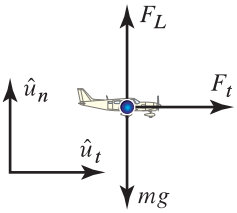
A 950 kg aerobatics plane initiates the basic loop maneuver at the bottom of a loop with radius $\rho = 110$ m and a speed of 225 km/h. At this instant, determine the magnitude of the plane's acceleration, expressed in terms of g, the acceleration due to gravity, and the magnitude of the lift provided by the wings.



Solution

Balance Principles:

$$\sum F_t: \quad F_t = ma_n,$$
$$\sum F_n: \quad F_L - mg = ma_n. \qquad (1)$$



Using the kinematic equation $a_n = v^2/\rho$ the magnitude of the acceleration in the normal direction is

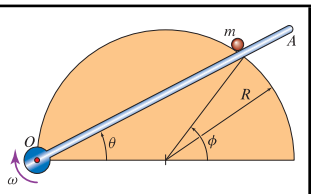
$$a_n = \frac{v^2}{\rho} = 35.51\,\text{m/s}^2 = 3.62\,\text{g}.$$

Eq. (1) becomes

$$F_L = mg + \frac{mv^2}{\rho} = 43{,}100\,\text{N}.$$

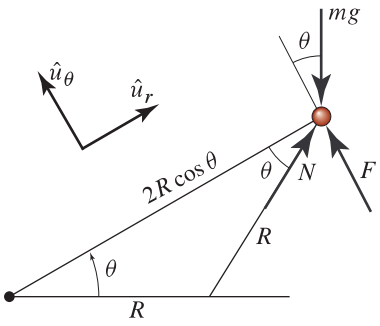
Problem 3.47

The ball of mass m is guided along the vertical circular path of radius $R = 1\,\text{m}$ using the arm OA. If the arm starts from $\phi = 90°$ and rotates clockwise with a constant angular velocity $\omega = 0.87\,\text{rad/s}$, determine the angle ϕ at which the particle starts to leave the surface of the semicylinder. Neglect all friction forces acting on the ball, neglect the thickness of the arm OA, and treat the ball as a particle.



Solution

Balance Principles:

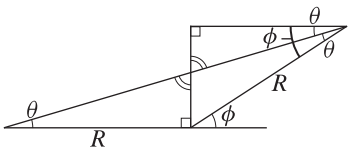


$$\sum F_r: \quad N\cos\theta - mg\sin\theta = ma_r, \tag{1}$$

$$\sum F_\theta: \quad F + N\sin\theta - mg\cos\theta = ma_\theta, \tag{2}$$

where N is the contact force of the semicylinder acting on the ball, and F is the contact force of the arm OA acting on the ball. Kinematics:

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}, \tag{3}$$



As long as the ball remains in contact with the semicylinder, we have a kinematic equation relating the radial position r to the angle θ. This equation and its derivatives with respect to time are

$$r = 2R\cos\theta, \quad \dot{r} = -2R\dot{\theta}\sin\theta, \quad \text{and} \quad \ddot{r} = -2R\dot{\theta}^2\cos\theta \tag{4}$$

where $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$. Combining Eq. (1), Eq. (3), and Eqs. (4) we get

$$N\cos\theta - mg\sin\theta = m(-2R\omega^2\cos\theta - 2R\omega^2\cos\theta) \tag{5}$$

When the ball separates $N = 0$ so we plug that into Eq. (5) and solve for θ yielding

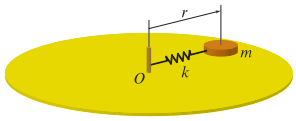
$$\theta = \tan^{-1}\left(\frac{4\omega^2 R}{g}\right) \quad \Rightarrow \quad \theta = 17.15°$$

From the geometry of the figure we see that $\phi = 2\theta$ so

$$\boxed{\phi = 2\tan^{-1}\left(\frac{4\omega^2 R}{g}\right) = 34.3°.}$$

Problem 3.48

Referring to Example 3.7, instead of using polar coordinates as was done in that example, work the problem using a Cartesian coordinate system with origin at O (cf. Fig. 1 of Example 3.7) and derive the problem's equations of motion.

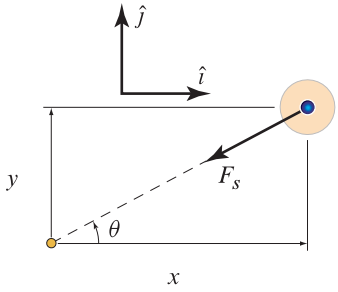


Solution

Balance Principles:

$$\sum F_x: \quad -F_s \cos\theta = ma_x, \tag{1}$$

$$\sum F_y: \quad -F_s \sin\theta = ma_y, \tag{2}$$



where

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}},$$

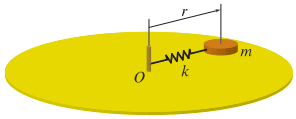
and the spring force is given by the material equation $F_s = k\delta$ with $\delta = \sqrt{x^2+y^2} - r_u$ in the given coordinates, and r_u is the unstretched length of the spring as given in Example 3.7. The kinematic equations are $a_x = \ddot{x}$ and $a_y = \ddot{y}$. Combining the kinematic equations and material equation with the Newton-Euler equations we obtain the equations of motion as

$$\ddot{x} + \frac{k}{m}x\left(1 - \frac{r_u}{\sqrt{x^2+y^2}}\right) = 0, \tag{3}$$

$$\ddot{y} + \frac{k}{m}y\left(1 - \frac{r_u}{\sqrt{x^2+y^2}}\right) = 0. \tag{4}$$

Problem 3.49

Continue Prob. 3.48 and using mathematical software, numerically solve the equations of motion. Use the same parameters and initial conditions that were used in Example 3.7, and compare your results with those presented in that example.



Solution

This problem is most easily solved using a package such as *Mathematica* or MATLAB. The differential equations we are solving are given in Eqs. (3) and (4) from the solution to Problem 3.48. The initial conditions from Example 3.7 are $r(0) = 0.35\,\text{m}$, $\theta(0) = 0\,\text{rad}$, $\dot{r}(0) = 0\,\text{m/s}$, and $\dot{\theta}(0) = 0.5\,\text{rad/s}$. We transform these initial conditions into cartesian coordinates as

$$x(0) = r(0)\cos(\theta(0)) = 0.3500\,\text{m}, \tag{1}$$

$$y(0) = r(0)\sin(\theta(0)) = 0\,\text{m}, \tag{2}$$

$$\dot{x}(0) = \dot{r}(0)\cos(\theta(0)) - r(0)\dot{\theta}(0)\sin(\theta(0)) = 0\,\text{m/s}, \tag{3}$$

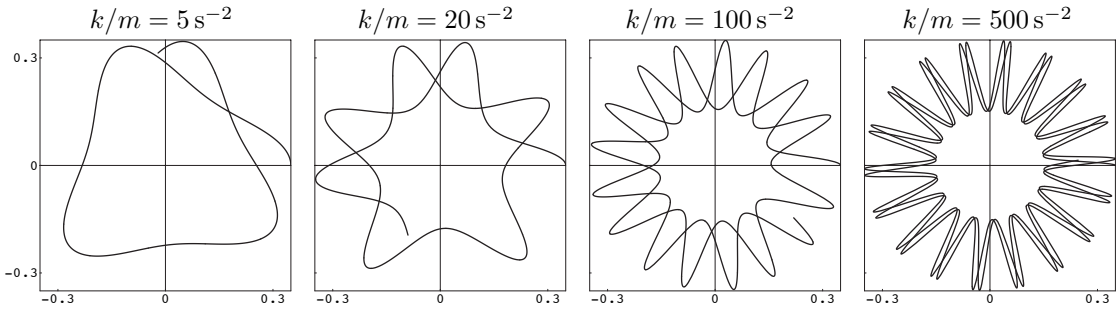
$$\dot{y}(0) = \dot{r}(0)\sin(\theta(0)) + r(0)\dot{\theta}(0)\cos(\theta(0)) = 0.1750\,\text{m/s}. \tag{4}$$

Example 3.7 gives the parameter $r_u = 0.25\,\text{m}$, and it gives four different values for the ratio of parameters k/m: 5, 20, 100, and $500\,\text{s}^{-2}$. The time range over which we are to solve the equations of motion is $0 \le t \le 10\,\text{s}$. To solve the equations of motion in *Mathematica* we use the `NDSolve` function and to plot the solution we use the `ParametricPlot` function as shown in the following code:

```
sol = NDSolve[{x''[t] + k/m x[t] (1 - ru/Sqrt[x[t]^2 + y[t]^2]) == 0,
    y''[t] + k/m y[t] (1 - ru/Sqrt[x[t]^2 + y[t]^2]) == 0,
    x[0] == 0.35, y[0] == 0, x'[0] == 0, y'[0] == 0.175} /.
  {ru → 0.25, m → 1, k → 5}, {x[t], y[t]}, {t, 0, 10}];

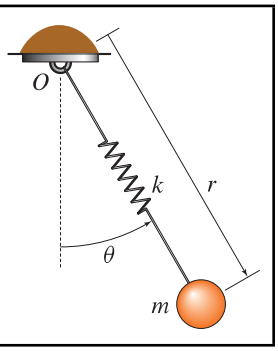
ParametricPlot[{x[t], y[t]} /. sol, {t, 0, 10}, AspectRatio → 1]
```

The above code shows the case where $k/m = 5\,\text{s}^{-2}$.
The computer solution generate trajectory plots that are similar to those shown in Example 3.7 for each value of k/m. Shown below are plots generated using the *Mathematica* code.



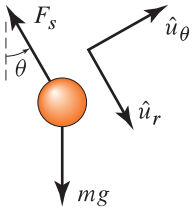
Problem 3.50

Derive the equations of motion for the pendulum supported by a linear spring of constant k and unstretched length r_u. Neglect friction at the pivot O, the mass of the spring, and air resistance. Treat the pendulum bob as a particle of mass m and use polar coordinates.



Solution

Based on the FBD shown at the right, the Newton-Euler equations for this system are



$$\sum F_r: \quad mg\cos\theta - F_s = ma_r, \tag{1}$$

$$\sum F_\theta: \quad -mg\sin\theta = ma_\theta, \tag{2}$$

We have the material equation $F_s = k(r - r_u)$. The kinematic equations are given by

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{3}$$

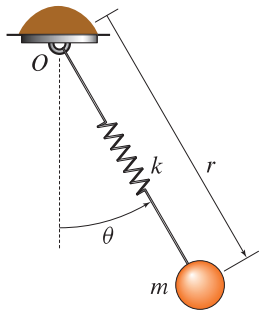
Substituting Eqs. (3) into Eq. (1) and Eq. (2) gives the desired equations of motion as

$$\ddot{r} - r\dot{\theta}^2 + \frac{k}{m}(r - r_u) - g\cos\theta = 0, \tag{4}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0. \tag{5}$$

Problem 3.51

Using mathematical software, solve the equations of motion for the pendulum supported by a linear spring of constant k (derived in Prob. 3.50). Plot the trajectory of the mass m in the vertical plane for a number of different values of k/m. The unstretched length of the spring is 0.25 m, and the mass is released when the pendulum is vertical, the spring is stretched 0.75 m, and the mass is moving to the right at 1 m/s.



Solution

The solution to the equations of motion (4) and (5) in the solution to Problem 3.50 should give us $r \to r(t)$ and $\theta \to \theta(t)$. Since the equations to be solved involve second derivative with respect to time, we need two initial conditions for each dependent variable, $r(t)$ and $\theta(t)$. So, we have $r(0) = 0.25 + 0.75 = 1$, $\dot{r}(0) = 0$, $\theta(0) = 0$ and $\dot{\theta}(0) = v(0)/r(0) = 1$. We vary k/m between 1 rad/s^2 and 26 rad/s^2 in steps of 5 rad/s^2 and use *Mathematica* to solve the equation of motions along with the initial conditions, using the code shown below:

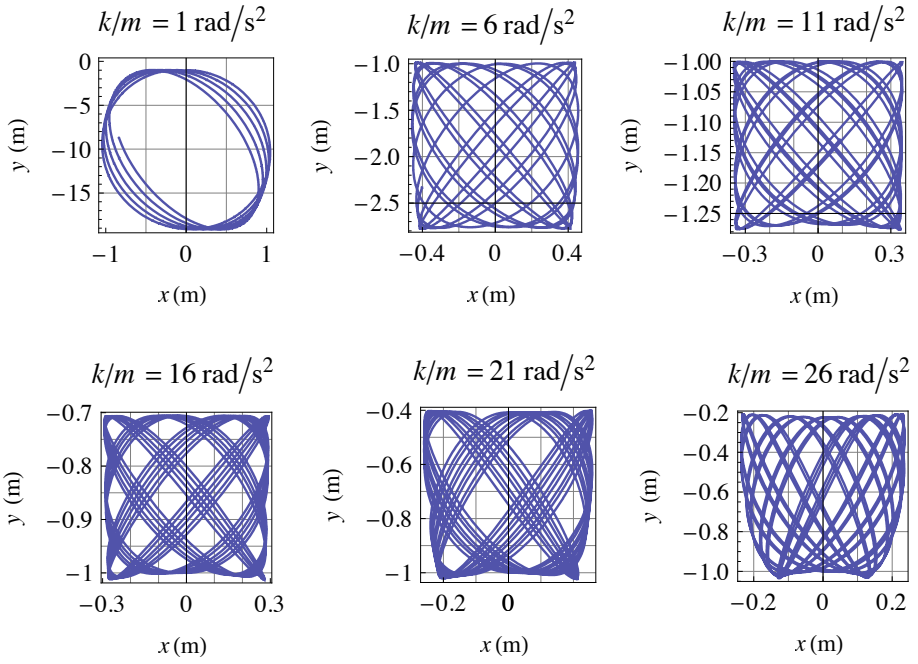
```
eq1 = r[t] θ''[t] + 2 r'[t] θ'[t] + g Sin[θ[t]] == 0;
eq2 = r''[t] - r[t] (θ'[t])^2 + γ (r[t] - ru) - g Cos[θ[t]] == 0;
IC = {γ → γset, ru → 0.25, g → 9.81, r0 → 1, rdot0 → 0, θ0 → 0, θdot0 → 1};

tickset = {{{Automatic, None}, {{-1, 0, 1}, None}},
   {{Automatic, None}, {{-0.4, 0, 0.4}, None}},
   {{Automatic, None}, {{-0.3, 0, 0.3}, None}},
   {{{-1, -0.9, -0.8, -0.7}, None}, {{-0.3, 0, 0.3}, None}},
   {{{-1, -0.8, -0.6, -0.4}, None}, {{-0.2, 0, 0, 2}, None}},
   {{Automatic, None}, {{-0.2, 0, 0.2}, None}}};

Table[
 ParametricPlot[
  Evaluate[{r[t] Sin[θ[t]], -r[t] Cos[θ[t]]} /.
    NDSolve[{eq1, eq2, r[0] == r0, r'[0] == rdot0, θ[0] == θ0, θ'[0] == θdot0} /. IC,
     {r[t], θ[t]}, {t, 0, 30}, MaxSteps → 10 000]], {t, 0, 30},

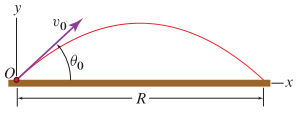
 PlotPoints → 350, GridLines → Automatic, AspectRatio → 1,
  PlotLabel → StringJoin["k/m = ", ToString[γset], " rad/s^2 "],
  FrameLabel → {"x (m) ", "y (m)"}, PlotRange → Automatic, Frame → True,
  FrameTicks → tickset[[(γset - 1) / 5 + 1]], PlotStyle → AbsoluteThickness[1]],
 {γset, {1, 6, 11, 16, 21, 26}}]
```

The trajectory plots for different k/m values are shown below :


$k/m = 1\ \text{rad/s}^2$
$k/m = 6\ \text{rad/s}^2$
$k/m = 11\ \text{rad/s}^2$
$k/m = 16\ \text{rad/s}^2$
$k/m = 21\ \text{rad/s}^2$
$k/m = 26\ \text{rad/s}^2$
y (m)
x (m)

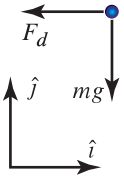
Problem 3.52

Revisiting Example 3.4, assume that the drag force in the x direction is proportional to the square of the x component of velocity, but that the ball's trajectory is shallow enough to neglect the drag force in the y direction. Using this assumption, letting O be the initial position of the ball, and letting the ball's initial velocity have a magnitude v_0 and form an angle θ_0 with the x axis, determine an expression for the trajectory of the ball of the form $y = y(x)$.



Solution

Balance Principles



$$\sum F_x: \quad -F_d = ma_x, \tag{1}$$

$$\sum F_y: \quad -mg = ma_y. \tag{2}$$

We have the force law $F_d = C_d\dot{x}^2$ and the kinematic equations $a_x = \ddot{x}$ and $a_y = \ddot{y}$. Eq. (1) and (2) become

$$-C_d\dot{x}^2 = m\ddot{x} \quad \text{and} \quad -g = \ddot{y}.$$

Acceleration is constant in the y direction so use the equation $s = s_0 + \dot{s}t + 0.5\,a_c t^2$ to obtain an expression for y as a function of t:

$$y = v_0 \sin\theta_0\, t - \tfrac{1}{2}gt^2. \tag{3}$$

In the x direction acceleration is a function of velocity so we must integrate it as follows:

$$\ddot{x} = \frac{d\dot{x}}{dt} \quad \Rightarrow \quad dt = \frac{d\dot{x}}{\ddot{x}} \quad \Rightarrow \quad \frac{-m}{C_d}\int_{\dot{x}_0}^{\dot{x}} \frac{d\dot{x}}{\dot{x}^2} = \int_0^t dt \quad \Rightarrow \quad t = \frac{m}{C_d}\left(\frac{1}{\dot{x}} - \frac{1}{\dot{x}_0}\right).$$

Plug in $\dot{x}_0 = v_0\cos\theta_0$, solve for $\dot{x}$, and integrate the results to obtain x as a function of t.

$$\frac{C_d t}{m} + \frac{1}{v_0\cos\theta_0} = \frac{1}{\dot{x}} \quad \Rightarrow \quad \dot{x} = \frac{mv_0\cos\theta_0}{m + v_0\cos\theta_0 C_d t} \quad \Rightarrow \quad \int_0^x dx = \int_0^t \frac{mv_0\cos\theta_0}{m + v_0\cos\theta_0 C_d t}\,dt$$

$$\Rightarrow \quad x = \frac{m}{C_d}\ln\left(1 + \frac{v_0}{m}\cos\theta_0 C_d t\right).$$

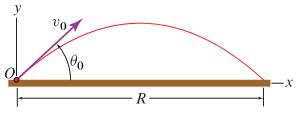
Solve this equation for t so we may substitute the results into Eq. (3).

$$t = \frac{m}{v_0\cos\theta_0 C_d}\left(e^{C_d x/m} - 1\right),$$

$$\boxed{y = \frac{m}{C_d}\tan\theta_0\left(e^{C_d x/m} - 1\right) - \frac{m^2 g}{2v_0^2\cos^2\theta_0 C_d^2}\left(e^{C_d x/m} - 1\right)^2.}$$

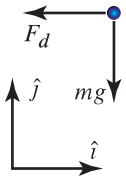
Problem 3.53

Revisit Example 3.4 and assume that the trajectory of the ball is shallow enough that the effects of the drag force in the y direction can be neglected and that the component of the drag force in the x direction is proportional to the square of the x component of velocity. Using this assumption and the same drag coefficient discussed in the example ($C_d = 4.71\times10^{-7}$ lb·s^2/ft^2), compute the horizontal distance R traveled by a 1.61 oz golf ball subject to the same initial conditions given in the example, i.e., the initial velocity has a magnitude $v_0 = 187$ mph and an initial orientation $\theta_0 = 11.2°$.



Solution

Balance Principles



$$\sum F_x: \quad -F_d = ma_x, \tag{1}$$

$$\sum F_y: \quad -mg = ma_y. \tag{2}$$

We have the force law $F_d = C_d\dot{x}^2$ and the kinematic equations $a_x = \ddot{x}$ and $a_y = \ddot{y}$. Eq. (1) and (2)

$$\ddot{x} = \frac{-C_d}{m}\dot{x}^2 \quad \text{and} \quad \ddot{y} = -g.$$

Acceleration is constant in the y direction so use the equation $s = s_0 + \dot{s}t + 0.5\,a_c t^2$ to obtain an expression for y as a function of t, then solve for t when $y = 0$.

$$y = v_0 \sin\theta_0\, t - \tfrac{1}{2}gt^2 \quad \Rightarrow \quad t_I = \frac{2v_0\sin\theta_0}{g}. \tag{3}$$

Where t_I is the time at impact.
In the x direction acceleration is a function of velocity so we must integrate it as follows:

$$\ddot{x} = \frac{d\dot{x}}{dt} \quad \Rightarrow \quad dt = \frac{d\dot{x}}{\ddot{x}} \quad \Rightarrow \quad \frac{-m}{C_d}\int_{\dot{x}_0}^{\dot{x}} \frac{d\dot{x}}{\dot{x}^2} = \int_0^t dt \quad \Rightarrow \quad t = \frac{m}{C_d}\left(\frac{1}{\dot{x}} - \frac{1}{\dot{x}_0}\right).$$

Plug in $\dot{x}_0 = v_0\cos\theta_0$, solve for $\dot{x}$, and integrate the results to obtain x as a function of t.

$$\frac{C_d t}{m} + \frac{1}{v_0\cos\theta_0} = \frac{1}{\dot{x}} \quad \Rightarrow \quad \dot{x} = \frac{mv_0\cos\theta_0}{m + v_0\cos\theta_0 C_d t} \quad \Rightarrow \quad \int_0^R dx = \int_0^{t_I} \frac{mv_0\cos\theta_0}{m + v_0\cos\theta_0 C_d t}\,dt$$

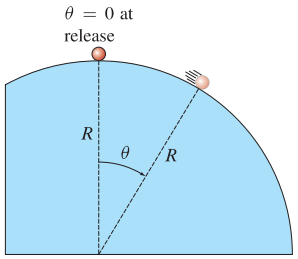
$$\Rightarrow \quad R = \frac{m}{C_d}\ln\left(1 + \frac{v_0}{m}\cos\theta_0 C_d t_I\right).$$

Substitute the second of Eq. (3) for t_I to obtain the final answer:

$$R = \frac{m}{C_d}\ln\left(1 + \frac{2v_0^2 C_d \sin\theta_0\cos\theta_0}{mg}\right) = 835\,\text{ft} = 278\,\text{yd}.$$

Problem 3.54

Revisit Example 3.6 on p. 216 by letting the sphere be released at $\theta = 0$ with a speed $v_0 = 0.5\,\text{m/s}$. Neglecting friction, compute the angle at which the sphere separates from the cylinder if $R = 1.35\,\text{m}$.

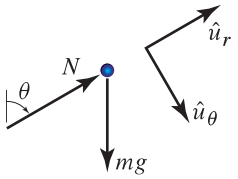


Solution

Balance Principles

$$\sum F_r: \quad N - mg\cos\theta = ma_r, \tag{1}$$

$$\sum F_\theta: \quad mg\sin\theta = ma_\theta. \tag{2}$$



Because R is constant, the kinematic equations are

$$a_r = \left(\ddot{r} - r\dot{\theta}^2\right) \quad \Rightarrow \quad a_r = -R\dot{\theta}^2,$$
$$a_\theta = \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \quad \Rightarrow \quad a_\theta = R\ddot{\theta}.$$

Eqs. (1) and (2) become

$$N - mg\cos\theta = -mR\dot{\theta}^2 \quad \text{and} \tag{3}$$

$$g\sin\theta = R\ddot{\theta}. \tag{4}$$

Integrate Eq. (4) to obtain $\dot{\theta}$ as a function of θ:

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \quad \Rightarrow \quad \frac{g}{R}\int_0^\theta \sin\theta\, d\theta = \int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}\, d\theta \quad \Rightarrow \quad \frac{-g}{R}(\cos\theta - 1) = \tfrac{1}{2}\dot{\theta}^2 - \tfrac{1}{2}\dot{\theta}_0^2$$

$$\Rightarrow \quad \dot{\theta}^2 = \dot{\theta}_0^2 + \frac{2g}{R}(1 - \cos\theta).$$

Substitute the expression for $\dot{\theta}^2$ into Eq. (3), then solve for N and set $N = 0$:

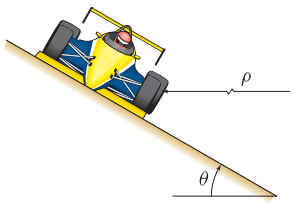
$$N = mg\cos\theta - mR\dot{\theta}_0^2 - 2mg(1 - \cos\theta) \quad \Rightarrow \quad 0 = 3\cos\theta_s - 2 - \frac{R\dot{\theta}_0^2}{g}.$$

Note that $\dot{\theta}_0 = v_0/R$. Solve for θ_s the angle where separation occurs.

$$3\cos\theta_s = 2 + \frac{R\dot{\theta}_0^2}{g} \quad \Rightarrow \quad \boxed{\theta_s = \cos^{-1}\left(\frac{2}{3} + \frac{v_0^2}{3gR}\right) = 47.7°.}$$

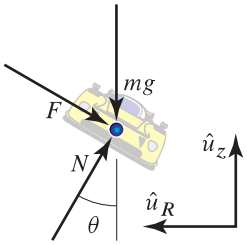
Problem 3.55

Using a cylindrical component system whose origin is at the center of curvature of the path of the car, derive the governing equations for Example 3.5 on p. 214. Solve the equations and verify that you get the same solution.



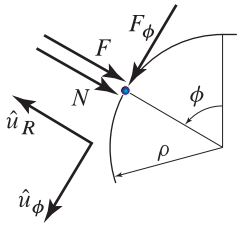
Solution

Balance Principles



$$\sum F_R: \quad -F\cos\theta - N\sin\theta = ma_R,$$
$$\sum F_\phi: \quad F_\phi = ma_\phi,$$
$$\sum F_z: \quad -F\sin\theta + N\cos\theta - mg = ma_z.$$

The force law is $F = \mu_s N$ because we assume that slip is impending. The car moves at constant speed so our kinematic equations are



$$a_R = \ddot{R} - R\dot{\phi}^2 \quad \Rightarrow \quad a_R = -\rho\frac{v^2}{\rho^2} = \frac{-v^2}{\rho},$$
$$a_\phi = R\ddot{\phi} + 2\dot{R}\dot{\phi} \quad \Rightarrow \quad a_\phi = 0,$$
$$a_z = 0.$$

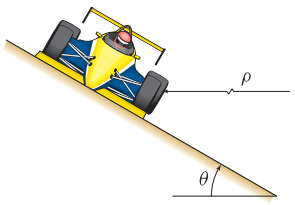
We have 7 equations in 7 unknowns. These equations can be solved by hand or with mathematical software to obtain

$$a_\phi = 0 = a_z,$$
$$a_R = \frac{-g(\mu_s + \tan\theta)}{1 - \mu_s\tan\theta} = -120\,\text{ft/s}^2,$$
$$F_\phi = 0,$$
$$F = \frac{\mu_s mg}{\cos\theta - \mu_s\sin\theta} = (83.2\,\text{ft/s}^2)\,m,$$
$$N = \frac{mg}{\cos\theta - \mu_s\sin\theta} = (92.4\,\text{ft/s}^2)\,m,$$
$$v_{\max} = \sqrt{\rho g}\sqrt{\frac{\mu_s + \tan\theta}{1 - \mu_s\tan\theta}} = 363\,\text{ft/s} = 248\,\text{mph}.$$

Note that a_R is negative whereas a_n was positive. This is because $\hat{u}_n = -\hat{u}_R$.

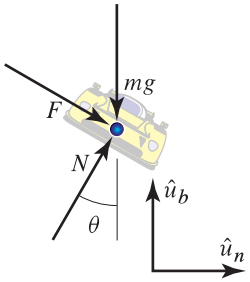
Problem 3.56

A race car is traveling at a constant speed over a circular banked turn. Oil on the track has caused the static friction coefficient between the tires and the track to be $\mu_s = 0.2$. If the radius of the car's trajectory is $\rho = 320\,\text{m}$ and the bank angle is $\theta = 33°$, determine the range of speeds within which the car must travel not to slide sideways.



Solution

To find the maximum speed we assume that slip is impending and friction acts to prevent the car from moving up the bank. Our balance equations for this case are



$$\sum F_n: \quad F\cos\theta + N\sin\theta = ma_n, \tag{1}$$

$$\sum F_b: \quad -mg + N\cos\theta - F\sin\theta = ma_b. \tag{2}$$

The force law is $F = \mu_s N$ because we assume that slip is impending.
The car moves at constant speed so our kinematic equations are

$$a_n = \frac{v^2}{\rho} \quad \text{and} \quad a_b = 0.$$

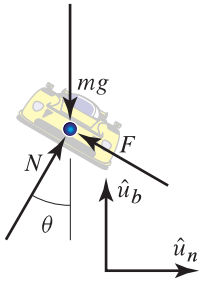
Substitute the force law and kinematic equation into Eq. (2) and solve for N to obtain

$$N = \frac{mg}{\cos\theta - \mu_s \sin\theta}.$$

Solve Eq. (1) for $v_{\max}$ and substitute the expression for N to obtain

$$\mu_s N\cos\theta + N\sin\theta = \frac{mv_{\max}}{\rho} \quad \Rightarrow \quad \boxed{v_{\max} = \sqrt{\rho g}\sqrt{\frac{\mu_s + \tan\theta}{1 - \mu_s\tan\theta}} = 55.36\,\text{m/s} = 199\,\text{km/hr}.}$$

To find the minimum speed we assume that slip is impending and friction acts to prevent the car from moving down the bank. Our balance equations for this case are



$$\sum F_n: \quad -F\cos\theta + N\sin\theta = ma_n, \tag{3}$$

$$\sum F_b: \quad -mg + N\cos\theta + F\sin\theta = ma_b. \tag{4}$$

The force law is still $F = \mu_s N$ because we assume that slip is impending.
The car moves at constant speed so our kinematic equations are

$$a_n = \frac{v^2}{\rho} \quad \text{and} \quad a_b = 0.$$

Substitute the force law and kinematic equation into Eq. (4) and solve for N to obtain

$$N = \frac{mg}{\cos\theta + \mu_s \sin\theta}.$$

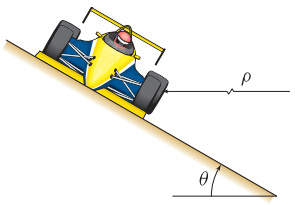
Solve Eq. (3) for $v_{\min}$ and substitute the expression for N to obtain

$$-\mu_s N\cos\theta + N\sin\theta = \frac{mv_{\min}}{\rho} \quad \Rightarrow \quad \boxed{v_{\min} = \sqrt{\rho g}\sqrt{\frac{\tan\theta - \mu_s}{1+\mu_s\tan\theta}} = 35.33\,\text{m/s} = 127\,\text{km/hr.}}$$

$$\boxed{127\,\text{km/hr} \le v \le 199\,\text{km/hr.}}$$

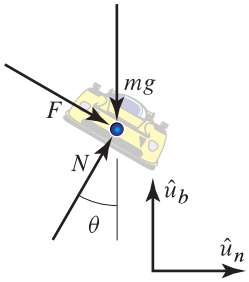
Problem 3.57

A race car is traveling at a constant speed $v = 200$ mph over a circular banked turn. Let the weight of the car be $W = 4000$ lb, the radius of the car's trajectory be $\rho = 1100$ ft, the bank angle be $\theta = 33°$, and the coefficient of static friction between the car and the track be $\mu_s = 1.7$. Determine the component of the friction force perpendicular to the direction of motion.



Solution

Balance Principles



$$\sum F_n: \quad F\cos\theta + N\sin\theta = ma_n, \tag{1}$$

$$\sum F_t: \quad F_t = ma_t, \tag{2}$$

$$\sum F_b: \quad -F\sin\theta - mg + N\cos\theta = ma_b. \tag{3}$$

Force Law: assume no slip:

$$|F| \le \mu_s |N|.$$

Kinematics:

$$a_n = \frac{v^2}{\rho}, \quad a_t = 0, \quad \text{and} \quad a_b = 0.$$

Eq. (2) becomes $F_t = 0$. Thus there is no friction in the $\hat{u}_t$ direction. Eq. (1) and (3) become

$$F\cos\theta + N\sin\theta = \frac{mv^2}{\rho} \quad \text{and} \quad N\cos\theta - F\sin\theta = mg.$$

We have 2 equations in 2 unknowns. Solving for F and N we obtain

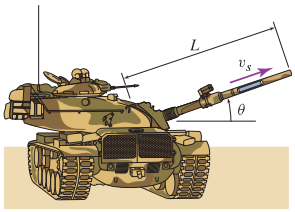
$$N = \frac{mv^2\sin\theta}{\rho} + mg\cos\theta = 8650\,\text{lb},$$

$$\boxed{F = \frac{mv^2\cos\theta}{\rho} - mg\sin\theta = 5970\,\text{lb}.}$$

Our assumption $|F| \le \mu_s |N|$ is correct.

Problem 3.58

The cutaway of the gun barrel shows a projectile moving through the barrel. If the projectile's exit speed is $v_s = 1675\,\text{m/s}$ (relative to the barrel), the projectile's mass is 18.5 kg, the length of the barrel is $L = 4.4\,\text{m}$, the acceleration of the projectile down the gun barrel is constant, and θ is increasing at a constant rate of 0.18 rad/s, determine



(a) The acceleration of the projectile.

(b) The pressure force acting on the back of the projectile.

(c) The normal force on the gun barrel due to the projectile.

as the projectile leaves the gun, but while it is still in the barrel. Assume that the projectile exits the barrel when $\theta = 20°$, and ignore friction between the projectile and the barrel.

Solution

Part (a) The acceleration of the projectile is given by

$$\vec{a} = a_r\,\hat{u}_r + a_\theta\,\hat{u}_\theta = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta, \tag{1}$$

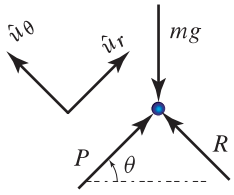
where $\dot{\theta} = 0.18\,\text{rad/s}$ and $\ddot{\theta} = 0$. When the shell exits the barrel, $r = L$ and $\dot{r} = v_s$. To find $\ddot{r}$ when the shell exits the barrel, we use the constant acceleration equation $\dot{s}^2 = \dot{s}_0^2 + 2a_c(s - s_0)$.

$$v_s^2 = 0 + 2\ddot{r}L \quad \Rightarrow \quad \ddot{r} = \frac{v_s^2}{2L}. \tag{2}$$

Substituting the value of $\ddot{r}$ from Eq. (2) and other given quantitites into Eq. (1), we get

$$\boxed{\vec{a} = (319\times10^3\,\hat{u}_r + 603\,\hat{u}_\theta)\,\text{m/s}^2.} \tag{3}$$

Part (b) Referring to the FBD, the Newton-Euler equations for the shell are given by



$$\sum F_r: \quad P - mg\sin\theta = ma_r, \tag{4}$$

$$\sum F_\theta: \quad R - mg\cos\theta = ma_\theta, \tag{5}$$

where m is the mass of the shell, P is the pressure force propelling the shell and R is the force exerted by the gun barrel on the shell. Now, using Eq. (4), we get

$$\boxed{P = ma_r + mg\sin\theta = 5.90\times10^6\,\text{N},} \tag{6}$$

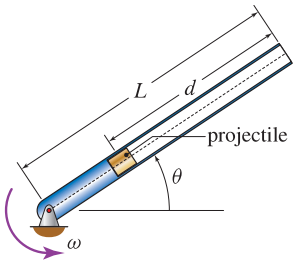
where $\theta = 20°$.

Part (c) Using Eq. (5), we get

$$\boxed{R = ma_\theta + mg\cos\theta = 11.3\times10^3\,\text{N}.} \tag{7}$$

Problems 3.59 and 3.60

A simple sling can be built by placing a projectile in a tube and then spinning it. Consider a simple model in which the tube is pinned as shown and is rotated about the pin in the *horizontal* plane at constant angular velocity ω. Assume that there is no friction between the projectile and the inside of the tube and that the projectile is initially kept fixed at a distance d from the open end of the tube.

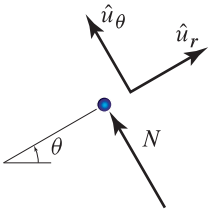


Problem 3.59 After the projectile is released, compute the normal force exerted by the inside of the tube on the projectile as a function of position of the projectile along the tube.

Problem 3.60 Leting $d = 3$ ft and $L = 7$ ft, determine the value of the tube's (constant) angular velocity ω if, after release, the projectile exits the tube with a (total) speed of 90 ft/s.

Solution to 3.59

Balance Principles:



$$\sum F_r: \quad 0 = ma_r, \tag{1}$$

$$\sum F_\theta: \quad N = ma_\theta. \tag{2}$$

Kinematics:

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta},$$

where $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$. Integrate Eq. (1). Using the chain rule we rewrite $\ddot{r}$ as $\dot{r}\frac{d\dot{r}}{dr}$

$$\ddot{r} = r\omega^2 \quad \Rightarrow \quad \int_d^r r\omega^2\, dr = \int_0^{\dot{r}} \dot{r}\, d\dot{r} \quad \Rightarrow \quad \tfrac{1}{2}\omega^2\left(r^2 - d^2\right) = \tfrac{1}{2}\dot{r}^2 \quad \Rightarrow \quad \dot{r} = \omega\sqrt{r^2 - d^2}.$$

Eq. (2) becomes

$$N = 2m\dot{r}\omega \quad \Rightarrow \quad \boxed{N = 2m\omega^2\sqrt{r^2 - d^2}.}$$

Solution to 3.60

This solution is identical to that of Problem 3.59 except we integrate Eq. (1) from d to L to obtain the radial component of velocity at L. Using the chain rule we rewrite $\ddot{r}$ as $\dot{r}\frac{d\dot{r}}{dr}$

$$\ddot{r} = r\omega^2 \quad \Rightarrow \quad \int_d^L r\omega^2\, dr = \int_0^{\dot{r}_L} \dot{r}\, d\dot{r} \quad \Rightarrow \quad \tfrac{1}{2}\omega^2\left(L^2 - d^2\right) = \tfrac{1}{2}\dot{r}_L{}^2 \quad \Rightarrow \quad \dot{r}_L = \omega\sqrt{L^2 - d^2}.$$

The transverse component of velocity at L is ωL. Therefore the speed at L is

$$v_L = \sqrt{\dot{r}^2 + \omega^2 L^2} = 90\,\text{ft/s}.$$

Substitute the equation for $\dot{r}_L$ and solve for ω

$$v_L^2 = \omega^2\left(L^2 - d^2\right) + \omega^2 L^2 \quad \Rightarrow \quad \boxed{\omega = \frac{v_L}{\sqrt{2L^2 - d^2}} = 9.54\,\text{rad/s}.}$$

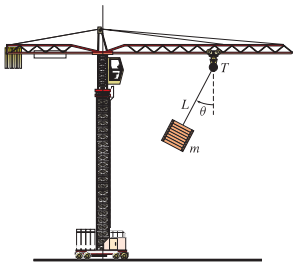
Problem 3.61

The trolley T moves along rails on the horizontal truss of the tower crane. The trolley and the load of mass m are both initially at rest (with $\theta = 0$) when the trolley starts moving to the right with constant acceleration $a_T = g$. Determine

(a) The maximum angle $\theta_{\max}$ achieved by the load m.

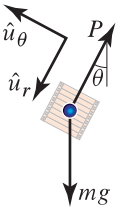
(b) The tension in the supporting cable as a function of θ.

Treat the load m as a particle, and ignore the mass of the supporting cable.



Solution

Balance Principles:



$$\sum F_r: \quad -P + mg\cos\theta = ma_r, \tag{1}$$

$$\sum F_\theta: \quad -mg\sin\theta = ma_\theta, \tag{2}$$

where P is the tension in the cable. Because the frame attached to the trolly is translating relative to the underlying inertial reference frame, the acceleration of the mass is

$$\vec{a}_m = \vec{a}_T + \vec{a}_{m/T}, \tag{3}$$

where $\vec{a}_T$ is the acceleration of the trolley, which is written in polar coordinates as $\vec{a}_T = -a_T\sin\theta\,\hat{u}_r - a_T\cos\theta\,\hat{u}_\theta$, and $\vec{a}_{m/T}$ is the acceleration of the mass relative to the trolley, which is written as $\vec{a}_{m/T} = (\ddot{L} - L\dot{\theta}^2)\,\hat{u}_r + (L\ddot{\theta} + 2\dot{L}\,\dot{\theta})\,\hat{u}_\theta$. Because the cable is inextensible, we have $\dot{L} = \ddot{L} = 0$. Then Eq. (3) gives us the acceleration components as

$$a_r = -a_T\sin\theta - L\dot{\theta}^2, \quad \text{and} \quad a_\theta = -a_T\cos\theta + L\ddot{\theta}. \tag{4}$$

Combining Eqs. (4) with Eq. (1) and Eq. (2) we obtain the equations of motion as

$$\dot{\theta}^2 + \frac{a_T}{L}\sin\theta + \frac{g}{L}\cos\theta = \frac{P}{mL}, \tag{5}$$

$$\ddot{\theta} - \frac{a_T}{L}\cos\theta + \frac{g}{L}\sin\theta = 0. \tag{6}$$

The solution to Eq. (6) would give us θ as a function of time; however, Eq. (6) is non-linear and has solutions in terms of elliptic functions. Fortunately, to find the maximum angle $\theta_{\max}$ we only need to know the angular velocity $\dot{\theta}$ as a function of θ. By the chain rule, $\ddot{\theta} = \dot{\theta}d\dot{\theta}/d\theta$ so we can integrate Eq. (6) as

$$\int_0^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = \frac{1}{L}\int_0^{\theta} a_T\cos\theta - g\sin\theta\,d\theta, \tag{7}$$

where the lower bounds on the integrals come from the fact that $\dot{\theta} = 0$ and $\theta = 0$ initially. From Eq. (7) we obtain an expression for $\dot{\theta}$ in terms of θ as

$$\dot{\theta}^2 = \frac{2a_T}{L}\sin\theta + \frac{2g}{L}(\cos\theta - 1). \tag{8}$$

The mass will reach its maximum height at the angle where $\dot{\theta} = 0$ so we set the left hand side of Eq. (8) to zero and solve for θ as

$$\frac{1-\cos\theta}{\sin\theta} = \frac{a_T}{g} \quad \Rightarrow \quad \tan\frac{\theta}{2} = \frac{a_T}{g}, \tag{9}$$

where we used the trigonometric identity $\tan(\theta/2) = (1-\cos\theta)/\sin\theta$ to get the second equation above. Then from Eq. (9) we have the solution for part (a) as

$$\boxed{\theta_{\max} = 2\tan^{-1}\left(\frac{a_T}{g}\right) = 90^\circ,} \tag{10}$$

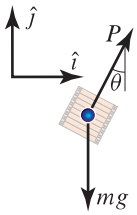
where $a_T = g$ gives the final answer. To find the tension P as a function of θ we substitute Eq. (8) into Eq. (5) for $\dot{\theta}^2$ and get the solution to part (b) as

$$\boxed{P = m\left(3a_T\sin\theta + 3g\cos\theta - 2g\right) = mg\left(3\sin\theta + 3\cos\theta - 2\right),} \tag{11}$$

where $a_T = g$ gives the final answer.

This problem can also be solved using Cartesian components. However, we only provide the setup.

Balance Principles:



$$\sum F_x: \quad P\sin\theta = ma_x, \tag{12}$$

$$\sum F_y: \quad P\cos\theta = ma_y. \tag{13}$$

The position of the crate is

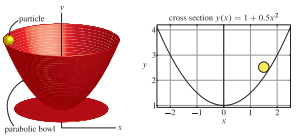
$$x = x_T - L\sin\theta \quad \text{and} \quad y = y_T - L\cos\theta.$$

Differentiate the position of the crate twice with respect to time

$$a_x = \ddot{x} \quad \Rightarrow \quad a_T + L\dot{\theta}^2\sin\theta - L\ddot{\theta}\cos\theta,$$
$$a_y = \ddot{y} \quad \Rightarrow \quad L\dot{\theta}^2\cos\theta + L\ddot{\theta}\sin\theta.$$

Problem 3.62

If the particle is constrained to only move back and forth in the plane of the center of the bowl, how many degrees of freedom must it have? Recall that this will also be the number of equations of motion of the particle.

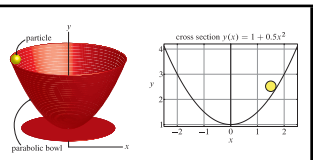


Solution

The particle has only one degree of freedom because the position of the particle can be completely known by specifying the x coordinate of the particle's position.

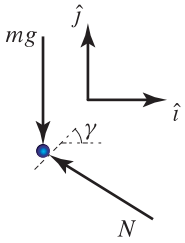
Problem 3.63

Derive the equation(s) of motion for a particle moving on the inner surface of the parabolic bowl shown. Assume that the particle only moves in the vertical xy plane that goes through the center of the bowl, that the equation of the parabola is $y(x) = 1 + 0.5x^2$, and that gravity is acting in the $-y$ direction. The bowl's cross section is shown on the right side of the figure.



Solution

Referring to the FBD, the Newton-Euler equations are given by



$$\sum F_x : \qquad -N\sin\gamma = ma_x, \tag{1}$$

$$\sum F_y : \quad N\cos\gamma - mg = ma_y, \tag{2}$$

where N is the normal force and m is the mass of the particle. In order to get the equation of motion, we need to eleminate N between Eqs. (1) and (2), and relate a_x and a_y. The kinematic equations are given by

$$a_x = \ddot{x} \quad \text{and} \quad a_y = \ddot{y}. \tag{3}$$

Now, we can relate $\ddot{x}$ and $\ddot{y}$ by noting that

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y'\dot{x}, \tag{4}$$

$$\ddot{y} = \frac{dy'}{dt}\dot{x} + y'\ddot{x} = \frac{dy'}{dx}\frac{dx}{dt}\dot{x} + y'\ddot{x} = y''\dot{x}^2 + y'\ddot{x}, \tag{5}$$

where $y' = 2(0.5)x = x$ and $y'' = 1$. Also, the slope γ can be found by noting that

$$\tan\gamma = y' = x. \tag{6}$$

Plugging in the kinematic equations into the Newton-Euler equations, we get

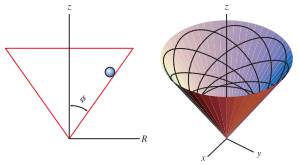
$$\begin{aligned} -N\sin\gamma &= m\ddot{x}, \\ N\cos\gamma &= mg + m(y''\dot{x}^2 + y'\ddot{x}) \end{aligned} \quad \Rightarrow \quad -\tan\gamma = -y' = \frac{\ddot{x}}{g + y''\dot{x}^2 + y'\ddot{x}}. \tag{7}$$

We can re-arrange Eq.(7) and substitute $y' = x$ and $y'' = 1$ to obtain the equation of motion

$$\left[1 + (y')^2\right]\ddot{x} + y'y''\dot{x}^2 + gy' = 0 \quad \Rightarrow \quad \boxed{(1 + x^2)\,\ddot{x} + x\dot{x}^2 + gx = 0.} \tag{8}$$

Problem 3.64

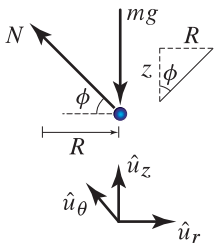
A particle moves over the inner surface of an inverted cone. Assuming that the cone's surface is frictionless and using the cylindrical coordinate system shown, show that the particle's equations of motion are



$$\ddot{R}\left(1+\cot^2\phi\right) - R\dot{\theta}^2 = -g\cot\phi,$$
$$R\ddot{\theta} + 2\dot{R}\dot{\theta} = 0.$$

Solution

Balance Principles:



$$\sum F_r: \quad -N\cos\phi = ma_r, \tag{1}$$
$$\sum F_\theta: \quad 0 = ma_\theta, \tag{2}$$
$$\sum F_z: \quad N\sin\phi - mg = ma_z, \tag{3}$$

where N is the normal force and m is the mass of the particle.
Kinematics:

$$a_r = (\ddot{R} - R\dot{\theta}^2), \quad a_\theta = (R\ddot{\theta} + 2\dot{R}\dot{\theta}), \quad \text{and} \quad a_z = \ddot{z}.$$

From Eq. (2), we obtain the first equation of motion as

$$\boxed{R\ddot{\theta} + 2\dot{R}\dot{\theta} = 0.} \tag{4}$$

Referring to the figure we state the relation $z = R\cot\phi$. Noting that ϕ is constant we calculate the second derivative of z with respect to time as $\ddot{R}\cot\phi$. Substituting this relationship into Eq. (3) we can express N as

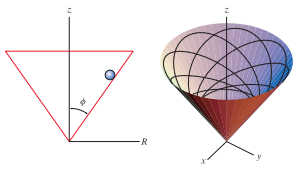
$$N = \frac{m}{\sin\phi}(g + \ddot{R}\cot\phi). \tag{5}$$

Substituting Eq. (5) into Eq. (1) and re-arranging, we get the other equation of motion as

$$\boxed{\ddot{R}(1+\cot^2\phi) - R\dot{\theta}^2 = -g\cot\phi.} \tag{6}$$

Problem 3.65

Continue Prob. 3.64 by integrating the particle's equations of motion and plotting its trajectory for $0 \le t \le 25$ s. Use the following parameter values and initial conditions: $g = 9.81\,\text{m/s}^2$, $\phi = 30°$, $\theta(0) = 0°$, $\dot{\theta}(0) = 1.00\,\text{rad/s}$, $R(0) = 5\,\text{m}$, $\dot{R}(0) = 0\,\text{m/s}$.



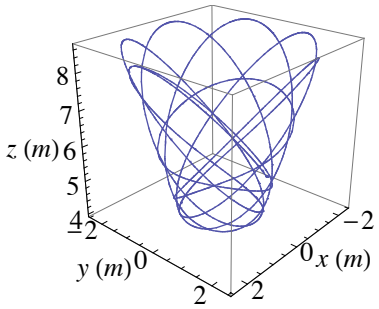
Solution

The non-linear differential equations of motion derived in the solution to Problem 3.64 can be solved using *Mathematica* with the initial conditions as given in the problem. The trajectory of the particle $(x(t), y(t), z(t))$ can be expressed in terms of $r(t)$ and $\theta(t)$ as $\{r(t)\sin(\phi)\cos[\theta(t)], r(t)\sin(\phi)\sin[\theta(t)], r(t)\cot\phi\}$. The following code plots the trajectory of the particle for $0 \le t \le 25$ s.

```
eq1 = r[t] θ''[t] + 2 r'[t] θ'[t] == 0;
eq2 = r''[t] (1 + (Cot[ϕ])^2) - r[t] (θ'[t])^2 == -g Cot[ϕ];
IC = {ϕ → 30 Degree, θ0 → 0, θdot0 → 1, r0 → 5, rdot0 → 0, g → 9.81};

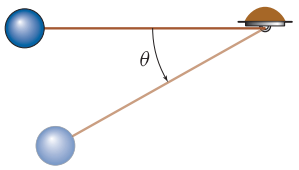
s = NDSolve[{eq1, eq2, r[0] == r0, r'[0] == rdot0, θ[0] == θ0, θ'[0] == θdot0} /. IC,
   {r, θ}, {t, 0, 25}];
ParametricPlot3D[
 Evaluate[{r[t] Cos[θ[t]] Sin[ϕ], r[t] Sin[θ[t]] Sin[ϕ], r[t] Cot[ϕ]} /. s /. IC],
 {t, 0, 25}, AxesLabel → {"x (m)", "y (m)", "z (m)"}, AspectRatio → 1,
 ImageSize → 150, ViewPoint → {2.318, 2.034, 1.393}]
```

The trajectory plot for $0 \le t \le 25$ s is shown below :



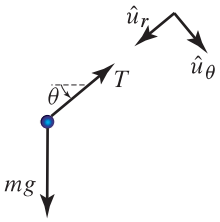
Problem 3.66

The pendulum is released from rest when $\theta = 0°$. If the string holding the pendulum bob breaks when the tension is twice the weight of the bob, at what angle does the string break? Treat the pendulum as a particle, ignore air resistance, and let the string be inextensible and massless.



Solution

Balance principles:



$$\sum F_r: \quad -T + mg\sin\theta = ma_r, \tag{1}$$

$$\sum F_\theta: \quad mg\cos\theta = ma_\theta, \tag{2}$$

where, T is the tension in the string and m is the mass of the bob. The kinematic equations are given by

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{3}$$

Since the length of the string $r = L$ is constant, we have $\dot{r} = \ddot{r} = 0$. The kinematic equations can be simplified to

$$a_r = -L\dot{\theta}^2 \quad \text{and} \quad a_\theta = L\ddot{\theta} \tag{4}$$

Substituting Eqs.(4) into Eqs. (1) and (2), we get

$$-T + mg\sin\theta = -mL\dot{\theta}^2, \tag{5}$$

$$mg\cos\theta = mL\ddot{\theta}. \tag{6}$$

From Eq. (6) we have

$$\ddot{\theta} = \frac{g}{L}\cos\theta. \tag{7}$$

Using $\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$ in Eq. (7), we get

$$\int_{\dot{\theta}(\theta_0)}^{\dot{\theta}(\theta)} \dot{\theta}\, d\dot{\theta} = \frac{g}{L}\int_{\theta_0}^{\theta} \cos\theta\, d\theta. \tag{8}$$

Now, using $\theta_0 = 0$ and $\dot{\theta}(0) = 0$, we get

$$\dot{\theta}^2(\theta) = 2\frac{g}{L}\sin\theta. \tag{9}$$

Substituting Eq. (9) into Eq. (5), we get

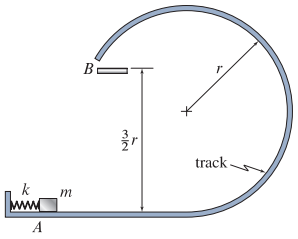
$$T(\theta) = mg\sin\theta + mL\left(2\frac{g}{L}\sin\theta\right) \quad \Rightarrow \quad T(\theta) = 3mg\sin\theta. \tag{10}$$

The string breaks when the tension $T(\theta_B) = 2mg$. Equating second of Eqs. (10) to $2mg$, we get

$$3mg\sin\theta_B = 2mg \quad \Rightarrow \quad \boxed{\theta_B = \sin^{-1}\left(\frac{2}{3}\right) = 41.8°.} \tag{11}$$

Problem 3.67

The package handling system is designed to launch the small package of mass m from A, using a compressed linear spring of constant k. After launch, the package slides along the track until it lands on the conveyer belt at B. The track has small, well-oiled rollers, making any friction between the packages and the track negligible. Modeling the package as a particle, determine the minimum initial compression of the spring so that the package gets to B without separating from the track, and determine the corresponding speed with which the package reaches the conveyer at B.

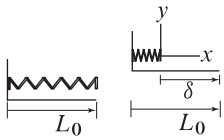


Solution

Balance principles during compression:

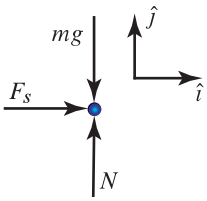
$$\sum F_x: \quad F_s = ma_x, \tag{1}$$

$$\sum F_y: \quad N - mg = ma_y, \tag{2}$$



The spring force $F_s = k(\delta - x)$, where x is the displacement of mass m from the point where the spring is initially compressed by an amount δ. The kinematic equations are given by

$$a_x = \ddot{x} \quad \text{and} \quad a_y = 0. \tag{3}$$



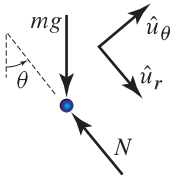
Using $\ddot{x} = v\frac{dv}{dx}$, Eq. (1) can be used to find out the velocity of launch v_0 of the package.

$$\int_0^{v_0} v\,dv = k\int_0^{\delta} (\delta - x)\,dx \quad \Rightarrow \quad v_0 = \delta\sqrt{\frac{k}{m}}. \tag{4}$$

Balance principles during circular motion along the track:

$$\sum F_r: \quad mg\cos\theta - N = ma_r, \tag{5}$$

$$\sum F_\theta: \quad -mg\sin\theta = ma_\theta, \tag{6}$$



Because r is constant the kinematic equations are given by

$$a_r = -r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta}. \tag{7}$$

Solve Eq. (5) for $\dot{\theta}^2$ with $N = 0$ and $\theta = \pi$ to find the angular velocity at the top of the track.

$$mg\cos\pi = -mr\dot{\theta}_t^2 \quad \Rightarrow \quad \dot{\theta}_t^2 = \frac{g}{r}. \tag{8}$$

Eq. (6) gives us $\ddot{\theta}$ as a function of θ. Integrate Eq. (6) using the chain rule

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \quad \Rightarrow \quad \frac{-g}{r}\int_0^{\theta} \sin\theta\,d\theta = \int_{\dot{\theta}_0}^{\dot{\theta}} d\dot{\theta} \quad \Rightarrow \quad \frac{2g}{r}(\cos\theta - 1) = \dot{\theta}^2 - \frac{v_0^2}{r^2}, \tag{9}$$

where we have substituted $\dot{\theta}_0 = \frac{v_0^2}{r^2}$. Substitute Eq. (8) into Eq. (9), evaluate at $\theta = \pi$ and solve for v_0.

$$\frac{2g}{r}(\cos\theta - 1) = \frac{g}{r} - \frac{v_0^2}{r^2} \quad \Rightarrow \quad v_0 = \sqrt{5gr}. \tag{10}$$

Now, using the expression for v_0 from second of Eqs. (4), we get

$$\boxed{\delta_{\min} = \sqrt{\frac{5mgr}{k}}.}$$

Solve the last of Eqs. (9) for $\dot{\theta}^2$, substitute $\dot{\theta}^2 = v^2/r^2$, solve for v,

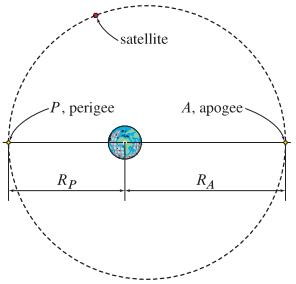
$$\dot{\theta}^2 = \frac{v_0^2}{r^2} + \frac{2g}{r}(\cos\theta - 1) \quad \Rightarrow \quad v = \sqrt{v_0^2 + 2gr\,(\cos\theta - 1)}.$$

Substitute Eq. (10) for v_0, and evaluate at $\theta = (4/3)\pi$. The velocity at B is

$$\boxed{v_B = \sqrt{2gr}.}$$

Problems 3.68 and 3.69

A satellite orbits the Earth as shown. Model the satellite as a particle, and assume that the center of the Earth can be chosen as the origin of an inertial frame of reference.

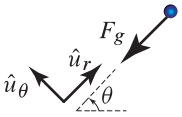


Problem 3.68 Using a polar coordinate system and letting m_e be the mass of the Earth, determine the equations of motion of the satellite.

Problem 3.69 The minimum and maximum distances from the center of the Earth are $R_P = 4.5\times10^7$ m and $R_A = 6.163\times10^7$ m, respectively, where the subscripts P and A stand for *perigee* (the point on the orbit closest to Earth) and *apogee* (the point on the orbit farthest from Earth), respectively. If the satellite's speed at P is $v_P = 3.2\times10^3$ m/s, determine the satellite's speed at A.

Solution to 3.68

Balance principles:



$$\sum F_r: \quad -F_g = ma_r, \tag{1}$$

$$\sum F_\theta: \quad 0 = ma_\theta, \tag{2}$$

Kinematics:

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{3}$$

Force law:

$$F_g = G\frac{mm_e}{r^2}, \tag{4}$$

where m_e is the mass of the earth, G is the universal constant of gravitation and r is the radial coordinate of the satellite. Substituting Eqs. (3) and (4) into Eq. (1) and re-writing Eq. (2), we get the equations of motion

$$\ddot{r} - r\dot{\theta}^2 + G\frac{m_e}{r^2} = 0, \tag{5}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{6}$$

Solution to 3.69

Noting that

$$\frac{d}{dt}(\dot{\theta}r^2) = r(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \tag{7}$$

Using Eq. (6) from the solution to Problem 3.68 and Eq. (7), we can write

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(\dot{\theta}r^2) = 0 \quad \Rightarrow \quad \dot{\theta}r^2 = \text{constant}. \tag{8}$$

So that

$$\dot{\theta}_A R_A^2 = \dot{\theta}_P R_P^2. \tag{9}$$

At points P and A, the velocity vector is perpendicular to the radial position vector. Hence, we have

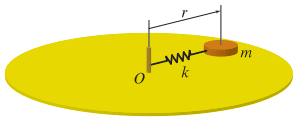
$$\left|\vec{v}\right|_P = R_P\dot{\theta}_P \quad \text{and} \quad \left|\vec{v}\right|_A = R_A\dot{\theta}_A. \tag{10}$$

Substituting $\dot{\theta}_P$ and $\dot{\theta}_A$ from first and second of Eqs. (10) into Eq. (9), we get

$$\boxed{\left|\vec{v}\right|_A = \left|\vec{v}\right|_P \frac{R_P}{R_A} = 2.34\times10^3 \text{ m/s}.} \tag{11}$$

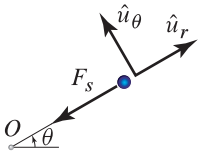
Problem 3.70

The disk shown, weighing 3 lb, rotates about O by sliding without friction over the horizontal surface shown. The spring is linear with constant k and unstretched length $L_0 = 0.75$ ft. The maximum distance achieved by the disk from point O is $d_{\max} = 1.85$ ft while traveling at a speed $v_0 = 20$ ft/s. Determine the value of k such that the minimum distance between the disk and O is $d_{\min} = d_{\max}/2$.



Solution

Balance Principles:



$$\sum F_r: \quad -F_s = ma_r, \tag{1}$$

$$\sum F_\theta: \quad 0 = ma_\theta. \tag{2}$$

Force Laws:

$$F_s = k\,(r - L_0)\,.$$

Kinematics:

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

Because the transverse component of acceleration is 0 we can say

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \quad \Rightarrow \quad r^2\dot{\theta} = \text{constant}.$$

Note: for $r = d_{\max}$ $\dot{r} = 0$ and for $r = d_{\min}$ $\dot{r} = 0$. Thus $\vec{v}$ has no component in the $\hat{u}_r$ direction for these two points. $v_\theta = v_0 = d_{\max}\dot{\theta}_o$ where $\dot{\theta}_o$ is the angular velocity at $r = d_{\max}$. Thus

$$v_0 = d_{\max}\dot{\theta}_o \quad \Rightarrow \quad \dot{\theta}_o = \frac{v_0}{d_{\max}} \quad \Rightarrow \quad r^2\dot{\theta} = d_{\max}^2\left(\frac{v_0}{d_{\max}}\right) \quad \Rightarrow \quad \dot{\theta} = \frac{v_0 d_{\max}}{r^2}.$$

From Eq. (1) we have

$$-\frac{k}{m}(r - L_o) = \ddot{r} - r\frac{v_0^2 d_{\max}^2}{r^4} \quad \Rightarrow \quad \ddot{r} = \frac{v_0^2 d_{\max}^2}{r^3} - \frac{k}{m}(r - L_0)\,.$$

We have acceleration as a function of position. Integrate it as follows:

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} \quad \Rightarrow \quad \int_{d_{\max}}^{d_{\min}}\left[\frac{v_0^2 d_{\max}^2}{r^3} - \frac{k}{m}(r - L_0)\right]dr = \int_0^0 \dot{r}\,d\dot{r}.$$

Evaluate the integrals and substitute $d_{\min} = d_{\max}/2$:

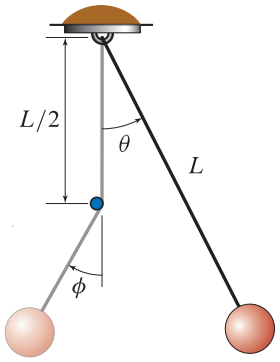
$$\frac{-v_0^2 d_{\max}^2}{2r^2}\Bigg|_{d_{\max}}^{d_{\min}} - \frac{k}{2m}(r - L_0)^2\Bigg|_{d_{\max}}^{d_{\min}} = 0 \quad \Rightarrow \quad v_0^2 - 4v_0^2 - \frac{k}{m}(d_{\max}/2 - L_0)^2 + \frac{k}{m}(d_{\max} - L_0)^2 = 0.$$

Solve for k

$$k = \frac{3v_0 m}{(d_{\max} - L_0)^2 - (d_{\max}/2 - L_0)^2} = 4.74\ \text{lb/ft}.$$

Problems 3.71 and 3.72

A pendulum with cord length $L = 6\,\text{ft}$ and a bob weighing $3\,\text{lb}$ is released from rest at an angle θ_i. Once the pendulum has swung to the vertical position (i.e., $\theta = 0$), its cord runs into a small fixed obstacle. In solving this problem, neglect the size of the obstacle; model the pendulum's bob as a particle and the pendulum's cord as massless and inextensible; and let gravity and the tension in the cord be the only relevant forces.

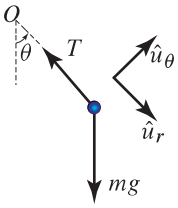


Problem 3.71 What is the maximum height reached by the pendulum, measured from its lowest point, if $\theta_i = 20°$?

Problem 3.72 If the bob is released from rest at $\theta_i = 90°$, at what angle ϕ does the cord go slack?

Solution to 3.71

For the period between the release of the pendulum and when the cord runs into the obstacle, the FBD of the bob is shown in the right. Referring to the FBD, the Newton-Euler equations for the bob are given by



$$\sum F_r: \quad -T + mg\cos\theta = ma_r, \tag{1}$$

$$\sum F_\theta: \quad -mg\sin\theta = ma_\theta, \tag{2}$$

where T is the tension in the cord and m is the mass of the bob. The kinematic equations are given by

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{3}$$

Since $r = L = \text{const.}$, we have $a_r = -L\dot{\theta}^2$ and $a_\theta = L\ddot{\theta}$. Substituting a_r and a_θ in Eqs. (1) and (2), we get

$$-T + mg\cos\theta = -mL\dot{\theta}^2, \tag{4}$$

$$-mg\sin\theta = mL\ddot{\theta}. \tag{5}$$

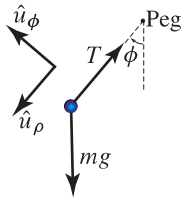
Now, from Eq. (5) we have $\ddot{\theta} = -g\sin\theta/L$. Re-writing $\ddot{\theta}$ as $\dot{\theta}d\dot{\theta}/d\theta$ and integrating both sides, we get

$$\int_0^{\dot{\theta}(0)} \dot{\theta}\,d\dot{\theta} = -\frac{g}{L}\int_{\theta_i}^{0^\circ} \sin\theta\,d\theta \quad \Rightarrow \quad \dot{\theta}(0) = -\sqrt{\frac{2g}{L}(1-\cos\theta_i)}. \tag{6}$$

Note that a minus sign has been introduced because θ is decreasing with time so that $\dot{\theta} < 0$ at all instants. Plugging in the known values we get $\dot{\theta}(0) = -0.8045\,\text{rad/s}$. The velocity of the bob at $\theta = 0°$ is given by

$$\vec{v}\big|_{\theta=0^\circ} = (L\dot{\theta}\hat{u}_\theta)\Big|_{\theta=0^\circ}. \tag{7}$$

Now, for the period between the cord running into the obstacle and the bob reaching its maximum height, the length of the pendulum is reduced to $L/2$. We set up another polar coordinate system for this period. Referring to the FBD for this period, the Newton-Euler equations for the bob are given by



$$\sum F_\rho : \quad -T + mg\cos\phi = ma_\rho, \tag{8}$$

$$\sum F_\phi : \quad -mg\sin\phi = ma_\phi. \tag{9}$$

Again, using the fact that $\rho = L/2 = \text{const.}$, we can write

$$a_\rho = -\frac{L}{2}\dot{\phi}^2 \quad \text{and} \quad a_\phi = \frac{L}{2}\ddot{\phi}. \tag{10}$$

The velocity at $\phi = 0°$ is given by

$$\vec{v}\big|_{\phi=0°} = \left(\frac{L}{2}\dot{\phi}\hat{u}_\phi\right)\Big|_{\phi=0°}. \tag{11}$$

Using the facts $\vec{v}\big|_{\phi=0°} = \vec{v}\big|_{\theta=0°}$ and $\hat{u}_\phi\big|_{\phi=0°} = -\hat{u}_\theta|_{\theta=0°}$, from Eqs. (7) and (11), we get

$$\frac{L}{2}\dot{\phi}\Big|_{\phi=0°} = -L\dot{\theta}\Big|_{\theta=0°} \quad \Rightarrow \quad \dot{\phi}\big|_{\phi=0°} = 1.609\,\text{rad/s}. \tag{12}$$

Substituting Eqs. (10) into Eqs. (8) and (9), we get

$$-T + mg\cos\phi = -m\frac{L}{2}\dot{\phi}^2, \tag{13}$$

$$-mg\sin\phi = m\frac{L}{2}\ddot{\phi}. \tag{14}$$

From Eq. (14), we have $\ddot{\phi} = -2g/L\sin\phi$. Re-writing $\ddot{\phi}$ as $\dot{\phi}d\dot{\phi}/d\phi$ and integrating both sides, we get

$$\int_{\dot{\phi}(0°)}^{\dot{\phi}(\phi_{\max})} \dot{\phi}\,d\dot{\phi} = -2\frac{g}{L}\int_{0°}^{\phi_{\max}} \sin\phi\,d\phi. \tag{15}$$

Noting that at $\phi_{\max}$, $\dot{\phi}(\phi_{\max}) = 0$ and using the value $\dot{\phi}(0°)$ from Eq. (12), we get

$$\cos\phi_{\max} = 1 - L\frac{\dot{\phi}^2(0°)}{4g} \quad \Rightarrow \quad \phi_{\max} = \cos^{-1}\left[1 - L\frac{\dot{\phi}^2(0°)}{4g}\right] = 28.43°. \tag{16}$$

Therefore, measured from the lowest point in the bob's trajectory, the maximum height reached by the bob is

$$h_{\max} = \frac{L}{2}(1 - \cos\phi_{\max}) \quad \Rightarrow \quad \boxed{h_{\max} = 0.362\,\text{ft.}} \tag{17}$$

Solution to 3.72

At the instant the cord encounters the peg the velocity of the bob does not change. $\dot{\theta}$ at that instant is found by using Eq. (6) with $\theta = 90°$.

$$\dot{\theta}^2\Big|_{\theta=0} = \frac{2g}{L}\left(1 - \cos 90°\right) \quad \Rightarrow \quad \dot{\theta}^2\Big|_{\theta=0} = \frac{2g}{L}.$$

We find $\dot{\phi}$ at that instant by equating two expressions for the velocity of the bob:

$$\frac{L}{2}\dot{\phi}\Big|_{\phi=0} = L\dot{\theta}\Big|_{\theta=0} \quad \Rightarrow \quad \dot{\phi}\Big|_{\phi=0} = \sqrt{\frac{8g}{L}}.$$

The integrals given by Eq. (15) are still valid, we simply change the limits of integration to be

$$\int_{\dot{\phi}(0^\circ)}^{\dot{\phi}} \dot{\phi}\, d\dot{\phi} = \frac{-2g}{L}\int_{0^\circ}^{\phi} \sin\phi\, d\phi \quad \Rightarrow \quad \dot{\phi}^2 - \frac{8g}{L} = \frac{4g}{L}(\cos\phi - 1) \quad \Rightarrow \quad \dot{\phi}^2 = \frac{4g}{L}(\cos\phi + 1)\,. \tag{18}$$

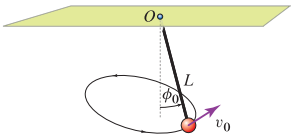
Substitute Eq. (18) into Eq. (13), set $T = 0$ and solve for ϕ_s the angle where $T = 0$

$$-T + mg\cos\phi = \frac{-mL}{2}\frac{4g}{L}(\cos\phi + 1) \quad \Rightarrow \quad \cos\phi_s = -2\cos\phi_s - 2,$$

$$\boxed{\phi_s = \cos^{-1}\left(\frac{-2}{3}\right) = 132^\circ.}$$

Problems 3.73 through 3.76

A spherical pendulum is suspended from point O and put in motion.



Problem 3.73 Derive the pendulum's equations of motion using Cartesian coordinates.

Problem 3.74 Derive the pendulum's equations of motion using cylindrical coordinates.

Problem 3.75 Derive the pendulum's equations of motion using spherical coordinates.

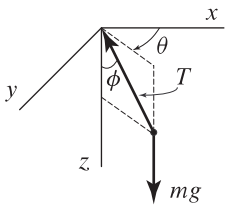
Problem 3.76 At the lowest point on its trajectory, the pendulum in the figure has a speed $v_0 = 2.5\,\text{ft/s}$ while $\phi_0 = 15°$. Letting $L = 2\,\text{ft}$, plot the trajectory of the pendulum bob for 5 s.

Solution to 3.73

Referring to the figure on the right, we first find the x, y and z components of $\vec{T}$, the tension in the cord, as

$$T_x = -T\sin\phi\cos\theta, \quad T_y = -T\sin\phi\sin\theta, \quad \text{and} \quad T_z = -T\cos\phi. \tag{1}$$

Referring to FBD, the Newton-Euler equations for the bob are given by



$$\sum F_x: \qquad T_x = m\ddot{x}, \tag{2}$$

$$\sum F_y: \qquad T_y = m\ddot{y}, \tag{3}$$

$$\sum F_z: \quad mg + T_z = m\ddot{z}, \tag{4}$$

where T is the magnitude of the tension in the cord and m is the mass of the bob. Now, we can express θ, ϕ, and z in terms of x, y, and the length of the pendulum L. Referring to the figure, we can write

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\phi = \frac{\sqrt{x^2+y^2}}{L}, \quad z = \sqrt{L^2-x^2-y^2}. \tag{5}$$

Using the third of Eqs. (5), we have

$$\dot{z} = -\frac{x\dot{x}+y\dot{y}}{\sqrt{L^2-x^2-y^2}} \quad \Rightarrow \quad \ddot{z} = -\frac{(x\dot{x}+y\dot{y})^2}{(L^2-x^2-y^2)^{3/2}} - \frac{(\dot{x}^2+\dot{y}^2+x\ddot{x}+y\ddot{y})}{\sqrt{L^2-x^2-y^2}} \tag{6}$$

Using Eq. (4) and T_z from third of Eqs. (1), we get

$$T = \frac{m(g-\ddot{z})}{\cos\phi}. \tag{7}$$

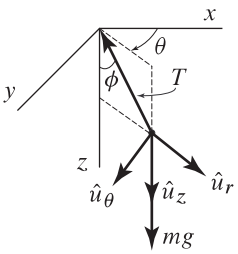
Substituting $\ddot{z}$ from Eq. (6) into Eq. (7) and then substituting the resulting equation along with the first and the second of Eqs. (5) into Eqs. (2) and (3), we get the equations of motion as

$$\ddot{x}(L^2-y^2) + \ddot{y}xy + x\frac{[\dot{x}^2(L^2-y^2)+\dot{y}^2(L^2-x^2)+2xy\dot{x}\dot{y}]}{(L^2-x^2-y^2)} + gx\sqrt{L^2-x^2-y^2} = 0, \tag{8}$$

$$\ddot{y}(L^2-x^2) + \ddot{y}xy + y\frac{[\dot{x}^2(L^2-y^2)+\dot{y}^2(L^2-x^2)+2xy\dot{x}\dot{y}]}{(L^2-x^2-y^2)} + gy\sqrt{L^2-x^2-y^2} = 0. \tag{9}$$

Solution to 3.74

Referring to the Figure and the FBD, the Newton-Euler equations for the bob are given by



$$\sum F_R: \quad -T\sin\phi = ma_R, \tag{10}$$

$$\sum F_\theta: \quad 0 = ma_\theta, \tag{11}$$

$$\sum F_z: \quad mg - T\cos\phi = m\ddot{z}, \tag{12}$$

The kinematic equations are

$$a_R = \ddot{R} - R\dot{\theta}^2 \quad \text{and} \quad a_\theta = R\ddot{\theta} + 2\dot{R}\dot{\theta}. \tag{13}$$

Referring to the figure again, we can express ϕ and z in terms of R and L as

$$\sin\phi = \frac{R}{L} \quad \text{and} \quad z = \sqrt{L^2 - R^2}. \tag{14}$$

Using second of Eqs. (14), we get

$$\dot{z} = -\frac{R\dot{R}}{\sqrt{L^2 - R^2}} \quad \Rightarrow \quad \ddot{z} = -\frac{R^2\dot{R}^2}{(L^2 - R^2)^{3/2}} - \frac{(R\ddot{R} + \dot{R}^2)}{\sqrt{L^2 - R^2}}. \tag{15}$$

Using Eq. (12), we find

$$T = \frac{m(g - \ddot{z})}{\cos\phi}. \tag{16}$$

Substituting $\ddot{z}$ from Eq. (15) into Eq. (16) and then substituting the resulting equation along with Eqs. (14) and first of Eqs. (13) into Eq. (10), we get the first equation of motion as

$$-\left[\frac{R^3\dot{R}^2}{(L^2 - R^2)^2} + \frac{(R^2\ddot{R} + R\dot{R}^2)}{(L^2 - R^2)} + \frac{gR}{\sqrt{L^2 - R^2}}\right] = \ddot{R} - R\dot{\theta}^2, \tag{17}$$

which can be simplified to

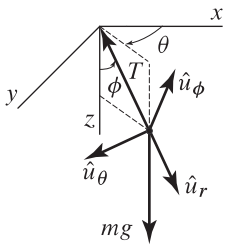
$$\boxed{\frac{\ddot{r}L^2}{(L^2 - R^2)} - R\dot{\theta}^2 + \frac{\dot{R}^2L^2R}{(L^2 - R^2)^2} + \frac{gR}{\sqrt{L^2 - R^2}} = 0.} \tag{18}$$

The second equation of motion can be obtained directly by substituting second of Eqs. (13) into Eq. (11), to get

$$\boxed{R\ddot{\theta} + 2\dot{R}\dot{\theta} = 0.} \tag{19}$$

Solution to 3.75

Referring to the figure and the FBD, the Newton-Euler equations of motion for the bob are given by



$$\sum F_r: \quad -T + mg\cos\phi = ma_r, \tag{20}$$

$$\sum F_\theta: \quad 0 = ma_\theta, \tag{21}$$

$$\sum F_\phi: \quad -mg\sin\phi = ma_\phi, \tag{22}$$

Noting that $r = L =$ constant , the kinematic equations can be written as

$$a_r = -L\dot{\phi}^2 - L\dot{\theta}^2\sin^2\phi, \tag{23}$$

$$a_\phi = L\ddot{\phi} - L\dot{\theta}^2\sin\phi\cos\phi, \tag{24}$$

$$a_\theta = L\ddot{\theta}\sin\phi + 2L\dot{\phi}\dot{\theta}\cos\phi. \tag{25}$$

Substituting Eq. (24) and Eq. (25) into Eq. (21) and Eq. (22), we get

$$\boxed{\begin{aligned} \ddot{\theta}\sin\phi + 2\dot{\phi}\dot{\theta}\cos\phi &= 0, \\ L\ddot{\phi} - L\dot{\theta}^2\sin\phi\cos\phi + g\sin\phi &= 0. \end{aligned}} \tag{26}$$

$$\tag{27}$$

Solution to 3.76

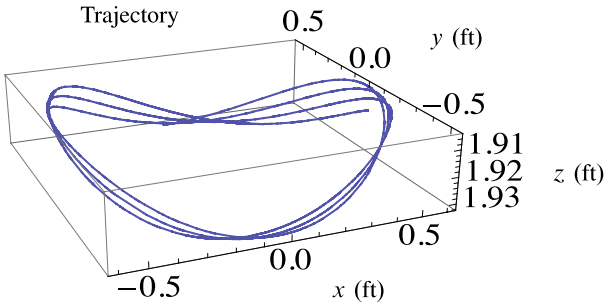
Solution to ??

We take advantage of the equations of motion derived in Problem 3.75. Using *Mathematica* we can solve these differential equations for ϕ and θ as functions of time.

```
v0 = 2.5; L = 2; g = 32.2;
InitialConditions = {θ[0] == 0, θ'[0] == v0/(L Sin[ϕ[0]]), ϕ[0] == 15 °, ϕ'[0] == 0};
Eqn1 = θ''[t] Sin[ϕ[t]] + 2 ϕ'[t] θ'[t] Cos[ϕ[t]] == 0;
Eqn2 = L ϕ''[t] - L (θ'[t])^2 Sin[ϕ[t]] Cos[ϕ[t]] + g Sin[ϕ[t]] == 0;
s = NDSolve[{Eqn1, Eqn2, InitialConditions}, {ϕ, θ}, {t, 0, 5}];
```

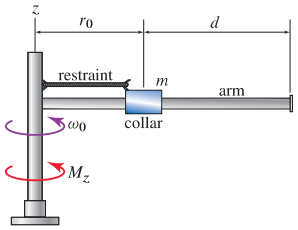
Then we must convert to Cartesian coordinates to plot the position of the bob.

```
x[t_] := Evaluate[L Sin[ϕ[t]] Cos[θ[t]] /. s[[1]]]
y[t_] := Evaluate[L Sin[ϕ[t]] Sin[θ[t]] /. s[[1]]]
z[t_] := Evaluate[L Cos[ϕ[t]] /. s[[1]]]
ParametricPlot3D[{x[t], y[t], z[t]}, {t, 0, 5}, ViewVertical → {0, 0, -1},
 BoxRatios -> {1, 1, 0.25}]
```



Problems 3.77 through 3.80

Consider a collar with mass m that is free to slide with no friction along a rotating arm, which has negligible mass. The system is initially rotating with an angular velocity ω_0 while the collar is kept a distance r_0 away from the z axis. At some point, the restraint keeping the collar in place is removed so that the collar is allowed to slide.



Problem 3.77 Derive the collar's equations of motion when $M_z = 0$.

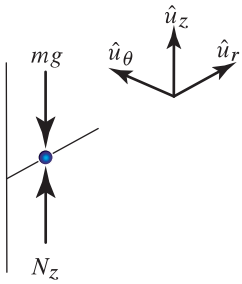
Problem 3.78 If no external forces and moments are applied to the system, what are the radial speed and the total speed of the collar when it reaches the end of the arm? Use $m = 2\,\text{kg}$, $\omega_0 = 1\,\text{rad/s}$, $r_0 = 0.5\,\text{m}$, and $d = 1\,\text{m}$.

Problem 3.79 Compute the moment M_z that you would need to apply to the arm, as a function of r, to keep the arm rotating at a *constant* angular velocity ω_0. In addition, determine the radial speed as well as the total speed with which the collar would reach the end of the arm with this moment applied. *Hint:* The moment M_z applied to the arm of negligible mass is equivalent to a force M_z/r applied to the collar in the plane of motion and perpendicular to the arm.

Problem 3.80 Compute and plot the collar's trajectory from the moment of release until the collar reaches the end of the arm. Use the parameters and initial conditions given in Prob. 3.78.

Solution to 3.77

Balance principles:



$$\sum F_r: \quad ma_r = 0, \tag{1}$$

$$\sum F_\theta: \quad ma_\theta = 0, \tag{2}$$

$$\sum F_z: \quad ma_z = N_z - mg. \tag{3}$$

Kinematics:

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}, \quad \text{and} \quad a_z = 0.$$

Thus the equations of motion expressed in cylindrical coordinates are

$$\boxed{\begin{aligned} 0 &= \ddot{r} - r\dot{\theta}^2, \\ 0 &= r\ddot{\theta} + 2\dot{r}\dot{\theta}. \end{aligned}} \tag{4, 5}$$

Solution to 3.78

Rewrite Eq. (5) as

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0. \tag{6}$$

thus, we can determine that

$$r^2\dot{\theta} = \text{constant} = C = 0.25\,\text{m}^2/\text{s} \tag{7}$$

Solving Eq. (7) for $\dot{\theta}$ and plugging it into Eq. (4) yields

$$\dot{\theta} = \frac{C}{r^2} \quad \Rightarrow \quad \ddot{r} - r\left(\frac{c^2}{r^4}\right) = 0. \tag{8}$$

Since, we can write $\ddot{r} = (d\dot{r}/dr)\dot{r}$, the above expression becomes

$$\frac{d\dot{r}}{dr}\dot{r} = \frac{C^2}{r^3}. \tag{9}$$

Solving the ordinary differential equation using separation of variables, from r_0 to r, we find

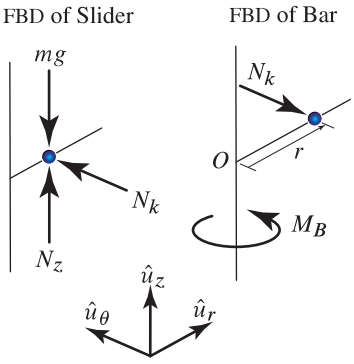
$$\dot{r}^2 = C^2\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right). \tag{10}$$

Now, evaluating Eq. (10) with the result of Eq. (7) and $r_0 = 0.5\,\text{m}$, we find

$$\boxed{\dot{r} = 0.471\,\text{m/s}}$$

Solution to 3.79

Balance Principles:



$$\sum F_r: \quad ma_r = 0, \tag{11}$$

$$\sum F_\theta: \quad ma_\theta = N_k, \tag{12}$$

$$\sum F_z: \quad ma_z = N_z - mg. \tag{13}$$

The relevant equation of motion for the bar is

$$\sum (M_0)_z = 0: \quad M_B - N_k r = 0, \tag{14}$$

where the right side of Eq. (14) is zero since the mass bar is massless.
Kinematics:

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}.$$

Substituting the kinematic relations into Eqs. (11) and (12) allows us to write

$$0 = m\left(\ddot{r} - r\dot{\theta}^2\right) \quad \Rightarrow \quad \ddot{r} = r\omega_0^2, \tag{15}$$

$$N_k = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \quad \Rightarrow \quad N_k = 2m\dot{r}\omega_0, \tag{16}$$

Therefore, we need $\dot{r}(r)$. We can find that from Eq. (15) by integration of

$$\frac{d\dot{r}}{dr}\dot{r} = r\omega_0^2 \quad \Rightarrow \quad \int_0^{\dot{r}} \dot{r}\,d\dot{r} = \omega_0^2 \int_{r_0}^{r} r\,dr, \tag{17}$$

$$\dot{r}^2 = \omega_0^2\left(r^2 - r_0^2\right). \tag{18}$$

Substituting Eq. (18) into Eq. (16), and solving for N_k, we find

$$N_k = 2m\omega_0^2\sqrt{r^2 - r_0^2}. \tag{19}$$

Substituting the above result into Eq. (14) yields the moment required to maintain a constant angular velocity,

$$\boxed{M_B = 2m\omega_0^2 r\sqrt{r^2 - r_0^2}.} \tag{20}$$

The radial speed of the collar upon reaching the end of the arm is calculated by evaluating Eq. (18) with the given parameters. The final radial speed is

$$\dot{r}(r) = \boxed{1.41\,\text{m/s}.}$$

In order to find the total speed of the collar, we also need to find the component of the velocity along the $\hat{u}_\theta$ direction.

$$v_\theta = r\dot{\theta} = r\omega \quad \Rightarrow \quad v_\theta = (1.5\,\text{m})\,(1\,\text{rad/s}) = 1.5\,\text{m/s}. \tag{21}$$

Thus the speed of the collar as it reaches the end of the bar is

$$\boxed{v = \sqrt{v_r^2 + v_\theta^2} = 2.06\,\text{m/s}.}$$

Solution to 3.80

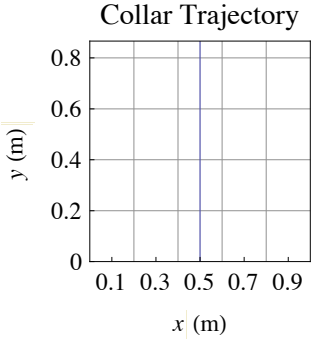
The trajectory is determined using Mathematica to numerically solve the system of equations of motion (4) and (5) with the initial conditions given in the problem statement. The equations of motion are numerically solved with the statement:

```
collar = NDSolve[{r''[t] - r[t] θ'[t]^2 == 0, 0 == r[t] θ''[t] + 2 r'[t] θ'[t],
   θ'[0] == 1, θ[0] == 0, r[0] == 0.5, r'[0] == 0}, {r[t], θ[t]}, {t, 0, 5}]
```

The trajectory of the collar is plotted using:

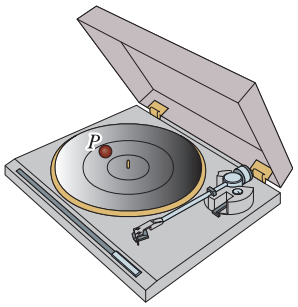
```
ParametricPlot[Evaluate[{r[t] Cos[θ[t]], r[t] Sin[θ[t]]} /. collar], {t, 0, 5},
 PlotRange → {{0, 1}, {0, 0.866}}, Frame → True,
 FrameTicks → {{{0, 0.2, 0.4, 0.6, 0.8}, None}, {{0.1, 0.3, 0.5, 0.7, 0.9}, None}},
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"x (m)", "y (m) "},
 PlotLabel → "Collar Trajectory"]
```

The result of executing the code above Mathematica statements results in the trajectory being plotted as:



Problem 3.81

The particle P is placed on the turntable, and both are initially at rest. The turntable is then turned on so that the disk starts spinning. Assuming that there is no friction between the turntable disk and the particle, what will be the motion of the particle after the disk starts spinning? Explain.

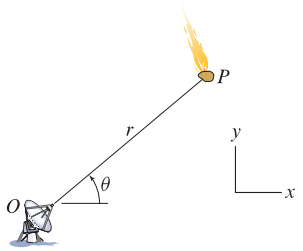


Solution

The particle P will remain stationary relative to its inertial frame. There is no friction between the particle P and the turntable so when the turntable accelerates there will be no frictional force between P and the turntable to promote the motion of P.

Problem 3.82

The radar station at O is tracking the meteor P as it moves through the atmosphere. The data measured by the radar station indicates that the acceleration vector of P is almost exactly in the opposite direction of the velocity vector. Explain why this is what we should expect.

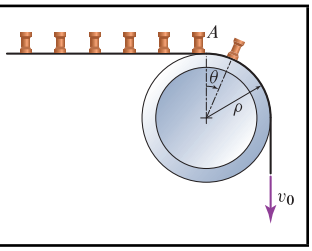


Solution

The forces acting on the meteor include the gravitational force and the drag force. The drag force acts opposite to the direction of the velocity. Considering only the drag force, one would expect that the direction of the acceleration should point exactly opposite that of the velocity. However, since gravity exerts a force on the meteor, the acceleration vector is not directed exactly opposite to the velocity. The effect of gravity is small because meteors enter the atmosphere at tremendous speeds and the effect of air resistance (proportional to v^2) dominates the effect of gravity (proportional to r^{-2}).

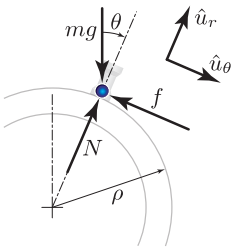
Problem 3.83

The conveyer belt moves parts, each with mass m, at a constant speed v_0. When the parts get to A, they begin moving over a circular path of radius ρ. If the coefficient of static friction between the belt and the parts is μ_s, determine the angle θ at which the parts will start to slide on the belt. Neglect the size of the parts. After determining θ in terms of μ_s, v_0, ρ, and g, evaluate θ for $\mu_s = 0.6$, $v_0 = 3$ mph, and $\rho = 14$ in.



Solution

Balance principles:



$$\sum F_r \qquad N - mg\cos\theta = ma_r, \tag{1}$$

$$\sum F_\theta : \quad -f + mg\sin\theta = ma_\theta. \tag{2}$$

Since the part is traveling on a circular path where $r = \rho$, $\dot{r} = 0$, and $\ddot{r} = 0$, we can employ the circular motion relations

$$\dot{\theta} = \frac{v}{\rho}, \quad \text{and} \quad \ddot{\theta} = \frac{\dot{v}}{\rho}. \tag{3}$$

Now, substituting the circular motion relations into Eqs. (1) and (2), we find:

$$a_r = -\rho\left(\frac{v_0^2}{\rho^2}\right) = -\frac{v_0^2}{\rho} \quad \text{and} \quad a_\theta = \rho\left(\frac{\dot{v}}{\rho}\right) = \dot{v} = 0, \tag{4}$$

where $v = v_0$ is a constant, thus $\dot{v} = 0$. The condition for the part to slip is that

$$f = \mu_s N. \tag{5}$$

Substituting the slip condition and Eq. (4) into the the Newton-Euler equations, we get

$$N - mg\cos\theta = -m\frac{v_0^2}{\rho}, \tag{6}$$

$$-\mu_s N + mg\sin\theta = 0. \tag{7}$$

Eliminating N from these two equations by solving the first for N and plugging the result into the second gives

$$-\mu_s\left(mg\cos\theta - m\frac{v_0^2}{\rho}\right) + mg\sin\theta = 0 \quad \Rightarrow \quad \cos\theta - \frac{1}{\mu_s}\sin\theta = \frac{v_0^2}{\rho g}. \tag{8}$$

Now, for a neat little trick let $1/\mu_s = \tan\phi$, so that:

$$\cos\theta - \tan\phi\sin\theta = \frac{v_0^2}{\rho g} \quad \Rightarrow \quad \cos\phi\cos\theta - \sin\phi\sin\theta = \cos\phi\frac{v_0^2}{\rho g}, \tag{9}$$

$$\cos(\theta + \phi) = \frac{v_0^2}{\rho g}\cos\phi, \tag{10}$$

$$\theta = \cos^{-1}\left(\frac{v_0^2}{\rho g}\cos\phi\right) - \phi. \tag{11}$$

But, we know that

$$\phi = \tan^{-1}\left(\frac{1}{\mu_s}\right) \quad \Rightarrow \quad \cos\phi = \frac{\mu_s}{\sqrt{1+\mu_s^2}},$$

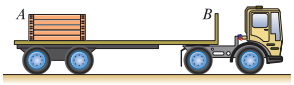
so that

$$\boxed{\theta = \cos^{-1}\left(\frac{v_0^2}{\rho g}\frac{\mu_s}{\sqrt{1+\mu_s^2}}\right) - \tan^{-1}\left(\frac{1}{\mu_s}\right).}$$

Using $v_0 = 3\,\text{mph} = 4.4\,\text{ft/s}$, $\mu_s = 0.6$, and $\rho = 14\,\text{in.} = 1.167\,\text{ft}$, we get $\boxed{\theta = 15.6°.}$

Problem 3.84

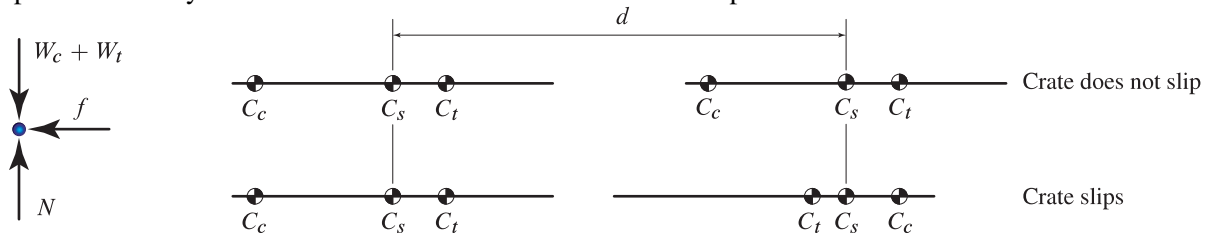
The driver of the truck suddenly applies the brakes, and the truck comes to a stop. During braking, either the crate slides or it does not. Considering the forces acting on the truck during braking, will the truck stop in a shorter distance (or time) if the crate slides, or will the distance (or time) be shorter if it does not? Justify your answer.



Solution

Consider the motion of the truck and crate relative to their common center of mass. The figure below shows the FBD of the truck and the crate taken together, where W_c is the weight of the crate, W_t is the weight of the truck, and f is the friction force between the truck's wheels and the ground (i.e., the force stopping the system). This FBD applies whether or not the crate slips and it tells us that the center of mass of the *system* takes the same distance d to stop regardless of whether or not the crate slips relative to the truck.

The figure below depicts the center of mass of the truck (C_t), crate (C_c), and system (C_s), relative to the bed of the truck (heavy black line). The figure also shows the position of the center of mass of each component of the system before and after the truck comes to a stop.

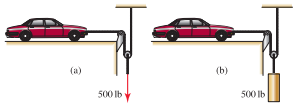


If the crate does not slip, then the center of mass of the system C_s will *not move* relative to the truck as shown in the top part of the figure. On the other hand, if the crate slips, then the center of mass of the system C_s will move forward (to the right) relative to the truck and the truck does not move as far for C_s to move the same inertial distance d. So, we can conclude that:

> The truck stops in a shorter distance if the crate slips relative to the truck.

Problem 3.85

A car is being pulled to the right in the two ways shown. Neglecting the inertia of the pulleys and rope as well as any friction in the pulleys, if the car is allowed to roll freely, will the acceleration of the car in (a) be smaller, equal to, or larger than the acceleration of the car in (b)?

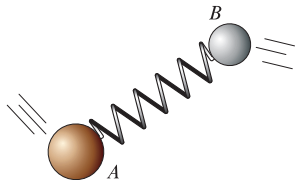


Solution

The acceleration of the car in case (a) will be larger than that in case (b). Until the car is released the tension in the cord is the same for case (a) and (b). When the car is released, in case (a), due to the lack of inertia in the pulleys and cord, the tension in the cord stays at the same value it had at equilibrium. In case (b) the equilibrium value of the tension is diminished due to the fact that the 500 lb weight will start accelerating downward. Hence, given that the acceleration of the car is directly proportional to the tension in the rope, the acceleration in case (a) will be larger than that in case (b).

Problem 3.86

Particles A and B, which are connected via a massless linear spring, have been thrown up in the air and are moving under the action of the spring force and their own weight. Assuming that no other forces are affecting the motion of the particles, what will be the acceleration of their center of mass?

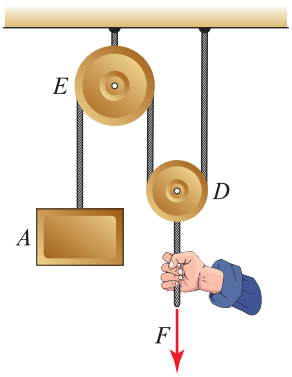


Solution

The acceleration of the center of mass of A and B is constant with magnitude equal to g, the acceleration due to gravity. The reason for this is that the weight force is the only external force acting on the system.

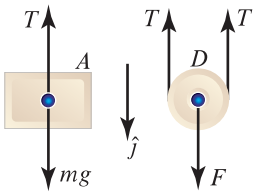
Problem 3.87

A person lifts the 80 kg load A by pulling down on the rope with a constant force F as shown. Neglecting any source of friction as well as the inertia of the ropes and the pulleys, determine F if A accelerates upward at 0.5 m/s^2.



Solution

Balance Principles:



$$\left(\sum F_j\right)_D: \quad -2T + F = m_D a_{Dy}, \tag{1}$$

$$\left(\sum F_j\right)_A: \quad -T + m_A g = m_A a_{Ay}. \tag{2}$$

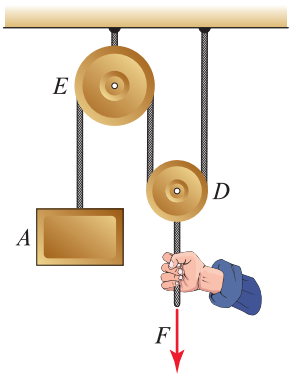
Recalling that $m_D = 0$ Eq. (1) tells us that $F = 2T$. Eq. (2) becomes

$$-\frac{F}{2} + m_A g = m_A a_{Ay} \quad \Rightarrow \quad \boxed{F = 2m_A\left(g - a_{Ay}\right) = 1650\,\text{N},}$$

where $a_{Ay} = -0.5\,\text{m/s}^2$ because A is accelerating upwards.

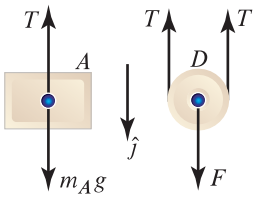
Problem 3.88

The load A weighs 185 lb. Neglecting any source of friction as well as the inertia of the ropes and the pulleys, determine the acceleration of A if a person pulls down on the rope with a constant force $F = 185$ lb as shown.



Solution

Balance Principles:



$$\left(\sum F_j\right)_D: \quad -2T + F = m_D a_{Dy}, \tag{1}$$

$$\left(\sum F_j\right)_A: \quad -T + m_A g = m_A a_{Ay}. \tag{2}$$

Kinematics:

$$L = 2y_D + y_A \quad \Rightarrow \quad a_{Ay} = -2a_{Dy}.$$

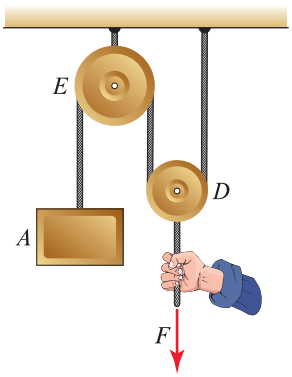
Recalling that $m_D = 0$ Eq. (1) tells us that $F = 2T$. Eq. (2) becomes

$$\boxed{a_{Ay} = \frac{-F}{2m_A} + g = 16.1\ \text{ft/s}^2.}$$

Since a_{Ay} is positive A accelerates downward.

Problem 3.89

A person lifts the 80 kg load A by pulling down on the rope with a constant force F as shown. Neglecting friction, the inertia of the ropes, and the rotational inertia of the pulleys, but accounting for the fact that pulley D has a mass $m_D = 8\,\text{kg}$, determine F if A accelerates upward at $2.5\,\text{m/s}^2$.

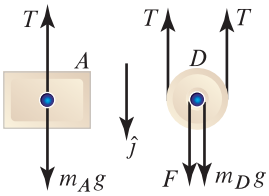


Solution

Balance Principles:

$$\left(\sum F_j\right)_D: \quad -2T + m_D g + F = m_D a_{Dy}, \tag{1}$$

$$\left(\sum F_j\right)_A: \quad -T + m_A g = m_A a_{Ay}. \tag{2}$$



Kinematics:

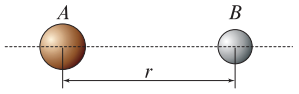
$$L = 2y_D + y_A \quad \Rightarrow \quad a_{Ay} = -2a_{Dy}.$$

Substitute the kinematic equation into Eq. (1) and eliminate T between Eqs. (1) and (2) yielding

$$F = g\,(2m_A - m_D) - a_{Ay}\left(2m_A + \tfrac{1}{2}m_D\right) = 1900\,\text{N}.$$

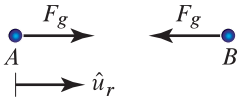
Problem 3.90

Two particles A and B with masses m_A and m_B, respectively, are placed at a distance r_0 from one another. Assuming that the only force acting on the masses is their mutual gravitational attraction, determine the acceleration of particle B as seen from particle A.



Solution

Balance principles:



$$\left(\sum F_r\right)_A : \quad F_g = m_A a_{Ar}, \tag{1}$$

$$\left(\sum F_r\right)_B : \quad -F_g = m_B a_{Br}. \tag{2}$$

The force of gravity is specified by Newton's Universal Law of Gravitation as

$$F_g = G\frac{m_A m_B}{r_{B/A}^2}. \tag{3}$$

The acceleration of B as seen by A is

$$a_{B/A} = \ddot{r}_{B/A} = \ddot{r}_B - \ddot{r}_A. \tag{4}$$

Now, substituting Eqs. (3) and (4) into Eqs. (1) and (2), we find

$$G\frac{m_A m_B}{r^2} = m_A \ddot{r}_A, \tag{5}$$

$$-G\frac{m_A m_B}{r^2} = m_B \ddot{r}_B. \tag{6}$$

We have replaced the distance between the particles, $r_{B/A}$, with r for convenience. Solving for the acceleration of each point, we find

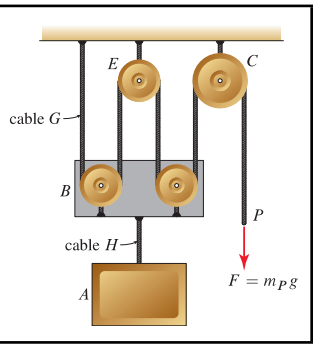
$$\boxed{\ddot{r}_A = G\frac{m_B}{r^2}, \quad \text{and} \quad \ddot{r}_B = -G\frac{m_A}{r^2}.} \tag{7}$$

Substituting the accelerations of each point into Eq. (4), we find

$$\boxed{\ddot{r}_{B/A} = -G\left(\frac{m_A + m_B}{r^2}\right).} \tag{8}$$

Problem 3.91

Revisit Example 3.10 and determine the expression for the acceleration of A if the load at P is replaced by a force with magnitude equal to the load's weight, i.e., $F = m_P g$.

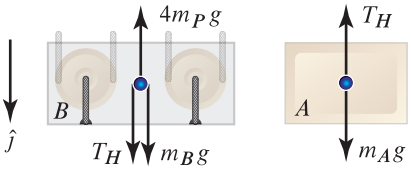


Solution

Use the same set up as for Example 3.10. Balance Principles:

$$\left(\sum F_y\right)_A : \quad m_A g - T_H = m_A a_A, \tag{1}$$

$$\left(\sum F_y\right)_B : \quad m_B g + T_H - 4m_P g = m_B a_B. \tag{2}$$

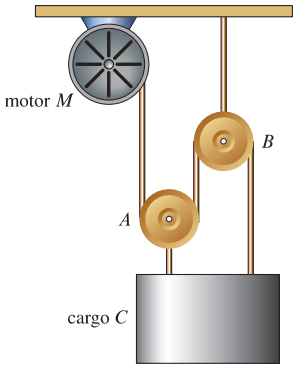


Kinematics: $a_A = a_B$.
Eliminate T_H from Eqs. (1) and (2) and solve for a_A.

$$a_A = g\left(1 - \frac{4m_P}{m_A + m_B}\right).$$

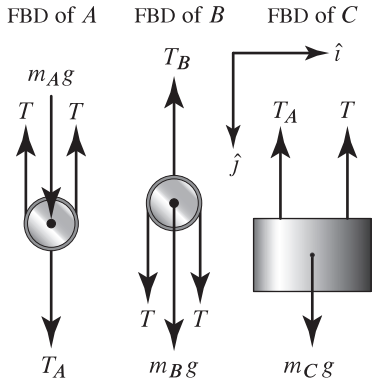
Problem 3.92

The motor M is at rest when someone flips a switch and it starts pulling in the rope. The acceleration of the rope is uniform and it takes 1 s to achieve a retraction rate of 4 ft/s. After 1 s the retraction rate becomes constant. Determine the tension in the cable during and after the initial 1 s interval. The cargo C weighs 130 lb, pulleys A and B each weigh 12 lb, and the weight of the ropes is negligible. Neglect friction in the pulleys and the rotational inertia of the pulleys.



Solution

Balance Principles:



$$\left(\sum F_y\right)_A: \quad T_A + m_A g - 2T = m_A\,(a_A)_y\,, \tag{1}$$

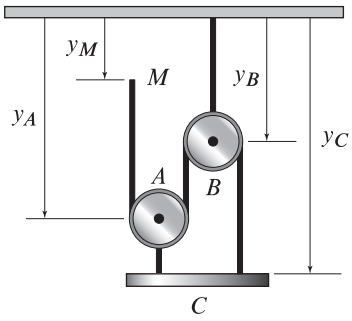
$$\left(\sum F_y\right)_B: \quad 2T + m_B g - T_B = m_B\,(a_B)_y\,, \tag{2}$$

$$\left(\sum F_y\right)_C: \quad m_C g - T_A - T = m_C\,(a_C)_y\,. \tag{3}$$

Since B is connected to the ceiling by an inextensible rope, $(a_B)_y = 0$. In addition, the pulley kinematics may be determined using the distances labeled on the diagram at the right. If L is the length of the *exposed* rope,

$$L = (y_C - y_B) + (y_A - y_B) + (y_A - y_M)\,, \tag{4}$$

$$= y_C - 2y_B + 2y_A - y_M\,. \tag{5}$$



Differentiating L with respect to time,

$$\dot{L} = \dot{y}_C - 2\dot{y}_B + 2\dot{y}_A - \dot{y}_M \quad \Rightarrow \quad \dot{L} = \dot{y}_C + 2\dot{y}_A, \tag{6}$$

since neither B nor M moves. Differentiating again and using the fact that $y_C = y_A + \text{Constant}$, we find

$$\ddot{L} = 3\ddot{y}_C\,. \tag{7}$$

In addition, since the motor accelerates the rope from 0 to 4 ft/s in 1 s,

$$\dot{L} = \dot{L}_0 + \ddot{L}t \quad \Rightarrow \quad -4\,\text{ft/s} = 0 + \ddot{L}(1\,\text{s}), \tag{8}$$

$$\text{thus,} \qquad \ddot{L} = -4\,\text{ft/s}^2. \tag{9}$$

Since $\ddot{y}_C = (a_C)_y$, Eq. (7) becomes

$$-4\,\text{ft/s}^2 = 3\,(a_C)_y \quad \Rightarrow \quad (a_C)_y = -\frac{4}{3}\,\text{ft/s}^2. \tag{10}$$

Using the facts that $(a_A)_y = (a_C)_y$, $(a_C)_y$ is known and that $(a_B)_y = 0$, Eqs. (1)–(3) become:

$$T_A + m_A g - 2T = m_A\,(a_C)_y\,, \tag{11}$$

$$2T + m_B g - T_B = 0, \tag{12}$$

$$m_C g - T_A - T = m_C\,(a_C)_y\,. \tag{13}$$

Solve Eqs. (11) and (13) for T as

$$T = \frac{1}{3}\left[(m_A + m_C)(g - (a_C)_y)\right] \tag{14}$$

Using Eq. (10) and the substitutions $m_A = 0.3727\,\text{slug}$, $m_C = 4.037\,\text{slug}$, and $g = 32.2\,\text{ft/s}^2$ the tension in the cable during the acceleration is

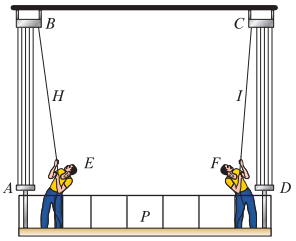
$$\boxed{T = 49.3\,\text{lb.}} \tag{15}$$

After the motor is finished accelerating, T can be found by substituting $(a_C)_y = 0$ into Eq. (14) to find

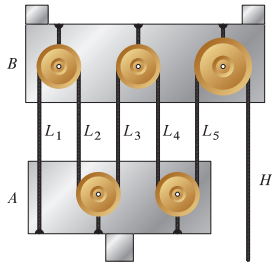
$$T = \frac{g}{3}\,(m_A + m_C) \quad \Rightarrow \quad \boxed{47.3\,\text{lb.}} \tag{16}$$

Problem 3.93

As seen in Fig. P3.93(a), a window washing platform is controlled via the two pulley systems at AB and CD. The workers E and F can raise and lower the platform P by pulling on the cables H and I, respectively. The weight of each of the workers is 185 lb, and the platform P weighs 200 lb. A schematic representation of the pulley system is shown in Fig. P3.93(b). If the workers start from rest and, in 1.5 s, uniformly start pulling the cable in at 2.5 ft/s, determine the force each worker must exert on the cables H and I during that 1.5 s. Neglect the mass of the pulleys, friction in the pulleys, and the mass of the cable.



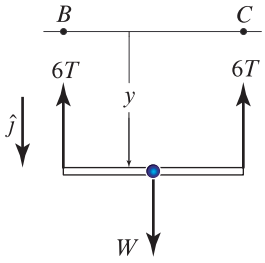
(a) the platform



(b) the pulleys

Solution

Because each worker has the same weight and because they both retract the cables at the same rate, the loading on the platform is symmetric. Balance Principles:



$$\sum F_y: \quad -12T + W = \frac{W}{g}\ddot{y}, \tag{1}$$

Kinematics:

$$L = L_1 + L_2 + L_3 + L_4 + L_5 + y,$$
$$\dot{L} = 6\dot{y} \quad \text{and} \quad \ddot{L} = 6\ddot{y}. \tag{2}$$

The acceleration of the cable as

$$\dot{L}_f = \dot{L}_0 + \ddot{L}(t_f - t_0) \quad \Rightarrow \quad \ddot{L} = \frac{\dot{L}_f}{t_f - t_0}. \tag{3}$$

Substitute the second of Eqs. (2) into Eq. (1) eliminating $\ddot{y}$ to obtain

$$-12T + W = \frac{W}{g}\frac{\ddot{L}}{6}, \tag{4}$$

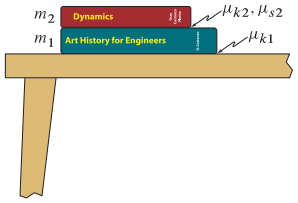
and then combine Eq. (4) with Eq. (3) to eliminate $\ddot{L}$ and solve for T to obtain

$$T = \frac{W}{12}\left[1 - \frac{\dot{L}_f}{6g\,(t_f - t_0)}\right] \quad \Rightarrow \quad \boxed{T = 47.9\,\text{lb},}$$

where $W = 570$ lb, $\dot{L}_f = -2.5$ ft/s, $(t_f - t_0) = 1.5$ s, and $g = 32.2$ ft/s^2.

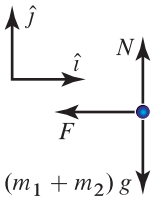
Problem 3.94

Revisit Example 3.11 and assume that the static coefficient of friction between the two books is $\mu_{s2} = 0.55$, while all other parameters stay as specified in the example. Determine the acceleration of each of the books.



Solution

Based on the analysis in Example 3.11 we know that the books do not slide relative to each other if $\mu_{S2} > \mu_{k1}$. Hence the two books behave as if they are a single body.
Balance Principles:



$$\sum F_x : \quad -F = (m_1 + m_2)\, a_x, \tag{1}$$

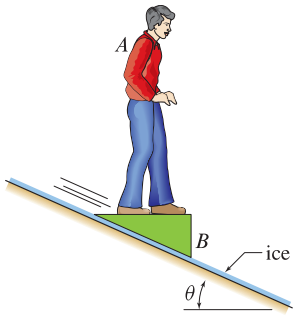
$$\sum F_y : \quad N - (m_1 + m_2)\, g = a_y. \tag{2}$$

The force equation is $F = \mu_{k1} N$. The kinematic equation is $a_y = 0$. Thus

$$N = (m_1 + m_2)\, g \quad \Rightarrow \quad \boxed{a_x = -\mu_{k1} g = -4.41 \text{ m/s}^2.}$$

Problem 3.95

A person A is trying to keep his balance while on a sled B that is sliding down an icy incline. Letting $m_A = 78\,\text{kg}$ and $m_B = 25\,\text{kg}$ be the masses of A and B, respectively, and assuming that there is enough friction between A and B for A not to slide with respect to B, determine the value of the normal reaction force between A and B as well as the magnitude of their acceleration if $\theta = 20°$. Friction between the sled and the incline is negligible.



Solution

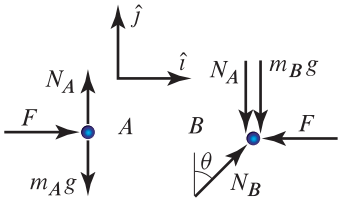
Balance Principles:

$$\left(\sum F_x\right)_A: \quad F = m_A a_{Ax}, \tag{1}$$

$$\left(\sum F_y\right)_A: \quad N_A - m_A g = m_A a_{Ay}, \tag{2}$$

$$\left(\sum F_x\right)_B: \quad N_B \sin\theta - F = m_B a_{Bx}, \tag{3}$$

$$\left(\sum F_y\right)_B: \quad N_B \cos\theta - N_A - m_B g = m_B a_{By}. \tag{4}$$



Kinematics, because there is no slip between A and B we can say

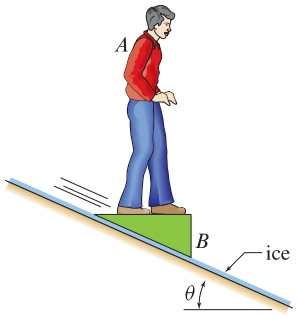
$$\vec{a}_A = \vec{a}_B, \quad a_{Ax} = a_{Bx} = a\cos\theta, \quad \text{and} \quad a_{Ay} = a_{By} = -a\sin\theta.$$

We have 8 equations in 8 unknowns. It is tedious but these equations can be solved by hand to find

$$N_A = m_A g\cos^2\theta = 676\,\text{N},$$
$$a = g\sin\theta = 3.36\,\text{m/s}^2.$$

Problem 3.96

A person A is trying to keep his balance while on a sled B that is sliding down an icy incline. The weights of A and B are $W_A = 181\,\text{lb}$ and $W_B = 50\,\text{lb}$, respectively. Determine the minimum coefficient of static friction μ_s between A and B required for A not to slide with respect to B if $\theta = 23°$. Friction between the sled and the incline is negligible.



Solution

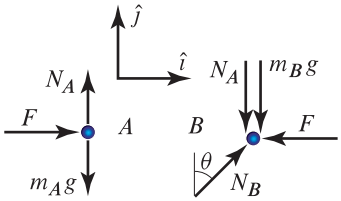
Balance Principles:

$$\left(\sum F_x\right)_A: \quad F = m_A a_{Ax}, \tag{1}$$

$$\left(\sum F_y\right)_A: \quad N_A - m_A g = m_A a_{Ay}, \tag{2}$$

$$\left(\sum F_x\right)_B: \quad N_B \sin\theta - F = m_B a_{Bx}, \tag{3}$$

$$\left(\sum F_y\right)_B: \quad N_B \cos\theta - N_A - m_B g = m_B a_{By}. \tag{4}$$



Kinematics, because there is no slip between A and B we can say

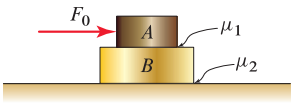
$$\vec{a}_A = \vec{a}_B, \quad a_{Ax} = a_{Bx} = a\cos\theta, \quad \text{and} \quad a_{Ay} = a_{By} = -a\sin\theta.$$

For impending slip the force law is $F = \mu_s N_A$. We have 9 equations in 9 unknowns. Solving these equations for μ_s yields

$$\boxed{\mu_s = \tan\theta = 0.424.}$$

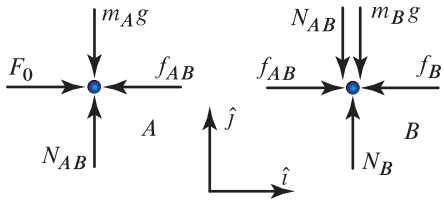
Problem 3.97

A force F_0 of 400 lb is applied to block A. Letting the weights of A and B be 55 and 73 lb, respectively, and letting the static *and* kinetic friction coefficients between blocks A and B be $\mu_1 = 0.25$, and the static *and* kinetic friction coefficients between block B and the ground be $\mu_2 = 0.45$, determine the accelerations of both blocks.



Solution

We first assume that A slides over B and B slides over the ground.
We solution must have $a_{Ax} > 0$ and $a_{Bx} > 0$.
Balance Principles for A and B:



$$\left(\sum F_x\right)_A: \quad F_0 - f_{AB} = m_A a_{Ax}, \tag{1}$$

$$\left(\sum F_y\right)_A: \quad N_{AB} - m_A g = m_A a_{Ay}, \tag{2}$$

$$\left(\sum F_x\right)_B: \quad f_{AB} - f_B = m_B a_{Bx}, \tag{3}$$

$$\left(\sum F_y\right)_B: \quad N_B - m_B g - N_{AB} = m_B a_{By}, \tag{4}$$

Because of the assumptions we made about the motion of A and B, the force laws are

$$f_{AB} = \mu_1 N_{AB} \quad \text{and} \quad f_B = \mu_2 N_B, \tag{5}$$

Kinematics:

$$a_{Ay} = 0 \quad \text{and} \quad a_{By} = 0. \tag{6}$$

Substituting the kinematic equations into the y components of the Newton-Euler Eqs. (2) and (4), we get the normal forces

$$N_{AB} = m_A g \quad \text{and} \quad N_B = N_{AB} + m_B g \quad \Rightarrow \quad N_{AB} = m_A g \quad \text{and} \quad N_B = (m_A + m_B)g. \tag{7}$$

Substituting these relations into Eqs. (5), we get

$$f_{AB} = \mu_1 m_A g \quad \text{and} \quad f_B = \mu_2 (m_A + m_B)g. \tag{8}$$

Substituting these friction forces into Eqs. (1) and (3), we get

$$F_0 - \mu_1 m_A g = m_A (a_A)_x \quad \text{and} \quad \mu_1 m_A g - \mu_2 (m_A + m_B)g = m_B (a_B)_x. \tag{9}$$

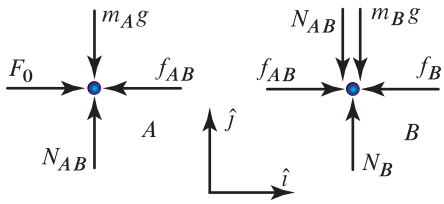
Solving for the accelerations:

$$a_{Ax} = [F_0 - \mu_1 m_A g]/m_A = 226\,\text{ft/s}^2, \tag{10}$$

$$a_{Bx} = [\mu_1 m_A g - \mu_2 (m_A + m_B)g]/m_B = -19.3\,\text{ft/s}^2. \tag{11}$$

We observe that one of our assumptions is not true, i.e. $a_{Bx} \not> 0$. So we now assume that A slides to the right relative to B; and B does not slide. The following inequalities must hold:

$$a_{Ax} > 0 \quad \text{and} \quad |f_B| \leq \mu_2 N_B. \tag{12}$$



Balance Principles for A and B:

$$\left(\sum F_x\right)_A : \qquad F_0 - f_{AB} = m_A (a_A)_x, \tag{13}$$

$$\left(\sum F_y\right)_A : \qquad N_{AB} - m_A g = m_A (a_A)_y, \tag{14}$$

$$\left(\sum F_x\right)_B : \qquad f_{AB} - f_B = m_B (a_B)_x, \tag{15}$$

$$\left(\sum F_y\right)_B : \quad N_B - m_B g - N_{AB} = m_B (a_B)_y. \tag{16}$$

Because of the assumptions we made on the motion of A and B, the force laws are

$$f_{AB} = \mu_1 N_{AB} \quad \text{and} \quad |f_B| \leq \mu_2 N_B. \tag{17}$$

Kinematics:

$$a_{Ay} = 0, \quad a_{Bx} = 0 \quad \text{and} \quad a_{By} = 0. \tag{18}$$

Substituting first and third of Eqs. (18) into Newton-Euler Eqs. (14) and (16), we get the normal forces

$$N_{AB} = m_A g \quad \text{and} \quad N_B = N_{AB} + m_B g \quad \Rightarrow \quad N_{AB} = m_A g \quad \text{and} \quad N_B = (m_A + m_B) g. \tag{19}$$

Substituting these relations into material Eqs. (17), we get

$$f_{AB} = \mu_1 m_A g \quad \text{and} \quad |f_B| \leq \mu_2 (m_A + m_B) g. \tag{20}$$

Next, substituting the relations $f_{AB} = \mu_1 m_A g$ and $a_{Bx} = 0$ into Newton-Euler Eqs. (13) and (15), we have

$$F_0 - \mu_1 m_A g = m_A (a_A)_x \quad \text{and} \quad \mu_1 m_A g - f_B = 0. \tag{21}$$

Solving for a_{Ax} and f_B, we have

$$a_{Ax} = [F_0 - \mu_1 m_A g] / m_A = 226 \, \text{ft/s}^2, \tag{22}$$

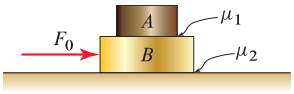
$$f_B = \mu_1 m_A g = 13.8 \, \text{lb}. \tag{23}$$

The solution just computed gives $a_{Ax} > 0$ and $f_B = 13.8 \, \text{lb} < \mu_2 N_B = 32.9 \, \text{lb}$. Therefore, this solution is fully consistent with the assumptions under which it was obtained. Hence, the accelerations are

$$\boxed{\vec{a}_A = (226 \, \hat{\imath}) \ \text{ft/s}^2 \quad \text{and} \quad \vec{a}_B = \vec{0} \, \text{ft/s}^2.} \tag{24}$$

Problem 3.98

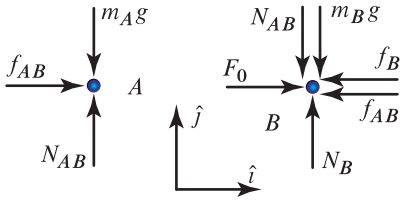
A force F_0 of 400 lb is applied to block B. Letting the weights of A and B be 55 and 73 lb, respectively, and letting the static *and* kinetic friction coefficients between blocks A and B be $\mu_1 = 0.25$, and the static *and* kinetic friction coefficients between block B and the ground be $\mu_2 = 0.45$, determine the accelerations of both blocks.



Solution

We first assume that B slides to the right and A also moves to the right but slides to relative to B. We expect the following inequalities to hold

$$a_{Bx} > 0, \quad a_{Ax} > 0 \quad \text{and} \quad a_{Ax} < a_{Bx}. \tag{1}$$



Balance Principles for A:

$$\left(\sum F_x\right)_A : \qquad f_{AB} = m_A a_{Ax}, \tag{2}$$

$$\left(\sum F_y\right)_A : \quad N_{AB} - m_A g = m_A a_{Ay}, \tag{3}$$

Balance Principles for B:

$$\left(\sum F_x\right)_B : \qquad F_0 - f_B - f_{AB} = m_B a_{Bx}, \tag{4}$$

$$\left(\sum F_y\right)_B : \quad N_B - N_{AB} - m_B g = m_B a_{By}, \tag{5}$$

Due to the assumptions our force laws are

$$f_{AB} = \mu_1 N_{AB} \quad \text{and} \quad f_B = \mu_2 N_B. \tag{6}$$

Kinematics:

$$a_{Ay} = 0 \quad \text{and} \quad a_{By} = 0. \tag{7}$$

Substituting Eqs. (7) into Newton-Euler Eqs. (3) and (5), we get

$$N_{AB} = m_A g \quad \text{and} \quad N_B = (m_A + m_B)g. \tag{8}$$

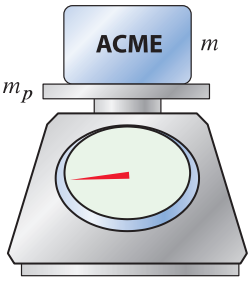
Substituting first of Eqs. (6) into (2) and using first of Eqs. (8), we get

$$\mu_1 m_A g = m_A a_{Ax} \quad \Rightarrow \quad \boxed{a_{Ax} = \vec{a}_A = \mu_1 g\,\hat{\imath} = (8.05\,\hat{\imath})\ \text{ft/s}^2.} \tag{9}$$

Proceeding in a similar way for B, we have

$$F_0 - \mu_2(m_A + m_B)g - \mu_1 m_A g = m_B a_{Bx} \quad \Rightarrow \quad \boxed{a_{Bx} = \vec{a}_B = (145\,\hat{\imath})\ \text{ft/s}^2.} \tag{10}$$

Problems 3.99 and 3.100

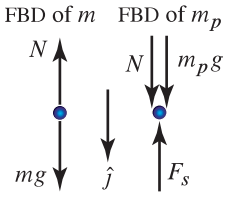


Spring scales work by measuring the displacement of a spring that supports both the platform of mass m_p and the object, of mass m, whose weight is being measured. Most scales read zero when no mass m has been placed on them; that is, they are calibrated so that the weight reading neglects the mass of the platform m_p. Assume that the spring is linear elastic with spring constant k.

Problem 3.99 If the mass m is gently placed on the spring scale (i.e., it is dropped from zero height above the scale), determine the *maximum reading* on the scale after the mass is released.

Problem 3.100 If the mass m is gently placed on the spring scale (i.e., it is dropped from zero height above the scale), determine the expression for *maximum speed* attained by the mass m as the spring compresses.

Solution to 3.99



We let W be the reading on the scale, where $W = k\delta_e$. δ_e is the *extra* compression of the spring, i.e., the compression caused by placing m on the platform and that is in addition to the initial compression δ_0 caused by the platform. Now, when m is placed on the scale, the platform will move. The maximum reading occurs when the scale first stops moving. The FBDs of the mass and the plate are shown at the right.
Balance Principles:

$$\left(\sum F_y\right)_m : \qquad mg - N = ma_{my}, \tag{1}$$

$$\left(\sum F_y\right)_p : \quad N + m_p g - F_s = ma_{py}. \tag{2}$$

We will model the spring as linear, so that $F_s = k\delta$, where

$$\delta = \delta_0 + \delta_e, \quad \text{and} \quad \delta_0 = (m_p g)/k. \tag{3}$$

Let y denote the position of m. At the initial time $y = 0$. Noting that m and m_p move together we can write

$$a_{my} = \ddot{y} = a_{py}, \quad \text{and} \quad \delta_e = y. \tag{4}$$

Combining Eqs. (3), and (4), with the Newton-Euler Equations (Eqs. (1), and (2)) we find

$$mg - N = m\ddot{y}, \tag{5}$$

$$N + m_p g - k\left(\frac{m_p g}{k} + y\right) = m_p \ddot{y}. \tag{6}$$

Solving, we find

$$\ddot{y} = \frac{(mg - ky)}{(m + m_p)}, \tag{7}$$

We have acceleration as a function of position. Use the following equation to find velocity as a function of position:

$$\dot{y}^2(y) = \dot{y}^2(0) + \frac{2}{(m + m_p)} \int_0^y (mg - ky)\, dy. \tag{8}$$

Recalling that $\dot{y}(0) = 0$, at the point of the maximum reading of the scale, we have

$$\dot{y}^2(y) = \frac{2}{(m + m_p)}\left(mgy - \frac{1}{2}ky^2\right). \tag{9}$$

At $y = y_{\max}$, $\dot{y}\,(y_{\max}) = 0$ so that

$$mgy_{\max} - \frac{1}{2}ky_{\max}^2 = 0 \quad \Rightarrow \quad y_{\max} = \frac{2mg}{k}. \tag{10}$$

Thus, we can determine that

$$\boxed{W_{\max} = ky_{\max} = 2mg.}$$

Solution to 3.100

We can find the position where velocity is maximum by setting Eq. (7) equal to zero:

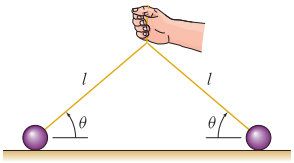
$$\ddot{y}(y) = 0 = \frac{(mg - ky)}{(m + m_p)} \quad \Rightarrow \quad y = \frac{mg}{k}. \tag{11}$$

Plug Eq. (11) into Eq. (9) to find

$$\boxed{\dot{y}(y) = \pm mg\sqrt{\frac{1}{k\,(m + m_p)}}} \tag{12}$$

Problem 3.101

Two identical balls, each of mass m, are connected by a string of negligible mass and length $2l$. A short string is attached to the middle of the string connecting the two balls and is pulled vertically with a constant force P. If the system starts from rest at $\theta = \theta_0$ and assuming that the balls only move in the horizontal direction, determine the expression for the speed of the two balls as θ approaches 90°. Neglect the size of the balls as well as friction between the balls and the surface on which they slide.

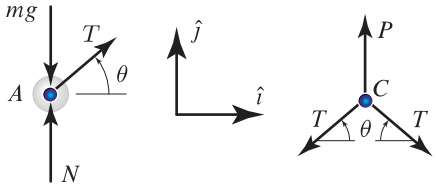


Solution

Balance Principles for A :

$$\left(\sum F_x\right)_A : \qquad T\cos\theta = ma_{Ax}, \qquad (1)$$

$$\left(\sum F_y\right)_A : \quad N + T\sin\theta - mg = ma_{Ay}. \qquad (2)$$



The strings are assumed to be inextensible and massless. Balance Principles for C :

$$\left(\sum F_x\right)_C : \quad -T\cos\theta + T\cos\theta = m_C a_{Cx}, \qquad (3)$$

$$\left(\sum F_y\right)_C : \qquad -2T\sin\theta + P = m_C a_{Cy}. \qquad (4)$$

Kinematics

$$a_{Ax} = \ddot{x}_A \quad \text{and} \quad a_{Ay} = 0. \qquad (5)$$

From the geometry of the problem, we can write

$$\sin\theta = \frac{\sqrt{l^2 - x_A^2}}{l} \quad \text{and} \quad \cos\theta = -\frac{x_A}{l}. \qquad (6)$$

Substituting $m_C = 0$ into Eqs. (3) and (4), we obtain

$$2T\sin\theta = P \quad \Rightarrow \quad T = \frac{P}{2\sin\theta}. \qquad (7)$$

Substituting Eq. (7) into Eq. (1), we have

$$T\cos\theta = m(a_A)_x \quad \Rightarrow \quad \frac{P\cos\theta}{2\sin\theta} = m\ddot{x}_A \quad \Rightarrow \quad \ddot{x}_A = -\frac{P}{2m}\frac{x_A}{\sqrt{l^2 - x_A^2}}, \qquad (8)$$

where we have used the definitions from Eq. (6). Eq. (8) is an expression for acceleration in terms of position so we will integrate it. Since $\theta = \theta_0$ and the system is at rest when P is first applied,

$$(x_A)_{\text{initial}} = -l\cos\theta_0 \quad \text{and} \quad (\dot{x}_A)_{\text{initial}} = 0. \qquad (9)$$

Therefore, in going from $(x_A)_{\text{initial}}$ to $x_A = 0$, we have

$$\dot{x}_A^2(x_A = 0) = -\frac{P}{m}\int_{-l\cos\theta_0}^{0} \frac{x_A}{\sqrt{l^2 - x_A^2}}\, dx_A = \frac{P}{m}\left[\sqrt{l^2 - x_A^2}\right]_{-l\cos\theta_0}^{0}, \qquad (10)$$

which gives

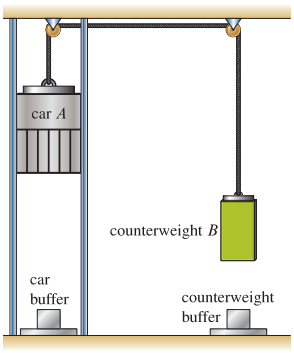
$$\dot{x}_A^2(x_A = 0) = \frac{P}{m}(l - \sqrt{l^2 - l^2\cos^2\theta_0}) = \frac{P}{m}(l - l\sqrt{1 - \cos^2\theta_0}) = \frac{Pl}{m}(1 - \sin\theta_0). \tag{11}$$

Finally, we have

$$\boxed{v_A(x_A = 0) = \sqrt{\dot{x}_A^2(x_A = 0)} = \sqrt{\frac{Pl}{m}(1 - \sin\theta_0)}.}$$

Problem 3.102

A simple elevator consists of a 15,000 kg car A connected to a 12,000 kg counterweight B. Suppose that a failure occurs when the car is at rest and 50 m above its buffer, causing the elevator car to fall. Model the car and the counterweight as particles and the cord as massless and inextensible; and model the action of the emergency brakes using a Coulomb friction model with kinetic friction coefficient $\mu_k = 0.5$ and a normal force equal to 35% of the car's weight. Determine the speed with which the car impacts the buffer.



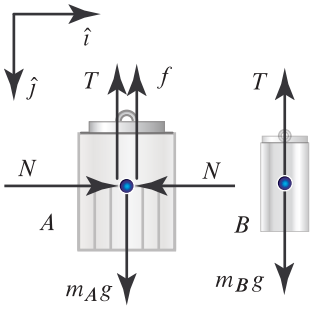
Solution

Balance Principles:

$$\left(\sum F_x\right)_A : N - N = m_A a_{Ax} \qquad (1)$$

$$\left(\sum F_y\right)_A : -T - f + m_A g = m_A a_{Ay} \qquad (2)$$

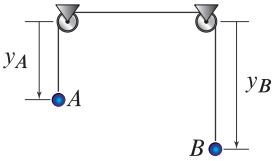
$$\left(\sum F_y\right)_B : -T + m_B g = m_B a_{By}. \qquad (3)$$



The force laws are $N = 0.35 m_A g$ and $f = \mu_k N$.

Pulley kinematics

$$L = y_A + y_B + \text{Constants}. \qquad (4)$$



Taking the second derivative with respect to time yields

$$\ddot{L} = 0 = \ddot{y}_A + \ddot{y}_B \quad \Rightarrow \quad a_{Ay} = -a_{By}. \qquad (5)$$

Substituting the expressions for N and f into Eqs. (2) and (3) yields,

$$-T - 0.35\mu_k m_A g + m_A g = m_A \left(a_A\right)_y, \qquad (6)$$

$$-T + m_B g = -m_B \left(a_A\right)_y. \qquad (7)$$

Eliminating T from the previous two equations, we find

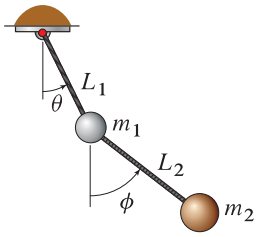
$$\left(a_A\right)_y = \frac{-m_B g - m_A g\left(0.35\mu_k - 1\right)}{m_A + m_B} = 0.1363\,\text{m/s}^2. \qquad (8)$$

Apply the constant acceleration equation $\dot{s}^2 = \dot{s}_0^2 + 2a_c(s - s_0)$ to find the speed of impact:

$$v^2 = v_0^2 + 2a\Delta y \quad \Rightarrow \quad v_i = \boxed{3.69\,\text{m/s}.} \qquad (9)$$

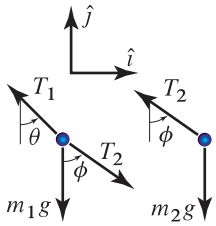
Problem 3.103

The double pendulum shown consists of two particles with masses $m_1 = 7.5\,\text{kg}$ and $m_2 = 12\,\text{kg}$ connected by two inextensible cords of length $L_1 = 1.4\,\text{m}$ and $L_2 = 2\,\text{m}$ and negligible mass. If the system is released from rest when $\theta = 10°$ and $\phi = 20°$, determine the tension in the two cords at the instant of release.



Solution

Balance Principles:



$$\left(\sum F_x\right)_1 : \quad -T_1 \sin\theta + T_2 \sin\phi = m_1 a_{1x}, \tag{1}$$

$$\left(\sum F_y\right)_1 : \quad T_1 \cos\theta - T_2 \cos\phi - m_1 g = m_1 a_{1y}, \tag{2}$$

$$\left(\sum F_x\right)_2 : \quad -T_2 \sin\phi = m_2 a_{2x}, \tag{3}$$

$$\left(\sum F_y\right)_2 : \quad T_2 \cos\phi - m_2 g = m_2 a_{2y}. \tag{4}$$

Kinematics for $t = 0$:

$$\begin{aligned}
x_1 &= L_1 \sin\theta && \Rightarrow && \ddot{x}_1 = L_1 \ddot{\theta} \cos\theta, \\
y_1 &= -L_1 \cos\theta && \Rightarrow && \ddot{y}_1 = L_1 \ddot{\theta} \sin\theta, \\
x_2 &= x_1 + L_2 \sin\phi && \Rightarrow && \ddot{x}_2 = L_1 \ddot{\theta} \cos\theta + L_2 \ddot{\phi} \cos\phi, \\
y_2 &= y_1 - L_2 \cos\phi && \Rightarrow && \ddot{y}_2 = L_1 \ddot{\theta} \sin\theta + L_2 \ddot{\phi} \sin\phi.
\end{aligned}$$

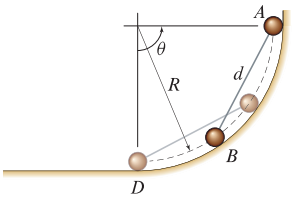
Note: These kinematic equations are only valid at $t = 0$ if the system is released from rest because the $\dot{\theta}$ and $\dot{\phi}$ terms have been set to 0. We have 8 equations in 8 unknowns. Using *Mathematica* we obtain

$$T_1 = \frac{2m_1 g\,(m_1 + m_2)\cos\theta}{2m_1 + m_2 - m_2 \cos(2\theta - 2\phi)} = 180\,\text{N},$$

$$T_2 = \frac{2m_1 g m_2 \cos\theta \cos(\theta - \phi)}{2m_1 + m_2 - m_2 \cos(2\theta - 2\phi)} = 109\,\text{N}.$$

Problems 3.104 and 3.105

Two small spheres A and B, each of mass m, are attached at either end of a rod of length d. The system is released from rest in the position shown. Neglect friction, treat the spheres as particles (assume that their diameter is negligible), neglect the mass of the rod, assume the rod is rigid, and assume that $d < R$. *Hint:* The force that the rod exerts on either ball has the same direction as the rod itself.

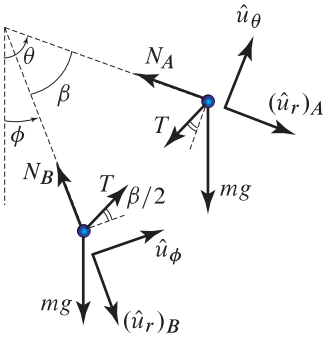


Problem 3.104 Using the angle θ as the dependent variable, derive the equation of motion for the particle system from the moment of release until particle B reaches point D.

Problem 3.105 Determine the expression for the speed of the spheres immediately before B reaches point D.

Solution to 3.104

Balance Principles for A:



$$\sum F_r: \quad -N_A - T\sin\frac{\beta}{2} + mg\cos\theta = ma_{Ar}, \tag{1}$$

$$\sum F_\theta: \quad -mg\sin\theta - T\cos\frac{\beta}{2} = ma_{A\theta}, \tag{2}$$

where T is the tension in the rod and N_A is the normal force on A.
Balance Principles for B :

$$\sum F_r: \quad -N_B - T\sin\frac{\beta}{2} + mg\cos\phi = ma_{Br}, \tag{3}$$

$$\sum F_\phi: \quad -mg\sin\phi + T\cos\frac{\beta}{2} = ma_{B\phi}, \tag{4}$$

Kinematics:

$$a_{Ar} = -R\dot{\theta}^2 \quad \text{and} \quad a_{A\theta} = R\ddot{\theta}, \tag{5}$$

$$a_{Br} = -R\dot{\phi}^2 \quad \text{and} \quad a_{B\phi} = R\ddot{\phi}. \tag{6}$$

Using the second of Eqs. (5) and (6), adding Eqs. (2) and (4), and noting that $\ddot{\phi} = \ddot{\theta}$, we get

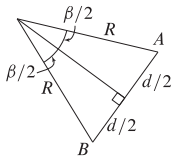
$$-g(\sin\theta + \sin\phi) = 2R\ddot{\theta}, \tag{7}$$

However, we know that $\phi = \theta - \beta$, so that we can write

$$\sin\phi = \sin\theta\cos\beta - \cos\theta\sin\beta. \tag{8}$$

An expression for $\cos\beta$ is found with the law of cosines and $\sin\beta = \sqrt{1-\cos^2\beta}$

$$\cos\beta = 1 - \frac{1}{2}(d/R)^2 \quad \text{and} \quad \sin\beta = \frac{d}{R^2}\sqrt{R^2 - (d/2)^2}. \tag{9}$$



Substituting these relations into Eq. (7) and then into Eq. (8), we get the equation of motion as

$$\boxed{gd\sqrt{4R^2 - d^2}\cos\theta - g(4R^2 - d^2)\sin\theta = 4R^3\ddot{\theta}.} \tag{10}$$

Solution to 3.105

The FBD and Newton-Euler equations for this problem are the same as in the solution to Problem 3.104. Using Eq. (7) from the solution to Problem 3.104, the fact that $\theta = \phi + \beta$, and the trig identity $\sin a + \sin b = 2\sin[(a+b)/2]\cos[(a-b)/2]$ we get

$$-g[\sin(\phi + \beta) + \sin\phi] = 2R\ddot{\phi} \quad \Rightarrow \quad -g\sin(\phi + \beta/2)\cos(\beta/2) = R\ddot{\phi}. \tag{11}$$

Using $\ddot{\phi} = \dot{\phi}d\dot{\phi}/d\phi$, we can find $\dot{\phi}$ when B reaches D as

$$\int_0^{\dot{\phi}_D} \dot{\phi}\, d\dot{\phi} = -\frac{g}{R}\cos(\beta/2)\int_{\pi/2-\beta}^{0} \sin(\phi + \beta/2)\, d\phi, \tag{12}$$

$$\Rightarrow \quad \dot{\phi}_D^2 = \frac{2g}{R}\cos(\beta/2)[\cos(\beta/2) - \sin(\beta/2)], \tag{13}$$

where $\sin(\beta/2) = d/(2R)$ and $\cos(\beta/2) = \sqrt{4R^2 - d^2}/(2R)$. Substituting these values in Eq. (13), we get

$$\dot{\phi}_D^2 = \frac{g}{2R^3}\sqrt{4R^2 - d^2}\left[\sqrt{4R^2 - d^2} - d\right] \tag{14}$$

The speed at D is given by $v_D = R\dot{\phi}_D$. So, using Eq. (14), we get the speed

$$\boxed{v_D = \sqrt{2gR - \frac{gd}{2R}\sqrt{4R^2 - d^2} - \frac{gd^2}{2R}}.} \tag{15}$$

Since $\dot{\phi} = \dot{\theta}$ at all instants, the speeds of sphere A and sphere B are the same and equal to v_D. Using Eq. (3) from the solution to Problem 3.104, we can find T for $\phi = 0$ as

$$T|_{\phi=0} = \frac{mR}{\cos(\beta/2)}\,\ddot{\phi}|_{\phi=0} \tag{16}$$

From Eq. (11), we have

$$\ddot{\phi}|_{\phi=0} = -\frac{g}{R}\sin(\beta/2)\cos(\beta/2). \tag{17}$$

Substituting Eq. (17) into Eq. (16), we get

$$T|_{\phi=0} = -mg\sin(\beta/2). \tag{18}$$

Noting that $\dot{\phi}|_{\phi=0} = \dot{\phi}_D$, we substitute Eqs. (13) and (18) into Eq. (3) of the solution to Problem 3.104, to get

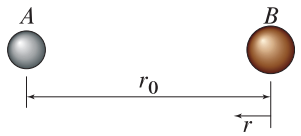
$$-N_B|_{\phi=0} + mg[1 + \sin^2(\beta/2)] = -mR\frac{2g}{R}[\cos^2(\beta/2) - \cos(\beta/2)\sin(\beta/2)]. \tag{19}$$

Simplifying, we get

$$\boxed{N_B|_{\phi=0} = mg\left[3 - \frac{d^2}{4R^2} - \frac{d}{2R^2}\sqrt{4R^2 - d^2}\right].} \tag{20}$$

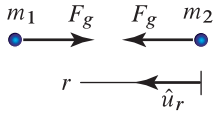
Problem 3.106

Two particles A and B with masses m_A and m_B, respectively, are a distance r_0 apart, and both masses are initially at rest. Using Eq. (1.6) on p. 5, determine the amount of time it takes for the two masses to come into contact if $m_A = 1\,\text{kg}$, $m_B = 2\,\text{kg}$, and $r_0 = 1\,\text{m}$. Assume that the two masses are out in space and infinitely far from any other mass.



Solution

Balance Principles:



$$\sum F_r: \quad -F_g = m_1 a_1, \tag{1}$$

$$\sum F_r: \quad F_g = m_2 a_2. \tag{2}$$

Force law:

$$F_g = G\frac{m_1 m_2}{r_{12}^2}, \tag{3}$$

where r_{12} is the distance between m_1 and m_2. Kinematics:

$$a_1 = \ddot{r}_1, \quad a_2 = \ddot{r}_2, \quad \text{and} \quad r_{12} = r_1 - r_2 \quad \Rightarrow \quad \ddot{r}_{12} = \ddot{r}_1 - \ddot{r}_2. \tag{4}$$

Substituting the material and the first set of kinematic eqns into the N-E equations we get:

$$-G\frac{m_1 m_2}{r_{12}^2} = m_1\ddot{r}_1 \quad \text{and} \quad G\frac{m_1 m_2}{r_{12}^2} = m_2\ddot{r}_2 \quad \Rightarrow \quad -G\frac{m_2}{r_{12}^2} = \ddot{r}_1 \quad \text{and} \quad G\frac{m_1}{r_{12}^2} = \ddot{r}_2. \tag{5}$$

Substituting this result into Eq. (4), yields

$$\ddot{r}_{12} = -G\frac{m_2}{r_{12}^2} - G\frac{m_1}{r_{12}^2} \quad \Rightarrow \quad \ddot{r}_{12} = -G\left(\frac{m_1 + m_2}{r_{12}^2}\right). \tag{6}$$

We need to find velocity as a function of position.

$$\dot{r}_{12}^2(r_{12}) = \dot{r}_{12}^2(r_0) - 2G(m_1 + m_2)\int_{r_0}^{r_{12}} \frac{dr_{12}}{r_{12}^2} = 2G(m_1 + m_2)\left(\frac{1}{r_{12}} - \frac{1}{r_0}\right).$$

$$\dot{r}_{12}(r_{12}) = \pm\sqrt{2G(m_1 + m_2)}\sqrt{\frac{1}{r_{12}} - \frac{1}{r_0}} \quad \Rightarrow \quad \dot{r}_{12}(r_{12}) = -\sqrt{2G(m_1 + m_2)}\sqrt{\frac{r_0 - r_{12}}{r_{12} r_0}},$$

Now, we can use the following equation to find time as a function of position

$$t(s) = t_0 + \int_{s_0}^{s} \frac{ds}{v(s)} \quad \Rightarrow \quad t_{\text{contact}} = 0 - \frac{1}{\sqrt{2G(m_1 + m_2)}}\int_{r_0}^{0} \sqrt{\frac{r_{12} r_0}{r_0 - r_{12}}}\, dr_{12}, \tag{7}$$

where we have used $t_0 = 0$. Now,

$$\int \sqrt{\frac{r_{12} r_0}{r_0 - r_{12}}}\, dr_{12} = \sqrt{r_0}\left[r_0 \tan^{-1}\left(\sqrt{\frac{r_{12}}{r_0 - r_{12}}}\right) - \sqrt{r_{12}(r_0 - r_{12})}\right] \tag{8}$$

Therefore, we can write

$$\int_{r_0}^{0} \sqrt{\frac{r_{12}r_0}{r_0 - r_{12}}}\, dr_{12} = 0 - \lim_{r_{12}\to r_0} \sqrt{r_0}\left[r_0 \tan^{-1}\left(\sqrt{\frac{r_{12}}{r_0 - r_{12}}}\right) - \sqrt{r_{12}(r_0 - r_{12})}\right] = -r_0^{3/2}\frac{\pi}{2}. \quad (9)$$

Substituting this result into Eq. (7) yields

$$t_{\text{contact}} = -\frac{1}{\sqrt{2G(m_1+m_2)}}\int_{r_0}^{0} \sqrt{\frac{r_{12}r_0}{r_0 - r_{12}}}\, dr_{12} = -\frac{1}{\sqrt{2G(m_1+m_2)}}\left(-r_0^{3/2}\frac{\pi}{2}\right) \quad (10)$$

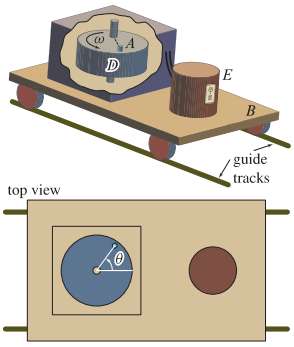
$$= \frac{\pi r_0^{3/2}}{2\sqrt{2G(m_1+m_2)}}. \quad (11)$$

Since $G = 6.674\times10^{-11}\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, and $r_0 = 1\,\text{m}$ we get

$$t_{\text{contact}} = \frac{\pi r_0^{3/2}}{2\sqrt{2G(m_1+m_2)}} = 78.5\times10^3\ \text{s} = \boxed{21.8\,\text{h.}}$$

Problem 3.107

Energy storage devices that use spinning flywheels to store energy are becoming available. To store as much energy as possible, it is important that the flywheel spin as fast as possible. Unfortunately, if it spins too fast, internal stresses in the flywheel cause it to come apart catastrophically. Therefore, it is important to keep the speed at the periphery of the flywheel below about 1000 m/s. In addition, it is critical that the flywheel be as balanced as possible to avoid the tremendous vibrations that would otherwise result. With this in mind, let the flywheel D, whose diameter is 0.3 m, rotate at $\omega = 60{,}000$ rpm. In addition, assume that the cart B is constrained to move rectilinearly along the guide tracks. Given that the flywheel is not perfectly balanced, that the unbalanced weight A has mass m_A, and that the total mass of the flywheel D, cart B, and electronics package E is m_B, determine the constraint force between the wheels of the cart and the guide tracks as a function of θ, the masses, the diameter, and the angular speed of the flywheel. What is the *maximum* constraint force between the wheels of the cart and the guide tracks? Finally, evaluate your answers for $m_A = 1$ g (about the mass of a paper clip) and $m_B = 70$ kg (the mass of the flywheel might be about 40 kg). Assume that the unbalanced mass is at the periphery of the flywheel.

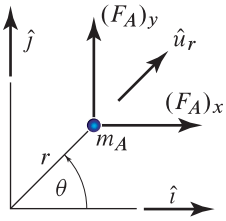


Solution

Choose the x-axis to be aligned with the direction of the guide tracks.
Balance Principles for A:

$$\left(\sum F_x\right)_A: \quad (F_A)_x = m_A\,(a_A)_x\,, \tag{1}$$

$$\left(\sum F_y\right)_A: \quad (F_A)_y = m_A\,(a_A)_y\,, \tag{2}$$



Kinematics:

$$\vec{a}_A = \vec{a}_B + \vec{a}_{A/B} \quad \Rightarrow \quad \vec{a}_A = (a_B)_x\,\hat{\imath} - r\dot{\theta}^2\hat{u}_r,$$

where $(a_B)_x$ is the acceleration of cart B, $r = d/2$, and $\dot{\theta} = \omega$. The components of $\vec{a}_A$ are

$$(a_A)_x = (a_B)_x - \frac{d\omega^2}{2}\cos\theta, \quad \text{and} \quad (a_A)_y = -\frac{d\omega^2}{2}\sin\theta.$$

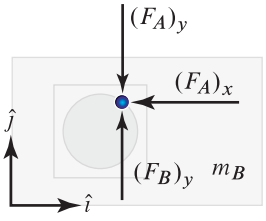
The solution for $(F_A)_y$ is

$$(F_A)_y = -\frac{m_A d\omega^2}{2}\sin\theta. \tag{3}$$

Balance Principles for B:

$$\left(\sum F_x\right)_B: \qquad -(F_A)_x = m_B\,(a_B)_x\,, \tag{4}$$

$$\left(\sum F_y\right)_B: \quad (F_B)_y - (F_A)_y = m_B\,(a_B)_y\,, \tag{5}$$



Kinematics: $(a_B)_y = 0$. Solving Eqs. (3) and (5), the force exerted by the rails on the cart is

$$\boxed{(F_B)_y = -\frac{m_A d\omega^2}{2}\sin\theta.} \tag{6}$$

Since $|\sin\theta| \leq 1$, the largest magnitude of $(F_B)_y$ occurs when $|\sin\theta| = 1$, or $\theta = n\pi/2$ where $n = 1, 3, 5, \ldots$. Letting $\sin\theta = 1$ in Eq. (6) the maximum reaction force exerted by the track, $(F_B)_{\max}$, is found to be

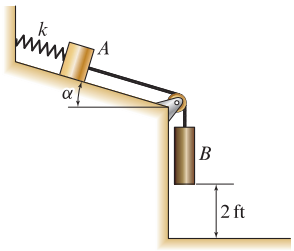
$$(F_B)_{\max} = \left|(F_B)_y\left(\theta = \frac{n\pi}{2}\right)\right| = \boxed{\frac{m_A d\omega^2}{2}.} \tag{7}$$

Evaluating the maximum force with $m_A = 1\times10^{-3}$ kg, $d = 0.3$ m, and $\omega = 6283$ rad/s we obtain

$$(F_B)_{\max} = \boxed{5.92\,\text{kN}.} \tag{8}$$

Problems 3.108 through 3.111

Two blocks A and B weighing 123 and 234 lb, respectively, are released from rest as shown. At the moment of release the spring is unstretched. In solving these problems, model A and B as particles, neglect air resistance, and assume that the cord is inextensible. *Hint:* If B hits the ground, then its maximum displacement is equal to the distance between the initial position of B and the ground.



Problem 3.108 Determine the maximum displacement and the maximum speed of block B if $\alpha = 0°$, the contact between A and the surface is frictionless, and the spring constant is $k = 30\,\text{lb/ft}$.

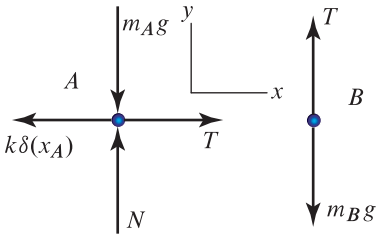
Problem 3.109 Determine the maximum displacement and the maximum speed of block B if $\alpha = 20°$, the contact between A and the incline is frictionless, and the spring constant is $k = 30\,\text{lb/ft}$.

Problem 3.110 Determine the maximum displacement and the maximum speed of block B if $\alpha = 20°$, the contact between A and the incline is frictionless, and the spring constant is $k = 300\,\text{lb/ft}$.

Problem 3.111 Determine the maximum displacement and the maximum speed of block B if $\alpha = 35°$, the static and kinetic friction coefficients are $\mu_s = 0.25$ and $\mu_k = 0.2$, respectively, and the spring constant is $k = 25\,\text{lb/ft}$.

Solution to 3.108

Balance Principles for A:



$$\sum F_x: \quad T - k\delta(x_A) = m_A \ddot{x}_A, \tag{1}$$

$$\sum F_y: \quad N - m_A g = 0, \tag{2}$$

For $x_A > 0$, $\delta(x_A) = x_A$. Balance Principles for B:

$$\sum F_y: \quad T - m_B g = m_B \ddot{y}_B, \tag{3}$$

Kinematics:

$$y_B = -x_A \quad \Rightarrow \quad \ddot{y}_B = -\ddot{x}_A. \tag{4}$$

Eliminating T between Eqs. (1) and (3) and using Eq. (4), we get

$$(m_A + m_B)\ddot{y}_B = -m_B g - k y_B \quad \Rightarrow \quad (m_A + m_B)\int_0^{v_B} \dot{y}_B \, d\dot{y}_B = -\int_0^{y_B} (m_B g + k y_B)\, dy_B, \tag{5}$$

$$\Rightarrow \quad v_B^2 = \frac{-1}{(m_A + m_B)}(k y_B^2 + 2 m_B g y_B). \tag{6}$$

For maximum displacement $v_B = 0$. So using Eq. (6), we get

$$v_B = 0 \quad \Rightarrow \quad \boxed{|(y_B)_{\max}| = \left|-\frac{2 m_B g}{k}\right|} = 15.6\,\text{ft}. \tag{7}$$

Since 15.6 ft > 2 ft, we conclude that

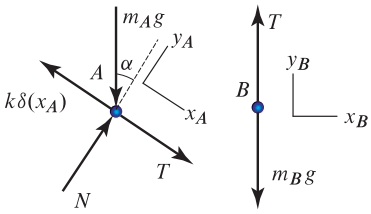
$$\boxed{|(y_B)_{\max}| = 2\,\text{ft}.} \tag{8}$$

From Eq. (6), we realize that v_B is a monotonically decreasing function for $y_B > -\frac{m_B g}{k} = -7.8\,\text{ft}$. Therefore, maximum velocity must occur at maximum y_B. So substituting $y_B = -2\,\text{ft}$ into Eq. (6), we find

$$\boxed{(v_B)_{\max} = v_B[(y_B)_{\max}] = 8.58\,\text{ft/s}.} \tag{9}$$

Solution to 3.109

Balance Principles for A:



$$\sum F_x: \quad T + m_A g \sin\alpha - k\delta(x_A) = m_A \ddot{x}_A, \tag{10}$$

$$\sum F_y: \quad N - m_A g \cos\alpha = 0, \tag{11}$$

Balance Principle for B:

$$\sum F_y: \quad T - m_B g = m_B \ddot{y}_B, \tag{12}$$

Kinematics:

$$y_B = -x_A \quad \Rightarrow \quad \ddot{y}_B = -\ddot{x}_A. \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (10), we get

$$(m_A + m_B)\ddot{y}_B = -k y_B - g(m_A \sin\alpha + m_B), \tag{14}$$

$$(m_A + m_B)\int_0^{v_B} \dot{y}_B\, d\dot{y}_B = -\int_0^{y_B} [k y_B + g(m_A \sin\alpha + m_B)]\, dy_B, \tag{15}$$

$$v_B^2 = \frac{-1}{(m_A + m_B)}\left[k y_B^2 + 2g y_B (m_A \sin\alpha + m_B)\right]. \tag{16}$$

For maximum displacement $v_B = 0$. So using Eq. (16), we get

$$v_B = 0 \quad \Rightarrow \quad \boxed{|y_B| = \left|-\frac{2}{k}(m_A \sin\alpha + m_B)g\right|} = 18.4\,\text{ft}. \tag{17}$$

Since 18.4 ft > 2 ft, we conclude that

$$\boxed{|(y_B)_{\max}| = 2\,\text{ft}.} \tag{18}$$

From Eq. (16), we realize that v_B is a monotonically decreasing function for $y_B > -\frac{g}{k}(m_A \sin\alpha + m_B) = -9.202\,\text{ft}$. Therefore, maximum velocity must occur at maximum y_B. So substituting $y_B = -2\,\text{ft}$ into Eq. (16), we find

$$\boxed{(v_B)_{\max} = v_B[(y_B)_{\max}] = 9.42\,\text{ft/s}.} \tag{19}$$

Solution to 3.110

The FBD and the Newton-Euler equations for the solution to this problem are same as that of the solution to Problem 3.109. Hence, using the expression for velocity v_B in Eq. (16), for maximum displacement we have

$$v_B = 0 \quad \Rightarrow \quad \boxed{|(y_B)_{\max}| = \left| -\frac{2}{k}(m_A \sin\alpha + m_B)g \right| = 1.84\,\text{ft},} \tag{20}$$

which is acceptable, since it is less that 2 ft. To find the maximum velocity, we differentiate the expression for v_B^2, obtained in Eq. (16), with respect to y_B to get

$$\frac{d(v_B^2)}{dy_B} = 0 \quad \Rightarrow \quad y_B = -\frac{1}{k}(m_A \sin\alpha + m_B)g = -0.9202\,\text{ft}. \tag{21}$$

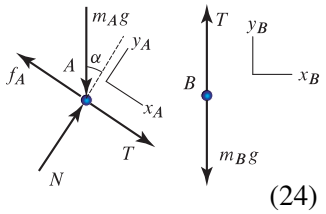
Using Eq. (16) from the solution to problem 3.109 to calculate the velocity v_B at $y_B = -0.9202$ ft, we get

$$\boxed{(v_B)_{\max} = v_B|_{(y_B=-0.9202)} = 4.79\,\text{ft/s}.} \tag{22}$$

Solution to 3.111

First, we need to check whether or not A slips at all. Referring to the FBD and assuming that A does not slip, the following inequality must be satisfied

$$\left|\frac{f_A}{N_A}\right| \leq \mu_s \tag{23}$$



For no slip, the Newton-Euler equations for A are given by

$$\sum F_x = 0: \quad T + m_A g \sin\alpha = f_A, \tag{24}$$

$$\sum F_y = 0: \quad N = m_A g \cos\alpha, \tag{25}$$

where T is the tension in the cord, m_A is the mass of A, N is the normal force and f_A is the friction force. For B, we have

$$\sum F_y = 0: \quad T = m_B g. \tag{26}$$

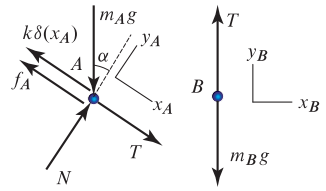
Using Eqs. (24), (25) and (26),we get

$$\left|\frac{f_A}{N_A}\right| = \left|\frac{(m_B/m_A + \sin\alpha)}{\cos\alpha}\right| = 3.022 > 0.25. \tag{27}$$

Therefore, we conclude that our assumption is incorrect and that A does slip. Now, for the situation when A slips, the Newton-Euler equations for A are given by

$$\sum F_x: \quad T + m_A g \sin\alpha - k\delta(x_A) - f_A = m_A \ddot{x}_A, \tag{28}$$

$$\sum F_y: \quad N - m_A g \cos\alpha = 0, \tag{29}$$



Balance Principles for B:

$$\sum F_y = m_B(a_B)_y: \quad T - m_B g = m_B \ddot{y}_B. \tag{30}$$

According to the material model, $f_A = \mu_k N$. Using Eq. (29), we get

$$f_A = \mu_k m_A g \cos\alpha. \tag{31}$$

Kinematics:

$$y_B = -x_A \quad \Rightarrow \quad \ddot{y}_B = -\ddot{x}_A. \tag{32}$$

Eliminating T between Eqs. (28) and (30) and using Eqs. (31) and (32), we get

$$(m_A + m_B)\ddot{y}_B = -\left[k y_B + (m_B g + m_A g \sin\alpha) - \mu_k m_A g \cos\alpha\right], \tag{33}$$

$$\int_0^{v_B} \dot{y}_B \, d\dot{y}_B = -\int_0^{y_B} \left[k y_B + (m_B g + m_A g \sin\alpha) - \mu_k m_A g \cos\alpha\right] dy_B, \tag{34}$$

$$v_B^2 = \frac{-1}{(m_A + m_B)}\left[k y_B^2 + 2g y_B (m_A \sin\alpha + m_B - \mu_k m_A \cos\alpha)\right]. \tag{35}$$

For maximum displacement $v_B = 0$. Using Eq. (35), we get

$$v_B = 0 \quad \Rightarrow \quad \boxed{|y_B| = \left|-\frac{2}{k}(m_A \sin\alpha + m_B - \mu_k m_A \cos\alpha)g\right|} = 22.75\,\text{ft}. \tag{36}$$

Since $22.75\,\text{ft} > 2\,\text{ft}$, we conclude that

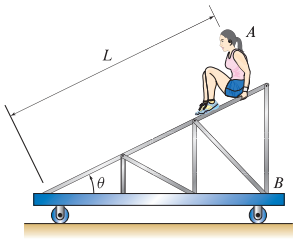
$$\boxed{|(y_B)_{\max}| = 2\,\text{ft}.} \tag{37}$$

From Eq. (35), we realize that velocity v_B is a monotonically decreasing function for $y_B > -\frac{g}{k}(m_A \sin\alpha + m_B - \mu_k m_A \cos\alpha) = -11.38\,\text{ft}$. Therefore, maximum velocity must occur at maximum y_B. So substituting $y_B = -2\,\text{ft}$ into Eq. (35), we find

$$\boxed{(v_B)_{\max} = v_B[(y_B)_{\max}] = 9.67\,\text{ft/s}.} \tag{38}$$

Problem 3.112

A 62 kg woman A sits atop the 60 kg cart B, both of which are initially at rest. If the woman slides down the frictionless incline of length $L = 3.5$ m, determine the velocity of both the woman and the cart when she reaches the bottom of the incline. Ignore the mass of the wheels on which the cart rolls and any friction in their bearings. The angle $\theta = 26°$.

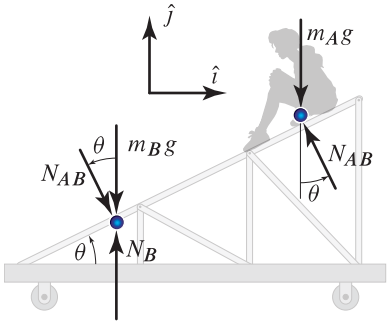


Solution

Balance Principles for A:

$$\left(\sum F_x\right)_A: \qquad -N_{AB}\sin\theta = m_A a_{Ax}, \tag{1}$$

$$\left(\sum F_y\right)_A: \qquad N_{AB}\cos\theta - m_A g = m_A a_{Ay}. \tag{2}$$



Balance Principles for B :

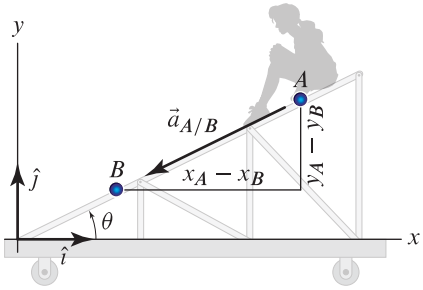
$$\left(\sum F_x\right)_B: \qquad N_{AB}\sin\theta = m_B a_{Bx}, \tag{3}$$

$$\left(\sum F_y\right)_B: \qquad N_B - N_{AB}\cos\theta - m_B g = m_B a_{By}. \tag{4}$$

Kinematics:

$$a_{Ax} = \ddot{x}_A, \qquad a_{Ay} = \ddot{y}_A, \tag{5}$$

$$a_{Bx} = \ddot{x}_B, \qquad a_{By} = 0. \tag{6}$$



Observe that $\vec{a}_{A/B} = \vec{a}_A - \vec{a}_B$ must have a specific direction: along the incline with angle θ. Therefore we must have

$$y_{A/B} = x_{A/B}\tan\theta. \quad \Rightarrow \quad y_A - y_B = \tan\theta(x_A - x_B). \tag{7}$$

Differentiating with respect to time and recalling that $\dot{y}_B = 0$ and $\ddot{y}_B = 0$, we have

$$\dot{y}_A = \tan\theta(\dot{x}_A - \dot{x}_B) \quad \text{and} \quad \ddot{y}_A = \tan\theta(\ddot{x}_A - \ddot{x}_B). \tag{8}$$

Substituting Eqs. (5) and (8) into Eqs. (1) and (2), we find

$$-N_{AB}\sin\theta = m_A\ddot{x}_A \quad \text{and} \quad N_{AB}\cos\theta - m_A g = m_A\tan\theta(\ddot{x}_A - \ddot{x}_B). \tag{9}$$

Substituting Eq. (6) into Eqs. (3) and (4), we find

$$N_{AB}\sin\theta = m_B\ddot{x}_B \quad \text{and} \quad N_B - N_{AB}\cos\theta - m_B g = 0. \tag{10}$$

Therefore, we have the following system of 4 equations:

$$-N_{AB}\sin\theta = m_A\ddot{x}_A, \tag{11}$$

$$N_{AB}\cos\theta - m_A g = m_A\tan\theta(\ddot{x}_A - \ddot{x}_B), \tag{12}$$

$$N_{AB}\sin\theta = m_B\ddot{x}_B, \tag{13}$$

$$N_B - N_{AB}\cos\theta - m_B g = 0. \tag{14}$$

Solving this system of equations, we find that

$$\ddot{x}_A = -\frac{m_B g\sin(2\theta)}{m_A[1-\cos(2\theta)]+2m_B} = -2.978\,\text{m/s}^2 \tag{15}$$

$$\ddot{x}_B = \frac{m_A g\sin(2\theta)}{m_A[1-\cos(2\theta)]+2m_B} = 4.616\,\text{m/s}^2, \tag{16}$$

$$N_{AB} = \frac{2m_A m_B g\cos\theta}{m_A[1-\cos(2\theta)]+2m_B} = 421.2\,\text{N}, \tag{17}$$

$$N_B = \frac{2m_B(m_A+m_B)g}{m_A[1-\cos(2\theta)]+2m_B} = 771.0\,\text{N}. \tag{18}$$

Notice that both $\ddot{x}_A$ and $\ddot{x}_B$ are constant.
Knowing values for $\ddot{x}_A$ and $\ddot{x}_B$, and that A and B start from rest, the constant acceleration equations give

$$(v_A)_x = -(2.978\,\text{m/s}^2)t \quad \text{and} \quad (v_B)_x = (4.616\,\text{m/s}^2)t. \tag{19}$$

In addition, recalling that $\ddot{y}_A$ from Eq. (8), we have

$$\ddot{y}_A = -3.704\,\text{m/s}^2 \quad \Rightarrow \quad (v_A)_y = -(3.704\,\text{m/s}^2)t. \tag{20}$$

We need to use the constant acceleration equation $s = s_0 + \dot{s}_0 t + 0.5a_c t^2$ to find the time for A to travel down the slide.

$$y_A(t_f) = L\sin\theta + \dot{y}_A(0)t_f + \tfrac{1}{2}\ddot{y}_A t_f^2. \quad \Rightarrow \quad 0 = L\sin\theta + \frac{1}{2}\ddot{y}_A t_f^2, \tag{21}$$

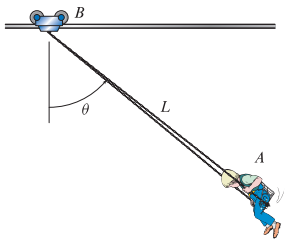
since $\dot{y}_A(0) = 0$. Hence, recalling that $L = 3.5\,\text{m}$, we have

$$t_f = \sqrt{-2L\sin\theta/\ddot{y}_A} = 0.9102\,\text{s}. \tag{22}$$

Finally, this means that

$$\boxed{\vec{v}_A(t_f) = -(2.71\,\hat{\imath} + 3.37\,\hat{\jmath})\,\text{m/s} \quad \text{and} \quad \vec{v}_B(t_f) = (4.20\,\hat{\imath})\,\text{m/s}.}$$

Problems 3.113 and 3.114



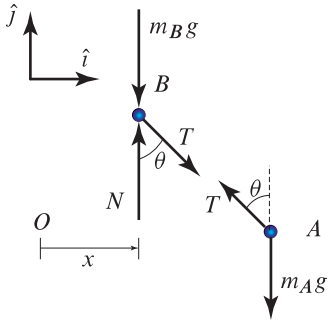
In the ride shown, a person A sits in a seat that is attached via a cable of length L to a freely moving trolley B of mass m_B. The total mass of the person and the seat is m_A. The trolley is constrained by the beam to move in only the horizontal direction. The system is released from rest at the angle $\theta = \theta_0$ and is allowed to swing in the vertical plane. Neglect the mass of the cable, and treat the person and the seat as a single particle.

Problem 3.113 Derive the system's equations of motion, using the position of the trolley and the angle θ as dependent variables.

Problem 3.114 Derive the system's equations of motion, using the position of the trolley and the angle θ as dependent variables, and then use a computer to solve these equations for one full period/cycle of the motion. Plot the speed of the trolley and the speed of the person vs. the angle θ for $m_A = 45\,\text{kg}$, $m_B = 10\,\text{kg}$, $L = 3\,\text{m}$, and $\theta_0 = 70°$.

Solution to 3.113

Balance Principles for B:



$$\left(\sum F_x\right)_B: \qquad T\sin\theta = m_B a_{Bx}, \tag{1}$$

$$\left(\sum F_y\right)_B: \quad N - T\cos\theta - m_B g = m_B a_{By}, \tag{2}$$

Balance Principles for A :

$$\left(\sum F_x\right)_A: \qquad -T\sin\theta = m_A a_{Ax}, \tag{3}$$

$$\left(\sum F_y\right)_A: \quad T\cos\theta - m_A g = m_A a_{Ay}, \tag{4}$$

Kinematics:

$$a_{Bx} = \ddot{x} \quad \text{and} \quad a_{By} = 0. \tag{5}$$

$$\vec{r}_A = (x + L\sin\theta)\,\hat{\imath} - (L\cos\theta)\,\hat{\jmath}. \tag{6}$$

$$a_{Ax} = \ddot{x} + \ddot{\theta}L\cos\theta - \dot{\theta}^2 L\sin\theta \quad \text{and} \quad a_{Ay} = \ddot{\theta}L\sin\theta + \dot{\theta}^2 L\cos\theta. \tag{7}$$

Combine Eqs. (1) and (3), the first of Eqs. (5), and the first of Eqs. (7) to obtain one equation of motion as

$$\boxed{(m_A + m_B)\ddot{x} + m_A L\ddot{\theta}\cos\theta - m_A L\dot{\theta}^2\sin\theta = 0.} \tag{8}$$

Then combine Eqs. (1) and (4), the first of Eqs. (5), and the second of Eqs. (7) to obtain a second equation of motion as

$$\boxed{m_B\ddot{x}\cot\theta - m_A L\ddot{\theta}\sin\theta - m_A L\dot{\theta}^2\cos\theta - m_A g = 0.} \tag{9}$$

Solution to 3.114

Equations of motion are derived in the solution to Problem 3.113 in Eqs. (8) and (9). The equations are solved numerically with the following *Mathematica* code:

```
eom1 = (mA + mB) x''[t] + mA L θ''[t] Cos[θ[t]] - mA L (θ'[t])^2 Sin[θ[t]] == 0;
eom2 = mB x''[t] Cot[θ[t]] - mA L θ''[t] Sin[θ[t]] - mA L (θ'[t])^2 Cos[θ[t]] - mA g == 0;
params = {mA → 45, mB → 10, L → 3, g → 9.81};
IC = {θ[0] == 70 °, θ'[0] == 0, x[0] == 0, x'[0] == 0};
tf = 10;
solution = NDSolve[Flatten[{eom1, eom2, IC}] /. params,
  {x[t], θ[t], x'[t], θ'[t]}, {t, 0, tf}]
```

Use the following *Mathematica* code to plot the angular rate of the swing and then use the `FindRoot` function to determine a precise value for the period:

```
Plot[θ'[t] /. solution /. params, {t, 0, tf}]
tps = FindRoot[θ'[t] /. solution, {t, 3}];
tp = (t /. tps)
```

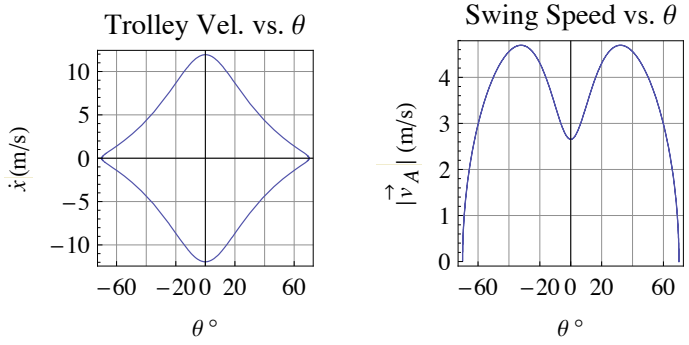
The following *Mathematica* code plots the trolley velocity $\dot{x}$ versus θ (plot shown at the end of this solution):

```
ParametricPlot[{θ[t] / °, x'[t]} /. solution, {t, 0, tp}, Frame → True,
 FrameTicks → {{Automatic, None}, {{-60, -20, 0, 20, 60}, None}},
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"θ °", "ẋ (m/s)"},
 PlotLabel → "Trolley Vel. vs. θ"]
```

The speed of the swing can also be plotted on a parametric plot. The following code specifies the speed of the swing in terms of θ, $\dot{\theta}$, and $\dot{x}$ and plots it versus θ (plot shown at the end of this solution):

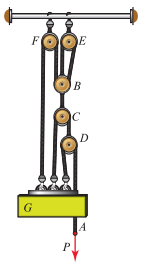
```
vA[t_] = {x'[t] + L Cos[θ[t]] θ'[t], L Sin[θ[t]] θ'[t]};
speedA[t_] = √(vA[t] .vA[t]) // Simplify;
ParametricPlot[{θ[t] / °, speedA[t]} /. solution /. params, {t, 0, tp}, Frame → True,
 FrameTicks → {{Automatic, None}, {{-60, -20, 0, 20, 60}, None}},
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"θ °", "|v⃗A| (m/s)"},
 PlotLabel → "Swing Speed vs. θ"]
```

Below are the plots of trolley and swing speeds vs. θ.



Problem 3.115

A constant force P is applied at A to the rope running behind the load G, which has a mass of 300 kg. Assuming that any source of friction and the inertia of the pulleys can be neglected, determine P such that G has an upward acceleration of 1 m/s^2.



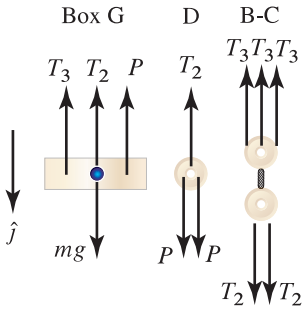
Solution

Balance Principles:

$$\left(\sum F_j\right)_G: \quad mg - P - T_2 - T_3 = ma_y, \tag{1}$$

$$\left(\sum F_j\right)_D: \quad 2P - T_2 = m_D a_{Dy}, \tag{2}$$

$$\left(\sum F_j\right)_{BC}: \quad 2T_2 - 3T_3 - m_{BC} a_{BCy}. \tag{3}$$



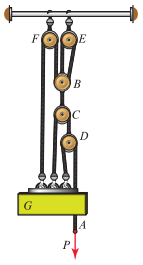
Because we neglect the mass of the pulleys Eqs. (2) and (3) tell us that $T_2 = 2P$ and $T_3 = \frac{4}{3}P$. Eq. (1) becomes

$$mg - P - 2P - \tfrac{4}{3}P = ma_y \quad \Rightarrow \quad \boxed{P = \tfrac{3}{13}m\left(g - a_y\right) = 748\,\text{N},}$$

where $a_y = -1\,\text{m/s}^2$ because G is accelerating upward.

Problem 3.116

A constant force $P = 300$ lb is applied at A to the rope running behind the load G, which weighs 1000 lb. If each of the pulleys weighs 7 lb, and assuming that any source of friction and the rotational inertia of the pulleys can be neglected, determine the acceleration of G and the tension in the rope connecting pulleys B and C.



Solution

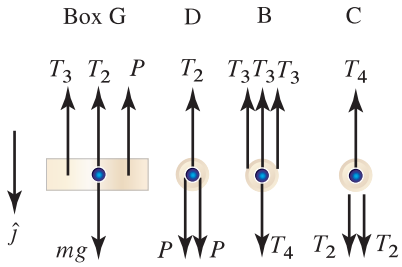
Balance Principles:

$$\left(\sum F_j\right)_G: \quad mg - P - T_2 - T_3 = m_G a_{Gy}, \tag{1}$$

$$\left(\sum F_j\right)_D: \quad 2P - T_2 = m_D a_{Dy}, \tag{2}$$

$$\left(\sum F_j\right)_B: \quad T_4 - 3T_3 = m_B a_{By}, \tag{3}$$

$$\left(\sum F_j\right)_C: \quad 2T_2 - T_4 = m_C a_{Cy}. \tag{4}$$



Kinematics:

$$L_3 = y_G + 3y_B \quad \Rightarrow \quad a_{Gy} = -3a_{By},$$
$$L_4 = y_C - y_B \quad \Rightarrow \quad a_{Cy} = a_{By},$$
$$L_2 = y_G - y_C + y_D - y_C \quad \Rightarrow \quad a_{Gy} + a_{Dy} = 2a_{Cy}.$$

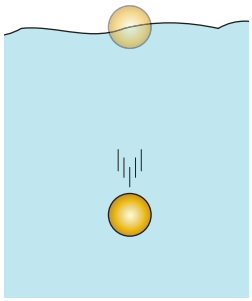
We have 7 equations in 7 unknowns. Solving these equations with *Mathematica* we get

$$a_{Gy} = \frac{9m_G g - 39P}{m_B + m_C + 25m_D + 9m_G} = -9.46\,\text{ft/s},$$
$$T_4 = \frac{3g\,(m_C + 10m_D)\,m_G + (4m_B - 9m_C - 30m_D + 36m_G)\,P}{m_B + m_C + 25m_D + 9m_G} = 1190\,\text{lb}.$$

a_{Gy} is negative, therefore G will accelerate upward.

Problem 3.117

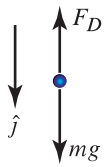
A metal ball weighing 0.2 lb is dropped from rest in a fluid. If the magnitude of the resistance due to the fluid is given by $C_d v$, where $C_d = 0.5\,\text{lb}\cdot\text{s/ft}$ is a drag coefficient and v is the ball's speed, determine the depth at which the ball will have sunk when the ball achieves a speed of 0.3 ft/s.



Solution

Balance Principles:

$$\sum F_y: \quad mg - F_D = ma_y. \tag{1}$$



Force Law: $F_D = C_D v$.
Kinematics: $a_y = \ddot{y}$ and $v = \dot{y}$.
Eq. (1) gives us acceleration as a function of velocity. Integrate as follows:

$$\ddot{y} = \dot{y}\frac{d\dot{y}}{dy} \quad \Rightarrow \quad \int_0^d dy = \int_0^{v_f} \frac{\dot{y}\,d\dot{y}}{g - (C_D/m)\dot{y}}.$$

Now rearrange the integral:

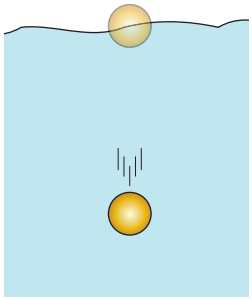
$$d = \int_0^{v_f} \frac{(C_d/m)\dot{y} + g - g}{(C_d/m)[g - (C_d/m)\dot{y}]}\,d\dot{y} \quad \Rightarrow \quad d = \int_0^{v_f} \left[\frac{mg/C_d}{g - (C_d/m)\dot{y}} - \frac{m}{C_d}\right] d\dot{y}.$$

Evaluate the integral and simplify

$$\boxed{d = -\frac{m}{C_d}\left[v_f + \frac{mg}{C_d}\ln\left(1 - \frac{C_d v_f}{gm}\right)\right] = 3.16\times10^{-3}\ \text{ft}.}$$

Problem 3.118

A metal ball weighing 0.2 lb is dropped from rest in a fluid. It is observed that after falling 1 ft, the ball has a speed of 2.25 ft/s. If the magnitude of the resistance due to the fluid is given by $C_d v$, where C_d is a drag coefficient and v is the ball's speed, determine the value of C_d.



Solution

We take advantage of the equation for d derived in the solution to Problem 3.117. Solving for C_d using *Mathematica*

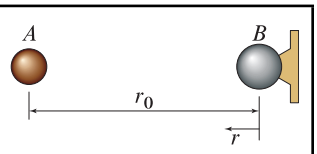
```
g = 32.2; v = 2.25; m = 0.2 / 32.2; d = 1;
FindRoot[(-m/Cd)(v + (m g)/Cd Log[1 - (Cd v)/(m g)]) == d, {Cd, 0.01}]
```

we find

$$C_d = 0.0888 \text{ lb}\cdot\text{s/ft}.$$

Problem 3.119

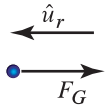
Two particles A and B, with masses m_A and m_B, respectively, are a distance r_0 apart. Particle B is fixed in space, and A is initially at rest. Using Eq. (1.6) on p. 5 and assuming that the diameters of the masses are negligible, determine the time it takes for the two particles to come into contact if $m_A = 1\,\text{kg}$, $m_B = 2\,\text{kg}$, and $r_0 = 1\,\text{m}$. Assume that the two masses are infinitely far from any other mass.



Solution

Balance Principles for m_A:

$$\sum F_r: \quad -F_G = m_A a_r. \tag{1}$$



Force law:

$$F_G = G\frac{m_A m_B}{r^2}.$$

Kinematics: $a_r = \ddot{r}.$

Eq. (1) gives us acceleration as a function of position. Integrate it to obtain velocity as a function of positon.

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} \quad \Rightarrow \quad -Gm_B\int_{r_0}^{r}\frac{dr}{r^2} = \int_{\dot{r}_0}^{\dot{r}}\dot{r}\,d\dot{r} \quad \Rightarrow \quad \dot{r} = -\sqrt{2Gm_B}\sqrt{\frac{r_0 - r}{rr_0}}. \tag{2}$$

Rewrite Eq. (2), make the substitution $z = r/r_0 \quad dz = dr/r_0$, and integrate the results to obtain time as a function of position.

$$\dot{r} = -\sqrt{2Gm_B}\sqrt{\frac{r_0 - r}{rr_0}} \quad \Rightarrow \quad \dot{r} = -\sqrt{\frac{2Gm_B}{r_0}}\sqrt{\frac{1 - r/r_0}{r/r_0}},$$

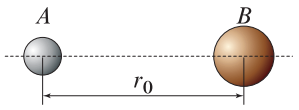
$$\dot{r} = \frac{dr}{dt} \quad \Rightarrow \quad \int_0^{t_c} dt = -\sqrt{\frac{r_0}{2Gm_B}}\int_{r_0}^{0}\sqrt{\frac{r/r_0}{1 - r/r_0}}\,dr \quad \Rightarrow \quad t_c = \frac{-r_0^{3/2}}{\sqrt{2Gm_B}}\int_1^0\sqrt{\frac{z}{1-z}}\,dz.$$

$$\boxed{t_c = \frac{\pi r_0^{3/2}}{2\sqrt{2Gm_B}} = 96{,}100\,\text{s} = 26.7\,\text{hr}.}$$

Where $G = 6.6742\times10^{-11}\,\text{m}^3/\text{kg}\cdot\text{s}^2$.

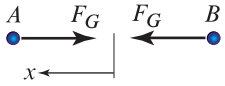
Problem 3.120

The centers of two spheres A and B, with weights $W_A = 3\,\text{lb}$ and $W_B = 7\,\text{lb}$, respectively, are a distance $r_0 = 5\,\text{ft}$ apart when they are released from rest. Using Eq. (1.6) on p. 5, determine the speed with which they collide if the diameters of spheres A and B are $d_A = 2.5\,\text{in.}$ and $d_B = 4\,\text{in.}$, respectively. Assume that the two masses are infinitely far from any other mass.



Solution

x is measured with respect to an inertial frame. Balance Principles:



$$\left(\sum F_x\right)_A: \quad -F_G = m_A a_{Ax}, \tag{1}$$

$$\left(\sum F_x\right)_B: \quad F_G = m_B a_{Bx}. \tag{2}$$

Force Law:

$$F_G = G\frac{m_A m_B}{x_{A/B}^2}.$$

Kinematics: $a_{Ax} = \ddot{x}_A$ and $a_{Bx} = \ddot{x}_B$.

Eqs. (1) and (2) become

$$\ddot{x}_A = -G\frac{m_B}{x_{A/B}^2} \quad \text{and} \quad \ddot{x}_B = G\frac{m_A}{x_{A/B}^2} \quad \Rightarrow \quad \ddot{x}_{A/B} = -G\frac{m_A + m_B}{x_{A/B}^2}. \tag{3}$$

Integrate Eq. (3) . Recall that $\ddot{x}_{A/B} = \dot{x}\frac{d\dot{x}_{A/B}}{dx_{A/B}}$.

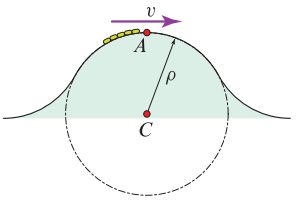
$$\int_0^{v_c} \dot{x}_{A/B}\, d\dot{x}_{A/B} = -G\,(m_A + m_B)\int_{r_0}^{(d_A+d_B)/2} \frac{dx_{A/B}}{x_{A/B}^2},$$

$$v_c = \sqrt{2G\,(m_A + m_B)\left(\frac{2}{d_A + d_B} - \frac{1}{r_0}\right)} = 276\times10^{-6}\,\text{ft/s}.$$

Where $G = 3.4388\times10^{-8}\,\text{ft}^3/\,\text{slug}\cdot\text{s}^2$, $m_A = 3/32.2\,\text{slug}$, and $m_B = 7/32.2\,\text{slug}$.

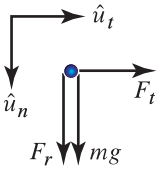
Problem 3.121

A roller coaster goes over the top A of the track shown with a speed $v = 135\,\text{km/h}$. If the radius of curvature at A is $\rho = 60\,\text{m}$, determine the minimum force that a restraint must apply to a person with a mass of 85 kg to keep the person on his or her seat.



Solution

Balance Principles:



$$\sum F_t: \quad F_t = ma_t, \tag{1}$$

$$\sum F_n: \quad F_r + mg = ma_n. \tag{2}$$

Kinematics:

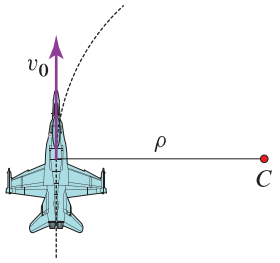
$$a_n = \frac{v^2}{\rho}.$$

Eq. (2) becomes

$$F_r = m\left(\frac{v^2}{\rho} - g\right) = 1160\,\text{N}.$$

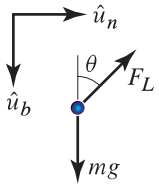
Problem 3.122

A 50,000 lb aircraft is flying along a rectilinear path at a constant altitude with a speed $v = 720$ mph when the pilot initiates a turn by banking the plane 20° to the right. Assuming that the initial rate of change of speed is negligible, determine the components of the acceleration of the aircraft right at the beginning of the turn if the pilot does not adjust the attitude of the aircraft so that the magnitude of the lift remains the same as when the plane is flying straight and the aerodynamic drag remains in the horizontal plane. In addition, determine the radius of curvature at the beginning of the turn.



Solution

Note: $\hat{u}_t$ is into the page. Balance Principles:



$$\sum F_t: \quad F_t = ma_t, \tag{1}$$

$$\sum F_n: \quad F_L \sin\theta = ma_n, \tag{2}$$

$$\sum F_b: \quad -F_L \cos\theta + mg = ma_b. \tag{3}$$

Kinematics:

$$a_t = \dot{v} = 0 \quad \text{and} \quad a_n = \frac{v^2}{\rho}.$$

Before the turn, by equilibrium in the vertical direction, we know that the lift is $F_L = 50{,}000\,\text{lb} = W$, where W is the weight of the aircraft. Solving Eq. (3) for a_b

$$a_b = \frac{-F_L \cos\theta}{m} + g \quad \Rightarrow \quad \boxed{a_b = g\,(1 - \cos\theta) = 1.94\,\text{ft/s}^2.}$$

Solving Eq. (2) for a_n

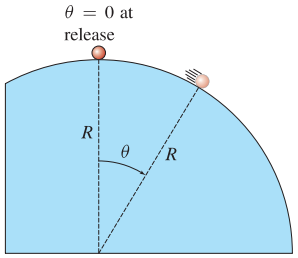
$$a_n = \frac{F_L \sin\theta}{m} \quad \Rightarrow \quad \boxed{a_n = g \sin\theta = 11\,\text{ft/s}^2.}$$

The radius of curvature at the beginning of the turn is

$$\rho = \frac{v^2}{a_n} \quad \Rightarrow \quad \boxed{\rho = \frac{v^2}{g \sin\theta} = 101 \times 10^3\,\text{ft} = 19.2\,\text{mi}.}$$

Problem 3.123

Referring to Example 3.6 on p. 216, let $R = 1.25$ ft and let the angle at which the sphere separates from the cylinder be $\theta_s = 34°$. If the sphere were placed in motion at the very top of the cylinder, determine the sphere's initial speed.

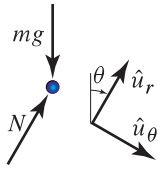


Solution

Balance Principles:

$$\sum F_r: \quad N - mg\cos\theta = ma_r, \tag{1}$$

$$\sum F_\theta: \quad mg\sin\theta = ma_\theta. \tag{2}$$



Because the radius does not change, the kinematics are:

$$a_r = -R\dot{\theta}^2 \quad \text{and} \quad a_\theta = R\ddot{\theta}.$$

Eq. (2) gives us angular acceleration as a function of angular position. Integrate it to obtain angular velocity as a function of angular position. Recall that $\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$.

$$\ddot{\theta} = \frac{g}{R}\sin\theta \quad \Rightarrow \quad \frac{g}{R}\int_0^\theta \sin\theta\, d\theta = \int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} \quad \Rightarrow \quad \dot{\theta}^2 = \dot{\theta}_0^2 - \frac{2g}{R}(\cos\theta - 1).$$

At $\theta = 34°$ separation occurs and $N = 0$. Eq. (1) becomes

$$N = mg\cos\theta - mR\dot{\theta}^2 \quad \Rightarrow \quad g\cos\theta = R\left[\dot{\theta}_0^2 - \frac{2g}{R}(\cos\theta - 1)\right].$$

Solve for $\dot{\theta}_0$:

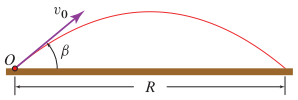
$$\dot{\theta}_0 = \sqrt{\frac{g}{R}(3\cos\theta - 2)}.$$

Recall that $v = R\dot{\theta}$.

$$\boxed{v_0 = \sqrt{gR(3\cos\theta - 2)} = 4.43 \text{ ft/s}.}$$

Problem 3.124

Revisit Example 3.4 and assume that the drag force acting on the ball has the form $\vec{F}_d = -\eta\vec{v}$, where $\vec{v}$ is the velocity of the ball and η is a drag coefficient. Determine the trajectory of the ball, expressing it in the form $y = y(x)$.

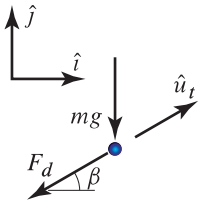


Solution

Balance Principles:

$$\sum F_x: \quad -F_d\cos\beta = ma_x, \tag{1}$$

$$\sum F_y: \quad -F_d\sin\beta - mg = ma_y. \tag{2}$$



Force law: $\vec{F}_d = -\eta\vec{v}$.

Kinematics: $a_x = \ddot{x}, \quad a_y = \ddot{y}, \quad$ and $\quad \vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$.

Eqs. (1) and (2) become

$$-\eta\dot{x} = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = \frac{-\eta}{m}\dot{x}, \tag{3}$$

$$-\eta\dot{y} - mg = m\ddot{y} \quad \Rightarrow \quad \ddot{y} = \frac{-\eta}{m}\dot{y} - g. \tag{4}$$

Both equations give us acceleration as a function of velocity. Integrate Eq. (4) to obtain $t(\dot{y})$.

$$\ddot{y} = \frac{d\dot{y}}{dt} \quad \Rightarrow \quad \int_0^t dt = -\int_{\dot{y}_0}^{\dot{y}} \frac{d\dot{y}}{(\eta/m)\dot{y} + g} \quad \Rightarrow \quad t = \frac{-m}{\eta}\ln\frac{(\eta/m)\dot{y} + g}{(\eta/m)\dot{y}_0 + g}.$$

Invert this equation to find $\dot{y}(t)$.

$$\frac{\eta}{m}\dot{y} + g = \left[\frac{\eta}{m}\dot{y}_0 + g\right]e^{-\eta t/m} \quad \Rightarrow \quad \dot{y} = \dot{y}_0 e^{-\eta t/m} + \frac{mg}{\eta}\left(e^{-\eta t/m} - 1\right).$$

Integrate this equation to find $y(t)$.

$$\dot{y} = \frac{dy}{dt} \quad \Rightarrow \quad \int_0^y dy = \dot{y}_0\int_0^t e^{-\eta t/m}\,dt + \frac{mg}{\eta}\int_0^t \left(e^{-\eta t/m} - 1\right)dt \quad \Rightarrow$$

$$y = \frac{-m\dot{y}_0}{\eta}\left(e^{-\eta t/m} - 1\right) - \frac{mgt}{\eta} - \frac{m^2 g}{\eta^2}\left(e^{-\eta t/m} - 1\right) \quad \Rightarrow$$

$$y = \left(1 - e^{-\eta t/m}\right)\left[\frac{m^2 g}{\eta^2} + \frac{m\dot{y}_0}{\eta}\right] - \frac{mgt}{\eta}. \tag{5}$$

Integrate Eq. (3) to obtain $t(\dot{x})$.

$$\ddot{x} = \frac{d\dot{x}}{dt} \quad \Rightarrow \quad \int_0^t dt = \frac{-m}{\eta}\int_{\dot{x}_0}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} \quad \Rightarrow \quad t = \frac{-m}{\eta}\ln\frac{\dot{x}}{\dot{x}_0}.$$

Invert this equation to find $\dot{x}(t)$.

$$\dot{x} = \dot{x}_0 e^{-\eta t/m}.$$

Integrate this equation to find $x(t)$.

$$\dot{x} = \frac{dx}{dt} \quad \Rightarrow \quad \int_0^x dx = \dot{x}_0\int_0^t e^{-\eta t/m}\,dt \quad \Rightarrow \quad x = \frac{-m\dot{x}_0}{\eta}\left(e^{-\eta t/m} - 1\right)$$

Invert this equation to find $t(x)$.

$$t = \frac{-m}{\eta} \ln\left(1 - \frac{\eta x}{m\dot{x}_0}\right)$$

Substitute this expression for t into Eq. (5) and simplify to find $y(x)$.

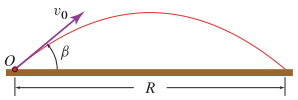
$$y = \frac{x}{\dot{x}_0}\left[\frac{mg}{\eta} + \dot{y}_0\right] + \frac{m^2 g}{\eta^2} \ln\left[1 - \frac{\eta x}{m\dot{x}_0}\right],$$

$$\boxed{y = \frac{x}{v_0 \cos\beta}\left[\frac{mg}{\eta} + v_0 \sin\beta\right] + \frac{m^2 g}{\eta^2} \ln\left[1 - \frac{\eta x}{m v_0 \cos\beta}\right],}$$

where v_0 is the initial speed of the ball.

Problem 3.125

Revisit Example 3.4 and assume that the drag force acting on the ball has the form $\vec{F}_d = -\eta\vec{v}$, where $\vec{v}$ is the velocity of the ball and η is a drag coefficient. Determine the value of η such that a 1.61 oz ball has a range $R = 270$ yd when put in motion with an initial velocity of magnitude $v_0 = 186$ mph and initial direction $\beta = 11.2°$.



Solution

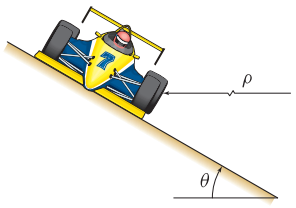
We take advantage of the trajectory equation derived in the solution to Problem 3.124. Using *Mathematica* we find η with the following code:

```
rules = {β → 11.2 °, v0 → 186 * 5280/3600, dx0 → v0 Cos[β], dy0 → v0 Sin[β], m → 1.61 / 16 / 32.2,
    g → 32.2, x → 270 * 3};
FindRoot[x/dx0 (m g/η + dy0) + m^2 g/η^2 Log[1 - η x/(m dx0)] == 0 //. rules, {η, 0.0001}]
```

$$\eta = 1.23\times10^{-4}\ \text{lb·s/ft.}$$

Problem 3.126

Referring to Example 3.5 on p. 214, show that, for $\theta = 33°$ and under the assumption that $\mu_s > 1/\tan\theta$, the no slip solution in Eqs. (15) and (16) satisfies the no slip condition $|F| \le \mu_s |N|$ for any value of the car's speed.



Solution

Recall that $F = \frac{mv^2}{\rho}\cos\theta - mg\sin\theta$ and $N = \frac{mv^2}{\rho}\sin\theta + mg\cos\theta$. Observe that θ, ρ, m, and g are given constants. Define the following function of v:

$$H(v) = \frac{F}{N} = \frac{\frac{mv^2}{\rho}\cos\theta - mg\sin\theta}{\frac{mv^2}{\rho}\sin\theta + mg\cos\theta}$$

Divide the numerator and denominator by $mg\cos\theta$.

$$H(v) = \frac{\frac{v^2}{\rho g} - \tan\theta}{\frac{v^2}{\rho g}\tan\theta + 1} \quad \Rightarrow \quad H(v) = \frac{Av^2 - \tan\theta}{Av^2\tan\theta + 1},$$

where $A = \frac{1}{\rho g}$. H(v) is defined for $0 \le v < \infty$. Next, observe that

$$\frac{dH}{dv} = \frac{2}{\left(Av^2\tan\theta + 1\right)^2}\left[Av\left(Av^2\tan\theta + 1\right) - Av\tan\theta\left(Av^2 - \tan\theta\right)\right] \quad \Rightarrow$$

$$\frac{dH}{dv} = \frac{2Av\left(\tan^2\theta + 1\right)}{\left(Av^2\tan\theta + 1\right)^2}$$

Observe that $\frac{dH}{dv} \ge 0$ for all v where $0 \le v < \infty$. Therefore $H(t)$ is a monotonic function of v.

$$\text{For} \quad v = 0 \qquad H(t) = -\tan\theta, \quad \text{and} \quad \frac{dH}{dv} = 0.$$

$$\text{For} \quad v \to \infty \qquad H(t) \to \frac{1}{\tan\theta}, \quad \text{and} \quad \frac{dH}{dv} \to 0.$$

Hence $-\tan\theta \le H \le \frac{1}{\tan\theta}$ for $0 \le v < \infty$. The no slip condition requires that

$$\left|\frac{F}{N}\right| = |H(v)| \le \mu_s \quad \text{for all possible } v.$$

Based on the analysis of the behavior of H(v), we have

$$0 \le |H(v)| < \frac{1}{\tan\theta} \quad \text{for} \quad 0 \le v < \infty.$$

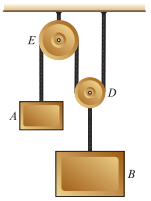
Because $\theta = 33°$ $\tan\theta < \frac{1}{\tan\theta}$. Since $\mu_s < \frac{1}{\tan\theta}$ by assumption, then

$$\left|\frac{F}{N}\right| = |H(v)| < \mu_s \quad \text{for all possible } v,$$

which is what we needed to prove.

Problem 3.127

The load B has a mass $m_B = 250\,\text{kg}$, and the load A has a mass $m_A = 120\,\text{kg}$. Let the system be released from rest, and neglecting any source of friction as well as the inertia of the ropes and the pulleys, determine the acceleration of A and the tension in the cord to which A is attached.



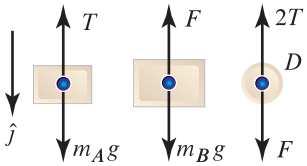
Solution

Balance Principles:

$$\left(\sum F_y\right)_A: \quad m_A g - T = m_A a_A, \tag{1}$$

$$\left(\sum F_y\right)_B: \quad m_B g - F = m_B a_B, \tag{2}$$

$$\left(\sum F_y\right)_D: \quad F - 2T = m_D a_D. \tag{3}$$



Kinematics:

$$L = 2y_D + y_A \quad \Rightarrow \quad a_A = -2a_D \quad \Rightarrow \quad a_A = -2a_B.$$

Because we are neglecting the mass of the pulleys Eq. (3) tells us that $F = 2T$. Eliminate T from Eqs. (1) and (2) to find

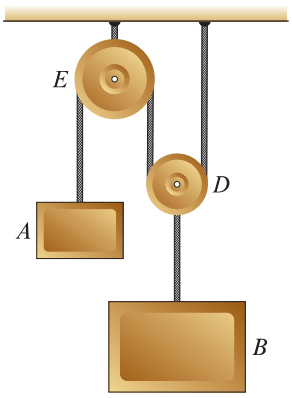
$$\begin{aligned} m_A g - T &= m_A a_A \\ m_B g - 2T &= -\tfrac{1}{2} m_B a_A \end{aligned} \quad \Rightarrow \quad \boxed{a_A = \frac{-2g\,(m_B - 2m_A)}{4m_A + m_B} = -0.269\,\text{m/s}^2.}$$

Solve Eq. (1) for T:

$$\boxed{T = m_A g\left[1 + \frac{2\,(m_B - 2m_A)}{4m_A + m_B}\right] = 1210\,\text{N}.}$$

Problem 3.128

The load B weighs 300 lb. Neglecting any source of friction as well as the inertia of the ropes and the pulleys, determine the weight of A if, after the system is released from rest, B moves upward with an acceleration of 0.75 ft/s^2.



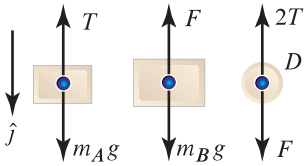
Solution

Balance Principles:

$$\left(\sum F_y\right)_A: \quad m_A g - T = m_A a_A, \tag{1}$$

$$\left(\sum F_y\right)_B: \quad m_B g - F = m_B a_B, \tag{2}$$

$$\left(\sum F_y\right)_D: \quad F - 2T = m_D a_D. \tag{3}$$



Kinematics:

$$L = 2y_D + y_A \quad \Rightarrow \quad a_A = -2a_D \quad \Rightarrow \quad a_A = -2a_B.$$

Because we are neglecting the mass of the pulleys Eq. (3) tells us that $F = 2T$. Eliminate T from Eqs. (1) and (2) to find

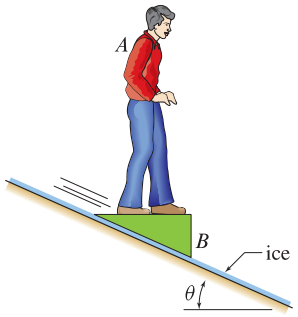
$$\begin{aligned} m_A g - T &= -2m_A a_B \\ m_B g - 2T &= m_B a_B \end{aligned} \quad \Rightarrow \quad m_A = \frac{m_B\,(g - a_B)}{2g + 4a_B}.$$

Mass = weight / gravity. Thus the weight of A is

$$\boxed{W_A = \frac{W_B\,(g - a_B)}{2g + 4a_B} = 161 \text{ lb}.}$$

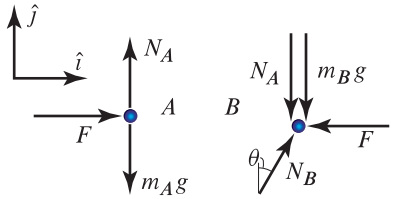
Problem 3.129

A person A is trying to keep his balance while on a sled B that is sliding down an icy incline. Let $m_A = 78\,\text{kg}$ and $m_B = 25\,\text{kg}$ be the masses of A and B, respectively. In addition, let the static and kinetic friction coefficients between A and B be $\mu_s = 0.4$ and $\mu_k = 0.35$, respectively. Determine the acceleration of A if $\theta = 23°$. Friction between the sled and the incline is negligible.



Solution

Working assumption: A slips to the left relative to B.
Balance Principles:



$$\left(\sum F_x\right)_A: \quad F = m_A a_{Ax}, \tag{1}$$

$$\left(\sum F_y\right)_A: \quad N_A - m_A g = m_A a_{Ay}, \tag{2}$$

$$\left(\sum F_x\right)_B: \quad -F + N_B \sin\theta = m_A a_{Bx}, \tag{3}$$

$$\left(\sum F_y\right)_B: \quad N_B \cos\theta - N_A - m_B g = m_B a_{By}. \tag{4}$$

Due to our working assumption: $F = \mu_k N_A$.
Kinematics: $a_{Ay} = a_{By}$. $a_{Bx} = a\cos\theta$, $a_{By} = -a\sin\theta$.
We have 8 equations in 8 unknowns. Solving this system of equations we find the acceleration of A to be

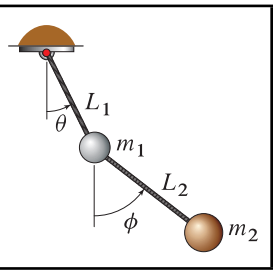
$$\vec{a}_A = \left[\frac{2g m_B \mu_k \cos^2\theta}{m_A + 2m_B - m_A(\cos 2\theta + \mu_k \sin 2\theta)}\right]\hat{\imath} + \left\{\frac{2g\sin\theta\,[(m_A + m_B)\sin\theta - m_A\mu_k\cos\theta]}{m_A(\cos 2\theta + \mu_k \sin 2\theta - m_A - 2m_B)}\right\}\hat{\jmath}$$

$$\vec{a}_A = (2.68\,\hat{\imath} - 2.14\,\hat{\jmath})\ \text{m/s}^2.$$

Solving for the other terms we see that $N_A > 0$, $(a_{A/B})_x < 0$, $a_{Ay} < 0$, and $a_{Ax} < 0$. Thus our working assumption is correct.

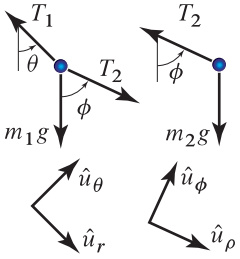
Problem 3.130

Derive the equations of motion for the double pendulum shown.



Solution

Balance Principles:



$$\left(\sum F_r\right)_1 : \quad m_1 g\cos\theta - T_1 + T_2\cos(\phi-\theta) = m_1 a_r, \tag{1}$$

$$\left(\sum F_\theta\right)_1 : \quad -m_1 g\sin\theta + T_2\sin(\phi-\theta) = m_1 a_\theta, \tag{2}$$

$$\left(\sum F_\rho\right)_2 : \quad m_2 g\cos\phi - T_2 = m_2 a_\rho, \tag{3}$$

$$\left(\sum F_\phi\right)_2 : \quad -m_2 g\sin\phi = m_2 a_\phi. \tag{4}$$

The quantity $\phi-\theta$ appears often so we define $\gamma = \phi-\theta$ for convenience.
The acceleration of m_1 is expressed in polar coordinates as

$$\vec{a}_1 = \left(-L_1\dot{\theta}^2\right)\hat{u}_r + \left(L_1\ddot{\theta}\right)\hat{u}_\theta, \tag{5}$$

The acceleration of m_2 is

$$\vec{a}_2 = \vec{a}_{2/1} + \vec{a}_1 = -\left(L_2\dot{\phi}^2\right)\hat{u}_\rho + \left(L_2\ddot{\phi}\right)\hat{u}_\phi - \left(L_1\dot{\theta}^2\right)\hat{u}_r + \left(L_1\ddot{\theta}\right)\hat{u}_\theta. \tag{6}$$

The conversion between coordinate systems is

$$\hat{u}_r = (\cos\gamma)\,\hat{u}_\rho - (\sin\gamma)\,\hat{u}_\phi \tag{7}$$

$$\hat{u}_\theta = (\sin\gamma)\,\hat{u}_\rho + (\cos\gamma)\,\hat{u}_\phi. \tag{8}$$

Substituting these relations into Eq. (6), we can write the absolute acceleration of m_2 as

$$\vec{a}_2 = \left(-L_1\dot{\theta}^2\cos\gamma + L_1\ddot{\theta}\sin\gamma - L_2\dot{\phi}^2\right)\hat{u}_\rho + \left(L_1\dot{\theta}^2\sin\gamma + L_1\ddot{\theta}\cos\gamma + L_2\ddot{\phi}\right)\hat{u}_\phi. \tag{9}$$

Now, since we want the equations of motion, we need to get rid of the force term T_2. The most efficient way to do this is to solve for T_2 from Eq. (3) in order to determine

$$T_2 = m_2 g\cos\phi - m_2\left[L_1\ddot{\theta}\sin\gamma - L_1\dot{\theta}^2\cos\gamma - L_2\dot{\phi}^2\right]. \tag{10}$$

Substituting the expression for T_2 into Eq. (2) results in the first equation of motion:

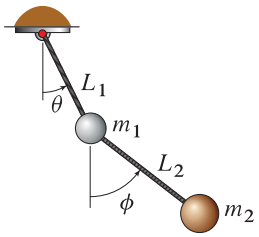
$$\boxed{m_1 L_1\ddot{\theta} = m_2 g\cos\phi\sin\gamma - m_2\sin\gamma\left[L_1\ddot{\theta}\sin\gamma - L_1\dot{\theta}^2\cos\gamma - L_2\dot{\phi}^2\right] - m_1 g\sin\theta.} \tag{11}$$

Find the second equation of motion (containing $\ddot{\phi}$) by substituting the acceleration from Eq. (9) into Eq. (4).

$$\boxed{L_2\ddot{\phi} = -g\sin\phi - L_1\ddot{\theta}\cos\gamma - L_1\dot{\theta}^2\sin\gamma.} \tag{12}$$

Problem 3.131

Derive the equations for the double pendulum shown. After doing so, let $L_1 = 1.4\,\text{m}$, $L_2 = 2\,\text{m}$, $m_1 = 7.5\,\text{kg}$, and $m_2 = 12\,\text{kg}$, and release the pendulum from rest with $\theta(0) = 25°$ and $\phi(0) = -37°$. Integrate the equations of motion and plot the trajectory of each of the particles for at least 5 s.



Solution

The equations of motion from problem 3.130 derived for the double pendulum and initial conditions are numerically solved by *Mathematica* with the following code:

```
eqθ =
  (m1 + m2) L1 θ''[t] + (m1 + m2) g Sin[θ[t]] +
    m2 L2 (ϕ''[t] Cos[ϕ[t] - θ[t]] - Sin[ϕ[t] - θ[t]] (ϕ'[t])^2) == 0;
eqϕ = L2 ϕ''[t] + g Sin[ϕ[t]] + L1 (θ''[t] Cos[ϕ[t] - θ[t]] + (θ'[t])^2 Sin[ϕ[t] - θ[t]]) == 0;
IC = {ϕ[0] == -37 °, ϕ'[0] == 0, θ[0] == 25 °, θ'[0] == 0};
params = {L1 → 1.4, L2 → 2, m1 → 7.5, m2 → 12, g → 9.81};
Solution = NDSolve[Flatten[{eqθ, eqϕ, IC}] /. params, {θ[t], ϕ[t]}, {t, 0, 5}]
```

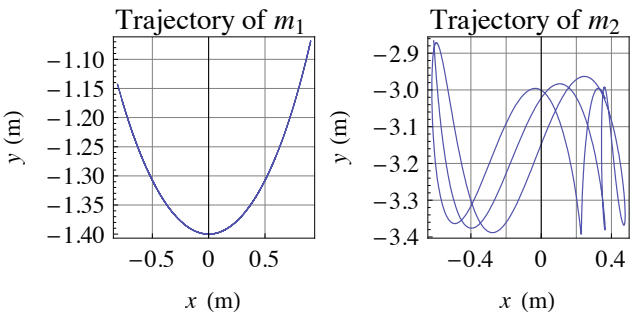
The following Mathematica code generates the plots of the trajectory of m_1 in the Cartesian plane:

```
ParametricPlot[Evaluate[{L1 Sin[θ[t]], -L1 Cos[θ[t]]} /. Solution /. params], {t, 0, 5},
 AspectRatio → 1, Frame → True, PlotRange → {Automatic, Automatic}, Axes → True,
 FrameTicks → {{Automatic, None}, {{-1.0, -0.5, 0, 0.5, 1.0}, None}},
 FrameLabel → {"x (m)", "y (m)"}, PlotLabel → "Trajectory of m1 ", GridLines → Automatic]
```

The following Mathematica code generates the plots of the trajectory of m_2 in the Cartesian plane:

```
ParametricPlot[
 Evaluate[{L1 Sin[θ[t]] + L2 Sin[ϕ[t]], -L1 Cos[θ[t]] - L2 Cos[ϕ[t]]} /. params /. Solution],
 {t, 0, 5}, AspectRatio → 1, Frame → True, PlotRange → {Automatic, Automatic},
 Axes → True, FrameTicks → {{Automatic, None}, {{-0.4, 0, 0.4}, None}},
 FrameLabel → {"x (m)", "y (m)"}, PlotLabel → "Trajectory of m2 ", GridLines → Automatic]
```

The resulting plots for the trajectories of m_1 and m_2 are:



Solutions Manual

Engineering Mechanics: Dynamics

1st Edition

Gary L. Gray
The Pennsylvania State University

Francesco Costanzo
The Pennsylvania State University

Michael E. Plesha
University of Wisconsin–Madison

With the assistance of:
Chris Punshon
Andrew J. Miller
Justin High
Chris O'Brien
Chandan Kumar
Joseph Wyne

Version: August 10, 2009

The McGraw-Hill Companies, Inc.

Copyright © 2002–2010
Gary L. Gray, Francesco Costanzo, and Michael E. Plesha

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Important Information about this Solutions Manual

Even though this solutions manual is nearly complete, we encourage you to visit

`http://www.mhhe.com/pgc`

often to obtain the most up-to-date version. In particular, as of July 30, 2009, please note the following:

- The solutions for Chapters 1 and 2 have been accuracy checked and have been edited by us. They are in their final form.
- The solutions for Chapters 4 and 7 have been accuracy checked and should be error free. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapters 3, 6, 8, and 9 are being accuracy checked and the accuracy checked versions should be available by the end of August 2009. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapter 10 should be available in their entirety by the end of August 2009.

All of the figures in Chapters 6–10 are in color. Color will be added to the figures in Chapters 1–5 over the coming weeks.

Contact the Authors

If you find any errors and/or have questions concerning a solution, please do not hesitate to contact the authors and editors via email at:

`dyn_solns@email.esm.psu.edu`

We welcome your input.

Accuracy of Numbers in Calculations

Throughout this solutions manual, we will generally assume that the data given for problems is accurate to 3 significant digits. When calculations are performed, all intermediate numerical results are reported to 4 significant digits. Final answers are usually reported with 3 significant digits. If you verify the calculations in this solutions manual using the rounded intermediate numerical results that are reported, you should obtain the final answers that are reported to 3 significant digits.

Chapter 4 Solutions

Problem 4.1

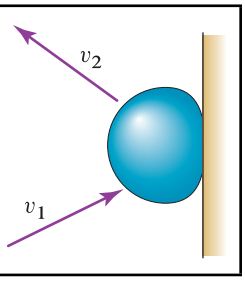
A rocket lifts off with an acceleration a. During liftoff, in terms of absolute values, is the work done on an astronaut by gravity larger than, equal to, or smaller than the work done by the normal reaction between the astronaut and her or his seat?

Solution

During liftoff the normal reaction between the astronaut and his seat does more work than gravity. For an astronaut to accelerate upward, the absolute value of the normal reaction between the astronaut and his seat must exceed the astronauts weight.

Problem 4.2

A soft rubber ball bounces against a wall. Assuming that the wall's deformation due to the ball's impact is negligible, does the contact force due to the wall do positive work, no work, or negative work on the ball?



Solution

Because the wall is viewed as rigid, the point of application of the contact force between the wall and the ball does not move. Therefore the contact force applied to the ball by the wall does no work.

Problem 4.3

Determine the kinetic energy of the bodies listed below, when modeled as particles. Express all answers using both U.S. Customary units and SI units.

(a) A .30-06 bullet weighing 150 gr (1 lb = 7000 gr) and traveling at 3000 ft/s.

(b) A 25 kg child traveling in a car at 45 km/h.

(c) A 415,000 lb locomotive traveling at 75 mph.

(d) A 20 g metal fragment from a space vehicle traveling at 8000 km/s.

(e) A 3000 lb car traveling at 60 mph.

Solution

The kinetic energy T is given by

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(W/g)v^2, \tag{1}$$

where m is the mass, W is the weight, g is the acceleration due to gravity, and v is the speed of the body. For the 45 caliber bullet,

$$T = \tfrac{1}{2}\left[(150\,\text{grain})\left(\frac{1\,\text{lb}}{7000\,\text{gr}}\right)/(32.2\,\text{ft/s}^2)\right](3000\,\text{ft/s})^2 = \boxed{3000\,\text{ft·lb},} \tag{2}$$

$$T = (2995\,\text{ft·lb})\left(\frac{0.3048\,\text{m}}{\text{ft}}\right)\left(\frac{4.448\,\text{N}}{\text{lb}}\right) = \boxed{4060\,\text{N·m.}} \tag{3}$$

For the child travelling in a car,

$$T = \tfrac{1}{2}(25\,\text{kg})\left[45\,\text{km/h}\left(\frac{1\times10^3\,\text{m}}{\text{km}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right)\right]^2 = \boxed{1950\,\text{N·m},} \tag{4}$$

$$T = (1953\,\text{N·m})\left(\frac{1\,\text{ft}}{0.3048\,\text{m}}\right)\left(\frac{1\,\text{lb}}{4.448\,\text{N}}\right) = \boxed{1440\,\text{ft·lb.}} \tag{5}$$

For the locomotive,

$$T = \tfrac{1}{2}\left[(415{,}000\,\text{lb})/(32.2\,\text{ft/s}^2)\right]\left[(75\,\text{mi/h})\left(\frac{5280\,\text{ft}}{\text{mi}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right)\right]^2 = \boxed{7.80\times10^7\,\text{ft·lb},} \tag{6}$$

$$T = (7.797\times10^7\,\text{ft·lb})\left(\frac{0.3048\,\text{m}}{\text{ft}}\right)\left(\frac{4.448\,\text{N}}{\text{lb}}\right) = \boxed{1.06\times10^8\,\text{N·m.}} \tag{7}$$

For the metal fragment,

$$T = \frac{1}{2}\left[(20\,\text{g})\left(\frac{1\,\text{kg}}{10^3\,\text{g}}\right)\right]\left[8000\,\text{km/s}\left(\frac{10^3\,\text{m}}{\text{km}}\right)\right]^2 = \boxed{6.40\times10^{11}\,\text{N·m},} \tag{8}$$

$$T = (6.400\times10^{11}\,\text{N·m})\left(\frac{1\,\text{ft}}{0.3048\,\text{m}}\right)\left(\frac{1\,\text{lb}}{4.448\,\text{N}}\right) = \boxed{4.72\times10^{11}\,\text{ft·lb.}} \tag{9}$$

For the car,

$$T = \tfrac{1}{2}\left[(3000\,\text{lb})/(32.2\,\text{ft/s}^2)\right]\left[(60\,\text{mi/h})\left(\frac{5280\,\text{ft}}{\text{mi}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right)\right]^2 = \boxed{3.61\times10^5\ \text{ft·lb},} \tag{10}$$

$$T = (3.607\times10^5\ \text{ft·lb})\left(\frac{0.3048\,\text{m}}{\text{ft}}\right)\left(\frac{4.448\,\text{N}}{\text{lb}}\right) = \boxed{4.89\times10^5\ \text{N·m}.} \tag{11}$$

Problems 4.4 and 4.5

Consider a 3000 lb car whose speed is increased by 30 mph.



Problem 4.4 Modeling the car as a particle and assuming that the car is traveling on a rectilinear and horizontal stretch of road, determine the amount of work done on the car throughout the acceleration process if the car starts from rest.

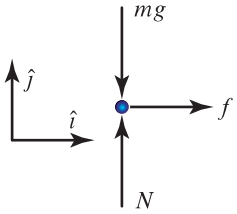
Problem 4.5 Modeling the car as a particle and assuming that the car is traveling on a rectilinear and horizontal stretch of road, determine the amount of work done on the car throughout the acceleration process if the car has an initial speed of 45 mph.

Solution to 4.4

Work-energy principle:

$$T_1 + U_{1\text{-}2} = T_2 \quad \Rightarrow \quad 0 + U_{1\text{-}2} = \tfrac{1}{2} m v_2^2, \qquad (1)$$

$$U_{1\text{-}2} = \frac{W}{2g} v_2^2 \quad = \boxed{9.02 \times 10^4 \text{ ft·lb},} \qquad (2)$$



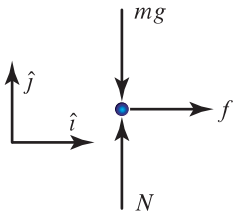
where $m = W/g$, $W = 3000$ lb, $g = 32.2$ ft/s^2, and $v_2 = 30$ mi/h $= 44$ ft/s.

Solution to 4.5

Work-energy principle:

$$T_1 + U_{1\text{-}2} = T_2 \quad \Rightarrow \quad \tfrac{1}{2}mv_1^2 + U_{1\text{-}2} = \tfrac{1}{2}mv_2^2, \tag{3}$$

$$U_{1\text{-}2} = \frac{W}{2g}\left(v_2^2 - v_1^2\right) = \boxed{3.61\times10^5\ \text{ft·lb.}} \tag{4}$$



where $m = W/g$, $W = 3000\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $v_1 = 45\,\text{mi/h} = 66\,\text{ft/s}$, and $v_2 = 75\,\text{mi/h} = 110\,\text{ft/s}$.

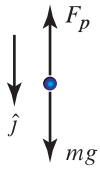
Problem 4.6

A 75 kg skydiver is falling at a speed of 250 km/h when the parachute is deployed, allowing the skydiver to land at a speed of 4 m/s. Modeling the skydiver as a particle, determine the total work done on the skydiver from the moment of parachute deployment until landing.



Solution

Taking ① to be where the skydiver has a velocity $v_1 = 250\,\text{km/h}$ and ② to be where the skydiver lands with a velocity $v_2 = 4\,\text{m/s}$, we apply the work-energy principle between ① and ②



$$T_1 + U_{1\text{-}2} = T_2, \tag{1}$$

where $U_{1\text{-}2}$ is the work done on the skydiver between ① and ②. The kinetic energies are given by

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2, \tag{2}$$

where $m = 75\,\text{kg}$, $v_1 = 250\,\text{km/h} = 69.44\,\text{m/s}$ and $v_2 = 4\,\text{m/s}$. Substituting Eq. (2) into Eq. (1), we get

$$\boxed{U_{1\text{-}2} = \tfrac{1}{2}m(v_2^2 - v_1^2) = -180\,\text{kJ}.}$$

Problems 4.7 and 4.8

Consider a 1500 kg car whose speed is increased by 45 km/h over a distance of 50 m while traveling up an incline with a 15% grade.



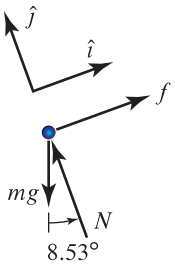
Problem 4.7 Modeling the car as a particle, determine the work done on the car if the car starts from rest.

Problem 4.8 Modeling the car as a particle, determine the work done on the car if the car has an initial speed of 60 km/h.

Solution to 4.7

Work-energy principle:

$$T_1 + U_{1\text{-}2} = T_2 \quad \Rightarrow \quad \boxed{U_{1\text{-}2} = \tfrac{1}{2}mv_2^2 = 117\,\text{kJ},} \qquad (1)$$

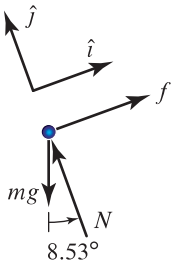


where $m = 1500\,\text{kg}$ and $v_2 = 45\,\text{km/h} = 12.5\,\text{m/s}$.

Solution to 4.8

Work-energy principle:

$$T_1 + U_{1\text{-}2} = T_2 \;\Rightarrow\; \tfrac{1}{2}mv_1^2 + U_{1\text{-}2} = \tfrac{1}{2}mv_2^2 \;\Rightarrow\; U_{1\text{-}2} = \tfrac{1}{2}m(v_2^2 - v_1^2), \qquad (2)$$



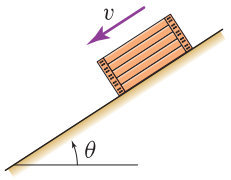
which can be evaluated to obtain

$$\boxed{U_{1\text{-}2} = 4.30\times10^5\ \text{J},}$$

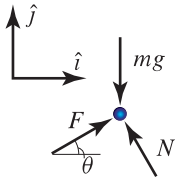
where we have used the following numerical data: $m = 1500\,\text{kg}$, $v_1 = 60\,\text{km/hr} = 16.67\,\text{m/s}$ and $v_2 = 105\,\text{km/h} = 29.17\,\text{m/s}$.

Problem 4.9

A 350 kg crate is sliding down a rough incline with a constant speed $v = 7\,\text{m/s}$. Assuming that the angle of the incline is $\theta = 33°$ and that the only forces acting on the crate are gravity, friction, and the normal force between the crate and the incline, determine the work done by friction over every meter slid by the crate.



Solution



Work-energy principle:

$$T_1 + U_{1\text{-}2} = T_2.$$

Kinetic energies:

$$T_1 = T_2.$$

Force laws:

$$U_{1\text{-}2} = 0 = (U_{1\text{-}2})_{\text{gravity}} + (U_{1\text{-}2})_{\text{friction}}.$$

Therefore

$$(U_{1\text{-}2})_{\text{friction}} = -(U_{1\text{-}2})_{\text{gravity}}$$

$$\boxed{(U_{1\text{-}2})_{\text{friction}} = -Lmg\sin\theta = -1870\,\text{J},}$$

where $L = 1\,\text{m}$, $\theta = 33°$, $m = 350\,\text{kg}$, and $g = 9.81\,\text{m/s}^2$.

Problem 4.10



A vehicle A is stuck on the railroad tracks as a train B approaches with a speed of 120 km/h. As soon as the problem is detected, the train's emergency brakes are activated, locking the wheels and causing the train's wheels to slide relative to the tracks. If the coefficient of kinetic friction between the wheels and the track is 0.2, determine the minimum distance $d_{\min}$ at which the brakes must be applied to avoid a collision under the following circumstances:

(a) The train consists of just a 195,000 kg locomotive.

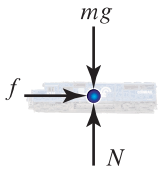
(b) The train consists of a 195,000 kg locomotive and a string of cars whose mass is 10×10^6 kg, all of which are able to apply brakes and lock their wheels.

(c) The train consists of a 195,000 kg locomotive and a string of cars whose mass is 10×10^6 kg, but only the locomotive is able to apply its brakes and lock its wheels.

Treat the train as a particle, and assume that the railroad tracks are rectilinear and horizontal.

Solution

Let ① be the point where the brakes are initially applied and let ② be the point where the train comes to rest. The work-energy principle between ① and ② is



$$T_1 + U_{1\text{-}2} = T_2, \tag{1}$$

Kinetic energies: $T_1 = \frac{1}{2}mv^2$ and $T_2 = 0$.

Force law: $f = \mu_k N = \mu_k mg$.

$$U_{1\text{-}2} = \int_{\mathscr{L}_{1\text{-}2}} \vec{f}\cdot d\vec{r} = \int_0^d -f\hat{\imath}\cdot\hat{\imath}\,dx = -fd = -\mu_k mgd, \tag{2}$$

Part (a) For this case, $T_1 = \frac{1}{2}m_l v^2$ and $U_{1\text{-}2} = -\mu_k m_l g d_{\min}$, where m_l is the mass of the locomotive. Solve Eq. (1) for $d_{\min}$ to obtain

$$d_{\min} = \frac{m_l v^2}{2\mu_k m_l g} = \frac{v^2}{2\mu_k g} = \boxed{283\text{ m},}$$

where $v = 120\text{ km/h} = 33.33\text{ m/s}$, $\mu_k = 0.2$, and $g = 9.81\text{ m/s}^2$.

Part (b) For this case, $T_1 = \frac{1}{2}(m_l + m_c)v^2$ and $U_{1\text{-}2} = -\mu_k(m_l + m_c)g d_{\min}$, where m_l is the mass of the locomotive, and m_c is the mass of the cars. Solve Eq. (1) for $d_{\min}$ to obtain

$$d_{\min} = \frac{(m_l + m_c)v^2}{2\mu_k(m_l + m_c)g} = \frac{v^2}{2\mu_k g} = \boxed{283\text{ m},}$$

where $v = 120\text{ km/h} = 33.33\text{ m/s}$, $\mu_k = 0.2$, and $g = 9.81\text{ m/s}^2$.

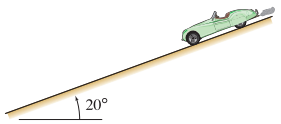
Part (c) For this case, $T_1 = \frac{1}{2}(m_l + m_c)v^2$ and, because only the locomotive is braking, $U_{1\text{-}2} = -\mu_k m_l g d_{\min}$, where m_l is the mass of the locomotive, and m_c is the mass of the cars. Solve Eq. (1) for $d_{\min}$ to obtain

$$d_{\min} = \frac{(m_l + m_c)v^2}{2\mu_k m_l g} = \boxed{14.8\,\text{km},}$$

where $m_l = 195{,}000\,\text{kg}$, $m_c = 10 \times 10^6\,\text{kg}$, $v = 120\,\text{km/h} = 33.33\,\text{m/s}$, $\mu_k = 0.2$, and $g = 9.81\,\text{m/s}^2$.

Problems 4.11 and 4.12

A classic car is driving down a 20° incline at 45 km/h when the brakes are applied. Treat the car as a particle, and neglect all forces except gravity and friction.

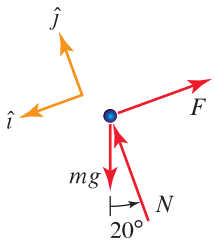


Problem 4.11 Determine the stopping distance if the tires slide and the coefficient of kinetic friction between the tires and the road is 0.7.

Problem 4.12 Determine the minimum stopping distance if the car is retrofitted with antilock brakes and the tires do not slide. Use 0.9 for the coefficient of static friction between the tires and the road.

Solution to 4.11

Let ① and ② be the positions occupied by the car when the brakes are first applied and when coming to a stop, respectively. The FBD on the right shows the forces acting between ① and ②. Applying the work-energy principle between ① and ②, we have



$$T_1 + (U_{1\text{-}2})_f + (U_{1\text{-}2})_g = T_2, \tag{1}$$

where $(U_{1\text{-}2})_f$ is the work done by friction and $(U_{1\text{-}2})_g$ is the work done by gravity. Summing forces in the direction perpendicular to the incline, we have

$$\sum F_y: \quad N - mg\cos 20^\circ = 0 \quad \Rightarrow \quad N = mg\cos 20^\circ. \tag{2}$$

If the tires slide, then the force of friction is given by the following relation:

$$F = \mu_k N = \mu_k mg\cos 20^\circ. \tag{3}$$

We now observe that the F is constant. Hence, letting d be the distance traveled by the car between ① and ②, the work of friction is given by

$$(U_{1\text{-}2})_f = -Fd = -\mu_k mgd\cos 20^\circ. \tag{4}$$

We now observe that gravity is also constant. Therefore, the work of gravity is given by the product of the component of the force $-mg\,\hat{\jmath}$ along the incline and the distance traveled, i.e.,

$$(U_{1\text{-}2})_g = mgd\sin 20^\circ. \tag{5}$$

The kinetic energies at ① and ② are given by, respectively,

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = 0. \tag{6}$$

Substituting Eqs. (4)–(6) into Eq. (1) and solving for d, we obtain

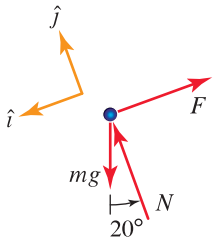
$$d = \frac{1}{2}\frac{v_1^2}{(\mu_k\cos 20^\circ - \sin 20^\circ)\,g}. \tag{7}$$

Next, recalling that we have $\mu_k = 0.7$, $v_1 = 45\,\text{km/h} = 12.50\,\text{m/s}$, and $g = 9.81\,\text{m/s}^2$, we can evaluate the above expression to obtain

$$\boxed{d = 25.2\,\text{m}.}$$

Solution to 4.12

Let ① and ② be the positions occupied by the car when the brakes are first applied and when coming to a stop, respectively. The FBD on the right shows the forces acting between ① and ②. Applying the work-energy principle between ① and ②, we have



$$T_1 + (U_{1\text{-}2})_f + (U_{1\text{-}2})_g = T_2, \tag{8}$$

where $(U_{1\text{-}2})_f$ is the work done by friction and $(U_{1\text{-}2})_g$ is the work done by gravity. Summing forces in the direction perpendicular to the incline, we have

$$\sum F_y: \quad N - mg\cos 20^\circ = 0 \quad \Rightarrow \quad N = mg\cos 20^\circ. \tag{9}$$

If the tires do not slide, then the minimum stopping distance is obtained by assuming that slip is impending, i.e.,

$$F = \mu_s N = \mu_s mg\cos 20^\circ. \tag{10}$$

We now observe that the F is constant. Hence, letting d be the distance traveled by the car between ① and ②, the work of friction is given by

$$(U_{1\text{-}2})_f = -Fd = -\mu_s mgd\cos 20^\circ. \tag{11}$$

We now observe that gravity is also constant. Therefore, the work of gravity is given by the product of the component of the force $-mg\,\hat{\jmath}$ along the incline and the distance traveled, i.e.,

$$(U_{1\text{-}2})_g = mgd\sin 20^\circ. \tag{12}$$

The kinetic energies at ① and ② are given by, respectively,

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = 0. \tag{13}$$

Substituting Eqs. (11)–(13) into Eq. (8) and solving for d, we obtain

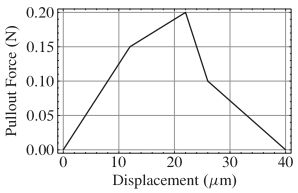
$$d = \frac{1}{2}\frac{v_1^2}{(\mu_s\cos 20^\circ - \sin 20^\circ)\,g}. \tag{14}$$

Next, recalling that we have $\mu_s = 0.9$, $v_1 = 45\,\text{km/h} = 12.50\,\text{m/s}$, and $g = 9.81\,\text{m/s}^2$, we can evaluate the above expression to obtain

$$\boxed{d = 15.8\,\text{m}.}$$

Problem 4.13

Many advanced materials consist of fibers (made of glass, Kevlar, carbon, etc.) placed within a matrix (such as epoxy, a titanium alloy, etc.). For these materials it is important to assess the bond strength between fibers and matrix, and this is often done via a *pullout test*, in which the tip of a fiber is exposed, the material sampled is properly clamped, and the fiber is pulled out of the matrix. The data collected often consists of a graph like the one shown, in which the force exerted on the fiber is recorded as a function of pullout displacement. With this in mind, the interface toughness assessment process may require a measure of the *energy* expended to pull out the fiber. Use the force-displacement graph shown, which is typical for a glass-fiber reinforced epoxy, to measure the total pullout energy. *Hint:* The work of the pullout force is given by the area under the curve.



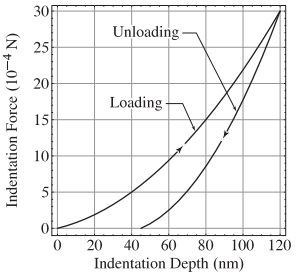
Solution

Calculate the area under the curve given by the force versus displacement plot:

$$\begin{aligned} U_{1\text{-}2} &= \int_{\mathcal{L}_{1\text{-}2}} F\,dx \\ &= \tfrac{1}{2}(12\,\mu\text{m})(0.15\,\text{N}) + (22-12)\left[0.15 + \tfrac{1}{2}(0.20-0.15)\right] \\ &\quad + (26-22)\left[0.10 + \tfrac{1}{2}(0.20-0.10)\right] + \tfrac{1}{2}(40-26)(0.1) \\ &= \boxed{3.95\times10^{-6}\,\text{J}.} \end{aligned}$$

Problem 4.14

Structural components subjected to large contact forces (e.g., brake disks) are often made of high-grade steel coated with a thin film of a very hard material such as diamond. A common test to assess the mechanical properties (hardness, elastic moduli, etc.) of the coating is the *nanoindentation test*, which, roughly speaking, consists of making a controlled dent in the film using a nail-like object, called an *indentor*. During the test one measures the force applied to the indentor and the indentation depth. In the graph shown, you can see the curves interpolating the loading and unloading data for a diamond film on steel. Since the unloading curve does not go back to the origin of the plot, the film is permanently deformed during the indentation process (i.e., an actual permanent dent remains). Determine the energy lost to permanent deformation if the indentation force is given by



$$F_I = \begin{cases} \frac{25}{4}x + \frac{5}{32}x^2 & \text{during loading,} \\ 300 - \frac{145}{6}x + \frac{7}{18}x^2 & \text{during unloading,} \end{cases}$$

where F_I is expressed in μN and x (indentation depth) in nm.

Solution

If ① corresponds to the beginning of the test, ② corresponds to the point of maximum indentation, and ③ corresponds to the end of the test (when the indentor is removed), it follows that the total energy that goes into permanently deforming the specimen is $U_{1\text{-}3}$, where

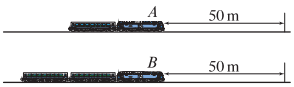
$$U_{1\text{-}3} = U_{1\text{-}2} + U_{2\text{-}3} = \int_{x_1}^{x_2} F_L \, dx + \int_{x_2}^{x_3} F_U \, dx, \tag{1}$$

and F_L is the loading force, F_U is the unloading force, x_1 equals 0 nm, x_2 equals 120 nm, and $x_3 = 45$ nm . Using Eq. (1), we see that

$$\begin{aligned} U_{1\text{-}3} &= \left(\frac{25}{4}\frac{1}{2}x^2 + \frac{5}{32}\frac{1}{3}x^3 \right)\Big|_{x_1}^{x_2} + \left(300x - \frac{145}{6}\frac{1}{2}x^2 + \frac{7}{18}\frac{1}{3}x^3 \right)\Big|_{x_2}^{x_3} \\ &= \frac{25}{8}\left(x_2^2 - x_1^2\right) + \frac{5}{96}\left(x_2^3 - x_1^3\right) + 300(x_3 - x_2) - \frac{145}{12}\left(x_3^2 - x_2^2\right) + \frac{7}{54}\left(x_3^3 - x_2^3\right) \\ &= 4.98\times10^4 \, \mu\text{N}\cdot\text{nm} = \boxed{4.98\times10^{-11} \text{ J.}} \end{aligned}$$

Problem 4.15

Two identical locomotives A and B are coupled with 1 and 2 passenger cars, respectively. Suppose that each passenger car is identical to the others in all respects. If each train (locomotive plus cars) starts from rest, each locomotive exerts the maximum tractive effort (traction force), and assuming that gravity and the tractive effort are the only relevant forces acting on the trains, which of the two trains will have greater kinetic energy after the locomotives have moved 50 m along a horizontal and rectilinear stretch?



Solution

The two trains will have exactly the same kinetic energy because the work done on each train is the same.

Problem 4.16



An F/A-18 Hornet takes off from an aircraft carrier, using two separate propulsion systems: its two jet engines and a steam-powered catapult. During launch, a fully loaded Hornet weighing about 50,000 lb goes from 0 mph (relative to the aircraft carrier) to 165 mph (measured relative to surface of the Earth) in a distance of 300 ft (measured relative to the aircraft carrier) while each of its two engines is at full power generating about 22,000 lb of thrust. Assuming that the aircraft carrier is traveling in the same direction as the takeoff direction and at a speed of 30 knot, determine:

(a) The total work done on the aircraft during launch.

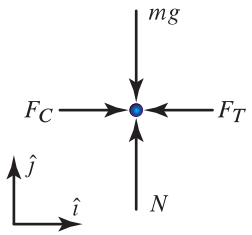
(b) The work done by the catapult on the aircraft during launch.

(c) The force exerted by the catapult on the aircraft.

In solving this problem, model the aircraft as a particle; assume that its trajectory is horizontal and that the catapult assists the aircraft the full 300 ft needed for takeoff; and finally, let all forces be constant and neglect air resistance and friction.

Solution

The FBD of the fighter jet is shown at the right where F_C is the force exerted by the catapult and F_T is the total thrust of the engines. The work-energy principle



$$T_1 + U_{1\text{-}2} = T_2, \tag{1}$$

where ① is the position of the plane at the beginning of take-off and ② is when the plane has traveled 300 ft relative to the carrier. In this problem the work done by non-conservative forces is accounted for in the $U_{1\text{-}2}$ term which becomes

$$U_{1\text{-}2} = (U_{1\text{-}2})_{\text{Engines}} + (U_{1\text{-}2})_{\text{Catapult}}\,, \tag{2}$$

and

$$(U_{1\text{-}2})_{\text{Engines}} = \int_{\mathscr{L}_{1\text{-}2}} F_C\, dx = F_C\,(x_2 - x_1)\,. \tag{3}$$

The kinetic energy of the plane at ① and ② is found to be

$$T_1 = \frac{1}{2} m v_1^2, \quad \text{and} \quad T_2 = \frac{1}{2} m v_2^2. \tag{4}$$

where v_1 and v_2 are the *inertial* speed of A at ① and ②, respectively. Recall that $v_2 = 165\,\text{mph} = 242\,\text{ft/s}$. Because the motion is completely in the positive x-direction, we have that

$$v_1 = v_{1/C} + v_C\,, \tag{5}$$

where $v_{1/C}$ is the velocity of A relative to the carrier at ①. Using the conversion factor 1 kt = 1.6878 ft/s we have,

$$v_1 = 0 + 30\,\text{kn} = 50.63\,\text{ft/s}. \tag{6}$$

Next, we need to determine x_1 and x_2. The value of x_1 can be taken to be equal to zero. With this in mind, we have $x_2 \neq d$ because the carrier is moving in the same direction as the plane. To compute x_2, observe that: the carrier is moving at a constant velocity, F_T and F_C have been assumed to be constant so that $a_A = \text{constant}$, $a_{A/C} = a_A$ because $a_C = 0$. Therefore, using Eq. (2.43) on page 59, we have

$$v_2^2 = v_1^2 + 2a_A\,(x_2 - x_1)\,, \quad \text{and,} \tag{7}$$

$$v_{2/C}^2 = v_{1/C}^2 + 2a_A d\,. \tag{8}$$

where,

$$v_{2/C} = v_2 - v_C = 191.4\,\text{ft/s}, \tag{9}$$

$$v_{1/C} = 0, \quad \text{with} \quad d = 300\,\text{ft}. \tag{10}$$

Thus,

$$a_A = 61.06\,\text{ft/s}^2, \tag{11}$$

and

$$x_2 = \frac{v_2^2 - v_1^2}{2a_A} = 458.6\,\text{ft} \tag{12}$$

Thus, the total work done on A is

$$U_{1\text{-}2} = T_2 - T_1 \quad \Rightarrow \quad U_{1\text{-}2} = \boxed{43.5\times10^6\,\text{ft}\cdot\text{lb}.} \tag{13}$$

The work done by the catapult is found by subtracting the work done by the thrust from the engines given by $F_T x_2 = 20.18\times10^6\,\text{ft}\cdot\text{lb}$. Now, we can find

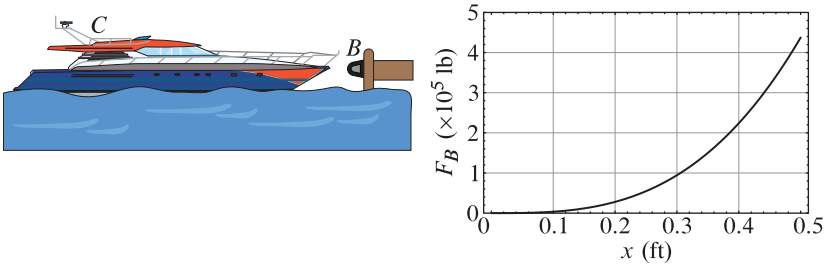
$$(U_{1\text{-}2})_{\text{Catapult}} = (U_{1\text{-}2}) - (U_{1\text{-}2})_{\text{Engines}} = \boxed{23.3\times10^6\,\text{ft}\cdot\text{lb}.} \tag{14}$$

Thus, the force exerted by the catapult is

$$F_C = \frac{(U_{1\text{-}2})_{\text{Catapult}]}}{x_2} = \boxed{50.8\times10^3\,\text{lb}.} \tag{15}$$

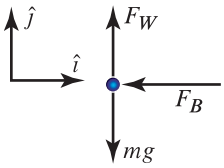
Problem 4.17

Rubber bumpers are commonly used in marine applications to keep boats and ships from getting damaged by docks. Treating the boat C as a particle, neglecting its vertical motion, and neglecting the drag force between the water and the boat C, what is the maximum speed of the boat at impact with the bumper B so that the deflection of the bumper is limited to 6 in.? The weight of the boat is 70,000 lb, and the force compression profile for the rubber bumper is given by $F_B = \beta x^3$, where $\beta = 3.5\times10^6$ lb/ft^3 and x is the compression of the bumper.



Solution

Consider ① to be at the moment of impact between the boat and the bumper at B and consider ② to be at the moment when the bumper has deflected a distance $d = 6$ in.. Work energy principle:



$$T_1 + U_{1\text{-}2} = T_2, \tag{1}$$

where,

$$U_{1\text{-}2} = \int_{\mathscr{L}_{1\text{-}2}} \vec{F}_B \cdot d\vec{r} = \int_0^d -\beta x^3\,\hat{\imath}\cdot dx\,\hat{\imath} = -\beta\,\frac{x^4}{4}\bigg|_0^d = -\frac{1}{4}\beta d^4. \tag{2}$$

The initial and final kinetic energy is

$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(W/g)v_1^2, \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2 = 0\,\text{ft}\cdot\text{lb}, \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1), we get

$$\tfrac{1}{2}(W/g)v_1^2 - \frac{1}{4}\beta d^4 = 0 \quad \Rightarrow \quad \boxed{v_1 = d^2\sqrt{\frac{g\beta}{2W}} = 7.09\,\text{ft/s}.} \tag{4}$$

Where $\beta = 3.5\times10^6$ lb/ft^3, $W = 70{,}000$ lb, $g = 32.2$ ft/s^2, and $d = 6$ in. $= 0.5$ ft.

Problem 4.18

Two identical cars travel at a speed of 60 mph, one along a newly asphalt-paved straight and horizontal road and the other on a straight and horizontal dirt road. If brakes are applied and if the second car slips during the braking process, what difference will there be in the amount of work done to stop each car?

Solution

The work done to stop each car is identical in the two cases.

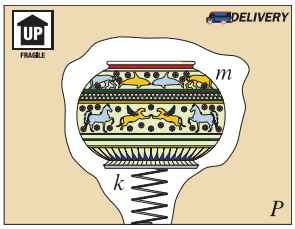
The reason for this is that, applying the work-energy principle, we have

$$T_2 - T_1 = U_{1\text{-}2}.$$

Therefore, so long as the initial and final values of kinetic energy for the two cars are the same, the corresponding values of $U_{1\text{-}2}$ will also be the same.

Problems 4.19 and 4.20

Packages for transporting delicate items (e.g., a laptop or glass) are designed to "absorb" some of the energy of the impact in order to protect their contents. These energy absorbers can get pretty complicated (e.g., the mechanics of Styrofoam peanuts is not easy), but we can begin to understand how they work by modeling them as a linear elastic spring of constant k that is placed between the contents (an expensive vase) of mass m and the package P. Assume that the vase weighs 6 lb and that the box is dropped from a height of 5 ft. Treat the vase as a particle, and neglect all forces except for gravity and the spring force.



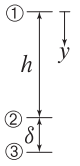
Problem 4.19 Determine the maximum displacement of the vase relative to the box and the maximum force on the vase if $k = 264\,\text{lb/ft}$.

Problem 4.20 Plot the maximum displacement of the vase relative to the box and the maximum force on the vase as a function of the linear elastic spring constant k. What do these plots tell you about the problem you would encounter in trying to minimize the force on the vase?

Solution to 4.19

Let ① be where the vase is at a height $h = 5$ ft and ③ be where the vase comes to rest after compressing the spring by an amount δ. We apply the work-energy principle between ① and ③ to get

$$T_1 + U_{1\text{-}3} = T_3. \tag{1}$$

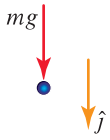


The work done by the forces acting on the vase can be calculated as

$$U_{1\text{-}3} = U_{1\text{-}2} + U_{2\text{-}3}, \tag{2}$$

where ② is where the package containing the vase just hits the ground. The work done by all the forces acting on the vase between ① and ② is given by

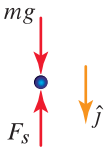
$$U_{1\text{-}2} = \int_0^h mg\,dy = mgh. \tag{3}$$



FBD of the vase between ① and ②

The work done by all forces acting on the vase between ② and ③ is given by

$$U_{2\text{-}3} = \int_h^{h+\delta} [mg - k(y-h)]\,dy = mg\delta - \tfrac{1}{2}k\delta^2. \tag{4}$$



FBD of the vase between ② and ③

Since the package is dropped from rest and is again at rest at ③, we have

$$T_1 = 0 \quad \text{and} \quad T_3 = 0. \tag{5}$$

Substituting Eqs. (2) and (5) into Eq. (1) and using Eqs. (3) and (4), we get

$$mg(h+\delta) = \tfrac{1}{2}k\delta^2 \quad \Rightarrow \quad \delta = \frac{mg + \sqrt{mg(mg+2hk)}}{k} = 0.5000\,\text{ft}, \tag{6}$$

where we have discarded the root with the minus sign in fron of the square root to satisfy the condition $\delta > 0$, and where we have used the following numerical data $mg = 6\,\text{lb}$, $h = 5\,\text{ft}$, and $k = 264\,\text{lb/ft}$. Expressing our answer to 3 significant digits, we have

$$\boxed{\delta = 0.500\,\text{ft}.}$$

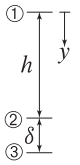
Using the result in Eq. (6) to determine the maximum force on the vase due to the sping, we have

$$\boxed{(F_s)_{\text{max}} = k\delta = 132\,\text{lb}.}$$

Solution to 4.20

Let ① be where the vase is at a height $h = 5$ ft and ③ be where the vase comes to rest after compressing the spring by an amount δ. We apply the work-energy principle between ① and ③ to get

$$T_1 + U_{1\text{-}3} = T_3. \tag{7}$$

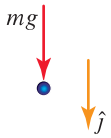


The work done by the forces acting on the vase can be calculated as

$$U_{1\text{-}3} = U_{1\text{-}2} + U_{2\text{-}3}, \tag{8}$$

where ② is where the package containing the vase just hits the ground. The work done by all the forces acting on the vase between ① and ② is given by

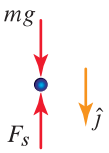
$$U_{1\text{-}2} = \int_0^h mg\, dy = mgh. \tag{9}$$



FBD of the vase between ① and ②

The work done by all forces acting on the vase between ② and ③ is given by

$$U_{2\text{-}3} = \int_h^{h+\delta} [mg - k(y-h)]\, dy = mg\delta - \tfrac{1}{2}k\delta^2. \tag{10}$$



FBD of the vase between ② and ③

Since the package is dropped from rest and is again at rest at ③, we have

$$T_1 = 0 \quad \text{and} \quad T_3 = 0. \tag{11}$$

Substituting Eqs. (8) and (11) into Eq. (7) and using Eqs. (9) and (10), we get

$$mg(h+\delta) = \tfrac{1}{2}k\delta^2 \quad \Rightarrow \quad \delta = \frac{mg + \sqrt{mg(mg+2hk)}}{k}. \tag{12}$$

where we have discarded the root with the minus sign in fron of the square root to satisfy the condition $\delta > 0$.

Substituting the values $h = 5$ ft and $m = 0.1863$ slug in the expression for δ in Eq. (12), we get

$$\delta = \left(\frac{6 + \sqrt{36 + 60k}}{k}\right) \text{ft}. \tag{13}$$

The force on the vase due to the spring is given by

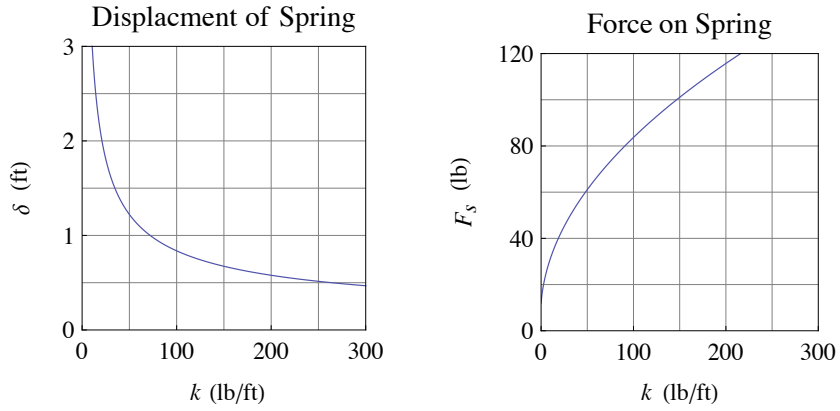
$$F_s = k\delta \quad \Rightarrow \quad F_s = \left(6 + \sqrt{36 + 60k}\right) \text{lb}. \tag{14}$$

The expressions for δ and F_s can be plotted with any appropriate mathematical software. We have used *Mathematica* with the following code:

```
Plot[{(6 + Sqrt[60 k + 36])/k}, {k, 0, 300}, PlotRange → {{0, 300}, {0, 3}}, PlotPoints → 250,
 GridLines → {Automatic, Automatic}, Frame → True,
 FrameTicks → {{{0, 1, 2, 3}, None}, {{0, 100, 200, 300}, None}}, AspectRatio → 1,
 PlotLabel → "Displacment of Spring", FrameLabel → {"k (lb/ft) ", "δ  (ft)"}]
```

```
Plot[{6 + Sqrt[60 k + 36]}, {k, 0, 300}, PlotRange → {{0, 300}, {0, 120}}, PlotPoints → 250,
 GridLines → {Automatic, Automatic}, Frame → True,
 FrameTicks → {{{0, 40, 80, 120}, None}, {{0, 100, 200, 300}, None}}, AspectRatio → 1,
 PlotLabel → "Force on Spring", FrameLabel → {"k  (lb/ft)", "F_s  (lb)"}]
```

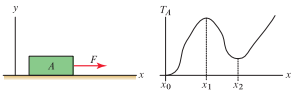
The code above yields the following two plots:



As can be seen, as the force is reduced by reducing the spring constant, the displacement becomes very large. Since the size of the package containing the vase has to accommodate the for the displacement of the package contained within, minimizing the force on the vase would require a large package.

Problem 4.21

A block A moves horizontally under the action of a force F whose line of action is parallel to the motion. If the kinetic energy of A as a function of x is that shown (x_1 and x_2 are extrema for T_A), what can you say about the *sign* of F for $x_0 < x < x_1$ and $x_1 < x < x_2$?

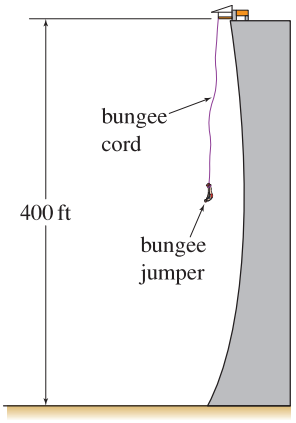


Solution

Block A moves in the positive x direction and the force F is always parallel to the block's motion. For $x_0 < x < x_1$ the block's kinetic energy is increasing so F acts in the *positive* x direction. For $x_1 < x < x_2$ the block's kinetic energy is decreasing so F acts in the *negative* x direction.

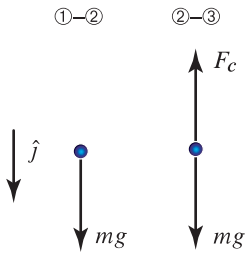
Problem 4.22

While the stiffness of an elastic cord can be quite constant (i.e., the force versus displacement curve is a straight line) over a large range of stretch, as a bungee cord is stretched, it softens; that is, the cord tends to get less stiff as it gets longer. Assuming a softening force-displacement relation of the form $k\delta - \beta\delta^3$, where δ (measured in ft) is the displacement of the cord from its unstretched length, considering a bungee cord whose unstretched length is 150 ft, and letting $k = 2.58$ lb/ft, determine the value of the constant β such that a bungee jumper weighing 170 lb and starting from rest gets to the bottom of a 400 ft tower with zero speed.



Solution

Let ① be the top of the tower where $y = 0$, let ② be the point at which the bungee cord becomes taut where $y = L_0 = 150$ ft, and let ③ be the base of the tower where $y = h = 400$ ft. From the FBD between ①–②, the only force acting on the jumper is gravity, and from the FBD between ②–③ both gravity and the force from the cord $F_c = k\delta - \beta\delta^3$ act on the jumper. Work-energy principle:



$$T_1 + U_{1\text{-}3} = T_3, \tag{1}$$

where $T_1 = 0$ because the jumper falls from rest, and $T_3 = 0$ because the jumper reaches the base with zero speed. The work done on the jumper $U_{1\text{-}3} = (U_{1\text{-}3})_g + (U_{2\text{-}3})_c$ where the work done by gravity is

$$(U_{1\text{-}3})_g = \int_0^h mg\hat{\jmath}\cdot\hat{\jmath}\,dy = \int_0^h mg\,dy = mgh, \tag{2}$$

and the work done on the jumper by the bungee cord is

$$\begin{aligned}(U_{2\text{-}3})_c = \int_{L_0}^h -\left(k\delta - \beta\delta^3\right)\hat{\jmath}\cdot\hat{\jmath}\,dy &= \int_{L_0}^h -k(y-L_0) + \beta(y-L_0)^3\,dy \\ &= -\frac{k}{2}(h-L_0)^2 + \frac{\beta}{4}(h-L_0)^4,\end{aligned} \tag{3}$$

where $\delta = y - L_0$ has been substituted. From Eq. (1) we have $U_{1\text{-}3} = 0$, and since $U_{1\text{-}3} = (U_{1\text{-}3})_g + (U_{2\text{-}3})_c$, we have

$$mgh - \frac{k}{2}(h-L_0)^2 + \frac{\beta}{4}(h-L_0)^4 = 0, \tag{4}$$

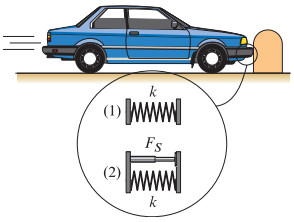
which we solve for β to obtain

$$\beta = \frac{2k\,(h-L_0)^2 - 4mgh}{(h-L_0)^4} = \boxed{1.29\times10^{-5}\ \text{lb/ft}^3,} \tag{5}$$

where $k = 2.58$ lb/ft, $mg = 170$ lb, $h = 400$ ft, and $L_0 = 150$ ft.

Problems 4.23 through 4.25

Car bumpers are designed to limit the extent of damage to the car in the case of low-velocity collisions. Consider a 1420 kg passenger car impacting a concrete barrier while traveling at a speed of 5.0 km/h. Model the car as a particle and consider two bumper models: (1) a simple linear spring with constant k and (2) a linear spring of constant k in parallel with a shock absorbing unit generating a nearly constant force $F_S = 2000$ N over 10 cm.



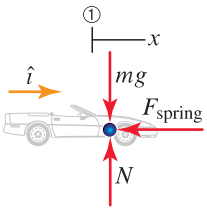
Problem 4.23 If the bumper is of type (1) and if $k = 9\times10^4$ N/m, find the spring compression (distance) necessary to stop the car.

Problem 4.24 If the bumper is of type (1), find the value of k necessary to stop the car when the bumper is compressed 10 cm.

Problem 4.25 If the bumper is of type (2), find the value of k necessary to stop the car when the bumper is compressed 10 cm.

Solution to 4.23

Referring to the FBD on the right, we model the car as a particle and, given that the motion of the car is only in the horizontal direction, we use an x axis with origin at the position of the car when it first impacts the barrier. We denote by ① the position of the car when it first impacts the barrier. We denote by ② the position of the car when it stops moving. Applying the work-energy principle between ① and ②, we have



$$T_1 + U_{1\text{-}2} = T_2, \tag{1}$$

where, recalling that the car comes to a stop in ②, we have

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = 0, \tag{2}$$

where $v_1 = 5.0$ km/h $= 1.389$ m/s is the initial speed of the car. We now observe that the car moves only in the horizontal direction. Therefore the term $U_{1\text{-}2}$ is the work done by the spring force in going from ① to ②. Letting δ_s denote the distance traveled by the car in going from ① to ② and recalling that the spring is uncompressed in ①, we compute the work in question as follows:

$$U_{1\text{-}2} = \int_{\mathscr{L}_{1\text{-}2}} \vec{F}_{\text{spring}} \cdot d\vec{r} = \int_0^{\delta_s} (-kx)\,\hat{\imath} \cdot (dx)\,\hat{\imath} = \int_0^{\delta_s} -kx\,dx = -\tfrac{1}{2}k\delta_s^2. \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1), we have

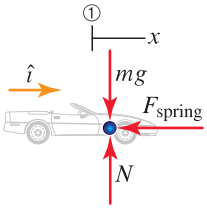
$$\tfrac{1}{2}mv_1^2 - \tfrac{1}{2}k\delta_s^2 = 0, \tag{4}$$

which we can solve for δ_s to obtain

$$\delta_s = \sqrt{\frac{mv_1^2}{k}} \quad \Rightarrow \quad \boxed{\delta_s = 0.174\,\text{m},}$$

where we have used the following numerical data: $m = 1420$ kg, $k = 9\times10^4$ N/m, and $v_1 = 5$ km/h $= 1.389$ m/s.

Solution to 4.24



Referring to the FBD on the right, we model the car as a particle and, given that the motion of the car is only in the horizontal direction, we use an x axis with origin at the position of the car when it first impacts the barrier. We denote by ① the position of the car when it first impacts the barrier. We denote by ② the position of the car when it stops moving. Applying the work-energy principle between ① and ②, we have

$$T_1 + U_{1\text{-}2} = T_2, \tag{5}$$

where, recalling that the car comes to a stop in ②, we have

$$T_1 = \tfrac{1}{2} m v_1^2 \quad \text{and} \quad T_2 = 0, \tag{6}$$

where $v_1 = 5.0\,\text{km/h} = 1.389\,\text{m/s}$ is the initial speed of the car. We now observe that the car moves only in the horizontal direction. Therefore the term $U_{1\text{-}2}$ is the work done by the spring force in going from ① to ②. Letting δ_s denote the distance traveled by the car in going from ① to ② and recalling that the spring is uncompressed in ①, we compute the work in question as follows:

$$U_{1\text{-}2} = \int_{\mathscr{L}_{1\text{-}2}} \vec{F}_{\text{spring}} \cdot d\vec{r} = \int_0^{\delta_s} (-kx)\,\hat{\imath} \cdot (dx)\,\hat{\imath} = \int_0^{\delta_s} -kx\,dx = -\tfrac{1}{2} k \delta_s^2. \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (5), we have

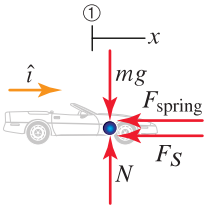
$$\tfrac{1}{2} m v_1^2 - \tfrac{1}{2} k \delta_s^2 = 0, \tag{8}$$

which we can solved for k to obtain

$$k = \frac{m v_1^2}{\delta_s^2} = 2.740 \times 10^5\,\text{N/m} \quad \Rightarrow \quad \boxed{k = 2.74 \times 10^5\,\text{N/m},}$$

where we have used the following numerical data: $m = 1420\,\text{kg}$, $v_1 = 5\,\text{km/h} = 1.389\,\text{m/s}$, and $\delta_s = 10\,\text{cm} = 0.1000\,\text{m}$.

Solution to 4.25



Referring to the FBD on the right, we model the car as a particle and, given that the motion of the car is only in the horizontal direction, we use an x axis with origin at the position of the car when it first impacts the barrier. We denote by ① the position of the car when it first impacts the barrier. We denote by ② the position of the car when it stops moving. Applying the work-energy principle between ① and ②, we have

$$T_1 + U_{1\text{-}2} = T_2, \tag{9}$$

where, recalling that the car comes to a stop in ②, we have

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = 0, \tag{10}$$

where $v_1 = 5.0\,\text{km/h} = 1.389\,\text{m/s}$ is the initial speed of the car. We now observe that the car moves only in the horizontal direction. Therefore the term $U_{1\text{-}2}$ is the work done by the spring force and by the force provided by the shock absorber in going from ① to ②, i.e.,

$$U_{1\text{-}2} = (U_{1\text{-}2})_{\text{spring}} + (U_{1\text{-}2})_{\text{shock}}. \tag{11}$$

Letting δ_s denote the distance traveled by the car in going from ① to ② and recalling that the spring is uncompressed in ①, we compute the work of the spring question as follows:

$$U_{1\text{-}2} = \int_{\mathscr{L}_{1\text{-}2}} \vec{F}_{\text{spring}} \cdot d\vec{r} = \int_0^{\delta_s} (-kx)\,\hat{\imath} \cdot (dx)\,\hat{\imath} = \int_0^{\delta_s} -kx\,dx = -\tfrac{1}{2}k\delta_s^2. \tag{12}$$

Recalling that the force provided by the shock absorber is assumed to be constant throughout the motion of the car, we have

$$(U_{1\text{-}2})_{\text{shock}} = \int_{\mathscr{L}_{1\text{-}2}} \vec{F}_S \cdot d\vec{r} = \int_0^{\delta_s} (-F_S)\,\hat{\imath} \cdot (dx)\,\hat{\imath} = \int_0^{\delta_s} -F_S\,dx = -F_S\delta_s. \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (11), we have

$$U_{1\text{-}2} = -\tfrac{1}{2}k\delta_s^2 - F_S\delta_s. \tag{14}$$

Substituting Eqs. (10) and (14) into Eq. (9), we have

$$\tfrac{1}{2}mv_1^2 - \tfrac{1}{2}k\delta_s^2 - F_S\delta_s = 0, \tag{15}$$

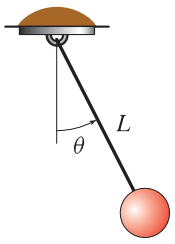
which we can solve for k to obtain

$$k = \frac{mv_1^2}{\delta_s^2} - \frac{2F_S}{\delta_s} \quad \Rightarrow \quad \boxed{k = 2.34\times10^5\,\text{N/m},}$$

where we have used the following numerical data: $m = 1420\,\text{kg}$, $v_1 = 5\,\text{km/h} = 1.389\,\text{m/s}$, $F_S = 2000\,\text{N}$, and $\delta_s = 10\,\text{cm} = 0.1000\,\text{m}$.

Problem 4.26

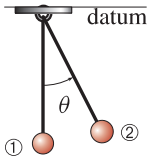
The pendulum shown is put in motion with a speed v_0 when $\theta = 0°$. Letting $L = 2\,\text{ft}$, determine v_0 if the pendulum first comes to a stop at $\theta = 47°$.



Solution

All forces doing work are conservative. Work-energy principle:

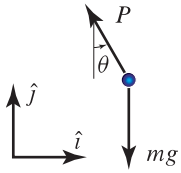
$$T_1 + V_1 = T_2 + V_2 \qquad (1)$$



Kinetic energies: $T_1 = \frac{1}{2}mv_1^2$ and $T_2 = 0$.

Force laws:

$$V_1 = -mgL \quad \text{and} \quad V_2 = -mgL\cos\theta$$

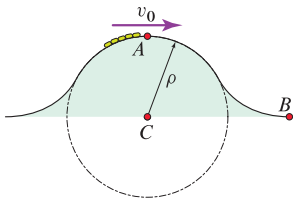


Eq. (1) becomes:

$$\tfrac{1}{2}mv_0^2 - mgL = -mgL\cos\theta \quad \Rightarrow \quad \boxed{v_0 = \sqrt{2gL\,(1-\cos\theta)} = 6.40\,\text{ft/s}.}$$

Problem 4.27

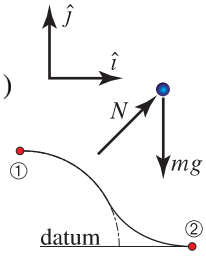
Point A is the highest point along the roller coaster ride section shown. The inscribed circle at A has radius $\rho = 25\,\text{m}$ and center at C. If point B is on a horizontal line going through C and if the roller coaster has a speed $v_0 = 45\,\text{km/h}$ at A, neglecting friction and air drag and treating the roller coaster as a particle, determine the speed of the roller coaster at B.



Solution

All forces doing work are conservative. Work-energy principle:

$$T_1 + V_1 = T_2 + V_2. \qquad (1)$$



Kinetic energies: $T_1 = \frac{1}{2}mv_0^2$ and $T_2 = \frac{1}{2}mv_2^2$.

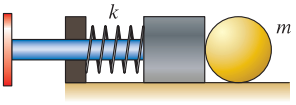
Force laws: $V_1 = mg\rho$ and $V_2 = 0$.

Eq. (1) becomes

$$\tfrac{1}{2}mv_0^2 + mg\rho = \tfrac{1}{2}mv_2^2 \quad \Rightarrow \quad \boxed{v_2 = \sqrt{v_0^2 + 2g\rho} = 25.4\,\text{m/s}.}$$

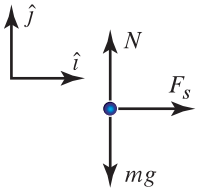
Problem 4.28

Assuming that the plunger of a pinball machine has negligible mass and that friction is negligible, determine the spring constant k such that a 2.85 oz ball is released with a speed $v = 15\,\text{ft/s}$, after pulling back the plunger 2 in. from its rest position, i.e., from the position in which the spring is uncompressed.



Solution

Let ① be when the spring is compressed and ② be when the ball is released. All forces doing work are conservative. Work-energy principle:



$$T_1 + V_1 = T_2 + V_2. \tag{1}$$

Kinetic energies: $T_1 = 0$ and $T_2 = \frac{1}{2}mv^2$.

Force laws

$$V_1 = \tfrac{1}{2}k\delta^2 \quad \text{and} \quad V_2 = 0.$$

Eq. (1) becomes

$$\tfrac{1}{2}k\delta^2 = \tfrac{1}{2}mv^2 \quad \Rightarrow \quad \boxed{k = \frac{mv^2}{\delta^2} = 44.8\,\text{lb/ft},}$$

where $\delta = 2\,\text{in.}$, $m = 5.532\times10^{-3}$ slug, and $v = 15\,\text{ft/s}$.

Problems 4.29 and 4.30



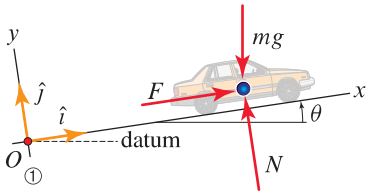
Consider a 3300 lb car whose speed is increased by 35 mph over a distance of 200 ft while traveling up a rectilinear incline with a 15% grade. Model the car as a particle, assume that the tires do not slip, and neglect *all* sources of frictional losses and drag.

Problem 4.29 Determine the work done on the car by the engine if the car starts from rest.

Problem 4.30 Determine the work done on the car by the engine if the car has an initial speed of 30 mph.

Solution to 4.29

Let ① be the initial position of the car and ② be the position of the car when its speed is increased by 35 mph. Referring to the FBD shown on the right and applying the work-energy principle between ① and ②, we have



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{np}} = T_2 + V_2, \qquad (1)$$

where, recalling that the car is at rest in ①, we have

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2, \qquad (2)$$

with $v_2 = 35$ mph. We will account for the work of gravity via its potential energy. Hence, choosing the datum at ① and letting $\ell = 200$ ft denote the distance covered by the car between ① and ②, we have

$$V_1 = 0 \quad \text{and} \quad V_2 = mg\ell \sin\theta, \qquad (3)$$

where $\theta = \tan^{-1}(15/100) = 8.531°$. Because we assume that there are no frictional losses of any kind, the work of the engine is equal to the the work of the force F propelling the car in the direction of motion. We account for this work as follows:

$$(U_{1\text{-}2})_{\text{np}} = (U_{1\text{-}2})_{\text{Engine}}. \qquad (4)$$

Substituting Eqs. (2)–(4) into Eq. (1), we have

$$(U_{1\text{-}2})_{\text{Engine}} = \tfrac{1}{2}mv_2^2 + mg\ell \sin\theta, \qquad (5)$$

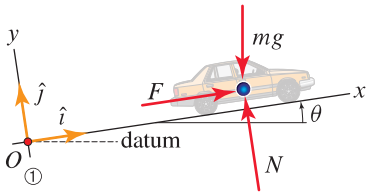
which can be evaluated to obtain

$$\boxed{(U_{1\text{-}2})_{\text{Engine}} = 2.33\times10^5 \text{ ft·lb},} \qquad (6)$$

where we have used the following numerical values: $m = (3300\,\text{lb})/(32.2\,\text{ft/s}^2) = 102.5$ slug, $v_2 = 35$ mph $= 51.33$ ft/s, $\theta = 8.531°$, and $\ell = 200$ ft.

Solution to 4.30

Let ① be the initial position of the car and ② be the position of the car when its speed is increased by 35 mph. Referring to the FBD shown on the right and applying the work-energy principle between ① and ②, we have



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{np}} = T_2 + V_2, \tag{7}$$

where

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2, \tag{8}$$

with $v_1 = 30\,\text{mph}$ and $v_2 = (30 + 35)\,\text{mph} = 65\,\text{mph}$. We will account for the work of gravity via its potential energy. Hence, choosing the datum at ① and letting $\ell = 200\,\text{ft}$ denote the distance covered by the car between ① and ②, we have

$$V_1 = 0 \quad \text{and} \quad V_2 = mg\ell\sin\theta, \tag{9}$$

where $\theta = \tan^{-1}(15/100) = 8.531°$. Because we assume that there are no frictional losses of any kind, the work of the engine is equal to the the work of the force F propelling the car in the direction of motion. We account for this work as follows:

$$(U_{1\text{-}2})_{\text{np}} = (U_{1\text{-}2})_{\text{Engine}}. \tag{10}$$

Substituting Eqs. (8)–(10) into Eq. (7), we have

$$\tfrac{1}{2}mv_1^2 + (U_{1\text{-}2})_{\text{Engine}} = \tfrac{1}{2}mv_2^2 + mg\ell\sin\theta \quad \Rightarrow \quad (U_{1\text{-}2})_{\text{Engine}} = \tfrac{1}{2}m\left(v_2^2 - v_1^2\right) + mg\ell\sin\theta, \tag{11}$$

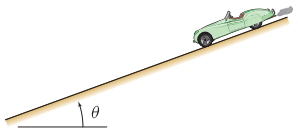
which can be evaluated to obtain

$$\boxed{(U_{1\text{-}2})_{\text{Engine}} = 4.64\times10^5\,\text{ft·lb},} \tag{12}$$

where we have used the following numerical values: $m = (3300\,\text{lb})/(32.2\,\text{ft/s}^2) = 102.5\,\text{slug}$, $v_1 = 30\,\text{mph} = 44.00\,\text{ft/s}$, $v_2 = 65\,\text{mph} = 95.33\,\text{ft/s}$, $\theta = 8.531°$, and $\ell = 200\,\text{ft}$.

Problem 4.31

A classic car is driving down an incline at 60 km/h when its brakes are applied. Treating the car as a particle, neglecting all forces except gravity and friction, and assuming that the tires slip, determine the coefficient of kinetic friction if the car comes to a stop in 55 m and $\theta = 20°$.

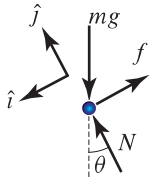


Solution

Taking ① to be where the brakes are applied and ② to be where the car comes to a stop, we apply the work-energy principle

$$T_1 + V_1 + (U_{1\text{-}2})_{\text{np}} = T_2 + V_2. \tag{1}$$

Referring to the FBD, the Newton-Euler equation in the y direction is given by



$$\sum F_y = ma_y: \quad N - mg\cos\theta = 0 \quad \Rightarrow \quad N = mg\cos\theta, \tag{2}$$

Taking the datum line to be at ②, the potential energies are given by

$$V_1 = mg\ell\sin\theta \quad \text{and} \quad V_2 = 0, \tag{3}$$

where ℓ is the distance travelled by the car along the incline before it comes to a stop. Since the tires slip, the friction force $f = \mu_k N$, where μ_k is the kinetic friction coefficient. Using the expression for N from Eq. (2), the work done by friction can then be calculated as

$$(U_{1\text{-}2})_{\text{np}} = \int_{\mathcal{L}_{1\text{-}2}} \vec{f}\cdot d\vec{r} = -\int_0^{\ell} f\,dx = -\int_0^{\ell} \mu_k mg\cos\theta\,dx = -\ell\mu_k mg\cos\theta, \tag{4}$$

where $(U_{1\text{-}2})_{\text{np}}$ is negative because f acts opposite to the displacement of the car. The kinetic energies are

$$T_1 = \tfrac{1}{2}mv^2 \quad \text{and} \quad T_2 = 0, \tag{5}$$

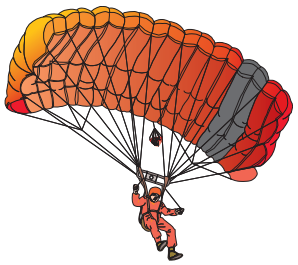
Substituting Eqs. (3)–(5) into Eq. (1), we obtain

$$\tfrac{1}{2}mv^2 + mg\ell\sin\theta - \ell\mu_k mg\cos\theta = 0 \quad \Rightarrow \quad \boxed{\mu_k = \frac{v^2 + 2g\ell\sin\theta}{2\ell g\cos\theta} = 0.638,} \tag{6}$$

where $v = 16.67$ m/s, $\ell = 55$ m, $g = 9.81$ m/s^2, $\theta = 20°$.

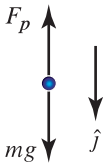
Problem 4.32

A 75 kg skydiver is falling at a speed of 250 km/h when, at a height of 245 m, the parachute is deployed, allowing the skydiver to land at a speed of 4 m/s. Modeling the skydiver as a particle and assuming that the skydiver follows a perfectly vertical trajectory, determine the average force exerted by the parachute from the moment of deployment until landing.



Solution

Let ① be where the skydiver has a velocity $v_1 = 250\,\text{km/h}$ and ② to be where the skydiver lands with a velocity $v_2 = 4\,\text{m/s}$. Work-energy principle:



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{np}} = T_2 + V_2. \tag{1}$$

Taking the datum line to be at ②, the potential energies are given by

$$V_1 = mgh \quad \text{and} \quad V_2 = 0, \tag{2}$$

where h is the starting height of the skydiver. Assuming that the skydiver's trajectory is perfectly vertical, the work done by the force exerted by the parachute, F_p, is given by

$$(U_{1\text{-}2})_{\text{np}} = \int_{\mathscr{L}_{1\text{-}2}} \vec{F}_p \cdot d\vec{r} = \int_0^h -F_p\,\hat{\jmath} \cdot dy\,\hat{\jmath} = -\int_0^h F_p\,dy = -F_{\text{avg}}h \tag{3}$$

The kinetic energies are given by

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2, \tag{4}$$

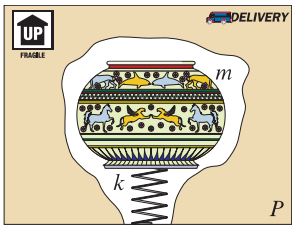
Substituting Eqs. (2)–(4) into Eq. (1), we get

$$\tfrac{1}{2}mv_1^2 + mgh - F_{\text{avg}}h = \tfrac{1}{2}mv_2^2 \quad \Rightarrow \quad \boxed{F_{\text{avg}} = mg + \frac{m}{2h}(v_1^2 - v_2^2) = 1.47\,\text{kN}.} \tag{5}$$

where $m = 75\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $v_1 = 250\,\text{km/h} = 69.44\,\text{m/s}$, $v_2 = 4\,\text{m/s}$, and $h = 245\,\text{m}$.

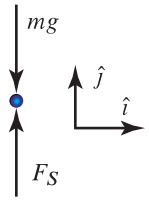
Problem 4.33

Packages for transporting delicate items (e.g., a laptop or glass) are designed to "absorb" some of the energy of the impact in order to protect their contents. These energy absorbers can get very complicated (e.g., the mechanics of Styrofoam peanuts can be complex), but we can begin to understand how they work by modeling them as a linear elastic spring of constant k that is placed between the contents (an expensive vase) of mass m and the package P. Assume that the vase's mass is 3 kg and that the box is dropped from rest from a height of 1.5 m. Treating the vase as a particle and neglecting all forces except for gravity and the spring force, determine the value of the spring constant k so that the maximum displacement of the vase relative to the box is 0.15 m. Assume that the spring relaxes after the box is dropped and that it does not oscillate.



Solution

Define ① to be the initial height of the vase $h + L_0$ where $h = 1.5$ m and L_0 is the unstretched length of the spring. Define ② to be when the vase comes to a complete stop for the first time after the spring is compressed by length δ. The kinetic and potential energies of the vase for ① and ② are



$$T_1 = 0, \quad \text{and} \quad T_2 = 0, \tag{1}$$

$$V_1 = mg(h + L_0), \quad \text{and} \quad V_2 = mg(L_0 - \delta) + \tfrac{1}{2}k\delta^2 \tag{2}$$

Plug Eqs. (1) and (2) into the conservation of mechanical energy equation,

$$T_1 + V_1 = V_2 + T_2 \quad \Rightarrow \quad mg(h + L_0) = \tfrac{1}{2}k\delta^2 + mg(L_0 - \delta) \tag{3}$$

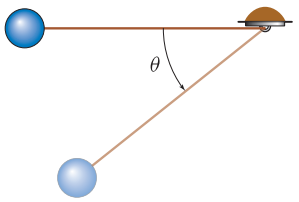
Solve for k

$$k = \frac{2mg\,(h + \delta)}{\delta^2} = \boxed{4320\,\text{N/m.}} \tag{4}$$

where $m = 3$ kg, $g = 9.81$ m/s^2, $h = 1.5$ m, and $\delta = 0.15$ m

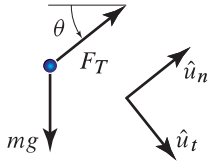
Problem 4.34

The pendulum is released from rest when $\theta = 0°$. If the string holding the pendulum bob breaks when the tension is twice the weight of the bob, at what angle does the string break? Treat the pendulum as a particle, ignore air resistance, and let the string be inextensible and massless.



Solution

Balance principles:



$$\sum F_n: \quad F_T - mg\sin\theta = m\frac{v^2}{\rho}, \tag{1}$$

$$\sum F_t: \quad mg\cos\theta = m\dot{v}. \tag{2}$$

Use the work-energy principle to obtain $v^2(\theta)$. Let ① be the initial position of the bob and ② represent the position of the bob at any value of θ. The kinetic energies are:

$$T_1 = 0, \quad \text{and} \quad T_2 = \tfrac{1}{2}mv^2. \tag{3}$$

If the datum line is at ① then

$$V_1 = 0, \quad \text{and} \quad V_2 = -mg\rho\sin\theta. \tag{4}$$

Work-energy principle:

$$T_1 + V_1 = T_2 + V_2 \quad \Rightarrow \quad 0 = -mg\rho\sin\theta + \tfrac{1}{2}mv^2 \quad \Rightarrow \quad v^2 = 2g\rho\sin\theta. \tag{5}$$

Using the expression for velocity from Eq. (5) and Eq. (1) solved for F_T, we find that

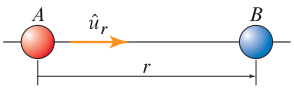
$$F_T = 2mg\sin\theta + mg\sin\theta. \tag{6}$$

Now we simply replace F_T with $2mg$

$$2mg = 2mg\sin\theta + mg\sin\theta \quad \Rightarrow \quad \boxed{\theta = \sin^{-1}\left(\tfrac{2}{3}\right) = 41.8°.} \tag{7}$$

Problem 4.35

The force acting on a stationary electric charge q_A interacting with a charge q_B is described by Coulomb's law and takes the form



$$\vec{F} = k\frac{q_A q_B}{r^2}\hat{u}_r,$$

where $k = 8.9875\times10^9\ \mathrm{N\cdot m^2/C^2}$ (C is the symbol for coulomb, the unit used to measure electric charge) is a constant and r is the distance between A and B. This force law is mathematically very similar to Newton's universal gravitation law. With this in mind, determine an expression of the electrostatic potential energy, choosing the datum at infinity, i.e., such that the potential energy is equal to zero when the two charges are separated by an infinite distance.

Solution

The potential energy of the electrostatic force can be derived by calculating the work done in moving two particles from the arbitrary distances r_1 to r_2. The potential energy can be expressed as

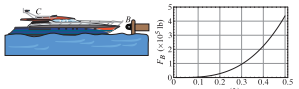
$$U_{1\text{-}2} = \int_{r_1}^{r_2}\left(k\frac{q_A q_B}{r^2}\right)\hat{u}_r\cdot(dr)\,\hat{u}_r = -k\frac{q_A q_B}{r}\bigg|_{r_1}^{r_2} = -kq_A q_B\left(\frac{1}{r_2}-\frac{1}{r_1}\right).$$

Therefore, using Eq. (4.31) from page 279, and taking the datum to be zero as $r\to\infty$, it follows that

$$U_{1\text{-}2} = -(V_2 - V_1) \quad\Rightarrow\quad \boxed{V = k\frac{q_A q_B}{r}.} \tag{1}$$

Problem 4.36

The force-compression profile of a rubber bumper B is given by $F_B = \beta x^3$, where $\beta = 3.5\times10^6\,\text{lb/ft}^3$ and x is the bumper's compression measured in the horizontal direction. Determine the expression for the potential energy of the bumper B. In addition, if the cruiser C weighs 70,000 lb and impacts B with a speed of 5 ft/s, determine the compression required to bring C to a stop. Model C as a particle and neglect C's vertical motion as well as the drag force between the water and the cruiser C.

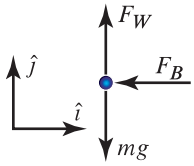


Solution

$\vec{F}_B$ is the only force doing work. Calculate the work of the force $\vec{F}_B$:

$$U_{1\text{-}2} = \int_{\mathcal{L}_{1\text{-}2}} \vec{F}_B \cdot d\vec{r} = \int_{\mathcal{L}_{1\text{-}2}} (-F_B)\,\hat{\imath}\cdot(dx)\,\hat{\imath} \tag{1}$$

$$\Rightarrow \quad U_{1\text{-}2} = \int_{x_1}^{x_2} -\left(\beta x^3\right)\,dx = -\beta\frac{x^4}{4}\bigg|_{x_1}^{x_2} = -\frac{\beta}{4}\left(x_2^4 - x_1^4\right).$$



Comparing this expression with the equation in the textbook, we can write

$$U_{1\text{-}2} = -\left[\,(V_B)_2 - (V_B)_1\,\right], \quad \text{where} \quad \boxed{V_B = \tfrac{1}{4}\beta x^4.}$$

Consider ① to be at the moment of impact and ② to be at the moment when the boat has stopped. Conservation of mechanical energy:

$$T_1 + (V_B)_1 = T_2 + (V_B)_2\,. \tag{2}$$

The kinetic energies are

$$T_1 = \tfrac{1}{2}mv_1^2 = \frac{W}{2g}v_1^2, \quad T_2 = 0.$$

Force laws: $\qquad (V_B)_1 = 0 \quad \text{and} \quad (V_B)_2 = \tfrac{1}{4}\beta d^2,$

where d is the compression required to stop the boat. Eq. (2) becomes

$$\frac{W}{2g}v_1^2 = \frac{\beta d^4}{4} \quad \Rightarrow \quad d = \sqrt[4]{\frac{2Wv_1^2}{g\beta}} \quad \Rightarrow \quad \boxed{d = 0.420\,\text{ft},} \tag{3}$$

where $W = 70{,}000\,\text{lb}$, $v_1 = 5\,\text{ft/s}$, $\beta = 3.5\times10^6\,\text{lb/ft}^3$, and $g = 32.2\,\text{ft/s}^2$.

Problem 4.37

The force law between two Ni atoms was given in Eq. (3.37) on p. 232. Using this equation, determine the potential between two Ni atoms. Assume the potential between the two atoms is zero when the distance between those two atoms is infinite.

Solution

Eqs. (4.12) and (3.37) are repeated for your convenience:

$$U_{1\text{-}2} = \int_{\mathscr{L}_{1\text{-}2}} \vec{F} \cdot d\vec{r} \quad \text{and} \quad \vec{f}_{ij} = 24\epsilon\left(\frac{\sigma^6}{r_{ij}^7} - \frac{2\sigma^{12}}{r_{ij}^{13}}\right)\hat{u}_{j/i}.$$

The Lennard-Jones force law is a function of the distance between two particles.

$$U_{1\text{-}2} = \int_{r_1}^{r_2} 24\epsilon\left(\frac{\sigma^6}{r^7} - \frac{2\sigma^{12}}{r^{13}}\right)\hat{u}_r \cdot (dr)\,\hat{u}_r = -4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]_{r_1}^{r_2}$$

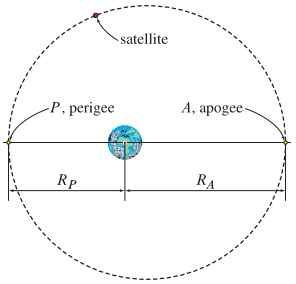
$$= -4\epsilon\left\{\left[\left(\frac{\sigma}{r_2}\right)^{12} - \left(\frac{\sigma}{r_2}\right)^{6}\right] - \left[\left(\frac{\sigma}{r_1}\right)^{12} - \left(\frac{\sigma}{r_1}\right)^{6}\right]\right\},$$

for two separation distances r_1 and r_2. Comparing this expression with Eq. (4.31) we see that

$$U_{1\text{-}2} = -(V_2 - V_1) \quad \Rightarrow \quad \boxed{V = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right].}$$

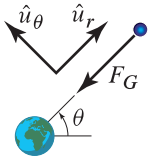
Problem 4.38

A satellite orbits the Earth along the orbit shown. The minimum and maximum distances from the center of the Earth are $R_P = 4.5\times10^7$ m and $R_A = 6.163\times10^7$ m, respectively, where the subscripts P and A stand for *perigee* (the point on the orbit closest to Earth) and *apogee* (the point on the orbit farthest from Earth), respectively. Modeling the satellite as a particle and assuming that the center of the Earth can be chosen as the origin of an inertial frame of reference, if the satellite's speed at P is $|\vec{v}|_P = 3.2\times10^3$ m/s, determine the satellite's speed at A.



Solution

Let ① be when the satellite is at P and ② be when the satellite is at A. Work-energy principle:



$$T_P + V_P = T_A + V_A, \tag{1}$$

The kinetic energies are

$$T_P = \tfrac{1}{2}m_S\,|\vec{v}|_P^2\,, \quad \text{and} \quad T_A = \tfrac{1}{2}m_S\,|\vec{v}|_A^2\,. \tag{2}$$

Force laws

$$V_P = -G\frac{m_E m_S}{R_P} \quad \text{and} \quad V_A = -G\frac{m_E m_S}{R_A}. \tag{3}$$

Combining Eqs. (1)–(3) we find

$$\tfrac{1}{2}m_S\,|\vec{v}|_P^2 - G\frac{m_E m_S}{R_P} = \tfrac{1}{2}m_S\,|\vec{v}|_A^2 - G\frac{m_E m_S}{R_A}. \tag{4}$$

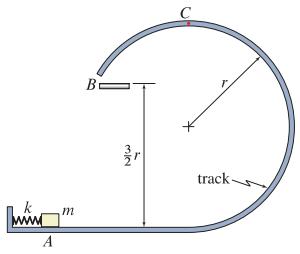
Solving for the speed at the apogee, we find that

$$|\vec{v}|_A = \sqrt{|\vec{v}|_P^2 - 2Gm_E\left(\frac{1}{R_P} - \frac{1}{R_A}\right)} = 2.34\times10^3\ \text{m/s},$$

where the mass of the Earth is $m_E = 5.9736\times10^{24}$ kg and $G = 6.6742\times10^{-11}$ m^3/(kg $\cdot$ s^2).

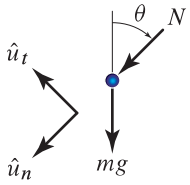
Problem 4.39

The package handling system is designed to launch the small package of mass m from A by using a compressed linear spring of constant k. After launch, the package slides along the track until it lands on the conveyor belt at B. The track has small, well-oiled rollers so that you can neglect any energy loss due to the movement of the package along the track. Modeling the package as a particle, determine the minimum initial compression of the spring so that the package gets to B without separating from the track at C. Finally, determine the speed with which the package reaches the conveyor at B.



Solution

From the FBD for $\theta = 0$, we find that the only force acting on the package is normal to the track so that



$$\sum F_n: \quad N + mg = ma_n. \tag{1}$$

All forces doing work are conservative. Applying the work-energy principle between points A and C:

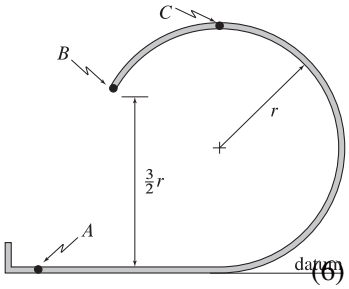
$$T_A + V_A = T_C + V_C. \tag{2}$$

Applying the work-energy principle between points C and B:

$$T_C + V_C = T_B + V_B. \tag{3}$$

Force laws:

$$V_A = \tfrac{1}{2}k\delta^2, \quad V_B = \tfrac{3}{2}mgr, \quad V_C = 2mgr. \tag{4}$$



Kinematics:

$$a_n = \frac{v_c^2}{r} \quad \text{and} \quad v_A = 0. \tag{5}$$

The kinetic energy expressions at A, B and C are:

$$T_A = 0, \quad T_B = \tfrac{1}{2}mv_B^2, \quad T_C = \tfrac{1}{2}mv_C^2. \tag{6}$$

In order to find the minimum spring compression necessary to ensure that the package does not separate from the track at C, we must meet the condition that $N = 0$. Substituting the first of Eqs. (5) into (1), we find that

$$N + mg = m\frac{v_C^2}{r}. \tag{7}$$

Now, we solve for N and set it equal to zero to find the minimum velocity required at point C to be

$$v_C^2 = gr. \tag{8}$$

Using the work-energy principle from Eq. (2), along with expression for the potential and kinetic energies from Eqs. (4) and (6), we find the minimum compression in the spring by solving

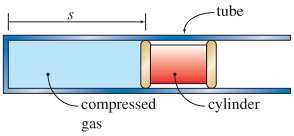
$$\tfrac{1}{2}k\delta_{\min}^2 = \tfrac{1}{2}m\,(gr) + 2mgr \quad \Rightarrow \quad \boxed{\delta_{\min} = \sqrt{\frac{5mgr}{k}}.} \tag{9}$$

To find the speed at B, we rewrite Eq. (3), by substituting the appropriate expressions from the potential and kinetic energies in Eqs. (4) and (6). We can now find the speed at B by solving

$$\tfrac{1}{2}mgr + 2mgr = \tfrac{1}{2}mv_B^2 + \tfrac{3}{2}mgr \quad \Rightarrow \quad \boxed{v_B = \sqrt{2gr}.} \tag{10}$$

Problem 4.40

Compressed gas is used in many circumstances to propel objects within tubes. For example, you can still find pneumatic tubes in use in many banks to send and receive items to drive-through tellers,and it is compressed gas that propels a bullet out of a gun barrel. Let the cross-sectional area of the tube be given by A and the position of the cylinder by s, and assume that the compressed gas is an ideal gas at constant temperature so that the pressure P times the volume Ω is a constant, i.e., $P\Omega =$ constant. Show that the potential energy of this compressed gas is given by $V = -P_0 s_0 A \ln(s/s_0)$, where P_0 is the initial pressure and s_0 is the initial value of s. Model the cylinder as a particle, and assume that the forces resisting the motion of the cylinder are negligible.



Solution

We first realize that with $P\Omega = C$. The force exerted on the cylinder can be written as

$$F = A\frac{C}{\Omega}. \tag{1}$$

Since $\Omega = As$, we can simplify the above equation to

$$F = \frac{C}{s}. \tag{2}$$

The work done by the gas can now be calculated using

$$U_{0\text{-}s} = \int_{s_0}^{s} F\,ds = \int_{s_0}^{s} \frac{C}{s}\,ds \quad \Rightarrow \quad U_{0\text{-}s} = C \ln s\Big|_{s_0}^{s}. \tag{3}$$

Knowing that we can express C as the product of the initial pressure P_0 and the initial volume $\Omega = s_0 A$, we can simplify the expression for the work done by the gas to

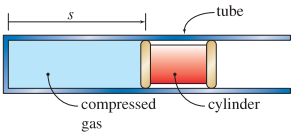
$$U_{0\text{-}s} = P_0 s_0 A \ln\left(\frac{s}{s_0}\right). \tag{4}$$

Therefore we can write the potential energy as

$$U_{1\text{-}2} = \int_{\mathcal{L}_{1\text{-}2}} \vec{F}\cdot d\vec{r} = -(V_2 - V_1) \quad \Rightarrow \quad \boxed{V = -P_0 s_0 A \ln\left(\frac{s}{s_0}\right).}$$

Problem 4.41

When a gun fires a bullet, the gun barrel acts as the tube, and the bullet acts as the cylinder in Prob. 4.40. Using the assumptions and result of that problem, determine the velocity of a bullet at the end of a 24 in. gun barrel, given that the bore diameter is 0.458 in., the bullet weight is 300 gr (7000 gr = 1 lb), the initial firing pressure is 27,000 psi, and the initial distance between the back of the bullet and the back wall of the firing chamber (i.e., s_0 in Prob. 4.40) is



(a) 1.855 in. (this distance is realistic and accurate),

(b) 1.5 in., and

(c) explain why the velocity of the bullet at the end of the gun barrel is lower in Part (b) when compared to Part (a).

Solution

The only force on the bullet, that we consider, is F_G from the compressed gas. Thus, the conservation of mechanical energy equation can be written as



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where ① is when the gun is fired, and ② is when the bullet reaches the end of the barrel. Force laws:

$$V_1 = -P_0 s_0 A \ln\left(\frac{s_0}{s_0}\right), \quad \text{and} \quad V_2 = -P_0 s_0 A \ln\left(\frac{s_2}{s_0}\right), \tag{2}$$

The kinetic energies at ① and ② are

$$T_1 = 0, \quad \text{and} \quad T_2 = \tfrac{1}{2} m v_2^2 \tag{3}$$

Plug Eqs. (2) and Eqs. (3) into Eq. (1)

$$0 + 0 = \tfrac{1}{2} m v_2^2 - P_0 s_0 A \ln\left(\frac{s_2}{s_0}\right) \quad \Rightarrow \quad v_2 = \sqrt{\frac{2 P_0 s_0 A \ln(s_2/s_0)}{m}} \tag{4}$$

Part (a)

$$v_2 = \sqrt{\frac{2 P_0 s_0 A \ln(s_2/s_0)}{m}} \quad = \quad \boxed{1.63 \times 10^3 \text{ ft/s},} \tag{5}$$

where $P_0 = 27{,}000$ psi $= 3.888 \times 10^6$ lb/ft^2, $s_0 = 1.855$ in $= 0.1546$ ft, $s_2 = 24$ in $= 2$ ft, $A = 1.144 \times 10^{-3}$ ft^2, and $m = 300$ grain $= 1.331 \times 10^{-3}$ slug.

Part (b)

$$v_2 = \sqrt{\frac{2 P_0 s_0 A \ln(s_2/s_0)}{m}} \quad = \quad \boxed{1.52 \times 10^3 \text{ ft/s},} \tag{6}$$

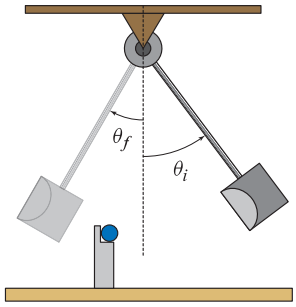
where $P_0 = 27{,}000$ psi $= 3.888 \times 10^6$ lb/ft^2, $s_0 = 1.5$ in $= 0.125$ ft, $s_2 = 24$ in $= 2$ ft $A = 1.144 \times 10^{-3}$ ft^2, and $m = 300$ grain $= 1.331 \times 10^{-3}$ slug.

Part (c)

In Part (b) there is simply less gas to expand and push the bullet down the barrel. Thus, the smaller amount of gas cannot push as hard for as long through the same length of barrel at the same initial pressure.

Problem 4.42

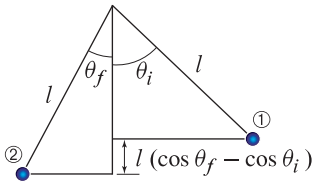
The resistance of a material to fracture is assessed via a fracture test. One such test is the *Charpy impact test*, in which the fracture toughness is assessed by measuring the energy required to break a specimen of a specified geometry. This is done by releasing a heavy pendulum from rest at an angle θ_i and by measuring the maximum swing angle θ_f reached by the pendulum after the specimen is broken. Suppose that in an experiment $\theta_i = 45°$, $\theta_f = 23°$, the weight of the pendulum's bob is 3 lb, and the length of the pendulum is 3 ft. Neglecting the mass of any other component of the testing apparatus, assuming that the pendulum's pivot is frictionless, and treating the pendulum's bob as a particle, determine the fracture energy of the specimen tested. Assume that the fracture energy is the energy required to break the specimen.



Solution

Work-energy principle:

$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2$$



The kinetic energies are

$$T_1 = 0, \quad \text{and} \quad T_2 = 0, \tag{1}$$

Set the datum line at ①. Force laws:

$$V_1 = 0, \quad V_2 = -mgl(\cos\theta_f - \cos\theta_i),$$

and $(U_{1\text{-}2})_{\text{nc}}$ represents the energy absorbed by the impact with the specimen. Eq. (1) becomes

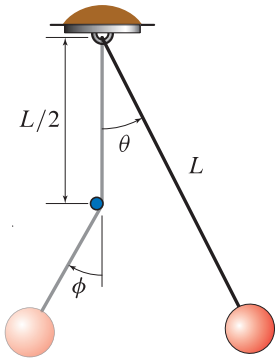
$$(U_{1\text{-}2})_{\text{nc}} = V_2 \quad \Rightarrow \quad (U_{1\text{-}2})_{\text{nc}} = -mgl(\cos\theta_f - \cos\theta_i) = -1.92\,\text{ft·lb}, \tag{2}$$

where $mg = 3$ lb, $l = 3$ ft, $\theta_f = 23°$, and $\theta_i = 45°$. The fracture energy is equal to the amount of energy removed from the pendulum:

$$\boxed{1.92\,\text{ft·lb}.} \tag{3}$$

Problems 4.43 and 4.44

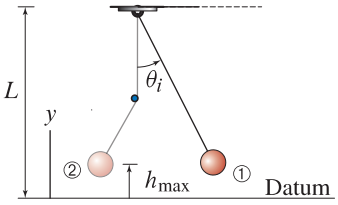
A pendulum with mass $m = 1.4\,\text{kg}$ and length $L = 1.75\,\text{m}$ is released from rest at an angle θ_i. Once the pendulum has swung to the vertical position (i.e., $\theta = 0$), its cord runs into a small fixed obstacle. In solving this problem, neglect the size of the obstacle, model the pendulum's bob as a particle, model the pendulum's cord as massless and inextensible, and let gravity and the tension in the cord be the only relevant forces.



Problem 4.43 What is the maximum height, measured from its lowest point, reached by the pendulum if $\theta_i = 20°$?

Problem 4.44 If the bob is released from rest at $\theta_i = 90°$, at what angle ϕ does the cord go slack?

Solution to 4.43



Referring to the figure on the right, we denote by ① the position of the pendulum at release and by ② the position of the pendulum when the pendulum bob achieves the maximum height. Referring to the FBD shown in the lower right figure, and recalling that the pendulum cord is inextensible, we have that the only work done is that due to gravity, which is a conservative force. Since all the work done on the pendulum bob is done by a conservative force, we can say that the system is conservative and that the work-energy principle can be applied as follows:

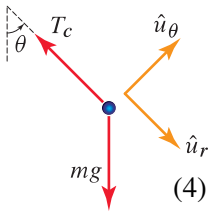
$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where recalling that the system is released from rest and that the maximum height is achieved when the bob comes to rest, we have

$$T_1 = 0 \quad \text{and} \quad T_2 = 0. \tag{2}$$

Choosing the datum for gravity to correspond to the position of the bob when the bob is at the lowest point in its trajectory, and using the y coordinate defined in the upper right figure, we have

$$V_1 = mgy_1 \quad \text{and} \quad V_2 = mgy_2. \tag{3}$$



Substituting Eqs. (2) and (3) into Eq. (1), we have

$$0 + mgy_1 + 0 = 0 + mgy_2 \quad \Rightarrow \quad y_1 = y_2,$$

where we recall that

$$y_1 = L(1 - \cos\theta_i) \quad \text{and} \quad y_2 = h_{\text{max}}. \tag{4}$$

Hence, we have say that

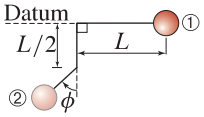
$$h_{\text{max}} = L(1 - \cos\theta_i), \tag{5}$$

which can be evaluated to obtain

$$\boxed{h_{\text{max}} = 0.106\,\text{m},}$$

where we have used the following numerical data: $L = 1.75\,\text{m}$ and $\theta_i = 20°$.

Solution to 4.44



Referring to the figure on the right, to determine the position of the pendulum when the cord goes slack we need to find an expression for the tension in the cord as a function of ϕ. Once the pendulum cord hits the peg, the pendulum will swing in a circle of radius $L/2$ and center at the peg. This will cause the tension in the cord to depend on the normal component of the acceleration of the pendulum bob. In turn, as is the case for circular motion, this component of acceleration depends on the speed of the pendulum. Therefore, to express the tension in the cord as a function of ϕ we need to express the speed as a function of ϕ and this will be done using the work-energy principle. Let ① denote the position at release and ② denote a position corresponding to the a generic value of ϕ. As shown in the FBD on the lower right, the only force doing work is gravity (since the cord is inextensible) and we can write the work-energy principle as follows:

$$T_1 + V_1 = T_2 + V_2, \tag{6}$$

where, recalling that the pendulum is released from rest, we have

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2. \tag{7}$$

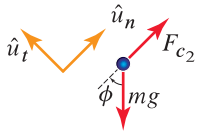
Given the choice of datum indicated on the upper right, for the potential energy of gravity we have

$$V_1 = 0 \quad \text{and} \quad V_2 = -\tfrac{1}{2}mgL(1+\cos\phi). \tag{8}$$

Substituting Eqs. (8)–(7) into Eq. (6), we obtain

$$0 = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mgL(1+\cos\phi) \quad \Rightarrow \quad v_2^2 = gL(1+\cos\phi). \tag{9}$$

Referring to the FBD on the right, we now sum forces in the normal direction to obtain



$$\sum F_n: \quad F_{c_2} - mg\cos\phi = ma_n. \tag{10}$$

where F_{c_2} is the tension in the cord at ②. Recalling that the radius of the path of the pendulum bob after the cord hits the peg is $L/2$, the normal component of acceleration is given by

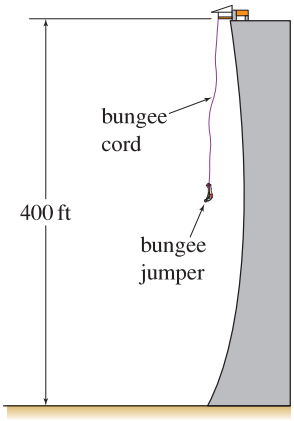
$$a_n = \frac{v^2}{L/2}. \tag{11}$$

Substituting the expression for v^2 in the last of Eqs. (9) into Eq. (11), then substituting the result into Eq. (10), and enforcing the condition that for the cord to become slack at ②, F_{c_2} must equal zero, we have

$$-mg\cos\phi = m\frac{gL(1+\cos\phi)}{L/2} \quad \Rightarrow \quad 2+3\cos\phi = 0 \quad \Rightarrow \quad \boxed{\phi = \cos^{-1}\left(\tfrac{-2}{3}\right) = 132°.}$$

Problem 4.45

While the stiffness of an elastic cord can be nearly constant (i.e., the force versus displacement curve is a straight line) over a large range of stretch, as a bungee cord is stretched, it softens; that is, the cord tends to get less stiff as it gets longer. Assuming a softening force-displacement relation of the form $k\delta - \beta\delta^3$, where $k = 2.58\,\text{lb/ft}$ and $\beta = 0.000013\,\text{lb/ft}^3$ and where δ (measured in ft) is the displacement of the cord from its unstretched length, and considering a bungee cord whose unstretched length is 150 ft, determine



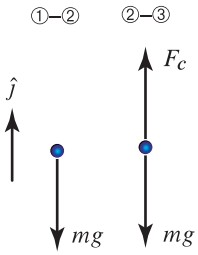
(a) the expression of the cord's potential energy as a function of δ;

(b) the velocity at the bottom of a 400 ft tower of a bungee jumper weighing 170 lb and starting from rest;

(c) the maximum acceleration, expressed in g's, felt by the bungee jumper in question.

Solution

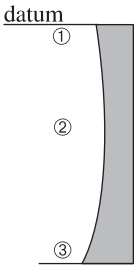
Part (a) Calculate the work $U_{2\text{-}3}$ of $\vec{F}_c$ for an arbitrary downward displacement δ



$$(U_{2\text{-}3})_c = \int_{-\delta_2}^{-\delta_3} \left(k\delta - \beta\delta^3\right)\hat{\jmath}\cdot(d\delta)\,\hat{\jmath} = -\int_{\delta_2}^{\delta_3} k\delta - \beta\delta^3\,d\delta$$

$$\Rightarrow \quad -\frac{k\delta^2}{2} + \frac{\beta\delta^4}{4}\bigg|_{\delta_2}^{\delta_3} = \left(-\frac{k\delta_3^2}{2} + \frac{\beta\delta_3^4}{4}\right) - \left(-\frac{k\delta_2^2}{2} + \frac{\beta\delta_2^4}{4}\right).$$

Thus the potential energy function is



$$\boxed{V = \frac{k\delta^2}{2} - \frac{\beta\delta^4}{4}.} \qquad (1)$$

Part (b) Taking the datum to be the top of the tower, and using Eq. (1), the initial and final kinetic and potential energy terms are

$$T_1 = 0, \quad \text{and} \quad T_3 = \tfrac{1}{2}mv_3^2 \qquad (2)$$

$$V_1 = 0, \quad \text{and} \quad V_3 = -mgh + \tfrac{1}{2}k\delta_3^2 - \tfrac{1}{4}\beta\delta_3^4 \qquad (3)$$

The balance of work and energy equation is

$$T_1 + V_1 = T_3 + V_3 \quad \Rightarrow \quad 0 = -mgh + \tfrac{1}{2}mv_3^2 + \tfrac{1}{2}k\delta_3^2 - \tfrac{1}{4}\beta\delta_3^4 \qquad (4)$$

Solve Eq. (4) for v_3

$$v_3 = \sqrt{\frac{\beta\delta_3^4 - 2k\delta_3^2 + 4ghm}{2m}} \quad \Rightarrow \quad \boxed{5.16\,\text{ft/s},} \qquad (5)$$

where $\beta = 0.000013\,\text{lb/ft}^3$, $\delta_3 = 250\,\text{ft}$, $k = 2.58\,\text{lb/ft}$, $g = 32.2\,\text{ft/s}^2$, $h = 400\,\text{ft}$, and $m = 5.280\,\text{slug}$.

Part (c) Between ①–②, from the FBD, the acceleration is simply $a_{1\text{-}2} = 1g$ downwards. Between ②–③, from the FBD, the acceleration is

$$a_{2\text{-}3} = g - F_c/m = g(1 - F_c/W) = g\left(1 - \frac{k\delta - \beta\delta^3}{W}\right) \tag{6}$$

where W is the weight of the jumper. Differentiating $a_{2\text{-}3}$ with respect to δ and setting the result equal to 0, we obtain

$$\frac{da_{2\text{-}3}}{d\delta} = \frac{g}{W}\left(-k + 3\beta\delta^2\right) = 0, \tag{7}$$

which we solve for δ to obtain

$$\delta = \pm\sqrt{\frac{k}{3\beta}} = \pm 257.2\,\text{ft}, \tag{8}$$

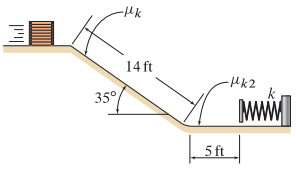
where $k = 2.58\,\text{lb/ft}$, and $\beta = 1.3\times10^{-5}\,\text{lb/ft}^3$. Notice that ±257.2 ft are outside the range ②–③ for which $a_{2\text{-}3}$ is defined, therefore the maximum absolute acceleration between ②–③ must occur either at $\delta = 0$ or at $\delta = 250\,\text{ft}$. Letting $\delta = 0$ in Eq. (6) we get $a_{2\text{-}3} = 1g$ which is the same as $a_{1\text{-}2}$. On the other hand, substituting $\delta = 250\,\text{ft}$ into Eq. (6) we have the maximum absolute acceleration between ②–③

$$a_{2\text{-}3} = \left|g\left(1 - \frac{k\delta - \beta\delta^3}{W}\right)\right| = \boxed{1.60g,} \tag{9}$$

where $W = 170\,\text{lb}$, $k = 2.58\,\text{lb/ft}$, and $\beta = 1.3\times10^{-5}\,\text{lb/ft}^3$. This result is also the largest absolute acceleration during the entire jump since it is larger than $a_{1\text{-}2} = 1g$.

Problems 4.46 through 4.48

A crate, initially traveling horizontally with a speed of 18 ft/s, is made to slide down a 14 ft chute inclined at 35°. The surface of the chute has a coefficient of kinetic friction μ_k, and at its lower end, it smoothly lets the crate onto a horizontal trajectory. The horizontal surface at the end of the chute has a coefficient of kinetic friction μ_{k2}. Model the crate as a particle, and assume that gravity and the contact forces between the crate and the sliding surface are the only relevant forces.



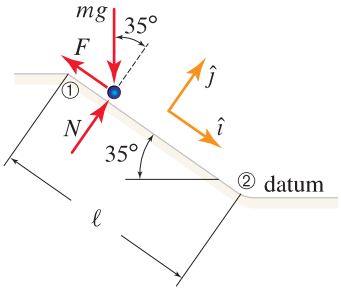
Problem 4.46 If $\mu_k = 0.35$, what is the speed with which the crate reaches the bottom of the chute (immediately before the crate's trajectory becomes horizontal)?

Problem 4.47 Find μ_k such that the crate's speed at the bottom of the chute (immediately before the crate's trajectory becomes horizontal) is 15 ft/s.

Problem 4.48 Let $\mu_k = 0.5$ and suppose that once the crate reaches the bottom of the chute and after sliding horizontally for 5 ft, the crate runs into a bumper. If the weight of the crate is $W = 110$ lb, $\mu_{k2} = 0.33$, and you model the bumper as a linear spring with constant k and neglect the mass of the bumper, determine the value of k so that the crate comes to a stop 2 ft after impacting with the bumper.

Solution to 4.46

We will solve the problem using the work-energy principle. Referring to figure on the right, we denote by ① the position of the crate when the crate starts sliding along the incline and we denote by ② the position of the crate when the crate reaches the bottom of the incline. We write the work-energy principle between ① and ② as follows:



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{np}} = T_2 + V_2, \tag{1}$$

where

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2, \tag{2}$$

with $v_1 = 18$ ft/s and v_2 the value of speed we want to determine. We observe that gravity is the only conservative force in this problem. Given the choice of datum, we have

$$V_1 = mg\ell \sin 35^\circ \quad \text{and} \quad V_2 = 0, \tag{3}$$

where $\ell = 14$ ft is the distance the crate slides along the inclined surface. To compute the work of friction, we need to determine the friction force F. We do this by first determining the normal force N. Summing forces in the direction perpendicular to the incline, we have

$$\sum F_y: \quad N - mg\cos 35^\circ = 0 \quad \Rightarrow \quad N = mg\cos 35^\circ \quad \Rightarrow \quad F = mg\mu_k \cos 35^\circ, \tag{4}$$

where we have used the kinetic friction law $F = \mu_k N$ to derive the last of Eqs. (4). Since the friction force is constant and it acts parallel and opposite to the direction of motion, its work between ① and ② is given by

$$(U_{1\text{-}2})_{\text{np}} = -\mu_k mg\ell \cos 35^\circ. \tag{5}$$

Substituting Eqs. (2), (3), and (5), into Eq. (1), we have

$$\tfrac{1}{2}mv_1^2 + mg\ell \sin 35^\circ - \mu_k mg\ell \cos 35^\circ = \tfrac{1}{2}mv_2^2, \tag{6}$$

which can be solved for v_2 to obtain,

$$v_2 = \sqrt{v_1^2 + 2g\ell \sin 35^\circ - 2\ell\mu_k g \cos 35^\circ}. \tag{7}$$

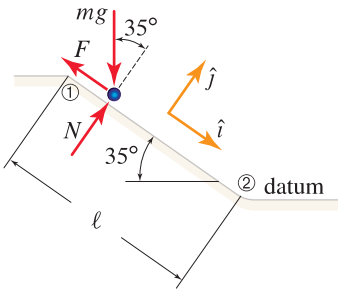
The above expression can evaluated to obtain

$$\boxed{v_2 = 24.1 \text{ ft/s},} \tag{8}$$

where we have used the following numerical data: $v_1 = 18\,\text{ft/s}$, $\ell = 14\,\text{ft}$, $g = 32.2\,\text{ft/s}^2$, and $\mu_k = 0.35$.

Solution to 4.47

We will solve the problem using the work-energy principle. Referring to figure on the right, we denote by ① the position of the crate when the crate starts sliding along the incline and we denote by ② the position of the crate when the crate reaches the bottom of the incline. We write the work-energy principle between ① and ② as follows:



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{np}} = T_2 + V_2, \tag{9}$$

where

$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2, \tag{10}$$

with $v_1 = 18\,\text{ft/s}$ and $v_2 = 15\,\text{ft/s}$. We observe that gravity is the only conservative force in this problem. Given the choice of datum, we have

$$V_1 = mg\ell \sin 35^\circ \quad \text{and} \quad V_2 = 0, \tag{11}$$

where $\ell = 14\,\text{ft}$ is the distance the crate slides along the inclined surface. To compute the work of friction, we need to determine the friction force F. We do this by first determining the normal force N. Summing forces in the direction perpendicular to the incline, we have

$$\sum F_y: \quad N - mg\cos 35^\circ = 0 \quad \Rightarrow \quad N = mg\cos 35^\circ \quad \Rightarrow \quad F = mg\mu_k \cos 35^\circ, \tag{12}$$

where we have used the kinetic friction law $F = \mu_k N$ to derive the last of Eqs. (12). Since the friction force is constant and it acts parallel and opposite to the direction of motion, its work between ① and ② is given by

$$(U_{1\text{-}2})_{\text{np}} = -\mu_k mg\ell \cos 35^\circ. \tag{13}$$

Substituting Eqs. (10), (11), and (13), into Eq. (9), we have

$$\tfrac{1}{2}mv_1^2 + mg\ell \sin 35^\circ - \mu_k mg\ell \cos 35^\circ = \tfrac{1}{2}mv_2^2, \tag{14}$$

which can be solved for μ_k to obtain,

$$\mu_k = \frac{v_1^2 - v_2^2 + 2g\ell \sin 35^\circ}{2\ell g \cos 35^\circ}. \tag{15}$$

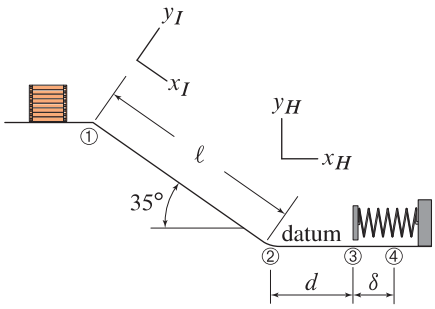
The above expression can evaluated to obtain

$$\boxed{\mu_k = 0.834,} \tag{16}$$

where we have used the following numerical data: $v_1 = 18\,\text{ft/s}$, $v_2 = 15\,\text{ft/s}$, $\ell = 14\,\text{ft}$, and $g = 32.2\,\text{ft/s}^2$.

Solution to 4.48

Referring to the figure on the right, we define four positions along the path of the crate. We will solve the problem by applying the work-energy principle between ① and ④, and we write



$$T_1 + V_1 + (U_{1\text{-}4})_{\text{np}} = T_4 + V_4, \tag{17}$$

where ① is where the crate starts sliding down the incline and ④ is where the crate comes to stop 2 ft after impacting with the bumper, so that we have

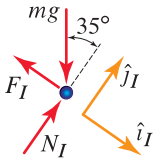
$$T_1 = \tfrac{1}{2}mv_1^2 \quad \text{and} \quad T_4 = 0, \tag{18}$$

with $v_1 = 18\,\text{ft/s}$. As shown in the FBDs on the right, gravity and the spring force are the only conservative forces acting on the crate. Hence, setting the datum for gravity at ④, we have

$$V_1 = mg\ell \sin 35^\circ \quad \text{and} \quad V_4 = \tfrac{1}{2}k\delta^2, \tag{19}$$

where $\ell = 14\,\text{ft}$ is the distance the crate travels along the incline and δ is the maximum compression of the bumper.

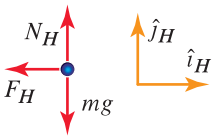
We now compute the work done by friction. To do so, we first need to determine the normal force acting on the crate. Between ① and ②, summing forces in the direction perpendicular to the incline, we have



FBD of the crate between ① and ②

$$\sum F_y: \; N_I - mg\cos 35^\circ = 0 \;\Rightarrow\; N_I = mg\cos 35^\circ \;\Rightarrow\; F_I = \mu_k mg\cos 35^\circ, \tag{20}$$

where we have used the force law $F_I = \mu_k N_I$. Summing forces in the vertical direction between ② and ④, we have

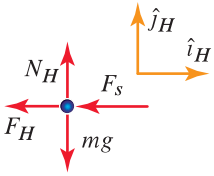


FBD of the crate between ② and ③

$$\sum F_y: \quad N_H - mg = 0 \quad \Rightarrow \quad N_H = mg \quad \Rightarrow \quad F_H = \mu_{k2}mg, \tag{21}$$

where we have used the force law $F_H = \mu_{k2}N_H$. Observing that both F_I and F_H are constant, the work done by friction between ① and ④ is

$$(U_{1\text{-}4})_{\text{np}} = -\mu_k mg\ell\cos 35^\circ - \mu_k mg(d+\delta), \tag{22}$$



FBD of the crate between ③ and ④

where $d = 5\,\text{ft}$. Substituting Eqs. (18), (19), and (22) into (17), we obtain

$$\tfrac{1}{2}mv_1^2 + mg\ell\sin 35^\circ - \mu_k mg\ell\cos 35^\circ - \mu_k mg(d+\delta) = \tfrac{1}{2}k\delta^2, \tag{23}$$

which we can solve for k to obtain

$$k = \frac{m}{\delta^2}\left\{v_1^2 + 2g\left[\ell\sin 35^\circ - \mu_k\ell\cos 35^\circ - \mu_{k2}(d+\delta)\right]\right\}. \tag{24}$$

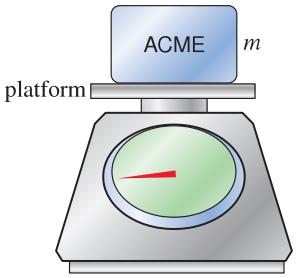
The above expression can be evaluated to obtain

$$\boxed{k = 276\,\text{lb/ft},} \tag{25}$$

where we have used the following numerical data: $m = (110\,\text{lb})/(32.2\,\text{ft/s}^2) = 3.416\,\text{slug}$, $\delta = 2\,\text{ft}$, $v_1 = 18\,\text{ft/s}$, $g = 32.2\,\text{ft/s}^2$, $\ell = 14\,\text{ft}$, $\mu_k = 0.5$, $\mu_{k2} = 0.33$, and $d = 5\,\text{ft}$.

Problems 4.49 and 4.50

Spring scales work by measuring the displacement of a spring that supports both the platform and the object, of mass m, whose weight is being measured. Neglect the mass of the platform on which the mass sits, and assume that the spring is uncompressed before the mass is placed on the platform. In addition, assume that the spring is linear elastic with spring constant k. You may have solved these same problems using Newton's second law when doing Prob. 3.16 and 3.17 — here use the work-energy principle to solve them.



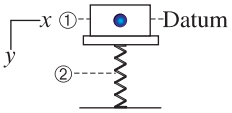
Problem 4.49 If the mass m is gently placed on the spring scale (i.e., it is dropped from zero height above the scale), determine the *maximum reading* on the scale after the mass is released.

Problem 4.50 If the mass m is gently placed on the spring scale (i.e., it is dropped from zero height above the scale), determine the expression for *maximum velocity* attained by the mass m as the spring compresses.

Solution to 4.49

All forces doing work are conservative. Work-energy principle:

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

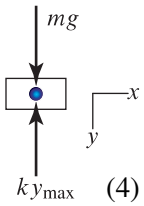


Force laws:

$$V_1 = 0 \quad \text{and} \quad V_2 = -mgy_{\max} + \tfrac{1}{2}ky_{\max}^2. \tag{2}$$

Kinetic energies:

$$T_1 = 0 \quad \text{and} \quad T_2 = 0 \tag{3}$$



Substituting Eqs. (2) and (3) into Eq. (1) and using $(U_{1\text{-}2})_{\text{np}} = 0$, we get

$$\tfrac{1}{2}ky_{\max}^2 = mgy_{\max} \quad \Rightarrow \quad y_{\max}\frac{2mg}{k}. \tag{4}$$

The maximum reading on the scale will be the value of the spring force:

$$F_{\max} = ky_{\max} \quad \Rightarrow \quad \boxed{F_{\max} = 2mg.}$$

Solution to 4.50

Define ① to be the position of release of m from rest and ② to be any intermediate position before the mass m comes to rest. All forces doing work are conservative. Work-energy principle:

$$T_1 + V_1 = T_2 + V_2. \tag{5}$$

Force laws:

$$V_1 = 0 \quad \text{and} \quad V_2 = -mgy + \tfrac{1}{2}ky^2. \tag{6}$$

The kinetic energies are

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_y^2(y) \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (5) we get

$$v_y^2(y) = 2gy - \frac{k}{m}y^2. \tag{8}$$

To find the maximum velocity, we differentiate $v_y^2(y)$ with respect to y, so that

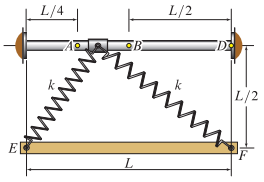
$$\frac{dv_y^2(y)}{dy} = 2g - 2\frac{ky}{m} = 0 \quad \Rightarrow \quad y_{(v_y)_{\max}} = \frac{mg}{k}. \tag{9}$$

Substituting $y = y_{(v_y)_{\max}}$ from Eq. (9) into Eq. (8), we get

$$\boxed{(v_y)_{\max} = g\sqrt{\frac{m}{k}}.} \tag{10}$$

Problem 4.51

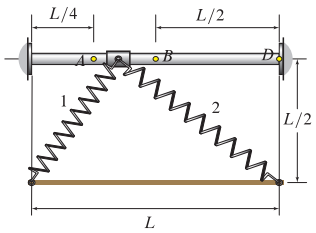
A 6 lb collar is constrained to travel along a rectilinear and frictionless bar of length $L = 5$ ft. The springs attached to the collar are identical, and they are unstretched when the collar is at B. Treating the collar as a particle, neglecting air resistance, and knowing that at A the collar is moving to the right with a speed of 11 ft/s, determine the linear spring constant k so that the collar reaches D with zero speed. Points E and F are fixed.



Solution

All forces doing work are conservative. Work-energy principle between A and D:

$$T_A + V_A + (U_{\text{A-D}})_{\text{np}} = T_D + V_D. \tag{1}$$



Denoting the two springs by 1 and 2, the potential energies are given by

$$V_A = (V_1)_A + (V_2)_A = \tfrac{1}{2}k\,(\delta_1)_A^2 + \tfrac{1}{2}k\,(\delta_2)_A^2\,, \tag{2}$$

$$V_D = (V_1)_D + (V_2)_D = \tfrac{1}{2}k\,(\delta_1)_D^2 + \tfrac{1}{2}k\,(\delta_2)_D^2\,, \tag{3}$$

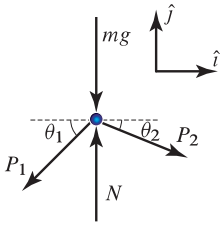
where $(\delta_1)_A = (L_1)_A - L_0$, $(\delta_1)_D = (L_1)_D - L_0$, $(\delta_2)_A = (L_2)_A - L_0$ and $(\delta_2)_D = (L_2)_D - L_0$. The lengths $(L_1)_A$, $(L_1)_D$, $(L_2)_A$ and $(L_2)_D$ are given by

$$(L_1)_A = \sqrt{L^2/4 + L^2/16} = \frac{\sqrt{5}}{4}L, \quad (L_1)_D = \sqrt{L^2 + L^2/4} = \frac{\sqrt{5}}{2}L, \tag{4}$$

$$(L_2)_A = \sqrt{L^2/4 + 9L^2/16} = \frac{\sqrt{13}}{4}L, \quad (L_2)_D = \frac{L}{2}. \tag{5}$$

The kinetic energies are given by

$$T_A = \tfrac{1}{2}mv_A^2 \quad \text{and} \quad T_D = 0. \tag{6}$$



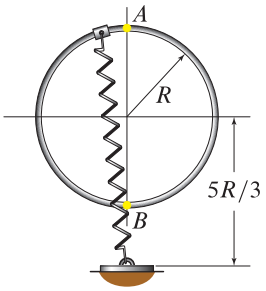
Finally, $L_0 = \sqrt{L^2/4 + L^2/4} = L/\sqrt{2}$. Substituting these relations and Eqs. (2)–(6) into Eq. (1) and solving for k

$$k = \frac{mv^2}{(0.1521)\,L^2} \quad = \quad \boxed{5.93\ \text{lb/ft},} \tag{7}$$

where $m = 0.1863$ slug, $v = 11$ ft/s, and $L = 5$ ft.

Problem 4.52

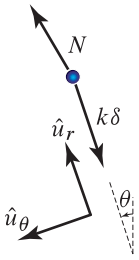
A 3 kg collar is constrained to travel in the *horizontal plane* along a frictionless ring of radius $R = 0.75$ m. The spring attached to the collar has a spring constant $k = 21$ N/m. Treating the collar as a particle, neglecting air resistance, and knowing that at A the collar is at rest, determine the spring's unstretched length if the collar is to reach point B with a speed of 2 m/s.



Solution

All forces doing work are conservative. Applying the work-energy principle between A and B, we obtain

$$T_A + V_A = T_B + V_B. \tag{1}$$



The potential energies are given by

$$V_A = \frac{1}{2}k\left(\frac{8R}{3} - L_0\right)^2 \quad \text{and} \quad V_B = \frac{1}{2}k\left(\frac{2R}{3} - L_0\right)^2. \tag{2}$$

The kinetic energies are given by

$$T_A = 0 \quad \text{and} \quad T_B = \tfrac{1}{2}mv_B^2, \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1) we get

$$\frac{1}{2}k\left(\frac{8R}{3} - L_0\right)^2 = \frac{1}{2}mv_B^2 + \frac{1}{2}k\left(\frac{2R}{3} - L_0\right)^2 \tag{4}$$

$$\left(\frac{8R}{3} - L_0\right)^2 - \left(\frac{2R}{3} - L_0\right)^2 = m\frac{v_B^2}{k}. \tag{5}$$

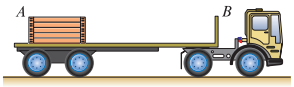
Taking $\left(\frac{8R}{3} - L_0\right) = a$ and $\left(\frac{2R}{3} - L_0\right) = b$ and using $a^2 - b^2 = (a+b)(a-b)$, we get

$$\left(\frac{10R}{3} - 2L_0\right)(2R) = \frac{mv_B^2}{k} \quad \Rightarrow \quad \boxed{L_0 = \frac{5R}{3} - \frac{mv_B^2}{4kR} = 1.06\,\text{m},} \tag{6}$$

where $v_B = 2$ m/s, $m = 3$ kg, $R = 0.75$ m, and $k = 21$ N/m.

Problem 4.53

The driver of the truck suddenly applies its brakes, and the truck comes to a stop under the action of a constant braking force. During braking, either the crate slides or it does not. Considering the work-energy principle applied to the truck during braking, will the truck stop in a shorter distance (or time) if the crate slides, or will the distance (or time) be shorter if it does not? Assume that the truck bed is long enough that you don't have to worry about whether or not the crate hits the truck. Justify your answer.



Solution

The balance of work and energy for this system of particles is

$$T_1 + (U_{1\text{-}2})_{\text{ext}} + (U_{1\text{-}2})_{\text{int}} = T_2. \tag{1}$$

T_1 and T_2 are the same whether or not the crate slides. Therefore

$$[(U_{1\text{-}2})_{\text{ext}} + (U_{1\text{-}2})_{\text{int}}]_{\text{ slide}} = [(U_{1\text{-}2})_{\text{ext}} + (U_{1\text{-}2})_{\text{int}}]_{\text{ no slide}}. \tag{2}$$

In the no slide case $(U_{1\text{-}2})_{\text{int}} = 0$, so

$$[(U_{1\text{-}2})_{\text{ext}} + (U_{1\text{-}2})_{\text{int}}]_{\text{ slide}} = [(U_{1\text{-}2})_{\text{ext}}]_{\text{ no slide}}. \tag{3}$$

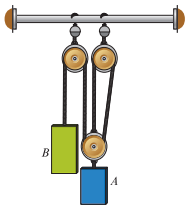
Since each of these terms has the same sign, we must have that:

$$[(U_{1\text{-}2})_{\text{ext}}]_{\text{ slide}} < [(U_{1\text{-}2})_{\text{ext}}]_{\text{ no slide}}. \tag{4}$$

Assuming the same external braking force in both cases, this means that the truck in the slide case stops in a shorter distance.

Problem 4.54

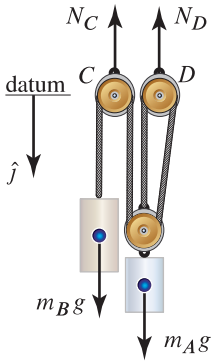
Consider a pulley system in which bodies A and B have masses $m_A = 2\,\text{kg}$ and $m_B = 10\,\text{kg}$. If the system is released from rest, neglecting all sources of friction as well as the inertia of the pulleys, determine the speeds of A and B after B has displaced a distance of 0.6 m downward.



Solution

Let ① be at release and ② be when B displaces $d = 0.6\,\text{m}$ downward. All forces doing work are conservative. Work-energy principle:

$$T_1 + V_1 = T_2 + V_2. \tag{1}$$



Kinetic energies:

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2} m_A v_{A2}^2 + \tfrac{1}{2} m_B v_{B2}^2.$$

Force laws:

$$V_1 = -m_A g y_{A1} - m_B g y_{B1} \quad \text{and} \quad V_2 = -m_A g y_{A2} - m_B g y_{B2}.$$

Pulley kinematics:

$$y_B + 3y_A = L \quad \Rightarrow \quad \dot{y}_B = -3\dot{y}_A \quad \text{and} \quad y_{B2} - y_{B1} = d = 3\,(y_{A1} - y_{A2})\,.$$

Eq. (1) becomes

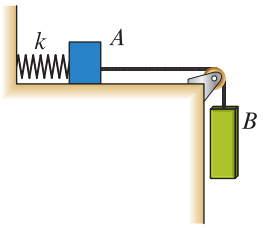
$$\begin{aligned} -m_A g y_{A1} - m_B g y_{B1} &= \tfrac{1}{2} m_A v_{A2}^2 + \tfrac{1}{2} m_B v_{B2}^2 - m_A g y_{A2} - m_B g y_{B2} \\ \Rightarrow \quad -m_A g\,(y_{A1} - y_{A2}) + m_B g\,(y_{B2} - y_{B1}) &= \tfrac{1}{2}\,(m_A + 9m_B)\, v_{A2}^2 \\ \Rightarrow \quad g d\left(m_B - \tfrac{1}{3} m_A\right) &= \tfrac{1}{2}\,(m_A + 9m_B)\, v_{A2}^2 \end{aligned}$$

$$v_{A2} = \sqrt{\frac{2gd\left(m_B - \frac{1}{3}m_A\right)}{m_A + 9m_B}} = 1.09\,\text{m/s},$$

$$v_{B2} = 3\sqrt{\frac{2gd\left(m_B - \frac{1}{3}m_A\right)}{m_A + 9m_B}} = 3.28\,\text{m/s}.$$

Problem 4.55

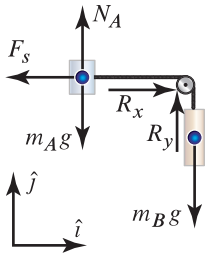
Blocks A and B, weighing 7 and 15 lb, respectively, are released from rest when the spring is unstretched. If all sources of friction are negligible and $k = 12$ lb/ft, determine the maximum vertical displacement of B from the release position, assuming that A never leaves the horizontal surface shown and the cord connecting A and B is inextensible.



Solution

Let ① be at release and ② be at maximum vertical displacement. All forces doing work are conservative. Work-energy principle:

$$T_1 + V_1 = T_2 + V_2. \tag{1}$$



The kinetic energies are

$$T_1 = 0 \quad \text{and} \quad T_2 = 0.$$

Chose a different datum for each mass corresponding to its initial position. Let δ represent the stretch in the spring. Force laws:

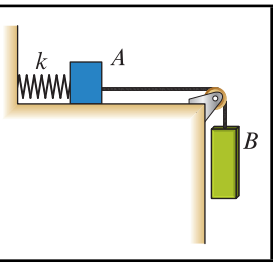
$$V_1 = 0 \quad \text{and} \quad V_2 = \tfrac{1}{2}k\delta^2 - m_B g\delta.$$

A and B will displace by an amount δ due to the in extensibility of the cord. Eq. (1) becomes

$$0 = \tfrac{1}{2}k\delta^2 - m_B g\delta \quad \Rightarrow \quad \boxed{\delta = \frac{2m_B g}{k} = 2.50\,\text{ft}.}$$

Problem 4.56

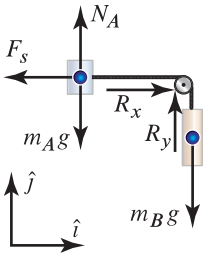
Blocks A and B are released from rest when the spring is unstretched. Block A has a mass $m_A = 2\,\text{kg}$, and the linear spring has stiffness $k = 7\,\text{N/m}$. If all sources of friction are negligible, determine the mass of block B such that B has a speed $v_B = 1.5\,\text{m}$ after moving 1.2 m downward, assuming that A never leaves the horizontal surface shown and the cord connecting A and B is inextensible.



Solution

Let ① be at release and ② be when $v_B = 1.5\,\text{m/s}$. All forces doing work are conservative. Work-energy principle:

$$T_1 + V_1 = T_2 + V_2. \tag{1}$$



The kinetic energies are

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2} m_A v_{A2}^2 + \tfrac{1}{2} m_B v_{B2}^2.$$

Chose a different datum for each mass corresponding to its initial position. Let δ represent the stretch in the spring. Force laws:

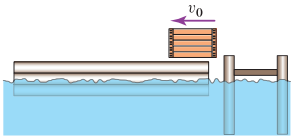
$$V_1 = 0 \quad \text{and} \quad V_2 = \tfrac{1}{2} k\delta^2 - m_B g\delta.$$

A and B will displace by an amount δ due to the in extensibility of the cord. Note that $v_{A2} = v_{B2}$. Eq. (1) becomes

$$0 = \tfrac{1}{2} m_A v_{A2}^2 + \tfrac{1}{2} m_B v_{B2}^2 + \tfrac{1}{2} k\delta^2 - m_B g\delta \quad \Rightarrow \quad \boxed{m_B = \frac{m_A v_{B2}^2 + k\delta^2}{2g\delta - v_{B2}^2} = 0.685\,\text{kg}.}$$

Problem 4.57

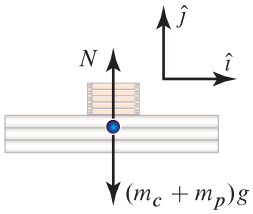
A 700 lb floating platform is at rest when a 200 lb crate is thrown onto it with a horizontal speed $v_0 = 12\,\text{ft/s}$. Once the crate stops sliding relative to the platform, the platform and the crate move with a speed $v = 2.667\,\text{ft/s}$. Neglecting the vertical motion of the system as well as any resistance due to the relative motion of the platform with respect to the water, determine the distance that the crate slides relative to the platform if the coefficient of kinetic friction between the platform and the crate is $\mu_k = 0.25$.



Solution

The only force doing work is the internal friction force. Let ① represent the crate first striking the platform and ② represent when the crate stops sliding .Work-energy principle:

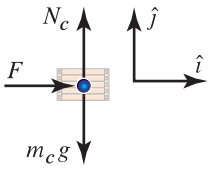
$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2 \qquad (1)$$



The kinetic energies are

$$T_1 = \tfrac{1}{2} m_c v_0^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}\left(m_c + m_p\right) v^2.$$

Force laws:



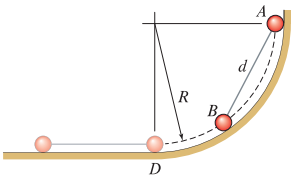
$$V_1 = 0, \quad V_2 = 0, \quad \text{and} \quad (U_{1\text{-}2})_{\text{nc}} = -Fd = -\mu_k m_c g d,$$

where d is the distance the crate slides relative to the platform. Eq. (1) becomes

$$\tfrac{1}{2} m_c v_0^2 - \mu_k m_c g d = \tfrac{1}{2}\left(m_c + m_p\right) v^2 \quad \Rightarrow \quad \boxed{d = \frac{m_c v_0^2 - \left(m_c + m_p\right) v^2}{2\mu_k m_c g} = 6.96\,\text{ft.}}$$

Problem 4.58

Two small spheres A and B, each of mass m, are attached at either end of a stiff light rod of length d. The system is released from rest in the position shown. Determine the speed of the spheres when A has reached point D, and determine the normal force between sphere A and the surface on which it is sliding immediately before it reaches point D. Neglect friction, treat the spheres as particles (assume that their diameter is negligible), and neglect the mass of the rod.



Solution

Let ① be when sphere A is at the top of the semicircle and ② be when sphere A reaches point D. Since the system is conservative, the balance of work and energy is given by

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

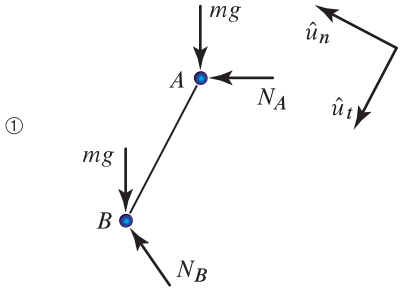
Balance principle:

$$\sum F_n: \quad N_A - mg = ma_n. \tag{2}$$

The datum for determining the potential energy is the line passing horizontally through D. Therefore, the potential energy is

$$V_1 = V_{A1} + V_{B1} = mgR + mg\left[R - d\sqrt{1 - d^2/(2R)^2}\right], \tag{3}$$

$$V_2 = V_{A2} + V_{B2} = 0, \tag{4}$$



The kinetic energy terms are given by

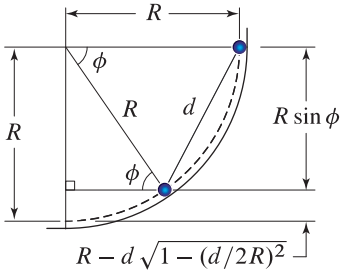
$$T_1 = T_{A1} + T_{B1} = 0, \quad T_2 = T_{A2} + T_{B2} = \tfrac{1}{2}mv_{A2}^2 + \tfrac{1}{2}mv_{B2}^2. \tag{5}$$

When sphere A reaches point D the speeds must be equal, such that $v_{A2} = v_{B2} = v_2$. Eq. (1) becomes

$$mgR + mg\left[R - d\sqrt{1 - d^2/(2R)^2}\right] = mv_2^2, \tag{6}$$

which may be solved to obtain v_2 to obtain

$$\boxed{v_2 = \sqrt{2gR - dg\sqrt{1 - d^2/(2R)^2}}.} \tag{7}$$



In order to find the normal force N_A substitute the kinematic relation $a_n = v^2/R$ into Eq. (2) to obtain

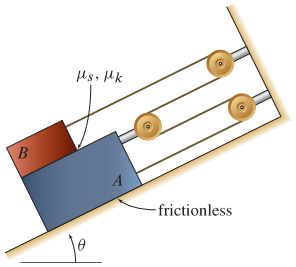
$$N_A - mg = \frac{mv_2^2}{R}, \tag{8}$$

since $v = v_2$ at ②. Applying Eq. (7) to this equation, we can solve for N_A to obtain

$$\boxed{N_A = mg\left(3 - \frac{d}{R}\sqrt{1 - d^2/(2R)^2}\right).} \tag{9}$$

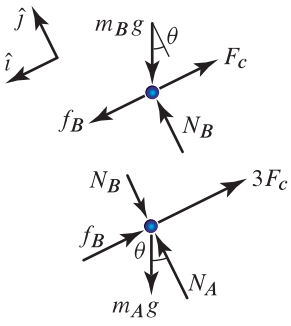
Problem 4.59

Solve Example 4.12 by applying the work-energy principle to each block individually, and show that the net work done by the cord on the two blocks is zero.



Solution

Let ① be the starting point and ② be after the blocks have moved a relative distance d. The work-energy principle for each block:



$$T_{A1} + V_{A1} + (U_{1\text{-}2})^A_{\text{np}} = T_{A2} + V_{A2}, \tag{1}$$

$$T_{B1} + V_{B1} + (U_{1\text{-}2})^B_{\text{np}} = T_{B2} + V_{B2}, \tag{2}$$

Balance Principle:

$$\left(\sum F_y\right)_B : \quad N_B - m_B g \cos\theta = 0. \tag{3}$$

Before writing out the potential energy terms, we need to consider the action of the tension on each block separately. If we consider the motions for each block, Δx_A for block A and Δx_B for block B, to be positive when moving down the incline at the angle θ, then for the tension T we may write

$$(U_{1\text{-}2})^A_T = \int_{x_{A_1}}^{x_{A_2}} -3F_c\, dx_A, \qquad (U_{1\text{-}2})^B_T = \int_{x_{B_1}}^{x_{B_2}} -F_c\, dx_B. \tag{4}$$

Note that the tension F_c is going to be constant and so the integral *can* be evaluated. The point here is that we don't need to know the form of F_c to show that its net work is zero. The datum for each block is taken to be at position ①. This means that the potential energy is due to gravity, such that

$$V_{A1} = 0, \quad V_{A2} = m_A g\,(-\Delta x_A \sin\theta), \quad V_{B1} = 0, \quad V_{B2} = m_B g\,(-\Delta x_B \sin\theta). \tag{5}$$

It is also necessary to account for the work from the friction force f_B acting on each block such that

$$(U_{1\text{-}2})^A_{\text{np}} = \int_0^{\Delta x_A} -f_B\, dx_A = -f_B \Delta x_A, \qquad (U_{1\text{-}2})^B_{\text{np}} = \int_0^{\Delta x_B} f_B\, dx_B = f_B \Delta x_B. \tag{6}$$

One final material relation is necessary to express f_B, such that with Eq. (3), we may write

$$f_B = \mu_k N_B = \mu_k m_B g \cos\theta. \tag{7}$$

The kinetic energy for each block at positions ① and ② is

$$T_{A1} = 0, \qquad T_{A2} = \tfrac{1}{2} m_A v_{A2}^2, \qquad T_{B1} = 0, \qquad T_{B2} = \tfrac{1}{2} m_B v_{B2}^2. \tag{8}$$

We will also utilize the following kinematic relations:

$$\frac{dx_B}{dt} = -3\frac{dx_A}{dt} \quad \Rightarrow \quad dx_b = -3dx_A. \tag{9}$$

Applying Eqs. (4)–(9) to Eqs. (1) and (2), we obtain

$$\tfrac{1}{4}gd\,(m_A \sin\theta - \mu_k m_B \cos\theta) - 3\int_{x_{A_1}}^{x_{A_2}} F_c\, dx_A = \tfrac{1}{2}m_A v_{A2}^2, \tag{10}$$

$$-\tfrac{3}{4}m_B gd\,(\sin\theta + \mu_k \cos\theta) - \int_{x_{A_1}}^{x_{A_2}} F_c\,(-3dx_A) = \tfrac{1}{2}m_B (v_{B2})^2. \tag{11}$$

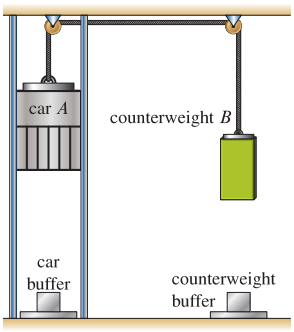
Next, summing Eq. (10) and (11) we have

$$\tfrac{1}{4}dg\,(m_A \sin\theta - \mu_k m_B \cos\theta) - \tfrac{3}{4}m_B gd\,(\sin\theta + \mu_k \cos\theta) = \tfrac{1}{2}m_A v_{A2}^2 + \tfrac{1}{2}m_B v_{B2}^2. \tag{12}$$

The work done by the tension in the cord is zero because it has been eliminated from this equation.

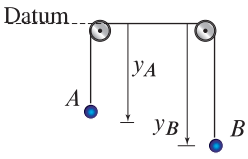
Problem 4.60

Consider a simple elevator design in which a 15,000 kg car A is connected to a 12,000 kg counterweight B. Suppose that a failure of the drive system occurs (the failure does not affect the rope connecting A and B) when the car is at rest and 50 m above its buffer, causing the elevator car to fall. Model the car and the counterweight as particles and the cord as massless and inextensible, and model the action of the emergency brakes using a Coulomb friction model with kinetic friction coefficient $\mu_k = 0.5$ and a normal force equal to 35% of the car's weight. Determine the speed with which the car impacts the buffer.



Solution

Taking ① to be the point where the car is at rest and ② be the point where the car impacts the buffer, we apply the work-energy principle between ① and ② to obtain

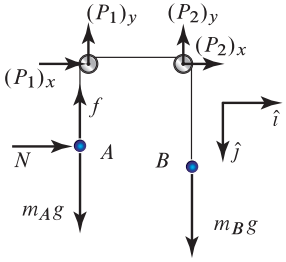


$$T_1 + V_1 + (U_{1\text{-}2})_{\text{np}} = T_2 + V_2. \tag{1}$$

The potential energies are given by

$$V_1 = -m_A g y_{A0} - m_B g y_{B0}, \tag{2}$$

$$V_2 = -m_A g (y_{A0} + h) - m_B g (y_{B0} - h), \tag{3}$$



The friction force is written as $f = \mu_k N$, where N is the normal force. In addition, the work done by friction force is found to be

$$(U_{1\text{-}2})_{\text{np}} = \int_{\mathscr{L}_{1\text{-}2}} -f\,\hat{\jmath}\cdot d\vec{r} = -\int_{y_{A0}}^{y_{A0}+h} f\,\hat{\jmath}\cdot dy\,\hat{\jmath} = -\mu_k N h, \tag{4}$$

The kinetic energies are given by

$$T_1 = 0, \quad \text{and} \quad T_2 = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2. \tag{5}$$

Pulley kinematics:

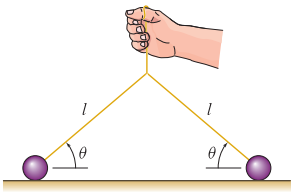
$$L = y_A + y_B \quad \Rightarrow \quad v_A = -v_B. \tag{6}$$

Eq. (1) becomes

$$v_A = \sqrt{\frac{2hg\,[m_A(1 - 0.35\mu_k) - m_B]}{m_A + m_B}} = 3.69\,\text{m/s}. \tag{7}$$

Problem 4.61

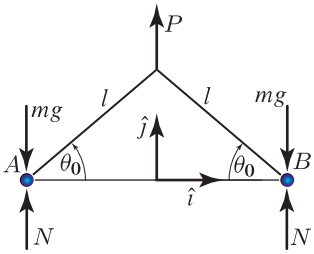
Two identical balls, each of mass m, are connected by a string of negligible mass and length $2l$. A short string is attached to the first at its middle and is pulled vertically with a constant force P (exerted by the hand). If the system starts at rest when $\theta = \theta_0$, determine the speed of the two balls as θ approaches 90°. Neglect the size of the balls as well as friction between the balls and the surface on which they slide.



Solution

Position ① is the starting point, $\theta = \theta_0$, and ② is the point where θ approaches 90°. The balance of work and energy for this problem will be given by

$$T_1 + (U_{1\text{-}2})^{\text{int}}_{\text{np}} + (U_{1\text{-}2})^{\text{ext}}_{\text{np}} = T_2, \tag{1}$$



Because the string is inextensible $(U_{1\text{-}2})^{\text{int}}_{\text{np}} = 0$. By symetry $v_A = v_b = v_2$. Therefore, $(U_{1\text{-}2})^{\text{ext}}_{\text{np}}$ is

$$(U_{1\text{-}2})^{\text{ext}}_{\text{np}} = \int_0^{l(1-\sin\theta_0)} P\, dy = Pl(1-\sin\theta_0), \tag{2}$$

where $l(1-\sin\theta_0)$ is the displacement over which P acts. The kinetic energies are

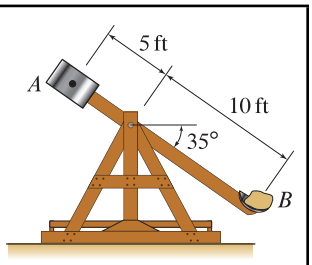
$$T_1 = 0, \quad \text{and,} \quad T_2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}mv_2^2 = mv_2^2,$$

Eq. (1) becomes

$$Pl(1-\sin\theta_0) = mv_2^2 \quad \Rightarrow \quad \boxed{v_2 = \sqrt{Pl(1-\sin\theta_0)/m}.} \tag{3}$$

Problem 4.62

Consider the simple catapult shown in the figure with an 800 lb counterweight A and a 150 lb projectile B. If the system is released from rest as shown, determine the speed of the projectile after the arm rotates (counterclockwise) through an angle of 110°. Model A and B as particles, neglect the mass of the catapult's arm, and assume that friction is negligible. The catapult's frame is fixed with respect to the ground, and the projectile does not separate from the arm during the motion considered.

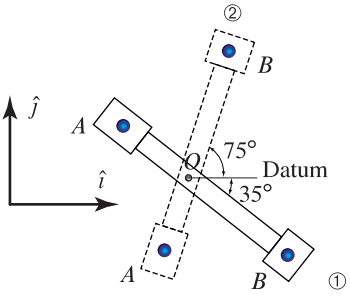


Solution

Since the system is conservative we apply the principle of work and energy in the form

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where ① is when $\theta = -35°$ and ② is when $\theta = 75°$. Choosing the datum line as shown, we have



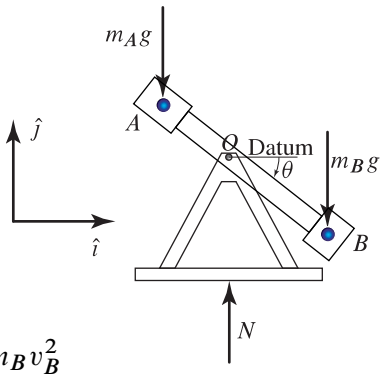
$$V_1 = m_A g \ell_{AO} \sin 35° - m_B g \ell_{OB} \sin 35°, \tag{2}$$

$$V_2 = -m_A g \ell_{AO} \sin 75° + m_B g \ell_{OB} \sin 75°, \tag{3}$$

where $\ell_{AO} = 5$ ft and $\ell_{OB} = 10$ ft. For the kinetic energy we have

$$T_1 = 0 \quad \text{and} \quad T_2 = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2. \tag{4}$$

Because the arm AB is rigid, A and B rotate about O with the same angular velocity $\dot{\theta}$. Thus we can say



$$\dot{\theta} = \frac{v_A}{\ell_{AO}} = \frac{v_B}{\ell_{OB}} \quad \Rightarrow \quad v_A = \frac{\ell_{AO}}{\ell_{OB}} v_B = \frac{v_B}{2}. \tag{5}$$

Using Eqs. (2)–(5), Eq. (1) becomes

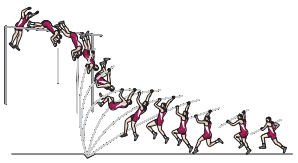
$$m_A g \ell_{AO} \sin 35° - m_B g \ell_{OB} \sin 35° = \frac{1}{2} m_A \left(\frac{v_B}{2}\right)^2 + \frac{1}{2} m_B v_B^2 - m_A g \ell_{AO} \sin 75° + m_B g \ell_{OB} \sin 75°.$$

So that

$$v_B = \sqrt{8g \frac{m_A \ell_{OA} - m_B \ell_{OB}}{m_A + 4m_B} (\sin 35° + \sin 75°)} = 26.6 \text{ ft/s}. \tag{6}$$

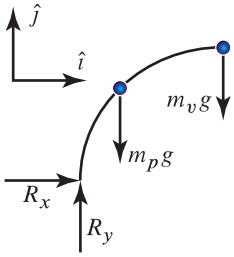
Problem 4.63

Revisit Example 4.6 on p. 288 and determine the maximum height reached by the pole vaulter, but this time include the mass of the pole in your analysis. To solve the problem, use the following data: the maximum speed achieved by the pole vaulter and pole at the time the vault begins is $v_{\max} = 34.19\,\text{ft/s}$; the pole vaulter is 6 ft tall with mass center at 55% of body height (as measured from the ground) and weighs 190 lb; the pole is uniform, has a weight of 5.8 lb, and is 17.1 ft long. In addition, assume that as the pole vaulter sprints before the vault, the pole is carried horizontally at the same height as the vaulter's center of mass relative to the ground. Explain why the vault height you find when the pole is included is higher than the vault height determined in Example 4.6, when the pole was not included.



Solution

The only forces doing work are conservative. Let ① be when the vault begins and ② be when the vaulter achieves maximum height. Work-energy principle:



$$T_1 + V_1 = T_2 + V_2. \tag{1}$$

Let m_v be the mass of the pole vaulter and m_p be the mass of the pole. The kinetic energies are:

$$T_1 = \tfrac{1}{2}\left(m_v + m_p\right) v_{\max}^2 \quad \text{and} \quad T_2 = 0.$$

Set the datum at ground level. Let ℓ be the height of the pole vaulter and L be the length of the pole. Force laws:

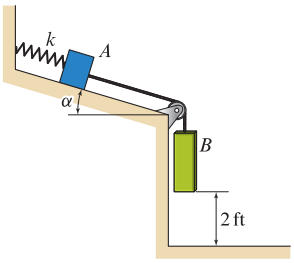
$$V_1 = 0.55\,\ell g\left(m_v + m_p\right) \quad \text{and} \quad V_2 = \tfrac{1}{2} m_p g L + m_v g h_{\max}.$$

Eq. (1) becomes

$$\tfrac{1}{2}\left(m_v + m_p\right) v_{\max}^2 + 0.55\,\ell g\left(m_v + m_p\right) = \tfrac{1}{2} m_p g L + m_v g h_{\max},$$

$$\boxed{h_{\max} = \frac{\left(m_v + m_p\right) v_{\max}^2 + 1.1\,\ell g\left(m_v + m_p\right) - m_p g L}{2 m_v g} = 21.8\,\text{ft}.}$$

Problems 4.64 and 4.65



Two blocks A and B weighing 123 and 234 lb, respectively, are released from rest as shown. At the moment of release the spring is unstretched. In solving these problems, model A and B as particles, neglect air resistance, and assume that the cord is inextensible.

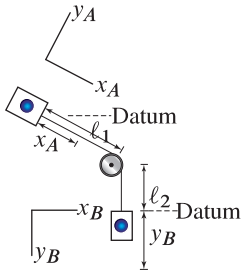
Problem 4.64 Determine the maximum speed attained by block B and the distance from the floor where the maximum speed is achieved if $\alpha = 20°$, the contact between A and the incline is frictionless, and the spring constant is $k = 30\,\text{lb/ft}$.

Problem 4.65 Determine the maximum displacement of block B if $\alpha = 20°$, the contact between A and the incline is frictionless, and the spring constant is $k = 300\,\text{lb/ft}$.

Solution to 4.64

Letting ① to when the system is released and ② be when A moves an arbitrary distance x_A down the incline and B moves an arbitrary distance y_B vertically downwards, we apply the work-energy principle between ① and ② to obtain

$$T_1 + V_1 + (U_{1\text{-}2})_{\text{np}}^{\text{int}} + (U_{1\text{-}2})_{\text{np}}^{\text{ext}} = T_2 + V_2. \tag{1}$$

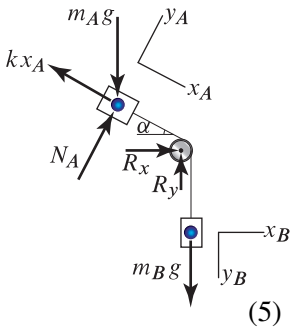


We fix two datum lines, for A and B each, at the initial positions of release. The potential energies are then given by

$$V_1 = 0 \quad \text{and} \quad V_2 = -m_A g x_A \sin\alpha - m_B g y_B + \tfrac{1}{2} k x_A^2. \tag{2}$$

Since there are no external non-conservative forces that do work and assuming that the cord is inextensible, we have

$$(U_{1\text{-}2})_{\text{np}}^{\text{int}} = 0 \quad \text{and} \quad (U_{1\text{-}2})_{\text{np}}^{\text{ext}} = 0. \tag{3}$$



The kinetic energies are given by

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2. \tag{4}$$

We need to relate y_A to y_B and v_A to v_B. Using the constraint that the rope length remains constant, we can write

$$l_1 + l_2 = L, \tag{5}$$

where L is a constant. After A moves a distance x_A down the incline and B moves a distance y_B vertically down, we have

$$l_1 - x_A + l_2 + y_B = L. \tag{6}$$

Substituting Eq. (5) into Eq. (6), we get

$$x_A = y_B. \tag{7}$$

Differentiating the above relation with respect to time, we get

$$v_A = v_B. \tag{8}$$

Substituting Eqs. (2)–(4) into Eq. (1) and using Eqs. (7) and (8), we get

$$v_B^2 = \frac{1}{(m_A + m_B)}\left[2gy_B(m_A \sin\alpha + m_B) - ky_B^2\right]. \tag{9}$$

To find the maximum velocity, we differentiate v_B^2 in Eq. (9) with respect to y_B

$$\frac{dv_B^2}{dy_B} = 0 \quad \Rightarrow \quad \boxed{y_B = \frac{1}{k}(m_A \sin\alpha + m_B)g} = 9.202\,\text{ft}. \tag{10}$$

However, $9.202\,\text{ft} > 2\,\text{ft}$ and so is not acceptable. From Eq. (9) we realize that v_B is a monotonically increasing function for $y_B < \frac{1}{k}(m_A \sin\alpha + m_B)g = 9.202\,\text{ft}$. Therefore, $(v_B)_{\max}$ must occur at $y_B = 2\,\text{ft}$, so that using Eq. (9), we have

$$\boxed{(v_B)_{\max} = v_B(y_B = 2) = 9.42\,\text{ft/s}.} \tag{11}$$

Denoting the position of B with reference to the floor as $d_B = 2 - y_B$, we find that for maximum velocity

$$\boxed{d_B = 0\,\text{ft}.} \tag{12}$$

Solution to 4.65

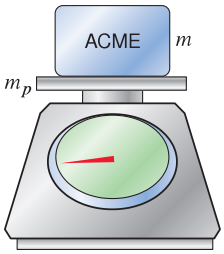
The FBD and balance of work and energy for this problem is the same as done in the solution to Problem 4.65. Maximum displacement must occur when the velocity v_B goes to zero. So, using Eq. (9) from the solution to Problem 4.65, we get

$$v_B = 0 \quad \Rightarrow \quad \boxed{y_B = \frac{2}{k}(m_A \sin\alpha + m_B)g = 1.84\,\text{ft},} \tag{13}$$

where $m_A g = 123\,\text{lb}$, $m_B g = 234\,\text{lb}$ and $k = 300\,\text{lb/ft}$.

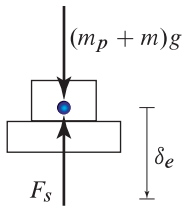
Problem 4.66

Spring scales work by measuring the displacement of a spring that supports both the platform of mass m_p and the object of mass m, whose weight is being measured. In your solution, note that most scales read zero when no mass m has been placed on them; that is, they are calibrated so that the weight reading neglects the mass of the platform m_p. Assume that the spring is linear elastic with spring constant k. If the mass m is gently placed on the spring scale (i.e., it is dropped from zero height above the scale), determine the *maximum reading* on the scale after the mass is released.



Solution

Let ① be when the spring is compressed an amount δ_0 by only the weight of the platform, and ② be after release when the system has stopped for the first time. F_s is the spring force and δ_e is the distance that the mass center travels between ① and ②. Let W be the maximum reading on the scale, then



$$W = k\delta_e. \tag{1}$$

The work-energy principle is

$$T_1 + V_1 = T_2 + V_2. \tag{2}$$

We take the datum line to be at ①. Since the the scale reads zero with only the platform of mass m_p, the spring must initially be compressed by an amount $\delta_0 = \frac{m_p g}{k}$. The kinetic and potential energies at ① and ② are

$$V_1 = \tfrac{1}{2}k\delta_0^2 \quad \text{and} \quad V_2 = -(m_p + m)g\delta_e + \tfrac{1}{2}k\left(\delta_e + \delta_0\right)^2. \tag{3}$$

$$T_1 = 0 \quad \text{and} \quad T_2 = 0. \tag{4}$$

Substituting Eqs. (3)–(4) into Eq. (2), we get

$$\tfrac{1}{2}k(\delta_0 + \delta_e)^2 - \tfrac{1}{2}k\delta_0^2 = (m_p + m)g\delta_e \quad \Rightarrow \quad 2\delta_0 + \delta_e = 2\frac{(m_p + m)g}{k}. \tag{5}$$

Now, substituting $\delta_0 = \frac{m_p g}{k}$ in the above expression, we have

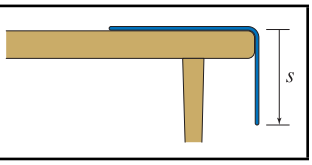
$$\delta_e = 2\frac{mg}{k}. \tag{6}$$

Substituting Eq. (6) into Eq. (1), we get

$$\boxed{W = 2mg.} \tag{7}$$

Problem 4.67

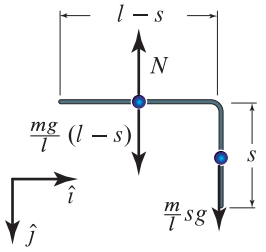
The rope of mass m and length l is released from rest with a *very* small amount of it hanging over the edge (i.e., $s > 0$ but it is very close to zero). Determine the speed of the rope as a function of s. Assume the surface is smooth.



Solution

Since the system is conservative the work-energy principle is

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$



where position ① is taken to be when $s \approx 0$ and position ② occurs at some point when $s > 0$. With the datum as the table surface, it follows that

$$V_1 = 0 \quad \text{and} \quad V_2 = W_v\,(-s/2)\,, \tag{2}$$

where W_v refers to the weight of the vertical portion of the rope. The weight of the horizontal portion of the rope is W_h, the constraint force perpendicular to the table is N and the rope's mass is m; it follows that

$$W_{h1} = mg, \quad W_{h2} = [(l-s)/l]\,mg, \quad \text{and} \quad W_{v2} = (s/l)\,mg. \tag{3}$$

Applying Eqs. (3) to Eqs. (2) leads to the following potential energy terms:

$$V_1 = 0 \quad \text{and} \quad V_2 = -s^2 mg/2l. \tag{4}$$

The kinetic energies are given by

$$T_1 = 0, \quad T_2 = \tfrac{1}{2}\,[(l-s)/l]\,m v_{h2}^2 + \tfrac{1}{2}\,(s/l)\,m v_{v2}^2 = \tfrac{1}{2} m v_2^2, \tag{5}$$

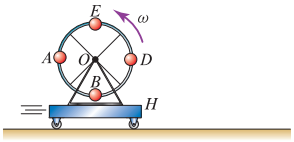
where we note that $v_{h2} = v_{v2} = v_2$. Applying Eqs. (4) and (5) to Eq. (1), the answer for part (a) is

$$0 = m v_2^2/2 - s^2 mg/2l \quad \Rightarrow \quad \boxed{v_2 = s\sqrt{\frac{g}{l}},} \tag{6}$$

for $s > 0$.

Problem 4.68

Consider a system of four identical particles A, B, D, and E, each with a mass of 2.3 kg. These particles are mounted on a wheel of negligible mass whose hub is at O and whose radius is $R = 0.75$ m. The wheel is mounted on a cart H of negligible mass. Suppose that the wheel is spinning counterclockwise with an angular speed $\omega = 3$ rad/s and that the cart is moving to the right with a speed of $v_H = 11$ m/s. Compute the kinetic energy of the system, using



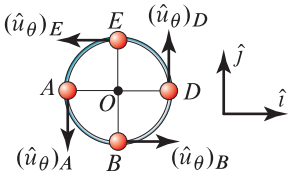
(a) Eq. (4.71) on p. 305,

(b) Eq. (4.74) on p. 306.

Show that the result is the same in both cases.

Solution

Part (a). The velocity of the particles are



$$\vec{v}_A = \vec{v}_{A/O} + \vec{v}_O = R\omega\,(\hat{u}_\theta)_A + v_O\,\hat{\imath} = -R\omega\,\hat{\jmath} + v_O\,\hat{\imath}, \tag{1}$$

$$\vec{v}_B = \vec{v}_{B/O} + \vec{v}_O = R\omega\,(\hat{u}_\theta)_B + v_O\,\hat{\imath} = (R\omega + v_O)\,\hat{\imath}, \tag{2}$$

$$\vec{v}_D = \vec{v}_{D/O} + \vec{v}_O = R\omega\,(\hat{u}_\theta)_D + v_O\,\hat{\imath} = R\omega\,\hat{\jmath} + v_O\,\hat{\imath}, \tag{3}$$

$$\vec{v}_E = \vec{v}_{E/O} + \vec{v}_O = R\omega\,(\hat{u}_\theta)_E + v_O\,\hat{\imath} = (v_O - R\omega)\,\hat{\imath}. \tag{4}$$

Using Eq. (4.71) the kinetic energy of the system is

$$T = \sum_{i=1}^{n} \tfrac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \tfrac{1}{2} m (v_A^2 + v_B^2 + v_D^2 + v_E^2), \tag{5}$$

where $m_A = m_B = m_D = m_E = m$. Using Eqs. (1)–(4), we have

$$v_A^2 = R^2\omega^2 + v_O^2, \quad v_B^2 = (R\omega + v_O)^2, \quad v_D^2 = R^2\omega^2 + v_O^2 \quad \text{and} \quad v_E^2 = (v_O - R\omega)^2. \tag{6}$$

Substituting Eqs. (6) into Eq. (5), we get

$$\boxed{T = 2m(v_O^2 + R^2\omega^2) = 580\,\text{J},} \tag{7}$$

where $v_O = v_H = 11$ m/s.

Part (b). Using Eq. (4.74) the total kinetic energy of the system is

$$T = \tfrac{1}{2} m_T v_C^2 + \tfrac{1}{2} \sum_{i=1}^{n} m_i v_{i/C}^2.$$

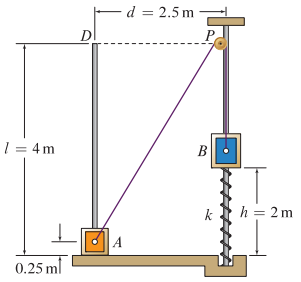
Here the center of mass of the system is clearly at O. Therefore, $v_C = v_O$ and $v_{A/C} = v_{B/C} = v_{D/C} = v_{E/C} = R\omega$. The total kinetic energy of the system is then given by

$$\boxed{T = \tfrac{1}{2}(4m)v_O^2 + \tfrac{1}{2} m(4R^2\omega^2) = 2m(v_O^2 + R^2\omega^2) = 580\,\text{J}.} \tag{8}$$

By comparing Eqs. (7) and (8), and recognizing that $v_C = v_H$, we see that same result is obtained in both cases.

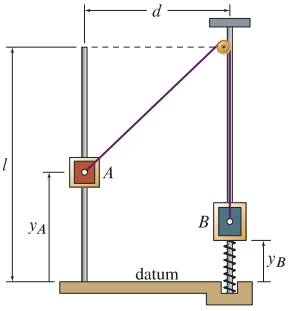
Problem 4.69

Reexamine Example 4.10 by choosing ① to be the same as that in the example and ② to be an arbitrary position after release. Determine the speed of A as a function of y_A, and plot the kinetic energy of A as a function of y_A. Use this plot to argue that the cord connecting A and B never goes slack between the instant of release and the impact of B with the ground. *Hint:* To argue that the cord remains taut between the positions of interest, you may want to look at concept Prob. 4.21 on p. 274.



Solution

The following figure and equations from Example 4.10 are repeated for your convenience.



$$L = \sqrt{(l - y_{A1})^2 + d^2} + (l - h) = \sqrt{(l - y_{A2})^2 + d^2} + (l - y_{B2}).$$

$$v_B = \left| -\frac{(l - y_A)}{\sqrt{(l - y_A)^2 + d^2}} \right| v_A.$$

Solve the first equation for y_{B2} letting $y_{B1} = h$.

$$y_B = \sqrt{(l - y_{A2})^2 + d^2} - \sqrt{(l - y_{A1})^2 + d^2} + h.$$

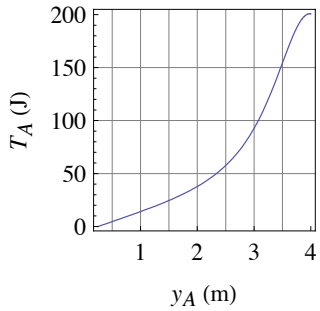
All forces doing work are conservative. Use the following *Mathematica* code to solve the work-energy equation for the velocity as a function of y_A then plot the kinetic energy of A as a function of y_A.

```
rules = {ya1 → 0.25, d → 2.5, l → 4, h → 2, ma → 1, mb → 8, g → 9.81, k → 40};

kinematics = {yb → Sqrt[(l - ya)^2 + d^2] - Sqrt[(l - ya1)^2 + d^2] + h, vb → (l - ya)/Sqrt[(l - ya)^2 + d^2] va};

KE =
  Solve[1/2 k h^2 + ma g ya1 + mb g h == ma g ya + mb g yb + 1/2 ma va^2 + 1/2 mb vb^2 + 1/2 k (yb)^2 /.
    kinematics //. rules, va];

Plot[1/2 ma va^2 /. KE[[2]] /. rules, {ya, 0.25, 4}, AxesOrigin → {0.25, 0}, Frame → True,
 FrameTicks → {{Automatic, None}, {{1, 2, 3, 4}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"y_A (m)", "T_A (J)"}]
```



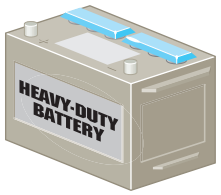
Referring to the plot for T_A, we conclude that $\frac{dT_A}{dy_A} > 0$ for all positions of interest or in other words, $T_{A2} > T_{A1}$. Thus, we can write

$$\int_{y_{A1}}^{y_{A2}} \vec{T}_c \cdot dy_A \hat{j} > 0 \quad \Rightarrow \quad \vec{T}_c \neq 0 \tag{1}$$

between any two postions ① and ②, before B hits the ground. Hence, the rope never goes slack between the instant of release and the impact of B with the ground.

Problem 4.70

Consider a battery that can power a machine whose operation requires 200 W of power. If the battery can keep the machine operating for 8 h and supply energy at a constant rate, determine the amount of energy initially stored in the battery pack.



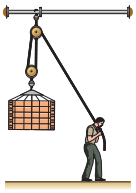
Solution

Use the definition of power. Because the energy is supplied at a constant rate we solve for the energy with

$$P = \frac{dU}{dt} \quad \Rightarrow \quad U = Pt \quad \Rightarrow \quad U = (200\,\text{W})\,(8 * 3600\,\text{s}) \quad \Rightarrow \quad \boxed{U = 5760\,\text{kJ}.}$$

Problem 4.71

A person takes 16 s to lift a 200 lb crate to a height of 30 ft. Determine the average power supplied by the person.



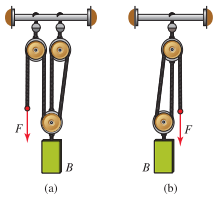
Solution

The average power is found by dividing the work of the lift by the time of the lift.

$$W = mgh \quad \Rightarrow \quad P_{\text{avg}} = \frac{(200\,\text{lb})\,(30\,\text{ft})}{16\,\text{s}} \quad \Rightarrow \quad \boxed{P_{\text{avg}} = 375\,\text{ft}\cdot\text{lb/s} = 0.682\,\text{hp}.}$$

Problem 4.72

The weights B in the pulley systems shown are identical. Assume that the two systems are released from rest, there are no energy losses in the pulleys, and the same constant force F is applied. After 1 s, will the power developed by the force F for system (a) be smaller than, equal to, or greater than that of system (b)?

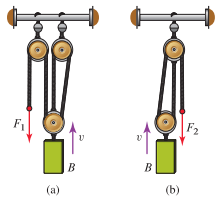


Solution

The power in case (a) is greater than in case (b) since the force in case (a) causes B to accelerate faster than in case (b).

Problem 4.73

The weights B in the pulley systems shown are identical and are being lifted at the same constant speed v. Assume there are no energy losses in the pulleys. Is the power developed by force F_1 in (a) smaller than, equal to, or greater than that developed by force F_2 in (b)?

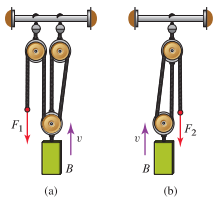


Solution

The power of the force F_1 is the same as the power of the force F_2 because the power in both cases is equal to the power of the weight of the block.

Problem 4.74

For the two pulley systems shown, the block B has a mass $m_B = 254\,\text{kg}$ and is being lifted with a constant speed $v = 3\,\text{m/s}$. Determine the power developed by the forces F_1 and F_2.



Solution

Because the blocks are being lifted at constant speed the forces are

$$F_1 = \tfrac{1}{3} m_B g \quad \text{and} \quad F_2 = \tfrac{1}{2} m_B g.$$

From the pulley kinematics we see that the speed of the point of application of F_1 is $3v$ and that the speed of the point of application of F_2 is $2v$.

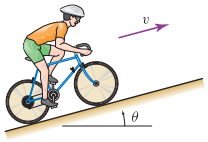
$$P_1 = 3F_1 v = m_B g v \quad \text{and} \quad P_2 = 2F_2 v = m_B g v.$$

Recalling that we have $m_B = 254\,\text{kg}$, $g = 981\,\text{m/s}^2$, and $v = 3\,\text{m/s}$, we have

$$\boxed{P_1 = P_2 = 7.48\,\text{kW}.}$$

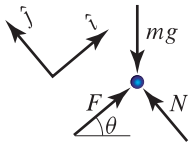
Problem 4.75

A cyclist is riding with a speed $v = 18$ mph over an inclined road with $\theta = 15°$. Neglecting aerodynamic drag, if the cyclist were to keep his power output constant, what speed would he attain if θ were equal to 20°?



Solution

Balance Principles:



$$\sum F_x: \quad F - mg\sin\theta = 0 \quad \Rightarrow \quad F = mg\sin\theta.$$

Because the power is constant we can say

$$P = F_1 v_1 = F_2 v_2 \quad \Rightarrow \quad mg\,v_1 \sin\theta_1 = mg\,v_2 \sin\theta_2$$

$$\boxed{v_2 = v_1 \frac{\sin\theta_1}{\sin\theta_2} = 13.6\,\text{mph}.}$$

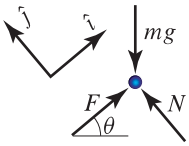
Problem 4.76

A 1500 kg car is traveling up the slope shown at a constant speed. Knowing that the maximum power output of the car is 160 hp, at what speed can the car travel if air resistance is negligible? Also, knowing that 1 L of regular gasoline provides 34.8 MJ of energy, how many liters of gasoline will be required in 1 h if the engine has an efficiency $\epsilon = 0.20$?



Solution

Balance Principles:



$$\sum F_x: \quad F - mg\sin\theta = 0 \quad \Rightarrow \quad F = mg\sin\theta.$$

Recalling that 1 hp = 745.7 W, the speed the car can travel is

$$P = Fv \quad \Rightarrow \quad v = \frac{P}{mg\sin\theta} \quad \Rightarrow \quad \boxed{v = 54.7\,\text{m/s} = 197\,\text{km/h}.}$$

From the definition of efficiency we find the power input of the car with

$$P_i = \frac{P_o}{\epsilon} = 596.6\,\text{kW} = 596.6\,\text{kJ/s}.$$

The consumption rate of the car is

$$CR = \frac{0.5966\,\text{MJ/s}}{34.8\,\text{MJ/L}} \quad \Rightarrow \quad \boxed{CR = 0.0171\,\text{L/s} = 61.7\,\text{L/h}.}$$

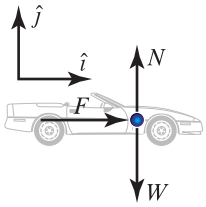
Problem 4.77

A 2600 lb car on straight horizontal road goes from 0 to 60 mph in 7 s. Neglecting aerodynamic drag, if the force propelling the car is constant during the acceleration phase, determine the power developed by this force 7 s after the beginning of the motion.



Solution

Neglecting drag, a constant propulsive force leads to a constant acceleration. That acceleration is



$$a_c = \frac{v_f}{t_f}.$$

The force that causes this acceleration is

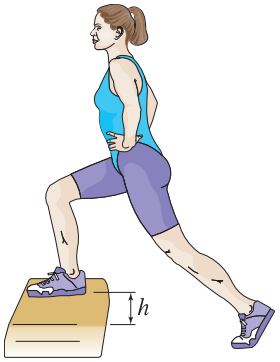
$$F = ma_c = \frac{mv_f}{t_f}.$$

The Power of this force at 7 s is

$$P = Fv_f \quad \Rightarrow \quad \boxed{P = \frac{mv_f^2}{t_f} = 162\,\text{hp}.}$$

Problem 4.78

The height h of the aerobic stepper shown is 25 cm. A 65 kg person goes up and down the step once every 2 s. Accounting only for the work done in lifting her body and assuming that a muscle efficiency is 25%, determine the calories (1 C = 4.184 kJ) burned by the person in 1 h.



Solution

The work per cycle is

$$W = mgh.$$

The number of cycles preformed in one hour is $N_c = 1800$. The work done in one hour is

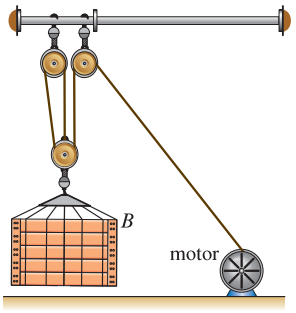
$$W_T = N_c mgh.$$

The energy required by her body is

$$\text{Energy Burned} = \frac{N_c mgh}{\epsilon} \quad \Rightarrow \quad \boxed{\text{Energy Burned} = 274\,\text{C}.}$$

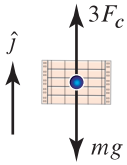
Problem 4.79

The motor powering the pulley system shown is drawing 6 kW of power. The crate B has a mass of 250 kg. Determine the speed of the crate if the crate is lifted at a constant speed and the motor's efficiency is $\epsilon = 0.87$.



Solution

Because the velocity is constant the force in the cord is



$$\sum F_y : \quad 3F_c - mg = 0 \quad \Rightarrow \quad F_c = \tfrac{1}{3}mg.$$

Let v_c be the velocity of the cord being pulled in by the motor. The power output of the motor is

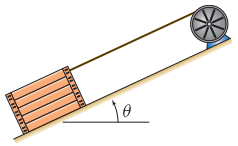
$$P_o = F_c v_c = P_i \epsilon \quad \Rightarrow \quad v_c = \frac{3P_i\epsilon}{mg}.$$

The pulley kinematics tells us that the speed of the crate is $\frac{1}{3}$ the speed of the cord being pulled in.

$$\boxed{v_B = \frac{P_i\epsilon}{mg} = 2.13\,\text{m/s}.}$$

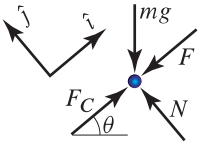
Problem 4.80

Assuming that the motor shown has an efficiency $\epsilon = 0.80$, determine the power to be supplied to the motor if it is to pull a 250 lb crate up the incline with a constant speed $v = 7$ ft/s. Assume that the kinetic friction coefficient between the slide and the crate is $\mu_k = 0.25$ and that $\theta = 28°$.



Solution

Balance Principles:



$$\sum F_x: \quad F_c - F - mg\sin\theta = 0, \tag{1}$$

$$\sum F_y: \quad N - mg\cos\theta = 0. \tag{2}$$

Force law:

$$F = \mu_k N \quad \Rightarrow \quad F = \mu_k mg\cos\theta.$$

Eq. (1) becomes

$$F_c = mg\left(\mu_k\cos\theta + \sin\theta\right).$$

The power delivered by the motor is

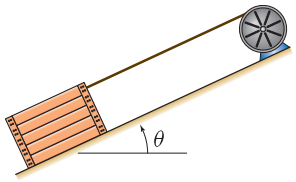
$$P_o = F_c v \quad \Rightarrow \quad P_o = mgv\left(\mu_k\cos\theta + \sin\theta\right).$$

The power supplied to the motor is

$$P_i = \frac{P_o}{\epsilon} \quad \Rightarrow \quad \boxed{P_i = \frac{mgv\left(\mu_k\cos\theta + \sin\theta\right)}{\epsilon} = 1.51\times10^3 \text{ ft·lb/s} = 2.75 \text{ hp.}}$$

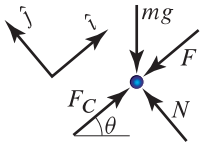
Problem 4.81

Assume that the motor shown has an efficiency $\epsilon = 0.82$ and that it draws 6 hp. Determine the constant speed with which the motor can pull a 300 lb crate up the incline if the kinetic friction coefficient between the slide and the crate is $\mu_k = 0.45$ and $\theta = 32°$.



Solution

Balance Principles:



$$\sum F_x: \quad F_c - F - mg\sin\theta = 0, \tag{1}$$

$$\sum F_y: \quad N - mg\cos\theta = 0. \tag{2}$$

Force law:

$$F = \mu_k N \quad \Rightarrow \quad F = \mu_k mg\cos\theta.$$

Eq. (1) becomes

$$F_c = mg\left(\mu_k\cos\theta + \sin\theta\right).$$

The power delivered by the motor is

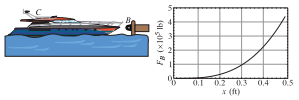
$$P_o = F_c v \quad \Rightarrow \quad v = \frac{P_o}{F_c}.$$

Since $P_o = \epsilon P_i$ the velocity is

$$\boxed{v = \frac{\epsilon P_i}{mg\left(\mu_k\cos\theta + \sin\theta\right)} = 9.90\,\text{ft/s}.}$$

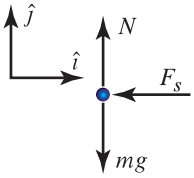
Problem 4.82

Rubber bumpers are commonly used in marine applications to keep boats and ships from getting damaged by docks. Consider a boat with a mass of 35,000 kg and a bumper with a force compression profile given by $F_B = \beta x^3$, where β is a constant and x is the compression of the bumper. Treating the boat C as a particle, neglecting its vertical motion, and neglecting the drag force between the water and the boat C, determine β so that if the boat were to impact the bumper with a speed of 4 m/s, the maximum compression of the bumper would be 18 cm.



Solution

We take advantage of the potential energy function for the bumper derived in Problem 4.36.



$$V = \tfrac{1}{4}\beta x^4.$$

Let ① be at impact and ② be when the boat stops. Work-energy principle:

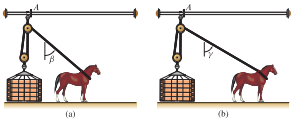
$$T_1 + V_1 = T_2 + V_2 \quad \Rightarrow \quad \tfrac{1}{2}mv_1^2 = \tfrac{1}{4}\beta x^4 \quad \Rightarrow$$

$$\boxed{\beta = \frac{2mv_1^2}{x_2^4} = 1.07\times 10^9\ \text{kg/m}^2\text{s}^2,}$$

where $x_2 = 0.18\,\text{m}$ and $v_1 = 4\,\text{m/s}$.

Problem 4.83

Starting from the position shown, assume that each horse moves to the right in such a way that the tension in the cord is the same in cases (a) and (b) and remains constant. Knowing that $\beta < \gamma$ and that in both cases (a) and (b) the horse advances by an equal amount L, determine which of the following statements is true: (1) the tension in the cord does more work in (a) than in (b); (2) the tension in the cord does exactly the same amount of work in (a) as in (b); (3) the tension in the cord does less work in (a) than in (b).

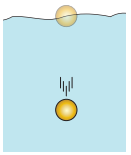


Solution

The tension in the cord does more work in case (b) because the component of this force in the direction of motion is larger than the same component in case (a).

Problem 4.84

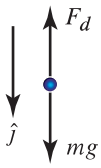
A metal ball with mass $m = 0.15\,\text{kg}$ is dropped from rest in a fluid. The magnitude of the resistance due to the fluid is given by $C_d v$, where C_d is a drag coefficient and v is the ball's speed. If $C_d = 2.1\,\text{kg/s}$, determine the total work done on the ball from the moment of release until the ball achieves 99% of terminal velocity.



Solution

Work-energy principle

$$U_{1\text{-}2} = T_2 - T_1. \tag{1}$$



The kinetic energies are

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2} m v_2^2,$$

where v_2 is 99% of the terminal velocity v_t. Force law: $F_d = C_d v$.

The terminal velocity of the ball is found with the following balance equation:

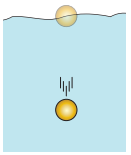
$$\sum F_y: \quad mg - F_d = 0 \quad \Rightarrow \quad C_d v_t = mg \quad \Rightarrow \quad v_t = \frac{mg}{C_d}.$$

Eq. (1) becomes

$$\boxed{U_{1\text{-}2} = \tfrac{1}{2} m \left(\frac{0.99\, mg}{C_d} \right)^2 = 0.0361\,\text{J}.}$$

Problem 4.85

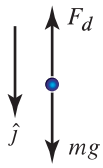
A metal ball weighing 0.2 lb is dropped from rest in a fluid. If the magnitude of the resistance due to the fluid is given by $C_d v$, where $C_d = 0.5\,\text{lb}\cdot\text{s/ft}$ is a drag coefficient and v is the ball's speed, determine the work done by the drag force during the first 2 s of the ball's motion.



Solution

Let ① be at release and ② be after 2 s of motion. Work-energy principle

$$T_1 + V_1 + (U_{1\text{-}2})_d = T_2 + V_2, \tag{1}$$



where $(U_{1\text{-}2})_d$ is the work of the drag force. We need to find the speed of the ball and the distance traveled after 2 s. Use the following balance equation:

$$\sum F_y : \quad mg - F_d = m\ddot{y}.$$

Force law:

$$F_d = C_d \dot{y}.$$

We have acceleration as a function of velocity. Integrate it as follows:

$$\ddot{y} = \frac{d\dot{y}}{dt} \quad \Rightarrow \quad \int_0^t dt = \int_0^{\dot{y}} \frac{d\dot{y}}{g - (C_d/m)\dot{y}} \quad \Rightarrow \quad t = \frac{-m}{C_d} \ln\left(1 - \frac{C_d}{mg}\dot{y}\right).$$

Invert this equation to find $\dot{y}(t)$.

$$\dot{y} = \frac{mg}{C_d}\left(1 - e^{-C_d t/m}\right). \tag{2}$$

Integrate this equation to find $y(t)$. Let d be the distance traveled from $t = 0$ to $t = t_2$, where t_2 is 2 s.

$$\dot{y} = \frac{dy}{dt} \quad \Rightarrow \quad \int_0^d dy = \frac{mg}{C_d}\int_0^{t_2}\left(1 - e^{-C_d t/m}\right)dt \quad \Rightarrow \quad d = \frac{mg}{C_d}t_2 + \frac{m^2 g}{C_d^2}\left(e^{-C_d t_2/m} - 1\right)$$

Evaluate Eq. (2) at $t = t_2$ and plug this expression into Eq. (1)

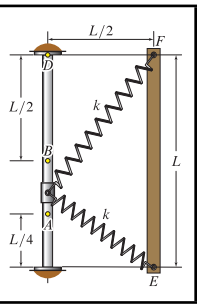
$$(U_{1\text{-}2})_d = \tfrac{1}{2}mv_2^2 - mgd \quad \Rightarrow$$

$$(U_{1\text{-}2})_d = \tfrac{1}{2}m\left[\frac{mg}{C_d}\left(1 - e^{-C_d t_2/m}\right)\right]^2 - mg\left[\frac{mg}{C_d}t_2 + \frac{m^2 g}{C_d^2}\left(e^{-C_d t_2/m} - 1\right)\right]$$

$$(U_{1\text{-}2})_d = \frac{m^3 g^2}{2C_d^2}\left(1 - e^{-C_d t_2/m}\right)^2 - \frac{m^2 g^2 t_2}{C_d} - \frac{m^3 g^2}{C_d^2}\left(e^{-C_d t_2/m} - 1\right) = -0.159\,\text{ft}\cdot\text{lb}.$$

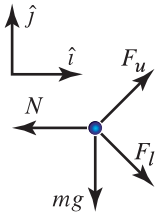
Problem 4.86

An 11 kg collar is constrained to travel along a rectilinear and frictionless bar of length $L = 2$ m that lies in the vertical plane. The springs attached to the collar are identical, and they are unstretched when the collar is at B. Treating the collar as a particle, neglecting air resistance, and knowing that at A the collar is moving upward with a speed of 23 m/s, determine the spring constant k so that the collar reaches D with zero speed. Points E and F are fixed.



Solution

Let ① be when the collar is at A and ② be when the collar is at D. All forces doing work are conservative. Work-energy principle



$$T_1 + V_1 = T_2 + V_2. \tag{1}$$

The kinetic energies are

$$T_1 = \tfrac{1}{2}mv^2 \quad \text{and} \quad T_2 = 0.$$

The unstretched length of the springs is $L_0 = L/\sqrt{2}$. The stretches in the upper and lower springs at ① and ② are

$$\delta_{l1} = L\left(\frac{\sqrt{5}}{4} - \frac{1}{\sqrt{2}}\right), \qquad \delta_{u1} = L\left(\frac{\sqrt{13}}{4} - \frac{1}{\sqrt{2}}\right), \tag{2}$$

$$\delta_{l2} = L\left(\frac{\sqrt{5}}{2} - \frac{1}{\sqrt{2}}\right), \qquad \delta_{u2} = L\left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right). \tag{3}$$

Choose the datum to be at D. Force laws:

$$V_1 = \tfrac{1}{2}k\delta_{u1}^2 + \tfrac{1}{2}k\delta_{l1}^2 - \tfrac{3}{4}mgL \quad \text{and} \quad V_2 = \tfrac{1}{2}k\delta_{u2}^2 + \tfrac{1}{2}k\delta_{l2}^2.$$

Eq. (1) becomes

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}k\delta_{u1}^2 + \tfrac{1}{2}k\delta_{l1}^2 - \tfrac{3}{4}mgL = \tfrac{1}{2}k\delta_{u2}^2 + \tfrac{1}{2}k\delta_{l2}^2.$$

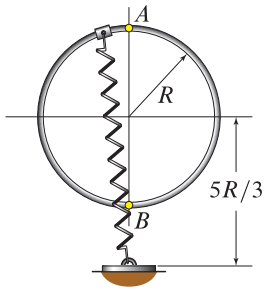
Solve for k:

$$k = \frac{mv^2 - \frac{3}{2}mgL}{\delta_{u2}^2 + \delta_{l2}^2 - \delta_{u1}^2 - \delta_{l1}^2} = 9030\,\text{N/m},$$

where the stretches are given by Eqs. (2) through (3).

Problem 4.87

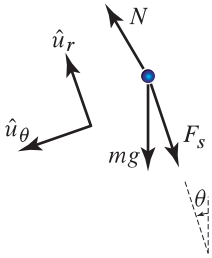
A 7 lb collar is constrained to travel along a frictionless vertical ring of radius $R = 1$ ft. The spring attached to the collar has a spring constant $k = 20$ lb/ft. Treating the collar as a particle, neglecting air resistance, and knowing that, while at rest at A, the collar is displaced gently to the left, determine the spring's unstretched length if the collar is to reach point B with a speed of 15 ft/s.



Solution

Let ① be at A and ② be at B. All forces doing work are conservative. Work-energy principle:

$$T_1 + V_1 = T_2 + V_2. \tag{1}$$



The kinetic energies are

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2.$$

Set the datum at B. Force laws:

$$V_1 = 2mgR + \tfrac{1}{2}k\left(\tfrac{8}{3}R - L_0\right)^2 \quad \text{and} \quad V_2 = \tfrac{1}{2}k\left(\tfrac{2}{3}R - L_0\right)^2.$$

Eq. (1) becomes

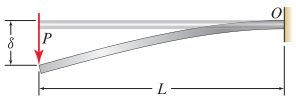
$$2mgR + \tfrac{1}{2}k\left(\tfrac{8}{3}R - L_0\right)^2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}k\left(\tfrac{2}{3}R - L_0\right)^2 \quad \Rightarrow$$

$$\boxed{L_0 = \frac{mg}{k} + \frac{5R}{3} - \frac{mv_2^2}{4Rk} = 1.41 \text{ ft},}$$

where $g = 32.2$ ft/s^2, $v_2 = 15$ ft/s, and $m = \frac{7}{32.2}$ slug.

Problem 4.88

Strength of materials tells us that if a load P is applied at the free end of a cantilevered beam, then the tip displacement δ is given by $\delta = PL^3/(3EI_{cs})$, where L is the length of the beam and E and I_{cs} are constants that depend on the material makeup and the geometry of the cross section, respectively. Determine an expression for the potential energy of a cantilevered beam loaded as shown.



Solution

We are given that

$$P = \frac{3EI_{cs}}{L^3}\delta.$$

We can interpret this relation by saying that the beam behaves like a linear spring with constant

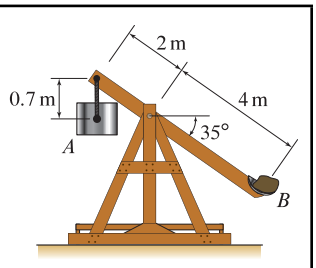
$$k = \frac{3EI_{cs}}{L^3}.$$

Therefore the potential energy is

$$\boxed{V = \frac{3EI_{cs}}{2L^3}\delta^2.}$$

Problem 4.89

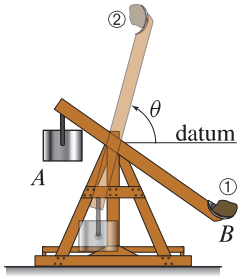
Consider the catapult shown in the figure with a 1200 kg counterweight A and a 330 kg projectile B. If the system is released from rest as shown, determine the speed of the projectile after the arm rotates (counterclockwise) through an angle of 110°. Model A and B as particles; assume that the catapult's arm has negligible mass and that friction is negligible. In addition, assume that the cord has negligible mass, is inextensible, and is always vertical. The catapult's frame is fixed with respect to the ground, and the projectile does not separate from the arm during the motion considered.



Solution

Let ① be at release from rest and ② be after rotating 110°. Because of the assumptions all forces doing work are conservative. Work-energy principle:

$$T_1 + V_1 = T_2 + V_2. \tag{1}$$

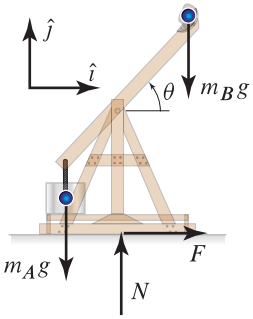


The kinetic energies are

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2} m_A v_{A_2}^2 + \tfrac{1}{2} m_B v_{B_2}^2.$$

Let $L = 4$ m. Kinematics:

$$v_{A_2} = \tfrac{1}{2} L \dot{\theta}_2^2 \quad \text{and} \quad v_{B_2} = L \dot{\theta}_2^2 \quad \Rightarrow \quad v_{A_2} = \tfrac{1}{2} v_{B_2}.$$



At ① $\theta_1 = -35°$ and at ② $\theta_2 = 75°$. Force laws:

$$V_1 = m_B g L \sin\theta_1 - \tfrac{1}{2} m_A g L \sin\theta_1 \quad \text{and} \quad m_B g L \sin\theta_2 - \tfrac{1}{2} m_A g L \sin\theta_2.$$

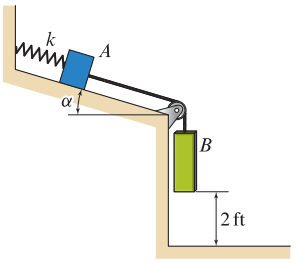
Note: because the rope is always vertical its length does not appear in the potential energy terms. Eq. (1) becomes

$$\begin{aligned} m_B g L \sin\theta_1 - \tfrac{1}{2} m_A g L \sin\theta_1 &= \tfrac{1}{2} m_A v_{A_2}^2 + \tfrac{1}{2} m_B v_{B_2}^2 + m_B g L \sin\theta_2 - \tfrac{1}{2} m_A g L \sin\theta_2 \quad \Rightarrow \\ g L \sin\theta_1 \left(m_B - \tfrac{1}{2} m_A\right) &= \tfrac{1}{8} m_A v_{B_2}^2 + \tfrac{1}{2} m_B v_{B_2}^2 + g L \sin\theta_2 \left(m_B - \tfrac{1}{2} m_A\right) \quad \Rightarrow \\ v_{B_2}^2 \left(4 m_B + m_A\right) &= 8 g L \left(m_B - \tfrac{1}{2} m_A\right) \left(\sin\theta_1 - \sin\theta_2\right) \quad \Rightarrow \end{aligned}$$

$$v_{B_2} = \sqrt{\frac{8 g L \left(m_B - \tfrac{1}{2} m_A\right) \left(\sin\theta_1 - \sin\theta_2\right)}{4 m_B + m_A}} = 7.20 \text{ m/s}.$$

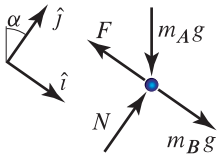
Problem 4.90

Two blocks A and B weighing 123 and 234 lb, respectively, are released from rest as shown. At the moment of release the spring is unstretched. Model A and B as particles, neglect air resistance, and assume that the cord is inextensible. Determine the maximum speed attained by block B and the distance from the floor where the maximum speed is achieved if $\alpha = 35°$, the static and kinetic friction coefficients are $\mu_s = 0.25$ and $\mu_k = 0.2$, respectively, and the spring constant is $k = 25\,\text{lb/ft}$.



Solution

First we must establish whether or not A slides. Assuming that A does not slide we have:



$$\sum F_x: \quad m_A g \sin\alpha - F + m_B g = 0,$$
$$\sum F_y: \quad N - m_A g \cos\alpha = 0,$$

where $F = \mu_k N$. Solving these equations for μ_k

$$\mu_k = \tan\alpha + \frac{m_B}{m_A \cos\alpha} = 3.023.$$

This represents the minimum value of μ_k to stop A from sliding. Thus A will slide.
Let ① be when A and B start moving and ② be an arbitrary position after ①. Work-energy principle:

$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2. \tag{1}$$

The kinetic energies are

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}(m_A + m_B)\, v_2^2.$$

Set a datum at each blocks starting position. For an arbitrary displacement δ the force laws are:

$$V_1 = 0, \qquad V_2 = \tfrac{1}{2}k\delta^2 - m_A g\delta\sin\alpha - m_B g\delta, \quad \text{and} \quad (U_{1\text{-}2})_{\text{nc}} = -\mu_k m_A g\delta\cos\alpha.$$

Eq. (1) becomes

$$-\mu_k m_A g\delta\cos\alpha = \tfrac{1}{2}(m_A + m_B)\, v_2^2 + \tfrac{1}{2}k\delta^2 - m_A g\delta\sin\alpha - m_B g\delta. \tag{2}$$

Differentiate this equation with respect to δ. Note that $\frac{dv_2^2}{d\delta} = 0$ for maximum velocity.

$$\mu_k m_A g\cos\alpha + (m_A + m_B)\, v_2 \frac{dv_2^2}{d\delta} + k\delta - g\,(m_A\sin\alpha + m_B) = 0 \quad \Rightarrow$$
$$\delta_{v_{\max}} = \frac{g}{k}(m_A\sin\alpha + m_B) - \frac{\mu_k m_A g\cos\alpha}{k} = 11.38\,\text{ft}.$$

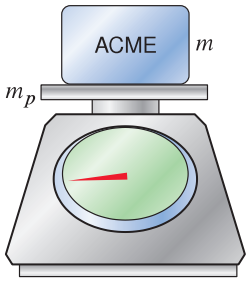
Thus the maximum speed occurs when B strikes the ground. Solve Eq. (2) for v_2 with $\delta = 2\,\text{ft}$.

$$(v_2)_{\max} = \sqrt{\frac{2g\delta(m_A\sin\alpha + m_B - \mu_k m_A\cos\alpha) - k\delta^2}{m_A + m_B}} = 9.67\,\text{ft/s}$$

The distance between B and the ground is 0 ft.

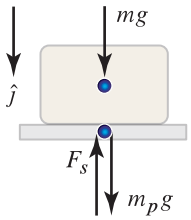
Problem 4.91

Spring scales work by measuring the displacement of a spring that supports both the platform of mass m_p and the object, of mass m, whose weight is being measured. In your solution, note that most scales read zero when no mass m has been placed on them; that is, they are calibrated so that the weight reading neglects the mass of the platform m_p. Assume that the spring is linear elastic with spring constant k. If the mass m is gently placed on the spring scale (i.e., it is dropped from zero height above the scale), determine the *maximum velocity* attained by the mass m as the spring compresses.



Solution

Let ① be at release and ② be a generic position after ①. All forces doing work are conservative. Work-energy principle:



$$T_1 + V_1 = T_2 + V_2. \tag{1}$$

The kinetic energies are

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}\left(m + m_p\right) v_2^2.$$

Set a datum at each masses initial position and let δ be the initial compression of the spring. For an additional displacement d the force laws are:

$$V_1 = \tfrac{1}{2}k\delta^2 \quad \text{and} \quad V_2 = \tfrac{1}{2}k\left(\delta + d\right)^2 - gd\left(m + m_p\right).$$

Eq. (1) becomes

$$\tfrac{1}{2}k\delta^2 = \tfrac{1}{2}\left(m + m_p\right) v_2^2 + \tfrac{1}{2}k\left(\delta + d\right)^2 - gd\left(m + m_p\right). \tag{2}$$

At maximum velocity the acceleration is zero. Therefore $F_s = (m + m_p)g$. The initial deflection δ is equal to $m_p g/k$. Therefore

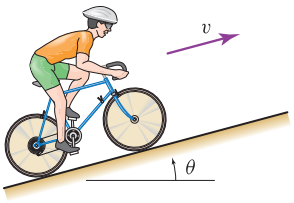
$$k\left(\delta + d_{v_{\max}}\right) = \left(m + m_p\right) g \quad \Rightarrow \quad d_{v_{\max}} = \frac{mg}{k}.$$

Solve Eq. (2) for v_2 with $\delta = m_p g/k$ and $d = mg/k$. After a little algebra we find

$$\boxed{v_{\max} = mg\sqrt{\frac{1}{k\left(m + mp\right)}}.}$$

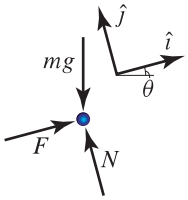
Problem 4.92

A cyclist is riding with a constant speed $v = 25\,\text{km/h}$ over an inclined road with $\theta = 15°$. The combined mass of the cyclist and bicycle is 85 kg. Neglecting aerodynamic drag, determine the number of calories ($1\,\text{C} = 4.184\,\text{kJ}$) burned by the cyclist in the course of 15 min if his muscle efficiency is 25%.



Solution

For constant speed we have the balance equation:



$$\sum F_x: \quad F - mg\sin\theta = 0 \quad \Rightarrow \quad F = mg\sin\theta.$$

The power of F is $\qquad F = mgv\sin\theta.$

The energy output of F over 15 minutes is $\qquad E_o = mgvt\sin\theta,$

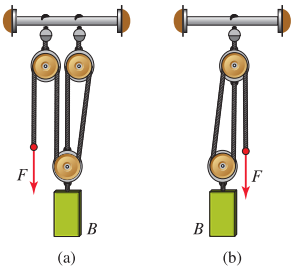
The energy burned is

$$E_b = \frac{E_o}{\epsilon} \quad \Rightarrow \quad \boxed{E_b = \frac{mgvt\sin\theta}{\epsilon} = 5.395\times10^3\,\text{kJ} = 1290\,\text{C},}$$

where $\epsilon = 0.25$ and $t = 900\,\text{s}$.

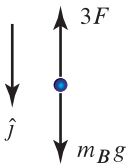
Problem 4.93

For the two pulley systems shown, the same block B, weighing 150 lb, is being lifted by applying the same constant force $F = 80$ lb. If the two systems are released from rest, determine the power developed by the force F in the two cases 1 s after release.



Solution

Balance Principles:

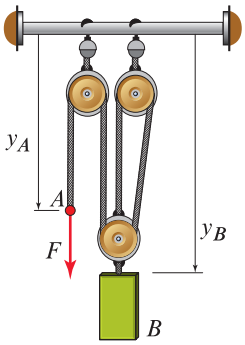


$$\sum F: \quad m_B g - 3F = m_B a_B.$$

We see that the acceleration of B is constant. The velocity at $t_1 = 1$ s is

$$v_B = a_c t \quad \Rightarrow \quad v_B = \left(g - \frac{3F}{m_B}\right) t_1.$$

Referring to the figure on the right, the kinematics for case (a) is



$$L = y_A + 3y_B \quad \Rightarrow \quad v_A = -3v_B.$$

The power of F in case (a) is

$$P_F = 3t_1 F \left(\frac{3F}{m_B} - g\right) = 4640 \text{ ft·lb/s}.$$

Similarly for case (b)

$$P_F = 2t_1 F \left(\frac{2F}{m_B} - g\right) = 343 \text{ ft·lb/s}.$$

Solutions Manual

Engineering Mechanics: Dynamics

1st Edition

Gary L. Gray
The Pennsylvania State University

Francesco Costanzo
The Pennsylvania State University

Michael E. Plesha
University of Wisconsin–Madison

With the assistance of:
Chris Punshon
Andrew J. Miller
Justin High
Chris O'Brien
Chandan Kumar
Joseph Wyne

Version: August 10, 2009

The McGraw-Hill Companies, Inc.

Copyright © 2002–2010
Gary L. Gray, Francesco Costanzo, and Michael E. Plesha

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Important Information about this Solutions Manual

Even though this solutions manual is nearly complete, we encourage you to visit

`http://www.mhhe.com/pgc`

often to obtain the most up-to-date version. In particular, as of July 30, 2009, please note the following:

- The solutions for Chapters 1 and 2 have been accuracy checked and have been edited by us. They are in their final form.
- The solutions for Chapters 4 and 7 have been accuracy checked and should be error free. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapters 3, 6, 8, and 9 are being accuracy checked and the accuracy checked versions should be available by the end of August 2009. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapter 10 should be available in their entirety by the end of August 2009.

All of the figures in Chapters 6–10 are in color. Color will be added to the figures in Chapters 1–5 over the coming weeks.

Contact the Authors

If you find any errors and/or have questions concerning a solution, please do not hesitate to contact the authors and editors via email at:

`dyn_solns@email.esm.psu.edu`

We welcome your input.

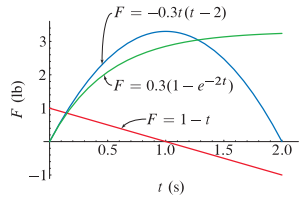
Accuracy of Numbers in Calculations

Throughout this solutions manual, we will generally assume that the data given for problems is accurate to 3 significant digits. When calculations are performed, all intermediate numerical results are reported to 4 significant digits. Final answers are usually reported with 3 significant digits. If you verify the calculations in this solutions manual using the rounded intermediate numerical results that are reported, you should obtain the final answers that are reported to 3 significant digits.

Chapter 5 Solutions

Problem 5.1

Use the definition of impulse given in Eq. (5.5) to compute the impulse of the forces shown during the interval $0 \le t \le 2\,\text{s}$.



Solution

Using the definition of impulse, we have

$$\text{Impulse} = \int_{t_1}^{t_2} \vec{F}(t)\,dt.$$

Hence, for the *blue* curve we have

$$\text{Impulse} = \int_0^{2\,\text{s}} -0.3\left(t^2 - 2t\right)dt, \quad \Rightarrow \quad \text{Impulse} = -0.3\left(\frac{t^3}{3} - t^2\right)\Big|_0^{2\,\text{s}} = \boxed{0.4\,\text{lb·s.}}$$

For the *green* curve we have

$$\text{Impulse} = \int_0^{2\,\text{s}} 0.3\left(1 - e^{-2t}\right)dt \quad \Rightarrow \quad \text{Impulse} = 0.3t + 0.3\frac{e^{-2t}}{2}\Big|_0^{2\,\text{s}} = \boxed{0.453\,\text{lb·s.}}$$

For the *red* curve we have

$$\text{Impulse} = \int_0^{2\,\text{s}} (1-t)\,dt \quad \Rightarrow \quad \text{Impulse} = t - \frac{t^2}{2}\Big|_0^{2\,\text{s}} = \boxed{0\,\text{lb·s.}}$$

Problem 5.2

The total mass of the Earth is $m_e = 5.9736\times10^{24}$ kg. Modeling the Earth (with everything in and on it) as an isolated system and assuming that the center of the Earth is also the center of mass of the Earth, determine the displacement of the center of the Earth due to

(a) a 2 m jump off the surface by a 85 kg person;

(b) the Space Shuttle, with a mass of 124,000 kg, reaching an orbit of 200 km;

(c) 170,000 km^3 of water being elevated 50 m (these numbers are estimates based on publicly available information about the Aswan Dam at the border between Egypt and Sudan). Use 1 g/cm^3 for the density of water.

Solution

We begin by choosing the center of mass of the system as the fixed origin of an inertial reference frame. Since the system is isolated, the velocity of the system's center of mass must remain constant and therefore equal to zero. This implies that the center of mass of the system must remain at the origin of our coordinate system. Consequently, assuming that the motion of the Earth and of an object P occurs along a line, and designating this line as the x axis of a coordinate system, we must have

$$0 = \frac{m_e x_e + m_P x_P}{m_e + m_P} \quad \Rightarrow \quad x_e = \frac{-m_P x_P}{m_e}, \tag{1}$$

where x_e and x_P are the x coordinates of the center of the Earth and of the object P, respectively, and where m_P is the mass of the object P.

Now, let d denote the distance by which the Earth is displaced. Then the displacement of the Earth is simply the absolute value of the result obtained in Eq. (1), i.e.,

$$d = \left|\frac{-m_P x_P}{m_e}\right|. \tag{2}$$

Using Eq. (2) in the three cases listed in the problem statement, and recalling that $m_e = 5.9736\times10^{24}$ kg, we have

(a) $\boxed{x_e = 28.5\times10^{-24}\text{ m},}$ where we set $m_P = 85.0$ kg and $x_P = 2.00$ m;

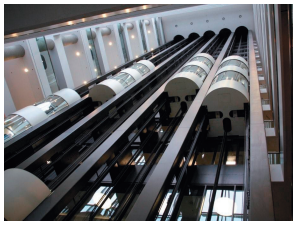
(b) $\boxed{x_e = 4.15\times10^{-12}\text{ m},}$ where we set $m_P = 124{,}000$ kg and $x_P = 200\text{ km} = 200\times10^3$ m;

(c) $\boxed{x_e = 1.42\times10^{-6}\text{ m},}$ where we set

$$m_P = (170{,}000\,\text{km}^3)\left(1000\,\frac{\text{m}}{\text{km}}\right)^3\left(1.00\,\frac{\text{g}}{\text{cm}^3}\right)\frac{0.001\,\text{kg/g}}{(0.01\,\text{m/cm})^3} = 170.0\times10^{15}\text{ kg} \quad \text{and} \quad x_P = 50\text{ m}.$$

Problem 5.3

Consider an elevator that moves with an operating speed of 2.5 m/s. Suppose that a person who boards the elevator on the ground floor gets off on the fifth floor. Assuming that the elevator has achieved operating speed by the time it reaches the second floor and that it is still moving at its operating speed as it passes the fourth floor, determine the momentum change of a person with a mass of 80 kg between the second and fourth floors if each floor is 4 m high. In addition, determine the impulse of the person's weight during the same time interval.



Solution

Let the subscripts 1 and 2 denote the time instants at which the elevator goes by floors 1 and 2, respectively. Observe that the person riding in the elevator moves with the same velocity of the elevator. Hence, denoting this velocity by $\vec{v}$ and observing that this velocity is constant, we have we have

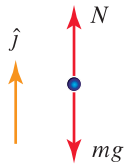
$$m\vec{v}_1 = m\vec{v}_2,$$

that is,

The change in momentum is equal to zero.

The impulse of the weight force is

$$\int_{t_1}^{t_2} (-mg\,\hat{\jmath})\,dt \quad \Rightarrow \quad -mg\,(t_2 - t_1)\,\hat{\jmath}.$$



Letting d be the distance between floors 2 and 4, and letting v be the operating speed of the elevator, then, since v is constant, we have that the time interval in traveling from the second to the fourth floor is

$$t_2 - t_1 = \frac{d}{v}.$$

Consequently, the impulse of the weight force acting on the person during the time interval (t_1, t_2) is

$$\int_{t_1}^{t_2} (-mg\,\hat{\jmath})\,dt = -\frac{mgd}{v}\,\hat{\jmath},$$

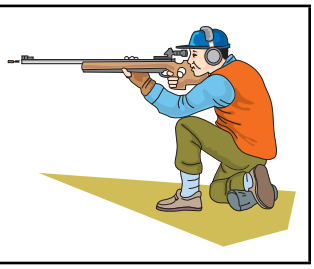
so that we have

Impulse of the weight $= (-2510\,\text{N}\cdot\text{s})\,\hat{\jmath}$ 

where we have used the following numerical data: $m = 80.0\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $d = 8.00\,\text{m}$, and $v = 2.5\,\text{m/s}$.

Problem 5.4

A 180 gr (7000 gr = 1 lb) bullet goes from rest to 3300 ft/s in 0.0011 s. Determine the magnitude of the impulse imparted to the bullet during the given time interval. In addition, determine the magnitude of the average force acting on the bullet.



Solution

Let the subscripts 1 and 2 denote the time instants when the bullet is at rest and when the bullet has a speed $v_2 = 3300\,\text{ft/s}$, respectively. Let $\vec{F}$ represent the total force acting on the bullet. The magnitude of the impulse of $\vec{F}$ is equal to the magnitude of the change in momentum of the bullet during the time interval (t_1, t_2). Hence, we have

$$\left|\int_{t_1}^{t_2} \vec{F}\,dt\right| = \left|m\vec{v}_2 - m\vec{v}_1\right| = \left|m\vec{v}_2\right| \quad \Rightarrow \quad \boxed{\left|\int_{t_1}^{t_2} \vec{F}\,dt\right| = mv_2 = 2.64\,\text{lb}\cdot\text{s},}$$

where we have used the following numerical data:

$$m = (180\,\text{gr})\frac{1}{(7000\,\text{gr/lb})}\frac{1}{(32.2\,\text{ft/s}^2)} = 7.986\times10^{-4}\,\text{lb}\cdot\text{s}^2/\text{ft} \quad \text{and} \quad v_2 = 3300\,\text{ft/s}. \tag{1}$$

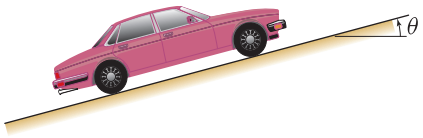
Recalling that the average force over the time interval (t_1, t_2) can be computed by dividing the change in momentum during the time interval by $t_2 - t_1$, then the magnitude of the average force acting on the bullet is

$$\boxed{\left|\vec{F}_{\text{avg}}\right| = \frac{mv_2}{t_2 - t_1} = 2400\,\text{lb},}$$

where, in addition to the data in Eq. (1) we have used the fact that $t_2 - t_1 = 0.0011\,\text{s}$.

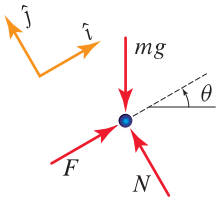
Problem 5.5

A 3400 lb car is parked as shown. Determine the impulse of the normal reaction force acting on the car during the span of an hour if $\theta = 15°$.



Solution

Using the FBD on the right and writing the equilibrium equation in the y direction, we have



$$\sum F_y: \quad N - mg\cos\theta = 0 \quad \Rightarrow \quad N = mg\cos\theta.$$

Then the impulse of the normal reaction for the time interval of duration $\Delta t = 1$ h is

$$\text{Impulse of } N = \int_0^{\Delta t} (N\,\hat{\jmath})\,dt = mgt\cos\theta\,\hat{\jmath}.$$

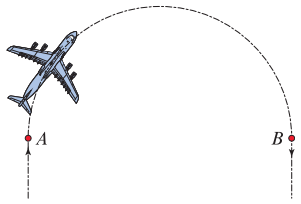
therefore, using the above equation we have

$$\boxed{\text{Impulse of } N = (11.8\times 10^6\ \text{lb·s})\,\hat{\jmath}\ \ \hat{\jmath}\ \angle\ \hat{\imath}\ @\ 15°,}$$

where we have used the following numerical data: $mg = 3400$ lb, $\Delta t = 3600$ s, and $\theta = 15.0°$.

Problem 5.6

An airplane performs a turn at constant speed and elevation so as to change its course by 180°. Let A and B designate the beginning and end points of the turn. Assuming that the change in mass of the plane due to fuel consumption is negligible, is the airplane's momentum at A different from the airplane's momentum at B? In addition, again neglecting the change in mass between A and B, is the total work done on the plane between A and B positive, negative, or equal to zero?



Solution

Yes the momentum is different because the momentum of an object is a vector quantity. Hence, even if the mass and the speed of the plane are constant, the fact that the direction of the plane at B is different from that at A implies that the momentum of the plane has changed. By contrast the plane's kinetic energy at A and B is the same and therefore the total work done on the plane between A and B is equal to zero.

Problems 5.7 and 5.8

The takeoff runway on carriers is much too short for a modern jetplane to take off on its own. For this reason, the takeoff of carrier planes is assisted by *hydraulic catapults* (Fig. A). The catapult system is housed below the deck except for a relatively small *shuttle* that slides along a rail in the middle of the runway (Fig. B). The front landing gear of carrier planes is equipped with a *tow bar* that, at takeoff, is attached to the catapult shuttle (Fig. C). When the catapult is activated, the shuttle pulls the airplane along the runway and helps the plane reach its takeoff speed. The takeoff runway is approximately 300 ft long, and most modern carriers have three or four catapults.

Problem 5.7 In a catapult-assisted takeoff, assume that a 45,000 lb plane goes from 0 to 165 mph in 2 s while traveling along a rectilinear and horizontal trajectory. Also assume that throughout the takeoff the plane's engines are providing 32,000 lb of thrust.

(a) Determine the average force exerted by the catapult on the plane.

(b) Now suppose that the takeoff order is changed so that a small trainer aircraft must take off first. If the trainer's weight and thrust are 13,000 and 5850 lb, respectively, and if the catapult is not reset to match the takeoff specifications for the smaller aircraft, estimate the average acceleration to which the trainer's pilots would be subjected and express the answer in terms of g. What do you think would happen to the trainer's pilot?

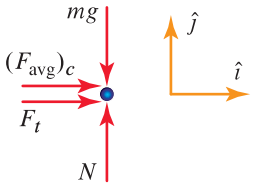
Problem 5.8 If the carrier takeoff of a 45,000 lb plane subject to the 32,000 lb thrust of its engines were not assisted by a catapult, estimate how long it would take for a plane to safely take off, i.e., to reach a speed of 165 mph starting from rest. Also, how long a runway would be needed under these conditions?



A B C

Solution to 5.7

Part (a). Let t_1 and t_2 be the time instants at which the plane starts and reaches the takeoff speed, respectively. Using the expression of average force in terms of change in momentum, the average average force acting on the airplane during the time interval of duration $t_2 - t_1$ is



$$\vec{F}_{\text{avg}} = \frac{\vec{p}(t_2) - \vec{p}(t_1)}{t_2 - t_1}, \tag{1}$$

where $\vec{p}(t_1)$ and $\vec{p}(t_2)$ are the linear momenta of the airplane at time instants t_1 and t_2, respectively. Since the motion is in the horizontal direction and using the FBD on the right, the x component of Eq. (1) is

$$(F_{\text{avg}})_c + F_t = \frac{m_f(v_{x2} - v_{x1})}{t_2 - t_1} \quad \Rightarrow \quad (F_{\text{avg}})_c = \frac{m_f(v_{x2} - v_{x1})}{t_2 - t_1} - F_t = 137{,}100\,\text{lb}, \tag{2}$$

where $(F_{\text{avg}})_c$ is the average force due to the catapult, $F_t = 32{,}000$ lb is the engines' thrust, $v_{x1} = 0$, $v_{x2} = 165\,\text{mph} = 242.0\,\text{ft/s}$, and $m_f = (45{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 1398\,\text{slug}$ is the mass of the first

airplane. Expressing $(F_{\text{avg}})_c$ to three significant digits, we have

$$\boxed{(F_{\text{avg}})_c = 137{,}000\,\text{lb}.} \tag{3}$$

Part (b). Repeating the calculations in Part (a) for the small trainer, we have

$$(F_{\text{avg}})_c + F_t = \frac{m_t(v_{x2} - v_{x1})}{t_2 - t_1}, \tag{4}$$

where F_t and m_t are the thrust and the mass of the trainer, respectively. Observe that the average acceleration of the trainer is given by

$$(a_{\text{avg}})_t = \frac{v_{x2} - v_{x1}}{t_2 - t_1}. \tag{5}$$

Therefore, using Eq. (4), we have

$$(a_{\text{avg}})_t = \frac{(F_{\text{avg}})_c + F_t}{m_t}, \tag{6}$$

from which we compute

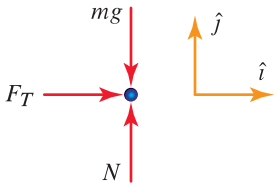
$$\boxed{(a_{\text{avg}})_t = 354\,\text{ft/s}^2 = 11.0g,}$$

where we have used the following numerical data: $(F_{\text{avg}})_c = 137{,}100\,\text{lb}$ (see Eq (2)), $F_t = 5850\,\text{lb}$, and $m_t = (13{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 403.7\,\text{slug}$.

As far as answering what would happen to the trainer's pilot is concerned, we can speculate that the pilot would likely black out.

Solution to 5.8

Let t_1 and t_2 be the time instants at which the plane starts and reaches the takeoff speed, respectively. Then, using the linear impulse–momentum principle, we have that the average force acting on the plane is



$$\vec{F}_{\text{avg}} = \frac{\vec{p}(t_2) - \vec{p}(t_1)}{t_2 - t_1} \quad \Rightarrow \quad F_T = \frac{m(v_{x2} - v_{x1})}{t_2 - t_1}, \tag{7}$$

where, referring to the FBD shown, we have used the impulse–momentum principle only in the horizontal direction.

Letting $t_1 = 0$ and recalling that $v_{x1} = 0$, solving Eq. (7) for t_2, we have

$$\boxed{t_2 = \frac{m v_{x2}}{F_T} = 10.6\,\text{s},}$$

where we have used the following numerical data: $m = (45{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 1398\,\text{slug}$, $v_{x2} = 165\,\text{mph} = 242.0\,\text{ft/s}$, and $F_T = 32{,}000\,\text{lb}$.

Next, using the work-energy principle to find the total distance d needed to take off, we have

$$T_1 + U_{1\text{-}2} = T_2, \tag{8}$$

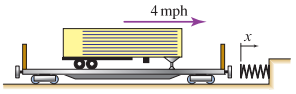
where $T_1 = 0$, $U_{1\text{-}2} = F_T d$, and $T_2 = \frac{1}{2} m v_2^2$, so that we have

$$F_T d = \tfrac{1}{2} m v_2^2 \quad \Rightarrow \quad d = \frac{m v_2^2}{2 F_T} \quad \Rightarrow \quad \boxed{d = 1280\,\text{ft},}$$

where we have used the following numerical data: $m = (45{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 1398\,\text{slug}$, $v_2 = v_{x2} = 165\,\text{mph} = 242.0\,\text{ft/s}$, and $F_T = 32{,}000\,\text{lb}$.

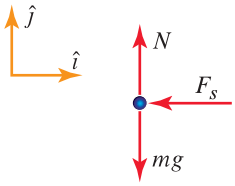
Problem 5.9

A 60 ton railcar and its cargo, a 27 ton trailer, are moving to the right at 4 mph when they come into contact with a bumper that is able to bring the system to a stop in 0.78 s. Determine the magnitude of the average force exerted on the railcar by the bumper.



Solution

Let the subscripts 1 and 2 denote the time instants when the train first comes into contact with the spring and when the train first comes to a stop, respectively. Then, using the relation between the change in linear momentum of a particle over a time interval of duration $t_2 - t_1$ and the average force over the same time interval, we have that the magnitude of the average force is

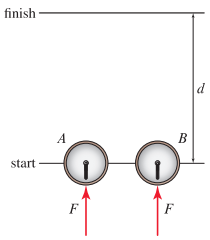


$$\left|\vec{F}_{\text{avg}}\right| = \left|\frac{m\vec{v}_2 - m\vec{v}_1}{t_2 - t_1}\right| \quad \Rightarrow \quad \boxed{F_{\text{avg}} = \frac{mv_1}{t_2 - t_1} = 40{,}600\,\text{lb},}$$

where we have used the fact that $v_2 = 0$ and we have used the following numerical data: $m = (60.0\,\text{ton} + 27.0\,\text{ton})(2000\,\text{lb/ton})(32.2\,\text{ft/s}^2)^{-1} = 5404\,\text{slug}$, $v_1 = 4.00\,\text{mph} = 5.867\,\text{ft/s}$, and $t_2 - t_1 = 0.780\,\text{s}$.

Problem 5.10

In a simple force-controlled experiment, two curling stones A and B are made to slide over a sheet of ice. Initially, A and B are at rest on the start line. Then they are acted upon by identical and constant forces $\vec{F}$, which continually push A and B all the way to the finish line. Let $\vec{p}_{A_{\text{FL}}}$ and $\vec{p}_{B_{\text{FL}}}$ denote the momentum of A and B *at the finish line*, respectively, assuming that the forces $\vec{F}$ are the only nonnegligible forces acting in the plane of motion. If $m_A < m_B$, which of the following statements is true?



(a) $\left|\vec{p}_{A_{\text{FL}}}\right| < \left|\vec{p}_{B_{\text{FL}}}\right|$.

(b) $\left|\vec{p}_{A_{\text{FL}}}\right| = \left|\vec{p}_{B_{\text{FL}}}\right|$.

(c) $\left|\vec{p}_{A_{\text{FL}}}\right| > \left|\vec{p}_{B_{\text{FL}}}\right|$.

(d) There is not enough information given to make a comparison between $\left|\vec{p}_{A_{\text{FL}}}\right|$ and $\left|\vec{p}_{B_{\text{FL}}}\right|$.

Solution

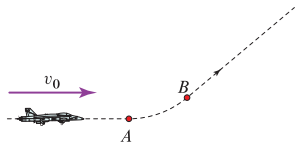
Since $m_B > m_A$, B will take more time to cross the finish line. From the definition of impulse for a constant force

$$\int_{t_1}^{t_2} \vec{F}\,(t)\,dt = \vec{p}\,(t_2) - \vec{p}\,(t_1) \quad \Rightarrow \quad \vec{F}\,(t_2 - t_1) = \vec{p}\,(t_2) - \vec{p}\,(t_1)\,,$$

we see that a constant force acting over a longer time interval will cause B to have more momentum than A at the finish line: (i) $\left|\vec{p}_{A_{\text{FL}}}\right| < \left|\vec{p}_{B_{\text{FL}}}\right|$ is true.

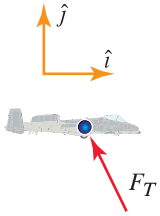
Problem 5.11

A 30,000 lb airplane is flying on a horizontal trajectory with a speed $v_0 = 650$ mph when, at point A, it maneuvers so that at point B it is set on a steady climb with $\theta = 40°$ and a speed of 600 mph. Assuming that the change in mass of the plane between A and B is negligible, determine the impulse that had to be exerted on the plane in going from A to B.



Solution

Referring to the figure on the right, we can describe the velocitities if the plane at A and B, respectively, as $\vec{v}_A = v_0\,\hat{\imath}$ and $\vec{v}_B = v_B(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$, v_0 is the speed at A and v_B is the speed at B. Then, applying the linear impulse–momentum principle, we have that the total impulse exerted on the plane between A and B is



$$\int_{t_A}^{t_B} \vec{F}_T\,dt = m(\vec{v}_B - \vec{v}_A),$$

where m is the mass of the plane and $\vec{F}_T$ is the total external force acting on the plane. Hence, using the component system shown, we have

$$\int_{t_A}^{t_B} \vec{F}_T\,dt = \frac{W}{g}\left[(v_B\cos\theta - v_0)\,\hat{\imath} + v_B\sin\theta\,\hat{\jmath}\right],$$

so that our numerical answer is

$$\int_{t_A}^{t_B} \vec{F}_T\,dt = (-260\times10^3\,\hat{\imath} + 527\times10^3\,\hat{\jmath})\ \text{lb·s}$$

where we have used the following numerical data:

$$\begin{aligned} W &= 30{,}000\ \text{lb},\\ g &= 32.2\ \text{ft/s}^2,\\ v_B &= (600\ \text{mph})(5280\ \text{ft/mi})(3600\ \text{s/h})^{-1} = 880.0\ \text{ft/s},\\ \theta &= 40.0°,\\ v_0 &= (650\ \text{mph})(5280\ \text{ft/mi})(3600\ \text{s/h})^{-1} = 953.3\ \text{ft/s}. \end{aligned}$$

Problem 5.12

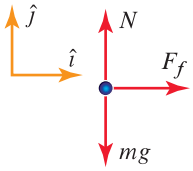
A 1600 kg car, when on a rectilinear and horizontal stretch of road, can go from rest to 100 km/h in 5.5 s.



(a) Assuming that the car travels on such a road, estimate the average value of the force acting on the car for the car to match the expected performance.

(b) Recalling that the force propelling a car is caused by the friction between the driving wheels and the road, and again assuming that the car travels on a rectilinear and horizontal stretch of road, estimate the average value of the friction force acting on the car for the car to match the expected performance. Also estimate the coefficient of friction required to generate such a force.

Solution



Part (a). Let the subscripts 1 and 2 denote the time instants when the car starts from rest and reaches the speed of 100 km/h, respectively. Then, using the relation between average force and changes in linear momentum over a given time interval, we have

$$\vec{F}_{\text{avg}} = \frac{m\vec{v}_2 - m\vec{v}_1}{t_2 - t_1}. \tag{1}$$

Using the component system shown on the right and since the car moves along a straight line, we have

$$\vec{v}_1 = \vec{0} \quad \text{and} \quad \vec{v}_2 = v_2\,\hat{\imath}, \tag{2}$$

where $v_2 = 100\,\text{km/h}$. Therefore, substituting Eqs. (2) into Eq. (1) and evaluating the magnitude of the result, we have

$$\left|\vec{F}_{\text{avg}}\right| = 8081\,\text{N} \tag{3}$$

where we have used the following numerical data: $m = 1600\,\text{kg}$, $v_2 = (100\,\text{km/h})(3600\,\text{s/hr})^{-1} = 27.78\,\text{m/s}$, and $t_2 - t_1 = 5.5\,\text{s}$. Expressing the above result using three significant figures, the answer to Part (a) of the problem is ad follows:

$$\boxed{\left|\vec{F}_{\text{avg}}\right| = 8080\,\text{N}.}$$

Part (b). The only force acting on the car is the force generated by friction between the wheels and the road. Thus, the magnitude of the average force needed to accelerate the car from rest to 100 km/h is that in Eq. (3). With this in mind, from the definition of static friction (since we are assuming no slip) we have that

$$|F_f| \leq \mu_s |N|, \tag{4}$$

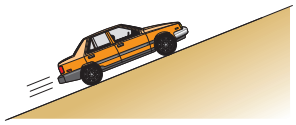
where N is the normal reaction force between the car and the ground and $F_f = \left|\vec{F}_{\text{avg}}\right|$. Since the car does not move in the vertical direction, the normal force N will have to equilibrate the car's weight, i.e., $N = mg$. Substituting this result into Eq. (4), we have

$$\boxed{\mu_s \geq \frac{\left|\vec{F}_{\text{avg}}\right|}{mg} = 0.515,}$$

where we have used the following numerical data: $\left|\vec{F}_{\text{avg}}\right| = 8081\,\text{N}$ (see Eq. (3)), $m = 1600\,\text{kg}$, and $g = 9.81\,\text{m/s}^2$.

Problem 5.13

A 1600 kg car, when on a rectilinear and horizontal stretch of road and when the tires do not slip, can go from rest to 100 km/h in 5.5 s. Assuming that the car travels on a straight stretch of road with a 40% slope and that no slip occurs, determine how long it would take to attain a speed of 100 km/h if the car were propelled by the same maximum average force that can be generated on a horizontal road.



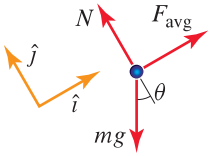
Solution

We start the solution of this problem by determining the average force that acts on the car when it travels on a horizontal surface going from zero to 100 km/s in 5.5 s. Let the subscripts 1 and 2 denote the time instants when the car starts from rest and reaches the speed of 100 km/h, respectively. Then, letting v_{h1} and v_{h2} be the horizontal components of velocity corresponding to the the time instants t_1 and t_2 respectively, using the relation between the notion of average force and the change in momentum (over a given time interval), we have

$$F_{\text{avg}} = \frac{mv_{h2} - mv_{h1}}{t_2 - t_1} \quad \Rightarrow \quad F_{\text{avg}} = 8081\,\text{N}, \tag{1}$$

where we have used the following numerical data: $m = 1600\,\text{kg}$, $v_{h2} = (100\,\text{km/h})(3600\,\text{s/hr})^{-1} = 27.78\,\text{m/s}$, $v_{h1} = 0$, and $t_2 - t_1 = 5.50\,\text{s}$.

Now, we turn the analysis of the motion of the car along the incline. Referring to the FBD shown on the right, as indicated by the problem statement, we assume that the car is subject to a force equal to f_{avg} acting in the direction of motion. Hence, applying Newton's second law in the x direction, we have



$$\sum F_x: \quad -mg\sin\theta + F_{\text{avg}} = ma_x, \quad \Rightarrow \quad a_x = \frac{F_{\text{avg}}}{m} - g\sin\theta.$$

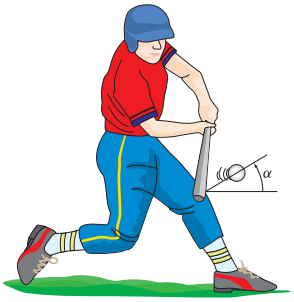
Next, since a_x is constant, we can apply the following constant acceleration equation: $v_{x2} = v_{x1} + a_x(t_2 - t_1)$. Let the initial time $t_1 = 0$ so that $v_{x1} = 0$. Then, letting $v_{x2} = 100\,\text{km/s}$, we can solve for t_2, namely the time needed to achieve v_{x2}. Specifically, we have

$$v_{x2} = a_x t_2 \quad \Rightarrow \quad \frac{v_{x2}}{a_x} = t_2 \quad \Rightarrow \quad \boxed{t_2 = \frac{mv_{x2}}{F_{\text{avg}} - mg\sin\theta} = 19.7\,\text{s},}$$

where we have used the following numerical data: $m = 1600\,\text{kg}$, $v_{x2} = (100\,\text{km/h})(3600\,\text{s/hr})^{-1} = 27.78\,\text{m/s}$, $F_{\text{avg}} = 8081\,\text{N}$ (see Eq. (1)), $g = 9.81\,\text{m/s}^2$, $\theta = \tan^{-1}(40/100) = 21.80°$.

Problem 5.14

A $5\frac{1}{8}$ oz baseball traveling at 80 mph rebounds off a bat with a speed of 160 mph. The ball is in contact with the bat for roughly 10^{-3} s. The incoming velocity of the ball is horizontal, and the outgoing trajectory forms an angle $\alpha = 31°$ angle with respect to the incoming trajectory.

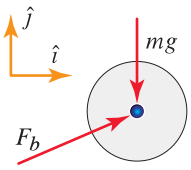


(a) Determine the impulse provided to the baseball by the bat.

(b) Determine the average force exerted by the bat on the ball.

(c) Determine how much the angle α would change (with respect to 31°) if we were to neglect the effects of the force of gravity on the ball.

Solution



Part (a). let the subscripts 1 and 2 denote the time instants immediately before and after the bat his the ball, respectively. Hence, v_1 and v_2 are the pre and postimpact speeds and the corresponding pre and, using the component system on the right, the postimpact velocities can be written as

$$\vec{v}_1 = -v_1\,\hat{\imath} = (-117.3\,\text{ft/s})\,\hat{\imath}, \tag{1}$$

$$\vec{v}_2 = v_2\,(\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}) = (201.2\,\hat{\imath} + 120.9\,\hat{\jmath})\,\text{ft/s}, \tag{2}$$

where we have used the following numerical data: $v_1 = 80\,\text{mph} = 117.3\,\text{ft/s}$, $v_2 = 160\,\text{mph} = 234.7\,\text{ft/s}$, and $\alpha = 31°$. The impulse on the ball is given by the change in momentum. That is, using the FBD shown on the right,

$$\text{Impulse} = \int_0^{t_f} \left(\vec{F}_b - mg\,\hat{\jmath}\right)\,dt = m\vec{v}_2 - m\vec{v}_1 = \left(3.168\,\hat{\imath} + 1.203\,\hat{\jmath}\right)\text{lb·s}, \tag{3}$$

where in addition to the results in Eqs. (1) and (2), we have used the following numerical data $m = (5\frac{1}{8}\,\text{oz})(16.0\,\text{oz/lb})^{-1}(32.2\,\text{ft/s}^2)^{-1} = 9.948\times10^{-3}$ slug. Expressing the answer to Part (a) using three significant digits, we have

$$\boxed{\text{Impulse} = (3.17\,\hat{\imath} + 1.20\,\hat{\jmath})\,\text{lb·s}}$$

Part (b). Using the first of Eqs. (3), we can write

$$\text{Impulse} = \int_0^{t_f} \left(\vec{F}_b - mg\,\hat{\jmath}\right)\,dt = \int_0^{t_f} \vec{F}_b\,dt - \int_0^{t_f} mg\,\hat{\jmath}\,dt = t_f(\vec{F}_b)_{\text{avg}} - \int_0^{t_f} mg\,\hat{\jmath}\,dt \tag{4}$$

$$\Rightarrow \quad (\vec{F}_b)_{\text{avg}} = \frac{1}{t_f}\left(\text{Impulse} + \int_0^{t_f} mg\,\hat{\jmath}\,dt\right) = \frac{1}{t_f}\left(\text{Impulse} + mgt_f\,\hat{\jmath}\right). \tag{5}$$

Using the value of the impulse in Eq. (3), and recalling that $mg = 5\frac{1}{8}\,\text{oz} = 0.3203\,\text{lb}$, and $t_f = 10^{-3}$ s, we then have

$$(\vec{F}_b)_{\text{avg}} = \left(3168\,\hat{\imath} + 1203\,\hat{\jmath}\right)\text{lb}, \tag{6}$$

which, when expressed using three significant digits, becomes

$$\boxed{(\vec{F}_b)_{\text{avg}} = (3170\,\hat{\imath} + 1200\,\hat{\jmath})\,\text{lb}}$$

Part (c). To answer the final question, we can again use Eq. (3), solve it for $\vec{v}_2$, and neglect mg to find

$$\vec{v}_2 = \vec{v}_1 + \frac{1}{m}\int_0^{t_f} \vec{F}_b\, dt. \tag{7}$$

Now, taking advantage of the average force from Eq. (6), we have

$$\int_0^{t_f} \vec{F}_b\, dt = t_f \left(\vec{F}_b\right)_{\text{avg}} = [(3168\,\hat{\imath} + 1203\,\hat{\jmath})\,\text{lb}](10^{-3}\,\text{s}) = (3.168\,\hat{\imath} + 1.203\,\hat{\jmath})\,\text{lb·s}, \tag{8}$$

so that,

$$\vec{v}_2 = (201.2\,\hat{\imath} + 120.9\,\hat{\jmath})\,\text{ft/s}. \tag{9}$$

The above value for $\vec{v}_2$ is identical, to four significant figures, to that in Eq. (2). Therefore, within the accuracy of our calculation, the resulting value of α would be unchanged with respect to the given value of 31°.

Within the accuracy of our calculation (4 significant figures), α remains equal to 31°.

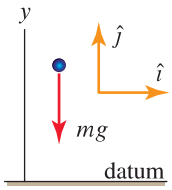
Problem 5.15

In an unfortunate incident, a 2.75 kg laptop computer is dropped onto the floor from a height of 1 m. Assuming that the laptop starts from rest, that it rebounds off the floor up to a height of 5 cm, and that the contact with the floor lasts 10^{-3} s, determine the impulse provided by the floor to the laptop and the average acceleration to which the laptop is subjected when in contact with the floor (express this result in terms of g, the acceleration of gravity).



Solution

Modeling the laptop as a particle of mass m, to determine the impulse provided by the floor to the laptop we need to determine the change in velocity of the laptop due to its collision with the ground. The velocity with which the laptop hits the ground can be related to the height from which the laptop is dropped using the work-energy principle. Let ① and ② be the positions of the laptop when it is first dropped and when it reached the ground, respectively. Referring to the FBD shown, assuming that the laptop is only subject to gravity, and setting the datum from gravity at the ground, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where

$$T_1 = 0, \quad V_1 = mgh_1, \quad T_2 = \tfrac{1}{2}v_2^2, \quad V_2 = 0. \tag{2}$$

Substituting Eqs. (2) into Eq. (1) and solving for v_2, we have

$$v_2 = \sqrt{2gh_1} = 4.429\,\text{m/s}, \tag{3}$$

where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$ and $h_1 = 1.00\,\text{m}$. Similarly, letting ③ and ④ be the position of the laptop right after impact and when reaching the maximum rebound height, respectively, then applying the work-energy principle again, we have that have

$$v_3 = \sqrt{2gh_4} = 0.9905\,\text{m}, \tag{4}$$

where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$ and $h_4 = 5.00\,\text{cm} = 0.0500\,\text{m}$. Hence, using the component system shown on the right, the velocity of the laptop right before impact and right after impact are

$$\vec{v}_2 = -(4.429\,\text{m/s})\,\hat{\jmath} \quad \text{and} \quad \vec{v}_3 = (0.9905\,\text{m})\,\hat{\jmath}. \tag{5}$$

Now, applying the impulse–momentum principle, we can say that the impulse provided by the floor is

$$\text{Impulse} = \vec{p}_3 - \vec{p}_2 = m\vec{v}_3 - m\vec{v}_2.$$

Recalling that $m = 2.75\,\text{kg}$, and using the numerical results in Eqs. (5), we have

$$\boxed{\text{Impulse} = (14.9\,\hat{\jmath})\,\text{N·s}}$$



Next, applying the formula for the average acceleration, we have

$$\vec{a}_{\text{avg}} = \frac{\vec{v}_3 - \vec{v}_2}{t_2 - t_1}, \tag{6}$$

which, recalling that $t_2 - t_1 = 1 \times 10^{-3}\,\text{s}$ and using again the numerical results in Eqs. (5), gives

$$\boxed{\vec{a}_{\text{avg}} = (5420\,\hat{\jmath})\,\text{m/s} = 552g\,\hat{\jmath}}$$



Problem 5.16

A train is moving at a constant speed v_t relative to the ground, when a person who initially at rest (relative to the train) starts running and gains a speed v_0 (relative to the train) after a time interval Δt. Had the person started from rest on the ground (as opposed to the moving train), would the magnitude of the total impulse exerted on the person during Δt be smaller than, equal to, or larger than the impulse needed to cause the same change in relative velocity in the same amount of time on the moving train? Assume that the person always moves in the direction of motion of the train.



Solution

The two impulses in question are the same. This is because the train is moving at a constant velocity with respect to the ground. Hence, if the ground can be chosen as an inertial reference frame then the train can also be chosen as an inertial reference frame. Therefore a given change in velocity in a given amount of time, either on the train or on the ground, will require the same impulse.

Problem 5.17

A train is decelerating at a constant rate, when a person who initially at rest (relative to the train) starts running and gains a speed v_0 (again relative to the train) after a time interval Δt. Had the person started from rest on the ground (as opposed to the moving train), would the magnitude of the total impulse exerted on the person during Δt be smaller than, equal to, or larger than the impulse needed to cause the same change in velocity in the same amount of time on the moving train? Assume that the person always moves in the direction of motion of the train and that the train does not reverse its motion during the time interval Δt.



Solution

The magnitude of the total impulse in question would be greater. Suppose the ground is chosen as an inertial frame. Then the train cannot be taken as an inertial frame. The use of the impulse-momentum principle requires that velocity changes be measured by an inertial observer. One such observer would perceive the change in speed of the person on the train to be smaller than it would have been on the ground due to the fact that the train is decelerating. Therefore, for a given change in speed over a given time interval, a person on a decelerating train requires a smaller impulse than a person on the ground.

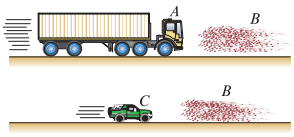
Problem 5.18

A car of mass m collides head-on with a truck of mass $50m$. What is the ratio between the magnitude of the impulse provided by the car to the truck and the magnitude of the impulse provided by the truck to the car during the collision?

Solution

The ratio of the impulse imparted onto the truck by the car is necessarily equal and opposite to the impulse provided by the truck. This result is a direct consequence of Newton's third law.

Problems 5.19 through 5.21



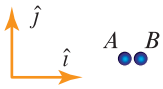
These problems are an introduction to perfectly plastic impact (which we will cover in Section 5.2). In each problem, model the vehicles A and C as particles and treat the swarm of bugs B hitting the vehicles as a single particle. Also assume that the swarm of bugs sticks perfectly to each vehicle (this is what is meant by a *perfectly plastic impact*).

Problem 5.19 An 80,000 lb semitruck A (the maximum weight allowed in many states) is traveling at 70 mph when it encounters a swarm of mosquitoes B. The swarm is traveling at 1 mph in the opposite direction of the truck. Assuming that the entire swarm sticks to the truck, the mass of each mosquito is 2 mg, and that all of these mosquitoes do not significantly damage the truck, how many mosquitoes must have hit the truck if it slows down by 2 mph on impact? If the same number of mosquitoes hit a small SUV C weighing 3000 lb, by how much will the SUV slow down?

Problem 5.20 An 80,000 lb semitruck A (the maximum weight allowed in many states) is traveling at 70 mph when it encounters a swarm of worker bees B. The swarm is traveling at 12 mph in the opposite direction of the truck. Assuming that the entire swarm sticks to the truck, the mass of each bee is 0.1 g, and that all of these bees do not significantly damage the truck, how many bees must have hit the truck if it slows down by 2 mph on impact? If the same number of bees hit a small SUV C weighing 3000 lb, by how much will the SUV slow down?

Problem 5.21 An 80,000 lb semitruck A (the maximum weight allowed in many states) is traveling at 70 mph when it encounters a swarm of dragonflies B. The swarm is traveling at 33 mph in the opposite direction of the truck. Assuming that the entire swarm sticks to the truck, the mass of each dragonfly is 0.25 g, and that all of these dragonflies do not significantly damage the truck, how many dragonflies must have hit the truck if it slows down by 2 mph on impact? If the same number of dragonflies hit a small SUV C weighing 3000 lb, by how much will the SUV slow down?

Solution to 5.19



Referring to the FBD shown, the system's linear momentum is conserved. Using the subscripts 1 and 2 to denote the pre- and postimpact conditions, respectively, in the x direction we have

$$m_A(v_{Ax})_1 + nm_m(v_{Bx})_1 = m_A(v_{Ax})_2 + nm_m(v_{Bx})_2 = (m_A + nm_m)v_{x2}, \tag{1}$$

where m_m is the mass of a single mosquito, n is the number of mosquitoes, and where we have set $(v_{Ax})_2 = (v_{Bx})_2 = v_{x2}$ since the mosquitoes will move with the truck after impact. Solving Eq. (1) for n and recalling that $v_{x2} = (v_{Ax})_1 - 2\,\text{mph}$, we have

$$n = \frac{m_A[(v_{Ax})_1 - v_{x2}]}{m_m[v_{x2} - (v_{Bx})_1]} = 5.317\times10^8 \text{ mosquitos}, \tag{2}$$

where we have used the following numerical data: $m_A = (80{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 2484\,\text{slug}$, $(v_{Ax})_1 = 70.0\,\text{mph} = 102.7\,\text{ft/s}$, $v_{x2} = 68.0\,\text{mph} = 99.73\,\text{ft/s}$, $m_m = (2.00\times10^{-6}\,\text{kg})(14.59\,\text{kg/slug})^{-1} =$

1.371×10^{-7} slug, and $(v_{Bx})_1 = -1.00\,\text{mph} = -1.467\,\text{ft/s}$. Expressing the above result to three significant figures, we have

$$\boxed{n = 5.32\times10^{8}\ \text{mosquitos.}} \tag{3}$$

To determine the effect of the mosquitoes on the SUV, we solve Eq. (1) for v_{x2} after having replaced m_A with m_C. This gives,

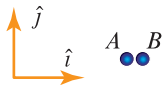
$$v_{x2} = \frac{m_C(v_{Cx})_1 + n m_m (v_{Bx})_1}{m_C + n m_m} = 56.98\,\text{ft/s} = 38.85\,\text{mph}, \tag{4}$$

where $(v_{Cx})_1 = (v_{Ax})_1$ and where we have used the following numerical data: $m_C = \frac{3000\,\text{lb}}{32.2\,\text{ft/s}^2} = 93.17\,\text{slug}$, $(v_{Cx})_1 = 70.0\,\text{mph} = 102.7\,\text{ft/s}$, $n = 5.317\times10^{8}$ mosquitos (see Eq. (2)), $m_m = 2.00\,\text{mg} = (10^{-6}\,\text{kg})(14.59\,\text{kg/slug})^{-1} = 1.371\times10^{-7}$ slug, and $(v_{Bx})_1 = -1.00\,\text{mph} = -1.467\,\text{ft/s}$. Thus the SUV will slow down by $(v_{Cx})_1 - v_{x2}$, i.e.,

$$\boxed{\text{Slowdown} = 45.7\,\text{ft/s} = 31.2\,\text{mph.}}$$

Solution to 5.20

Referring to the FBD shown, the system's linear momentum is conserved. Using the subscripts 1 and 2 to denote the pre- and postimpact conditions, respectively, in the x direction we have



$$m_A(v_{Ax})_1 + nm_b(v_{Bx})_1 = m_A(v_{Ax})_2 + nm_b(v_{Bx})_2 = (m_A + nm_b)v_{x2}, \tag{5}$$

where m_b is the mass of a single bee, n is the number of bees, and where we have set $(v_{Ax})_2 = (v_{Bx})_2 = v_{x2}$ since the bees will move with the truck after impact. Solving Eq. (5) for n and recalling that $v_{x2} = (v_{Ax})_1 - 2\,\text{mph}$, we have

$$n = \frac{m_A[(v_{Ax})_1 - v_{x2}]}{m_b[v_{x2} - (v_{Bx})_1]} = 9.174\times10^6\ \text{bees}, \tag{6}$$

where we have used the following numerical data: $m_A = (80{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 2484\,\text{slug}$, $(v_{Ax})_1 = 70.0\,\text{mph} = 102.7\,\text{ft/s}$, $v_{x2} = 68.0\,\text{mph} = 99.73\,\text{ft/s}$, $m_b = 0.100\,\text{g} = (10.0^{-4}\,\text{kg})(14.59\,\text{kg/slug})^{-1} = 6.854\times10^{-6}\,\text{slug}$, and $(v_{Bx})_1 = -12.0\,\text{mph} = -17.60\,\text{ft/s}$. Expressing the above result to three significant figures, we have

$$\boxed{n = 9.170\times10^6\ \text{bees}.} \tag{7}$$

To determine the effect of the bees on the SUV, we solve Eq. (5) for v_{x2} after having replaced m_A with m_C. This gives,

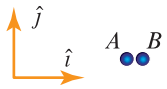
$$v_{x2} = \frac{m_C(v_{Cx})_1 + nm_b(v_{Bx})_1}{m_C + nm_b} = 54.23\,\text{ft/s} = 36.98\,\text{mph},$$

where $(v_{Cx})_1 = (v_{Ax})_1$ and where we have used the following numerical data: $m_C = \frac{3000\,\text{lb}}{32.2\,\text{ft/s}^2} = 93.17\,\text{slug}$, $(v_{Cx})_1 = 70.0\,\text{mph} = 102.7\,\text{ft/s}$, $n = 9.174\times10^6$ bees (see Eq. (6)), $m_b = 0.100\,\text{g} = (10^{-3}\,\text{kg})(14.59\,\text{kg/slug})^{-1} = 6.854\times10^{-6}\,\text{slug}$, and $(v_{Bx})_1 = -12.0\,\text{mph} = -17.60\,\text{ft/s}$. Thus the SUV will slow down by $(v_{Cx})_1 - v_{x2}$, i.e.,

$$\boxed{\text{Slowdown} = 48.5\,\text{ft/s} = 33.0\,\text{mph}.}$$

Solution to 5.21

Referring to the FBD shown, the system's linear momentum is conserved. Using the subscripts 1 and 2 to denote the pre- and postimpact conditions, respectively, in the x direction we have



$$m_A(v_{Ax})_1 + nm_d(v_{Bx})_1 = m_A(v_{Ax})_2 + nm_d(v_{Bx})_2 = (m_A + nm_d)v_{x2}, \tag{8}$$

where m_d is the mass of a single dragonfly, n is the number of dragonflies, and where we have set $(v_{Ax})_2 = (v_{Bx})_2 = v_{x2}$ since the dragonflies will move with the truck after impact. Solving Eq. (5) for n and recalling that $v_{x2} = (v_{Ax})_1 - 2\,\text{mph}$, we have

$$n = \frac{m_A[(v_{Ax})_1 - v_{x2}]}{m_d[v_{x2} - (v_{Bx})_1]} = 2.906\times10^6 \text{ dragonflies}, \tag{9}$$

where we have used the following numerical data: $m_A = (80{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 2484\,\text{slug}$, $(v_{Ax})_1 = 70.0\,\text{mph} = 102.7\,\text{ft/s}$, $v_{x2} = 68.0\,\text{mph} = 99.73\,\text{ft/s}$, $m_d = 0.250\,\text{g} = (10^{-3}\,\text{kg})(14.59\,\text{kg/slug})^{-1} = 17.14\times10^{-6}\,\text{slug}$, and $(v_{Bx})_1 = -33.0\,\text{mph} = -48.40\,\text{ft/s}$. Expressing the above result to three significant figures, we have

$$\boxed{n = 2.91\times10^6 \text{ dragonflies}.} \tag{10}$$

To determine the effect of the dragonflies on the SUV, we solve Eq. (5) for v_{x2} after having replaced m_A with m_C. This gives,

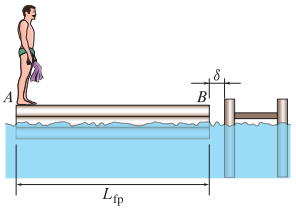
$$v_{x2} = \frac{m_C(v_{Cx})_1 + nm_d(v_{Bx})_1}{m_C + nm_d} = 50.06\,\text{ft/s} = 34.13\,\text{mph},$$

where $(v_{Cx})_1 = (v_{Ax})_1$ and where we have used the following numerical data: $m_C = \frac{3000\,\text{lb}}{32.2\,\text{ft/s}^2} = 93.17\,\text{slug}$, $(v_{Cx})_1 = 70.0\,\text{mph} = 102.7\,\text{ft/s}$, $n = 2.906\times10^6$ dragonflies (see Eq. (9)), $m_d = 0.250\,\text{g} = (10^{-3}\,\text{kg})(14.59\,\text{kg/slug})^{-1} = 17.14\times10^{-6}\,\text{slug}$, and $(v_{Bx})_1 = -33.0\,\text{mph} = -48.40\,\text{ft/s}$. Thus the SUV will slow down by $(v_{Cx})_1 - v_{x2}$, i.e.,

$$\boxed{\text{Slowdown} = 52.6\,\text{ft/s} = 35.9\,\text{mph}.}$$

Problem 5.22

Solve Example 5.4 by directly applying Eq. (5.14), using the same assumptions made in that solution. Note that, unlike Example 5.4, the velocity of the person relative to the platform does not appear in the solution.

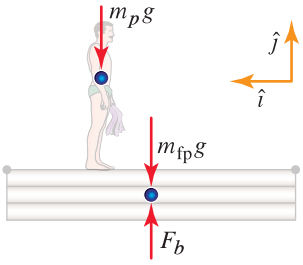


Solution

Equation (5.14) is a direct consequence of the impulse–momentum principle for a system, i.e.,

$$\vec{F} = \frac{d}{dt}(m\vec{v}_G), \tag{1}$$

where $\vec{v}_G$ is the velocity of the system's center of mass G. Observe that $\vec{F}_x = 0$. Hence, using Eq. (1), we have



$$0 = mv_{Gx} \quad \Rightarrow \quad C = mv_{Gx}, \tag{2}$$

where C is a constant. Since the system is initially stationary, we have $C = 0$. In turn, this implies that

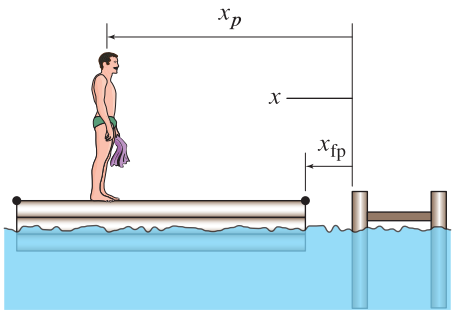
$$0 = mv_{Gx} = m\frac{x_G}{dt} \quad \Rightarrow \quad x_G = \text{constant}. \tag{3}$$

If the position of the G does not change with time, then letting the subscripts 1 and 2 denote the initial time instant and a generic subsequent time instant, respectively, we must have

$$x_{G1} = x_{G2}. \tag{4}$$

Applying the definition of center of mass to the platform-person system, we have that, at any instant,

$$x_G = \frac{x_A m_A + x_B m_B}{m_A + m_B}. \tag{5}$$



Substituting Eq. (5) into Eq. (4), and using the coordinate system shown to the right, we have

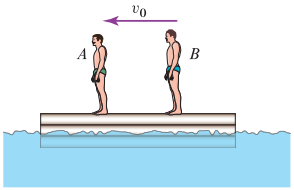
$$\frac{L_{fp} m_p + 0 m_{fp}}{m_p + m_{fp}} = \frac{x_p m_p + x_p m_{fp}}{m_p + m_{fp}}. \tag{6}$$

Letting the time t_2 be the instant when the person reaches the right end of the platform, i.e., letting $x_p = x_{fp}$ in Eq. (6) and then solving for $(x_{fp})_2$, we have

$$\boxed{(x_{fp})_2 = \frac{L_{fp} m_p}{m_p + m_{fp}},}$$

which is exactly the same result expressed by Eq. (10) in Example 5.4.

Problems 5.23 through 5.25



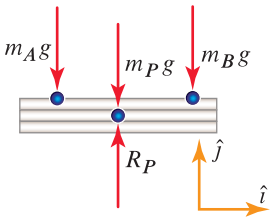
Two persons A and B weighing 140 and 180 lb, respectively, jump off a floating platform (in the same direction) with a velocity relative to the platform that is completely horizontal and with magnitude $v_0 = 6\,\text{ft/s}$ for both A and B. The floating platform weighs 800 lb. Assume that A, B, and the platform are initially at rest.

Problem 5.23 Neglecting the water resistance to the horizontal motion of the platform, determine the speed of the platform after A and B jump at the same time.

Problem 5.24 Neglecting the water resistance to the horizontal motion of the platform, and knowing that B jumps first, determine the speed of the platform after both A and B have jumped.

Problem 5.25 Neglecting the water resistance to the horizontal motion of the platform, and knowing that A jumps first, determine the speed of the platform after both A and B have jumped.

Solution to 5.23



Referring to the FBD on the right, there are no external forces acting on the system. Therefore, the momentum is conserved in the x direction. Let the subscripts 1 and 2 represent the time instants immediately before and after the jump, respectively. Then we must have

$$m_A(v_{Ax})_1 + m_B(v_{Bx})_1 + m_P(v_{Px})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2 + m_P(v_{Px})_2. \quad (1)$$

All masses are initially at rest and $(v_{Ax})_2 = (v_{Bx})_2$. Because the velocity v_0 is a relative velocity we must have

$$-v_0 = (v_{Ax})_2 - (v_{Px})_2 \quad \Rightarrow \quad (v_{Ax})_2 = (v_{Px})_2 - v_0.$$

Consequently, Eq. (1) becomes

$$0 = \big[(v_{Px})_2 - v_0\big](m_A + m_B) + m_P(v_{Px})_2,$$

which can be solved for $(v_{Px})_2$ to obtain

$$(v_{Px})_2 = \frac{(m_A + m_B)v_0}{m_A + m_B + m_P} = \frac{(W_A + W_B)v_0}{W_A + W_B + W_P}, \quad (2)$$

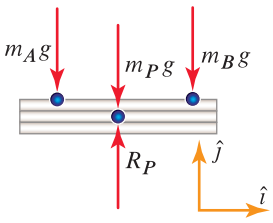
where we have multiplied both the numerator and denominator by the acceleration due to gravity g to obtain the last expression. Observing that the platform moves only in the x direction, then $v_{P2} = |(v_{Px})_2|$ so that we have

$$\boxed{v_{P2} = 1.71\,\text{ft/s},}$$

where we have used the following numerical data: $W_A = 140\,\text{lb}$, $W_B = 180\,\text{lb}$, $W_P = 800\,\text{lb}$, and $v_0 = 6.00\,\text{ft/s}$.

Solution to 5.24

There are no external forces acting on the system. Therefore, the momentum is conserved in the x direction. Let the subscripts 1, 2, 3, and 4 represent the time instants immediately before B jumps, right after B jumps, right before A jumps, and right after A jumps, respectively. Then we must have



$$m_A(v_{Ax})_1 + m_B(v_{Bx})_1 + m_P(v_{Px})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2 + m_P(v_{Px})_2. \quad (3)$$

All masses are at rest at time t_1 and $(v_{Ax})_2 = (v_{Px})_2$. Because v_0 is a relative velocity, we have

$$(v_{Bx})_2 - (v_{Px})_2 = -v_0 \quad \Rightarrow \quad (v_{Bx})_2 = (v_{Px})_2 - v_0.$$

Consequently, Eq. (3) becomes

$$0 = m_B\big[(v_{Px})_2 - v_0\big] + (m_A + m_P)(v_{Px})_2,$$

which can be solved for $(v_{Px})_2$ to obtain

$$(v_{Px})_2 = \frac{m_B v_0}{m_A + m_B + m_P} = (v_{Px})_3 = (v_{Ax})_3.$$

Since momentum is conserved in the x direction between time instants t_3 and t_4, we have

$$m_A(v_{Ax})_3 + m_P(v_{Px})_3 = m_A(v_{Ax})_4 + m_P(v_{Px})_4. \quad (4)$$

Again, because v_0 is a relative velocity, we must write

$$(v_{Ax})_4 - (v_{Px})_4 = -v_0 \quad \Rightarrow \quad (v_{Ax})_4 = (v_{Px})_4 - v_0.$$

Then, Eq. (4) can be rewritten as

$$\frac{(m_A + m_P)m_B v_0}{m_A + m_B + m_P} = m_A\big[(v_{Px})_4 - v_0\big] + m_P(v_{Px})_4,$$

which can be solved for $(v_{Px})_4$ to obtain

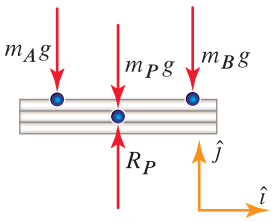
$$(v_{Px})_4 = \frac{m_A v_0}{m_A + m_P} + \frac{m_B v_0}{m_A + m_B + m_P} = \frac{W_A v_0}{W_A + W_P} + \frac{W_B v_0}{W_A + W_B + W_P}, \quad (5)$$

where we have multiplied both the numerator and denominator of each fraction by the acceleration due to gravity g to obtain the last expression. Observing that the platform moves only in the x direction, then $v_{P4} = |(v_{Px})_4|$ so that we have

$$\boxed{v_{P4} = 1.86\,\text{ft/s},}$$

where we have used the following numerical data: $W_A = 140\,\text{lb}$, $W_B = 180\,\text{lb}$, $W_P = 800\,\text{lb}$, and $v_0 = 6.00\,\text{ft/s}$.

Solution to 5.25



There are no external forces acting on the system. Therefore, the momentum is conserved in the x direction. Let the subscripts 1, 2, 3, and 4 represent the time instants immediately before A jumps, right after A jumps, right before B jumps, and right after B jumps, respectively. Then we must have

$$m_A(v_{Ax})_1 + m_B(v_{Bx})_1 + m_P(v_{Px})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2 + m_P(v_{Px})_2. \quad (6)$$

All masses are at rest at time t_1 and $(v_{Bx})_2 = (v_{Px})_2$. Because v_0 is a relative velocity, we have

$$(v_{Ax})_2 - (v_{Px})_2 = -v_0 \quad \Rightarrow \quad (v_{Ax})_2 = (v_{Px})_2 - v_0.$$

Consequently, Eq. (6) becomes

$$0 = m_A\big[(v_{Px})_2 - v_0\big] + (m_B + m_P)(v_{Px})_2,$$

which can be solved for $(v_{Px})_2$ to obtain

$$(v_{Px})_2 = \frac{m_A v_0}{m_A + m_B + m_P} = (v_{Px})_3 = (v_{Bx})_3.$$

Since momentum is conserved in the x direction between time instants t_3 and t_4, we have

$$m_B(v_{Bx})_3 + m_P(v_{Px})_3 = m_B(v_{Bx})_4 + m_P(v_{Px})_4. \quad (7)$$

Again, because v_0 is a relative velocity, we must write

$$(v_{Bx})_4 - (v_{Px})_4 = -v_0 \quad \Rightarrow \quad (v_{Bx})_4 = (v_{Px})_4 - v_0.$$

Then, Eq. (7) becomes

$$\frac{(m_B + m_P)m_A v_0}{m_A + m_B + m_P} = m_B\big[(v_{Px})_4 - v_0\big] + m_P(v_{Px})_4,$$

which can be solved for $(v_{Px})_4$ to obtain

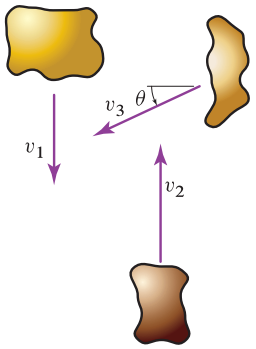
$$(v_{Px})_4 = \frac{m_B v_0}{m_B + m_P} + \frac{m_A v_0}{m_A + m_B + m_P} = \frac{W_B v_0}{W_B + W_P} + \frac{W_A v_0}{W_A + W_B + W_P}, \quad (8)$$

where we have multiplied both the numerator and denominator by the acceleration due to gravity g to obtain the last expression. Observing that the platform moves only in the x direction, then $v_{P4} = |(v_{Px})_4|$ so that we have

$$\boxed{v_{P4} = 1.85\,\text{ft/s}.}$$

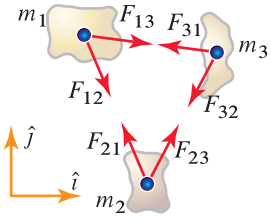
Problem 5.26

At the instant shown a group of three space-junk fragments with masses $m_1 = 7.45\,\text{kg}$, $m_2 = 3.22\,\text{kg}$, and $m_3 = 8.45\,\text{kg}$ are traveling as shown with $v_1 = 7701\,\text{m/s}$, $v_2 = 6996\,\text{m/s}$, and $v_3 = 6450\,\text{m/s}$. Assume that the velocity vectors of the fragments are coplanar, that $\vec{v}_1$ and $\vec{v}_2$ are parallel, and that θ is measured with respect to a line perpendicular to the direction of both $\vec{v}_1$ and $\vec{v}_2$. Furthermore, assume that the system is *isolated* and that, because of gravity, the fragments will eventually form a single body. Determine the common velocity of the fragments after they come together if $\theta = 25°$.



Solution

Since the system is isolated, the velocity of the center of mass of the system is conserved. When the objects form a single body they will travel with a velocity equal to the initial velocity of the mass center. We write the initial velocities as



$$\vec{v}_1 = -v_1\,\hat{\jmath}, \quad \vec{v}_2 = v_2\,\hat{\jmath}, \quad \text{and} \quad v_3 = -v_3\cos\theta\,\hat{\imath} - v_3\sin\theta\,\hat{\jmath}. \qquad (1)$$

The velocity of the center of mass of the system is therefore

$$\vec{v}_G = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3}{m_1 + m_2 + m_3} = \frac{-m_3 v_3\cos\theta\,\hat{\imath} + (m_2 v_2 - m_1 v_1 - m_3 v_3\sin\theta)\hat{\jmath}}{m_1 + m_2 + m_3}.$$

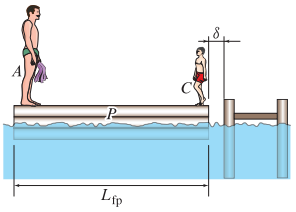
Using the problem's given we then have

$$\vec{v}_G = (-2580\,\hat{\imath} - 3030\,\hat{\jmath})\,\text{m/s},$$

where we have used the following numerical data , $m_1 = 7.45\,\text{kg}$, $m_2 = 3.22\,\text{kg}$, $m_3 = 8.45\,\text{kg}$, $v_1 = 7701\,\text{m/s}$, $v_2 = 6996\,\text{m/s}$, $v_3 = 6450\,\text{m/s}$, and $\theta = 25°$.

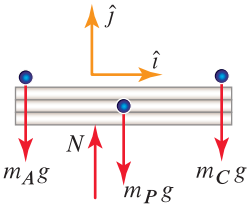
Problem 5.27

A 180 lb man A and a 40 lb child C are at the opposite ends of a 250 lb floating platform P with a length $L_{\text{fp}} = 15$ ft. The man, child, and platform are initially at rest at a distance $\delta = 1$ ft from a mooring dock. The child and the man move toward each other with the same speed v_0 relative to the platform. Determine the distance d from the mooring dock where the child and man will meet. Assume that the resistance due to the water to the horizontal motion of the platform is negligible.



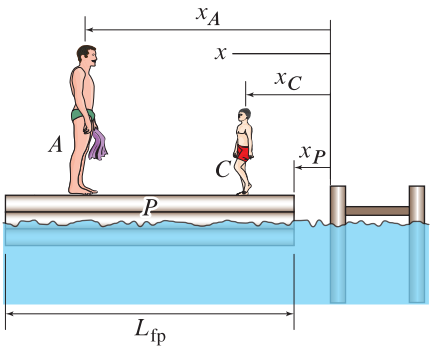
Solution

There are no external forces in the x direction so momentum is conserved in this direction. Since the system is initially at rest, the position of the center of mass must remain constant. The man and child will meet in the center of the platform because they move with equal speed relative to the platform. Let the subscripts 1 and 2 denote the initial time and the time at which A and C meet, respectively. Then we must have,



$$m_A x_{A1} + m_C x_{C1} + m_P x_{P1} = m_A x_{A2} + m_C x_{C2} + m_P x_{P2}, \qquad (1)$$

where, referring to the coordinate system shown to the below,



$x_{C1} = \delta = 1$ ft, $m_{P1} = x_{C1} + \frac{1}{2}L_{\text{fp}}$, $x_{A1} = x_{C1} + L_{\text{fp}}$, and $x_{A2} = x_{C2} = x_{P2} = d + \frac{1}{2}L_{\text{fp}}$. Consequently, Eq. (1) becomes

$$m_A(x_{C1} + L_{\text{fp}}) + m_C x_{C1} + m_P\left(x_{C1} + \tfrac{1}{2}L_{\text{fp}}\right) = (m_A + m_C + m_P)d. \qquad (2)$$

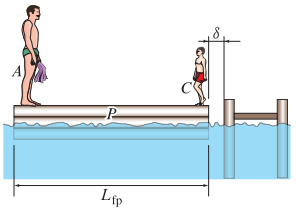
Multiplying Eq. (2) through by the acceleration due to gravity g and solving for d we obtain

$$d = \frac{W_A(x_{C1} + L_{\text{fp}}) + W_C x_{C1} + W_P\left(x_{C1} + \frac{1}{2}L_{\text{fp}}\right)}{W_A + W_C + W_P} = 10.7\,\text{ft},$$

where we have used the following numerical data: $W_A = 180$ lb, $W_C = 40.0$ lb, $W_P = 250$ lb, $L_{\text{fp}} = 15.0$ ft, $x_{C1} = 1.00$ ft.

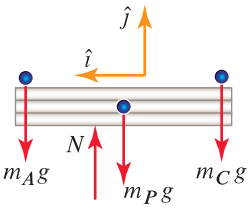
Problem 5.28

A man A, with a mass $m_A = 85\,\text{kg}$, and a child C, with a mass $m_C = 18\,\text{kg}$, are at the opposite ends of a floating platform P, with a mass $m_P = 150\,\text{kg}$ and a length $L_{\text{fp}} = 6\,\text{m}$. Assume that the man, child, and platform are initially at rest and that the resistance due to the water to the horizontal motion of the platform is negligible. Suppose that the man and child start moving toward each other in such a way that the platform does not move relative to the water. Determine the distance covered by the child until meeting the man.



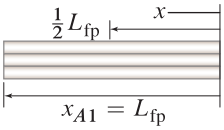
Solution

There are no external forces in the x direction so momentum is conserved in this direction. Since the system is initially at rest, the center of mass does not move. Let the subscripts 1 and 2 denote the time instants corresponding to the initial and final positions, respectively. Since we want the platform to remain stationary we can use it as an inertial frame of reference. Choosing a coordinate system with origin at x_{C1}, as shown, we have



$$m_A x_{A1} + m_P x_{P1} + m_C x_{C1} = m_A x_{A2} + m_P x_{P2} + m_C x_{C2}, \qquad (1)$$

$$\Rightarrow \quad m_A x_{A1} = m_A x_{A2} + m_C x_{C2}, \qquad (2)$$



given that $x_{C1} = 0$. When the man and child meet $x_{A2} = x_{C2}$ and Eq. (2) becomes

$$m_A x_{A1} = (m_A + m_C) x_{C2},$$

which, recalling that $x_{A1} = L_{\text{fp}}$, can be solved for x_{C2} to obtain

$$x_{C2} = \frac{m_A L_{\text{fp}}}{m_A + m_C}. \qquad (3)$$

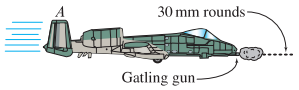
Observing that the distance covered by the child is $d = |x_{C2}|$, we have

$$\boxed{d = \frac{m_A L_{\text{fp}}}{m_A + m_C} = 4.95\,\text{m},}$$

where we have used the following numerical data: $m_A = 85.0\,\text{kg}$, $m_C = 18.0\,\text{kg}$, $m_P = 150.0\,\text{kg}$, and $L_{\text{fp}} = 6.00\,\text{m}$.

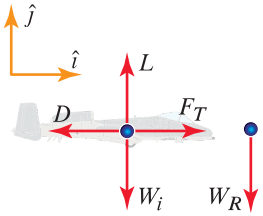
Problem 5.29

The 28,000 lb A-10 Thunderbolt is flying at a constant speed of 375 mph when it fires a 4 s burst from its forward-facing seven-barrel Gatling gun. The gun fires 13.2 oz projectiles at a rate of 4200 rounds/min. The muzzle velocity of each projectile is 3250 ft/s. Assuming that each of the plane's two jet engines maintains a constant thrust of 9000 lb, that the plane is subject to a constant air resistance while the gun is firing (equal to that before the burst), and that the plane flies straight and level, determine the plane's change in velocity at the end of the 4 s burst.



Solution

Referring to the FBD shown, F_T , D, and L are the engine thrust, aerodynamic drag, and the aerodynamic lift, respectively. Based on the information given, we have that the forces F_T and D balance each other exactly so that the momentum in the x direction is conserved. Let W_i be the weight of the airplane and the remaining rounds, after the i^{th} round has been fired. Let $W_R = 13.2\,\text{oz.} = 0.8250\,\text{lb}$ be the weight of a single round. The quantities v_i and m_i refer to the velocity and mass of the plane (including the rounds yet to be fired) when the i^{th} round, with velocity v_{Ri} and mass m_R, is fired. Thus, the balance of impulse and momentum gives



$$m_{i-1}(v_{i-1})_x = m_i(v_i)_x + m_R\,(v_{Ri})_x\,. \tag{1}$$

The weight of the plane after after the i^{th} round is fired is given by $W_i = W_T - iW_R$, where $W_T = 28{,}000\,\text{lb}$. Observe that, for $i = 0$, $v_0 = 550\,\text{ft/s}$. ALso, when a round is fired, its velocity relative to the plane is equal to the muzzle velocity $v_M = 3250\,\text{ft/s}$, so that

$$(v_{Ri})_x = (v_i)_x + v_M, \tag{2}$$

Recalling that 4200 rounds are fired each minute, we have that N_{TR}, the total number of rounds fired in 4 s is

$$N_{TR} = \frac{4200}{60}4 = 280.$$

Substituting Eq. (2) into Eq. (1), we obtain

$$\frac{W_T - (i-1)W_R}{g}(v_{i-1})_x = \frac{W_T - iW_R}{g}(v_i)_x + \frac{W_R}{g}[(v_i)_x + v_M]. \tag{3}$$

Finally, solving for $(v_i)_x$, we obtain

$$(v_i)_x = \frac{1}{W_T + (1-i)W_R}\left\{[W_T - (i-1)W_R](v_{i-1})_x - W_R v_M\right\}. \tag{4}$$

This is a recursion relation. To find the velocity of the plane after 280 rounds have been fired will require a short computer program. Using one such program, we obtain that airplane's change in speed (Δv), is

$$(v_{280})_x = 523.1\,\text{ft/s} \quad \Rightarrow \quad \boxed{\Delta v = -26.9\,\text{ft/s}.}$$

This result was obtained using Mathematica with the following two alternative codes.

Recursion via a For Loop

```
vInitial = 375.0 · 5280./3600.;
For[i = 1, i < 281, i++, If[i == 1, vFinal = vInitial;];
  vFinal = 1/(28 000.0 + (1 - i) 13.2/16.) ((28 000.0 + (1 - i) 13.2/16.) vFinal - 13.2/16. 3250.);]
Δv = vFinal - vInitial
```

−26.9233

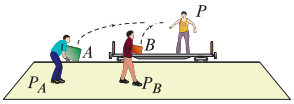
Recursion via Function Definition

```
vAirplane[i_] :=
  1/(28 000.0 + (1 - i) 13.2/16.) ((28 000.0 + (1 - i) 13.2/16.) vAirplane[i - 1] - 13.2/16. 3250.);
vAirplane[0] = 375.0 · 5280./3600.;
Δv = Block[{$RecursionLimit = Infinity}, vAirplane[280]] - vAirplane[0]
```

−26.9233

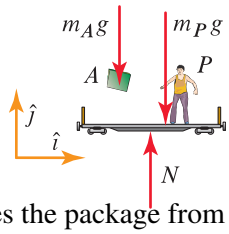
Problem 5.30

A person P on a cart on rails is receiving packages from people standing on a stationary platform. Assume that person P and the cart have a combined weight of 350 lb and start from rest. In addition suppose that a person P_A throws a package A weighing 60 lb, which is received by person P with a horizontal speed $v_A = 4.5$ ft/s. After person P has received the package from person P_A, a second person P_B throws a package B weighing 80 lb, which is received by person P with a horizontal speed relative to P and in the same direction as the velocity of P of 5.25 ft/s. Determine the final velocity of the person P and the cart. Neglect any friction or air resistance acting on P and the cart.



Solution

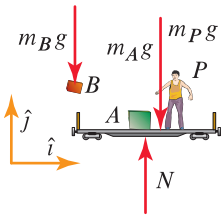
The FBD of the system when P receives the package from P_A shows that there are no forces in the horizontal direction. Then, recalling that the cart starts from rest, we must have



$$m_A(v_{Ax})_1 = (m_P + m_A)(v_{Px})_2 \quad \Rightarrow \quad (v_{Px})_2 = \frac{m_A}{m_P + m_A}(v_{Ax})_1, \qquad (1)$$

where the subscripts 1 and 2 identify the states of the system before and after P receives the package from P_A, respectively, $(v_{Ax})_1 = 4.5$ ft/s, and where we have accounted for the fact that, after P receives the package, P and the package have a common velocity.

Let 3 denote the system after P receives the package from P_B. The FBD of the system in going from 2 to 3 also indicates that the linear momentum of the system in the x direction is conserved, so that we have



$$(m_P + m_A)(v_{Px})_2 + m_B(v_{Bx})_2 = (m_P + m_A + m_B)(v_{Px})_3, \qquad (2)$$

where, based on the problem statement

$$(v_{Bx})_2 - (v_{Px})_2 = (v_{B/P})_2 = 5.25 \text{ ft/s} \quad \Rightarrow \quad (v_{Bx})_2 = (v_{B/P})_2 + (v_{Px})_2. \qquad (3)$$

Substituting Eq. (3) into Eq. (2) and then using the second of Eqs. (1), we have

$$m_A(v_{Ax})_1 + m_B(v_{B/P})_2 + \frac{m_A m_B}{m_P + m_A}(v_{Ax})_1 = (m_P + m_A + m_B)(v_{Px})_3, \qquad (4)$$

which can be rewritten as

$$m_A(v_{Ax})_1 \frac{m_P + m_A + m_B}{m_P + m_A} + m_B(v_{B/P})_2 = (m_P + m_A + m_B)(v_{Px})_3. \qquad (5)$$

Multiplying each mass terms in Eq. (5) by the acceleration due to gravity g, i.e., replacing each mass term by the corresponding weight term, and solving for $(v_{Px})_3$, we have

$$(v_{Px})_3 = (v_{Px})_{\text{final}} = \frac{W_A}{W_P + W_A}(v_{Ax})_1 + \frac{W_B}{W_P + W_A + W_B}(v_{B/P})_2, \qquad (6)$$

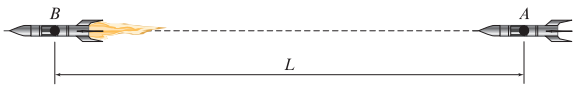
which, given that the motion fo P is only in the x direction gives

$$(\vec{v}_P)_{\text{final}} = (1.52 \text{ ft/s})\,\hat{\imath}$$

where we have used the following numerical data: $W_A = 60.0$ lb, $W_P = 350$ lb, $W_B = 80.0$ lb, $(v_{Ax})_1 = 4.50$ ft/s, and $(v_{B/P})_2 = 5.25$ ft/s.

Problem 5.31

The spacecraft shown is out in space and is far enough from any other mass (e.g., planets, etc.) so as not to be affected by any gravitational influence (i.e., the net external force on the rocket is approximately zero). The system (i.e., the spacecraft *and all* its fuel) is at rest when it starts at A, and it thrusts all the way to B along the straight line shown using internal chemical rockets (which work by ejecting the fuel mass at very high speeds out the tail of the rocket). We are given that the mass of the system at A is m and that it has ejected half of its mass in thrusting from A to B. What will be the location of the system's mass center when the spacecraft reaches B?

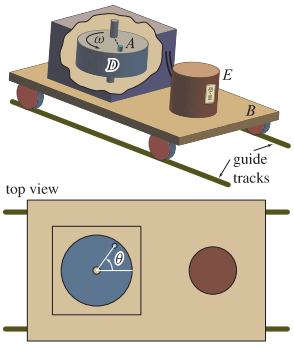


Solution

The system's center of mass will not move from its initial position. the reason for this is that the rocket *and* its fuel form an isolated system, that is, a system with no external forces acting on it. The linear momentum of an isolated system is conserved. Since the momentum is conserved we know that the center of mass cannot change its velocity. Since the system was initially at rest, then the center of mass of the system will have to remain in its initial position.

Problem 5.32

Energy storage devices that use spinning flywheels to store energy are starting to become available. To store as much energy as possible, it is important that the flywheel spin as fast as possible. Unfortunately, if it spins too fast, internal stresses in the flywheel cause it to come apart catastrophically. Therefore, it is important to keep the speed at the edge of the flywheel below about 1000 m/s. In addition, it is critical that the flywheel be almost perfectly balanced to avoid the tremendous vibrations that would otherwise result. With this in mind, let the flywheel D, whose diameter is 0.3 m, rotate at $\omega = 60{,}000$ rpm. In addition, assume that the cart B is constrained to move rectilinearly along the guide tracks. Given that the flywheel is not perfectly balanced, that the unbalanced weight A has mass m_A, and that the total mass of the flywheel D, cart B, and electronics package E is m_B, determine the following as a function θ, the masses, the diameter, and the angular speed of the flywheel:



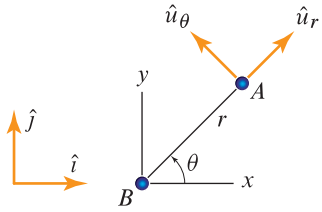
(a) the amplitude of the motion of the cart,

(b) the maximum speed achieved by the cart.

Neglect the mass of the wheels, assume that initially everything is at rest, and assume that the unbalanced mass is at the edge of the flywheel. Finally, evaluate your answers to Parts (a) and (b) for $m_A = 1$ g (about the mass of a paper clip) and $m_B = 70$ kg (the mass of the flywheel might be about 40 kg).

Solution

Part (a). The position of the unbalanced mass A attached to the flywheel is described in polar coordinates. Expressing the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ in terms of $\hat{\imath}$ and $\hat{\jmath}$, we have



$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \tag{1}$$

$$\hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{2}$$

The velocity A relative to B can now be expressed as

$$\vec{v}_{A/B} = r\dot{\theta}\,\hat{u}_\theta = -r\dot{\theta}\sin\theta\,\hat{\imath} + r\dot{\theta}\cos\theta\,\hat{\jmath}, \tag{3}$$

where $r = d/2 = 0.1500$ m is the radius of the flywheel. We can determine the absolute velocities of A and B as

$$\vec{v}_B = v_B\,\hat{\imath} \quad \text{and} \quad \vec{v}_A = \vec{v}_B + \vec{v}_{A/B} = (v_B - r\omega\sin\theta)\hat{\imath} + (r\omega\cos\theta)\hat{\jmath},$$

where the angular velocity $\dot{\theta}$ is a constant: ω. Therefore, the total momentum of the system is

$$\vec{p} = m_A\vec{v}_A + m_B\vec{v}_B = [m_A(v_B - r\omega\sin\theta) + m_B v_B]\,\hat{\imath} + m_A r\omega\cos\theta\,\hat{\jmath} \tag{4}$$

Since there are no external forces acting the in the x direction, the component of momentum in this direction is conserved. Thus,

$$(m_A + m_B)v_B - m_A r\omega\sin\theta = C, \tag{5}$$

where C is a constant. We can express the angle θ as ωt since the angular velocity ω is a constant. Thus, we can rewrite Eq. (5) as

$$(m_A + m_B)\, v_B - m_A r\omega \sin\omega t = C, \tag{6}$$

Since we know the cart starts from rest, we have

$$\vec{v}_B(0) = 0, \quad \Rightarrow \quad C = 0. \tag{7}$$

Knowing the value of the constant in Eq. (6) from Eq. (7), we can solve for the velocity of the cart

$$v_B = \frac{m_A r\omega \sin\omega t}{m_A + m_B}. \tag{8}$$

Integrating the velocity of the cart with respect to time, we find that

$$x_B = -\frac{m_A r}{m_A + m_B}\cos\omega t + K, \tag{9}$$

where K is a constant. Recalling that the amplitude of the oscillation is the term in front of the term $\cos\omega t$, we have

$$\boxed{\text{Amplitude} = \frac{m_A r}{m_A + m_B} = 2.14\times10^{-6}\,\text{m},}$$

where we have used the following numerical data: $m_A = 1\,\text{g} = 1.00\times10^{-3}\,\text{kg}$, $r = 0.1500\,\text{m}$, and $m_B = 70.0\,\text{kg}$.

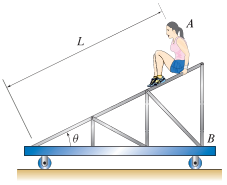
Part (b). The maximum speed achieved by the cart occurs when $\sin\omega t = 1$ in Eq. (8). This gives

$$\boxed{(v_B)_{\text{max}} = \frac{m_A r\omega}{m_A + m_B} = 0.0135\,\text{m/s},}$$

where we have used the following numerical data: $m_A = 1\,\text{g} = 1.00\times10^{-3}\,\text{kg}$, $r = 0.1500\,\text{m}$, $\omega = (60{,}000\,\text{rpm})(2\pi\,\text{rad/rev})/(60\,\text{s/min}) = 6283\,\text{rad/s}$, and $m_B = 70.0\,\text{kg}$.

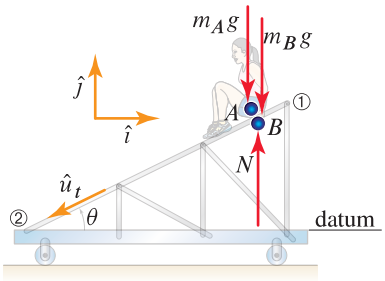
Problem 5.33

The 135 lb woman A sits atop the 90 lb cart B, both of which are initially at rest. If the woman slides down the frictionless incline of length $L = 11$ ft, determine the velocity of both the woman and the cart when she reaches the bottom of the incline. Ignore the mass of the wheels on which the cart rolls and any friction in their bearings. The angle $\theta = 26°$.



Solution

Let ① be when the person is at the top of the incline and ② be when the person is at the bottom of the incline. All forces doing work are conservative. Choosing the datume for the potential energy of gravity at the ②, then at ① and ② the kinetic and potential energies of the person and the cart are:



$$T_1 = 0, \quad T_2 = \tfrac{1}{2}m_A v_{A2}^2 + \tfrac{1}{2}m_B v_{B2}^2, \quad V_1 = m_A g L \sin\theta, \quad V_2 = 0.$$

Hence, using the work-energy principle, we have

$$T_1 + V_1 = T_2 + V_2 \quad \Rightarrow \quad 2g m_A L \sin\theta = m_A v_{A2}^2 + m_B v_{B2}^2. \tag{1}$$

The above equation has two unknowns, namely the speeds v_{A2} and v_{B2}. Now observe that the problem requires that we find the *velocities* of the cart and the woman. Hence, we need to derive additional equations to accomplish this task. Clearly, we need to take into account the slope of the incline. Second, referring to the FBD on the right, we notice that there are no external forces acting in the x direction so that the linear momentum of the system is conserved in the x direction, i.e.,

$$m_A(v_{Ax})_1 + m_B(v_{Bx})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2 \quad \Rightarrow \quad 0 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2, \tag{2}$$

where we have used the fact that A and B are initially at rest.

Next we need to deal with the kinematics of the problem. Using the component system shown, observing that the cart will be moving in the positive x direction, and that the relative velocity of A with respect to be must B must be in the direction of the unit vector $\hat{u}_t$, we have

$$\vec{v}_{B2} = v_{B2}\,\hat{\imath} \quad \text{and} \quad (\vec{v}_{A/B})_2 = (v_{A/B})_2\,\hat{u}_t = -(v_{A/B})_2 \cos\theta\,\hat{\imath} - (v_{A/B})_2 \sin\theta\,\hat{\jmath}, \tag{3}$$

where $(v_{A/B})_2$ is the component of the relative velocity of A with respect to B in the direction of $\hat{u}_t$. Then, using relative kinematics, we have

$$\vec{v}_{A2} = (\vec{v}_{A/B})_2 + \vec{v}_{B2} \quad \Rightarrow \quad \vec{v}_{A2} = \left[v_{B2} - (v_{A/B})_2 \cos\theta\right]\hat{\imath} - (v_{A/B})_2 \sin\theta\,\hat{\jmath}. \tag{4}$$

The last of Eqs. (4) implies that

$$v_{A2}^2 = |\vec{v}_{A2}|^2 = \left[v_{B2} - (v_{A/B})_2 \cos\theta\right]^2 + \left[(v_{A/B})_2 \sin\theta\right]^2 = v_{B2}^2 - 2v_{B2}(v_{A/B})_2 \cos\theta + (v_{A/B})_2^2. \tag{5}$$

Furthermore, the first of Eqs. (3) along with the last of Eqs. (4) imply that

$$(v_{Ax})_2 = v_{B2} - (v_{A/B})_2 \cos\theta \quad \text{and} \quad (v_{Bx})_2 = v_{B2}. \tag{6}$$

Finally, substituting the result in Eq. (5) into the last of Eqs. (1) and substituting Eqs. (6) into the last of Eqs. (2) we have

$$2m_A gL\sin\theta = m_A\left[v_{B2}^2 - 2v_{B2}\left(v_{A/B}\right)_2\cos\theta + \left(v_{A/B}\right)_2^2\right] + m_B v_{B2}^2, \tag{7}$$

$$0 = m_A\left[v_{B2} - \left(v_{A/B}\right)_2\cos\theta\right] + m_B v_{B2}, \tag{8}$$

which is a system of two equations in the two unknowns $\left(v_{A/B}\right)_2$ and v_{B2}. We can solve this system by first solving Eq. (8) with respect to $\left(v_{A/B}\right)_2$. This gives

$$\left(v_{A/B}\right)_2 = \frac{m_A + m_B}{m_A\cos\theta}v_{B2}. \tag{9}$$

Substituting the result in Eq. (9) into Eq. (7) and simplifying, we have

$$2m_A gL\sin\theta = \frac{(m_A + m_B)(m_A + m_B - m_A\cos^2\theta)}{m_A\cos^2\theta}v_{B2}^2$$

$$\Rightarrow \quad v_{B2} = \sqrt{\frac{2gm_A^2 L\sin\theta\cos^2\theta}{(m_A + m_B)(m_A + m_B - m_A\cos^2\theta)}} = 13.24\,\text{ft/s}, \tag{10}$$

where the quantity v_{B2} is necessarily positive since it is a speed, and where we have used the following numerical data: $m_A = (135\,\text{lb})/(32.2\,\text{ft/s}^2) = 4.193\,\text{slug}$, $m_B = (90.0\,\text{lb})/(32.2\,\text{ft/s}^2) = 2.795\,\text{slug}$, $\theta = 26.0°$, $L = 11.0\,\text{ft}$, $g = 32.2\,\text{ft/s}^2$. Substituting this result into Eq. (9), we then obtain

$$\left(v_{A/B}\right)_2 = 24.55\,\text{ft/s}, \tag{11}$$

where we have used the appropriate values listed right below Eq. (10). Finally, substituting the results of Eqs. (10) and (11) into the first of Eqs. (3) and the last of Eqs. (4), we have

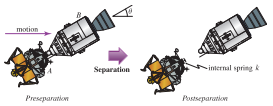
$$\boxed{\vec{v}_A = (-8.83\,\hat{\imath} - 10.8\,\hat{\jmath})\,\text{ft/s} \quad \text{and} \quad \vec{v}_B = (13.2\,\text{ft/s})\,\hat{\imath}}$$



where, again, we have used the appropriate values listed right below Eq. (10).

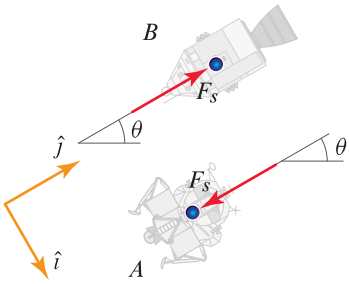
Problem 5.34

An Apollo Lunar Module A and Command and Service Module B are moving through space far from any other bodies (so that their gravitational effects can be ignored). When $\theta = 30°$, the two craft are separated using an internal linear elastic spring whose constant is $k = 200{,}000\,\text{N/m}$ and is precompressed 0.5 m. Noting that the mass of the Command and Service Module is about 29,000 kg and that the mass of the Lunar Module is about 15,100 kg, determine their post-separation velocities if their common preseparation velocity is 11,000 m/s.



Solution

Let ① be when the spring is compressed and ② be post-separation. Referring to the FBD on the right, observe that the system is isolated. Furthermore, considering the individual FBDs of A and B, we observe that no force is acting on either A or B in the x direction. These observations, imply that the x components of the moment of A and B individually are conserved and that the y component of the linear momentum of the system is conserved, i.e.,



$$m_A(v_{Ax})_1 = m_A(v_{Ax})_2, \quad m_A(v_{Bx})_1 = m_A(v_{Bx})_2, \tag{1}$$
$$m_A(v_{Ay})_1 + m_B(v_{By})_1 = m_A(v_{Ay})_2 + m_B(v_{By})_2. \tag{2}$$

Next, we observe that the system is conservative so that the work-energy principle gives:

$$T_1 + V_1 = T_2 + V_2, \tag{3}$$

where

$$V_1 = \tfrac{1}{2}k\delta_1^2, \quad T_1 = \tfrac{1}{2}m_A v_{A_1}^2 + \tfrac{1}{2}m_B v_{B_1}^2 = \tfrac{1}{2}(m_A + m_B)v_0^2, \tag{4}$$
$$V_2 = 0, \quad T_2 = \tfrac{1}{2}m_A v_{A_2}^2 + \tfrac{1}{2}m_B v_{B_2}^2 = \tfrac{1}{2}m_A\big[(v_{Ax})_2^2 + (v_{Ay})_2^2\big] + \tfrac{1}{2}m_B\big[(v_{Bx})_2^2 + (v_{By})_2^2\big], \tag{5}$$

and where the x and y components of the velocities for A and B at ① are:

$$(v_{Ax})_1 = v_0 \sin\theta = (v_{Bx})_1 \quad \text{and} \quad (v_{Ay})_1 = v_0 \cos\theta = (v_{By})_1. \tag{6}$$

Substituting the first of Eqs. (6) into Eqs. (1) and simplifying, we have

$$(v_{Ax})_2 = v_0 \sin\theta = (v_{Bx})_2. \tag{7}$$

Next, substituting Eqs. (4) and (5) into Eq. (3), we have

$$\tfrac{1}{2}(m_A + m_B)v_0^2 + \tfrac{1}{2}k\delta_1^2 = \tfrac{1}{2}m_A\big[(v_{Ax})_2^2 + (v_{Ay})_2^2\big] + \tfrac{1}{2}m_B\big[(v_{Bx})_2^2 + (v_{By})_2^2\big] \tag{8}$$

The, substituting the last of Eqs. (6) into Eq. (2), and substituting Eqs. (7) into Eq. (8), we obtain the following system of two equations in the two unknowns $(v_{Ay})_2$ and $(v_{By})_2$:

$$(m_A + m_B)v_0 \cos\theta = m_A(v_{Ay})_2 + m_B(v_{By})_2, \tag{9}$$
$$\tfrac{1}{2}(m_A + m_B)v_0^2 + \tfrac{1}{2}k\delta_1^2 = \tfrac{1}{2}m_A\big[v_0^2 \sin^2\theta + (v_{Ay})_2^2\big] + \tfrac{1}{2}m_B\big[v_0^2 \sin^2\theta + (v_{By})_2^2\big]. \tag{10}$$

To solve this system, we being with solving Eq. (9) for $(v_{By})_2$ to obtain

$$(v_{By})_2 = \frac{m_A + m_B}{m_B}v_0 \cos\theta - \frac{m_A}{m_B}(v_{Ay})_2. \tag{11}$$

Substituting the result in Eq. (11) into Eq. (10), simplifying, and rearranging, we have

$$\underbrace{\tfrac{1}{2}m_A\left(1+\frac{m_A}{m_B}\right)}_{\alpha}(v_{Ay})_2^2-\underbrace{\frac{m_A}{m_B}(m_A+m_B)v_0\cos\theta}_{\beta}(v_{Ay})_2+\underbrace{\frac{m_A}{2m_B}\cos^2\theta(m_A+m_B)v_0^2-\tfrac{1}{2}k\delta_1^2}_{\gamma}=0, \quad (12)$$

which, upon recognizing that the terms α, β, and γ are constants, can simply be given the form of the familiar the second order algebraic equation

$$\alpha(v_{Ay})_2^2-\beta(v_{Ay})_2+\gamma=0 \quad \Rightarrow \quad (v_{Ay})_2=\begin{cases}\frac{\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}=9528\,\text{m/s},\\ \frac{\beta-\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}=9525\,\text{m/s},\end{cases} \quad (13)$$

where we have used the definition of the parameters α, β, and γ given in Eq. (12), and where we have used the following numerical data: $m_A = 15{,}100\,\text{kg}$, $m_B = 29{,}000\,\text{kg}$, $v_0 = 11{,}000\,\text{m/s}$, $\theta = 30°$, $k = 200{,}000\,\text{N/m}$, and $\delta_1 = 0.5\,\text{m}$. Next, using Eq. (11), we have

$$(v_{By})=\begin{cases}9525\,\text{m/s} & \text{for } (v_{Ay})_2=9528\,\text{m/s},\\ 9527\,\text{m/s} & \text{for } (v_{Ay})_2=9525\,\text{m/s}.\end{cases} \quad (14)$$

Now, we observe that for our solution to be meaningful, we must have $(v_{Ay})_2 < (v_{By})_2$. Hence, the only acceptable solutions is as follows:

$$(v_{Ay})_2=9525\,\text{m/s} \quad \text{and} \quad (v_{By})_2=9527\,\text{m/s}.$$

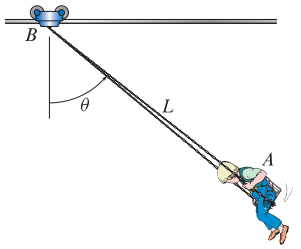
Putting everything together, the final result for the velocities of A and B after separation is given by

$$\boxed{\vec{v}_{A2}=(5500\,\hat{\imath}+9520\,\hat{\jmath})\,\text{m/s} \quad \text{and} \quad \vec{v}_{B2}=(5500\,\hat{\imath}+9530\,\hat{\jmath})\,\text{m/s} \quad \hat{\jmath}\,\angle\,\hat{\imath}=-60°,}$$

where the final results have been expressed to three significant digits and we have used Eqs. (7) along with the numerical data listed below Eq. (13) to evaluate the x components of the velocities of A and B after separation.

Problems 5.35 through 5.38

In the ride shown, a person A sits in a seat that is attached via a cable of length L to a freely moving trolley B of mass m_B. The total mass of the person and the seat is m_A. The trolley is constrained by the beam to move only in the horizontal direction. The system is released from rest at the angle $\theta = \theta_0$ and it is allowed to swing in the vertical plane. Neglect the mass of the cable and treat the person and the seat as a single particle.



Problem 5.35 Determine the velocities of the trolley and the rider the first time that $\theta = 0°$. Evaluate your solution for $W_A = 100\,\text{lb}$, $W_B = 20\,\text{lb}$, $L = 15\,\text{ft}$, and $\theta_0 = 70°$.

Problem 5.36 As in Prob. 5.35, determine the velocities of the trolley and the rider the first time that $\theta = 0°$. After doing so, for given g, L, m_A, and θ_0, determine the maximum velocity achievable by the rider at $\theta = 0°$ and the corresponding value of m_B. Evaluate your solution for $W_A = 100\,\text{lb}$, $L = 15\,\text{ft}$, and $\theta_0 = 70°$. What would be the motion of B for this value of m_B?

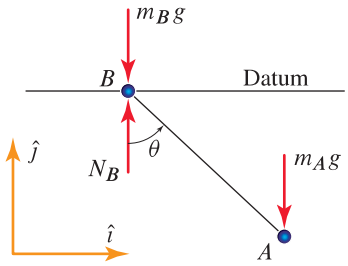
Problem 5.37 Determine the velocity of the trolley and the speed of the rider for any arbitrary value of θ.

Problem 5.38 Determine the equations needed to find the velocity of the trolley and the rider for any arbitrary value of θ. Clearly label all equations and list the corresponding unknowns, showing that you have as many equations as you have unknowns. Solve the equations for the unknowns, and then plot the velocity of the trolley and the speed of the rider as a function of the angle θ for both halves of a full swing of the rider. Use $W_A = 100\,\text{lb}$, $W_B = 20\,\text{lb}$, $L = 15\,\text{ft}$, and $\theta_0 = 70°$.

Solution to 5.35

Let ① be at release and ② be when $\theta = 0$ for the first time. All forces doing work are conservative. Hence, the work-energy principle can be written as:

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$



where, observing that at ② $\vec{v}_{A2}$ and $\vec{v}_{B2}$ do not have a component in the y direction, i.e., $\vec{v}_{A2} = (v_{Ax})_2\,\hat{\imath}$ and $\vec{v}_{B2} = (v_{Bx})_2\,\hat{\imath}$,

$$T_1 = 0, \qquad T_2 = \tfrac{1}{2}m_A(v_{Ax})_2^2 + \tfrac{1}{2}m_B(v_{Bx})_2^2,$$
$$V_1 = -m_A g L\cos\theta_0, \qquad V_2 = -m_A g L,$$

so that Eq. (1) becomes

$$-m_A g L\cos\theta_0 = \tfrac{1}{2}m_A(v_{Ax})_2^2 + \tfrac{1}{2}m_B(v_{Bx})_2^2 - m_A g L.$$

Observe that the linear momentum of the system is conserved in the x direction. Hence we must have

$$0 = m_B(v_{Bx})_2 + m_A(v_{Ax})_2.$$

We have two equations for $(v_{Ax})_2$ and $(v_{Bx})_2$ whose solution is

$$\vec{v}_{A2} = (v_{Ax})_2\,\hat{\imath} = -\sqrt{\frac{2m_B g L\,(1-\cos\theta_0)}{m_A + m_B}}\,\hat{\imath} = (-10.3\,\hat{\imath})\ \text{ft/s}$$

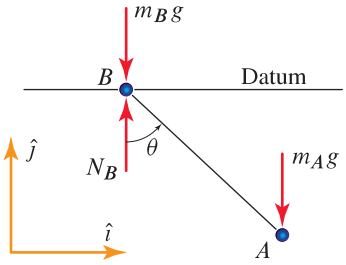
$$\vec{v}_{B2} = (v_{Bx})_2\,\hat{\imath} = \sqrt{\frac{2m_A^2 g L\,(1-\cos\theta_0)}{m_B\,(m_A + m_B)}}\,\hat{\imath} = (51.5\,\hat{\imath})\ \text{ft/s}$$

where we have used the following numerical data: $m_A = (100\,\text{lb})/(32.2\,\text{ft/s}^2) = 3.106\,\text{slug}$, $m_B = (20\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.6211\,\text{slug}$, $g = 32.2\,\text{ft/s}^2$, $L = 15\,\text{ft}$, and $\theta_0 = 70°$.

Solution to 5.36

Let ① be at release and ② be when $\theta = 0$ for the first time. All forces doing work are conservative. Hence, the work-energy principle can be written as:

$$T_1 + V_1 = T_2 + V_2, \tag{2}$$



where, observing that in ② $\vec{v}_{A2}$ and $\vec{v}_{B2}$ do not have a component in the y direction, i.e., $\vec{v}_{A2} = (v_{Ax})_2\,\hat{\imath}$ and $\vec{v}_{B2} = (v_{Bx})_2\,\hat{\imath}$,

$$\begin{aligned} T_1 &= 0, & T_2 &= \tfrac{1}{2}m_A(v_{Ax})_2^2 + \tfrac{1}{2}m_B(v_{Bx})_2^2, \\ V_1 &= -m_A g L\cos\theta_0, & V_2 &= -m_A g L, \end{aligned}$$

so that Eq. (1) becomes

$$-m_A g L\cos\theta_0 = \tfrac{1}{2}m_A(v_{Ax})_2^2 + \tfrac{1}{2}m_B(v_{Bx})_2^2 - m_A g L.$$

Observe that the linear momentum of the system is conserved in the x direction. Hence we must have

$$0 = m_B(v_{Bx})_2 + m_A(v_{Ax})_2.$$

We have two equations for $(v_{Ax})_2$ and $(v_{Bx})_2$ whose solution is

$$(v_{Ax})_2 = -\sqrt{\frac{2m_B g L\,(1-\cos\theta_0)}{m_A + m_B}} \quad \text{and} \quad (v_{Bx})_2 = \sqrt{\frac{2m_A^2 g L\,(1-\cos\theta_0)}{m_B\,(m_A + m_B)}}. \tag{3}$$

Focusing on $(v_{Ax})_2$, observe that for fixed values of m_A, g, L, and θ_0, m_B is the only variable in first of Eqs. (3). Next observe that, the argument of the square root in the first of Eqs. (3) is a monotonic increasing function of m_B. That is, the larger m_B the larger the absolute value of $(v_{Ax})_2$. Therefore, the largest absolute value of $(v_{Ax})_2$ is achieved for $m_B \to \infty$. Taking the limit of the first of Eqs. (3) as $m_B \to \infty$ gives

$$\lim_{m_B\to\infty}(v_{Ax})_2 = -\sqrt{2gL(1-\cos\theta_0)}.$$

Finally, using the second of Eqs. (3) observe that

$$\lim_{m_B\to\infty}(v_{Bx})_2 = 0.$$

In conclusion, using the problem's data we have

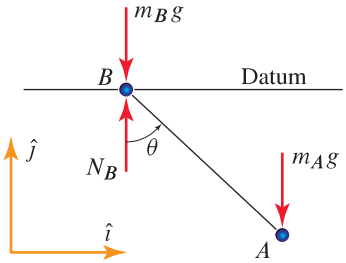
$$\boxed{(v_A)_{\max} = 25.2\,\text{ft/s} \quad \text{and} \quad (v_B) = 0 \quad \text{for} \quad m_B \to \infty,}$$

where we have used the following numerical data: $m_A = (100\,\text{lb})/(32.2\,\text{ft/s}^2) = 3.106\,\text{slug}$, $m_B = (20\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.6211\,\text{slug}$, $g = 32.2\,\text{ft/s}^2$, $L = 15\,\text{ft}$, and $\theta_0 = 70°$.

Solution to 5.37

Let ① be at release and ② be an arbitrary value of θ. All forces doing work are conservative. Hence, the work-energy principle can be written as:

$$T_1 + V_1 = T_2 + V_2, \tag{4}$$



where

$$T_1 = 0, \qquad T_2 = \tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2,$$
$$V_1 = -m_A g L\cos\theta_0, \qquad V_2 = -m_A g L\cos\theta,$$

so that Eq. (4) becomes

$$m_A g L\,(\cos\theta - \cos\theta_0) = \tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2 - m_A g L.$$

Since B can only move in the x direction and since the length of the cable connecting A and B is constant, we must have

$$\vec{v}_{A/B} = L\dot{\theta}\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \quad \Rightarrow \quad v_{Ax} = v_{Bx} + L\dot{\theta}\cos\theta \quad \text{and} \quad v_{Ay} = L\dot{\theta}\sin\theta.$$

Observing that the system's linear momentum is conserved in the x direction, we have

$$0 = m_B v_{Bx} + m_A v_{Ax}.$$

The equations developed so far can be combined in the following system of 3 equations in the 3 unknowns $\dot{\theta}$, v_A, andv_{Bx}:

$$\begin{aligned} m_A g L(\cos\theta - \cos\theta_0) &= \tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_{Bx}^2 - m_A g L, \\ 0 &= m_B v_{Bx} + m_A(v_{Bx} + L\dot{\theta}\cos\theta), \\ v_A^2 &= \left(v_{Bx} + L\dot{\theta}\cos\theta\right)^2 + \left(L\dot{\theta}\sin\theta\right)^2. \end{aligned}$$

Letting $r = \frac{m_A}{m_B}$, the solution to these equations can be written as

$$\dot{\theta} = \mp\frac{1+r}{L\cos\theta}\sqrt{\frac{2gL(\cos\theta - \cos\theta_0)}{r^2 + r + (1+r)^2\tan^2\theta}},$$

and

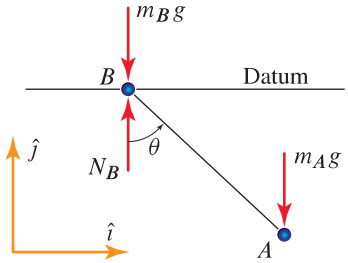
$$\vec{v}_B = v_{Bx}\,\hat{\imath} = \pm\sqrt{\frac{2gL(\cos\theta - \cos\theta_0)}{r^2 + r + (1+r)^2\tan^2\theta}}\,\hat{\imath}$$

$$v_A = \sqrt{2gL}\sqrt{\frac{[1 + 2r(1+r) - (1+2r)\cos 2\theta](\cos\theta - \cos\theta_0)}{(1+r)(1+2r-\cos 2\theta)}}.$$

Solution to 5.38

Let ① be at release and ② be an arbitrary value of θ. All forces doing work are conservative. Hence, the work-energy principle can be written as:

$$T_1 + V_1 = T_2 + V_2, \tag{5}$$



where

$$T_1 = 0, \qquad T_2 = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2,$$
$$V_1 = -m_A g L \cos\theta_0, \qquad V_2 = -m_A g L \cos\theta,$$

so that Eq. (5) becomes

$$m_A g L\,(\cos\theta - \cos\theta_0) = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2 - m_A g L.$$

Since B can only move in the x direction and since the length of the cable connecting A and B is constant, we must have

$$\vec{v}_{A/B} = L\dot{\theta}\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \quad \Rightarrow \quad v_{Ax} = v_{Bx} + L\dot{\theta}\cos\theta \quad \text{and} \quad v_{Ay} = L\dot{\theta}\sin\theta.$$

Observing that the system's linear momentum is conserved in the x direction, we have

$$0 = m_B v_{Bx} + m_A v_{Ax}.$$

The equations developed so far can be combined in the following system of 3 equations in the 3 unknowns $\dot{\theta}$, v_A, andv_{Bx}:

$$\begin{aligned} m_A g L(\cos\theta - \cos\theta_0) &= \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_{Bx}^2 - m_A g L, \\ 0 &= m_B v_{Bx} + m_A (v_{Bx} + L\dot{\theta}\cos\theta), \\ v_A^2 &= \left(v_{Bx} + L\dot{\theta}\cos\theta\right)^2 + \left(L\dot{\theta}\sin\theta\right)^2. \end{aligned}$$

Using *Mathematica* to solve these equations and generate the plots, we have:

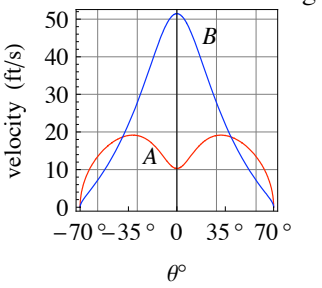
```
T1 = 0; V1 = -mA g L Cos[θ0]; T2 = 1/2 mA vA^2 + 1/2 mB vBx^2; V2 = -mA g L Cos[θ];

vAx = vBx + L θdot Cos[θ]; vA = Sqrt[(vBx + L θdot Cos[θ])^2 + (L θdot Sin[θ])^2];
WEP = T1 + V1 == T2 + V2 // FullSimplify; IMP = 0 == mB vBx + mA vAx // FullSimplify;
sol = FullSimplify[Solve[{WEP, IMP}, {θdot, vBx}],
    Assumptions → mA + mB > 0 && mA ∈ Reals && mB ∈ Reals];

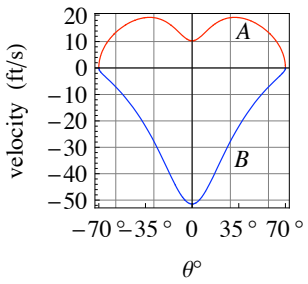
params = {θ0 → 70°, mA → 100 / 32.2, mB → 20 / 32.2, L → 15, g → 32.2};
Plot[{vA /. sol[[1]] /. params, vBx /. sol[[1]] /. params}, {θ, 70°, -70°},
 PlotStyle → {Red, Blue}, Frame → True,
 FrameTicks → {{Automatic, None}, {{-70°, -35°, 0°, 35°, 70°}, None}},
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"θ°", "velocity  (ft/s)"},
 PlotLabel → "First half of a full swing"]

Plot[{vA /. sol[[2]] /. params, vBx /. sol[[2]] /. params}, {θ, 70°, -70°},
 PlotStyle → {Red, Blue}, Frame → True,
 FrameTicks → {{Automatic, None}, {{-70°, -35°, 0°, 35°, 70°}, None}},
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"θ°", "velocity  (ft/s)"},
 PlotLabel → "Second half of a full swing"]
```

First half of a full swing

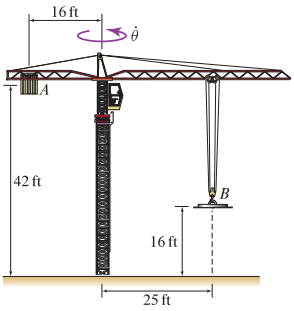


Second half of a full swing



Problem 5.39

A tower crane is lifting a 10,000 lb object B at a constant rate of 7 ft/s while rotating at a constant rate of $\dot{\theta} = 0.15$ rad/s. In addition, B is moving outward with a radial velocity of 1.5 ft/s. Assume that the object B does not swing relative to the crane (i.e., it always hangs vertically) and that the crane is fixed to the ground at O.



(a) Determine the radial velocity required of the 20 ton counterweight A to prevent the horizontal motion of the system's center of mass.

(b) Find the total force acting on A and on B.

(c) Determine the velocity and acceleration of the mass center of the system when A moves as determined in Part (a).

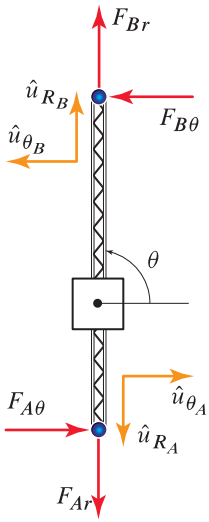
Solution

Part (a). By definition of center of mass of a system of particles, we must have

$$m_A\vec{v}_A + m_B\vec{v}_B = (m_A + m_B)\vec{v}_G. \tag{1}$$

Considering the figure to the right, we have

$$\hat{u}_{R_A} = -\hat{u}_{R_B} \quad \text{and} \quad \hat{u}_{\theta_A} = -\hat{u}_{\theta_B}. \tag{2}$$



To prevent the horizontal motion of the system's center of mass, the sum of the radial components of momentum of A and B must be equal to zero. This gives

$$m_A(\vec{v}_A \cdot \hat{u}_{R_A}) + m_B(\vec{v}_B \cdot \hat{u}_{R_B}) = 0 \quad \Rightarrow \quad v_{AR} = \frac{m_B}{m_A} v_{BR} = 0.3751 \text{ ft/s}, \tag{3}$$

where v_{AR} and v_{BR} are the radial components of velocity for A and B, respectively, and where we have used the following additional numerical values: $v_{BR} = 1.5$ ft/s, $m_A = (20 \text{ ton})(2000 \text{ lb/ton})/(32.2 \text{ ft/s}^2) = 1242$ slug and $m_B = (10{,}000 \text{ lb})/(32.2 \text{ ft/s}^2) = 310.6$ slug. Expressing v_{AR} to three significant digits, we have

$$\boxed{v_{AR} = 0.375 \text{ ft/s}.}$$

Part (b). Applying Newton's 2nd law to A and B, we have

$$\left(\sum F_R\right)_A: \quad F_{AR} = m_A a_{AR}, \qquad \left(\sum F_R\right)_B: \quad F_{BR} = m_B a_{BR}, \tag{4}$$

$$\left(\sum F_\theta\right)_A: \quad F_{A\theta} = m_A a_{A\theta}, \qquad \left(\sum F_\theta\right)_B: \quad F_{B\theta} = m_B a_{B\theta}, \tag{5}$$

$$\left(\sum F_z\right)_A: \quad F_{Az} = m_A a_{Az}, \qquad \left(\sum F_z\right)_B: \quad F_{Bz} = m_B a_{Bz}, \tag{6}$$

where, using the acceleration kinematics equations in cylindrical coordinates, we have

$$a_{AR} = -\dot{\theta}^2 R_A, \quad a_{BR} = -\dot{\theta}^2 R_B, \quad a_{A\theta} = 2v_{AR}\dot{\theta}, \quad a_{B\theta} = 2v_{BR}\dot{\theta}, \quad a_{Az} = 0, \quad a_{Bz} = 0, \tag{7}$$

where, denoting by R_{A0} and R_{B0} the initial radial coordinates of A and B, and recalling that the radial velocities of A and B are constant, we must have

$$R_A = R_{A0} + v_{AR}t \quad \text{and} \quad R_B = R_{B0} + v_{BR}t. \tag{8}$$

Substituting the kinematics relations in Eqs. (7) and (8) into Eqs. (4)–(6), we then obtain the following expression for the forces acting on A and B:

$$\vec{F}_A = -m_A[\dot{\theta}^2(R_{A0} + v_{AR}t)]\,\hat{u}_{R_A} + 2m_A\dot{\theta}v_{AR}\,\hat{u}_{\theta_A}, \tag{9}$$

$$\vec{F}_B = -m_B[\dot{\theta}^2\,(R_{B0} + v_{BR}t)]\,\hat{u}_{R_B} + 2m_B\dot{\theta}v_{BR}\,\hat{u}_{\theta_B}. \tag{10}$$

Recalling that v_{AR} is given by the last of Eqs. (3), and recalling that $m_A = 1242\,\text{slug}$, $m_B = 310.6\,\text{slug}$, $\dot{\theta} = 0.15\,\text{rad/s}$, $v_{RB} = 1.5\,\text{ft/s}$, $R_{A0} = 16\,\text{ft}$, and $R_{B0} = 25\,\text{ft}$, we can evaluate Eqs. (9) and (10)

$$\boxed{\begin{aligned}\vec{F}_A &= -[(447\,\text{lb}) + (10.5\,\text{lb/s})t]\,\hat{u}_{R_A} + (140\,\text{lb})\,\hat{u}_{\theta_A},\\ \vec{F}_B &= -[(175\,\text{lb}) + (10.5\,\text{lb/s})t]\,\hat{u}_{R_B} + (140\,\text{lb})\,\hat{u}_{\theta_B}.\end{aligned}}$$

Part (c). Recall that the origin of the chosen coordinate system is on the crane's axis of rotation. Then, because $m_A \gg m_B$, we anticipate that the center of mass G of the system will be between A and the origin of the coordinate system. We now define the following cylindrical component system attached to the mass center:

$$\hat{u}_{R_G} = \hat{u}_{R_A}, \quad \hat{u}_{\theta_G} = \hat{u}_{\theta_A}, \quad \text{and} \quad \hat{k} = \hat{u}_{R_G} \times \hat{u}_{\theta_G}.$$

Recalling that v_{Bz} is constant, the coordinates of the mass center are

$$R_G = \frac{m_A R_{A0} - m_B R_{B0}}{m_A + m_B} = 7.798\,\text{ft}, \quad \theta_G = \theta, \quad z_G = z_0 + v_{Gz}\,t, \tag{11}$$

where the numerical values of m_A, m_B, R_{A0}, and R_{B0} have already been given, the fact that $R_G > 0$ confirms that G is located somewhere between the origin and A, and where z_0 and v_{Gz} are constants that can be determined as follows:

$$z_0 = \frac{m_A z_{A0} + m_B z_{B0}}{m_A + m_B}, \quad \text{and} \quad v_{Gz} = \frac{m_A v_{Az} + m_B v_{Bz}}{m_A + m_B} = \frac{m_B v_{Bz}}{m_A + m_B}, \tag{12}$$

where we have used the fact that $v_{Az} = 0$, and where we recall that $v_{Bz} = 7\,\text{ft/s}$.

Next, we recall that, in cylindrical components, the velocity of G has the following generic form:

$$\vec{v}_G = \dot{R}_G\,\hat{u}_{R_G} + R\dot{\theta}_G\,\hat{u}_{\theta_G} + \dot{z}_G\,\hat{k}. \tag{13}$$

Therefore, taking the time derivatives of Eqs. (11) and taking advantage of the relations in Eq. (12), Eq. (13) can be written as follows:

$$\boxed{\vec{v}_G = \dot{\theta}_G \frac{m_A R_{A0} - m_B R_{B0}}{m_A + m_B}\,\hat{u}_{\theta_G} + \frac{m_B v_{Bz}}{m_A + m_B}\,\hat{k} \quad \Rightarrow \quad \vec{v}_G = (1.17\,\text{ft/s})\,\hat{u}_{\theta_G} + (1.40\,\text{ft/s})\,\hat{k},}$$

where we have used the following numerical data: $m_A = 1242\,\text{slug}$, $m_B = 310.6\,\text{slug}$, $\dot{\theta}_G = 0.15\,\text{rad/s}$, $g = 32.2\,\text{ft/s}^2$, $v_{RB} = 1.5\,\text{ft/s}$, $R_{A0} = 16\,\text{ft}$, and $R_{B0} = 25\,\text{ft}$. Similarly, for the acceleration we have

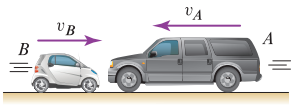
$$\vec{a}_G = \left(\ddot{R}_G - R_G\dot{\theta}_G^2\right)\hat{u}_{R_G} + \left(R_G\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_{\theta_G} + \ddot{z}\,\hat{k} \quad \Rightarrow \quad \vec{a}_G = -R_G\dot{\theta}_G^2\hat{u}_{R_G},$$

which implies that

$$\boxed{\vec{a}_G = -\dot{\theta}_G^2 \frac{m_A R_{A0} - m_B R_{B0}}{m_A + m_B}\,\hat{u}_{R_G} = (-0.176\,\text{ft/s})\,\hat{u}_{R_G}.}$$

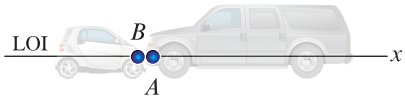
Problem 5.40

An 8600 lb Ford Excursion A traveling with a speed $v_A = 55$ mph collides head-on with a 1990 lb Smart Fortwo B traveling in the opposite direction with a speed $v_B = 35$ mph. Determine the postimpact velocity of the two cars if the impact is perfectly plastic.



Solution

The collision can be modeled as a (one dimensional) direct central impact. Using the impact-relevant FBD shown, we have conservation of momentum along the LOI, i.e.,



$$m_A v_{Ax}^- + m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+, \tag{1}$$

where m_A and m_B are the masses of A and B, respectively, and where $v_{Ax}^- = -55$ mph and $v_{Bx}^- = 35$ mph. Because the impact is perfectly plastic we must have

$$v_{Ax}^+ = v_{Bx}^+. \tag{2}$$

Equations (1) and (2) form a system of two equations in the two unknowns v_{Ax}^+ and v_{Bx}^+ whose solution is

$$v_{Ax}^+ = v_{Bx}^+ = \frac{m_A v_{Ax}^- + m_B v_{Bx}^-}{m_A + m_B} = \frac{W_A v_{Ax}^- + W_B v_{Bx}^-}{W_A + W_B}, \tag{3}$$

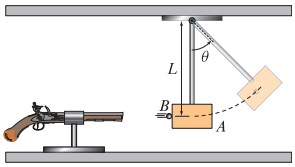
where the last expression has been obtained by multiplying each mass term by the acceleration due to gravity g to replace the terms in question with their corresponding weights. Recalling that the motion is only in the x direction, we have

$$\vec{v}_A^+ = \vec{v}_B^+ = (-55.9\,\text{ft/s})\,\hat{\imath}\,,$$

where we have used the following numerical data: $W_A = 8600$ lb, $W_B = 1990$ lb, $v_{Ax}^- = -v_A = -55$ mph $= -80.67$ ft/s, and $v_{Bx}^- = v_B = 35$ mph $= 51.33$ ft/s.

Problems 5.41 through 5.43

The ballistic pendulum used to be a common tool for the determination of the muzzle velocity of bullets as a measure of the performance of firearms and ammunition (nowadays, the ballistic pendulum has been replaced by the ballistic chronograph, an electronic device). The ballistic pendulum is a simple pendulum that allows one to record the maximum swing angle of the pendulum arm caused by the firing of a bullet into the pendulum bob.



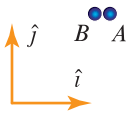
Problem 5.41 Letting L be the length of the pendulum's arm (whose mass is assumed to be negligible), m_A be the bob's mass, and m_B be the mass of the bullet, and assuming that the pendulum is at rest when the weapon is fired, derive the formula that relates the pendulum's maximum swing angle to the impact velocity of the bullet.

Problem 5.42 Let $L = 1.5\,\text{m}$ and $m_A = 6\,\text{kg}$. For George Washington's 0.58 caliber pistol, which fired a roundball of mass $m_B = 87\,\text{g}$, it is found that the maximum swing angle of the pendulum is $\theta_{\max} = 46°$. Determine the preimpact speed of the bullet B.

Problem 5.43 Suppose we want to build a ballistic pendulum to test rifles using standard NATO 7.62 mm ammunition, i.e., ammunition for which a (single) cartridge weighs roughly 147 gr (1 lb = 7000 gr) and the muzzle speed is typically 2750 ft/s. If the pendulum's length is taken to be 5 ft, and if we are to fire from a short distance so that there is a negligible decrease in speed before the bullet reaches the pendulum, what is the minimum weight we need to give to the pendulum bob to avoid having the pendulum swing to an angle greater than 90°?

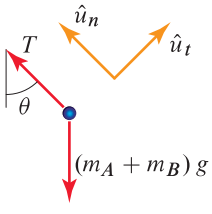
Solution to 5.41

Let v_B^- denote the speed of the bullet B before impact. Since B is initially traveling in the horizontal direction and the block A is stationary, we can write $v_{Bx}^- = v_B^-$ and $v_{Ax}^- = 0$. Since B becomes embedded in A, using the component system shown, we have $v_{Ax}^+ = v_{Bx}^+ = v_2$, where v_2 denotes the common speed of A and B after impact. Next, referring to the impact-relevant FBD shown, impulse-momentum principle in the x direction gives



$$m_B v_B^- = (m_A + m_B)v_2 \quad \Rightarrow \quad v_2 = \frac{m_B v_B^-}{m_A + m_B}. \tag{1}$$

Let ② and ③ be the positions right after impact and when the pendulum is at maximum swing angle, respectively. The only force doing work between ② and ③ is gravity. Hence, we have



$$T_2 + V_2 = T_3 + V_3, \tag{2}$$

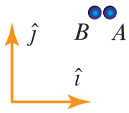
where, given the last of Eqs. (1) and our choice of datum,

$$T_2 = \tfrac{1}{2}(m_A + m_B)\left(\frac{m_B v_B^-}{m_A + m_B}\right)^2, \quad V_2 = 0, \quad T_3 = 0, \quad V_3 = (m_A + m_B)gL(1 - \cos\theta_m). \tag{3}$$

Substituting Eqs. (3) into Eq. (2), and solving for θ_m, we have

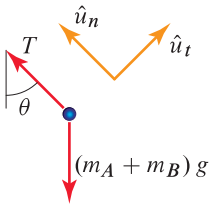
$$\theta_m = \cos^{-1}\left[1 - \frac{m_B^2(v_B^-)^2}{2gL(m_A + m_B)^2}\right].$$

Solution to 5.42



Let v_B^- denote the speed of the bullet B before impact. Since B is initially traveling in the horizontal direction and the block A is stationary, we can write $v_{Bx}^- = v_B^-$ and $v_{Ax}^- = 0$. Since B becomes embedded in A, using the component system shown, we have $v_{Ax}^+ = v_{Bx}^+ = v_2$, where v_2 denotes the common speed of A and B after impact. Next, referring to the impact-relevant FBD shown, impulse-momentum principle in the x direction gives

$$m_B v_B^- = (m_A + m_B) v_2 \quad \Rightarrow \quad v_2 = \frac{m_B v_B^-}{m_A + m_B}. \tag{4}$$



Let ② and ③ be the positions right after impact and when the pendulum is at maximum swing angle, respectively. The only force doing work between ② and ③ is gravity. Hence, we have

$$T_2 + V_2 = T_3 + V_3, \tag{5}$$

where, given the last of Eqs. (4) and our choice of datum,

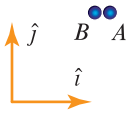
$$T_2 = \tfrac{1}{2}(m_A + m_B)\left(\frac{m_B v_B^-}{m_A + m_B}\right)^2, \quad V_2 = 0, \quad T_3 = 0, \quad V_3 = (m_A + m_B) g L (1 - \cos\theta_m). \tag{6}$$

Substituting Eqs. (6) into Eq. (5), solving for v_B^-, we have

$$\frac{m_B^2 (v_B^-)^2}{(m_A + m_B)^2} = 2gL(1 - \cos\theta_m) \quad \Rightarrow \quad \boxed{v_B^- = \frac{m_A + m_B}{m_B}\sqrt{2gL(1 - \cos\theta_m)} = 210\,\text{m/s},}$$

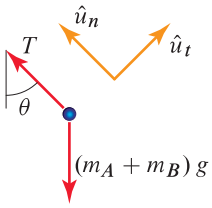
where we have used the following numerical data: $m_A = 6\,\text{kg}$, $m_B = 87\,\text{g} = 87 \times 10^{-3}\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $L = 1.5\,\text{m}$, and $\theta_m = 46°$.

Solution to 5.43



Let v_B^- denote the speed of the bullet B before impact. Since B is initially traveling in the horizontal direction and the block A is stationary, we can write $v_{Bx}^- = v_B^-$ and $v_{Ax}^- = 0$. Since B becomes embedded in A, using the component system shown, we have $v_{Ax}^+ = v_{Bx}^+ = v_2$, where v_2 denotes the common speed of A and B after impact. Next, referring to the impact-relevant FBD shown, impulse-momentum principle in the x direction gives

$$m_B v_B^- = (m_A + m_B)\, v_2 \quad \Rightarrow \quad v_2 = \frac{m_B v_B^-}{m_A + m_B}. \tag{7}$$



Let ② and ③ be the positions right after impact and when the pendulum is at maximum swing angle, respectively. The only force doing work between ② and ③ is gravity. Hence, we have

$$T_2 + V_2 = T_3 + V_3, \tag{8}$$

where, given the last of Eqs. (7) and our choice of datum,

$$T_2 = \tfrac{1}{2}(m_A + m_B)\left(\frac{m_B v_B^-}{m_A + m_B}\right)^2, \quad V_2 = 0, \quad T_3 = 0, \quad V_3 = (m_A + m_B) g L (1 - \cos\theta_m). \tag{9}$$

Substituting Eqs. (9) into Eq. (8), recalling that v_{Bx}^- is equal to the preimpact speed of the bullet, and solving for m_A with $\theta_m = 90°$, we have

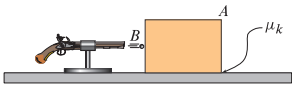
$$\frac{m_B^2 \left(v_B^-\right)^2}{(m_A + m_B)^2} = 2gL \quad \Rightarrow \quad \boxed{m_A = \frac{m_B v_B^-}{\sqrt{2gL}} - m_B = 0.0993\,\text{slug},}$$

where we have used the following numerical data:

$$\begin{aligned} m_B &= \frac{147\,\text{gr}}{7000\,\text{gr/lb}} \frac{1}{32.2\,\text{ft/s}^2} = 0.0006522\,\text{slug}, \\ v_B^- &= 2750\,\text{ft/s}, \\ g &= 32.2\,\text{ft/s}^2, \\ L &= 5\,\text{ft}. \end{aligned}$$

Problem 5.44

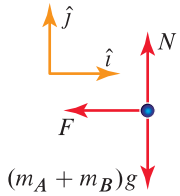
A 323 gr bullet (1 lb = 7000 gr) hits a 2 kg block that is initially at rest. After the collision, the bullet becomes embedded in the block, and they slide a distance of 0.31 m. If the coefficient of friction between the block and the ground is $\mu_k = 0.7$, determine the preimpact speed of the bullet. Although the definition of the unit "grain" is given in terms of pounds, express the answer in SI units.



Solution

Let ② and ③ be the positions of A is right after impact and at the end of sliding, respectively. Applying the work-energy principle between ② and ③, we have

$$T_2 + V_2 + (U_{1\text{-}2})_{\text{np}} = T_3 + V_3, \tag{1}$$



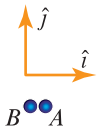
where, letting d be the distance over which A slides and referring to the FBD shown,

$$T_2 = \tfrac{1}{2}(m_A+m_B)v_2^2, \quad V_2 = 0, T_3 = 0, \quad V_3 = 0, \quad \text{and} \quad (U_{1\text{-}2})_{\text{np}} = -Fd = -\mu_k(m_A+m_B)gd. \tag{2}$$

Substituting Eqs. (2) into Eq. (1), and solving for v_2, we have

$$\tfrac{1}{2}(m_A + m_B)v_2^2 - \mu_k(m_A + m_B)gd = 0 \quad \Rightarrow \quad v_2 = \sqrt{2\mu_k gd}. \tag{3}$$

Now that we know the postimpact speed of A, we consider the impact between the block and the bullet. Let's observe that



$$v_{Ax}^- = 0, \quad v_{Ax}^+ = v_{Bx}^+ = v_2.$$

Conserving momentum in the x direction, we have

$$m_B v_{Bx}^- = (m_A + m_B)v_2 \quad \Rightarrow \quad m_B v_{Bx}^- = (m_A + m_B)\sqrt{2\mu_k gd}, \tag{4}$$

which, observing that v_{Bx}^- coincides with the preimpact speed of the bullet, yields the following result

$$\boxed{v_B^- = \frac{m_A + m_B}{m_B}\sqrt{2\mu_k gd} = 199\,\text{m/s},}$$

where we have used the following numerical data:

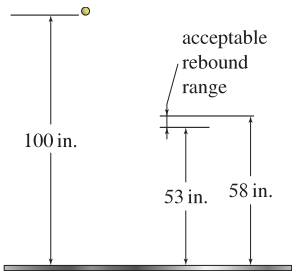
$$\begin{aligned} m_B &= \frac{323\,\text{gr}}{7000\,\text{gr/lb}}\frac{1}{32.2\,\text{ft/s}^2}(14.59\,\text{kg/slug}) = 0.02091\,\text{kg}, \\ \mu_k &= 0.7, \\ m_A &= 2\,\text{kg}, \\ g &= 9.81\,\text{m/s}^2, \\ d &= 0.31\,\text{m}. \end{aligned}$$

Problem 5.45

The official rules of tennis specify that

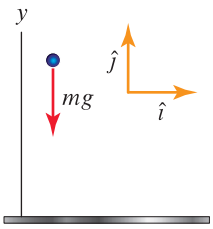
> "The ball shall have a [re]bound of more than 53 in. (134.62 cm) and less than 58 in. (147.32 cm) when dropped 100 in. (254.00 cm) upon a concrete base."

Understanding the expression "when dropped" as "when dropped from rest," determine the range of acceptable CORs for the collision of a tennis ball with concrete.



Solution

We assume that the ball is only subject to (constant) gravity and, when in contact with the ground, to a reaction force normal to the ground. Hence, the preimpact velocity can be determined using constant acceleration equations, as follows:



$$(v_y^-)^2 = 0 + 2g(h_i - 0) \quad \Rightarrow \quad (v_y^-)^2 = 2gh_i \quad \Rightarrow \quad v_y^- = -\sqrt{2gh_i}. \qquad (1)$$

where $h_i = 100.0$ in. We assume that the ground does not move due to the collision with the ball. Hence, given that the motion in only in the y direction, the collision in question is completely governed by the COR equation. Hence, we have

$$v_y^+ - (v_y^+)_{\text{ground}} = e[(v_y^-)_{\text{ground}} - v_y^-] \quad \Rightarrow \quad v_y^+ = -ev_y^- \quad \Rightarrow \quad v_y^+ = e\sqrt{2gh_i}, \qquad (2)$$

where we have used the expression for v_y^- derived in the last of Eqs. (1). We now consider two cases. One in which the ball rebounds to the minimum allowable height $h_{\text{RL}} = 53.00$ in., and the other in which the ball rebounds to the maximum allowable height $h_{\text{RU}} = 58.00$ in. Using the same method employed to derive Eq. (1), we have that h_{RL} and h_{RU} are related to v_y^+ as follows:

$$(v_y^+)_L = \sqrt{2gh_{\text{RL}}} \quad \text{and} \quad (v_y^+)_U = \sqrt{2gh_{\text{RU}}}, \qquad (3)$$

where $(v_y^+)_U$ and $(v_y^+)_L$ are the values of v_y^+ corresponding to the two cases considered, respectively. In both of the cases considered, we still have that v_y^+ must also conform to the result in the last of Eqs. (2). Therefore, for the case in which the rebound height is h_{RL}, letting $e = e_L$, we must have

$$e_L\sqrt{2gh_i} = \sqrt{2gh_{\text{RL}}} \quad \Rightarrow \quad e_L = \sqrt{\frac{h_{\text{RL}}}{h_i}} = 0.7280, \qquad (4)$$

where we have used the following numerical data: $h_{\text{RL}} = 53$ in. and $h_i = 100$ in. For the case in which the rebound height is h_{RU}, letting $e = e_U$ we must have

$$e_U\sqrt{2gh_i} = \sqrt{2gh_{\text{RU}}} \quad \Rightarrow \quad e_U = \sqrt{\frac{h_{\text{RU}}}{h_i}} = 0.7616, \qquad (5)$$

where we have used the following numerical data: $h_{\text{RU}} = 58$ in. and $h_i = 100$ in. Thus, expressing the final result using three significant figures, we must have

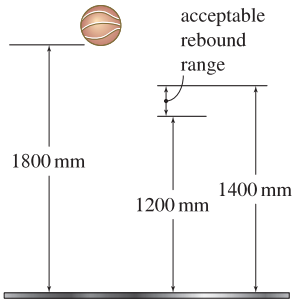
$$\boxed{0.728 \leq e \leq 0.762.}$$

Problem 5.46

The official rules of basketball specify that a basketball is properly inflated

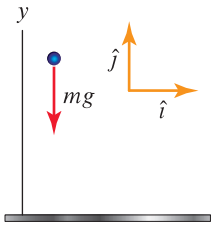
> "such that when it is dropped onto the playing surface from a height of about 1800 mm measured from the bottom of the ball, it will rebound to a height, measured to the top of the ball, of not less than about 1200 mm nor more than about 1400 mm."

Based on this rule, and understanding the expression "when it is dropped" as "when it is dropped from rest," determine the range of acceptable CORs for the collision between the ball and the court's surface.



Solution

We assume that the ball is only subject to (constant) gravity and, when in contact with the ground, to a reaction force normal to the ground. Hence, the preimpact velocity can be determined using constant acceleration equations, as follows:



$$(v_y^-)^2 = 0 + 2g(h_i - 0) \quad \Rightarrow \quad (v_y^-)^2 = 2gh_i \quad \Rightarrow \quad v_y^- = -\sqrt{2gh_i}. \tag{1}$$

where $h_i = 1800\,\text{mm}$. We assume that the ground does not move due to the collision with the ball. Hence, given that the motion in only in the y direction, the collision in question is completely governed by the COR equation. Hence, we have

$$v_y^+ - (v_y^+)_{\text{ground}} = e[(v_y^-)_{\text{ground}} - v_y^-] \quad \Rightarrow \quad v_y^+ = -ev_y^- \quad \Rightarrow \quad v_y^+ = e\sqrt{2gh_i}, \tag{2}$$

where we have used the expression for v_y^- derived in the last of Eqs. (1). We now consider two cases. One in which the ball rebounds to the minimum allowable height $h_{\text{RL}} = 1.2\,\text{m}$, and the other in which the ball rebounds to the maximum allowable height $h_{\text{RU}} = 1.4\,\text{m}$. Using the same method employed to derive Eq. (1), we have that h_{RL} and h_{RU} are related to v_y^+ as follows:

$$(v_y^+)_L = \sqrt{2gh_{\text{RL}}} \quad \text{and} \quad (v_y^+)_U = \sqrt{2gh_{\text{RU}}}, \tag{3}$$

where $(v_y^+)_U$ and $(v_y^+)_L$ are the values of v_y^+ corresponding to the two cases considered, respectively. In both of the cases considered, we still have that v_y^+ must also conform to the result in the last of Eqs. (2). Therefore, for the case in which the rebound height is h_{RL}, letting $e = e_L$, we must have

$$e_L\sqrt{2gh_i} = \sqrt{2gh_{\text{RL}}} \quad \Rightarrow \quad e_L = \sqrt{\frac{h_{\text{RL}}}{h_i}} = 0.8165, \tag{4}$$

where we have used the following numerical data: $h_{\text{RL}} = 1.2\,\text{m}$ and $h_i = 1.8\,\text{m}$. For the case in which the rebound height is h_{RU}, letting $e = e_U$ we must have

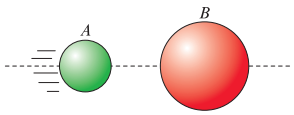
$$e_U\sqrt{2gh_i} = \sqrt{2gh_{\text{RU}}} \quad \Rightarrow \quad e_U = \sqrt{\frac{h_{\text{RU}}}{h_i}} = 0.8819, \tag{5}$$

where we have used the following numerical data: $h_{\text{RU}} = 1.4\,\text{m}$ and $h_i = 1.8\,\text{m}$. Thus, expressing the final result using three significant figures, we must have

$$\boxed{0.817 \leq e \leq 0.882.}$$

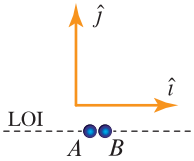
Problem 5.47

Consider a direct central impact for two spheres. Let m_A, m_B, and e denote the mass of sphere A, the mass of sphere B, and the COR, respectively. If sphere B is at rest before the collision, determine the relation that m_A, m_B, and e need to satisfy in order for A to come to a complete stop right after impact.



Solution

The impact-relevant FBD of the system is shown to the right. This FBD implies the conservation of linear momentum of the system in the x direction (which is also the LOI), i.e.,



$$m_A v_{Ax}^- + m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+ \quad \Rightarrow \quad m_A v_A^- = m_B v_B^+, \tag{1}$$

where we have accounted for the fact that B is initially at rest and that A comes to a stop right after impact. In addition to the above equation, we must enforce the COR equation, i.e., The COR equation is

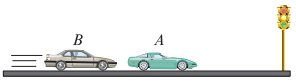
$$v_{Bx}^+ - v_{Ax}^+ = e\left(v_{Ax}^- - v_{Bx}^-\right) \quad \Rightarrow \quad v_{Bx}^+ = e v_{Ax}^-, \tag{2}$$

where, again, we have accounted for the fact that B is initially at rest and that A comes to a stop right after impact. Substituting the last of Eqs. (2) into the last of Eqs. (1) and canceling the term v_{Ax}^-, we see that the relation m_A, m_B, and e must satisfy is

$$\boxed{m_A = e m_B.}$$

Problems 5.48 and 5.49

Car A, with $m_A = 1550\,\text{kg}$, is stopped at a red light. Car B, with $m_B = 1865\,\text{kg}$ and a speed of $40\,\text{km/h}$, fails to stop before impacting car A. After impact, cars A and B slide over the pavement with a coefficient of friction $\mu_k = 0.65$.

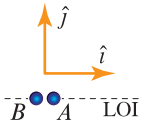


Problem 5.48 How far will the cars slide if the cars become entangled?

Problem 5.49 How far will the cars slide if the COR for the impact is $e = 0.2$?

Solution to 5.48

We take the LOI to be parallel horizontal. Hence, the impact-relevant FBD is as shown to the right and it implies conservation of linear momentum in the x direction, i.e,



$$m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+, \tag{1}$$

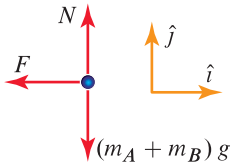
where we have accounted for the fact that $\vec{v}_A^- = \vec{0}$. If the cars become entangled, then

$$v_{Ax}^+ = v_{Bx}^+ = v_x^+, \tag{2}$$

where v_x^+ is the common value of the horizontal component of the velocity of the two cars. Substituting Eq. (2) into Eq. (1) and solving for v_x^+, we have

$$v_x^+ = \frac{m_B v_{Bx}^-}{m_A + m_B}. \tag{3}$$

Now, let ① and ② be the positions of the two cars right after the impact and when they come to a stop, respectively. The FBD of the system between ① and ② is shown to the right, where we have modeled the two entangled cars as a single particle. applying the work-energy principle between ① and ② we have



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2, \tag{4}$$

where, using the result in Eq. (3),

$$T_1 = \tfrac{1}{2}(m_A + m_B)\left(\frac{m_B v_{Bx}^-}{m_A + m_B}\right)^2, \quad V_1 = 0, \quad T_2 = 0, \quad V_2 = 0, \tag{5}$$

and where, calling d the distance A and B slide into the intersection, and accounting for the fact that $F = \mu_k N = \mu_k (m_A + m_B) g$, we have

$$(U_{1\text{-}2})_{\text{nc}} = -\mu_k (m_A + m_B) g d. \tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (4) we have

$$\tfrac{1}{2}(m_A + m_B)\left(\frac{m_B v_{Bx}^-}{m_A + m_B}\right)^2 - \mu_k (m_A + m_B) g d = 0,$$

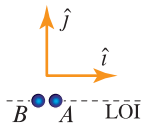
which, recalling that $v_{Bx}^- = v_B^-$, can be solved for d to obtain

$$\boxed{d = \frac{m_B^2 (v_B^-)^2}{2\mu_k g (m_A + m_B)^2} = 2.89\,\text{m},}$$

where we have used the following numerical data: $m_A = 1550\,\text{kg}$, $m_B = 1865\,\text{kg}$, $v_B^- = 40\,\text{km/h} = 11.11\,\text{m/s}$, $\mu_k = 0.65$, and $g = 9.81\,\text{m/s}^2$.

Solution to 5.49

We will assume that the LOI is parallel to the street and the street is horizontal. Hence, the impact-relevant FBD is as shown to the right and it implies conservation of linear momentum in the x direction, i.e,



$$m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+, \tag{7}$$

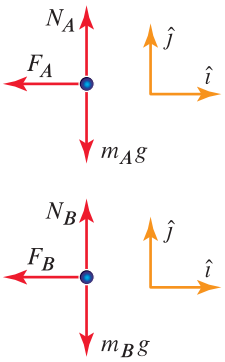
where we have accounted for the fact that the preimpact velocity of A is equal to zero. Given that the two cars rebound off of one another, we must enforce the COR equation, which can be written as

$$v_{Ax}^+ - v_{Bx}^+ = e v_{Bx}^-, \tag{8}$$

where again we have accounted for the fact that A does not move before impact. Equations (7) and (8) form a system of two equations in the two unknowns v_{Ax}^+ and v_{Bx}^+ whose solution is

$$v_{Ax}^+ = \frac{m_B v_{Bx}^-(1+e)}{m_A + m_B} \quad \text{and} \quad v_{Bx}^+ = \frac{v_{Bx}^-(m_B - e m_A)}{m_A + m_B}. \tag{9}$$

After the impact A and B no longer interact. Let ① and ② be the positions of the two cars right after the impact and when they come to a stop, respectively. The FBDs of the A and B between ① and ② are shown to the right.



Aplying the work-energy principle between ① and ② for A, we have

$$T_{A1} + V_{A1} + (U_{1\text{-}2})_{A\text{nc}} = T_{A2} + V_{A2}. \tag{10}$$

where, using the result in Eq. (3),

$$T_{A1} = \tfrac{1}{2} m_A \left[\frac{m_B v_{Bx}^-(1+e)}{m_A + m_B}\right]^2, \quad V_{A1} = 0, \quad T_{A2} = 0, \quad V_{A2} = 0, \tag{11}$$

and where, calling d_A the distance A slides into the intersection, and accounting for the fact that $F_A = \mu_k N_A = \mu_k m_A g$, we have

$$(U_{1\text{-}2})_{A\text{nc}} = -\mu_k m_A g d_A. \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (10) we have

$$\tfrac{1}{2} m_A \left[\frac{m_B v_{Bx}^-(1+e)}{m_A + m_B}\right]^2 - \mu_k m_A g d_A = 0,$$

which can be solved for d_A to obtain

$$\boxed{d_A = \frac{1}{2\mu_k g}\left[\frac{m_B v_{Bx}^-(1+e)}{m_A + m_B}\right]^2 = 4.16\,\text{m},}$$

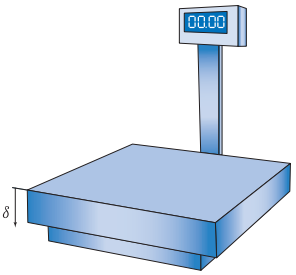
where we have used the following numerical data: $m_A = 1550\,\text{kg}$, $m_B = 1865\,\text{kg}$, $v_B^- = 40\,\text{km/h} = 11.11\,\text{m/s}$, $e = 0.2$, $\mu_k = 0.65$, and $g = 9.81\,\text{m/s}^2$. Repeating the process for the determination of d_A for the case of B, we obtain

$$\boxed{d_B = \frac{1}{2\mu_k g}\left[\frac{v_{Bx}^-(m_B - e m_A)}{m_A + m_B}\right]^2 = 2.01\,\text{m},}$$

where we have used the numerical data listed below the previous result.

Problems 5.50 and 5.51

A platform bench scale consists of a 120 lb plate resting on linear elastic springs whose combined spring constant is $k = 5000\,\text{lb/ft}$. Let $W = k(\delta - \delta_0)$ be the weight measurement actually provided by the scale (that is, it reads zero pounds when nothing is on the plate), where δ_0 is the spring's compression due to the weight of the scale's plate.

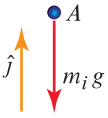


Problem 5.50 A 50 lb sack of portland cement is dropped (from rest) onto the scale from a height $h = 4\,\text{ft}$ measured from the scale's plate (there is no rebound of the sack). Determine the maximum weight displayed by the scale.

Problem 5.51 Repeat Prob. 5.50 with $h = 0\,\text{ft}$.

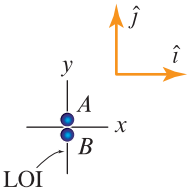
Solution to 5.50

We denote the cement sack by A and the scale plate by B. Referring to the FBD to the right, before impact we assume that A moves only due to gravity. Hence, the preimpact velocity of A can be computed via constant acceleration equations, as follows:



$$\dot{s}^2 = \dot{s}_0^2 + 2a_c(s - s_0) \quad \Rightarrow \quad (v_{Ay}^-)^2 = 2gh \quad \Rightarrow \quad v_{Ay}^- = -\sqrt{2gh}. \tag{1}$$

Since spring forces are not impulsive, the impact-relevant FBD is that shown to the right. The A-B impact is an unconstrained perfectly plastic impact with LOI coinciding with the y axis. This implies that the system's momentum is conserved through the impact, i.e.,

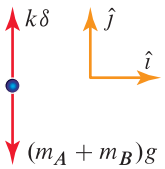


$$m_A v_{Ay}^- + m_B v_{By}^- = m_A v_{Ay}^+ + m_B v_{By}^+. \tag{2}$$

Recalling that B is initially at rest, that the preimpact velocity of A is given in the last of Eqs. (1), and that the impact is perfectly plastic, i.e., $v_{Ay}^+ = v_{By}^+$, we can solve Eq. (2) for v_{Ay}^+ and v_{By}^+ to obtain

$$v_{Ay}^+ = v_{By}^+ = -\frac{m_A\sqrt{2gh}}{m_A + m_B}. \tag{3}$$

The maximum weight displayed corresponds to the maximum displacement of the scale's plate, which we can determine using the work-energy principle. Let ② be just after impact and ③ be at maximum spring compression. Referring to the FBD to the right, we assume that the only forces acting on the system between ② and ③ are gravity and the spring force. This allows to write the work-energy principle as follows:



$$T_2 + V_2 = T_3 + V_3, \tag{4}$$

where, letting v denote the common speed of A and B and choosing the datum for gravity at ②,

$$T_2 = \tfrac{1}{2}(m_A + m_B)v_2^2, \quad V_2 = \tfrac{1}{2}k\delta_2^2, \quad T_3 = 0, \quad V_3 = \tfrac{1}{2}k\delta_3^2 - (m_A + m_B)g(\delta_3 - \delta_2). \tag{5}$$

Observing that $\delta_2 = \delta_0 = m_B g/k$ (to equilibrate the weight of the scale's plate), recalling that $v_2 = |v_{Ay}^+|$ (given given by Eq. (3)), substituting Eqs. (5) into Eq. (4), yields an equation in δ_3 whose solutions is

$$\delta_3 = (m_A + m_B)\frac{g}{k} \pm \frac{\sqrt{(m_A + m_B)m_A^2 g k^2[2hk + (m_A + m_B)g]}}{(m_A + m_B)k^2}. \tag{6}$$

We observe that the term in Eq. (6) to the right of $\pm$ symbol is larger than the term to left of the symbol. This observation is important because it tell us that the only meaningful root of the problem is that characterized by the $+$ sign, i.e.,

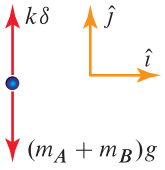
$$\delta_3 = (m_A + m_B)\frac{g}{k} + \frac{\sqrt{(m_A + m_B)m_A^2 g k^2 [2hk + (m_A + m_B)g]}}{(m_A + m_B)k^2}. \tag{7}$$

Consequently, given that the maximum weight displayed by the scale is $W_{\max} = k(\delta_3 - \delta_0)$, we have

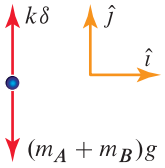
$$\boxed{W_{\max} = m_A g + \frac{\sqrt{(m_A + m_B)m_A^2 g k^2 [2hk + (m_A + m_B)g]}}{(m_A + m_B)k} = 819\,\text{lb},}$$

where we have used the following numerical data: $m_A = (50\,\text{lb})/(32.2\,\text{ft/s}^2) = 1.553\,\text{slug}$, $g = 32.2\,\text{ft/s}^2$, $m_B = (120\,\text{lb})/(32.2\,\text{ft/s}^2) = 3.727\,\text{slug}$, $k = 5000\,\text{lb/ft}$, and $h = 4\,\text{ft}$.

Solution to 5.51



If h were different from zero, we would have to solve the problem by first computing the speed with which the cement sack strikes the scale's plate. Then we would have to solve the impact problem associated to the collision between the cement sack and the scale's plate. This would give us the velocity with which the sack and plate move right after impact. Finally, we would need to compute the distance that the plate moves after impact in order for the sack and the plate to come to rest. With the above in mind, if $h = 0$, then the impact part of the problem is characterized by a trivial solution, that is, the postimpact velocity of the sack and the plate is equal to zero. Hence, the only part of the problem that remains to be solved, concerns the calculation of the maximum displacement achieved by the plate in order to stop the motion of the sack.



We denote the sake and the plate with A and B, respectively. We will determine the maximum displacement of the scale's plate using the work-energy principle. Let ① be right after the A is placed on B and ② be at maximum spring compression. Referring to the FBD to the right, we assume that the only forces acting on the system between ① and ② are gravity and the spring force. This allows to write the work-energy principle as follows:

$$T_1 + V_1 = T_2 + V_2, \tag{8}$$

where, letting v denote the common speed of A and B and choosing the datum for gravity at ①,

$$T_1 = 0, \quad V_1 = \tfrac{1}{2}k\delta_1^2, \quad T_2 = 0, \quad V_2 = \tfrac{1}{2}k\delta_2^2 - (m_A + m_B)g(\delta_2 - \delta_1), \tag{9}$$

where we have accounted for the fact that A and B start from rest and that the system is again at rest in ②. Observing that $\delta_1 = \delta_0 = m_B g/k$ (to equilibrate the weight of the scale's plate), Eqs. (9) into Eq. (8), yields a second order algebraic equation in δ_2 with the following two roots:

$$(\delta_2)_1 = \frac{m_B g}{k} \quad \text{and} \quad (\delta_2)_2 = \frac{m_B g}{k} + 2\frac{m_A g}{k}. \tag{10}$$

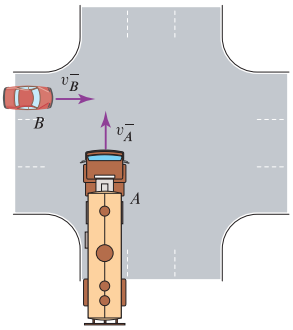
The first root corresponds to the release position of the system. Hence, the only acceptable solution is given by the second root. Using such a root, and recalling that the maximum weight displayed by the scale is $W_{\max} = k(\delta_2 - \delta_0)$, we have

$$\boxed{W_{\max} = 2m_A g = 100\,\text{lb},}$$

where we have used the fact that $m_A g = 50\,\text{lb}$.

Problems 5.52 through 5.54

A 31,000 lb truck A and a 3970 lb sports car B collide at an intersection. Right before the collision the truck and the sports car are traveling at $v_A^- = 60$ mph and $v_B^- = 50$ mph. Assume that the entire intersection forms a horizontal surface.



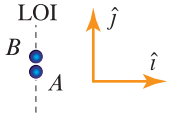
Problem 5.52 Letting the line of impact be parallel to the ground and to the preimpact velocity of the truck, determine the postimpact velocities of A and B if A and B become entangled. Furthermore, assuming that the truck and the car slide after impact and that the coefficient of kinetic friction is $\mu_k = 0.7$, determine the position at which A and B come to a stop relative to the position they occupied at the instant of impact.

Problem 5.53 Letting the line of impact be parallel to the ground and to the preimpact velocity of the truck, determine the postimpact velocities of A and B if the contact between A and B is frictionless and the COR $e = 0$. Furthermore, assuming that the truck and the car slide after impact and that the coefficient of kinetic friction is $\mu_k = 0.7$, determine the position at which A and B come to a stop relative to the position they occupied at the instant of impact.

Problem 5.54 Letting the line of impact be parallel to the ground and to the preimpact velocity of the truck, determine the postimpact velocities of A and B if the contact between A and B is frictionless and the COR $e = 0.1$. Furthermore, assuming that the truck and the car slide after impact and that the coefficient of kinetic friction is $\mu_k = 0.7$, determine the position at which A and B come to a stop relative to the position they occupied at the instant of impact.

Solution to 5.52

We model A and B as particle and the overall impact as an unconstrained perfectly plastic impact with LOI parallel to the y direction. Therefore, the impact is governed by the following equations:



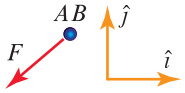
$$m_A v_{Ax}^- + m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+, \tag{1}$$
$$m_A v_{Ay}^- + m_B v_{By}^- = m_A v_{Ay}^+ + m_B v_{By}^+, \tag{2}$$
$$v_{Ax}^+ = v_{Bx}^+, \tag{3}$$
$$v_{Ay}^+ = v_{By}^+, \tag{4}$$

where Eqs. (1) and (2) represent the conservation of linear momentum for the system along the x and y directions, respectively, and where Eqs. (3) and (4) express the fact that A and B move together after the impact. The preimpact velocity components are known. Specifically, we have $v_{Ax}^- = 0$, $v_{Ay}^- = v_A^- = 60$ mph, $v_{Bx}^- = v_B^- = 50$ mph, and $v_{By}^- = 0$. Therefore, Eqs. (1)–(4) form a system of four equations in the four unknowns v_{Ax}^+, v_{Ay}^+, v_{Bx}^+, and v_{By}^+, whose solution is

$$v_{Ax}^+ = v_{Bx}^+ = \frac{m_B v_B^-}{m_A + m_B} = 8.326\,\text{ft/s} \quad \text{and} \quad v_{Ay}^+ = v_{Bx}^+ = \frac{m_A v_A^-}{m_A + m_B} = 78.01\,\text{ft/s}, \tag{5}$$

where we have used the following numerical data: $m_A = (31{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 962.7\,\text{slug}$, $m_B = (3970\,\text{lb})/(32.2\,\text{ft/s}^2) = 123.3\,\text{slug}$, $v_A^- = 60\,\text{mph} = 88.00\,\text{ft/s}$, and $v_A^- = 50\,\text{mph} = 73.33\,\text{ft/s}$. Hence, when expressed to three significant digits, the postimpact velocities of A and B are

$$\boxed{\vec{v}_A^+ = \vec{v}_B^+ = (8.33\,\hat{\imath} + 78.0\,\hat{\jmath})\,\text{ft/s}} \quad (6)$$



Right after impact, A and B slide as a single particle along a straight line with the same direction as the postimpact velocity vector of A and B. Let ① be immediately after impact and ② be when the system comes to rest at a distance r relative to ① (in the direction of $\vec{v}_A^+ = \vec{v}_B^+$). As shown in the FBD to the right (top view of A and B as a single particle), A and B are subject only to the friction force F, which will act opposite to the direction of motion. Since A and B do not move in the direction perpendicular to the ground, the combined weight of A and B will be equilibrate by a normal force N, i.e., $N = (m_A + m_B)g$. Consequently, the magnitude of the friction force F is

$$F = \mu_k N = \mu_k (m_A + m_B)g. \quad (7)$$

Observing that F in Eq. (7) is constant, the application of the work-energy principle between ① and ② gives

$$T_1 - \mu_k (m_A + m_B)gr = T_2, \quad (8)$$

where

$$T_1 = \tfrac{1}{2}(m_A + m_B)(v^+)^2 \quad \text{and} \quad T_2 = 0, \quad (9)$$

where, referring to Eq. (5),

$$v^+ = \sqrt{(v_{Ax}^+)^2 + (v_{Ay}^+)^2} = \sqrt{(v_{Bx}^+)^2 + (v_{By}^+)^2} = 78.45\,\text{ft/s} \quad (10)$$

is the common postimpact speed of A and B. Substituting Eqs. (9) in Eq. (8) and solving for r, we have

$$r = \frac{(v^+)^2}{2\mu_k g} = 136.5\,\text{ft}, \quad (11)$$

where we have used the result in Eq. (10) and the given numerical value of $\mu_k = 0.7$.

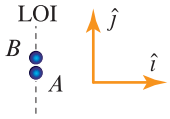
As already discussed, the postimpact motion of A and B is in the direction of the postimpact (common) velocity of A and B. Referring to Eqs. (5) and (10), this direction is described by the following unit vector

$$\hat{u}_r = \frac{v_{Ax}^+\,\hat{\imath} + v_{Ax}^+\,\hat{\jmath}}{\sqrt{(v_{Ay}^+)^2 + (v_{Ay}^+)^2}} = \frac{v_{Bx}^+\,\hat{\imath} + v_{Bx}^+\,\hat{\jmath}}{\sqrt{(v_{By}^+)^2 + (v_{By}^+)^2}} = 0.1061\,\hat{\imath} + 0.9944\,\hat{\jmath}. \quad (12)$$

Now that the unit vector $\hat{u}_r$ is known, we can express the vector describing ② relative to ① as $\vec{r} = r\,\hat{u}_r$, which gives

$$\boxed{\vec{r} = (14.5\,\hat{\imath} + 136\,\hat{\jmath})\,\text{ft}}$$

Solution to 5.53



We model A and B as particles and the overall impact as an unconstrained elastic impact with LOI parallel to the y direction. Therefore, the impact is governed by the following equations:

$$m_A v_{Ay}^- + m_B v_{By}^- = m_A v_{Ay}^+ + m_B v_{By}^+, \tag{13}$$

$$v_{Ax}^- = v_{Ax}^+ \tag{14}$$

$$v_{Bx}^- = v_{Bx}^+, \tag{15}$$

$$v_{By}^+ - v_{Ay}^+ = e(v_{Ay}^- - v_{By}^-), \tag{16}$$

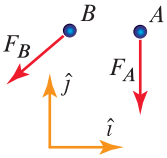
where Eqs. (13)–(16) represent the conservation of linear momentum for the system along the LOI, the conservation of linear momentum for particle A perpendicular to the LOI, the conservation of linear momentum for particle B perpendicular to the LOI, and the COR equation for the system, respectively. The preimpact velocity components are known. Specifically, we have $v_{Ax}^- = 0$, $v_{Ay}^- = v_A^- = 60\,\text{mph} = 88.00\,\text{ft/s}$, $v_{Bx}^- = v_B^- = 50\,\text{mph} = 73.33\,\text{ft/s}$, and $v_{By}^- = 0$. Therefore, Eqs. (25)–(28) form a system of four equations in the four unknowns v_{Ax}^+, v_{Ay}^+, v_{Bx}^+, and v_{By}^+, whose solution is

$$v_{Ax}^+ = v_{Ax}^- = 0, \qquad v_{Ay}^+ = \frac{m_A v_{Ay}^-}{m_A + m_B} = 78.01\,\text{ft/s}, \tag{17}$$

$$v_{Bx}^+ = v_{Bx}^- = 73.33\,\text{ft/s}, \qquad v_{By}^+ = \frac{m_A v_{Ay}^-}{m_A + m_B} = 78.01\,\text{ft/s}, \tag{18}$$

where, in addition to the data already indicated, we also used the following numerical data: $m_A = (31{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 962.7\,\text{slug}$, $m_B = (3970\,\text{lb})/(32.2\,\text{ft/s}^2) = 123.3\,\text{slug}$, and $e = 0$. Hence, the postimpact velocities of A and B are

$$\boxed{\vec{v}_A^+ = 78.0\,\hat{\jmath}\,\text{ft/s} \quad \text{and} \quad \vec{v}_B^+ = (73.3\,\hat{\imath} + 78.0\,\hat{\jmath})\,\text{ft/s}}$$ 



Right after impact, A and B slide as along straight lines with the same direction as their respective postimpact velocity vectors. Let ① be immediately after impact and ② be when the system comes to rest at a distance. In going from ① to ② A and B will travel the distances r_A and r_B respectively. As shown in the FBD to the right (top view of A and B), A and B are subject only to the friction forces F_A and F_B, respectively. The friction on A will act opposite to the direction of the motion of A. Similarly, the friction on B will act opposite to the direction of the motion of B. Since A and B do not move in the direction perpendicular to the ground, the weights of A and B will be equilibrate by corresponding normal forces N_A and N_B such that $N_A = m_A g$ and $N_N = m_B g$. Consequently, the magnitudes of the friction forces F_A and F_B are

$$F_A = \mu_k N_A = \mu_k m_A g \quad \text{and} \quad F_B = \mu_k N_B = \mu_k m_B g. \tag{19}$$

Observing that F_A and F_B in Eq. (19) are constant, the application of the work-energy principle between ① and ② for A and B individually gives

$$T_{A1} - \mu_k m_A g r_A = T_{A2} \quad \text{and} \quad T_{B1} - \mu_k m_B g r_B = T_{B2}, \tag{20}$$

where the kinetic energy terms are

$$T_{A1} = \tfrac{1}{2} m_A (v_A^+)^2, \quad T_{A2} = 0, \quad T_{B1} = \tfrac{1}{2} m_A (v_B^+)^2, \quad T_{B2} = 0, \tag{21}$$

and where v_A^+ and v_B^+ are the postimpact speeds of A and B, respectively, which, using Eqs. (17) and (18), have the following values:

$$v_A^+ = \sqrt{(v_{Ax}^+)^2 + (v_{Ay}^+)^2} = 78.01\,\text{ft/s} \quad \text{and} \quad v_B^+ = \sqrt{(v_{Bx}^+)^2 + (v_{By}^+)^2} = 107.1\,\text{ft/s}. \tag{22}$$

Substituting Eqs. (21) into (the appropriate) Eqs. (20) and solving for r_A and r_B we have

$$r_A = \frac{(v_A^+)^2}{2\mu_k g} = 135.0\,\text{ft} \quad \text{and} \quad r_B = \frac{(v_B^+)^2}{2\mu_k g} = 254.4\,\text{ft}, \tag{23}$$

where we have used Eqs. (22) and the given value of μ_k.

As already discussed, the postimpact motion of A and B is in the direction of $\vec{v}_A^+$ and $\vec{v}_B^+$, respectively. Referring to Eqs. (17) and (18), these directions are described by the following unit vectors

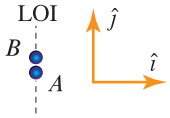
$$\hat{u}_{r_A} = \hat{\jmath} \quad \text{and} \quad \hat{u}_{r_B} = \frac{v_{Bx}^+\,\hat{\imath} + v_{By}^+\,\hat{\jmath}}{\sqrt{(v_{Bx}^+)^2 + (v_{By}^+)^2}} = 0.6849\,\hat{\imath} + 0.7286\,\hat{\jmath}. \tag{24}$$

Now that the unit vectors $\hat{u}_{r_A}$ and $\hat{u}_{r_B}$ are known, we can express the vectors describing ② relative to ① for A and B as $\vec{r}_A = r_A\,\hat{u}_{r_A}$ and $\vec{r}_B = r_B\,\hat{u}_{r_B}$, respectively. Therefore, using eqs. (23) and (24), we can write (using three significant figures)

$$\boxed{\vec{r}_A = 135\,\hat{\jmath}\,\text{ft} \quad \text{and} \quad \vec{r}_B = (174\,\hat{\imath} + 185\,\hat{\jmath})\,\text{ft}}$$



Solution to 5.54



We model A and B as particles and the overall impact as an unconstrained elastic impact with LOI parallel to the y direction. Therefore, the impact is governed by the following equations:

$$m_A v_{Ay}^- + m_B v_{By}^- = m_A v_{Ay}^+ + m_B v_{By}^+, \tag{25}$$

$$v_{Ax}^- = v_{Ax}^+ \tag{26}$$

$$v_{Bx}^- = v_{Bx}^+, \tag{27}$$

$$v_{By}^+ - v_{Ay}^+ = e(v_{Ay}^- - v_{By}^-), \tag{28}$$

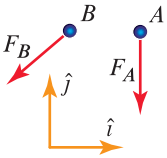
where Eqs. (25)–(28) represent the conservation of linear momentum for the system along the LOI, the conservation of linear momentum for particle A perpendicular to the LOI, the conservation of linear momentum for particle B perpendicular to the LOI, and the COR equation for the system, respectively. The preimpact velocity components are known. Specifically, we have $v_{Ax}^- = 0$, $v_{Ay}^- = v_A^- = 60\,\text{mph} = 88.00\,\text{ft/s}$, $v_{Bx}^- = v_B^- = 50\,\text{mph} = 73.33\,\text{ft/s}$, and $v_{By}^- = 0$. Therefore, Eqs. (25)–(28) form a system of four equations in the four unknowns v_{Ax}^+, v_{Ay}^+, v_{Bx}^+, and v_{By}^+, whose solution is

$$v_{Ax}^+ = v_{Ax}^- = 0, \qquad v_{Ay}^+ = \frac{(m_A - e m_B)v_{Ay}^-}{m_A + m_B} = 77.01\,\text{ft/s}, \tag{29}$$

$$v_{Bx}^+ = v_{Bx}^- = 73.33\,\text{ft/s}, \qquad v_{By}^+ = \frac{m_A(1+e)v_{Ay}^-}{m_A + m_B} = 85.81\,\text{ft/s}, \tag{30}$$

where, in addition to the data already indicated, we also used the following numerical data: $m_A = (31{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 962.7\,\text{slug}$, $m_B = (3970\,\text{lb})/(32.2\,\text{ft/s}^2) = 123.3\,\text{slug}$, and $e = 0.1$. Hence, the postimpact velocities of A and B are

$$\boxed{\vec{v}_A^+ = 77.0\,\hat{\jmath}\,\text{ft/s} \quad \text{and} \quad \vec{v}_B^+ = (73.3\,\hat{\imath} + 85.8\,\hat{\jmath})\,\text{ft/s}}$$



Right after impact, A and B slide as along straight lines with the same direction as their respective postimpact velocity vectors. Let ① be immediately after impact and ② be when the system comes to rest at a distance. In going from ① to ② A and B will travel the distances r_A and r_B respectively. As shown in the FBD to the right (top view of A and B), A and B are subject only to the friction forces F_A and F_B, respectively. The friction on A will act opposite to the direction of the motion of A. Similarly, the friction on B will act opposite to the direction of the motion of B. Since A and B do not move in the direction perpendicular to the ground, the weights of A and B will be equilibrate by corresponding normal forces N_A and N_B such that $N_A = m_A g$ and $N_N = m_B g$. Consequently, the magnitudes of the friction forces F_A and F_B are

$$F_A = \mu_k N_A = \mu_k m_A g \quad \text{and} \quad F_B = \mu_k N_B = \mu_k m_B g. \tag{31}$$

Observing that F_A and F_B in Eq. (31) are constant, the application of the work-energy principle between ① and ② for A and B individually gives

$$T_{A1} - \mu_k m_A g r_A = T_{A2} \quad \text{and} \quad T_{B1} - \mu_k m_B g r_B = T_{B2}, \tag{32}$$

where the kinetic energy terms are

$$T_{A1} = \tfrac{1}{2} m_A (v_A^+)^2, \quad T_{A2} = 0, \quad T_{B1} = \tfrac{1}{2} m_A (v_B^+)^2, \quad T_{B2} = 0, \tag{33}$$

and where v_A^+ and v_B^+ are the postimpact speeds of A and B, respectively, which, using Eqs. (29) and (30), have the following values:

$$v_A^+ = \sqrt{(v_{Ax}^+)^2 + (v_{Ay}^+)^2} = 77.01\,\text{ft/s} \quad \text{and} \quad v_B^+ = \sqrt{(v_{Bx}^+)^2 + (v_{By}^+)^2} = 112.9\,\text{ft/s}. \tag{34}$$

Substituting Eqs. (33) into (the appropriate) Eqs. (32) and solving for r_A and r_B we have

$$r_A = \frac{(v_A^+)^2}{2\mu_k g} = 131.6\,\text{ft} \quad \text{and} \quad r_B = \frac{(v_B^+)^2}{2\mu_k g} = 282.8\,\text{ft}, \tag{35}$$

where we have used Eqs. (34) and the given value of μ_k.

As already discussed, the postimpact motion of A and B is in the direction of $\vec{v}_A^+$ and $\vec{v}_B^+$, respectively. Referring to Eqs. (29) and (30), these directions are described by the following unit vectors

$$\hat{u}_{r_A} = \hat{\jmath} \quad \text{and} \quad \hat{u}_{r_B} = \frac{v_{Bx}^+\,\hat{\imath} + v_{By}^+\,\hat{\jmath}}{\sqrt{(v_{Bx}^+)^2 + (v_{By}^+)^2}} = 0.6497\,\hat{\imath} + 0.7602\,\hat{\jmath}. \tag{36}$$

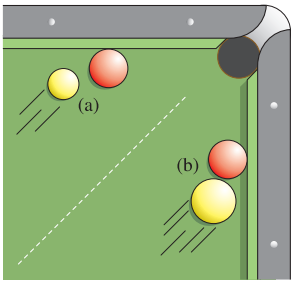
Now that the unit vectors $\hat{u}_{r_A}$ and $\hat{u}_{r_B}$ are known, we can express the vectors describing ② relative to ① for A and B as $\vec{r}_A = r_A\,\hat{u}_{r_A}$ and $\vec{r}_B = r_B\,\hat{u}_{r_B}$, respectively. Therefore, using eqs. (35) and (36), we can write (using three significant figures)

$$\boxed{\vec{r}_A = 132\,\hat{\jmath}\,\text{ft} \quad \text{and} \quad \vec{r}_B = (184\,\hat{\imath} + 215\,\hat{\jmath})\,\text{ft}}$$



Problem 5.55

Although competition rules prohibit significant difference in size, typical coin-operated pool tables may present players with a significant difference in diameter between the typical object ball (i.e., a colored ball) and the cue ball (i.e., the white ball). In fact, once an object ball goes into a pocket, it is captured by the table whereas a cue ball must always be returned to the player; and it is not uncommon for the return mechanism to use the difference in ball diameter to separate the cue ball from the rest. Given this, suppose we want to hit a ball resting against the bumper in such a way that, after the collision, it moves along the bumper. Modeling the contact between balls as frictionless, establish whether or not it is possible to execute the shot in question with (a) an undersized cue ball and (b) an oversized cue ball.



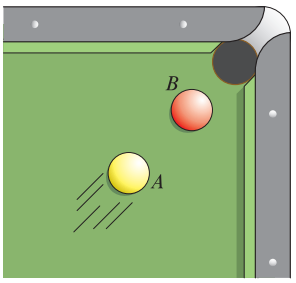
Solution

It is possible to execute the shot in case (a) (undersized cue ball) but not in case (b) (oversized cue ball). The reason is that to execute the shot in question, the LOI of the impact must be parallel to the bumper. Since the object ball is assumed to be at rest and touching the bumper, we have that the LOI can be parallel to the bumper if the cue ball is either of the same size as the object ball or smaller than the object ball. If the cue ball is larger than the object ball then the LOI will not be directed "into the bumper." Consequently, right after impact, the object ball will tend to rebound off the bumber.

Problem 5.56

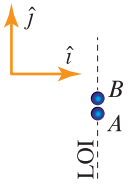
Competition billiard balls and tables need to adhere to strict standards (see the Billiard Congress of America for standards in the United States). Specifically, billiard balls must weigh between 5.5 and 6 oz, and they must be 2.25 ± 0.005 in. in diameter.

Using the theory presented in this section, establish whether or not it is possible to have a moving ball A hit a stationary ball B so that A stops right after the impact, if A and B have the same diameter but not the same weight (since it appears possible to have a weight difference of up to 0.5 oz while staying within regulations). Assume that the COR $e = 1$.



Solution

If the pre-impact velocity of A had a non-zero component of velocity perpendicular to the LOI, then this component of velocity would be conserved through the impact. This consideration implies that for A to stop after impact, at the very least, its preimpact velocity must be entirely parallel to the LOI. Working under this assumption and using the component system shown, the preimpact velocities of A and B are



$$\vec{v}_A^- = v_A^- \, \hat{j} \quad \text{and} \quad \vec{v}_B^- = \vec{0},$$

where v_A^- is the preimpact speed of A, having assumed that A is initially moving in the positive y direction. Consequently, the conservation of linear momentum in the y direction reads

$$m_A v_A^- = m_A v_{Ay}^+ + m_B v_{By}^+, \tag{1}$$

and the COR equation reads

$$v_{By}^+ - v_{Ay}^+ = e v_A^-. \tag{2}$$

Equations (1) and (2) are two equations in the two unknowns v_{Ay}^+ and v_{By}^+, whose solution is

$$v_{Ay}^+ = \frac{(m_A - m_B e) v_A^-}{m_A + m_B} \quad \text{and} \quad v_{By}^+ = \frac{m_A v_A^- (e+1)}{m_A + m_B}. \tag{3}$$

To check whether or not it is possible for A to stop, we set $v_{Ay}^+ = 0$ in Eq. (3), then we have

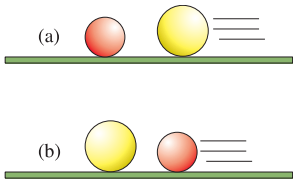
$$\frac{(m_A - m_B e) v_A^-}{m_A + m_B} = 0 \quad \Rightarrow \quad m_A = m_B e. \tag{4}$$

Recalling that the COR $e = 1$, we have that v_{Ay}^+ can only be equal zero if the masses are identical.

Given the assumptions, it is not possible to have a moving ball A hit a stationary ball B so that A stops right after the impact.

Problem 5.57

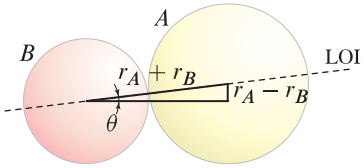
Competition billiard balls and tables need to adhere to strict standards (see the Billiard Congress of America for standards in the United States). Specifically, billiard balls must weigh between 5.5 and 6 oz, and they must be 2.25 ± 0.005 in. in diameter.



Professional billiard players can easily impart to a ball a speed of 20 mph. Assume the tolerance on the ball diameter to be 1/100 in. instead of 5/1000 in. and determine the outcome of the collision between (a) a 2.26 in. diameter ball traveling at 20 mph with a stationary 2.24 in. diameter ball (i.e., each ball is at the extreme limit of tolerance relative to the nominal diameter) and (b) a 2.24 in. diameter ball traveling at 20 mph with a stationary 2.26 in. diameter ball. Assume that the COR $e = 1$ and that the weights of the two balls are identical. Furthermore, assume that the contact between the balls and the table can be treated as essentially frictionless.

Solution

Part (a). Let A be the incoming ball and B the stationary ball. Also, let θ be the angle the LOI forms with the horizontal.

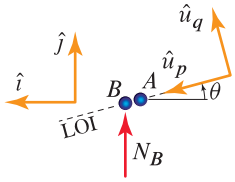


$$\theta = \sin^{-1}\frac{r_A - r_B}{r_A + r_B} = 0.2546^\circ, \tag{1}$$

where $r_A = 2.24$ in. and $r_B = 2.26$ in. The preimpact velocity components of A and B are:

$$v_{Ax}^- = v_A^- = 20\,\text{mph} = 29.33\,\text{ft/s}, \quad v_{Ay}^- = 0, \quad v_{Bx}^- = 0, \quad \text{and} \quad v_{By}^- = 0. \tag{2}$$

The impact-relevant FBD shows that there are no external impulsive forces on the system in the x direction. Hence, recalling that $m_A = m_B$ and accounting for the preimpact conditions in Eqs. (2), we have



$$m_A v_{Ax}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+ \quad \Rightarrow \quad v_{Ax}^- = v_{Ax}^+ + v_{Bx}^+. \tag{3}$$

The COR equation holds only along the LOI and it is most easily written using the pq component system. This gives $v_{Bp}^+ - v_{Ap}^+ = e v_{Ap}^-$. Observe that

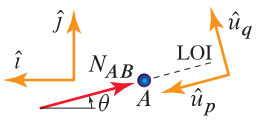
$$v_{Ap} = v_{Ax}\cos\theta - v_{Ay}\sin\theta, \qquad v_{Bp} = v_{Bx}\cos\theta - v_{By}\sin\theta, \tag{4}$$

$$v_{Aq} = v_{Ax}\sin\theta + v_{Ay}\cos\theta \qquad v_{Bq} = v_{Bx}\sin\theta + v_{By}\cos\theta. \tag{5}$$

Using Eqs. (4) and since that $v_{By}^+ = 0$, for the COR equation in the xy component system we have

$$\left(v_{Bx}^+ - v_{Ax}^+\right)\cos\theta + v_{Ay}^+\sin\theta = e v_{Ax}^-\cos\theta. \tag{6}$$

The impact-relevant FBD of A implies that $v_{Aq}^- = v_{Aq}^+$, which, with Eqs. (5), gives



$$v_{Ax}^-\sin\theta = v_{Ax}^+\sin\theta + v_{Ay}^+\cos\theta. \tag{7}$$

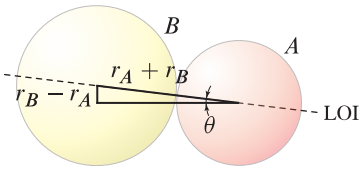
Equations (3), (6), and (7) form a system of three equations in the three unknowns v_{Ax}^+, v_{Ay}^+, v_{Bx}^+ whose solution is

$$v_{Ax}^+ = \frac{v_{Ax}^-(2 - e - e\cos 2\theta)}{3 + \cos 2\theta}, \quad v_{Ay}^+ = \frac{(1+e)v_{Ax}^- \sin 2\theta}{3 + \cos 2\theta}, \quad v_{Bx}^+ = \frac{2(1+e)v_{Ax}^- \cos^2\theta}{3 + \cos 2\theta}.$$

Recalling that A and B are the large and small balls, respectively, $e = 1$ and using the numerical data in Eq. (1) and the first of Eqs. (2), the above equations yield the following result:

$$\vec{v}^+_{\text{large ball}} = (2.90\times 10^{-4}\,\hat{\imath} + 0.130\,\hat{\jmath})\ \text{ft/s} \quad \text{and} \quad \vec{v}^+_{\text{small ball}} = 29.3\,\hat{\imath}\ \text{ft/s}$$

Part (b). Again, let A be the incoming ball and B the stationary ball and again let θ be the angle the LOI forms with the horizontal. This time this angle is given by



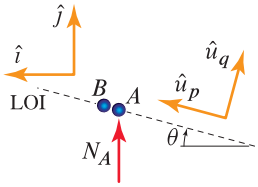
$$\theta = \sin^{-1}\frac{r_B - r_A}{r_A + r_B} = 0.2546^\circ, \tag{8}$$

where, again, $r_A = 2.24$ in. and $r_B = 2.26$ in.

The preimpact velocity components of A and B are:

$$v_{Ax}^- = v_A^- = 20\,\text{mph} = 29.33\,\text{ft/s}, \quad v_{Ay}^- = 0, \quad v_{Bx}^- = 0, \quad \text{and} \quad v_{By}^- = 0. \tag{9}$$

The impact-relevant FBD, shows that there are no external impulsive forces on the system in the x direction. Hence, recalling that $m_A = m_B$ and accounting for the preimpact conditions in Eqs. (9),



$$m_A v_{Ax}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+ \quad \Rightarrow \quad v_{Ax}^- = v_{Ax}^+ + v_{Bx}^+. \tag{10}$$

The COR equation holds only along the LOI and it is most easily written using the pq component system. This gives $v_{Bp}^+ - v_{Ap}^+ = e v_{Ap}^-$. Observe that

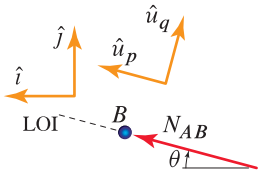
$$v_{Ap} = v_{Ax}\cos\theta + v_{Ay}\sin\theta, \qquad v_{Bp} = v_{Bx}\cos\theta + v_{By}\sin\theta, \tag{11}$$

$$v_{Aq} = -v_{Ax}\sin\theta + v_{Ay}\cos\theta \qquad v_{Bq} = -v_{Bx}\sin\theta + v_{By}\cos\theta. \tag{12}$$

Using Eqs. (11), the preimpact conditions in Eqs. (9), and the fact that $v_{Ay}^+ = 0$, the COR equation can be rewritten in the xy component system to give

$$\left(v_{Bx}^+ - v_{Ax}^+\right)\cos\theta + v_{By}^+ \sin\theta = e v_{Ax}^- \cos\theta. \tag{13}$$

The impact-relevant FBD of just B implies that $v_{Bq}^- = v_{Bq}^+$. Hence, using the second of Eqs. (12) and the preimpact conditions in Eqs. (9), we have



$$0 = -v_{Bx}^- \sin\theta + v_{By}^+ \cos\theta. \tag{14}$$

Equations (10), (13), and (14) form a system of three equations in the three unknowns v_{Ax}^+, v_{Bx}^+, v_{By}^+ whose solution is

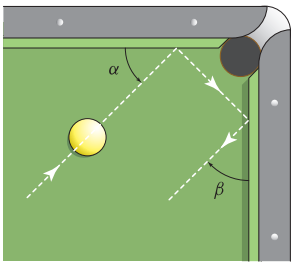
$$v_{Ax}^+ = \frac{v_{Ax}^-(2 - e - e\cos 2\theta)}{3 + \cos 2\theta}, \quad v_{Bx}^+ = \frac{2(1+e)v_{Ax}^- \cos^2\theta}{3 + \cos 2\theta}, \quad v_{By}^+ = \frac{(1+e)v_{Ax}^- \sin 2\theta}{3 + \cos 2\theta}.$$

Recalling that this time A and B are the small and large balls, respectively, $e = 1$, and using the numerical data in Eq. (8) and the first of Eqs. (9), the above equations yield the following result:

$$\vec{v}^+_{\text{large ball}} = (29.3\,\hat{\imath} + 0.130\,\hat{\jmath})\ \text{ft/s} \quad \text{and} \quad \vec{v}^+_{\text{small ball}} = 2.90\times 10^{-4}\,\hat{\imath}\ \text{ft/s}.$$

Problem 5.58

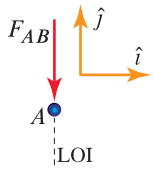
On a billiard table, the COR for the impact between a ball and any of the four bumpers should be the same. Assuming that this is the case, determine the angle β after two banks as a function of the initial incidence angle α.



Solution

In studying the collision between the ball and the bumper, we model the bumper as a stationary object. We denote the ball by A and the table by B. In addition, we denote by v_0 the initial speed of the ball. We will denote the first and second collision via the subscripts 1 and 2, respectively.

We now study the first collision. Referring to the FBD on the right, we see that the linear momentum of A is conserved in the direction perpendicular to the LOI, i.e.,



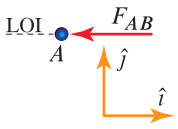
$$(v_{Ax}^-)_1 = (v_{Ax}^+)_1 \quad \Rightarrow \quad (v_{Ax}^+)_1 = v_0 \cos\alpha. \tag{1}$$

In addition to the conservation of linear momentum, the collision between the ball and the bumper is governed by the COR equation, i.e.,

$$(v_{Ay}^+)_1 - (v_{By}^+)_1 = e[(v_{By}^-)_1 - (v_{Ay}^-)_1] \quad \Rightarrow \quad (v_{Ay}^+)_1 = -e(v_{Ay}^-)_1 \quad \Rightarrow \quad (v_{Ay}^+)_1 = -ev_0 \sin\alpha, \tag{2}$$

where we have accounted for the fact that the table is stationary.

We now observe that the postimpact velocity components of A after the first impact are the preimpact velocity components of the velocity of A for the second impact, i.e.,



$$(v_{Ax}^-)_2 = v_0 \cos\alpha. \quad \text{and} \quad (v_{Ay}^-)_2 = -ev_0 \sin\alpha. \tag{3}$$

Next, referring to the FBD to the right, we see that the LOI of the second impact is parallel to the x axis. Furthermore, we see that the linear momentum of A is conserved along the y direction, i.e.,

$$(v_{Ay}^-)_2 = (v_{Ay}^+)_2 \quad \Rightarrow \quad (v_{Ay}^+)_2 = -ev_0 \sin\alpha. \tag{4}$$

The COR equation for the second impact is

$$(v_{Ax}^+)_2 - (v_{Bx}^+)_2 = e[(v_{Bx}^-)_2 - (v_{Ax}^-)_2] \quad \Rightarrow \quad (v_{Ax}^+)_2 = -ev_0 \cos\alpha, \tag{5}$$

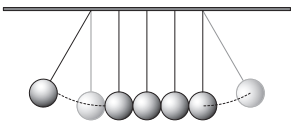
where, we have accounted for the fact that B remains stationary.

Now that the velocity components of A after the second impact are known, we can now proceed to compute the angle β as follows:

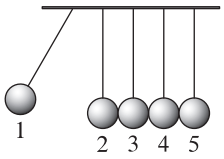
$$\beta = \tan^{-1}\left[\frac{(v_{Ax}^+)_2}{(v_{Ay}^+)_2}\right] = \tan^{-1}\left(\frac{-ev_0\cos\alpha}{-ev_0\sin\alpha}\right) \quad \Rightarrow \quad \boxed{\beta = \tan^{-1}(\cot\alpha).}$$

Problem 5.59

Newton's cradle is a common desk toy consisting of a number of identical pendulums with steel balls as bobs. These pendulums are arranged in a row in such a way that, when at rest, each ball is tangent to the next and the cords are all vertical. Assume that the COR for the impact of a ball with the next is $e = 1$. Explain why if you release the ball to the far left from a certain angle, the ball in question comes to a stop after impact while all the other balls do not seem to move except for the ball to the far right, which swings upward and achieves a maximum swing angle equal to the initial release angle of the ball to the far left.



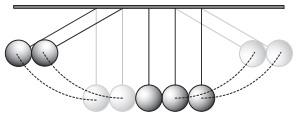
Solution



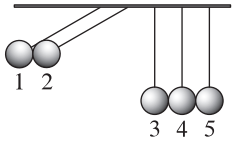
Treat each impact as only involving two balls. Because the COR $e = 1$ and the masses are identical we see from the solution to Problem 5.47 that ball 1 will come to a complete stop after impacting with ball 2. We also see that ball 2 will have a post impact velocity identical to the pre impact velocity of ball 1. Each ball in the train is tangent to the next so it will not appear to move at all during its impact with the next ball. Ball 4 impacts ball 5 which is free to move. Ball 5 will have a post impact velocity equal to the pre impact velocity of ball 1. The work-energy principle tells us that ball 5 will stop moving when it has reached the initial height ball 1 was released from. Finally, since the lengths of the pendulums are identical the maximum swing angle of ball 5 is equal to the initial release angle of ball 1.

Problem 5.60

Newton's cradle is a common desk toy consisting of a number of identical pendulums with steel balls as bobs. These pendulums are arranged in a row in such a way that, when at rest, each ball is tangent to the next and the cords are all vertical. Assume that the COR for the impact of a ball with the next is $e = 1$. Explain why if you release the two balls to the far left from a certain angle, the balls in question come to a stop right after impact while all the other balls do not seem to move except for the two balls to the far right, which swing upward and achieve a maximum swing angle equal to the initial release angle of the two balls to the far left.



Solution



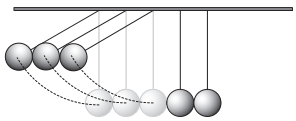
Treat each impact as only involving two balls. The first impact will occur between balls 2 and 3. Because the COR $e = 1$ and the masses are identical we see from the solution to Problem 5.47 that ball 2 will come to a complete stop after impacting with ball 3. We also see that ball 3 will have a post impact velocity identical to the pre impact velocity of ball 2. Call this velocity v_0. At the same instant ball 2 impacts ball 3 ball 1 impacts ball 2. Ball 1 will stop and ball 2 will have a post impact velocity v_0.

Now balls 2 and 3 have velocity v_0. When ball 3 impacts ball 4 ball 3 stops and ball 4 has a post impact velocity v_0, ball 3 is impacted by ball 2, ball 2 stops and ball 3 has a post impact velocity v_0.

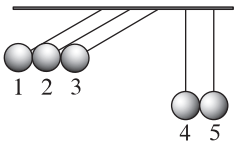
Now balls 3 and 4 have velocity v_0. When ball 4 impacts ball 5 ball 4 stops and ball 5 has a post impact velocity v_0, ball 4 is impacted by ball 3, ball 3 stops and ball 4 has a post impact velocity v_0. The work-energy principle tells us that balls 4 and 5 will stop moving when they have reached the initial height balls 1 and 2 were released from. Finally, since the lengths of the pendulums are identical the maximum swing angle of balls 4 and 5 are equal to the initial release angle of balls 1 and 2.

Problem 5.61

Newton's cradle is a common desk toy consisting of a number of identical pendulums with steel balls as bobs. These pendulums are arranged in a row in such a way that, when at rest, each ball is tangent to the next and the cords are all vertical. Assume that the COR for the impact of a ball with the next is $e = 1$. Predict the swing pattern of the system in the figure if you release from rest, and from a given angle, three of the five balls.



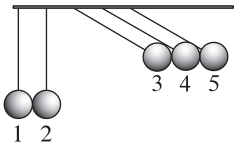
Solution



Treat each impact as only involving two balls. The first impact will occur between balls 3 and 4. Because the COR $e = 1$ and the masses are identical we see from the solution to Problem 5.47 that ball 3 will come to a complete stop after impacting with ball 4. We also see that ball 4 will have a post impact velocity identical to the pre impact velocity of ball 3. Call this velocity v_0. At the same instant ball 3 impacts ball 4 ball 2 impacts ball 3. Ball 2 will stop and ball 3 will have a post impact velocity v_0. Also at the same instant ball 1 will impact ball 2. Ball 1 will stop and ball 2 will have a post impact velocity v_0.

Now balls 2, 3, and 4 have velocity v_0. When ball 4 impacts ball 5 ball 4 stops and ball 5 has a post impact velocity v_0, ball 4 is impacted by ball 3, ball 3 stops and ball 4 has a post impact velocity v_0. At the same instant ball 2 impacts ball 3. Ball 2 stops and ball 3 has a post impact velocity v_0.

Now balls 3, 4, and 5 have velocity v_0 and there are no other balls in the train to impact so they will all swing together as a unit. The work-energy principle tells us that balls 3, 4, and 5 will stop moving when they have reached the initial height balls 1, 2, and 3 were released from. Finally, since the lengths of the pendulums are identical the maximum swing angle of balls 3, 4, and 5 are equal to the initial release angle of balls 1, 2, and 3.



Balls 3, 4, and 5 will swing up as a single unit until reaching the height that balls 1, 2, and 3 were released from while balls 1 and 2 will hang motionless.

Problem 5.62

If an impact is an event spanning an infinitesimally small time interval, is the total potential energy of two colliding objects conserved through the impact? What about the potential energy of each individual object?

Solution

Answer to the first question. In an impact the potential energy of the system is conserved. The reason is that in an infinitesimal time interval our impact model allows objects to change velocity but not position. No change in position implies that there cannot be a change in potential energy through an impact for any of the colliding objects and therefore for the system as a whole.

Answer to the second question. In an impact the potential energy of each colliding object is conserved. The reason is that in an infinitesimal time interval our impact model allows objects to change velocity but not position. No change in position implies that there cannot be a change in potential energy through an impact for any of the colliding objects.

Problem 5.63

If an impact is an event spanning an infinitesimally small time interval, is the total kinetic energy of two colliding objects conserved through an impact? What about the kinetic energy of each individual object?

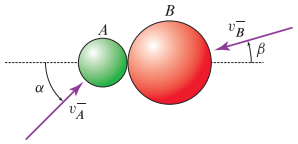
Solution

Answer to the first question. In general, the kinetic energy of a two colliding particles is not conserved during the impact. To explain why this is the case, let's begin with observing that we model impacts as events that take place in an infinitesimal time interval and that cause the colliding objects to change velocity but not position. This implies that there cannot be a change in potential energy through an impact for any of the colliding objects. With this in mind, applying the work energy principle, the difference between the total pre- and postimpact kinetic energies of the system measures the work done during the impact by the impulsive forces acting on the system. The total kinetic energy would be conserved if the total work done by the impulsive forces were equal to zero. In turn this would happen if the work done during the deformation phase of the impact were equal and opposite to the work done during the restitution phase. Such a case corresponds to the case in which the collision is perfectly elastic, i.e., the COR $e = 1$. Hence, whenever the impact is not perfectly elastic, the total kinetic energy of the system is not conserved.

Answer to the second question. In general, the kinetic energy of the individual particles is not conserved during the impact. The explanation for this answer can be given by consider an example in which a moving particle collides with a stationary particle. After the collision, the particle that was initially stationary would be moving, and this clearly indicates that the kinetic energy of that particle has changed. In general, such an impact would cause a change in the speed of the particle that was initially moving so that the kinetic energy of his particle also changes from before to after the impact.

Problems 5.64 and 5.65

Two spheres, A and B, with masses $m_A = 1.35\,\text{kg}$ and $m_B = 2.72\,\text{kg}$, respectively, collide with $v_A^- = 26.2\,\text{m/s}$, and $v_B^- = 22.5\,\text{m/s}$.

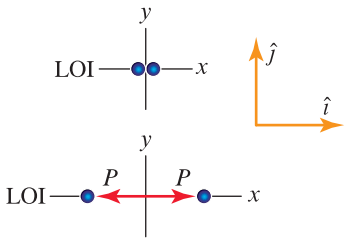


Problem 5.64 Compute the postimpact velocities of A and B if $\alpha = 45°$, $\beta = 16°$, the COR is $e = 0.57$, and the contact between A and B is frictionless.

Problem 5.65 Compute the postimpact velocities of A and B if $\alpha = 45°$, $\beta = 16°$, the COR is $e = 0$, and the contact between A and B is frictionless.

Solution to 5.64

The impact in this problem is a typical two-dimensional oblique central impact. We have selected an xy coordinate system with the x axis aligned with the LOI. The impact is therefore characterized by the following four equations:



$$m_A v_{Ax}^- + m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+, \quad (1)$$

$$v_{Ay}^- = v_{Ay}^+, \quad (2)$$

$$v_{By}^- = v_{By}^+, \quad (3)$$

$$v_{Ax}^+ - v_{Bx}^+ = e(v_{Bx}^- - v_{Ax}^-). \quad (4)$$

The above four equations express, in order, the conservation of the linear momentum of the system along the LOI, the conservation of the linear momentum of particle A in the direction normal to the LOI, the conservation of the linear momentum of particle B in the direction normal to the LOI, and the COR equation.

Observe that the preimpact velocities are given and are

$$v_{Ax}^- = v_A^- \cos\alpha, \quad v_{Ay}^- = v_A^- \sin\alpha, \quad v_{Bx}^- = -v_B^- \cos\beta, \quad v_{By}^- = -v_B^- \sin\beta. \quad (5)$$

Substituting Eqs. (5) into Eqs. (1)–(4) and solving for the postimpact velocities of A and B, we have

$$v_{Ax}^+ = -\frac{1}{m_A + m_B}[-m_A v_{Ax}^- \cos\alpha + e m_B (v_{Ax}^- \cos\alpha + v_{Bx}^- \cos\beta) + m_B v_{Bx}^- \cos\beta], \quad (6)$$

$$v_{Ay}^- = v_A^- \sin\alpha \quad (7)$$

$$v_{Bx}^+ = -\frac{1}{m_A + m_B}[-m_A v_{Ax}^- \cos\alpha - e m_A (v_{Ax}^- \cos\alpha + v_{Bx}^- \cos\beta) + m_B v_{Bx}^- \cos\beta], \quad (8)$$

$$v_{By}^+ = -v_B^- \sin\beta, \quad (9)$$

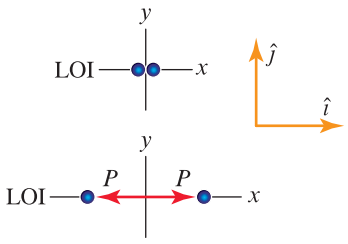
Substituting the problem's data in Eqs. (6)–(9), we then obtain

$$\vec{v}_A^+ = (-23.6\,\hat{\imath} + 18.5\,\hat{\jmath})\,\text{m/s} \quad \text{and} \quad \vec{v}_B^+ = (-0.717\,\hat{\imath} - 6.20\,\hat{\jmath})\,\text{m/s}$$

where we have used the following numerical data: $m_A = 1.35\,\text{kg}$, $m_B = 2.72\,\text{kg}$, $v_A^- = 26.2\,\text{m/s}$, $v_B^- = 22.5\,\text{m/s}$, $\alpha = 45°$, $\beta = 16°$, and $e = 0.57$.

Solution to 5.65

The impact in this problem is a typical two-dimensional oblique central impact. We have selected an xy coordinate system with the x axis aligned with the LOI. The impact is therefore characterized by the following four equations:



$$m_A v_{Ax}^- + m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+, \tag{10}$$

$$v_{Ay}^- = v_{Ay}^+, \tag{11}$$

$$v_{By}^- = v_{By}^+, \tag{12}$$

$$v_{Ax}^+ - v_{Bx}^+ = e(v_{Bx}^- - v_{Ax}^-). \tag{13}$$

The above four equations express, in order, the conservation of the linear momentum of the system along the LOI, the conservation of the linear momentum of particle A in the direction normal to the LOI, the conservation of the linear momentum of particle B in the direction normal to the LOI, and the COR equation.

Observe that the preimpact velocities are given and are

$$v_{Ax}^- = v_A^- \cos\alpha, \quad v_{Ay}^- = v_A^- \sin\alpha, \quad v_{Bx}^- = -v_B^- \cos\beta, \quad v_{By}^- = -v_B^- \sin\beta. \tag{14}$$

Substituting Eqs. (14) into Eqs. (10)–(13) and solving for the postimpact velocities of A and B, we have

$$v_{Ax}^+ = -\frac{1}{m_A + m_B}[-m_A v_{Ax}^- \cos\alpha + e m_B (v_{Ax}^- \cos\alpha + v_{Bx}^- \cos\beta) + m_B v_{Bx}^- \cos\beta], \tag{15}$$

$$v_{Ay}^- = v_A^- \sin\alpha \tag{16}$$

$$v_{Bx}^+ = -\frac{1}{m_A + m_B}[-m_A v_{Ax}^- \cos\alpha - e m_A (v_{Ax}^- \cos\alpha + v_{Bx}^- \cos\beta) + m_B v_{Bx}^- \cos\beta], \tag{17}$$

$$v_{By}^+ = -v_B^- \sin\beta, \tag{18}$$

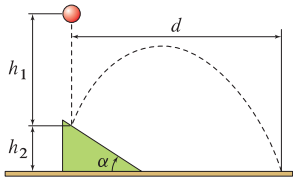
Substituting the problem's data in Eqs. (15)–(18), we then obtain

$$\vec{v}_A^+ = (-8.31\,\hat{\imath} + 18.5\,\hat{\jmath})\ \text{m/s} \quad \text{and} \quad \vec{v}_B^+ = (-8.31\,\hat{\imath} - 6.20\,\hat{\jmath})\ \text{m/s}$$

where we have used the following numerical data: $m_A = 1.35\,\text{kg}$, $m_B = 2.72\,\text{kg}$, $v_A^- = 26.2\,\text{m/s}$, $v_B^- = 22.5\,\text{m/s}$, $\alpha = 45°$, $\beta = 16°$, and $e = 0$.

Problems 5.66 and 5.67

A 1.34 lb ball is dropped on a 10 lb incline with $\alpha = 33°$. The ball's release height is $h_1 = 5$ ft, and the height of the impact point relative to the ground is $h_2 = 0.3$ ft. Assume that the contact between the ball and the incline is frictionless, and let the COR for the impact be $e = 0.88$.



Problem 5.66 Compute the distance d at which the ball will hit ground for the first time if the incline cannot move relative to the floor.

Problem 5.67 Compute the distance d at which the ball will hit ground for the first time if the incline can slide without friction relative to the floor.

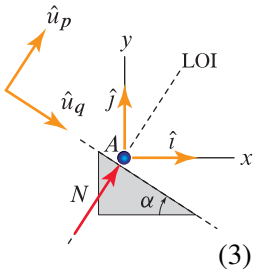
Solution to 5.66

When ball A is dropped (from rest) over the incline, ball A first falls under the action of gravity and therefore the velocity with which ball A impacts the incline is found using the following constant acceleration equation:

$$\dot{s}^2 = \dot{s}_0^2 + 2a_c\,(s - s_0) \quad \Rightarrow \quad v_{Ay}^- = -\sqrt{2gh_1}. \tag{1}$$

Thus, referring to the figure shown to the right, the component of the pre-impact velocity of A are

$$v_{Ax}^- = 0 \quad \text{and} \quad v_{Ay}^- = -\sqrt{2gh_1}. \tag{2}$$



Since the incline cannot move, then A is subject to the external impulsive force N. Consequently, the only component of the momentum of A that is conserved is that along the q direction and we can write

$$v_{Aq}^- = v_{Aq}^+. \tag{3}$$

The COR equation for the collision is only applicable along the LOI. Hence, letting B denote the incline, we have

$$v_{Ap}^+ - v_{Bp}^+ = e(v_{Bp}^- - v_{Ap}^-) \quad \Rightarrow \quad v_{Ap}^+ = -ev_{Ap}^-, \tag{4}$$

where we have accounted for the fact that the incline B does not move. We now rewrite Eqs. (3) and (4) in the xy component system. To do this, observe that the components of the velocity of A in the p and q can be expressed in terms of the x and y components as follows:

$$v_{Ap} = \vec{v}_A \cdot \hat{u}_p = v_{Ax}\,\hat{\imath} \cdot \hat{u}_p + v_{Ay}\,\hat{\jmath} \cdot \hat{u}_p, = v_{Ax}\sin\alpha + v_{Ay}\cos\alpha, \tag{5}$$

$$v_{Aq} = \vec{v}_A \cdot \hat{u}_q = v_{Ax}\,\hat{\imath} \cdot \hat{u}_q + v_{Ay}\,\hat{\jmath} \cdot \hat{u}_q, = v_{Ax}\cos\alpha - v_{Ay}\sin\alpha. \tag{6}$$

Using the result in Eqs. (5) and (6) to rewrite Eqs. (3) and (4), allows us to rewrite (3) and (4) as follows

$$\sqrt{2gh_1}\sin\alpha = v_{Ax}^+\cos\alpha - v_{Ay}^+\sin\alpha, \tag{7}$$

$$v_{Ax}^+\sin\alpha + v_{Ay}^+\cos\alpha = e\sqrt{2gh_1}\cos\alpha, \tag{8}$$

Equations (7) and (8) form a system of two equations in the two unknowns v_{Ay}^+ and v_{Ax}^+ whose solution is

$$v_{Ax}^+ = (1+e)\sqrt{2gh_1}\cos\alpha\sin\alpha \quad \text{and} \quad v_{Ay}^+ = \sqrt{\frac{gh_1}{2}}[(e-1) + (1+e)\cos 2\alpha]. \tag{9}$$

Now that we have the postimpact velocity of A we can find d solving a projectile problem. Hence, we first find the time A takes to reach the floor after the initial impact using the constant acceleration equation $s = s_0 + \dot{s}_0 t + 0.5 a_c t^2$ with s in the y direction, where $s = 0$, $s_0 = h_2$, $\dot{s}_0 = v_{Ay}^+$, and $a_c = -g$. This gives

$$0 = h_2 + v_{Ay}^+ t - \tfrac{1}{2} g t^2, \quad \Rightarrow \quad t = \frac{1}{g}\left(v_{Ay}^+ \pm \sqrt{2gh_2 + (v_{Ay}^+)^2}\right). \tag{10}$$

Observe that the argument of the square root in the result of Eq. (10) is larger than v_{Ay}^+. Hence, the only physically acceptable root is

$$t = \frac{1}{g}\left[v_{Ay}^+ + \sqrt{2gh_2 + (v_{Ay}^+)^2}\right]. \tag{11}$$

Next, since the velocity of A is constant in the x direction, we can now say that

$$d = v_{Ax}^+ t. \tag{12}$$

Substituting the Eq. (11) into Eq. (12), we find

$$d = \sqrt{\frac{2}{gh_1}}(1+e)h_1 \cos\alpha \sin\alpha \left\{\sqrt{\frac{gh_1}{2}}[(e-1) + (1+e)\cos 2\alpha] + \sqrt{2gh_2 + \tfrac{1}{2}gh_1[(e-1) + (1+e)\cos 2\alpha]^2}\right\} = 6.24\,\text{ft},$$

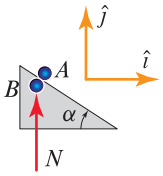
where we have used the parameters: $g = 32.2\,\text{ft/s}^2$, $h_1 = 5\,\text{ft}$, $h_2 = 0.3\,\text{ft}$, $e = 0.88$, and $\alpha = 33°$.

Solution to 5.67

When ball A is dropped (from rest) over the incline, ball A first falls under the action of gravity and therefore the velocity with which ball A impacts the incline is found using the following constant acceleration equation:

$$\dot{s}^2 = \dot{s}_0^2 + 2a_c\,(s - s_0) \quad \Rightarrow \quad v_{Ay}^- = -\sqrt{2gh_1}. \tag{13}$$

Thus, referring to the figure shown to the right, the component of the pre-impact velocity of A are



$$v_{Ax}^- = 0 \quad \text{and} \quad v_{Ay}^- = -\sqrt{2gh_1}. \tag{14}$$

Recall that the incline is initially at rest and that, after impact, it can only move in the horizontal direction. Hence, we must have

$$v_{Bx}^- = 0, \quad v_{By}^- = 0, \quad \text{and} \quad v_{By}^+ = 0. \tag{15}$$

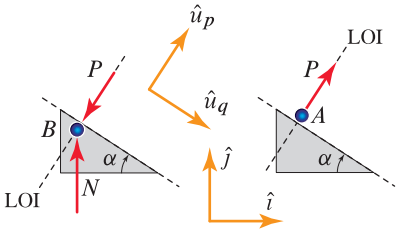
Referring to the impact-relevant FBD shown, we see that the system is acted upon by an external impulsive force in the y direction. Hence, the only component of the system's linear momentum that is conserved is that in the x direction, i.e.,

$$m_A v_{Ax}^- + m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+. \tag{16}$$

Substituting the first of Eqs. (14) and the first of Eqs. (15) into Eq. (16), we have

$$m_A v_{Ax}^+ = -m_B v_{Bx}^+. \tag{17}$$

We now consider the impact-relevant FBDs of A and B individually (shown to the right). From the FBD of A we see that the momentum of A must be conserved in the direction perpendicular to the LOI. This condition is more easily expressed in the pq component system and is given by



$$v_{Aq}^- = v_{Aq}^+. \tag{18}$$

Additionally, we can enforce the COR equation. This equation concerns velocity components along the LOI and, again, is more easily expressed in the pq component system as

$$v_{Ap}^+ - v_{Bp}^+ = -e v_{Ap}^-, \tag{19}$$

where we have accounted for the fact that B is at rest before the impact. We now need to rewrite Eqs. (18) and (19) in the xy component system. To do so, observe that, for A we have

$$v_{Ap} = \vec{v}_A \cdot \hat{u}_p = v_{Ax}\,\hat{\imath} \cdot \hat{u}_p + v_{Ay}\,\hat{\jmath} \cdot \hat{u}_p = v_{Ax} \sin\alpha + v_{Ay} \cos\alpha, \tag{20}$$

$$v_{Aq} = \vec{v}_A \cdot \hat{u}_q = v_{Ax}\,\hat{\imath} \cdot \hat{u}_q + v_{Ay}\,\hat{\jmath} \cdot \hat{u}_q = v_{Ax} \cos\alpha - v_{Ay} \sin\alpha. \tag{21}$$

For B we have similar relations. However, recalling that B can only move in the x direction, the only relation we need is as follows:

$$v_{Bp} = v_{Bx} \sin\alpha. \tag{22}$$

Substituting the results in Eqs. (20)–(22) into Eqs. (18) and (19), gives

$$\sqrt{2gh_1} \sin\alpha = v_{Ax}^+ \cos\alpha - v_{Ay}^+ \sin\alpha, \tag{23}$$

$$v_{Ax}^+ \sin\alpha + v_{Ay}^+ \cos\alpha - v_{Bx}^+ \sin\alpha = e\sqrt{2gh_1} \cos\alpha, \tag{24}$$

where we have enforced Eqs. (14). Now observe that Eqs. (17), (23), and (24) form a system of three equations in the three unknowns v_{Ax}^+, v_{Bx}^+, and v_{Ay}^+ whose solution is

$$v_{Ax}^+ = \frac{\sqrt{2gh_1}(1+e)m_B \sin 2\alpha}{m_A + 2m_B - m_A \cos 2\alpha}, \tag{25}$$

$$v_{Ay}^+ = \frac{\sqrt{2gh_1}\{m_A + m_B(1-e) - [m_A + m_B(1+e)]\cos 2\alpha\}}{-m_A - 2m_B + m_A \cos 2\alpha}, \tag{26}$$

$$v_{Bx}^+ = \frac{\sqrt{2gh_1}(1+e)m_A \sin 2\alpha}{-m_A - 2m_B + m_A \cos 2\alpha}. \tag{27}$$

Now that we have the postimpact velocity of A we can find d solving a projectile problem. Hence, we first find the time A takes to reach the floor after the initial impact using the constant acceleration equation $s = s_0 + \dot{s}_0 t + 0.5 a_c t^2$ with s in the y direction, where $s = 0$, $s_0 = h_2$, $\dot{s}_0 = v_{Ay}^+$, and $a_c = -g$. This gives

$$0 = h_2 + v_{Ay}^+ t - \tfrac{1}{2} g t^2 \quad \Rightarrow \quad t = \frac{1}{g}\left[v_{Ay}^+ \pm \sqrt{2gh_2 + (v_{Ay}^+)^2}\right]. \tag{28}$$

Observe that the argument of the square root in the result of Eq. (28) is larger than v_{Ay}^+. Hence, the only physically acceptable root is

$$t = \frac{1}{g}\left[v_{Ay}^+ + \sqrt{2gh_2 + (v_{Ay}^+)^2}\right]. \tag{29}$$

Since the velocity of A is constant in the x direction, we can now say that

$$d = v_{Ax}^+ t. \tag{30}$$

Substituting Eq. (29) into Eq. (30) along with the results in Eqs. (25) and (26), we find

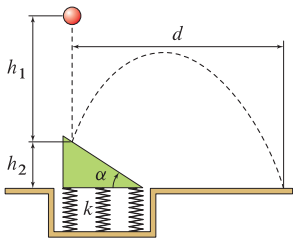
$$\begin{aligned} d = {} & \frac{\sqrt{2}(1+e)h_1 m_B \sin 2\alpha}{\sqrt{gh_1}\,(m_A + 2m_B - m_A \cos 2\alpha)} \left\{ \frac{\sqrt{2gh_1}[m_A + m_B - em_B - (m_A + m_B + em_B)\cos 2\alpha]}{-m_A - 2m_B + m_A \cos 2\alpha} \right. \\ & \left. + \sqrt{2gh_2 + \frac{2gh_1[m_A + m_B - em_B - (m_A + m_B + em_B \cos 2\alpha)^2]}{(-m_A - 2m_B + m_A \cos 2\alpha)^2}} \right\} \\ = {} & 5.27\,\text{ft}, \end{aligned}$$

where we have used the parameters:

$$\begin{aligned} g &= 32.2\,\text{ft/s}^2, \\ h_1 &= 5\,\text{ft}, \\ h_2 &= 0.3\,\text{ft}, \\ e &= 0.88, \\ m_A &= \frac{1.34\,\text{lb}}{32.2\,\text{ft/s}^2} = 0.04161\,\text{slug}, \\ m_B &= \frac{10\,\text{lb}}{32.2\,\text{ft/s}^2} = 0.3106\,\text{slug}, \\ \alpha &= 33^\circ. \end{aligned}$$

Problem 5.68

A 1.34 lb ball is dropped on a 10 lb incline with $\alpha = 33°$. The ball's release height is $h_1 = 5$ ft, and the height of the impact point relative to the ground is $h_2 = 0.3$ ft. Assume that the contact between the ball and the incline is frictionless, and let the COR for the impact be $e = 0.88$. Compute the distance d at which the ball will hit ground for the first time if the combined stiffness of the supporting springs is $k = 50$ lb/in. Assume that the incline can move only vertically.



Solution

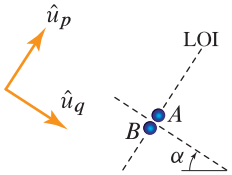
When ball A is dropped (from rest) over the incline, ball A first falls under the action of gravity and therefore the velocity with which ball A impacts the incline is found using the following constant acceleration equation:

$$\dot{s}^2 = \dot{s}_0^2 + 2a_c(s - s_0) \quad \Rightarrow \quad v_{Ay}^- = -\sqrt{2gh_1}. \tag{1}$$

Thus, denoting the incline by B, the component of the pre-impact velocities of A and B are

$$v_{Ax}^- = 0, \quad v_{Ay}^- = -\sqrt{2gh_1}, \quad \text{and} \quad \vec{v}_B^- = 0. \tag{2}$$

Referring to the FBD shown to the right, because the incline is supported by springs (as opposed to a rigid surface), the collision between A and the incline can be modeled as an unconstrained oblique impact with LOI perpendicular to the incline surface. Hence, the system's momentum is conserved along the LOI and the individual momenta of A and the incline, respectively, are conserved in the direction perpendicular to the LOI. These considerations yield the following three equations:



$$m_A v_{Ap}^- = m_A v_{Ap}^+ + m_B v_{Bp}^+, \tag{3}$$

$$v_{Aq}^- = v_{Aq}^+, \tag{4}$$

$$0 = v_{Bq}^+, \tag{5}$$

where we have accounted for the pre-impact condition expressed by the third of Eqs. (2). In addition to the consequence of the impulse–momentum principle, we can write the COR equation, which holds only along the LOI and therefore is written as

$$v_{Ap}^+ - v_{Bp}^+ = -e v_{Ap}^-, \tag{6}$$

where, again, we have accounted for the fact that the incline is at rest before the impact.

Now we rewrite Eqs. (3)–(6) in the xy component system. To do so, observe that we have

$$v_{Ap} = \vec{v}_A \cdot \hat{u}_p = v_{Ax}\,\hat{\imath} \cdot \hat{u}_p + v_{Ay}\,\hat{\jmath} \cdot \hat{u}_p = v_{Ax} \sin\alpha + v_{Ay} \cos\alpha, \tag{7}$$

$$v_{Aq} = \vec{v}_A \cdot \hat{u}_q = v_{Ax}\,\hat{\imath} \cdot \hat{u}_q + v_{Ay}\,\hat{\jmath} \cdot \hat{u}_q = v_{Ax} \cos\alpha - v_{Ay} \sin\alpha. \tag{8}$$

Similarly, for B we have

$$v_{Bp} = v_{Bx} \sin\alpha + v_{By} \cos\alpha \quad \text{and} \quad v_{Aq} = v_{Ax} \cos\alpha - v_{Ay} \sin\alpha. \tag{9}$$

Using Eqs. (7)–(9), Eqs. (3)–(6) can be rewritten as

$$-m_A\sqrt{2gh_1}\cos\alpha = m_A(v_{Ax}^+ \sin\alpha + v_{Ay}^+ \cos\alpha) + m_B(v_{Bx}^+ \sin\alpha + v_{By}^+ \cos\alpha), \tag{10}$$

$$-\sqrt{2gh_1}\sin\alpha = v_{Ax}^+\cos\alpha - v_{Ay}^+\sin\alpha, \tag{11}$$

$$0 = v_{Bx}^+\cos\alpha - v_{By}^+\sin\alpha, \tag{12}$$

$$e\sqrt{2gh_1}\cos\alpha = v_{Ax}^+\sin\alpha + v_{Ay}^+\cos\alpha - v_{Bx}^+\sin\alpha - v_{By}^+\cos\alpha, \tag{13}$$

where we have accounted for the first two of Eqs. (2). Equations (10)–(13) form a system of four equations in the four unknowns v_{Ax}^+, v_{Ay}^+, v_{Bx}^+, and v_{By}^+. Eliminating v_{Bx}^+, and v_{By}^+ and solving for v_{Ax}^+, and v_{Ay}^+, we have

$$v_{Ax}^+ = \sqrt{2gh_1}\frac{(1+e)m_B\cos\alpha\sin\alpha}{m_A+m_B} = 13.59\,\text{ft/s}, \tag{14}$$

$$v_{Ay}^+ = \sqrt{\frac{gh_1}{2}}\frac{(1+e)m_B\cos 2\alpha + m_B(e-1) - 2m_A}{m_A+m_B} = 2.981\,\text{ft/s}, \tag{15}$$

where we have used the following numerical data: $g = 32.2\,\text{ft/s}^2$, $h_1 = 5\,\text{ft}$, $e = 0.88$, $m_A = \frac{1.34\,\text{lb}}{32.2\,\text{ft/s}^2} = 0.04161\,\text{slug}$, $m_B = \frac{10\,\text{lb}}{32.2\,\text{ft/s}^2} = 0.3106\,\text{slug}$, and $\alpha = 33°$.

Now that we have the components of the postimpact velocity of A, we proceed to solve a projectile problem to determine the required distance d. We begin by finding the time t_f that A takes to reach the floor via the constant acceleration equation $y = y_0 + \dot{y}_0 t_f + \frac{1}{2}a_c t_f^2$ where $y = 0$, $y_0 = h_2$, $\dot{y}_0 = v_{Ay}^+$, and $a_c = -g$. By doing so, we obtain

$$t_f = \frac{1}{g}\left[v_{Ay}^+ \pm \sqrt{2gh_2 + (v_{Ay}^+)^2}\right]. \tag{16}$$

Observing that the square root term in Eq. (16) is larger than v_{Ay}^+, the only meaningful solution for t_f is

$$t_f = \frac{1}{g}\left[v_{Ay}^+ + \sqrt{2gh_2 + (v_{Ay}^+)^2}\right] = 0.2575\,\text{s}. \tag{17}$$

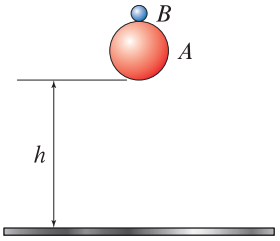
Finally, since the x component of the velocity of A is constant, we have

$$d = v_{Ax}^+ t_f \quad \Rightarrow \quad \boxed{d = 3.50\,\text{ft},} \tag{18}$$

where we have used the result in Eqs. (14) and (17).

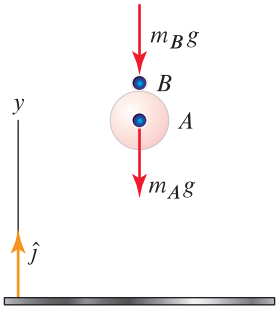
Problem 5.69

Consider two balls A and B that are stacked one on top of the other and dropped from rest from a height h. Let $e_{AG} = 1$ be the COR for the collision of ball A with the ground, and let $e_{AB} = 1$ be the COR for the collision between balls A and B. Finally, assume that the balls can move only vertically and that $m_A \gg m_B$, that is, that $m_B/m_A \approx 0$. Model the combined collision as a sequence of impacts, and predict the rebound speed of ball B as a function of h and g, the acceleration due to gravity.



Solution

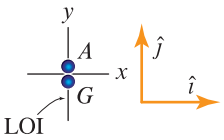
We denote the Earth as "ball G" and we treat it as being initially stationary and as having a mass far larger than the mass of ball A. Next, we observe that both A and B drop due to gravity by the same height h. Using the FBD to the right, in which we have neglected air resistance, the preimpact speed of either ball can be computed using constant acceleration equations as follows:



$$\dot{y}^2 = \dot{y}_0^2 - 2g(y - y_0) \;\Rightarrow\; (v_y^-)^2 = 2gh \;\Rightarrow\; v_{Ay}^- = v_{By}^- = -\sqrt{2gh}. \qquad (1)$$

To model the overall impact as a sequence of impacts means that the first impact is between ball A, traveling with a downward speed equal to $\sqrt{2gh}$, and the Earth, which is stationary. The second impact is between ball A, traveling upward after having rebounded off of the Earth, and ball B traveling with a downward speed equal to $\sqrt{2gh}$.

We start with the analysis of the first impact. The motion is completely in the y direction, which is also the LOI. Each impact is an unconstrained perfectly elastic impact. Therefore the linear momentum of the system formed by balls A and G is conserved through the impact, i.e.,



$$m_A(v_{Ay}^-)_1 + m_G(v_{Gy}^-)_1 = m_A(v_{Ay}^+)_1 + m_G(v_{Gy}^+)_1, \qquad (2)$$

where the subscript 1 denotes the first impact. Dividing Eq. (2) by m_G we have

$$\frac{m_A}{m_G}(v_{Ay}^-)_1 + (v_{Gy}^-)_1 = \frac{m_A}{m_G}(v_{Ay}^+)_1 + (v_{Gy}^+)_1. \qquad (3)$$

By assumption $m_A/m_G \approx 0$, and therefore Eq. (3) can be written as

$$(v_{Gy}^+)_1 = (v_{Gy}^-)_1 = 0, \qquad (4)$$

which implies that the Earth is unaffected by the collision.

In addition to the conservation of momentum, the impact between A and G is also governed by the COR equation, i.e,

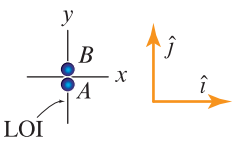
$$(v_{Ay}^+)_1 - (v_{Gy}^+)_1 = (v_{Gy}^-)_1 - (v_{Ay}^-)_1. \qquad (5)$$

Substituting Eq. (4) into Eq. (5) and rearranging terms, we have

$$(v_{Ay}^+)_1 = -(v_{Ay}^-)_1 = \sqrt{2gh}, \qquad (6)$$

where we have used the result in Eq. (1).

We will now examine the impact between the rebounding ball A with and the falling ball B. Again, we can invoke conservation of linear momentum along the LOI, i.e.,



$$m_A(v_{Ay}^-)_2 + m_B(v_{By}^-)_2 = m_A(v_{Ay}^+)_2 + m_B(v_{By}^+)_2, \tag{7}$$

where the subscript 2 denotes the second impact in the sequence. Dividing Eq. (7) by m_A, we have

$$(v_{Ay}^-)_2 + \frac{m_B}{m_A}(v_{By}^-)_2 = (v_{Ay}^+)_2 + \frac{m_B}{m_A}(v_{By}^+)_2. \tag{8}$$

Recalling that $(m_B/m_A) \approx 0$, Eq. (8) yields the following result:

$$(v_{Ay}^+)_2 = (v_{Ay}^-)_2 = (v_{Ay}^+)_1 = \sqrt{2gh}, \tag{9}$$

where we have used the fact that $(v_{Ay}^-)_2$ coincides with $(v_{Ay}^+)_1$, and where we have used the result in Eq. (6). Again, we wee that the velocity of A is essentially unaffected by the collision with B.

Next, we write the COR equation for the second impact, i.e.,

$$(v_{Ay}^+)_y - (v_{By}^+)_2 = e_{AB}\left[(v_{By}^-)_2 - (v_{Ay}^-)_2\right]. \tag{10}$$

Recalling that $e_{AB} = 1$ and that, from Eq. (1), $(v_{By}^-)_2 = -\sqrt{2gh}$, substituting the result in Eq. (9) into Eq. (10), and solving for $(v_{By}^+)_2$, we obtain

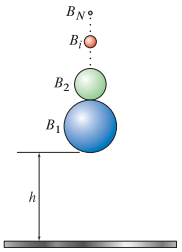
$$(v_{By}^+)_2 = 3\sqrt{2gh}. \tag{11}$$

Consequently, denoting by v_B^+ the final postimpact speed of B, we have

$$\boxed{v_B^+ = 3\sqrt{2gh}.}$$

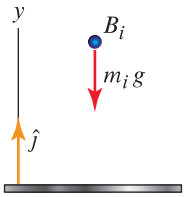
Problem 5.70

Consider a stack of N balls dropped from rest from a height h. Let all impacts be perfectly elastic, and assume that $m_i \gg m_{i+1}$, that is, that $m_{i+1}/m_i \approx 0$, with $i = 1, \ldots, N-1$ and m_i being the mass of the ith ball. Model the combined collision as a sequence of impacts, and predict the rebound speed of the topmost ball. Assume that the balls can move only vertically.



Solution

We denote the Earth as "ball zero" and we treat it as being initially stationary and as having a mass far larger than the mass of the first ball in the stack, i.e., $m_0 \gg m_1$. Next, we observe that each ball drops due to gravity by the same height h as the stack as a whole. Using the FBD to the right, in which we have neglected air resistance, the preimpact speed of ball i ($i > 0$) can be computed using constant acceleration equations as follows:

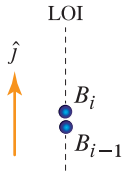


$$\dot{y}^2 = \dot{y}_0^2 - 2g(y - y_0) \quad \Rightarrow \quad (v_i^-)_y^2 = 2gh \quad \Rightarrow \quad (v_i^-)_y = -\sqrt{2gh}. \tag{1}$$

To model the overall impact as a sequence of impacts means that

- the first impact is between ball 1, traveling with a downward speed equal to $\sqrt{2gh}$, and the Earth, which is stationary.
- The second impact is between ball 1, traveling upward after having rebounded off of the Earth, and ball 2 traveling with a downward speed equal to $\sqrt{2gh}$. Once ball 2 rebounds off of ball 1, it will collide with ball 3, and so on.

We can determine the outcome of the entire sequence of impacts by studying the impact between ball i and ball $i-1$ (with $i > 0$). The LOI for all impacts coincides with the y axis. Each impact is an unconstrained perfectly elastic impact. Therefore the linear momentum of the system formed by balls i and $i-1$ is conserved through the impact, i.e,



$$m_{i-1}(v_{i-1}^-)_y + m_i(v_i^-)_y = m_{i-1}(v_{i-1}^+)_y + m_i(v_i^+)_y \tag{2}$$

Diving Eq. (2) by the mass of ball m_{i-1} we have

$$(v_{i-1}^-)_y + \frac{m_i}{m_{i-1}}(v_i^-)_y = (v_{i-1}^+)_y + \frac{m_i}{m_{i-1}}(v_i^+)_y. \tag{3}$$

By assumption $m_i/m_{i-1} \approx 0$, and therefore Eq. (3) can be written as

$$(v_{i-1}^+)_y = (v_{i-1}^-)_y, \tag{4}$$

which implies that the velocity of ball $i-1$ is unaffected by the collision.

In addition to the conservation of momentum, each impact is also governed by the COR equation, i.e,

$$(v_i^+)_y - (v_{i-1}^+)_y = (v_{i-1}^-)_y - (v_i^-)_y. \tag{5}$$

Substituting Eq. (4) into Eq. (5) and rearranging terms, we have

$$(v_i^+)_y = 2(v_{i-1}^+)_y - (v_i^-)_y. \tag{6}$$

Recall that the index i is such that $i > 0$. In this case, we can replace the expression for $(v_i^-)_y$ with that given in the last of Eqs. (1). Hence, Eq. (6) can be written as

$$(v_i^+)_y = 2(v_{i-1}^+)_y + \sqrt{2gh}. \tag{7}$$

We are now ready to determine the outcome of the collision sequence. In doing so, we must keep in mind that the meaning of the superscripts $+$ and $-$ is confined to an individual impact. Therefore, for example, the postimpact velocity of, say, ball 3 with ball 2 is also the preimpact velocity of ball 3 for the impact between ball 4 and ball 3. Consequently, recalling that the preimpact velocity of ball 0 is equal to zero, we then have

Impact between Balls 0 and 1: By Eq. (4), $(v_0^+)_y = 0$, so that, by Eq. (7), $(v_1^+)_y = \sqrt{2gh}$;

Impact between Balls 1 and 2: By Eq. (4), $(v_1^+)_y = \sqrt{2gh}$, so that, by Eq. (7) $(v_2^+)_y = 3\sqrt{2gh}$;

Impact between Balls 2 and 3: By Eq. (4), $(v_2^+)_y = 3\sqrt{2gh}$, so that, by Eq. (7) $(v_3^+)_y = 7\sqrt{2gh}$;

Impact between Balls 3 and 4: By Eq. (4), $(v_3^+)_y = 7\sqrt{2gh}$, so that, by Eq. (7) $(v_4^+)_y = 15\sqrt{2gh}$;

. . . and so on.

By carefully reviewing the sequence of solutions just generated, we conclude that the velocity of ball i right after the collision with ball $i - 1$ is given by

$$(v_i^+)_y = \left(2^i - 1\right)\sqrt{2gh}.$$

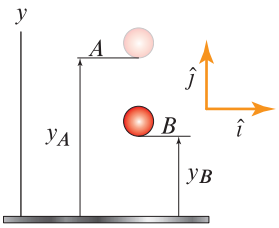
Consequently, recalling that the motion of each ball is only in the y direction, for the Nth ball we have

$$\boxed{v_N^+ = \left(2^N - 1\right)\sqrt{2gh}.}$$

Problem 5.71

A ball is dropped from rest from a height $h_0 = 1.5\,\text{m}$. The impact between the ball and the floor has a COR $e = 0.92$. Find the formula that allows you to compute the rebound height h_i of the ith rebound. Furthermore, find the formula that provides the total time required to complete i rebounds. Finally, compute the time t_{stop} that the ball will take to stop bouncing. *Hint:* A formula you may find useful in the solution of this problem is that of the limit value of a geometric series: $\sum_{i=0}^{N-1} e^i = (e^N - 1)/(e - 1)$, with $|e| < 1$.

Solution



Computation of the formula for h_i. We assume that the ball is only subject to gravity (assumed to be constant) and, when in contact with the ground, to a reaction force normal to the ground. Consequently, referring to the figure to the right, consider a ball dropping from a position y_A to a position y_B, and let v_A and v_B be the speeds of the ball at A and B respectively. Using constant acceleration equations, we have

$$v(y_B)^2 = v(y_A)^2 - 2g(y_A - y_B), \tag{1}$$

In our problem the balls rebounds various times. We will use the index i to denote the ith rebound, whereby this rebound is the event that causes the ball to drop from rest from a height h_{i-1} and rebound to a maximum height h_i. Hence, assuming that the motion of the ball is only in the vertical direction, applying Eq. (1) and observing that for $y = h_{i-1}$ we have $v(h_{i-1}) = 0$, the vertical component of velocity with which the ball strikes the ground (i.e., reaches the point with $y = 0$) is

$$v_{iy}^- = -\sqrt{2gh_{i-1}}. \tag{2}$$

We will treat the ground as stationary. Hence, because the motion is completely along the LOI, the impact between the ball and the ground is governed by the COR equation so that

$$v_{iy}^+ = -ev_{iy}^- \quad \Rightarrow \quad v_i^+ = e\sqrt{2gh_{i-1}}. \tag{3}$$

The maximum height reached after the ith rebound is h_i and recalling that we must have $v(h_i) = 0$, this height can be computed using Eq. (1) again as follows:

$$v^2(h_i) = v^2(0) - 2g(h_i - 0) \quad \Rightarrow \quad 0 = (v_i^+)^2 - 2gh_i \quad \Rightarrow \quad h_i = e^2 h_{i-1}, \tag{4}$$

where the last expression was obtained by substituting the result in Eqs. (3). Recalling that h_0 is the height from which the ball is dropped before the first rebound, then a repeated application of the result in Eq. (4) gives

$$h_1 = e^2 h_0 \quad \Rightarrow \quad h_2 = e^2 h_1 = e^4 h_0 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \boxed{h_i = e^{2i} h_0.} \tag{5}$$

Keeping in mind that $0 < e < 1$, notice that the above solution indicates for it is necessary that $i \to \infty$ for $h_i \to 0$.

Computation of the formula for the total time. Using constant acceleration formulas, recall that the vertical position y of the ball is related to time by the following formula:

$$y = y(t_0) + \dot{y}(t - t_0) - \tfrac{1}{2}g(t - t_0)^2, \tag{6}$$

where t_0 is a reference time and where $y(t_0)$ and $\dot{y}(t_0)$ are the position and velocity in the y direction at time t_0. We now consider the ith rebound again. Applying Eq. (6), for when the ball drops from $y = h_{i-1}$ and hits the ground, we have

$$0 = h_{i-1} - \tfrac{1}{2}g[(\Delta t_{\text{drop}})_i]^2 \quad \Rightarrow \quad (\Delta t_{\text{drop}})_i = \sqrt{\frac{2h_{i-1}}{g}}. \tag{7}$$

When the ball rebounds off of the ground it will leave the ground with a velocity $v_{iy}^+ = e\sqrt{2gh_{i-1}}$ (see Eq. (3)) and then it will reach a maximum height $h_i = e^2 h_{i-1}$ (see Eq. (4)). Hence, applying Eq. (6) for when the ball goes from the ground and reaches $y = h_{i+1}$, we have

$$h_i = 0 + v_{iy}^+ \Delta t_{\text{raise}} - \tfrac{1}{2}g(\Delta t_{\text{raise}})^2 \quad \Rightarrow \quad e^2 h_{i-1} = 0 + e\sqrt{2gh_{i-1}}(\Delta t_{\text{raise}})_i - \tfrac{1}{2}g[(\Delta t_{\text{raise}})_i]^2$$

$$\Rightarrow \quad (\Delta t_{\text{raise}})_i = e\sqrt{\frac{2h_{i-1}}{g}}. \tag{8}$$

Therefore, the overall time taken by the ith rebound is

$$\Delta t_i = (\Delta t_{\text{drop}})_i + (\Delta t_{\text{raise}})_i = (1 + e)\sqrt{\frac{2h_{i-1}}{g}} = (1 + e)e^{i-1}\sqrt{\frac{2h_0}{g}}, \tag{9}$$

where we have used the last of Eqs. (5) to express h_{i-1} in terms of h_0. Given the result in Eq. (9), and denoting t_i the total time taken by i rebounds (i.e., the time needed for the ball to achieve the height h_i), t_i is found by the summation

$$t_i = \sum_{j=1}^{i}\left[(1 + e)e^{j-1}\sqrt{\frac{2h_0}{g}}\right] = (1 + e)\sqrt{\frac{2h_0}{g}}\sum_{j=1}^{i} e^{j-1} = (1 + e)\sqrt{\frac{2h_0}{g}}\sum_{j=0}^{i-1} e^j. \tag{10}$$

Using the suggestion made in the problem statement, we can then write the final expression for t_i in Eq. (10) as follows:

$$\boxed{t_i = (1 + e)\sqrt{\frac{2h_0}{g}}\frac{(e^i - 1)}{(e - 1)}.} \tag{11}$$

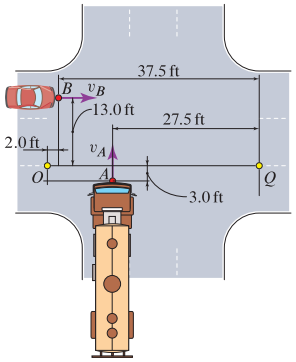
We have already observed that the ball will require an infinite number of rebounds to come to rest. Hence, the elapsed time until the ball comes to rest is found by computing the limit of t_i as $i \to \infty$. Observing again that $0 < e < 1$ and that $e^i \to 0$ for $i \to \infty$, we have

$$\boxed{t_{\text{stop}} = \lim_{i\to\infty}(1 + e)\sqrt{\frac{2h_0}{g}}\frac{(e^i - 1)}{(e - 1)} = \frac{1 + e}{1 - e}\sqrt{\frac{2h_0}{g}} = 13.3\,\text{s},}$$

where we have used the data $e = 0.92$, and $h_0 = 1.5\,\text{m}$.

Problems 5.72 through 5.74

At the instant shown, a truck A, of weight $W_A = 31{,}000\,\text{lb}$, and a car B, of weight $W_B = 3970\,\text{lb}$, are traveling with speeds $v_A = 35\,\text{mph}$ and $v_B = 34\,\text{mph}$, respectively.



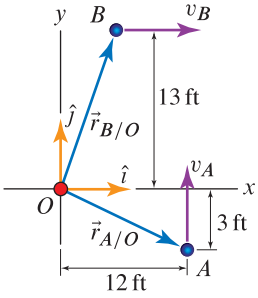
Problem 5.72 Choosing point O as the moment center, determine the angular momentum (with respect to O) of A and B individually at this instant.

Problem 5.73 Choosing point O as the moment center, determine the angular momentum (with respect to O) of the *particle system* formed by A and B at this instant.

Problem 5.74 Choosing point Q as the moment center, determine the angular momentum (with respect to Q) of the *particle system* formed by A and B at this instant.

Solution to 5.72

Using the cartesian coordinate system shown to the right, we have that



$$\vec{r}_{A/O} = (12\,\hat{\imath} - 3\,\hat{\jmath})\,\text{ft}, \qquad \vec{r}_{B/O} = (2\,\hat{\imath} + 13\,\hat{\jmath})\,\text{ft}, \tag{1}$$

$$\vec{v}_A = (35\,\hat{\jmath})\,\text{mph} = (51.33\,\hat{\jmath})\,\text{ft/s}, \qquad \vec{v}_B = (34\,\hat{\imath})\,\text{mph} = (49.87\,\hat{\imath})\,\text{ft/s}. \tag{2}$$

The angular momentum of A about O is:

$$(\vec{h}_O)_A = \vec{r}_{A/O} \times m_A \vec{v}_A = \left[(r_{A/O})_x m_A v_{Ay}\right]\hat{k}. \tag{3}$$

Hence, recalling that $m_A = (31{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 962.7\,\text{slug}$, we have

$$\boxed{(\vec{h}_O)_A = \left(593\times10^3\,\hat{k}\right)\,\text{slug}\cdot\text{ft}^2/\text{s}}$$

Similarly, the angular momentum of B about O is

$$(\vec{h}_O)_B = \vec{r}_{B/O} \times m_B \vec{v}_B = \left[-(r_{B/O})_y m_B v_{Bx}\right]\hat{k}. \tag{4}$$

Hence, recalling that $m_B = (3970\,\text{lb})/(32.2\,\text{ft/s}^2) = 123.3\,\text{slug}$, we have

$$\boxed{(\vec{h}_O)_A = \left(-79.9\times10^3\,\hat{k}\right)\,\text{slug}\cdot\text{ft}^2/\text{s}}$$

Solution to 5.73

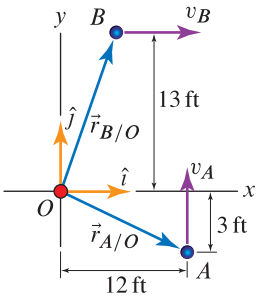
The definition of angular momentum for the given system of particles about point O is

$$\vec{h}_O = \sum_{i=1}^{N} \vec{h}_{Oi} \quad \Rightarrow \quad \vec{h}_O = (\vec{h}_O)_A + (\vec{h}_O)_B, \tag{5}$$

where

$$(\vec{h}_O)_A = \vec{r}_{A/O} \times m_A \vec{v}_A \quad \text{and} \quad (\vec{h}_O)_B = \vec{r}_{B/O} \times m_B \vec{v}_B, \tag{6}$$

where $(\vec{h}_O)_A$ and $(\vec{h}_O)_B$ are the angular momenta of A and B about O, respectively.



To perform the required calculation, using the cartesian coordinate system shown to the right, we begin with describing the following position vectors:

$$\vec{r}_{A/O} = (12\,\hat{\imath} - 3\,\hat{\jmath})\,\text{ft}, \qquad \vec{r}_{B/O} = (2\,\hat{\imath} + 13\,\hat{\jmath})\,\text{ft}, \tag{7}$$

$$\vec{v}_A = (35\,\hat{\jmath})\,\text{mph} = (51.33\,\hat{\jmath})\,\text{ft/s}, \qquad \vec{v}_B = (34\,\hat{\imath})\,\text{mph} = (49.87\,\hat{\imath})\,\text{ft/s}. \tag{8}$$

Then, the angular momentum of A about O is

$$(\vec{h}_O)_A = \vec{r}_{A/O} \times m_A \vec{v}_A = \left[(r_{A/O})_x m_A v_{Ay}\right]\hat{k}. \tag{9}$$

Hence, recalling that $m_A = (31{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 962.7\,\text{slug}$, we have

$$(\vec{h}_O)_A = \left(593.0\times10^3\,\hat{k}\right)\text{slug}\cdot\text{ft}^2/\text{s}. \tag{10}$$

Similarly the angular momentum of B about O is

$$(\vec{h}_O)_B = \vec{r}_{B/O} \times m_B \vec{v}_B = \left[-(r_{B/O})_y m_B v_{Bx}\right]\hat{k}. \tag{11}$$

Hence, recalling that $m_B = (3970\,\text{lb})/(32.2\,\text{ft/s}^2) = 123.3\,\text{slug}$, we have

$$(\vec{h}_O)_B = \left(-79.94\times10^3\,\hat{k}\right)\text{slug}\cdot\text{ft}^2/\text{s}. \tag{12}$$

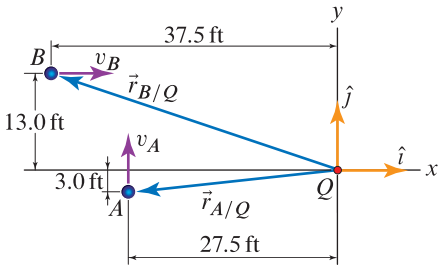
Substituting the results in Eqs. (10) and (12) into Eq. (5) and expressing the final result to three significant digits, we have

$$\boxed{\vec{h}_O = 513\times10^3\,\hat{k}\ \text{slug}\cdot\text{ft}^2/\text{s}}$$ 

Solution to 5.74

The definition of angular momentum for the given system of particles about point Q is

$$\vec{h}_Q = \sum_{i=1}^{N} \vec{h}_{Qi} \quad \Rightarrow \quad \vec{h}_Q = (\vec{h}_Q)_A + (\vec{h}_Q)_B, \qquad (13)$$



where

$$(\vec{h}_Q)_A = \vec{r}_{A/Q} \times m_A \vec{v}_A \quad \text{and} \quad (\vec{h}_Q)_B = \vec{r}_{B/Q} \times m_B \vec{v}_B, \qquad (14)$$

where $(\vec{h}_Q)_A$ and $(\vec{h}_Q)_B$ are the angular momenta of A and B about Q, respectively. To perform the required calculation, using the cartesian coordinate system shown to the right, we begin with describing the following position vectors:

$$\vec{r}_{A/Q} = (-27.5\,\hat{\imath} - 3\,\hat{\jmath})\,\text{ft}, \qquad \vec{r}_{B/Q} = (-37.5\,\hat{\imath} + 13\,\hat{\jmath})\,\text{ft}, \qquad (15)$$

$$\vec{v}_A = (35\,\hat{\jmath})\,\text{mph} = (51.33\,\hat{\jmath})\,\text{ft/s}, \qquad \vec{v}_B = (34\,\hat{\imath})\,\text{mph} = (49.87\,\hat{\imath})\,\text{ft/s}. \qquad (16)$$

Then, the angular momentum of A about Q is

$$(\vec{h}_Q)_A = \vec{r}_{A/Q} \times m_A \vec{v}_A = \left[(r_{A/Q})_x m_A v_{Ay}\right]\hat{k}. \qquad (17)$$

Hence, recalling that $m_A = (31{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 962.7\,\text{slug}$, we have

$$(\vec{h}_Q)_A = \left(1.359\times10^6\,\hat{k}\right)\text{slug}\cdot\text{ft}^2/\text{s}. \qquad (18)$$

Similarly the angular momentum of B about O is

$$(\vec{h}_Q)_B = \vec{r}_{B/Q} \times m_B \vec{v}_B = \left[-(r_{B/Q})_y m_B v_{Bx}\right]\hat{k}. \qquad (19)$$

Hence, recalling that $m_B = (3970\,\text{lb})/(32.2\,\text{ft/s}^2) = 123.3\,\text{slug}$, we have

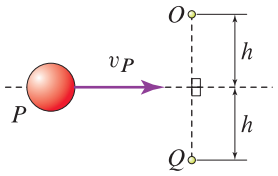
$$(\vec{h}_Q)_B = \left(-79.94\times10^3\,\hat{k}\right)\text{slug}\cdot\text{ft}^2/\text{s}. \qquad (20)$$

Substituting the results in Eqs. (18) and (20) into Eq. (13) and expressing the final result to three significant digits, we have

$$\boxed{\vec{h}_Q = -1.44\times10^6\,\hat{k}\ \text{slug}\cdot\text{ft}^2/\text{s}\ \ \hat{\jmath},\ \hat{\imath}.}$$

Problem 5.75

Consider the situation depicted in the figure. At the instant shown, how are the angular momenta of particle P with respect to O and Q related?



Solution

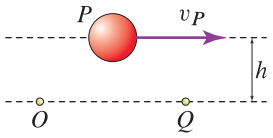
At the instant shown $\vec{h}_O = -\vec{h}_Q$.The reason is that $\left|\vec{h}_O\right| = \left|\vec{h}_Q\right|$ since it is easy to see that

$$\left|\vec{h}_O\right| = mv_P h, \quad \text{and} \quad \left|\vec{h}_Q\right| = mv_P h,$$

Since the direction of $\vec{h}$ is given by the right-hand rule, we see that $\vec{h}_O$ points out of the page and $\vec{h}_Q$ points into the page.

Problem 5.76

Consider the situation depicted in the figure. At the instant shown, how are the angular momenta of particle P with respect to O and Q related?



Solution

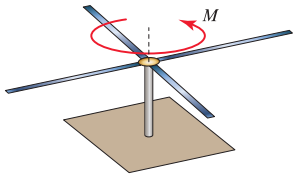
At the instant shown $\vec{h}_O = \vec{h}_Q$. The reason is that $\left|\vec{h}_O\right| = \left|\vec{h}_Q\right|$ since it is easy to see that

$$\left|\vec{h}_O\right| = mv_P h, \quad \text{and} \quad \left|\vec{h}_Q\right| = mv_P h,$$

where h is the perpendicular distance between the line between line OQ, and the velocity vector $\vec{v}_P$. Since the direction of $\vec{h}$ is given by the right-hand rule, we see that both $\vec{h}_O$ and $\vec{h}_Q$ point into the page and so must be equal.

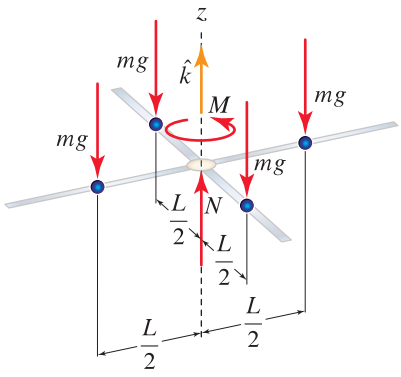
Problem 5.77

A rotor consists of four horizontal blades each of length $L = 4\,\text{m}$ and mass $m = 90\,\text{kg}$ cantilevered off of a vertical shaft. Assume that each blade can be modeled as having its mass concentrated at its midpoint. The rotor is initially at rest when it is subjected to a moment $M = \beta t$, with $\beta = 60\,\text{N}\cdot\text{m/s}$. Determine the angular speed of the rotor after 10 s.



Solution

From the rotor's FBD we see that the weight of each blade and the force N do not contribute a moment about the z axis because they are parallel to the z axis. Hence, applying the angular impulse–momentum principle in the z direction we have



$$\int_{t_1}^{t_2} M\,dt = h_{Oz}(t_2) - h_{Oz}(t_1), \tag{1}$$

where $t_1 = 0$ and $t_2 = 10\,\text{s}$ and where

$$h_{Oz}(t_1) = 0 \quad \text{and} \quad h_{Oz}(t_2) = 4\left[m\frac{L}{2}\left(\dot{\theta}(t_2)\frac{L}{2}\right)\right], \tag{2}$$

where $\dot{\theta}$ is the angular velocity of the rotor, assumed positive if in the positive z direction, and where we have accounted for the fact that the system starts from rest and that, due to the symmetry of the system, the total angular momentum about the z axis of the system is 4 times the angular momentum about the z axis of a single blade.

Now observe that

$$\int_{t_1}^{t_2} M\,dt = \int_{t_1}^{t_2} \beta t\,dt = \tfrac{1}{2}\beta\left(t_2^2 - t_1^2\right)^2 = \tfrac{1}{2}\beta t_2^2. \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1) and then solving for $\dot{\theta}(t_2)$, we have

$$\dot{\theta}(t_2) = \frac{\beta t_2^2}{2mL^2}, \tag{4}$$

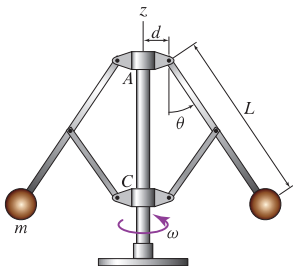
which gives

$$\boxed{\dot{\theta}(t_2) = 2.08\,\text{rad/s},}$$

after substituting the following numerical data: $\beta = 60\,\text{N}\cdot\text{m/s}$, $t_2 = 10\,\text{s}$, $m = 90\,\text{kg}$, and $L = 4\,\text{m}$.

Problem 5.78

The object shown is called a *speed governor*, a mechanical device for the regulation and control of the speed of mechanisms. The system consists of two arms of negligible mass at the end of which are attached two spheres, each of mass m. The upper end of each arm is attached to a fixed collar A. The system is then made to spin with a given angular speed ω_0 at a set opening angle θ_0. Once it is in motion, the opening angle of the governor can be varied by adjusting the position of the collar C (by the application of some force). Let θ represent the generic value of the governor opening angle. If the arms are free to rotate, that is, if no moment is applied to the system about the spin axis after the system is placed in motion, determine the expression of the angular velocity ω of the system as a function of ω_0, θ_0, m, d, and L, where L is the length of each arm and d is the distance of the top hinge point of each arm from the spin axis. Neglect any friction at A and C.



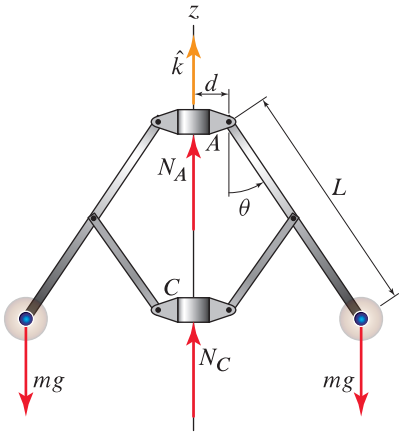
Solution

The FBD shown implies that the moment of the external forces about the z axis is equal to zero. Because the z axis is fixed we can then say that the angular momentum in the z direction is conserved, i.e.,

$$h_{Oz}(0) = h_{Oz}, \tag{1}$$

where $h_{Oz}(0)$ and h_{Oz} denote the z component of the angular momentum of the system at the initial time and at a generic subsequent time, respectively. Due to the symmetry of the system, the system's angular momentum is

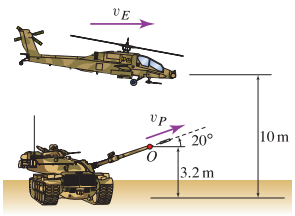
$$h_{Oz} = 2m(L\sin\theta + d)\omega(L\sin\theta + d), \tag{2}$$



where ω is the angular velocity of the system. Recalling that at the initial time $\omega(0) = \omega_0$ and $\theta(0) = \theta_0$, substituting Eq. (2) into Eq. (1) and solving for ω, we have

$$\boxed{\omega = \frac{(L\sin\theta_0 + d)^2}{(L\sin\theta + d)^2}\omega_0.}$$

Problems 5.79 through 5.81



Consider the motion of a projectile P of mass $m_P = 18.5\,\text{kg}$, which is shot with an initial speed $v_P = 1675\,\text{m/s}$ as shown in the figure. Ignore aerodynamic drag forces.

Problem 5.79 Compute the projectile's angular momentum with respect to the point O as a function of time from the time it exits the barrel until the time it hits the ground.

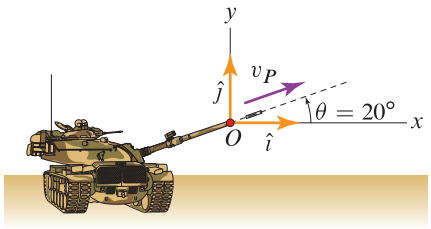
Problem 5.80 Choose point O as moment center. Then verify the validity of the angular impulse-momentum principle as given in Eq. (5.65) by showing that the time derivative of the angular momentum does, in fact, equal the moment.

Problem 5.81 Knowing that the helicopter E happens to have the same horizontal coordinate of the projectile at the instant the projectile leaves the gun and that it moves at a constant speed $v_E = 15\,\text{m/s}$ as shown, and treating E as a moving moment center, verify the angular impulse-momentum principle as given in Eq. (5.64).

Solution to 5.79

By definition, the angular momentum of P with respect to O is

$$\vec{h}_O = \vec{r}_{P/O} \times m_P \vec{v}_P. \tag{1}$$



Hence, to provide the answer to this problem, we need to find expressions for $\vec{r}_{P/O}$ and $\vec{v}_P$ as functions of time. To do so, for convenience, we let $\theta = 20°$ and we define a Cartesian coordinate system with origin at O, as shown in the figure the right. Then, applying standard solutions methods for projectile problems, we have that the x and y coordinates of point P are

$$x_P = \dot{x}_P(0)t \quad \text{and} \quad y_P = \dot{y}_P(0)t - \tfrac{1}{2}gt^2, \tag{2}$$

where

$$\dot{x}_P(0) = v_P(0)\cos\theta \quad \text{and} \quad \dot{y}_P(0) = v_P(0)\sin\theta, \tag{3}$$

where $v_P(0) = 1675\,\text{m/s}$.

Next, observing that $\vec{r}_{P/O} = x_P\,\hat{\imath} + y_P\,\hat{\jmath}$ and that, since O is a fixed point, $\vec{v}_P = \dot{x}_P\,\hat{\imath} + \dot{y}_P\,\hat{\jmath}$, using Eqs. (2) and (3) into Eq. (1), we have

$$\vec{h}_O = \{v_P(0)t\cos\theta\,\hat{\imath} + [v_P(0)t\sin\theta - \tfrac{1}{2}gt^2]\,\hat{\jmath}\} \times m\{v_P(0)\cos\theta\,\hat{\imath} + [v_P(0)\sin\theta - gt]\,\hat{\jmath}\}. \tag{4}$$

Carrying out the cross products, simplifying, and recalling that time is being measured in seconds, we then have

$$\boxed{\vec{h}_O = -\tfrac{1}{2}m_P g t^2 v_P(0)\cos\theta\,\hat{k} = -\left(143\times10^3\,\text{kg}\cdot\text{m}^2/\text{s}^3\right)t^2\,\hat{k}}$$

where we have used the following numerical data: $m_P = 18.5\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $v_P(0) = 1675\,\text{m/s}$, and $\theta = 20°$.

Solution to 5.80

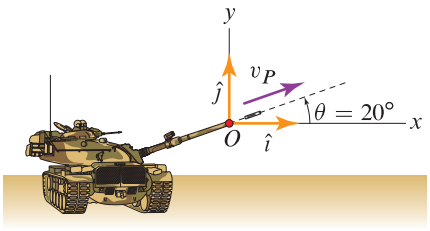
Choosing the fixed point O as moment center, the application of Eq. (5.65) on p. 390 of the textbook, reads

$$\vec{M}_O = \dot{\vec{h}}_O. \tag{5}$$

In order to verify the relationship for the projectile, we need to calculate $\vec{M}_O$ and $\dot{\vec{h}}_O$ independently of one another, and finally check that they are equal.

We begin with calculating $\dot{\vec{h}}_O$. To do so, we recall that, by definition, the angular momentum of P with respect to O is

$$\vec{h}_O = \vec{r}_{P/O} \times m_P \vec{v}_P. \tag{6}$$



To find expressions for $\vec{r}_{P/O}$ and $\vec{v}_P$ as functions of time, we let $\theta = 20°$ and we define a Cartesian coordinate system with origin at O, as shown in the figure the right. Then, applying standard solutions methods for projectile problems, we have that the x and y coordinates of point P are

$$x_P = \dot{x}_P(0)t \quad \text{and} \quad y_P = \dot{y}_P(0)t - \tfrac{1}{2}gt^2, \tag{7}$$

where

$$\dot{x}_P(0) = v_P(0)\cos\theta \quad \text{and} \quad \dot{y}_P(0) = v_P(0)\sin\theta, \tag{8}$$

where $v_P(0) = 1675\,\text{m/s}$.

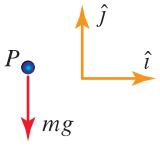
Next, observing that $\vec{r}_{P/O} = x_P\,\hat{\imath} + y_P\,\hat{\jmath}$ and that, since O is a fixed point, $\vec{v}_P = \dot{x}_P\,\hat{\imath} + \dot{y}_P\,\hat{\jmath}$, substituting Eqs. (7) and (8) into Eq. (6), we have

$$\vec{h}_O = \{v_P(0)t\cos\theta\,\hat{\imath} + [v_P(0)t\sin\theta - \tfrac{1}{2}gt^2]\,\hat{\jmath}\} \times m\{v_P(0)\cos\theta\,\hat{\imath} + [v_P(0)\sin\theta - gt]\,\hat{\jmath}\}. \tag{9}$$

Carrying out the cross products and simplifying, we then have

$$\vec{h}_O = -\tfrac{1}{2}m_P g t^2 v_P(0)\cos\theta\,\hat{k} \quad \Rightarrow \quad \dot{\vec{h}}_O = -m_P g v_P(0)t\cos\theta. \tag{10}$$

We now proceed to compute the moment of the forces acting on P with respect to O. Referring to the FBD to the right, consistently with the calculation carried out so far, we treat this problem as a projectile problem and therefore we assume that the only force acting on P is its weight $m_P g$. Then, we have



$$\vec{M}_O = \vec{r}_{P/O} \times (-mg)\,\hat{\jmath}. \tag{11}$$

Hence, recalling again that $\vec{r}_{P/O} = x_P\,\hat{\imath} + y_P\,\hat{\jmath}$ and that, since O is a fixed point, $\vec{v}_P = \dot{x}_P\,\hat{\imath} + \dot{y}_P\,\hat{\jmath}$, using Eqs. (2) and (3) into Eq. (11), we have

$$\vec{M}_O = (x_P\,\hat{\imath} + y_P\,\hat{\jmath}) \times (-mg\,\hat{\jmath}) = -mgx_P\,\hat{k} \quad \Rightarrow \quad \vec{M}_O = -m_P g v_P(0)t\cos\theta\,\hat{k}. \tag{12}$$

Comparing the results in Eqs. (10) and (12) we see that indeed Eq. (5) is verified. We formalize this result as follows:

> Since $\dot{\vec{h}}_O = -m_P g v_P(0)t\cos\theta$ and $\vec{M}_O = -m_P g v_P(0)t\cos\theta\,\hat{k}$, it is indeed true that $\vec{M}_O = \dot{\vec{h}}_O$.

Solution to 5.81

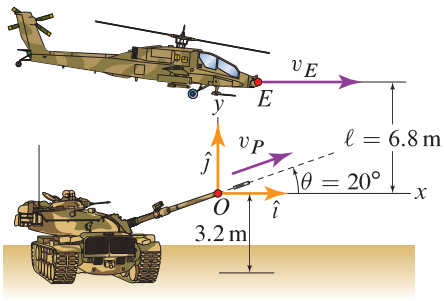
Choosing the point E as moment center, the application of Eq. (5.64) on p. 390 of the textbook, reads

$$\vec{M}_E = \dot{\vec{h}}_E + \vec{v}_E \times m\vec{v}_P. \tag{13}$$

In order to verify this relationship for the projectile, we need to calculate the terms $\vec{M}_E$ and $\dot{\vec{h}}_E + \vec{v}_E \times m\vec{v}_P$ independently of one another, and finally check that they are equal.

We begin with calculating $\dot{\vec{h}}_E$. To do so, we recall that, by definition, the angular momentum of P with respect to E is

$$\vec{h}_E = \vec{r}_{P/E} \times m_P \vec{v}_P. \tag{14}$$



To find expressions for $\vec{r}_{P/E}$ and $\vec{v}_P$ as functions of time, we let $\theta = 20°$ and we define a Cartesian coordinate system with origin at O, as shown in the figure the right. We model P as a projectile. Hence, applying standard solutions methods for projectile problems, we have that the x and y coordinates of point P are

$$x_P = \dot{x}_P(0)t \quad \text{and} \quad y_P = \dot{y}_P(0)t - \tfrac{1}{2}gt^2, \tag{15}$$

where

$$\dot{x}_P(0) = v_P(0)\cos\theta \quad \text{and} \quad \dot{y}_P(0) = v_P(0)\sin\theta, \tag{16}$$

where $v_P(0) = 1675$ m/s. In addition, since E moves at constant velocity, we have

$$x_E = v_E t \quad \text{and} \quad y_E = \ell. \tag{17}$$

Next, observing that $\vec{r}_{P/E} = (x_P - x_E)\,\hat{\imath} + (y_P - y_E)\,\hat{\jmath}$ and that $\vec{v}_P = \dot{x}_P\,\hat{\imath} + \dot{y}_P\,\hat{\jmath}$, substituting Eqs. (15)–(17) into Eq. (14), we have

$$\vec{h}_E = \{[v_P(0)\cos\theta - v_E]t\,\hat{\imath} + [v_P(0)t\sin\theta - \tfrac{1}{2}gt^2 - \ell]\,\hat{\jmath}\} \times m\{v_P(0)\cos\theta\,\hat{\imath} + [v_P(0)\sin\theta - gt]\,\hat{\jmath}\}. \tag{18}$$

Carrying out the cross products and simplifying, we then have

$$\vec{h}_E = \{\tfrac{1}{2}m(2\ell - gt^2)v_P(0)\cos\theta + mv_E t[gt - v_P(0)\sin\theta]\}\,\hat{k}. \tag{19}$$

Then, taking the time derivative of the above expression we obtain

$$\dot{\vec{h}}_E = [2mgv_E t - mgv_P(0)t\cos\theta - mv_E v_P(0)\sin\theta]\,\hat{k}. \tag{20}$$

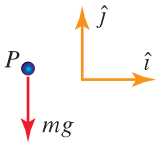
Next, we calculate the term $\vec{v}_E \times m\vec{v}_P$. Revalling that $\vec{v}_P = \dot{x}_P\,\hat{\imath} + \dot{y}_P\,\hat{\jmath}$ and $\vec{v}_E = \dot{x}_E\,\hat{\imath} + \dot{y}_E\,\hat{\jmath}$, using the kinematics relations in Eqs. (16) and (17), we have

$$\vec{v}_E \times m\vec{v}_P = [mv_E v_P(0)\sin\theta - mgv_E t]\,\hat{k}.$$

Hence, we have that the term $\dot{\vec{h}}_E + \vec{v}_E \times m\vec{v}_P$ is given by

$$\dot{\vec{h}}_E + \vec{v}_E \times m\vec{v}_P = mgt[v_E - v_P(0)\cos\theta]\,\hat{k}. \tag{21}$$

We now proceed to compute the moment of the forces acting on P with respect to E. Referring to the FBD to the right, consistently with the calculation carried out so far, we treat this problem as a projectile problem and therefore we assume that the only force acting on P is its weight $m_P g$. Then, we have



$$\vec{M}_E = \vec{r}_{P/E} \times (-mg)\,\hat{j}. \tag{22}$$

Hence, recalling that $\vec{r}_{P/E} = (x_P - x_E)\,\hat{\imath} + (y_P - y_E)\,\hat{j}$, substituting Eqs. (15) and (17) into Eq. (22), and simplifying, we have

$$\vec{M}_E = mgt[v_E - v_P(0)\cos\theta]\,\hat{k}. \tag{23}$$

Comparing the results in Eqs. (21) and (23) we see that indeed Eq. (13) is verified. We formalize this result as follows:

Since

$$\dot{\vec{h}}_E + \vec{v}_E \times m\vec{v}_P = mgt[v_E - v_P(0)\cos\theta]\,\hat{k}$$

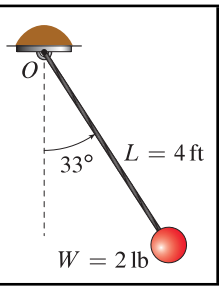
and

$$\vec{M}_E = mgt[v_E - v_P(0)\cos\theta]\,\hat{k},$$

it is indeed true that $\vec{M}_E = \dot{\vec{h}}_E + \vec{v}_E \times m\vec{v}_P$.

Problems 5.82 and 5.83

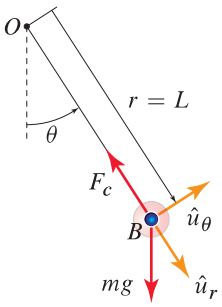
The simple pendulum in the figure is released from rest as shown.



Problem 5.82 Knowing that the bob's weight is $W = 2\,\text{lb}$, determine its angular momentum computed with respect to O as a function of the angle θ.

Problem 5.83 Use the angular impulse-momentum principle in Eq. (5.65) to determine the equations of motion of the pendulum bob.

Solution to 5.82



Let B denote the pendulum bob. To describe the angular momentum of B as a function of the angle θ, we need to first describe the velocity of B as a function of θ. To do so, choosing O as moment center, we will apply the angular impulse–momentum principle to B, which states that

$$\vec{M}_O = \dot{\vec{h}}_O, \tag{1}$$

where, referring to figure to the right,

$$\vec{M}_O = L\,\hat{u}_r \times [-F_c\,\hat{u}_r + mg(\cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta)] = -mgL\sin\theta\,\hat{k} \tag{2}$$

and

$$\vec{h}_O = L\,\hat{u}_r \times m\vec{v} = L\,\hat{u}_r \times mL\dot{\theta}\,\hat{u}_\theta = mL^2\dot{\theta}\,\hat{k} \quad \Rightarrow \quad \dot{\vec{h}}_O = mL^2\ddot{\theta}\,\hat{k}, \tag{3}$$

where we have used the fact that, in polar coordinates, $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$ and the fact that, in this problem $r = L = \text{constant}$.

Substituting the last expression in Eq. (2) and the last expression in Eq. (3) into Eq. (1) and simplifying, we obtain

$$\ddot{\theta} = -\frac{g}{L}\sin\theta. \tag{4}$$

Recalling that we can write $\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{dr}$, the above equation can be rewritten as follows:

$$\dot{\theta}\frac{d\dot{\theta}}{dr} = -\frac{g}{L}\sin\theta \quad \Rightarrow \quad \int_0^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = \int_{33^\circ}^{\theta} -\frac{g}{L}\sin\theta\,d\theta \quad \Rightarrow \quad \tfrac{1}{2}\dot{\theta}^2 = \frac{g}{L}(\cos\theta - \cos 33^\circ). \tag{5}$$

Hence, solving for $\dot{\theta}$, we obtain

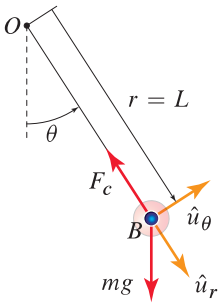
$$\dot{\theta} = \pm\sqrt{2\frac{g}{L}(\cos\theta - \cos 33^\circ)}. \tag{6}$$

Finally, substituting this result into the expression for $\vec{h}_O$ given in the first of Eqs. (3), recalling that $m = W/g$, and simplifying, we have

$$\vec{h}_O = \pm W\sqrt{\frac{2L^3}{g}(\cos\theta - \cos 33^\circ)}\,\hat{k} = \pm(3.99\,\text{lb·ft·s})\sqrt{\cos\theta - 0.839}\,\hat{k} \quad \hat{u}_\theta \perp \hat{u}_r @ -57^\circ,$$

where we have used the following data: $W = 2\,\text{lb}$, $L = 4\,\text{ft}$, and $g = 32.2\,\text{ft/s}^2$.

Solution to 5.83



Let B denote the pendulum bob. To apply the impulse–momentum principle to B, first set up a polar coordinate system with origin at O. Then, referring to the FBD to the right, choosing O as moment center, the application of the angular impulse–momentum principle to B yields

$$\vec{M}_O = \dot{\vec{h}}_O, \tag{7}$$

where, referring to figure to the right,

$$\vec{M}_O = L\,\hat{u}_r \times [-F_c\,\hat{u}_r + mg(\cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta)] = -mgL\sin\theta\,\hat{k} \tag{8}$$

and

$$\vec{h}_O = L\,\hat{u}_r \times m\vec{v} = L\,\hat{u}_r \times mL\dot{\theta}\,\hat{u}_\theta = mL^2\dot{\theta}\,\hat{k} \quad \Rightarrow \quad \dot{\vec{h}}_O = mL^2\ddot{\theta}\,\hat{k}, \tag{9}$$

where we have used the fact that, in polar coordinates, $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$ and the fact that, in this problem $r = L =$ constant.

Substituting the last expression in Eq. (8) and the last expression in Eq. (9) into Eq. (7) and simplifying, we obtain

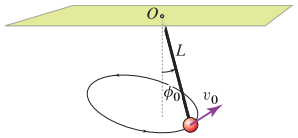
$$\ddot{\theta} = -\frac{g}{L}\sin\theta. \tag{10}$$

Given that the pendulum cord has constant length, the pendulum cord is a physical system with one degree of freedom. Hence, collecting all the terms in the dependent variable θ on one side of the equal sign in Eq. (10), we have that the equation of motion of the pendulum bob is

$$\boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = 0.}$$

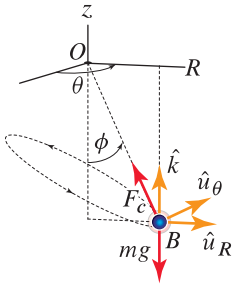
Problem 5.84

At the lowest and highest points on its trajectory, the pendulum cord, with a length $L = 2\,\text{ft}$, forms angles $\phi_1 = 15°$ and $\phi_2 = 50°$ with the vertical direction, respectively. Determine the speed of the pendulum bob corresponding to ϕ_1 and ϕ_2.



Solution

The FBD shown applies to any configuration of the pendulum bob B. Using this FBD, we see that the moment of the forces acting on B about the fixed point O is given by



$$\begin{aligned}\vec{M}_O &= \vec{r}_{P/O} \times \vec{F} \\ &= L(\sin\phi\,\hat{u}_R - \cos\phi\,\hat{k}) \times [-mg\,\hat{k} + F_c(\cos\phi\,\hat{k} - \sin\phi\,\hat{u}_R)] \\ &= mgL\sin\phi\,\hat{u}_\theta, \end{aligned} \tag{1}$$

where $\vec{F}$ is the total force acting on B and F_c is the tension in the cord. Equation (1) shows that the z component of $\vec{M}_O$ is equal to zero. This fact and the fact that the z axis is fixed, imply that the z component of the angular momentum of B is conserved.

Let ① and ② denote the minimum and maximum height positions achieved by B, respectively. Both at ① and ② the velocity of B can only be in the transverse direction, that is, the R and z components of the velocity of B must be equal to zero. Observing that B is moving in the (positive) transverse direction, then at ① and ② the velocity vectors of B are, respectively,

$$\vec{v}_1 = v_1\,\hat{u}_{\theta_1} \quad \text{and} \quad \vec{v}_2 = v_2\,\hat{u}_{\theta_2}, \tag{2}$$

where v_1 and v_2 are the speeds of B at ① and ②. respectively. Using Eq. (2) we then have that the angular momenta of B at ① and ② are

$$\vec{h}_{O1} = (\vec{r}_{B/O})_1 \times m\vec{v}_1 = mL(\sin\phi_1\,\hat{u}_{R_1} - \cos\phi_1\,\hat{k}) \times v_1\,\hat{u}_{\theta_1} = mv_1L\sin\phi_1\,\hat{k} + mv_1L\cos\phi_1\,\hat{u}_{R_1} \tag{3}$$

and

$$\vec{h}_{O2} = (\vec{r}_{B/O})_2 \times m\vec{v}_2 = mL(\sin\phi_2\,\hat{u}_{R_2} - \cos\phi_2\,\hat{k}) \times v_2\,\hat{u}_{\theta_2} = mv_2L\sin\phi_2\,\hat{k} + mv_2L\cos\phi_2\,\hat{u}_{R_2}. \tag{4}$$

As argued above, the z components of the the angular momenta just computed must be equal to each other (since h_{Oz} is a conserved quantity), so that we have

$$mv_1L\sin\phi_1 = mv_2L\sin\phi_2. \tag{5}$$

To find v_1 and v_2 we need a second equation. Observe that all the forces doing work are conservative. Hence, the energy of B is conserved and, setting the datum at O, we must have

$$T_1 + V_1 = T_2 + V_2 \quad \Rightarrow \quad -mgL\cos\phi_1 + \tfrac{1}{2}mv_1^2 = -mgL\cos\phi_2 + \tfrac{1}{2}mv_2^2. \tag{6}$$

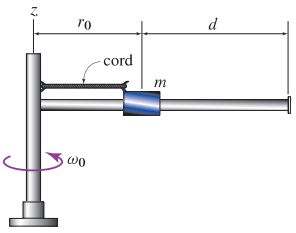
Equation (5) and the last of Eqs. (6) form a system of 2 equations in the 2 unknowns v_1 and v_2 whose solution is

$$\begin{aligned} v_1 &= \sin\phi_2\sqrt{\frac{2gL(\cos\phi_2 - \cos\phi_1)}{\sin^2\phi_1 - \sin^2\phi_2}} = 6.85\,\text{ft/s}, \\ v_2 &= \sin\phi_1\sqrt{\frac{2gL(\cos\phi_2 - \cos\phi_1)}{\sin^2\phi_1 - \sin^2\phi_2}} = 2.32\,\text{ft/s}, \end{aligned}$$

where we have used the following numerical data: $\phi_1 = 15°$, $\phi_2 = 50°$, $g = 32.2\,\text{ft/s}^2$, and $L = 2\,\text{ft}$.

Problems 5.85 and 5.86

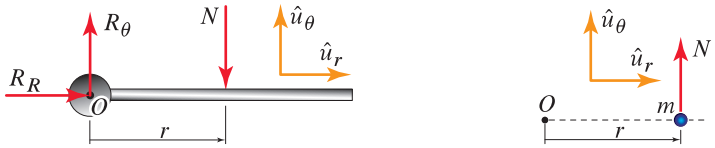
A collar with mass $m = 2\,\text{kg}$ is mounted on a rotating arm of negligible mass that is initially rotating with an angular velocity $\omega_0 = 1\,\text{rad/s}$. The collar's initial distance from the z axis is $r_0 = 0.5\,\text{m}$ and $d = 1\,\text{m}$. At some point, the restraint keeping the collar in place is removed so that the collar is allowed to slide. Assume that the friction between the arm and the collar is negligible.



Problem 5.85 If no external forces and moments are applied to the system, with what speed will the collar impact the end of the arm?

Problem 5.86 Compute the moment that must be applied to the arm, as a function of position along the arm, to keep the arm rotating at a constant angular velocity while the collar travels toward the end of the arm.

The "speed" with which the collar impacts the end of the arm must be understood as the magnitude of the relative velocity of the collar with respect to the end of the arm. Using a polar component system with radial direction aligned with the arm, the relative velocity in question coincides with the radial component of the velocity of the collar. To compute the required velocity, choosing point O to be on the axis of rotation, we first sketch the FBDs of the arm and collar m, whose top views are displayed in the figures below.



The forces R_θ and R_R are the reactions that the ground exerts on the vertical shaft. The force N is the contact force between the sliding mass and the arm. Focusing on the FBD of the arm (left), because the arm's mass is negligible, the sum of the moments about O (for the arm), must be equal to zero, i.e.,

$$\sum M_O: \quad -Nr = 0 \quad \Rightarrow \quad N = 0. \tag{1}$$

Equation (1) implies that the moment about O of the external forces (i.e., N) acting on m is equal to zero. Since O is a fixed point, this means that the angular momentum of the collar about O is conserved. Letting the subscripts 1 and 2 denote the instant at which m is released and a generic time instant thereafter, respectively, we have

$$\vec{h}_{O1} = \vec{h}_{O2} \quad \Rightarrow \quad \vec{r}_1 \times m\vec{v}_1 = \vec{r}_2 \times m\vec{v}_2, \tag{2}$$

where

$$\vec{r}_1 = r_0\hat{u}_{r_1}, \quad \vec{v}_1 = \dot{r}_1\,\hat{u}_{r_1} + r_0\dot{\theta}_0\,\hat{u}_{\theta_1}, \quad \vec{r}_2 = r\,\hat{u}_{r_2}, \quad \vec{v}_2 = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta \tag{3}$$

Substituting Eq. (3) into Eq. (2), recalling that $\dot{\theta} = \omega_0$, and simplifying, we have

$$r_0^2\omega_0\,\hat{k} = r_2^2\dot{\theta}_2\,\hat{k} \quad \Rightarrow \quad \dot{\theta}_2 = \left(\frac{r_0}{r}\right)^2\omega_0. \tag{4}$$

The result in Eq. (4) allows us to compute the transverse component of the velocity. To compute the radial component of the velocity we need to determine $\dot{r}$ as a function of r. To do so, we consider the FBD of m and we sum forces in the radial direction. This yields

$$\sum F_r: \quad 0 = ma_r. \tag{5}$$

Recalling that, in polar coordinates, $a_r = \ddot{r} - r\dot{\theta}^2$, Eq. (5) can be rewritten as

$$\ddot{r} = r\dot{\theta}^2 \quad \Rightarrow \quad \dot{r}\frac{d\dot{r}}{dr} = \dot{\theta}^2 r \quad \Rightarrow \quad \dot{r}\,d\dot{r} = \dot{\theta}^2 r\,dr \quad \Rightarrow \quad \dot{r}\,d\dot{r} = r_0^4\omega_0^2\frac{dr}{r^3}, \tag{6}$$

where we have used Eq. (4) to derive the last expression. Recalling that at time t_1 we have $\dot{r} = 0$ and $r = r_0$, the expression in question can be integrated as follows

$$\int_0^{\dot{r}} \dot{r}\,d\dot{r} = r_0^4\omega_0^2\int_{r_0}^{r}\frac{dr}{r^3} \quad \Rightarrow \quad \dot{r}^2 = r_0^4\omega_0^2\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right). \tag{7}$$

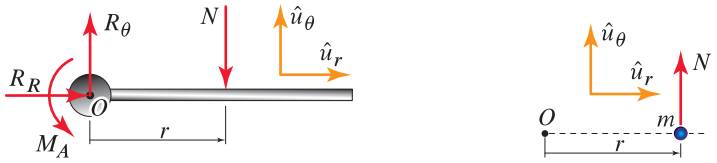
Substituting the result in Eqs. (4) into Eq. (7), and evaluating the result for $r = r_0 + d$, we have that the speed with which the collar impact the end of arm is

$$\boxed{|\dot{r}_{r=r_0+d}| = r_0^2\omega_0\sqrt{\frac{1}{r_0^2} - \frac{1}{(r_0+d)^2}} = 0.471\,\text{m/s},}$$

where we have used the following numerical data: $r_0 = 0.5\,\text{m}$, $omega_0 = 1\,\text{rad/s}$, and $d = 1\,\text{m}$.

Solution to 5.85

We start the solution of the problem by sketching the FBDs of the arm and collar m, whose top view is displayed below and where point O is a point on the axis of rotation.



Since the mass of the arm is negligible, we must have that the sum of the moments about O for the arm must be equal to zero. This implies that we have

$$\sum (M_O)_{\text{arm}}: \quad M_A - Nr = 0 \quad \Rightarrow \quad N = \frac{M_A}{r}. \tag{8}$$

Focusing now on the collar, observing that O is a fixed point, the application of angular impulse–momentum principle yields,

$$\vec{M}_O = \dot{\vec{h}}_O \quad \Rightarrow \quad Nr\,\hat{k} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{v} \times m\vec{v} + \vec{r} \times m\vec{a} \quad \Rightarrow \quad Nr\,\hat{k} = \vec{r} \times m\vec{a}. \tag{9}$$

Next, recall that in polar coordinates we have

$$\vec{r} = r\,\hat{u}_r, \quad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta. \tag{10}$$

In addition, we have that

$$\dot{\theta} = \omega_0 = \text{constant} \quad \Rightarrow \quad \ddot{\theta} = 0. \tag{11}$$

Substituting Eqs. (10) and Eq. (10) into the last of Eqs. (9) we have

$$Nr\,\hat{k} = r\,\hat{u}_r \times m[(\ddot{r} - r\omega_0^2)\,\hat{u}_r + 2\dot{r}\omega_0\,\hat{u}_\theta] = 2m\omega_0 r\dot{r}\,\hat{k} \quad \Rightarrow \quad N = 2m\omega_0\dot{r}. \tag{12}$$

Hence, substituting the result in Eq. (8) into Eq. (12) we then have

$$M_A = 2m\omega_0 r\dot{r}. \tag{13}$$

Given the result in Eq. (13), we will obtain M_A as a function of r after we manage to express $\dot{r}$ as a function of r. To do so, referring to the FBD of the mass m, summing forces in the radial direction we have

$$\sum F_r: \quad 0 = ma_r \quad \Rightarrow \quad \ddot{r} = \omega_0^2 r, \tag{14}$$

where we have used the fact that $a_r = \ddot{r} - r\dot{\theta}^2 = \ddot{r} - r\omega_0^2$. Recalling that we can write $\ddot{r} = \dot{r}(d\dot{r}/dr)$, Eq. (14) yields the following result:

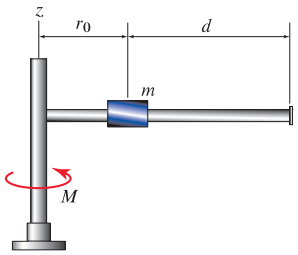
$$\dot{r}\frac{d\dot{r}}{dr} = \omega_0^2 r \quad \Rightarrow \quad \int_0^{\dot{r}} \dot{r}\,d\dot{r} = \int_{r_0}^{r} r\,dr \quad \Rightarrow \quad \dot{r} = \omega_0\sqrt{r^2 - r_0^2}, \tag{15}$$

where we have used the fact that $\dot{r} =$ when $r = r_0$. Substituting the result in Eq. (15) into Eq. (13), we finally have

$$\boxed{M_A = 2m\omega_0^2 r\sqrt{r^2 - r_0^2}.}$$

Problems 5.87 and 5.88

A collar of mass m is initially at rest on a horizontal arm when a constant moment M is applied to the system to make it rotate. Assume that the mass of the horizontal arm is negligible and that the collar is free to slide without friction.



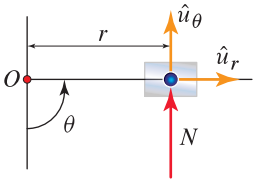
Problem 5.87 Derive the equations of motion of the system, taking advantage of the angular impulse-momentum principle. *Hint:* Applying the angular impulse momentum principle yields only one of the needed equations of motion.

Problem 5.88 Continue Prob. 5.87 by integrating the collar's equations of motion and determine the time the collar takes to reach the end of the arm. Assume that the collar weighs 1.2 lb and that $M = 20$ ft·lb. Also, at the initial time let $r_0 = 1$ ft and $d = 3$ ft.

Solution to 5.87

The FBD shown, which shows a view of the collar from above. The only contribution to a moment in the z direction (out of the page) is that of the force N. Hence, we must have that the applied moment M is such that

top view



$$M = Nr, \tag{1}$$

With this in mind, choosing O as moment center and observing that the z is a fixed axis, the application of the angular impulse–momentum principle in the z direction gives

$$M = \dot{h}_{Oz}, \tag{2}$$

where

$$\vec{h}_O = \vec{r} \times m\vec{v}, \tag{3}$$

and where

$$\vec{r} = r\hat{u}_r \quad \text{and} \quad \vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{4}$$

Substituting Eqs. (4) into Eq. (3) and carrying out the cross-product, yields

$$h_{Oz} = mr^2\dot{\theta} \quad \Rightarrow \quad \dot{h}_{Oz} = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}. \tag{5}$$

Substituting Eq. (5) into Eq. (2), we then obtain

$$M = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}, \tag{6}$$

where we recall that M is a given quantity. Now observe that the collar is not subject to any force in the r direction, we must have $ma_r = 0$. Since $a_r = \ddot{r} - r\dot{\theta}^2$, this then implies

$$\ddot{r} - r\dot{\theta}^2 = 0. \tag{7}$$

Summarizing, the equations of motion of the collar are

$$\boxed{\ddot{r} - r\dot{\theta}^2 = 0 \quad \text{and} \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = M.}$$

Solution to 5.88

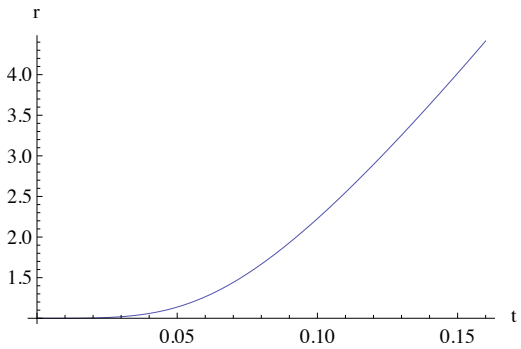
Using the solution of Problem 5.87 along with the problem's numerical data and appropriate mathematical software, we can find the function $r(t)$ that describes the position of the collar along the arm as a function of time. Plotting such a function, we then see that $r = r_0 + d = 4\,\text{ft}$ in the vicinity of $t = 0.15\,\text{s}$. This observation can be confirmed by using a numerical root finding algorithm. The code to perform these operations using *Mathematica* and the corresponding output is illustrated below.

```
EOMs = {20.0 == 2 (1.2/32.2) r[t] r'[t] θ'[t] + (1.2/32.2) (r[t])^2 θ''[t], r''[t] - r[t] (θ'[t])^2 == 0};
ICs = {r[0] == 1.0, r'[0] == 0, θ[0] == 0, θ'[0] == 0};
Solution = NDSolve[{EOMs, ICs}, {r, θ}, {t, 0, 0.15}];

Solution = NDSolve[{EOMs, ICs}, {r, θ}, {t, 0, 0.16}]
```

$\{\{r \to \text{InterpolatingFunction}[(\,0.\quad 0.16\,),\ <>],\ \theta \to \text{InterpolatingFunction}[(\,0.\quad 0.16\,),\ <>]\}\}$

```
Plot[r[t] /. Solution, {t, 0, 0.16}, AxesLabel → {"t", "r"}]
```




```
FindRoot[r[t] == 4 /. Solution, {t, 0.14}]
```

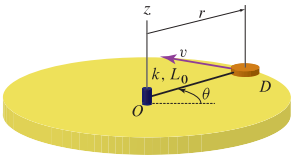
$\{t \to 0.149561\}$

Expressing the result of the root finding algorithm using three significant figures then gives us the following final answer.

time to reach end of arm $= 0.150\,\text{s}$.

Problem 5.89

A simple model of orbital motion under a central force can be constructed by considering the motion of a disk D sliding with no friction over a horizontal surface while connected to a fixed point O by a linear elastic cord of constant k and unstretched length L_0. Let the mass of D be $m = 0.45\,\text{kg}$ and $L_0 = 1\,\text{m}$. Suppose that when D is at its maximum distance from O, this distance is $r_0 = 1.75\,\text{m}$ and the corresponding speed of D is $v_0 = 4\,\text{m/s}$. Determine the elastic cord constant k such that the minimum distance between D and O is equal to the unstretched length L_0.



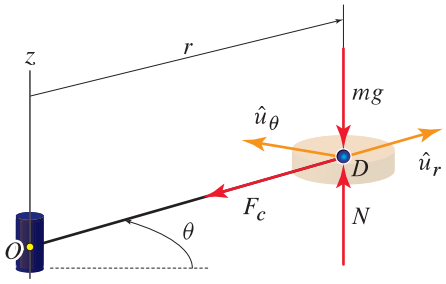
Solution

The FBD to the right implies that the angular momentum of D about O in the z direction is conserved, i.e.,

$$(h_{Oz})_1 = (h_{Oz})_2, \tag{1}$$

where the subscripts 1 and 2 denote the instants in which D achieves the maximum and minimum distance from O, respectively. Also,

$$\vec{h}_O = \vec{r} \times m\vec{v} = r\,\hat{u}_r \times m(v_r\,\hat{u}_r + v_\theta\,\hat{u}_\theta) = mrv_\theta\,\hat{k}. \tag{2}$$



When D is at the maximum or minimum distance from O, v_r is equal to zero (i.e., D is no longer increasing or decreasing its distance from O). Hence, given that $v_\theta > 0$, we have

$$(h_{Oz})_1 = mr_1v_1 = mr_0v_0 \quad \text{and} \quad (h_{Oz})_2 = mr_2v_2 = mL_0v_2, \tag{3}$$

where v_1 and v_2 are the speeds of D at the instants 1 and 2, respectively, and where we have enforced the fact that $r_2 = L_0$. Substituting Eqs. (3) into Eq. (1) and solving for v_2 we have

$$v_2 = (r_0/L_0)v_0. \tag{4}$$

Since the only force doing work on D is the force $F_c$due to due to the elastic cord, applying the work-energy principle, we have

$$T_1 + V_1 = T_2 + V_2, \tag{5}$$

where

$$T_1 = \tfrac{1}{2}mv_0^2, \quad V_1 = \tfrac{1}{2}k(r_0 - L_0)^2, \quad T_2 = \tfrac{1}{2}m(r_0/L_0)^2v_0^2, \quad V_2 = 0, \tag{6}$$

where we have accounted for the fact that at ② $r_2 = L_0$ and the elastic cord is unstretched. Substituting Eq. (6) into Eq. (5) we have

$$\tfrac{1}{2}mv_0^2 + \tfrac{1}{2}k(r_0 - L_0)^2 = \tfrac{1}{2}m(r_0/L_0)^2v_0^2, \tag{7}$$

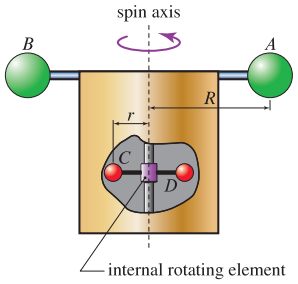
where k is the elastic spring constant of the elastic cord. Solving Eq. (7) for k we have

$$\boxed{k = \frac{m(r_0 + L_0)v_0^2}{L_0^2(r_0 - L_0)} = 26.4\,\text{N/m},}$$

where we have used the following numerical data: $m = 0.45\,\text{kg}$, $r_0 = 1.75\,\text{m}$, $L_0 = 1\,\text{m}$, and $v_0 = 4\,\text{m/s}$.

Problem 5.90

The body of the satellite shown has a weight that is negligible with respect to the two spheres A and B that are rigidly attached to it, which weigh 150 lb each. The distance between A and B from the spin axis of the satellite is $R = 3.5$ ft. Inside the satellite there are two spheres C and D weighing 4 lb mounted on a motor that allows them to spin about the axis of the cylinder at a distance $r = 0.75$ ft from the spin axis. Suppose that the satellite is released from rest and that the internal motor is made to spin up the internal masses at a constant time rate of 5.0 rad/s^2 for a total of 10 s. Treating the system as isolated, determine the angular speed of the satellite at the end of spin-up.



Solution

The final angular velocity of the internal masses is

$$\omega_i = (5.00\,\text{rad/s}^2)(10.0\,\text{s}) = 50.00\,\text{rad/s}, \tag{1}$$

where the subscript i stands for internal. Since the system is isolated, the angular momentum of the system must be conserved throughout the motion of the system. In particular, assuming that the spin axis of the system does not change orientation, then the angular momentum about the spin axis must be conserved, so that we have

$$(m_A + m_B)R^2\omega_{\text{satellite}} = (m_C + m_D)r^2\omega_i \quad \Rightarrow \quad \omega_{\text{satellite}} = \frac{(m_C + m_D)r^2}{(m_A + m_B)R^2}\omega_i. \tag{2}$$

Using the problem's numerical data, we have

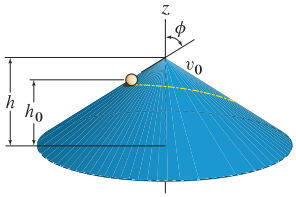
$$\boxed{\omega_{\text{satellite}} = 0.0612\,\text{rad/s},}$$

where, in addition to the result in Eq. (1), we have used the following numerical data:

$$\begin{aligned}
m_C &= \frac{4\,\text{lb}}{32.2\,\text{ft/s}^2} = 0.1242\,\text{slug},\\
m_D &= \frac{4\,\text{lb}}{32.2\,\text{ft/s}^2} = 0.1242\,\text{slug},\\
m_A &= \frac{150\,\text{lb}}{32.2\,\text{ft/s}^2} = 4.658\,\text{slug},\\
m_B &= \frac{150\,\text{lb}}{32.2\,\text{ft/s}^2} = 4.658\,\text{slug},\\
r &= 0.75\,\text{ft},\\
R &= 3.5\,\text{ft}.
\end{aligned}$$

Problem 5.91

A sphere of mass m slides over the outer surface of a cone with angle ϕ and height h. The sphere was released at a height h_0 with a velocity of magnitude v_0 and a direction that was completely horizontal. Assume that the opening angle of the cone and the value of v_0 are such that the sphere does not separate from the surface of the cone once put in motion. In addition, assume that the friction between the sphere and the cone is negligible. Determine the vertical component of the sphere's velocity as a function of the vertical position z (measured from the base of the cone), v_0, h, h_0, and ϕ.



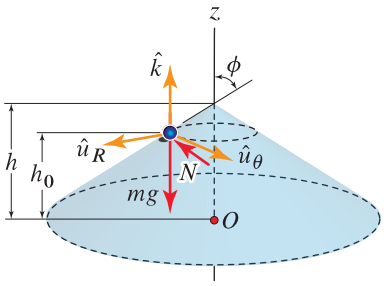
Solution

The z component of the moment about O of the forces acting on the sphere is equal to zero. Since the z axis is fixed, we have

$$h_{Oz} = \text{constant} \quad \Rightarrow \quad mv_\theta R = mv_\theta(0)R(0), \tag{1}$$

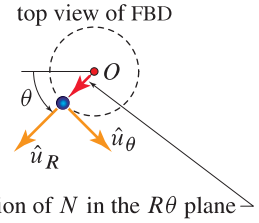
where, $v_\theta(0) = v_0$ and

$$R = (h - z)\tan\phi \quad \Rightarrow \quad R(0) = (h - h_0)\tan\phi. \tag{2}$$



Substituting both of Eqs. (2) in Eq. (1), and solving for v_θ, we have

$$v_\theta = \frac{h - h_0}{h - z} v_0. \tag{3}$$



Differentiating the first of Eqs. (2) with respect to time gives

$$\dot{R} = -\dot{z}\tan\phi \quad \Rightarrow \quad v_R = -v_z\tan\phi, \tag{4}$$

where $v_R = \dot{R}$ and $v_z = \dot{z}$. Now notice that gravity is the only force doing work on the sphere. Letting ① be the position of the particle at release and ② a generic position thereafter, we must have

$$T_1 + V_1 = T_2 + V_2, \tag{5}$$

where, setting the datum as shown and letting $v_2 = v$, we have

$$T_1 = \tfrac{1}{2}mv_0^2, \quad V_1 = mgh_0. \quad T_2 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(v_R^2 + v_\theta^2 + v_z^2), \quad V_2 = mgz. \tag{6}$$

Substituting the expressions in Eqs. (6) in Eq. (5) we obtain

$$\tfrac{1}{2}mv_0^2 + mgh_0 = \tfrac{1}{2}m(v_R^2 + v_\theta^2 + v_z^2) + mgz. \tag{7}$$

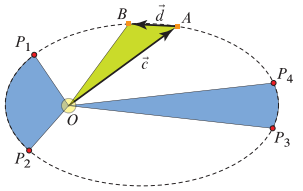
Substituting the expressions in Eq. (3) and the second of Eqs. (4) into Eq. (7) and solving for v_z, after simplifying, we obtain

$$v_z = -\cos\phi\sqrt{2g(h_0 - z) + v_0^2\left[1 - \left(\frac{h - h_0}{h - z}\right)^2\right]},$$

where the minus sign in front of $\cos\phi$ indicates that the sphere is moving downward.

Problem 5.92

Consider a planet orbiting the Sun, and let P_1, P_2, P_3, and P_4 be the planet's position at four corresponding time instants t_1, t_2, t_3, and t_4 such that $t_2 - t_1 = t_4 - t_3$. Letting O denote the position of the Sun, determine the ratio between the areas of the orbital sectors P_1OP_2 and P_3OP_4. *Hint:* (1) The area of triangle OAB defined by the two planar vectors $\vec{c}$ and $\vec{d}$ as shown is given by $\text{Area}(ABC) = |\vec{c} \times \vec{d}|$; (2) the solution of this problem is a demonstration of Kepler's second law (see Section 1.1).



Solution

Gravity is a central force so $\vec{M}_O = 0$ and there is conservation of angular momentum.

$$r_1\,\hat{u}_r \times m\vec{v}_1 = r_2\,\hat{u}_r \times m\vec{v}_2.$$

The term $\vec{v}\,dt$ represents the differential displacement of the planet so we can multiply both sides by dt to get

$$r_1\,\hat{u}_r \times m\vec{v}_1\,dt = r_2\,\hat{u}_r \times m\vec{v}_2\,dt \quad \Rightarrow \quad \int_{t_1}^{t_2} r_1\,\hat{u}_r \times m\vec{v}_1\,dt = \int_{t_3}^{t_4} r_2\,\hat{u}_r \times m\vec{v}_2\,dt.$$

Using the hint we see that these integrals represent the differential area of an orbital sector. Thus we see that for $t_2 - t_1 = t_4 - t_3$ the ratio of the orbital sectors is equal to 1.

The ratio of the orbital sectors is equal to 1.

Problem 5.93

Starting with Eq. (5.113) and using Eqs. (5.106), (5.108), and (5.109), show that the radius at apoapsis r_A can be written as shown in Eq. (5.114).

Solution

We start by repeating here for convenience Eq. (5.113) (on p. 413 of the textbook), Eq. (5.106) (on p. 412 of the textbook), Eq. (5.108) (on p. 412 of the textbook), and Eq. (5.109) (on p. 413 of the textbook), which, in order, are

$$r_A = \frac{r_P}{2Gm_B/(r_P v_P^2) - 1}, \tag{1}$$

$$e = \frac{C\kappa^2}{Gm_B}, \tag{2}$$

$$\kappa = r_P v_P, \tag{3}$$

$$C = \frac{1}{r_P}\left(1 - \frac{Gm_B}{r_P v_P^2}\right). \tag{4}$$

Substituting Eq. (3) into Eq. (2), we have

$$e = \frac{C r_P^2 v_P^2}{Gm_B}. \tag{5}$$

Next, we view Eqs. (4) and (5) as two equations in the two unknowns G and C, whose solution is

$$G = \frac{r_P v_P^2}{(1+e)\, m_B} \quad \text{and} \quad C = \frac{e}{(1+e)\, r_P}.$$

we then take the expression for G and substitute it into Eq. (1), which yields

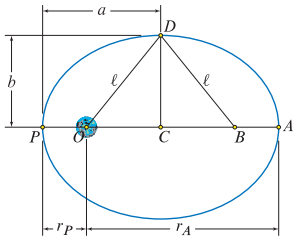
$$r_A = \frac{r_P}{\left[\frac{2m_B}{r_P v_P^2}\right]\left[\frac{r_P v_P^2}{(1+e)m_B}\right] - 1} \quad \Rightarrow \quad r_A = \frac{r_P}{\frac{2}{1+e} - 1}.$$

This equation can be rewritten as Eq. (5.114) (on p. 413 of the textbook), i.e.,

$$\boxed{r_A = r_P\left(\frac{1+e}{1-e}\right).}$$

Problem 5.94

Using the lengths shown as well as the property of an ellipse that states that the sum of the distances from each of the foci (i.e., points O and B) to any point on the ellipse is a constant, prove Eq. (5.122), that is, that the length of the semiminor axis can be related to the periapsis and apoapsis radii via $b = \sqrt{r_P r_A}$.



Solution

Using the property of an ellipse given in the problem statement, we can write that:

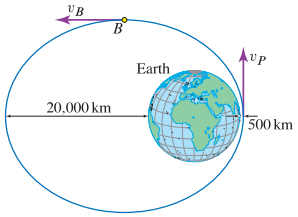
$$\ell + \ell = \overline{OP} + \overline{BP} = 2a \quad \Rightarrow \quad \ell = a.$$

Now, we can also see that:

$$\left(\overline{OC}\right)^2 + \left(\overline{CD}\right)^2 = \ell^2 \quad \Rightarrow \quad (a - r_P)^2 + b^2 = \ell^2 = a^2 \quad \Rightarrow \quad b^2 = a^2 - (a - r_P)^2 \quad \Rightarrow$$
$$b^2 = a^2 - \left(a^2 - 2ar_P + r_P^2\right) \quad \Rightarrow \quad b^2 = 2ar_P - r_P^2 \quad \Rightarrow \quad b^2 = r_P\,(2a - r_P) \quad \Rightarrow$$
$$(2a - r_P) = r_A \quad \text{so} \quad \boxed{b = \sqrt{r_P r_A}.}$$

Problem 5.95

An artificial satellite is launched from an altitude of 500 km with a velocity v_P that is parallel to the surface of the Earth. Requiring that the altitude at apogee be 20,000 km, determine the velocity at B, that is, the position in the orbit when the velocity is first orthogonal to the launch velocity.



Solution

The radius of Earth is $r_e = 6.371\times10^6$ m so

$$\begin{aligned} r_P &= r_e + 500\,\text{km} \quad &\Rightarrow \quad r_P &= 6.871\times10^6\,\text{m}, \\ r_A &= r_e + 20{,}000\,\text{km} \quad &\Rightarrow \quad r_A &= 26.37\times10^6\,\text{m}, \\ a &= \tfrac{1}{2}(r_A + r_P) \quad &\Rightarrow \quad a &= 16.62\times10^6\,\text{m}, \\ b &= \sqrt{r_A r_P} \quad &\Rightarrow \quad b &= 13.46\times10^6\,\text{m}. \end{aligned}$$

The distance from the center of Earth to B is

$$r_B = \sqrt{(a - r_P)^2 + b^2} = 16.62\times10^6\,\text{m}.$$

We can now apply Eq. (5.134) on p. 416 of the textbook directly to find the velocity at B. This gives

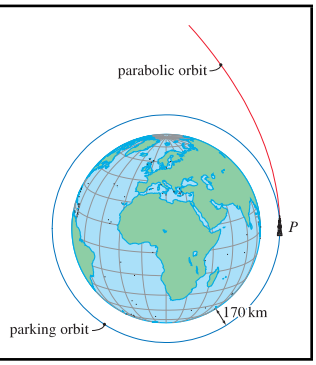
$$v_B = \sqrt{Gm_B\left(\frac{2}{r_B} - \frac{1}{a}\right)} = \sqrt{gr_e^2\left(\frac{2}{r_B} - \frac{1}{a}\right)}, \tag{1}$$

where $g = 9.81\,\text{m/s}^2$ is the acceleration due to gravity on the surface of the Earth, and where the last expression has been obtained by recalling that the term Gm_B can be replaced by t he term gr_e^2 when the body B is the Earth. Upon substitution of the numerical results derived earlier, Eq. (1) gives

$v_B = 4.89\times10^3$ m/s in the direction described in the problem statement.

Problem 5.96

The S-IVB third stage of the Saturn V rocket, which was used for the Apollo missions, would burn for about 2.5 min to place the spacecraft into a "parking orbit" and then, after several orbits, would burn for about 6 min to accelerate the spacecraft to escape velocity to send it to the Moon. Assuming a circular parking orbit with an altitude of 170 km, determine the change in speed needed at P to go from the parking orbit to escape velocity. Assume that the change in speed occurs instantaneously so that you need not worry about changes in orbital position during the engine thrust.



Solution

For a circular orbit with an altitude of 170 km above the Earth, the radius of the orbit is

$$r_c = r_e + 170\,\text{km} \quad \Rightarrow \quad r_c = 6.541\times10^6\,\text{m}. \tag{1}$$

where $r_e = 6.371\times10^6$ m is the radius of the Earth. To determine the speed of the rocket stage while on the circular orbit in question, we can use Eq. (5.111) on p. 413 of the textbook, which gives

$$v_c = \sqrt{\frac{Gm_e}{r_c}} \quad \Rightarrow \quad v_c = \sqrt{\frac{gr_e^2}{r_c}}, \tag{2}$$

where we have replaced the term Gm_e with the term gr_e^2, in which $g = 9.81\,\text{m/s}^2$ is the acceleration due to gravity on the surface of the Earth. To determine the escape velocity for a satellite in circular orbit, we can use Eq. (5.128) on p. 415 of the textbook, which gives

$$v_{\text{esc}} = \sqrt{\frac{2Gm_e}{r_c}} \quad \Rightarrow \quad v_{\text{esc}} = \sqrt{\frac{2gr_e^2}{r_c}} \tag{3}$$

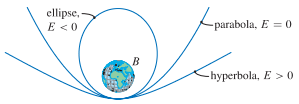
Consequently, using Eqs. (2) and (3), the change in speed is

$$\Delta v = v_{\text{esc}} - v_c \quad \Rightarrow \quad \boxed{\Delta v = \sqrt{\frac{gr_e^2}{r_c}}(\sqrt{2}-1) = 3.23\times10^3\,\text{m/s},} \tag{4}$$

where we have used the numerical following numerical values: $g = 9.81\,\text{m/s}^2$, $r_e = 6.371\times10^6$ m, and $r_c = 6.541\times10^6$ m (see the last of Eqs. (1)).

Problem 5.97

Using the last of Eqs. (5.131), along with Eq. (5.132), solve for the eccentricity e as a function of E, κ, and Gm_B.



(a) Using that result, along with fact that $e \geq 0$, show that $E < 0$ corresponds to an elliptical orbit, $E = 0$ corresponds to a parabolic trajectory, and $E > 0$ corresponds to a hyperbolic trajectory.

(b) Show that for $e = 0$, the expression you found for e leads to Eq. (5.111).

Solution

We start by repeating here for convenience Eqs. (5.131) (on p. 416 of the textbook), Eq. (5.132) (on p. 416 of the textbook), and Eq. (5.111) (on p. 413 of the textbook), which, in order, are

$$\kappa^2 = Gm_B a(1-e^2), \quad E = -\frac{Gm_B}{2a}, \quad \text{and} \quad v_c = \sqrt{\frac{Gm_B}{r_P}}. \tag{1}$$

Next, we solve the second of Eqs. (1) for a and substitute the result into the first of Eqs. (1) to obtain

$$a = -\frac{Gm_B}{2E} \quad \Rightarrow \quad \kappa^2 = -Gm_B\frac{Gm_B}{2E}(1-e^2) \quad \Rightarrow \quad \kappa^2 = \frac{(Gm_B)^2}{2E}(e^2-1)$$
$$\Rightarrow \quad e^2 = 1 + \frac{2E\kappa^2}{(Gm_B)^2} \quad \Rightarrow \quad e = \sqrt{1+\frac{2E\kappa^2}{(Gm_B)^2}}.$$

Part (a). Now that we have an expression for e, observe that

$$E < 0 \quad \Rightarrow \quad 1 + \frac{2E\kappa^2}{(Gm_B)^2} < 1 \quad \Rightarrow \quad e < 1 \quad \Rightarrow \quad \text{elliptical orbit.}$$
$$E = 0 \quad \Rightarrow \quad 1 + \frac{2E\kappa^2}{(Gm_B)^2} = 1 \quad \Rightarrow \quad e = 1 \quad \Rightarrow \quad \text{parabolic trajectory.}$$
$$E > 0 \quad \Rightarrow \quad 1 + \frac{2E\kappa^2}{(Gm_B)^2} > 1 \quad \Rightarrow \quad e > 1 \quad \Rightarrow \quad \text{hyperbolic orbit.}$$

Part (b). If $e = 0$, then

$$1 + \frac{2E\kappa^2}{(Gm_B)^2} = 0, \quad \text{but} \quad E = -\frac{Gm_B}{2a} \quad \text{so that} \quad \kappa^2 = Gm_B a.$$

Since $e = 0$ corresponds to a circular orbit, recall that for such an orbit, $\kappa = r_c v_c$, so that we can write

$$\kappa = r_c v_c \quad \Rightarrow \quad \kappa^2 = r_c^2 v_c^2 \quad \Rightarrow \quad r_c^2 v_c^2 = Gm_B a. \tag{2}$$

In addition, since the orbit is circular, the semimajor a axis coincides with the orbit's radius, so that we have $a = r_c = r_P$, which allows us to write

$$r_c^2 v_c^2 = Gm_B r_c \quad \Rightarrow \quad \boxed{v_c = \sqrt{\frac{Gm_B}{r_c}}.}$$

Problem 5.98

Assuming that the Sun is the only significant body in the solar system (the mass of the Sun accounts for 99.8% of the mass of the solar system), determine the escape velocity from the Sun as a function of the distance r from its center. What is the value of the escape velocity (expressed in km/h) when r is equal to the radius of Earth's orbit? Use 1.989×10^{30} kg for the mass of the Sun and 150×10^{6} km for the radius of Earth's orbit.

Solution

Recall that the expression for the escape velocity (strictly speaking this is a speed) of a satellite P orbiting a body B is given by Eq. (5.128) on p. 415 of the textbook, which gives

$$v_{\text{esc}} = \sqrt{\frac{2Gm_B}{r_P}}, \tag{1}$$

where $G = 6.674\times10^{-11}\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$ is the universal gravitational constant, m_B is the mass of B and r_P is the distance of P from B. In our problem the body B is the Sun. Hence, we have

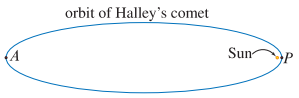
$$v_{\text{esc}} = \sqrt{\frac{2Gm_{\text{Sun}}}{r}} = \frac{(1.692\times10^{10}\ \text{m}^{3/2}/\text{s})}{\sqrt{r}} \quad \Rightarrow \quad \boxed{v_{\text{esc}} = \frac{(1.69\times10^{10}\ \text{m}^{3/2}/\text{s})}{\sqrt{r}},}$$

where $m_{\text{Sun}} = 1.989\times10^{30}$ kg is the mass of the Sun. Now, the distance of the Earth from the Sun is $r = 150\times10^{6}\ \text{km} = 150.0\times10^{9}\ \text{m}$. Therefore,the escape velocity of Earth is

$$\boxed{(v_{\text{esc}})_{\text{Earth}} = 4.21\times10^{4}\ \text{m/s} = 151{,}000\ \text{km/h}.}$$

Problem 5.99

In 1705, Edmund Halley (1656–1742), an English astronomer, claimed that the comet sightings of 1531, 1607, and 1682 were all the same comet and that, after some rough calculations accounting for the influence of the larger planets, this comet would return again in 1758. Halley did not live to see the comet's return, but it did return late in 1758 and reached perihelion in March 1759. In honor of his prediction, this comet was named "Halley." Each elliptical orbit of Halley is slightly different, but the average value of the semimajor axis a is about 17.95 AU. Using this value, along with the fact that its orbital eccentricity is 0.967 (the orbit is drawn to scale, but the Sun is shown to be 36 times bigger than it should be), determine



(a) the orbital period in years of Halley's comet, and

(b) its distance, in AU, from the Sun at perihelion P and at aphelion A. Look up the orbits of the planets of our solar system on the Web. What planetary orbits is Halley near to at perihelion and aphelion?

Use 1.989×10^{30} kg for the mass of the Sun.

Solution

Part (a). Given a and the mass of the sun m_S we calculate τ with using the result in Eq. (5.126) on p. 415 of the textbook. This yields the following result

$$\tau = \sqrt{\frac{4\pi^2 a^3}{G m_S}} \quad \Rightarrow \quad \boxed{\tau = 2.40\times10^9 \text{ s} = 76.1 \text{ yr},}$$

where where $G = 6.674\times10^{-11}$ m^3/(kg·s^2) is the universal gravitational constant, $m_S = 1.989\times10^{30}$ kg is the mass of the Sun, and $a = 17.95$ AU $= 2.685\times10^{12}$ m (1 AU $= 1.496\times10^{11}$ m).

Part (b). Let the distances from the Sun at P and A be denoted by r_P and r_A, respectively. We can determine r_P and r_A as direct application of Eqs. (5.119) on p. 414 of the textbook, i.e.,

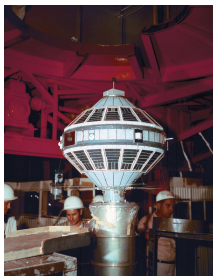
$$r_P = a\,(1-e) \quad \text{and} \quad r_A = a(1+e),$$

which give

$$\boxed{\begin{aligned} r_P &= 0.592 \text{ AU}: \quad r_P \text{ is between the orbits of Mercury and Venus,} \\ r_A &= 35.3 \text{ AU}: \quad r_A \text{ is outside the orbit of Neptune,} \end{aligned}}$$

where we have used the following numerical data: $a = 17.95$ AU and $e = 0.967$.

Problems 5.100 and 5.101



Explorer 7 was launched on October 13, 1959, with an apogee altitude above the Earth's surface of 1073 km and a perigee altitude of 573 km above the Earth's surface. Its orbital period was 101.4 min.

Problem 5.100 Using this information, calculate Gm_e for the Earth and compare it with gr_e^2.

Problem 5.101 Determine the eccentricity of the Explorer 7's orbit as well as its speeds at perigee and apogee.

Solution to 5.100

Recalling that the radius of the Earth $r_e = 6371$ km, we begin with computing the orbit's the radii at perigee and apogee. Specifically, we have

$$r_A = 1073\,\text{km} + 6371\,\text{km} = 7444\,\text{km} = 7.444\times10^6\,\text{m},$$
$$r_P = 573\,\text{km} + 6371\,\text{km} = 6944\,\text{km} = 6.944\times10^6\,\text{m}.$$

Therefore, the orbit's semimajor axis is given by

$$a = \tfrac{1}{2}(r_A + r_P) = 7194\,\text{km} = 7.194\times10^6\,\text{m}.$$

As stated in the problem, the orbital period is $\tau = 101.4\,\text{min} = 6084\,\text{s}$. Now, using Kepler's third law with $m_B = m_e$ (m_e is the mass of the Earth), we have

$$\tau^2 = \frac{4\pi^2}{Gm_e}a^3 \quad \Rightarrow \quad \boxed{Gm_e = \frac{4\pi^2 a^3}{\tau^2} = 3.97\times10^{14}\,\text{m}^3/\text{s}^2.}$$

Comparing this result with the following result:

$$gr_e^2 = (9.81\,\text{m/s}^2)(6371\times10^3\,\text{m})^2 = 3.98\times10^{14}\,\text{m}^3/\text{s}^2,$$

we see that they differ by about 0.3%.

Solution to 5.101

Recalling that the radius of the Earth $r_e = 6371\,\text{km}$, we begin with computing the orbit's the radii at perigee and apogee. Specifically, we have

$$r_A = 1073\,\text{km} + 6371\,\text{km} = 7444\,\text{km} = 7.444\times10^6\,\text{m}, \tag{1}$$

$$r_P = 573\,\text{km} + 6371\,\text{km} = 6944\,\text{km} = 6.944\times10^6\,\text{m}. \tag{2}$$

Therefore, the orbit's semimajor axis is given by

$$a = \tfrac{1}{2}(r_A + r_P) = 7194\,\text{km} = 7.194\times10^6\,\text{m}. \tag{3}$$

We now make use of Eq. (5.114) on p. 413 of the textbook, which states that

$$r_A = r_P\left(\frac{1+e}{1-e}\right) \quad \Rightarrow \quad \boxed{e = \frac{r_A - r_P}{r_A + r_P} = 0.0348,} \tag{4}$$

where we have used the numerical value for r_A and r_P determined earlier in Eqs. (1) and (2).

The required speeds along the orbit are calculated using Eq. (5.134) on p. 416 os the textbook, which states that

$$v = \sqrt{Gm_B\left(\frac{2}{r} - \frac{1}{a}\right)}, \tag{5}$$

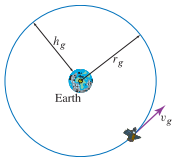
where G is the universal constant of gravitation, m_B is the mass of the body B about which the satellite is orbiting, r is the distance between the satellite and the body B, and a is the semimajor axis of the satellite's orbit. Recalling that when the body B is the Earth, the term Gm_B can be replaced by the the term gr_e^2, where $g = 9.81\,\text{m/s}^2$ is the acceleration due to gravity on the surface of the Earth, for the speeds at perigee and apogee we have

$$\boxed{\begin{aligned} v_P &= \sqrt{gr_e^2\left(\frac{2}{r_P} - \frac{1}{a}\right)} = 7.71\times10^3\,\text{m/s} = 27{,}700\,\text{km/h}, \\ v_A &= \sqrt{gr_e^2\left(\frac{2}{r_A} - \frac{1}{a}\right)} = 7.19\times10^3\,\text{m/s} = 25{,}900\,\text{km/h}, \end{aligned}}$$

where, in addition to the value of g and r_e, we have used the values of r_A, r_P, and a in Eqs. (1)–(3).

Problem 5.102

A *geosynchronous equatorial orbit* is a circular orbit above the Earth's equator that has a period of 1 day (these are sometimes called *geostationary orbits*). These geostationary orbits are of great importance for telecommunications satellites because a satellite orbiting with the same angular rate as the rotation rate of the Earth will appear to hover in the same point in the sky as seen by a person standing on the surface of the Earth. Using this information, determine the altitude h_g and radius r_g of a geostationary orbit (in miles). In addition, determine the speed v_g of a satellite in such an orbit (in miles per hour).



Solution

The orbital period of a geostationary satellite is

$$\tau_g = 1\,\text{day} = 8.640\times10^4\,\text{s}. \tag{1}$$

Recall that the orbital period τ for an elliptical orbit with semimajor axis equal to a is given by

$$\tau = 2\pi\frac{a^3}{Gm_B}.$$

Therefore, recalling that the quantity $Gm_B = gr_e^2$, where $g = 32.2\,\text{ft/s}^2$ and $r_e = 3959\,\text{mi} = 2.090\times10^7\,\text{ft}$ is the radius of the Earth, for a geosynchronous circular orbit of radius r_g the orbital period τ_g is

$$\tau_g = 2\pi\sqrt{\frac{r_g^3}{gr_e^2}} \quad\Rightarrow\quad r_g = \left(\frac{\tau_g^2 g r_e^2}{4\pi^2}\right)^{1/3} = 1.385\times10^8\,\text{ft}. \tag{2}$$

To find h_g observe that

$$h_g = r_g - r_e = 1.176\times10^8\,\text{ft}. \tag{3}$$

Finally, observing that the eccentricity e of a circular orbit is equal to zero, the speed of the satellite is given by

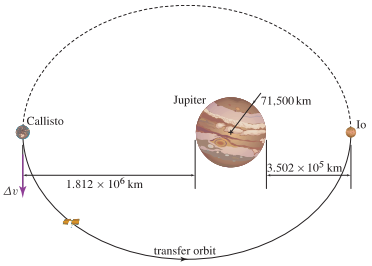
$$v_c = \sqrt{\frac{Gm_B}{r_P}} = \sqrt{\frac{gr_e^2}{r_g}} = 1.008\times10^4\,\text{ft/s}. \tag{4}$$

Expressing the results in Eqs. (2)–(4) to three significant figures, we have

$$r_g = 1.39\times10^8\,\text{ft} = 26{,}200\,\text{mi}, \quad h_g = 1.18\times10^8\,\text{ft} = 22{,}300\,\text{mi}, \quad v_c = 10{,}100\,\text{ft/s} = 6870\,\text{mph}.$$

Problem 5.103

The mass of the planet Jupiter is 318 times that of Earth, and its equatorial radius is 71,500 km. If a space probe is in a circular orbit about Jupiter at the altitude of the Galilean moon Callisto (orbital altitude 1.812×10^6 km), determine the change in speed Δv needed in the outer orbit so that the probe reaches a minimum altitude at the orbital radius of the Galilean moon Io (orbital altitude 3.502×10^5 km). Assume that the probe is at the maximum altitude in the transfer orbit when the change in speed occurs and that change in speed is impulsive, that is, it occurs instantaneously.



Solution

Let r_c and r_i denote the radii of Callisto's and Io's orbits, respectively. The radius r_c is also the radius of the initial circular orbit and the radius r_i is also the radius of the final circular orbit. Hence, we have

$$r_c = 1.812\times10^6 \text{ km} + 71{,}500 \text{ km} = 1.884\times10^9 \text{ m} \tag{1}$$

and

$$r_i = 3.502\times10^5 \text{ km} + 71{,}500 \text{ km} = 4.217\times10^8 \text{ m}. \tag{2}$$

Now, let a_1 and a_2 be the semimajor axes of the initial and transfer orbits, respectively. Specifically, we have

$$a_1 = r_c = 1.884\times10^9 \text{ m}, \quad \text{and} \quad a_2 = \tfrac{1}{2}(r_c + r_i) = 1.153\times10^9 \text{ m}. \tag{3}$$

Next, let v_1 and v_2 denote the speeds of the satellite corresponding to the initial circular orbit and the speed needed at apojove to move onto the the transfer orbit, respectively. These speeds are related to the semimajor axes of their respective orbits by Eq. (5.134) on p. 416 of the textbook, which states

$$v = \sqrt{Gm_B\left(\frac{2}{r} - \frac{1}{a}\right)}, \tag{4}$$

where G is the universal gravitational constant, m_B is the mass of the body about which the satellite is orbiting, and r is the distance of the satellite from B. In this problem B is the planet Jupiter. We now recall that for a satellite orbiting the Earth, the term Gm_B can be replaced by the term gr_e^2 where $r_e = 6371$ km is the radius of the Earth and $g = 9.81 \text{ m/s}^2$ is the acceleration due to gravity on the surface of the Earth. The problem states that

$$Gm_B = G318m_e = 318gr_e^2, \tag{5}$$

where m_e is the mass of the Earth. Hence, applying Eq. (4), we can determine the speeds v_1 and v_2 as follows:

$$v_1 = \sqrt{\frac{318gr_e^2}{r_c}} = 8199 \text{ m/s} = 29{,}520 \text{ km/h} \tag{6}$$

and

$$v_2 = \sqrt{318gr_e^2\left(\frac{2}{r_c} - \frac{1}{a_2}\right)} = 4959 \text{ m/s} = 17{,}850 \text{ km/h}, \tag{7}$$

where we have used the values for g, r_e, r_c and a_2 determined earlier in the problem. Thus the change in speed at apojove is $\Delta v = v_2 - v_1$, i.e.,

$$\boxed{\Delta v = -3240 \text{ m/s} = -11{,}700 \text{ km/h}.}$$

Problem 5.104

The on-orbit assembly of the International Space Station (ISS) began in 1998 and continues today. The ISS has an apogee altitude above the Earth's surface of 341.9 km and a perigee altitude of 331.0 km above the Earth's surface. Determine its maximum and minimum speeds in orbit, its orbital eccentricity, and its orbital period. Research its actual orbital period and compare it with your calculated value.



Solution

Recall that the speed of a satellite on an elliptical orbit about a body B is given by Eq. (5.134) on p. 416 of the textbook, i.e.,

$$v = \sqrt{Gm_B\left(\frac{2}{r} - \frac{1}{a}\right)}, \tag{1}$$

where r is the distance of the satellite from B, G is the universal gravitational constant, m_B is the mass of B, and a is the semimajor axis of the satellite's (elliptical) orbit. If B is the Earth, then the term Gm_B can be written as gr_e^2, where $r_e = 6371\,\text{km} = 6.371\times10^6\,\text{m}$ is the radius of the Earth and $g = 9.81\,\text{m/s}^2$ is the acceleration due to gravity on the surface of the Earth.

To apply the above formula, we now use the given data starting from the determination of the radii at apogee and perigee as well as the orbit's semimajor axis, i.e.,

$$\begin{aligned} r_A &= (6371\,\text{km} + 341.9\,\text{km})10^3\,\frac{\text{m}}{\text{km}} = 6.713\times10^6\,\text{m},\\ r_P &= (6371\,\text{km} + 331.0\,\text{km})10^3\,\frac{\text{m}}{\text{km}} = 6.702\times10^6\,\text{m},\\ a &= \tfrac{1}{2}(r_A + r_P) = 6.708\times10^6\,\text{m}. \end{aligned}$$

Hence, using the above results, the maximum orbital speed is at perigee and is given by

$$\boxed{v_P = \sqrt{gr_e^2\left(\frac{2}{r_P} - \frac{1}{a}\right)} = 7.71\times10^3\,\text{m/s} = 27{,}800\,\text{km/h}.}$$

Again, suing the above results, the minimum orbital speed is at apogee and is given by

$$\boxed{v_A = \sqrt{gr_e^2\left(\frac{2}{r_A} - \frac{1}{a}\right)} = 7.70\times10^3\,\text{m/s} = 27{,}700\,\text{km/h}.}$$

To find the eccentricity e we can use the second of Eqs. (5.119) on p. 414 of the textbook, which gives

$$r_A = a\,(1+e) \quad \Rightarrow \quad \boxed{e = \frac{r_A}{a} - 1 = 8.13\times10^{-4}.}$$

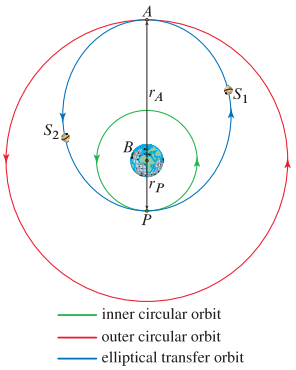
The orbital period is the obtained by using Eq. (5.126) on p. 415 of the textbook, which gives

$$\tau^2 = \frac{4\pi^2 a^3}{Gm_e} \quad \Rightarrow \quad \boxed{\tau = 2\pi\sqrt{\frac{a^3}{gr_e^2}} = 5470\,\text{s} = 91.2\,\text{min}.}$$

Different sources give different answers for the orbital period of the ISS: between 91.2 min and 91.34 min.

Problems 5.105 through 5.107

The optimal way (from an energy standpoint) to transfer from one circular orbit about a primary body B to another circular orbit is via the so-called *Hohmann transfer*, which involves transferring from one circular orbit to another using an elliptical orbit that is tangent to both at the periapsis and apoapsis of the ellipse. The ellipse is uniquely defined because we know r_P (the radius of the inner circular orbit) and r_A (the radius of the outer circular orbit), and therefore we know the semimajor axis a via Eq. (5.117) and the eccentricity e via Eq. (5.114) or Eqs. (5.119). Performing a Hohmann transfer requires two maneuvers, the first to leave the inner (outer) circular orbit and enter the transfer ellipse and the second to leave the transfer ellipse and enter the outer (inner) circular orbit.



Problem 5.105 A spacecraft S_1 needs to transfer from circular low Earth parking orbit with altitude 120 mi above the surface of the Earth to a circular geosynchronous orbit with altitude 22,240 mi. Determine the change in speed Δv_P required at perigee P of the elliptical transfer orbit and the change in speed Δv_A required at apogee A. In addition, compute the time required for the orbital transfer. Assume that the changes in speed are impulsive, that is, they occur instantaneously.

Problem 5.106 A spacecraft S_2 needs to transfer from a circular Earth orbit whose period is 12 h (i.e., it is overhead twice per day) to a low Earth circular orbit with an altitude of 110 mi. Determine the change in speed Δv_A required at apogee A of the elliptical transfer orbit and the change in speed Δv_P required at perigee P. In addition, compute the time required for the orbital transfer. Assume that the changes in speed are impulsive; that is, they occur instantaneously.

Problem 5.107 A spacecraft S_1 is transferring from circular low Earth parking orbit with altitude 100 mi to a circular orbit with radius r_A. Plot, as a function of r_A for $r_P \leq r_A \leq 100 r_P$, the change in speed Δv_P required at perigee of the elliptical transfer orbit as well as the change in speed Δv_A required at apogee. In addition, plot the time as a function of r_A, again for $r_P \leq r_A \leq 100 r_P$, required for the orbital transfer. Assume that the changes in speed are impulsive; that is, they occur instantaneously.

Solution to 5.105

Recalling that the radius of the Earth is $r_e = 3959\,\text{mi} = 2.090\times10^7\,\text{ft}$, the radii at perigee and apogee are

$$r_P = (5280\,\text{ft/mi})(3959\,\text{mi} + 120\,\text{mi}) = 2.154\times10^7\,\text{ft}, \tag{1}$$

$$r_A = (5280\,\text{ft/mi})(3959\,\text{mi} + 22{,}240\,\text{mi}) = 1.383\times10^8\,\text{ft}. \tag{2}$$

Therefore, we have that the semimajor axis a is given by

$$a = \tfrac{1}{2}(r_A + r_P) = 7.993\times10^7\,\text{ft}. \tag{3}$$

Next, we recall that the speed in a circular orbit of radius r is (see Eq. (5.111) on p. 413 of the textbook)

$$v = \sqrt{\frac{Gm_B}{r}} = \sqrt{\frac{gr_e^2}{r}}, \tag{4}$$

whereas, the speed in an elliptical orbit at distance r from B is (see Eq. (5.134) on p. 416 of the textbook)

$$v = \sqrt{Gm_B\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{gr_e^2\left(\frac{2}{r} - \frac{1}{a}\right)}. \tag{5}$$

Let v_1 be the speed when the spacecraft is in a circular orbit with radius r_P, v_2 be the speed at perigee for the transfer orbit, v_3 be the speed at apogee for the transfer orbit, and v_4 be the speed at the circular orbit with radius r_A. Then, applying Eqs. (4) and (5) as appropriate, we have

$$\begin{aligned}
v_1 &= \sqrt{\frac{gr_e^2}{r_P}} = 2.556\times10^4 \text{ ft/s} = 17{,}430 \text{ mph},\\
v_2 &= \sqrt{gr_e^2\left(\frac{2}{r_P} - \frac{1}{a}\right)} = 3.362\times10^4 \text{ ft/s} = 22{,}930 \text{ mph},\\
v_3 &= \sqrt{gr_e^2\left(\frac{2}{r_A} - \frac{1}{a}\right)} = 5235 \text{ ft/s} = 3569 \text{ mph},\\
v_4 &= \sqrt{\frac{gr_e^2}{r_A}} = 1.009\times10^4 \text{ ft/s} = 6876 \text{ mph},
\end{aligned}$$

where, in addition to using the values of r_P, r_A, and a indicated in Eqs. (1)–(3), we have used $g = 32.2 \text{ ft/s}^2$ and $r_e = 2.090\times10^7$ ft. Thus the changes in speed at perigee and apogee are:

$$\boxed{\begin{aligned}
\Delta v_P &= v_2 - v_1 = 8060 \text{ ft/s} = 5500 \text{ mph},\\
\Delta v_A &= v_4 - v_3 = 4850 \text{ ft/s} = 3310 \text{ mph}.
\end{aligned}}$$

Since the spacecraft only completes one half of the elliptical orbit during transfer, the time required for the orbital transfer t is half of the orbital period τ. In turn, the orbital period is found using Eq. (5.126) on p. 415 of the textbook, which gives,

$$\tau = 2\pi\sqrt{\frac{a^3}{gr_e^2}}.$$

Hence, we have

$$\boxed{t = \pi\sqrt{\frac{a^3}{gr_e^2}} = 18{,}930 \text{ s} = 5.26 \text{ hr},}$$

where, again we have used $g = 32.2 \text{ ft/s}^2$, $r_e = 2.090\times10^7$ ft, and the value of a in Eq. (3).

Solution to 5.106

The orbital period τ for an elliptical orbit about a body B is related to the orbit's semimajor axis a via Eq. (5.126) on p. 415 of the textbook, which gives $\tau^2 = 4\pi^2 a^3/(Gm_B)$, where G is the universal gravitational constant. In our problem, the body B is the Earth and the term Gm_B can be replaced by the term gr_e^2 where $r_e = 3959\,\text{mi}$ is the radius of the Earth and $g = 32.2\,\text{ft/s}^2$ is the acceleration due to gravity on the surface of the Earth. Therefore, for this problem we have

$$\tau^2 = \frac{4\pi a^3}{gr_e^2}. \tag{6}$$

The satellite's initial orbit is circular so that the orbit's semimajor axis a coincides with the radius of the orbit, which also coincides with the radius at Apogee, r_A. Thus, letting $a = r_A$ in Eq. (6) and solving for r_A, we have

$$r_A = \sqrt[3]{\frac{\tau^2 g r_e^2}{4\pi^2}} = 8.729\times10^7\,\text{ft} = 16{,}530\,\text{mi}, \tag{7}$$

where we have used the fact that, initially, $\tau = 12\,\text{h} = 4.320\times10^4\,\text{s}$. Next, the radius at periapsis is

$$r_P = 3959\,\text{mi} + 110\,\text{mi} = 4069\,\text{mi} = 2.148\times10^7\,\text{ft}, \tag{8}$$

and the semimajor axis a_e of the elliptical transfer orbit is

$$a_e = \tfrac{1}{2}(r_A + r_P) = 10{,}300\,\text{mi} = 5.438\times10^7\,\text{ft}. \tag{9}$$

Next, the speed v_{Ac} of the satellite while on the initial circular orbit, and the speed v_{Ae} that the satellite must have to get onto the transfer orbit at Apogee are computed using Eq. (5.111) on p. 413 of the textbook and Eq. (5.134) on p. 416, respectively, which give

$$v_{Ac} = \sqrt{\frac{gr_e^2}{r_A}} = 1.270\times10^4\,\text{ft/s} \quad \text{and} \quad v_{Ae} = \sqrt{gr_e^2\left(\frac{2}{r_A} - \frac{1}{a_e}\right)} = 7980\,\text{ft/s}. \tag{10}$$

Therefore, the change in speed at apogee is $\Delta v_A = v_{Ae} - v_{Ac}$, whose value (to three significant digits) is

$$\boxed{\Delta v_A = -4720\,\text{ft/s} = -3220\,\text{mph}.}$$

Replacing r_A in Eqs. (10) with r_P we can compute the speeds at perigee corresponding to the final circular destination orbit and the transfer orbit, respectively. This gives

$$v_{Pc} = \sqrt{\frac{gr_e^2}{r_P}} = 2.559\times10^4\,\text{ft/s} \quad \text{and} \quad v_{Pe} = \sqrt{gr_e^2\left(\frac{2}{r_P} - \frac{1}{a_e}\right)} = 3.242\times10^4\,\text{ft/s}. \tag{11}$$

Therefore, the change in speed at perigee is $\Delta v_P = v_{Pc} - v_{Pe}$, whose numerical value (to three significant digits) is

$$\boxed{\Delta v_A = -6830\,\text{ft/s} = -4660\,\text{mph}.}$$

Finally, observe that the spacecraft only completes half of the transfer orbit. Hence, the time t needed for the orbital transfer is $\tau_e/2$, where τ_e is the period of the elliptical transfer orbit. Letting $a = a_e$ in Eq. (6), solving for the corresponding τ_e, and dividing by 2, we have

$$\boxed{t = \pi\sqrt{\frac{a_e^3}{gr_e^2}} = 10{,}600\,\text{s} = 2.95\,\text{hr}.}$$

Solution to 5.107

To provide the plots required by the problem statement, we need general expressions for the quantities to be plotted. We will first derive the expressions in question and then provide the corresponding plots.

As stated in the problem, we denote the radii at perigee and apogee for the transfer orbit by r_P and r_A, respectively. The radius r_P is also the radius of the initial circular parking orbit and the radius r_A is also the radius of the final circular orbit. Now , let a_i, a_e, and a_f be the semimajor axes of the initial, transfer, and final orbits, respectively. Therefore we have

$$a_i = r_P, \quad a_e = \tfrac{1}{2}(r_P + r_A), \quad \text{and} \quad a_f = r_A. \tag{12}$$

Next, let v_1, v_2, v_3 and v_4 denote the speeds of the satellite corresponding to the initial circular orbit, the speed needed at perigee to move onto the the transfer orbit, the speed on the transfer orbit at apogee, and the speed at apogee needed to move onto the final orbit. Also, recall that the relation between speed, orbita radius r, and semimajor axis a (for an elliptic orbit) is given by Eq. (5.134) on p. 416 of the textbook, which states

$$v = \sqrt{Gm_B\left(\frac{2}{r} - \frac{1}{a}\right)}, \tag{13}$$

where G is the universal gravitational constant and m_B is the mass of the body about which the satellite is orbiting. In our problem, the body B is the Earth and the term Gm_B can be replaced by the term gr_e^2 where $r_e = 3959\,\text{mi}$ is the radius of the Earth and $g = 32.2\,\text{ft/s}^2$ is the acceleration due to gravity on the surface of the Earth. Hence, applying Eq. (13), we can determine the speeds v_1, v_2, v_3 and v_4 as follows:

$$v_1 = \sqrt{\frac{gr_e^2}{r_P}}, \quad v_2 = \sqrt{gr_e^2\left(\frac{2}{r_P} - \frac{1}{a_e}\right)}, \quad v_3 = \sqrt{gr_e^2\left(\frac{2}{r_A} - \frac{1}{a_e}\right)}, \quad \text{and} \quad v_4 = \sqrt{\frac{gr_e^2}{r_A}}. \tag{14}$$

The above results allow us to derive expressions for $\Delta v_P = v_2 - v_1$ and $\Delta v_A = v_4 - v_3$. Specifically, using the second of Eqs. (12), we have

$$\Delta v_P = \sqrt{gr_e^2\left(\frac{2}{r_P} - \frac{2}{r_P + r_A}\right)} - \sqrt{\frac{gr_e^2}{r_P}} \quad \text{and} \quad \Delta v_A = \sqrt{\frac{gr_e^2}{r_A}} - \sqrt{gr_e^2\left(\frac{2}{r_A} - \frac{2}{r_P + r_A}\right)}. \tag{15}$$

Now that we have derived relations for Δv_P and Δv_A, we need to derive a relation for the time t needed for the orbital transfer. Since the spacecraft only completes one half of the elliptical orbit during transfer, then t is half of the orbital period τ. In turn, the orbital period is found using Eq. (5.126) on p. 415 of the textbook, which gives,

$$\tau = 2\pi\sqrt{\frac{a^3}{gr_e^2}}.$$

So that we have

$$t = \pi\sqrt{\frac{(r_P + r_A)^3}{8gr_e^2}}. \tag{16}$$

Using a mathematical software package we can now plot the quantities Δv_P, Δv_A, and t. We have used Mathematica and the code below.

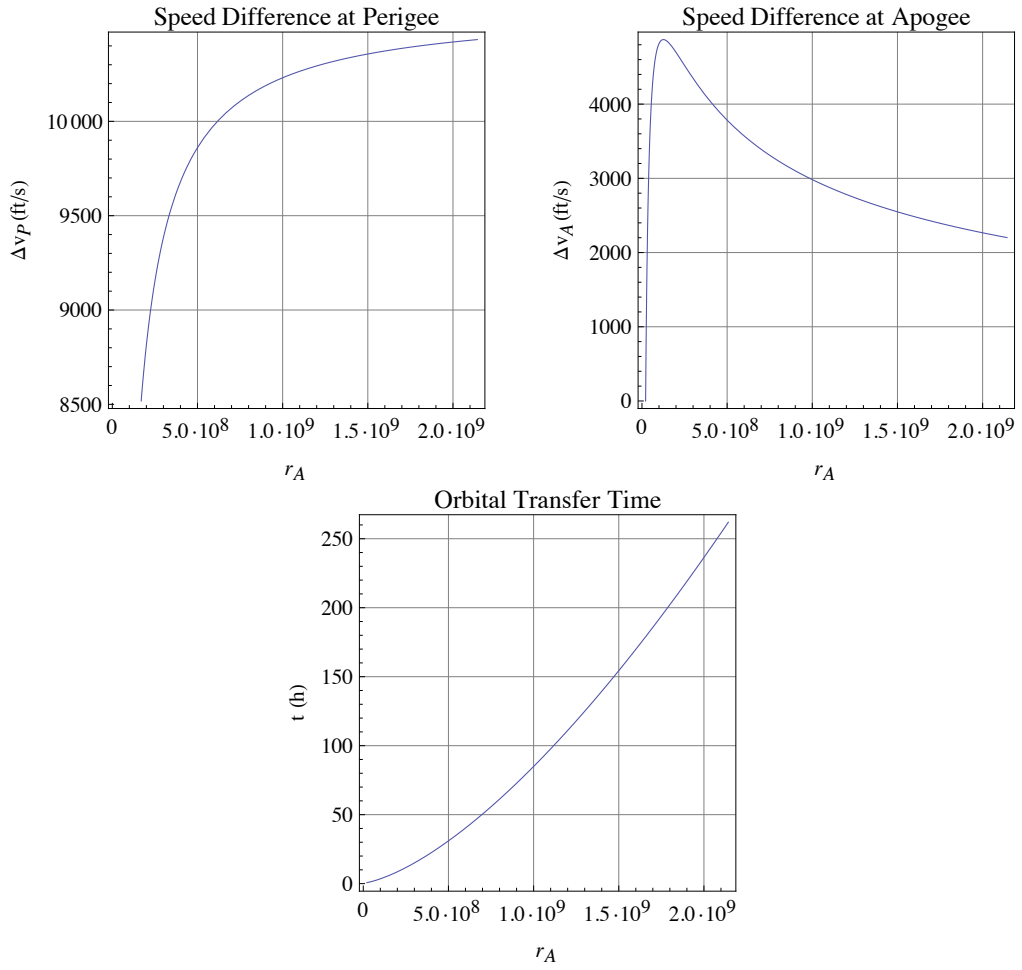
```
Parameters = {rP → (100. + 3959.) 5280., g → 32.2, re -> 3959. · 5280.};

ΔvP = √(g re² (2/rP - 2/(rP + rA))) - √(g re²/rP) ; ΔvA = √(g re²/rA) - √(g re² (2/rA - 2/(rP + rA))) ;

t = π √((rP + rA)³/(8 g re²)) ;

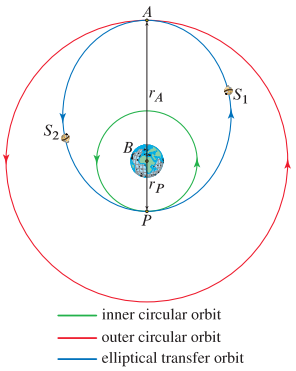
Plot[ΔvP /. Parameters, {rA, (100. + 3959.) 5280., 100. (100. + 3959.) 5280.},
 Frame → True, FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"r_A (ft)", "Δv_P (ft/s)"},
 PlotLabel → "Speed Difference at Perigee"]
Plot[ΔvA /. Parameters, {rA, (100. + 3959.) 5280., 100. (100. + 3959.) 5280.},
 Frame → True, FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"r_A (ft)", "Δv_A (ft/s)"},
 PlotLabel → "Speed Difference at Apogee"]
Plot[(t / 3600.) /. Parameters, {rA, (100. + 3959.) 5280., 100. (100. + 3959.) 5280.},
 Frame → True, FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"r_A (ft)", "t (h)"}, PlotLabel → "Orbital Transfer Time"]
```

Using the above plot commands, we obtain the required plots.



Problem 5.108

Referring to the description given for Probs. 5.105–5.107, for a Hohmann transfer from an inner circular orbit to an outer circular orbit, what would you expect to be the signs on the change in speed at periapsis and at apoapsis?

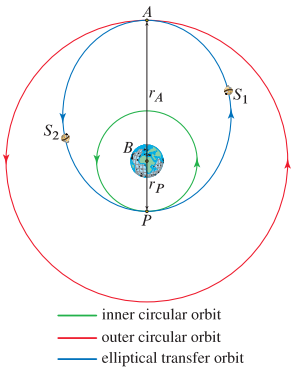


Solution

In both cases we expect the signs to be positive. In fact, when going from an inner circular orbit to an outer circular orbit via a Hohmann transfer, the spacecraft is going from periapsis to apoapsis in the elliptical transfer orbit. To move to a larger distance from the central body, the spacecraft must speed up during the periapsis maneuver. Once the spacecraft reaches apoapsis, if it does not speed up again to go into the outer orbit, it will fall back toward the central body—so it must gain energy again at apoapsis.

Problem 5.109

Referring to the description given for Probs. 5.105–5.107, for a Hohmann transfer from an outer circular orbit to an inner circular orbit, what would you expect to be the signs on the change in speed at periapsis and at apoapsis?

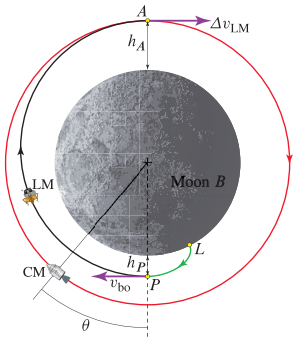


Solution

In both cases we expect the signs to be negative. In fact, when going from an outer circular orbit to an inner circular orbit via a Hohmann transfer, the spacecraft is going from apoapsis to periapsis in the elliptical transfer orbit. To begin to "fall" toward the central body during the apoapsis maneuver, the spacecraft must loose energy and slow down. At periapsis, if the spacecraft does not reduce its energy and slow down again, it will head back out to the apoapsis point — therefore to move into the inner circular orbit it must slow sown again.

Problem 5.110

During the Apollo missions, while the astronauts were on the Moon with the lunar module (LM), the command module (CM) would fly in a circular orbit around the Moon at an altitude of 60 mi. After the astronauts were done exploring the Moon, the LM would launch from the Moon's surface (at L) and undergo powered flight until burnout at P, which occurred when the LM was approximately 15 mi above the surface of the Moon with its velocity v_{bo} parallel to the surface of the Moon (i.e., at periapsis). It would then fly under the influence of the Moon's gravity until reaching apoapsis A, at which point it would rendezvous with the CM. The radius of the Moon is 1079 mi, and its mass is 0.0123 times that of the Earth.



(a) Determine the required speed v_{bo} at burnout P.

(b) What is the change in speed Δv_{LM} required of the LM at the rendezvous point A?

(c) Determine the time it takes the LM to travel from P to A.

(d) In terms of the angle θ, where should the CM be when the LM reaches P so that they can rendezvous at A?

Assume that the changes in speed are impulsive; that is, they occur instantaneously.

Solution

Part (a). Using Eq. (5.134) on p. 416 of the textbook, the speed at burnout v_{bo} is the speed at periapsis in the transfer orbit, i.e.,

$$v_{\text{bo}} = \sqrt{Gm_m\left(\frac{2}{r_P} - \frac{1}{a}\right)}, \tag{1}$$

where m_m is the mass of the moon. Now,

$$Gm_m = G(0.0123m_e) = 0.0123Gm_e = 0.0123gr_e^2, \tag{2}$$

m_e is the mass of the Earth, $g = 32.2\,\text{ft/s}^2$ is the acceleration due to gravity on the surface of the Earth, and $r_e = 3959\,\text{mi} = 2.090\times10^7$ ft is the radius of the Earth. In addition,

$$r_P = r_m + h_P = 5.776\times10^6\,\text{ft} \quad \text{and} \quad r_A = r_m + h_A = 6.014\times10^6\,\text{ft}, \tag{3}$$

where we have used the following numerical data: $r_m = 1079\,\text{mi} = 5.697\times10^6$ ft, $h_P = 15\,\text{mi} = 7.920\times10^4$ ft, and $h_A = 60\,\text{mi} = 3.168\times10^5$ ft. Therefore, since

$$a = (r_A + r_P)/2 = 5.895\times10^6\,\text{ft}, \tag{4}$$

going back to Eqs. (1) and (2), we have

$$\boxed{v_{\text{bo}} = 5530\,\text{ft/s} = 3770\,\text{mph}.}$$

Part (b). Now the LM must go from its speed at apoapsis to the speed of the CM, which is in a circular orbit about the Moon with an altitude of 60 mi. Replacing r_P with r_A in Eq. (1) (and using Eq. (2) as well), the speed of the LM at apoapsis is

$$(v_{\text{LM}})_A = 5309\,\text{ft/s} = 3620\,\text{mph}, \tag{5}$$

where we have used the numerical values indicated in Part (a). The speed of the CM in the circular orbit is obtained from Eq. (1) (and Eq. (2) as well) by setting both r_P and a equal to r_A. This gives

$$v_{\text{CM}} = 5363\,\text{ft/s} = 3657\,\text{mph}. \tag{6}$$

Therefore, the change in speed of LM is

$$\boxed{\Delta v_{\text{LM}} = 54.0\,\text{ft/s} = 37.0\,\text{mph}.}$$

Part (c). Let the time it takes for the LM to go from P to A be denoted by t. This time is equal to half of the period τ of the elliptical transfer orbit. In turn, the period of an elliptical orbit is given by is given by Eq. (5.126) on p. 415 of the textbook. Hence, we have

$$t = \tfrac{1}{2}\tau = \pi\sqrt{\frac{a^3}{Gm_m}} = 3419\,\text{s} = 0.9496\,\text{h}, \tag{7}$$

where we have used Eqs. (2) and (4), as well as the fact that $g = 32.2\,\text{ft/s}^2$ and $r_e = 3959\,\text{mi} = 2.090{\times}10^7$ ft. Expressing t to three significant figures, we have

$$\boxed{t = 3420\,\text{s} = 0.950\,\text{h}}$$

Part (d). Now we know that the CM must travel for 3419 s in a circular orbit with a speed of 5363 ft/s. Therefore the distance traveled d is

$$d = v_{\text{CM}}t = 1.834{\times}10^7\,\text{ft}. \tag{8}$$

To determine the angle traveled β, we write

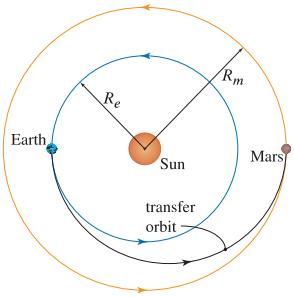
$$d = r_A\beta \quad \Rightarrow \quad \beta = \frac{d}{r_A} = 3.050\,\text{rad} = 174.7^\circ, \tag{9}$$

where we have used the numerical values in the last of Eqs. (3) and (8). Consequently we have

$$\boxed{\theta = 180^\circ - \beta = 5.30^\circ.}$$

Problem 5.111

One option when traveling to Mars from the Earth is to use a Hohmann transfer orbit like that described in Probs. 5.105–5.109. Assuming that the Sun is the primary gravitational influence and ignoring the gravitational influence of Earth and Mars (since the Sun accounts for 99.8% of the mass of the solar system), determine the change in speed required at the Earth Δv_e (perihelion in the transfer orbit) and the required change in speed at Mars Δv_m (aphelion in the transfer orbit) to accomplish the mission to Mars using a Hohmann transfer. In addition, determine the amount of time τ it would take for orbital transfer. Use 1.989×10^{30} kg for the mass of the Sun, assume that the orbits of Earth and Mars are circular, and assume that the changes in speed are impulsive, that is, they occur instantaneously. In addition, use 150×10^{6} km for the radius of Earth's orbit and 228×10^{6} km for the radius of Mars' orbit.



Solution

We start with determining the circular orbit speed (see Eq. (5.111) on p. 413 of the textbook) and the elliptical orbit speed (see Eq. (5.134) on p. 416 of the textbook) at aphelion corresponding to a distance from the Sun equal to that between the Earth and the Sun, i.e.,

$$(v_e)_{\text{circ}} = \sqrt{\frac{Gm_s}{R_e}} = 29{,}750\,\text{m/s} = 107{,}100\,\text{km/h}, \tag{1}$$

$$(v_e)_{\text{ell}} = \sqrt{Gm_s\left(\frac{2}{R_e} - \frac{2}{R_e + R_m}\right)} = 32{,}670\,\text{m/s} = 117{,}600\,\text{km/h}, \tag{2}$$

where we have used the following numerical data: $G = 6.674\times10^{-11}\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$, $m_s = 1.989\times10^{30}$ kg, $R_e = 150\times10^{9}$ m, and $R_m = 228\times10^{9}$ m. Therefore we have

$$\boxed{\Delta v_e = 2930\,\text{m/s} = 10{,}500\,\text{km/h}.}$$

Next we determine the circular orbit and aphelion speeds at Mars, i.e.,

$$(v_m)_{\text{circ}} = \sqrt{\frac{Gm_s}{R_m}} = 24{,}130\,\text{m/s} = 86{,}870\,\text{km/h}, \tag{3}$$

$$(v_m)_{\text{ell}} = \sqrt{Gm_s\left(\frac{2}{R_e} - \frac{2}{R_e + R_m}\right)} = 24{,}130\,\text{m/s} = 86{,}870\,\text{km/h}, \tag{4}$$

where we have used the appropriate numerical values listed earlier. Therefore we have

$$\boxed{\Delta v_m = 2630\,\text{m/s} = 9480\,\text{km/h}.}$$

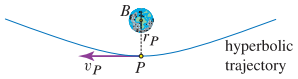
Finally, we determine the time needed for the orbital transfer. This time is half of the full elliptical orbit period. Hence, using Eq. (5.126) on p. 415 of the textbook, we have

$$\boxed{\tau = 2\pi\sqrt{\frac{(R_e + R_m)^3}{8Gm_s}} = 2.24\times10^{7}\,\text{s} = 259\,\text{days},}$$

where we have used the data listed earlier again.

Problem 5.112

Use the work-energy principle applied between periapsis P and $r = \infty$, along with the potential energy of the force of gravity given in Eq. (4.35).



(a) Show that a satellite on a hyperbolic trajectory arrives at $r = \infty$ with speed

$$v_\infty = \sqrt{\frac{r_P v_P^2 - 2Gm_B}{r_P}}.$$

(b) In addition, using Eqs. (5.106) and (5.109), show that for a hyperbolic trajectory, $r_P v_P^2 > 2Gm_B$, which means that the square root in the above equation must always yield a real value.

Solution

Part (a). Applying the work-energy principle between periapsis and $r = \infty$ in a hyperbolic orbit, we get

$$T_P + V_P = T_\infty + V_\infty, \tag{1}$$

where the subscripts P and ∞ denote periapsis and $r = \infty$, respectively. The terms in Eq. (1) are given by

$$T_P = \tfrac{1}{2}mv_P^2, \quad T_\infty = \tfrac{1}{2}mv_\infty^2, \quad V_P = -\frac{Gm_B m}{r_P}, \quad \text{and} \quad V_\infty = 0. \tag{2}$$

Substituting the terms in Eq. (2) into Eq. (1), and the solving for for v_∞, we have

$$\tfrac{1}{2}v_P^2 - \frac{Gm_B}{r_P} = \tfrac{1}{2}v_\infty^2 \quad \Rightarrow \quad \boxed{v_\infty = \sqrt{\frac{r_P v_P^2 - 2Gm_B}{r_P}}.}$$

Part (b). We start by repeating here for convenience Eq. (5.106) (on p. 412 of the textbook) and Eq. (5.109) (on p. 413 of the textbook), which, in order, are

$$e = \frac{C\kappa^2}{Gm_B} \quad \text{and} \quad C = \frac{1}{r_P}\left(1 - \frac{Gm_B}{r_P v_P^2}\right). \tag{3}$$

Now recall that, for a hyperbolic orbit $e > 1$. Hence, from the first of Eqs. (3) we have

$$C\kappa^2 > Gm_B.$$

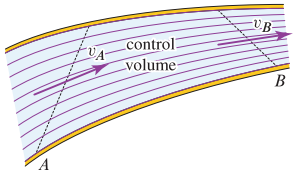
Using this result and the fact that $\kappa = r_P v_P$ (see, for example, Eq. (5.108) on p. 412 of the textbook), the second of Eqs. (3) can be written as

$$C\kappa^2 = \frac{\kappa^2}{r_P}\left(1 - \frac{Gm_B}{r_P v_P^2}\right) > Gm_B \quad \Rightarrow \quad C\kappa^2 = r_P v_P^2\left(1 - \frac{Gm_B}{r_P v_P^2}\right) > Gm_B \quad \Rightarrow$$

$$\boxed{r_P v_P^2 > 2Gm_B.}$$

Problem 5.113

A fluid is in steady motion in the conduit shown. The lines depicted are tangent to the velocity of the fluid particles in the conduit (these lines are called streamlines). Explain whether or not the control volume defined by the cross sections A and B in the figure is consistent with the assumptions laid out in this section.

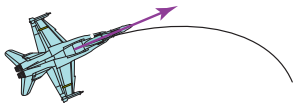


Solution

The control volume defined in the figure is *not* consistent with the assumptions laid out in this section. The theory presented demands that cross sections be perpendicular to the velocity of the fluid particles. In this case the cross section should be perpendicular to each streamline.

Problem 5.114

A hydraulic system is being used to actuate the control surfaces of a plane. Suppose that there is a time interval during which (a) the speed of the hydraulic fluid within a particular line is constant relative to the line itself and (b) the plane is performing a turn. Explain whether or not the force balance for control volumes presented in this section is applicable to the analysis of the hydraulic fluid in question.

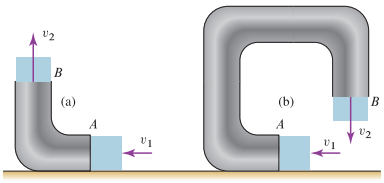


Solution

The theory presented in Section 5.5 (starting on p. 426) of the textbook is not applicable to the case discussed in the problem because the hydraulic line in question is accelerating due to the maneuver of the plane. Therefore, no control volume can be defined that is stationary relative to an inertial frame.

Problem 5.115

The cross sections labeled A and B in case (a) are identical to the corresponding cross sections in case (b). Assume that, in both (a) and (b), a fluid in steady motion flows through A with speed v_1 and exits the system at B with a speed v_2. If the pipe sections are to remain stationary and if the mass flow rate is identical in the two cases, determine whether the magnitude of the horizontal force acting on the pipes due to the water flow in case (a) is smaller than, equal to, or larger than that in case (b). In addition, for both (a) and (b), establish the direction of the force.

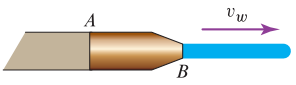


Solution

The horizontal force is the same in both cases because the change in horizontal momentum of the fluid is the same in both cases. In both cases the horizontal force acting on the pipes due to the water flow is directed to the left.

Problem 5.116

Experience tells us that when a steady water jet comes out of a nozzle, the line attached to the nozzle is in tension, that is, the nozzle exerts a force on the line that is in the direction of the flow. If the end of the nozzle at B were capped to stop the water flow, would the force exerted by the nozzle on the line decrease, stay the same, or increase?

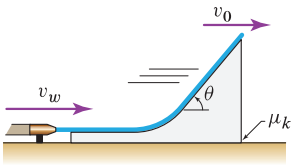


Solution

The force will increase. The force that the nozzle exerts on the line is equal and opposite to the force that the water exerts on the nozzle. When the water is not allowed to flow, the water exerts a force on the nozzle that must balance the effect of the static water pressure. When the water is stopped from flowing, and given that the cross section at B is smaller than that at A, the change in momentum of the water will cause an increase in the force that the water exerts on the nozzle. This causes an increase in the force that the nozzle exerts on the line.

Problem 5.117

Revisit Example 5.17 and use the numerical result in Eq. (13) of the example, along with the fact that the specific weight of water is $62.4\,\text{lb/ft}^3$, to determine the volumetric flow rate at the nozzle and the nozzle diameter.



Solution

Recall that the if $\dot{m}_f$ is the mass flow rate and Q is the corresponding volumetric flow rate, these quantities are related by the equation $\dot{m}_f = \rho Q$, where ρ is the mass density of the fluid. Applying this relation to the problem at hand, we have

$$(\dot{m}_f)_{\text{nz}} = \rho Q_{\text{nz}} \quad \Rightarrow \quad Q_{\text{nz}} = \frac{g(\dot{m}_f)_{\text{nz}}}{\gamma}, \tag{1}$$

where $\gamma = \rho g$ is the specific weight of the fluid. From Eq. (13) (p. 433 of the textbook) in Example 5.17, we have that $(\dot{m}_f)_{\text{nz}} = 7.10\,\text{slug/s}$. In addition, the problem s statement gives $\gamma = 62.4\,\text{lb/ft}^3$. Hence, recalling that $g = 32.2\,\text{ft/s}^2$, we have that

$$Q_{\text{nz}} = 3.664\,\text{ft}^3/\text{s}, \tag{2}$$

which, when expressed via three significant figures, becomes

$$\boxed{Q_{\text{nz}} = 3.66\,\text{ft}^3/\text{s}.}$$

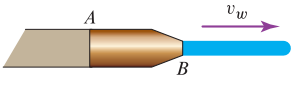
Next, recall that the volumetric flow rate is related to the speed v of the flow and the area of the cross section S via the relation $Q = vS$. In the case of our nozzle, the cross section is circular, so that $S = \pi d^2/4$. Finally, recall that in statement of Example 13 (on p. 433 of the textbook), the speed of the water flow emitted by the nozzle was given as $v_w = 65\,\text{ft/s}$. Hence, we have

$$Q_{\text{nz}} = v_w S \quad \Rightarrow \quad Q_{\text{nz}} = \tfrac{1}{4} v_w \pi d^2 \quad \Rightarrow \quad \boxed{d = 2\sqrt{\frac{Q_{\text{nz}}}{v_w \pi}} = 0.268\,\text{ft},}$$

where we have used the numerical value for Q_{nz} given in Eq. (2).

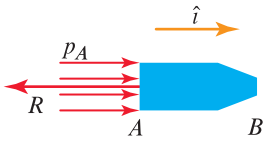
Problem 5.118

The tip B of a nozzle is 1.5 in. in diameter whereas the diameter at A where the line is attached is 3 in. If water is flowing through the nozzle at 95 gpm ("gpm" stands for gallons per minute, 1 U.S. gallon is defined as 231 in.^{3}) and the water static pressure in the line is 300 psi, determine the force necessary to hold the nozzle stationary. Recall that the specific weight of water is $\gamma = 62.4\,\text{lb/ft}^3$, and neglect the atmospheric pressure at B.



Solution

Since we need to determine a force acting in the horizontal direction, and given the size of the nozzle, we neglect the weight of the water contained in the nozzle. Also, observing that the water flow is steady, we select as our control volume the interior of volume of the nozzle, delimited by the cross sections at A and B. The FBD for such a system is that shown on the right, where p_A is the static pressure at A (the static pressure at B is atmospheric pressure and it is neglected). The force R is the force exerted by the nozzle on the control volume. In turn, $-R$ would be the force that the water in the control volume exerts on the nozzle. This force would have to be equilibrated by an equal and opposite force exerted on the nozzle by the line. In conclusion, the force R coincides with the reaction force exerted by the line on the nozzle.



Based on our modeling choices, the force balance for our control volume in the horizontal direction yields

$$\sum F_x: \quad p_A S_A - R = \dot{m}_f (v_{Bx} - v_{Ax}), \tag{1}$$

where S_A is the area of the cross sections at A, $\dot{m}_f$ is the mass flow rate through the nozzle, and v_{Ax} and v_{Bx} are the horizontal components of the flow's velocities at A and B, respectively. Since we are given information about the volumetric flow rate Q, and given that the mass density ρ of water is constant, we can relate the problem's data to the mass flow rate as follows:

$$\dot{m}_f = \rho Q = \frac{\gamma}{g} Q, \tag{2}$$

where $\rho = \gamma/g$ since γ is the specific weight and g is the acceleration due to gravity.

For later use we report here the conversion of the value of Q from gallons per minute into cubic feet per second:

$$Q = 95\,\text{gpm} = \left(\frac{95.00\,\text{gal}}{1.000\,\text{min}}\right)\left(\frac{1.000\,\text{min}}{60.00\,\text{s}}\right)\left(\frac{0.1337\,\text{ft}^3}{1.000\,\text{gal}}\right) = 0.2117\,\text{ft}^3/\text{s}. \tag{3}$$

Now, since the volumetric rate is also constant and the cross sections A and B are perpendicular to the horizontal direction, we have

$$Q = v_{Ax} S_A = v_{Bx} S_B \quad \Rightarrow \quad v_{Ax} = \frac{4Q}{\pi d_A^2} \quad \text{and} \quad v_{Bx} = \frac{4Q}{\pi d_B^2}. \tag{4}$$

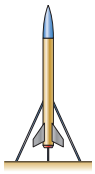
Substituting the result in Eq. (2) and the expressions for v_{Ax} and v_{Bx} in Eqs. (4) into Eq. (1), we obtain an equation in the unknown R whose solution is

$$R = \tfrac{1}{4} p_A \pi d_A^2 + \frac{4\gamma Q^2}{\pi g}\left(\frac{1}{d_A^2} - \frac{1}{d_B^2}\right) = 2120\,\text{lb},$$

where we have used the following numerical values for the problem's data: $p_A = 300\,\text{psi} = 43{,}200\,\text{lb/ft}^2$, $d_A = 3.00\,\text{in.} = 0.2500\,\text{ft}$, $d_B = 1.50\,\text{in.} = 0.125\,\text{ft}$, $\gamma = 62.4\,\text{lb/ft}^3$, and $g = 32.2\,\text{ft/s}^2$.

Problem 5.119

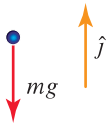
The rocket shown has 7 lb of propellant with a burnout time (time required to burn all the fuel) of 7 s. Assume that the mass flow rate is constant and that the speed of the exhaust relative to the rocket is also constant and equal to 6500 ft/s. If the rocket is fired from rest, determine the initial weight of the body for the rocket if the rocket is to experience an initial acceleration of $6g$.



Solution

We model the rocket as an open system with losing mass. Letting m denote the mass of the rocket, we have

$$m = \frac{W_b}{g} + \frac{W_f}{g}, \tag{1}$$



where W_b is the (constant) weight of the body and W_f is the (variable) weight of the fuel. Referring to the FBD shown to the right, and writing the force balance (in the force appropriate for variable mass systems) we have

$$\sum F_j: \quad -mg = ma_y + \dot{m}_o \vec{v}_o \cdot \hat{\jmath} \quad \Rightarrow \quad -mg = ma_y - \dot{m}_o v_o. \tag{2}$$

where m_o is the mass outflow rate, $\vec{v}_o = -v_o\,\hat{\jmath}$, with $v_o = 6500$ ft/s, is the velocity of the exhaust relative to the rocket. Since we are to assume that the rate of fuel consumption $\dot{m}_o$ is constant, denoting $t_{\text{bo}} = 7$ s the burnout time, we have

$$\dot{m}_o = \frac{m_f}{t_{\text{bo}}} = \frac{W_f}{g t_{\text{bo}}}, \tag{3}$$

where $W_f = 7$ lb is the fuel's weight.

Substituting Eqs. (1) and (3) into the last of Eqs. (2) and solving for for W_b, we have

$$W_b = \frac{g}{a_y + g}\left(\frac{v_o}{g t_{\text{bo}}} - 1 - \frac{a_y}{g}\right) W_f. \tag{4}$$

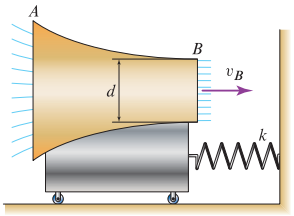
Recalling that at $t = 0$, $W_f = 7$ lb and $a_y = 6g$, the above equation yields the following result:

$$\boxed{W_b = 21.8 \text{ lb},}$$

where we have used the following additional numerical data: $v_o = 6500$ ft/s and $t_{\text{bo}} = 7$ s.

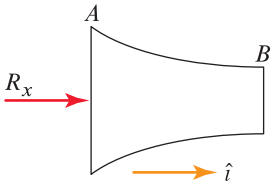
Problem 5.120

An intubed fan is mounted on a cart connected to a fixed wall via a linear elastic spring with constant $k = 50\,\text{lb/ft}$. Assume that in a test the fan draws air at A with essentially zero speed and that the outgoing flow causes the cart to displace to the left so that the spring is stretched by 0.5 ft from its unstretched position. Assuming that the specific weight of the air $\gamma = 7.5\times10^{-2}\,\text{lb/ft}^3$ is constant, and letting the diameter of the tube at B be $d = 4\,\text{ft}$ (the cross section is assumed circular), determine the airspeed at B.



Solution

We begin by observing that the airflow is steady. We select as our control volume the interior volume of the tube delimited by the (vertical) cross sections at A and B. The FBD of the chosen control volume is as shown on the right, where R_x is the horizontal force acting on the airflow do to the fan. Hence, focusing on the horizontal direction, the force balance for the chosen control volume is



$$\sum F_x: \quad R_x = \dot{m}_f(v_{Bx} - v_{Ax}) \quad \Rightarrow \quad R_x = \dot{m}_f v_{Bx}, \tag{1}$$

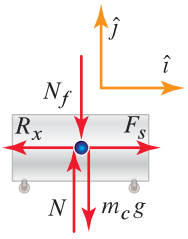
where $\dot{m}_f$ is the mass flow rate, v_{Ax} and v_{Bx} are the horizontal components of the the airflow velocities at A and B, respectively, and where we have set $v_A = 0$ since the fan draws air with essentially zero speed. The mass flow rate is related to the geometry and specific weight of air γ as follows:

$$\dot{m}_f = \rho Q = \frac{\gamma}{g}\tfrac{1}{4}\pi d_B^2 v_{Bx}, \tag{2}$$

where $\rho = \gamma/g$ is the mass density of air, g is the acceleration due to gravity, $\frac{1}{4}\pi d_B^2$ is the cross-sectional area at B, and $Q = \frac{1}{4}\pi d_B^2 v_{Bx}$ is the volumetric air flow through the cross section at B. Substituting Eq. (2) into Eq. (1), we have

$$R_x = \tfrac{1}{4}\pi d_B^2 \frac{\gamma}{g} v_{Bx}^2. \tag{3}$$

Now we consider the FBD (shown on the right) of the cart, modeled as a particle, that supports the intubed fan shown. Here R_x is the force exerted by the tube on the cart, F_s is the force due to the spring, N is the reaction perpendicular to the flow, $m_c g$ is the weight of the cart, and N_f is the normal reaction between the tube and the cart. Focusing on the horizontal direction only, the application of Newton's second law yields



$$\sum F_x: \quad F_s - R_x = 0 \quad \Rightarrow \quad R_x = k\delta, \tag{4}$$

where δ is the stretch of the spring, and we have expressed the spring force as $F_s = k\delta$. Substituting the expression for R_x in Eq. (3) into the last of Eqs. (4) we obtain an equation in v_{Bx} whose solution is

$$v_{Bx} = \frac{2}{d}\sqrt{\frac{gk\delta}{\pi\gamma}}. \tag{5}$$

Finally, given that at B the airflow is all in the horizontal direction, then v_{Bx} coincides with the airspeed at B. Using this consideration along with Eq. (5), we then have

$$\boxed{v_B = 29.2\,\text{ft/s},}$$

where we have used the following numerical values for the quantities appearing on the right-hand side of Eq. (5): $d = 4.00\,\text{ft}$, $g = 32.2\,\text{ft/s}^2$, $k = 50.0\,\text{lb/ft}$, $\delta = 0.50\,\text{ft}$, $\gamma = 7.50\times10^{-2}\,\text{lb/ft}^3$.

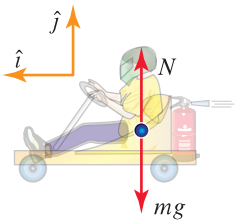
Problem 5.121

A test is conducted in which an 80 kg person sitting in a 15 kg cart is propelled by the jets emitted by two household fire extinguishers with a combined initial mass of 18 kg. The cross section of the exhaust nozzles is 3 cm in diameter, and the density of the exhaust is $\rho = 1.98\,\text{kg/m}^3$. The vehicle starts from rest, and it is determined that the initial acceleration of the "jet cart" is $1.8\,\text{m/s}^2$. Recalling that the mass flow rate out of the nozzle is given by $\dot{m}_o = \rho S v_o$, where S is the area of the nozzle cross section and v_o is the exhaust speed, determine v_o at the initial time. Ignore any resistance to the horizontal motion of the cart.



Solution

We model the cart and all that travels with it as an open system losing mass. Neglecting any resistance to the horizontal motion of the system., we have that the system's FBD is that shown on the right. Accordingly, the force balance in the horizontal direction reads



$$\sum F_x: \quad 0 = ma_x + 2\dot{m}_o \vec{v}_0 \cdot \hat{\imath} \quad \Rightarrow \quad 0 = ma_x - 2\dot{m}_o v_0, \tag{1}$$

where m is the current combined mass of the cart, the rider, and the extinguishers, $\dot{m}_o$ is the mass outflow rate of each extinguisher, and $\vec{v}_o$ is the velocity of the matter ejected by each extinguisher relative to the extinguishers themselves.

To characterize the mass flow rate, we can use the given information about exhaust's density and the area of the cross section of extinguishers' nozzles. Specifically, we can write

$$\dot{m}_o = \rho S v_0 \quad \Rightarrow \quad \dot{m}_o = \tfrac{1}{4}\pi d^2 \rho v_0. \tag{2}$$

Hence, substituting the last of Eqs. (2) into Eq. (1), solving for v_o, we have

$$v_o = \sqrt{\frac{2ma_x}{\rho \pi d^2}}. \tag{3}$$

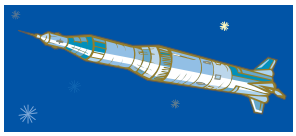
Evaluating the above expression at the initial time, we have

$$\boxed{v_o = 270\,\text{m/s},}$$

where we have used the following numerical data: $m = 80\,\text{kg} + 15\,\text{kg} + 18\,\text{kg} = 113.0\,\text{kg}$, $a_x = 1.80\,\text{m/s}^2$, $\rho = 1.98\,\text{kg/m}^3$, $d = 3.00\,\text{cm} = 0.03000\,\text{m}$.

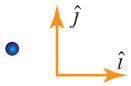
Problem 5.122

Consider a rocket in space so that it can be assumed that no external forces act on the rocket. Let v_o be the speed of the exhaust gases relative to the rocket. In addition, let $m_b + m_f$ and m_b be the total mass of the rocket and its fuel at the initial time and the mass of the body after all the fuel is burned, respectively. If the rocket is fired from rest, determine an expression for the maximum speed that the rocket can achieve.



Solution

We model the rocket as an open system with losing mass. Under this assumption, choosing a coordinate system so the rocket accelerates in the positive $\hat{\imath}$ direction, we see that no external force acts on the rocket (to better understand why the thrust due to the jet pack is not included in the FBD see the discussion of Example 5.19 on p. 436 of the textbook). Hence, summing forces in the x direction for the chosen system we have



$$\sum F_x: \quad 0 = ma_x + \dot{m}_o \vec{v}_o \cdot \hat{\imath} \quad \Rightarrow \quad 0 = ma_x - \dot{m}_o v_o, \tag{1}$$

where m is the current mass of the rocket, $\dot{m}_o$ is the mass outflow rate, and where $\vec{v}_0$ is the velocity of the outflow gases relative to the rocket. Because $\dot{m}_o$ is treated as a positive quantity, re have

$$\dot{m} = -\dot{m}_o. \tag{2}$$

Hence, substituting Eq. (2) into Eq. (1), we have

$$m\ddot{x} = -\dot{m} v_o, \tag{3}$$

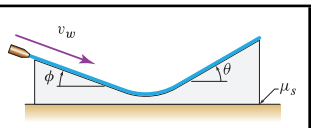
where we used the kinematic equation $\ddot{x} = a_x$. Letting $v_{\max}$ be the final speed achieved by the rocket and recalling that the rocket starts from rest, Eq. (3) can be integrated as follows:

$$\frac{d\dot{x}}{dt} = \frac{-v_o}{m}\frac{dm}{dt} \quad \Rightarrow \quad \int_0^{v_{\max}} d\dot{x} = -v_o \int_{m_b+m_f}^{m_b} \frac{dm}{m} \quad \Rightarrow \quad v_{\max} = -v_o \ln\left(\frac{m_b}{m_b + m_f}\right)$$

$$\Rightarrow \quad \boxed{v_{\max} = v_o \ln\left(1 + \frac{m_f}{m_b}\right).}$$

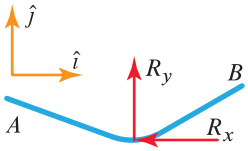
Problem 5.123

A stationary 4 cm diameter nozzle emits a water jet with a speed of 30 m/s. The water jet impinges on a vane with a mass of 15 kg. Recalling that water has a mass density of 1000 kg/m^3, determine the minimum static friction coefficient with the ground such that the vane does not move if $\phi = 20°$ and $\theta = 30°$. Neglect the weight of the water layer in contact with the vane as well as friction between the water and the vane.



Solution

The flow of water out of the nozzle is steady. We select as our control volume the part of the jet that is in contact with the vane. Then, referring to the FBD on the right and applying the force balance for control volumes with steady flows, we have



$$\sum F_x: \quad -R_x = \dot{m}_f(v_{Bx} - v_{Ax}), \tag{1}$$

$$\sum F_y: \quad R_y = \dot{m}_f(v_{By} - v_{Ay}), \tag{2}$$

where R_x and R_y are the forces applied by the vane on the water jet, $\dot{m}_f$ is the water mass flow rate through the control volume, and where $\vec{v}_A = v_{Ax}\,\hat{\imath} + v_{Ay}\,\hat{\jmath}$ and $\vec{v}_B = v_{Bx}\,\hat{\imath} + v_{By}\,\hat{\jmath}$ are the velocity of the water jet at points A and B. We denote the speed of the water jet emitted by the nozzle by v_w. Because the speed of the water jet is not reduced by any friction between the jet and the vane, we have that the speed of the water jet remains constant in going from A to B. In turn, this allows us to write

$$v_{Ax} = v_w \cos\phi, \quad v_{Ay} = -v_w \sin\phi, \quad v_{Bx} = v_w \cos\theta, \quad \text{and} \quad v_{By} = v_w \sin\theta. \tag{3}$$

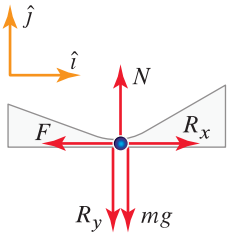
Since we are given information about the water mass density and the diameter of the nozzle, then we can express the mass flow rate $\dot{m}_f$ via the volumetric flow rate Q as follows:

$$\dot{m}_f = \rho Q = \tfrac{1}{4}\rho\pi d^2 v_w. \tag{4}$$

Substituting Eqs. (3) and (4) into Eqs. (1) and (2), we have

$$R_x = \tfrac{1}{4}\rho\pi d^2 v_w^2(\cos\phi - \cos\theta) \quad \text{and} \quad R_y = \tfrac{1}{4}\rho\pi d^2 v_w^2(\sin\theta + \sin\phi). \tag{5}$$

To relate R_x and R_y to the friction needed to hold the vane stationary, referring to the vane's FBD on the right, and writing the equilibrium equations for the vane, we have



$$\sum F_x: \quad R_x - F = 0, \tag{6}$$

$$\sum F_y: \quad N - mg - R_y = 0. \tag{7}$$

Because we need to determine the minimum friction coefficient, we consider an impending slip condition we substitute $F = \mu_s N$ in Eq. (6). Then, solving Eq. (7) for N and substituting the result in Eq. (6), we have

$$R_x = \mu_s(mg + R_y). \tag{8}$$

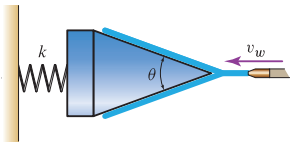
Substituting Eqs. (5) into Eq. (8), we obtain an equation for μ_s whose solution is

$$\mu_s = \frac{\rho\pi d^2 v_w^2(\cos\phi - \cos\theta)}{4mg + \rho\pi d^2 v_w^2(\sin\theta + \sin\phi)} = 0.0758,$$

where we have used the following numerical data: $\rho = 1000\,\text{kg/m}^3$, $d = 4.00\,\text{cm} = 0.0400\,\text{m}$, $\phi = 20.0°$, $\theta = 30.0°$, $m = 15.0\,\text{kg}$, and $v_w = 30.0\,\text{m/s}$.

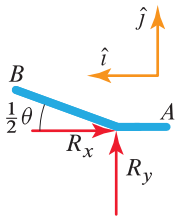
Problem 5.124

A diffuser is attached to a structure whose rigidity in the horizontal direction can be modeled via a linear spring with constant k. The diffuser is hit by a water jet issued with a speed $v_w = 55\,\text{ft/s}$ from a 2 in. diameter nozzle. Assume that the friction between the jet and the diffuser is negligible and that the diffuser's motion in the vertical direction can be neglected. Recalling that the specific weight of water is $\gamma = 62.4\,\text{lb/ft}^3$, if the opening angle of the diffuser is $\theta = 40°$, determine k such that the horizontal displacement of the diffuser does not exceed 0.25 in. from the diffuser's rest position. Assume that the water jet splits symmetrically over the diffuser.



Solution

Because of the symmetry we can analyze one of the two jets in which the main jet splits. Referring to the figure on the right, we model the chosen half-jet as a steady flow system whose control volume is defined by the cross sections at A and B. Then, applying the balance of forces pertaining to the chosen control volume, we have



$$\sum F_x: \quad -R_x = \tfrac{1}{2}\dot{m}_f(v_{Bx} - v_{Ax}), \tag{1}$$

$$\sum F_y: \quad R_y = \tfrac{1}{2}\dot{m}_f(v_{By} - v_{Ay}), \tag{2}$$

where R_x and R_y are the reaction forces exerted y the diffuser on the half-jet, and where $\dot{m}_f$ is the mass flow rate through the cross sections of the control volume. Ultimately, the horizontal reaction is due to the spring on the left side of the diffuser. Hence, keeping in mind that the R_x is only half of the total force provided by the spring, we have

$$R_x = \tfrac{1}{2}k\delta_{\max}, \tag{3}$$

where $\delta_{\max} = 0.25\,\text{in.}$ The kinematic equations for the problem consist in the description of the velocity of the jet at the two cross sections of the control volume, i.e.,

$$v_{Ax} = v_w, \quad v_{Ay} = 0, \quad v_{Bx} = v_w\cos(\theta/2), \quad \text{and} \quad v_{By} = v_w\sin(\theta/2). \tag{4}$$

As far as the mass flow rate $\dot{m}_f$ is concerned, letting ρ be the density of the fluid and Q be the volume flow rate, we recall $\dot{m}_f$ can be written as

$$\dot{m}_f = \rho Q \quad \Rightarrow \quad \dot{m}_f = \frac{\gamma\pi d^2 v_w}{4g}, \tag{5}$$

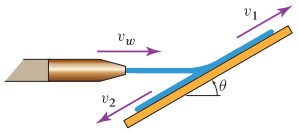
where $d = 2\,\text{in.}$ is the diameter of the nozzle.

Substituting Eqs. (3)–(5) into Eq. (1), we have

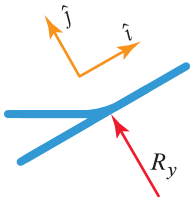
$$k\delta_{\max} = \frac{\gamma\pi d^2 v_w^2}{4g}[1 - \cos(\theta/2)] \quad \Rightarrow \quad \boxed{k = \frac{\gamma\pi d^2 v_w^2}{4g\delta_{\max}}[1 - \cos(\theta/2)] = 370\,\text{lb/ft}.}$$

Problem 5.125

A water jet with a mass flow rate $\dot{m}_f$ at the nozzle impinges with a speed v_w on a fixed flat vane inclined at an angle θ with respect to the horizontal. Assuming that there is no friction between the water jet and the vane, the jet will split into two flows with mass flow rates $\dot{m}_{f1}$ and $\dot{m}_{f2}$. Neglecting the weight of the water, determine how $\dot{m}_{f1}$ and $\dot{m}_{f2}$ depend on $\dot{m}_f$, v_w, and θ. *Hint:* Due to the no-friction assumption, there is no force that slows down the water in the direction tangent to the vane, and this implies that the momentum in that direction is conserved.



Solution



If there were friction between the vane and the split water jet flowing over the vane, there would be forces acting on the split jet that would be tangent to the vane and that therefore would reduce the velocity of water in each of the branches of the split water jet. Therefore, neglecting friction has two important consequences. The first is that, under the assumption that the weight of the water is negligible in this problem, the FBD of a control volume that includes both the incoming jet and the split jet is as shown on the right, i.e., is characterized by a force perpendicular to the vane itself. The second is that, because the flow of water tangent to the vane in not hindered by friction, and given that the reaction force R_y performs no work on the water jet, then the speed of the particles of water in both branches of the split water jet remains equal to the speed they had in the jet that comes out of the nozzle.

Since there is no external force on the control volume in the direction tangent to the vane, then the component of momentum in that direction must be conserved, i.e.,

$$\dot{m}_f v_w \cos\theta = \dot{m}_{f1} v_1 - \dot{m}_{f2} v_2. \tag{1}$$

In addition, as a consequence of the second observation made above, have

$$v_1 = v_2 = v_w. \tag{2}$$

Finally, since the overall mass flow rate must remain constant, we have

$$\dot{m}_f = \dot{m}_{f1} + \dot{m}_{f2}. \tag{3}$$

Substituting Eq. (2) into Eq. (1) we obtain

$$\dot{m}_f \cos\theta = \dot{m}_{f1} - \dot{m}_{f2}. \tag{4}$$

Next, we observe that Eq. (3) and (4) form a system of two equations in the two unknowns $\dot{m}_{f1}$ and $\dot{m}_{f2}$ whose solution is

$$\dot{m}_{f1} = \tfrac{1}{2}\dot{m}_f(1+\cos\theta),$$
$$\dot{m}_{f2} = \tfrac{1}{2}\dot{m}_f(1-\cos\theta).$$

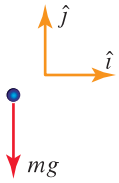
Problem 5.126

A person wearing a jet pack lifts off from rest and ascends along a straight vertical trajectory. Let M denote the initial combined mass of the pilot and the equipment, including the fuel in the pack. Assume that the mass flow rate m_o and exhaust gas speed v_o are known constants and that the pilot can take off as soon as the rocket engine is started. If the exhaust engine is completely directed in the direction of gravity, determine the expression of the pilot's speed as a function of time, M, m_o, v_o, and g (the acceleration due to gravity) while the pack is providing a thrust. Neglect air resistance and assume that gravity is constant.



Solution

We model the person with the jet pack as a system with losing mass. The FBD of the system is shown on the right (to better understand why the thrust due to the jet pack is not included in the FBD see the discussion of Example 5.19 on p. 436 of the textbook). Then, focusing only on the vertical direction, applying the force balance to the open system at hand, we have



$$\sum F_y: \quad -mg = ma_y + \dot{m}_o \vec{v}_o \cdot \hat{j} \quad \Rightarrow \quad -mg = ma_y - \dot{m}_o v_o, \tag{1}$$

where m is the current mass of the system, $\dot{m}_o$ is the mass outflow rate, and $\vec{v}_o$ is the velocity of the outflow gases relative to the system considered. Observe that because $\dot{m}_o$ is understood to be a positive quantity, we must have:

$$\dot{m} = -\dot{m}_o. \tag{2}$$

In addition, since $\dot{m}_o$ is constant, so is $\dot{m}$ and therefore we have that the current value of $m(t)$ is given by

$$m(t) = M - \dot{m}_o t, \tag{3}$$

where M is the initial mass of the system.

Next, recalling that $a_y = \ddot{y}$, substituting Eqs. (2) into Eq. (1), dividing by m and rearranging, we have

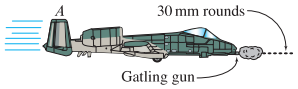
$$-g = \ddot{y} + \frac{\dot{m}}{m} v_o \quad \Rightarrow \quad -g = \frac{d\dot{y}}{dt} + \frac{v_o}{m}\frac{dm}{dt} \quad \Rightarrow \quad -g\,dt = d\dot{y} + \frac{v_o}{m}\,dm \quad \Rightarrow$$
$$-g\int_0^t dt = \int_0^v d\dot{y} + v_o \int_M^{m(t)} \frac{dm}{m} \quad \Rightarrow \quad -gt = v + v_o \ln\left[\frac{m(t)}{M}\right].$$

Substituting $m(t)$ with Eq. (3) and solving for v we obtain

$$\boxed{v = v_o \ln\left(\frac{M}{M - \dot{m}_o t}\right) - gt.}$$

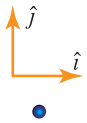
Problem 5.127

A 28,000 lb A-10 Thunderbolt is flying at a constant speed of 375 mph when it fires a 4 s burst from its forward-facing seven-barrel Gatling gun. The gun fires 13.2 oz projectiles at a constant rate of 4200 rounds/min. The muzzle velocity of each projectile is 3250 ft/s. Assume that each of the plane's two jet engines maintains a constant thrust of 9000 lb, that the plane is subject to a constant air resistance while the gun is firing (equal to that before the burst), and that the plane flies straight and level during that time. Determine the plane's change in velocity at the end of the 4 s burst, modeling the airplane's change of mass due to firing as a continuous mass loss.



Solution

From the problem statement we know that the net external force on the A-10 during firing is equal to zero. In addition, we are to model the system at hand an an *open system*. Hence, focusing on the motion in the horizontal direction and summing forces in that direction, we have



$$\sum F_x: \quad 0 = ma_x + \dot{m}_o \vec{v}_o \cdot \hat{\imath} \quad \Rightarrow \quad 0 = ma_x + \dot{m}_o v_o, \tag{1}$$

where $\dot{m}_o$ is the mass outflow rate and $\vec{v}_o$ is the velocity at which the outflowing mass is being ejected from the plane *relative* to the velocity of the plane. Next, since the mass outflow is must be modeled as being continuous, let t_R be the time interval between two consecutive rounds and W_R be the weight of the outflowing matter per round, i.e.,

$$t_R = \frac{60\,\text{s/min}}{4200\,\text{round/min}} = 0.01429\,\text{s/round} \quad \text{and} \quad W_R = \frac{13.2\,\text{oz/round}}{16\,\text{oz/lb}} = 0.8250\,\text{lb/round}. \tag{2}$$

Hence, using the above expressions, we have that the (continuous) mass outflow $\dot{m}_o$ is given by

$$\dot{m}_o = \frac{W_R}{g t_R} = 1.793\,\text{slug/s}. \tag{3}$$

As the plane ejects the rounds at a constant mass rate $\dot{m}_o$, we can then say that the mass of the plane as a function of time is

$$m(t) = m(0) - \dot{m}_o t, \tag{4}$$

where $m(0) = (28{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 869.6\,\text{slug}$ is the initial mass of the plane. Substituting Eqs. (3) and (4) into Eq. (1), solving for the horizontal acceleration of the plane a_x, we have

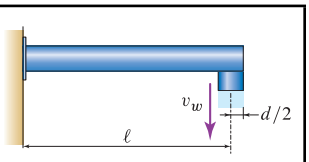
$$a_x = -\frac{\dot{m}_0 v_o}{m(0) - \dot{m}_o t} \quad \Rightarrow \quad \frac{dv_x}{dt} = -\frac{\dot{m}_o v_o}{m(0) - \dot{m}_o t} \quad \Rightarrow \quad \int_{v_{1x}}^{v_{2x}} dv_x = -\dot{m}_o v_o \int_0^{t_f} \frac{dt}{m(0) - \dot{m}_o t} \quad \Rightarrow$$

$$v_{2x} - v_{1x} = v_o \ln\left[\frac{m(0) - \dot{m}_o t_f}{m(0)}\right] \quad \Rightarrow \quad \boxed{\Delta v_x = v_o \ln\left[1 - \frac{\dot{m}_o t_f}{m(0)}\right] = -26.9\,\text{ft/s},}$$

where, in addition to the value indicated in Eq. (3), we have used the following numerical data: $v_o = 3250\,\text{ft/s}$, $t_f = 4\,\text{s}$, and $m(0) = 869.6\,\text{slug}$.

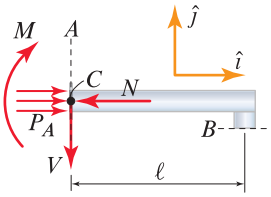
Problem 5.128

A faucet is letting out water at a rate of 15 L/min. Assume that the internal diameter d of the faucet is uniform and equal to 1.5 cm, the distance $\ell = 20$ cm, and the static water pressure at the wall is 0.30 MPa. Neglecting the weight of the water inside the faucet as well as the weight of the faucet itself, determine the forces and the moment that the wall exerts on the faucet. Recall that the density of water is $\rho = 1000\,\text{kg/m}^3$, and neglect the atmospheric pressure at the spout. *Hint:* Define your control volume using a section along the wall.



Solution

The flow of water through the faucet is steady. We select as our control volume the volume defined by the cross sections at A and B. Then, referring to the FBD on the right, applying the balance of forces and moments pertaining to control volumes with steady flows, we have



$$\sum F_x: \quad -N + P_A S_A = \dot{m}_f (v_{Bx} - v_{Ax}), \tag{1}$$

$$\sum F_y: \quad -V = \dot{m}_f (v_{By} - v_{Ay}), \tag{2}$$

$$\sum M_C: \quad -M = \dot{m}_f v_{By} \ell, \tag{3}$$

where N, V, and M represent the force system, with reference point C on the cross section at A, applied to the fluid by the faucet, and where P_A is the static pressure of the fluid at A, S_A is the area of the cross section at A, and $\dot{m}_f$ if the fluid's mass flow rate. The force system consisting of N, V, and M is equal and opposite to the force system exerted on the faucet by the fluid. In turn, this force system must be equilibrated by the force system exerted by the wall on the faucet. In other words, N, V, and M coincide with the force system exerted by the wall on the faucet.

Going back to the solution of the problem, recall that the quantity $\dot{m}_f$ can be related to the given volumetric flow rate $Q = 15\,\text{L/min}$ and the density ρ of the fluid as follows:

$$\dot{m}_f = \rho Q, \quad \text{where} \quad Q = \frac{15\,\text{L/min}}{1000\,\text{L/m}^3} \frac{1}{60\,\text{s/min}} = 0.0002500\,\text{m}^3/\text{s}. \tag{4}$$

In addition, recall that the volumetric flow rate can be related to the velocity of the fluid at A and B as follows:

$$v_{Ax} = \frac{Q}{S_A} = \frac{4Q}{\pi d^2}, \quad v_{Ay} = 0, \quad v_{Bx} = 0, \quad v_{By} = -\frac{Q}{S_B} = -\frac{4Q}{\pi d^2}. \tag{5}$$

Substituting Eqs. (4) and (5) into Eqs. (1)–(3), we obtain a system of three equations in the unknowns N, V, and M whose solution is

$$N = \tfrac{1}{4}\pi P_A d^2 + \frac{4\rho Q^2}{\pi d^2} = 53.4\,\text{N},$$

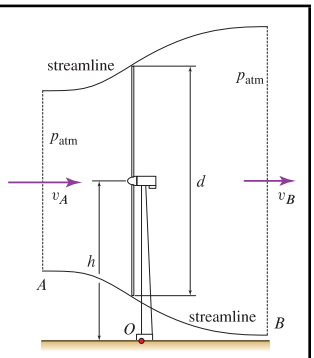
$$V = \frac{4\rho Q^2}{\pi d^2} = 0.354\,\text{N},$$

$$M = \frac{4\rho Q^2 \ell}{\pi d^2} = 0.0707\,\text{N·m},$$

where, in addition to the value of Q in Eq. (4), we have used the following numerical data: $P_A = 0.30\,\text{MPa}$, $d = 1.5\,\text{cm} = 0.01500\,\text{m}$, $\rho = 1000\,\text{kg/m}^3$, and $\ell = 20\,\text{cm} = 0.2000\,\text{m}$.

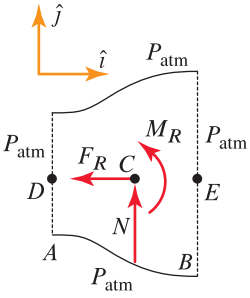
Problem 5.129

Consider a wind turbine with a diameter $d = 110\,\text{m}$ and the airflow streamlines shown, which are symmetric relative to the axis of the turbine. Since the airflow is tangent to the streamlines (by definition), these lines can be taken to define the top and bottom surfaces of a control volume. Suppose that pressure measurements indicate that the flow experiences atmospheric pressure at the cross sections A and B (as well as outside the control volume) where the wind speed is $v_A = 7\,\text{m/s}$ and $v_B = 2.5\,\text{m/s}$, respectively. Furthermore, assume that the average pressure along the streamlines defining the control volume is also atmospheric. Finally, assume that the diameter of the flow cross section at A is 85% of the rotor diameter and that the rotor hub is at a distance $h = 75\,\text{m}$ above the ground. If the density of air is constant and equal to $\rho = 1.25\,\text{kg/m}^3$, determine the force exerted by the air on the wind turbine and the reaction moment at the base of the support.



Solution

The problem statement clearly identifies the control volume to study., whose FBD is shown on the right. Observe that N, F_R, and M_R are reactions acting on the air due to the rotor, where the moment center used for the definition of such force system is point C, i.e., the rotor'shub. Then, using the force and moment balance pertaining to control volumes with steady flows, we have



$$\sum F_x: \quad -F_R = \dot{m}_f(v_{Bx} - v_{Ax}), \tag{1}$$

$$\sum F_y: \quad N = \dot{m}_f(v_{By} - v_{Ay}), \tag{2}$$

$$\sum M_C: \quad M_R = \dot{m}_f(\vec{r}_{E/C} \times \vec{v}_B - \vec{r}_{D/C} \times \vec{v}_A) \cdot \hat{k}, \tag{3}$$

where $\dot{m}_f$ is the air mass flow rate through the control volume, and D and E are the centers of the cross sections A and B, respectively.

The airflow velocity components at A and B are

$$v_{Bx} = v_B, \quad v_{Ax} = v_A, \quad v_{By} = 0, \quad \text{and} \quad v_{Ay} = 0. \tag{4}$$

In addition, observe that the vectors $\vec{r}_{D/C}$ and $\vec{r}_{E/C}$ are parallel to $\vec{v}_A$ and $\vec{v}_B$, respectively. Therefore, we have

$$\vec{r}_{D/C} \times \vec{v}_A = \vec{0} \quad \text{and} \quad (\vec{r}_{E/C} \times \vec{v}_B = \vec{0}. \tag{5}$$

Fianlly, observe that the mass flow rate $\dot{m}_f$ can be related to the volumetric flow rate (for which the problem provides information) as follows:

$$\dot{m}_f = \rho Q = \tfrac{1}{4}\rho\pi(0.85d)^2 v_A. \tag{6}$$

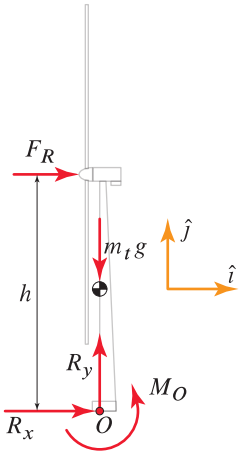
Substituting Eqs. (4)–(6) into Eqs. (1)–(3), we have, $N = 0$, $M_R = 0$, and

$$F_R = \tfrac{1}{4}\rho\pi(0.85d)^2 v_A(v_A - v_B) = 270.4\,\text{kN}, \tag{7}$$

where we have used the following numerical data: $\rho = 1.25\,\text{kg/m}^3$, $d = 110\,\text{m}$, $v_A = 7\,\text{m/s}$, and $v_B = 2.5\,\text{m/s}$. Expressing the above result to three significant figures, we have

$$\boxed{F_R = 270\,\text{kN}.}$$

To determine the reaction moment at the base of the of the turbine's support, we now consider the turbine's FBD shown on the right. In this FBD, we have indicated the force acting on the rotor due to the airflow as well as the reaction force system between the turbine's support and the ground. The moment center for the reaction force system at the turbine's base is point O, which we chose on the line of action of the combined weight of the rotor and the turbine's support. Consequently, the weight of the rotor and its support (the force $m_t g$) provides no moment about point O and writing the equilibrium equation for moments about O yields the following result:

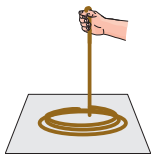


$$\sum M_O: \quad M_O - F_R h = 0 \quad \Rightarrow \quad \boxed{M_O = F_R h = 20.3\times 10^6\,\text{N}\cdot\text{m},}$$

where we have used the following numerical data: $F_R = 270.4\,\text{kN}$ (see Eq. (7)) and $h = 75.0\,\text{m}$.

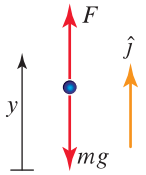
Problem 5.130

A rope with weight per unit length of 0.1 lb/ft is lifted at a constant upward speed $v_0 = 8$ ft/s. Treating the rope as inextensible, determine the force applied to the top end of the rope after it is lifted 9 ft. Assume that the top end of the rope is initially at rest and on the floor. In addition, disregard the horizontal motion associated with the uncoiling of the rope.



Solution

To solve this problem we model the portion of the rope that has been lifted as a variable mass system gaining mass. Hence, referring to the FBD shown to the right, using the appropriate force balance for variable mass systems and focusing only on the vertical motion, we have



$$\sum F_y: \quad F - mg = ma_y - \dot{m}_i \vec{v}_i \cdot \hat{j}, \tag{1}$$

where F is the force with which the rope is being lifted, $\vec{v}_i$ is the velocity with which new mass is being added relative to the velocity of the rope, and $\dot{m}_i$ is the time rate at which mass is being added to the rope. To describe $\dot{m}_i$, let $\rho = (0.1\,\text{lb/ft})(32.2\,\text{ft/s}^2) = 0.003106\,\text{slug/ft}$ be the mass of the rope per unit length. In addition, let the origin of the y axis be correspond to the surface on which the rope is initially resting. Hence, we can express the mass of the lifted rope as a function of y and the corresponding inflow mass rate as follows:

$$m(y) = \rho y \quad \text{and} \quad \dot{m}_i = \dot{m}(y) = \rho \dot{y} = \rho v_0. \tag{2}$$

Now recall that the vertical acceleration of the rope is equal to zero because the rope is lifted at a constant vertical rate. In addition, let $\vec{v}_{\text{mass added}}$ be the absolute velocity of the particles of rope when these particles become part of the part of the rope being lifted. Since the particles in question are at rest at $y = 0$ when they become part of the lifted part of the rope, we can then write

$$\vec{v}_i = \vec{v}_{\text{mass added}} - \vec{v}_{\text{rope}} \quad \Rightarrow \quad \vec{v}_i = \vec{0} - v_0\,\hat{j} \quad \Rightarrow \quad \vec{v}_i = -v_0\,\hat{j}. \tag{3}$$

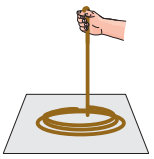
Substituting the expressions for m and $\dot{m}_i$ from Eqs. (2) and the expression for $\vec{v}_i$ from the last of Eqs. (3) into Eq. (1), we have

$$F = \rho g y + \rho v_0^2 \quad \Rightarrow \quad \boxed{F = \rho\left(gL + v_0^2\right) = 1.10\,\text{lb}.}$$

where we have used the following numerical data: $\rho = (0.1\,\text{lb/ft})/(32.2\,\text{ft/s2}) = 0.003106\,\text{slug/ft}$, $g = 32.2\,\text{ft/s}^2$, $y = L = 9\,\text{ft}$, and $v_0 = 8\,\text{ft/s}$.

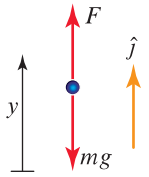
Problem 5.131

A rope with mass per unit length of 0.05 kg/m is lifted at a constant upward acceleration $a_0 = 6\,\text{m/s}^2$. Treating the rope as inextensible, determine the force that must be applied at the top end of the rope after it is lifted 3 m. Assume that the top end of the rope is initially at rest and on the floor. In addition, disregard the horizontal motion associated with the uncoiling of the rope.



Solution

To solve this problem we model the portion of the rope that has been lifted as a variable mass system gaining mass. Hence, referring to the FBD shown to the right, using the appropriate force balance for variable mass systems and focusing only on the vertical motion, we have



$$\sum F_y: \quad F - mg = ma_y - \dot{m}_i \vec{v}_i \cdot \hat{j}, \tag{1}$$

where F is the force with which the rope is being lifted, $\vec{v}_i$ is the velocity with which new mass is being added relative to the velocity of the rope, and $\dot{m}_i$ is the time rate at which mass is being added to the rope. To describe $\dot{m}_i$, let $\rho = 0.05\,\text{kg/m}$ be the mass of the rope per unit length. In addition, let the origin of the y axis be correspond to the surface on which the rope is initially resting. Hence, we can express the mass of the lifted rope as a function of y and the corresponding inflow mass rate as follows:

$$m(y) = \rho y \quad \text{and} \quad \dot{m}_i = \dot{m}(y) = \rho \dot{y}. \tag{2}$$

Now recall that the vertical acceleration of the rope is constant, i.e., $\ddot{y} = a_0$. Hence, recalling that the rope is initially at rest at $y = 0$, that the absolute velocity of the lifted part of the rope is $\vec{v} = \dot{y}\,\hat{j}$, and using constant acceleration equations, we have

$$\dot{y}^2(y) = \dot{y}^2(0) + 2a_0[y - y(0)] = 2a_0 y \quad \Rightarrow \quad \dot{y} = \sqrt{2a_0 y}, \tag{3}$$

where $\dot{y}$ was chosen to be positive since the rope is being lifted. Now, let $\vec{v}_{\text{mass added}}$ be the absolute velocity of the particles of rope when these particles become part of the part of the rope being lifted. Since the particles in question are at rest at $y = 0$ when they become part of the lifted part of the rope, we can then write

$$\vec{v}_i = \vec{v}_{\text{mass added}} - \vec{v} = \vec{v}_{\text{mass added}} - \dot{y}\,\hat{j} \quad \Rightarrow \quad \vec{v}_i = -\sqrt{2a_0 y}\,\hat{j}. \tag{4}$$

Substituting into Eq. (1) the expressions for $m(y)$ and $\dot{m}_i$ in Eqs. (2) and the expression for $\vec{v}_i$ in Eqs. (4), and solving for F, we have

$$F = \rho y(g + 3a_0). \tag{5}$$

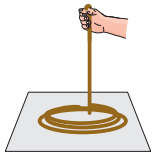
Therefore, for $y = L = 3.00\,\text{m}$, we have

$$\boxed{F = \rho L\,(3a_0 + g) = 4.17\,\text{N},}$$

where we have used the following numerical data: $\rho = 0.05\,\text{kg/m}$, $L = 3\,\text{m}$, $a_0 = 6\,\text{m/s}^2$, and $g = 9.81\,\text{m/s}^2$.

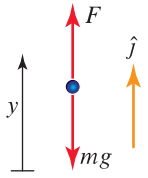
Problem 5.132

A rope with mass per unit length of 0.05 kg/m is lifted by applying a constant vertical force $F = 10$ N. Treating the rope as inextensible, plot the velocity and position of the top end of the string as a function of time for $0 \le t \le 3$ s. Assume that the top end of the rope is initially at rest and 1 mm off the floor. In addition, disregard the horizontal motion associated with the uncoiling of the rope.



Solution

To solve this problem we model the portion of the rope that has been lifted as a variable mass system gaining mass. Hence, referring to the FBD shown to the right, using the appropriate force balance for variable mass systems and focusing only on the vertical motion, we have



$$\sum F_y: \quad F - mg = ma_y - \dot{m}_i \vec{v}_i \cdot \hat{j}, \tag{1}$$

where F is the force with which the rope is being lifted, $\vec{v}_i$ is the velocity with which new mass is being added relative to the velocity of the rope, and $\dot{m}_i$ is the time rate at which mass is being added to the rope. To describe $\dot{m}_i$, let $\rho = 0.05$ kg/m be the mass of the rope per unit length. In addition, let the origin of the y axis be correspond to the surface on which the rope is initially resting. Hence, we can express the mass of the lifted rope as a function of y and the corresponding inflow mass rate as follows:

$$m(y) = \rho y \quad \text{and} \quad \dot{m}_i = \dot{m}(y) = \rho \dot{y}. \tag{2}$$

We now observe that the vertical acceleration of the lifted part of the rope is simply

$$a_y = \ddot{y}. \tag{3}$$

Finally, let $\vec{v}_{\text{mass added}}$ be the absolute velocity of the particles of rope when these particles become part of the part of the rope being lifted. Since the particles in question are at rest at $y = 0$ when they become part of the lifted part of the rope, recalling that the velocity of the lifted part of the rope is $\vec{v} = \dot{y}\,\hat{j}$, we can then write

$$\vec{v}_i = \vec{v}_{\text{mass added}} - \vec{v} = -\dot{y}\,\hat{j}. \tag{4}$$

Substituting into Eq. (1) the expressions for $m(y)$ and $\dot{m}_i$ in Eqs. (2), the expression for a_y in Eqs. (4), and the expression for $\vec{v}_i$ in Eq. (4), we obtain

$$F - \rho g y = \rho y \ddot{y} + \rho \dot{y}^2 \quad \Rightarrow \quad y\ddot{y} + \dot{y}^2 + gy = \frac{F}{\rho},$$

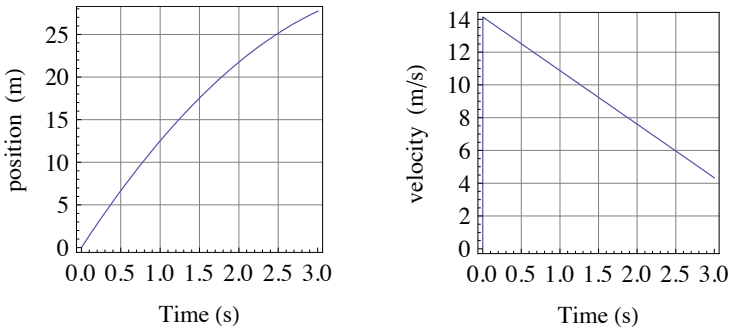
which is the equation of motion for the lifted part of the rope. By solving this equation, we can then predict the position and velocity of the rope as a function of time. This equation, along with its initial conditions, according to which $y(0) = 0$ and $\dot{y}(0) = 0$, can be solved using some mathematical software.

The plots at the end of this problem are based on a numerical solution obtained via *Mathematica* using the code reported here below.

```
rules = {ρ → 0.05, g → 9.81, F → 10};
s = NDSolve[{y[t] y''[t] + y'[t]^2 + g y[t] == F/ρ, y[0] == 1/1000, y'[0] == 0} /. rules,
   {y[t], y'[t]}, {t, 0, 3}];
```

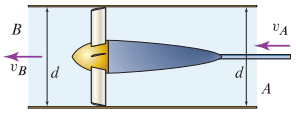
Once the equation was solved, we used the Mathematica code here below to generate the required plots.

```
Plot[Evaluate[y[t] /. s], {t, 0, 3}, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"Time (s)", "position  (m)"}]
Plot[Evaluate[y'[t] /. s], {t, 0, 3}, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"Time (s)", "velocity  (m/s)"}]
```



Problem 5.133

Let p_A and p_B be given static pressure measurements at the cross sections A and B in the air duct shown. Assume that any cross section between A and B is circular with diameter d. Assume that the flow is steady and that the mass density ρ_A at A is known along with v_A, the speed of the flow at A, and v_B, the speed of the flow at B. Determine the expression of mass density at B and the expression of the force F acting on the fan.

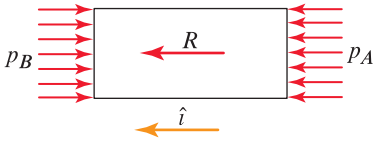


Solution

The flow under consideration is steady and therefore we will solve this problem applying concepts pertaining to steady flows systems. We begin with observing that since the mass flow rate $\dot{m}_f$ is constant and the cross sectional area at A and B are equal, we must have

$$\rho_A v_A S_A = \rho_B v_B S_B \quad \Rightarrow \quad \boxed{\rho_B = \frac{v_A}{v_B}\rho_A.}$$

Next, for the force analysis, we choose as control volume the system contained between the cross sections at A and B. Then, since the duct's walls are parallel and horizontal, no force is exerted in the $\hat{\imath}$ direction by the duct. Therefore R is the force acting on the flow due to the fan. Hence, summing forces in the x direction for the control volume, we have



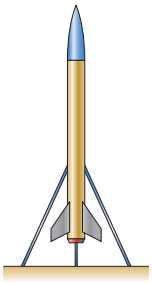
$$\sum F_x: \quad R + p_A S_A - p_B S_B = \dot{m}_f (v_B - v_A) \quad \Rightarrow$$
$$R + \tfrac{1}{4} p_A \pi d^2 - \tfrac{1}{4} p_B \pi d^2 = \tfrac{1}{4} \rho_A v_A \pi d^2 (v_B - v_A).$$

Letting the force applied by the flow to the fan be F and recognizing that $F = -R$, we have

$$\boxed{F = \tfrac{1}{4}\pi d^2 (p_A - p_B) - \tfrac{1}{4}\rho_A v_A \pi d^2 (v_B - v_A).}$$

Problem 5.134

An amateur rocket with a body weight of 6.5 lb is equipped with a rocket engine holding 2.54 lb of solid propellant with a burnout time (time required to burn all the fuel) of 5.25 s (this is the typical data made available by amateur rocket engine manufacturers). The initial thrust is 68 lb. Assuming that the mass flow rate and the speed of the exhaust relative to the rocket remain constant, determine the exhaust mass flow rate m_o and the speed relative to the rocket v_o. In addition, determine the maximum speed achieved by the rocket $v_{\max}$ if the rocket is launched from rest and moves in the direction opposite to gravity. Neglect air resistance and assume that gravity does not change with elevation.



Solution

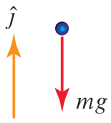
Let the mass of the (empty) body of the rocket be $m_b = W_b/g = (6.5\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.2019\,\text{slug}$. Also, let the initial mass of fuel be $m_f = W_f/g = (2.54\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.07888\,\text{slug}$. Since the burnout time $t_{\text{bo}} = 5.25\,\text{s}$, then the rate of fuel consumption must be

$$\dot{m}_o = m_f/t_{\text{bo}} = 0.01502\,\text{slug/s}, \tag{1}$$

which, expressed to three significant figures is

$$\boxed{\dot{m}_o = 0.0150\,\text{slug/s}.}$$

We now turn to the analysis of the motion of the rocket, which we model an an open system losing mass. The FBD of the rocket is sketched on the right, where, consistent with our modeling of open systems, includes only the weight of the system (to better understand why the thrust due to the jet pack is not included in the FBD see the discussion of Example 5.19 on p. 436 of the textbook). Hence, applying the force balance in the vertical direction, we have



$$\sum F_y: \quad -mg = ma_y + \dot{m}_o \vec{v}_o \cdot \hat{\jmath}, \tag{2}$$

where m is the current mass of the rocket, $\dot{m}_o$ is the mass outflow rate, and $\vec{v}_o$ is the velocity of the outflow gases relative to the body of the rocket. Observe that because $\dot{m}_o$ is understood to be a positive quantity, we must have:

$$\dot{m} = -\dot{m}_o. \tag{3}$$

Next, to determine $\vec{v}_o$, as discussed in the textbook, recall that the term $-\dot{m}_o\vec{v}_o$ is equal to the thrust of the rocket engine, which is a given of the problem. Let the thrust of the rocket be $\vec{T} = T\,\hat{\jmath}$, with $T = 68.0\,\text{lb}$. Then observing that the relative velocity of the exhaust is all in the y direction, i.e., $\vec{v}_o = -v_o\,\hat{\jmath}$, we must have

$$\vec{T} = -\dot{m}_o\vec{v}_o \quad \Rightarrow \quad T = m_o v_o \quad \Rightarrow \quad v_o = T/\dot{m}_o \quad \Rightarrow \quad v_o = 4527\,\text{ft/s}, \tag{4}$$

where we have used the value of $\dot{m}_o$ in Eq. (1). Expressing the result for v_o to three significant figures, we have

$$\boxed{v_o = 4530\,\text{ft/s}.}$$

Going back to the analysis of the motion of the rocket, recalling that $a_y = \dot{v}_y$, substituting the expressions for $\vec{v}_o$ and $\dot{m}_o$ into Eq. (2), we can then write

$$-mg = m\dot{v}_y + \dot{m}v_o \quad \Rightarrow \quad \frac{dv_y}{dt} + \frac{v_o}{m}\frac{dm}{dt} = -g. \tag{5}$$

Letting $v_{\max}$ be the speed at burnout, multiplying both sides of the above equation by dt and integrating, we have

$$\int_0^{v_{\max}} dv_y + v_o \int_{m_b+m_f}^{m_b} \frac{dm}{m} = -\int_0^{t_{\text{bo}}} g\, dt \quad \Rightarrow \quad v_{\max} + v_o \ln\left(\frac{m_b}{m_b+m_f}\right) = -gt_{\text{bo}}.$$

Solving for $v_{\max}$ and simplifing, we then obtain

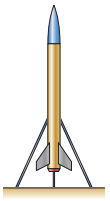
$$\boxed{v_{\max} = v_o \ln\left(1 + \frac{m_f}{m_b}\right) - gt_{\text{bo}} = 1320\,\text{ft/s},}$$

where we have used the following values for the quantities on the right hand side of the above equation: $v_o = 4527\,\text{ft/s}$ (see Eq. (4)), $m_b = 0.2019\,\text{slug}$, $m_f = 0.07888\,\text{slug}$, and $t_{\text{bo}} = 5.25\,\text{s}$.

Problem 5.135

Continue Prob. 5.134 and determine the maximum height reached by the rocket, again neglecting air resistance and changes of gravity with elevation. *Hint:* For $0 < t < t_0$,

$$\int \ln\left(1-\frac{t}{t_0}\right) dt = (t_0 - t)\left[1 - \ln\left(1-\frac{t}{t_0}\right)\right] + C.$$

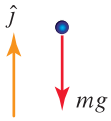


Solution

With reference to the statement of Problem 5.134, let the mass of the (empty) body of the rocket be $m_b = W_b/g = (6.5\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.2019\,\text{slug}$. Also, let the initial mass of fuel be $m_f = W_f/g = (2.54\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.07888\,\text{slug}$. Since the burnout time $t_{\text{bo}} = 5.25\,\text{s}$, then the rate of fuel consumption must be

$$\dot{m}_o = m_f/t_{\text{bo}} = 0.01502\,\text{slug/s}. \tag{1}$$

We now turn to the analysis of the motion of the rocket, which we model an an open system losing mass. The FBD of the rocket is sketched on the right, where, consistent with our modeling of open systems, includes only the weight of the system (to better understand why the thrust due to the jet pack is not included in the FBD see the discussion of Example 5.19 on p. 436 of the textbook). Hence, applying the force balance in the vertical direction, we have



$$\sum F_y: \quad -mg = ma_y + \dot{m}_o \vec{v}_o \cdot \hat{\jmath}, \tag{2}$$

where m is the current mass of the rocket, $\dot{m}_o$ is the mass outflow rate, and $\vec{v}_o$ is the velocity of the outflow gases relative to the body of the rocket. Observe that because $\dot{m}_o$ is understood to be a positive quantity, we must have:

$$\dot{m} = -\dot{m}_o. \tag{3}$$

Next, to determine $\vec{v}_o$, as discussed in the textbook, recall that the term $-\dot{m}_o\vec{v}_o$ is equal to the thrust of the rocket engine, which is a given of the problem. Let the thrust of the rocket be $\vec{T} = T\,\hat{\jmath}$, with $T = 68.0\,\text{lb}$. Then observing that the relative velocity of the exhaust is all in the y direction, i.e., $\vec{v}_o = -v_o\,\hat{\jmath}$, we must have

$$\vec{T} = -\dot{m}_o\vec{v}_o \quad \Rightarrow \quad T = m_o v_o \quad \Rightarrow \quad v_o = T/\dot{m}_o \quad \Rightarrow \quad v_o = 4527\,\text{ft/s}, \tag{4}$$

where we have used the value of $\dot{m}_o$ in Eq. (1). Going back to the analysis of the motion of the rocket, recalling that $a_y = \dot{v}_y$, substituting the expressions for $\vec{v}_o$ and $\dot{m}_o$ into Eq. (2), we can then write

$$-mg = m\dot{v}_y + \dot{m}v_o \quad \Rightarrow \quad \frac{dv_y}{dt} + \frac{v_o}{m}\frac{dm}{dt} = -g. \tag{5}$$

After multiplying the last of Eqs. (5) by dt, recalling that the initial mass of the system is $m(0) = m_b + m_f$ and that the system starts from rest, this equation can be integrated as follows:

$$\int_0^{v_y} dv_y + v_o \int_{m_b+m_f}^{m(t)} \frac{dm}{m} = -g\int_0^t dt \quad \Rightarrow \quad v_y + v_o \ln\left[\frac{m(t)}{m_b+m_f}\right] = -gt,$$

which, given the the fact that $\dot{m}_o$ is constant and therefore we have $m(t) = m_b + m_f - \dot{m}_o t$, can be solved for v_y to obtain

$$v_y = -v_o \ln\left(\frac{m_b + m_f - \dot{m}_o t}{m_b + m_f}\right) - gt.$$

Now observe that the ratio $(m_b + m_f)/\dot{m}_o$ has the dimensions of time. Hence, defining (for convenience) the time t_o as

$$t_o = \frac{m_b + m_f}{\dot{m}_o}, \tag{6}$$

we can rewrite the preceding equation as

$$v_y = -v_o \ln\left(1 - \frac{t}{t_o}\right) - gt \quad \Rightarrow \quad \frac{dy}{dt} = -v_o \ln\left(1 - \frac{t}{t_o}\right) - gt, \tag{7}$$

where we hav used the fact that $v_y = dy/dt$. Multiplying Eq. (7) by dt we can then integrate it with respect to time from $t = 0$ to $t = t_{\text{bo}}$ as follows:

$$\int_0^{y_{\text{bo}}} dy = -v_o \int_0^{t_{\text{bo}}} \ln\left(1 - \frac{t}{t_o}\right) dt - g \int_0^{t_{\text{bo}}} t\, dt.$$

Taking advantage of the hint, we then have

$$y_{\text{bo}} = -v_o \left\{(t_o - t)\left[1 - \ln\left(1 - \frac{t}{t_o}\right)\right]_0^{t_{\text{bo}}}\right\} - \tfrac{1}{2} g t_{\text{bo}}^2$$

$$\Rightarrow \quad y_{\text{bo}} = v_o \left\{(t_o - t_{\text{bo}})\left[\ln\left(1 - \frac{t_{\text{bo}}}{t_o}\right) - 1\right] + t_o\right\} - \tfrac{1}{2} g t_{\text{bo}}^2. \tag{8}$$

Now, recalling that $\dot{m}_o = m_f/t_{\text{bo}} = W_f/(g t_{\text{bo}})$ and recalling the definition of t_o in Eq. (6), we have

$$t_o = t_{\text{bo}}\left(1 + \frac{W_b}{W_f}\right) \quad \text{and} \quad \frac{t_{\text{bo}}}{t_o} = \frac{W_f}{W_b + W_f}. \tag{9}$$

Then, substituting Eqs. (9) into Eq. (8) and simplifying we have

$$y_{\text{bo}} = v_o t_{\text{bo}}\left[1 - \frac{W_b}{W_f} \ln\left(1 + \frac{W_f}{W_b}\right)\right] - \tfrac{1}{2} g t_{\text{bo}}^2 = 3260\,\text{ft}, \tag{10}$$

where we have used the following numerical data: $v_o = 4527\,\text{ft/s}$ (see Eq. (4)), $t_{\text{bo}} = 5.25\,\text{s}$, $W_b = 6.5\,\text{lb}$, $W_f = 2.54\,\text{lb}$, and $g = 32.2\,\text{ft/s}^2$.

Now observe that y_{bo} is the altitude at burnout and, unless the velocity of the rocket at burnout is equal to zero, then the rocket will be able to climb beyond y_{bo} while moving solely under the action of gravity. The velocity at burnout is found by evaluating Eq. (7) at $t = t_{\text{bo}}$, which gives

$$v_{\text{bo}} = v_o \ln\left(1 + \frac{W_f}{W_b}\right) - g t_{\text{bo}} = 1324\,\text{ft/s}, \tag{11}$$

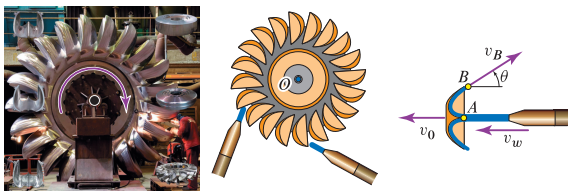
where, again, we have used the following numerical data: $v_o = 4527\,\text{ft/s}$ (see Eq. (4)), $t_{\text{bo}} = 5.25\,\text{s}$, $W_b = 6.5\,\text{lb}$, $W_f = 2.54\,\text{lb}$, and $g = 32.2\,\text{ft/s}^2$. After burnout the vertical acceleration of the rocket is $\ddot{y} = -g$. Therefore, using the constant acceleration equation $\dot{y}^2 = \dot{y}_0^2 - 2g(y - y_0)$, and recalling that $\dot{y} = 0$ for $y = y_{\text{max}}$, and $\dot{y} = v_{\text{bo}}$ for $y_0 = y_{\text{bo}}$, we have

$$0 = v_{\text{bo}}^2 - 2g\,(y_{\text{max}} - y_{\text{bo}}) \quad \Rightarrow \quad \boxed{y_{\text{max}} = \frac{v_{\text{bo}}^2}{2g} + y_{\text{bo}} = 30{,}500\,\text{ft},}$$

where we have used the following numerical data: $y_{\text{bo}} = 3260\,\text{ft}$ (see Eq. (10)), $v_{\text{bo}} = 1324\,\text{ft/s}$ (see Eq. (11)), and $g = 32.2\,\text{ft/s}^2$.

Problem 5.136

A Pelton impulse wheel, as shown in Fig. P5.136(a), typically found in hydroelectric power plants consists of a wheel at the periphery of which are attached a series of buckets. As shown in Fig. P5.136(b), water jets impinge on the buckets and cause the wheel to spin about its axis (labeled O). Let v_w and $(\dot{m}_f)_{\text{nz}}$ be the speed and the mass flow rate of the water jets at the nozzles (the nozzles are stationary), respectively. As the wheel spins, a given water jet will impinge on a given bucket only for a very small portion of the bucket's trajectory. This fact allows us to model the motion of a bucket relative to a given jet (during the time the bucket interacts with that jet) as essentially rectilinear and with constant relative speed, as was done in Example 5.17. Although each bucket moves away from the jet, the fact that they are arranged in a wheel is such that the effective mass flow rate experienced by the vanes is $(\dot{m}_f)_{\text{nz}}$ instead of the reduced mass flow rate computed in Eq. (6). With this in mind, consider a bucket, as shown in Fig. P5.136(c), that is moving with a speed v_0 horizontally away from a fixed nozzle but subject to a mass flow rate $(\dot{m}_f)_{\text{nz}}$. The inside of the bucket is shaped so as to redirect the water jet laterally out (away from the plane of the wheel). The angle θ describes the orientation of the velocity of the fluid relative to the (moving) bucket at B, the point at which the water leaves the bucket. Determine θ and v_0 such that the power transmitted by the water to the wheel is maximum. Express v_0 in terms of v_w.

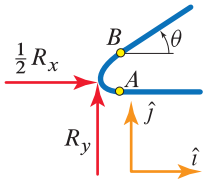


(a) (b) (c)

Solution

Due to the symmetry of the shape of a bucket, we can study the flow over half of a bucket. Using the arguments presented in Example 5.17, under the assumption that the bucket is moving at constant velocity, we can choose a control volume moving with the bucket. It is sufficient to study the motion only in the horizontal direction. As explained in Example 5.17, the velocity of the water flow over the vanes must be understood as *relative* velocity of the water with respect to the vanes.

Based on our choice of control volume, the FBD of our system is shown on the right. Then, summing forces in the x direction, we have



$$\sum F_x: \quad \tfrac{1}{2}R_x = \tfrac{1}{2}(\dot{m}_f)_{\text{nz}}(v_{Bx} - v_{Ax}), \tag{1}$$

where, as discussed in the problem statement, we have used the full mass flow rate instead of the reduced mass flow rate that is perceived by a single bucket. The kinematic equations for this problem are as follows:

$$v_{Ax} = -(v_w - v_0) \quad \text{and} \quad v_{Bx} = (v_w - v_0)\cos\theta. \tag{2}$$

Substituting Eq. (2) into Eq. (1), we have

$$R_x = (\dot{m}_f)_{\text{nz}}(v_{Bx} - v_{Ax}) \quad \Rightarrow \quad R_x = (\dot{m}_f)_{\text{nz}}(v_w - v_0)(\cos\theta + 1).$$

Note that we have ignored the forces in the y direction. This is due to the fact that the forces in the y direction on the half-bucket considered is canceled by the the force that acts on the other half of the bucket. Thus the force acting on the bucket is $\vec{F} = -R_x\,\hat{\imath}$.

$$\vec{F} = -(\dot{m}_f)_{\text{nz}}(v_w - v_0)(\cos\theta + 1)\hat{\imath}.$$

Now, observing that the velocity of the bucket is $\vec{v}_0 = -v_0\,\hat{\imath}$, then the power of the wheel is

$$P = \vec{F} \cdot \vec{v}_0 \quad \Rightarrow \quad P = (\dot{m}_f)_{\text{nz}} v_0 (v_w - v_0)(\cos\theta + 1). \tag{3}$$

To determine the conditions that allow to maximize the power, recall that $-1 < \cos\theta < 1$. Hence, the power is maximized with respect to θ when

$$\boxed{\theta_{\text{max power}} = 0^\circ.}$$

Now, substituting $\theta = 0^\circ$ in Eq. (3), we have

$$P = 2(\dot{m}_f)_{\text{nz}} v_0 (v_w - v_0).$$

Them, maximizing P by taking the derivative of P respect to v_0 and setting the result equal to zero, we have

$$\frac{dP}{dv_0} = 2\left(\dot{m}_f\right)_{\text{nz}} (v_w - 2v_0) = 0 \quad \Rightarrow \quad \boxed{v_0 = \tfrac{1}{2} v_w.}$$

Problem 5.137

In Major League Baseball, a pitched ball has been known to hit the head of the batter (sometimes unintentionally and sometimes not). Let the pitcher be, for example, Nolan Ryan who can throw a $5\frac{1}{8}$ oz baseball that crosses the plate at 100 mph. Studies have shown that the impact of a baseball with a person's head has a duration of about 1 ms. So using Eq. (5.9) on p. 335 and assuming that the rebound speed of the ball after the collision is negligible, determine the magnitude of the average force exerted on the person's head during the impact.

Solution

By a straightforward application of Eq. (5.9) on p. 335, we have

$$\left|\vec{F}_{\text{avg}}\right| = \left|\frac{\vec{p}(t_2) - \vec{p}(t_1)}{t_2 - t_1}\right|, \tag{1}$$

where $\vec{p}(t_2) = \vec{0}$ since we are assuming that the rebound velocity of the ball is negligible, $\vec{p}(t_1) = m\vec{v}(t_1)$, $t_2 = 1$ ms, $t_1 = 0$, and where m and $\vec{v}(t_1)$ is are the mass and initial velocity of the ball respectively. Hence, Eq. (1) simplifies to

$$\left|\vec{F}_{\text{avg}}\right| = \frac{mv(t_1)}{t_2}, \tag{2}$$

where $v(t_1)$ is the initial speed of the ball. Substituting the problem's data into Eq. (2) we have

$$\boxed{\left|\vec{F}_{\text{avg}}\right| = 1460\,\text{lb},}$$

where the numerical values of the data we have used are as follows:

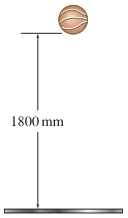
$$m = \left(5\tfrac{1}{8}\,\text{oz}\right)\frac{1}{16.00\,\text{oz/lb}}\frac{1}{32.2\,\text{ft/s}^2} = 0.009948\,\text{slug},$$

$$v(t_1) = 100\,\text{mph}\frac{5280\,\text{ft/mi}}{3600\,\text{s/h}} = 146.7\,\text{ft/s},$$

$$t_2 = 1\,\text{ms}\frac{1}{1000\,\text{ms/s}} = 1.000\times10^{-3}\,\text{s}.$$

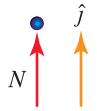
Problem 5.138

A 0.6 kg ball that is initially at rest is dropped on the floor from a height of 1.8 m and has a rebound height of 1.25 m. If the ball spends a total of 0.01 s in contact with the ground, determine the average force applied to the ball by the ground during the rebound. In addition, determine the ratio between the magnitude of the impulse provided to the ball by the ground and the magnitude of the impulse provided to the ball by gravity during the time interval that the ball is in contact with the ground. Neglect air resistance.



Solution

We assume that the ball is only subject to (constant) gravity and, when in contact with the ground, to a reaction force normal to the ground. Hence, the preimpact velocity can be determined using constant acceleration equations, as follows:



$$\dot{y}^2 = \dot{y}_0^2 - 2g(y - y_0) \quad \Rightarrow \quad v_y^- = -\sqrt{2gh_i}, \tag{1}$$

where $y = h_i = 1.8$ m is the height from which the ball is initially dropped, and where we have accounted for the fact that when $y = h_1$, $\dot{y} = 0$ (i.e., the ball is dropped from rest) and for the fact that the preimpact velocity must be in the negative y direction. Now, let v_y^+ be the post impact vertical velocity of the ball. In addition we observe that once the ball leaves the ground, the ball is again subject only to gravity. Hence, we can again relate the postimpact velocity of the ball to the final height reached $h_f = 1.25$ m using constant acceleration equations. This yields

$$\dot{y}^2 = \dot{y}_0^2 - 2g(y - y_0) \quad \Rightarrow \quad v_y^+ = \sqrt{2gh_f}, \tag{2}$$

where we have set $\dot{y}_0 = v_y^+$ for $y_0 = 0$ and $\dot{y} = 0$ for $y = h_f$, and where we have accounted for the fact that v_y^+ must be in the positive y direction.

Now that we have expressions for the pre- and postimact velocities of the ball we can then express the pre- and postimpact linear momenta of the ball as follows:

$$\vec{p}^- = -m\sqrt{2gh_i}\,\hat{j} \quad \text{and} \quad \vec{p}^+ = m\sqrt{2gh_f}\,\hat{j}.$$

Then, using Eq. (5.9) on p. 335 of the textbook, we have that the average force on the ball from the ground during impact is given by

$$\vec{F}_{\text{avg}} = \frac{\vec{p}^+ - \vec{p}^-}{t^+ - t^-} \quad \Rightarrow \quad \boxed{\vec{F}_{\text{avg}} = \frac{m}{\Delta t}\left(\sqrt{2gh_f} + \sqrt{2gh_i}\right)\hat{j} = (654\text{ N})\,\hat{j}} \tag{3}$$

where $\Delta t = t^+ = t^- = 0.01$ s and where we have used the following numerical data: $m = 0.60$ kg and $g = 9.81$ m/s^2.

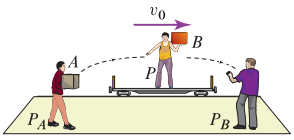
Based on the assumptions we typically use to solve impact problems, we have that the quantity $\vec{p}^+ - \vec{p}^-$ is equal to the impulse provided to the ball by the ground. In addition, using the definition of impulse, the impulse provided by gravity to the ball during the impact is $\Delta t(-mg\,\hat{j})$. Hence, referring to the expression of the force $\vec{F}_{\text{avg}}$, the ratio of the two impulses in question is given by

$$\boxed{\frac{|\vec{p}^+ - \vec{p}^-|}{|\Delta t(-mg\,\hat{j})|} = \frac{|\vec{F}_{\text{avg}}|}{mg} = \frac{\sqrt{2gh_f} + \sqrt{2gh_i}}{g\Delta t} = 111}$$

which indicates that the impulse provided by the ground during the impact is over two orders of magnitude larger than the corresponding impulse provided by gravity.

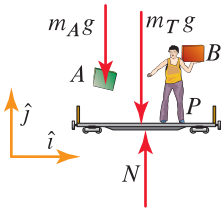
Problem 5.139

A person P is initially standing on a cart on rails, which is moving to the right with a speed $v_0 = 2\,\text{m/s}$. The cart is not being propelled by any motor. The combined mass of person P, the cart, and all that is being carried on the cart is 270 kg. At some point a person P_A standing on a stationary platform throws to person P a package A to the right with a mass $m_A = 50\,\text{kg}$. Package A is received by P with a horizontal speed $v_{A/P} = 1.5\,\text{m/s}$. After receiving the package from A, person P throws a package B with a mass $m_B = 45\,\text{kg}$ toward a second person P_B. The package intended for P_B is thrown to the right, i.e., in the direction of the motion of P, and with a horizontal speed $v_{B/P} = 4\,\text{m/s}$ relative to P. Determine the final velocity of the person P. Neglect any friction or air resistance acting on P and the cart.



Solution

The FBD shown is for the system consisting of the cart and the person P as he/she receives package A. No forces act on the system in the horizontal direction. Hence, we must have



$$m_A(v_{Ax})_1 + m_T(v_{Px})_1 = m_A(v_{Ax})_2 + m_T(v_{Px})_2, \tag{1}$$

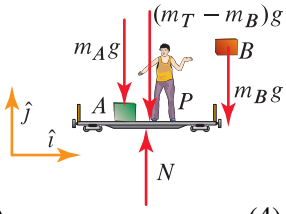
where $m_T = 270\,\text{kg}$ is the combined mass of the person P, the cart, and the rest of the objects being carried by the cart, and where the subscripts 1 and 2 refer to the time instants right before and right after the person P receives the package A, respectively. Observe that we have

$$(v_{Px})_1 = 2.00\,\text{m/s}, \quad (v_{Ax})_2 = (v_{Px})_2, \quad \text{and} \quad (v_{Ax})_1 - (v_{Px})_1 = v_{A/P} = 1.5\,\text{m/s}. \tag{2}$$

Substituting the symbolic forms of Eqs. (2) into Eq. (1) and solving for $(v_{Ax})_2$, we have

$$(v_{Px})_2 = (v_{Px})_1 + \frac{m_A}{m_A + m_T} v_{A/P}. \tag{3}$$

Next, we consider the case when person P throws a package of mass m_B to person P_B. The FBD for this case (shown to the right) again indicates that the linear momentum in the horizontal direction is conserved. Then, letting the subscripts 3 and 4 denote the time instants right before and right after the person P throws the package, we have



$$(m_T + m_A)(v_{Px})_3 = (m_T - m_B + m_A)(v_{Px})_4 + m_B(v_{Bx})_4. \tag{4}$$

Note that

$$(v_{Px})_3 = (v_{Px})_2 \quad \text{and} \quad (v_{Bx})_4 - (v_{Px})_4 = v_{B/P} = 4.00\,\text{m/s}. \tag{5}$$

Hence, substituting Eqs. (5) into Eq. (4), using the result in Eq. (3), and solving for $(v_{Px})_4$, we have

$$(v_{Px})_4 = (v_{Px})_1 + \frac{m_A}{m_T + m_A} v_{A/P} - \frac{m_B}{m_T + m_A}. \tag{6}$$

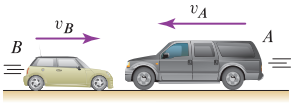
Recalling that $(\vec{v}_P)_{\text{final}} = (v_{Px})_4\,\hat{\imath}$, using the above equation, we have

$$\boxed{(\vec{v}_P)_{\text{final}} = (1.67\,\text{m/s})\,\hat{\imath}}$$

where we have used the following numerical data: $(v_{Px})_1 = 2.00\,\text{m/s}$, $m_A = 50\,\text{kg}$, $m_T = 270\,\text{kg}$, and $v_{A/P} = 4\,\text{m/s}$.

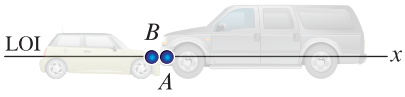
Problem 5.140

A Ford Excursion A, with a mass $m_A = 3900\,\text{kg}$, traveling with a speed $v_A = 85\,\text{km/h}$ collides head-on with a Mini Cooper B, with a mass $m_B = 1200\,\text{kg}$, traveling in the opposite direction with a speed $v_B = 40\,\text{km/h}$. Determine the postimpact velocities of the two cars if the impact's coefficient of restitution is $e = 0.22$. In addition, determine the percentage of kinetic energy loss.



Solution

The collision can be modeled as a (one dimensional) direct central impact. Using the impact-relevant FBD shown, we have conservation of momentum along the LOI, i.e.,



$$m_A v_{Ax}^- + m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+, \tag{1}$$

where m_A and m_B are the masses of A and B, respectively, and where $v_{Ax}^- = -85\,\text{km/h}$ and $v_{Bx}^- = 40\,\text{km/h}$. The COR equation is

$$v_{Ax}^+ - v_{Bx}^+ = e(v_{Bx}^- - v_{Ax}^-). \tag{2}$$

Equations (1) and (2) form a system of two equations in the two unknowns v_{Ax}^+ and v_{Bx}^+ whose solution is

$$v_{Ax}^+ = \frac{m_B[e(v_{Bx}^- - v_{Ax}^-) + v_{Bx}^-] + m_A v_{Ax}^-}{m_A + m_B} = -13.64\,\text{m/s}, \tag{3}$$

$$v_{Bx}^+ = \frac{m_A[e(v_{Ax}^- - v_{Bx}^-) + v_{Ax}^-] + m_B v_{Bx}^-}{m_A + m_B} = -21.28\,\text{m/s}, \tag{4}$$

where we have used the following numerical data: $m_A = 3900\,\text{kg}$, $m_B = 1200\,\text{kg}$, $e = 0.22$, $v_{Ax}^- = 85\,\text{km/h} = 23.61\,\text{m/s}$, and $v_{Bx}^- = 40\,\text{km/h} = 11.11\,\text{m/s}$. Hence, expressing our answer to three significant digits, we have

$$\vec{v}_A^+ = (-13.6\,\text{m/s})\,\hat{\imath} \quad \text{and} \quad \vec{v}_B^+ = (-21.3\,\text{m/s})\,\hat{\imath}.$$

Recalling that $v_A^+ = |v_{Ax}^+|$ and $v_B^+ = |v_{Bx}^+|$, and using the results in Eqs. (3) and (4), we have

$$T^- = \tfrac{1}{2}\left[m_A(v_A^-)^2 + m_B(v_B^-)^2\right] = 1.161\times10^6\,\text{J} \tag{5}$$

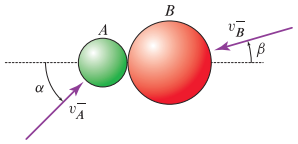
$$T^+ = \tfrac{1}{2}\left[m_A(v_A^+)^2 + m_B(v_B^+)^2\right] = 6.345\times10^5\,\text{J}, \tag{6}$$

where we have also used the following numerical data: $m_A = 3900\,\text{kg}$, $m_B = 1200\,\text{kg}$. Hence, using the results in the above two equations, we have that percentage of energy lost is given by

$$\frac{T^- - T^+}{T^-}100\% = 45.3\%.$$

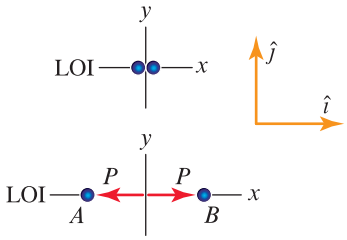
Problem 5.141

The two spheres, A and B, with masses $m_A = 1.35\,\text{kg}$ and $m_B = 2.72\,\text{kg}$, respectively, collide with $v_A^- = 26.2\,\text{m/s}$ and $v_B^- = 22.5\,\text{m/s}$. Let $\alpha = 45°$ and compute the value of β if the component of the postimpact velocity of B along the LOI is equal to zero and if the COR is $e = 0.63$.



Solution

We model the collision as an unconstrained oblique impact. As such the solution of the problem is governed by the following four equation:



$$m_A v_{Ax}^- + m_B v_{Bx}^- = m_A v_{Ax}^+ + m_B v_{Bx}^+, \tag{1}$$

$$v_{Ay}^- = v_{Ay}^+, \tag{2}$$

$$v_{By}^- = v_{By}^+, \tag{3}$$

$$v_{Ax}^+ = e(v_{Bx}^- - v_{Ax}^-), \tag{4}$$

which represent, in order, the conservation of linear momentum for the system as a whole along the LOI, the conservation of linear momentum for particle A in the direction perpendicular to the LOI, the conservation of linear momentum for particle B in the direction perpendicular to the LOI, and the COR equation along the LOI.

Observe that the preimpact velocities are given. Specifically, we have

$$v_{Ax}^- = v_A^- \cos\alpha = 18.53\,\text{m/s}, \qquad v_{Ay}^- = v_A^- \sin\alpha = 18.52\,\text{m/s}, \tag{5}$$

$$v_{Bx}^- = -v_B^- \cos\beta, \qquad v_{By}^- = -v_B^- \sin\beta, \tag{6}$$

where $\alpha = 45°$, $v_A^- = 26.2\,\text{m/s}$, and $v_B^- = 22.5\,\text{m/s}$. In addition, notice that the postimpact component of velocity along the LOI for particle B is

$$v_{Bx}^+ = 0. \tag{7}$$

Substituting the first of Eqs. (5), the first of Eqs. (6), and Eq. (7) into Eqs. (1) and (4) yields two equations in the two unknowns v_{Ax}^+ and β whose solution is

$$v_{Ax}^+ = \frac{e(m_A v_A^- \cos\alpha + m_B v_A^- \cos\alpha)}{e m_A - m_B} \quad \text{and} \quad \beta = \cos^{-1}\left[-\frac{(1+e)m_A v_A^- \cos\alpha}{(e m_A - m_B)v_B^-}\right] \tag{8}$$

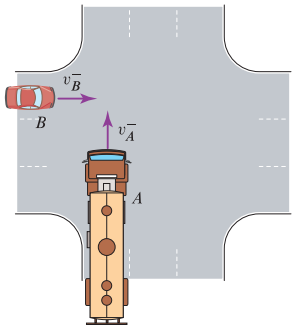
Focusing on the solution for β given by the second of Eqs. (8), we have

$$\boxed{\beta = 14.3°,}$$

where we have used the following numerical data: $e = 0.63$, $m_A = 1.35\,\text{kg}$, $v_A^- = 26.2\,\text{m/s}$, $\alpha = 45°$, $m_B = 2.72\,\text{kg}$, and $v_B^- = 22.5\,\text{m/s}$.

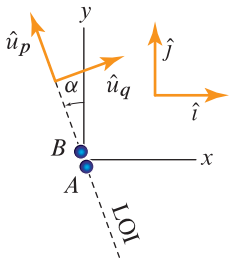
Problem 5.142

A 31,000 lb truck A and a 3970 lb sports car B collide at an intersection. At the moment of the collision the truck and the sports car are traveling with speeds $v_A^- = 60$ mph and $v_B^- = 50$ mph, respectively. Assume that the entire intersection forms a horizontal surface. Letting the line of impact be parallel to the ground and rotated counterclockwise by $\alpha = 20°$ with respect to the preimpact velocity of the truck, determine the postimpact velocities of A and B if the contact between A and B is frictionless and the COR $e = 0.1$. Furthermore, assuming that the truck and the car slide after impact and that the coefficient of kinetic friction is $\mu_k = 0.7$, determine the position at which A and B come to a stop relative to the position they occupied at the instant of impact.



Solution

We model A abd B as particle and the overall impact as an unconstrained elastic impact. Therefore, using the qp component system, the equations governing the impact are



$$m_A v_{Ap}^- + m_B v_{Bp}^- = m_A v_{Ap}^+ + m_B v_{Bp}^+, \tag{1}$$

$$v_{Aq}^+ = v_{Aq}^-, \tag{2}$$

$$v_{Bq}^+ = v_{Bq}^-, \tag{3}$$

$$v_{Bp}^+ - v_{Ap}^+ = e\left(v_{Ap}^- - v_{Bp}^-\right), \tag{4}$$

where Eqs. (1)–(4) represent the conservation of linear momentum for the system along the LOI, the conservation of linear momentum for particle A perpendicular to the LOI, the conservation of linear momentum for particle B perpendicular to the LOI, and the COR equation for the system, respectively. Observing that

$$\hat{u}_q = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}, \tag{5}$$

$$\hat{u}_p = -\sin\alpha\,\hat{\imath} + \cos\alpha\,\hat{\jmath}, \tag{6}$$

$$\hat{\imath} = \cos\alpha\,\hat{u}_q - \sin\alpha\,\hat{u}_p, \tag{7}$$

$$\hat{\jmath} = \sin\alpha\,\hat{u}_q + \cos\alpha\,\hat{u}_p. \tag{8}$$

Then we have that

$$v_{Aq}^- = v_{Ax}^- \cos\alpha + v_{Ay}^- \sin\alpha = v_A^- \sin\alpha, \tag{9}$$

$$v_{Ap}^- = -v_{Ax}^- \sin\alpha + v_{Ay}^- \cos\alpha = v_A^- \cos\alpha, \tag{10}$$

$$v_{Bq}^- = v_{Bx}^- \cos\alpha + v_{By}^- \sin\alpha = v_B^- \cos\alpha, \tag{11}$$

$$v_{Bp}^- = -v_{Bx}^- \sin\alpha + v_{By}^- \cos\alpha = -v_B^- \sin\alpha, \tag{12}$$

where we have accounted for the fact that

$$v_{Ax}^- = 0, \quad v_{Ay}^- = v_A^-, \quad v_{Bx}^- = v_B^-, \quad \text{and} \quad v_{By}^- = 0. \tag{13}$$

Substituting Eqs. (9) and (11) into Eqs. (2) and (3), we have

$$v_{Aq}^+ = v_A^- \sin\alpha = 30.10\ \text{ft/s} \quad \text{and} \quad v_{Bq}^+ = v_{Bx}^- \cos\alpha = 68.91\ \text{ft/s}. \tag{14}$$

Substituting Eqs. (10) and (12) into Eqs. (1) and (4) yields a system of two equations in the two unknowns v_{Ap}^+ and v_{Bp}^+ whose solution is

$$v_{Ap}^+ = \frac{(m_A - e m_B)v_A^- \cos\alpha - (1+e)m_B v_B^- \sin\alpha}{m_A + m_B} = 69.23\,\text{ft/s}, \tag{15}$$

$$v_{Bp}^+ = \frac{(1+e)m_A v_A^- \cos\alpha + (e m_A - m_B)v_B^- \sin\alpha}{m_A + m_B} = 80.01\,\text{ft/s}, \tag{16}$$

where we have used the following numerical data: $m_A = (31{,}000\,\text{lb})/(32.2\,\text{ft/s}^2) = 962.7\,\text{slug}$, $m_B = (3970\,\text{lb})/(32.2\,\text{ft/s}^2) = 123.3\,\text{slug}$, $v_A^- = 60\,\text{mph} = 88.00\,\text{ft/s}$, and $v_A^- = 50\,\text{mph} = 73.33\,\text{ft/s}$, $\alpha = 20°$, and $e = 0.1$.

Recalling Eqs. (7) and (8), the post impact velocities can be expressed as

$$\vec{v}_A^+ = (4.607\,\hat{\imath} + 75.35\,\hat{\jmath})\,\text{ft/s} \quad \text{and} \quad \vec{v}_B^+ = (37.39\,\hat{\imath} + 98.75\,\hat{\jmath})\text{ft/s}, \tag{17}$$

so that, when expressed using three significant figures, we have

$$\boxed{\vec{v}_A^+ = (4.61\,\hat{\imath} + 75.3\,\hat{\jmath})\,\text{ft/s} \quad \text{and} \quad \vec{v}_B^+ = (37.4\,\hat{\imath} + 98.8\,\hat{\jmath})\text{ft/s}}$$

Right after impact, A will move along the post impact direction of the velocity of A until the kinetic friction force due to sliding will cause A to stop. A similar consideration can be stated for B. Letting ① and ② denote the positions right after impact and the final positions of A and B, applying the work-energy principle we must have

$$T_{A1} - \mu_k m_A g d_A = T_{A2} \quad \text{and} \quad T_{B1} - \mu_k m_B g d_B = T_{B2} \tag{18}$$

where d_A and d_B are the stopping distances of A and B, respectively, we have accounted for the fact that the friction force on A and B are constant and equal to $\mu_k m_A g$ and $\mu_k m_B g$, respectively, and where the kinetic energy terms are

$$T_{A1} = \tfrac{1}{2}m_A(v_A^+)^2, \quad T_{A2} = 0, \quad T_{B1} = \tfrac{1}{2}m_B(v_B^+)^2, \quad \text{and} \quad T_{B2} = 0. \tag{19}$$

Substituting Eqs. (19) into Eqs. (18) and solving for the stopping distances d_A and d_B, we obtain

$$d_A = 126.4\,\text{ft} \quad \text{and} \quad d_B = 247.3\,\text{ft}, \tag{20}$$

where, in addition to the appropriate parameters listed below Eq. (16), we have used Eqs. (17) to compute the postimpact speeds, as well as the following nuerical data: $\mu_k = 0.7$ and $g = 32.2\,\text{ft/s}^2$. Since A moves along the direction of its postimpact velocity, then the position vector of A at ② relative to ① is given by

$$\boxed{\vec{r}_A = d_A \frac{\vec{v}_A^+}{v_A^+} = (7.71\,\hat{\imath} + 126\,\hat{\jmath})\,\text{ft}}$$

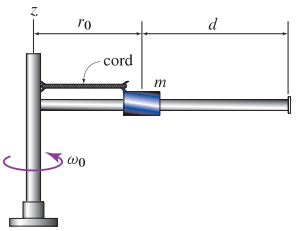
and, proceeding similarly for B, we have

$$\boxed{\vec{r}_B = d_B \frac{\vec{v}_B^+}{v_B^+} = (87.6\,\hat{\imath} + 231\,\hat{\jmath})\,\text{ft}}$$

where we have used the numerical results in Eqs. (17) and (20).

Problem 5.143

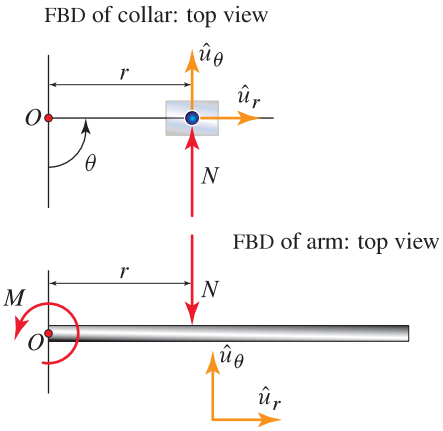
Consider a collar with mass m that is free to slide with no friction along a rotating arm of negligible mass. The system is initially rotating with a constant angular velocity ω_0 while the collar is kept at a distance r_0 from the z axis. At some point, the restraint keeping the collar in place is removed so that the collar is allowed to slide. Determine the expression for the moment that you need to apply to the arm, as a function of time, to keep the arm rotating at a constant angular velocity while the collar travels toward the end of the arm. *Hint:*



$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left(x+\sqrt{x^2-1}\right)+C.$$

Solution

The FBD of the collar is show. Also, referring to the FBD of the arm, since the mass of the arm is negligible, the equilibrium of moments about point O implies that we must have



$$M = Nr \quad \Rightarrow \quad N = \frac{M}{r}, \tag{1}$$

where r is the distance of the collar from the spin axis. Then, choosing O as moment center, the application of the angular impulse–momentum principle about O for the collar gives:

$$M = \frac{d}{dt}(mrv_\theta), \tag{2}$$

where mrv_θ is the component of the angular momentum of collar about O. In addition, applying Newton's second law in the radial direction, we have

$$0 = ma_r, \tag{3}$$

where a_r is the radial acceleration of the collar.

The kinematic equations for this problem are

$$v_\theta = r\dot{\theta}, \quad a_r = \ddot{r} - r\dot{\theta}^2, \quad \dot{\theta} = \omega_0 = \text{constant}. \tag{4}$$

Substituting the first of Eqs. (4) into Eq. (2), we obtain

$$M = 2m\omega_0 r\dot{r}. \tag{5}$$

Equation (5) indicates that to obtain M as a function of time we need to have both r and $\dot{r}$ as a functions of time. To obtain these expressions we begin by substituting the second of Eqs. (4) into Eq. (3), to obtain

$$\ddot{r} - r\omega^2 = 0 \quad \Rightarrow \quad \dot{r}d\dot{r} = \omega_0^2 r dr, \tag{6}$$

where we have used the relation $\ddot{r} = \dot{r}d\dot{r}/dr$. Integrating both sides of Eq. (6) with appropriate limits of integration we have

$$\int_0^{\dot{r}} \dot{r}d\dot{r} = \int_{r_0}^{r} \omega_0^2 r dr \quad \Rightarrow \quad \tfrac{1}{2}\dot{r}^2 = \tfrac{1}{2}\omega_0^2(r^2 - r_0^2) \quad \Rightarrow \quad \dot{r} = \omega_0\sqrt{r^2 - r_0^2}, \tag{7}$$

where we have accounted for the fact that, in this problem, $\dot{r} > 0$. Next, recalling that $\dot{r} = dr/dt$, we can rewrite the final result in Eq. (7) as

$$\frac{1}{r_0}\frac{dr}{\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} = \omega_0\, dt \quad \Rightarrow \quad \int_{r_0}^{r} \frac{1}{r_0}\frac{dr}{\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} = \int_0^t \omega_0\, dt \tag{8}$$

where we have used that fact that $\sqrt{r^2 - r_0^2} = \frac{1}{r_0}\sqrt{(r/r_0)^2 - 1}$. The last equality in Eq. (8) can be rewritten as

$$\int_1^{r/r_0} \frac{dx}{\sqrt{x^2 - 1}} = \omega_0 t, \tag{9}$$

where we have used the change of variables of integration $x = r/r_0$. Then, using the hint provided, we have

$$\ln\left(\frac{r}{r_0} + \sqrt{\left(\frac{r}{r_0}\right)^2 - 1}\right) = \omega_0 t \quad \Rightarrow \quad \frac{r}{r_0} + \sqrt{\left(\frac{r}{r_0}\right)^2 - 1} = e^{\omega_0 t}, \tag{10}$$

which can be solved for r to obtain

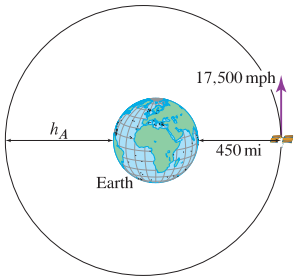
$$r = \tfrac{1}{2} r_0 \left(e^{\omega_0 t} + e^{-\omega_0 t}\right) \quad \text{and} \quad \dot{r} = \tfrac{1}{2} r_0 \omega_0 \left(e^{\omega_0 t} - e^{-\omega_0 t}\right), \tag{11}$$

where $\dot{r}$ was obtained by differentiating the expression for r with respect to time. Finally, substituting the above results in Eq. (5) and simplifying, we obtain

$$\boxed{M = \tfrac{1}{2} m \omega_0^2 r_0^2 \left(e^{2\omega_0 t} - e^{-2\omega_0 t}\right).}$$

Problem 5.144

A satellite is launched parallel to the Earth's surface at an altitude of 450 mi with a speed of 17,500 mph. Determine the apogee altitude h_A above the Earth's surface as well as the period of the satellite.



Solution

The satellite will orbit the Earth along an elliptical orbit. The launch conditions corresponds to the satellite being at the perigee of the elliptical orbit. Recalling that the radius of Earth is $r_e = 3959\,\text{mi} = 2.090\times10^7\,\text{ft}$, then the radius at perigee is

$$r_P = r_e + 450\,\text{mi} = 4409\,\text{mi} = 2.328\times10^7\,\text{ft}. \tag{1}$$

As stated in the problem, the speed at perigee is $v_P = 17{,}500\,\text{mph} = 25{,}670\,\text{ft/s}$. To relate the information we have at perigee to the value of the radius at apogee we now use Eq. (5.113) on p. (5.113) of the textbook, which states that

$$r_A = \frac{r_P}{2Gm_B/(r_P v_P^2) - 1}, \tag{2}$$

where G is the universal gravitational constant and m_B is the mass of the body about which the satellite is orbiting. In our case the body B is the Earth and the term Gm_B is equal to gr_e^2, where $g = 32.2\,\text{ft/s}^2$. Therefore, using the results obtained thus far, we have

$$r_A = \frac{r_P}{2gr_e^2/(r_P v_P^2) - 1} = 2.788\times10^7\,\text{ft} = 5281\,\text{mi}. \tag{3}$$

We are now in a position to determine the altitude h_A of the satellite at apogee. In fact, we have

$$\boxed{h_A = r_A - r_E = 6.98\times10^6\,\text{ft} = 1320\,\text{mi}.}$$

We now turn to the determination of the orbit's period, which we will denote by τ. To do so, we will use Eq. (5.123) on p. 415 of the textbook, which states that

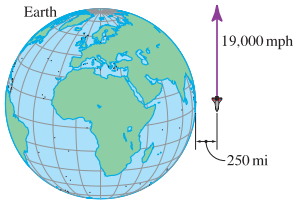
$$\tau = \frac{\pi}{\kappa}(r_P + r_A)\sqrt{r_P r_A}, \tag{4}$$

where the quantity κ is defined in Eq. (5.108) on p. 412 of the textbook, which states that $\kappa = r_P v_P$. Therefore, using the values of r_P in Eqs. (1) along with the value of v_P indicated right after Eq. (1), we have $\kappa = 5.976\times10^{11}\,\text{ft/s}^2$. Using this result, along with the values for r_P, r_A, and v_P already found, applying Eq. (4) we have

$$\boxed{\tau = 6850\,\text{s} = 1.90\,\text{hr}.}$$

Problem 5.145

A spacecraft is traveling at 19,000 mph parallel to the surface of the Earth at an altitude of 250 mi, when it fires a retrorocket to transfer to a different orbit. Determine the change in speed Δv necessary for the spacecraft to reach a minimum altitude of 110 mi during the ensuing orbit. Assume that the change in speed is impulsive; that is, it occurs instantaneously.



Solution

The spacecraft intends to transfer onto an orbit with a radius that is smaller than that of its initial orbit. Therefore, when the spacecraft fires its retrorockets it occupies the apogee of the ensuing elliptical transfer orbit. In addition, the radius of the destination orbit is also the radius at perigee of the elliptical transfer orbit. Therefore, recalling that the radius of the Earth is $r_e = 3959\,\text{mi} = 2.090\times10^7\,\text{ft}$, we can write

$$r_A = (3959\,\text{mi} + 250\,\text{mi}) = 2.222\times10^7\,\text{ft}, \tag{1}$$

$$r_P = (3959\,\text{mi} + 110\,\text{mi}) = 2.148\times10^7\,\text{ft}. \tag{2}$$

Consequently, the semimajor axis of the transfer orbit is

$$a = \tfrac{1}{2}(r_A + r_P) = 2.185\times10^7\,\text{ft}. \tag{3}$$

Now we observe that the initial speed at apogee is given and we denote it by v_1, i.e.,

$$v_1 = 19{,}000\,\text{mph}\frac{5280\,\text{ft/mi}}{3600\,\text{s/h}} = 27{,}870\,\text{ft/s}. \tag{4}$$

The new speed needed by the spacecraft at apogee to get into the transfer orbit is found by applying Eq. (5.134) on p. 416 of the textbook, which states

$$v = \sqrt{Gm_B\left(\frac{2}{r} - \frac{1}{a}\right)}, \tag{5}$$

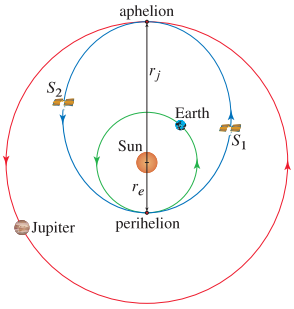
where G is the universal gravitational constant, m_B is the mass of the body about which the satellite is orbiting, and r is the distance of the satellite from B. In our case the body B is the Earth and the term Gm_B is equal to gr_e^2, where $g = 32.2\,\text{ft/s}^2$. In addition, $r = r_A$ given in Eq. (1) and a is given by Eq. (3). Therefore, denoting the new speed at apogee by v_2, we have

$$v_2 = \sqrt{gr_e^2\left(\frac{2}{r_A} - \frac{1}{a}\right)} = 2.495\times10^4\,\text{ft/s} = 17{,}010\,\text{mph}. \tag{6}$$

Finally, the change in speed to get the spacecraft on the transfer orbit is $\Delta v = v_2 - v_1$. Hence, using the results in Eqs. (4) and (6) and expressing the final result using three significant figures, we have

$$\boxed{\Delta v = -2.92\times10^3\,\text{ft/s} = -1990\,\text{mph}.}$$

Problems 5.146 and 5.147



The optimal way (from an energy standpoint) to transfer from one circular orbit about a primary body (in this case, the Sun) to another circular orbit is via the *Hohmann transfer*, which involves transferring from one circular orbit to another using an elliptical orbit that is tangent to both at the periapsis and apoapsis of the ellipse. This ellipse is uniquely defined because we know the perihelion radius r_e (the radius of the inner circular orbit) and the aphelion radius r_j (the radius of the outer circular orbit), and therefore we know the semimajor axis a via Eq. (5.117) and the eccentricity e via Eq. (5.114) or Eqs. (5.119). Performing a Hohmann transfer requires two maneuvers, the first to leave the inner (outer) circular orbit and enter the transfer ellipse and the second to leave the transfer ellipse and enter the outer (inner) circular orbit. Assume that the orbits of Earth and Jupiter are circular, use 150×10^6 km for the radius of Earth's orbit, use 779×10^6 km for the radius of Jupiter's orbit, and note that the mass of the Sun is 333,000 times that of the Earth.

Problem 5.146 A space probe S_1 is launched from Earth to Jupiter via a Hohmann transfer orbit. Determine the change in speed Δv_e required at the radius of Earth's orbit of the elliptical transfer orbit (perihelion) and the change in speed Δv_j required at the radius of Jupiter's orbit (aphelion). In addition, compute the time required for the orbital transfer. Assume that the changes in speed are impulsive; that is, they occur instantaneously.

Problem 5.147 A space probe S_2 is at Jupiter and is required to return to the radius of Earth's orbit about the Sun so that it can return samples taken from one of Jupiter's moons. Assuming that the mass of the probe is 722 kg, determine the change in kinetic energy required at Jupiter ΔT_j for the maneuver at aphelion. In addition, determine the change in kinetic energy required at Earth ΔT_e for the perihelion maneuver. Finally, what is the change in potential energy ΔV of the spacecraft in going from Jupiter to the Earth?

Solution to 5.146

We begin with the determination of the speed corresponding to a circular orbit with radius equal to that of the Earth's orbit. Using Eq. (5.111) on p. 413 of the textbook, we have

$$(v_{\text{circ}})_{\text{Earth}} = \sqrt{\frac{Gm_s}{r_e}} = \sqrt{\frac{333{,}000gR_E^2}{r_e}} = 29{,}730\,\text{m/s} = 107{,}000\,\text{km/h}, \tag{1}$$

where m_s is the mass of the Sun, $r_e = 150\times10^6$ km $= 150\times10^9$ m is the radius of the Earth's orbit, $R_E = 6371\times10^3$ m is the radius of the Earth, and $g = 9.81\,\text{m/s}^2$ is the acceleration due to gravity on the surface of the Earth.

Next, we compute the speed corresponding to the elliptical transfer orbit at perihelion. To do so, we use Eq. (5.134) on p. 416 of the textbook. To apply this formula, we must first compute the semimajor axis of the elliptical transfer orbit, which is given by

$$a = \tfrac{1}{2}(r_e + r_j) = 4.645\times10^{11}\,\text{m}, \tag{2}$$

where $r_j = 779\times10^6$ km $= 779\times10^9$ m is the radius of Jupiter's orbit. Now that a is known, the application of Eq. (5.134) on p. 416 of the textbook to the computation the speed corresponding to the elliptical transfer orbit at perihelion gives

$$v_{\text{perihelion}} = \sqrt{Gm_s\left(\frac{2}{r_e}-\frac{1}{a}\right)} = \sqrt{333{,}000gR_E^2\left(\frac{2}{r_e}-\frac{1}{a}\right)} = 38{,}500\,\text{m/s} = 138{,}600\,\text{km/h}, \quad (3)$$

Therefore denoting the change in speed at perihelion by $\Delta v_e = v_{\text{perihelion}} - (v_{\text{circ}})_{\text{Earth}}$, using the results in Eqs. (1) and (3), we have

$$\boxed{\Delta v_e = 8770\,\text{m/s} = 31{,}600\,\text{km/h}.}$$

We now compute the speed at aphelion. Applying Eq. (5.134) on p. 416 again, we have

$$v_{\text{aphelion}} = \sqrt{Gm_s\left(\frac{2}{r_j}-\frac{1}{a}\right)} = \sqrt{333{,}000gR_E^2\left(\frac{2}{r_j}-\frac{1}{a}\right)} = 7414\,\text{m/s} = 26{,}690\,\text{km/h}. \quad (4)$$

Next we compute the speed corresponding to a circular orbit with the same radius as Jupiter's orbit. Applying Eq. (5.111) on p. 413 of the textbook again, we have

$$(v_{\text{circ}})_{\text{Jupiter}} = \sqrt{\frac{Gm_s}{r_j}} = \sqrt{\frac{333{,}000gR_E^2}{r_j}} = 13{,}050\,\text{m/s} = 46{,}970\,\text{km/h}. \quad (5)$$

Hence, we have

$$\boxed{\Delta v_j = 5630\,\text{m/s} = 20{,}300\,\text{km/h}.}$$

Finally, the time t needed for the orbital transfer is equal to half of the obit's period τ, which is given by Eq. (5.126) on p. 415 of the textbook. Hence, applying this formula, we have

$$t = \tfrac{1}{2}\tau = \pi\sqrt{\frac{a^3}{333000gR_E^2}}, \quad (6)$$

which, upon substitution of the problem's numerical data , yields

$$\boxed{\tau = 8.64\times10^7\,\text{s} = 1000\,\text{days}.}$$

Solution to 5.147

We start with computing the speed corresponding to a circular orbit with the same radius as Jupiter's orbit. Using Eq. (5.111) on p. 413 of the textbook, we have

$$(v_{\text{circ}})_{\text{Jupiter}} = \sqrt{\frac{Gm_s}{r_j}} = \sqrt{\frac{333{,}000gR_E^2}{r_j}} = 13{,}050\,\text{m/s} = 46{,}970\,\text{km/h}, \tag{7}$$

where m_s is the mass of the Sun, $r_j = 779\times10^6\,\text{km} = 779\times10^9\,\text{m}$ is the radius of Jupiter's orbit, $R_E = 6371\times10^3\,\text{m}$ is the radius of the Earth, and $g = 9.81\,\text{m/s}^2$ is the acceleration due to gravity on the surface of the Earth.

Next we now compute the speed at aphelion. To do so, we use Eq. (5.134) on p. 416 of the textbook. To apply this formula, we must first compute the semimajor axis of the elliptical transfer orbit, which is given by

$$a = \tfrac{1}{2}(r_e + r_j) = 4.645\times10^{11}\,\text{m}, \tag{8}$$

$r_e = 150\times10^6\,\text{km} = 150\times10^9\,\text{m}$ is the radius of Earth's orbit. Now that a is known, the application of Eq. (5.134) on p. 416 of the textbook to the computation the speed corresponding to the elliptical transfer orbit at aphelion gives

$$v_{\text{aphelion}} = \sqrt{Gm_s\left(\frac{2}{r_j} - \frac{1}{a}\right)} = \sqrt{333{,}000gR_E^2\left(\frac{2}{r_j} - \frac{1}{a}\right)} = 7414\,\text{m/s} = 26{,}690\,\text{km/h}, \tag{9}$$

Using the results in Eqs. (7) and (9), letting $m = 722\,\text{kg}$ denote the mass of the probe in our problem, the change in kinetic energy of the probe at aphelion is

$$\boxed{\Delta T_j = \tfrac{1}{2}mv_{\text{aphelion}}^2 - \tfrac{1}{2}m(v_{\text{circ}})_{\text{Jupiter}}^2 = -4.16\times10^{10}\,\text{J}.}$$

Applying Eq. (5.134) on p. 416 again, for the second maneuver we have that the speed at perihelion is

$$v_{\text{perihelion}} = \sqrt{Gm_s\left(\frac{2}{r_e} - \frac{1}{a}\right)} = \sqrt{333{,}000gR_E^2\left(\frac{2}{r_e} - \frac{1}{a}\right)} = 38{,}500\,\text{m/s} = 138{,}600\,\text{km/h}. \tag{10}$$

In addition, using Eq. (5.111) on p. 413 of the textbook again, the speed in the circular orbit at the radius of the Earth is

$$(v_{\text{circ}})_{\text{Earth}} = \sqrt{\frac{Gm_s}{r_e}} = \sqrt{\frac{333{,}000gR_E^2}{r_e}} = 29{,}730\,\text{m/s} = 107{,}000\,\text{km/h}. \tag{11}$$

Hence, the change in kinetic energy of the probe at perihelion is

$$\boxed{\Delta T_j = \tfrac{1}{2}m(v_{\text{circ}})_{\text{Earth}}^2 - \tfrac{1}{2}mv_{\text{perihelion}}^2 = -2.16\times10^{11}\,\text{J}.}$$

To compute the change in potential energy of the probe, observe that, at Jupiter we have

$$V_j = -G\frac{m_s m}{r_j} = -\frac{333000gR_E^2 m}{r_j} = -1.229\times10^{11}\,\text{J}. \tag{12}$$

The potential energy at Earth is

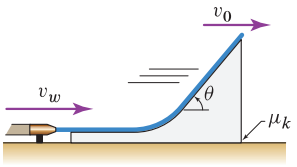
$$V_e = -G\frac{m_s m}{r_e} = -\frac{333000gR_E^2 m}{r_e} = -6.382\times10^{11}\,\text{J}. \tag{13}$$

Hence, the change in potential energy of the probe is $\Delta V = V_e - V_j$, whose numerical value is

$$\boxed{\Delta V = -5.15\times10^{11}\,\text{J}.}$$

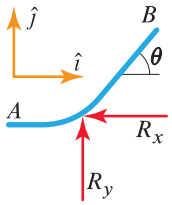
Problem 5.148

A water jet is emitted from a nozzle attached to the ground. The jet has a constant mass flow rate $(\dot{m}_f)_{\text{nz}} = 15\,\text{kg/s}$ and a speed v_w relative to the nozzle. The jet strikes a 12 kg incline and causes it to slide at a constant speed $v_0 = 2\,\text{m/s}$. The kinetic coefficient of friction between the incline and the ground is $\mu_k = 0.25$. Neglecting the effect of gravity and air resistance on the water flow, as well as friction between the water jet and the incline, determine the speed of the water jet at the nozzle if $\theta = 47°$.



Solution

In solving this problem we will follow the approach demonstrated in Example 5.17 on p. 432 of the textbook. Specifically, because both the speed of the water jet at the nozzle and the speed of the incline are constant, we can conclude thatsince the water flow over the moving incline is steady. Furthermore, again because the incline is moving with a constant velocity relative to the ground (which we view as an inertial reference frame) the incline can be chosen as an inertial reference frame. Using the frame of reference, we choose as our control volume, the volume occupied by the fluid in contact with the incline, delimited by cross sections at A and B. Hence, referring to the control volume FBD on the right, the force balance for the control volume gives



$$\sum F_x: \quad -R_x = \dot{m}_f(v_{Bx} - v_{Ax}), \tag{1}$$

$$\sum F_y: \quad R_y = \dot{m}_f(v_{By} - v_{Ay}), \tag{2}$$

where, as in Example 5.17, R_x and R_y are the components of the force applied by the incline to the water jet, $\dot{m}_f$ is the mass flow rate across the cross sections A and B, and where $\vec{v}_A = v_{Ax}\,\hat{\imath} + v_{Ay}\,\hat{\jmath}$ and $\vec{v}_B = v_{Bx}\,\hat{\imath} + v_{By}\,\hat{\jmath}$ are the velocities of the water jet at A and B, respectively, as measured by an observer moving with the incline. Now, since we are neglecting any friction between the water jet and the incline, there is no force that would slow down the water jet as it travels over the incline. This implies that the speed of the water jet remains the same at every point along the top surface of the incline and we therefore have

$$v_{Ax} = v_w - v_0, \quad v_{Ay} = 0, \quad v_{Bx} = (v_w - v_0)\cos\theta, \quad \text{and} \quad v_{By} = (v_w - v_0)\sin\theta. \tag{3}$$

As far as the flow rate is concerned, just like in Example 5.17, we need to distinguish between the mass flow rate at the nozzle, denoted by $(\dot{m}_f)_{\text{nz}}$ from the mass flow rate over the incline, denoted by $\dot{m}_f$. Recalling the relation between the mass flow rate and the volumetric flow rate, we can write

$$(\dot{m}_f)_{\text{nz}} = \rho S v_w \quad \text{and} \quad \dot{m}_f = \rho S(v_w - v_0) \quad \Rightarrow \quad \dot{m}_f = \frac{(\dot{m}_f)_{\text{nz}}(v_w - v_0)}{v_w}, \tag{4}$$

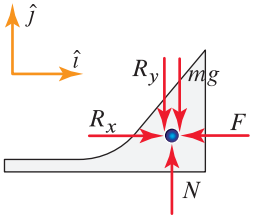
where we have assumed that the area of the cross section of the jet flowing over the incline remains constant and equal to the corresponding cross section at the nozzle.

Substituting Eqs. (3) and the last of Eqs. (4) into Eqs. (1) and (2), treating v_w as it it were known, we obtain a system of two equations in the two unknowns R_x and R_y whose solution is

$$R_x = \frac{(\dot{m}_f)_{\text{nz}}}{v_w}(1 - \cos\theta)(v_w - v_0)^2 \quad \text{and} \quad R_y = \frac{(\dot{m}_f)_{\text{nz}}}{v_w}\sin\theta(v_w - v_0)^2. \tag{5}$$

We can now consider the FBD of the incline and then relate the quantity v_w to the friction force acting on the incline.

We model the incline as a particle and subject to the forces depicted in the FBD on the right. Note that the forces R_x and R_y now represent the action of the water jet on the incline. Furthermore, the force F is the friction force between the incline and the ground. The application of Newton's second law to the the incline gives



$$\sum F_x: \qquad R_x - F = ma_x, \tag{6}$$

$$\sum F_y: \quad N - mg - R_y = ma_y, \tag{7}$$

where m is the mass of the incline. Now, since the incline is moving at a constant velocity, we have

$$a_x = 0 \quad \text{and} \quad a_y = 0. \tag{8}$$

Since the incline is sliding relative to the ground, the friction force is then related to the normal force N as

$$F = \mu_k N. \tag{9}$$

Substituting the second of Eqs. (5) and the second of Eqs. (8) into Eq. (7) and solving for N, we have

$$N = mg + \frac{(\dot{m}_f)_{\text{nz}}}{v_w} \sin\theta (v_w - v_0)^2. \tag{10}$$

Next, substituting Eq. (10) into Eq. (9), then substituting the result in Eq. (6), and solving for R_x, gives

$$R_x = \mu_k \left[mg + \frac{(\dot{m}_f)_{\text{nz}}}{v_w} \sin\theta (v_w - v_0)^2 \right]. \tag{11}$$

Setting the expressions for R_x given in the first of Eqs. (5) equal to that given in Eq. (11), we have

$$\frac{(\dot{m}_f)_{\text{nz}}}{v_w}(1 - \cos\theta)(v_w - v_0)^2 = \mu_k \left[mg + \frac{(\dot{m}_f)_{\text{nz}}}{v_w} \sin\theta (v_w - v_0)^2 \right], \tag{12}$$

which, after multiplication by v_w, can be rearranged to read

$$(\dot{m}_f)_{\text{nz}}(v_w - v_0)^2(1 - \cos\theta - \mu_k \sin\theta) - mg\mu_k v_w = 0. \tag{13}$$

Finally, dividing this equation by the term $(\dot{m}_f)_{\text{nz}}(1 - \cos\theta - \mu_k \sin\theta)$ and expanding the term in parenthesis, we have

$$v_w^2 - \left(2v_0 + \frac{mg\mu_k}{(\dot{m}_f)_{\text{nz}}(1 - \cos\theta - \mu_k \sin\theta)} \right) v_w + v_0^2 = 0. \tag{14}$$

Now observe that the term in parenthesis in the above equation consists of all known constant quantities. Hence, let the constant K be defined as the term in question, i.e.,

$$K = 2v_0 + \frac{mg\mu_k}{(\dot{m}_f)_{\text{nz}}(1 - \cos\theta - \mu_k \sin\theta)} = 18.52\,\text{m/s}, \tag{15}$$

where we have used the following numerical data: $v_0 = 2.00\,\text{m/s}$, $m = 12.0\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $\mu_k = 0.250$, $(\dot{m}_f)_{\text{nz}} = 15.0\,\text{kg/s}$, and $\theta = 47.0°$. Then Eq. (14) can be rewritten as

$$v_w^2 - Kv_w + v_0^2 = 0, \tag{16}$$

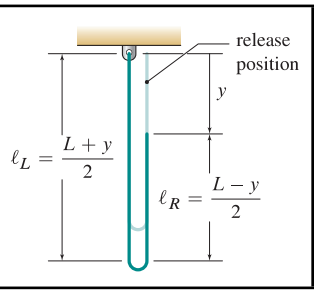
which is a quadratic equation in v_w whose solution is

$$\boxed{v_w = \begin{cases} \frac{1}{2}K - \frac{1}{2}\sqrt{K^2 - 4v_0^2} = 0.219\,\text{m/s}, \\ \frac{1}{2}K + \frac{1}{2}\sqrt{K^2 - 4v_0^2} = 18.3\,\text{m/s}, \end{cases}}$$

where we have used the value of K in Eq. (15) and $v_0 = 2.00\,\text{m/s}$.

Problem 5.149

Revisit Example 5.20 and derive the equation of motion of the free end of the string starting from the force balance for the right branch of the string modeled as a variable mass system.



Solution

We start with reporting Eq. (9) (on p. 439 of the textbook) from Example 5.20.

$$\sum F_{yR}: \quad \rho \ell_R g = \rho \ell_R a_{yR} + \dot{m}_o \vec{v}_o \cdot \hat{j},$$

In addition, we observe that Eqs. (10)–(12) of Example 5.20 are still valid, so that we can write

$$a_{yR} = \ddot{y}, \quad \dot{m}_o = -\dot{m}_R = -\rho \dot{\ell}_R = \rho \dot{y}/2, \quad \text{and} \quad \vec{v}_o = \dot{\ell}_R \, \hat{j} = -\tfrac{1}{2}\dot{y} \, \hat{j}. \tag{1}$$

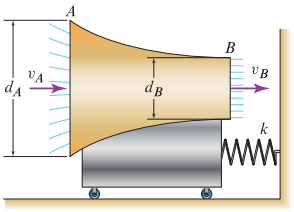
Substituting Eqs. (1) into Eq. () and simplifying we have Combining the above equations we have

$$\tfrac{1}{2}\rho(L-y)g = \tfrac{1}{2}\rho(L-y)\ddot{y} - \tfrac{1}{4}\rho\dot{y}^2 \quad \Rightarrow \quad \boxed{(L-y)(\ddot{y}-g) - \tfrac{1}{2}\dot{y}^2 = 0.}$$

This as a nonlinear ordinary differential equation whose solution as a function of time is the motion of the free end of the string.

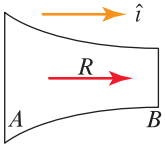
Problem 5.150

An intubed fan (a fan rotating within a tube or other conduit) is mounted on a cart that is connected to a fixed wall via a linear elastic spring with constant $k = 70\,\text{N/m}$. Assume that in a particular test the fan draws air that enters the tube at A with a speed v_A. The outgoing flow at B has a speed v_B. The flow of air through the tube causes the cart to displace to the left so that the spring is stretched by 0.25 m from its unstretched position. Assume that the density of air is constant throughout the tube and equal to $\rho = 1.25\,\text{kg/m}^3$. In addition let the tube's cross section be circular, and let the cross-sectional diameters at A and B be $d_A = 3\,\text{m}$ and $d = 1.5\,\text{m}$, respectively. Determine the velocities of the airflow at A and B.



Solution

Since the airflow is steady, we select as our control volume the interior volume of the tube delimited by the (vertical) cross sections at A and B. The FBD of the chosen control volume is as shown on the right, where R is the horizontal force acting on the airflow do to the fan (using symmetry arguments we can say that no net force acts on the airflow in the vertical direction). Hence, the force balance for the chosen control volume in horizontal direction is



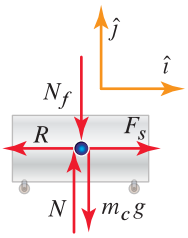
$$\sum F_x: \quad R = \dot{m}_f(v_{Bx} - v_{Ax}), \tag{1}$$

where $\dot{m}_f$ is the mass flow rate, v_{Ax} and v_{Bx} are the horizontal components of the airflow velocities at A and B, respectively. The term $\dot{m}_f$ is related to the volumetric flow rate Q as $\dot{m}_f = \rho Q$, where ρ is the density of the fluid. In turn, Q at a generic cross section of area S is related to the flow speed v at that cross section by $Q = vS$. Therefore, recalling that in this problem the density is constant, at cross sections A and B we have

$$\dot{m}_f = \rho Q_A = \rho Q_B \quad \Rightarrow \quad \dot{m}_f = \rho \tfrac{1}{4}\pi d_A^2 v_{Ax} = \rho \tfrac{1}{4}\pi d_B^2 v_{Bx} \quad \Rightarrow \quad d_A^2 v_{Ax} = d_B^2 v_{Bx}, \tag{2}$$

where d_A and d_B are the diameters of the cross sections at A and B, respectively.

The airflow applies to the tube a force equal and opposite to R. In turn, the tube will transmit this force to the cart. Therefore, the FBD of the cart is as shown on the right and applying Newton's second law to the cart in the horizontal direction yields



$$\sum F_x: \quad R - F_s = m_c a_x, \tag{3}$$

where m_c is the cart's mass and F_s is the spring force due to the spring, which is given by

$$F_s = k\delta. \tag{4}$$

Substituting Eq. (4) into Eq. (3), since the cart is stationary so that $a_x = 0$, and solving for R, we have

$$R = k\delta. \tag{5}$$

Next, referring to Eqs. (2), we substitute into Eq. (1) the expression for $\dot{m}_f$ in terms of v_{Ax}. We then substitute Eq. (5) into Eq. (1). The resulting equation, along with the last of Eqs. (2) gives

$$kd = \rho \tfrac{1}{4}\pi d_A^2 v_{Ax}(v_{Bx} - v_{Ax}), \tag{6}$$

$$d_A^2 v_{Ax} = d_B^2 v_{Bx}, \tag{7}$$

which is a system of two equations in the two unknowns v_{Ax} and v_{Bx} whose solution is

$$v_{Ax} = \frac{2d_B}{d_A}\sqrt{\frac{k\delta}{\pi\rho\left(d_A^2 - d_B^2\right)}} = 0.8125\,\text{m/s} \quad \text{and} \quad v_{Bx} = \frac{2d_A}{d_B}\sqrt{\frac{k\delta}{\pi\rho\left(d_A^2 - d_B^2\right)}} = 3.250\,\text{m/s}, \tag{8}$$

where we have used the following numerical data: $d_A = 3.00\,\text{m}$, $d_B = 1.50\,\text{m}$, $k = 70.0\,\text{N/m}$, $\delta = 0.250\,\text{m}$, and $\rho = 1.25\,\text{kg/m}^3$. Finally, using the component system shown and expressing our answer to three significant figures, we have

$$\boxed{\vec{v}_A = (0.813\,\text{m/s})\,\hat{\imath} \quad \text{and} \quad \vec{v}_B = (3.25\,\text{m/s})\,\hat{\imath}}$$

Solutions Manual

Engineering Mechanics: Dynamics

1st Edition

Gary L. Gray
The Pennsylvania State University

Francesco Costanzo
The Pennsylvania State University

Michael E. Plesha
University of Wisconsin–Madison

With the assistance of:
Chris Punshon
Andrew J. Miller
Justin High
Chris O'Brien
Chandan Kumar
Joseph Wyne

Version: August 10, 2009

The McGraw-Hill Companies, Inc.

Copyright © 2002–2010
Gary L. Gray, Francesco Costanzo, and Michael E. Plesha

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Important Information about this Solutions Manual

Even though this solutions manual is nearly complete, we encourage you to visit

`http://www.mhhe.com/pgc`

often to obtain the most up-to-date version. In particular, as of July 30, 2009, please note the following:

- The solutions for Chapters 1 and 2 have been accuracy checked and have been edited by us. They are in their final form.
- The solutions for Chapters 4 and 7 have been accuracy checked and should be error free. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapters 3, 6, 8, and 9 are being accuracy checked and the accuracy checked versions should be available by the end of August 2009. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapter 10 should be available in their entirety by the end of August 2009.

All of the figures in Chapters 6–10 are in color. Color will be added to the figures in Chapters 1–5 over the coming weeks.

Contact the Authors

If you find any errors and/or have questions concerning a solution, please do not hesitate to contact the authors and editors via email at:

`dyn_solns@email.esm.psu.edu`

We welcome your input.

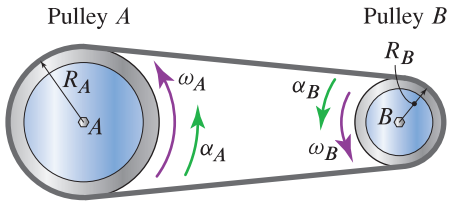
Accuracy of Numbers in Calculations

Throughout this solutions manual, we will generally assume that the data given for problems is accurate to 3 significant digits. When calculations are performed, all intermediate numerical results are reported to 4 significant digits. Final answers are usually reported with 3 significant digits. If you verify the calculations in this solutions manual using the rounded intermediate numerical results that are reported, you should obtain the final answers that are reported to 3 significant digits.

Chapter 6 Solutions

Problem 6.1

Letting $R_A = 7.2\,\text{in.}$ and $R_B = 4.6\,\text{in.}$, and assuming that the belt does not slip relative to pulleys A and B, determine the angular velocity and angular acceleration of pulley B when pulley A rotates at $340\,\text{rad/s}$ while accelerating at $120\,\text{rad/s}^2$.



Solution

Let Q be a point on the periphery of pulley A and let P be a point on the periphery of pulley B. The no slip condition between the pulleys and the belt requires

$$v_Q = v_P \quad \Rightarrow \quad R_A\omega_A = R_B\omega_B \quad \Rightarrow \quad \omega_B = \frac{R_A}{R_B}\omega_A. \tag{1}$$

Substituting in given data, we obtain:

$$\vec{\omega}_B = 532\,\hat{k}\ \text{rad/s}$$

Differentiating Eq. (1) with respect to time, we obtain

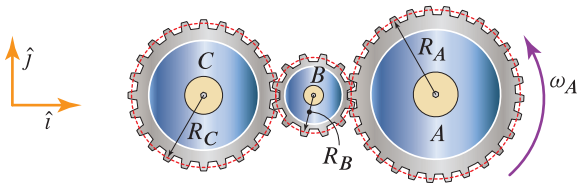
$$\alpha_B = \frac{R_A}{R_B}\alpha_A,$$

which, upon substituting in given data gives

$$\vec{\alpha}_B = 188\,\hat{k}\ \text{rad/s}^2$$

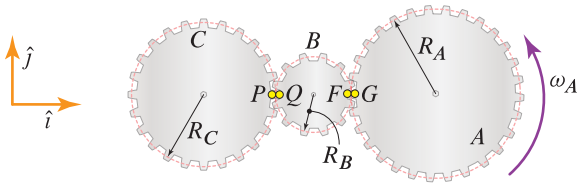
Problem 6.2

Letting $R_A = 8\,\text{in.}$, $R_B = 4.2\,\text{in.}$, and $R_C = 6.5\,\text{in.}$, determine the angular velocity of gears B and C when gear A has an angular speed $|\omega_A| = 945\,\text{rpm}$ in the direction shown.



Solution

Using a vector approach with the component system shown in the figure below, we define points G and F as the points on gears A and B, respectively, that are in contact at this instant. We also define points Q and P as the points on gears B and C, respectively, that are in contact at this instant.



For $\vec{v}_G$ and $\vec{v}_F$ we have

$$\vec{v}_G = \vec{\omega}_A \times \vec{r}_{G/A} = \omega_A \hat{k} \times R_A(-\hat{\imath}) = -\omega_A R_A \hat{\jmath},$$
$$\vec{v}_F = \vec{\omega}_B \times \vec{r}_{F/B} = \omega_B \hat{k} \times R_B \hat{\imath} = \omega_B R_B \hat{\jmath}.$$

Since $\vec{v}_G = \vec{v}_F$,

$$-\omega_A R_A = \omega_B R_B \quad \Rightarrow \quad \omega_B = -\frac{R_A}{R_B}\omega_A \quad \Rightarrow \quad \boxed{\vec{\omega}_B = -1800\,\hat{k}\,\text{rpm}.}$$

Similarly, we can find $\vec{v}_Q$ and $\vec{v}_P$ using

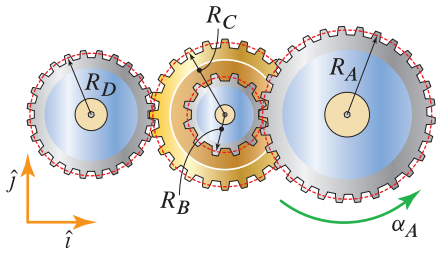
$$\vec{v}_Q = \vec{\omega}_B \times \vec{r}_{Q/B} = -\frac{R_A}{R_B}\omega_A \hat{k} \times R_B(-\hat{\imath}) = R_A \omega_A \hat{\jmath},$$
$$\vec{v}_P = \vec{\omega}_C \times \vec{r}_{P/C} = \omega_C \hat{k} \times R_C \hat{\imath} = \omega_C R_C \hat{\jmath}.$$

Since $\vec{v}_Q = \vec{v}_P$, we have

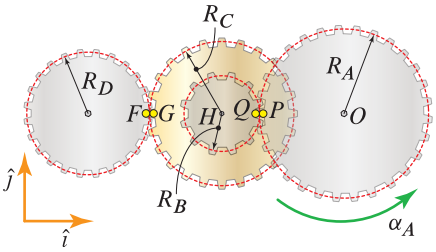
$$R_A \omega_A = \omega_C R_C \quad \Rightarrow \quad \omega_C = \frac{R_A}{R_C}\omega_A \quad \Rightarrow \quad \boxed{\vec{\omega}_B = 1160\,\hat{k}\,\text{rpm}.}$$

Problem 6.3

Letting $R_A = 203\,\text{mm}$, $R_B = 107\,\text{mm}$, $R_C = 165\,\text{mm}$, and $R_D = 140\,\text{mm}$, determine the angular acceleration of gears B, C, and D when gear A has an angular acceleration with magnitude $|\alpha_A| = 47\,\text{rad/s}^2$ in the direction shown. Note that gears B and C are mounted on the same shaft and they rotate together as a unit.



Solution



We let O and H be the centers of gears A and B respectively. Because the gears can not slip relative to each other, we can enforce the no slip condition $\vec{v}_{P/Q} = \vec{0}$ between the contact points P and Q. Using the rigid body velocity equation for gear A we have

$$\vec{v}_P = \vec{v}_O + \omega_A \hat{k} \times \vec{r}_{P/O} \quad \Rightarrow \quad \vec{v}_P = -\omega_A R_A \hat{\jmath}.$$

Doing the same for gear B we have

$$\vec{v}_Q = \vec{v}_H + \omega_B \hat{k} \times \vec{r}_{Q/H} \quad \Rightarrow \quad \vec{v}_Q = \omega_B R_B \hat{\jmath}.$$

Enforcing the no slip condition we must have

$$\vec{v}_P = \vec{v}_Q + \vec{v}_{P/Q} \quad \Rightarrow \quad \vec{v}_P = \vec{v}_Q \quad \Rightarrow \quad -\omega_A R_A = \omega_B R_B \quad \Rightarrow \quad \omega_B = -\frac{R_A}{R_B}\omega_A. \tag{1}$$

Differentiating Eq. (1) with respect to time we get

$$\alpha_B = -\frac{R_A}{R_B}\alpha_A$$

Plugging in known values we get

$$\boxed{\vec{\alpha}_B = -89.2\,\hat{k}\,\text{rad/s}^2.}$$

Because gears B and C are mounted on the same shaft, we have

$$\boxed{\vec{\alpha}_C = -89.2\,\hat{k}\,\text{rad/s}^2.}$$

Next, by considering gears C and D with contact points F and G and proceeding as we did for gears A and B, we get

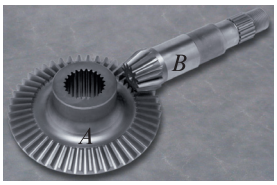
$$\alpha_D = -\frac{R_C}{R_D}\alpha_C. \tag{2}$$

Plugging in known values we get

$$\boxed{\vec{\alpha}_D = 105\,\hat{k}\,\text{rad/s}^2.}$$

Problem 6.4

The bevel gears A and B have nominal radii $R_A = 20\,\text{mm}$ and $R_B = 5\,\text{mm}$, respectively, and their axes of rotation are mutually perpendicular. If the angular speed of gear A is $\omega_A = 150\,\text{rad/s}$, determine the angular speed of gear B.



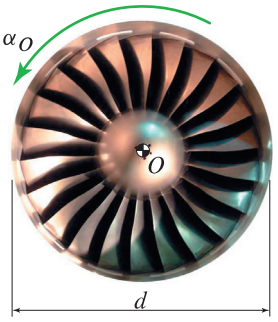
Solution

Since the gears do not slip over their nominal circles relative to one another, we can write

$$\omega_A R_A = \omega_B R_B \quad \Rightarrow \quad \boxed{\omega_B = \frac{R_A}{R_B}\omega_A = 600\,\text{rad/s}.}$$

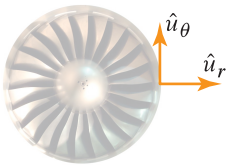
Problem 6.5

A rotor with a fixed spin axis identified by point O is accelerated from rest with an angular acceleration $\alpha_O = 0.5\,\text{rad/s}^2$. If the rotor's diameter is $d = 15\,\text{ft}$, determine the time it takes for the points at the outer edge of the rotor to reach a speed $v_0 = 300\,\text{ft/s}$. Finally, determine the magnitude of the acceleration of these points when the speed v_0 is achieved.



Solution

Points on the outer edge of the rotor have a speed $v = \frac{d}{2}\omega$, where ω is the angular speed of the rotor. Letting ω_0 be the angular speed corresponding to v_0, we must have



$$v_0 = \tfrac{1}{2} d\omega_0 \quad \Rightarrow \quad \omega_0 = \frac{2v_0}{d}.$$

If the angular acceleration is constant, and t_0 is the time taken to go from rest to ω_0, we have

$$\alpha_O t_0 = \omega_0 = \frac{2v_0}{d} \quad \Rightarrow \quad \boxed{t_0 = \frac{2v_0}{\alpha_O d}.}$$

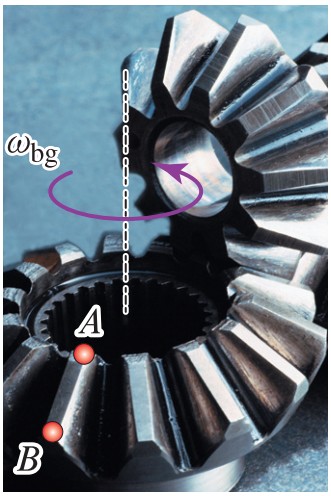
The angular acceleration of points a distance $\frac{d}{2}$ from O is

$$\vec{a}_d = \vec{\alpha}_O \times \vec{r}_{d/O} - \omega_0^2 \vec{r}_{d/O} \quad \Rightarrow \quad \vec{a}_d = \alpha_O \hat{k} \times \tfrac{1}{2} d\,\hat{u}_r - \frac{2v_0^2}{d}\hat{u}_r \quad \Rightarrow \quad \vec{a}_d = -\frac{2v_0^2}{d}\hat{u}_r + \tfrac{1}{2}\alpha_O d\,\hat{u}_\theta,$$

$$\boxed{|\vec{a}_d| = \sqrt{\frac{4v_0^4}{d^2} + \frac{\alpha_O^2 d^2}{4}} = 12{,}000\,\text{ft/s}^2.}$$

Problem 6.6

Do points A and B on the surface of the bevel gear (bg), which rotates with angular velocity ω_{bg}, move relative to each other? At what rate does the distance between A and B change?



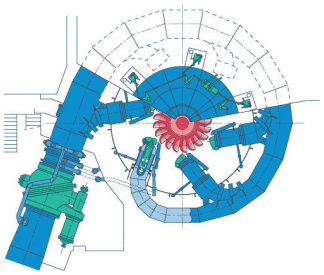
Solution

Points A and B move relative to one another in the sense that the velocity of A is different from the velocity of B. The difference between the velocity of A and the velocity of B arises from the fact that points A and B are at different distances from the axis of rotation. As far as the distance between A and B is concerned, if we model the gear as a rigid body, then that distance must remain constant.

Problem 6.7

A Pelton turbine (a type of turbine used in hydroelectric power generation) is spinning at 1100 rpm when the water jets acting on it are shut off, thus causing the turbine to slow down. Assuming that the angular deceleration rate is constant and equal to 1.31 rad/s^2, determine the time it takes for the turbine to stop. In addition, determine the number of revolutions of the turbine during the spin-down.





Solution

Letting ω and α be the angular velocity and angular acceleration of the turbine respectively, we have $\alpha = -1.31$ rad/s^2, and at $t = 0$, $\omega(0) = 1100$ rpm. Since the angular acceleration is constant we must have

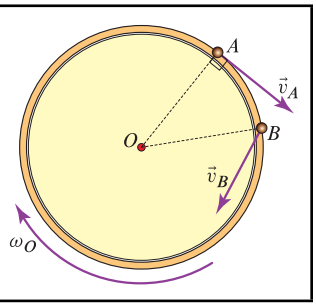
$$\omega(t) = \omega(0) + t\alpha. \tag{1}$$

At the stop time $\omega = 0$, so we have $\omega(t_{\text{stop}}) = 0$. Plugging this into Eq. (1) we have

$$t_{\text{stop}} = -\frac{\omega(0)}{\alpha} = 87.9\,\text{s}.$$

Problem 6.8

The velocities of points A and B on a disk, which is undergoing planar motion, are such that $|\vec{v}_A| = |\vec{v}_B|$. Is it possible for the disk to be rotating about a fixed axis going through the center of the disk at O? Explain.

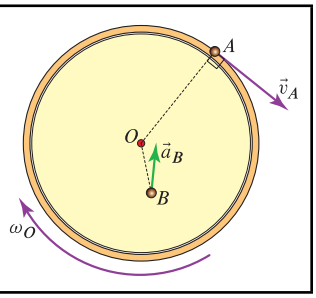


Solution

The answer is no. The reason is that if the disk were spinning about a fixed axis perpendicular to the plane of the figure, the velocity of B would have been perpendicular to the segment OB. However, the figure shows that the velocity vector of B is not perpendicular to OB and therefore the motion of the disk cannot be a rotation about a fixed axis perpendicular to the plane of the figure.

Problem 6.9

The velocity of a point A and the acceleration of a point B on a disk undergoing planar motion are shown. Is it possible for the disk to be rotating about a fixed axis going through the center of the disk at O? Explain.

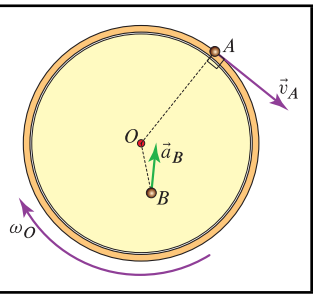


Solution

Yes, it is possible that the motion of the disk is a fixed-axis rotation about O. The velocity vector of point A is perpendicular to the segment OA, which does not guarantee that the motion of the disk is a fixed-axis rotation, but it does not exclude it either.

Problem 6.10

Assuming that the disk shown is rotating about a fixed axis going through its center at O, determine whether the disk's angular velocity is constant, increasing, or decreasing.

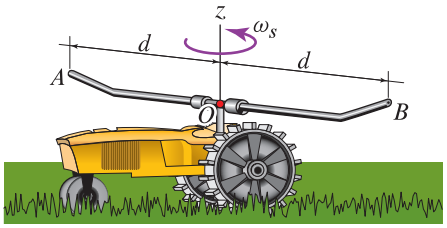


Solution

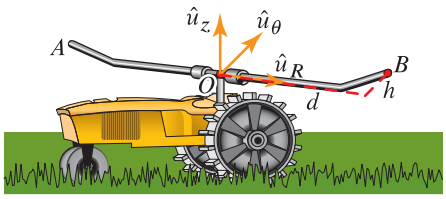
The angular velocity of the disk is decreasing. In fact, if the angular velocity were constant, then the acceleration of point B would be directed toward the center O. Since the figure indicates that this is not the case, we know that the angular velocity of the disk must either be increasing or decreasing. With this in mind, notice that the the vector $\vec{a}_B$ has a component that is opposite to the vector $\vec{v}_B = \vec{\omega}_O \times \vec{r}_{B/O}$. This in turn indicates that the speed of point B is decreasing and therefore that the spin rate of the disk as a whole is slowing down.

Problem 6.11

The sprinkler shown consists of a pipe AB mounted on a hollow vertical shaft. The water comes in the horizontal pipe at O and goes out the nozzles at A and B, causing the pipe to rotate. Letting $d = 7\,\text{in.}$, determine the angular velocity of the sprinkler ω_s, and $|\vec{a}_B|$, the magnitude of the acceleration of B, if B is moving with a constant speed $v_B = 20\,\text{ft/s}$. Assume that the sprinkler does not roll on the ground.



Solution



Because the arms of the sprinkler are modeled as rigid bodies, we know that

$$\vec{v}_B = \omega_S\,\hat{k} \times \vec{r}_{B/O}.$$

Defining the position vector from O to B as shown to the right, we also have

$$\vec{v}_B = \omega_S\,\hat{k} \times (d\,\hat{u}_R + h\,\hat{u}_z) = d\omega_S\,\hat{u}_\theta \quad \Rightarrow \quad d\omega_S = v_B.$$

Plugging in known values and solving, we have

$$\boxed{\vec{\omega}_S = \frac{v_B}{d} = 34.3\,\hat{u}_z\,\text{rad/s}.}$$

Since O is not in the plane of motion of B, we have

$$\vec{a}_B = \vec{\omega}_S \times (\vec{\omega}_S \times \vec{r}_{B/O}) = -\omega_S^2 d\,\hat{u}_r \quad \Rightarrow \quad |\vec{a}_B| = \frac{v_B^2}{d}.$$

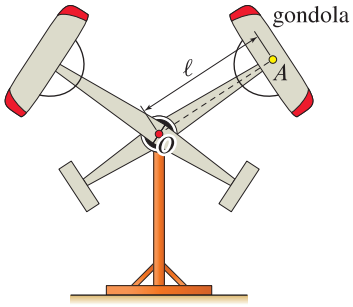
Plugging in known values we have

$$\boxed{|\vec{a}_B| = 686\,\text{ft/s}^2.}$$

Problem 6.12

In a carnival ride, two gondolas spin in opposite directions about a fixed axis. If $\ell = 4\,\text{m}$, determine the maximum constant angular speed of the gondolas if the magnitude of the acceleration of point A is not to exceed $2.5g$.





Solution

Under the stated conditions, point A would be moving in uniform circular motion. Therefore, we would have

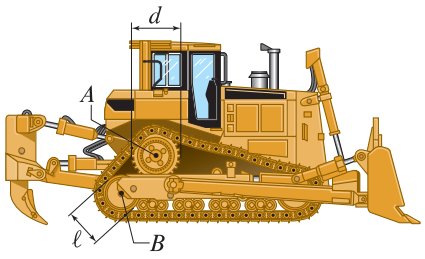
$$\left|\vec{a}_A\right| = \omega_{OA}^2 \ell \leq 2.5g \quad \Rightarrow \quad \omega_{OA} \leq \sqrt{2.5g/\ell}.$$

Using the given numerical value of ℓ, we therefore have

$$(\omega_{OA})_{\text{max}} = 2.48\,\text{rad/s}.$$

Problem 6.13

The tractor shown is stuck with its right track off the ground, and therefore the track is able to move without causing the tractor to move. Letting the radius of sprocket A be $d = 2.5$ ft and the radius of sprocket B be $\ell = 2$ ft, determine the angular speed of wheel B if the sprocket A is rotating at 1 rpm.



Solution

Because the track does not slip relative to the sprocket A we must have

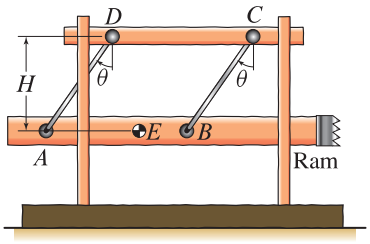
$$\omega_A \frac{d}{2} = \omega_B \frac{\ell}{2} \quad \Rightarrow \quad \omega_B = \omega_A \frac{d}{\ell}.$$

Plugging in known values we have

$$\boxed{\omega_B = 1.25 \text{ rpm}.}$$

Problem 6.14

A battering ram is suspended in its frame via bars AD and BC, which are identical and pinned at their endpoints. At the instant shown, point E on the ram moves with a speed $v_0 = 15\,\text{m/s}$. Letting $H = 1.75\,\text{m}$ and $\theta = 20°$, determine the magnitude of the angular velocity of the ram at this instant.



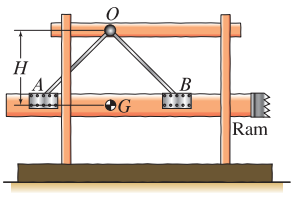
Solution

Since bars AD and BC are identical and parallel to one another, we must conclude that the velocities of points A and B are equal to one another. Since point A and B are coplanar and distinct, then we must conclude that the ram is simply translating, which implies that

$$|\vec{\omega}_{\text{ram}}| = 0.$$

Problem 6.15

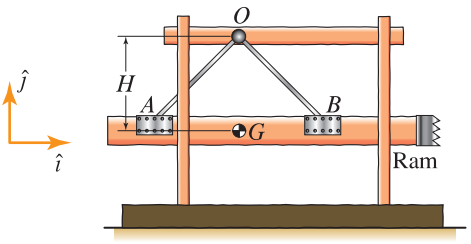
A battering ram is suspended in its frame via bars OA and OB, which are pinned at O. At the instant shown, point G on the ram is moving to the right with a speed $v_0 = 15\,\text{m/s}$. Letting $H = 1.75\,\text{m}$, determine the angular velocity of the ram at this instant.



Solution

The ram and its supports OA and OB form a single rigid body, therefore we must have

$$\vec{v}_G = \omega_{\text{ram}}\,\hat{k} \times \vec{v}_{G/O} = \omega\,\hat{k} \times (-H)\,\hat{\jmath} = \omega_{\text{ram}} H\,\hat{\imath}.$$



We are told that $|\vec{v}_G| = v_0$, which means that

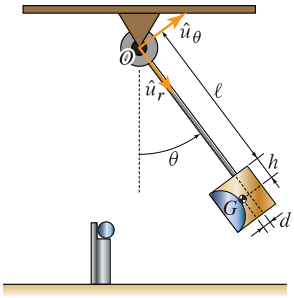
$$\omega_{\text{ram}} H = v_0.$$

Plugging in known values we get

$$\vec{\omega}_{\text{ram}} = \frac{v_0}{H}\,\hat{k} = 8.57\,\hat{k}\ \text{rad/s}$$ 

Problems 6.16 and 6.17

The hammer of a Charpy impact toughness test machine has the geometry shown, where G, is the mass center of the hammer head. Use Eqs. (6.8) and (6.13) and write your answers in terms of the component system shown.



Problem 6.16 Determine the velocity and acceleration of G, assuming $\ell = 500\,\text{mm}$, $h = 65\,\text{mm}$, $d = 25\,\text{mm}$, $\dot{\theta} = -5.98\,\text{rad/s}$, and $\ddot{\theta} = -8.06\,\text{rad/s}^2$.

Problem 6.17 Determine the velocity and acceleration of G as a function of the geometric parameters shown, $\dot{\theta}$, and $\ddot{\theta}$.

Solution to 6.16

Modeling the system as rigid and observing that point O is fixed, we have

$$\vec{v}_H = \vec{v}_O + \vec{\omega}_p \times \vec{r}_{H/O} = \vec{\omega}_p \times \vec{r}_{H/O}, \tag{1}$$

where the vectors $\vec{\omega}_p$ and $\vec{r}_{H/O}$ are given by

$$\vec{\omega}_p = \dot{\theta}\,\hat{k} = -5.98\,\hat{k}\,\text{rad/s} \quad \text{and} \quad \vec{r}_{H/O} = (\ell + h)\,\hat{u}_r - d\,\hat{u}_\theta = (0.565\,\text{m})\,\hat{u}_r - (0.025)\,\hat{u}_\theta. \tag{2}$$

Substituting the expressions in Eqs. (2) into Eq. (1) and carrying out the cross product, we have

$$\boxed{\vec{v}_H = (-0.15\,\hat{u}_r - 3.38\,\hat{u}_\theta)\,\text{m/s}.}$$

Again recalling that O is a fixed point, for the acceleration, we have

$$\vec{a}_H = \vec{\alpha}_p \times \vec{r}_{H/O} - \omega_p^2 \vec{r}_{H/O}, \tag{3}$$

where $\vec{\alpha}_p = \ddot{\theta}\,\hat{k} = -8.06\,\hat{k}\,\text{rad/s}$ is the angular acceleration of the pendulum. Using this expression for $\vec{\alpha}_p$ and the expression for $\vec{r}_{H/O}$ in the last of Eqs. (2), after simplifying, we have

$$\boxed{\vec{a}_H = (-20.4\,\hat{u}_r - 3.66\,\hat{u}_\theta)\,\text{m/s}^2.}$$

Solution to 6.17

Modeling the system as rigid and observing that point O is fixed, we have

$$\vec{v}_H = \vec{v}_O + \vec{\omega}_p \times \vec{r}_{H/O} = \vec{\omega}_p \times \vec{r}_{H/O}, \tag{4}$$

where the vectors $\vec{\omega}_p$ and $\vec{r}_{H/O}$ are given by

$$\vec{\omega}_p = \dot{\theta}\,\hat{k} \quad \text{and} \quad \vec{r}_{H/O} = (\ell + h)\,\hat{u}_r - d\,\hat{u}_\theta. \tag{5}$$

Substituting the expressions in Eqs. (5) into Eq. (4) and carrying out the cross product, we have

$$\boxed{\vec{v}_H = \dot{\theta} d\,\hat{u}_r + \dot{\theta}(\ell + h)\,\hat{u}_\theta.}$$

Again recalling that O is a fixed point, for the acceleration, we have

$$\vec{a}_H = \vec{\alpha}_p \times \vec{r}_{H/O} - \omega_p^2 \vec{r}_{H/O}, \tag{6}$$

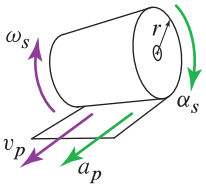
where $\vec{\alpha}_p = \ddot{\theta}\,\hat{k}$ is the angular acceleration of the pendulum. Using this expression for $\vec{\alpha}_p$ and the expression for $\vec{r}_{H/O}$ in the second of Eqs. (5), after simplifying, we have

$$\boxed{\vec{a}_H = \left[\ddot{\theta} d - \dot{\theta}^2(\ell + h)\right]\hat{u}_r + \left[\ddot{\theta}(\ell + h) + \dot{\theta}^2 d\right]\hat{u}_\theta.}$$

Problem 6.18

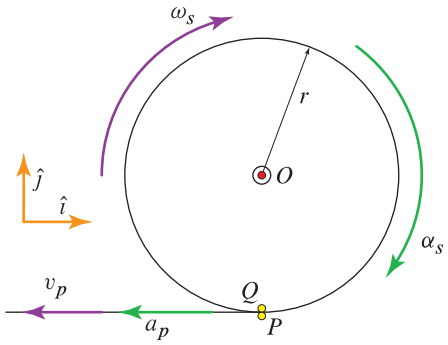
At the instant shown the paper is being unrolled with a speed $v_p = 7.5\,\text{m/s}$ and an acceleration $a_p = 1\,\text{m/s}^2$. If at this instant the outer radius of the roll is $r = 0.75\,\text{m}$, determine the angular velocity ω_s and acceleration α_s of the roll.





Solution

Referring to the figure shown to the right, the spool is in a fixed axis rotation about its center O. It is assumed that the paper is just coming off the spool does not slip relative to the paper that is still on the spool. Let P be a point on the outmost layer of paper that is just coming off the spool and let Q be the point on the paper layer immediately below the outmost layer and on the same radial line as P. Due to the no slip assumption, we have $\vec{v}_P = \vec{v}_Q$, i.e.,



$$\vec{v}_P = -v_P\,\hat{\imath} = \vec{v}_Q = \vec{\omega}_s \times \vec{r}_{Q/O}, \qquad (1)$$

where $\vec{\omega}_s = -\omega_s\,\hat{k}$ is the angular velocity of the spool and $\vec{r}_{Q/O} = -r\,\hat{\jmath}$. Substituting these expressions in Eq. (1) and solving for ω_s we have

$$\boxed{\vec{\omega}_s = -(v_P/r)\,\hat{k}}$$

Enforcing the no slip condition for the acceleration gives that $\vec{a}_Q \cdot \hat{\imath} = \vec{a}_P \cdot \hat{\imath}$, that is, the components of the accelerations of Q and P are the equal in the direction tangent to the contact. Now, observe that

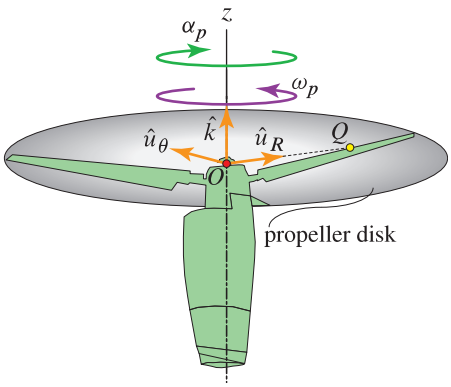
$$\vec{a}_P = -a_P\,\hat{\imath} \quad \text{and} \quad \vec{a}_Q = \vec{\alpha}_s \times (-r\,\hat{\jmath}) - \omega_s^2(-r\,\hat{\jmath}) = \alpha_s r\,\hat{\imath} - \frac{v_P^2}{r}\,\hat{\jmath}. \qquad (2)$$

Using the expressions in Eq. (2) to enforce the no slip condition we have $-\alpha_s r = -a_P$, which can be solved for α_s to obtain

$$\boxed{\vec{\alpha}_s = -(a_p/r)\,\hat{k}}$$

Problem 6.19

At the instant shown, the propeller is rotating with an angular velocity $\omega_p = 400\,\text{rpm}$ in the positive z direction and an angular acceleration $\alpha_p = -2\,\text{rad/s}^2$ in the negative z direction, where the z axis is also the spin axis of the propeller. Consider the cylindrical coordinate system shown, with origin O on the z axis and unit vector $\hat{u}_R$ pointing toward point Q on the propeller which is 14 ft away from the spin axis. Compute the velocity and acceleration of Q. Express your answer using the component system shown.



Solution

The propeller is a rigid body, so we must have

$$\vec{v}_Q = \vec{\omega}_P \times \vec{r}_{Q/O} = \omega_P\,\hat{u}_z \times R_Q\,\hat{u}_R = \omega_P R_Q\,\hat{u}_\theta$$

where $\omega_P = 400\frac{2\pi}{60} = 41.89\,\text{rad/s}$ and $R_Q = 14\,\text{ft}$. Plugging in known values we get

$$\boxed{\vec{v}_Q = 586\,\hat{u}_\theta\ \text{ft/s}.}$$

Similarly for $\vec{a}_Q$ we get

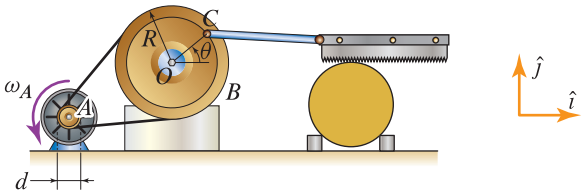
$$\begin{aligned}\vec{a}_Q &= -\omega_P^2\vec{r}_{Q/O} + \alpha_P\,\hat{k} \times \vec{r}_{Q/O}\\ &= -\omega_P^2 R_Q\,\hat{u}_R + \alpha_P\,\hat{k} \times R_Q\\ &= -\omega_P^2 R_Q\,\hat{u}_R + \alpha_P R_Q \hat{u}_\theta.\end{aligned}$$

Plugging in known values we get

$$\boxed{\vec{a}_Q = [-24.6\times10^3\,\hat{u}_R - 28.0\,\hat{u}_\theta]\,\text{ft/s}^2.}$$

Problem 6.20

The wheel A, with diameter $d = 5\,\text{cm}$, is mounted on the shaft of the motor shown and is rotating with a constant angular speed $\omega_A = 250\,\text{rpm}$. The wheel B, with center at the fixed point O, is connected to A with a belt, which does not slip relative to A or B. The radius of B is $R = 12.5\,\text{cm}$. At point C the wheel B is connected to a saw. If point C is at distance $\ell = 10\,\text{cm}$ from O, determine the velocity and acceleration of C when $\theta = 20°$. Express your answers using the component system shown.



Solution

Since the belt does not slip relative to A or B, we have

$$\omega_A \frac{d}{2} = \omega_B R \quad \Rightarrow \quad \omega_B = \frac{d}{2R}\omega_A. \tag{1}$$

Next, observing that the crank is in a fixed axis rotation about O, we have

$$\vec{v}_C = \vec{\omega}_B \times \vec{r}_{C/O}, \tag{2}$$

where the

$$\vec{\omega}_B = \frac{d\omega_A}{2R}\hat{k} \quad \text{and} \quad \vec{r}_{C/O} = \ell(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \tag{3}$$

Substituting Eqs. (3) into Eq. (2), after simplifying we obtain

$$\boxed{\vec{v}_C = \frac{d\ell\omega_A}{2R}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) = (-0.179\,\hat{\imath} + 0.492\,\hat{\jmath})\,\text{m/s}.}$$

Recalling that the angular velocity of the wheel B is constant and that O is a fixed point, we have

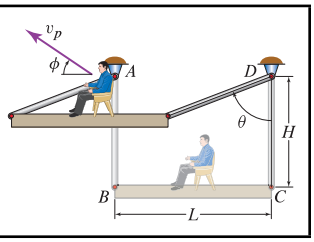
$$\vec{a}_C = -\omega_B^2 \vec{r}_{C/O}. \tag{4}$$

Hence we have

$$\boxed{\vec{a}_C = -\frac{d^2\ell\omega_A^2}{4R^2}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -(2.58\,\hat{\imath} + 0.938\,\hat{\jmath})\,\text{m/s}^2.}$$

Problem 6.21

In a contraption built by a fraternity, a person is sitting at the center of a swinging platform with length $L = 12\,\text{ft}$ that is suspended via two identical arms each of length $H = 10\,\text{ft}$. Determine the angle θ and the angular speed of the arms if the person is moving upward and to the left with a speed $v_p = 25\,\text{ft/s}$ at the angle $\phi = 33°$.



Solution

The platform and its rider are undergoing a curvilinear translation. Therefore, the velocity vector of the person, $\vec{v}_P$, is equal to the velocity vectors of points C and D.

$$\vec{v}_C = \vec{v}_P = \vec{v}_D \quad \Rightarrow \quad \phi = \theta \quad \Rightarrow \quad \boxed{\theta = 33.0°.}$$

Also, we must have

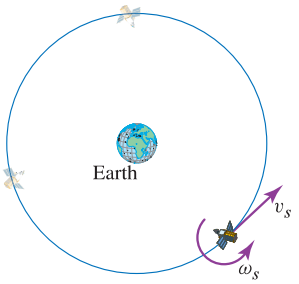
$$v_P = |\omega_{AB}|H = |\omega_{CD}|H.$$

Plugging in known values we get

$$\boxed{|\omega_{AB}| = |\omega_{CD}| = \frac{v_P}{H} = 2.50\,\text{rad/s}.}$$

Problem 6.22

A *geosynchronous equatorial orbit* is a circular orbit above the Earth's equator that has a period of 1 day (these are sometimes called *geostationary orbits*). These geostationary orbits are of great importance for telecommunications satellites because a satellite orbiting with the same angular rate as the rotation rate of the Earth will appear to hover in the same point in the sky as seen by a person standing on the surface of the Earth. Using this information, modeling a geosynchronous satellite as a rigid body, and noting that the satellite has been stabilized so that the same side always faces the Earth, determine the angular speed ω_s of the satellite.



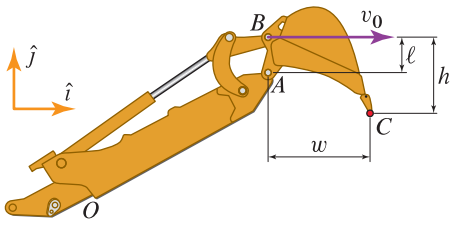
Solution

The motion of the satellite is a fixed-axis rotation about the center of the Earth where the axis is perpendicular to the plane of the satellite's orbit. For the satellite to be geosynchronous, the satellite's angular velocity must be such that it moves through 2π radians every day

$$\omega_s = \frac{2\pi \text{ rad}}{24\,\text{h}} \frac{1\,\text{h}}{3600\,\text{s}} = 72.7\times10^{-6}\,\text{rad/s}.$$

Problem 6.23

The bucket of a backhoe is being operated while holding the arm OA fixed. At the instant shown, point B has a horizontal component of velocity $v_0 = 0.25\,\text{ft/s}$ and is vertically aligned with point A. Letting $\ell = 0.9\,\text{ft}$, $w = 2.65\,\text{ft}$, and $h = 1.95\,\text{ft}$, determine the velocity of point C. In addition, assuming that, at the instant shown, point B is not accelerating in the horizontal direction, compute the acceleration of point C. Express your answers using the component system shown.



Solution

Since the bucket is in a fixed axis rotation about point A, we have

$$\vec{v}_B = \omega_{AB}\,\hat{k} \times \vec{r}_{B/A} = v_0\,\hat{\imath} \quad \Rightarrow \quad \omega_{AB}\,\hat{k} \times \ell\,\hat{\jmath} = v_0\,\hat{\imath} \quad \Rightarrow \quad -\ell\omega_{AB}\,\hat{\imath} = v_0\,\hat{\imath}$$

$$\Rightarrow \quad \omega_{AB} = -\frac{v_0}{\ell} = -0.2778\,\text{rad/s},$$

where ω_{AB} is the angular velocity of the bucket. Since B is not accelerating in the horizontal direction, $\alpha_{AB} = 0$ and we have

$$\vec{v}_C = \omega_{AB}\,\hat{k} \times \vec{r}_{C/A} = \omega_{AB} \times [w\,\hat{\imath} - (h-\ell)\,\hat{\jmath}] \quad \Rightarrow \quad \vec{v}_C = (h-\ell)\omega_{AB}\,\hat{\imath} + \omega_{AB} w\,\hat{\jmath}.$$

Upon substituting in given quantities, we have

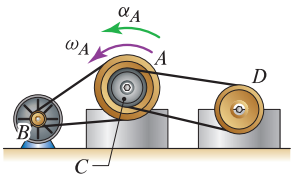
$$\boxed{\vec{v}_C = (-0.292\,\hat{\imath} - 0.736\,\hat{\jmath})\,\text{ft/s}.}$$

Since $\alpha_{AB} = 0$, $\vec{a}_C = -\omega_{AB}^2 \vec{r}_{C/A} = -v_0^2[w\,\hat{\imath} - (h-\ell)\,\hat{\jmath}]$, which means that

$$\boxed{\vec{a}_C = (-0.204\,\hat{\imath} + 0.0810\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 6.24

Wheels A and C are mounted on the same shaft and rotate together. Wheels A and B are connected via a belt and so are wheels C and D. The axes of rotation of all the wheels are fixed, and the belts do not slip relative to the wheels they connect. If, at the instant shown, wheel A has an angular velocity $\omega_A = 2\,\text{rad/s}$ and an angular acceleration $\alpha_A = 0.5\,\text{rad/s}^2$, determine the angular velocity and acceleration of wheels B and D. The radii of the wheels are $R_A = 1\,\text{ft}$, $R_B = 0.25\,\text{ft}$, $R_C = 0.6\,\text{ft}$, and $R_D = 0.75\,\text{ft}$.



Solution

Since the belts do not slip relative to the wheels they connect, we must have that

$$\omega_A R_A = \omega_B R_B, \qquad \alpha_A R_A = \alpha_B R_B,$$
$$\omega_C R_C = \omega_D R_D, \qquad \alpha_C R_C = \alpha_D R_D,$$

where

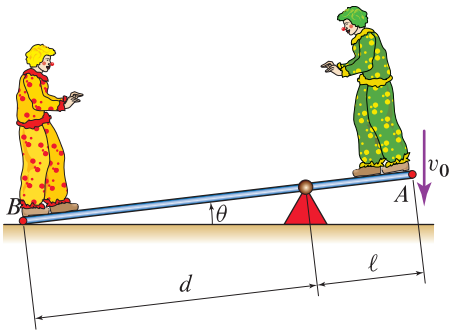
$$\omega_A = \omega_C, \qquad \text{and} \qquad \alpha_A = \alpha_C.$$

Using the given quantities, we have

$$\vec{\omega}_B = \frac{R_A}{R_B}\omega_A\,\hat{k} = 8.00\,\hat{k}\,\text{rad/s}$$
$$\vec{\alpha}_B = \frac{R_A}{R_B}\alpha_A\,\hat{k} = 2.00\,\hat{k}\,\text{rad/s}^2$$
$$\vec{\omega}_D = \frac{R_C}{R_D}\omega_C\,\hat{k} = 2.50\,\hat{k}\,\text{rad/s}$$
$$\vec{\alpha}_D = \frac{R_C}{R_D}\alpha_C\,\hat{k} = 0.0625\,\hat{k}\,\text{rad/s}^2$$

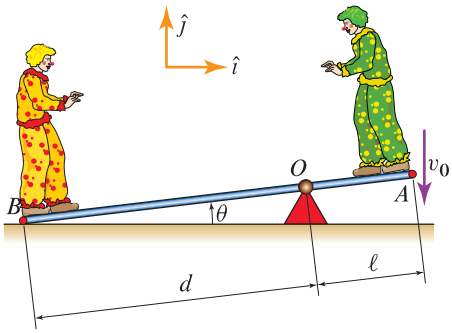
Problem 6.25

An acrobat lands at the end A of a board and, at the instant shown, point A has a downward vertical component of velocity $v_0 = 5.5\,\text{m/s}$. Letting $\theta = 15°$, $\ell = 1\,\text{m}$, and $d = 2.5\,\text{m}$, determine the vertical component of velocity of point B at this instant if the board is modeled as a rigid body.



Solution

Letting the pivot of the board point be O, and modeling the board as a rigid body, we know that



$$\begin{aligned}\vec{v}_A = \vec{\omega}_{AB} \times \vec{r}_{A/O} &= \omega_{AB}\,\hat{k} \times \ell(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= \omega_{AB}\ell(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}).\end{aligned}$$

Looking at the y component of $\vec{v}_A$ we have

$$v_{Ay} = -v_0 = \omega_{AB}\ell\cos\theta \quad \Rightarrow \quad \omega_{AB} = -\frac{v_0}{\ell\cos\theta}.$$

Similarly, for point B we know that

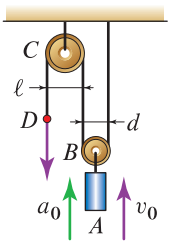
$$\vec{v}_B = \omega_{AB}\,\hat{k} \times \vec{r}_{B/O} = -\frac{v_0}{\ell\cos\theta}\,\hat{k} \times (-d\cos\theta\,\hat{\imath} - d\sin\theta\,\hat{\jmath}) = -\frac{v_0 d}{\ell}\tan\theta\,\hat{\imath} + \frac{v_0 d}{\ell}\,\hat{\jmath}.$$

Plugging known values into the y component of $\vec{v}_B$ we have

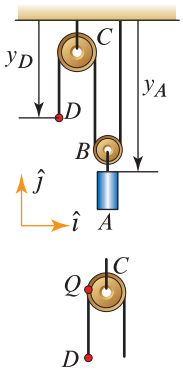
$$\boxed{v_{By} = \frac{v_0 d}{\ell} = 13.8\,\text{m/s}\ \hat{\jmath}\ \hat{\imath}.}$$

Problem 6.26

At the instant shown, A is moving upward with a speed $v_0 = 5\,\text{ft/s}$ and acceleration $a_0 = 0.65\,\text{ft/s}^2$. Assuming that the rope that connects the pulleys does not slip relative to the pulleys and letting $\ell = 6\,\text{in.}$ and $d = 4\,\text{in.}$, determine the angular velocity and angular acceleration of pulley C.



Solution



We can define the length of the rope as

$$L = y_D + 2y_A.$$

Taking two time derivatives of this equation, and remembering that the overall length is constant because the rope is inextensible, we have

$$0 = \dot{y}_D + 2\dot{y}_A \quad \Rightarrow \quad \dot{y}_D = -2\dot{y}_A = 2v_0$$
$$0 = \ddot{y}_D + 2\ddot{y}_A \quad \Rightarrow \quad \ddot{y}_D = -2\ddot{y}_A = 2a_0$$

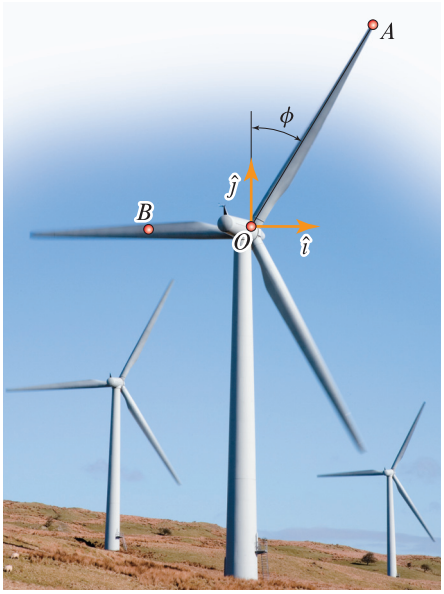
Because the point Q is contacting the rope, and the rope is not slipping, point Q has the same velocity and vertical component of acceleration as D.

$$v_Q = v_D = 2v_0 = \omega_C \frac{\ell}{2} \quad \Rightarrow \quad \boxed{\vec{\omega}_C = 4\frac{v_0}{\ell} = 40.0\,\hat{k}\ \text{rad/s}}$$

$$a_{Qy} = a_D = 2a_0 = \alpha_C \frac{\ell}{2} \quad \Rightarrow \quad \boxed{\vec{\alpha}_C = 4\frac{a_0}{\ell} = 5.20\,\hat{k}\ \text{rad/s}^2}$$

Problem 6.27

At the instant shown, the angle $\phi = 30°$, $|\vec{v}_A| = 292\,\text{ft/s}$, and the turbine is rotating clockwise. Letting $\overline{OA} = R$, $\overline{OB} = R/2$, $R = 182\,\text{ft}$, and treating the blades as being equally spaced, determine the velocity of point B at the given instant and express it using the component system shown.



Solution

At the instant shown $\phi = 30°$, so the positions of A and B in the given cartesian coordinate system are

$$\vec{r}_A = \vec{r}_{A/O} = R(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$$
$$\vec{r}_B = \vec{r}_{B/O} = -\frac{R}{2}\,\hat{\imath}.$$

Because the windmill is being treated as a rigid body, we also know

$$\vec{v}_A = \vec{\omega}_T \times \vec{r}_{A/O} = \omega_T\,\hat{k} \times \vec{r}_{A/O} = \omega_T R(-\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) \quad \Rightarrow \quad v_A = |\omega_T| R = 292\,\text{ft/s}$$

where $\vec{\omega}_T$ is the angular velocity of the turbine and v_A is the speed of A. Solving for ω_T we have

$$\omega_T = -\frac{v_A}{R}$$

because the turbine is rotating clockwise. The velocity of point B is then

$$\vec{v}_B = \vec{\omega}_T \times \vec{r}_{B/O} = -\frac{v_A}{R}\,\hat{k} \times -\frac{R}{2}\,\hat{\imath}$$

$$\boxed{\vec{v}_B = \frac{v_A}{2}\,\hat{\jmath} = 146\,\hat{\jmath}\,\text{ft/s}.}$$

Problem 6.28

At the instant shown, the angle $\phi = 30°$, the turbine is rotating clockwise, and $\vec{a}_B = (70.8\,\hat{\imath} - 12.8\,\hat{\jmath})\,\text{m/s}^2$. Letting $\overline{OA} = R$, $\overline{OB} = R/2$, $R = 55.5\,\text{m}$, and assuming the blades are equally spaced, determine the angular velocity and angular acceleration of the turbine blades as well as the acceleration of point A at the given instant.



Solution

At the instant shown $\phi = 30°$, so the positions of A and B in the given cartesian coordinate system are

$$\vec{r}_A = \vec{r}_{A/O} = R(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$$
$$\vec{r}_B = \vec{r}_{B/O} = -\frac{R}{2}\,\hat{\imath}.$$

Because the windmill is being treated as a rigid body, we also know

$$\begin{aligned}\vec{a}_B &= -\omega_T^2\vec{r}_{B/O} + \alpha_T\,\hat{k}\times\vec{r}_{B/O}\\ &= -\omega_T^2 - \frac{R}{2}\,\hat{\imath} + \alpha_T\,\hat{k}\times\left(-\frac{R}{2}\right)\hat{\imath}\\ &= \omega_T^2\frac{R}{2}\,\hat{\imath} - \alpha_T\frac{R}{2}\,\hat{\jmath} = a_{Bx}\,\hat{\imath} + a_{By}\,\hat{\jmath}.\end{aligned}$$

Therefore,

$$\boxed{\begin{aligned}\vec{\omega}_T &= -\sqrt{\frac{2a_{Bx}}{R}}\,\hat{k} = -1.60\,\hat{k}\,\text{rad/s},\\ \vec{\alpha}_T &= -\frac{2a_{By}}{R}\,\hat{k} = 0.461\,\hat{k}\,\text{rad/s}^2.\end{aligned}}$$

The negative root is chosen for ω_T because the turbine is rotating clockwise. The acceleration of A is

$$\begin{aligned}\vec{a}_A &= -\omega_T^2\vec{r}_{A/O} + \alpha_T\,\hat{k}\times\vec{r}_{A/O}\\ &= -\omega_T^2 R(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) + \alpha_T\,\hat{k}\times R(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})\\ &= -R(\omega_T^2\sin\phi + \alpha_T\cos\phi)\,\hat{\imath} + R(\alpha_T\sin\phi - \omega_T^2\cos\phi)\,\hat{\jmath}.\end{aligned}$$

Plugging in the known values of R, ω_T, α_T, and ϕ we get

$$\boxed{\vec{a}_A = (-93.0\,\hat{\imath} - 110\,\hat{\jmath})\,\text{m/s}^2.}$$

Problems 6.29 through 6.31





A bicycle has wheels 700 mm in diameter and a gear set with the dimensions given in the table below.

Crank			
Sprocket	C1	C2	C3
No. of Cogs	26	36	48
Radius (mm)	52.6	72.8	97.0

Cassette (9 speeds)									
Sprocket	S1	S2	S3	S4	S5	S6	S7	S8	S9
No. of Cogs	11	12	14	16	18	21	24	28	34
Radius (mm)	22.2	24.3	28.3	32.3	36.4	42.4	48.5	56.6	68.7

Problem 6.29 If a cyclist has a cadence of 1 Hz, determine the angular speed of the rear wheel in rpm when using the combination of C3 and S2. In addition, knowing that the speed of the cyclist is equal to the speed of a point on the tire relative to the wheel's center, determine the cyclist's speed in m/s.

Problem 6.30 If a cyclist has a cadence of 68 rpm, determine which combination of chain ring (a sprocket mounted on the crank) and (rear) sprocket would *most closely* make the rear wheel rotate with an angular speed of 127 rpm. Having found a chain ring/sprocket combination, determine the wheel's exact angular speed corresponding to the chosen chain ring/sprocket combination and the given cadence.

Problem 6.31 If a cyclist is pedaling so that the rear wheel rotates with an angular speed of 16 rad/s, determine all possible (rear) sprocket/chain ring (crank-mounted sprocket) combinations that would allow him or her to pedal with a frequency within the range 1.00–1.25 Hz.

Solution to 6.29

Using C3 and S2, $R_C = 97.0\,\text{mm}$, $R_S = 24.3\,\text{mm}$, and $R_W = 0.7\,\text{m}$ where the subscripts $C, S,$ and W denote the crank, sprocket and wheel respectively. We let $\omega_C, \omega_S, \omega_W$ be the angular velocities of the crank, sprocket, and wheel. The sprocket is fixed to the wheel, so

$$\omega_S = \omega_W .$$

Since the chain does not slip relative to the crank or sprocket, we have

$$\omega_S R_S = \omega_C R_C \quad \Rightarrow \quad \omega_S = \omega_W = \frac{R_C}{R_S}\omega_C .$$

Because the cyclist's cadence is 1 Hz, $\omega_C = 2\pi$ rad/s. Plugging in known values, we get

$$\boxed{\omega_W = 25.1\,\text{rad/s} = 240\,\text{rpm}.}$$

We are also told that $v_{\text{cyclist}} = v_{P/O}$ where $\vec{v}_{P/O}$ is the velocity of a point on the rim of the wheel relative to the wheel's center O. From this, we have

$$\vec{v}_{P/O} = \vec{\omega}_W \times \vec{r}_{P/O} \quad \Rightarrow \quad v_{\text{cyclist}} = v_{P/O} = \omega_W R_W .$$

Plugging in known values we get

$$v_{\text{cyclist}} = \frac{R_C}{R_S} R_W \omega_C = 17.6\,\text{m/s}.$$

Solution to 6.30

We let ω_C, ω_W, and ω_S be the angular velocities and R_C, R_W and R_S be the radii of the crank, wheel, and sprocket respectively. The chain does not slip with respect to the crank or sprocket, so we must have

$$\omega_C R_C = \omega_S R_S.$$

Given $\omega_C = 68\,\text{rpm}$, we are asked to pick a sprocket and crank combination such that ω_S is close to 128 rpm. In other words, $\frac{R_C}{R_S}$ must be close to

$$\frac{\omega_S}{\omega_C} = \frac{127}{68} = 1.868.$$

By trial and error we find that pairing C1 with S3 gives

$$\boxed{\frac{R_C}{R_S} = \frac{52.6\,\text{mm}}{28.3\,\text{mm}} = 1.86,}$$

which is the closest pairing to the desired ratio. For this choice of gears and the given cadence,

$$\boxed{\omega_W = \omega_S = \frac{R_C}{R_S}\omega_C = \frac{52.6\,\text{mm}}{28.3\,\text{mm}}68\,\text{rpm} = 126\,\text{rpm}.}$$

Solution to 6.31

We let ω_C, ω_W, and ω_S be the angular velocities and R_C, R_W, and R_S be the radii of the crank, wheel, and sprocket respectively. The chain does not slip with respect to the crank or sprocket, and $\omega_S = \omega_W$ so we must have

$$\omega_S R_S = \omega_W R_S = \omega_C R_C \quad \Rightarrow \quad \omega_C = \omega_W \frac{R_S}{R_C}. \tag{1}$$

We need to find the combinations of crank and sprocket such that

$$2\pi \text{ rad/s} \leq \omega_C \leq 2.5\pi \text{ rad/s} \tag{2}$$

Plugging Eq. (1) into Eq. (2) we must have that

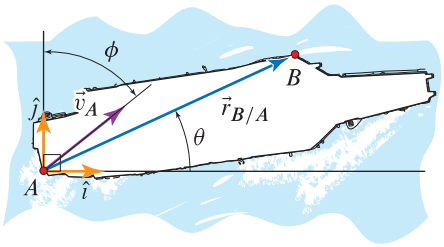
$$\boxed{0.393 \leq \frac{R_S}{R_C} \leq 0.491.}$$

By trial and error we find that all possible combinations include:

$$\boxed{\begin{aligned} \text{S1 and C1: } & \frac{R_S}{R_C} = 0.422 \\ \text{S2 and C1: } & \frac{R_S}{R_C} = 0.460 \\ \text{S4 and C2: } & \frac{R_S}{R_C} = 0.444 \\ \text{S6 and C3: } & \frac{R_S}{R_C} = 0.437. \end{aligned}}$$

Problem 6.32

A carrier is maneuvering so that, at the instant shown, $|\vec{v}_A| = 25$ knots (1 kn is *exactly* equal to 1.852 km/h) and $\phi = 33°$. Letting the distance between A and B be 220 m and $\theta = 22°$, determine $\vec{v}_B$ at the given instant if the ship's turning rate at this instant is $\dot{\theta} = 2°/\text{s}$ clockwise.



Solution

Since A and B are on the same rigid body, we must have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{\text{carrier}} \times \vec{r}_{B/A} \tag{1}$$

where

$$\vec{v}_A = v_A(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}), \tag{2}$$

$$\vec{\omega}_{\text{carrier}} = \omega\,\hat{k} = (-2\,°/\text{s})\,\hat{k} = (-0.03491\ \text{rad/s})\,\hat{k}, \tag{3}$$

and

$$\vec{r}_{B/A} = AB(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \tag{4}$$

Substituting Eqs. (2)–(4) into Eq. (1) we have

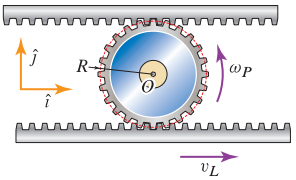
$$\begin{aligned}\vec{v}_B &= v_A(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) + \omega\,\hat{k} \times AB(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= (v_A\sin\phi - \omega AB\sin\theta)\,\hat{\imath} + (v_A\cos\phi + \omega AB\cos\theta)\,\hat{\jmath}.\end{aligned}$$

Substituting in known values we get

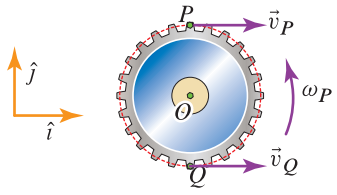
$$\boxed{\vec{v}_B = (9.88\,\hat{\imath} + 3.67\,\hat{\jmath})\ \text{m/s}.}$$

Problem 6.33

At the instant shown, the pinion is rotating between two racks with an angular velocity $\omega_P = 55\,\text{rad/s}$. If the nominal radius of the pinion is $R = 4\,\text{cm}$ and if the lower rack is moving to the right with a speed $v_L = 1.2\,\text{m/s}$, determine the velocity of the upper rack.



Solution



We define P and Q to be the contact points between the pinion and the top and bottom racks, respectively. Then we have

$$\vec{v}_Q = \vec{v}_L = (1.2\,\text{m/s})\,\hat{\imath}.$$

Because Q and P are points on the same rigid body, we must also have

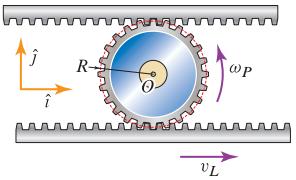
$$\begin{aligned}\vec{v}_P &= \vec{v}_Q + \vec{\omega}_P \times \vec{r}_{P/Q} \\ &= v_L\,\hat{\imath} + \omega_P\,\hat{k} \times (2R)\,\hat{\jmath} \\ &= (v_L - 2R\omega_P)\,\hat{\imath}.\end{aligned}$$

Substituting known values we get

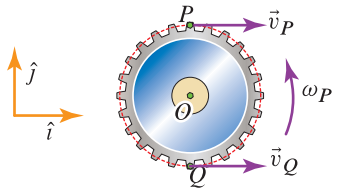
$$\boxed{\vec{v}_P = -3.2\,\hat{\imath}\,\text{m/s}.}$$

Problem 6.34

At the instant shown the lower rack is moving to the right with a speed of $v_L = 4\,\text{ft/s}$, while the upper rack is fixed. If the nominal radius of the pinion is $R = 2.5\,\text{in.}$, determine ω_P, the angular velocity of the pinion, as well as the velocity of point O, i.e., the center of the pinion.



Solution



We define P and Q to be the contact points between the pinion and the top and bottom racks, respectively. Then we have

$$\vec{v}_Q = \vec{v}_L = (1.2\,\text{m/s})\,\hat{\imath} \quad \text{and} \quad \vec{v}_P = \vec{0}.$$

Because Q and P are points on the same rigid body, we must also have

$$\vec{v}_Q = \vec{\omega}_P \times \vec{r}_{Q/P} = \vec{v}_L \quad \Rightarrow \quad \omega_P\,\hat{k} \times (-2R)\,\hat{\jmath} = \vec{v}_L \quad \Rightarrow \quad 2R\omega_P\,\hat{\imath} = \vec{v}_L \quad \Rightarrow \quad \omega_P = \frac{v_L}{2R}. \quad (1)$$

Plugging in known values we get

$$\boxed{\vec{\omega}_P = 9.60\,\hat{k}\,\text{rad/s}.}$$

Points O and P are also on the same rigid body, so we also must have

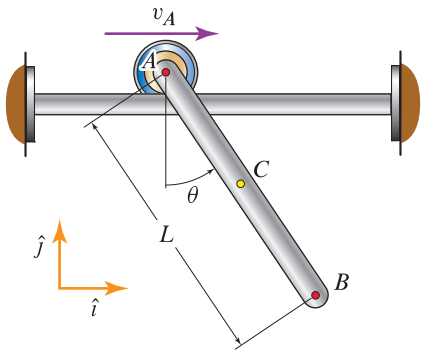
$$\vec{v}_O = \vec{\omega}_P \times \vec{r}_{O/P} = \omega_P\,\hat{k} \times (-R)\,\hat{\jmath} = \omega_P R\,\hat{\imath}. \quad (2)$$

Substituting Eq. (1) into Eq. (2) we have

$$\boxed{\vec{v}_O = 2.00\,\hat{\imath}\,\text{ft/s}.}$$

Problem 6.35

A bar of length $L = 2.5\,\text{m}$ is pin-connected to a roller at A. The roller is moving along a horizontal rail as shown with $v_A = 5\,\text{m/s}$. If at a certain instant $\theta = 33°$ and $\dot{\theta} = 0.4\,\text{rad/s}$, compute the velocity of the bar's midpoint C.



Solution

Because C and A are points on the same rigid body, we must have

$$\vec{v}_C = \vec{v}_A + \vec{\omega}_{AC} \times \vec{r}_{C/A}, \tag{1}$$

where

$$\vec{v}_A = v_A\,\hat{\imath}, \quad \vec{\omega}_{AC} = \dot{\theta}\,\hat{k}, \quad \vec{r}_{C/A} = \frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}). \tag{2}$$

Plugging Eqs. (2) into Eq. (1) we have

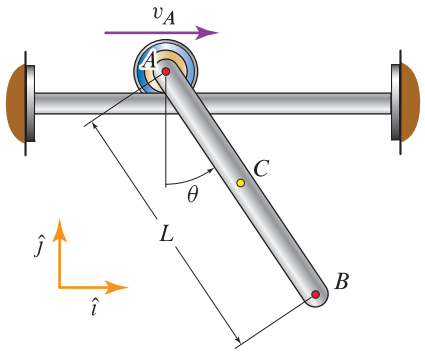
$$\begin{aligned}\vec{v}_C &= v_A\,\hat{\imath} + \omega_{AC}\,\hat{k} \times \frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) \\ &= (v_A + \frac{L}{2}\omega_{AC}\cos\theta)\,\hat{\imath} + \omega_{AC}\frac{L}{2}\sin\theta\,\hat{\jmath}.\end{aligned}$$

Plugging in known values we get

$$\boxed{\vec{v}_C = (5.42\,\hat{\imath} + 0.272\,\hat{\jmath})\,\text{m/s}.}$$

Problem 6.36

If the motion of the bar is planar, what would the speed of A need to be for $\vec{v}_C$ to be perpendicular to the bar AB? Why?



Solution

The kinematic relation between points C and A is

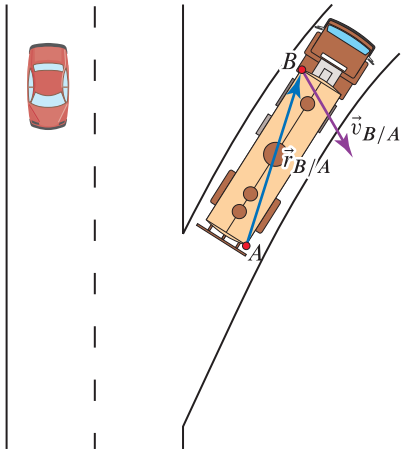
$$\vec{v}_C = \vec{v}_A + \vec{\omega}_{AC} \times \vec{r}_{A/C}.$$

For $\vec{v}_C$ to be perpendicular to $\vec{r}_{A/C}$ in planar motion, we must have

$$\vec{v}_A = \vec{0} \quad \Rightarrow \quad v_A = 0.$$

Problem 6.37

Points A and B are both on the trailer part of the truck. If the relative velocity of point B with respect to A is as shown, is the body undergoing a planar rigid body motion?

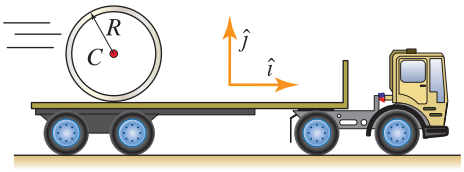


Solution

For the motion to be a planar rigid body motion, $\vec{v}_{B/A}$ must be perpendicular to the vector $\vec{r}_{B/A}$. The figure indicates that $\vec{v}_{B/A}$ and $\vec{r}_{B/A}$ are not perpendicular. Therefore, the motion of the trailer is not a planar rigid body motion.

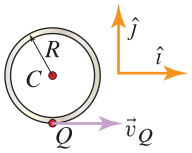
Problem 6.38

A truck is moving to the right with a speed $v_0 = 12\,\text{km/h}$ while the pipe section with radius $R = 1.25\,\text{m}$ and center at C rolls without slipping over the truck's bed. The center of the pipe section C is moving to the right at $2\,\text{m/s}$ relative to the truck. Determine the angular velocity of the pipe section and the absolute velocity of C.



Solution

We let $\vec{\omega}_P$ be the angular velocity of the pipe. Because C and Q can be considered points on the same rigid body, we must have



$$\vec{v}_C = \vec{v}_Q + \vec{\omega}_P \times \vec{r}_{C/Q} \tag{1}$$

where

$$\vec{\omega}_P = \omega_P\,\hat{k}, \quad \vec{v}_Q = \vec{v}_T = v_0\,\hat{\imath}, \quad \text{and} \quad \vec{r}_{C/Q} = R\,\hat{\jmath}. \tag{2}$$

Substituting Eqs. (2) into Eq. (1) we have

$$\vec{v}_C = (v_0 - \omega_P R)\,\hat{\imath}. \tag{3}$$

We are also told that

$$\vec{v}_{C/Q} = \vec{\omega}_P \times \vec{r}_{C/Q} = v_{\text{pipe/truck}}\,\hat{\imath} \quad \Rightarrow \quad -\omega_P R\,\hat{\imath} = 2.000\,\hat{\imath}\,\text{m/s}.$$

Plugging in known values we have $\vec{\omega}_P = -1.600$, which to three significant figures is

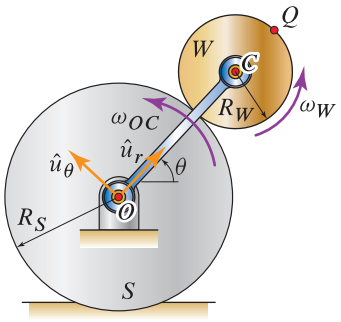
$$\boxed{\vec{\omega}_P = -1.60\,\hat{k}\,\text{rad/s}.}$$

Plugging the unrounded value for $\vec{\omega}_P$ and all known values into Eq. (3), and rounding to three significant figures we have

$$\boxed{\vec{v}_C = 5.33\,\hat{\imath}\,\text{m/s}.}$$

Problem 6.39

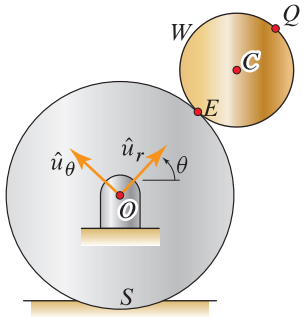
A wheel W of radius $R_W = 7\,\text{mm}$ is connected to point O via the rotating arm OC, and it rolls without slip over the stationary cylinder S of radius $R_S = 15\,\text{mm}$. If, at the instant shown, $\theta = 47°$ and $\omega_{OC} = 3.5\,\text{rad/s}$, determine the angular velocity of the wheel and the velocity of point Q, where point Q lies on the edge of W and along the extension of the line OC.



Solution

Points C and O are two points on the arm, which is a rigid body, so we must have

$$\vec{v}_C = \vec{v}_O + \vec{\omega}_{OC} \times \vec{r}_{C/O}, \tag{1}$$



where

$$\vec{\omega}_{OC} = \omega_{OC}\,\hat{k} \quad \text{and} \quad \vec{r}_{C/O} = (R_S + R_W)\,\hat{u}_r. \tag{2}$$

Substituting Eqs. (2) into Eq. (1) we have

$$\vec{v}_C = (R_S + R_W)\omega_{OC}\,\hat{u}_\theta. \tag{3}$$

We let E be the contact point between cylinder S and wheel W, so $\vec{v}_E = \vec{0}$, and

$$\vec{v}_C = \vec{v}_E + \vec{\omega}_W \times \vec{r}_{C/E} = \omega_W\,\hat{k} \times R_W\hat{u}_r = \omega_W R_W \hat{u}_\theta. \tag{4}$$

Equating Eqs. (3) and (4) we have

$$\omega_W R_W \hat{u}_\theta = \omega_{OC}(R_S + R_W)\hat{u}_\theta \quad \Rightarrow \quad \omega_W = \frac{(R_S + R_W)}{R_W}\omega_{OC}.$$

Plugging in known values we get

$$\boxed{\vec{\omega}_W = 11.0\,\hat{k}\,\text{rad/s}.}$$

Because Q is a point on the same rigid body as E, we also must have

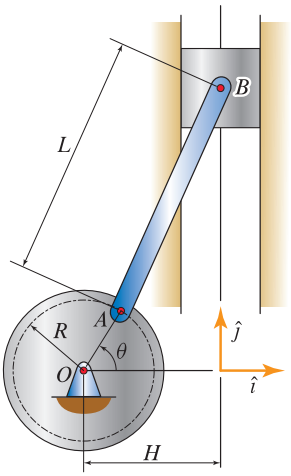
$$\begin{aligned}\vec{v}_Q &= \vec{v}_E + \omega_W\,\hat{k} \times 2R_W\hat{u}_r \\ &= 2R_W\omega_W\,\hat{u}_\theta = 2(R_S + R_W)\omega_{OC}\,\hat{u}_\theta.\end{aligned}$$

Plugging in known values we get

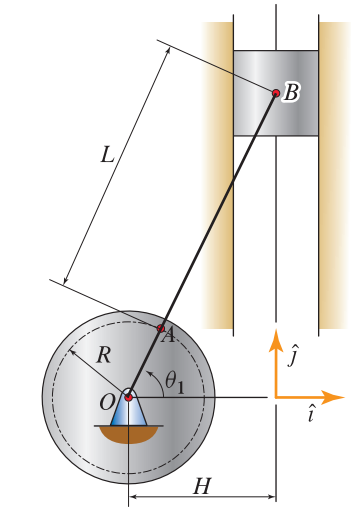
$$\boxed{\vec{v}_Q = 0.154\,\hat{u}_\theta\,\text{m/s}.}$$

Problem 6.40

For the slider-crank mechanism shown, let $R = 20\,\text{mm}$, $L = 80\,\text{mm}$, and $H = 38\,\text{mm}$. Use the concept of instantaneous center of rotation to determine the values of θ, with $0° \leq \theta \leq 360°$, for which $v_B = 0$. Also, determine the angular velocity of the connecting rod at these values of θ.



Solution



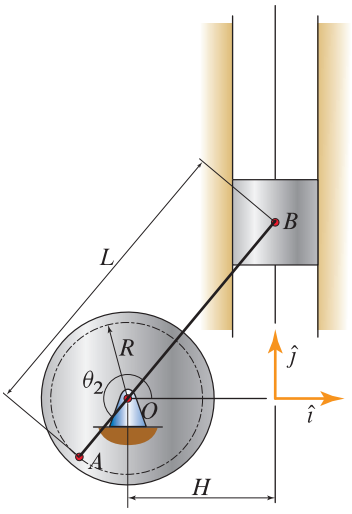
When $v_B = 0$, B is the instantaneous center of rotation of AB, and $\vec{v}_A$ must be perpendicular to the line BA. However, $\vec{v}_A$ must also be perpendicular to the line OA, since O is the center of rotation of the crank. Therefore, O, A, and B must be collinear. The two cases where this happens are shown to the left. In the first, case we have that

$$(L + R)\cos\theta_1 = H \quad \Rightarrow \quad \theta_1 = \cos^{-1}\frac{H}{L+R} \quad \Rightarrow \quad \boxed{\theta_1 = 67.7°.}$$

In the second case we have that

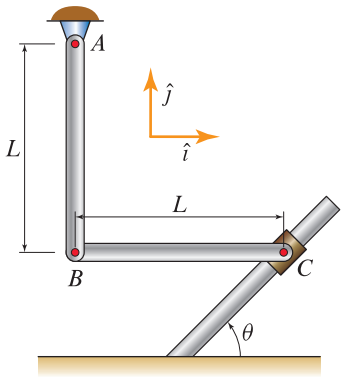
$$(L - R)\cos\theta_2 = H \quad \Rightarrow \quad \theta_2 = 180° + \cos^{-1}\frac{H}{L-R}$$

$$\Rightarrow \quad \boxed{\theta_2 = 231°.}$$



Problem 6.41

At the instant shown bars AB and BC are perpendicular to each other, and bar BC is rotating counterclockwise at 20 rad/s. Letting $L = 2.5$ ft and $\theta = 45°$, determine the angular velocity of bar AB as well as the velocity of the slider C.



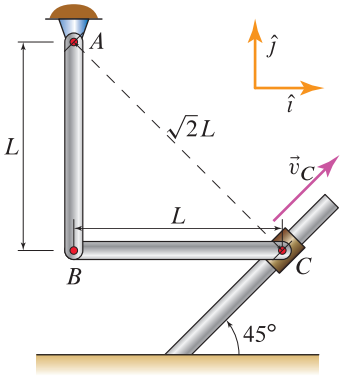
Solution

Because $\theta = 45°$, the instantaneous center of rotation of bar BC is point A. Therefore,

$$v_C = \sqrt{2}L\omega_{BC} = 70.7\,\text{ft/s}$$

Because C is constrained to move along the guide with vector $(\hat{\imath} + \hat{\jmath})$, and is moving up and to the right, we have

$$\vec{v}_C = (50.0\,\hat{\imath} + 50.0\,\hat{\imath})\,\text{ft/s}.$$



A is also the center of rotation of bar AB. Therefore, in the present configuration

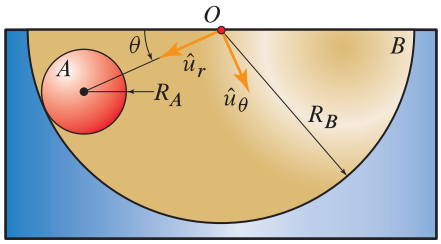
$$v_B = \omega_{AB}L = \omega_{BC}L \quad \Rightarrow \quad \omega_{AB} = \omega_{BC} \quad \Rightarrow \quad \vec{\omega}_{AB} = 20.0\,\hat{k}\,\text{rad/s}.$$

Problems 6.42 and 6.43

A ball of radius $R_A = 3\,\text{in.}$ is rolling without slip in a stationary spherical bowl of radius $R_B = 8\,\text{in.}$ Assume that the ball's motion is planar.

Problem 6.42 If the speed of the center of the ball is $v_A = 1.75\,\text{ft/s}$ and if the ball is moving down and to the right, determine the angular velocity of the ball.

Problem 6.43 If the angular speed of the ball $|\omega_A| = 4\,\text{rad/s}$ is counterclockwise, determine the velocity of the center of the ball.



Solution to 6.42

Converting R_A to feet we have $R_A = 0.2500\,\text{ft}$. Since the ball rolls without slip over a stationary surface, the point of contact between the ball and the bowl is the instantaneous center of rotation of the ball. Therefore,

$$v_A = |\vec{\omega}_A| R_A \quad \Rightarrow \quad \omega_A = \frac{v_A}{R_A} \quad \Rightarrow \quad \omega_A = 7.000\,\text{rad/s}$$

Observing that the ball will be rotating counterclockwise and given that $\hat{k} = \hat{u}_r \times \hat{u}_\theta$, to three significant figures we have

$$\boxed{\vec{\omega}_A = -7.00\,\hat{k}\,\text{rad/s}.}$$

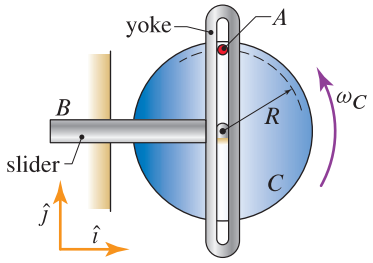
Solution to 6.43

Since the ball rolls without slip over a stationary surface, the point of contact between the ball and the bowl is the instantaneous center of rotation of the ball. Therefore,

$$v_A = |\vec{\omega}_A| R_A = 4 R_A \quad \Rightarrow \quad \boxed{\vec{v}_A = -12\,\hat{u}_\theta \text{ in/s.}}$$

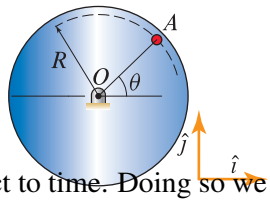
Problem 6.44

One way to convert rotational motion into linear motion and vice versa is via the use of a mechanism called a Scotch yoke, which consists of a crank C that is connected to a slider B via a pin A. The pin rotates with the crank while sliding within the yoke, which, in turn, rigidly translates with the slider. This mechanism has been used, for example, to control the opening and closing valves in pipelines. Letting the radius of the crank be $R = 1.5\,\text{ft}$, determine the angular velocity ω_C of the crank so that the maximum speed of the slider is $v_B = 90\,\text{ft/s}$.



Solution

Calling the center of the wheel the origin, from the geometry of the problem we have



$$\vec{r}_A = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \qquad (1)$$

By construction, the velocity of the slider is equal to the horizontal component of the velocity of point A. We can get $\vec{v}_A$ by taking the derivative of Eq. 1 with respect to time. Doing so we have

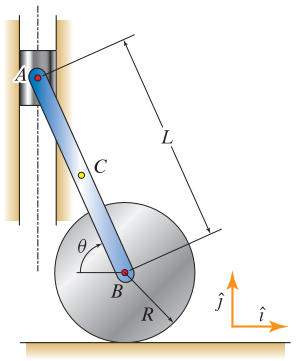
$$\vec{v}_A = R\dot{\theta}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \quad \Rightarrow \quad \vec{v}_B = -R\dot{\theta}\sin\theta\,\hat{\imath}.$$

Because $R\dot{\theta}\sin\theta$ varies as a function of θ, with its maximum value equal to $R|\dot{\theta}|$, for $(v_B)_{max}$ we have

$$(v_B)_{\max} = R|\dot{\theta}| = R\omega_C \quad \Rightarrow \quad \omega_C = \frac{(v_B)_{\max}}{R} \quad \Rightarrow \quad \boxed{\vec{\omega}_C = 60.0\,\hat{k}\,\text{rad/s}.}$$

Problems 6.45 through 6.47

The system shown consists of a wheel of radius $R = 14\,\text{in.}$ rolling on a horizontal surface. A bar AB of length $L = 40\,\text{in.}$ is pin-connected to the center of the wheel and to a slider A that is constrained to move along a vertical guide. Point C is the bar's midpoint.



Problem 6.45 If, when $\theta = 72°$, the wheel is moving to the right so that $v_B = 7\,\text{ft/s}$, determine the angular velocity of the bar as well as the velocity of the slider A.

Problem 6.46 If, when $\theta = 53°$, the slider is moving downward with a speed $v_A = 8\,\text{ft/s}$, determine the velocity of points B and C.

Problem 6.47 If the wheel rolls without slip with a constant counterclockwise angular velocity of $10\,\text{rad/s}$, determine the velocity of the slider A when $\theta = 45°$.

Solution to 6.45

Converting units on known values, we have

$$R = 1.167\,\text{ft} \quad L = 3.333\,\text{ft}.$$

We are told the wheel is moving to the right. Therefore, using the component system shown

$$\vec{v}_B = 7\,\hat{\imath}\,\text{ft/s}. \tag{1}$$

In addition, the slider must be moving down, so we have $\vec{v}_A = -v_A\,\hat{\jmath}$, and the angular velocity of the bar is $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$. Also, the position vector of B with respect to A is $\vec{r}_{B/A} = L(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$. From rigid body kinematics, we must have

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= -v_A\,\hat{\jmath} + \omega_{AB}\,\hat{k} \times L(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}) \\ &= \omega_{AB} L\cos\theta\,\hat{\imath} + (\omega_{AB} L\sin\theta - v_A)\,\hat{\jmath}. \end{aligned} \tag{2}$$

Equating Eq. (1) and Eq. (2) we have

$$v_B = \omega_{AB} L\sin\theta \quad \Rightarrow \quad \omega_{AB} = \frac{v_B}{L\sin\theta}, \tag{3}$$

$$v_A = \omega_{AB} L\cos\theta. \tag{4}$$

Plugging known values into Eq. (3) we get

$$\omega_{AB} = 2.208\,\text{rad/s}.$$

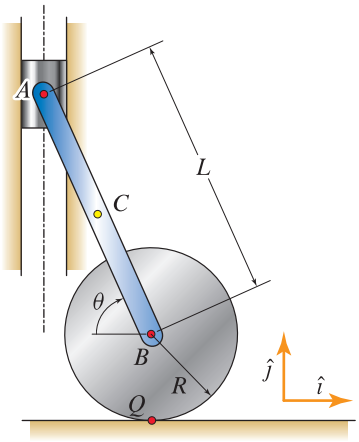
To three significant figures this is

$$\boxed{\vec{\omega}_{AB} = 2.21\,\hat{k}\,\text{rad/s}.}$$

Plugging the unrounded value for ω_{AB} into Eq. (4) we get

$$v_A = 2.27\,\text{ft/s} \quad \Rightarrow \quad \boxed{\vec{v}_A = -2.27\,\hat{\jmath}\,\text{ft/s}.}$$

Solution to 6.46



Converting units on known values, we have

$$R = 1.167\,\text{ft} \quad L = 3.333\,\text{ft}.$$

We are told the slider is moving down. Therefore, using the component system shown,

$$\vec{v}_A = -8\,\hat{\jmath}\,\text{ft/s}.$$

In addition, because the slider is moving down, the the wheel must be moving to the right, so we have $\vec{v}_B = v_B\hat{\imath}$, $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$, and $\vec{\omega}_W = \omega_W\,\hat{k}$. Also, the position vector of B relative to A is $\vec{r}_{B/A} = L(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$, and of B relative to Q is $\vec{r}_{B/Q} = R\,\hat{\jmath}$. From rigid body kinematics we must have

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= -v_A\,\hat{\jmath} + \omega_{AB}\,\hat{k} \times L(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}) \\ &= \omega_{AB} L\cos\theta\,\hat{\imath} + (\omega_{AB} L\sin\theta - v_A)\,\hat{\jmath}. \end{aligned} \tag{5}$$

We must also have

$$\begin{aligned}\vec{v}_B &= \vec{v}_Q + \vec{\omega}_W \times \vec{r}_{B/Q} \\ &= \omega_W\,\hat{k} \times R\,\hat{\jmath} \\ &= -\omega_W R\,\hat{\imath}. \end{aligned} \tag{6}$$

Equating components of Eq. (5) with those of Eq. (6) we have

$$-\omega_W R = \omega_{AB} L\sin\theta, \tag{7}$$

$$v_A = \omega_{AB} L\cos\theta. \tag{8}$$

Solving Eq. (8) gives $\omega_{AB} = 3.988\,\text{rad/s}$. Plugging ω_{AB} into Eq. (7) gives $\omega_W = -9.100\,\text{rad/s}$. Plugging ω_W into Eq. (6) gives

$$\boxed{\vec{v}_B = 10.6\,\hat{\imath}\,\text{ft/s}.}$$

Realizing that the position vector from A to C is $\vec{r}_{C/A} = \frac{\vec{r}_{B/A}}{2}$, from rigid body kinematics we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{C/A} \\ &= -v_A\,\hat{\jmath} + \omega_{AB}\,\hat{k} \times \frac{L}{2}(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}) \\ &= \omega_{AB} L\cos\theta\,\hat{\imath} + (\omega_{AB} L\sin\theta - v_A)\,\hat{\jmath}. \end{aligned} \tag{9}$$

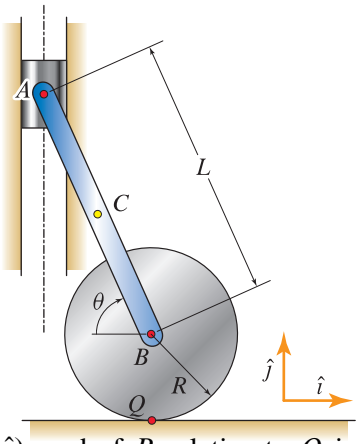
Plugging all known values into Eq. (9) gives

$$\boxed{\vec{v}_C = (5.31\,\hat{\imath} - 4\,\hat{\jmath})\,\text{ft/s}.}$$

Solution to 6.47

Converting units on known values, we have

$$R = 1.167\,\text{ft} \quad L = 3.333\,\text{ft}.$$



We are told that the wheel spins counterclockwise at 10 rad/s, so we have

$$\vec{\omega}_W = 10\,\hat{k}\,\text{rad/s}. \tag{10}$$

In addition, because the wheel is turning counterclockwise, the slider is moving up, and B is moving to the left, we must have

$$\vec{v}_A = v_A\,\hat{\jmath}, \quad \vec{v}_B = -v_B\hat{\imath}, \quad \text{and} \quad \vec{\omega}_{AB} = \omega_{AB}\,\hat{k}. \tag{11}$$

Also, the position vector of B relative to A is $\vec{r}_{B/A} = L(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$, and of B relative to Q is $\vec{r}_{B/Q} = R\,\hat{\jmath}$. From rigid body kinematics we must have

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= v_A\,\hat{\jmath} + \omega_{AB}\,\hat{k} \times L(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}) \\ &= \omega_{AB} L\cos\theta\,\hat{\imath} + (\omega_{AB} L\sin\theta + v_A)\,\hat{\jmath}. \end{aligned} \tag{12}$$

We must also have

$$\vec{v}_B = \vec{v}_Q + \vec{\omega}_W \times \vec{r}_{B/Q} = \omega_W\,\hat{k} \times R\,\hat{\jmath} = -\omega_W R\,\hat{\imath} = -11.67\,\hat{\imath}\,\text{ft/s}. \tag{13}$$

Equating components of Eq. (12) with those of Eq. (13) we have

$$-11.67 = \omega_{AB} L\sin\theta, \tag{14}$$

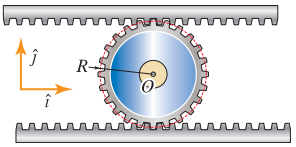
$$v_A = -\omega_{AB} L\cos\theta. \tag{15}$$

Solving Eq. (14) gives $\omega_{AB} = -4.952\,\text{rad/s}$. Substituting this value for ω_{AB} into Eq. (15), we have

$$\boxed{\vec{v}_A = 11.7\,\hat{\jmath}\,\text{ft/s}.}$$

Problem 6.48

At the instant shown, the lower rack is moving to the right with a speed of 2.7 m/s while the upper rack is moving to the left with a speed of 1.7 m/s. If the nominal radius of the pinion O is $R = 0.25$ m, determine the angular velocity of the pinion, as well as the position of the pinion's instantaneous center of rotation relative to point O.



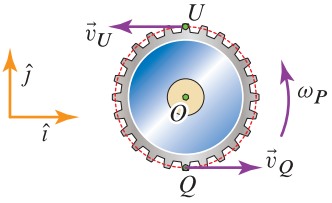
Solution

From the no slip condition at points U and Q we know that

$$\vec{v}_U = -1.7\,\hat{\imath}\ \text{m/s} \qquad (1)$$

and

$$\vec{v}_Q = 2.7\,\hat{\imath}\ \text{m/s}. \qquad (2)$$



From rigid body kinematics we have

$$\vec{v}_U = \vec{v}_Q + \vec{\omega}_P \times \vec{r}_{U/Q} = v_L\,\hat{\imath} + \omega_P\,\hat{k} \times 2R\,\hat{\jmath} = v_L\,\hat{\imath} - 2R\omega_P\,\hat{\imath}, \qquad (3)$$

where ω_P is the angular speed of the pinion. Comparing Eq. (3) with Eq. (1) component by component, we have

$$-1.7 = v_L - 2R\omega_P \quad \Rightarrow \quad \omega_P = \frac{v_L + 1.7}{2R} \quad \Rightarrow \quad \omega_P = 8.800\ \text{rad/s}.$$

In vector form, and to three significant figures this is

$$\boxed{\vec{\omega}_P = 8.80\,\hat{k}\ \text{rad/s}.}$$

We also know that

$$\begin{aligned}
\vec{v}_{IC} &= \vec{v}_Q + \vec{\omega}_P \times \vec{r}_{IC/Q} \\
\vec{0} &= v_L\,\hat{\imath} + \omega_P\,\hat{k} \times [(d_{IC/Q})_x\,\hat{\imath} + (d_{IC/Q})_y\,\hat{\jmath}] \\
\vec{0} &= [v_L - (d_{IC/Q})_y\omega_P]\,\hat{\imath} + (d_{IC/Q})_x\omega_P\,\hat{\jmath}. \qquad (4)
\end{aligned}$$

Breaking Eq. (4) into two scalar equations by components we have

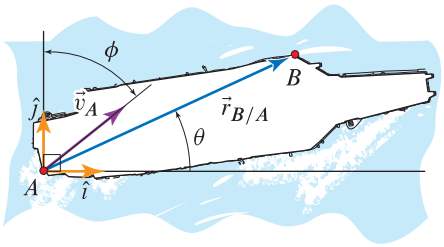
$$\begin{aligned}
0 &= v_L - (d_{IC/Q})_y\omega_P \quad \Rightarrow \quad (d_{IC/Q})_y = \frac{v_L}{\omega_P}, \\
0 &= (d_{IC/Q})_x\omega_P \quad \Rightarrow \quad (d_{IC/Q})_x = 0.
\end{aligned}$$

Substituting known values we have

$$\boxed{\vec{r}_{IC/Q} = 0.307\,\hat{\jmath}\ \text{rad/s}.}$$

Problem 6.49

A carrier is maneuvering so that, at the instant shown, $|\vec{v}_A| = 22\,\text{kn}$, $\phi = 35°$, $|\vec{v}_B| = 24\,\text{kn}$ (1 kn is equal to 1 nautical mile (nml) per hour or 6076 ft/h). Letting $\theta = 19°$ and the distance between A and B be 720 ft, determine the ship's turning rate at the given instant if the ship is rotating clockwise.



Solution

We let $\vec{v}_A = v_A(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$, $\vec{\omega}_A = \omega_A\,\hat{k}$, and $\vec{r}_{B/A} = d_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$. From rigid body kinematics we know

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= v_A(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) + \omega_{AB}\,\hat{k} \times d_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= (v_A \sin\phi - \omega_{AB} d_{AB} \sin\theta)\,\hat{\imath} + (v_A \cos\phi + \omega_{AB} d_{AB} \cos\theta)\,\hat{\jmath}. \qquad (1)\end{aligned}$$

Taking the magnitude of both sides of Eq. (1) we have

$$v_B^2 = (v_A \sin\phi - \omega_{AB} d_{AB} \sin\theta)^2 + (v_A \cos\phi + \omega_{AB} d_{AB} \cos\theta)^2. \qquad (2)$$

All values in Eq. (2) are known except for ω_{AB}. Solving, we have $\omega_{AB} = \pm 0.06805\,\text{rad/s}$. Since we are told that the ship is turning clockwise, using the component system shown we must have

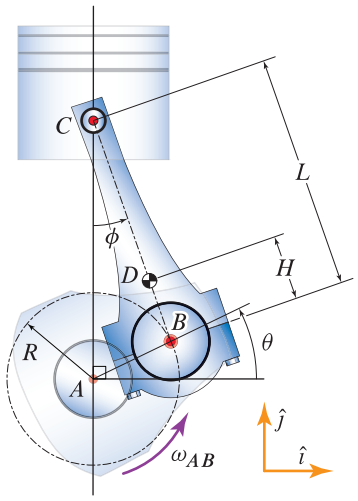
$$\boxed{\vec{\omega}_{AB} = -0.0681\,\hat{k}\,\text{rad/s}.}$$

Problems 6.50 and 6.51

For the slider-crank mechanism shown, let $R = 1.9$ in., $L = 6.1$ in., and $H = 1.2$ in. Also, at the instant shown, let $\theta = 27°$ and $\omega_{AB} = 4850$ rpm.

Problem 6.50 Determine the velocity of the piston at the instant shown.

Problem 6.51 Determine $\dot{\phi}$ and the velocity of point D at the instant shown.



Solution to 6.50

We let $\vec{r}_{B/A} = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$ and $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$. Converting all given values to ft or rad/s we have

$$R = 0.1583\,\text{ft}, \quad L = 0.5083\,\text{ft}, \quad H = 0.1000\,\text{ft}, \quad \text{and} \quad \omega_{AB} = 507.9\,\text{rad/s}.$$

From rigid body kinematics we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} = \omega_{AB}\,\hat{k} \times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -\omega_{AB} R \sin\theta\,\hat{\imath} + \omega_{AB} R \cos\theta\,\hat{\jmath}. \qquad (1)$$

If we let $\vec{\omega}_{BC} = \omega_{BC}\,\hat{k}$, $\vec{r}_{C/B} = L(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$, and $\vec{v}_C = v_C\,\hat{\jmath}$, then we have

$$\begin{aligned} \vec{v}_C = v_C\,\hat{\jmath} &= \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}. \\ &= \vec{v}_B + \omega_{BC}\,\hat{k} \times L(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) \\ &= \vec{v}_B - \omega_{BC} L \cos\phi\,\hat{\imath} + \omega_{BC} L \sin\phi\,\hat{\jmath}. \end{aligned} \qquad (2)$$

Substituting Eq. (1) for $\vec{v}_B$ in Eq. (2) we get

$$v_C\,\hat{\jmath} = -(\omega_{AB} R \sin\theta + \omega_{BC} L \cos\phi)\,\hat{\imath} + (\omega_{AB} R \cos\theta + \omega_{BC} L \sin\phi)\,\hat{\jmath}. \qquad (3)$$

From the geometry of the problem we also have

$$L\sin\phi = R\cos\theta \quad \Rightarrow \quad \phi = \sin^{-1}\left(\frac{R}{L}\cos\theta\right). \qquad (4)$$

Substituting Eq. (4) for ϕ in Eq. (3) we have

$$v_C\,\hat{\jmath} = -\left\{\omega_{AB} R \sin\theta + \omega_{BC} L \cos\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]\right\}\hat{\imath} + 2R\cos\theta(\omega_{AB} + \omega_{BC})\,\hat{\jmath}. \qquad (5)$$

Breaking Eq. (5) into two scalar equations by components we have

$$0 = -\omega_{AB} R \sin\theta - \omega_{BC} \cos\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right] \tag{6}$$

$$v_C = 2R\cos\theta(\omega_{AB} + \omega_{BC}). \tag{7}$$

Eqs. (6) and (7) are two equations in v_C, ω_{BC}, and known values. Solving these equations for v_C we get

$$\boxed{\vec{v}_C = 61.1\,\hat{j}\ \text{ft/s}.}$$

Solution to 6.51

We let $\vec{r}_{B/A} = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$ and $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$. Converting all given values to ft/s or rad/s we have

$$R = 0.1583\,\text{ft}, \quad L = 0.5083\,\text{ft}, \quad H = 0.1000\,\text{ft}, \quad \text{and} \quad \omega_{AB} = 507.9\,\text{rad/s}.$$

From rigid body kinematics we have

$$\vec{v}_B = \vec{\omega}_{AB} \times \vec{r}_{B/A} = \omega_{AB}\,\hat{k} \times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -\omega_{AB} R \sin\theta\,\hat{\imath} + \omega_{AB} R\cos\theta\,\hat{\jmath}. \tag{8}$$

If we let $\vec{\omega}_{BC} = \omega_{BC}\,\hat{k}$, $\vec{r}_{C/B} = L(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$, and $\vec{v}_C = v_C\,\hat{\jmath}$, then from rigid body kinematics we have

$$\begin{aligned}\vec{v}_C = v_C\,\hat{\jmath} &= \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}. \\ &= \vec{v}_B + \omega_{BC}\,\hat{k} \times L(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) \\ &= \vec{v}_B - \omega_{BC} L\cos\phi\,\hat{\imath} + \omega_{BC} L\sin\phi\,\hat{\jmath}. \end{aligned} \tag{9}$$

Substituting Eq. (8) into Eq. (9) we get

$$v_C\,\hat{\jmath} = -(\omega_{AB} R\sin\theta + \omega_{BC} L\cos\phi)\,\hat{\imath} + (\omega_{AB} R\cos\theta + \omega_{BC} L\sin\phi)\,\hat{\jmath}. \tag{10}$$

From the geometry of the problem we also have

$$L\sin\phi = R\cos\theta \quad \Rightarrow \quad \phi = \sin^{-1}\left(\frac{R}{L}\cos\theta\right). \tag{11}$$

Substituting Eq. (11) into Eq. (10) we have

$$v_C\,\hat{\jmath} = -\left\{\omega_{AB} R\sin\theta + \omega_{BC} L\cos\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]\right\}\hat{\imath} + 2R\cos\theta(\omega_{AB} + \omega_{BC})\,\hat{\jmath}. \tag{12}$$

Equating components of Eq. (12) we have

$$0 = -\omega_{AB} R\sin\theta - \omega_{BC}\cos\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right] \tag{13}$$

$$v_C = 2R\cos\theta(\omega_{AB} + \omega_{BC}). \tag{14}$$

Eqs. (13) and (14) are two equations in v_C, ω_{BC}, and known values. Solving these equations for ω_{BC}, we get

$$\omega_{BC} = -74.76\,\text{rad/s}.$$

Because ϕ measures the angle of BC with respect to the horizontal axis, $\dot{\phi} = \omega_{BC}$. Therefore,

$$\boxed{\dot{\phi} = -74.8\,\text{rad/s}.}$$

We let $\vec{r}_{D/B} = H(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$. Therefore, from rigid body kinematics we have

$$\begin{aligned}\vec{v}_D &= \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{D/B} \\ &= \vec{v}_B + \omega_{BC}\,\hat{k} \times H(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) \\ &= \vec{v}_B - \omega_{BC} H\cos\phi\,\hat{\imath} + \omega_{BC} H\sin\phi\,\hat{\jmath}. \end{aligned} \tag{15}$$

Substituting Eq. (8) into Eq. (15) we get

$$\vec{v}_D = -(\omega_{AB} R\sin\theta + \omega_{BC} H\cos\phi)\,\hat{\imath} + (\omega_{AB} R\cos\theta + \omega_{BC} H\sin\phi)\,\hat{\jmath}. \tag{16}$$

Combining Eq. (11) with Eq. (16) to eliminate ϕ gives

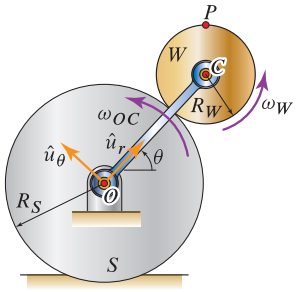
$$\vec{v}_D = -\left\{\omega_{AB} R \sin\theta + \omega_{BC} H \cos\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]\right\}\hat{\imath} + \left[R\cos\theta\left(\omega_{AB} + \omega_{BC}\frac{H}{L}\right)\right]\hat{\jmath}. \qquad (17)$$

Substituting all known values into Eq. (17) we get

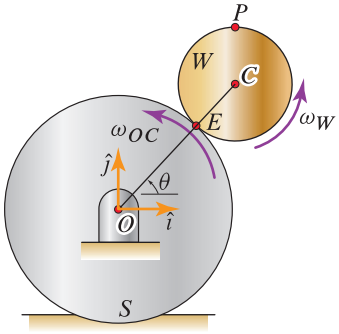
$$\boxed{\vec{v}_D = (-29.3\,\hat{\imath} + 69.6\,\hat{\jmath})\ \text{ft/s}.}$$

Problem 6.52

A wheel W of radius $R_W = 7\,\text{mm}$ is connected to point O via the rotating arm OC, and it rolls without slip over the stationary cylinder S of radius $R_S = 15\,\text{mm}$. If, at the instant shown, $\theta = 63°$ and $\omega_W = 9\,\text{rad/s}$, determine the angular velocity of the arm OC and the velocity of point P, where point P lies on the edge of W and is vertically aligned with point C.



Solution



Let E be the contact point between W and S. From the no slip condition, $\vec{v}_E = \vec{0}$. We let $\vec{\omega}_W = \omega_W \hat{k}$ and from the geometry of the problem $\vec{r}_{C/E} = R_W(\cos\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$. From rigid body kinematics we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_E + \vec{\omega}_W \times \vec{r}_{C/E} \\ &= \omega_W \hat{k} \times R_W(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= \omega_W R_W(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}). \end{aligned} \tag{1}$$

Points C and O are also points on the arm where $\vec{r}_{C/O} = (R_S + R_W)(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$. Therefore, we must have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{\omega}_{OC} \times \vec{r}_{C/O} \\ &= \omega_{OC} \hat{k} \times (R_S + R_W)(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= \omega_{OC}(R_S + R_W)(-\sin\theta\,\hat{\jmath} + \cos\theta\,\hat{\jmath}). \end{aligned} \tag{2}$$

Equating Eqs. (1) and (2) we have

$$\omega_W R_W = \omega_{OC}(R_S + R_W) \quad \Rightarrow \quad \boxed{\vec{\omega}_{OC} = \frac{R_W}{R_S + R_W}\omega_W \hat{k} = 2.86\,\hat{k}\,\text{rad/s}.}$$

Points P and C are both points on the wheel with $\vec{r}_{P/C} = R_W\,\hat{\jmath}$. Therefore, we must have

$$\vec{v}_P = \vec{v}_C + \vec{\omega}_W \times \vec{r}_{P/C}.$$

Substituting in Eq. (1) for $\vec{v}_C$ we have

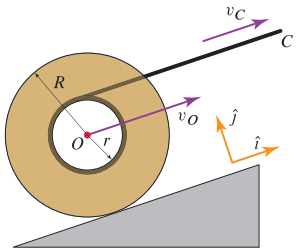
$$\begin{aligned}\vec{v}_P &= \omega_W R_W(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) + \omega_W \hat{k} \times R_W\,\hat{\jmath} \\ &= \omega_W R_W[(-\sin\theta - 1)\,\hat{\imath} + \cos\theta\,\hat{\jmath}]. \end{aligned} \tag{3}$$

Substituting known values into Eq. (3) we get

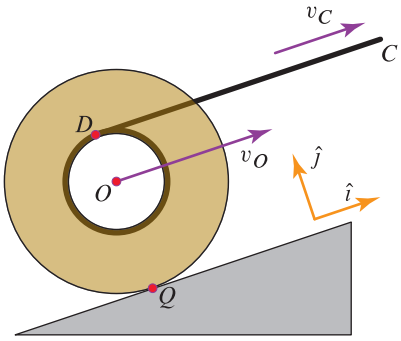
$$\boxed{\vec{v}_P = (-0.119\,\hat{\imath} + 0.0286\,\hat{\jmath})\,\text{m/s}.}$$

Problem 6.53

At the instant shown, the center O of a spool with inner and outer radii $r = 1\,\text{m}$ and $R = 2.2\,\text{m}$, respectively, is moving up the incline at speed $v_O = 3\,\text{m/s}$. If the spool does not slip relative to the ground or relative to the cable C, determine the rate at which the cable is wound or unwound, that is, the length of rope being wound or unwound per unit time.



Solution



From the no slip condition, $\vec{v}_Q = \vec{0}$. We also know $\vec{v}_O = v_O\,\hat{\imath}$, $\vec{r}_{O/Q} = R\,\hat{\jmath}$, and $\vec{\omega}_S = \omega_S\,\hat{k}$. From rigid body kinematics we have

$$\vec{v}_O = \vec{v}_Q + \vec{\omega}_S \times \vec{r}_{O/Q} \quad \Rightarrow \quad v_O\,\hat{\imath} = \omega_S\,\hat{k} \times R\,\hat{\jmath}$$

$$\Rightarrow \quad \omega_S = -\frac{v_O}{R}.$$

We also know that $\vec{r}_{D/O} = r\,\hat{\jmath}$, and $\vec{v}_C = \vec{v}_D$. From rigid body kinematics we have

$$\vec{v}_D = \vec{v}_O + \vec{\omega}_S \times \vec{r}_{D/O}$$

$$\vec{v}_C = v_O\,\hat{\imath} + \omega_S\,\hat{k} \times r\,\hat{\jmath}$$

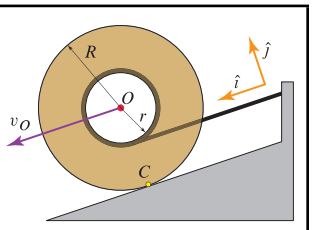
$$\vec{v}_C = (v_O - \omega_S r)\,\hat{\imath} \quad \Rightarrow \quad \vec{v}_C = v_O\left(1 + \frac{r}{R}\right)\hat{\imath}.$$

Since $\vec{v}_C$ and $\vec{v}_O$ are in the same direction, and $v_C > v_O$, the cord is being unwound at a rate of

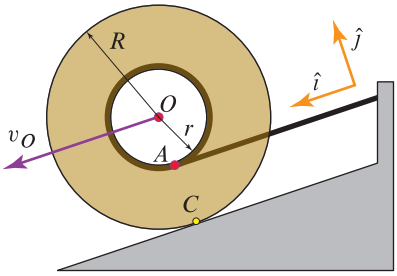
$$|\vec{v}_C - \vec{v}_O| = \frac{r}{R}v_O = 1.36\,\text{m/s}.$$

Problem 6.54

At the instant shown, the center O of a spool with inner and outer radii $r = 3\,\text{ft}$ and $R = 7\,\text{ft}$, respectively, is moving down the incline at a speed $v_O = 12.2\,\text{ft/s}$. If the spool does not slip relative to the rope and if the rope is fixed at one end, determine the velocity of point C (the point on the spool that is in contact with the incline) as well as the rope's unwinding rate, that is, the length of rope being unwound per unit time.



Solution



We let point A be the point where the rope is losing contact with the spool. Because the end of the rope not around the spool is fixed, the speed of the rope is 0, and therefore $v_A = 0$. We also know that $\vec{v}_O = 12.2\,\hat{\imath}\,\text{ft/s}$. From rigid body kinematics we have

$$\begin{aligned}\vec{v}_O &= \vec{v}_A + \vec{\omega}_O \times \vec{r}_{O/A}\\ v_O\,\hat{\imath} &= \omega_O\,\hat{k} \times r\,\hat{\jmath}\\ v_O\,\hat{\imath} &= -r\omega_O\,\hat{\imath} \quad \Rightarrow \quad \omega_O = -\frac{v_O}{r}.\end{aligned}$$

We must also have

$$\begin{aligned}\vec{v}_C &= \vec{v}_A + \vec{\omega}_O \times \vec{r}_{C/A}\\ &= -\frac{v_O}{r}\,\hat{k} \times (r - R)\,\hat{\jmath}\\ &= \frac{v_O(r - R)}{r}.\end{aligned}$$

Plugging in known values we have

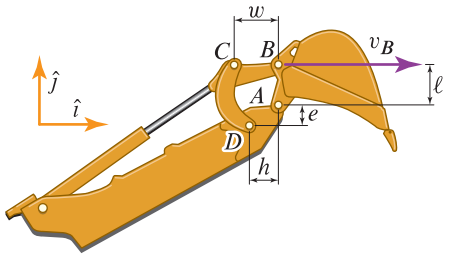
$$\boxed{\vec{v}_C = -16.3\,\hat{\imath}\,\text{ft/s}.}$$

Since $\vec{v}_A = \vec{0}$, the rope is being unwound, and its rate is

$$\boxed{|\vec{v}_O - \vec{v}_A| = 12.2\,\text{ft/s}.}$$

Problem 6.55

The bucket of a backhoe is the element AB of the four-bar linkage system $ABCD$. Assume that the points A and D are fixed and that, at the instant shown, point B is vertically aligned with point A, point C is horizontally aligned with point B, and point B is moving to the right with a speed $v_B = 1.2\,\text{ft/s}$. Determine the velocity of point C at the instant shown, along with the angular velocities of elements BC and CD. Let $h = 0.66\,\text{ft}$, $e = 0.46\,\text{ft}$, $l = 0.9\,\text{ft}$, and $w = 1.0\,\text{ft}$.



Solution

Point C is a point on both bar BC and bar CD, and therefore can be related to both points B and D through rigid body kinematics. We have that

$$\vec{v}_C = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{C/D} = \omega_{CD}\,\hat{k} \times [-(w-h)\,\hat{\imath} + (e+\ell)\,\hat{\jmath}] = -\omega_{CD}(e+\ell)\,\hat{\imath} - \omega_{CD}(w-h)\,\hat{\jmath} \quad (1)$$

Since $\vec{v}_B = v_B\,\hat{\imath}$, we ahve

$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} = v_B\,\hat{\imath} + \omega_{BC} \times (-w)\,\hat{\imath} = v_B\,\hat{\imath} - \omega_{BC}\,w\,\hat{\jmath}. \quad (2)$$

Equating Eqs. (1) and (2) component by component we have

$$-\omega_{CD}(e+\ell)\,\hat{\imath} = v_B\,\hat{\imath} \quad \text{and} \quad -\omega_{CD}(w-h)\,\hat{\jmath} = -\omega_{BC}\,w\,\hat{\jmath}. \quad (3)$$

Solving Eqs. (3) we have

$$\omega_{CD} = -\frac{v_B}{e+\ell} \quad \text{and} \quad \omega_{BC} = \frac{\omega_{CD}(w-h)}{w} \quad \Rightarrow \quad \omega_{BC} = -\frac{v_B}{e+\ell}\frac{w-h}{w}. \quad (4)$$

Substituting known values into Eq. (4), we have

$$\boxed{\vec{\omega}_{CD} = -0.882\,\hat{k}\,\text{rad/s}, \quad \vec{\omega}_{BC} = -0.300\,\hat{k}\,\text{rad/s}.}$$

Substituting Eqs. (4) into Eq. (2) we have

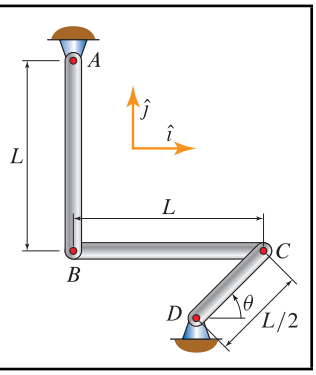
$$\vec{v}_C = v_B\,\hat{\imath} + \frac{v_B}{e+\ell}(w-h)\,\hat{\jmath}. \quad (5)$$

Substituting known values into Eq. (5) we have

$$\boxed{\vec{v}_C = (1.20\,\hat{\imath} + 0.300\,\hat{\jmath})\,\text{ft/s}.}$$

Problem 6.56

Bar AB is rotating counterclockwise with an angular velocity of 15 rad/s. Letting $L = 1.25$ m, determine the angular velocity of bar CD when $\theta = 45°$.



Solution

Because bar AB is rotating about the fixed point A, at the given instant point B is only moving in the $\hat{\imath}$ direction, and we have

$$\vec{v}_B = \omega_{AB} L \hat{\imath}.$$

Points B and C are both on the bar BC so we have

$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} = \omega_{AB} L \hat{\imath} + \omega_{BC} \hat{k} \times L \hat{\imath} = \omega_{AB} L \hat{\imath} + \omega_{BC} L \hat{\jmath}. \tag{1}$$

Points C and D are also both on the bar CD so we have

$$\vec{v}_C = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{C/D} = \omega_{CD} \hat{k} \times \frac{L}{2}(\cos\theta \hat{\imath} + \sin\theta \hat{\jmath}) = \frac{L}{2}\omega_{CD}(-\sin\theta \hat{\imath} + \cos\theta \hat{\jmath}). \tag{2}$$

Equating the $\hat{\imath}$ components of Eqs. (1) and (2) we have

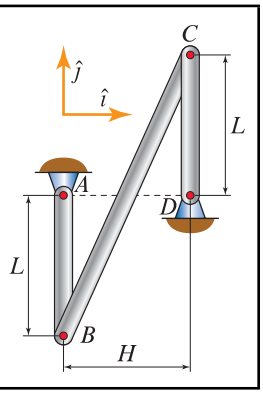
$$\omega_{AB} L = -\frac{L}{2}\omega_{CD} \sin\theta \quad \Rightarrow \quad \omega_{CD} = -\frac{2\omega_{AB}}{\sin\theta}.$$

Substituting in known values we have

$$\boxed{\vec{\omega}_{CD} = -42.4 \hat{k} \text{ rad/s}.}$$

Problem 6.57

At the instant shown, bars AB and CD are vertical and point C is moving to the left with a speed of 35 ft/s. Letting $L = 1.5$ ft and $H = 0.6$ ft, determine the velocity of point B.



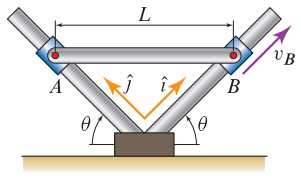
Solution

From the concept of instantaneous center of rotation, we see that lines ABA and CD are parallel, so the instantaneous center of rrotation of bar BC is at infinity, so $\omega_{BC} = 0$. Therefore, from rigid body kinematics we have $\vec{v}_B = \vec{v}_C$, which gives us

$$\boxed{\vec{v}_C = -35\,\hat{\imath}\ \text{ft/s}.}$$

Problem 6.58

Collars A and B are constrained to slide along the guides shown and are connected by a bar with length $L = 0.75\,\text{m}$. Letting $\theta = 45°$, determine the angular velocity of the bar AB at the instant shown if, at this instant, $v_B = 2.7\,\text{m/s}$.



Solution

From rigid body kinematics, we must have

$$\vec{v}_A = \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{A/B},$$

where $\vec{v}_A = -v_A\,\hat{j}$, $\vec{v}_B = v_B\,\hat{\imath}$, and $\vec{r}_{A/B} = -\frac{L}{\sqrt{2}}\,\hat{\imath} + \frac{L}{\sqrt{2}}\,\hat{j}$. Plugging these values in we have

$$-v_A\,\hat{j} = v_B\,\hat{\imath} + \omega_{AB}\,\hat{k} \times \left(-\frac{L}{\sqrt{2}}\,\hat{\imath} + \frac{L}{\sqrt{2}}\,\hat{j}\right)$$

$$-v_A\,\hat{j} = v_B\,\hat{\imath} - \omega_{AB}\frac{L}{\sqrt{2}}\,\hat{\imath} - \omega_{AB}\frac{L}{\sqrt{2}}\,\hat{j}. \tag{1}$$

From the $\hat{\imath}$ component of Eq. (1) we have

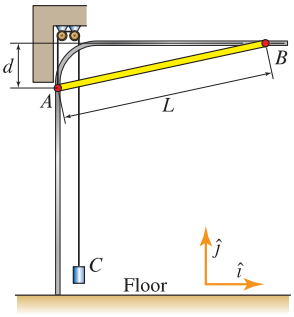
$$v_B = \omega_{AB}\frac{L}{\sqrt{2}} \quad \Rightarrow \quad \omega_{AB} = \frac{v_B}{L}\sqrt{2}.$$

Substituting known values we have

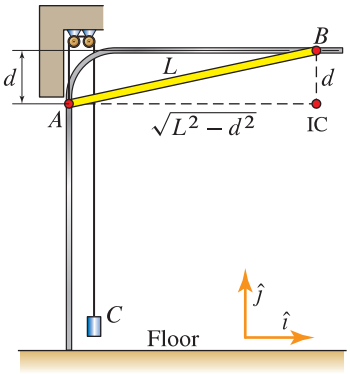
$$\boxed{\vec{\omega}_{AB} = 5.09\,\hat{k}\,\text{rad/s}.}$$

Problem 6.59

At the instant shown, an overhead garage door is being shut with point B moving to the left within the horizontal part of the door guide at a speed of 5 ft/s, while point A is moving vertically downward. Determine the angular velocity of the door and the velocity of the counterweight C at this instant if $L = 6$ ft and $d = 1.5$ ft.



Solution



Using the concept of instantaneous center of rotation we have

$$v_B = \omega_{AB} d, \tag{1}$$

$$v_A = \omega_{AB}\sqrt{L^2 - d^2}. \tag{2}$$

Solving Eq. (1) for ω_{AB} we have

$$\omega_{AB} = \frac{v_B}{d} \quad \Rightarrow \quad \omega_{AB} = 3.333 \text{ rad/s}.$$

In vector form and to three significant figures we have

$$\boxed{\vec{\omega}_{AB} = 3.33\,\hat{k} \text{ rad/s}.}$$

Substituting all known values into Eq. 2 we have

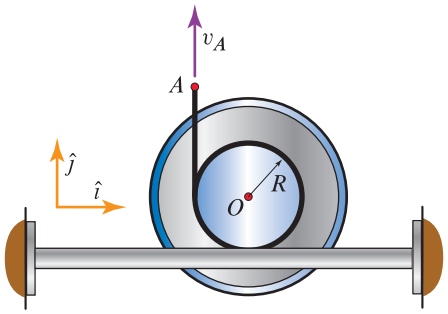
$$v_A = \frac{v_B}{d}\sqrt{L^2 - d^2} = -19.35 \text{ ft/s}.$$

Given the geometry of the pulley system, $v_C = -v_A$. In vector form, we have

$$\boxed{\vec{v}_C = 19.4\,\hat{j} \text{ ft/s}.}$$

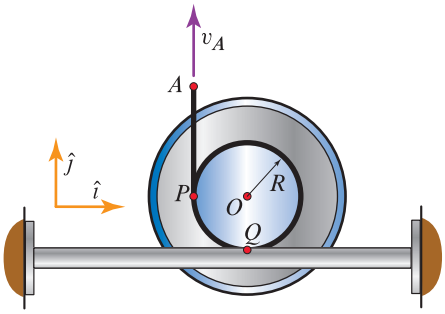
Problem 6.60

A spool with inner radius $R = 1.5\,\text{m}$ rolls without slip over a horizontal rail as shown. If the cable on the spool is unwound at a rate $v_A = 5\,\text{m/s}$ in such a way that the unwound cable remains perpendicular to the rail, determine the angular velocity of the spool and the velocity of the spool's center O.



Solution

We let P be the contact point between the rope and the spool. By the no slip condition, the $\hat{j}$ component of $\vec{v}_P$ is equal to v_A. Also, $v_Q = 0$ because it is the contact point between the spool and the rail. From rigid body kinematics we then have



$$\begin{aligned}\vec{v}_P &= \vec{v}_Q + \vec{\omega}_{\text{spool}} \times \vec{r}_{P/Q} \\ &= \omega_{\text{spool}}\,\hat{k} \times (-R\,\hat{\imath} + R\,\hat{j}) \\ &= -R\omega_{\text{spool}}(\hat{\imath} + \hat{j}).\end{aligned}$$

Equating v_{Py} to v_A we have

$$v_A = -R\omega_{\text{spool}} \quad \Rightarrow \quad \omega_{\text{spool}} = -\frac{v_A}{R} = -3.33\,\text{rad/s}.$$

In vector form this is

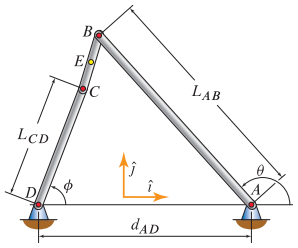
$$\boxed{\vec{\omega}_{\text{spool}} = -3.33\,\hat{k}\,\text{rad/s}.}$$

We must also have

$$\vec{v}_O = \vec{v}_Q + \vec{\omega}_{\text{spool}} \times \vec{r}_{O/Q} = \omega_{\text{spool}}\,\hat{k} \times R\,\hat{j} = -\omega_{\text{spool}} R\,\hat{\imath} \quad \Rightarrow \quad \boxed{\vec{v}_O = 5.00\,\hat{\imath}\,\text{rad/s}.}$$

Problem 6.61

In the four-bar linkage system shown, the lengths of the bars AB and CD are $L_{AB} = 46\,\text{mm}$ and $L_{CD} = 25\,\text{mm}$, respectively. In addition, the distance between points A and D is $d_{AD} = 43\,\text{mm}$. The dimensions of the mechanism are such that when the angle $\theta = 132°$, the angle $\phi = 69°$. For $\theta = 132°$ and $\dot{\theta} = 27\,\text{rad/s}$, determine the angular velocity of bars BC and CD as well as the velocity of the point E, the midpoint of bar BC. Note that the figure is drawn to scale and that bars BC and CD are not collinear.



Solution

We let point D be the origin. From the geometry of the problem we have

$$\begin{aligned}
\vec{r}_A &= d_{AD}\,\hat{\imath} = 0.04300\,\hat{\imath}\,\text{m},\\
\vec{r}_{B/A} &= L_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = (-0.03078\,\hat{\imath} + 0.03418\,\hat{\jmath})\,\text{m},\\
\vec{r}_B &= \vec{r}_A + \vec{r}_{B/A} = (0.01222\,\hat{\imath} + 0.03418\,\hat{\jmath})\,\text{m},\\
\vec{r}_C &= L_{CD}(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) = (0.008959\,\hat{\imath} + 0.02334\,\hat{\jmath})\,\text{m},
\end{aligned}$$

and

$$\begin{aligned}
\vec{r}_{C/B} = \vec{r}_C - \vec{r}_B &= (L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta)\,\hat{\imath}\\
&\quad + (L_{CD}\sin\phi - L_{AB}\sin\theta)\,\hat{\jmath} = (-0.003261\,\hat{\imath} - 0.01085\,\hat{\jmath})\,\text{m}.
\end{aligned}$$

From rigid body kinematics we have

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB}\times\vec{r}_{B/A}.\\
&= \omega_{AB}\,\hat{k}\times L_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\\
&= \omega_{AB}L_{AB}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) = (-0.9230\,\hat{\imath} - 0.8311\,\hat{\jmath})\,\text{m/s}.
\end{aligned}$$

We must also have

$$\begin{aligned}
\vec{v}_C &= \vec{v}_B + \vec{\omega}_{BC}\times\vec{r}_{C/B}\\
&= \omega_{AB}L_{AB}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) + \omega_{BC}\,\hat{k}\times(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta)\,\hat{\imath}\\
&\quad + (L_{CD}\sin\phi - L_{AB}\sin\theta)\,\hat{\jmath}\\
&= [\omega_{BC}(L_{AB}\sin\theta - L_{CD}\sin\phi) - \omega_{AB}L_{AB}\sin\theta]\,\hat{\imath}\\
&\quad + [\omega_{BC}(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta) + \omega_{AB}L_{AB}\cos\theta]\,\hat{\jmath}. \qquad (1)
\end{aligned}$$

Point C is also on bar CD where $\vec{r}_{C/D} = L_{CD}(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})$. From rigid body kinematics we then have

$$\begin{aligned}
\vec{v}_C &= \vec{v}_D + \vec{\omega}_{CD}\times\vec{r}_{C/D}\\
&= \omega_{CD}L_{CD}(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}). \qquad (2)
\end{aligned}$$

Equating Eqs. (1) and (2) component by component we have

$$\omega_{BC}(L_{AB}\sin\theta - L_{CD}\sin\phi) - \omega_{AB}L_{AB}\sin\theta = -\omega_{CD}L_{CD}\sin\phi \qquad (3)$$

and

$$\omega_{BC}(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta) + \omega_{AB}L_{AB}\cos\theta = \omega_{CD}L_{CD}\cos\phi. \qquad (4)$$

Eqs. (3) and (4) are two equations in ω_{BC} and ω_{CD}. Dividing the left hand side of Eq. (3) by the left hand side of Eq. (4) and the right hand side of Eq. (3) by the right hand side of Eq. (4) and switching the sides of the result we have

$$\begin{aligned} -\tan\phi &= \frac{\omega_{BC}(L_{AB}\sin\theta - L_{CD}\sin\phi) - \omega_{AB}L_{AB}\sin\theta}{\omega_{BC}(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta) + \omega_{AB}L_{AB}\cos\theta} \quad \Rightarrow \\ \omega_{AB}L_{AB}(\sin\theta - \tan\phi\cos\theta) &= \omega_{BC}[(L_{AB}\sin\theta - L_{CD}\sin\phi) \\ &\quad + \tan\phi(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta)] \\ \Rightarrow \quad \omega_{BC} &= \frac{\omega_{AB}L_{AB}(\sin\theta - \tan\phi\cos\theta)}{L_{AB}\sin\theta - L_{CD}\sin\phi + \tan\phi(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta)}. \end{aligned} \tag{5}$$

Substituting known values into Eq. (5) we get

$$\boxed{\vec{\omega}_{BC} = 1310\,\hat{k}\ \text{rad/s}.}$$

Solving Eq. (3) for ω_{CD} we have

$$\omega_{CD} = \frac{\omega_{AB}L_{AB}\sin\theta - \omega_{BC}(L_{AB}\sin\theta - L_{CD}\sin\phi)}{L_{CD}\sin\phi}. \tag{6}$$

Substituting Eq. (5) for ω_{BC} in Eq. (6) we have

$$\begin{aligned} \omega_{CD} = &\frac{\omega_{AB}L_{AB}\sin\theta}{L_{CD}\sin\phi} \\ &- \frac{\omega_{AB}L_{AB}(\sin\theta - \tan\phi\cos\theta)(L_{AB}\sin\theta - L_{CD}\sin\phi)}{L_{AB}\sin\theta - L_{CD}\sin\phi + \tan\phi(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta)L_{CD}\sin\phi} \end{aligned} \tag{7}$$

Substituting known values into Eq. (7) we have

$$\boxed{\vec{\omega}_{CD} = -571\,\hat{k}\ \text{rad/s}.}$$

We can calculate $\vec{v}_E$ from the rigid body kinematics of bar BC, where $\vec{r}_{E/B} = \frac{1}{2}\vec{r}_{C/D}$. We have

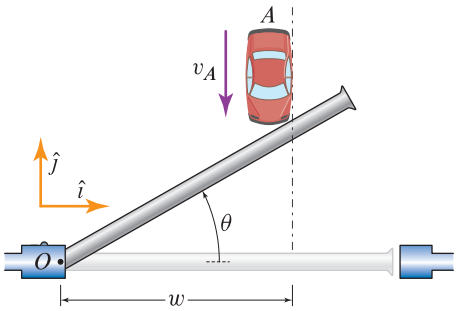
$$\begin{aligned} \vec{v}_E &= \vec{v}_B + \omega_{BC}\,\hat{k} \times \vec{r}_{E/B} \\ &= v_{Bx}\,\hat{\imath} + v_{By}\,\hat{\jmath} + \omega_{BC}\,\hat{k} \times \frac{1}{2}\vec{r}_{C/D} \\ &= v_{Bx}\,\hat{\imath} + v_{By}\,\hat{\jmath} + \frac{\omega_{BC}}{2}\,\hat{k} \times [(r_{C/B})_x\,\hat{\imath} + (r_{C/B})_y\,\hat{\jmath}] \\ &= [v_{Bx} - \frac{\omega_{AB}L_{AB}(\sin\theta - \tan\phi\cos\theta)}{2[L_{AB}\sin\theta - L_{CD}\sin\phi + \tan\phi(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta)]}(r_{C/B})_y]\,\hat{\imath} \\ &\quad + [v_{By} + \frac{\omega_{AB}L_{AB}(\sin\theta - \tan\phi\cos\theta)}{2[L_{AB}\sin\theta - L_{CD}\sin\phi + \tan\phi(L_{CD}\cos\phi - d_{AD} - L_{AB}\cos\theta)]}(r_{C/B})_x]\,\hat{\jmath}. \end{aligned} \tag{8}$$

Substituting all known values into Eq. (8) we have

$$\boxed{\vec{v}_E = (6.20\,\hat{\imath} - 2.97\,\hat{\jmath})\ \text{m/s}.}$$

Problem 6.62

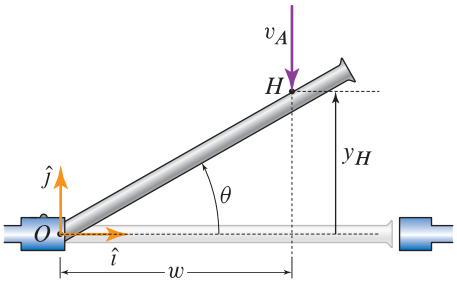
A person is closing a heavy gate with rusty hinges by pushing the gate with car A. If $w = 24\,\text{m}$ and $v_A = 1.2\,\text{m/s}$, determine the angular velocity of the gate when $\theta = 15°$.



Solution

From the geometry of the problem, we have

$$y_H = w \tan\theta. \tag{1}$$



Taking the first derivative of Eq. (1) we have

$$\dot{y}_H = \frac{w}{\cos^2\theta}\dot{\theta}.$$

Realizing that $\dot{y}_H = -v_A$ we have

$$v_A = -\frac{w}{\cos^2\theta}\dot{\theta}.$$

Solving for $\dot{\theta}$ we are left with

$$\dot{\theta} = v_A\frac{\cos^2\theta}{w}. \tag{2}$$

Finally, substituting known values into Eq. (2) we have

$$\boxed{\dot{\theta} = -0.0467\,\text{rad/s}.}$$

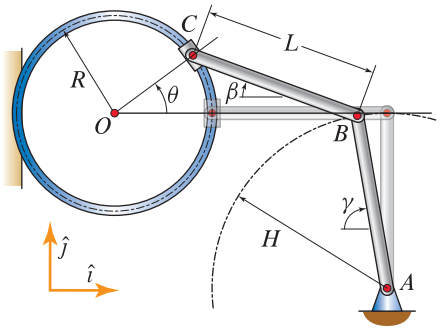
Problems 6.63 through 6.65

In the four-bar linkage system shown, let the circular guide with center at O be fixed and such that, when $\theta = 0°$, the bars AB and BC are vertical and horizontal, respectively. In addition, let $R = 2\,\text{ft}$, $L = 3\,\text{ft}$, and $H = 3.5\,\text{ft}$.

Problem 6.63 When $\theta = 0°$, the collar at C is sliding downward with a speed of $23\,\text{ft/s}$. Determine the angular velocities of the bars AB and BC at this instant.

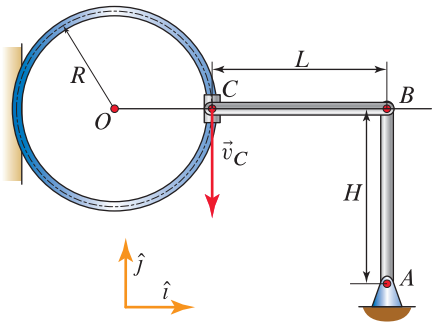
Problem 6.64 When $\theta = 37°$, $\beta = 25.07°$, $\gamma = 78.71°$, and the collar is sliding clockwise with a speed $v_C = 23\,\text{ft/s}$. Determine the angular velocities of the bars AB and BC.

Problem 6.65 Determine the general expression for the angular velocities of bars AB and BC as a function of θ, β, γ, R, L, H, and $\dot{\theta}$.



Solution to 6.63

Because point B is the instantaneous center of rotation of bar CB, it has zero velocity. Point A is pinned and cannot move, so bar AB cannot be moving. Therefore we have



$$\boxed{\vec{\omega}_{AB} = \vec{0}.}$$

From the instantaneous center of rotation method, we then have

$$v_C = L\omega_{BC} \quad \Rightarrow \quad \omega_{BC} = \frac{v_C}{L} = 7.667\,\text{rad/s}.$$

Following the given component system, in vector form, and to three significant figures we have

$$\boxed{\vec{\omega}_{BC} = 7.67\,\hat{k}\,\text{rad/s}.}$$

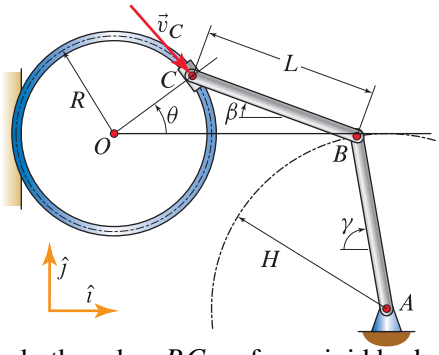
Solution to 6.64

From the geometry of the problem, we have the following position vectors:

$$\vec{r}_{C/O} = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}),$$
$$\vec{r}_{B/C} = L(\cos\beta\,\hat{\imath} - \sin\beta\,\hat{\jmath},$$
$$\vec{r}_{B/A} = H(-\cos\gamma\,\hat{\imath} + \sin\gamma\,\hat{\jmath}),$$

and the velocity vector

$$\vec{v}_C = v_C(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}).$$



We also define $\vec{\omega}_{BC} = \omega_{BC}\,\hat{k}$ and $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$. Points B and C are both on bar BC so from rigid body kinematics we have

$$\begin{aligned}\vec{v}_B &= \vec{v}_C + \omega_{BC}\,\hat{k} \times \vec{r}_{B/C} \\ &= v_C(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) + \omega_{BC}L(\sin\beta\,\hat{\imath} + \cos\beta\,\hat{\jmath}) \\ &= (v_C\sin\theta + \omega_{BC}L\sin\beta)\,\hat{\imath} + (\omega_{BC}L\cos\beta - v_C\cos\theta)\,\hat{\jmath}. \end{aligned} \quad (1)$$

Also, because points A and B are both on bar AB, we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} = -\omega_{AB}H(\sin\gamma\,\hat{\imath} + \cos\gamma\,\hat{\jmath}). \quad (2)$$

Equating Eq. (1) with Eq. (2) component by component we have

$$v_C\sin\theta + \omega_{BC}L\sin\beta = -\omega_{AB}H\sin\gamma \quad (3)$$
$$\omega_{BC}L\cos\beta - v_C\cos\theta = -\omega_{AB}H\cos\gamma. \quad (4)$$

Eqs. (3) and (4) are two equations in ω_{AB} and ω_{BC}. To solve them we divide the left hand side of Eq. (3) by the left hand side of Eq. (4) and the right hand side of Eq. (3) by the right hand side of Eq. (4). We have

$$\frac{v_C\sin\theta + \omega_{BC}L\sin\beta}{\omega_{BC}L\cos\beta - v_C\cos\theta} = \tan\gamma \quad \Rightarrow \quad v_C\sin\theta + \omega_{BC}L\sin\beta = \tan\gamma(\omega_{BC}L\cos\beta - v_C\cos\theta)$$

$$\Rightarrow \quad \omega_{BC} = v_C\frac{\sin\theta + \tan\gamma\cos\theta}{L(\tan\gamma\cos\beta - \sin\beta)}.$$

Substituting in known values, we have

$$\omega_{BC} = -7.210\,\text{rad/s}.$$

To three significant figures, and in vector form we have

$$\boxed{\vec{\omega}_{BC} = -7.21\,\hat{k}\,\text{rad/s}.}$$

Solving Eq. (3) for ω_{AB} we have

$$\omega_{AB} = -\frac{v_C\sin\theta + \omega_{BC}L\sin\beta}{H\sin\gamma}.$$

Plugging in known values, we have

$$\boxed{\vec{\omega}_{AB} = 8.58\,\hat{k}\,\text{rad/s}.}$$

Solution to 6.65

From the geometry of the problem, we have

$$L\sin\beta + H\sin\gamma = H + R\sin\theta \quad \text{and} \quad L\sin\beta + H\cos\gamma = R(1-\cos\theta) + L. \tag{5}$$

Taking the first derivative with respect to time of Eqs. (5) we have

$$L\dot{\beta}\cos\beta + H\dot{\gamma}\cos\gamma = R\dot{\theta}\cos\theta \tag{6}$$

$$-L\dot{\beta}\sin\beta - H\dot{\gamma}\sin\gamma = R\dot{\theta}\sin\theta. \tag{7}$$

Eqs. (6) and (7) can be viewed as two equations in $\dot{\beta}$ and $\dot{\gamma}$. Realizing that $\dot{\beta} = -\omega_{BC}$ and $\dot{\gamma} = -\omega_{AB}$ tells us that solving these two equations for $\dot{\beta}$ and $\dot{\gamma}$ will give us the angular velocities of the two bars as a function of the specified terms. Solving these equations we have

$$\omega_{BC} = -\dot{\beta} = \frac{R\dot{\theta}}{L\cos\beta}\left[\cos\theta - \frac{\cos\theta\tan\beta + \sin\theta}{\tan\beta - \tan\gamma}\right]$$

and

$$\omega_{AB} = -\dot{\gamma} = -\frac{R\dot{\theta}}{H}\frac{\sin\theta + \tan\beta\cos\theta}{\cos\gamma\tan\beta - \sin\gamma}.$$

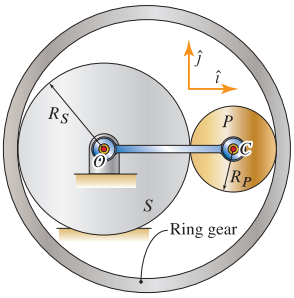
Simplifying these we end up with

and

$$\boxed{\begin{aligned}\vec{\omega}_{BC} &= \frac{R\dot{\theta}}{L}\csc(\beta-\gamma)\sin(\gamma+\theta)\,\hat{k}\\ \vec{\omega}_{AB} &= \frac{R\dot{\theta}}{H}\csc(\beta-\gamma)\sin(\beta+\theta)\,\hat{k}.\end{aligned}}$$

Problem 6.66

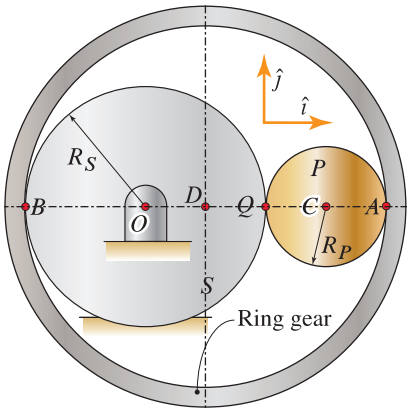
At the instant shown, the arm OC rotates counterclockwise with an angular velocity of 35 rpm about the fixed sun gear S of radius $R_S = 3.5$ in. The planet gear P with radius $R_P = 1.2$ in. rolls without slip over both the fixed sun gear and the outer ring gear. Finally, notice that the ring gear is not fixed and it rolls without slip over the sun gear. Determine the angular velocity of the ring gear and the velocity of the center of the ring gear at the instant shown.



Solution

Points C and O are both on the bar OC, so from rigid body kinematics we must have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{\omega} + {}_{OC} \times \vec{r}_{C/O} \\ &= \omega_{OC}\,\hat{k} \times (R_S + R_P)\,\hat{\imath} \\ &= \omega_{OC}(R_S + R_P)\,\hat{\jmath}. \end{aligned} \tag{1}$$



Point C is also on the planetary gear with point Q, so we must have

$$\begin{aligned}\vec{v}_C &= \vec{v}_Q + \vec{\omega}_P \times \vec{r}_{C/Q} \\ &= \omega_P\,\hat{k} \times R_P\,\hat{\imath} \\ &= \omega_P R_P\,\hat{\jmath}. \end{aligned} \tag{2}$$

Equating Eqs. (1) and (2) we have

$$(R_S + R_P)\omega_{OC} = R_P\omega_P \quad \Rightarrow \quad \omega_P = \frac{R_S + R_P}{R_P}\omega_{OC}.$$

Point A is also on the planetary gear, so we must have

$$\begin{aligned}\vec{v}_A &= \vec{v}_Q + \vec{\omega}_P \times \vec{r}_{A/Q} \\ &= \frac{R_S + R_P}{R_P}\omega_{OC}\,\hat{k} \times 2R_P\,\hat{\imath} \\ &= 2(R_S + R_P)\omega_{OC}\,\hat{\jmath}. \end{aligned} \tag{3}$$

Points A and B are both points on the ring gear, so we must have

$$\begin{aligned}\vec{v}_A &= \vec{v}_B + \vec{\omega}_R \times \vec{r}_{A/B} \\ &= \omega_R\,\hat{k} \times 2(R_S + R_P)\,\hat{\imath} \\ &= 2\omega_R(R_S + R_P)\,\hat{\jmath}. \end{aligned} \tag{4}$$

Equating Eqs. (3) and (4) we have

$$2(R_S + R_P)\omega_{OC} = 2\omega_R(R_S + R_P) \quad \Rightarrow \quad \omega_R = \omega_{OC}.$$

In vector form this is

$$\boxed{\vec{\omega}_R = 35.0\,\hat{k}\text{ rpm.}}$$

Calling the center of the ring point D we must have

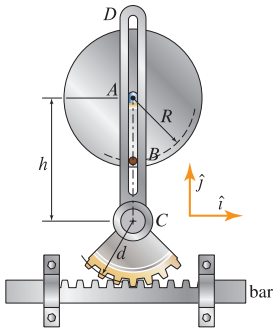
$$\vec{v}_D = \vec{v}_B + \vec{\omega}_R \times \vec{r}_{D/B} = \omega\,\hat{k} \times (R_S + R_P)\,\hat{\imath} = (R_S + R_P)\omega_{OC}\,\hat{\jmath}. \tag{5}$$

Substituting known values into Eq. (5) we have

$$\boxed{\vec{v}_D = 17.2\,\hat{\jmath}\text{ in./s.}}$$

Problem 6.67

The crank AB is rotating counterclockwise at a constant angular velocity of 12 rad/s while the pin B slides within the slot in the bar CD, which is pinned at C. Letting $R = 0.5$ m, $h = 1$ m, and $d = 0.25$ m, determine the angular velocity of CD at the instant shown (with points A, B, and C vertically aligned) as well as the velocity of the horizontal bar to which bar CD is connected.



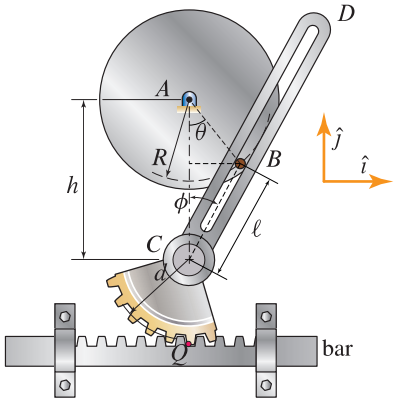
Solution

If we let ℓ be the distance between B and C, then from the geometry of the problem we have the constraint equation

$$R \sin\theta = \ell \sin\phi.$$

Differentiating this constraint with respect to time we have

$$R\dot{\theta}\cos\theta = \dot{\ell}\sin\phi + \ell\dot{\phi}\cos\phi. \qquad (1)$$



At the instant given in the problem, we have $\theta = 0°, \phi = 0°$, and $\ell = h - R \quad \Rightarrow \quad \dot{\ell} = 0$. Substituting these values into Eq. (1) we have

$$R\dot{\theta} = (h - R)\dot{\phi} \quad \Rightarrow \quad \dot{\phi} = \frac{R}{h - R}\dot{\theta}.$$

Making the observation that $\omega_{AB} = \dot{\theta}$ and $\omega_{CD} = -\dot{\phi}$, we have

$$\omega_{CD} = -\frac{R}{h - R}\omega_{AB} = -12.00 \text{ rad/s} \quad \Rightarrow \quad \boxed{\vec{\omega}_{CD} = -12.0\,\hat{k} \text{ rad/s}.}$$

Points Q and C are both on the arm CD so from rigid body kinematics we must have

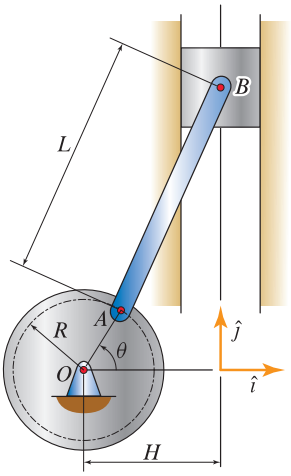
$$\vec{v}_Q = \vec{v}_C + \vec{\omega}_{CD} \times \vec{r}_{Q/C} = \omega_{CD}\,\hat{k} \times (-d)\,\hat{\jmath} = \omega_{CD} d\,\hat{\imath}.$$

Substituting known values we have

$$\boxed{\vec{v}_Q = 3.00\,\hat{\imath} \text{ m/s}.}$$

Problems 6.68 through 6.72

For the slider-crank mechanism shown, let $R = 20\,\text{mm}$, $L = 80\,\text{mm}$, and $H = 38\,\text{mm}$.



Problem 6.68 If $\dot{\theta} = 1700\,\text{rpm}$, determine the angular velocity of the connecting rod AB and the speed of the slider B for $\theta = 90°$.

Problem 6.69 Determine the angular velocity of the crank OA when $\theta = 20°$ and the slider is moving downward at $15\,\text{m/s}$.

Problem 6.70 Determine the general expression for the velocity of the slider B as a function of θ, $\dot{\theta}$, and the geometrical parameters R, H, and L, using the *vector approach*.

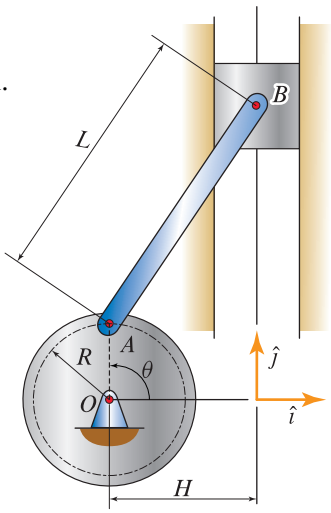
Problem 6.71 Determine the general expression for the velocity of the slider B as a function of θ, $\dot{\theta}$, and the geometrical parameters R, H, and L, using *differentiation of constraints*.

Problem 6.72 Plot the velocity of the slider C as a function of θ, for $0 \le \theta \le 360°$, and for $\dot{\theta} = 1000\,\text{rpm}$, $\dot{\theta} = 3000\,\text{rpm}$, and $\dot{\theta} = 5000\,\text{rpm}$.

Solution to 6.68

Converting units on known values we have

$$R = 20\,\text{mm} = 0.020\,\text{m}, \quad L = 80\,\text{mm} = 0.080\,\text{m}, \quad H = 38\,\text{mm} = 0.038\,\text{m}.$$



From the rigid body kinematics of the wheel, and letting $\omega_O = \dot{\theta}$ we have

$$\vec{v}_A = \vec{v}_O + \vec{\omega}_O \times \vec{r}_{A/O} = \dot{\theta}\,\hat{k} \times R\,\hat{\jmath} = -\dot{\theta} R\,\hat{\imath}.$$

From the rigid body kinematics of arm AB where $\vec{r}_{B/A} = H\,\hat{\imath} + \sqrt{L^2 - H^2}\,\hat{\jmath}$, we have

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= -R\dot{\theta}\,\hat{\imath} + \omega_{AB}\,\hat{k} \times (H\,\hat{\imath} + \sqrt{L^2 - H^2}\,\hat{\jmath}) \\ &= (-\dot{\theta} R - \omega_{AB}\sqrt{L^2 - H^2})\,\hat{\imath} + \omega_{AB} H\,\hat{\jmath}.\end{aligned}$$

Because B is confined to move vertically in the slot, $v_{Bx} = 0$. From this we have

$$\omega_{AB} = -\frac{R}{\sqrt{L^2 - H^2}}\dot{\theta}.$$

In vector form, and plugging in known values we have

$$\boxed{\vec{\omega}_{AB} = -50.6\,\hat{k}\ \text{rad/s}.}$$

From the y component of $\vec{v}_B$ we have

$$\vec{v}_B = v_{By}\,\hat{\jmath} = \omega_{AB} H\,\hat{\jmath} = \dot{\theta} H \frac{R}{\sqrt{L^2 - H^2}}\,\hat{\jmath}.$$

Plugging in known values we have

$$\boxed{\vec{v}_B = 1.92\,\hat{j}\ \text{m/s}.}$$

Solution to 6.69

Converting units on known values we have

$$R = 20\,\text{mm} = 0.020\,\text{m}, \quad L = 80\,\text{mm} = 0.080\,\text{m}, \quad H = 38\,\text{mm} = 0.038\,\text{m}.$$

From the rigid body kinematics of the wheel, and letting $\omega_O = \dot{\theta}$ we have

$$\vec{v}_A = \vec{v}_O + \vec{\omega}_O \times \vec{r}_{A/O} = \dot{\theta}\,\hat{k} \times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = \dot{\theta} R(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}).$$

From the rigid body kinematics of arm AB where $\vec{r}_{B/A} = (H - R\cos\theta)\,\hat{\imath} + \sqrt{L^2 - (H - R\cos\theta)^2}\,\hat{\jmath}$, we have

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\
&= -R\dot{\theta}\,\hat{\imath} + \omega_{AB}\,\hat{k} \times \left[(H - R\cos\theta)\,\hat{\imath} + \sqrt{L^2 - (H - R\cos\theta)^2}\,\hat{\jmath}\right] \\
&= \left(-\dot{\theta} R - \omega_{AB}\sqrt{L^2 - (H - R\cos\theta)^2}\right)\hat{\imath} + \omega_{AB}(H - R\cos\theta)\,\hat{\jmath}.
\end{aligned}$$

Because B is confined to move vertically in the slot, $v_{Bx} = 0$. From this we have

$$\dot{\theta} = \frac{-\omega_{AB}\sqrt{L^2 - (H - R\cos\theta)^2}}{R}. \tag{1}$$

We also know that $\vec{v}_B = -15\,\hat{\jmath}\,\text{m/s}$. From this we have

$$\omega_{AB}(H - R\cos\theta) = -15 \quad \Rightarrow \quad \omega_{AB} = \frac{15}{R\cos\theta - H}. \tag{2}$$

Substituting Eq. (2) into Eq. (1) we have

$$\dot{\theta} = \frac{-15\sqrt{L^2 - (H - R\cos\theta)^2}}{R(R\cos\theta - H)}.$$

Plugging in known values we have

$$\boxed{\vec{\omega}_O = \dot{\theta}\,\hat{k} = -877\,\hat{k}\,\text{rad/s}.}$$

Solution to 6.70

From the rigid body kinematics of the wheel, and letting $\omega_O = \dot{\theta}$ we have

$$\vec{v}_A = \vec{v}_O + \vec{\omega}_O \times \vec{r}_{A/O} = \dot{\theta}\,\hat{k} \times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = \dot{\theta} R(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}).$$

From the rigid body kinematics of arm AB, where $\vec{r}_{B/A} = (H - R\cos\theta)\,\hat{\imath} + \sqrt{L^2 - (H - R\cos\theta)^2}\,\hat{\jmath}$, we have

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\
&= -R\dot{\theta}\,\hat{\imath} + \omega_{AB}\,\hat{k} \times \left[(H - R\cos\theta)\,\hat{\imath} + \sqrt{L^2 - (H - R\cos\theta)^2}\,\hat{\jmath}\right] \\
&= \left(-\dot{\theta} R\sin\theta - \omega_{AB}\sqrt{L^2 - (H - R\cos\theta)^2}\right)\hat{\imath} + [\omega_{AB}(H - R\cos\theta) + R\dot{\theta}\cos\theta]\,\hat{\jmath}. \qquad (3)
\end{aligned}$$

Because B is confined to move vertically in the slot, $v_{Bx} = 0$. From this we have

$$\omega_{AB} = -\frac{R\dot{\theta}\sin\theta}{\sqrt{L^2 - (H - R\cos\theta)^2}}. \qquad (4)$$

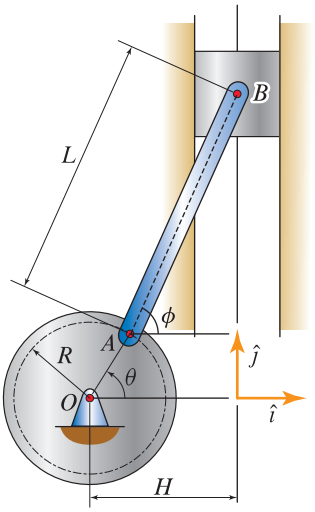
Plugging Eq. (4) into Eq. (3) we have

$$\boxed{\vec{v}_B = \left(R\dot{\theta}\cos\theta - \frac{R\dot{\theta}\sin\theta(H - R\cos\theta)}{\sqrt{L^2 - (H - R\cos\theta)^2}}\right)\hat{\jmath}.}$$

Solution to 6.71

From the geometry of the problem, we have

$$x_B = R\cos\theta + L\cos\phi = H \tag{5}$$
$$y_B = R\sin\theta + L\sin\phi.$$



Taking a time derivative of these two equations we have

$$-R\dot{\theta}\sin\theta - L\dot{\phi}\sin\phi = 0 \quad \Rightarrow \quad \dot{\phi} = -\frac{R\dot{\theta}\sin\theta}{L\sin\phi} \tag{6}$$

$$v_B = R\dot{\theta}\cos\theta + L\dot{\phi}\cos\phi. \tag{7}$$

Substituting Eq. (6) into Eq. (7) we have

$$v_B = R\dot{\theta}\cos\theta - R\dot{\theta}\sin\theta\cot\phi. \tag{8}$$

From Eq. (5) we know that

$$\cos\phi = \frac{H - R\cos\theta}{L}, \tag{9}$$

and from the Pythagorean theorem, we know that

$$\sin\phi = \sqrt{L^2 - (H - R\cos\theta)^2}. \tag{10}$$

Combining Eqs. (9) and (10) we have

$$\cot\phi = \frac{H - R\cos\theta}{L\sqrt{L^2 - (H - R\cos\theta)^2}}. \tag{11}$$

Substituting Eq. (11) into Eq. (8) we have

$$v_B = R\dot{\theta}\cos\theta - R\dot{\theta}\sin\theta\frac{H - R\cos\theta}{L\sqrt{L^2 - (H - R\cos\theta)^2}}.$$

Knowing that B is confined to the $\hat{j}$ direction, we have

$$\boxed{\vec{v}_B = \left(R\dot{\theta}\cos\theta - \frac{R\dot{\theta}\sin\theta(H - R\cos\theta)}{\sqrt{L^2 - (H - R\cos\theta)^2}}\right)\hat{j}.}$$

Solution to 6.72

Converting units on known values we have

$$\begin{aligned} R &= 0.020\,\text{m}, \\ L &= 0.080\,\text{m}, \\ H &= 0.038\,\text{m}. \end{aligned}$$

From the rigid body kinematics of the wheel, and letting $\omega_O = \dot{\theta}$ we have

$$\begin{aligned} \vec{v}_A &= \vec{v}_O + \vec{\omega}_O \times \vec{r}_{A/O} \\ &= \dot{\theta}\,\hat{k} \times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= \dot{\theta} R(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}). \end{aligned}$$

From the rigid body kinematics of arm AB, where $\vec{r}_{B/A} = (H - R\cos\theta)\,\hat{\imath} + \sqrt{L^2 - (H - R\cos\theta)^2}\,\hat{\jmath}$, we have

$$\begin{aligned} \vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= -R\dot{\theta}\,\hat{\imath} + \omega_{AB}\,\hat{k} \times \left[(H - R\cos\theta)\,\hat{\imath} + \sqrt{L^2 - (H - R\cos\theta)^2}\,\hat{\jmath}\right] \\ &= \left(-\dot{\theta} R\sin\theta - \omega_{AB}\sqrt{L^2 - (H - R\cos\theta)^2}\right)\hat{\imath} + [\omega_{AB}(H - R\cos\theta) + R\dot{\theta}\cos\theta]\,\hat{\jmath}. \end{aligned} \tag{12}$$

Because B is confined to move vertically in the slot, $v_{Bx} = 0$. From this we have

$$\omega_{AB} = -\frac{R\dot{\theta}\sin\theta}{\sqrt{L^2 - (H - R\cos\theta)^2}}. \tag{13}$$

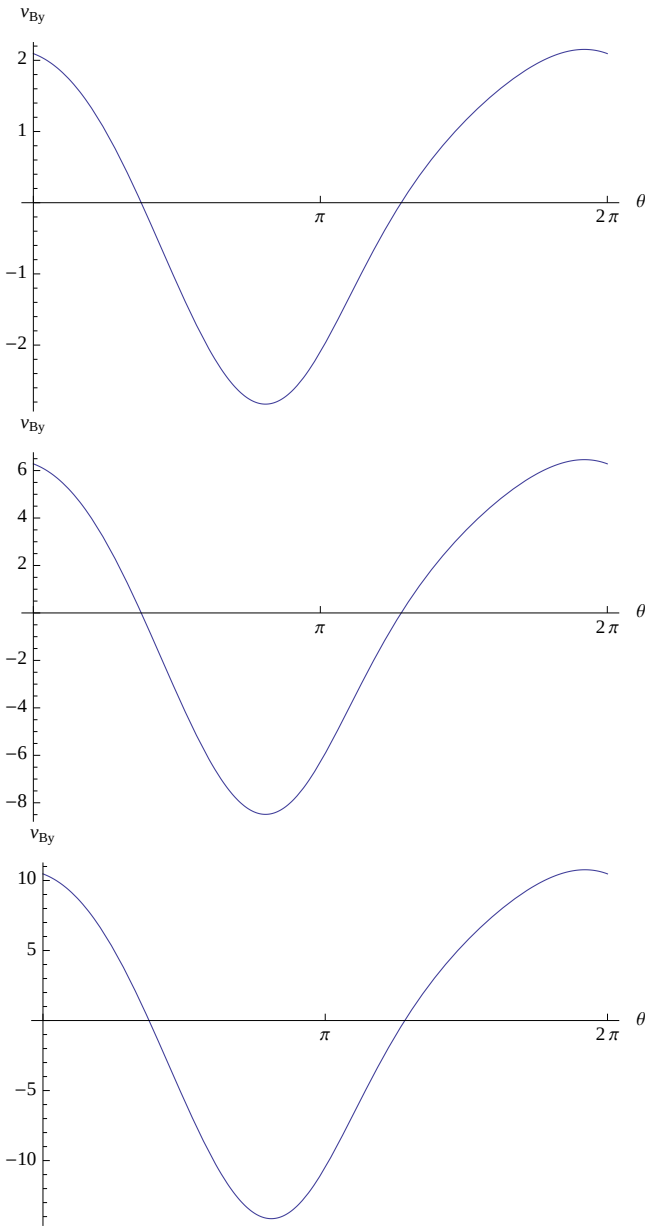
Plugging Eq. (13) into Eq. (12) we have

$$\boxed{\vec{v}_B = \left(R\dot{\theta}\cos\theta - \frac{R\dot{\theta}\sin\theta(H - R\cos\theta)}{\sqrt{L^2 - (H - R\cos\theta)^2}}\right)\hat{\jmath}.}$$

The expression for $\vec{v}_B$ just derived can be reduced to a function of θ and $\dot{\theta}$ by substituting in the given parameters. For each of the required values of $\dot{\theta}$, the function can be plotted with software such as *Mathematica* or MATLAB. The plots presented below were obtained using *Mathematica* with the following code:

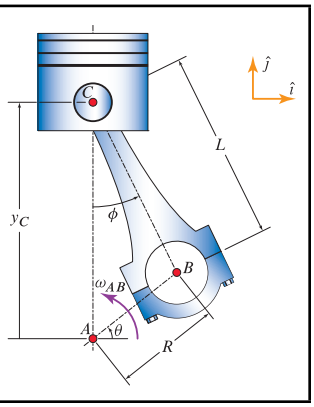
```
Parameters = {R → 0.02, L → 0.08, H → 0.038};
Velocity = R θdot Cos[θ] - R θdot Sin[θ] (H - R Cos[θ]) / Sqrt[L^2 - (H - R Cos[θ])^2] /. Parameters;
θdotval = 1/30 π {1000, 3000, 5000};
Table[Plot[Velocity /. {θdot → i}, {θ, 0, 2π}, AxesLabel → {"θ", "v_By"},
  Ticks → {{0, π, 2π, 3π}, Automatic}], {i, θdotval}]
```

Using the above code we obtain the following plots:


v_{By}
2
1
−1
−2
π
2π
θ
v_{By}
6
4
2
−2
−4
−6
−8
π
2π
θ
v_{By}
10
5
−5
−10
π
2π
θ

Problem 6.73

Complete the velocity analysis of the slider-crank mechanism, using differentiation of constraints that was outlined beginning on p. 478. That is, determine the velocity of the piston C and the angular velocity of the connecting rod as a function of the given quantities θ, ω_{AB}, R, and L. Use the component system shown for your answers.



Solution

One way to solve this problem is to write y_C as

$$y_C = R\cos\theta + L\cos\phi, \tag{1}$$

where for $\cos\phi$ and $\sin\phi$ we will *always* use

$$\sin\phi = \frac{R}{L}\sin\theta \quad \text{and} \quad \cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-\frac{R^2}{L^2}\sin^2\theta}. \tag{2}$$

Now we differentiate Eq. (1) with respect to time:

$$\dot{y}_C = -R\dot{\theta}\sin\theta - L\dot{\phi}\sin\phi, \tag{3}$$

where $\dot{\phi}$ is found by differentiating the first of Eq. (2) with respect to time:

$$\dot{\phi}\cos\phi = \frac{R}{L}\dot{\theta}\cos\theta \quad \Rightarrow \quad \dot{\phi} = \frac{R}{L}\dot{\theta}\frac{\cos\theta}{\cos\phi}. \tag{4}$$

Realizing that $\dot{\phi} = \omega_{BC}$ and $\dot{\theta} = \omega_{AB}$, we have

$$\omega_{BC} = \omega_{AB}\frac{R}{L}\frac{\cos\theta}{\cos\phi} \quad \Rightarrow \quad \boxed{\vec{\omega}_{BC} = \omega_{AB}\frac{R}{L}\frac{\cos\theta}{\cos\phi}\hat{k}.}$$

Also, noting that $\vec{v}_C = \dot{y}_C\,\hat{j}$, and substituting Eq. (4) for $\dot{\phi}$ in Eq. (3) we have

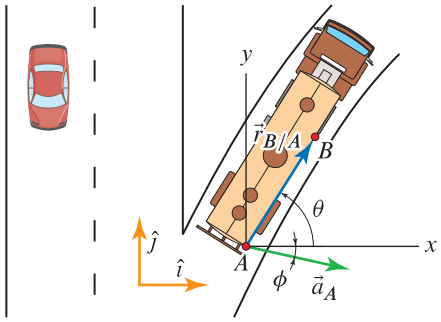
$$\dot{y}_C = -R\dot{\theta}\sin\theta - L\left(\frac{R\dot{\theta}\cos\theta}{L\cos\phi}\right)\left(\frac{R\sin\theta}{L}\right) = -R\dot{\theta}\sin\theta - \frac{R^2\dot{\theta}\sin\theta\cos\theta}{L\cos\phi}. \tag{5}$$

After substituting ω_{AB} into Eq. (5), and putting it in vector form we have

$$\boxed{\vec{v}_C = \left(-R\omega_{AB}\sin\theta - \frac{R^2\omega_{AB}\sin\theta\cos\theta}{L\cos\phi}\right)\hat{j}.}$$

Problem 6.74

A truck on an exit ramp is moving in such a way that, at the instant shown, $|\vec{a}_A| = 17\,\text{ft/s}^2$, $\dot{\theta} = -0.3\,\text{rad/s}$, and $\ddot{\theta} = -0.1\,\text{rad/s}^2$. If the distance between points A and B is $d_{AB} = 12\,\text{ft}$, $\theta = 57°$, and $\phi = 13°$, determine $\vec{a}_B$.



Solution

Based on the picture shown, we let $\vec{a}_A = |\vec{a}_A|(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath})$, $\vec{\omega}_{AB} = \dot{\theta}\,\hat{k}$, $\vec{\alpha}_{AB} = \ddot{\theta}\,\hat{k}$, and $\vec{r}_{B/A} = d_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$. From the vector approach to acceleration analysis we have

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} \\ &= |\vec{a}_A|(\cos\phi\,\hat{\imath} - \sin\phi) + \ddot{\theta}\,\hat{k} \times d_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) - \dot{\theta}^2 d_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= (|\vec{a}_A|\cos\phi - \ddot{\theta} d_{AB}\sin\theta - \dot{\theta}^2 d_{AB}\cos\theta)\,\hat{\imath} + (-|\vec{a}_A|\sin\phi + \ddot{\theta} d_{AB}\cos\theta - \dot{\theta}^2 d_{AB}\sin\theta)\,\hat{\jmath},\end{aligned}$$

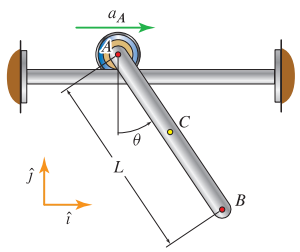
which can be evaluated to obtain

$$\boxed{\vec{a}_B = (17.0\,\hat{\imath} - 5.38\,\hat{\jmath})\,\text{ft/s}^2,}$$

using the given values $|\vec{a}_A| = 17\,\text{ft/s}^2$, $\dot{\theta} = -0.3\,\text{rad/s}$, $\ddot{\theta} = -0.1\,\text{rad/s}^2$, $d_{AB} = 12\,\text{ft}$, $\theta = 57°$, and $\phi = 13°$.

Problems 6.75 through 6.77

Let $L = 4\,\text{ft}$, let point A travel parallel to the guide shown, and let C be the midpoint of the bar.



Problem 6.75 If point A is accelerating to the right with $a_A = 27\,\text{ft/s}^2$ and $\dot{\theta} = 7\,\text{rad/s} = \text{constant}$, determine the acceleration of point C when $\theta = 24°$.

Problem 6.76 If point A is accelerating to the right with $a_A = 27\,\text{ft/s}^2$, $\dot{\theta} = 7\,\text{rad/s}$, and $\ddot{\theta} = -0.45\,\text{rad/s}^2$, determine the acceleration of point C when $\theta = 26°$.

Problem 6.77 If, when $\theta = 0°$, A is accelerating to the right with $a_A = 27\,\text{ft/s}^2$ and $\vec{a}_C = \vec{0}$, determine $\dot{\theta}$ and $\ddot{\theta}$.

Solution to 6.75

We have $\vec{a}_A = a_A\,\hat{\imath}, \vec{\omega}_{AB} = \dot{\theta}\,\hat{k}, \vec{\alpha}_{AB} = \ddot{\theta}\,\hat{k} = \vec{0}$, and $\vec{r}_{C/A} = \frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$. From the vector approach to acceleration analysis we have

$$\begin{aligned}\vec{a}_C &= \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{C/A} - \omega_{AB}^2 \vec{r}_{C/A} \\ &= a_A\,\hat{\imath} - \frac{L\dot{\theta}^2}{2}(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \\ &= \left(a_A - \frac{L}{2}\dot{\theta}^2 \sin\theta\right)\hat{\imath} + \dot{\theta}^2\frac{L}{2}\cos\theta\,\hat{\jmath},\end{aligned}$$

which can be evaluated to obtain

$$\boxed{\vec{a}_C = (-12.9\,\hat{\imath} + 89.5\,\hat{\jmath})\,\text{ft/s}^2.}$$

using the given values $L = 4\,\text{ft}$, $a_A = 27\,\text{ft/s}^2$, $\dot{\theta} = 7\,\text{rad/s}$, and $\theta = 24°$.

Solution to 6.76

We have $\vec{a}_A = a_A\,\hat{\imath}, \vec{\omega}_{AB} = \dot{\theta}\,\hat{k}$, $\vec{\alpha}_{AB} = \ddot{\theta}\,\hat{k} = \vec{0}$, and $\vec{r}_{C/A} = \frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$. From the vector approach to acceleration analysis we have

$$\begin{aligned}\vec{a}_C =& \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{C/A} - \omega_{AB}^2 \vec{r}_{C/A} \\ &= a_A\,\hat{\imath} + \ddot{\theta}\,\hat{k} \times \frac{L}{2}(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) - \frac{L\dot{\theta}^2}{2}(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \\ &= \left(a_A + \ddot{\theta}\frac{L}{2}\cos\theta - \frac{L}{2}\dot{\theta}^2\sin\theta\right)\hat{\imath} + \left(\ddot{\theta}\frac{L}{2}\sin\theta + \dot{\theta}^2\frac{L}{2}\cos\theta\right)\hat{\jmath},\end{aligned}$$

which can be evaluated to obtain

$$\boxed{(16.8\,\hat{\imath} + 87.7\,\hat{\jmath})\ \text{ft/s}^2.}$$

using the given values $L = 4\,\text{ft}$, $a_A = 27\,\text{ft/s}^2$, $\dot{\theta} = 7\,\text{rad/s}$, $\ddot{\theta} = -0.45\,\text{rad/s}^2$, and $\theta = 26°$.

Solution to 6.77

We have $\vec{a}_A = a_A\,\hat{\imath}$, $\vec{\omega}_{AB} = \dot{\theta}\,\hat{k}$, $\vec{\alpha}_{AB} = \ddot{\theta}\,\hat{k} = \vec{0}$, and $\vec{r}_{C/A} = -\frac{L}{2}\,\hat{\jmath}$. From the vector approach to acceleration analysis we have

$$\begin{aligned}\vec{a}_C &= \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{C/A} - \omega_{AB}^2 \vec{r}_{C/A} \\ \vec{0} &= a_A\,\hat{\imath} + \ddot{\theta}\,\hat{k} \times \left(-\frac{L}{2}\right)\hat{\jmath} + \frac{L\dot{\theta}^2}{2}\,\hat{\jmath} \\ \vec{0} &= \left(a_A + \ddot{\theta}\frac{L}{2}\right)\hat{\imath} + \dot{\theta}^2\frac{L}{2}\,\hat{\jmath}. \qquad (1)\end{aligned}$$

Separating the components of Eq. (1) we have

$$0 = a_A + \ddot{\theta}\frac{L}{2} \quad \Rightarrow \quad \boxed{\ddot{\theta} = -a_A\frac{2}{L} = -13.5\,\text{rad/s}^2,}$$

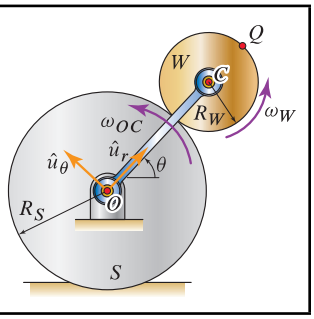
and

$$0 = \dot{\theta}^2\frac{L}{2} \quad \Rightarrow \quad \boxed{\dot{\theta} = 0,}$$

using the given values $L = 4\,\text{ft}$, $a_A = 27\,\text{ft/s}^2$, $a_C = 0\,\text{ft/s}^2$, and $\theta = 0°$.

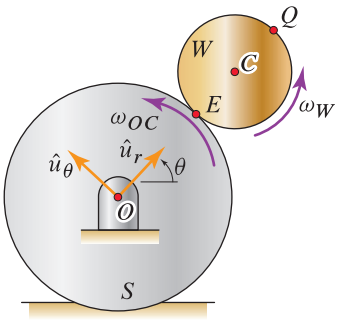
Problem 6.78

A wheel W of radius $R_W = 5\,\text{cm}$ rolls without slip over the stationary cylinder S of radius $R_S = 12\,\text{cm}$, and the wheel is connected to point O via the arm OC. If $\omega_{OC} = \text{constant} = 3.5\,\text{rad/s}$, determine the acceleration of point Q, which lies on the edge of W and along the extension of the line OC.



Solution

Let E be the contact point between W and S. From the no slip condition, $\vec{v}_E = \vec{0}$. We let $\vec{\omega}_W = \omega_W \hat{k}$ and from the geometry of the problem $\vec{r}_{C/E} = R_W(\cos\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$. From rigid body kinematics of the wheel we have



$$\begin{aligned}\vec{v}_C &= \vec{v}_E + \vec{\omega}_W \times \vec{r}_{C/E} \\ &= \omega_W \hat{k} \times R_W \hat{u}_r \\ &= \omega_W R_W \hat{u}_\theta. \end{aligned} \tag{1}$$

Points C and O are also points on the arm where $\vec{r}_{C/O} = (R_S + R_W)\,\hat{u}_r$. From rigid body kinematics we must have

$$\vec{v}_C = \vec{v}_O + \vec{\omega}_{OC} \times \vec{r}_{C/O} = \omega_{OC}\,\hat{k} \times (R_S + R_W)\hat{u}_r = \omega_{OC}(R_S + R_W)\hat{u}_\theta. \tag{2}$$

Equating Eqs. (1) and (2) we have

$$\omega_W R_W = \omega_{OC}(R_S + R_W) \quad \Rightarrow \quad \omega_W = \frac{R_S + R_W}{R_W}\omega_{OC}.$$

Because C is undergoing a uniform circular motion with radius $(R_S + R_W)$ about O, we have

$$\vec{a}_C = -\omega_{OC}^2 (R_S + R_W)\,\hat{u}_r.$$

From the vector approach to acceleration analysis we have

$$\begin{aligned}\vec{a}_Q &= \vec{a}_C + \vec{\alpha}_W \times \vec{r}_{Q/C} - \omega_W^2 \vec{r}_{Q/C} \\ &= -\omega_{OC}^2 (R_S + R_W)\,\hat{u}_r - \left(\frac{R_S + R_W}{R_W}\right)^2 \omega_{OC}^2 R_W\,\hat{u}_r \\ &= -\omega_{OC}^2 (R_S + R_W)\left(1 + \frac{R_S + R_W}{R_W}\right)\hat{u}_r \\ &= -\omega_{OC}^2 (R_S + R_W)\left(2 + \frac{R_S}{R_W}\right)\hat{u}_r,\end{aligned}$$

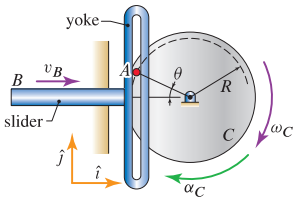
which can be evaluated to obtain

$$\boxed{-9.16\,\hat{u}_r\ \text{m/s}^2.}$$

using the given values $R_W = 5\,\text{cm} = 0.05000\,\text{m}$, $R_S = 12\,\text{cm} = 0.1200\,\text{m}$, and $\omega_{OC} = 3.5\,\text{rad/s}$.

Problem 6.79

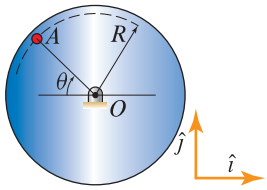
One way to convert rotational motion into linear motion and vice versa is via the use of a mechanism called the Scotch yoke, which consists of a crank C that is connected to a slider B via a pin A. The pin rotates with the crank while sliding within the yoke, which, in turn, rigidly translates with the slider. This mechanism has been used, for example, to control the opening and closing of valves in pipelines. Letting the radius of the crank be $R = 25\,\text{cm}$, determine the angular velocity ω_C and the angular acceleration α_C of the crank at the instant shown if $\theta = 25°$ and the slider is moving to the right with a constant speed $v_B = 40\,\text{m/s}$.



Solution

Choosing point O as the origin of the reference frame, from the geometry of the problem we have

$$\vec{r}_A = R(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \tag{1}$$



By construction, the velocity of the slider is equal to the horizontal component of the velocity of point A. We can get $\vec{v}_A$ by taking the derivative of Eq. (1) with respect to time. Doing so we have

$$\vec{v}_A = R\dot{\theta}(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \quad \Rightarrow \quad \vec{v}_B = R\dot{\theta}\sin\theta\,\hat{\imath} \quad \Rightarrow \quad \dot{\theta} = \frac{v_B}{R\sin\theta} = 378.6\,\text{rad/s},$$

where we have used the given values $R = 25\,\text{cm} = 0.2500\,\text{m}$, $v_B = 40\,\text{m/s}$, and $\theta = 25°$. Realizing that $\omega_C = -\dot{\theta}$ we have

$$\boxed{\vec{\omega}_C = -379\,\hat{k}\,\text{rad/s}.}$$

Because we are told $v_B = 0$ we must have

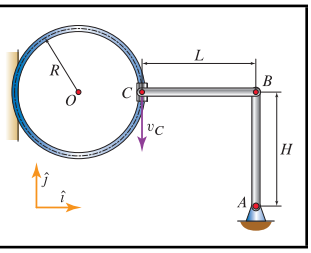
$$\frac{d}{dt}v_B = R\sin\theta\ddot{\theta} + R\cos\theta\dot{\theta}^2 = 0 \quad \Rightarrow \quad \ddot{\theta} = -\frac{\dot{\theta}^2}{\tan\theta}.$$

Plugging in known values we have

$$\boxed{\ddot{\theta} = -307\times10^3\,\hat{k}\,\text{rad/s}^2.}$$

Problem 6.80

Collar C moves along a circular guide with radius $R = 2\,\text{ft}$ with a constant speed $v_C = 18\,\text{ft/s}$. At the instant shown, the bars AB and BC are vertical and horizontal, respectively. Letting $L = 4\,\text{ft}$ and $H = 5\,\text{m}$, determine the angular accelerations of the bars AB and BC at this instant.



Solution

Note: The problem statement gives $H = 5\,\text{m}$, which is incorrect. The solution will use $H = 5\,\text{ft}$.

Realizing that B is the instantaneous center of rotation of bar AB and A is not moving, we must have $\omega_{AB} = 0$. Because B is also the instantaneous center of rotation of bar BC, we must have

$$\omega_{BC} = \frac{v_C}{L} \quad \Rightarrow \quad \vec{\omega}_{BC} = \frac{v_B}{L}\,\hat{k}.$$

Since C moves in a uniform circular motion about O at constant speed, we have

$$\vec{a}_C = -\frac{v_C^2}{R}\,\hat{\imath},$$

and from the vector approach to acceleration analysis where, $\vec{r}_{B/C} = L\,\hat{\imath}$ and $\vec{\alpha}_{BC} = \alpha_{BC}\,\hat{k}$, we have

$$\begin{aligned}\vec{a}_B &= \vec{a}_C + \vec{\alpha}_{BC} \times \vec{r}_{B/C} - \omega_{BC}^2 \vec{r}_{B/C} \\ &= -\frac{v_C^2}{R}\,\hat{\imath} + \alpha_{BC}\,\hat{k} \times L\,\hat{\imath} - \frac{v_C^2}{R^2} L\,\hat{\imath} \\ &= -\frac{v_C^2}{R}\left(1 + \frac{L}{R}\right)\hat{\imath} + \alpha_{BC} L\,\hat{\jmath}. \end{aligned} \tag{1}$$

Also from the vector approach to acceleration analysis where $\vec{r}_{B/A} = H\,\hat{\jmath}$ and $\vec{\alpha}_{AB} = \alpha_{AB}\,\hat{k}$ we have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} = \alpha_{AB}\,\hat{k} \times H\,\hat{\jmath} = -\alpha_{AB} H\,\hat{\imath}. \tag{2}$$

Equating components of Eqs. (1) and (2) we have

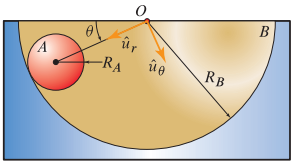
$$-\frac{v_C^2}{R}\left(1 + \frac{L}{R}\right) = -\alpha_{AB} H \quad \text{and} \quad \alpha_{BC} L = 0. \tag{3}$$

Solving Eq. (3) for α_{AB} and α_{BC}, we have

$$\alpha_{AB} = \frac{v_C^2}{R}\left(1 + \frac{L}{R}\right) \quad \Rightarrow \quad \boxed{\vec{\alpha}_{AB} = 97.2\,\hat{k}\,\text{rad/s}^2} \quad \text{and} \quad \boxed{\vec{\alpha}_{BC} = \vec{0}.}$$

Problems 6.81 and 6.82

A ball of radius $R_A = 5$ in. is rolling without slip inside a stationary spherical bowl of radius $R_B = 17$ in. Assume that the motion of the ball is planar.

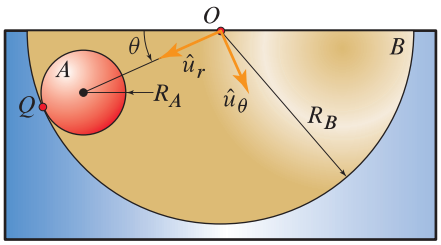


Problem 6.81 If, at the instant shown, the center of the ball is traveling counterclockwise with a speed $v_A = 32$ ft/s and such that $\dot{v}_A = 0$, determine the acceleration of the center of the ball as well as the acceleration of the point on the ball that is contact with the bowl.

Problem 6.82 If, at the instant shown, the center of the ball is traveling counterclockwise with a speed $v_A = 32$ ft/s and such that $\dot{v}_A = 24$ ft/s^2, determine the acceleration of the center of the ball as well as the acceleration of the point on the ball that is contact with the bowl.

Solution to 6.81

The center of the ball is making a uniform circular motion of radius $R_B - R_A$ about O at constant speed, so we must have



$$\vec{a}_A = -\frac{v_A^2}{R_B - R_A}\,\hat{u}_r \qquad (1)$$

Since the ball rolls without slip over a stationary surface, the point of contact between the ball and the bowl, point Q, is the instantaneous center of rotation of the ball. Therefore,

$$v_A = |\vec{\omega}_A| R_A \quad \Rightarrow \quad \omega_A = \frac{v_A}{R_A}.$$

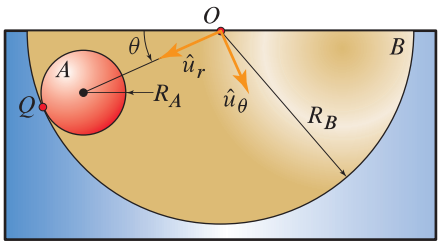
Notice that $\alpha_A = 0$ because ω_A is constant. From the vector approach to acceleration analysis where $\vec{r}_{Q/A} = R_A\,\hat{u}_r$ we must have

$$\vec{a}_Q = \vec{a}_A + \vec{\alpha}_A \times \vec{r}_{Q/A} - \omega_A^2 \vec{r}_{Q/A} = -\frac{v_A^2}{R_B - R_A}\,\hat{u}_r - \frac{v_A^2}{R_A^2} R_A\,\hat{u}_r = -v_A^2\left(\frac{1}{R_B - R_A} + \frac{1}{R_A}\right)\hat{u}_r \qquad (2)$$

Substituting given values $v_A = 32$ ft/s, $R_A = 5$ in. $= 0.4167$ ft, and $R_B = 17$ in. $= 1.417$ ft into Eqs. (1) and (2) we have

$$\vec{a}_A = -1020\,\hat{u}_r \text{ ft/s}^2,$$
$$\vec{a}_Q = -3480\,\hat{u}_r \text{ ft/s}^2.$$

Solution to 6.82



The center of the ball is moving in a uniform circular motion of radius $R_B - R_A$ about O, as well as accelerating in the θ direction, so we must have

$$\vec{a}_A = -\frac{v_A^2}{R_B - R_A}\,\hat{u}_r + \dot{v}_A\,\hat{u}_\theta \tag{3}$$

Substituting given values $v_A = 32\,\text{ft/s}$, $\dot{v}_A = 24\,\text{ft/s}^2$, $R_A = 0.4167\,\text{ft}$, and $R_B = 1.417\,\text{ft}$ into Eq. (3), we have

$$\boxed{\vec{a}_A = (-1020\,\hat{u}_r + 24.0\,\hat{u}_\theta)\,\text{ft/s}^2}$$

Since the ball rolls without slip over a stationary surface, the point of contact between the ball and the bowl, point Q, is the instantaneous center of rotation of the ball. Therefore,

$$v_A = |\vec{\omega}_A| R_A \quad \Rightarrow \quad \omega_A = \frac{v_A}{R_A}.$$

From the no slip condition, $a_{Q\theta} = 0$. From the vector approach to acceleration analysis where $\vec{r}_{Q/A} = R_A\,\hat{u}_r$ and $\vec{\alpha}_A = \alpha_A\,\hat{k}$ we must have

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_A + \vec{\alpha}_A \times \vec{r}_{Q/A} - \omega_A^2 \vec{r}_{Q/A} \\
&= -\frac{v_A^2}{R_B - R_A}\,\hat{u}_r + \dot{v}_A\,\hat{u}_\theta + \alpha_A R_A\,\hat{u}_\theta - \frac{v_A^2}{R_A^2} R_A\,\hat{u}_r \\
&= -v_A^2\left(\frac{1}{R_B - R_A} + \frac{1}{R_A}\right)\hat{u}_r + (\dot{v}_A + \alpha_A R_A)\,\hat{u}_\theta.
\end{aligned} \tag{4}$$

However, $\dot{v}_A + \alpha_A R_A = 0$, so we are left with

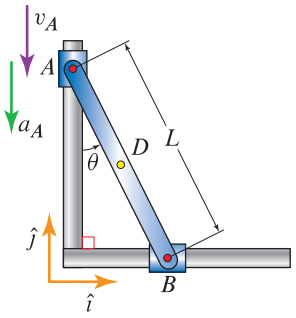
$$\vec{a}_Q = -v_A^2\left(\frac{1}{R_B - R_A} + \frac{1}{R_A}\right)\hat{u}_r. \tag{5}$$

Substituting given values $v_A = 32\,\text{ft/s}$, $\dot{v}_A = 24\,\text{ft/s}^2$, $R_A = 0.4167\,\text{ft}$, and $R_B = 1.417\,\text{ft}$ into Eq. (5), we have

$$\boxed{\vec{a}_Q = -3480\,\hat{u}_r\,\text{ft/s}^2.}$$

Problem 6.83

A bar of length $L = 2.5\,\text{m}$ is falling so that, when $\theta = 34°$, $v_A = 3\,\text{m/s}$ and $a_A = 8.7\,\text{m/s}^2$. At this instant, determine the angular acceleration of the bar AB and the acceleration of point D, where D is the midpoint of the bar.



Solution

We have $\vec{r}_{B/A} = L(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$ and $\vec{r}_{D/A} = \frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$. From the vector approach to velocity analysis where $\vec{v}_A = -v_A\,\hat{\jmath}$ and $\vec{v}_B = v_B\,\hat{\imath}$, we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \quad \Rightarrow \quad v_B\,\hat{\imath} = -v_A\,\hat{\jmath} + \omega_{AB}\,\hat{k} \times L(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$$
$$\Rightarrow \quad v_B\,\hat{\imath} = (L\omega_{AB}\cos\theta)\,\hat{\imath} + (L\omega_{AB}\sin\theta - v_A)\,\hat{\jmath}. \quad (1)$$

Equating the two sides of Eq. (1) component by component, we have

$$0 = L\omega_{AB}\sin\theta - v_A \quad \Rightarrow \quad \omega_{AB} = \frac{v_A}{L\sin\theta}, \quad \text{and} \quad v_B = L\omega_{AB}\cos\theta. \quad (2)$$

Substituting given values $v_A = 3\,\text{m/s}$, $L = 2.5\,\text{m}$, and $\theta = 34°$ into Eqs. (2), we have

$$\omega_{AB} = 2.146\,\text{rad/s} \quad \text{and} \quad v_B = 4.448\,\text{m/s}.$$

From the vector approach to acceleration analysis where $\vec{a}_A = -a_A\,\hat{\jmath}$ and $\vec{a}_B = a_B\,\hat{\imath}$, we have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2\vec{r}_{B/A}$$
$$\Rightarrow \quad a_B\,\hat{\imath} = -a_A\,\hat{\jmath} + \alpha_{AB}\,\hat{k} \times L(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) - \omega_{AB}^2 L(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$$
$$= (\alpha_{AB}L\cos\theta - \omega_{AB}^2 L\sin\theta)\,\hat{\imath} + (\alpha_{AB}L\sin\theta + \omega_{AB}^2 L\cos\theta - a_A)\,\hat{\jmath}. \quad (3)$$

From the $\hat{\jmath}$ component of Eq. (3) we have

$$\alpha_{AB}L\sin\theta + \omega_{AB}^2 L\cos\theta - a_A = 0 \quad \Rightarrow \quad \alpha_{AB} = \frac{a_A - \omega_{AB}^2 L\cos\theta}{L\sin\theta}. \quad (4)$$

Substituting known values $a_A = 8.7\,\text{m/s}^2$, $L = 2.5\,\text{m}$, $\omega_{AB} = 2.146\,\text{rad/s}$, and $\theta = 34°$ into Eq. (4), we have

$$\boxed{\vec{\alpha}_{AB} = -0.604\,\hat{k}\,\text{rad/s}^2.}$$

For D we have

$$\vec{a}_D = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{D/A} - \omega_{AB}^2\vec{r}_{D/A}$$
$$= -a_A\,\hat{\jmath} + \alpha_{AB}\,\hat{k} \times \frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) - \omega_{AB}^2\frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$$
$$= \left[\left(\frac{a_A - \omega_{AB}^2 L\cos\theta}{L\sin\theta}\right)\frac{L}{2}\cos\theta - \omega_{AB}^2\frac{L}{2}\sin\theta\right]\hat{\imath}$$

$$+\left[\left(\frac{a_A-\omega_{AB}^2 L\cos\theta}{L\sin\theta}\right)\frac{L}{2}\sin\theta+\omega_{AB}^2\frac{L}{2}\cos\theta-a_A\right]\hat{j}$$

$$=\left[\frac{1}{2}(a_A-\omega_{AB}^2 L\cos\theta)\cot\theta-\omega_{AB}^2\frac{L}{2}\sin\theta\right]\hat{\imath}$$

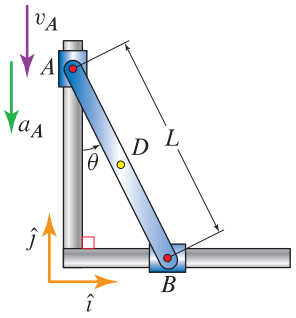
$$+\left[\frac{1}{2}(a_A-\omega_{AB}^2 L\cos\theta)+\omega_{AB}^2\frac{L}{2}\cos\theta-a_A\right]\hat{j} \tag{5}$$

Substituting known values $a_A = 8.7\,\text{m/s}^2$, $L = 2.5\,\text{m}$, $\omega_{AB} = 2.146\,\text{rad/s}$, and $\theta = 34°$ into Eq. (5), we have

$$\boxed{\vec{a}_D = (-3.84\,\hat{\imath} - 4.35\,\hat{j})\,\text{m/s}^2.}$$

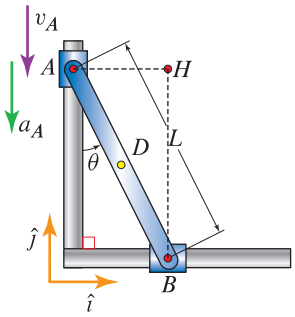
Problem 6.84

A bar of length $L = 8\,\text{ft}$ and midpoint D is falling so that when $\theta = 27°$, $|\vec{v}_D| = 18\,\text{ft/s}$, and the vertical acceleration of point D is $23\,\text{ft/s}^2$ downward. At this instant, compute the angular acceleration of the bar and the acceleration of point B.



Solution

We have $\vec{r}_{B/A} = L(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$ and $\vec{r}_{D/A} = \frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$. Realizing that point H is the instantaneous center of rotation of the line HD, where the distance between H and D is $d_{HD} = \frac{L}{2}$, we have



$$\omega_{AB} = \frac{v_D}{L/2} = 4.500\,\text{rad/s}.$$

From the vector approach to acceleration analysis where $\vec{\alpha}_{AB} = \alpha_{AB}\,\hat{k}$, $\vec{a}_A = a_A\,\hat{\jmath}$ and $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$ we have

$$\begin{aligned}\vec{a}_D &= \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{D/A} - \omega_{AB}^2 \vec{r}_{D/A} \\ &= a_A\,\hat{\jmath} + \alpha_{AB}\frac{L}{2}\cos\theta\,\hat{\imath} + \alpha_{AB}\frac{L}{2}\sin\theta\,\hat{\jmath} - \omega_{AB}^2\frac{L}{2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) \\ &= (\alpha_{AB}L\cos\theta - \omega_{AB}^2 L\sin\theta)\,\hat{\imath} + \left(a_A + \alpha_{AB}\frac{L}{2}\sin\theta + \omega_{AB}^2\frac{L}{2}\cos\theta\right)\hat{\jmath} \qquad (1)\end{aligned}$$

We also have

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} \\ &= a_A\,\hat{\jmath} + \alpha_{AB}\,\hat{k} \times L(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) - \omega_{AB}^2 L(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) \\ &= (\alpha_{AB}L\cos\theta - \omega_{AB}^2 L\sin\theta)\,\hat{\imath} + (a_A + \alpha_{AB}L\sin\theta + \omega_{AB}^2 L\cos\theta)\,\hat{\jmath} \qquad (2)\end{aligned}$$

Recalling that a_{Dy} is given and $a_{By} = 0$, we have

$$a_{By} = 0 = a_A + \alpha_{AB}L\sin\theta + \omega_{AB}^2 L\cos\theta \qquad (3)$$

$$a_{Dy} = -23.00\,\text{ft/s}^2 = a_A + \alpha_{AB}\frac{L}{2}\sin\theta + \omega_{AB}^2\frac{L}{2}\cos\theta \qquad (4)$$

Solving Eq. (3) for a_A and substituting into Eq. (4) we have

$$\begin{aligned}-23.00\,\text{ft/s}^2 &= -\left(\alpha_{AB}L\sin\theta + \omega_{AB}^2 L\cos\theta\right) + \alpha_{AB}\frac{L}{2}\sin\theta + \omega_{AB}^2\frac{L}{2}\cos\theta \\ &= -\alpha_{AB}\frac{L}{2}\sin\theta - \omega_{AB}^2\frac{L}{2}\cos\theta \quad \Rightarrow \quad \alpha_{AB} = \frac{46.00\,\text{ft/s}^2}{L\sin\theta} - \omega_{AB}^2\cot\theta \qquad (5)\end{aligned}$$

Substituting given values $L = 8\,\text{ft}$, $\theta = 27°$, and $\omega_{AB} = 4.500\,\text{rad/s}$ into Eq. (3), we have

$$\alpha_{AB} = -27.08\,\text{rad/s}^2 \quad \Rightarrow \quad \boxed{\vec{\alpha}_{AB} = -27.1\,\hat{k}\,\text{rad/s}^2.}$$

Substituting known values $L = 8\,\text{ft}$, $\theta = 27°$, $\alpha_{AB} = -27.08\,\text{rad/s}^2$, and $\omega_{AB} = 4.500\,\text{rad/s}$ into Eq. (3), we have

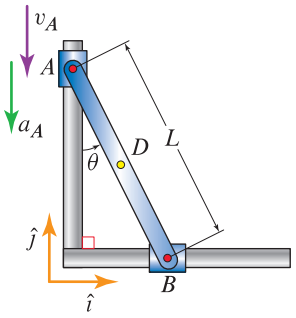
$$a_A = 45.99\,\text{ft/s}^2.$$

Substituting known values $L = 8\,\text{ft}$, $\theta = 27°$, $\alpha_{AB} = -27.08\,\text{rad/s}^2$, $\omega_{AB} = 4.500\,\text{rad/s}$, and $a_A = 45.99\,\text{ft/s}^2$ into Eq. (1) we have

$$\boxed{\vec{a}_D = (-133\,\hat{\imath} - 23.0\,\hat{\jmath})\,\text{ft/s}^2.}$$

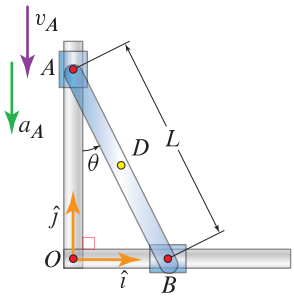
Problem 6.85

Assuming that, for $0° \leq \theta \leq 90°$, v_A is constant, compute the expression for the acceleration of point D, the midpoint of the bar, as a function of θ and v_A.



Solution

Calling O the origin of the reference frame, from the geometry of the problem we have



$$x_D = \frac{L}{2}\sin\theta, \quad y_D = L - \frac{L}{2}\cos\theta \quad \text{, and} \quad y_A = L\cos\theta. \tag{1}$$

Observing that $\vec{a}_D = \ddot{x}_D\,\hat{\imath} - \ddot{y}_D\,\hat{\jmath}$, by taking time derivatives of Eqs. (1) we have

$$\vec{a}_D = \left(-\frac{L}{2}\dot{\theta}^2\sin\theta + \frac{L}{2}\ddot{\theta}\cos\theta\right)\hat{\imath} - \left(\frac{L}{2}\dot{\theta}^2\cos\theta + \frac{L}{2}\ddot{\theta}\sin\theta\right)\hat{\jmath}, \tag{2}$$

$$\dot{y}_A = -L\dot{\theta}\sin\theta = v_A \quad \Rightarrow \quad \dot{\theta} = -\frac{v_A}{L\sin\theta}. \quad \Rightarrow \quad \ddot{\theta} = \frac{v_A}{L\sin^2\theta}\cos\theta\dot{\theta} = -\frac{v_A^2}{L^2\sin^3\theta}\cos\theta \tag{3}$$

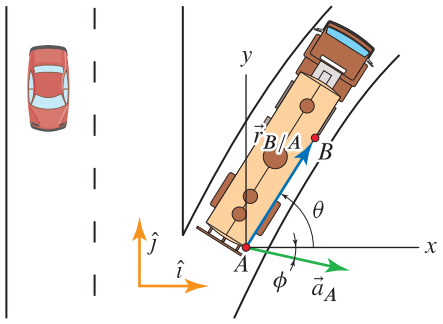
Substituting Eqs. (3) for $\dot{\theta}$ and $\ddot{\theta}$ in Eq. (2) we have

$$\vec{a}_D = \left(-\frac{v_A^2}{2L\sin\theta} - \frac{1}{2}\frac{v_A^2}{L\sin^3\theta}\cos^2\theta\right)\hat{\imath} - \left(\frac{v_A^2}{2L\sin^2\theta}\cos\theta - \frac{v_A^2}{2L\sin^2\theta}\cos\theta\right)\hat{\jmath}$$

$$\Rightarrow \quad \boxed{\vec{a}_D = -\frac{v_A^2}{2L\sin^3\theta}\,\hat{\imath}.}$$

Problem 6.86

A truck on an exit ramp is moving in such a way that, at the instant shown, $|\vec{a}_A| = 6\,\text{m/s}^2$ and $\phi = 13°$. Let the distance between points A and B be $d_{AB} = 4\,\text{m}$. If, at this instant, the truck is turning clockwise, $\theta = 59°$, $a_{Bx} = 6.3\,\text{m/s}^2$, and $a_{By} = -2.6\,\text{m/s}^2$, determine the angular velocity and angular acceleration of the truck.



Solution

From the information given in the problem we have

$$\vec{r}_{B/A} = d_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}), \quad \vec{\alpha}_{AB} = \alpha_{AB}\,\hat{k}, \quad \text{and} \quad \vec{a}_A = |\vec{a}_A|(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}). \tag{1}$$

From rigid body kinematics we have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} \tag{2}$$

Substituting Eqs. (1) into Eq. (2) we have

$$\begin{aligned}\vec{a}_B &= |\vec{a}_A|(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}) + \alpha_{AB}\,\hat{k} \times d_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) - \omega_{AB}^2 d_{AB}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= (|\vec{a}_A|\cos\phi - \alpha_{AB} d_{AB}\sin\theta - \omega_{AB}^2 d_{AB}\cos\theta)\,\hat{\imath} \\ &\quad + (-|\vec{a}_A|\sin\phi + \alpha_{AB} d_{AB}\cos\theta - \omega_{AB}^2 d_{AB}\sin\theta)\,\hat{\jmath}\end{aligned} \tag{3}$$

Breaking Eq. (3) into components we have

$$a_{Bx} = |\vec{a}_A|\cos\phi - \alpha_{AB} d_{AB}\sin\theta - \omega_{AB}^2 d_{AB}\cos\theta \tag{4}$$

$$a_{By} = -|\vec{a}_A|\sin\phi + \alpha_{AB} d_{AB}\cos\theta - \omega_{AB}^2 d_{AB}\sin\theta \tag{5}$$

Eqs. (4) and (5) are two equations in α_{AB} and ω_{AB}. Solving them we have

$$\omega_{AB} = \sqrt{\frac{|\vec{a}_A|(\cos\phi\cot\theta - \sin\phi) - a_{Bx}\cot\theta - a_{By}}{d_{AB}(\cos\theta\cot\theta - \sin\theta)}}, \tag{6}$$

and

$$\alpha_{AB} = \frac{[a_{By}\cos\theta|\vec{a}_A|(\sin\phi\cos\theta + \sin\theta\cos\phi) - a_{Bx}\sin\theta]}{d_{AB}}. \tag{7}$$

Substituting known values $d_{AB} = 4\,\text{m}$, $\phi = 13°$, $\theta = 59°$, $a_{Bx} = 6.3\,\text{m/s}^2$, $a_{By} = -2.6\,\text{m/s}^2$, and $|a_A| = 6\,\text{m/s}^2$ into Eqs. (6) and (7) we have

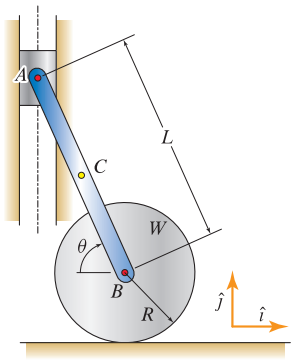
$$\omega_{AB} = -0.4577\,\text{rad/s} \quad \text{and} \quad \alpha_{AB} = -0.2582\,\text{rad/s}.$$

In vector form, and to three significant figure we have

$$\vec{\omega}_{AB} = -0.458\,\hat{k}\ \text{rad/s} \quad \vec{\alpha}_{AB} = -0.258\,\hat{k}\ \text{rad/s}^2.$$

Problems 6.87 through 6.89

The system shown consists of a wheel of radius $R = 1.4\,\text{m}$ rolling without slip on a horizontal surface. A bar AB of length $L = 3.7\,\text{m}$ is pin-connected to the center of the wheel and to a slider A constrained to move along a vertical guide. Point C is the bar's midpoint.

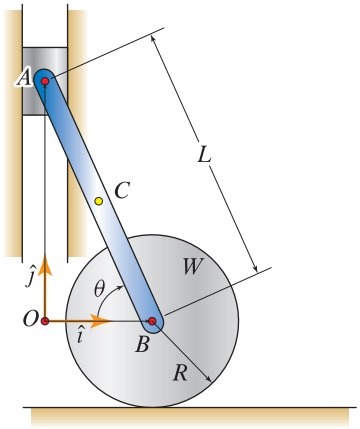


Problem 6.87 If the wheel is rolling clockwise with a constant angular speed of $2\,\text{rad/s}$, determine the angular acceleration of the bar when $\theta = 72°$.

Problem 6.88 If the slider A is moving downward with a constant speed $3\,\text{m/s}$, determine the angular acceleration of the wheel when $\theta = 53°$.

Problem 6.89 Determine the general relation expressing the acceleration of the slider A as a function of θ, L, R, the angular velocity of the wheel ω_W, and the angular acceleration of the wheel α_W.

Solution to 6.87



We assume the wheel does not leave the ground, hence B is constrained to move in the horizontal direction, and from the rolling without slip condition we have

$$\vec{v}_B = v_B\,\hat{\imath} = -R\omega_W\,\hat{\imath}. \tag{1}$$

Also, calling point O the origin of the reference frame, from the geometry of the problem we have

$$x_B = L\cos\theta. \tag{2}$$

Differentiating Eq. (2) with respect to time, and observing that $v_B = \dot{x}_B$ we have

$$\dot{x}_B = -L\sin\theta\dot{\theta} = -R\omega_W \quad \Rightarrow \quad \dot{\theta} = \frac{R\omega_W}{L\sin\theta} \tag{3}$$

Taking a time derivative of Eq. (3) we have

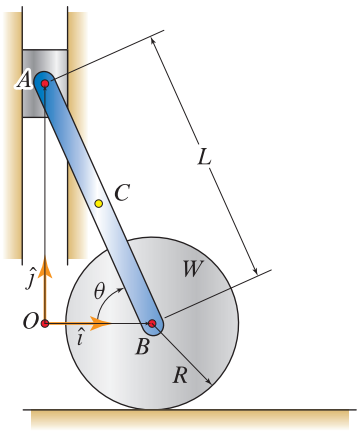
$$\ddot{\theta} = -\frac{R\omega_W}{L}\frac{\cos\theta}{\sin^2\theta}\dot{\theta} \quad \Rightarrow \quad \ddot{\theta} = \frac{R^2\omega_W^2}{L^2}\frac{\cos\theta}{\sin^3\theta} = 0.2057\,\text{rad/s}^2,$$

where we have substituted the known values $\omega_{AB} = 2\,\text{rad/s}$, $R = 1.4\,\text{m}$, $L = 3.7\,\text{m}$, and $\theta = 72°$. Observing that $\alpha_{AB} = -\ddot{\theta}$, in vector form and to three significant figures, we have

$$\boxed{\vec{\alpha}_{AB} = 0.206\,\hat{k}\,\text{rad/s}^2.}$$

Solution to 6.88

We assume the wheel does not leave the ground, hence B is constrained to move in the horizontal direction, and from the rolling without slip condition, we have



$$\vec{v}_B = v_B\,\hat{\imath} = -R\omega_W\,\hat{\imath}. \tag{4}$$

Also, calling point O the origin of the reference frame, from the geometry of the problem we have

$$x_B = L\cos\theta \quad \text{and} \quad y_A = R + L\sin\theta. \tag{5}$$

Differentiating Eqs. (5) with respect to time, and observing that $v_B = \dot{x}_B$ and $v_A = -\dot{y}_A$, we have

$$y_A = R + L\sin\theta \quad \Rightarrow \quad \dot{y}_A = -v_A = L\dot{\theta}\cos\theta \quad \Rightarrow \quad \dot{\theta} = -\frac{v_A}{L\cos\theta}$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{v_A^2\sin\theta}{L^2\cos\theta}\dot{\theta} = \frac{v_A^2\sin\theta}{L^2\cos^3\theta} \tag{6}$$

and

$$\dot{x}_B = -L\sin\theta\dot{\theta} = -R\omega_W \quad \Rightarrow \quad \omega_W = \dot{\theta}\frac{L}{R}\sin\theta = \frac{L}{R}\sin\theta\left(-\frac{v_A}{L\cos\theta}\right) = -\frac{v_A}{R}\tan\theta. \tag{7}$$

Taking a time derivative of Eq. (7), we have

$$\alpha_W = -\frac{L\cos\theta\dot{\theta}^2}{R} - \frac{L\sin\theta\ddot{\theta}}{R}. \tag{8}$$

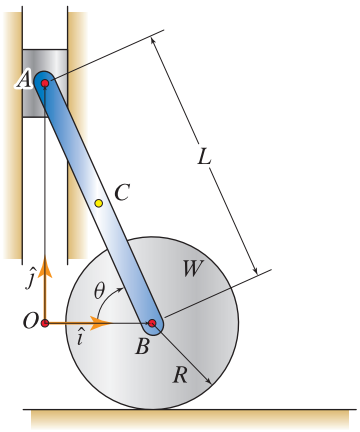
Substituting the expressions for $\dot{\theta}$ and $\ddot{\theta}$ in Eqs. (6) into Eq. (8), we have

$$\alpha_W = -\frac{L\cos\theta\left(\frac{v_A}{L\cos\theta}\right)^2}{R} - \frac{L\sin\theta\left(\frac{v_A^2\sin\theta}{L^2\cos^3\theta}\right)}{R} = -\frac{v_A^2}{LR\cos\theta} - \frac{v_A^2\sin^2\theta}{LR\cos^3\theta} = -\frac{v_A^2(1+\tan^2\theta)}{LR\cos\theta}. \tag{9}$$

Substituting known values $v_A = 3\,\text{m/s}$, $R = 1.4\,\text{m}$, $L = 3.7\,\text{m}$, and $\theta = 53°$ into Eq. (9), we have

$$\boxed{\alpha_W = -7.971 \quad \Rightarrow \quad \vec{\alpha}_W = -7.97\,\hat{k}\,\text{rad/s}^2}$$

Solution to 6.89



We assume the wheel does not leave the ground, hence B is constrained to move in the horizontal direction, and from the rolling without slip condition we have

$$\vec{v}_B = v_B\,\hat{\imath} = -R\omega_W\,\hat{\imath} \quad \text{and} \quad \vec{a}_B = -R\alpha_W\,\hat{\imath}. \tag{10}$$

From rigid body kinematics we have

$$\begin{aligned}\vec{v}_A = \vec{v}_B + \vec{\omega}_{AB}\times\vec{r}_{A/B} &= -R\omega_W + \omega_{AB}\,\hat{k}\times L(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= (-R\omega_W - L\omega_{AB}\sin\theta)\,\hat{\imath} + (-L\omega_{AB}\cos\theta)\,\hat{\jmath}. \end{aligned}\tag{11}$$

Because the slider is constrained to move in the slot, $v_{Ax} = 0$. Therefore, setting the x component of the last of Eqs. (11), we have

$$\omega_{AB} = -\omega_W\frac{R}{L\sin\theta}.$$

We must also have

$$\begin{aligned}\vec{a}_A &= \vec{a}_B + \vec{\alpha}_{AB}\times\vec{r}_{A/B} - \omega_{AB}^2\vec{r}_{A/B} \\ &= -R\alpha_W\,\hat{\imath} + \alpha_{AB}\,\hat{k}\times L(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) - \omega_W^2\frac{R^2}{L^2\sin^2\theta}L(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= \left(-R\alpha_W - \alpha_{AB}L\sin\theta + \frac{R^2\omega_W^2}{L\sin^2\theta}\cos\theta\right)\hat{\imath} + \left(-\alpha_{AB}L\cos\theta - \omega_W^2\frac{R^2}{L\sin\theta}\right)\hat{\jmath}. \end{aligned}\tag{12}$$

Because the slider is constrained to move in the slot, $a_{Ax} = 0$. Therefore, setting the x component of the last of Eqs. (11), we have

$$\alpha_{AB} = \frac{1}{L\sin\theta}\left(\frac{R^2\omega_W^2}{L\sin^2\theta}\cos\theta - R\alpha_W\right). \tag{13}$$

Substituting Eq. (13) for α_{AB} in Eq. (12) we have

$$\vec{a}_A = \left(\frac{R\alpha_W}{\tan\theta} - \frac{R^2\omega_W^2\cot^2\theta}{L\sin\theta} - \omega_W^2\frac{R^2}{L\sin\theta}\right)\hat{\jmath} = \left(\frac{R\alpha_W}{\tan\theta} - \frac{R^2\omega_W^2\cot^2\theta + 1}{L\sin\theta}\right)\hat{\jmath}$$

$$\Rightarrow \qquad \boxed{\vec{a}_A = \left(\frac{R\alpha_W}{\tan\theta} - \frac{R^2\omega_W^2}{L\sin^3\theta}\right)\hat{\jmath}.} \tag{14}$$

Problems 6.90 through 6.93

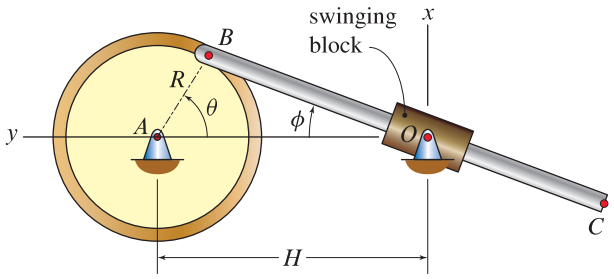
For the slider-crank mechanism shown, let $R = 0.75\,\text{m}$ and $H = 2\,\text{m}$, and let the length of bar BC be $L_{BC} = 3.25\,\text{m}$.

Problem 6.90 Assume that $\dot{\theta} = 50\,\text{rad/s} = \text{constant}$ and compute the angular acceleration of the slider for $\theta = 27°$.

Problem 6.91 Assume that, at the instant shown, $\theta = 27°$, $\dot{\theta} = 50\,\text{rad/s}$, and $\ddot{\theta} = 15\,\text{rad/s}^2$. Compute the angular acceleration of the slider at this instant as well as the acceleration of point C.

Problem 6.92 Assuming that $\dot{\theta}$ is constant, determine the expression of the angular acceleration of the slider as a function of θ and $\dot{\theta}$ (and the accompanying geometrical parameters).

Problem 6.93 Letting $\dot{\theta} = 300\,\text{rpm} = \text{constant}$, plot the angular acceleration of the slider as a function of θ for $0° \le \theta \le 360°$. In addition, plot the speed of point C for the same range of θ.



Solution to 6.90

Observing that $\alpha_{BC} = -\ddot{\phi}$, we can use differentiation of constraints to find an expression for α_{BC}. From the geometry of the problem we have the constraint equations

$$d_{BO}\sin\phi = R\sin\theta \quad \text{and} \quad d_{BO}\cos\phi + R\cos\theta = H. \tag{1}$$

Solving Eqs. (1) for d_{BO} and ϕ, we have

$$\phi = \tan^{-1}\left(\frac{R\sin\theta}{H - R\cos\theta}\right) \quad \text{and} \quad d_{BO} = \frac{H\sin\theta}{\cos\phi\sin\theta + \cos\theta\sin\phi}. \tag{2}$$

Using Eqs. (2) for $\theta = 27°$, we have

$$\phi = 0.2503\,\text{rad} \quad \text{and} \quad d_{BO} = 1.375\,\text{m}.$$

Taking time derivatives of Eqs. (1) we have

$$\dot{d}_{BO}\sin\phi + d_{BO}\dot{\phi}\cos\phi = R\dot{\theta}\cos\theta \quad \text{and} \quad \dot{d}_{BO}\cos\phi - d_{BO}\dot{\phi}\sin\phi - R\dot{\theta}\sin\theta = 0. \tag{3}$$

Solving Eqs. (3) for $\dot{d}_{BO}$ and $\dot{\phi}$ we have

$$\dot{d}_{BO} = R\sin\theta + \phi\dot{\theta} \quad \text{and} \quad \dot{\phi} = \frac{R\cos(\theta+\phi)\dot{\theta}}{d_{BO}}. \tag{4}$$

Substituting known values $R = 0.75\,\text{m}$, $\phi = 0.2503°$ $\theta = 27°$, $\dot{\theta} = 50\,\text{rad/s}$, and $d_{BO} = 1.375\,\text{m}$ into Eqs. (4), we have

$$\dot{d}_{BO} = 24.77\,\text{m/s} \quad \text{and} \quad \dot{\phi} = 20.48\,\text{rad/s}.$$

Taking a time derivative of $\dot{\phi}$ from Eq. (4) and letting $\ddot{\theta} = 0$, we have

$$\ddot{\phi} = \frac{R\cos(\theta+\phi)\dot{d}_{BO}\dot{\theta}}{d_{BO}^2} + \frac{R\sin(\theta+\phi)\dot{\theta}(\dot{\theta}+\dot{\phi})}{d_{BO}}. \tag{5}$$

Substituting known values $R = 0.75\,\text{m}$, $\phi = 0.2503°$, $\dot{\phi} = 20.48\,\text{rad/s}$, $\theta = 27°$, $\dot{\theta} = 50\,\text{rad/s}$, $d_{BO} = 1.375\,\text{m}$, and $\dot{d}_{BO} = 24.77\,\text{m/s}$ into Eq. (5), we have

$$\ddot{\phi} = -1639\,\text{rad/s}^2 \quad \Rightarrow \quad \boxed{\vec{\alpha}_{BO} = 1640\,\hat{k}\,\text{rad/s}^2.}$$

Solution to 6.91

Observing that $\alpha_{BC} = -\ddot{\phi}$, we can use differentiation of constraints to find an expression for α_{BC}. From the geometry of the problem we have the constraint equations

$$d_{BO}\sin\phi = R\sin\theta \quad \text{and} \quad d_{BO}\cos\phi + R\cos\theta = H. \tag{6}$$

Solving Eqs. (6) for d_{BO} and ϕ we have

$$\phi = \tan^{-1}\left(\frac{R\sin\theta}{H - R\cos\theta}\right) \quad \text{and} \quad d_{BO} = \frac{H\sin\theta}{\cos\phi\sin\theta + \cos\theta\sin\phi}. \tag{7}$$

Using Eqs. (7) for $\theta = 27°$, we have

$$\phi = 0.2503\,\text{rad} \quad \text{and} \quad d_{BO} = 1.375\,\text{m}.$$

Taking time derivatives of Eqs. (6) we have

$$\dot{d}_{BO}\sin\phi + d_{BO}\dot{\phi}\cos\phi = R\dot{\theta}\cos\theta \quad \text{and} \quad \dot{d}_{BO}\cos\phi - d_{BO}\dot{\phi}\sin\phi - R\dot{\theta}\sin\theta = 0. \tag{8}$$

Solving Eqs. (8) for $\dot{d}_{BO}$ and $\dot{\phi}$ we have

$$\dot{d}_{BO} = R\sin\theta + \phi\dot{\theta} \quad \text{and} \quad \dot{\phi} = \frac{R\cos(\theta+\phi)\dot{\theta}}{d_{BO}}. \tag{9}$$

Substituting known values $R = 0.75\,\text{m}$, $\phi = 0.2503°$ $\theta = 27°$, $\dot{\theta} = 50\,\text{rad/s}$, and $d_{BO} = 1.375\,\text{m}$ into Eqs. (9), we have

$$\dot{d}_{BO} = 24.77\,\text{m/s} \quad \text{and} \quad \dot{\phi} = 20.48\,\text{rad/s}.$$

Taking a time derivative of $\dot{\phi}$ from Eq. (9), we have

$$\ddot{\phi} = \frac{R\cos(\theta+\phi)\dot{d}_{BO}\dot{\theta}}{d_{BO}^2} + \frac{R\sin(\theta+\phi)\dot{\theta}(\dot{\theta}+\dot{\phi})}{d_{BO}} - \frac{R\cos(\theta+\phi)\ddot{\theta}}{d_{BO}}.$$

Substituting known values $R = 0.75\,\text{m}$, $\phi = 0.2503°$, $\dot{\phi} = 20.48\,\text{rad/s}$, $\theta = 27°$, $\dot{\theta} = 50\,\text{rad/s}$, $\ddot{\theta} = 15\,\text{rad/s}^2$, $d_{BO} = 1.375\,\text{m}$, and $\dot{d}_{BO} = 24.77\,\text{m/s}$ into Eq. (5), we have

$$\ddot{\phi} = -1632\,\text{rad/s}^2 \quad \Rightarrow \quad \boxed{\vec{\alpha}_{BO} = 1630\,\hat{k}\,\text{rad/s}^2.}$$

The acceleration of C can be found by applying rigid body kinematics. Observing that $\vec{r}_{B/A} = R(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$, $\omega_{AB} = \dot{\theta}$, and $\alpha_{AB} = \ddot{\theta}$, we have

$$\begin{aligned}
\vec{a}_B &= \vec{a}_A + \vec{\alpha}_{AB}\times\vec{r}_{B/A} - \omega_{AB}^2\vec{r}_{B/A} \\
&= \alpha_{AB}\,\hat{k}\times R(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) - \omega_{AB}^2 R(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) \\
&= (-R\dot{\theta}^2\sin\theta + R\ddot{\theta}\cos\theta)\,\hat{\imath} + (R\dot{\theta}^2\cos\theta + R\ddot{\theta}\sin\theta)\,\hat{\jmath} \\
&= (-841.2\,\hat{\imath} + 1676s\,\hat{\jmath})\,\text{m/s}^2,
\end{aligned} \tag{10}$$

where we have used the known values $R = 0.75\,\text{m}$, $\theta = 27°$, $\dot{\theta} = 50\,\text{rad/s}$, and $\ddot{\theta} = 15\,\text{rad/s}^2$. Observing that $\omega_{BC} = -\dot{\phi}$, $\alpha_{BC} = -\ddot{\phi}$, and $\vec{r}_{C/B} = L(-\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath})$ we must also have

$$\vec{a}_C = \vec{a}_B + \vec{\alpha}_{OB}\times\vec{r}_{C/B} - \omega_{OB}^2\vec{r}_{C/B}$$

$$
\begin{aligned}
&= (-841.2\,\hat{\imath} + 1676s\,\hat{\jmath})\,\mathrm{m/s^2} + (1632\,\hat{k}\,\mathrm{rad/s^2}) \times L(-\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath}) \\
&\quad - (20.48\,\mathrm{rad/s})L(-\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath}) \\
&= (-848.2\,\hat{\imath} + 7153\,\hat{\jmath})\,\mathrm{m/s},
\end{aligned}
$$

where we have used the expression for a_B from Eq. (), and known values $R = 0.75\,\mathrm{m}$, $\phi = 0.2503°$, $\dot{\phi} = 20.48\,\mathrm{rad/s}$, $\ddot{\phi} = 1632\,\mathrm{rad/s^2}$. To three significant figures we have

$$\boxed{\vec{a}_C = (-848\,\hat{\imath} + 7150\,\hat{\jmath})\,\mathrm{m/s}.}$$

Solution to 6.92

Observing that $\alpha_{BC} = -\ddot{\phi}$, we can use differentiation of constraints to find an expression for α_{BC}. From the geometry of the problem we have the constraint equations

$$d_{BO} \sin\phi = R \sin\theta \quad \text{and} \quad d_{BO} \cos\phi + R \cos\theta = H. \tag{11}$$

Solving Eqs. (11) for ϕ we have

$$\phi = \tan^{-1}\left(\frac{R \sin\theta}{H - R\cos\theta}\right). \tag{12}$$

Taking time derivatives of ϕ in Eq. (11) we have

$$\begin{aligned}
\dot{\phi} &= \frac{R\dot{\theta}(H\cos\theta - R)}{H^2 + R^2 - 2HR\cos\theta} \\
\ddot{\phi} &= -\frac{2HR^2(-R + H\cos\theta)\sin\theta\dot{\theta}}{(H^2 + R^2 - 2HR\cos\theta)^2} - \frac{HR\dot{\theta}\sin\theta}{H^2 + R^2 - 2HR\cos\theta}
\end{aligned} \tag{13}$$

For $\vec{\alpha}_{OB}$, we then have

$$\boxed{\vec{\alpha}_{OB} = \left(\frac{2HR^2(-R + H\cos\theta)\sin\theta\dot{\theta}}{(H^2 + R^2 - 2HR\cos\theta)^2} + \frac{HR\dot{\theta}\sin\theta}{H^2 + R^2 - 2HR\cos\theta}\right)\hat{k}.}$$

Solution to 6.93

Observing that $\alpha_{BC} = -\ddot{\phi}$, we can use differentiation of constraints to find an expression for α_{BC}. From the geometry of the problem we have the constraint equations

$$d_{BO} \sin\phi = R \sin\theta \quad \text{and} \quad d_{BO} \cos\phi + R \cos\theta = H. \tag{14}$$

Solving Eqs. (14) for ϕ we have

$$\phi = \tan^{-1}\left(\frac{R \sin\theta}{H - R\cos\theta}\right). \tag{15}$$

Taking time derivatives of ϕ in Eq. (14), we have

$$\begin{aligned} \dot{\phi} &= \frac{R\dot{\theta}(H\cos\theta - R)}{H^2 + R^2 - 2HR\cos\theta} \\ \ddot{\phi} &= -\frac{2HR^2(-R + H\cos\theta)\sin\theta\dot{\theta}}{(H^2 + R^2 - 2HR\cos\theta)^2} - \frac{HR\dot{\theta}\sin\theta}{H^2 + R^2 - 2HR\cos\theta}. \end{aligned} \tag{16}$$

For $\vec{\alpha}_{OB}$, we then have

$$\boxed{\vec{\alpha}_{OB} = \left(\frac{2HR^2(-R + H\cos\theta)\sin\theta\dot{\theta}}{(H^2 + R^2 - 2HR\cos\theta)^2} + \frac{HR\dot{\theta}\sin\theta}{H^2 + R^2 - 2HR\cos\theta}\right)\hat{k}.}$$

We know the position of C can be expressed as

$$\vec{r}_C = (R\sin\theta - L\sin\phi)\,\hat{\imath} - (R\cos\theta + L\cos\phi)\,\hat{\jmath}. \tag{17}$$

For the velocity of C, we take a time derivative of Eq. (17), which gives us

$$\vec{v}_C = (R\dot{\theta}\cos\theta - L\dot{\phi}\cos\phi)\,\hat{\imath} + (R\dot{\theta}\sin\theta + L\dot{\phi}\sin\phi)\,\hat{\jmath}. \tag{18}$$

Substituting ϕ from Eq. (15) and $\dot{\phi}$ from Eq. (16) into Eq. (18), we have

$$\begin{aligned} \vec{v}_C = &\left\{R\dot{\theta}\cos\theta - L\left[\frac{R\dot{\theta}(H\cos\theta - R)}{H^2 + R^2 - 2HR)}\right]\cos\left[\tan^{-1}\left(\frac{R\sin\theta}{H - R\cos\theta}\right)\right]\right\}\hat{\imath} \\ &+ \left\{R\dot{\theta}\sin\theta + L\left[\frac{R\dot{\theta}(H\cos\theta - R)}{H^2 + R^2 - 2HR}\right]\sin\left[\tan^{-1}\left(\frac{R\sin\theta}{H - R\cos\theta}\right)\right]\right\}\hat{\jmath}. \end{aligned}$$

The expressions for $\vec{\alpha}_{OB}$ and $\vec{v}_C$ just derived are reduced to functions of θ and $\dot{\theta}$. The function can be plotted with software such as *Mathematica* or MATLAB. The plots presented below were obtained using *Mathematica* with the following code:

```
Parameters = {R → 0.75, L_BC → 3.25, H → 2, θDot → 300 * π/30};

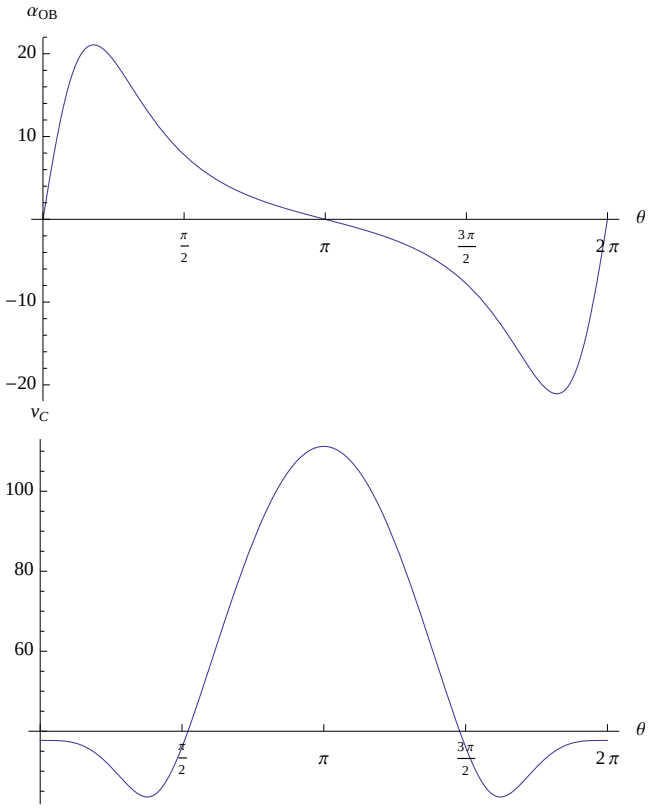
α_OB = (2 H R^2 (H Cos[θ] - R) Sin[θ] θDot / (H^2 + R^2 - 2 H R Cos[θ])^2 + H R θDot Sin[θ] / (H^2 + R^2 - 2 H R Cos[θ])) /. Parameters;

v_C =
  Sqrt[(R θDot Cos[θ] - L_BC R θDot (H Cos[θ] - R) / (H^2 + R^2 - 2 H R) Cos[ArcTan[R Sin[θ] / (H - R Cos[θ])]])^2 +
    (R θDot Sin[θ] + L_BC R θDot (H Cos[θ] - R) / (H^2 + R^2 - 2 H R) Sin[ArcTan[R Sin[θ] / (H - R Cos[θ])]])^2] /. Parameters;

Plot[α_OB, {θ, 0, 2π}, AxesLabel → {"θ", "α_OB"}, Ticks → {{0, π/2, π, 3π/2, 2π}, Automatic}]

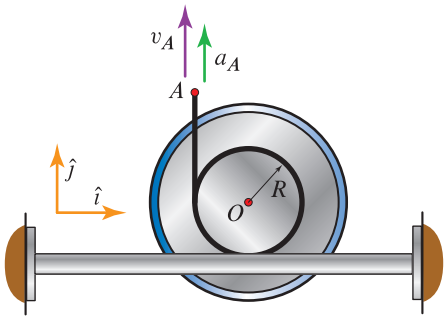
Plot[v_C, {θ, 0, 2π}, AxesLabel → {"θ", "v_C"}, Ticks → {{0, π/2, π, 3π/2, 2π}, Automatic}]
```

Using the above code we obtain the following plots:



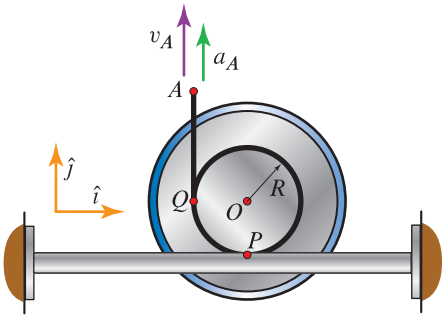
Problem 6.94

A spool with inner radius $R = 5\,\text{ft}$ is made to roll without slip over a horizontal rail as shown. If the cable on the spool is unwound in such a way that the free or vertical portion of cable remains perpendicular to the rail, determine the angular acceleration of the spool and the acceleration of the spool's center O. The vertical component of the velcocity of point A is $v_A = 12\,\text{ft/s}$, and the vertical component of its acceleration is $a_A = 2\,\text{ft/s}^2$.



Solution

We let point P be the contact point between the spool and rail, and point Q be the contact point between the spool and rope. From the no slip condition, we then have



$$\vec{v}_P = \vec{0} \quad \text{and} \quad v_{Qy} = v_A. \tag{1}$$

From rigid body kinematics, we must have

$$\begin{aligned}\vec{v}_Q &= \vec{v}_P + \vec{\omega}_{\text{spool}} \times \vec{r}_{Q/P} \\ &= \omega_{\text{spool}}\,\hat{k} \times (-R\,\hat{\imath} + R\,\hat{\jmath}) \\ &= -\omega_{\text{spool}} R(\hat{\imath} + \hat{\jmath}),\end{aligned} \tag{2}$$

Equating the y component of $\vec{v}_Q$ in Eq. (2) with v_{Qy} in Eq. (1), we have

$$-\omega_{\text{spool}} R = v_A \quad \Rightarrow \quad \omega_{\text{spool}} = -\frac{v_A}{R}.$$

Now that we have found the angular velocity of the spool, we look at the acceleration. From the no slip condition we have

$$a_{Px} = 0 \quad \Rightarrow \quad \vec{a}_P = a_{Py}\,\hat{\jmath}.$$

Also, because O can only move in the horizontal direction, we must have

$$\vec{a}_O = a_{Ox}\,\hat{\imath}.$$

From rigid body kinematics, we must have

$$\vec{a}_O = \vec{a}_P + \vec{\alpha}_{\text{spool}} \times \vec{r}_{O/P} - \omega_{\text{spool}}^2 \vec{r}_{O/P}$$

$$a_{Ox}\,\hat{\imath} = a_{Py}\,\hat{\jmath} + \alpha_{\text{spool}}\,\hat{k} \times R\,\hat{\jmath} - \left(\frac{v_A}{R}\right)^2 R\,\hat{\jmath}$$

$$a_{Ox}\,\hat{\imath} = -\alpha_{\text{spool}} R\,\hat{\imath} + \left[a_{Py} - \frac{v_A^2}{R}\right]\hat{\jmath}. \tag{3}$$

Writing Eq. (3) as two scalar equations we have

$$a_{Ox} = -\alpha_{\text{spool}} R \quad \text{and} \quad a_{Py} = \frac{v_A^2}{R} \quad \Rightarrow \quad \vec{a}_P = \frac{v_A^2}{R}\,\hat{\jmath}.$$

We now know the acceleration of two points on the spool, and we can relate them to find the angular acceleration. We have

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_P + \vec{\alpha}_{\text{spool}} \times \vec{r}_{Q/P} - \omega_{\text{spool}}^2 \vec{r}_{Q/P} \\
&= \frac{v_A^2}{R}\,\hat{\jmath} + \alpha_{\text{spool}}\,\hat{k} \times R(-\hat{\imath} + \hat{\jmath}) - \left(\frac{v_A}{R}\right)^2 R(-\hat{\imath} + \hat{\jmath}) \\
&= \left(\frac{v_A^2}{R} - \alpha_{\text{spool}} R\right)\hat{\imath} - \alpha_{\text{spool}} R\,\hat{\jmath}.
\end{aligned} \tag{4}$$

From the rolling without slip condition at Q we have

$$a_{Qy} = a_A. \tag{5}$$

Equating Eq. (4) with the expression for a_{Qy} in Eq. (5), we have

$$-\alpha_{\text{spool}} R = a_A \quad \Rightarrow \quad \alpha_{\text{spool}} = -\frac{a_A}{R} \quad \Rightarrow \quad \boxed{\vec{\alpha}_{\text{spool}} = -0.400\,\hat{k}\ \text{rad/s}^2.}$$

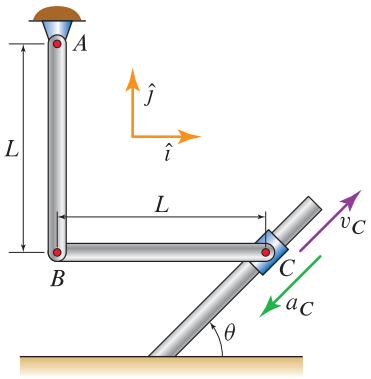
where we have used the given values $R = 5\,\text{ft}$ and $a_A = 2\,\text{ft/s}^2$. Substituting this expression for α_{spool} into Eq. (3) we have

$$\vec{a}_O = a_A\,\hat{\imath} \quad \Rightarrow \quad \boxed{\vec{a}_O = 2.00\,\hat{\imath}\ \text{m/s}^2,}$$

where we have used the given value $a_A = 2\,\text{ft/s}^2$.

Problem 6.95

At the instant shown, bars AB and BC are perpendicular to each other while the slider C has a velocity $v_C = 24\,\text{m/s}$ and an acceleration $a_C = 2.5\,\text{m/s}^2$ in the directions shown. Letting $L = 1.75\,\text{m}$ and $\theta = 45°$, determine the angular acceleration of bars AB and BC.



Solution

Because $\theta = 45°$, the instantaneous center of rotation of bar BC is point A. Therefore,

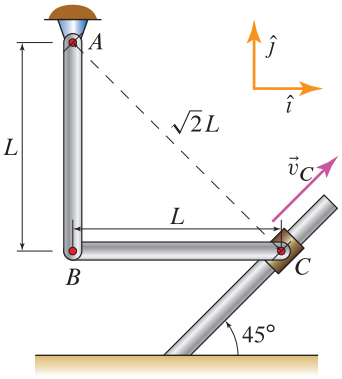
$$v_C = \sqrt{2}L\omega_{BC} \quad \Rightarrow \quad \omega_{BC} = \frac{v_C}{\sqrt{2}L} = 9.697\,\text{rad/s},$$

where we have used the given values $v_C = 24\,\text{m/s}$ and $L = 1.75\,\text{m}$. Because C is constrained to move along the guide oriented by the unit vector $\hat{u}_t = (\hat{\imath} + \hat{\jmath})/\sqrt{2}$, and is moving up and to the right, we have

$$\vec{v}_C = v_C(\cos 45°\,\hat{\imath} + \sin 45°\,\hat{\jmath}) = (16.97\,\hat{\imath} + 16.97\,\hat{\imath})\,\text{ft/s} \qquad (1)$$

and

$$\vec{a}_C = a_C(\cos 45°\,\hat{\imath} + \sin 45°\,\hat{\jmath}) = (-1.786\,\hat{\imath} - 1.786\,\hat{\jmath})\,\text{ft/s}^2, \qquad (2)$$



where we have used the given values $v_C = 24\,\text{m/s}$ and $a_C = 2.5\,\text{m/s}^2$. Point A is also the center of rotation of bar AB. Therefore, in the present configuration

$$v_B = \omega_{AB}L = \omega_{BC}L \quad \Rightarrow \quad \omega_{AB} = \omega_{BC} = 9.697\,\text{rad/s}.$$

From rigid body kinematics we have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2\vec{r}_{B/A} = \alpha_{AB}\,\hat{k} \times (-L\,\hat{\jmath}) + \omega_{AB}^2 L\,\hat{\jmath} = L\alpha_{AB}\,\hat{\imath} + \omega_{AB}^2 L\,\hat{\jmath}.$$

We must also have

$$\begin{aligned}\vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2\vec{r}_{C/B} \\ &= (L\alpha_{AB}\,\hat{\imath} + \omega_{AB}^2 L\,\hat{\jmath}) + \alpha_{BC}\,\hat{k} \times L\,\hat{\imath} - \omega_{AB}L\,\hat{\imath} \\ &= (\alpha_{AB}L - \omega_{AB}^2 L)\,\hat{\imath} + (\alpha_{BC}L + \omega_{AB}^2 L)\,\hat{\jmath} \qquad (3)\end{aligned}$$

Equating the expressions for $\vec{a}_C$ in Eqs. (2) and (3), on a component by component basis, we have

$$-1.786\,\text{m/s}^2 = \alpha_{AB}L - \omega_{AB}^2 L \quad \text{and} \quad -1.786\,\text{m/s}^2 = \alpha_{BC}L + \omega_{AB}^2 L. \qquad (4)$$

Eqs. (4) are two equations in the two unknowns α_{AB} and α_{BC}. Solving them we have

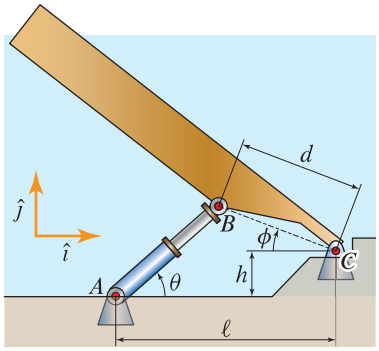
$$\alpha_{AB} = \frac{\omega_{AB}^2 L - 1.786\,\text{m/s}^2}{L} \quad \text{and} \quad \alpha_{BC} = \frac{-(1.786\,\text{m/s}^2 + \omega_{AB}^2 L)}{L}.$$

Substituting in known values $L = 1.75\,\text{m}$ and $\omega_{AB} = 9.697\,\text{rad/s}$, and putting the angular accelerations in vector form, we have

$$\boxed{\vec{\alpha}_{AB} = 93.0\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -95.1\,\hat{k}\,\text{rad/s}^2.}$$

Problem 6.96

A flood gate is controlled via the hydraulic cylinder AB. If the length of the cylinder is increased with a constant time rate of 2.5 ft/s, determine the angular acceleration of the gate when $\phi = 0°$. Let $\ell = 10$ ft, $h = 2.5$ ft, and $d = 5$ ft.



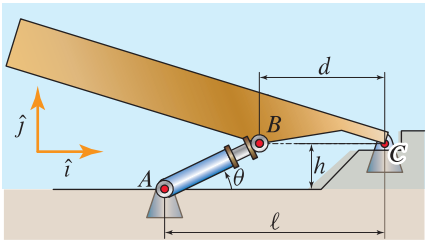
Solution

At any position of the gate, we have the constraint equations

$$\ell - L_{AB}\cos\theta = d\cos\phi \quad \text{and} \quad L_{AB}\sin\theta - h = d\sin\phi. \tag{1}$$

where L_{AB} is the distance between points A and B. For $\phi = 0$, we then have

$$\theta = \tan^{-1}\left(\frac{h}{\ell - d}\right) \quad \text{and} \quad L_{AB} = \sqrt{h^2 + (\ell - d)^2}. \tag{2}$$



Substituting known values $h = 2.5$ ft, $d = 5$ ft, and $\ell = 10$ ft into Eqs. (2), we have

$$\theta = 0.4636 \text{ rad} \quad \text{and} \quad L_{AB} = 5.590 \text{ ft}.$$

Taking time derivatives of Eqs. (1) we have

$$L_{AB}\dot{\theta}\sin\theta = d\dot{\phi}\cos\phi \quad \text{and} \quad L_{AB}\dot{\theta}\cos\theta = d\dot{\phi}\sin\phi. \tag{3}$$

Eqs. (3) are two equations in $\dot{\theta}$ and $\dot{\phi}$. Solving them, we have

$$\dot{\theta} = \frac{\cot(\theta + \phi)\dot{L}_{AB}}{L_{AB}} \quad \text{and} \quad \dot{\phi} = \frac{\csc(\theta + \phi)\dot{L}_{AB}}{d}. \tag{4}$$

Substituting known values $\theta = 0.4636$ rad, $\phi = 0$ rad, $d = 5$ ft, $L_{AB} = 5.590$ ft, and $\dot{L}_{AB} = 2.5$ ft/s into Eqs. (4) we have

$$\dot{\theta} = 0.8946 \text{ rad/s} \quad \text{and} \quad \dot{\phi} = 1.118 \text{ rad/s}.$$

Observing that $\alpha_{\text{gate}} = -\ddot{\phi}$, we take a time derivative of $\dot{\phi}$ from Eqs. (4), and remembering that $\ddot{L}_{AB} = 0$, we have

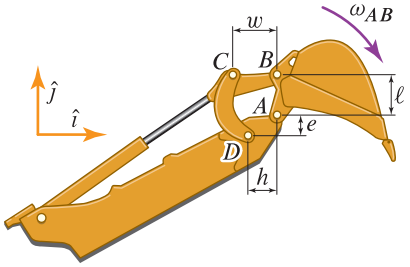
$$\alpha_{\text{gate}} = -\ddot{\phi} = \frac{\cot(\theta + \phi)\csc(\theta + \phi)\dot{L}_{AB}\left(\dot{\theta} + \dot{\phi}\right)}{d}$$

Substituting in known values $\theta = 0.4636$ rad, $\dot{\theta} = 0.8946$ rad/s, $\phi = 0$ rad, $\dot{\phi} = 1.118$ rad/s, $\dot{L}_{AB} = 2.5$ ft/s, and $d = 5$ ft, and expressing in vector form with three significant figures, we have

$$\boxed{\vec{\alpha}_{\text{gate}} = 4.50\,\hat{k} \text{ rad/s}^2.} \tag{5}$$

Problem 6.97

The bucket of a backhoe is the element AB of the four-bar linkage system $ABCD$. Assume that the points A and D are fixed and that the bucket rotates with a constant angular velocity $\omega_{AB} = 0.25\,\text{rad/s}$. In addition, suppose that, at the instant shown, point B is aligned vertically with point A, and C is aligned horizontally with B. Determine the acceleration of point C at the instant shown along with the angular accelerations of the elements BC and CD. Let $h = 0.66\,\text{ft}$, $e = 0.46\,\text{ft}$, $l = 0.9\,\text{ft}$, and $w = 1.0\,\text{ft}$.



Solution

Point C is a point on both bar BC and bar CD, and therefore can be related to both points B and D through rigid body kinematics. We have that

$$\vec{v}_C = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{C/D} = \omega_{CD}\,\hat{k} \times [-(w-h)\,\hat{\imath} + (e+\ell)\,\hat{\jmath}] = -\omega_{CD}(e+\ell)\,\hat{\imath} - \omega_{CD}(w-h)\,\hat{\jmath}. \quad (1)$$

Observing that A is the instantaneous enter of rotation of AB, $v_{By} = 0$, and $\vec{v}_B = v_{Bx}\,\hat{\imath}$. We then have

$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} = v_{Bx}\,\hat{\imath} + \omega_{BC} \times (-w)\,\hat{\imath} = v_{Bx}\,\hat{\imath} - \omega_{BC}\,w\,\hat{\jmath}. \quad (2)$$

Equating Eqs. (1) and (2) component by component we have

$$-\omega_{CD}(e+\ell)\,\hat{\imath} = v_{Bx}\,\hat{\imath} \quad \text{and} \quad -\omega_{CD}(w-h)\,\hat{\jmath} = -\omega_{BC}\,w\,\hat{\jmath}. \quad (3)$$

Observing that A is the instantaneous center of rotation of AB, we also have $v_B = |\omega_{AB}|\ell$. Solving Eqs. (3), for ω_{CD} and ω_{BC}, we have

$$-\omega_{CD}(e+\ell) = |\omega_{AB}|\ell \quad \Rightarrow \quad \omega_{CD} = -\frac{|\omega_{AB}|\ell}{e+\ell} \quad (4)$$

and

$$\omega_{BC} = \frac{\omega_{CD}(w-h)}{w} \quad \Rightarrow \quad \omega_{BC} = -\frac{|\omega_{AB}|\ell}{e+\ell}\frac{w-h}{w}. \quad (5)$$

Substituting known values $|\omega_{AB}| = 0.25\,\text{rad/s}$, $e = 0.46\,\text{ft}$, $\ell = 0.9\,\text{ft}$, $h = 0.66\,\text{ft}$, and $w = 1.0\,\text{ft}$ into Eqs. (4) and (5), we have

$$\vec{\omega}_{CD} = 0.1654\,\hat{k}\,\text{rad/s}, \quad \vec{\omega}_{BC} = 0.05625\,\hat{k}\,\text{rad/s}.$$

From rigid body kinematics we must also have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} = -\omega_{AB}^2 \ell\,\hat{\jmath}, \quad (6)$$

$$\vec{a}_C = \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2 \vec{r}_{C/B}$$

$$= -\omega_{AB}^2 \ell\,\hat{\jmath} + \alpha_{BC}\,\hat{k} \times (-w\,\hat{\imath}) - \left(\frac{\omega_{AB}\ell}{e+\ell}\frac{w-h}{w}\right)^2 (-w\,\hat{\imath})$$

$$= w\left(\frac{\omega_{AB}\ell}{e+\ell}\frac{w-h}{w}\right)^2 \hat{\imath} - (\omega_{AB}^2\ell + \alpha_{BC}\ell)\,\hat{\jmath}, \tag{7}$$

and

$$\begin{aligned}\vec{a}_C &= \vec{a}_D + \vec{\alpha}_{CD}\times\vec{r}_{C/D} - \omega_{CD}^2\vec{r}_{C/D}\\ &= \alpha_{CD}\,\hat{k}\times(e+\ell)\,\hat{\jmath} - \left(\frac{\omega_{AB}\ell}{e+\ell}\right)^2(e+\ell)\,\hat{\jmath}\\ &= -\alpha_{CD}(e+\ell)\,\hat{\imath} - \frac{(\omega_{AB}\ell)^2}{e+\ell}\,\hat{\jmath}\end{aligned} \tag{8}$$

Equating Eqs. (7) and (8) component by component we have

$$w\left(\frac{\omega_{AB}\ell}{e+\ell}\frac{w-h}{w}\right)^2 = -\alpha_{CD}(e+\ell) \quad\Rightarrow\quad \alpha_{CD} = \frac{[\omega_{AB}\ell(w-h)]^2}{(e+\ell)w} = -0.01749\,\text{rad/s}^2 \tag{9}$$

$$(\omega_{AB}^2\ell + \alpha_{BC}\ell) = \frac{(\omega_{AB}\ell)^2}{e+\ell} \quad\Rightarrow\quad \alpha_{BC} = \frac{\omega_{AB}^2\ell}{e+\ell} - \omega_{AB}^2 \quad\Rightarrow\quad \alpha_{BC} = \omega_{AB}^2\left(\frac{\ell}{e+\ell} - 1\right)$$
$$= 0.004516\,\text{rad/s}^2. \tag{10}$$

In vector form, we have

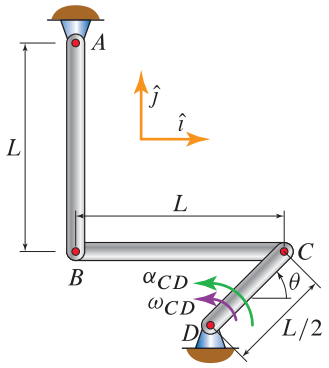
$$\boxed{\vec{\alpha}_{BC} = -0.0175\,\hat{\jmath}\,\text{rad/s}^2 \quad\text{and}\quad \vec{\alpha}_{CD} = 0.00452\,\hat{k}\,\text{rad/s}^2.}$$

Substituting known values $w = 1.0\,\text{ft}$, $\ell = 0.9\,\text{ft}$, $e = 0.46\,\text{ft}$, $h = 0.66\,\text{ft}$, $|\omega_{AB}| = 0.25\,\text{rad/s}$, and $\alpha_{AB} = 0.004516\,\text{rad/s}^2$ into Eq. (7) we have

$$\boxed{\vec{a}_C = (0.00316\,\hat{\imath} - 0.0388\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 6.98

At the instant shown, bar CD is rotating with an angular velocity 20 rad/s and with angular acceleration 2 rad/s^2 in the directions shown. Furthermore, at this instant $\theta = 45°$. Letting $L = 2.25$ ft, determine the angular accelerations of bars AB and BC.



Solution

Because bar AB is rotating about the fixed point A, at the given instant point B is only moving in the $\hat{\imath}$ direction, and we have

$$\vec{v}_B = \omega_{AB} L\,\hat{\imath}.$$

Points B and C are both on the bar BC so we have

$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} = \omega_{AB} L\,\hat{\imath} + \omega_{BC}\,\hat{k} \times L\,\hat{\imath} = \omega_{AB} L\,\hat{\imath} + \omega_{BC} L\,\hat{\jmath}. \tag{1}$$

Points C and D are also both on the bar CD so we have

$$\vec{v}_C = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{C/D} = \omega_{CD}\,\hat{k} \times \frac{L}{2}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = \frac{L}{2}\omega_{CD}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}). \tag{2}$$

Equating Eqs. (1) and (2) component by component, we have

$$\omega_{AB} L = -\frac{L}{2}\omega_{CD}\sin\theta \quad \Rightarrow \quad \omega_{AB} = -\frac{\omega_{CD}\sin\theta}{2} \tag{3}$$

$$\omega_{BC} L = \frac{L}{2}\omega_{CD}\cos\theta \quad \Rightarrow \quad \omega_{BC} = \frac{\omega_{CD}\cos\theta}{2}. \tag{4}$$

Similarly for the accelerations, we have

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} \\ &= \alpha_{AB}\,\hat{k} \times (-L\,\hat{\jmath}) - \frac{\omega_{CD}^2 \sin^2\theta}{4}(-L\,\hat{\jmath}) \\ &= L\alpha_{AB}\,\hat{\imath} + \frac{L\omega_{CD}^2 \sin^2\theta}{4}\,\hat{\jmath},\end{aligned}$$

as well as

$$\begin{aligned}\vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2 \vec{r}_{C/B} \\ &= \left(L\alpha_{AB}\,\hat{\imath} + \frac{L\omega_{CD}^2 \sin^2\theta}{4}\,\hat{\jmath}\right) + \alpha_{BC}\,\hat{k} \times L\,\hat{\imath} - \frac{\omega_{CD}^2 \cos^2\theta}{4} L\,\hat{\imath}\end{aligned}$$

$$= \left(L\alpha_{AB} - \frac{L\omega_{CD}^2 \cos^2\theta}{4} \right) \hat{\imath} + \left(\frac{L\omega_{CD}^2 \sin^2\theta}{4} + L\alpha_{BC} \right) \hat{\jmath}, \tag{5}$$

and

$$\begin{aligned} \vec{a}_C &= \vec{a}_D + \vec{\alpha}_{CD} \times \vec{r}_{C/D} - \omega_{CD}^2 \vec{r}_{C/D} \\ &= \alpha_{CD}\, \hat{k} \times \frac{L}{2}(\cos\theta\, \hat{\imath} + \sin\theta\, \hat{\jmath}) - \omega_{CD}^2 \frac{L}{2}(\cos\theta\, \hat{\imath} + \sin\theta\, \hat{\jmath}) \\ &= -\left(\alpha_{CD} \frac{L}{2} \sin\theta + \omega_{CD}^2 \frac{L}{2} \cos\theta \right) \hat{\imath} + \left(\alpha_{CD} \frac{L}{2} \cos\theta - \omega_{CD}^2 \frac{L}{2} \sin\theta \right) \hat{\jmath}. \end{aligned} \tag{6}$$

Equatings Eqs. (5) and (6) component by component we have

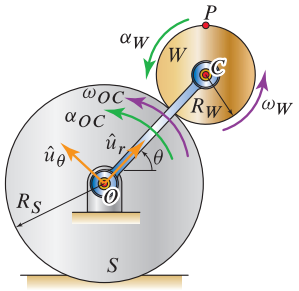
$$\begin{aligned} L\alpha_{AB} - \frac{L\omega_{CD}^2 \cos^2\theta}{4} &= -\left(\alpha_{CD} \frac{L}{2} \sin\theta + \omega_{CD}^2 \frac{L}{2} \cos\theta \right) \\ \Rightarrow \quad \alpha_{AB} &= \omega_{CD}^2 \frac{\cos^2\theta}{4} - \alpha_{CD} \frac{\sin\theta}{2} - \omega_{CD}^2 \frac{\cos\theta}{2} \\ \frac{L\omega_{CD}^2 \sin^2\theta}{4} + L\alpha_{BC} &= \alpha_{CD} \frac{L}{2} \cos\theta - \omega_{CD}^2 \frac{L}{2} \sin\theta \\ \Rightarrow \quad \alpha_{BC} &= -\frac{\omega_{CD}^2 \sin^2\theta}{4} + \alpha_{CD} \frac{\cos\theta}{2} - \omega_{CD}^2 \frac{\sin\theta}{2}. \end{aligned}$$

Substituting known values $\omega_{CD} = 20\,\text{rad/s}, \alpha_{CD} = 2\,\text{rad/s}^2, \theta = 45°$, and $L = 2.25\,\text{ft}$, in vector form we have

$$\boxed{\vec{\alpha}_{AB} = -92.1\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -191\,\hat{k}\,\text{rad/s}^2.}$$

Problems 6.99 through 6.101

A wheel W of radius $R_W = 2$ in. rolls without slip over the stationary cylinder S of radius $R_S = 5$ in., and the wheel is connected to point O via the arm OC.



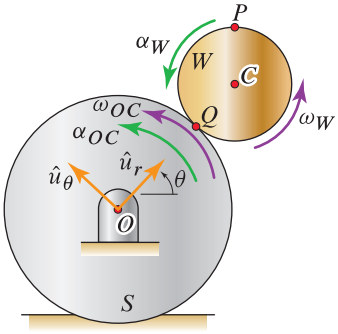
Problem 6.99 Determine the acceleration of the point on the wheel W that is in contact with S for $\omega_{OC} = 7.5$ rad/s = constant.

Problem 6.100 Determine the acceleration of the point on the wheel W that is in contact with S for $\omega_{OC} = 7.5$ rad/s and $\alpha_{OC} = 2$ rad/s^2.

Problem 6.101 If, at the instant shown, $\theta = 63°$, $\omega_W = 9$ rad/s, and $\alpha_W = -1.3$ rad/s^2, determine the angular acceleration of the arm OC and the acceleration of point P, where P lies on the edge of W and is aligned vertically with point C.

Solution to 6.99

We call the contact point between the wheel W and cylinder S point Q. From the geometry of the problem, we have



$$\vec{r}_{C/O} = (R_S + R_W)\,\hat{u}_r \quad \text{and} \quad \vec{r}_{C/Q} = R_W\,\hat{u}_r.$$

From rigid body kinematics we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{\omega}_{OC} \times \vec{r}_{C/O} \\ &= \omega_{OC}\,\hat{k} \times (R_S + R_W)\,\hat{u}_r \\ &= \omega_{OC}(R_S + R_W)\,\hat{u}_\theta. \end{aligned} \tag{1}$$

Due to rolling without slip, $\vec{v}_Q = \vec{0}$. Therefore, we must have

$$\vec{v}_C = \vec{v}_Q + \vec{\omega}_W \times \vec{r}_{C/Q} = \omega_W\,\hat{k} \times R_W\,\hat{u}_r = \omega_W R_W \hat{u}_\theta \tag{2}$$

Equating Eqs. (1) and (2) component by component we have

$$\omega_{OC}(R_S + R_W) = \omega_W R_W \quad \Rightarrow \quad \omega_W = \frac{(R_S + R_W)}{R_W}\omega_{OC}. \tag{3}$$

We must also have

$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{\alpha}_{OC} \times \vec{r}_{C/O} - \omega_{OC}^2 \vec{r}_{C/O} \\ &= -\omega_{OC}^2 (R_S + R_W)\,\hat{u}_r\end{aligned}$$

and

$$\begin{aligned}\vec{a}_Q &= \vec{a}_C + \vec{\alpha}_W \times \vec{r}_{Q/C} - \omega_W^2 \vec{r}_{Q/C} \\ &= -\omega_{OC}^2 (R_S + R_W)\,\hat{u}_r + \alpha_W\,\hat{k} \times (-R_W)\,\hat{u}_r + \omega_W^2 R_W\,\hat{u}_r \\ &= [-\omega_{OC}^2 (R_S + R_W) + \omega_W^2 R_W]\,\hat{u}_r - R_W \alpha_W\,\hat{u}_\theta \end{aligned} \tag{4}$$

From the no slip condition, we know that $\vec{a}_{Q\theta} = 0$. Substituting Eq. (3) for ω_W in Eq. (4) we have

$$\vec{a}_Q = \left[-\omega_{OC}^2(R_S + R_W) + \frac{(R_S + R_W)^2}{R_W}\omega_{OC}^2\right]\hat{u}_r = \omega_{OC}^2(R_S + R_W)\left[\frac{R_S + R_W}{R_W} - 1\right]\hat{u}_r. \quad (5)$$

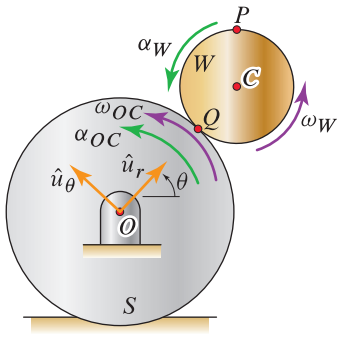
Substituting known values $R_W = 2\,\text{in.} = 0.1667\,\text{ft}$, $R_S = 5\,\text{in.} = 0.4167\,\text{ft}$, and $\omega_{OC} = 7.5\,\text{rad/s}$, we have

$$\boxed{\vec{a}_Q = 82.0\,\hat{u}_r\ \text{ft/s}^2.}$$

Solution to 6.100

We call the contact point between the wheel W and cylinder S point Q. From the geometry of the problem, we have

$$\vec{r}_{C/O} = (R_S + R_W)\,\hat{u}_r \quad \text{and} \quad \vec{r}_{C/Q} = R_W\,\hat{u}_r.$$



From rigid body kinematics we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{\omega}_{OC} \times \vec{r}_{C/O} \\ &= \omega_{OC}\,\hat{k} \times (R_S + R_W)\,\hat{u}_r \\ &= \omega_{OC}(R_S + R_W)\,\hat{u}_\theta. \end{aligned} \tag{6}$$

Due to rolling without slip, $\vec{v}_Q = \vec{0}$. Therefore, we must have

$$\vec{v}_C = \vec{v}_Q + \vec{\omega}_W \times \vec{r}_{C/Q} = \omega_W\,\hat{k} \times R_W\,\hat{u}_r = \omega_W R_W \hat{u}_\theta. \tag{7}$$

Equating Eqs. (6) and (7) component by component we have

$$\omega_{OC}(R_S + R_W) = \omega_W R_W \quad \Rightarrow \quad \omega_W = \frac{(R_S + R_W)}{R_W}\omega_{OC}. \tag{8}$$

We must also have

$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{\alpha}_{OC} \times \vec{r}_{C/O} - \omega_{OC}^2 \vec{r}_{C/O} \\ &= \alpha_{OC}\,\hat{k} \times (R_S + R_W)\,\hat{u}_r - \omega_{OC}^2 (R_S + R_W)\,\hat{u}_r \\ &= -\omega_{OC}^2 (R_S + R_W)\,\hat{u}_r + (R_S + R_W)\alpha_{OC}\,\hat{u}_\theta \end{aligned}$$

and

$$\begin{aligned}\vec{a}_Q &= \vec{a}_C + \vec{\alpha}_W \times \vec{r}_{Q/C} - \omega_W^2 \vec{r}_{Q/C} \\ &= [-\omega_{OC}^2 (R_S + R_W)\,\hat{u}_r + (R_S + R_W)\alpha_{OC}\,\hat{u}_\theta] + \alpha_W\,\hat{k} \times (-R_W)\,\hat{u}_r + \omega_W^2 R_W\,\hat{u}_r \\ &= [-\omega_{OC}^2 (R_S + R_W) + \omega_W^2 R_W]\,\hat{u}_r + [(R_S + R_W)\alpha_{OC} - R_W \alpha_W]\,\hat{u}_\theta. \end{aligned} \tag{9}$$

From the no slip condition, we know that

$$\vec{a}_{Q\theta} = 0 \quad \Rightarrow \quad \alpha_W = \frac{R_S + R_W}{R_W}\alpha_{OC}. \tag{10}$$

Substituting Eq. (8) for ω_W in Eq. (9) we have

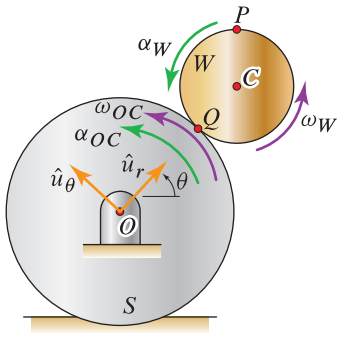
$$\vec{a}_Q = \left[-\omega_{OC}^2 (R_S + R_W) + \frac{(R_S + R_W)^2}{R_W}\omega_{OC}^2\right]\hat{u}_r = \omega_{OC}^2 (R_S + R_W)\left[\frac{R_S + R_W}{R_W} - 1\right]\hat{u}_r. \tag{11}$$

Substituting known values $R_W = 2\,\text{in.} = 0.1667\,\text{ft}$, $R_S = 5\,\text{in.} = 0.4167\,\text{ft}$, and $\omega_{OC} = 7.5\,\text{rad/s}$, we have

$$\boxed{\vec{a}_Q = 82.0\,\hat{u}_r\ \text{ft/s}^2.}$$

Solution to 6.101

We call the contact point between the wheel W and cylinder S point Q. From the geometry of the problem, we have



$$\vec{r}_{C/O} = (R_S + R_W)\,\hat{u}_r,$$
$$\vec{r}_{P/C} = R_W[\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta],$$

and

$$\vec{r}_{C/Q} = R_W\,\hat{u}_r.$$

From rigid body kinematics we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{\omega}_{OC} \times \vec{r}_{C/O}\\ &= \omega_{OC}\,\hat{k} \times (R_S + R_W)\,\hat{u}_r\\ &= \omega_{OC}(R_S + R_W)\,\hat{u}_\theta. \qquad (12)\end{aligned}$$

Due to rolling without slip, $\vec{v}_Q = \vec{0}$. Therefore, we must have

$$\vec{v}_C = \vec{v}_Q + \vec{\omega}_W \times \vec{r}_{C/Q} = \omega_W\,\hat{k} \times R_W\,\hat{u}_r = \omega_W R_W \hat{u}_\theta \qquad (13)$$

Equating Eqs. (12) and (13) component by component we have

$$\omega_{OC}(R_S + R_W) = \omega_W R_W \quad \Rightarrow \quad \omega_{OC} = \frac{R_W}{(R_S + R_W)}\omega_W. \qquad (14)$$

We must also have

$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{\alpha}_{OC} \times \vec{r}_{C/O} - \omega_{OC}^2 \vec{r}_{C/O}\\ &= \alpha_{OC}\,\hat{k} \times (R_S + R_W)\,\hat{u}_r - \omega_{OC}^2(R_S + R_W)\,\hat{u}_r\\ &= -\omega_{OC}^2(R_S + R_W)\,\hat{u}_r + (R_S + R_W)\alpha_{OC}\,\hat{u}_\theta\end{aligned}$$

and

$$\begin{aligned}\vec{a}_Q &= \vec{a}_C + \vec{\alpha}_W \times \vec{r}_{Q/C} - \omega_W^2 \vec{r}_{Q/C}\\ &= [-\omega_{OC}^2(R_S + R_W)\,\hat{u}_r + (R_S + R_W)\alpha_{OC}\,\hat{u}_\theta] + \alpha_W\,\hat{k} \times (-R_W)\,\hat{u}_r + \omega_W^2 R_W\,\hat{u}_r\\ &= [-\omega_{OC}^2(R_S + R_W) + \omega_W^2 R_W]\,\hat{u}_r + [(R_S + R_W)\alpha_{OC} - R_W\alpha_W]\,\hat{u}_\theta.\end{aligned}$$

From the no slip condition, we know that

$$\vec{a}_{Q\theta} = 0 \quad \Rightarrow \quad \alpha_{OC} = \frac{R_W}{R_S + R_W}\alpha_W. \qquad (15)$$

We must also have

$$\begin{aligned}\vec{a}_P &= \vec{a}_C + \vec{\alpha}_W \times \vec{r}_{P/C} - \omega_W^2 \vec{r}_{P/C}\\ &= -\omega_{OC}^2(R_S + R_W)\,\hat{u}_r + (R_S + R_W)\alpha_{OC}\,\hat{u}_\theta + \alpha_W\,\hat{k} \times R_W(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta)\\ &\quad - \omega_W^2 R_W(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta)\\ &= \left[-\omega_{OC}^2(R_S + R_W) - R_W\alpha_W\cos\theta - \omega_W^2 R_W\sin\theta\right]\hat{u}_r\\ &\quad + \left[(R_S + R_W)\alpha_{OC} + R_W\alpha_W\sin\theta - \omega_W^2 R_W\cos\theta\right]\hat{u}_\theta \qquad (16)\end{aligned}$$

Substituting Eqs. (14) and (15) into Eq. (16) we have

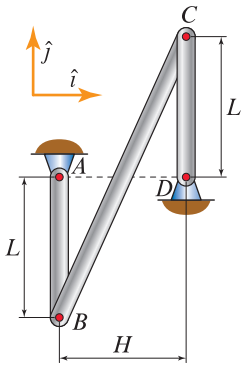
$$\vec{a}_P = \left[-\frac{R_W^2}{(R_S + R_W)}\omega_W^2 - R_W\alpha_W \cos\theta - \omega_W^2 R_W \sin\theta \right] \hat{u}_r + \left[R_W\alpha_W(1 + \sin\theta) - \omega_W^2 R_W \cos\theta \right] \hat{u}_\theta \tag{17}$$

Substituting known values $R_W = 2\,\text{in.} = 0.1667\,\text{ft}$, $R_S = 5\,\text{in.} = 0.4167\,\text{ft}$, $\theta = 63°$, $\omega_W = 9\,\text{rad/s}$, and $\alpha_W = -1.3\,\text{rad/s}^2$, into Eqs. (15) and (17), to three significant figures we have

$$\boxed{\vec{\alpha}_{OC} = -0.371\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{a}_P = (-15.8\,\hat{u}_r - 6.54\,\hat{u}_\theta)\,\text{ft/s}^2.}$$

Problem 6.102

At the instant shown, bars AB and CD are vertical. In addition, point C is moving to the left with a speed of 4 m/s, and the magnitude of the acceleration of C is 55 m/s^2. Letting $L = 0.5$ m and $H = 0.2$ m, determine the angular accelerations of bars AB and BC.



Solution

Realizing that if C is moving left, ω_{CD} must be counterclockwise, from the concept of instantaneous center of rotation, we have

$$\vec{\omega}_{CD} = \frac{v_C}{L}\hat{k} = 8.000\,\hat{k}\ \text{rad/s} \quad \text{and} \quad \vec{\alpha}_{CD} = \frac{a_C}{L}\hat{k}. \tag{1}$$

Similarly, because the center of rotation of bar BC is at infinity, we have

$$\vec{\omega}_{BC} = \vec{0}.$$

Therefore, the velocity of B is the same as the velocity of C, so we must have

$$\vec{\omega}_{AB} = -\frac{v_C}{L}\hat{k} = -8.000\,\hat{k}\ \text{rad/s}. \tag{2}$$

Observing that at the given instant we have the position vectors

$$\vec{r}_{B/A} = -L\,\hat{\jmath}, \quad \vec{r}_{B/C} = -(H\,\hat{\imath} + 2L\,\hat{\jmath}), \quad \text{and} \quad \vec{r}_{C/D} = L\,\hat{\jmath},$$

then from rigid body kinematics we must have

$$\vec{a}_C = \vec{a}_D + \vec{\alpha}_{CD} \times \vec{r}_{C/D} - \omega_{CD}^2 \vec{r}_{C/D} = \alpha_{CD}\,\hat{k} \times L\,\hat{\jmath} - \omega_{CD}^2 (L\,\hat{\jmath}) = -L\alpha_{CD}\,\hat{\imath} - \frac{v_C^2}{L}\,\hat{\jmath}, \tag{3}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} = \alpha_{AB}\,\hat{k} \times (-L)\,\hat{\jmath} - \omega_{AB}^2 (-L)\,\hat{\jmath} = L\alpha_{AB}\,\hat{\imath} + L\omega_{AB}^2\,\hat{\jmath},$$

and

$$\begin{aligned}\vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2 \vec{r}_{C/B} \\ &= L\alpha_{AB}\,\hat{\imath} + L\omega_{AB}^2\,\hat{\jmath} + \alpha_{BC}\,\hat{k} \times (H\,\hat{\imath} + 2L\,\hat{\jmath}) - \omega_{BC}^2 (H\,\hat{\imath} + 2L\,\hat{\jmath}) \\ &= (L\alpha_{AB} - 2L\alpha_{BC} - H\omega_{BC}^2)\,\hat{\imath} + (L\omega_{AB}^2 + H\alpha_{BC} - 2L\omega_{BC}^2)\,\hat{\jmath}.\end{aligned} \tag{4}$$

Equations (3) and (4) are two expressions for $\vec{a}_C$, and setting them equal component by component, we have

$$-L\alpha_{CD} = L\alpha_{AB} - 2L\alpha_{BC} - H\omega_{BC}^2$$

and

$$-\frac{v_C^2}{L} = L\omega_{AB}^2 + H\alpha_{BC} - 2L\omega_{BC}^2,$$

which are two equations in α_{AB} and α_{BC}. Solving them, we have

$$\alpha_{AB} = \frac{-2v_C^2 - L\left(\frac{H}{L}a_C + 2L\omega_{AB}^2\right) + \left(H^2 + 4L^2\right)\omega_{BC}^2}{HL}$$
$$\alpha_{BC} = -\frac{v_C^2 + L^2\left(\omega_{AB}^2 - 2\omega_{BC}^2\right)}{HL},$$

where we have let $\alpha_{CD} = \frac{a_C}{L}$. However, closer inspection, only the magnitude of a_C is given in the problem, so we must find answers for both the positive and negative values. Choosing $a_C = 55\,\text{m/s}^2$, and substituting the given values $v_C = 4\,\text{m/s}$, $a_C = 55\,\text{m/s}^2$, $L = 0.5\,\text{m}$, $H = 0.2\,\text{m}$, $\omega_{AB} = -8.000\,\text{rad/s}$, and $\omega_{BC} = 8.000\,\text{rad/s}$, in vector forma and to three significant figures, we have

$$\boxed{\vec{\alpha}_{AB} = -84.4\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -315\,\hat{k}\,\text{rad/s}^2.}$$

Choosing $a_C = -55\,\text{m/s}^2$ and substituting the given values $v_C = 4\,\text{m/s}$, $a_C = 55\,\text{m/s}^2$, $L = 0.5\,\text{m}$, $H = 0.2\,\text{m}$, $\omega_{AB} = -8.000\,\text{rad/s}$, and $\omega_{BC} = 8.000\,\text{rad/s}$, in vector forma and to three significant figures, we have

$$\boxed{\vec{\alpha}_{AB} = 136\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -315\,\hat{k}\,\text{rad/s}^2.}$$

Problems 6.103 through 6.106

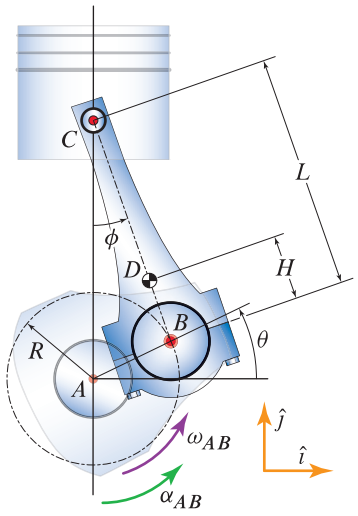
For the slider-crank mechanism shown, let $R = 1.9$ in., $L = 6.1$ in., and $H = 1.2$ in.

Problem 6.103 Assuming that $\omega_{AB} = 4850$ rpm and is constant, determine the angular acceleration of the connecting rod BC and the acceleration of point C at the instant when $\theta = 27°$.

Problem 6.104 Assuming that $\omega_{AB} = 4850$ rpm and is constant, determine the acceleration of point D at the instant when $\phi = 10°$.

Problem 6.105 Assuming that, at the instant shown, $\theta = 31°$, $\omega_{AB} = 4850$ rpm, and $\alpha_{AB} = \dot{\omega}_{AB} = -280\,\text{rad/s}^2$, determine the angular acceleration of the connecting rod and the acceleration of point C.

Problem 6.106 Determine the general expression of the acceleration of the piston C as a function of L, R, θ, $\omega_{AB} = \dot{\theta}$, and $\alpha_{AB} = \ddot{\theta}$.



Solution to 6.103

From the geometry of the problem we have

$$\vec{r}_{B/A} = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}),$$
$$\vec{r}_{C/B} = L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}),$$

and

$$L\sin\phi = R\cos\theta \quad \Rightarrow \quad \phi = \sin^{-1}\left(\frac{R}{L}\cos\theta\right).$$

From rigid body kinematics we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} = \omega_{AB}\,\hat{k} \times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -R\omega_{AB}\sin\theta\,\hat{\imath} + R\omega_{AB}\cos\theta\,\hat{\jmath}$$

and

$$\begin{aligned}\vec{v}_C &= \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} \\ &= -R\omega_{AB}\sin\theta\,\hat{\imath} + R\omega_{AB}\cos\theta\,\hat{\jmath} + \omega_{BC}\,\hat{k} \times L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) \\ &= (-R\omega_{AB}\sin\theta - L\omega_{BC}\cos\phi)\,\hat{\imath} + (R\omega_{AB}\cos\theta - L\omega_{BC}\sin\phi)\,\hat{\jmath}.\end{aligned}$$

Because C is constrained to move in the vertical direction, we have

$$-R\omega_{AB}\sin\theta - L\omega_{BC}\cos\phi = 0 \quad \Rightarrow \quad \omega_{BC} = -\frac{R\omega_{AB}\sin\theta}{L\cos\phi}.$$

Remembering that $\alpha_{AB} = 0$, we must also have

$$\begin{aligned}\vec{a}_B &= \vec{a}_A - \omega_{AB}^2\vec{r}_{B/A}\\ &= -\omega_{AB}^2 R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\\ &= -R\omega_{AB}^2\cos\theta\,\hat{\imath} - \omega_{AB}^2\sin\theta\,\hat{\jmath}\end{aligned}$$

and

$$\begin{aligned}\vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC}\times\vec{r}_{C/B} - \omega_{BC}^2\vec{r}_{C/B}\\ &= -R\omega_{AB}^2\cos\theta\,\hat{\imath} - \omega_{AB}^2\sin\theta\,\hat{\jmath}\\ &\quad + \alpha_{BC}\,\hat{k}\times L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) - \omega_{BC}^2 L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})\\ &= \left[-R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi - L\alpha_{BC}\cos\phi\right]\hat{\imath}\\ &\quad + \left[-R\omega_{AB}^2\sin\theta - L\alpha_{BC}\sin\phi - \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\cos\phi\right]\hat{\jmath} \qquad (1)\end{aligned}$$

Because C is constrained to move in the vertical direction, we have

$$-R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi - L\alpha_{BC}\cos\phi$$

$$\Rightarrow \quad \alpha_{BC} = \frac{-R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi}{L\cos\phi} \qquad (2)$$

Substituting known values $R = 1.9\,\text{in.} = 0.1583\,\text{ft}$, $L = 6.1\,\text{in.} = 0.5083\,\text{ft}$, $H = 1.2\,\text{in.} = 0.1000\,\text{ft}$, $\theta = 27°$, and $\omega_{AB} = 4850\,\text{rpm} = 507.9\,\text{rad/s}$ into Eqs. (1) and (2), rounding to three significant figures, and putting in vector form we have

$$\boxed{\vec{a}_C = -11000\,\hat{\jmath}\,\text{ft/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -72900\,\hat{k}\,\text{rad/s}^2.} \qquad (3)$$

Solution to 6.104

From the geometry of the problem we have

$$\begin{aligned}\vec{r}_{B/A} &= R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}),\\ \vec{r}_{C/B} &= L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}),\\ \vec{r}_{D/C} &= (L-H)(\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath}),\end{aligned}$$

and

$$L\sin\phi = R\cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{L}{R}\sin\phi\right) = 56.17^\circ.$$

From rigid body kinematics we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB}\times\vec{r}_{B/A} = \omega_{AB}\,\hat{k}\times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -R\omega_{AB}\sin\theta\,\hat{\imath} + R\omega_{AB}\cos\theta\,\hat{\jmath}$$

and

$$\begin{aligned}\vec{v}_C &= \vec{v}_B + \vec{\omega}_{BC}\times\vec{r}_{C/B}\\ &= -R\omega_{AB}\sin\theta\,\hat{\imath} + R\omega_{AB}\cos\theta\,\hat{\jmath} + \omega_{BC}\,\hat{k}\times L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})\\ &= (-R\omega_{AB}\sin\theta - L\omega_{BC}\cos\phi)\,\hat{\imath} + (R\omega_{AB}\cos\theta - L\omega_{BC}\sin\phi)\,\hat{\jmath}.\end{aligned}$$

Because C is constrained to move in the vertical direction, we have

$$-R\omega_{AB}\sin\theta - L\omega_{BC}\cos\phi = 0 \quad \Rightarrow \quad \omega_{BC} = -\frac{R\omega_{AB}\sin\theta}{L\cos\phi}.$$

Remembering that $\alpha_{AB} = 0$, we must also have

$$\vec{a}_B = \vec{a}_A - \omega_{AB}^2\vec{r}_{B/A} = -\omega_{AB}^2 R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -R\omega_{AB}^2\cos\theta\,\hat{\imath} - \omega_{AB}^2\sin\theta\,\hat{\jmath}$$

and

$$\begin{aligned}\vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC}\times\vec{r}_{C/B} - \omega_{BC}^2\vec{r}_{C/B}\\ &= -R\omega_{AB}^2\cos\theta\,\hat{\imath} - \omega_{AB}^2\sin\theta\,\hat{\jmath}\\ &\quad + \alpha_{BC}\,\hat{k}\times L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) - \omega_{BC}^2 L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})\\ &= \left[-R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi - L\alpha_{BC}\cos\phi\right]\hat{\imath}\\ &\quad + \left[-R\omega_{AB}^2\sin\theta - L\alpha_{BC}\sin\phi - \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\cos\phi\right]\hat{\jmath}. \qquad (4)\end{aligned}$$

Because C is constrained to move in the vertical direction, $a_{Cx} = 0$, and we have

$$-R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi - L\alpha_{BC}\cos\phi = 0$$

$$\Rightarrow \quad \alpha_{BC} = \frac{-R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi}{L\cos\phi}.$$

Substituting known values $R = 1.9\,\text{in.} = 0.1583\,\text{ft}$, $L = 6.1\,\text{in.} = 0.5083\,\text{ft}$, $H = 1.2\,\text{in.} = 0.1000\,\text{ft}$, $\phi = 10^\circ$, $\theta = 56.12^\circ$, and $\omega_{AB} = 4850\,\text{rpm} = 507.9\,\text{rad/s}$, we have

$$\vec{a}_C = -39070\,\hat{\jmath}\,\text{ft/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -42330\,\hat{k}\,\text{rad/s}^2.$$

Finally, we must have

$$\begin{aligned}
\vec{a}_D &= \vec{a}_C + \vec{\alpha}_{BC} \times \vec{r}_{D/C} - \omega_{BC}^2 \vec{r}_{D/C} \\
&= \vec{a}_C + \alpha_{BC}\,\hat{k} \times (L-H)(\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath}) - \omega_{BC}^2 (L-H)(\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath}) \\
&= \left[-R\omega_{AB}^2 \cos\theta - R\alpha_{AB} \sin\theta + \frac{HR^2\omega_{AB}^2 \sin^2\theta \tan\phi}{L^2 \cos\phi} \right. \\
&\quad \left. - \frac{H\left(-LR\omega_{AB}^2 \cos\theta - LR\alpha_{AB} \sin\theta + R^2\omega_{AB}^2 \sec\phi \sin\theta^2 \tan\phi\right)}{L^2} \right] \hat{\imath} \\
&\quad + \left[R\alpha_{AB} \cos\theta - R\omega_{AB}^2 \sin\theta - \frac{HR^2\omega_{AB}^2 \sec\phi \sin\theta^2}{L^2} \right. \\
&\quad \left. - \frac{1}{L^2} H \tan\phi \left(-LR\omega_{AB}^2 \cos\theta - LR\alpha_{AB} \sin\theta + R^2\omega_{AB}^2 \sec\phi \sin\theta^2 \tan\phi\right) \right] \hat{\jmath} \qquad (5)
\end{aligned}$$

Substituting known values $R = 1.9\,\text{in.} = 0.1583\,\text{ft}$, $L = 6.1\,\text{in.} = 0.5083\,\text{ft}$, $H = 1.2\,\text{in.} = 0.1000\,\text{ft}$, $\phi = 10°$, $\theta = 56.12°$, and $\omega_{AB} = 4850\,\text{rpm} = 507.9\,\text{rad/s}$ into Eq. (5), and rounding to three significant figures, we have

$$\boxed{\vec{a}_D = (31000\,\hat{\imath} - 46100\,\hat{\jmath})\,\text{ft/s}^2.}$$

Solution to 6.105

From the geometry of the problem we have

$$\vec{r}_{B/A} = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}), \quad \vec{r}_{C/B} = L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}),$$

and

$$L\sin\phi = R\cos\theta \quad \Rightarrow \quad \phi = \sin^{-1}\left(\frac{R}{L}\cos\theta\right).$$

From rigid body kinematics we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB}\times\vec{r}_{B/A} = \omega_{AB}\,\hat{k}\times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -R\omega_{AB}\sin\theta\,\hat{\imath} + R\omega_{AB}\cos\theta\,\hat{\jmath}$$

and

$$\begin{aligned}\vec{v}_C &= \vec{v}_B + \vec{\omega}_{BC}\times\vec{r}_{C/B}\\ &= -R\omega_{AB}\sin\theta\,\hat{\imath} + R\omega_{AB}\cos\theta\,\hat{\jmath} + \omega_{BC}\,\hat{k}\times L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})\\ &= (-R\omega_{AB}\sin\theta - L\omega_{BC}\cos\phi)\,\hat{\imath} + (R\omega_{AB}\cos\theta - L\omega_{BC}\sin\phi)\,\hat{\jmath}.\end{aligned}$$

Because C is constrained to move in the vertical direction, we have

$$-R\omega_{AB}\sin\theta - L\omega_{BC}\cos\phi = 0 \quad \Rightarrow \quad \omega_{BC} = -\frac{R\omega_{AB}\sin\theta}{L\cos\phi}.$$

We must also have

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + \vec{\alpha}_{AB}\times\vec{r}_{B/A} - \omega_{AB}^2\vec{r}_{B/A}\\ &= \alpha_{AB}\,\hat{k}\times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) - \omega_{AB}^2 R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\\ &= (-R\alpha_{AB}\sin\theta - R\omega_{AB}^2\cos\theta)\,\hat{\imath} + (R\alpha_{AB}\cos\theta - \omega_{AB}^2\sin\theta)\,\hat{\jmath}\end{aligned}$$

and

$$\begin{aligned}\vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC}\times\vec{r}_{C/B} - \omega_{BC}^2\vec{r}_{C/B}\\ &= (-R\alpha_{AB}\sin\theta - R\omega_{AB}^2\cos\theta)\,\hat{\imath} + (R\alpha_{AB}\cos\theta - \omega_{AB}^2\sin\theta)\,\hat{\jmath}\\ &\quad + \alpha_{BC}\,\hat{k}\times L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) - \omega_{BC}^2 L(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})\\ &= \left[-R\alpha_{AB}\sin\theta - R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi - L\alpha_{BC}\cos\phi\right]\hat{\imath}\\ &\quad + \left[R\alpha_{AB}\cos\theta - R\omega_{AB}^2\sin\theta - L\alpha_{BC}\sin\phi - \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\cos\phi\right]\hat{\jmath}. \qquad (6)\end{aligned}$$

Because C is constrained to move in the vertical direction we have

$$-R\alpha_{AB}\sin\theta - R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi - L\alpha_{BC}\cos\phi = 0$$

$$\Rightarrow \quad \alpha_{AB} = \frac{-R\omega_{AB}^2\cos\theta + \left(\frac{R^2\omega_{AB}^2\sin^2\theta}{L\cos^2\phi}\right)\sin\phi - L\alpha_{BC}\cos\phi}{R\sin\theta} \qquad (7)$$

Substituting known values $R = 1.9\,\text{in.} = 0.1583\,\text{ft}$, $L = 6.1\,\text{in.} = 0.5083\,\text{ft}$, $H = 1.2\,\text{in.} = 0.1000\,\text{ft}$, $\theta = 31°$, $\alpha_{AB} = -280\,\text{rad/s}^2$ and $\omega_{AB} = 4850\,\text{rpm} = 507.9\,\text{rad/s}$ into Eqs. (6) and (7), rounding to three significant figures, and putting it in vector form, we have

$$\boxed{\vec{\alpha}_{BC} = -69400\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{a}_C = -15200\,\hat{\jmath}\,\text{ft/s}^2.}$$

Solution to 6.106

Letting the origin of the reference frame be at A, at the indicated instant, we have that the vertical distance from A to C, y_C, is

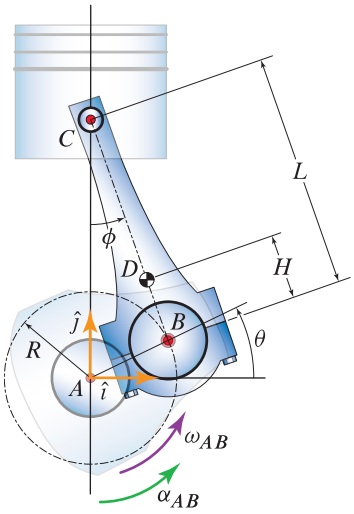
$$y_C = R\sin\theta + L\cos\phi.$$

We also know that

$$L\sin\phi = R\cos\theta.$$

Hence, y_C can be rewritten as

$$y_C = R\sin\theta + L\sqrt{1-\left(\frac{R}{L}\cos\theta\right)^2}.$$



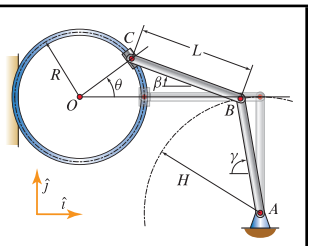
Because C is constrained to move vertically, have

$$\vec{v}_C = \dot{y}_C\,\hat{\jmath} = R\dot{\theta}\cos\theta\left[1+\frac{R\sin\theta}{L\sqrt{1-\frac{R^2\cos^2\theta}{L^2}}}\right]\hat{\jmath}$$

and differentiating again with respect to time, we have

$$\vec{a}_C = \left[L\left(\frac{R^2\cos^2\theta\dot{\theta}^2}{L^2\sqrt{1-\frac{R^2\cos\theta^2}{L^2}}} - \frac{R^4\cos^2\theta\sin^2\theta\dot{\theta}^2}{L^4\left(1-\frac{R^2\cos^2\theta}{L^2}\right)^{3/2}}\right)\right.$$
$$\left. -\left(\frac{R^2\sin\theta^2\dot{\theta}^2}{L^2\sqrt{1-\frac{R^2\cos^2\theta}{L^2}}} + \frac{R^2\cos\theta\sin\theta\ddot{\theta}}{L^2\sqrt{1-\frac{R^2\cos^2\theta}{L^2}}}\right) + R\left(-\sin\theta\dot{\theta}^2+\cos\theta\ddot{\theta}\right)\right]\hat{\jmath}.$$

Problems 6.107 and 6.108



In the four-bar linkage system shown, let the circular guide with center at O be fixed and such that, for $\theta = 0°$, the bars AB and BC are vertical and horizontal, respectively. In addition, let $R = 0.6\,\text{m}$, $L = 1\,\text{m}$, and $H = 1.25\,\text{m}$.

Problem 6.107 When $\theta = 37°$, $\beta = 25.07°$, and $\gamma = 78.71°$, assume collar C is sliding clockwise with a speed $7\,\text{m/s}$. Assuming that, at the instant in question, the speed is increasing and that $|\vec{a}_C| = 93\,\text{m/s}^2$, determine the angular accelerations of the bars AB and BC.

Problem 6.108 Use the method of differentiation of constraints to derive expressions for the angular accelerations of bars AB and BC as a function of θ, $\dot{\theta}$, and $\ddot{\theta}$. Finally, Let $\theta = (0.3\,\text{rad/s}^2)t^2$ and plot the angular accelerations of AB and BC for $0 \le t \le 1\,\text{s}$.

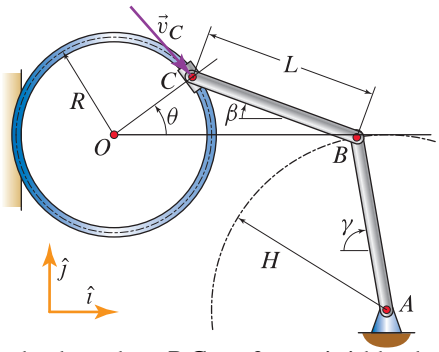
Solution to 6.107

From the geometry of the problem, we have the following position vectors:

$$\vec{r}_{C/O} = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}),$$
$$\vec{r}_{B/C} = L(\cos\beta\,\hat{\imath} - \sin\beta\,\hat{\jmath},$$
$$\vec{r}_{B/A} = H(-\cos\gamma\,\hat{\imath} + \sin\gamma\,\hat{\jmath}),$$

and the velocity vector

$$\vec{v}_C = v_C(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}).$$



We also define $\vec{\omega}_{BC} = \omega_{BC}\,\hat{k}$ and $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$. Points B and C are both on bar BC so from rigid body kinematics we have

$$\begin{aligned}\vec{v}_B &= \vec{v}_C + \omega_{BC}\,\hat{k} \times \vec{r}_{B/C} \\ &= v_C(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) + \omega_{BC}L(\sin\beta\,\hat{\imath} + \cos\beta\,\hat{\jmath}) \\ &= (v_C\sin\theta + \omega_{BC}L\sin\beta)\,\hat{\imath} + (\omega_{BC}L\cos\beta - v_C\cos\theta)\,\hat{\jmath}. \end{aligned} \tag{1}$$

Also, because points A and B are both on bar AB, we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} = -\omega_{AB}H(\sin\gamma\,\hat{\imath} + \cos\gamma\,\hat{\jmath}). \tag{2}$$

Equating Eq. (1) with Eq. (2) component by component we have

$$v_C\sin\theta + \omega_{BC}L\sin\beta = -\omega_{AB}H\sin\gamma \tag{3}$$
$$\omega_{BC}L\cos\beta - v_C\cos\theta = -\omega_{AB}H\cos\gamma. \tag{4}$$

Eqs. (3) and (4) are two equations in ω_{AB} and ω_{BC}. To solve them we divide the left hand side of Eq. (3) by the left hand side of Eq. (4) and the right hand side of Eq. (3) by the right hand side of Eq. (4). We have

$$\frac{v_C\sin\theta + \omega_{BC}L\sin\beta}{\omega_{BC}L\cos\beta - v_C\cos\theta} = \tan\gamma \quad \Rightarrow \quad v_C\sin\theta + \omega_{BC}L\sin\beta = \tan\gamma(\omega_{BC}L\cos\beta - v_C\cos\theta)$$

$$\Rightarrow \quad \omega_{BC} = v_C\frac{\sin\theta + \tan\gamma\cos\theta}{L(\tan\gamma\cos\beta - \sin\beta)}.$$

Substituting in given values $v_C = 7\,\text{m/s}$, $\theta = 37°$, $\gamma = 78.71°$, $\beta = 25.07°$, and $L = 1\,\text{m}$, we have

$$\omega_{BC} = 22.09\,\text{rad/s}.$$

Solving Eq. (3) for ω_{AB} and substituting in known values $v_C = 7\,\text{m/s}$, $\omega_{BC} = 22.09\,\text{rad/s}$, $\theta = 37°$, $\gamma = 78.71°$, $\beta = 25.07°$, $H = 1.25\,\text{m}$, and $L = 1\,\text{m}$, we have

$$\omega_{AB} = \frac{v_C \sin\theta + \omega_{BC} L \sin\beta}{H \sin\gamma} = 13.36\,\text{rad/s}.$$

We can now begin to look at the acceleration analysis. For the slider we have

$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{\alpha}_{OC} \times \vec{r}_{C/O} - \omega_{OC}^2 \vec{r}_{C/O} \\ &= \alpha_{OC}\,\hat{k} \times R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) - \frac{v_C^2}{R^2} R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= \left(-R\alpha_{OC}\sin\theta - \frac{v_C^2}{R}\cos\theta\right)\hat{\imath} + \left(R\alpha_{OC}\cos\theta - \frac{v_C^2}{R}\sin\theta\right)\hat{\jmath}. \end{aligned} \tag{5}$$

Taking the magnitude of $\vec{a}_C$ and solving for α_{OC} we have

$$a_C^2 = \left(-R\alpha_{OC}\sin\theta - \frac{v_C^2}{R}\cos\theta\right)^2 + \left(R\alpha_{OC}\cos\theta - \frac{v_C^2}{R}\sin\theta\right)^2 \quad \Rightarrow \quad \alpha_{OC} = \pm\sqrt{\frac{a_C^2}{R^2} - \frac{v_C^4}{R^4}} \tag{6}$$

Because the speed is increasing, the angular acceleration must be clockwise, and we choose the negative solution for α_{OC}. Substituting in known values $R = 0.6\,\text{m}$, $L = 1\,\text{m}$, $H = 1.25\,\text{m}$, $\theta = 37°$, $\beta = 25.07°$, and $\gamma = 78.71°$ we have

$$\alpha_{OC} = -74.15\,\text{rad/s}^2.$$

From rigid body kinematics we have

$$\begin{aligned}\vec{a}_B &= \vec{a}_C + \vec{\alpha}_{BC} \times \vec{r}_{B/C} - \omega_{BC}^2 \vec{r}_{B/C} \\ &= \left(-L\omega_{BC}^2\cos\beta - \frac{v_C^2\cos\theta}{R} + L\alpha_{BC}\sin\beta - R\alpha_{OC}\sin\theta\right)\hat{\imath} \\ &\quad + \left(L\alpha_{BC}\cos\beta + R\alpha_{OC}\cos\theta + L\omega_{BC}^2\sin\beta - \frac{v_C^2\sin\theta}{R}\right)\hat{\jmath} \end{aligned} \tag{7}$$

and

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} \\ &= (H\omega_{AB}^2\cos\gamma - H\alpha_{AB}\sin\gamma)\,\hat{\imath} + (-H\alpha_{AB}\cos\gamma - H\omega_{AB}^2\sin\gamma)\,\hat{\jmath}. \end{aligned} \tag{8}$$

Setting Eqs. (7) and (8) equal and solving for α_{AB} and α_{BC} we have

$$\alpha_{AB} = -\frac{\csc(\beta-\gamma)}{HR}\left\{HR\omega_{AB}^2\cos(\beta-\gamma) + v_C^2\cos(\beta+\theta) + R\left[L\omega_{BC}^2 + R\alpha_{OC}\sin(\beta+\theta)\right]\right\},$$

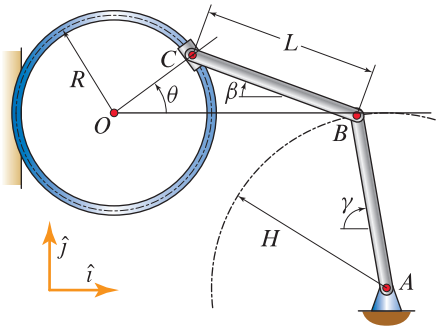
$$\alpha_{BC} = \frac{\csc(\beta-\gamma)}{LR}\left\{LR\omega_{BC}^2\cos(\beta-\gamma) + v_C^2\cos(\gamma+\theta) + R\left[H\omega_{AB}^2 + R\alpha_{OC}\sin(\gamma+\theta)\right]\right\}.$$

Substituting in known values as designated above, rounding to three significant figures, and putting in vector form, we have

$$\boxed{\vec{\alpha}_{AB} = -1090\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -1190\,\hat{k}\,\text{rad/s}^2.}$$

Solution to 6.108

To solve the problem we first assume that β and γ are known functions of θ (and of the geometric parameters R, L, and H). Then we proceed to determine β and γ as functions of θ.



Let $\vec{\alpha}_{AB} = \alpha_{AB}\,\hat{k}$ and $\vec{\alpha}_{BC} = \alpha_{BC}\,\hat{k}$ denote the angular accelerations of bars AB and BC, respectively, and therefore we have

$$\alpha_{AB} = -\ddot{\gamma} \quad \text{and} \quad \alpha_{BC} = -\ddot{\beta}. \tag{9}$$

Hence, the answer to the problem is found by providing expressions for $\ddot{\beta}$ and $\ddot{\gamma}$ as a function of the angle θ and its time derivatives. To do so, we observe that, using trigonometry, the motion of the mechanism is subject to the following two constraint equations:

$$L\sin\beta + H\sin\gamma = H + R\sin\theta, \tag{10}$$

$$L\cos\beta + H\cos\gamma = R + L - R\cos\theta. \tag{11}$$

Differentiating the above equations with respect to time, we have

$$L\dot{\beta}\cos\beta + H\dot{\gamma}\cos\gamma = R\dot{\theta}\cos\theta, \tag{12}$$

$$L\dot{\beta}\sin\beta + H\dot{\gamma}\sin\gamma = -R\dot{\theta}\sin\theta. \tag{13}$$

Equations (12) and (13) can be viewed as a system of 2 equations in the 2 unknowns $\dot{\beta}$ and $\dot{\gamma}$ whose solution is

$$\dot{\beta} = -\frac{R\dot{\theta}\sin(\gamma+\theta)}{L\sin(\beta-\gamma)} \quad \text{and} \quad \dot{\gamma} = \frac{R\dot{\theta}\sin(\beta+\theta)}{H\sin(\beta-\gamma)}, \tag{14}$$

where we have used the following trigonometric identities: $\sin(x+y) = \sin x\cos y + \cos x\sin y$ and $\sin(x-y) = \sin x\cos y - \cos x\sin y$. Differentiating the first of Eqs. (14) with respect to time we obtain

$$\begin{aligned}\ddot{\beta} &= -\frac{R\ddot{\theta}\sin(\gamma+\theta)}{L\sin(\beta-\gamma)} - \frac{R\dot{\theta}\big[(\dot{\gamma}+\dot{\theta})\cos(\gamma+\theta)\sin(\beta-\gamma) - (\dot{\beta}-\dot{\gamma})\sin(\gamma+\theta)\cos(\beta-\gamma)\big]}{L\sin^2(\beta-\gamma)} \\ &= -\frac{R\ddot{\theta}\sin(\gamma+\theta)}{L\sin(\beta-\gamma)} - \frac{R\dot{\theta}\big[\dot{\theta}\cos(\gamma+\theta)\sin(\beta-\gamma) + \dot{\gamma}\sin(\beta+\theta) - \dot{\beta}\sin(\gamma+\theta)\cos(\beta-\gamma)\big]}{L\sin^2(\beta-\gamma)},\end{aligned} \tag{15}$$

where we have used the trigonometric identity $\sin(x+y) = \sin x\cos y + \cos x\sin y$. Next, substituting Eqs. (14) into Eq. (15), simplifying, and recalling that $\alpha_{BC} = -\ddot{\beta}$, we have

$$\boxed{\alpha_{BC} = \frac{R\ddot{\theta}\sin(\gamma+\theta)}{L\sin(\beta-\gamma)} + \frac{R\dot{\theta}^2\big[\cos(\gamma+\theta)\sin^2(\beta-\gamma) + \frac{R}{H}\sin^2(\beta+\theta) + \frac{R}{L}\sin^2(\gamma+\theta)\cos(\beta-\gamma)\big]}{L\sin^3(\beta-\gamma)}.} \tag{16}$$

Repeating similar steps, i.e., differentiating with respect to time the second of Eqs. (14) to obtain an expression for $\ddot{\gamma}$ in terms of θ, $\dot{\theta}$, $\ddot{\theta}$, $\dot{\beta}$ and $\dot{\gamma}$, recalling that $\alpha_{AB} = -\ddot{\gamma}$, and finally using Eqs. (14) again to replace the expressions for $\dot{\beta}$ and $\dot{\gamma}$, after simplification, we have

$$\boxed{\alpha_{AB} = -\frac{R\ddot{\theta}\sin(\beta+\theta)}{H\sin(\beta-\gamma)} - \frac{R\dot{\theta}^2\big[\cos(\beta+\theta)\sin^2(\beta-\gamma) + \frac{R}{L}\sin^2(\gamma+\theta) + \frac{R}{H}\sin^2(\beta+\theta)\cos(\beta-\gamma)\big]}{H\sin^3(\beta-\gamma)}.} \tag{17}$$

We now need to provide β and γ as functions of θ. This can be done in (at least) two ways. The time interval of the needed plots is $0 \le t \le 1\,\text{s}$, which corresponds to $0 \le \theta \le 0.3\,\text{rad}$. Hence, we can use numerical software to obtain β and γ for $0 \le \theta \le 0.3\,\text{rad}$ and then plot the results. A second but challenging approach, is to find analytical expressions of β and γ as functions of θ. We will explore both approaches.

Determination of β and γ via a numerical approach. We begin by going back to Eqs. (10) and (11), which we now differentiate with respect to θ to obtain

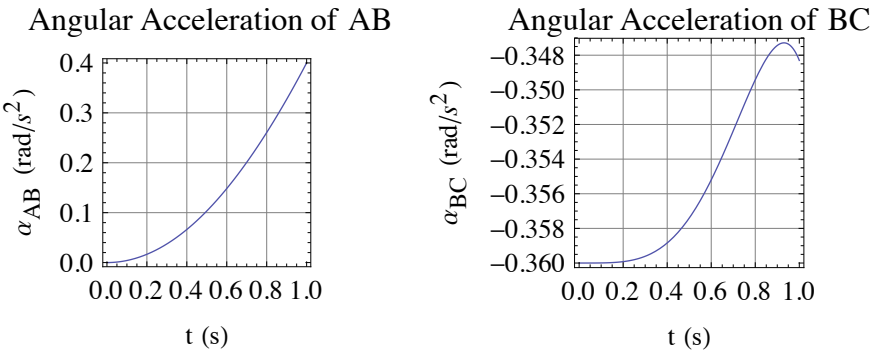
$$L\beta' \cos\beta + H\gamma' \cos\gamma = R\cos\theta, \tag{18}$$

$$L\beta' \sin\beta + H\gamma' \sin\gamma = -R\sin\theta, \tag{19}$$

where $\beta' = d\beta/d\theta$ and $\gamma' = d\gamma/d\theta$. The above equations can be solved for β' and γ' to obtain

$$\beta' = -\frac{R\sin(\gamma+\theta)}{L\sin(\beta-\gamma)} \quad \text{and} \quad \gamma' = \frac{R\sin(\beta+\theta)}{H\sin(\beta-\gamma)}. \tag{20}$$

Now observe that for $\theta = 0$, $\beta = 0$ and $\gamma = \pi/2$. Hence, we can obtain β and γ by numerically solving the system of first order differential equations in Eqs. (20) subject to the initial conditions $\beta(0) = 0$ and $\gamma(0) = \pi/2$. Once this is done, recalling that $H = 1.25\,\text{m}$, $L = 1\,\text{m}$, and $R = 0.6\,\text{m}$, $\theta = (0.3\,\text{rad/s}^2)t^2$, $\dot\theta = (0.6\,\text{rad/s}^2)t$, and $\ddot\theta = 0.6\,\text{rad/s}^2$, we can then plot the expressions for α_{AB} and α_{BC} found earlier. The plots in question are shown below.



These plots and the operations needed to generated them, can be obtained with appropriate mathematical software. The plots given below were obtained using Mathematica using the code below.

```
Parameters = {R → 0.6, L → 1., H → 1.25, ddθdtdt → 0.6, dθdt → 0.6 t, θ → 0.3 t^2};

αAB = -(R ddθdtdt Sin[β + θ])/(H Sin[β - γ]) - (R dθdt^2)/(H Sin[β - γ]^3) (Cos[β + θ] Sin[β - γ]^2 + R/L Sin[γ + θ]^2 + R/H Sin[β + θ]^2 Cos[β - γ]);

αBC = (R ddθdtdt Sin[γ + θ])/(L Sin[β - γ]) + (R dθdt^2)/(L Sin[β - γ]^3) (Cos[γ + θ] Sin[β - γ]^2 + R/H Sin[β + θ]^2 + R/L Sin[γ + θ]^2 Cos[β - γ]);

βγDifEqandICs = {β'[θ] == -R/L Sin[γ[θ] + θ]/Sin[β[θ] - γ[θ]], γ'[θ] == R/H Sin[β[θ] + θ]/Sin[β[θ] - γ[θ]], β[0] == 0, γ[0] == π/2} /.
    {R → 0.6, L → 1., H → 1.25};
βγSolution = NDSolve[βγDifEqandICs, {β, γ}, {θ, 0, 0.3}];
Plot[αAB /. β → (β[θ] /. βγSolution) /. γ → (γ[θ] /. βγSolution) /. Parameters, {t, 0, 1},
 Frame → True, GridLines → Automatic, AspectRatio → 1, FrameLabel → {"t (s)", "αAB (rad/s^2)"},
 PlotLabel → "Angular Acceleration of AB" ]
Plot[αBC /. β → (β[θ] /. βγSolution) /. γ → (γ[θ] /. βγSolution) /. Parameters, {t, 0, 1},
 Frame → True, GridLines → Automatic, AspectRatio → 1, FrameLabel → {"t (s)", "αBC (rad/s^2)"},
 PlotLabel → "Angular Acceleration of BC"]
```

Determination of β and γ via a purely analytical approach. To determine β and γ in terms of θ analytically, for convenience, we begin by dividing Eqs. (10) and (11) by L and then rewriting them as follows:

$$\sin\beta + \kappa\sin\gamma = A \quad \Rightarrow \quad \sin\beta = A - \kappa\sin\gamma \quad \text{and} \quad \kappa\sin\gamma = A - \sin\beta, \tag{21}$$

$$\cos\beta + \kappa\cos\gamma = B \quad \Rightarrow \quad \cos\beta = B - \kappa\cos\gamma \quad \text{and} \quad \kappa\cos\gamma = B - \cos\beta, \tag{22}$$

where the terms κ, A, and B are

$$\kappa = \frac{H}{L}, \quad A = \frac{H}{L} + \frac{R}{L}\sin\theta, \quad \text{and} \quad B = \frac{R}{L}(1-\cos\theta) + 1. \tag{23}$$

Squaring the second of Eqs. (21) and (22) and summing, we have

$$\sin^2\beta + \cos^2\beta = (A - \kappa\sin\gamma)^2 + (B - \kappa\cos\gamma)^2 \quad \Rightarrow \quad 1 = A^2 + B^2 + \kappa^2 - 2A\kappa\sin\gamma - 2B\kappa\cos\gamma, \tag{24}$$

where we have used the identity $\sin^2 x + \cos^2 x = 1$. The last of Eqs. (24) can then be rewritten as

$$A\sin\gamma + B\cos\gamma = \frac{A^2 + B^2 + \kappa^2 - 1}{2\kappa}. \tag{25}$$

We now introduce the following definition:

$$\tan\psi = \frac{B}{A} \quad \Rightarrow \quad \sin\psi = \frac{B}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \cos\psi = \frac{A}{\sqrt{A^2 + B^2}}, \tag{26}$$

where we observe that, since A and B are known functions of θ, the angle ψ is also a known function of θ. Then, dividing Eqs. (25) through by $\sqrt{A^2 + B^2}$, and using the last two of Eqs. (26), we have

$$\sin\gamma\cos\psi + \cos\gamma\sin\psi = \frac{A^2 + B^2 + \kappa^2 - 1}{2\kappa\sqrt{A^2 + B^2}} \quad \Rightarrow \quad \sin(\gamma + \psi) = \frac{A^2 + B^2 + \kappa^2 - 1}{2\kappa\sqrt{A^2 + B^2}}, \tag{27}$$

where we have used the following trigonometric identity: $\sin(x + y) = \sin x\cos y + \cos x\sin y$. Finally, solving Eq. (27) for γ and recalling that transcendental equations admit infinite solutions, we have

$$\gamma = \begin{cases} \sin^{-1}\left(\frac{A^2+B^2+\kappa^2-1}{2\kappa\sqrt{A^2+B^2}}\right) \pm 2n_1\pi - \tan^{-1}\left(\frac{B}{A}\right) \pm n_2\pi, \text{ or} \\ \pi - \sin^{-1}\left(\frac{A^2+B^2+\kappa^2-1}{2\kappa\sqrt{A^2+B^2}}\right) \pm 2n_1\pi - \tan^{-1}\left(\frac{B}{A}\right) \pm n_2\pi, \end{cases} \tag{28}$$

where $n_1 = 0, 1, 2, \ldots, n_2 = 0, 1, 2, \ldots$ and where it is understood that ranges of the inverse sine and inverse tangent functions are $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$, respectively. To select the solution that pertains to our problem, we observe that for $\theta = 0$, we must have $\gamma = \frac{\pi}{2}$ rad. In addition, we observe that, for $\theta = 0$, we have

$$\sin^{-1}\left(\frac{A^2 + B^2 + \kappa^2 - 1}{2\kappa\sqrt{A^2 + B^2}}\right)\bigg|_{\theta=0} = 0.8961\,\text{rad} \quad \text{and} \quad \tan^{-1}\left(\frac{B}{A}\right)\bigg|_{\theta=0} = 0.6630\,\text{rad}, \tag{29}$$

where we have used the definitions in Eq. (23) and the following numerical data: $H = 1.25\,\text{m}$, $L = 1\,\text{m}$, and $R = 0.6\,\text{m}$. Therefore, in order for us to obtain $\gamma = \frac{\pi}{2}\,\text{rad} = 1.571\,\text{rad}$ using the results in Eqs. (29) we must choose the solution for γ given by the lower expression in Eq. (28) and with $n_1 = n_2 = 0$. That is, the correct expression for γ in our problem is

$$\gamma = \pi - \sin^{-1}\left(\frac{A^2 + B^2 + \kappa^2 - 1}{2\kappa\sqrt{A^2 + B^2}}\right) - \tan^{-1}\left(\frac{B}{A}\right). \tag{30}$$

Proceeding in a similar manner, we can obtain an expression for the angle β. That is, squaring the last of Eqs. (21) and (22) and summing them we have

$$\kappa^2 = A^2 + B^2 + 1 - 2A\sin\beta - 2B\cos\beta \quad \Rightarrow \quad A\sin\beta + B\cos\beta = \frac{A^2 + B^2 + 1 - \kappa^2}{2}$$
$$\Rightarrow \ \sin\beta\cos\psi + \cos\beta\sin\psi = \frac{A^2 + B^2 + 1 - \kappa^2}{2\sqrt{A^2 + B^2}} \ \Rightarrow \ \sin(\beta + \psi) = \frac{A^2 + B^2 + 1 - \kappa^2}{2\sqrt{A^2 + B^2}}, \quad (31)$$

which can be solved for β to obtain

$$\beta = \begin{cases} \sin^{-1}\left(\frac{A^2+B^2+1-\kappa^2}{2\sqrt{A^2+B^2}}\right) \pm 2n_3\pi - \tan^{-1}\left(\frac{B}{A}\right) \pm n_4\pi. \\ \pi - \sin^{-1}\left(\frac{A^2+B^2+1-\kappa^2}{2\sqrt{A^2+B^2}}\right) \pm 2n_3\pi - \tan^{-1}\left(\frac{B}{A}\right) \pm n_4\pi. \end{cases} \quad (32)$$

Then, in order for us to obtain $\beta = 0$ when $\theta = 0$, we must choose the upper formula in Eq. (32) with $n_3 = n_4 = 0$. In summary, recalling the definitions in Eqs. (26), we have

$$\gamma = \pi - \sin^{-1}\left(\frac{H^2 + R(L+R) - R(L+R)\cos\theta + HR\sin\theta}{H\sqrt{(L+R-R\cos\theta)^2 + (H+R\sin\theta)^2}}\right) - \tan^{-1}\left(\frac{L+R-R\cos\theta}{H+R\sin\theta}\right),$$
$$\beta = \sin^{-1}\left(\frac{L^2 + R(L+R) - R(L+R)\cos\theta + HR\sin\theta}{L\sqrt{(L+R-R\cos\theta)^2 + (H+R\sin\theta)^2}}\right) - \tan^{-1}\left(\frac{L+R-R\cos\theta}{H+R\sin\theta}\right).$$

Combining the above equations with the expressions for α_{AB} and α_{BC} found earlier, and recalling that $H = 1.25\,\text{m}$, $L = 1\,\text{m}$, and $R = 0.6\,\text{m}$, $\theta = (0.3\,\text{rad/s}^2)t^2$, $\dot\theta = (0.6\,\text{rad/s}^2)t$, and $\ddot\theta = 0.6\,\text{rad/s}^2$, using mathematical software we obtain precisely the same plots given earlier. However, this time we have used the Mathematica code given below.

```
Parameters = {R → 0.6, L → 1., H → 1.25, ddθdtdt → 0.6, dθdt → 0.6 t, θ → 0.3 t^2};

αAB = -(R ddθdtdt Sin[β + θ])/(H Sin[β - γ]) - (R dθdt^2)/(H Sin[β - γ]^3) (Cos[β + θ] Sin[β - γ]^2 + R/L Sin[γ + θ]^2 + R/H Sin[β + θ]^2 Cos[β - γ]);

αBC = (R ddθdtdt Sin[γ + θ])/(L Sin[β - γ]) + (R dθdt^2)/(L Sin[β - γ]^3) (Cos[γ + θ] Sin[β - γ]^2 + R/H Sin[β + θ]^2 + R/L Sin[γ + θ]^2 Cos[β - γ]);

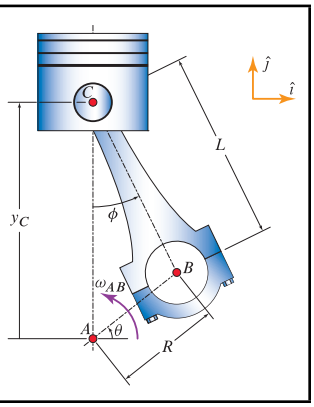
β = ArcSin[(L^2 + L R + R^2 - R (L + R) Cos[θ] + H R Sin[θ])/(L Sqrt[(L + R - R Cos[θ])^2 + (H + R Sin[θ])^2])] - ArcTan[(L + R - R Cos[θ])/(H + R Sin[θ])];

γ = π - ArcSin[(H^2 + R (L + R) - R (L + R) Cos[θ] + H R Sin[θ])/(H Sqrt[(L + R - R Cos[θ])^2 + (H + R Sin[θ])^2])] - ArcTan[(L + R - R Cos[θ])/(H + R Sin[θ])];

Plot[αAB /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic, AspectRatio → 1,
 FrameLabel → {"t (s)", "α_AB (rad/s^2)"}, PlotLabel → "Angular Acceleration of AB" ]
Plot[αBC /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic, AspectRatio → 1,
 FrameLabel → {"t (s)", "α_BC (rad/s^2)"}, PlotLabel → "Angular Acceleration of BC"]
```

Problem 6.109

Complete the acceleration analysis of the slider-crank mechanism using differentiation of constraints that was outlined beginning on p. 498. That is, determine the accleration of the piston C and the angular acceleration of the connecting rod as a function of the given quantities θ, ω_{AB}, R, and L. Assume that ω_{AB} is constant, and use the component system shown for your answers.



Solution

We have that the vertical distance from A to C, y_C is

$$y_C = R\sin\theta + L\cos\phi.$$

We also know that

$$L\sin\phi = R\cos\theta. \tag{1}$$

Hence, from the trigonometry of ϕ, y_C can be rewritten as

$$y_C = R\sin\theta + L\sqrt{1-\left(\frac{R}{L}\cos\theta\right)^2}.$$

Because C is constrained to move vertically, we know that

$$\vec{v}_C = \dot{y}_C\,\hat{j} = R\dot{\theta}\cos\theta\left[1+\frac{R\sin\theta}{L\sqrt{1-\frac{R^2\cos^2\theta}{L^2}}}\right]\hat{j}$$

and

$$\vec{a}_C = \left[L\left(\frac{R^2\cos^2\theta\dot{\theta}^2}{L^2\sqrt{1-\frac{R^2\cos\theta^2}{L^2}}}-\frac{R^4\cos^2\theta\sin^2\theta\dot{\theta}^2}{L^4\left(1-\frac{R^2\cos^2\theta}{L^2}\right)^{3/2}}\right)\right.$$
$$\left.-\left(\frac{R^2\sin\theta^2\dot{\theta}^2}{L^2\sqrt{1-\frac{R^2\cos^2\theta}{L^2}}}+\frac{R^2\cos\theta\sin\theta\ddot{\theta}}{L^2\sqrt{1-\frac{R^2\cos^2\theta}{L^2}}}\right)+R\left(-\sin\theta\dot{\theta}^2+\cos\theta\ddot{\theta}\right)\right]\hat{j}.$$

Furthermore, taking two time derivatives of Eq. (1), we have

$$L\dot{\phi}\cos\phi = -R\dot{\theta}\sin\theta \quad\Rightarrow\quad \dot{\phi} = -\frac{R\dot{\theta}\sin\theta}{L\cos\phi} \tag{2}$$

$$\ddot{\phi} = -\frac{R(\dot{\theta}^2\cos\theta+\ddot{\theta}\sin\theta)}{L\cos\phi}+\dot{\phi}^2\tan\phi. \tag{3}$$

Substituting Eq. (2) for $\dot{\phi}$ in Eq. (3), we have

$$\begin{aligned}\ddot{\phi} &= -\frac{R(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta)}{L\cos\phi} + \left(\frac{R\dot{\theta}\sin\theta}{L\cos\phi}\right)^2 \tan\phi \\ &= \frac{-LR\cos\phi(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta) + R^2\dot{\theta}^2\sin^2\theta\tan\phi}{L^2\cos^2\phi}. \end{aligned} \tag{4}$$

Realizing that $\dot{\theta} = \omega_{AB}$, $\ddot{\theta} = 0$, and from Eq. (1), $\phi = \sin^{-1}\left(\frac{R}{L}\cos\theta\right)$, Eq. (4) becomes

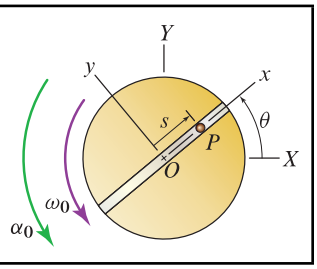
$$\ddot{\phi} = \frac{-LR\omega_{AB}^2\cos\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]\cos\theta + R^2\omega_{AB}^2\sin^2\theta\tan\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]}{L^2\cos^2\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]}.$$

Realizing that $\alpha_{BC} = \ddot{\phi}$, in vector form we have

$$\boxed{\vec{\alpha}_{BC} = \frac{-LR\omega_{AB}^2\cos\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]\cos\theta + R^2\omega_{AB}^2\sin^2\theta\tan\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]}{L^2\cos^2\left[\sin^{-1}\left(\frac{R}{L}\cos\theta\right)\right]}\hat{k}.}$$

Problem 6.110

Obtain the acceleration of point P using the information from the mini-example on p. 522; that is, obtain Eq. (6.71), using polar coordinates.



Solution

We let xy be a rotating reference frame with origin at O that rotates with the disc. For the acceleration of P, we have

$$\vec{a}_P = \vec{a}_O + \vec{a}_{P\text{rel}} + 2\vec{\Omega} \times \vec{v}_{P\text{rel}} + \dot{\vec{\Omega}} \times \vec{r}_{P/O} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{P/O}) \tag{1}$$

where

$$\vec{a}_O = \vec{0}, \quad \vec{r}_{P/O} = s\,\hat{\imath}, \quad \vec{v}_{P\text{rel}} = \dot{s}\,\hat{\imath}, \quad \vec{a}_{P\text{rel}} = \ddot{s}\,\hat{\imath}, \quad \vec{\Omega} = \omega_0\,\hat{k}, \quad \text{and} \quad \dot{\vec{\Omega}} = \alpha_0\,\hat{k}. \tag{2}$$

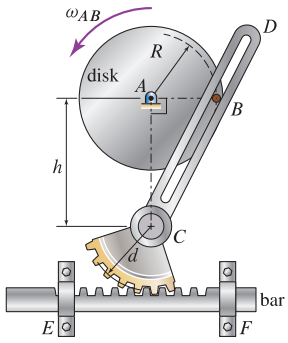
Substituting Eqs. (2) into Eq. (1), we have

$$\begin{aligned}
\vec{a}_P &= \ddot{s}\,\hat{\imath} + 2\omega_0\,\hat{k} \times \dot{s}\,\hat{\imath} + \alpha_0\,\hat{k} \times s\,\hat{\imath} + \omega_0\,\hat{k} \times \left(\omega_0\,\hat{k} \times s\,\hat{\imath}\right) \\
&= \ddot{s}\,\hat{\imath} + 2\omega_0\dot{s}\,\hat{\jmath} + \alpha_0 s\,\hat{\jmath} + \omega_0\,\hat{k} \times (s\omega_0)\,\hat{\jmath} \\
&= \ddot{s}\,\hat{\imath} + 2\omega_0\dot{s}\,\hat{\jmath} + \alpha_0 s\,\hat{\jmath} - s\omega_0^2\,\hat{\imath}
\end{aligned}$$

$$\boxed{\vec{a}_P = \left(\ddot{s} - s\omega_0^2\right)\hat{\imath} + (2\omega_0\dot{s} + \alpha_0 s)\,\hat{\jmath}.}$$

Problem 6.111

The *reciprocating rectilinear motion* mechanism consists of a disk pinned at its center at A that rotates with a constant angular velocity ω_{AB}, a slotted arm CD that is pinned at C, and a bar that can oscillate within the guides at E and F. As the disk rotates, the peg at B moves within the slotted arm, causing it to rock back and forth. As the arm rocks, it provides a slow advance and a quick return to the reciprocating bar due to the change in distance between C and B. For the position shown, determine

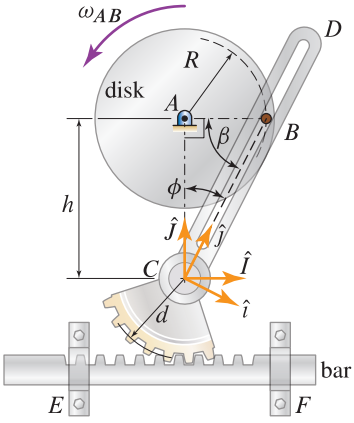


(a) The angular velocity of the slotted arm CD and the velocity of the bar

(b) The angular acceleration of the slotted arm CD and the acceleration of the bar

Evaluate your results for $\omega_{AB} = 120\,\text{rpm}$, $R/h = 0.5$, and $d = 0.12\,\text{m}$.

Solution



Part (a). Given a fixed reference frame XY and a rotating reference frame xy attached to the bar CD with origin at point C, we have the relationships

$$\hat{I} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath} \quad \text{and} \quad \hat{J} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}. \tag{1}$$

From rigid body kinematics, we have

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= \omega_{AB}\,\hat{k} \times R\,\hat{I} \\ &= R\omega_{AB}\,\hat{J} \\ &= R\omega_{AB}(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}).\end{aligned} \tag{2}$$

Relating point B to point C, we must also have

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B\text{rel}} + \vec{\Omega} \times \vec{r}_{B/C} = \dot{r}_{B/C}\,\hat{\jmath} + \omega_{CD}\,\hat{k} \times r_{B/C}\,\hat{\jmath} = -r_{B/C}\omega_{CD}\,\hat{\imath} + \dot{r}_{B/C}\,\hat{\jmath}, \tag{3}$$

where we have used the fact that $\vec{v}_C = \vec{0}$. Equating Eqs. (2) and (3) component by component, we have

$$\omega_{CD} = R\omega_{AB}\frac{\sin\phi}{r_{B/C}} = R\omega_{AB}\frac{\sin\phi}{h^2+R^2} = \frac{R^2\omega_{AB}}{h^2+R^2} = \frac{\delta^2\omega_{AB}}{1+\delta^2} \tag{4}$$

and

$$\dot{r}_{B/C} = R\omega_{AB}\cos\phi, \tag{5}$$

where we have let $\frac{R}{h} = \delta$. From the geometry of the problem at this instant, we have that

$$\tan\phi = \frac{R}{h}, \quad \sin\phi = \frac{R}{\sqrt{h^2+R^2}}, \quad \text{and} \quad \cos\phi = \frac{h}{\sqrt{h^2+R^2}}. \tag{6}$$

Substituting Eqs. (6) into Eq. (5) we have

$$\dot{r}_{B/C} = \frac{Rh\omega_{AB}}{\sqrt{h^2+R^2}}.$$

In vector form, for $\vec{\omega}_{CD}$, we have

$$\vec{\omega}_{CD} = \frac{\delta^2 \omega_{AB}}{1+\delta^2}\,\hat{k}. \tag{7}$$

Because there is no slip between arm CD and the bar, $v_{\text{bar}} = \omega_{CD} d$, and so we have

$$\vec{v}_{\text{bar}} = \frac{\delta^2 \omega_{AB}}{1+\delta^2} d\,\hat{I}. \tag{8}$$

Substituting the given values $\omega_{AB} = 120\,\text{rpm} = 12.57\,\text{rad/s}$, $\delta = 0.5$, and $d = 0.12\,\text{m}$ into Eqs. (7) and (8), we have

$$\boxed{\vec{\omega}_{CD} = 2.51\,\hat{k}\,\text{rad/s} \quad \text{and} \quad \vec{v}_{\text{bar}} = 0.302\,\hat{I}\,\text{m/s}.}$$

Part (b). Obtaining the acceleration of point B by relating it to point A, we must have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} = -R\omega_{AB}^2\,\hat{I} = -R\omega_{AB}^2(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}), \tag{9}$$

where we have used the fact that ω_{AB} is constant, so $\alpha_{AB} = 0$. Relating the acceleration of B to C using the rotating reference frame, for $\vec{a}_B$ we must also have

$$\begin{aligned}
\vec{a}_B &= \vec{a}_C + \vec{a}_{B\text{rel}} + 2\vec{\omega}_{CD} \times \vec{v}_{B\text{rel}} + \vec{\alpha}_{CD} \times \vec{r}_{B/C} - \omega_{CD}^2 \vec{r}_{B/C} \\
&= \ddot{r}_{B/C}\,\hat{\jmath} + 2\omega_{CD}\,\hat{k} \times \dot{r}_{B/C}\,\hat{\jmath} + \alpha_{CD}\,\hat{k} \times r_{B/C}\,\hat{\jmath} - \omega_{CD}^2 r_{B/C}\,\hat{\jmath} \\
&= \ddot{r}_{B/C}\,\hat{\jmath} + 2\frac{\delta^2\omega_{AB}}{1+\delta^2}\,\hat{k} \times \frac{Rh\omega_{AB}}{\sqrt{h^2+R^2}}\,\hat{\jmath} \\
&\qquad + \alpha_{CD}\,\hat{k} \times \sqrt{h^2+R^2}\,\hat{\jmath} - \left(\frac{\delta^2\omega_{AB}}{1+\delta^2}\right)^2 \sqrt{h^2+R^2}\,\hat{\jmath} \\
&= -\left[2\left(\frac{\delta^2\omega_{AB}}{1+\delta^2}\right)\left(\frac{Rh\omega_{AB}}{\sqrt{h^2+R^2}}\right) + \alpha_{CD}\sqrt{h^2+R^2}\right]\hat{\imath} \\
&\qquad + \left[\ddot{r}_{B/C} - \left(\frac{\delta^2\omega_{AB}}{1+\delta^2}\right)^2 \sqrt{h^2+R^2}\right]\hat{\jmath},
\end{aligned} \tag{10}$$

where we have substituted in for ω_{CD} from Eq. (4). Equating Eqs. (9) and (10), solving for α_{CD}, and substituting the relationships from Eqs. (6), we have

$$\begin{aligned}
-R\omega_{AB}^2\cos\phi &= -\left[2\left(\frac{\delta^2\omega_{AB}}{1+\delta^2}\right)\left(\frac{Rh\omega_{AB}}{\sqrt{h^2+R^2}}\right) + \alpha_{CD}\sqrt{h^2+R^2}\right] \\
\alpha_{CD} &= \frac{1}{\sqrt{h^2+R^2}}\left[R\omega_{AB}^2\cos\phi - 2\left(\frac{\delta^2\omega_{AB}}{1+\delta^2}\right)\left(\frac{Rh\omega_{AB}}{\sqrt{h^2+R^2}}\right)\right] \\
&= \frac{Rh(h^2-R^2)\omega_{AB}^2}{(h^2+R^2)^2} \\
&= \frac{\frac{R}{h}\left(1-\frac{R^2}{h^2}\right)\omega_{AB}^2}{\left(1+\frac{R^2}{h^2}\right)^2} \\
\Rightarrow \quad \vec{\alpha}_{CD} &= \frac{\delta\left(1-\delta^2\right)\omega_{AB}^2}{\left(1+\delta^2\right)^2}\,\hat{k}.
\end{aligned} \tag{11}$$

Because there is no slip between arm CD and the bar, $a_{\text{bar}} = d\alpha_{CD}$, and we can write

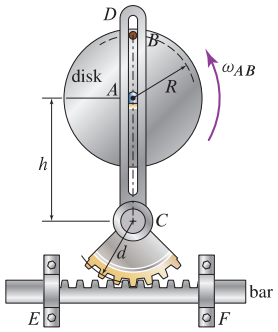
$$\vec{a}_{\text{bar}} = d\,\frac{\delta\left(1-\delta^2\right)\omega_{AB}^2}{\left(1+\delta^2\right)^2}\,\hat{k}. \tag{12}$$

Substituting known values $\omega_{AB} = 120\,\text{rpm} = 12.57\,\text{rad/s}$, $\delta = 0.5$, and $d = 0.12\,\text{m}$ into Eqs. (11) and (12), we have

$$\boxed{\vec{\alpha}_{CD} = 37.9\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{a}_{\text{bar}} = 4.55\,\hat{I}\,\text{m/s}^2.}$$

Problem 6.112

The *reciprocating rectilinear motion* mechanism consists of a disk pinned at its center at A that rotates with a constant angular velocity ω_{AB}, a slotted arm CD that is pinned at C, and a bar that can oscillate within the guides at E and F. As the disk rotates, the peg at B moves within the slotted arm, causing it to rock back and forth. As the arm rocks, it provides a slow advance and a quick return to the reciprocating bar due to the change in distance between C and B. For the position shown, determine

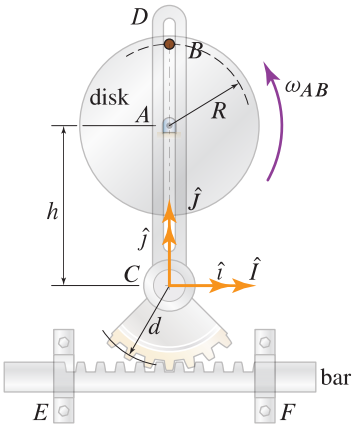


(a) The angular velocity of the slotted arm CD and the velocity of the bar

(b) The angular acceleration of the slotted arm CD and the acceleration of the bar

Evaluate your results for $\omega_{AB} = 90\,\text{rpm}$, $R/h = 0.5$, and $d = 0.12\,\text{m}$.

Solution



Part (a). Given a fixed coordinate system XY and a rotating coordinate system xy attached to the bar CD at point C, we have the relationships

$$\hat{I} = \hat{\imath} \quad \text{and} \quad \hat{J} = \hat{\jmath}. \tag{1}$$

From rigid body kinematics, we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} = \omega_{AB}\,\hat{k} \times R\,\hat{J} = -R\omega_{AB}\,\hat{I} = -R\omega_{AB}\,\hat{\imath} \tag{2}$$

Relating point B to point C, we must also have

$$\begin{aligned} \vec{v}_B &= \vec{v}_C + \vec{v}_{B\text{rel}} + \vec{\Omega} \times \vec{r}_{B/C} \\ &= \dot{r}_{B/C}\,\hat{\jmath} + \omega_{CD}\,\hat{k} \times r_{B/C}\,\hat{\jmath} \\ &= -r_{B/C}\omega_{CD}\,\hat{\imath} + \dot{r}_{B/C}\,\hat{\jmath}, \end{aligned} \tag{3}$$

where we have used the fact that $\vec{v}_C = \vec{0}$. Equating Eqs. (2) and (3) component by component, we have

$$\omega_{CD} = \frac{R\omega_{AB}}{r_{B/C}} = \frac{R}{h+R}\omega_{AB} = \frac{\delta}{1+\delta}\omega_{AB} \quad \text{and} \quad \dot{r}_{B/C} = 0, \tag{4}$$

where we have left $\delta = \frac{R}{h}$. In vector form, for $\vec{\omega}_{CD}$, we have

$$\vec{\omega}_{CD} = \frac{\delta}{1+\delta}\omega_{AB}\,\hat{k}. \tag{5}$$

Because there is no slip between arm CD and the bar, we have $v_{\text{bar}} = \omega_{CD}d$, and so

$$\vec{v}_{\text{bar}} = \frac{\delta}{1+\delta}\omega_{AB}d\,\hat{I}. \tag{6}$$

Substituting the given values $\omega_{AB} = 90\,\text{rpm} = 9.425\,\text{rad/s}$, $\delta = 0.5$, and $d = 0.12\,\text{m}$ into Eqs. (5) and (6), we have

$$\boxed{\vec{\omega}_{CD} = 7.07\,\hat{k}\,\text{rad/s} \quad \text{and} \quad \vec{v}_{\text{bar}} = 0.848\,\hat{I}\,\text{m/s}.}$$

Part (b). Obtaining the acceleration of point B by relating it to point A, we must have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} = -R\omega_{AB}^2 \, \hat{J} = -R\omega_{AB}^2 \, \hat{\jmath}. \tag{7}$$

Relating the acceleration of B to C using rotating reference frames, for $\vec{a}_B$ we must also have

$$\begin{aligned} \vec{a}_B &= \vec{a}_C + \vec{a}_{B\text{rel}} + 2\vec{\omega}_{CD} \times \vec{v}_{B\text{rel}} + \vec{\alpha}_{CD} \times \vec{r}_{B/C} - \omega_{CD}^2 \vec{r}_{B/C} \\ &= \ddot{r}_{B/C} \, \hat{\jmath} + 2\omega_{CD} \, \hat{k} \times \dot{r}_{B/C} \, \hat{\jmath} + \alpha_{CD} \, \hat{k} \times r_{B/C} \, \hat{\jmath} - \omega_{CD}^2 r_{B/C} \, \hat{\jmath} \\ &= \ddot{r}_{B/C} \, \hat{\jmath} + \alpha_{CD} \, \hat{k} \times (h + R) \, \hat{\jmath} - \left(\frac{\delta}{1+\delta} \omega_{AB} \right)^2 (h + R) \, \hat{\jmath}. \end{aligned} \tag{8}$$

Setting Eqs. (7) and (8) equal, and solving for α_{CD}, we have

$$R\omega_{AB}^2 \, \hat{\jmath} = \alpha_{CD}(h + R) \, \hat{\imath} + \left[\left(\frac{\delta}{1+\delta} \omega_{AB} \right)^2 (h + R) \right] \hat{\jmath} \quad \Rightarrow \quad \alpha_{CD}(h + R) = 0$$

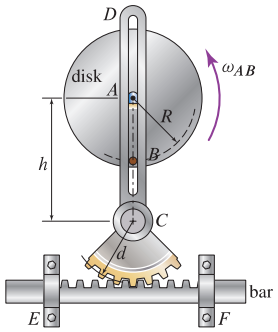
$$\Rightarrow \quad \boxed{\vec{\alpha}_{CD} = \vec{0} \, \text{rad/s}^2.}$$

Because there is no slip between arm CD and the bar, and the bar is confined to move in the horizontal direction, $\vec{a}_{\text{bar}} = d\alpha_{CD} \, \hat{I}$, so we must have

$$\boxed{\vec{a}_{\text{bar}} = \vec{0} \, \text{m/s}^2.}$$

Problem 6.113

The *reciprocating rectilinear motion* mechanism consists of a disk pinned at its center at A that rotates with a constant angular velocity ω_{AB}, a slotted arm CD that is pinned at C, and a bar that can oscillate within the guides at E and F. As the disk rotates, the peg at B moves within the slotted arm, causing it to rock back and forth. As the arm rocks, it provides a slow advance and a quick return to the reciprocating bar due to the change in distance between C and B. For the position shown, determine



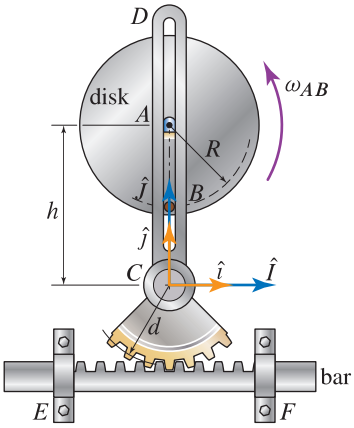
(a) The angular velocity of the slotted arm CD and the velocity of the bar

(b) The angular acceleration of the slotted arm CD and the acceleration of the bar

Evaluate your results for $\omega_{AB} = 60\,\text{rpm}$, $R/h = 0.5$, and $d = 0.12\,\text{m}$.

Solution

Part (a). Given a fixed coordinate system XY and a rotating coordinate system xy attached to the bar CD at point C, we have the relationships



$$\hat{I} = \hat{\imath} \quad \text{and} \quad \hat{J} = \hat{\jmath}. \tag{1}$$

From rigid body kinematics, we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} = \omega_{AB}\,\hat{k} \times (-R)\,\hat{J} = R\omega_{AB}\,\hat{I} = R\omega_{AB}\,\hat{\imath} \tag{2}$$

Relating point B to point A, we must also have

$$\begin{aligned} \vec{v}_B &= \vec{v}_C + \vec{v}_{B\text{rel}} + \vec{\Omega} \times \vec{r}_{B/C} \\ &= \dot{r}_{B/C}\,\hat{\jmath} + \omega_{CD}\,\hat{k} \times r_{B/C}\,\hat{\jmath} \\ &= -r_{B/C}\omega_{CD}\,\hat{\imath} + \dot{r}_{B/C}\,\hat{\jmath}. \end{aligned} \tag{3}$$

Equating Eqs. (2) and (3) component by component, we have

$$\omega_{CD} = \frac{R\omega_{AB}}{r_{B/C}} = \frac{R}{h-R}\omega_{AB} = \frac{\delta}{1-\delta}\omega_{AB} \quad \text{and} \quad \dot{r}_{B/C} = 0. \tag{4}$$

In vector form, for $\vec{\omega}_{CD}$, we have

$$\vec{\omega}_{CD} = \frac{\delta}{1-\delta}\omega_{AB}\,\hat{k}. \tag{5}$$

From the concept of instantaneous center of rotation, $v_{\text{bar}} = \omega_{CD} d$. Observing that if $\omega_{CD} > 0$ then $v_{\text{bar}} > 0$, we have

$$\vec{v}_{\text{bar}} = \frac{\delta}{1-\delta}\omega_{AB} d\,\hat{I}. \tag{6}$$

Substituting known values $\omega_{AB} = 60\,\text{rpm} = 6.283\,\text{rad/s}$, $\delta = 0.5$, and $d = 0.12\,\text{m}$ into Eqs. (5) and (6), we have

$$\boxed{\vec{\omega}_{CD} = 6.28\,\hat{k}\,\text{rad/s} \quad \text{and} \quad \vec{v}_{\text{bar}} = 0.754\,\hat{I}\,\text{m/s}.}$$

Part (b). Obtaining the acceleration of point B by relating it to point A, we must have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} = R\omega_{AB}^2 \, \hat{J} = R\omega_{AB}^2 \, \hat{\jmath}. \tag{7}$$

Using the idea of moving reference frames, for $\vec{a}_B$ we must also have

$$\begin{aligned} \vec{a}_B &= \vec{a}_C + \vec{a}_{B\text{rel}} + 2\vec{\omega}_{CD} \times \vec{v}_{B\text{rel}} + \vec{\alpha}_{CD} \times \vec{r}_{B/C} - \omega_{CD}^2 \vec{r}_{B/C} \\ &= \ddot{r}_{B/C} \, \hat{\jmath} + 2\omega_{CD} \, \hat{k} \times \dot{r}_{B/C} \, \hat{\jmath} + \alpha_{CD} \, \hat{k} \times r_{B/C} \, \hat{\jmath} - \omega_{CD}^2 r_{B/C} \, \hat{\jmath} \\ &= \ddot{r}_{B/C} \, \hat{\jmath} + \alpha_{CD} \, \hat{k} \times (h - R) \, \hat{\jmath} - \left(\frac{\delta}{1+\delta} \omega_{AB} \right)^2 (h - R) \, \hat{\jmath}. \end{aligned} \tag{8}$$

Setting Eqs. (7) and (8) equal, and solving for α_{CD}, we have

$$R\omega_{AB}^2 \, \hat{\jmath} = -\alpha_{CD}(h - R) \, \hat{\imath} + \left[\ddot{r}_{B/C} - \left(\frac{\delta}{1+\delta} \omega_{AB} \right)^2 (h - R) \right] \hat{\jmath} \quad \Rightarrow \quad -\alpha_{CD}(h - R) = 0$$

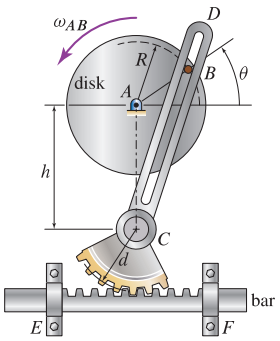
$$\Rightarrow \quad \boxed{\vec{\alpha}_{CD} = \vec{0} \, \text{rad/s}^2.}$$

Because there is no slip between the arm CD and the bar, and the bar is confined the move in the horizontal direction, $a_{\text{bar}} = d\alpha_{CD}$, so we must have

$$\boxed{\vec{a}_{\text{bar}} = \vec{0} \, \text{m/s}^2.}$$

Problem 6.114

The *reciprocating rectilinear motion* mechanism consists of a disk pinned at its center at A that rotates with a constant angular velocity ω_{AB}, a slotted arm CD that is pinned at C, and a bar that can oscillate within the guides at E and F. As the disk rotates, the peg at B moves within the slotted arm, causing it to rock back and forth. As the arm rocks, it provides a slow advance and a quick return to the reciprocating bar due to the change in distance between C and B. For the arbitrary position shown, determine

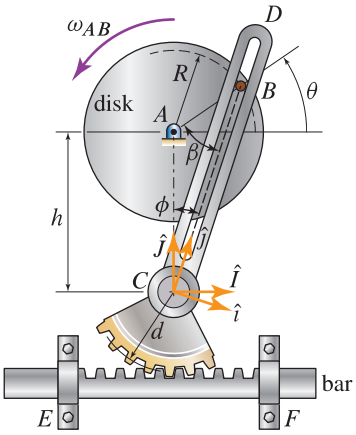


(a) The angular velocity of the slotted arm CD and the velocity of the bar

(b) The angular acceleration of the slotted arm CD and the acceleration of the bar

as functions of θ, $\delta = R/h$, d, and ω_{AB}

Solution



Part (a). Given a fixed coordinate system XY and a rotating coordinate system xy attached to the bar CD at point C, we have the relationships

$$\hat{I} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath} \quad \text{and} \quad \hat{J} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}. \tag{1}$$

From rigid body kinematics, we have

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\
&= \omega_{AB}\,\hat{k} \times R(\cos\theta\,\hat{I} + \sin\theta\,\hat{J}) \\
&= -R\omega_{AB}\sin\theta\,\hat{I} + R\omega_{AB}\cos\theta\,\hat{J} \\
&= -R\omega_{AB}\sin\theta(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) \\
&\quad + R\omega_{AB}\cos\theta(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) \\
&= -(R\omega_{AB}\sin\theta\cos\phi + R\omega_{AB}\cos\theta\sin\phi)\,\hat{\imath} + (R\omega_{AB}\cos\theta\cos\phi - R\omega_{AB}\sin\theta\sin\phi)\,\hat{\jmath}
\end{aligned} \tag{2}$$

Relating point B to point C, we must also have

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B\text{rel}} + \vec{\Omega} \times \vec{r}_{B/C} = \dot{r}_{B/C}\,\hat{\jmath} + \omega_{CD}\,\hat{k} \times r_{B/C}\,\hat{\jmath} = -r_{B/C}\omega_{CD}\,\hat{\imath} + \dot{r}_{B/C}\,\hat{\jmath}. \tag{3}$$

Equating Eqs. (2) and (3) component by component, from the $\hat{\imath}$ component we have

$$\begin{aligned}
-r_{B/C}\omega_{CD} &= -(R\omega_{AB}\sin\theta\cos\phi + R\omega_{AB}\cos\theta\sin\phi) \\
\omega_{CD} &= R\omega_{AB}\frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{r_{B/C}} \\
&= R\omega_{AB}\frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\sqrt{h^2 + R^2 + 2hR\sin\theta}}
\end{aligned} \tag{4}$$

where we have replaced $r_{B/C}$ by its equivalent expression from the Law of Cosines. Also from the geometry of the problem we have

$$\tan\phi = \frac{R\cos\theta}{h + R\sin\theta}, \text{ and we also have the trig identities}$$

$$\sin\phi = \frac{\tan\phi}{\sqrt{1+\tan^2\phi}} \quad \text{and} \quad \cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}}. \tag{5}$$

Substituting Eqs. (5) into Eq. (4) we have

$$\omega_{CD} = \frac{R\omega_{AB}(R + h\sin\theta)}{\sqrt{h^2+R^2+2hR\sin\theta}} = \frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta} \tag{6}$$

and

$$\dot{r}_{B/C} = R\omega_{AB}(-\sin\theta\sin\phi + \cos\theta\cos\phi). \tag{7}$$

From the geometry of the problem, we have that Substituting Eqs. (5) into Eq. (7) we have

$$\dot{r}_{B/C} = \frac{Rh\omega_{AB}\cos\theta}{\sqrt{h^2+R^2+2Rh\sin\theta}}.$$

In vector form, for $\vec{\omega}_{CD}$, we have

$$\boxed{\vec{\omega}_{CD} = \frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\,\hat{k}.}$$

Because the arm CD doesn't slip relative to the bar, we have $v_{\text{bar}} = \omega_{CD}d$, which gives us

$$\boxed{\vec{v}_{\text{bar}} = \frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}d\,\hat{I}.}$$

Part (b). Obtaining the acceleration of point B by relating it to point A, we must have

$$\begin{aligned}
\vec{a}_B &= \vec{a}_A + \vec{\alpha}_{AB}\times\vec{r}_{B/A} - \omega_{AB}^2\vec{r}_{B/A} \\
&= -R\omega_{AB}^2(\cos\theta\,\hat{I} + \sin\theta\,\hat{J}) \\
&= -R\omega_{AB}^2(\cos\theta\cos\phi - \sin\theta\sin\phi)\,\hat{\imath} - R\omega_{AB}(\cos\theta\sin\phi + \sin\theta\cos\phi)\,\hat{\jmath}.
\end{aligned} \tag{8}$$

Using the idea of moving reference frames, for $\vec{a}_B$ we must also have

$$\begin{aligned}
\vec{a}_B &= \vec{a}_C + \vec{a}_{B\text{rel}} + 2\vec{\omega}_{CD}\times\vec{v}_{B\text{rel}} + \vec{\alpha}_{CD}\times\vec{r}_{B/C} - \omega_{CD}^2\vec{r}_{B/C} \\
&= \ddot{r}_{B/C}\,\hat{\jmath} + 2\omega_{CD}\,\hat{k}\times\dot{r}_{B/C}\,\hat{\jmath} + \alpha_{CD}\,\hat{k}\times r_{B/C}\,\hat{\jmath} - \omega_{CD}^2 r_{B/C}\,\hat{\jmath} \\
&= \ddot{r}_{B/C}\,\hat{\jmath} + 2\left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)\hat{k}\times\frac{Rh\omega_{AB}\cos\theta}{\sqrt{h^2+R^2+2Rh\sin\theta}}\,\hat{\jmath} \\
&\quad + \alpha_{CD}\,\hat{k}\times\sqrt{h^2+R^2}\,\hat{\jmath} - \left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)^2\sqrt{h^2+R^2}\,\hat{\jmath} \\
&= \left[-2\left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)\frac{Rh\omega_{AB}\cos\theta}{\sqrt{h^2+R^2+2Rh\sin\theta}}\right]\hat{\imath} \\
&\quad + \left[\alpha_{CD}\sqrt{h^2+R^2} + \ddot{r}_{B/C} - \left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)^2\sqrt{h^2+R^2}\right]\hat{\jmath}.
\end{aligned} \tag{9}$$

Setting Eqs. (8) and (9) equal, and solving for α_{CD}, and substituting the relationships from Eqs. (5), we have

$$-R\omega_{AB}^2(\cos\theta\cos\phi - \sin\theta\sin\phi) = \left[-2\left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)\frac{Rh\omega_{AB}\cos\theta}{\sqrt{h^2+R^2+2Rh\sin\theta}}\right]$$

$$-R\omega_{AB}^2(\cos\theta\sin\phi+\sin\theta\cos\phi)=\left[\alpha_{CD}\sqrt{h^2+R^2}+\ddot{r}_{B/C}-\left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)^2\sqrt{h^2+R^2}\right]$$

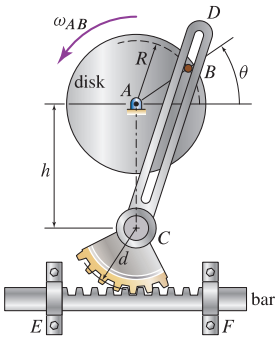
$$\alpha_{CD}=\frac{Rh(h^2-R^2)\omega_{AB}^2\cos\theta}{(h^2+R^2+2Rh\sin\theta^2}=\frac{\frac{R}{h}\left(1-\frac{R^2}{h^2}\right)\omega_{AB}^2\cos\theta}{\left(1+\frac{R^2}{h^2}+2\frac{R}{h}\sin\theta\right)^2}\quad\Rightarrow\quad\boxed{\vec{\alpha}_{CD}=\frac{\delta\left(1-\delta^2\right)\omega_{AB}^2\cos\theta}{\left(1+\delta^2+2\delta\sin\theta\right)^2}\hat{k}.}$$

Because the arm CD doesn't slip relative to the bar, we have $a_{\text{bar}}=d\alpha_{CD}$, and so we have

$$\boxed{\vec{a}_{\text{bar}}=d\frac{\delta\left(1-\delta^2\right)\omega_{AB}^2\cos\theta}{\left(1+\delta^2+2\delta\sin\theta\right)^2}\hat{k}.}$$

Problem 6.115

The *reciprocating rectilinear motion* mechanism consists of a disk pinned at its center at A that rotates with a constant angular velocity ω_{AB}, a slotted arm CD that is pinned at C, and a bar that can oscillate within the guides at E and F. As the disk rotates, the peg at B moves within the slotted arm, causing it to rock back and forth. As the arm rocks, it provides a slow advance and a quick return to the reciprocating bar due to the change in distance between C and B. Determine the angular velocity and angular acceleration of the slotted arm CD as functions of θ, $\delta = R/h$, d, and ω_{AB}. After doing so, plot the velocity and acceleration of the bar as a function of the disk angle θ for one full cycle of the disk's motion and for $\omega_{AB} = 90$ rpm, $d = 0.12$ m, and



(a) $\delta = R/h = 0.1$

(b) $\delta = R/h = 0.3$

(c) $\delta = R/h = 0.6$

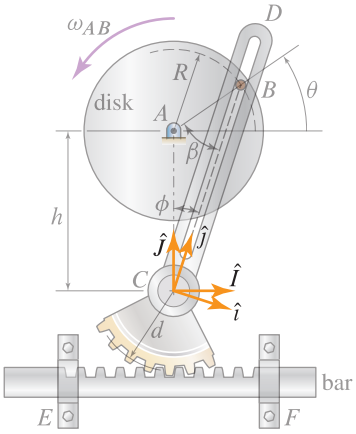
(d) $\delta = R/h = 0.9$

Explain why this mechanism is often referred to as a *quick return mechanism*.

Solution

Given a fixed coordinate system XY and a rotating coordinate system xy attached to the bar CD at point C, we have the relationships

$$\hat{I} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}, \quad \hat{J} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}, \quad \text{and} \quad \hat{k} = \hat{K}. \tag{1}$$



From rigid body kinematics, we have

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\
&= \omega_{AB}\,\hat{k} \times R(\cos\theta\,\hat{I} + \sin\theta\,\hat{J}) \\
&= -R\omega_{AB}\sin\theta\,\hat{I} + R\omega_{AB}\cos\theta\,\hat{J} \\
&= -R\omega_{AB}\sin\theta(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) \\
&\quad + R\omega_{AB}\cos\theta(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) \\
&= -(R\omega_{AB}\sin\theta\cos\phi + R\omega_{AB}\cos\theta\sin\phi)\,\hat{\imath} \\
&\quad + (R\omega_{AB}\cos\theta\cos\phi - R\omega_{AB}\sin\theta\sin\phi)\,\hat{\jmath}.
\end{aligned} \tag{2}$$

Relating point B to point C, we must also have

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B\text{rel}} + \vec{\Omega} \times \vec{r}_{B/C} = \dot{r}_{B/C}\,\hat{\jmath} + \omega_{CD}\,\hat{k} \times r_{B/C}\,\hat{\jmath} = -r_{B/C}\omega_{CD}\,\hat{\imath} + \dot{r}_{B/C}\,\hat{\jmath}, \tag{3}$$

where we have used the fact that $\vec{v}_C = \vec{0}$. Equating the x components of Eqs. (2) and (3), we have

$$-r_{B/C}\omega_{CD} = -(R\omega_{AB}\sin\theta\cos\phi + R\omega_{AB}\cos\theta\sin\phi)$$

$$\omega_{CD} = R\omega_{AB}\frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{r_{B/C}}$$

$$= R\omega_{AB}\frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\sqrt{h^2 + R^2 + 2hR\sin\theta}}. \tag{4}$$

where we have replaced $r_{B/C}$ by its equivalent expression from the law of cosines. Also from the geometry of the problem we have

$$\tan\phi = \frac{R\cos\theta}{h + R\sin\theta}, \quad \sin\phi = \frac{\tan\phi}{\sqrt{1+\tan^2\phi}}, \quad \text{and} \quad \cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}}. \tag{5}$$

Substituting Eqs. (5) into Eq. (4) we have

$$\omega_{CD} = \frac{R\omega_{AB}(R + h\sin\theta)}{\sqrt{h^2 + R^2 + 2hR\sin\theta}} = \frac{\delta\omega_{AB}(\delta + \sin\theta)}{1 + \delta^2 + 2\delta\sin\theta} \tag{6}$$

and

$$\dot{r}_{B/C} = R\omega_{AB}(-\sin\theta\sin\phi + \cos\theta\cos\phi). \tag{7}$$

Substituting Eqs. (5) into Eq. (7) we have

$$\dot{r}_{B/C} = \frac{Rh\omega_{AB}\cos\theta}{\sqrt{h^2 + R^2 + 2Rh\sin\theta}}.$$

In vector form, for $\vec{\omega}_{CD}$, we have

$$\boxed{\vec{\omega}_{CD} = \frac{\delta\omega_{AB}(\delta + \sin\theta)}{1 + \delta^2 + 2\delta\sin\theta}\,\hat{K}}$$



Because the arm CD does not slip relative to the bar, we have $v_{\text{bar}} = \omega_{CD}d$, which gives us

$$\boxed{\vec{v}_{\text{bar}} = \frac{\delta\omega_{AB}(\delta + \sin\theta)}{1 + \delta^2 + 2\delta\sin\theta}d\,\hat{I}}$$



Obtaining the acceleration of point B by relating it to point A, we must have

$$\begin{aligned}
\vec{a}_B &= \vec{a}_A + \vec{\alpha}_{AB}\times\vec{r}_{B/A} - \omega_{AB}^2\vec{r}_{B/A} \\
&= -R\omega_{AB}^2(\cos\theta\,\hat{I} + \sin\theta\,\hat{J}) \\
&= -R\omega_{AB}^2(\cos\theta\cos\phi - \sin\theta\sin\phi)\,\hat{\imath} - R\omega_{AB}^2(\cos\theta\sin\phi + \sin\theta\cos\phi)\,\hat{\jmath}.
\end{aligned} \tag{8}$$

Using the idea of moving reference frames, for $\vec{a}_B$ we must also have

$$\begin{aligned}
\vec{a}_B &= \vec{a}_C + \vec{a}_{B\text{rel}} + 2\vec{\omega}_{CD}\times\vec{v}_{B\text{rel}} + \vec{\alpha}_{CD}\times\vec{r}_{B/C} - \omega_{CD}^2\vec{r}_{B/C} \\
&= \ddot{r}_{B/C}\,\hat{\jmath} + 2\omega_{CD}\,\hat{k}\times\dot{r}_{B/C}\,\hat{\jmath} + \alpha_{CD}\,\hat{k}\times r_{B/C}\,\hat{\jmath} - \omega_{CD}^2 r_{B/C}\,\hat{\jmath} \\
&= \ddot{r}_{B/C}\,\hat{\jmath} + 2\left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)\hat{k}\times\frac{Rh\omega_{AB}\cos\theta}{\sqrt{h^2+R^2+2Rh\sin\theta}}\,\hat{\jmath} \\
&\quad + \alpha_{CD}\,\hat{k}\times\sqrt{h^2+R^2}\,\hat{\jmath} - \left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)^2\sqrt{h^2+R^2}\,\hat{\jmath} \\
&= \left[-2\left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)\frac{Rh\omega_{AB}\cos\theta}{\sqrt{h^2+R^2+2Rh\sin\theta}}\right]\hat{\imath} \\
&\quad + \left[\alpha_{CD}\sqrt{h^2+R^2} + \ddot{r}_{B/C} - \left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)^2\sqrt{h^2+R^2}\right]\hat{\jmath}.
\end{aligned} \tag{9}$$

Setting Eqs. (8) and (9) equal, and solving for α_{CD}, and substituting the relationships from Eqs. (5), we have

$$-R\omega_{AB}^2(\cos\theta\cos\phi-\sin\theta\sin\phi)=\left[-2\left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)\frac{Rh\omega_{AB}\cos\theta}{\sqrt{h^2+R^2+2Rh\sin\theta}}\right]$$

$$-R\omega_{AB}^2(\cos\theta\sin\phi+\sin\theta\cos\phi)=\left[\alpha_{CD}\sqrt{h^2+R^2}+\ddot{r}_{B/C}-\left(\frac{\delta\omega_{AB}(\delta+\sin\theta)}{1+\delta^2+2\delta\sin\theta}\right)^2\sqrt{h^2+R^2}\right]$$

$$\alpha_{CD}=\frac{Rh(h^2-R^2)\omega_{AB}^2\cos\theta}{(h^2+R^2+2Rh\sin\theta^2}=\frac{\frac{R}{h}\left(1-\frac{R^2}{h^2}\right)\omega_{AB}^2\cos\theta}{\left(1+\frac{R^2}{h^2}+2\frac{R}{h}\sin\theta\right)^2}\quad\Rightarrow\quad\boxed{\vec{\alpha}_{CD}=\frac{\delta\left(1-\delta^2\right)\omega_{AB}^2\cos\theta}{\left(1+\delta^2+2\delta\sin\theta\right)^2}\hat{K}.}$$

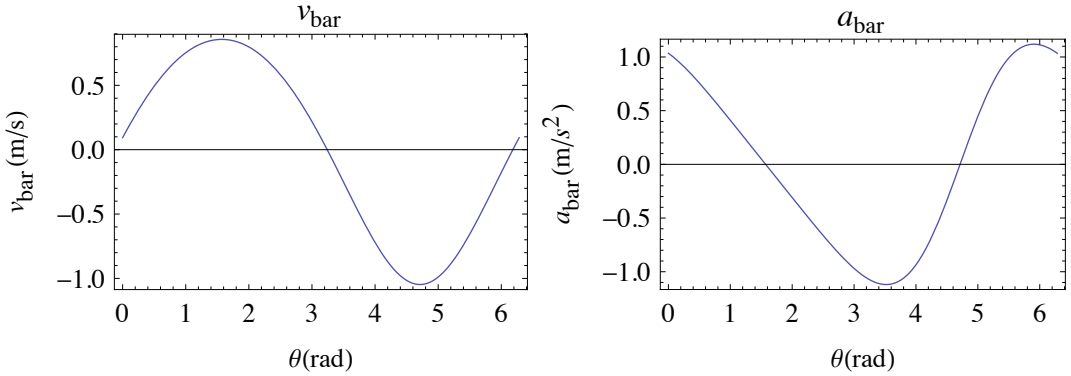
Because the arm CD does not slip relative to the bar, we have $a_{\text{bar}}=d\alpha_{CD}$, and so we have

$$\boxed{\vec{a}_{\text{bar}}=d\,\frac{\delta\left(1-\delta^2\right)\omega_{AB}^2\cos\theta}{\left(1+\delta^2+2\delta\sin\theta\right)^2}\hat{I}\ \hat{j}\ \hat{i}\,.}$$

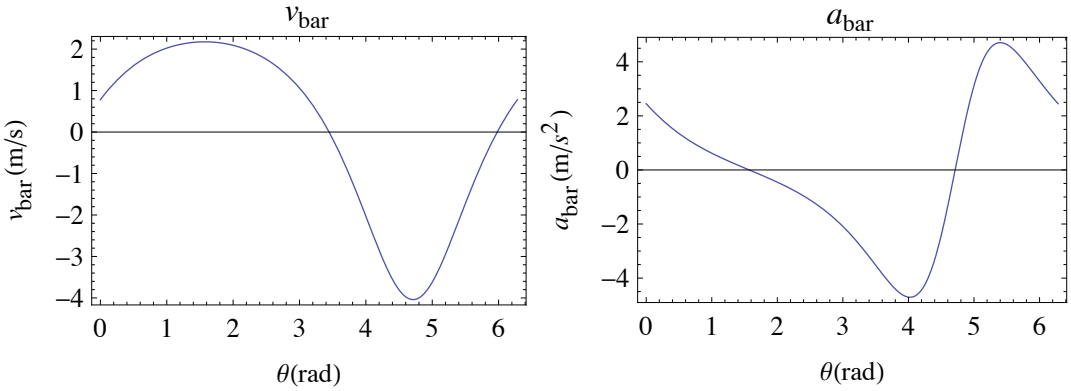
The expressions for $\vec{v}_{\text{bar}}$ and $\vec{a}_{\text{bar}}$ just derived is a function of θ when ω_{AB}, d, and δ are specified. For each of the several values of δ, the function can be plotted with software such as *Mathematica* or MATLAB. Using *Mathematica*, the following code was used, with the value for δ changed for each part:

```
Parameters = {δ → 0.1, d → 0.12, ωAB → 9.425};
vbar = δ ωAB (δ + Sin[θ]) / (1 + δ^2 + 2 δ Sin[θ]) /. Parameters;
abar = d δ (1 - δ^2) ωAB^2 Cos[θ] / (1 + δ^2 + 2 δ Sin[θ])^2 /. Parameters;
Plot[vbar, {θ, 0, 2π}, PlotLabel → "vbar", Frame → True,
 FrameLabel → {"θ(rad)", "vbar (m/s)"}, ImageSize → Medium]
Plot[abar, {θ, 0, 2π}, PlotLabel → "abar", Frame → True,
 FrameLabel → {"θ(rad)", "abar (m/s^2)"}, ImageSize → Medium]
```

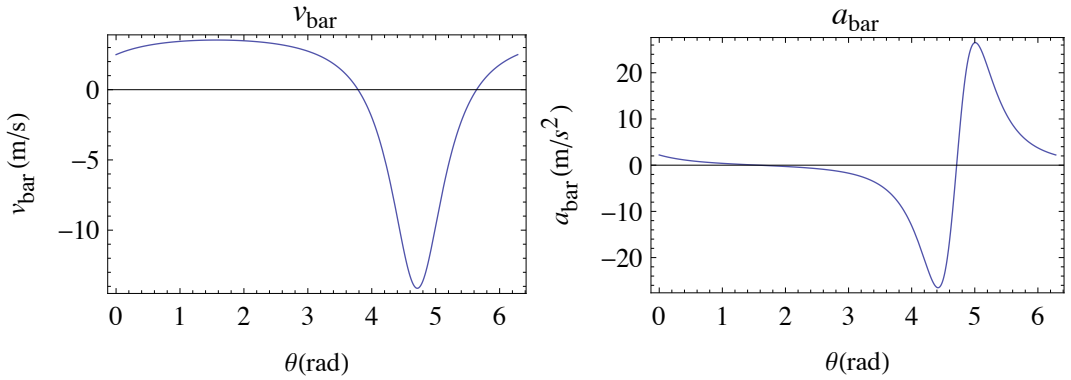
Part (a). We let $\delta = 0.1$. Using the above code we obtain the following plots:



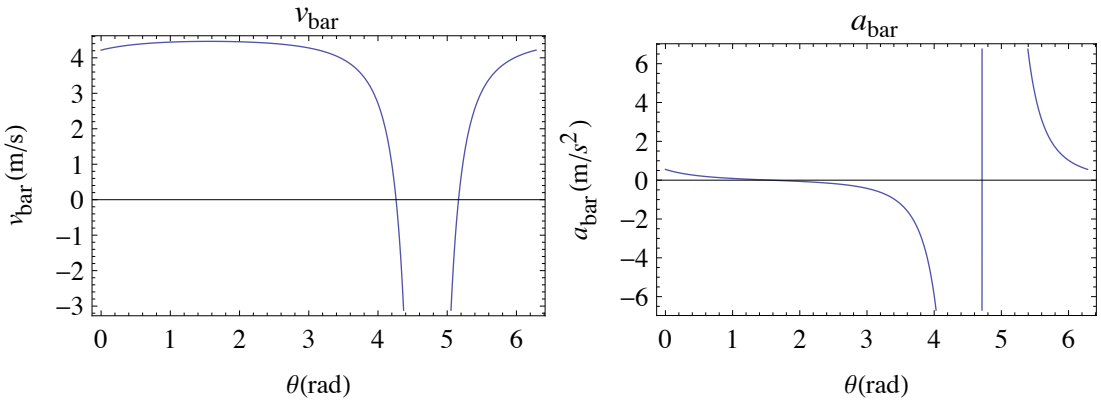
Part (b). We let $\delta = 0.3$. Using the above code we obtain the following plots:



Part (c). We let $\delta = 0.6$. Using the above code we obtain the following plots:

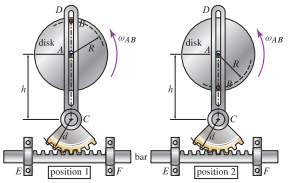


Part (d). We let $\delta = 0.9$. Using the above code we obtain the following plots:


v_{bar}
v_{bar} (m/s)
θ(rad)
a_{bar}
a_{bar} (m/s^2)
θ(rad)

Problem 6.116

The *reciprocating rectilinear motion* mechanism of Probs. 6.111–6.115 is often referred to as a *quick return mechanism* since it can move much more quickly in one direction than the other. To see this, we will determine the velocity of the bar in each of the two positions shown (position 1 when B is farthest from C and position 2 when B is closest to C) under the assumption that the disk, whose center is at A, rotates with a constant angular velocity $\omega_{AB} = 120$ rpm. Find the velocity of the bar in positions 1 and 2 for $d = 0.12$ m and for



(a) $R/h = 0.3$

(b) $R/h = 0.8$

Comment on which of the two R/h values would provide a better quick return and why.

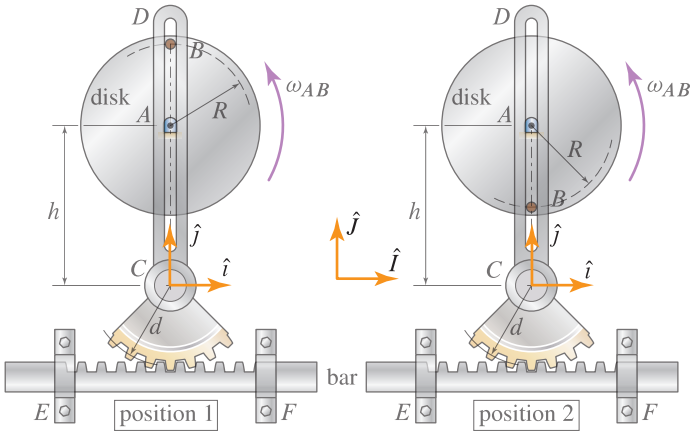
Solution

Position 1 Given a fixed coordinate system XY and a rotating coordinate system xy attached to the bar CD at point C, we have the relationships

$$\hat{I} = \hat{\imath}, \quad \hat{J} = \hat{\jmath}, \quad \text{and} \quad \hat{K} = \hat{k}.$$

From rigid body kinematics, we have

$$\begin{aligned} \vec{v}_B &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= \omega_{AB}\,\hat{k} \times R\,\hat{J} \\ &= -R\omega_{AB}\,\hat{I} \\ &= -R\omega_{AB}\,\hat{\imath} \end{aligned} \qquad (1)$$



Relating point B to point C, we must also have

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B\text{rel}} + \vec{\Omega} \times \vec{r}_{B/C} = \dot{r}_{B/C}\,\hat{\jmath} + \omega_{CD}\,\hat{k} \times r_{B/C}\,\hat{\jmath} = -r_{B/C}\omega_{CD}\,\hat{\imath} + \dot{r}_{B/C}\,\hat{\jmath}, \qquad (2)$$

where we have used the fact that $\vec{v}_C = \vec{0}$. Equating Eqs. (1) and (2) component by component, we have

$$\omega_{CD} = \frac{R\omega_{AB}}{r_{B/C}} = \frac{R}{h+R}\omega_{AB} = \frac{\delta}{1+\delta}\omega_{AB} \quad \text{and} \quad \dot{r}_{B/C} = 0, \qquad (3)$$

where we have let $\delta = \frac{R}{h}$. In vector form, for $\vec{\omega}_{CD}$, we have

$$\vec{\omega}_{CD} = \frac{\delta}{1+\delta}\omega_{AB}\,\hat{k}.$$

Because there is no slip between arm CD and the bar, we have $v_{\text{bar}} = \omega_{CD}d$, and so

$$\boxed{\vec{v}_{\text{bar}} = \frac{\delta}{1+\delta}\omega_{AB}d\,\hat{I}}$$

Position 2 Given a fixed coordinate system XY and a rotating coordinate system xy attached to the bar CD at point C, we have the relationships

$$\hat{I} = \hat{\imath}, \quad \hat{J} = \hat{\jmath} \quad \text{and} \quad \hat{K} = \hat{k}.$$

From rigid body kinematics, we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} = \omega_{AB}\,\hat{k} \times (-R)\,\hat{J} = R\omega_{AB}\,\hat{I} = R\omega_{AB}\,\hat{\imath} \tag{4}$$

Relating point B to point A, we must also have

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B\text{rel}} + \vec{\Omega} \times \vec{r}_{B/C} = \dot{r}_{B/C}\,\hat{\jmath} + \omega_{CD}\,\hat{k} \times r_{B/C}\,\hat{\jmath} = -r_{B/C}\omega_{CD}\,\hat{\imath} + \dot{r}_{B/C}\,\hat{\jmath}. \tag{5}$$

Equating Eqs. (4) and (5) component by component, we have

$$\omega_{CD} = \frac{R\omega_{AB}}{r_{B/C}} = \frac{R}{h-R}\omega_{AB} = \frac{\delta}{1-\delta}\omega_{AB} \quad \text{and} \quad \dot{r}_{B/C} = 0.$$

In vector form, for $\vec{\omega}_{CD}$, we have

$$\vec{\omega}_{CD} = \frac{\delta}{1-\delta}\omega_{AB}\,\hat{k}.$$

From the concept of instantaneous center of rotation, $v_{\text{bar}} = \omega_{CD} d$. Observing that if $\omega_{CD} > 0$ then $v_{\text{bar}} > 0$, we have

$$\boxed{\vec{v}_{\text{bar}} = \frac{\delta}{1-\delta}\omega_{AB} d\,\hat{I}}$$



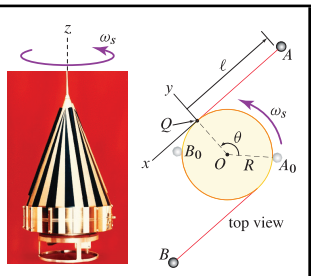
Part (a) For $\frac{R}{h} = 0.3$ and $\omega_{AB} = 12.57\,\text{rad/s}$, we have $\boxed{v_{\text{bar}} = 2.90\,\text{m/s}}$ at position 1 and $\boxed{v_{\text{bar}} = 5.39\,\text{m/s}}$ at position 2.

Part (b) For $\frac{R}{h} = 0.8$ and $\omega_{AB} = 12.57\,\text{rad/s}$, we have $\boxed{v_{\text{bar}} = 5.59\,\text{m/s}}$ at position 1 and $\boxed{v_{\text{bar}} = 50.3\,\text{m/s}}$ at position 2.

The bar is moving through the return phase when it is in position 2. Therefore, the $\frac{R}{h}$ value from Part (b) would provide a better quick return because it is moving back into position faster than the $\frac{R}{h}$ value from Part (a).

Problem 6.117

The Pioneer 3 spacecraft was a spin-stabilized spacecraft launched on December 6, 1958, by the U.S. Army Ballistic Missile agency in conjunction with NASA. It was designed with a despin mechanism consisting of two equal masses A and B that could be spooled out to the end of two wires of variable length $\ell(t)$ when triggered by a hydraulic timer. As a prelude to Probs. 7.64, 7.65, and 8.77, we will find the velocity and acceleration of each the two masses. To do this, assume that masses A and B are initially at positions A_0 and B_0, respectively. After the masses are released, they begin to unwind symmetrically, and the length of the cord attaching each mass to the spacecraft of radius R is $\ell(t)$. Given that the angular velocity of the spacecraft at each instant is $\omega_s(t)$, determine



(a) The velocity of mass A

(b) The acceleration of mass A

in components expressed in the rotating reference frame whose origin is at Q, as well as R, $\ell(t)$, and $\omega_s(t)$. Note that the rotating frame is always aligned with the unwinding cord, and Q is the point on the cord that is about to unwind at time t.

Solution

We let θ be the unwind angle of the rope relative to the satellite. As stated, we will use a rotating reference frame attached at Q, as indicated on the diagram. For the velocity of A, we have

$$\vec{v}_A = \vec{v}_Q + \dot{\vec{\rho}}_{\text{rel}} + \vec{\omega}_{OQ} \times \vec{\rho}_{A/Q} \tag{1}$$

where

$$\vec{v}_Q = (\omega_S + \dot{\theta}) R\,\hat{\imath},\ \dot{\vec{\rho}}_{\text{rel}} = -\dot{\ell}\,\hat{\imath},\ \vec{\omega}_{OQ} = -(\omega_S + \dot{\theta})\,\hat{k}, \vec{\rho}_{A/Q} = -\ell\,\hat{\imath},\ \text{and } \theta = \frac{\ell}{R} \quad \Rightarrow \quad \dot{\theta} = \frac{\dot{\ell}}{R}. \tag{2}$$

Substituting Eqs (2) into Eq. (1), we have

$$\vec{v}_A = R\left(\omega_S + \frac{\dot{\ell}}{R}\right)\hat{\imath} - \dot{\ell}\,\hat{\imath} - \left(\omega_S + \frac{\dot{\ell}}{R}\right)\hat{k} \times (-\ell)\,\hat{\imath}$$

$$\Rightarrow \qquad \boxed{\vec{v}_A = R\omega_S(t)\,\hat{\imath} + \ell(t)\left(\omega_S(t) + \frac{\dot{\ell}(t)}{R}\right)\hat{\jmath}.} \tag{3}$$

For the acceleration we have

$$\vec{a}_A = \vec{a}_Q + \ddot{\vec{\rho}}_{\text{rel}} + 2\vec{\omega}_{OQ} \times \dot{\vec{\rho}}_{\text{rel}} + \vec{\alpha}_{OQ} \times \vec{\rho}_{A/Q} - \omega_{OQ}^2 \vec{\rho}_{A/Q}, \tag{4}$$

where

$$\vec{a}_Q = -R(\omega_S + \dot{\theta})^2\,\hat{\jmath} + R(\dot{\omega}_S + \ddot{\theta})\,\hat{\imath}, \quad \ddot{\vec{\rho}}_{\text{rel}} = -\ddot{\ell}\,\hat{\imath}, \quad \vec{\alpha}_{OQ} = -(\dot{\omega}_S + \ddot{\theta}), \quad \text{and} \quad \ddot{\theta} = \frac{\ddot{\ell}}{R}\hat{k} \tag{5}$$

Substituting Eqs. (5) into Eq. (4), we have

$$\vec{a}_A = -R(\omega_S + \dot{\theta})^2\,\hat{\jmath} + R(\dot{\omega}_S + \ddot{\theta})\,\hat{\imath} - \ddot{\ell}\,\hat{\imath} - 2(\omega_S + \dot{\theta})\,\hat{k} \times (-\dot{\ell}\,\hat{\imath}) - (\dot{\omega}_S + \ddot{\theta})\,\hat{k} \times (-\ell\,\hat{\imath}) - (\omega_S + \dot{\theta})^2(-\ell\,\hat{\imath})$$

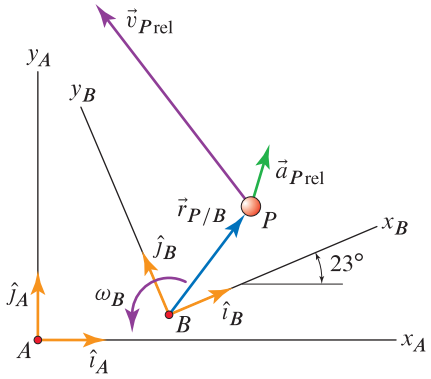
$$\boxed{\vec{a}_A = \left[\frac{\ell(t)}{R^2}(\dot{\ell}(t) + R\omega_S(t))^2 + R\dot{\omega}_S(t)\right]\hat{\imath} + \left[\frac{\ell(t)}{R}\ddot{\ell}(t) + \frac{\dot{\ell}(t)^2}{R} - R\omega_S(t)^2 + \ell\dot{\omega}_S(t)\right]\hat{\jmath}.}$$

Problem 6.118

Let frames A and B be the frames with origins at points A and B, respectively. Point B does not move relative to point A. The velocity and acceleration of point P relative to frame B are

$$\vec{v}_{P\text{rel}} = (-6.14\,\hat{\imath}_B + 23.7\,\hat{\jmath}_B)\,\text{ft/s} \quad \text{and} \quad \vec{a}_{P\text{rel}} = (3.97\,\hat{\imath}_B + 4.79\,\hat{\jmath}_B)\,\text{ft/s}^2.$$

Knowing that, at the instant shown, frame B rotates relative to frame A at a constant angular velocity $\omega_B = 1.2\,\text{rad/s}$, that the position of point P relative to frame B is $\vec{r}_{P/B} = (8\,\hat{\imath}_B + 4.5\,\hat{\jmath}_B)\,\text{ft}$, and that frame A is fixed, determine the velocity and acceleration of P at the instant shown and express the results using the frame A component system.



Solution

From the kinematic equations for the velocity of a point relative to a rotating reference frame we have

$$\vec{v}_P = \vec{v}_B + \vec{v}_{P\text{rel}} + \vec{\Omega} \times \vec{r}_{P/B} = \dot{\vec{P}}_{\text{rel}} + \omega_B\,\hat{k} \times \vec{r}_{P/B}. \tag{1}$$

We have known values $\dot{\vec{P}}_{\text{rel}} = (-6.14\,\hat{\imath}_B + 23.7\,\hat{\jmath}_B)\,\text{ft/s}$, $\omega_B = 1.2\,\hat{k}_B\,\text{rad/s}$, and $\vec{r}_{P/B} = (8\,\hat{\imath}_B + 4.5\,\hat{\jmath}_B)\,\text{ft}$, but they must be transformed from the B component system to the A component system where $\hat{\imath}_B = \cos 23^\circ\,\hat{\imath}_A + \sin 23^\circ\,\hat{\jmath}_A$ and $\hat{\jmath}_B = -\sin 23^\circ\,\hat{\imath}_A + \cos 23^\circ\,\hat{\jmath}_A$. Therefore, in the A component system, our known values are

$$\dot{\vec{P}}_{\text{rel}} = (-14.91\,\hat{\imath}_A + 19.42\,\hat{\jmath}_A), \quad \vec{r}_{P/B} = (5.606\,\hat{\imath}_A + 7.268\,\hat{\jmath}_A)\,\text{ft}, \quad \text{and} \quad \vec{\omega}_B = 1.2\,\hat{k}_A\,\text{rad/s}.$$

Substituting these known values into Eq. (1) we have

$$\boxed{\vec{v}_P = (-23.6\,\hat{\imath}_A + 26.1\,\hat{\jmath}_A)\,\text{ft/s}.}$$

Similarly, for the acceleration of point P we have

$$\begin{aligned}\vec{a}_P &= \vec{a}_B + \vec{a}_{P\text{rel}} + 2\vec{\Omega} \times \vec{v}_{P\text{rel}} + \dot{\vec{\Omega}} \times \vec{r}_{P/B} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{P/B}) \\ &= \ddot{\vec{P}}_{\text{rel}} + 2\omega_B\,\hat{k} \times \dot{\vec{P}}_{\text{rel}} + \dot{\omega}_B\,\hat{k} \times \vec{r}_{P/B} + \omega_B\,\hat{k} \times (\omega_B\,\hat{k} \times \vec{r}_{B/P}).\end{aligned} \tag{2}$$

We must also transform $\ddot{\vec{P}}_{\text{rel}}$ into the A component system. Doing so we have

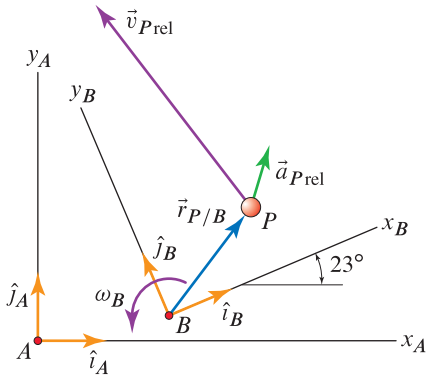
$$\ddot{\vec{P}}_{\text{rel}} = (2.103\,\hat{\imath}_A + 6.447\,\hat{\jmath}_A)\,\text{ft/s}^2. \tag{3}$$

Substituting this value, as well as the known values $\dot{\vec{P}}_{\text{rel}} = (-14.91\,\hat{\imath}_A + 19.42\,\hat{\jmath}_A)$, $\vec{r}_{P/B} = (5.606\,\hat{\imath}_A + 7.268\,\hat{\jmath}_A)$ ft, and $\vec{\omega}_B = 1.2\,\hat{k}_A$ rad/s into Eq. (2), we have

$$\boxed{\vec{a}_P = (-52.9\,\hat{\imath}_A - 40.3\,\hat{\jmath}_A)\ \text{ft/s}^2.}$$

Problem 6.119

Repeat Prob. 6.118, but express the results using the component system of frame B.



Solution

From the kinematic equations for the velocity of a point relative to a rotating reference frame we have

$$\begin{aligned}\vec{v}_P &= \vec{v}_B + \vec{v}_{P\text{rel}} + \vec{\Omega} \times \vec{r}_{P/B} \\ &= \dot{\vec{P}}_{\text{rel}} + \omega_B \hat{k} \times \vec{r}_{P/B}. \end{aligned} \tag{1}$$

Substituting known values $\dot{\vec{P}}_{\text{rel}} = (-6.14\,\hat{\imath}_B + 23.7\,\hat{\jmath}_B)$ ft/s, $\vec{\omega}_B = 1.2\,\hat{k}$ rad/s, and $\vec{r}_{P/B} = (8\,\hat{\imath}_B + 4.5\,\hat{\jmath}_B)$ ft into Eq. (1) we have

$$\boxed{\vec{v}_P = (-11.5\,\hat{\imath}_B + 33.3\,\hat{\jmath}_B)\ \text{ft/s}.}$$

Similarly, for the acceleration of point P we have

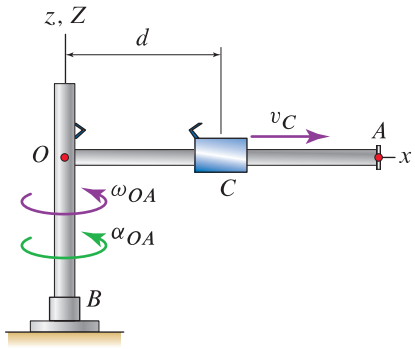
$$\begin{aligned}\vec{a}_P &= \vec{a}_B + \vec{a}_{P\text{rel}} + 2\vec{\Omega} \times \vec{v}_{P\text{rel}} + \dot{\vec{\Omega}} \times \vec{r}_{P/B} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{P/B}) \\ &= \ddot{\vec{P}}_{\text{rel}} + 2\omega_B \hat{k} \times \dot{\vec{P}}_{\text{rel}} + \dot{\omega}_B \hat{k} \times \vec{r}_{P/B} + \omega_B \hat{k} \times (\omega_B \hat{k} \times \vec{r}_{B/P}) \end{aligned} \tag{2}$$

Substituting the known values $\dot{\vec{P}}_{\text{rel}} = (-6.14\,\hat{\imath}_B + 23.7\,\hat{\jmath}_B)$ ft/s, $\ddot{\vec{P}}_{\text{rel}} = (3.97\,\hat{\imath}_B + 4.79\,\hat{\jmath}_B)$ ft/s^2, $\vec{\omega}_B = 1.2\,\hat{k}$ rad/s, and $\vec{r}_{P/B} = (8\,\hat{\imath}_B + 4.5\,\hat{\jmath}_B)$ ft into Eq. (2), we have

$$\boxed{\vec{a}_P = (-64.4\,\hat{\imath}_B - 16.4\,\hat{\jmath}_B)\ \text{ft/s}^2.}$$

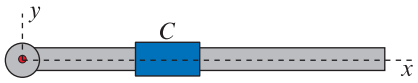
Problem 6.120

A vertical shaft has a base B that is stationary relative to an inertial reference frame with vertical axis Z. Arm OA is attached to the vertical shaft and rotates about the Z axis with an angular velocity $\omega_{OA} = 5\,\text{rad/s}$ and an angular acceleration $\alpha_{OA} = 1.5\,\text{rad/s}^2$. The z axis is coincident with the Z axis but is part of a reference frame that rotates with the arm OA. The x axis of the rotating reference frame coincides with the axis of the arm OA. At the instant shown, the y axis is perpendicular to the page and directed into the page. At this instant, the collar C is sliding along OA with a constant speed $v_C = 5\,\text{ft/s}$ and is at a distance $d = 1.2\,\text{ft}$ from the z axis. Compute the inertial acceleration of the collar, and express it relative to the rotating coordinate system.



Solution

Using the idea of rotating reference frames, we have



$$\vec{v}_C = \vec{v}_O + \dot{\vec{r}}_{C/O} + \vec{\Omega} \times \vec{r}_{C/O} = v_C\,\hat{\imath} + \omega_{OA}\,\hat{k} \times d\,\hat{\imath} = v_C\,\hat{\imath} + d\omega_{OA}\,\hat{\jmath},$$

where we have used the fact that $v_O = 0$. For the acceleration, we then have

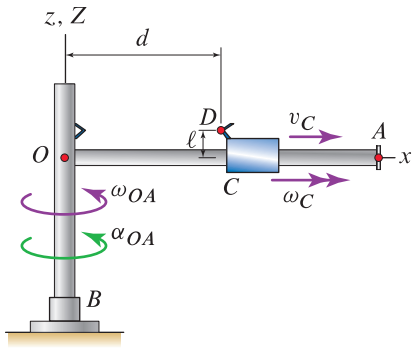
$$\begin{aligned}
\vec{a}_C &= \vec{a}_O + \vec{a}_{C/O} + 2\vec{\Omega}_{OC} \times \vec{v}_{C/O} + \dot{\vec{\Omega}}_{OC} \times \vec{r}_{C/O} + \vec{\Omega}_{OC} \times (\vec{\Omega}_{OC} \times \vec{r}_{C/O}) \\
&= 2\omega_{OA}\,\hat{k} \times v_C\,\hat{\imath} + \alpha_{OA}\,\hat{k} \times d\,\hat{\imath} + \omega_{OA}\,\hat{k} \times (\omega_{OA}\,\hat{k} \times d\,\hat{\imath}) \\
&= 2\omega_{OA} v_C\,\hat{\jmath} + d\alpha_{OA}\,\hat{\jmath} + \omega_{OA}\,\hat{k} \times d\omega_{OA}\,\hat{\jmath} \\
&= -d\omega_{OA}^2\,\hat{\imath} + (2\omega_{OA} v_C + d\alpha_{OA})\,\hat{\jmath}, \qquad (1)
\end{aligned}$$

where we have used the fact that $a_O = 0$, $v_C = \text{constant}$, $\vec{\Omega}_{OC} = \omega_{OA}\,\hat{k}$, and $\dot{\vec{\Omega}}_{OC} = \alpha_{OA}\,\hat{k}$. Substituting the given values $d = 1.2\,\text{ft}$, $v_C = 5\,\text{ft/s}$, $\omega_{OA} = 5\,\text{rad/s}$, and $\alpha_{OA} = 1.5\,\text{rad/s}^2$ into Eq. (1), and rounding to three significant figures, we have

$$\boxed{\vec{a}_C = (-30.0\,\hat{\imath} + 51.8\,\hat{\jmath})\,\text{ft/s}^2.}$$

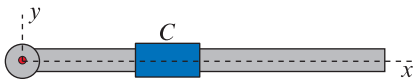
Problem 6.121

A vertical shaft has a base B that is stationary relative to an inertial reference frame with vertical axis Z. Arm OA is attached to the vertical shaft and rotates about the Z axis with an angular velocity $\omega_{OA} = 5\,\text{rad/s}$ and an angular acceleration $\alpha_{OA} = 1.5\,\text{rad/s}^2$. The z axis is coincident with the Z axis but is part of a reference frame that rotates with the arm OA. The x axis of the rotating reference frame coincides with the axis of the arm OA. At the instant shown, the y axis is perpendicular to the page and directed into the page. At this instant, the collar C is sliding along OA with a constant speed $v_C = 3.32\,\text{m/s}$ and is rotating with a constant angular velocity $\omega_C = 2.3\,\text{rad/s}$ relative to the arm OA. At the instant shown, point D happens to be in the xz plane and is at a distance $\ell = 0.05\,\text{m}$ from the x axis and at a distance $d = 0.75\,\text{m}$ from the z axis. Compute the inertial acceleration of point D, and express it relative to the rotating reference frame.



Solution

Using the idea of rotating reference frames, we have



$$\begin{aligned}\vec{v}_D &= \vec{v}_O + \dot{\vec{r}}_{D/O} + \vec{\Omega} \times \vec{r}_{D/O} \\ &= (v_C\,\hat{\imath} - \ell\omega_C\,\hat{\jmath}) + (\omega_C\,\hat{\imath} + \omega_{OA}\,\hat{k}) \times d\,\hat{\imath} \\ &= v_C\,\hat{\imath} + (d\omega_{OA} - \ell\omega_C)\,\hat{\jmath},\end{aligned}$$

where we have used the fact that $v_O = 0$ and D is making a circular motion about the bar OA, and moving out of the page at the instant shown. For the acceleration, we then have

$$\begin{aligned}\vec{a}_D &= \vec{a}_O + \vec{a}_{D/O} + 2\vec{\Omega} \times \vec{v}_{D/O} + \dot{\vec{\Omega}} \times \vec{r}_{D/O} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}_{D/O}\right) \\ &= 2\left(\omega_C\,\hat{\imath} + \omega_{OA}\,\hat{k}\right) \times (v_C\,\hat{\imath} - \ell\omega_C\,\hat{\jmath}) + \alpha_{OA}\,\hat{k} \times \left(d\,\hat{\imath} + \ell\,\hat{k}\right) \\ &\quad + \left(\omega_C\,\hat{\imath} + \omega_{OA}\,\hat{k}\right) \times \left[\left(\omega_C\,\hat{\imath} + \omega_{OA}\,\hat{k}\right) \times d\,\hat{\imath}\right] \\ &= -2\ell\omega_C^2\,\hat{k} + 2\omega_{OA}\,(v_C\,\hat{\jmath} + \ell\omega_C\,\hat{\imath}) + d\alpha_{OA}\,\hat{\jmath} + \left(\omega_C\,\hat{\imath} + \omega_{OA}\,\hat{k}\right) \times (d\omega_{OA}\,\hat{\jmath}) \\ &= \left(2\ell\omega_C\omega_{OA} - d\omega_{OA}^2\right)\hat{\imath} + (d\alpha_{OA} + 2v_C\omega_{OA})\,\hat{\jmath} + \left(-\ell\omega_C^2\right)\hat{k} \qquad (1)\end{aligned}$$

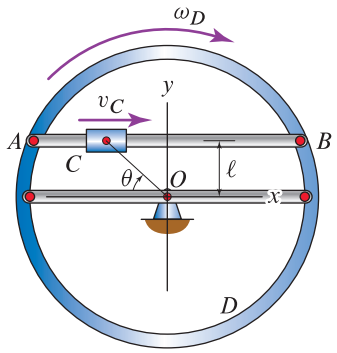
where we have used the fact that $a_O = 0$, $v_C = \text{constant}$, $\omega_C = \text{constant}$, D is making a circular motion about the bar OA, and moving out of the page at the instant shown, $\vec{\Omega} = \left(\omega_C\,\hat{\imath} + \omega_{OA}\,\hat{k}\right)$, and $\dot{\vec{\Omega}}_{OC} = \alpha_{OA}\,\hat{k}$. Substituting the given values $d = 0.75\,\text{m}$, $\ell = 0.05\,\text{m}$, $v_C = 3.32\,\text{m/s}$, $\omega_C = 2.3\,\text{rad/s}$,

$\omega_{OA} = 5\,\text{rad/s}$, and $\alpha_{OA} = 1.5\,\text{rad/s}^2$ into Eq. (1), and rounding to three significant figures, we have

$$\boxed{\vec{a}_C = \left(-17.6\,\hat{\imath} + 34.3\,\hat{\jmath} + -0.264\,\hat{k}\right)\ \text{m/s}^2.}$$

Problem 6.122

The wheel D rotates with a constant angular velocity $\omega_D = 14\,\text{rad/s}$ about the fixed point O, which is assumed to be stationary relative to an inertial frame of reference. The xy frame rotates with the wheel. Collar C slides along the bar AB with a constant velocity $v_C = 4\,\text{ft/s}$ relative to the xy frame. Letting $\ell = 0.25\,\text{ft}$, determine the inertial velocity and acceleration of C when $\theta = 25°$. Express the result with respect to the xy frame.



Solution

From the geometry of the problem, we have $\vec{r}_{C/O} = (-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath})$. From the concept of rotating reference frames, we have

$$\vec{v}_C = \vec{v}_O + \vec{v}_{C/O} + \vec{\Omega}\times\vec{r}_{C/O} = v_C\,\hat{\imath} + \omega_D\,\hat{k}\times(\ell\cot\theta\,\hat{\imath} - \ell\,\hat{\jmath}) = (v_C + \ell\omega_D)\,\hat{\imath} + \ell\omega_D\cot\theta\,\hat{\jmath}, \quad (1)$$

where we have used the fact that $v_O = 0$ and $\vec{\Omega} = \omega_D\,\hat{k}$. Substituting known values $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (1), and rounding to three significant figures we have

$$\boxed{\vec{v}_C = (7.50\,\hat{\imath} + 7.51\,\hat{\jmath})\,\text{ft/s}.}$$

For the acceleration, we then have

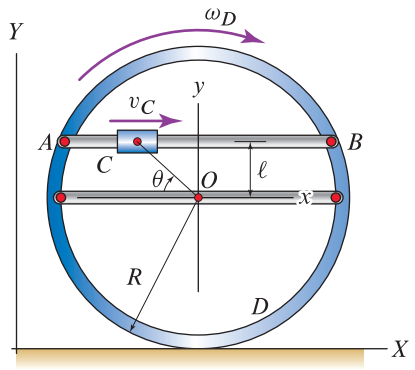
$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{a}_{C/O} + 2\vec{\Omega}\times\vec{v}_{C/O} + \dot{\vec{\Omega}}\times\vec{r}_{C/O} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}_{C/O}\right)\\ &= -2\omega_D\,\hat{k}\times v_C\,\hat{\imath} + \omega_D\,\hat{k}\times\left[\omega_D\,\hat{k}\times(-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath})\right]\\ &= \left(\ell\omega_D^2\cot\theta\right)\hat{\imath} - \left(2v_C\omega_D + \ell\omega_D^2\right)\hat{\jmath}, \qquad (2)\end{aligned}$$

where we have used the fact that $v_O = 0$, $\vec{\Omega} = \omega_D\,\hat{k}$, $\dot{\vec{\Omega}} = \vec{0}$. Substituting the known values $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (2), and rounding to three significant figures we have

$$\boxed{\vec{a}_C = (105\,\hat{\imath} - 161\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 6.123

The wheel D rotates without slipping over a flat surface. The XY frame shown is inertial, whereas the xy frame is attached to D at O and rotates with it at a constant angular velocity $\omega_D = 14\,\text{rad/s}$. Collar C slides along the bar AB with a constant velocity $v_C = 4\,\text{ft/s}$ relative to the xy frame. Letting $\ell = 0.25\,\text{ft}$ and $R = 1\,\text{ft}$, determine the inertial velcity and acceleration of C when $\theta = 25°$ and the xy frame is parallel to the XY frame as shown. Express your result in both the xy and XY frames.



Solution

From the geometry of the problem, we have $\vec{r}_{C/O} = (-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath})$. From the concept of rotating reference frames, we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{v}_{C/O} + \vec{\Omega}\times\vec{r}_{C/O}\\ &= (R\omega_D + v_C)\hat{\imath} + \omega_D\,\hat{k}\times(\ell\cot\theta\,\hat{\imath} - \ell\,\hat{\jmath})\\ &= (R\omega_D + v_C + \ell\omega_D)\,\hat{\imath} + \ell\omega_D\cot\theta\,\hat{\jmath},\end{aligned}\tag{1}$$

where we have used the fact that $\vec{v}_O = R\omega_D\,\hat{\imath}$ and $\vec{\Omega} = \omega_D\,\hat{k}$. Substituting known values $R = 1\,\text{ft}$, $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (1), and rounding to three significant figures we have

$$\boxed{\vec{v}_C = (21.5\,\hat{\imath} + 7.51\,\hat{\jmath})\,\text{ft/s}.}$$

For the acceleration, we then have

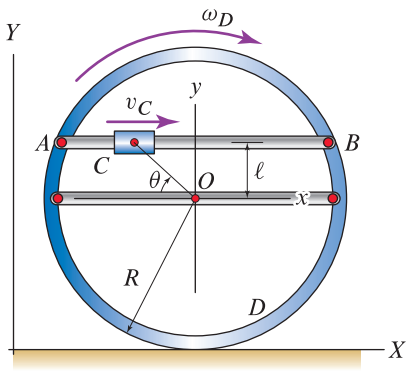
$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{a}_{C/O} + 2\vec{\Omega}\times\vec{v}_{C/O} + \dot{\vec{\Omega}}\times\vec{r}_{C/O} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}_{C/O}\right)\\ &= -2\omega_D\,\hat{k}\times v_C\,\hat{\imath} + \omega_D\,\hat{k}\times\left[\omega_D\,\hat{k}\times(-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath})\right]\\ &= \left(\ell\omega_D^2\cot\theta\right)\hat{\imath} - \left(2v_C\omega_D + \ell\omega_D^2\right)\hat{\jmath},\end{aligned}\tag{2}$$

where we have used the fact that $v_O = 0$, $\vec{\Omega} = \omega_D\,\hat{k}$, $\dot{\vec{\Omega}} = \vec{0}$. Substituting the known values $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (2), and rounding to three significant figures we have

$$\boxed{\vec{a}_C = (105\,\hat{\imath} - 161\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 6.124

The wheel D rotates without slipping over a flat surface. The XY frame shown is inertial whereas the xy frame is attached to D and rotates with it at a constant angular velocity ω_D. Collar C slides along the bar AB with a velocity v_C relative to the xy frame. Suppose that ℓ, θ, and R are given and that we want to determine the inertial acceleration of C when the xy frame is parallel to the XY frame as shown. Would the expression of the inertial acceleration of the collar in the two frames be different or the same?

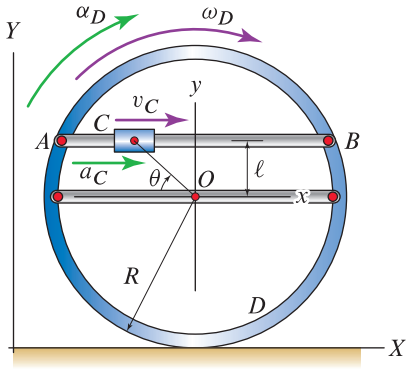


Solution

At the instant shown, the two reference frames are coincidental, i.e. $\hat{\imath} = \hat{I}$ and $\hat{\jmath} = \hat{J}$. Therefore, a vector will have exactly the same representation with respect to the two frames.

Problem 6.125

At the instant shown, the wheel D rotates without slipping over a flat surface with an angular velocity $\omega_D = 14\,\text{rad/s}$ and an angular acceleration $\alpha_D = 1.1\,\text{rad/s}^2$. The XY frame shown is inertial whereas the xy frame is attached to D. At the instant shown, the collar C is sliding along the bar AB with a velocity $v_C = 4\,\text{ft/s}$ and acceleration $a_C = 7\,\text{ft/s}^2$, both relative to the xy frame. Letting $\ell = 0.25\,\text{ft}$ and $R = 1\,\text{ft}$, determine the inertial velocity and acceleration of C when $\theta = 25°$ and the xy frame is parallel to the XY frame as shown. Express your result in both the xy and the XY frames.



Solution

From the geometry of the problem, we have $\vec{r}_{C/O} = (-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath})$. From the concept of rotating reference frames, we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{v}_{C/O} + \vec{\Omega}\times\vec{r}_{C/O}\\ &= (R\omega_D + v_C)\hat{\imath} + \omega_D\,\hat{k}\times(\ell\cot\theta\,\hat{\imath} - \ell\,\hat{\jmath})\\ &= (R\omega_D + v_C + \ell\omega_D)\,\hat{\imath} + \ell\omega_D\cot\theta\,\hat{\jmath},\end{aligned}\tag{1}$$

where we have used the fact that $\vec{v}_O = R\omega_D\,\hat{\imath}$ and $\vec{\Omega} = \omega_D\,\hat{k}$. Substituting known values $R = 1\,\text{ft}$, $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (1), and rounding to three significant figures we have

$$\boxed{\vec{v}_C = (21.5\,\hat{\imath} + 7.51\,\hat{\jmath})\,\text{ft/s}.}$$

For the acceleration, we then have

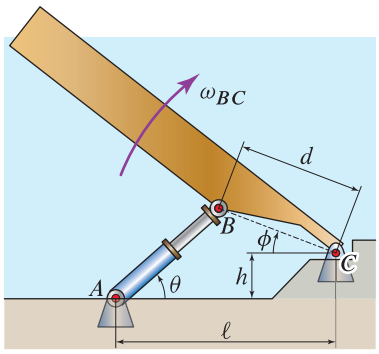
$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{a}_{C/O} + 2\vec{\Omega}\times\vec{v}_{C/O} + \dot{\vec{\Omega}}\times\vec{r}_{C/O} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}_{C/O}\right)\\ &= -3\omega_D\,\hat{k}\times v_C\,\hat{\imath} - \alpha_D\,\hat{k}\times(-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath}) + \omega_D\,\hat{k}\times\left[\omega_D\,\hat{k}\times(-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath})\right]\\ &= \left(a_C + \ell\alpha_D + R\alpha_D + \ell\omega_D^2\cot\theta\right)\hat{\imath} + \left(-2v_C\omega_D - \ell\omega_D^2 + \ell\alpha_D\cot\theta\right)\hat{\jmath},\end{aligned}\tag{2}$$

where we have used the fact that $v_O = 0$, $\vec{\Omega} = \omega_D\,\hat{k}$, $\dot{\vec{\Omega}} = R\alpha_D\,\hat{\imath}$. Substituting the known values $R = 1\,\text{ft}$, $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $\alpha_D = 1.1\,\text{rad/s}^2$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (2), and rounding to three significant figures we have

$$\boxed{\vec{a}_C = (113\,\hat{\imath} - 160\,\hat{\jmath})\,\text{ft/s}^2.}$$

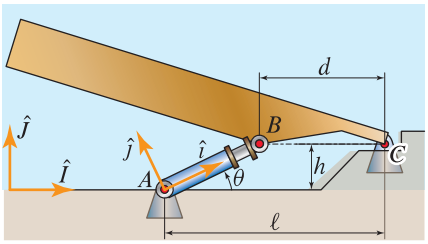
Problem 6.126

A floodgate is controlled by the motion of the hydraulic cylinder AB. If the gate BC is to be lifted with a constant angular velocity $\omega_{BC} = 0.5\,\text{rad/s}$, determine $\dot{d}_{AB}$ and $\ddot{d}_{AB}$, where d_{AB} is the distance between points A and B when $\phi = 0$. Let $\ell = 10\,\text{ft}$, $h = 2.5\,\text{ft}$, and $d = 5\,\text{ft}$.



Solution

We define a fixed reference frame XY such that X is horizontal and pointing to the right, and Y is vertical and pointing up, and a rotating reference frame xy attached at A that rotates with the bar. We are asked to find $\dot{d}_{AB}$ and $\ddot{d}_{AB}$, which correspond to the $\hat{\imath}$ component of $\dot{r}_{B/A}$ and $\ddot{r}_{B/A}$, respectively. If we write the velocity of B using the rotating reference frame, we have



$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} + \vec{\Omega} \times \vec{r}_{B/A} \\ &= \dot{d}_{B/A}\,\hat{\imath} + \omega_{AB}\,\hat{k} \times \sqrt{(\ell-d)^2 + h^2}\,\hat{\imath} \\ &= \dot{d}_{B/A}\,\hat{\imath} + \omega_{AB}\sqrt{(\ell-d)^2 + h^2}\,\hat{\jmath}, \qquad (1)\end{aligned}$$

where we used the fact that $v_A = 0$ and $\vec{r}_{B/A} = \sqrt{(\ell-d)^2 + h^2}\,\hat{\jmath}$. We must also have

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B/C} + \vec{\Omega} \times \vec{r}_{B/C} = -\omega_{BC}\,\hat{k} \times (-d)\,\hat{I} = -d\omega_{BC}\,\hat{J} = d\omega_{BC}\sin\theta\,\hat{\imath} + d\omega_{BC}\cos\theta\,\hat{\jmath}, \qquad (2)$$

where we used the fact that $v_C = 0$, $\vec{r}_{B/A} = \sqrt{(\ell-d)^2 + h^2}\,\hat{\jmath}$, and $\hat{J} = \sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$. Setting Eqs. (1) and (2) equal component by component, we have

$$\dot{d}_{A/B} = d\omega_{BC}\sin\theta, \qquad (3)$$

$$\omega_{AB}\sqrt{(\ell-d)^2 + h^2} = d\omega_{BC}\cos\theta \quad \Rightarrow \quad \omega_{AB} = \frac{d\omega_{BC}\cos\theta}{\sqrt{(\ell-d)^2 + h^2}}. \qquad (4)$$

Substituting the known values $d = 5\,\text{ft}$, $\omega_{BC} = 0.5\,\text{rad/s}$, and $\theta = \tan^{-1}\left(\frac{h}{\ell-d}\right) = 26.57°$ into Eq. (3), we have

$$\boxed{\dot{d}_{AB} = 1.12\,\text{ft/s}.}$$

Similarly for the acceleration we have

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} + 2\vec{\Omega} \times \vec{v}_{B/A} + \dot{\vec{\Omega}} \times \vec{r}_{B/A} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{B/A})$$

$$= \ddot{d}_{B/A} + 2\omega_{AB}\,\hat{k} \times \dot{d}_{B/A}\,\hat{\imath} + \alpha_{AB}\,\hat{k} \times d_{B/A} + \omega_{AB}\,\hat{k} \times (\omega_{AB}\,\hat{k} \times d_{B/A}\,\hat{\imath})$$
$$= \left(\ddot{d}_{AB} - \omega_{AB}^2\sqrt{(\ell-d)^2+h^2}\right)\hat{\imath} + \left(\alpha_{AB}\sqrt{(\ell-d)^2+h^2} + 2d\omega_{BC}\omega_{AB}\sin\theta\right)\hat{\jmath} \qquad (5)$$

where we have used the fact that $a_A = 0$, $\Omega = \omega_{AB}$, $\dot{\Omega} = \alpha_{AB}\,\hat{k}$, and $d_{AB} = \omega_{AB}$. We must also have

$$\begin{aligned}\vec{a}_B &= \vec{a}_C + \vec{a}_{B/C} + 2\vec{\Omega} \times \vec{v}_{B/C} + \dot{\vec{\Omega}} \times \vec{r}_{B/C} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{B/C}) \\ &= \omega_{BC}\,\hat{k} \times \left[\omega_{BC} \times (-r_{B/C})\hat{I}\right] \\ &= \omega_{BC}^2 r_{B/C}\,\hat{I} \\ &= d\omega_{BC}^2(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) \\ &= d\omega_{BC}^2\sin\theta\,\hat{\imath} - d\omega_{BC}^2\cos\theta\,\hat{\jmath}, \end{aligned} \qquad (6)$$

where we have used $\Omega = \omega_{BC}$ and $\hat{I} = \cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}$. Setting the y components of Eqs. (5) and (6) equal to each other, we have

$$\ddot{d}_{AB} = d\omega_{BC}^2\sin\theta + \omega_{AB}^2\sqrt{(\ell-d)^2+h^2} \qquad (7)$$

Substituting the expression for ω_{AB} from Eq. (4) into Eq. (7), we have

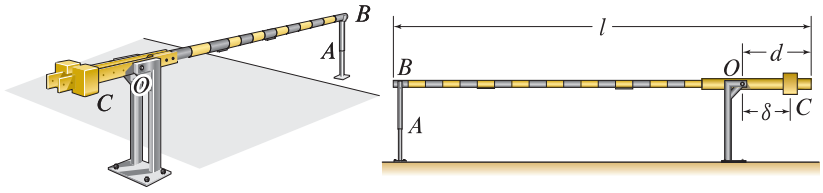
$$\ddot{d}_{AB} = d\omega_{BC}^2\sin\theta + \left(\frac{d\omega_{BC}\cos\theta}{\sqrt{(\ell-d)^2+h^2}}\right)^2\sqrt{(\ell-d)^2+h^2} = d\omega_{BC}^2\sin\theta + \frac{(d\omega_{BC}\cos\theta)^2}{\sqrt{(\ell-d)^2+h^2}} \qquad (8)$$

Substituting the known values $h = 2.5\,\text{ft}$, $d = 5\,\text{ft}$, $\ell = 10\,\text{ft}$, $\omega_{BC} = 0.5\,\text{rad/s}$, and $\theta = \tan^{-1}\left(\frac{h}{\ell-d}\right) = 26.57°$ into Eq. (8) and rounding to three significant figures, we have

$$\boxed{\ddot{d}_{AB} = 2.01\,\text{ft/s}^2.}$$

Problem 6.127

The manually operated road barrier shown has a total length $l = 15.7\,\text{ft}$ and has a counterweight C whose position is identified by the distances $d = 2.58\,\text{ft}$ and $\delta = 1.4\,\text{ft}$. The barrier is pinned at O and can move in the vertical plane. Suppose that the barrier is raised and then released so that the end B of the barrier hits the support with a speed $v_B = 1.5\,\text{ft/s}$. Determine the speed of the counterweight C when B hits A.



Solution

The barrier rotates about the fixed point O. Since both points B and C are on the barrier, ie.e, belong to the same rigid body, we must have

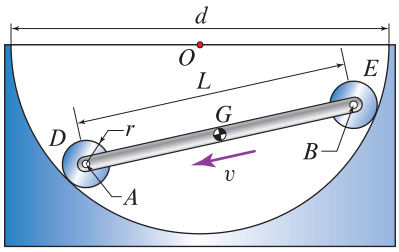
$$\omega_{\text{barrier}} = \frac{v_B}{\ell - d} \quad \text{and} \quad \omega_{\text{barrier}} = \frac{v_C}{\delta} \quad \Rightarrow \quad v_C = \delta \frac{v_B}{\ell - d}. \tag{1}$$

Substituting given values $\delta = 1.4\,\text{ft}$, $\ell = 15.7\,\text{ft}$, $d = 2.58\,\text{ft}$, and $v_B = 1.5\,\text{ft/s}$ into the last of Eqs. (1), we have

$$\boxed{v_C = 0.160\,\text{ft/s}.}$$

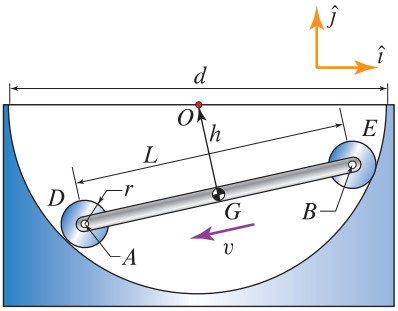
Problem 6.128

A slender bar AB of length $L = 1.45\,\text{ft}$ is mounted on two identical disks D and E pinned at A and B, respectively, and of radius $r = 1.5\,\text{in}$. The bar is allowed to move within a cylindrical bowl with center at O and diameter $d = 2\,\text{ft}$. At the instant shown, the center G of the bar is moving with a speed $v = 7\,\text{ft/s}$. Determine the angular velocity of the bar at the instant shown.



Solution

By construction, the bar rotates about the fixed point O. Therefore, its angular speed is given by $|\vec{\omega}_{\text{bar}}| = \frac{v}{h}$, where h is the vertical distance between G and O. From the geometry of the problem we have that



$$h = \sqrt{\left(\frac{d}{2} - r\right)^2 - \frac{L^2}{4}}.$$

Given the direction of v, G is moving clockwise about O, so we must have

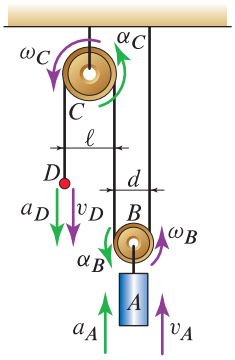
$$\vec{\omega}_{\text{bar}} = \frac{-v}{\sqrt{\left(\frac{d}{2} - r\right)^2 - \frac{L^2}{4}}}\,\hat{k}$$

Substituting in the known values $v = 7\,\text{ft/s}$, $d = 2\,\text{ft}$, $r = 1.9\,\text{in.} = 0.1583\,\text{ft}$, and $L = 1.45\,\text{ft}$, we have

$$\vec{\omega}_{\text{bar}} = -14.3\,\hat{k}\ \text{rad/s}$$

Problem 6.129

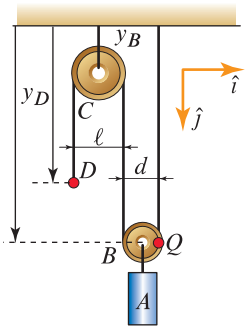
Assuming that the rope does not slip relative to any of the pulleys in the system, determine the velocity and acceleration of A and D, knowing that the angular velocity and acceleration of pulley B are $\omega_B = 7\,\text{rad/s}$ and $\alpha_B = 3\,\text{rad/s}^2$, respectively. The diameters of pulleys B and C are $d = 25\,\text{cm}$ and $\ell = 34\,\text{cm}$, respectively.



Solution

We let Q be the point of contact between pulley B and the cord to the right of pulley B. This cord is not moving because it is fixed to the wall, so from the no slip condition at Q, we must have

$$v_A = \omega_B \frac{d}{2} \quad \text{and} \quad a_A = \alpha_B \frac{d}{2}.$$



Substituting given values $d = 25\,\text{cm} = 0.25\,\text{m}$, $\omega_B = 7\,\text{rad/s}$, and $\alpha_B = 3\,\text{rad/s}^2$, we have

$$v_A = 0.8750\,\text{m/s} \quad \text{and} \quad a_A = 0.3750\,\text{m/s}^2. \tag{1}$$

Using the directions shown in the problem statement, in vector form and with three significant figures, we have

$$\boxed{\vec{v}_A = 0.875\,\hat{\jmath}\,\text{m/s} \quad \text{and} \quad \vec{a}_A = 0.375\,\hat{\jmath}\,\text{m/s}^2 \;\; \hat{\jmath}\;\hat{\imath}.}$$

By pulley kinematics, we have

$$y_D + 2y_B = \text{constant} \quad \Rightarrow \quad \dot{y}_D = -2\dot{y}_B \quad \text{and} \quad \ddot{y}_D = -2\ddot{y}_B$$

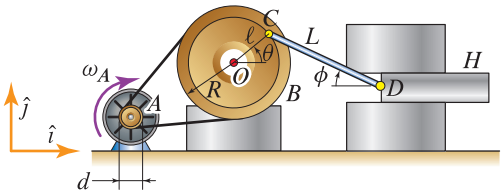
$$\Rightarrow \quad \dot{y}_D = -2v_A \quad \text{and} \quad \ddot{y}_D = -2a_B, \tag{2}$$

where we have used that fact that A and B must have the same velocity and acceleration. Substituting the values from Eqs. (1) into Eqs. (2), for D we have

$$\boxed{\vec{v}_D = -1.75\,\hat{\jmath}\,\text{m/s} \quad \text{and} \quad \vec{a}_D = -0.750\,\hat{\jmath}\,\text{m/s}^2 \;\; \hat{\jmath}\;\hat{\imath}.}$$

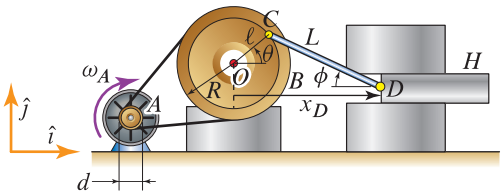
Problem 6.130

At the instant shown, the hammer head H is moving to the right with a speed $v_H = 45\,\text{ft/s}$ and the angle $\theta = 20°$. Assuming that the belt does not slip relative to wheels A and B and assuming that wheel A is mounted on the shaft of the motor shown, determine the angular velocity of the motor at the instant shown. The diameter of wheel A is $d = 0.25\,\text{ft}$, the radius of wheel B is $R = 0.75\,\text{ft}$, and point C is at a distance $\ell = 0.72\,\text{ft}$ from O, which is the center of the wheel A. Finally, let CD have a length $L = 2\,\text{ft}$, and assume that, at the instant shown, $\phi = 25°$.



Solution

From the geometry of the problem, we have the geometrical constraints



$$x_D = \ell\cos\theta + L\cos\phi \quad \text{and} \quad L\sin\phi = \ell\sin\theta + h, \quad (1)$$

where h is the vertical distance between O and D. Taking the first time derivative of Eqs. (1), we have

$$\dot{x}_D = v_{Hx} = -\left(\ell\dot{\theta}\sin\theta + L\dot{\phi}\sin\phi\right) \quad (2)$$

and

$$\dot{\phi}L\cos\phi = \dot{\theta}\ell\cos\theta \quad \Rightarrow \quad \dot{\phi} = \dot{\theta}\frac{\ell\cos\theta}{L\cos\phi}. \quad (3)$$

Substituting the expresion for $\dot{\phi}$ from Eq. (3) into Eq. (2), we have

$$v_{Hx} = -\frac{\ell\dot{\theta}}{\cos\phi}(\sin\theta\cos\phi + \sin\phi\cos\theta) = -\frac{\ell\dot{\theta}}{\cos\phi}\sin(\theta+\phi) \quad \Rightarrow \quad \dot{\theta} = -\frac{v_{Hx}\cos\phi}{\ell\sin(\theta+\phi)}$$

Since the belt does not slip, and because ω_A is indicated as being clockwise, we have

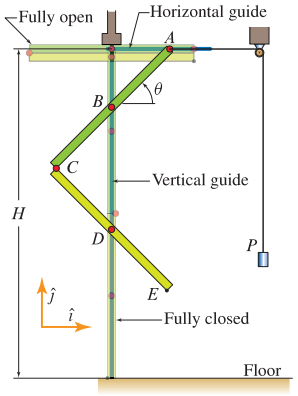
$$d\omega_A = 2R\omega_B \quad \Rightarrow \quad \vec{\omega}_A = -\frac{2R}{d}\frac{v_{Hx}\cos\phi}{\ell\sin(\theta+\phi)}\hat{k}, \quad (4)$$

where we have used the fact that $\omega_B = \dot{\theta}$. Subsituting given values $R = 0.75\,\text{ft}$, $d = 0.25\,\text{ft}$, $\ell = 0.72\,\text{ft}$, $v_{Hx} = 45\,\text{ft/s}$, $\phi = 25°$, and $\theta = 20°$ into Eq. (4), we have

$$\boxed{\vec{\omega}_A = -481\,\hat{k}\,\text{rad/s}.}$$

Problems 6.131 and 6.132

An overhead fold-up door with height $H = 30\,\text{ft}$ consists of two identical sections hinged at C. The roller at A moves along a horizontal guide, whereas the rollers at B and D, which are the midpoints of sections AC and CE, move along a vertical guide. The door's operation is assisted by a counterweight P. Express your answers using the component system shown.



Problem 6.131 If at the instant shown, the angle $\theta = 55°$ and P is moving upward with a speed $v_P = 15\,\text{ft/s}$, determine the velocity of point E as well as the angular velocities of sections AC and CE.

Problem 6.132 If at the instant shown, $\theta = 45°$, and A is moving to the right with a speed $v_A = 2\,\text{ft/s}$ while decelerating at a rate of $1.5\,\text{ft/s}^2$, determine the acceleration of point E.

Solution to 6.131

From the geometry of the problem, we have the constraints

$$x_A = \frac{H}{4}\cos\theta, \quad x_E = x_A, \quad \text{and} \quad y_E = -H\sin\theta. \tag{1}$$

Taking a time derivative of x_A from the first of Eq. (1) and observing that $\dot{x}_A = v_A = -v_P$, we have

$$v_A = -\frac{H}{4}\dot{\theta}\sin\theta = -v_P \quad \Rightarrow \quad \dot{\theta} = \frac{4v_P}{H\sin\theta}. \tag{2}$$

Observing that $\omega_{AC} = \dot{\theta}$, and $\vec{\omega}_{AC} = -\vec{\omega}_{CE}$ by symmetry, substituting given values $v_P = 15\,\text{ft/s}$, $H = 30\,\text{ft}$, and $\theta = 55°$ gives us

$$\dot{\theta} = 2.442\,\text{rad/s} \quad \Rightarrow \quad \boxed{\vec{\omega}_{AC} = 2.44\,\hat{k}\,\text{rad/s}} \quad \text{and} \quad \boxed{\vec{\omega}_{CE} = -2.44\,\hat{k}\,\text{rad/s}.}$$

For the position of E, we have

$$\vec{r}_E = \frac{H}{4}\cos\theta\,\hat{\imath} - H\sqrt{1-\cos^2\theta} \tag{3}$$

Taking a time derivative of Eq. (3), we have

$$\vec{v}_E = -\frac{H}{4}\dot{\theta}\sin\theta\,\hat{\imath} - H\dot{\theta}\cos\theta\,\hat{\jmath} = -v_P\,\hat{\imath} - \frac{4v_P}{\tan\theta}\,\hat{\jmath}.$$

Substituting known values $v_P = 15\,\text{ft/s}$ and $\theta = 55°$, we have

$$\boxed{\vec{v}_E = (-15.0\,\hat{\imath} - 42.0\,\hat{\jmath})\,\text{ft/s}.}$$

Solution to 6.132

From the geometry of the problem, we have the constraints

$$x_E = x_A = \frac{H}{4}\cos\theta = 5.303\,\text{ft} \quad \text{and} \quad y_E = y_E = -H\sin\theta.$$

For the position of E, we have

$$\vec{r}_E = \frac{H}{4}\cos\theta\,\hat{\imath} - H\sqrt{1-\cos^2\theta} \tag{4}$$

Taking two time derivatives of Eq. (4), we have

$$\vec{v}_E = \vec{v}_E = -\frac{H}{4}\dot{\theta}\sin\theta\,\hat{\imath} - H\dot{\theta}\cos\theta\,\hat{\jmath} = -v_P\,\hat{\imath} - \frac{4v_P}{\tan\theta}\,\hat{\jmath}.$$

and

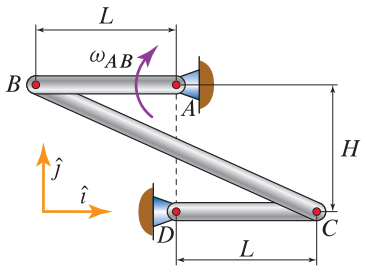
$$\vec{a}_E = a_A\,\hat{\imath} + H\left(\frac{256x_A^2v_A^2}{H^4\left(1-\frac{16x_A^2}{H^2}\right)^{3/2}} + \frac{16v_A^2}{H^2\sqrt{1-\frac{16x_A^2}{H^2}}} + \frac{16x_Aa_A}{H^2\sqrt{1-\frac{16x_A^2}{H^2}}}\right)\hat{\jmath}.$$

Substituting known values $x_A = 5.303\,\text{ft}$, $v_A = 2\,\text{ft/s}$, $a_A = 1.5\,\text{ft/s}^2$, and $\theta = 45°$, we have

$$\boxed{\vec{a}_E = (-1.50\,\hat{\imath} - 0.320\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 6.133

At the instant shown, bar AB rotates with a constant angular velocity $\omega_{AB} = 24\,\text{rad/s}$. Letting $L = 0.75\,\text{m}$ and $H = 0.85\,\text{m}$, determine the angular acceleration of bar BC when bars AB and CD are as shown, i.e., parallel and horizontal.



Solution

From the concept of instantaneous center of rotation, we know that the center of rotation of bar BC is at infinity, so

$$\omega_{BC} = 0.$$

Therefore, the velocity of B is the same as that of C, so we have

$$\omega_{CD} = -\omega_{AB} \quad \Rightarrow \quad \vec{\omega}_{CD} = 24\,\hat{k}\,\text{rad/s}.$$

From rigid body kinematics, we know that

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} = L\omega_{AB}^2\,\hat{\imath} - L\alpha_{AB}\,\hat{\jmath}$$

where we have used the fact that $a_A = 0$. We must also have

$$\begin{aligned}\vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2 \vec{r}_{C/B} \\ &= L\omega_{AB}^2\,\hat{\imath} - L\alpha_{AB}\,\hat{\jmath} + \alpha_{BC}\,\hat{k} \times (2L\,\hat{\imath} - H\,\hat{\jmath}) - \omega_{BC}^2 (2L\,\hat{\imath} - H\,\hat{\jmath}) \\ &= (L\omega_{AB}^2 + H\alpha_{BC})\,\hat{\imath} + (2L\alpha_{BC} - L\alpha_{AB})\,\hat{\jmath}. \end{aligned} \tag{1}$$

We must also have

$$\vec{a}_C = \vec{a}_D + \vec{\alpha}_{CD} \times \vec{r}_{C/D} - \omega_{CD}^2 \vec{r}_{C/D} = -L\omega_{AB}^2\,\hat{\imath} - L\alpha_{CD}\,\hat{\jmath}. \tag{2}$$

Eqsuations (1) and (2) are both expressions for $\vec{a}_C$, and equating them component by component yields

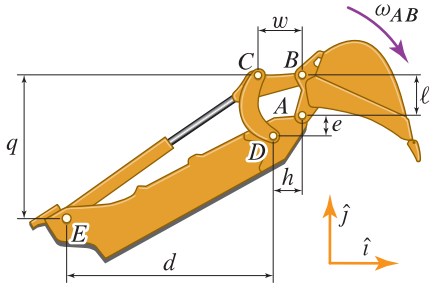
$$\begin{aligned} L\omega_{AB}^2 + H\alpha_{BC} &= -L\omega_{AB}^2 \quad \Rightarrow \quad \alpha_{BC} = -\frac{2L\omega_{AB}^2}{H} \\ 2L\alpha_{BC} - L\alpha_{AB} &= -L\alpha_{CD} \quad \Rightarrow \quad \alpha_{CD} = -\alpha_{AB} - \frac{4L\omega_{AB}^2}{H}. \end{aligned} \tag{3}$$

Substituting known values $L = 0.75\,\text{m}$, $\omega_{AB} = 24\,\text{rad/s}$, and $H = 0.85\,\text{m}$, in vector form to three significant figures, we have

$$\boxed{\vec{\alpha}_{BC} = -1020\,\hat{k}\,\text{rad/s}^2.}$$

Problem 6.134

The bucket of a backhoe is the element AB of the four-bar linkage system $ABCD$. The bucket's motion is controlled by extending or retracting the hydraulic arm EC. Assume that the points A, D, and E are fixed and that the bucket is made to rotate with a constant angular velocity $\omega_{AB} = 0.25\,\text{rad/s}$. In addition, suppose that, at the instant shown, point B is vertically aligned with point A and point C is horizontally aligned with B. Letting d_{EC} denote the distance between points E and C, determine $\dot{d}_{EC}$ and $\ddot{d}_{EC}$ at the instant shown. Let $h = 0.66\,\text{ft}$, $e = 0.46\,\text{ft}$, $l = 0.9\,\text{ft}$, $w = 1.0\,\text{ft}$, $d = 4.6\,\text{ft}$, and $q = 3.2\,\text{ft}$.



Solution

Because point A is stationary, it is the center of rotation of bar AB, and therefore we have

$$\vec{v}_B = \ell\omega_{AB}\,\hat{\imath}.$$

Then, for point C, we must have

$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} = \ell\omega_{AB}\,\hat{\imath} + \omega_{BC}\,\hat{k} \times (-w)\,\hat{\imath} = \ell\omega_{AB}\,\hat{\imath} - w\omega_{BC}\,\hat{\jmath}, \tag{1}$$

and also

$$\vec{v}_C = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{C/D} = \omega_{CD}\,\hat{k} \times [(h-w)\,\hat{\imath} + (\ell+e)\,\hat{\jmath}] = -(\ell+e)\omega_{CD}\,\hat{\imath} + (h-w)\omega_{CD}\,\hat{\jmath}. \tag{2}$$

Equations (1) and (2) are two equations in ω_{BC} and ω_{CD}, and when solved give

$$\omega_{BC} = -\frac{\ell(w-h)\omega_{AB}}{w(e+\ell)} \quad \text{and} \quad \omega_{CD} = -\frac{\ell\omega_{AB}}{e+\ell}. \tag{3}$$

Substituting ω_{BC} from Eq. (3) into Eq. (1), we have

$$\vec{v}_C = \ell\omega_{AB}\,\hat{\imath} + \frac{\ell(w-h)\omega_{AB}}{(e+\ell)}\,\hat{\jmath} = (0.2250\,\hat{\imath} + 0.05625\,\hat{\jmath})\,\text{ft/s}, \tag{4}$$

where we have used the given values $\ell = 0.9\,\text{ft}$, $w = 1.0\,\text{ft}$, $e = 0.46\,\text{ft}$, $h = 0.66\,\text{ft}$, and $\omega_{AB} = 0.25\,\text{rad/s}$. Similarly for the accelerations, we have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2\vec{r}_{B/A} = -\omega_{AB}^2\ell\,\hat{\jmath}, \tag{5}$$

where we have used the fact that $a_A = \alpha_{AB} = 0$. Relating point C to point B gives us

$$\begin{aligned}\vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2\vec{r}_{C/B} \\ &= -\omega_{AB}^2\ell\,\hat{\jmath} + \alpha_{BC}\,\hat{k}(-w)\,\hat{\imath} - \omega_{BC}^2(-w)\,\hat{\imath}\end{aligned}$$

$$= w\left(\frac{\ell(w-h)\omega_{AB}}{w(e+\ell)}\right)^2 \hat{\imath} - (\ell\omega_{AB}^2 + w\alpha_{BC})\,\hat{\jmath}$$
$$= \frac{\ell^2(w-h)^2\omega_{AB}^2}{w(e+\ell)^2}\,\hat{\imath} - (\ell\omega_{AB}^2 + w\alpha_{BC})\,\hat{\jmath}. \tag{6}$$

Similarly, relating point C to point D we can have

$$\begin{aligned}\vec{a}_C &= \vec{a}_D + \vec{\alpha}_{CD}\times\vec{r}_{C/D} - \omega_{CD}^2\vec{r}_{C/D}\\ &= \alpha_{CD}\,\hat{k}\times[(h-w)\,\hat{\imath} + (e+\ell)\,\hat{\jmath}] - \omega_{CD}^2[(h-w)\,\hat{\imath} + (e+\ell)\,\hat{\jmath}]\\ &= \left(-e\alpha_{CD} - \ell\alpha_{CD} - \frac{\ell^2(h-w)\omega_{AB}^2}{(e+\ell)^2}\right)\hat{\imath} + \left(h\alpha_{CD} - w\alpha_{CD} - \frac{\ell^2\omega_{AB}^2}{e\ell}\right)\hat{\jmath}.\end{aligned} \tag{7}$$

Equations (6) and (7) are two equations in α_{BC} and α_{CD}, which when solved give

$$\alpha_{BC} = -\frac{\ell[-h^3\ell + 2h^2w\ell + e(e+\ell)^2w - h\ell w^2]\omega_{AB}^2}{(e+\ell)^3w^2} \tag{8}$$

and

$$\alpha_{CD} = \frac{h\ell^2(w-h)\omega_{AB}^2}{(e+\ell)w}. \tag{9}$$

Substituting α_{BC} from Eq. (8) into Eq. (6), we have

$$a_C = \frac{\ell^2(w-h)^2\omega_{AB}^2}{w(e+\ell)^2}\,\hat{\imath} - \left[\ell\omega_{AB}^2 - \frac{\ell[-h^3\ell + 2h^2w\ell + e(e+\ell)^2w - h\ell w^2]\omega_{AB}^2}{(e+\ell)^3w}\right]\hat{\jmath}$$
$$= (0.003164\,\hat{\imath} + 0.03876\,\hat{\jmath})\ \text{ft/s}^2, \tag{10}$$

where we have used the given values $\ell = 0.9$ ft, $w = 1.0$ ft, $e = 0.46$ ft, $h = 0.66$ ft, and $\omega_{AB} = 0.25$ rad/s. Now that the motion of both ends of the hydraulic arm EC is known, we can determine its time rate of change in length. We begin by defining the horizontal distance from E to C as x_{EC}, and the vertical distance from E to C as y_{EC}. The distance between E and C is then

$$d_{EC} = \sqrt{x_{EC}^2 + y_{EC}^2}, \tag{11}$$

where $x_{EC} = d + h - w = 4.260$ ft and $y_{EC} = q = 3.200$ ft. Taking two time derivatives of Eq. (11) we have

$$\dot{d}_{EC} = \frac{2x_{EC}\dot{x}_{EC} + 2y_{EC}\dot{y}_{EC}}{2\sqrt{x_{EC}^2 + y_{EC}^2}}$$

and

$$\ddot{d}_{EC} = -\frac{(2x_{EC}\dot{x}_{EC} + 2y_{EC}\dot{y}_{EC})^2}{4\left(x_{EC}^2 + y_{EC}^2\right)^{3/2}} + \frac{2\dot{x}_{EC}^2 + 2\dot{y}_{EC}^2 + 2x_{EC}\ddot{x}_{EC} + 2y_{EC}\ddot{y}_{EC}}{2\sqrt{x_{EC}^2 + y_{EC}^2}}. \tag{12}$$

Observing that $\dot{x}_{EC} = v_{Cx} = 0.2250$ ft/s, $\dot{y}_{EC} = v_{Cy} = 0.05625$ ft/s, $\ddot{x}_{EC} = a_{Cx} = 0.003164$ ft/s^2, and $\ddot{y}_{EC} = a_{Cy} = 0.03876$ ft/s^2, we have

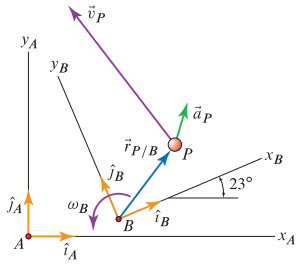
$$\boxed{\dot{d}_{EC} = 0.214\ \text{ft/s} \quad \text{and} \quad \ddot{d}_{EC} = -0.0192\ \text{ft/s}^2.}$$

Problem 6.135

Let frames A and B have their origins at points A and B, respectively. Point B does not move relative to point A. The velocity and acceleration of point P relative to frame A, which is fixed, are

$$\vec{v}_P = (-14.9\,\hat{\imath}_A + 19.4\,\hat{\jmath}_A)\ \text{ft/s} \quad \text{and} \quad \vec{a}_P = (1.78\,\hat{\imath}_A + 5.96\,\hat{\jmath}_A)\ \text{ft/s}^2.$$

Knowing that frame B rotates relative to frame A at a constant angular velocity $\omega_B = 1.2\,\text{rad/s}$ and that the position of point P relative to frame B is $\vec{r}_{P/B} = (8\,\hat{\imath}_B + 4.5\,\hat{\jmath}_B)$ ft, determine the velocity and acceleration of P relative to frame B at the instant shown and express the results using the frame A component system.



Solution

Given the two reference frames in the problem statement, we have the following relationships between them:

$$\hat{\imath}_B = \cos 23^\circ \hat{\imath}_A + \sin 23^\circ \hat{\jmath}_A \quad \text{and} \quad \hat{\jmath}_B = -\sin 23^\circ \hat{\imath}_A + \cos 23^\circ \hat{\jmath}_A. \tag{1}$$

From the concept of rotating reference frames, we have

$$\begin{aligned}
\vec{v}_P &= \vec{v}_B + \vec{v}_{Prel} + \vec{\Omega} \times \vec{r}_{P/B} \\
\Rightarrow \quad v_{PxA}\,\hat{\imath}_A + v_{PyA}\,\hat{\jmath}_A &= v_{PxB}\,\hat{\imath}_B + v_{PyB}\,\hat{\jmath}_B + \omega_B\,\hat{k} \times (8\,\hat{\imath}_B + 4.5\,\hat{\jmath}_B) \\
\Rightarrow \quad v_{PxA}\,\hat{\imath}_A + v_{PyA}\,\hat{\jmath}_A &= v_{PxB}(\cos 23^\circ \hat{\imath}_A + \sin 23^\circ \hat{\jmath}_A) + v_{PyB}(-\sin 23^\circ \hat{\imath}_A + \cos 23^\circ \hat{\jmath}_A) \\
&\quad + \omega_B\,\hat{k} \times (8(\cos 23^\circ \hat{\imath}_A + \sin 23^\circ \hat{\jmath}_A) + 4.5(-\sin 23^\circ \hat{\imath}_A + \cos 23^\circ \hat{\jmath}_A)) \\
\Rightarrow \quad (-14.9\,\hat{\imath}_A + 19.4\,\hat{\jmath}_A) &= (v_{PxB}\cos 23^\circ - v_{PyB}\sin 23^\circ - 8\omega_B \sin 23^\circ - 4.5\omega_B \sin 23^\circ)\,\hat{\imath}_A \\
&\quad + (v_{PxB}\sin 23^\circ + v_{PyB}\cos 23^\circ 8\omega_B \cos 23^\circ - 4.5\omega_B \sin 23^\circ)\,\hat{\jmath}_A,
\end{aligned} \tag{2}$$

where we've used the fact that $v_B = 0$. Solving Eq. (2) for v_{PxB} and v_{PyB} using the known value $\omega_B = 1.2\,\text{rad/s}$, we have

$$v_{PxB} = -6.178\,\text{ft/s} \quad \text{and} \quad v_{PyB} = 12.67\,\text{ft/s} \quad \Rightarrow \quad \boxed{\vec{v}_{P\text{rel}} = (-6.18\,\hat{\imath}_A + 12.7\,\hat{\jmath}_A)\ \text{ft/s}.}$$

Similarly for the acceleration, we have

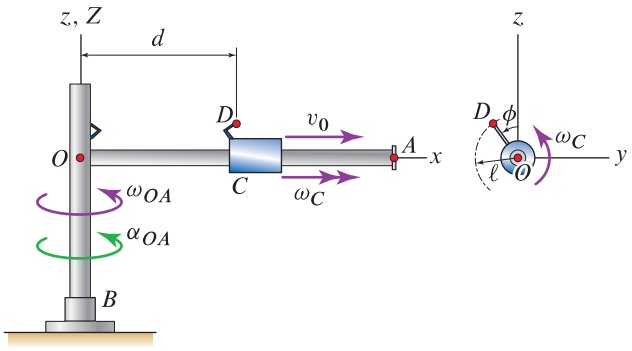
$$\begin{aligned}
\vec{a}_P &= \vec{a}_B + \vec{a}_{P\text{rel}} + 2\vec{\Omega} \times \vec{v}_{P\text{rel}} + \dot{\vec{\Omega}} \times \vec{r}_{P/B} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{P/B}) \\
\Rightarrow \quad (1.78\,\hat{\imath}_A + 5.96\,\hat{\jmath}_A)\ \text{ft/s}^2 &= a_{PxB}\,\hat{\imath}_B + a_{PyB}\,\hat{\imath}_B + 2\omega_B\,\hat{k} \times (-6.18\,\hat{\imath}_A + 12.7\,\hat{\jmath}_A) \\
&\quad + \omega_B\,\hat{k} \times [\omega_B\,\hat{k} \times (8\,\hat{\imath}_B + 4.5\,\hat{\jmath}_B)] \\
\Rightarrow \quad (1.78\,\hat{\imath}_A + 5.96\,\hat{\jmath}_A)\ \text{ft/s}^2 &= a_{PxB}\,\hat{\imath}_B + a_{PyB}\,\hat{\imath}_B - (25.4\omega_B\,\hat{\imath}_A + \omega_B 12.36\,\hat{\jmath}_A) \\
&\quad + \omega_B\,\hat{k} \times (-4.5\omega_B\,\hat{\imath}_B + 8\omega_B\,\hat{\jmath}_B).
\end{aligned} \tag{3}$$

Substituting Eqs. (1) and the given value $\omega_B = 1.2\,\text{rad/s}$ into Eq. (3), and solving for a_{PxB} and a_{PyB}, we have

$$a_{PxB} = 40.27\,\text{ft/s}^2 \quad \text{and} \quad a_{PyB} = 31.25\,\text{ft/s}^2 \quad \Rightarrow \quad \boxed{\vec{a}_{P\text{rel}} = (40.3\,\hat{\imath}_A + 31.3\,\hat{\jmath}_A).}$$

Problem 6.136

A vertical shaft has a base B that is stationary relative to an inertial reference frame with vertical axis Z. Arm OA is attached to the vertical shaft and rotates about the Z axis with an angular velocity ω_{OA} and an angular acceleration α_{OA}. The z axis is coincident with the Z axis but is part of a reference frame that rotates with the arm OA. The x axis of the rotating reference frame coincides with the axis of the arm OA. At the instant shown, the y axis is perpendicular to the page and directed into the page. At this instant, the collar C is at a distance d from the Z axis, is sliding along OA with a constant speed v_C (relative to the arm OA), and is rotating with a constant angular velocity ω_C (relative to the xyz frame). Point D is attached to the collar and is at a distance ℓ from the x axis. At the instant shown, point D happens to be in the plane that is rotated by an angle ϕ from the xz plane. Compute the expression of the inertial velocity and acceleration of point D at the instant shown in terms of the parameters given, and express it relative to the rotating coordinate system.



Solution

Using the concept of rotating reference frames, we have

$$\vec{v}_D = \vec{v}_O + \vec{v}_{Drel} + \vec{\Omega} \times \vec{r}_{D/O}, \tag{1}$$

where

$$\vec{v}_O = \vec{0}, \quad \vec{v}_{Drel} = v_C\,\hat{\imath} - \ell\omega_C \cos\phi\,\hat{\jmath} - \ell\omega_C \sin\phi\,\hat{k}, \quad \vec{\Omega} = \omega_{OA}\,\hat{k}, \tag{2}$$

and

$$\vec{r}_{D/O} = d\,\hat{\imath} + \ell(-\sin\phi\,\hat{\jmath} + \cos\phi\,\hat{k}). \tag{3}$$

Substituting Eqs. (3) into Eq. (1), we have

$$\vec{v}_D = v_C\,\hat{\imath} - \ell\omega_C \cos\phi\,\hat{\jmath} - \ell\omega_C \sin\phi\,\hat{k} + \omega_{OA}\,\hat{k} \times [d\,\hat{\imath} + \ell(-\sin\phi\,\hat{\jmath} + \cos\phi\,\hat{k})], \tag{4}$$

which can be simplified to read

$$\boxed{\vec{v}_D = (v_C + \omega_{OA}\ell \sin\phi)\,\hat{\imath} + (d\omega_{OA} - \ell\omega_C \cos\phi)\,\hat{\jmath} - \ell\omega_C \sin\phi\,\hat{k}.}$$

Similarly, for the accelerations, we have

$$\vec{a}_D = \vec{a}_O + \vec{a}_{D\text{rel}} + 2\vec{\Omega} \times \vec{v}_{D\text{rel}} + \dot{\vec{\Omega}} \times \vec{r}_{D/O} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{D/O}), \tag{5}$$

where

$$\vec{a}_O = \vec{0}, \quad \vec{v}_{D\text{rel}} = v_C\,\hat{\imath} - \ell\omega_C \cos\phi\,\hat{\jmath} - \ell\omega_C \sin\phi\,\hat{k}, \quad \vec{a}_{D\text{rel}} = \omega_C^2 \ell \sin\phi\,\hat{\jmath} - \omega_C^2 \cos\phi\,\hat{k},$$
$$\vec{\Omega} = \omega_{OA}\,\hat{k}, \quad \dot{\vec{\Omega}} = \alpha_{OA}\,\hat{k}, \quad \text{and} \quad \vec{r}_{D/O} = d\,\hat{\imath} + \ell(-\sin\phi\,\hat{\jmath} + \cos\phi\,\hat{k}). \quad (6)$$

Substituting Eqs. (6) into Eq. (5), we have

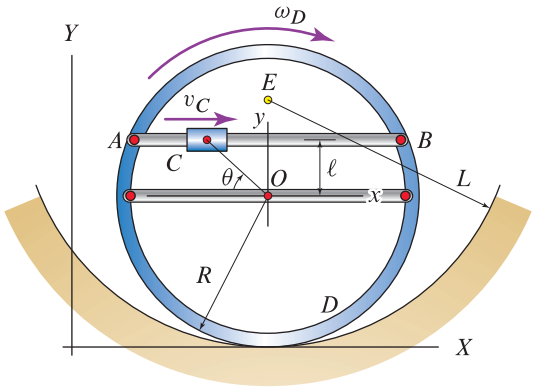
$$\vec{a}_D = \omega_C^2 \ell \sin\phi\,\hat{\jmath} - \omega_C^2 \cos\phi\,\hat{k} + 2\omega_{OA}\,\hat{k} \times [v_C\,\hat{\imath} - \ell\omega_C \cos\phi\,\hat{\jmath} - \ell\omega_C \sin\phi\,\hat{k}]$$
$$+ \alpha_{OA}\,\hat{k} \times [d\,\hat{\imath} + \ell(-\sin\phi\,\hat{\jmath} + \cos\phi\,\hat{k})] + \omega_{OA}\,\hat{k} \times [\omega_{OA}\,\hat{k} \times (d\,\hat{\imath} - \ell \sin\phi\,\hat{\jmath} + \ell \cos\phi\,\hat{k})],$$

which can be simplified to read

$$\vec{a}_D = (-d\omega_{OA}^2 + 2\omega_{OA}\ell\omega_C \cos\phi + \ell\alpha_{OA} \sin\phi)\,\hat{\imath} + (d\alpha_{OA} + (\omega_C^2 + \omega_{OA}^2)\ell \sin\phi + 2\omega_{OA} v_C)\,\hat{\jmath}$$
$$- \omega_C^2 \cos\phi\,\hat{k}.$$

Problem 6.137

The wheel D rotates without slip over a curved cylindrical surface with constant radius of curvature $L = 1.6\,\text{ft}$ and center at the fixed point E. The XY frame is attached to the rolling surface, and it is inertial. The xy frame is attached to D at O and rotates with it at a constant angular velocity $\omega_D = 14\,\text{rad/s}$. Collar C slides along the bar AB with a constant velocity $v_C = 4\,\text{ft/s}$ relative to the xy frame. At the instant shown, points O and E are vertically aligned. Letting $\ell = 0.25\,\text{ft}$ and $R = 1\,\text{ft}$, determine the inertial velocity and acceleration of C at the instant shown when $\theta = 25°$ and the xy frame is parallel to the XY frame. Express your result in the xy and XY frames.



Solution

From the geometry of the problem, we have $\vec{r}_{C/O} = -\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath}$. From the concept of rotating reference frames, we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{v}_{C/O} + \vec{\Omega}\times\vec{r}_{C/O}\\ &= (R\omega_D + v_C)\hat{\imath} + \omega_D\,\hat{k}\times(\ell\cot\theta\,\hat{\imath} - \ell\,\hat{\jmath})\\ &= (R\omega_D + v_C + \ell\omega_D)\,\hat{\imath} + \ell\omega_D\cot\theta\,\hat{\jmath},\end{aligned}\tag{1}$$

where we have used the fact that $\vec{v}_O = R\omega_D\,\hat{\imath}$ and $\vec{\Omega} = \omega_D\,\hat{k}$. Substituting known values $R = 1\,\text{ft}$, $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (1), and rounding to three significant figures we have

$$\boxed{\vec{v}_C = (21.5\,\hat{\imath} + 7.51\,\hat{\jmath})\,\text{ft/s}.}$$

For the acceleration, we then have

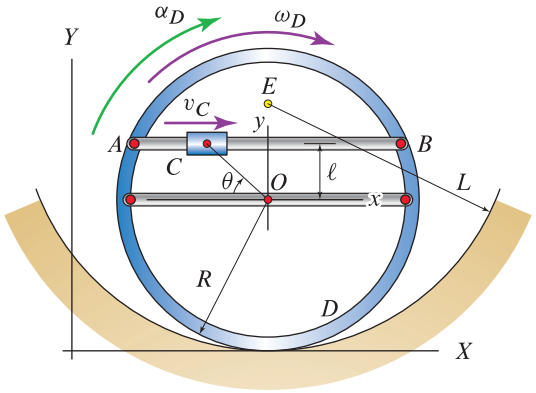
$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{a}_{C/O} + 2\vec{\Omega}\times\vec{v}_{C/O} + \dot{\vec{\Omega}}\times\vec{r}_{C/O} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}_{C/O}\right)\\ &= \frac{R^2\omega_D^2}{L-R}\,\hat{\jmath} - 2\omega_D\,\hat{k}\times v_C\,\hat{\imath} + \alpha_D\,\hat{k}\times(-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath}) + \omega_D\,\hat{k}\times\left[\omega_D\,\hat{k}\times(-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath})\right]\\ &= \left(\ell\omega_D^2\cot\theta\right)\hat{\imath} + \left(-2v_C\omega_D - \ell\omega_D^2 + \frac{R^2\omega_D^2}{L-R}\right)\hat{\jmath},\end{aligned}\tag{2}$$

where we have used the fact that $v_O = 0$, $\vec{\Omega} = -\omega_D\,\hat{k}$, $\dot{\vec{\Omega}} = -\alpha_D\,\hat{k} = \vec{0}$, and $\vec{a}_O = \frac{R^2\omega_D^2}{L-R}\,\hat{\jmath}$. Substituting known values $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (2), and rounding to three significant figures we have

$$\boxed{\vec{a}_C = (105\,\hat{\imath} - 166\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 6.138

The wheel D rotates without slip over a curved cylindrical surface with constant radius of curvature $L = 1.6\,\text{ft}$ and center at the fixed point E. The XY frame is attached to the rolling surface, and it is inertial. The xy frame is attached to D at O and rotates with it at an angular velocity $\omega_D = 14\,\text{rad/s}$ and an angular acceleration $\alpha_D = 1.3\,\text{rad/s}^2$. Collar C slides along the bar AB with a constant velocity $v_C = 4\,\text{ft/s}$ relative to the xy frame. Letting $\ell = 0.25\,\text{ft}$ and $R = 1\,\text{ft}$, determine the inertial velocity and acceleration of C, at the instant shown, when $\theta = 25°$ and the xy frame is parallel to the XY frame. Express your result in the xy and XY frames.



Solution

From the geometry of the problem, we have $\vec{r}_{C/O} = -\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath}$. From the concept of rotating reference frames, we have

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{v}_{C/O} + \vec{\Omega}\times\vec{r}_{C/O}\\ &= (R\omega_D + v_C)\hat{\imath} + \omega_D\,\hat{k}\times(\ell\cot\theta\,\hat{\imath} - \ell\,\hat{\jmath})\\ &= (R\omega_D + v_C + \ell\omega_D)\,\hat{\imath} + \ell\omega_D\cot\theta\,\hat{\jmath},\end{aligned}\tag{1}$$

where we have used the fact that $\vec{v}_O = R\omega_D\,\hat{\imath}$ and $\vec{\Omega} = \omega_D\,\hat{k}$. Substituting known values $R = 1\,\text{ft}$, $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (1), and rounding to three significant figures we have

$$\boxed{\vec{v}_C = (21.5\,\hat{\imath} + 7.51\,\hat{\jmath})\,\text{ft/s}.}$$

For the acceleration, we then have

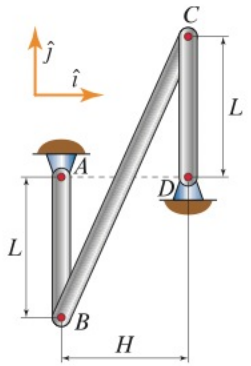
$$\begin{aligned}\vec{a}_C &= \vec{a}_O + \vec{a}_{C/O} + 2\vec{\Omega}\times\vec{v}_{C/O} + \dot{\vec{\Omega}}\times\vec{r}_{C/O} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}_{C/O}\right)\\ &= \frac{R^2\omega_D^2}{L-R}\,\hat{\jmath} - 2\omega_D\,\hat{k}\times v_C\,\hat{\imath} + \alpha_D\,\hat{k}\times(-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath}) + \omega_D\,\hat{k}\times\left[\omega_D\,\hat{k}\times(-\ell\cot\theta\,\hat{\imath} + \ell\,\hat{\jmath})\right]\\ &= \left(\ell\alpha_D + \ell\omega_D^2\cot\theta\right)\hat{\imath} + \left(-2v_C\omega_D - \ell\omega_D^2 + \frac{R^2\omega_D^2}{L-R} + \ell\alpha_D\cos\theta\right)\hat{\jmath},\end{aligned}\tag{2}$$

where we have used the fact that $v_O = 0$, $\vec{\Omega} = -\omega_D\,\hat{k}$, $\dot{\vec{\Omega}} = -\alpha_D\,\hat{k}$, and $\vec{a}_O = \frac{R^2\omega_D^2}{L-R}\,\hat{\jmath}$. Substituting known values $\ell = 0.25\,\text{ft}$, $\omega_D = 14\,\text{rad/s}$, $v_C = 4\,\text{ft/s}$, $\theta = 25°$ into Eq. (2), and rounding to three significant figures we have

$$\boxed{\vec{a}_C = (107\,\hat{\imath} - 166\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 6.102

At the instant shown, bars AB and CD are vertical. In addition, point C is moving to the left with a speed of 4 m/s, and the magnitude of the acceleration of C is 55 m/s^2. Letting $L = 0.5$ m and $H = 0.2$ m, determine the angular accelerations of bars AB and BC.



Solution

To determine the angular accelerations of bars AB and BC using the vector approach to rigid body acceleration analysis, we first need to determine the angular velocities of these bars. Hence, we begin the solution of this problem with the velocity analysis.

The fixed point D is the center of rotation for the bar CD. At the instant shown, $\vec{v}_C = -v_C\,\hat{\imath}$, with $v_C = 4$ m/s. Letting $\vec{\omega}_{CD} = \omega_{CD}\,\hat{k}$ be the angular velocity of the bar CD, since $\vec{v}_C = \vec{\omega}_{CD} \times \vec{r}_{C/D}$, with $\vec{r}_{C/D} = L\,\hat{\jmath}$, we have

$$v_C = \omega_{CD} L \quad \Rightarrow \quad \omega_{CD} = v_C/L. \tag{1}$$

Now we observe that the fixed point A is the center of rotation of the bar AB. Therefore, at the instant shown the velocity of B is horizontal. This observation is important because it implies that, at the instant shown, points B and C are moving along parallel lines. In turn, this implies that the instantaneous center of rotation of bar BC is at infinity and, consequently,

$$\vec{\omega}_{BC} = \vec{0}. \tag{2}$$

The result in Eq. (2) implies that the bar BC is translating and that therefore $\vec{v}_B = \vec{v}_C = -v_C\,\hat{\imath}$. Since we must also have $\vec{v}_B = \vec{\omega}_{AB} \times \vec{r}_{B/A}$, where $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$ and $\vec{r}_{B/A} = -L\,\hat{\jmath}$, we can conclude that

$$-v_C = \omega_{AB} L \quad \Rightarrow \quad \omega_{AB} = -v_C/L. \tag{3}$$

This concludes the velocity analysis and we now turn to the velocity analysis.

Letting $\vec{\alpha}_{CD} = \alpha_{CD}\,\hat{k}$ be the angular acceleration of CD, for the acceleration of C we can write

$$\vec{a}_C = \vec{a}_D + \vec{\alpha}_{CD} \times \vec{r}_{C/D} - \omega_{CD}^2 \vec{r}_{C/D} = \alpha_{CD}\,\hat{k} \times L\,\hat{\jmath} - \omega_{CD}^2 (L\,\hat{\jmath}) = -L\alpha_{CD}\,\hat{\imath} - \frac{v_C^2}{L}\,\hat{\jmath}, \tag{4}$$

where we have accounted for the second of Eqs. (1). Recall that one of the given of the problem is the magnitude of the acceleration of C, namely, $|\vec{a}_C| = 55$ m/s^2. Hence, using Eq. (4), we have

$$|\vec{a}_C|^2 = L^2\alpha_{CD}^2 + \frac{v_C^4}{L^2} \quad \Rightarrow \quad \alpha_{CD} = \pm\sqrt{\frac{|\vec{a}_C|^2 L^2 - v_C^4}{L^2}} = \pm 89.47\ \text{rad/s}^2. \tag{5}$$

Note that, since the problem statement does not indicate whether α_{CD} is positive or negative, our solution will have to take into account both of the possible values of α_{CD}.

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Letting $\vec{\alpha}_{AB} = \alpha_{AB}\,\hat{k}$ be the angular acceleration of AB, for the acceleration of B we have

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A} = \alpha_{AB}\,\hat{k} \times (-L)\,\hat{\jmath} - \omega_{AB}^2 (-L)\,\hat{\jmath} = L\alpha_{AB}\,\hat{\imath} + \frac{v_C^2}{L}\,\hat{\jmath}, \qquad (6)$$

where we have accounted for the second of Eqs. (3). Then, letting $\vec{\alpha}_{BC} = \alpha_{BC}\,\hat{k}$ be the angular acceleration of BC, we can use $\vec{a}_B$ to express $\vec{a}_C$ as follows:

$$\begin{aligned} \vec{a}_C &= \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2 \vec{r}_{C/B} \\ &= L\alpha_{AB}\,\hat{\imath} + \frac{v_C^2}{L}\,\hat{\jmath} + \alpha_{BC}\,\hat{k} \times (H\,\hat{\imath} + 2L\,\hat{\jmath}) \\ &= (L\alpha_{AB} - 2L\alpha_{BC})\,\hat{\imath} + \left(\frac{v_C^2}{L} + H\alpha_{BC}\right)\hat{\jmath}, \end{aligned} \qquad (7)$$

where we have accounted for Eq. (2). Setting Eqs. (4) and (7) equal to each other component by component, we have

$$-L\alpha_{CD} = L\alpha_{AB} - 2L\alpha_{BC} \quad \text{and} \quad -\frac{v_C^2}{L} = \frac{v_C^2}{L} + H\alpha_{BC}. \qquad (8)$$

The above two equations form a system in the two unknowns α_{AB} and α_{BC} whose solution is

$$\alpha_{AB} = -\frac{4v_C^2}{HL} - \alpha_{CD} \quad \text{and} \quad \alpha_{BC} = -\frac{2v_C^2}{HL}. \qquad (9)$$

As mention earlier, α_{CD} can take on the two distinct values given in the second of Eqs. (5). Recalling that $v_C = 4\,\text{m/s}$, $L = 0.5\,\text{m}$, and $H = 0.2\,\text{m}$, for $\alpha_{CD} = -89.47\,\text{rad/s}^2$, i.e., for C accelerating downward and to the right,

$$\boxed{\vec{\alpha}_{AB} = -551\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -320\,\hat{k}\,\text{rad/s}^2.}$$

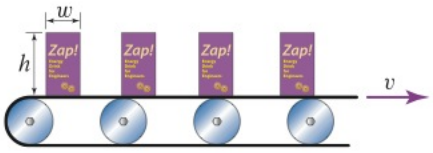
For $\alpha_{CD} = 89.47\,\text{rad/s}^2$, i.e., for C accelerating downward and to the left,

$$\boxed{\vec{\alpha}_{AB} = -729\,\hat{k}\,\text{rad/s}^2 \quad \text{and} \quad \vec{\alpha}_{BC} = -320\,\hat{k}\,\text{rad/s}^2.}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

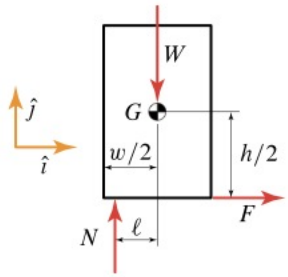
Problem 7.8

A conveyor belt must accelerate the cans from rest to $v = 18\,\text{ft/s}$ as quickly as possible. Treating each can as a uniform circular cylinder weighing 1.1 lb, find the minimum possible time to reach v so that the cans do not tip or slip on the conveyer. Assume that the acceleration is uniform and use $w = 2.71\,\text{in.}$, $h = 5\,\text{in.}$, and $\mu_s = 0.5$.



Solution

The FBD shown assumes that each can does not tip. The corresponding Newton-Euler equations are



$$\sum F_x: \qquad F = (W/g)a_{Gx}, \tag{1}$$

$$\sum F_y: \qquad N - W = (W/g)a_{Gy}, \tag{2}$$

$$\sum M_G: \quad F(h/2) - N\ell = I_G\alpha_{\text{can}}. \tag{3}$$

The maximum acceleration is achieved under the assumption that the motion occur in an impending slip condition, i.e.,

$$F = \mu_s N. \tag{4}$$

Due to the no tipping assumption, the kinematic equations are

$$a_{Gy} = 0 \quad \text{and} \quad \alpha_{\text{can}} = 0. \tag{5}$$

After substituting the kinematic equations into the Newton-Euler equations, Eqs. (1)–(4) yield a system of four equations in the four unknowns F, N, a_{Gx}, and ℓ. The solution of these equations yields the following result:

$$a_{Gx} = \mu_s g, \quad \ell = \tfrac{1}{2}h\mu_s, \quad \text{and} \quad N = W. \tag{6}$$

Using the given parameters, we have $\ell = 0.1042\,\text{ft}$ which is smaller than $w/2 = 0.1129\,\text{ft}$. Hence, the solution is acceptable.

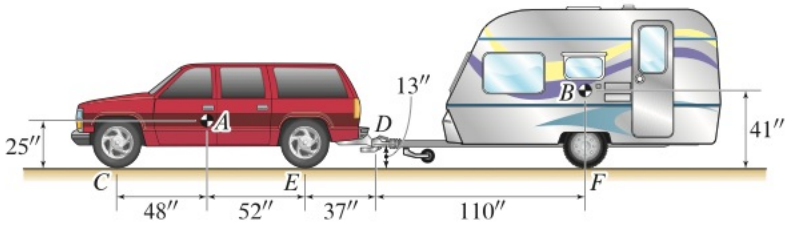
Since the horizontal acceleration is constant, applying constant acceleration equations, we have

$$\boxed{t_{\min} = v/(a_{Gx}) = v/(g\mu_s) = 1.12\,\text{s}.} \tag{7}$$

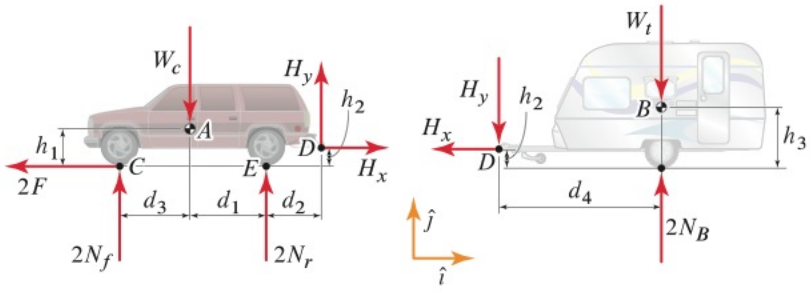
This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problem 7.10

The 3300 lb front-wheel-drive car whose mass center is at A is pulling a 4300 lb trailer whose mass center is at B. The car and trailer start from rest and accelerate uniformly to 60 mph in 18 s. Determine the forces on all tires as well as the total force acting on the car due to the trailer. In addition, determine the friction required so that the wheels of the car do not slip. Assume that the car and trailer are laterally symmetric and that the rotational inertia of the wheels is negligible. Note that the mass center of the trailer is directly above the axle of the rear wheel.



Solution



Referring to the FBD of the car, the Newton-Euler equations for the car are

$$\sum F_x: \qquad H_x - 2F = (W_c/g)a_{Ax}, \tag{1}$$

$$\sum F_y: \qquad 2N_f + 2N_r + H_y - W_c = (W_c/g)a_{Ay}, \tag{2}$$

$$\sum M_A: \quad 2N_r d_1 + H_x(h_1 - h_2) + H_y(d_1 + d_2) - 2Fh_1 - 2N_f d_3 = I_A\alpha_c, \tag{3}$$

where I_A is the mass moment of the car about the car's mass center A. The Newton-Euler equations for the trailer are

$$\sum F_x: \qquad -H_x = (W_t/g)a_{Bx}, \tag{4}$$

$$\sum F_y: \qquad 2N_B - H_y - W_t = (W_t/g)a_{By}, \tag{5}$$

$$\sum M_B: \quad H_y d_4 - H_x(h_3 - h_2) = I_B\alpha_t, \tag{6}$$

where I_B is the mass moment of inertia of the trailer about the trailer's center of mass B.

Recalling that the acceleration in the horizontal direction is constant and that the the vehicles start from rest, the kinematic equations for the car are

$$a_{Ax} = -(v_f/t_f), \quad a_{Ay} = 0, \quad \text{and} \quad \alpha_c = 0, \tag{7}$$

where $v_f = 60\,\text{mph} = 88.00\,\text{ft/s}$ and $t_f = 18\,\text{s}$. For the trailer, the kinematic equations are

$$a_{Bx} = -(v_f/t_f), \quad a_{By} = 0, \quad \text{and} \quad \alpha_t = 0. \tag{8}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Substituting the kinematic equations into the Newton-Euler equations, Eqs. (1)–(6) yield a system of six equations in the six unknowns N_f, N_r, F, H_x, H_y, and N_B. The solution of these equations yields the following result:

$$N_f = \frac{W_c d_4(d_1 g t_f - h_1 v_f) + W_t v_f(d_2 h_3 - d_2 h_2 - d_4 h_2)}{2 g t_f d_4(d_1 + d_3)} = 783.7\,\text{lb}, \tag{9}$$

$$N_r = \frac{W_c d_4(d_3 g t_f + h_1 v_f) + W_t v_f[(d_1 + d_2 + d_3)(h_2 - h_3) + h_2 d_4]}{2 g t_f d_4(d_1 + d_3)} = 783.2\,\text{lb}, \tag{10}$$

$$F = \frac{(W_t + W_c) v_f}{2 g t_f} = 576.9\,\text{lb}, \tag{11}$$

$$N_B = \frac{[v_f(h_3 - h_2) + g t_f d_4] W_t}{2 g t_f d_4} = 2233\,\text{lb}, \tag{12}$$

$$H_x = \frac{W_t v_f}{g t_f} = 652.9\,\text{lb}, \tag{13}$$

$$H_y = \frac{h_3 - h_2}{d_4} \frac{W_t v_f}{g t_f} = 166.2\,\text{lb}, \tag{14}$$

where we have used the following numerical data:

$$W_c = 3300\,\text{lb}, \quad W_t = 4300\,\text{lb}, \quad v_f = 60\,\text{mph} = 88\,\text{ft/s}, \quad t_f = 18\,\text{s}, \tag{15}$$

$$d_1 = 52\,\text{in.} = 4.333\,\text{ft}, \quad d_2 = 37\,\text{in.} = 3.083\,\text{ft}, \tag{16}$$

$$d_3 = 48\,\text{in.} = 4.000\,\text{ft}, \quad d_4 = 110\,\text{in.} = 9.167\,\text{ft}, \tag{17}$$

$$h_1 = 25\,\text{in.} = 2.083\,\text{ft}, \quad h_2 = 13\,\text{in.} = 1.083\,\text{ft}, \quad h_3 = 41\,\text{in.} = 3.417\,\text{ft}, \quad g = 32.2\,\text{ft/s}^2. \tag{18}$$

Using the above results, we have

$$(\mu_s)_{\min} = \frac{F}{N_f} = 0.7361. \tag{19}$$

Referring to the FBD and expressing our answers to 3 significant figures, we have that the forces on all tires are

$$\boxed{F = 577\,\text{lb}, \quad N_f = 784\,\text{lb}, \quad N_r = 783\,\text{lb}, \quad \text{and} \quad N_B = 2230\,\text{lb}.}$$

The force acting on the car due to the trailer is

$$\boxed{\vec{H} = H_x\,\hat{\imath} + H_y\,\hat{\jmath} = (653\,\hat{\imath} + 166\,\hat{\jmath})\,\text{lb} \quad \Rightarrow \quad |\vec{H}| = 674\,\text{lb}.}$$

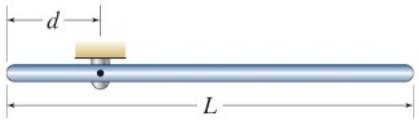
Finally, minimum static friction coefficient required not to slip is

$$\boxed{(\mu_s)_{\min} = 0.736.}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

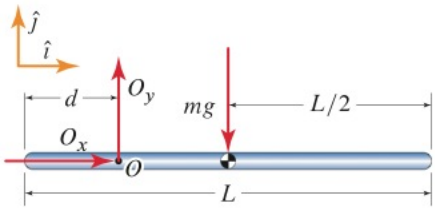
Problem 7.18

The uniform thin bar of length L and mass m is released from rest in the horizontal position shown. Determine the distance d at which the pin should be located from the end of the bar so that it has the maximum possible angular acceleration $\alpha_{\max}$. In addition, determine the value of $\alpha_{\max}$.



Solution

Based on the FBD shown, the Newton-Euler equations are



$$\sum F_x: \quad O_x = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad O_y - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad -O_y\left(\frac{L}{2} - d\right) = I_G\alpha_{g\,\text{bar}}, \tag{3}$$

where α_{bar} is the angular acceleration of the bar and where

$$I_G = \frac{1}{12}mL^2, \tag{4}$$

is the mass moment of the bar about G. Since the bar is in a fixed axis rotation about O, we have

$$a_{Gx} = -\left(\frac{L}{2} - d\right)\omega_{\text{bar}}^2 = 0 \quad \text{and} \quad a_{Gy} = \left(\frac{L}{2} - d\right)\alpha_{\text{bar}}, \tag{5}$$

where ω_{bar} is the angular velocity of the bar. Substituting the kinematic equations and the expression for I_G into the Newton-Euler equations, we have a system of three equations in the three unknowns O_x, O_y, and α_{bar} whose solution is

$$O_x = 0, \quad O_y = \frac{1}{4}\frac{mgL^2}{3d^2 - 3dL + L^2}, \quad \alpha_{\text{bar}} = \frac{6dg - 3gL}{6d^2 - 6dL + 2L^2}. \tag{6}$$

We now proceed to find the value of d such that α_{bar} is maximum. Differentiating the expression for α_{bar} with respect to d and setting the result equal to zero, we have

$$-\frac{3}{2}g\frac{6d^2 - 6dL + L^2}{(3d^2 - 3dL + L^2)^2} = 0 \quad \Rightarrow \quad d_1 = \tfrac{1}{6}(3 - \sqrt{3})L \quad \text{and} \quad d_2 = \tfrac{1}{6}(3 + \sqrt{3})L. \tag{7}$$

There are two distinct values of d for which a relative extremum of α_{bar} is achieved. The values of α_{bar} corresponding to the two values of d just found are

$$\alpha_{\text{bar}} = \begin{cases} -\sqrt{3}g/L & \text{for } d = d_1, \\ \sqrt{3}g/L & \text{for } d = d_2. \end{cases}$$

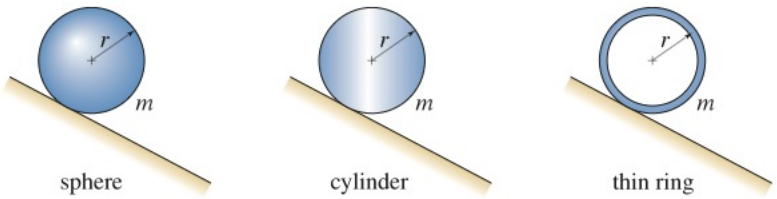
Hence, the relative minimum and maximum values of α_{bar} are equal in absolute value and, if we do not distinguish between a clockwise and a counterclockwise rotation of the bar, both results are acceptable. Before providing the final answer, we need to verify whether or not absolute maximum values of α_{bar} are obtained for $d = 0$ and $d = L$. Indeed, for $d = 0$ and $d = L$, $|\alpha_{\text{bar}}| = 3g/(2L) < \sqrt{3}g/L$. Hence, the final result is

$$|\alpha_{\text{bar}}|_{\max} = \sqrt{3}\frac{g}{L} \quad \text{for } d = \tfrac{1}{6}(3 - \sqrt{3})L \text{ or } d = \tfrac{1}{6}(3 + \sqrt{3})L.$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problem 7.22

The sphere, cylinder, and thin ring each have mass m and radius r. Each is released from rest on identical inclines. Assuming they all roll without slipping, which will have the largest initial angular acceleration? In addition, which will reach the bottom of the incline first?



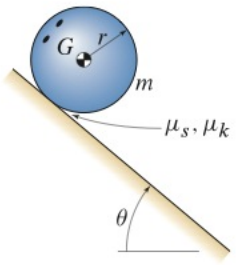
Solution

Since the three object would be subjected to the same force system, the object that will have the largest angular acceleration is the object with the smallest mass moment of inertia. The mass moment of inertia about its mass center for a sphere is $\frac{2}{5}mr^2$, that for a cylinder is $\frac{1}{2}mr^2$, and that for a thin ring is mr^2. Hence, the object with the smallest mass moment of inertia is the sphere. Given that the motion is assumed to occur without any slip, the linear acceleration of the mass center of each of the three object is directly proportional to the angular acceleration. Hence, the sphere will reach the bottom of the incline first.

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problems 7.23 and 7.24

A bowling ball of radius r, mass m, and radius of gyration k_G is released from rest on a rough surface that is inclined at the angle θ with respect to the horizontal. The coefficients of static and kinetic friction between the ball and the incline are μ_s and μ_k, respectively. Assume that the mass center G is at the geometric center.

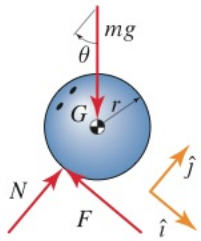


Problem 7.23 Assuming the ball rolls without slip, determine the angular acceleration of the ball and the friction and normal force between the ball and the incline. In addition, find the minimum value of μ_s that is compatible with this motion.

Problem 7.24 Let the weight of the ball be 14 lb, the radius be 4.25 in., and the radius of gyration be $k_G = 2.6$ in. If the incline is 10 ft long, determine the time it takes the ball to reach the bottom of the incline and the speed of G when it reaches the bottom. Use $\theta = 40°$, $\mu_s = 0.2$, and $\mu_k = 0.15$.

Solution to 7.23

Based on the FBD shown, the Newton-Euler equations are



$$\sum F_x: \quad -F + mg\sin\theta = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg\cos\theta = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad -Fr = I_G\alpha_b, \tag{3}$$

where α_b is the angular velocity of the ball and where, modeling the ball as a uniform sphere, we have

$$I_G = mk_G^2. \tag{4}$$

Since the ball is assumed to roll without slip, the kinematic equations are

$$a_{Gx} = -\alpha_b t \quad \text{and} \quad a_{Gy} = 0. \tag{5}$$

Substituting the kinematic equations and the expression for the mass moment of inertia into the Newton-Euler equations gives three equations in the three unknowns F, N, and α_b whose solution is

$$F = \frac{k_G^2}{k_G^2 + r^2} mg\sin\theta, \quad N = mg\cos\theta, \quad \text{and} \quad \alpha_b = -\frac{gr}{k_G^2 + r^2}\sin\theta.$$

The minimum static friction coefficient is therefore

$$(\mu_s)_{\min} = F/N = \frac{k_G^2}{k_G^2 + r^2}\tan\theta.$$

Solution to 7.24

The FBD for this problem is the same as that for Problem 7.23 and therefore the Newton-Euler equations for this problem are the same as those given in Eqs. (1)–(3). The expression for the mass moment of inertia is also identical to that used in to be used in Problem 7.23, i.e., $I_G = mk_G^2$. Contrary to what was assumed in

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problem 7.23, here it is assumed that the ball slips relative to the incline. We therefore have the following force law:

$$F = \mu_k N. \tag{6}$$

Finally, having assumed that the ball is slipping, the kinematic equations reduce to

$$a_{Gy} = 0. \tag{7}$$

Substituting the force law, the expression for the mass moment of inertia, and the kinematic equations into the Newton-Euler equations yields a system of three equations in the unknowns N, a_{Gx}, and α_b whose solution is

$$N = mg\cos\theta, \quad a_{Gx} = g(\sin\theta - \mu_k\cos\theta), \quad \text{and} \quad \alpha_b = -\frac{gr}{k_G^2}\mu_k\cos\theta. \tag{8}$$

To verify that our assumption concerning the direction of slip (to match the direction of the force F on the chosen FBD) was indeed correct, we need to verify that the point of contact between the ball and the incline accelerates *down* the incline. Letting D denote the point of contact between the incline and the ball, we have

$$\vec{a}_D = \vec{a}_G + \alpha_b\hat{k}\times\vec{r}_{D/G} \quad \Rightarrow \quad \vec{a}_D = \left[g(\sin\theta - \mu_k\cos\theta) - \frac{gr^2}{k_G^2}\mu_k\cos\theta\right]\hat{\imath} = (7.11\,\text{ft/s})\,\hat{\imath}. \tag{9}$$

The above results confirms that our assumption was indeed correct.

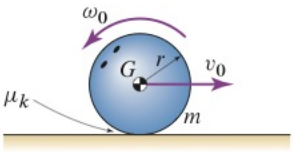
We can now proceed to complete the solution to our problem. Since the acceleration of the center of mass of the ball is constant and since the angular acceleration of the ball is constant, applying constant acceleration equations, and recalling that the ball is released from rest, we have

$$\boxed{\begin{aligned}(v_G)_{\text{final}} &= \sqrt{2gL(\sin\theta - \mu_k\cos\theta + \sin\theta)} = 18.4\,\text{ft/s},\\ t_{\text{final}} &= (v_G)_{\text{final}}/a_{Gx} = \frac{\sqrt{2gL(\sin\theta - \mu_k\cos\theta + \sin\theta)}}{g(\sin\theta - \mu_k\cos\theta)} = 1.08\,\text{s}.\end{aligned}}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problems 7.25 and 7.26

A bowling ball is thrown onto a lane with a backspin ω_0 and forward velocity v_0. The mass of the ball is m, its radius is r, its radius of gyration is k_G, and the coefficient of kinetic friction between the ball and the lane is μ_k. Assume the mass center G is at the geometric center.

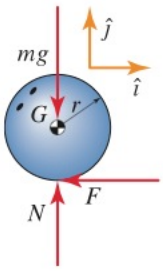


Problem 7.25 Find the acceleration of G and the ball's angular acceleration while the ball is slipping.

Problem 7.26 For a 14 lb ball with $r = 4.25$ in., $k_G = 2.6$ in, $\omega_0 = 10$ rad/s, and $v_0 = 17$ mph, determine the time it takes for the ball to start rolling without slip and its speed when it does so. In addition, determine the distance it travels before it starts rolling without slip. Use $\mu_k = 0.10$.

Solution to 7.25

Based on the FBD shown, the Newton-Euler equations are



$$\sum F_x: \quad -F = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad -Fr = I_G\alpha_b, \tag{3}$$

where α_b is the angular velocity of the ball and where

$$I_G = mk_G^2. \tag{4}$$

Since the ball slides (at least at first) over the lane, the force law for this problem is

$$F = \mu_k N. \tag{5}$$

The kinematic equations consist only of the following relation:

$$a_{Gy} = 0. \tag{6}$$

Substituting the force law, the kinematic equations, and the expression for the mass moment of inertia into the Newton-Euler equations gives three equations in the three unknowns N, a_{Gx}, and α_b whose solution is

$$N = mg \quad a_{Gx} = -\mu_k g, \quad \text{and} \quad \alpha_b = -\frac{gr\mu_k}{k_G^2}.$$

Hence, the final answer is

$$\boxed{a_{Gx} = -\mu_k g \quad \text{and} \quad \alpha_b = -\frac{gr\mu_k}{k_G^2}.}$$

Solution to 7.26

In this problem, the governing equations are identical to those of Problem 7.25. Therefore, we have that the solution of this problem in terms of acceleration of the mass center and angular acceleration of the ball are the same as those of Problem 7.25.

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

To determine when the ball starts slipping, we need to determine the velocity of the point on the ball that is in contact with the ground as a function of time and then determine how long it takes for that velocity to become equal to zero. Observing that the computed accelerations are constant, applying constant acceleration equations, we have that the velocity of the mass center G as a function of time is

$$\vec{v}_G = (v_0 - \mu_k g t)\,\hat{\imath}. \tag{7}$$

Again using constant acceleration equations, we have that the angular velocity of the ball is

$$\vec{\omega}_b = \left(\omega_0 - \frac{\mu_k r g}{k_G^2} t\right)\hat{k}. \tag{8}$$

Denoting the point on the ball that is in contact with the ground by D and observing that the position of D relative to G is $\vec{r}_{D/G} = -r\,\hat{\jmath}$, we have

$$\vec{v}_D = \vec{v}_G + \vec{\omega}_b \times \vec{r}_{D/G} = \left[v_0 + r\omega_0 - g t \mu_k \left(1 + \frac{r^2}{k_G^2}\right)\right]\hat{\imath}. \tag{9}$$

The velocity of point D is the velocity with which the ball slips relative to the floor. The time required to achieve rolling without slip is therefore the time it takes for $\vec{v}_D = \vec{0}$, i.e.,

$$\vec{v}_D = \vec{0} \quad \Rightarrow \quad v_0 + r\omega_0 - g t_r \mu_k \left(1 + \frac{r^2}{k_G^2}\right) = 0 \quad \Rightarrow \quad t_r = \frac{k_G^2 v_0 + k_G^2 r\omega_0}{g\mu_k (k_G^2 + r^2)}, \tag{10}$$

where t_r is the time at which the ball starts rolling without slip. As far as the distance traveled by the ball from $t = 0$ until t_r, applying constant acceleration equations, we have

$$d = v_0 t_r + \tfrac{1}{2} a_{Gx} t_r^2 = \frac{k_G^2 (v_0 + \omega_0 r)[2v_0 r^2 + k_G^2 (v_0 - r\omega_0)]}{2g\mu_k (k_G^2 + r^2)^2}. \tag{11}$$

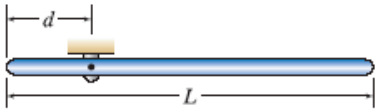
Substituting the values provided we have

$$\boxed{t_r = 2.41\,\text{s} \quad \text{and} \quad d = 50.7\,\text{ft}.}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

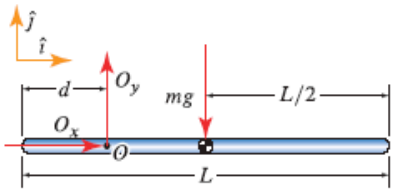
Problem 7.18

The uniform thin bar of length L and mass m is released from rest in the horizontal position shown. Determine the distance d at which the pin should be located from the end of the bar so that it has the maximum possible angular acceleration α_{max}. In addition, determine the value of α_{max}.



Solution

Based on the FBD shown, the Newton-Euler equations are



$$\sum F_x: \quad O_x = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad O_y - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad -O_y\left(\frac{L}{2} - d\right) = I_G\alpha_{g\,\text{bar}}, \tag{3}$$

where α_{bar} is the angular acceleration of the bar and where

$$I_G = \frac{1}{12}mL^2, \tag{4}$$

is the mass moment of the bar about G. Since the bar is in a fixed axis rotation about O, we have

$$a_{Gx} = -\left(\frac{L}{2} - d\right)\omega_{\text{bar}}^2 = 0 \quad \text{and} \quad a_{Gy} = \left(\frac{L}{2} - d\right)\alpha_{\text{bar}}, \tag{5}$$

where ω_{bar} is the angular velocity of the bar. Substituting the kinematic equations and the expression for I_G into the Newton-Euler equations, we have a system of three equations in the three unknowns O_x, O_y, and α_{bar} whose solution is

$$O_x = 0, \quad O_y = \frac{1}{4}\frac{mgL^2}{3d^2 - 3dL + L^2}, \quad \alpha_{\text{bar}} = \frac{6dg - 3gL}{6d^2 - 6dL + 2L^2}. \tag{6}$$

We now proceed to find the value of d such that α_{bar} is maximum. Differentiating the expression for α_{bar} with respect to d and setting the result equal to zero, we have

$$-\frac{3}{2}g\frac{6d^2 - 6dL + L^2}{(3d^2 - 3dL + L^2)^2} = 0 \quad \Rightarrow \quad d_1 = \tfrac{1}{6}(3 - \sqrt{3})L \quad \text{and} \quad d_2 = \tfrac{1}{6}(3 + \sqrt{3})L. \tag{7}$$

There are two distinct values of d for which a relative extremum of α_{bar} is achieved. The values of α_{bar} corresponding to the two values of d just found are

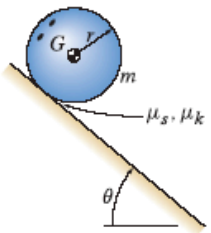
$$\alpha_{\text{bar}} = \begin{cases} -\sqrt{3}g/L & \text{for } d = d_1, \\ \sqrt{3}g/L & \text{for } d = d_2. \end{cases}$$

Hence, the relative minimum and maximum values of α_{bar} are equal in absolute value and, if we do not distinguish between a clockwise and a counterclockwise rotation of the bar, both results are acceptable. Before providing the final answer, we need to verify whether or not absolute maximum values of α_{bar} are obtained for $d = 0$ and $d = L$. Indeed, for $d = 0$ and $d = L$, $|\alpha_{\text{bar}}| = 3g/(2L) < \sqrt{3}g/L$. Hence, the final result is

$$|\alpha_{\text{bar}}|_{\max} = \sqrt{3}\frac{g}{L} \quad \text{for } d = \tfrac{1}{6}(3 - \sqrt{3})L \text{ or } d = \tfrac{1}{6}(3 + \sqrt{3})L.$$

Problems 7.23 and 7.24

A bowling ball of radius r, mass m, and radius of gyration k_G is released from rest on a rough surface that is inclined at the angle θ with respect to the horizontal. The coefficients of static and kinetic friction between the ball and the incline are μ_s and μ_k, respectively. Assume that the mass center G is at the geometric center.

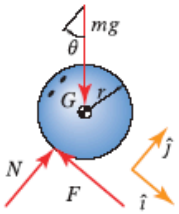


Problem 7.23 Assuming the ball rolls without slip, determine the angular acceleration of the ball and the friction and normal force between the ball and the incline. In addition, find the minimum value of μ_s that is compatible with this motion.

Problem 7.24 Let the weight of the ball be 14 lb, the radius be 4.25 in., and the radius of gyration be $k_G = 2.6$ in. If the incline is 10 ft long, determine the time it takes the ball to reach the bottom of the incline and the speed of G when it reaches the bottom. Use $\theta = 40°$, $\mu_s = 0.2$, and $\mu_k = 0.15$.

Solution to 7.23

Based on the FBD shown, the Newton-Euler equations are



$$\sum F_x: \quad -F + mg\sin\theta = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg\cos\theta = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad -Fr = I_G\alpha_b, \tag{3}$$

where α_b is the angular velocity of the ball and where, modeling the ball as a uniform sphere, we have

$$I_G = mk_G^2. \tag{4}$$

Since the ball is assumed to roll without slip, the kinematic equations are

$$a_{Gx} = -\alpha_b t \quad \text{and} \quad a_{Gy} = 0. \tag{5}$$

Substituting the kinematic equations and the expression for the mass moment of inertia into the Newton-Euler equations gives three equations in the three unknowns F, N, and α_b whose solution is

$$F = \frac{k_G^2}{k_G^2 + r^2} mg\sin\theta, \quad N = mg\cos\theta, \quad \text{and} \quad \alpha_b = -\frac{gr}{k_G^2 + r^2}\sin\theta.$$

The minimum static friction coefficient is therefore

$$(\mu_s)_{\min} = F/N = \frac{k_G^2}{k_G^2 + r^2}\tan\theta.$$

Solution to 7.24

The FBD for this problem is the same as that for Problem 7.23 and therefore the Newton-Euler equations for this problem are the same as those given in Eqs. (1)–(3). The expression for the mass moment of inertia is also identical to that used in to be used in Problem 7.23, i.e., $I_G = mk_G^2$. Contrary to what was assumed in

Problem 7.23, here it is assumed that the ball slips relative to the incline. We therefore have the following force law:

$$F = \mu_k N. \tag{6}$$

Finally, having assumed that the ball is slipping, the kinematic equations reduce to

$$a_{Gy} = 0. \tag{7}$$

Substituting the force law, the expression for the mass moment of inertia, and the kinematic equations into the Newton-Euler equations yields a system of three equations in the unknowns N, a_{Gx}, and α_b whose solution is

$$N = mg\cos\theta, \quad a_{Gx} = g(\sin\theta - \mu_k\cos\theta), \quad \text{and} \quad \alpha_b = -\frac{gr}{k_G^2}\mu_k\cos\theta. \tag{8}$$

To verify that our assumption concerning the direction of slip (to match the direction of the force F on the chosen FBD) was indeed correct, we need to verify that the point of contact between the ball and the incline accelerates *down* the incline. Letting D denote the point of contact between the incline and the ball, we have

$$\vec{a}_D = \vec{a}_G + \alpha_b\hat{k}\times\vec{r}_{D/G} \quad \Rightarrow \quad \vec{a}_D = \left[g(\sin\theta - \mu_k\cos\theta) - \frac{gr^2}{k_G^2}\mu_k\cos\theta\right]\hat{\imath} = (7.11\,\text{ft/s})\,\hat{\imath}. \tag{9}$$

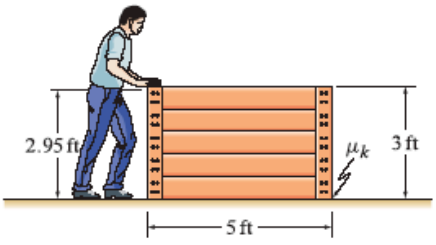
The above results confirms that our assumption was indeed correct.

We can now proceed to complete the solution to our problem. Since the acceleration of the center of mass of the ball is constant and since the angular acceleration of the ball is constant, applying constant acceleration equations, and recalling that the ball is released from rest, we have

$$\begin{aligned}(v_G)_{\text{final}} &= \sqrt{2gL(\sin\theta - \mu_k\cos\theta + \sin\theta)} = 18.4\,\text{ft/s},\\ t_{\text{final}} &= (v_G)_{\text{final}}/a_{Gx} = \frac{\sqrt{2gL(\sin\theta - \mu_k\cos\theta + \sin\theta)}}{g(\sin\theta - \mu_k\cos\theta)} = 1.08\,\text{s}.\end{aligned}$$

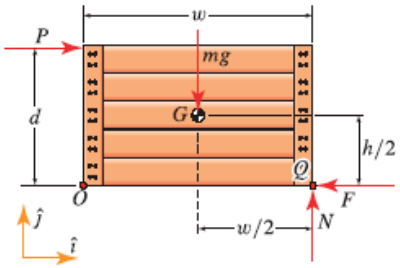
Problem 7.29

Solve Example 7.3 on p. 562 by assuming that the crate slips *and* tips. In doing so, show that this motion is not possible for the given conditions since part of your solution will not be physically admissible.



Solution

If we assume that the crate slips *and* tips, then its FBD is as shown. The Newton-Euler equations corresponding to this FBD are:



$$\sum F_x: \qquad P - F = ma_{Gx}, \tag{1}$$

$$\sum F_y: \qquad N - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad \frac{w}{2}N - \frac{h}{2}F - P\left(d - \frac{h}{2}\right) = I_G\alpha, \tag{3}$$

where, treating the crate as a uniform body,

$$I_G = \tfrac{1}{12}m\left(w^2 + h^2\right). \tag{4}$$

Since the crate slips, we have

$$F = \mu_k N. \tag{5}$$

If the body is to tip, it does so about point Q, which is then the only point on the crate that moves along the x axis. Hence, the acceleration of G must have the form

$$\vec{a}_G = \vec{a}_Q + \vec{\alpha}_c \times \vec{r}_{G/Q} - \omega_c^2 \vec{r}_{G/Q}, \tag{6}$$

where $\vec{a}_Q = a_{Qx}\,\hat{\imath}$ and $\vec{r}_{G/Q} = -(w/2)\,\hat{\imath} + (h/2)\,uj$. Recalling that the crate is initially at rest, i.e., $\omega_c = 0$, we have

$$a_{Gx} = a_{Qx} - \tfrac{1}{2}h\alpha_c \quad \text{and} \quad a_{Gy} = -\tfrac{1}{2}w\alpha_c. \tag{7}$$

Substituting the expression for the mass moment of inertia, the force laws, and the kinematic equations into the Newton-Euler equations, we obtain a system of three equations in the three unknowns N, a_{Qx}, and α_c whose solution is

$$a_{Qx} = \frac{3mghw + 4h^2(P - mg\mu_k) + w^2(4P - mg\mu_k) - 6dP(h + w\mu_k)}{m(h^2 + 4w^2 - 3hw\mu_k)} = 22.4\ \text{ft/s}^2,$$

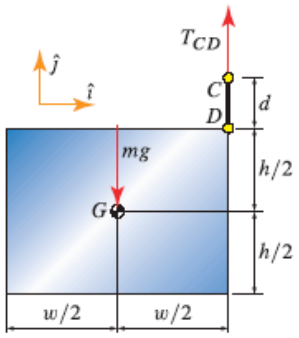
$$N = \frac{6dPw - 3hPw + mg(h^2 + w^2)}{h^2 + 4w^2 - 3hw\mu_k} = 88.1\ \text{lb},$$

$$\alpha_c = \frac{6[mgw - 2Pd + h(P - gm\mu_k)]}{m(h^2 + 4w^2 - 3hw\mu_k)} = 3.43\ \text{rad/s}^2.$$

Notice that the sign of the angular acceleration α_c contradicts our assumption that the crate tips and slips, since the only physically meaningful tipping would have been in the clockwise direction. Hence, this solution shows that the crate cannot tip and slip.

Solution to 7.31

Based on the FBD shown, the Newton-Euler equations are



$$\sum F_x: \qquad 0 = ma_{Gx}, \tag{8}$$

$$\sum F_y: \qquad T_{CD} - mg = ma_{Gy}, \tag{9}$$

$$\sum M_G: \qquad T_{CD}(w/2) = I_G\alpha_s, \tag{10}$$

where α_s is the angular velocity of the sign and where, since the sign is uniform,

$$I_G = \tfrac{1}{12}m(w^2 + h^2). \tag{11}$$

Since the cord CD is extensible and since the expression "right after release" implies that the sign has not moved appreciably from its position before the break of the cord AB, we have that the force in the cord CD right after release remains equal to that in the cord CD right before release. Given the symmetry of the problem, then the force in the cord CD is simply equal to half of the weight of the sign, i.e.,

$$T_{CD} = \tfrac{1}{2}mg. \tag{12}$$

Since the cord CD is elastic, it does not impose any kinematic restrictions to the motion of the sign and therefore to the motion of the sign's center of mass. Hence, after substituting Eqs. (11) and (12) into Eqs. (8)–(10) we have a system of three equations in the unknowns a_{Gx}, a_{Gy}, and α_s whose solution is

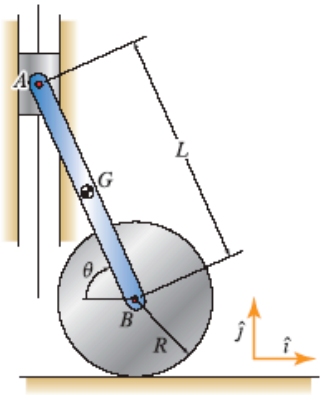
$$a_{Gx} = 0, \quad a_{Gy} = -\tfrac{1}{2}g, \quad \text{and} \quad \alpha_s = \frac{3gw}{h^2 + w^2}. \tag{13}$$

We now recall that $g = 9.81\,\text{m/s}^2$ and that we are given the following numerical data: $m = 30\,\text{kg}$, $h = 1.5\,\text{m}$, $w = 2\,\text{m}$, $d = 0.6\,\text{m}$. Therefore, expressing the acceleration in vector form and expressing all the answers using 3 significant figures, we have

$$\boxed{T_{CD} = \tfrac{1}{2}mg = 147\,\text{N}, \quad \text{and} \quad \vec{a}_G = a_{Gx}\,\hat{\imath} + a_{Gy}\,\hat{\jmath} = (-4.90\,\text{m/s}^2)\,\hat{\jmath}.} \tag{14}$$

Problems 7.39 through 7.42

The uniform thin rod AB couples the slider A, which moves along a frictionless guide, to the wheel B, which rolls without slip over a horizontal surface.



Problem 7.39 Assuming that A and B have negligible mass, that the mass of AB is m_{AB}, and that the system is released from rest at the angle θ, determine, immediately after release, the angular acceleration of the rod AB, the acceleration of the center of the wheel at B, and the angular acceleration of the wheel.

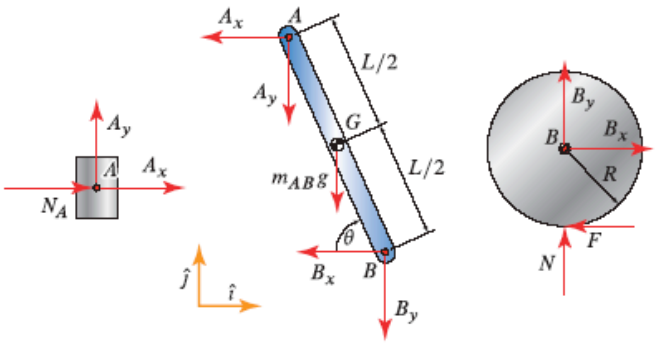
Problem 7.40 Assuming that A has negligible mass, B is a uniform disk of mass m_B, the mass of AB is m_{AB}, and the system is released from rest at the angle θ, determine, immediately after release, the angular acceleration of the rod AB, the acceleration of the center of the wheel at B, and the angular acceleration of the wheel.

Problem 7.41 Assuming that AB has negligible mass, the mass of A is m_A, B is a uniform disk of mass m_B, and the system is released from rest at the angle θ, determine, immediately after release, the angular acceleration of the rod AB, the acceleration of the center of the wheel at B, and the angular acceleration of the wheel.

Problem 7.42 Assuming that the mass of A is m_A, B is a uniform disk of mass m_B, the mass of AB is m_{AB}, and that the system is released from rest at the angle θ, determine, immediately after release, the angular acceleration of the rod AB, the acceleration of the center of the wheel at B, and the angular acceleration of the wheel.

Solution to 7.39

We consider the FBD of each element of the system. Notice that since the slider and the wheel are assumed to have negligible mass, their weights has been neglected. Based on the FBDs shown, the Newton-Euler equations for the slider are



$$\sum F_x: \quad A_x + N_A = 0, \qquad (1)$$

$$\sum F_y: \quad A_y = 0, \qquad (2)$$

where the right-hand side of the above equations is set to zero since the mass of A has been neglected. The Newton-Euler equations for the bar AB are

$$\sum F_x: \quad -A_x - B_x = m_{AB}a_{Gx}, \qquad (3)$$

$$\sum F_y: \quad -A_y - B_y - m_{AB}g = m_{AB}a_{Gy}, \qquad (4)$$

$$\sum M_G: \quad A_x \frac{L}{2}\sin\theta + A_y \frac{L}{2}\cos\theta - B_x \frac{L}{2}\sin\theta - B_y \frac{L}{2}\cos\theta = I_G \alpha_{AB}, \tag{5}$$

where the mass moment of inertia of the bar with respect to G is

$$I_G = \tfrac{1}{12} m_{AB} L^2. \tag{6}$$

Finally, the Newton-Euler equations for the wheel are

$$\sum F_x: \quad B_x - F = 0, \tag{7}$$

$$\sum F_y: \quad N + B_y = 0, \tag{8}$$

$$\sum M_B: \quad -FR = 0, \tag{9}$$

where we have neglected the mass and therefore the rotational inertia of the wheel.

Observing that A is constrained to move vertically, that B is constrained to move horizontally, and that the system is released from rest, enforcing planar rigid body kinematics at the time of release for the bar AB, i.e., enforcing $\vec{a}_A = \vec{a}_B + \alpha_{AB}\hat{k} \times \vec{r}_{A/B} - \omega_{AB}^2 \vec{r}_{A/B}$, with $\vec{r}_{A/B} = L(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$, we have

$$a_{Ay}\,\hat{\jmath} = a_{Bx}\,\hat{\imath} + \alpha_{AB}\hat{k} \times L(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \;\Rightarrow\; a_{Bx} - \alpha_{AB} L\sin\theta = 0 \text{ and } a_{Ay} = -\alpha_{AB} L\cos\theta. \tag{10}$$

Using these equations, the fact that $\omega_{AB} = 0$ at release, and the fact that $\vec{a}_G = \vec{a}_B + \alpha_{AB}\hat{k} \times \vec{r}_{G/B} - \omega_{AB}^2 \vec{r}_{G/B}$, with $\vec{r}_{G/B} = (L/2)(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$, we have

$$a_{Gx} = \frac{a_{Bx}}{2}, \quad a_{Gy} = -\frac{a_{Bx}\cos\theta}{2\sin\theta}, \quad \text{and} \quad \alpha_{AB} = \frac{a_{Bx}}{L\sin\theta}. \tag{11}$$

Substituting the expression for I_G in Eq. (6) and the kinematic equations in Eqs. (11) into the eight Newton-Euler equations written earlier, we have a system of 8 equations in the 8 unknowns A_x, A_y, N_A, B_x, B_y, F, N, and a_{Bx} whose solution is

$$A_x = -\tfrac{3}{8}\sin(2\theta) m_{AB} g, \quad A_y = 0, \quad N_A = \tfrac{3}{8}\sin(2\theta) m_{AB} g, \tag{12}$$

$$B_x = 0, \quad B_y = \tfrac{1}{8}[3\cos(2\theta) - 5] m_{AB} g, \quad N = \tfrac{1}{8}[5 - 3\cos(2\theta)] m_{AB} g, \quad F = 0, \tag{13}$$

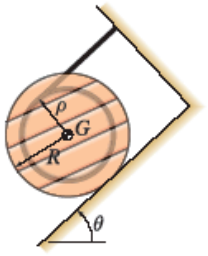
$$a_{Bx} = \tfrac{3}{4} g \sin(2\theta). \tag{14}$$

Using the result for a_{Bx} and the last of Eqs. (11), along with the fact that if the wheels rolls without slip we must have $a_{Bx} = -\alpha_w R$, where α_w is the angular acceleration of the wheel, the problems final answers are

$$\boxed{\alpha_{AB} = \tfrac{3}{2}(g/L)\cos\theta, \quad \vec{a}_B = \tfrac{3}{4} g \sin(2\theta)\,\hat{\imath}, \quad \text{and} \quad \alpha_w = -\tfrac{3}{4}(g/R)\sin(2\theta).}$$

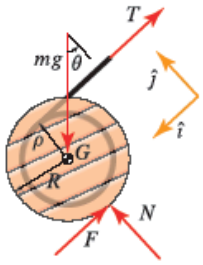
Problem 7.45

A spool of mass $m = 300\,\text{kg}$, inner and outer radii $\rho = 1.5\,\text{m}$ and $R = 2\,\text{m}$, respectively, and radius of gyration $k_G = 1.8\,\text{m}$, is placed on an incline with $\theta = 43°$. The cable that is wrapped around the spool and attached to the wall is initially taut. If the static and kinetic friction coefficients between the incline and the spool are $\mu_s = 0.35$ and $\mu_k = 0.3$, respectively, determine the acceleration of G, the angular acceleration of the spool, and the tension in the cable once the spool is released from rest.



Solution

Since the spool will roll without slip over the rope, the spool will slip over the incline. A downward motion of the center of spool corresponds to a clockwise rotation of the spool. In turn, this implies that the point of contact of the spool with the incline is slipping down the incline thus causing the friction force to point up the incline, as shown in the figure. Based on the FBD shown, the Newton-Euler equations for the spool are



$$\sum F_x: \quad -F - T + mg\sin\theta = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg\cos\theta = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad -FR + T\rho = I_G\alpha_s, \tag{3}$$

where the mass moment of inertia of the spool with respect to G is

$$I_G = mk_G^2. \tag{4}$$

Since the spool slips against the incline, the force law for this problem is

$$F = \mu_k N. \tag{5}$$

Since the spool rolls without slip over the rope and travels along the incline, we have

$$a_{Gx} = \rho\alpha_s \quad \text{and} \quad a_{Gy} = 0. \tag{6}$$

Substituting the expression for I_G, the force law, and the kinematic equations into the Newton-Euler equations, we have a system of three equations in the three unknowns N, T, and α_s whose solution is

$$\alpha_s = \frac{g[\rho\sin\theta - \mu_k(R+\rho)\cos\theta]}{\rho^2 + k_G^2}, \tag{7}$$

$$N = mg\cos\theta, \tag{8}$$

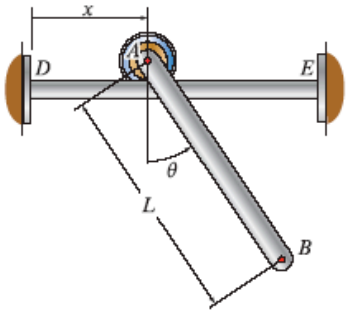
$$T = \frac{k_G^2\sin\theta - \mu_k(k_G^2 - R\rho)\cos\theta}{\rho^2 + k_G^2}mg. \tag{9}$$

Recalling that $\vec{a}_G = \rho\alpha_s\,\hat{\imath}$, using the given parameters, we have

$$\vec{a}_G = (0.684\,\text{m/s}^2)\,\hat{\imath}, \quad \alpha_s = 0.456\,\text{rad/s}^2. \quad \text{and} \quad T = 1160\,\text{N}.$$

Problems 7.47 through 7.49

The thin uniform bar AB of mass m and length L hangs from a wheel at A, which rolls freely on the horizontal bar DE. In the following problems, neglect the mass of the wheel and assume that the wheel never separates from the horizontal bar.



Problem 7.47 If the bar is released from rest at the angle θ, determine, immediately after release, the angular acceleration of the bar, the force on the bar at A, and the acceleration of end A.

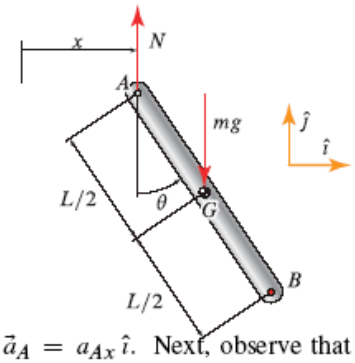
Problem 7.48 Find the equation(s) of motion of the bar, using the coordinates x and θ shown on the figure as the dependent variables.

Problem 7.49 Find the equation(s) of motion of the bar using the coordinates x and θ shown on the figure as the dependent variables and then simulate the system's behavior by numerically solving the equations of motion for 5 s, using $m = 2\,\text{kg}$, $L = 0.6\,\text{m}$, $x(0) = 0\,\text{m}$, $\dot{x}(0) = 0\,\text{m/s}$, $\theta(0) = 60°$, and $\dot{\theta}(0) = 0\,\text{rad/s}$. Plot x and θ for $0 \le t \le 5\,\text{s}$.

Solution to 7.47

Solution

Based on the FBD shown, the Newton-Euler equations for the spool are



$$\sum F_x: \quad 0 = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad -N\frac{L}{2}\sin\theta = I_G\alpha_b, \tag{3}$$

where the mass moment of inertia of the spool with respect to G is

$$I_G = \tfrac{1}{12}mL^2. \tag{4}$$

The acceleration of A can only be in the horizontal direction, i.e., $\vec{a}_A = a_{Ax}\,\hat{\imath}$. Next, observe that $\vec{r}_{G/A} = \frac{1}{2}L(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})$. Hence, applying planar rigid body kinematics, we have

$$\vec{a}_G = \vec{a}_A + \alpha_b\hat{k}\times\vec{r}_{G/A} - \omega_b^2\vec{r}_{G/A} \quad \Rightarrow \quad \vec{a}_G = \left(a_{Ax} + \tfrac{1}{2}L\alpha_b\cos\theta\right)\hat{\imath} + \tfrac{1}{2}L\alpha_b\sin\theta\,\hat{\jmath}, \tag{5}$$

where, given that the system is released from rest, we have accounted for the fact that, at the instant of release, $\omega_b = 0$. Hence, the kinematic equations for this problem are

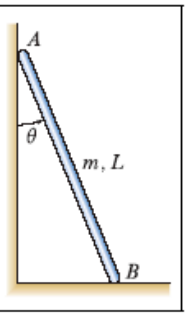
$$a_{Gx} = a_{Ax} + \tfrac{1}{2}L\alpha_b\cos\theta \quad \text{and} \quad a_{Gy} = \tfrac{1}{2}L\alpha_b\sin\theta. \tag{6}$$

Substituting the expression for I_G and the kinematic equations into the Newton-Euler equations, we have a system of three equations in the three unknowns α_b, N, and a_{Ax} whose solution is

$$\alpha_b = -\frac{6g\sin\theta}{L(1+3\sin^2\theta)}, \quad N = \frac{mg}{1+3\sin^2\theta}, \quad \text{and} \quad a_{Ax} = \frac{3g\sin\theta\cos\theta}{1+3\sin^2\theta}.$$

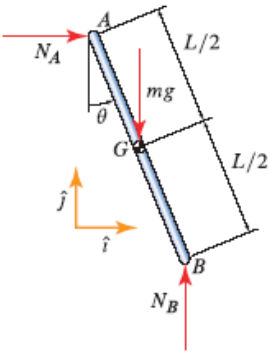
Problem 7.53

The uniform bar AB of mass m and length L is leaning against the corner with $\theta \approx 0$ when end B is given a slight nudge so that end A starts sliding down the wall as B slides along the floor. Assuming that friction is negligible between the bar and the two surfaces against which it is sliding, determine the angle θ at which end A will lose contact with the vertical wall.



Solution

Given the FBD shown, the corresponding Newton-Euler equations are:



$$\sum F_x: \qquad N_A = ma_{Gx}, \tag{1}$$

$$\sum F_y: \qquad N_B - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad N_B \frac{L}{2}\sin\theta - N_A \frac{L}{2}\cos\theta = I_G \alpha_{AB}, \tag{3}$$

where α_{AB} is the angular acceleration of the bar, and where the mass moment of inertia of the bar is given by

$$I_G = \tfrac{1}{12} mL^2. \tag{4}$$

Observe that point B will be moving only in the horizontal direction so that we must have $\vec{a}_B = a_{Bx}\,\hat{\imath}$. In addition, until the bar separates from the wall, the horizontal component of the acceleration of the bar must be equal to zero. At the same time, rigid body kinematica demands that

$$\vec{a}_A = \vec{a}_B + \alpha_{AB}\,\hat{k} \times \vec{r}_{A/B} - \omega_{AB}^2 \vec{r}_{A/B}, \tag{5}$$

where $\omega_{AB}\,\hat{k}$ is the angular velocity of the bar, and where $\vec{r}_{A/B} = L(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$. Substituting the expressions for $\vec{a}_B$ and $\vec{r}_{A/B}$ into Eq. (5) and expanding the cross-product, we have

$$\vec{a}_A = (a_{Bx} - L\alpha_{AB}\cos\theta + L\omega_{AB}^2 \sin\theta)\,\hat{\imath} - L(\omega_{AB}^2\cos\theta + \alpha_{AB}\sin\theta)\,\hat{\jmath}. \tag{6}$$

Using the result in Eq. (6) and enforcing the condition $a_{Ax} = 0$ we obtain that

$$a_{Bx} = L(\alpha_{AB}\cos\theta - \omega_{AB}^2 \sin\theta). \tag{7}$$

Now recall that we can express the acceleration of the mass center of the bar as $\vec{a}_G = \vec{a}_B + \alpha_{AB}\,\hat{k} \times \vec{r}_{G/B} - \omega_{AB}^2 \vec{r}_{G/B}$, where $\vec{r}_{G/B} = (L/2)(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$. Doing so, recalling that $\vec{a}_B = a_{Bx}\,\hat{\imath}$, and taking advantage of the result in Eq. (7), we obtain

$$\vec{a}_G = \tfrac{1}{2}L(\alpha_{AB}\cos\theta - \omega_{AB}^2\sin\theta)\,\hat{\imath} - \tfrac{1}{2}L(\alpha_{AB}\sin\theta + \omega_{AB}^2\cos\theta)\,\hat{\jmath}. \tag{8}$$

Substituting Eqs. (4) and (8) into the Newton-Euler equations, we obtain a system of three equations, which we can solve for the three unknowns N_A, N_B, and α_{AB} and obtain

$$N_A = \tfrac{1}{4}m(3g\cos\theta - 2L\omega_{AB}^2)\sin\theta \tag{9}$$

$$N_B = \tfrac{1}{8}m\{[5 + 3\cos(2\theta)]g - 4L\omega_{AB}^2\cos\theta\} \tag{10}$$

$$\alpha_{AB} = \frac{3g}{2L}\sin\theta. \tag{11}$$

The value of θ_{sep} for which the bar separates from the wall is the value of θ for which the force N_A becomes equal to zero. From Eq. (9), we see that the determination of θ_{sep} requires the calculation of ω_{AB}^2 as a function of θ. In turn, this can be done by observing that $\omega_{AB} = d\theta/dt$ and then applying the chain rule to the result obatined for α_{AB} in Eq. (11), i.e.,

$$\alpha_{AB} = \frac{d\omega_{AB}}{dt} = \frac{d\omega_{AB}}{d\theta}\frac{d\theta}{dt} = \omega_{AB}\frac{d\omega_{AB}}{d\theta} \quad \Rightarrow \quad \omega_{AB}d\omega_{AB} = \frac{3g}{2L}\sin\theta d\theta. \tag{12}$$

Recalling that the system is released from rest when $\theta = 0$, integrating the final expression in Eq. (12), we have

$$\int_0^{\omega_{AB}} \omega_{AB}\, d\omega_{AB} = \int_0^{\theta} \frac{3g}{2L}\sin\theta\, d\theta \quad \Rightarrow \quad \omega_{AB}^2 = \frac{3g}{L}(1 - \cos\theta). \tag{13}$$

Substituting the expression for ω_{AB}^2 given by Eq. (13) into Eq. (9) and setting the result to zero, we have

$$\tfrac{3}{4}mg\sin\theta(3\cos\theta - 2) = 0. \tag{14}$$

The solution of Eq. (14) yields the value of θ for which the end A of the bar looses contact with the wall. Equation (14) admits the following solutions:

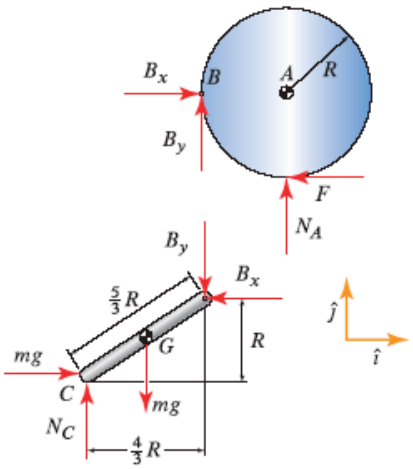
$$\theta = 0^\circ \pm n180^\circ,\ n = 1, 2, \ldots, \quad \text{and} \quad \theta = \pm\cos^{-1}(2/3) \pm m360^\circ,\ m = 1, 2, \ldots. \tag{15}$$

Clearly, not all the above solutions make physical sense. Observing that the solution $\theta = 0$ corresponds to the instant of release, the first physically meaningful solution is given by $\theta = \cos^{-1}(2/3)$. Hence, we conclude that the end A of the bar separates from the wall for the following value of θ

$$\boxed{\theta = \cos^{-1}\left(\tfrac{2}{3}\right) = 48.2^\circ.}$$

Solution to 7.57

Since both the mass of the disk is negligible, the mass of the bar is equal to m, and the force $P = mg$, using the assumptions stated in the problem, the FBDs of each component of the system are as shown. The Newton-Euler equations for the disk are:



$$\sum F_x: \qquad B_x - F = 0, \tag{17}$$

$$\sum F_y: \qquad B_y + N_A = 0, \tag{18}$$

$$\sum M_A: \quad -B_y R - FR = 0, \tag{19}$$

where the right-hand sides of the Newton-Euler equations are equal to zero because the mass of disk is assumed to be negligible. The Newton-Euler equations for the bar are

$$\sum F_x: \qquad -B_x + mg = ma_{Gx}, \tag{20}$$

$$\sum F_y: \qquad -B_y + N_C - mg = ma_{Gy}, \tag{21}$$

$$\sum M_G: \quad B_x\frac{R}{2} + mg\frac{R}{2} - B_y\frac{2}{3}R - N_C\frac{2}{3}R = I_G\alpha_b, \tag{22}$$

where α_b is the angular acceleration of the bar, and where the mass moment of inertia of the bar (about its mass center) is given by

$$I_G = \tfrac{1}{12}m\left(\frac{5}{3}R\right)^2 = \frac{25}{108}mR^2. \tag{23}$$

Since the disk must roll without slip on a horizontal surface, we have

$$a_{Ay} = 0, \quad \text{and} \quad \alpha_d = -\frac{a_{Ax}}{R}, \tag{24}$$

where α_d is the angular acceleration of the disk. As for the kinematic equations for the bar, observe that point B is attached to the disk and therefore we must have

$$\vec{a}_B = \vec{a}_A + \alpha_d\,\hat{k}\times\vec{r}_{B/A} - \omega_d^2\vec{r}_{B/A} \quad \Rightarrow \quad \vec{a}_B = a_{Ax}\,\hat{\imath} + a_{Ax}\,\hat{\jmath}, \tag{25}$$

where $\vec{r}_{B/A} = -R\,\hat{\imath}$, ω_d is the angular velocity of the disk, and where we have accounted for the fact that at the instant considered, i.e., at the instant of release, $\omega_d = 0$ because the system is released from rest. Next, we have that the acceleration of point C is

$$\vec{a}_C = \vec{a}_B + \alpha_b\,\hat{k}\times\vec{r}_{C/B} - \omega_b^2\vec{r}_{C/B} \quad \Rightarrow \quad \vec{a}_C = (a_{Ax} + R\alpha_b)\,\hat{\imath} + \left(a_{Ax} - \tfrac{4}{3}R\alpha_b\right)\,\hat{\jmath}, \tag{26}$$

where $\vec{r}_{C/B} = -\frac{4}{3}R\,\hat{\imath} - R\,\hat{\jmath}$, ω_b is the angular velocity of the bar, and where we have accounted for the fact that, at the time instant considered, $\omega_b = 0$. Now we observe that point C must move only in the x direction. Hence we have

$$a_{Cy} = 0 \quad \Rightarrow \quad \alpha_b = \frac{3a_{Ax}}{4R}. \tag{27}$$

The acceleration of the mass center of the bar is given by

$$\vec{a}_G = \vec{a}_B + \alpha_b\,\hat{k}\times\vec{r}_{G/B} - \omega_b\vec{r}_{G/B} \quad \Rightarrow \quad \vec{a}_G = \tfrac{11}{8}a_{Ax}\,\hat{\imath} + \tfrac{1}{2}a_{Ax}\,\hat{\jmath} \tag{28}$$

where $\vec{r}_{G/B} = -\frac{2}{3}R\,\hat{\imath} - \frac{1}{2}R\,\hat{\jmath}$, and where we have used the results in Eqs. (25) and (27), as well as the fact that, at the time instant considered, $\omega_b = 0$.

We can summarize the kinematic analysis of the bar via the following relations:

$$a_{Gx} = \tfrac{11}{8} a_{Ax}, \quad a_{Gy} = \tfrac{1}{2} a_{Ax}, \quad \text{and} \quad \alpha_b = \frac{3a_{Ax}}{4R}. \tag{29}$$

Substituting Eqs. (23), (24), and (29) into the Newton-Euler equations, we obtain a system of six equations in the six unknowns a_{Ax}, B_x, B_y, N_A, F N_C. Solving the system of equations in question we obtain

$$a_{Ax} = \tfrac{60}{109} g, \quad B_x = \tfrac{53}{218} mg, \quad B_y = -\tfrac{53}{218} mg, \tag{30}$$

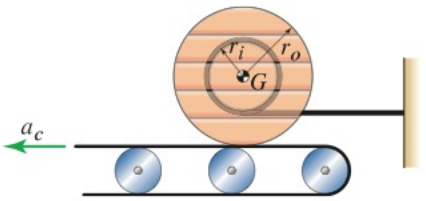
$$N_A = \tfrac{53}{218} mg, \quad F = \tfrac{53}{218} mg, \quad \text{and} \quad N_C = \tfrac{225}{218} mg. \tag{31}$$

Hence, recalling Eq. (27), we have that the solution of this problem consists of the following two results:

$$\boxed{a_{Ax} = \frac{60}{109} g \quad \text{and} \quad \alpha_b = \frac{45g}{109R}.}$$

Problem 7.46

The spool of mass m, radius of gyration k_G, inner radius r_i, and outer radius r_o is placed on a horizontal conveyer belt. The cable that is wrapped around the spool and attached to the wall is initially taut. Both the spool and the conveyer belt are initially at rest when the conveyer belt starts moving with acceleration a_c. If the coefficient of static friction between the conveyer belt and spool is μ_s, determine



(a) The maximum acceleration of the conveyer belt so that the spool rolls without slipping on the belt

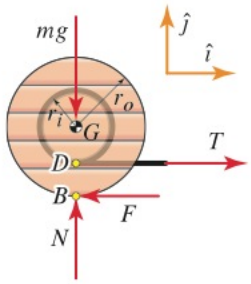
(b) The initial tension in the cable that attaches the spool to the wall

(c) The angular acceleration of the spool

Evaluate your answers for $m = 500\,\text{kg}$, $k_G = 1.3\,\text{m}$, $\mu_s = 0.5$, $r_i = 0.8\,\text{m}$, and $r_o = 1.6\,\text{m}$.

Solution

In this problem the spool will roll without slip over the rope as well as the conveyer belt. Based on the FBD shown, the Newton-Euler equations for the spool are



$$\sum F_x: \quad -F + T = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad -Fr_o + Tr_i = I_G\alpha_s, \tag{3}$$

where the mass moment of inertia of the spool with respect to G is

$$I_G = mk_G^2. \tag{4}$$

Since we are seeking the maximum acceleration of the spool corresponding to the spool rolling without slip over the conveyer belt, we assume impending slip and we must have

$$F = \mu_s N. \tag{5}$$

Letting $\vec{a}_G = a_{Gx}\,\hat{\imath} + a_{Gy}\,\hat{\jmath}$ be the acceleration of the G, recalling that the instant considered the angular velocity of the spool is equal to zero, and observing that the position of points B and D relative to the center of mass G are $\vec{r}_{B/G} = -r_o\,\hat{\jmath}$ and $\vec{r}_{D/G} = -r_i\,\hat{\jmath}$, respectively, we have B and D are

$$\vec{a}_B = \vec{a}_G + \alpha_s\,\hat{k} \times \vec{r}_{B/G} = (a_{Gx} + \alpha_s r_o)\,\hat{\imath} \tag{6}$$

$$\vec{a}_D = \vec{a}_G + \alpha_s\,\hat{k} \times \vec{r}_{D/G} = (a_{Gx} + \alpha_s r_i)\,\hat{\imath}. \tag{7}$$

Enforcing rolling without slip at both B and D, we have that $a_{Bx} = -a_c$ and $a_{Dx} = 0$, which implies that

$$a_{Gx} = \frac{r_i}{r_o - r_i}a_c \quad \text{and} \quad \alpha_s = -\frac{a_c}{r_o - r_i}. \tag{8}$$

In addition to the above equations, since the spool must move parallel to the conveyer belt, we must have $a_{Gy} = 0$. Hence, summarizing, the kinematic equations for this problem are

$$a_{Gx} = \frac{r_i}{r_o - r_i}a_c, \quad a_{Gy} = 0, \quad \text{and} \quad \alpha_s = -\frac{a_c}{r_o - r_i}. \tag{9}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Substituting the expression for I_G, the force law, and the kinematic equations into the Newton-Euler equations, we have a system of three equations in the three unknowns N, T, and a_c whose solution is

$$a_c = \mu_s \frac{(r_o - r_i)^2}{r_i^2 + k_G^2} g, \quad T = \mu_s \frac{r_o r_i + k_G^2}{r_i^2 + k_G^2} mg, \quad N = mg. \tag{10}$$

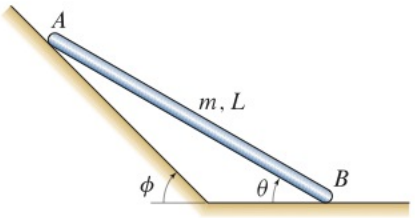
Recalling that $\alpha_s = -a_c/(r_o - r_i)$, and using the given numerical values, we have

$$\boxed{a_c = 1.35\,\text{m/s}^2, \quad T = 3130\,\text{N}, \quad \text{and} \quad \alpha_s = -1.68\,\text{rad/s}.}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

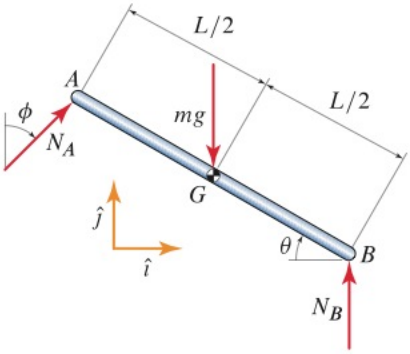
Problem 7.54

The uniform thin bar, which is leaning on the incline, is released from rest in the position shown and slides in the vertical plane. The contacts between the bar and the surface at ends A and B have negligible friction. Determine the angular acceleration of the bar immediately after it is released. Evaluate your answer for $m = 3\,\text{kg}$, $L = 0.75\,\text{m}$, $\phi = 45°$, and $\theta = 30°$.



Solution

Given the FBD shown, the corresponding Newton-Euler equations are:



$$\sum F_x: \qquad N_A \sin\phi = ma_{Gx}, \tag{1}$$

$$\sum F_y: \qquad N_A \cos\phi + N_B - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad \frac{L}{2}(N_B \cos\theta - N_A \cos\phi \cos\theta - N_A \sin\phi \sin\theta) = I_G \alpha_{AB}, \tag{3}$$

where α_{AB} is the angular acceleration of the bar, and where the mass moment of inertia of the bar is given by

$$I_G = \tfrac{1}{12} m L^2. \tag{4}$$

Since point A must move down the incline, letting a_A be the component of the acceleration of A down the incline, we must have

$$\vec{a}_A = a_A(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}). \tag{5}$$

Recall that rigid body kinematics allows us to write

$$\vec{a}_B = \vec{a}_A + \alpha_{AB}\,\hat{k} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A}, \tag{6}$$

where $\omega_{AB}\,\hat{k}$ is the angular velocity of the bar, and where $\vec{r}_{B/A} = L(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$. Recalling that at the time instant of interest $\omega_{AB} = 0$, substituting the expressions for $\vec{a}_A$ and $\vec{r}_{B/A}$ into Eq. (6), and expanding the cross-products we have

$$\vec{a}_B = (a_A \cos\phi + L\alpha_{AB} \sin\theta)\,\hat{\imath} + (L\alpha_{AB} \cos\theta - a_A \sin\phi)\,\hat{\jmath}. \tag{7}$$

Since point B can only move in the horizontal direction, we must have $a_{By} = 0$. Hence, setting the y component of $\vec{a}_B$ in Eq. (7) equal to zero and solving for a_A, we obtain

$$a_A = L\alpha_{AB}\frac{\cos\theta}{\sin\phi} \quad \Rightarrow \quad \vec{a}_A = L\alpha_{AB}\frac{\cos\theta}{\sin\phi}(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}). \tag{8}$$

Now recall that we can express the acceleration of the mass center of the bar as $\vec{a}_G = \vec{a}_A + \alpha_{AB}\,\hat{k} \times \vec{r}_{G/A} - \omega_{AB}^2 \vec{r}_{G/A}$, where $\vec{r}_{G/A} = (L/2)(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$. Doing so, using the expression for $\vec{a}_A$ in Eq. (8), we obtain

$$\vec{a}_G = L\alpha_{AB}\left(\frac{\cos\theta}{\sin\phi}\cos\phi + \frac{\sin\theta}{2}\right)\hat{\imath} - \frac{L\alpha_{AB}\cos\theta}{2}\,\hat{\jmath}. \tag{9}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Treating the components of the acceleration of $\vec{a}_G$ as if they were known, using Eq. (4), the Newton-Euler equations can be regarded as a system of three equations in the unknowns unknowns N_A, N_B, and α_{AB}, whose solution is

$$N_A = \frac{ma_{Gx}}{\sin\phi}, \tag{10}$$

$$N_B = m\left(g + a_{Gy} - a_{Gx}\frac{\cos\phi}{\sin\phi}\right), \tag{11}$$

$$\alpha_{AB} = \frac{6}{L}\left[\left(g + a_{Gy} - 2a_{Gx}\frac{\cos\phi}{\sin\phi}\right)\cos\theta - a_{Gx}\sin\theta\right]. \tag{12}$$

From Eq. (9) we have

$$a_{Gx} = L\alpha_{AB}\left(\frac{\cos\theta}{\sin\phi}\cos\phi + \frac{\sin\theta}{2}\right) \quad \text{and} \quad a_{Gy} = -\frac{L\alpha_{AB}\cos\theta}{2}. \tag{13}$$

Substituting Eqs. (13) into Eq. (12), we obtain

$$\alpha_{AB} = \frac{6}{L}\left[\left(g - \frac{L\alpha_{AB}\cos\theta}{2} - 2L\alpha_{AB}\left(\frac{\cos\theta}{\sin\phi}\cos\phi + \frac{\sin\theta}{2}\right)\frac{\cos\phi}{\sin\phi}\right)\cos\theta - L\alpha_{AB}\left(\frac{\cos\theta}{\sin\phi}\cos\phi + \frac{\sin\theta}{2}\right)\sin\theta\right]. \tag{14}$$

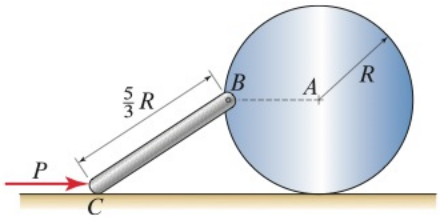
which is an equation in the only unknown α_{AB} whose solution is

$$\boxed{\alpha_{AB} = \frac{g}{2L}\frac{\cos\theta}{\frac{1}{3} + \cos^2\theta(\cos\phi/\sin\phi)^2 + \sin\theta\cos\theta(\cos\phi/\sin\phi)} = 3.74\,\text{rad/s}^2.}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problems 7.56 and 7.57

The disk A rolls without slipping on a horizontal surface. End B of bar BC is pinned to the edge of the disk A, and end C of the bar BC can slide freely along the horizontal surface. In addition, bar BC is pushed by the force $P = mg$ at its left end. The mass of bar BC is m_{BC}, and the mass of the disk A is m_A. The system is initially at rest.

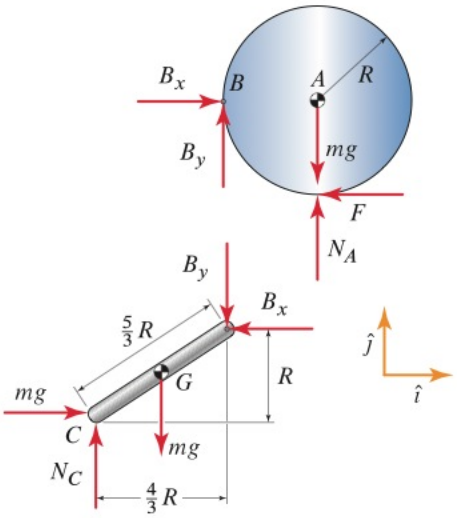


Problem 7.56 Determine the acceleration of the center of the disk A and the angular acceleration of bar BC immediately after the force P is applied if $m_A = m_{BC} = m$.

Problem 7.57 Determine the acceleration of the center of disk A and the angular acceleration of bar BC immediately after force P is applied if $m_{BC} = m$ and the mass of the disk m_A is negligible.

Solution to 7.56

Since the mass of the disk and the mass of the bar are equal to m, and since the force $P = mg$, using the assumptions stated in the problem, the FBDs of each component of the system are as shown. Observing that point A is both the geometric center of the disk and the center of mass of the disk, the Newton-Euler equations for the disk are:



$$\sum F_x: \quad B_x - F = ma_{Ax}, \tag{1}$$

$$\sum F_y: \quad B_y + N_A - mg = ma_{Ay}, \tag{2}$$

$$\sum M_A: \quad -B_y R - FR = I_A \alpha_d, \tag{3}$$

where α_d is the angular acceleration of the disk, and where the mass moment of inertia of the disk (about its mass center) is given by

$$I_A = \tfrac{1}{2} m R^2. \tag{4}$$

The Newton-Euler equations for the bar are

$$\sum F_x: \quad -B_x + mg = ma_{Gx}, \tag{5}$$

$$\sum F_y: \quad -B_y + N_C - mg = ma_{Gy}, \tag{6}$$

$$\sum M_G: \quad B_x \frac{R}{2} + mg\frac{R}{2} - B_y \frac{2}{3}R - N_C \frac{2}{3}R = I_G \alpha_b, \tag{7}$$

where α_b is the angular acceleration of the bar, and where the mass moment of inertia of the bar (about its mass center) is given by

$$I_G = \tfrac{1}{12} m \left(\frac{5}{3}R\right)^2 = \frac{25}{108} m R^2. \tag{8}$$

Since the disk must roll without slip on a horizontal surface, we have

$$a_{Ay} = 0, \quad \text{and} \quad \alpha_d = -\frac{a_{Ax}}{R}. \tag{9}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

As for the kinematic equations for the bar, observe that point B is attached to the disk and therefore we must have

$$\vec{a}_B = \vec{a}_A + \alpha_d\,\hat{k} \times \vec{r}_{B/A} - \omega_d^2 \vec{r}_{B/A} \quad \Rightarrow \quad \vec{a}_B = a_{Ax}\,\hat{\imath} + a_{Ax}\,\hat{\jmath}, \tag{10}$$

where $\vec{r}_{B/A} = -R\,\hat{\imath}$, ω_d is the angular velocity of the disk, and where we have accounted for the fact that at the instant considered, i.e., at the instant of release, $\omega_d = 0$ because the system is released from rest. Next, we have that the acceleration of point C is

$$\vec{a}_C = \vec{a}_B + \alpha_b\,\hat{k} \times \vec{r}_{C/B} - \omega_b^2 \vec{r}_{C/B} \quad \Rightarrow \quad \vec{a}_C = (a_{Ax} + R\alpha_b)\,\hat{\imath} + \left(a_{Ax} - \tfrac{4}{3}R\alpha_b\right)\,\hat{\jmath}, \tag{11}$$

where $\vec{r}_{C/B} = -\frac{4}{3}R\,\hat{\imath} - R\,\hat{\jmath}$, ω_b is the angular velocity of the bar, and where we have accounted for the fact that, at the time instant considered, $\omega_b = 0$. Now we observe that point C must move only in the x direction. Hence we have

$$a_{Cy} = 0 \quad \Rightarrow \quad \alpha_b = \frac{3a_{Ax}}{4R}. \tag{12}$$

The acceleration of the mass center of the bar is given by

$$\vec{a}_G = \vec{a}_B + \alpha_b\,\hat{k} \times \vec{r}_{G/B} - \omega_b \vec{r}_{G/B} \quad \Rightarrow \quad \vec{a}_G = \tfrac{11}{8}a_{Ax}\,\hat{\imath} + \tfrac{1}{2}a_{Ax}\,\hat{\jmath} \tag{13}$$

where $\vec{r}_{G/B} = -\frac{2}{3}R\,\hat{\imath} - \frac{1}{2}R\,\hat{\jmath}$, and where we have used the results in Eqs. (10) and (12), as well as the fact that, at the time instant considered, $\omega_b = 0$.

We can summarize the kinematic analysis of the bar via the following relations:

$$a_{Gx} = \tfrac{11}{8}a_{Ax}, \quad a_{Gy} = \tfrac{1}{2}a_{Ax}, \quad \text{and} \quad \alpha_b = \frac{3a_{Ax}}{4R}. \tag{14}$$

Substituting Eqs. (4), (8), (9), and (14) into the Newton-Euler equations, we obtain a system of six equations in the six unknowns a_{Ax}, B_x, B_y, N_A, F N_C. Solving the system of equations in question we obtain

$$a_{Ax} = \tfrac{60}{181}g, \quad B_x = \tfrac{197}{362}mg, \quad B_y = -\tfrac{17}{362}mg, \tag{15}$$

$$N_A = \tfrac{379}{362}mg, \quad F = \tfrac{77}{362}mg, \quad \text{and} \quad N_C = \tfrac{405}{362}mg. \tag{16}$$

Hence, recalling Eq. (12), we have that the solution of this problem consists of the following two results:

$$\boxed{a_{Ax} = \frac{60}{181}g \quad \text{and} \quad \alpha_b = \frac{45g}{181R}.}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problems 7.58 through 7.61

The uniform ball of radius ρ and mass m is gently placed in the bowl B with inner radius R and is released. The angle ϕ measures the position of the center of the ball at G with respect to a vertical line, and the angle θ measures the rotation of the ball with respect to a vertical line. Assume that the system lies in the vertical plane. *Hint:* In working the following problems, we recommend using the $r\phi$ coordinate system shown.

Problem 7.58 Assuming that the ball rolls without slip, determine the acceleration of the center of the ball at G, the angular acceleration of the ball, and the force on the ball due to the bowl immediately after the ball is released.

Problem 7.59 Assuming that the ball rolls without slip, that it weighs 3 lb, is at the position $\phi = 40°$, and is moving clockwise at 10 ft/s, determine the acceleration of the center of the ball at G and the normal and friction force between the ball and the bowl. Use $R = 4$ ft and $\rho = 1.2$ ft.

Problem 7.60 Assuming that friction is sufficient to prevent slipping, derive the equation(s) of motion of the ball in terms of the angle ϕ.

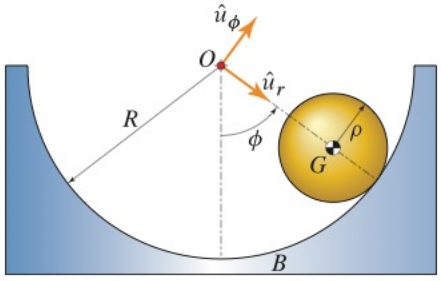
Problem 7.61 Assume that friction is sufficient to prevent slipping.

(a) Derive the equation(s) of motion of the ball in terms of the angle ϕ.

(b) Determine the friction force as a function of ϕ.

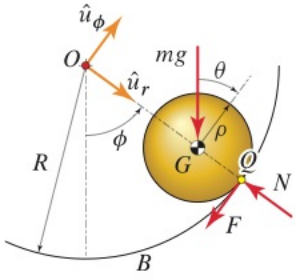
(c) Letting $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$, with $0° < \phi_0 < 90°$ integrate the equation(s) of motion to determine the normal force as a function of ϕ.

(d) Using the results of Parts (b) and (c), given a value for μ_s, determine the maximum value of $\phi(0) = \phi_0$ so that the ball does not slip.



Solution to 7.58

Given the FBD shown, the corresponding Newton-Euler equations are:



$$\sum F_r: \quad mg\cos\phi - N = ma_{Gr}, \tag{1}$$

$$\sum F_\phi: \quad -F - mg\sin\phi = ma_{G\phi}, \tag{2}$$

$$\sum M_G: \quad -F\rho = I_G\alpha_b, \tag{3}$$

where α_b is the angular acceleration of the ball, and where the mass moment of inertia of the ball is given by

$$I_G = \tfrac{2}{5}m\rho^2. \tag{4}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Because the center of the ball is in circular motion about point O, the acceleration of point G can be given the form

$$\vec{a}_G = -(R-\rho)\dot{\phi}^2\,\hat{u}_r + (R-\rho)\ddot{\phi}\,\hat{u}_\phi = (R-\rho)\ddot{\phi}\,\hat{u}_\phi, \tag{5}$$

where we have accounted for the fact that at the time instant of interest, which is the instant of release, $\dot{\phi} = 0$ given that the system is released from rest. Now, letting Q denote the point of contact between the ball and the bowl, we have

$$\vec{a}_Q = \vec{a}_G + \alpha_b\,\hat{k} \times (\rho\,\hat{u}_r) - \omega_b^2(\rho\,\hat{u}_r) = [\alpha_b\rho + (R-\rho)\ddot{\phi}]\,\hat{u}_\phi, \tag{6}$$

where ω_b is the angular velocity of the ball, $\rho\,\hat{u}_r = \vec{r}_{Q/G}$, and where we have used Eq. (5) as well as the fact that $\omega_b = 0$ at the instant considered.

Because the ball rolls without slip on a stationary surface, we must have $a_{Q\phi} = 0$, i.e.,

$$\alpha_b\rho + (R-\rho)\ddot{\phi} = 0 \quad \Rightarrow \quad \alpha_b = -\frac{R-\rho}{\rho}\ddot{\phi}. \tag{7}$$

Substituting Eqs. (5) and the result in Eq. (7), and using Eq. (4), the Newton-Euler equations yield a system of three equations in the three unknowns N, F, and $\ddot{\phi}$, whose solution is

$$N = mg\cos\phi, \quad F = -\tfrac{2}{7}mg\sin\phi, \quad \text{and} \quad \ddot{\phi} = -\frac{5g\sin\phi}{7(R-\rho)}. \tag{8}$$

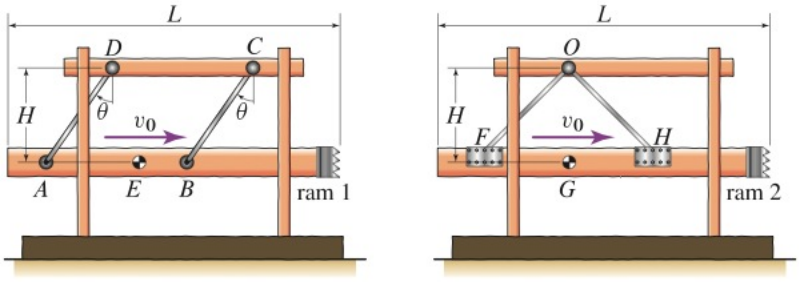
Observe that the force exerted by the bowl on the ball is given by $-F\,\hat{u}_\phi - N\,\hat{u}_r$. In addition, using the last result in Eq. (8) along with the result in Eqs. (5) and (7) we obtain the acceleration of the center of the ball as well as the angular acceleration of the ball. Hence, in summary, the solution of the problem is

$$\boxed{\vec{a}_G = -\tfrac{5}{7}g\sin\phi\,\hat{u}_\phi, \quad \alpha_b = \frac{5g\sin\phi}{7\rho}, \quad \text{and} \quad \vec{F}_{\text{due to bowl}} = -mg\cos\phi\,\hat{u}_r + \tfrac{2}{7}mg\sin\phi\,\hat{u}_\phi}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problem 8.3

Two identical battering rams are mounted on their respective frames as shown. Bars BC and AD are identical and pinned at B and C and at A and D, respectively. Bars FO and HO are rigidly attached to the ram and are pinned at O. At the instant shown, the mass centers of rams 1 and 2, at E and G, respectively, are moving horizontally with speed v_0. Letting T_1 and T_2 be the kinetic energies of rams 1 and 2, respectively, state which of the following statements is true and why: (a) $T_1 < T_2$; (b) $T_1 = T_2$; (c) $T_1 > T_2$.



Solution

Modeling the rams as rigid bodies, the kinetic energy of ram 2 is larger than the kinetic energy of ram 1.

Exaplanation. Recall that the kinetic energy of a rigid body has the following form:

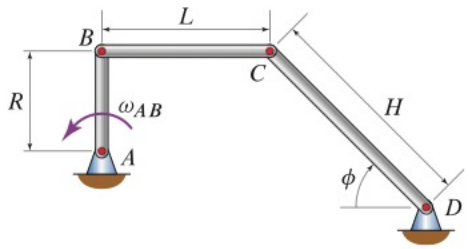
$$T = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}I_G\omega_0^2,$$

where m is the mass of the body, v_0 is the speed of the center of mass of the body, I_G is the mass moment of inertia of the body about its mass center, and ω_0 is the angular speed of the body. Notice that ram 1 and ram 2 have the mass and the same mass moment of inertia. In addition, their mass centers have the same speed and therefore their kinetic energies have the same terms $\frac{1}{2}mv_0^2$. Now observe that ram 1 is in curvilinear translation and therefore the contribution to the kinetic energy due to the body's angular velocity is equal to zero. By contrast, ram 2 does have a nonzero angular velocity and therefore its kinetic energy has a corresponding nonzero contribution due to the kinetic energy.

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problem 8.9

The weights of the uniform thin pin-connected bars AB, BC, and CD are $W_{AB} = 4\,\text{lb}$, $W_{BC} = 6.5\,\text{lb}$, and $W_{CD} = 10\,\text{lb}$, respectively. Letting $\phi = 47°$, $R = 2\,\text{ft}$, $L = 3.5\,\text{ft}$, and $H = 4.5\,\text{ft}$, and knowing that bar AB rotates at an angular velocity $\omega_{AB} = 4\,\text{rad/s}$, compute the kinetic energy T of the system at the instant shown.



Solution

The kinetic energy of the system is the sum of the kinetic energies of each individual component, i.e.,

$$T = T_{AB} + T_{BC} + T_{CD}. \tag{1}$$

Bars AB and CD are in fixed axis rotations about A and D respectively. Hence, letting G denote the mass center of bar BC, we have

$$T_{AB} = \tfrac{1}{2} I_A \omega_{AB}^2, \quad T_{BC} = \tfrac{1}{2} m_{BC} v_G^2 + \tfrac{1}{2} I_G \omega_{BC}^2, \quad \text{and} \quad T_{CD} = \tfrac{1}{2} I_D \omega_{CD}^2, \tag{2}$$

where I_A is the mass moment of inertia of bar AB about A, I_G is the mass moment of inertia of bar BC about its own mass center G, and I_D is the mass moment of inertia of bar CD about D, i.e.,

$$I_A = \tfrac{1}{12} m_{AB} R^2 + m_{AB}(R/2)^2 = 0.1656\,\text{slug}\cdot\text{ft}^2, \tag{3}$$

$$I_G = \tfrac{1}{12} m_{BC} L^2 = 0.2061\,\text{slug}\cdot\text{ft}^2, \tag{4}$$

$$I_D = \tfrac{1}{12} m_{CD} H^2 + m_{CD}(H/2)^2 = 2.097\,\text{slug}\cdot\text{ft}^2, \tag{5}$$

where, recalling that the acceleration due to gravity is $g = 32.2\,\text{ft/s}^2$, we have used the following numerical data: $m_{AB} = W_{AB}/g = (4\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.1242\,\text{slug}$, $R = 2\,\text{ft}$, $m_{BC} = W_{BC}/g = (6.5\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.2019\,\text{slug}$, $L = 3.5\,\text{ft}$, $m_{CD} = W_{CD}/g = (10\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.3106\,\text{slug}$, and $H = 4.5\,\text{ft}$.

We now determine ω_{BC}, v_G, and ω_{CD}. The position of vectors of B with relative to A, C relative to B, and D relative to C are, respectively,

$$\vec{r}_{B/A} = R\,\hat{\jmath}, \quad \vec{r}_{C/B} = L\,\hat{\imath}, \quad \text{and} \quad \vec{r}_{D/C} = H(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}). \tag{6}$$

Then we have

$$\vec{v}_B = \omega_{AB}\,\hat{k} \times \vec{r}_{B/A} = -R\omega_{AB}\,\hat{\imath}, \tag{7}$$

$$\vec{v}_C = \vec{v}_B + \omega_{BC}\,\hat{k} \times \vec{r}_{C/B} = -R\omega_{AB}\,\hat{\imath} + \omega_{BC} L\,\hat{\jmath}, \tag{8}$$

$$\vec{v}_D = \vec{v}_C + \omega_{CD}\,\hat{k} \times \vec{r}_{D/C} = (H\omega_{CD}\sin\phi - R\omega_{AB})\,\hat{\imath} + (L\omega_{BC} + H\omega_{CD}\cos\phi)\,\hat{\jmath}. \tag{9}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Since D is a fixed point, we must have $\vec{v}_D = \vec{0}$, which then implies

$$\omega_{BC} = -\omega_{AB}\frac{R\cos\phi}{L\sin\phi} = -2.131\,\text{rad/s} \quad \text{and} \quad \omega_{CD} = \omega_{AB}\frac{R}{H\sin\phi} = 2.431\,\text{rad/s}, \tag{10}$$

where we have used the following numerical data: $\omega_{AB} = 4\,\text{rad/s}$, $R = 2\,\text{ft}$, $L = 3.5\,\text{ft}$, and $\phi = 47°$. Then, using the first of Eqs. (10), we have that the velocity of G is

$$\vec{v}_G = \vec{v}_B + \omega_{BC}\,\hat{k} \times \tfrac{1}{2}\vec{r}_{C/B} = -R\omega_{AB}\,\hat{\imath} - \tfrac{1}{2}R\omega_{AB}\frac{\cos\phi}{\sin\phi}\,\hat{\jmath}$$

$$\Rightarrow \quad v_G^2 = R^2\omega_{AB}^2\left(1 + \tfrac{1}{4}\frac{\cos^2\phi}{\sin^2\phi}\right) = 77.91\,\text{ft}^2/\text{s}^2, \tag{11}$$

where we have used some of the same numerical data listed earlier. Substituting Eq. (3) into the first of Eq. (2) and recalling that $\omega_{AB} = 4\,\text{rad/s}$, we have

$$T_{AB} = 1.325\,\text{ft·lb}. \tag{12}$$

Substituting the numerical results in Eq. (4), the first of Eqs. (10), and Eq. (11) into the second of Eq. (2), we have

$$T_{BC} = 8.333\,\text{ft·lb}. \tag{13}$$

Finally, substituting the numerical results in Eq. (5) and in the second of Eqs. (10), we have

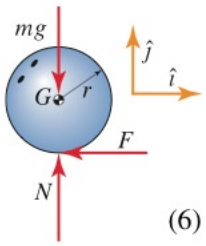
$$T_{CD} = 6.196\,\text{ft·lb}. \tag{14}$$

Hence, summing the last three results, we have

$$\boxed{T = 15.9\,\text{ft·lb}.}$$

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Solution to 8.14



Letting ① be the position at which the ball first makes contact with the lane and ② be the position at which the ball starts rolling without slip, and letting $U_{1\text{-}2}$ be the work of friction, we have

$$U_{1\text{-}2} = T_2 - T_1, \tag{5}$$

where

$$T_1 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}I_G\omega_1^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}I_G\omega_2^2, \tag{6}$$

wre the kinetic energies of the ball at ① and ②, respectively.

Observe that we have

$$m = W/g, \quad I_G = (W/g)k_G^2, \quad v_1 = v_0, \quad \omega_1 = \omega_0, \quad v_2 = v_f, \quad \text{and} \quad \omega_2 = v_f/r. \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (5), after simplifying we have

$$U_{1\text{-}2} = \frac{W}{2gr^2}\Big[v_f^2k_G^2 - r^2\big(v_0^2 - v_f^2 + k_G^2\omega_0^2\big)\Big], \tag{8}$$

Now, referring to the FBD on the right, observing that $F = \mu_k N$ because the ball is slipping, summing forces in the vertical direction, and observing that the ball does not move in the vertical direction, we must have

$$F = \mu_k mg = \mu_k W. \tag{9}$$

Since F is constant, $U_{1\text{-}2} = -L_f F$, where $U_{1\text{-}2}$ is that given in Eq. (8). Hence,

$$L_f = U_{1\text{-}2}/F = \frac{1}{2\mu_k gr^2}\big[v_f^2k_G^2 - r^2(v_0^2 - v_f^2 + k_G^2\omega_0^2),\big] \tag{10}$$

which can be evaluated to obtain

$$\boxed{L_f = 34.1\,\text{ft},}$$

where we have used the following numerical data: $\mu_k = 0.1$, $W = 14\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $r = 4.25\,\text{in.} = 0.3542\,\text{ft}$, $v_f = 17.2\,\text{ft/s}$, $k_G = 2.6\,\text{in.} = 0.2167\,\text{ft}$, $v_0 = 17.0\,\text{mph} = 24.93\,\text{ft/s}$, and $\omega_0 = 10\,\text{rad/s}$.

In general, the distance L_f is not the distance traveled by the center of ball. This is because the center of wheel goes from a speed of v_0 to the speed v_f whereas the point of contact between the wheel and the ground must slow down from a speed of $v_0 + \omega_0 r$ to 0. Hence, depending on the initial conditions, the distance over which the friction force acts during slip may not be the same as that traveled by the center of the ball.

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Problem 8.17

One of the basement doors is left open in the vertical position when it is given a nudge and allowed to freely fall to the closed position. Given that the door has mass m and that it is modeled as a uniform thin plate of width w and length d, determine its angular velocity when it reaches the closed position. *Hint:* Treat the problem as symmetric with respect to a plane of motion in which the acceleration due to gravity is $g\cos\theta$ rather than g.



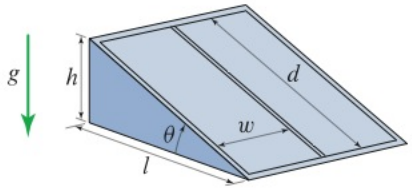
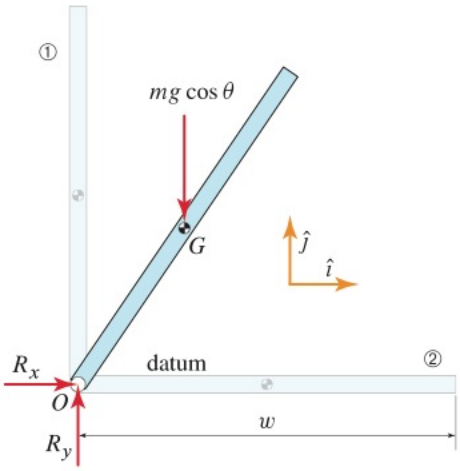


Photo credit: © McGraw-Hill, Photo by Lucinda Dowell

Solution

In the FBD shown, the direction perpendicular to the plane of motion is parallel to the axis of rotation of the door. Following the given hint, we see that the door is subject only to a component of the force of gravity. The forces R_x and R_y are the (fixed) pin reaction forces and do not work. Hence, letting ① be the position at which the door is released and ② be the position at which the door first achieves its closed position, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where

$$T_1 = 0, \quad V_1 = \tfrac{1}{2}wmg\cos\theta, \quad T_2 = \tfrac{1}{2}I_O\omega_{d2}^2, \quad V_2 = 0, \tag{2}$$

and where ω_d is the angular speed of the door and I_O is the mass moment of inertia of the door computed relative to the axis of rotation and is given by

$$I_O = \tfrac{1}{12}mw^2 + m\left(\frac{w}{2}\right)^2 = \tfrac{1}{3}mw^2. \tag{3}$$

Notice that, given the geometry of the door, we have

$$\cos\theta = \frac{l}{\sqrt{h^2 + l^2}} \tag{4}$$

Substituting Eqs. (2)–(4) into Eq. (1) and solving for ω_d, we have

$$\omega_{d2} = \sqrt{\frac{3gl}{w\sqrt{h^2 + l^2}}}. \tag{5}$$

Given our choice of component system, then the angular velocity of the door at ② is

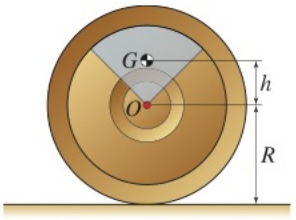
$$\vec{\omega}_{d2} = -\sqrt{\frac{3gl}{w\sqrt{h^2 + l^2}}}\,\hat{k},$$

where $\hat{k}$ is parallel to the rotation axis of the door and pointing toward the ground.

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

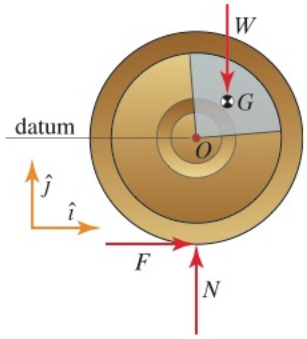
Problem 8.24

An eccentric wheel with weight $W = 250\,\text{lb}$, mass center G, and radius of gyration $k_G = 1.32\,\text{ft}$ is initially at rest in the position shown. Letting $R = 1.75\,\text{ft}$ and $h = 0.8\,\text{ft}$, and assuming that the wheel is gently nudged to the right and rolls without slip, determine the speed of O when G is closest to the ground.



Solution

Referring to the FBD shown, since the wheel rolls without slip, work is done only by gravity. We define ① as the position of the system at release. We define ② as the position of the system when G is closest to the ground. Hence, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, since the system is released from rest and due to our choice of datum, we have

$$T_1 = 0, \quad V_1 = Wh, \quad T_2 = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_{w2}^2, \quad V_2 = -Wh, \tag{2}$$

where v_G is the speed of the mass center of the wheel, ω_w is the angular speed of the wheel, and where I_G, which is the mass moment of inertia of the wheel about its own mass center. The mass moment of inertia I_G is given by

$$I_G = (W/g)k_G^2. \tag{3}$$

Since the wheel rolls without slip, we have

$$\omega_{w2} = \frac{v_O}{R} \quad \text{and} \quad v_{G2} = \frac{v_O}{R}(R-h). \tag{4}$$

Substituting Eqs. (2)–(4) into Eq. (1) and solving for v_O, we have

$$v_O = 2R\sqrt{\frac{gh}{(R-h)^2 + k_G^2}}. \tag{5}$$

Evaluating Eq. (5), we have

$$\boxed{v_O = 10.9\,\text{ft/s},}$$

where we have used the following numerical data: $R = 1.75\,\text{ft}$, $g = 32.2\,\text{ft/s}^2$, $h = 0.8\,\text{ft}$, $k_G = 1.32\,\text{ft}$.

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Solutions Manual

Engineering Mechanics: Dynamics

1st Edition

Gary L. Gray
The Pennsylvania State University

Francesco Costanzo
The Pennsylvania State University

Michael E. Plesha
University of Wisconsin–Madison

With the assistance of:
Chris Punshon
Andrew J. Miller
Justin High
Chris O'Brien
Chandan Kumar
Joseph Wyne

Version: August 10, 2009

The McGraw-Hill Companies, Inc.

Copyright © 2002–2010
Gary L. Gray, Francesco Costanzo, and Michael E. Plesha

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Important Information about this Solutions Manual

Even though this solutions manual is nearly complete, we encourage you to visit

`http://www.mhhe.com/pgc`

often to obtain the most up-to-date version. In particular, as of July 30, 2009, please note the following:

- The solutions for Chapters 1 and 2 have been accuracy checked and have been edited by us. They are in their final form.
- The solutions for Chapters 4 and 7 have been accuracy checked and should be error free. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapters 3, 6, 8, and 9 are being accuracy checked and the accuracy checked versions should be available by the end of August 2009. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapter 10 should be available in their entirety by the end of August 2009.

All of the figures in Chapters 6–10 are in color. Color will be added to the figures in Chapters 1–5 over the coming weeks.

Contact the Authors

If you find any errors and/or have questions concerning a solution, please do not hesitate to contact the authors and editors via email at:

`dyn_solns@email.esm.psu.edu`

We welcome your input.

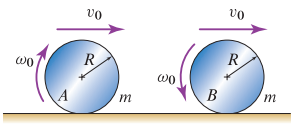
Accuracy of Numbers in Calculations

Throughout this solutions manual, we will generally assume that the data given for problems is accurate to 3 significant digits. When calculations are performed, all intermediate numerical results are reported to 4 significant digits. Final answers are usually reported with 3 significant digits. If you verify the calculations in this solutions manual using the rounded intermediate numerical results that are reported, you should obtain the final answers that are reported to 3 significant digits.

Chapter 8 Solutions

Problem 8.1

At the instant shown, the centers of the two identical uniform disks A and B are moving to the right with the same speed v_0. In addition, disk A is rolling clockwise with an angular speed ω_0 while disk B has a backspin with angular speed equal to ω_0. Letting T_A and T_B be the kinetic energies of A and B, respectively, state which of the following statements is true and why: (a) $T_A < T_B$; (b) $T_A = T_B$; (c) $T_A > T_B$.



Solution

At the instant shown, the kinetic energies of A and B are equal to one another.

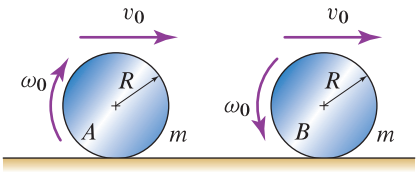
Explanation. The kinetic energy of a rigid body in planar motion is given by

$$T = \tfrac{1}{2}mv_0^2\tfrac{1}{2}I_G\omega_0^2,$$

where v_0 is the speed of the mass center and I_G is the mass moment of inertia with respect to the mass center. Observe that the quantities that contribute to the kinetic energy are the same for bodies A and B. The fact that the direction of rotation of A os opposite to the direction of rotation of B is irrelevant from the viewpoint of kinetic energy calculations.

Problem 8.2

At the instant shown, the centers of the two identical uniform disks A and B, each with mass m and radius R, are moving to the right with the same speed $v_0 = 4\,\text{m/s}$. In addition, disk A is rolling clockwise with an angular speed $\omega_0 = 5\,\text{rad/s}$, while disk B has a backspin with angular speed $\omega_0 = 5\,\text{rad/s}$. Letting $m = 45\,\text{kg}$ and $R = 0.75\,\text{m}$, determine the kinetic energy of each disk.



Solution

Let C and D denote the mass centers of disks A and B, respectively. Then, applying the definition of kinetic energy for a rigid body, for A we have

$$T_A = \tfrac{1}{2} m_A v_C^2 + \tfrac{1}{2} I_C \omega_A^2,$$

and for B we have

$$T_B = \tfrac{1}{2} m_B v_D^2 + \tfrac{1}{2} I_D \omega_D^2.$$

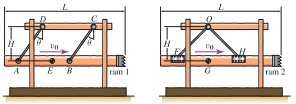
Observing that $m_A = m_B = m$, $v_C = v_D = v_0$, and that $\omega_A = -\omega_0 = -\omega_B$, $I_C = I_D = \frac{1}{2} m R^2$, we have

$$\boxed{T_A = T_B = 518\,\text{J},}$$

where we have used the following numerical data: $m = 45\,\text{kg}$, $v_0 = 4\,\text{m/s}$, $\omega_0 = 5\,\text{rad/s}$, and $R = 0.75\,\text{m}$.

Problem 8.3

Two identical battering rams are mounted on their respective frames as shown. Bars BC and AD are identical and pinned at B and C and at A and D, respectively. Bars FO and HO are rigidly attached to the ram and are pinned at O. At the instant shown, the mass centers of rams 1 and 2, at E and G, respectively, are moving horizontally with speed v_0. Letting T_1 and T_2 be the kinetic energies of rams 1 and 2, respectively, state which of the following statements is true and why: (a) $T_1 < T_2$; (b) $T_1 = T_2$; (c) $T_1 > T_2$.



Solution

Modeling the rams as rigid bodies, the kinetic energy of ram 2 is larger than the kinetic energy of ram 1.

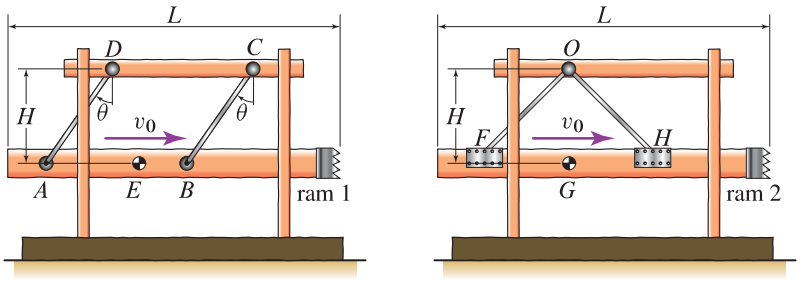
Exaplanation. Recall that the kinetic energy of a rigid body has the following form:

$$T = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}I_G\omega_0^2,$$

where m is the mass of the body, v_0 is the speed of the center of mass of the body, I_G is the mass moment of inertia of the body about its mass center, and ω_0 is the angular speed of the body. Notice that ram 1 and ram 2 have the mass and the same mass moment of inertia. In addition, their mass centers have the same speed and therefore their kinetic energies have the same terms $\frac{1}{2}mv_0^2$. Now observe that ram 1 is in curvilinear translation and the therefore the contribution to the kinetic energy due to the body's angular velocity is equal to zero. By contrast, ram 2 does have a nonzero angular velocity and therefore its kinetic energy has a corresponding nonzero contribution due to the kinetic energy.

Problem 8.4

Two identical battering rams are mounted on their respective frames as shown. Bars BC and AD are identical and pinned at B and C and at A and D, respectively. Bars FO and HO are rigidly attached to the ram and are pinned at O. At the instant shown, the centers of mass of rams 1 and 2, at E and G, respectively, are moving horizontally with a speed $v_0 = 20\,\text{ft/s}$. Treating the rams as slender bars with length $L = 10\,\text{ft}$ and weight $W = 1250\,\text{lb}$, and letting $H = 3\,\text{ft}$, compute the kinetic energy of the two rams.



Solution

The kinetic energy of ram 1 is given by

$$T_{R1} = \tfrac{1}{2} m_{R1} v_E^2 + \tfrac{1}{2} I_E \omega_{R1}^2,$$

where m_{R1} is the mass of ram 1, v_E is the speed of the center of mass of ram 1, I_E is the mass moment of inertia of ram 1 relative to E, and ω_{R1} is the angular speed of ram 1. Similarly, for ram 2 we have

$$T_{R2} = \tfrac{1}{2} m_{R2} v_G^2 + \tfrac{1}{2} I_G \omega_{R2}^2,$$

where m_{R2} is the mass of ram 2, v_G is the speed of the center of mass of ram 2, I_G is the mass moment of inertia of ram 2 relative to G, and ω_{R2} is the angular speed of ram 2.

Now, observe for ram 1 we have

$$m_{R1} = W/g = 38.82\,\text{slug}, \quad v_E = v_0 = 20.00\,\text{ft/s}, \quad I_E = \tfrac{1}{12}(W/g)L^2 = 323.5\,\text{slug}\cdot\text{ft}^2, \quad \omega_{R1} = 0,$$

where $\omega_{R1} = 0$ reflects the fact the motion of ram 1 is a translation, and where we have used the following numerical values: $W = 1250\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, and $L = 10\,\text{ft}$.

For ram 2 we have

$$m_{R2} = W/g = 38.82\,\text{slug}, \quad v_G = v_0 = 20.00\,\text{ft/s},$$
$$I_G = \tfrac{1}{12}(W/g)L^2 323.5\,\text{slug}\cdot\text{ft}^2, \quad \omega_{R2} = \frac{v_0}{H} = 6.667\,\text{rad/s},$$

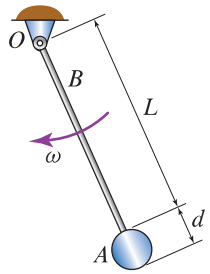
where $\omega_{R2} = v_0/H$ reflects the fact that ram 2 in a fixed axis rotation about O, and where, in addition to the numerical data listed earlier, we have used the fact that $H = 3\,\text{ft}$.

In conclusion, substituting the numerical values just listed into the top two equations, we have

$$T_{R1} = 7760\,\text{ft}\cdot\text{lb} \quad \text{and} \quad T_{R2} = 15{,}000\,\text{ft}\cdot\text{lb},$$

Problem 8.5

A pendulum consists of a uniform disk A of diameter $d = 0.15\,\text{m}$ and mass $m_A = 0.35\,\text{kg}$ attached at the end of a uniform bar B of length $L = 0.75\,\text{m}$ and mass $m_B = 0.8\,\text{kg}$. At the instant shown, the pendulum is swinging with an angular velocity $\omega = 0.24\,\text{rad/s}$ clockwise. Determine the kinetic energy of the pendulum at this instant, using Eq. (8.10) on p. 593.



Solution

The pendulum is undergoing a fixed axis rotation about O. Hence, the kinetic energy of the pendulum is

$$T = \tfrac{1}{2} I_O \omega^2,$$

where I_O is the mass moment of inertia of the system about O and is therefore given by

$$I_O = \tfrac{1}{12} m_B L^2 + m_B (\tfrac{1}{2} L)^2 + \tfrac{1}{2} m_A (\tfrac{1}{2} d)^2 + m_A (L + \tfrac{1}{2} d)^2 = 0.3892\,\text{kg}\cdot\text{m}^2,$$

where we have used the following numerical data: $m_A = 0.35\,\text{kg}$, $m_B = 0.8\,\text{kg}$, $L = 0.75\,\text{m}$, and $d = 0.15\,\text{m}$.

Substituting the above numerical value for I_O into the expression for the kinetic energy, and recalling that $\omega = 0.24\,\text{rad/s}$, we have

$$\boxed{T = 0.0112\,\text{J}.}$$

Problem 8.6

A 2570 lb car (this includes the weight of the wheels) is traveling on a horizontal flat road at 60 mph. If each wheel has a diameter $d = 24.3$ in. and a mass moment of inertia with respect to its mass center equal to 0.989 slug·ft^2, determine the kinetic energy of the car. Neglect the rotational energy of all parts of the car except for the wheels.



Solution

The kinetic energy of the car is

$$T = \tfrac{1}{2}mv^2 + 4\left(\tfrac{1}{2}I_{Gw}\omega_w^2\right),$$

where $m = W/g = (2570\,\text{lb})/(32.2\,\text{ft/s}^2) = 79.81$ slug is the mass of the car, $v = 60$ mph $= 88.00$ ft/s is the speed of the car, $I_{Gw} = 0.989$ slug·ft^2 is the mass moment of inertia of each wheel relative to the wheel's mass center, and where ω_w is the angular speed of each wheel. Assuming rolling without slip, the angular speed of each wheel is

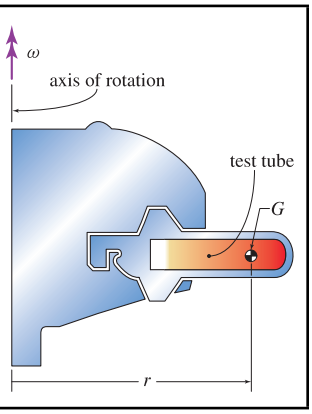
$$\omega_w = 2v/d = 86.91\,\text{rad/s},$$

where we have used the fact that $d = 24.3$ in. $= 2.025$ ft. Hence, using the given data, we have

$$\boxed{T = 324{,}000\,\text{ft·lb}.}$$

Problem 8.7

In Example 7.2 on p. 560 we analyzed the forces acting on a test tube in an ultra-centrifuge. Recalling that the center of mass G of the test tube was assumed to be at a distance $r = 0.0918\,\text{m}$ from the centrifuge's spin axis, that the test tube had a mass $m = 0.01\,\text{kg}$ and a mass moment of inertia $I_G = 2.821\times10^{-6}\,\text{kg}\cdot\text{m}^2$, determine the kinetic energy of the test tube when it is spun at $\omega = 60{,}000\,\text{rpm}$. In addition, if you were to convert the computed kinetic energy to gravitational potential energy, at what height (expressed in meters) relative to the ground could you lift a 10 kg weight?



Solution

The test tube is undergoing a fixed axis rotation about the axis of rotation of the centrifuge. Hence, the kinetic energy of the test tube is

$$T = \tfrac{1}{2} I_O \omega^2, \tag{1}$$

where O is the center of the path of point G and I_O is the mass moment of inertia of the tube computed about O. Using the parallel axis theorem, we have

$$I_O = I_G + mr^2 = 0.00008709\,\text{kg}\cdot\text{m}^2, \tag{2}$$

where we have used the following numerical data: $I_G = 2.821\times10^{-6}\,\text{kg}\cdot\text{m}^2$, $m = 0.01\,\text{kg}$, and $r = 0.0918\,\text{m}$. Hence, using the given data, we have

$$T = 1719\,\text{J}, \tag{3}$$

which, when expressed to three significant figures, is

$$\boxed{T = 1720\,\text{J}.}$$

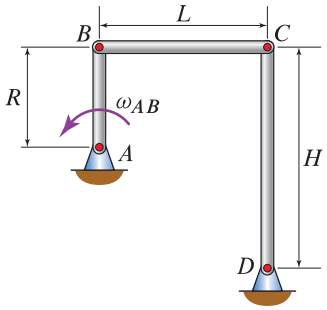
Given a body of mass m_B, the potential energy of such a body placed at a height h above the ground is $m_B g h$. Hence, setting such an energy amount equal to the computed kinetic energy, letting $m_B = 10.0\,\text{kg}$, and solving for h gives

$$\boxed{h = \frac{T}{m_B g} = 17.5\,\text{m},}$$

where we have used the value of T in Eq. (3) and $g = 9.81\,\text{m/s}^2$.

Problem 8.8

The uniform thin bars AB, BC, and CD have masses $m_{AB} = 2.3\,\text{kg}$, $m_{BC} = 3.2\,\text{kg}$, and $m_{CD} = 5.0\,\text{kg}$, respectively. The connections at A, B, C, and D are pinned joints. Letting $R = 0.75\,\text{m}$, $L = 1.2\,\text{m}$, and $H = 1.55\,\text{m}$, and knowing that bar AB rotates at an angular velocity $\omega_{AB} = 4\,\text{rad/s}$, compute the kinetic energy T of the system at the instant shown.



Solution

The kinetic energy of the system is the sum of the kinetic energy of each individual component of the system. Bar AB is undegoing a fixed axis rotation about A. Hence, the kinetic energy of bar AB is

$$T_{AB} = \tfrac{1}{2} I_A \omega_{AB}^2, \tag{1}$$

where I_A is the mass moment of inertia of bar AB about A. Using the concept of instantaneous center of rotation, we see that the motion of bar BC is a curvilinear translation. Hence, the speed of every point on the bar BC is the same as the speed of point B. In turn, this means that the kinetic energy of bar BC is

$$T_{BC} = \tfrac{1}{2} m_{BC} (\omega_{AB} R)^2, \tag{2}$$

where $\omega_{AB} R$ is the speed of point B. Next, we observe that bar CD is in a fixed axis rotation about point D. Since the speed of point C must be equal to the speed of point B, we have that the angular speed of bar CD is $\omega_{CD} = \omega_{AB}(R/H)$, so that the kinetic energy of bar CD is

$$T_{CD} = \tfrac{1}{2} I_D \left(\omega_{AB} \frac{R}{H} \right)^2, \tag{3}$$

where I_D is the mass moment of inertia of bar CD about point D. The mass moments of inertia needed to evaluate the above kinetic energies are

$$I_A = \tfrac{1}{12} m_{AB} R^2 + m_{AB} (R/2)^2 = 0.4312\,\text{kg}\cdot\text{m}^2 \tag{4}$$

and

$$I_D = \tfrac{1}{12} m_{CD} H^2 + m_{CD} (H/2)^2 = 4.004\,\text{kg}\cdot\text{m}^2, \tag{5}$$

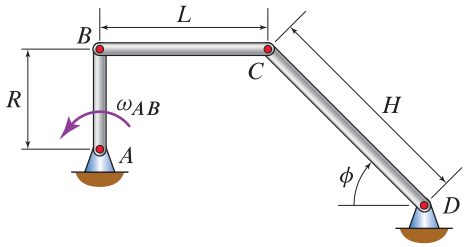
where we have used the following numerical data: $m_{AB} = 2.3\,\text{kg}$, $R = 0.75\,\text{m}$, $m_{CD} = 5.0\,\text{kg}$, and $H = 1.55\,\text{m}$. Hence, using Eqs. (1)–(3), the total kinetic energy of the system is

$$\boxed{T = \frac{1}{2H^2} [I_D R^2 + H^2 (I_A + m_{BC} R^2)] \omega_{AB}^2 = 25.3\,\text{J},}$$

where, in addition to those in Eqs. (4) and (5), we have used the following numerical data: $H = 1.55\,\text{m}$, $R = 0.75\,\text{m}$, $m_{BC} = 3.2\,\text{kg}$, and $\omega_{AB} = 4\,\text{rad/s}$.

Problem 8.9

The weights of the uniform thin pin-connected bars AB, BC, and CD are $W_{AB} = 4\,\text{lb}$, $W_{BC} = 6.5\,\text{lb}$, and $W_{CD} = 10\,\text{lb}$, respectively. Letting $\phi = 47°$, $R = 2\,\text{ft}$, $L = 3.5\,\text{ft}$, and $H = 4.5\,\text{ft}$, and knowing that bar AB rotates at an angular velocity $\omega_{AB} = 4\,\text{rad/s}$, compute the kinetic energy T of the system at the instant shown.



Solution

The kinetic energy of the system is the sum of the kinetic energies of each individual component, i.e.,

$$T = T_{AB} + T_{BC} + T_{CD}. \tag{1}$$

Bars AB and CD are in fixed axis rotations about A and D respectively. Hence, letting G denote the mass center of bar BC, we have

$$T_{AB} = \tfrac{1}{2} I_A \omega_{AB}^2, \quad T_{BC} = \tfrac{1}{2} m_{BC} v_G^2 + \tfrac{1}{2} I_G \omega_{BC}^2, \quad \text{and} \quad T_{CD} = \tfrac{1}{2} I_D \omega_{CD}^2, \tag{2}$$

where I_A is the mass moment of inertia of bar AB about A, I_G is the mass moment of inertia of bar BC about its own mass center G, and I_D is the mass moment of inertia of bar CD about D, i.e.,

$$I_A = \tfrac{1}{12} m_{AB} R^2 + m_{AB}(R/2)^2 = 0.1656\,\text{slug}\cdot\text{ft}^2, \tag{3}$$

$$I_G = \tfrac{1}{12} m_{BC} L^2 = 0.2061\,\text{slug}\cdot\text{ft}^2, \tag{4}$$

$$I_D = \tfrac{1}{12} m_{CD} H^2 + m_{CD}(H/2)^2 = 2.097\,\text{slug}\cdot\text{ft}^2, \tag{5}$$

where, recalling that the acceleration due to gravity is $g = 32.2\,\text{ft/s}^2$, we have used the following numerical data: $m_{AB} = W_{AB}/g = (4\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.1242\,\text{slug}$, $R = 2\,\text{ft}$, $m_{BC} = W_{BC}/g = (6.5\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.2019\,\text{slug}$, $L = 3.5\,\text{ft}$, $m_{CD} = W_{CD}/g = (10\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.3106\,\text{slug}$, and $H = 4.5\,\text{ft}$.

We now determine ω_{BC}, v_G, and ω_{CD}. The position of vectors of B with relative to A, C relative to B, and D relative to C are, respectively,

$$\vec{r}_{B/A} = R\,\hat{\jmath}, \quad \vec{r}_{C/B} = L\,\hat{\imath}, \quad \text{and} \quad \vec{r}_{D/C} = H(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}). \tag{6}$$

Then we have

$$\vec{v}_B = \omega_{AB}\,\hat{k} \times \vec{r}_{B/A} = -R\omega_{AB}\,\hat{\imath}, \tag{7}$$

$$\vec{v}_C = \vec{v}_B + \omega_{BC}\,\hat{k} \times \vec{r}_{C/B} = -R\omega_{AB}\,\hat{\imath} + \omega_{BC} L\,\hat{\jmath}, \tag{8}$$

$$\vec{v}_D = \vec{v}_C + \omega_{CD}\,\hat{k} \times \vec{r}_{D/C} = (H\omega_{CD}\sin\phi - R\omega_{AB})\,\hat{\imath} + (L\omega_{BC} + H\omega_{CD}\cos\phi)\,\hat{\jmath}. \tag{9}$$

Since D is a fixed point, we must have $\vec{v}_D = \vec{0}$, which then implies

$$\omega_{BC} = -\omega_{AB}\frac{R\cos\phi}{L\sin\phi} = -2.131\,\text{rad/s} \quad \text{and} \quad \omega_{CD} = \omega_{AB}\frac{R}{H\sin\phi} = 2.431\,\text{rad/s}, \tag{10}$$

where we have used the following numerical data: $\omega_{AB} = 4\,\text{rad/s}$, $R = 2\,\text{ft}$, $L = 3.5\,\text{ft}$, and $\phi = 47°$. Then, using the first of Eqs. (10), we have that the velocity of G is

$$\vec{v}_G = \vec{v}_B + \omega_{BC}\,\hat{k} \times \tfrac{1}{2}\vec{r}_{C/B} = -R\omega_{AB}\,\hat{\imath} - \tfrac{1}{2}R\omega_{AB}\frac{\cos\phi}{\sin\phi}\,\hat{\jmath}$$

$$\Rightarrow \quad v_G^2 = R^2\omega_{AB}^2\left(1 + \tfrac{1}{4}\frac{\cos^2\phi}{\sin^2\phi}\right) = 77.91\,\text{ft}^2/\text{s}^2, \tag{11}$$

where we have used some of the same numerical data listed earlier. Substituting Eq. (3) into the first of Eq. (2) and recalling that $\omega_{AB} = 4\,\text{rad/s}$, we have

$$T_{AB} = 1.325\,\text{ft·lb}. \tag{12}$$

Substituting the numerical results in Eq. (4), the first of Eqs. (10), and Eq. (11) into the second of Eq. (2), we have

$$T_{BC} = 8.333\,\text{ft·lb}. \tag{13}$$

Finally, substituting the numerical results in Eq. (5) and in the second of Eqs. (10), we have

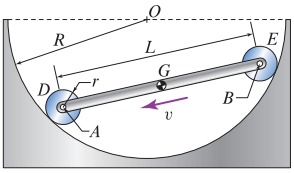
$$T_{CD} = 6.196\,\text{ft·lb}. \tag{14}$$

Hence, summing the last three results, we have

$$\boxed{T = 15.9\,\text{ft·lb}.}$$

Problem 8.10

The uniform slender bar AB has length $L = 1.45\,\text{ft}$ and weight $W_{AB} = 20\,\text{lb}$. Rollers D and E, which are pinned at A and B, respectively, can be modeled as two identical uniform disks, each with radius $r = 1.5\,\text{in.}$ and weight $W_r = 0.35\,\text{lb}$. Rollers D and E roll without slip on the surface of a cylindrical bowl with center at O and radius $R = 1\,\text{ft}$. Determine the system's kinetic energy when G (the center of mass of bar AB) moves with a speed $v = 7\,\text{ft/s}$.



Solution

The kinetic energy of the system is

$$T = T_{AB} + T_D + T_E, \tag{1}$$

where

$$T_{AB} = \tfrac{1}{2} m_{AB} v_G^2 + \tfrac{1}{2} I_G \omega_{AB}^2 \quad \text{and} \quad T_D = T_E = \tfrac{1}{2} m_r v_A^2 + \tfrac{1}{2} I_r \omega_r^2, \tag{2}$$

where ω_{AB} is the angular speed of bar AB, I_G is the mass moment of inertia of bar AB relative to its own mass center G, v_A is the speed of A, I_r is the mass moment of inertia of each roller about its own mass center, and ω_r is the angular speed of each roller.

Observe that bar AB is in a fixed axis rotation about O. In addition, the distance d_{GO} between points G and O is

$$d_{GO} = \sqrt{(R-r)^2 - (L/2)^2}. \tag{3}$$

Hence, we have

$$\omega_{AB} = \frac{v_G}{d_{GO}} = \frac{v_G}{\sqrt{(R-r)^2 - (L/2)^2}} = 14.29\,\text{rad/s} \tag{4}$$

and

$$v_A = (R-r)\omega_{AB} = \frac{v_G (R-r)}{\sqrt{(R-r)^2 - (L/2)^2}} = 12.50\,\text{ft/s}, \tag{5}$$

where we have used the following numerical data: $v_G = 7\,\text{ft/s}$, $R = 1\,\text{ft}$, $r = 1.5\,\text{in.} = 0.1250\,\text{ft}$, and $L = 1.45\,\text{ft}$. Recalling that each roller rolls without slip, we must have

$$v_A = r\omega_r \quad \Rightarrow \quad \omega_r = \frac{v_A}{r} = 100.0\,\text{rad/s}. \tag{6}$$

The needed mass moments of inertia are

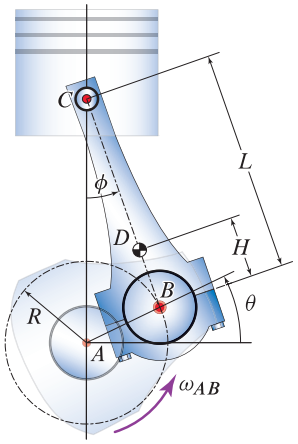
$$I_G = \tfrac{1}{12} m_{AB} L^2 = 0.1088\,\text{slug}\cdot\text{ft}^2 \quad \text{and} \quad I_r = \tfrac{1}{2} m_r r^2 = 84.92\times 10^{-6}\,\text{slug}\cdot\text{ft}^2, \tag{7}$$

where, in addition to the values of L and r that have been already listed, we have used the following numerical data: $m_{AB} = (20\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.6211\,\text{slug}$ and $m_r = (0.35\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.01087\,\text{slug}$. Substituting the numerical values we have computed thus far, we can say that

$$\boxed{T = 28.9\,\text{ft}\cdot\text{lb}.}$$

Problems 8.11 and 8.12

For the slider-crank mechanism shown, let $L = 141\,\text{mm}$, $R = 48.5\,\text{mm}$, and $H = 36.4\,\text{mm}$. In addition, observing that D is the center of mass of the connecting rod, let the mass moment of inertia of the connecting rod be $I_D = 0.00144\,\text{kg}\cdot\text{m}^2$ and the mass of the connecting rod be $m = 0.439\,\text{kg}$.



Problem 8.11 Letting $\omega_{AB} = 2500\,\text{rpm}$, compute the kinetic energy of the connecting rod for $\theta = 90°$ and for $\theta = 180°$.

Problem 8.12 Plot the kinetic energy of the connecting rod as a function of the crank angle θ over one full cycle of the crank for $\omega_{AB} = 2500\,\text{rpm}$, 5000 rpm, and 7500 rpm.

Solution to 8.11

The kinetic energy of the connecting rod is

$$T_{BC} = \tfrac{1}{2}mv_D^2 + \tfrac{1}{2}I_D\omega_{BC}^2, \tag{1}$$

where v_D is the speed of the center of mass of the connecting rod, ω_{BC} is the angular velocity of the connecting rod, and where I_D is the mass moment of inertia of the connecting rod relative to its own mass center. Using the results of Example 6.5 on p. 484 of the textbook, we have.

$$\omega_{BC} = -\frac{\omega_{AB}\sin\theta}{\sqrt{(L/R)^2 - \cos^2\theta}}, \tag{2}$$

and

$$\vec{v}_D = R\omega_{AB}\left[\sin\theta\left(\frac{H}{L} - 1\right)\hat{\imath} + \cos\theta\left(1 + \frac{H\sin\theta}{L\sqrt{(L/R)^2 - \cos^2\theta}}\right)\hat{\jmath}\right]. \tag{3}$$

Recalling that $v_D^2 = \vec{v}_D \cdot \vec{v}_D$, substituting Eqs. (2) and (3) into Eq. (1), after simplification, we have

$$T_{BC} = \tfrac{1}{2}R^2\omega_{AB}^2\left[\frac{I_D\sin^2\theta}{L^2 - R^2\cos^2\theta} + m\left(\frac{H}{L} - 1\right)^2 + m\cos^2\theta\left(1 + \frac{HR\sin\theta}{L\sqrt{L^2 - R^2\cos^2\theta}}\right)^2\right]. \tag{4}$$

evaluating the above expression, we have

$$\boxed{T_{BC}\big|_{\theta=90°} = 25.3\,\text{J} \quad \text{and} \quad T_{BC}\big|_{\theta=180°} = 35.4\,\text{J},}$$

where we have used the following numerical data: $R = 48.5\,\text{mm} = 48.50\times10^{-3}\,\text{m}$, $\omega_{AB} = 2500\,\text{rpm} = 261.8\,\text{rad/s}$, $I_D = 0.00144\,\text{kg}\cdot\text{m}^2$, $L = 141\,\text{mm} = 141.0\times10^{-3}\,\text{m}$, $m = 0.439\,\text{kg}$, and $H = 36.4\,\text{mm} = 36.40\times10^{-3}\,\text{m}$.

Solution to 8.12

This problem simply requires plots of the function derived in Eq. (4) of the solution to Problem 8.11. The plots shown below were obtained with the following Mathematica code.

```
In[1]:= KE = 1/2 R^2 ω_AB^2 ( I_D Sin[θ]^2 / (L^2 - R^2 Cos[θ]^2) + m ( (-1 + H/L)^2 Sin[θ]^2 + Cos[θ]^2 ( 1 + H Sin[θ] / (L Sqrt[L^2/R^2 - Cos[θ]^2]) )^2 ) );

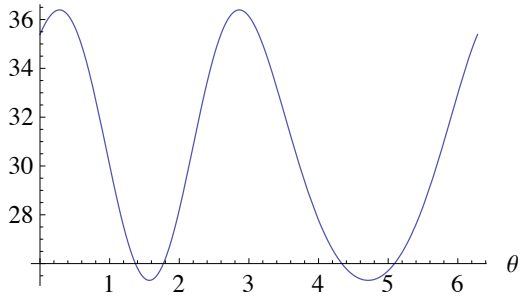
Parameters = { L → 141.0/1000.0 , R → 48.5/1000.0 , H → 36.4/1000.0 , I_D → 0.00144 , m → 0.439 };

Plot[ KE /. ω_AB → 2500 · 2 π / 60.0 /. Parameters, {θ, 0, 2 π}, AxesLabel → {"θ", "T_BC"} ]

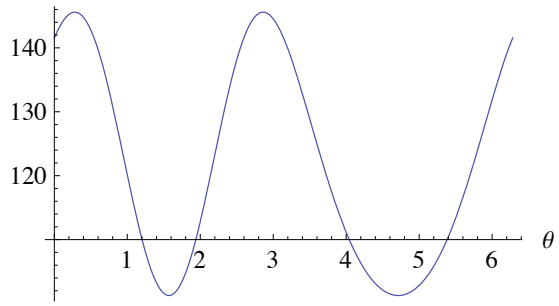
Plot[ KE /. ω_AB → 5000 · 2 π / 60.0 /. Parameters, {θ, 0, 2 π}, AxesLabel → {"θ", "T_BC"} ]

Plot[ KE /. ω_AB → 7500 · 2 π / 60.0 /. Parameters, {θ, 0, 2 π}, AxesLabel → {"θ", "T_BC"} ]
```

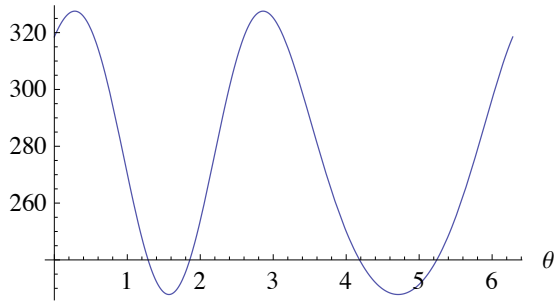
T_{BC}; ω_{AB} =2500 rpm



T_{BC}; ω_{AB} =5000 rpm

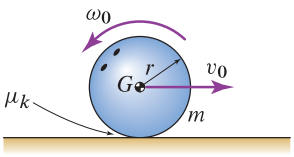


T_{BC}; ω_{AB} =7500 rpm



Problems 8.13 and 8.14

A 14 lb bowling ball is thrown onto a lane with a backspin angular speed $\omega_0 = 10$ rad/s and forward velocity $v_0 = 17$ mph. After a few seconds, the ball starts rolling without slip and moving forward with a speed $v_f = 17.2$ ft/s. Let $r = 4.25$ in. be the radius of the ball, and let $k_G = 2.6$ in. be its radius of gyration.

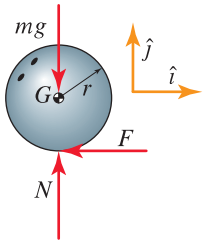


Problem 8.13 Determine the work done by friction on the ball from the initial time until the time that the ball starts rolling without slip.

Problem 8.14 Knowing that the coefficient of kinetic friction between the lane and the ball is $\mu_k = 0.1$, determine the length L_f over which the friction force acts in order to slow down the ball from v_0 to v_f. Does L_f also represent the distance traveled by the center of the ball? Explain.

Solution to 8.13

Given the FBD shown and that the ball does not move in the vertical direction, F is the only force doing work on the ball. Letting ① be the position at which the ball first makes contact with the lane and ② be the position at which the ball starts rolling without slip, and letting $U_{1\text{-}2}$ be the work of friction, we have



$$U_{1\text{-}2} = T_2 - T_1, \tag{1}$$

where

$$T_1 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}I_G\omega_1^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}I_G\omega_2^2, \tag{2}$$

wre the kinetic energies of the ball at ① and ②, respectively.

Observe that we have

$$m = W/g, \quad I_G = (W/g)k_G^2, \quad v_1 = v_0, \quad \omega_1 = \omega_0, \quad v_2 = v_f, \quad \text{and} \quad \omega_2 = v_f/r. \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1), after simplifying we have

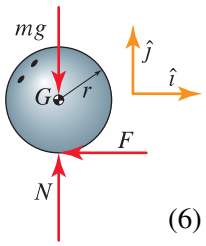
$$U_{1\text{-}2} = \frac{W}{2gr^2}\left[v_f^2k_G^2 - r^2\left(v_0^2 - v_f^2 + k_G^2\omega_0^2\right)\right], \tag{4}$$

which can be evaluated to obtain

$$\boxed{U_{1\text{-}2} = -47.8 \text{ ft·lb},}$$

where we have used the following numerical data: $W = 14$ lb, $g = 32.2$ ft/s^2, $r = 4.25$ in. $= 0.3542$ ft, $v_f = 17.2$ ft/s, $k_G = 2.6$ in. $= 0.2167$ ft, $v_0 = 17.0$ mph $= 24.93$ ft/s, and $\omega_0 = 10$ rad/s.

Solution to 8.14



Letting ① be the position at which the ball first makes contact with the lane and ② be the position at which the ball starts rolling without slip, and letting $U_{1\text{-}2}$ be the work of friction, we have

$$U_{1\text{-}2} = T_2 - T_1, \tag{5}$$

where

$$T_1 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}I_G\omega_1^2 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}I_G\omega_2^2, \tag{6}$$

wre the kinetic energies of the ball at ① and ②, respectively.

Observe that we have

$$m = W/g, \quad I_G = (W/g)k_G^2, \quad v_1 = v_0, \quad \omega_1 = \omega_0, \quad v_2 = v_f, \quad \text{and} \quad \omega_2 = v_f/r. \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (5), after simplifying we have

$$U_{1\text{-}2} = \frac{W}{2gr^2}\Big[v_f^2k_G^2 - r^2\big(v_0^2 - v_f^2 + k_G^2\omega_0^2\big)\Big], \tag{8}$$

Now, referring to the FBD on the right, observing that $F = \mu_k N$ because the ball is slipping, summing forces in the vertical direction, and observing that the ball does not move in the vertical direction, we must have

$$F = \mu_k mg = \mu_k W. \tag{9}$$

Since F is constant, $U_{1\text{-}2} = -L_f F$, where $U_{1\text{-}2}$ is that given in Eq. (8). Hence,

$$L_f = U_{1\text{-}2}/F = \frac{1}{2\mu_k gr^2}\big[v_f^2k_G^2 - r^2(v_0^2 - v_f^2 + k_G^2\omega_0^2),\big] \tag{10}$$

which can be evaluated to obtain

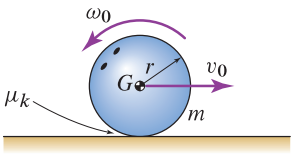
$$\boxed{L_f = 34.1\,\text{ft},}$$

where we have used the following numerical data: $\mu_k = 0.1$, $W = 14\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $r = 4.25\,\text{in.} = 0.3542\,\text{ft}$, $v_f = 17.2\,\text{ft/s}$, $k_G = 2.6\,\text{in.} = 0.2167\,\text{ft}$, $v_0 = 17.0\,\text{mph} = 24.93\,\text{ft/s}$, and $\omega_0 = 10\,\text{rad/s}$.

In general, the distance L_f is not the distance traveled by the center of ball. This is because the center of wheel goes from a speed of v_0 to the speed v_f whereas the point of contact between the wheel and the ground must slow down from a speed of $v_0 + \omega_0 r$ to 0. Hence, depending on the initial conditions, the distance over which the friction force acts during slip may not be the same as that traveled by the center of the ball.

Problem 8.15

A bowling ball is thrown onto a lane with a forward velocity v_0 and no angular velocity ($\omega_0 = 0$). Because of friction between the lane and the ball, after a short time, the ball starts rolling without slip and moving forward with speed v_f. Let L_G be the distance traveled by the center of the ball while slowing down from v_0 to v_f. In addition, let L_f be the length over which the friction force had to act in order to slow down the ball from v_0 to v_f. State which of the following relations is true and why: (a) $L_G < L_f$; (b) $L_G = L_f$; (c) $L_G > L_f$.



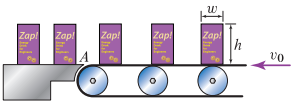
Solution

Answer The problem solution is (c) $L_G > L_f$.

Explanation Begin with observing that the point of application of the friction force is not an individual point but it changes with time. With this in mind, we can think of the point of application of the friction force as a point whose initial speed is v_0 and its final speed if zero, where this value of final speed is achieved when the ball starts rolling without slip. In the same amount of time that the point of application of the friction force goes from v_0 to zero, the center of the ball goes from v_0 to some value $v_f \neq 0$. This indicates that the magnitude of the acceleration of the point of application of the friction force is larger than that of the center of the wheel. In turn, recalling that the points in question have the same initial speed, this indicates that the center of wheel travels farther than the point of application of the friction force.

Problem 8.16

A conveyor is moving cans at a constant speed v_0 when, to proceed to the next step in packaging, the cans are transferred onto a stationary surface at A. The cans each have mass m, width w, and height h. Assuming that there is friction between each can and the stationary surface, under what conditions would we be able to compute the stopping distance of the cans, using the work-energy principle for a particle?



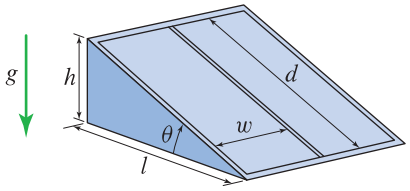
Solution

A rigid body motion problem can be reduced to the solution of a particle problem if the rigid body is a translation. Therefore, as long as the cans do not tip once they slide over the fixed surface to the left of the conveyor, then we would be able to compute the slipping distance using only a particle model.

Problem 8.17

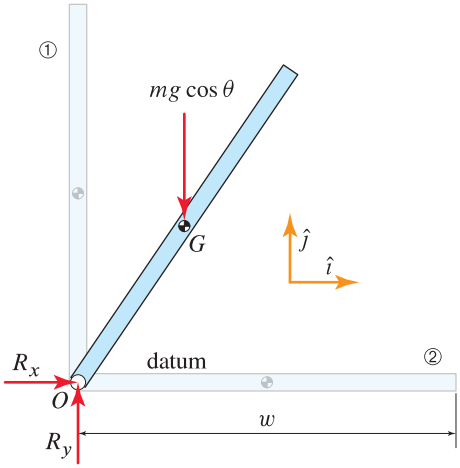
One of the basement doors is left open in the vertical position when it is given a nudge and allowed to freely fall to the closed position. Given that the door has mass m and that it is modeled as a uniform thin plate of width w and length d, determine its angular velocity when it reaches the closed position. *Hint:* Treat the problem as symmetric with respect to a plane of motion in which the acceleration due to gravity is $g\cos\theta$ rather than g.





Solution

In the FBD shown, the direction perpendicular to the plane of motion is parallel to the axis of rotation of the door. Following the given hint, we see that the door is subject only to a component of the force of gravity. The forces R_x and R_y are the (fixed) pin reaction forces and do not work. Hence, letting ① be the position at which the door is released and ② be the position at which the door first achieves its closed position, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where

$$T_1 = 0, \quad V_1 = \tfrac{1}{2}wmg\cos\theta, \quad T_2 = \tfrac{1}{2}I_O\omega_{d2}^2, \quad V_2 = 0, \tag{2}$$

and where ω_d is the angular speed of the door and I_O is the mass moment of inertia of the door computed relative to the axis of rotation and is given by

$$I_O = \tfrac{1}{12}mw^2 + m\left(\frac{w}{2}\right)^2 = \tfrac{1}{3}mw^2. \tag{3}$$

Notice that, given the geometry of the door, we have

$$\cos\theta = \frac{l}{\sqrt{h^2 + l^2}} \tag{4}$$

Substituting Eqs. (2)–(4) into Eq. (1) and solving for ω_d, we have

$$\omega_{d2} = \sqrt{\frac{3gl}{w\sqrt{h^2 + l^2}}}. \tag{5}$$

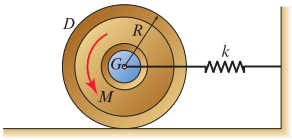
Given our choice of component system, then the angular velocity of the door at ② is

$$\vec{\omega}_{d2} = -\sqrt{\frac{3gl}{w\sqrt{h^2 + l^2}}}\,\hat{k},$$

where $\hat{k}$ is parallel to the rotation axis of the door and pointing toward the ground.

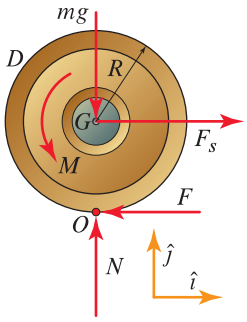
Problem 8.18

The disk D, which has mass m, center of mass G, and radius of gyration k_G, is at rest on a flat horizontal surface when the constant moment M is applied to it. The disk is attached at its center to a vertical wall by a linear elastic spring of constant k. The spring is unstretched when the system is at rest. Assuming that the disk rolls without slipping and that it has not yet come to a stop, determine the angular velocity of the disk after its center G has moved a distance d. After doing so, determine the distance d_s that the disk moves before it comes to a stop.



Solution

Referring to the FBD shown and recalling that the disk rolls without slip, work is done only by the spring force and by the applied moment. Letting position 1 be the position of the disk at release and letting any generic position following the position of release be position 2, we have



$$T_1 + V_1 + U_{1\text{-}2} = T_2 + V_2, \tag{1}$$

where, recalling that M is constant,

$$T_1 = 0, \quad V_1 = 0, \quad U_{1\text{-}2} = M(d/R), \quad T_2 = \tfrac{1}{2} I_O \omega_{d2}^2, \quad V_2 = \tfrac{1}{2} k d^2, \tag{2}$$

and where ω_d is the angular speed of the disk, I_O is the mass moment of inertia of the disk relative to point O which is in contact with the ground, and d/R measures the angular displacement of the disk corresponding to an horizontal displacement of the center of the disk of an amount d.

Using the parallel axis theorem, the quantity I_O is given by

$$I_O = mk_G^2 + mR^2 = m(k_G^2 + R^2). \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1), and solving for ω_2, we have

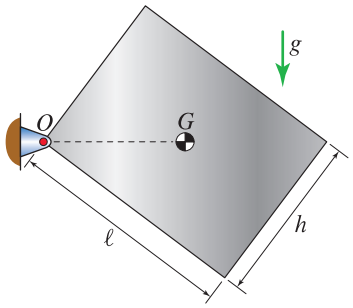
$$\boxed{\omega_2 = \sqrt{\frac{(2Md/R) - kd^2}{m(k_G^2 + R^2)}}.} \tag{4}$$

Setting $\omega_{d2} = 0$ in Eq. (4) and we obtain the value of d_s, which is

$$\boxed{d_s = \frac{2M}{kR}.} \tag{5}$$

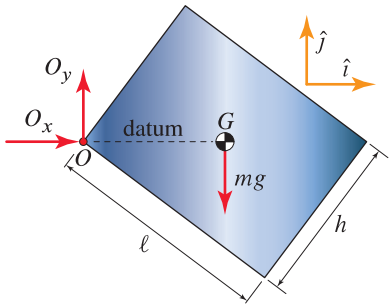
Problem 8.19

The uniform rectangular plate of length ℓ, height h, and mass m lies in the vertical plane and is pinned at one corner. If the plate is released from rest in the position shown, determine its angular velocity when the center of mass G is directly below the pivot O. Neglect any friction at the pin at O.



Solution

Referring to the FBD shown, work is done only by gravity. Hence, letting position 1 be the position of the plate at release and letting position 2 be when point G is immediately below O, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where

$$T_1 = 0, \quad V_1 = 0, \quad T_2 = \tfrac{1}{2} I_O \omega_{p2}^2, \quad V_2 = -\tfrac{1}{2} mg\sqrt{\ell^2 + h^2}, \tag{2}$$

and where ω_p is the angular speed of the plate, and where I_O, which is the mass moment of inertia of the plate about point O, is given by

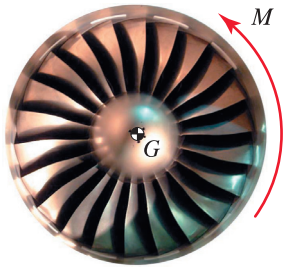
$$I_O = \tfrac{1}{12} m(\ell^2 + h^2) + \tfrac{1}{4} m(\ell^2 + h^2) = \tfrac{1}{3} m(\ell^2 + h^2). \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1) and solving for ω_{p2} we have that the plate rotates clockwise with the following angular speed

$$\omega_{p2} = \sqrt{\frac{3g}{\sqrt{\ell^2 + h^2}}}.$$

Problem 8.20

A turbine rotor with weight $W = 3000\,\text{lb}$, center of mass at the fixed point G, and radius of gyration $k_G = 15\,\text{ft}$ is brought from rest to an angular velocity $\omega = 1500\,\text{rpm}$ in 20 revolutions by applying a constant torque M. Neglecting friction, determine the value of M needed to spin up the rotor as described.



Solution

Observe that work is done only by the applied moment M. Letting ① be the initial position and ② be the position of the rotor after 20 revolutions, we have

$$T_1 + U_{1\text{-}2} = T_2, \tag{1}$$

where

$$T_1 = 0, \quad T_2 = \tfrac{1}{2} I_G \omega^2, \quad U_{1\text{-}2} = M\Delta\theta, \tag{2}$$

where ω is the angular speed of the rotor at ②, $\Delta\theta$ is the angular displacement of the rotor between positions ① and ②, and $I_G = (W/g)k_G^2$ is the rotor's mass moment of inertia.

Substituting the expressions in Eq. (2) and the expression for I_G into Eq. (1), and solving for M, we obtain

$$M = \frac{k_G^2 W \omega^2}{2g\Delta\theta}, \tag{3}$$

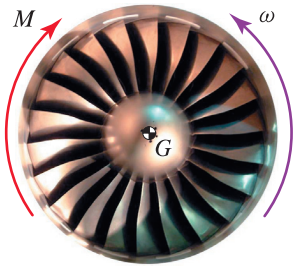
which can be evaluated to obtain

$$\boxed{M = 2.06\times10^6\,\text{ft·lb},}$$

where we have used the following numerical data: $k_G = 15\,\text{ft}$, $W = 3000\,\text{lb}$, $\omega = 1500\,\text{rpm} = 157.1\,\text{rad/s}$, $g = 32.2\,\text{ft/s}^2$, and $\Delta\theta = 20\,\text{rev} = 125.7\,\text{rad}$.

Problem 8.21

A turbine rotor with weight $W = 3000\,\text{lb}$, center of mass at the fixed point G, and radius of gyration $k_G = 15\,\text{ft}$ is spinning with an angular speed $\omega = 1200\,\text{rpm}$ when a braking system is engaged that applies a constant torque $M = 3000\,\text{ft·lb}$. Determine the number of revolutions needed to bring the rotor to a stop.



Solution

Observe that work is done only by the applied moment M. Letting ① be the position at which the brakes are first applied and ② be the position of the rotor when it first comes to a stop, we have

$$T_1 + U_{1\text{-}2} = T_2, \tag{1}$$

where

$$T_1 = \tfrac{1}{2} I_G \omega^2, \quad T_2 = 0, \quad U_{1\text{-}2} = M\Delta\theta, \tag{2}$$

where ω is the angular speed of the rotor at ①, $\Delta\theta$ is the angular displacement of the rotor between positions ① and ②, and $I_G = (W/g)k_G^2$ is the rotor's mass moment of inertia.

Substituting the expressions in Eq. (2) and the expression for I_G into Eq. (1), and solving for $\Delta\theta$, we obtain

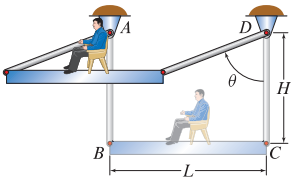
$$\Delta\theta = \frac{k_G^2 W \omega^2}{2gM}, \tag{3}$$

which can be evaluated to obtain

$$\boxed{\Delta\theta = 8790\,\text{rev},}$$

where we have used the following numerical data: $k_G = 15\,\text{ft}$, $W = 3000\,\text{lb}$, $\omega = 1200\,\text{rpm} = 125.7\,\text{rad/s}$, $g = 32.2\,\text{ft/s}^2$, and $M = 3000\,\text{ft·lb}$.

Problems 8.22 and 8.23

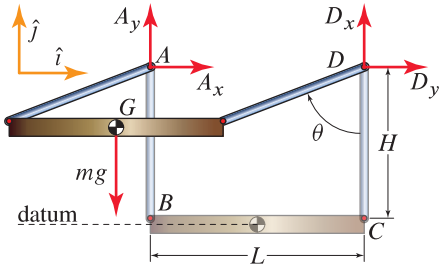


In a contraption built by a fraternity, a person sits at the center of a swinging platform with mass $m = 400\,\text{kg}$ and length $L = 4\,\text{m}$ suspended via two identical arms of length $H = 3\,\text{m}$.

Problem 8.22 Neglecting the mass of the arms and of the person, neglecting friction, and assuming that the platform is released from rest when $\theta = 180°$, compute the speed of the person as a function θ for $0° \leq \theta \leq 180°$. In addition, find the speed of the person for $\theta = 0°$.

Problem 8.23 Neglecting the mass of the person, neglecting friction, letting the mass of each arm be $m_A = 150\,\text{kg}$, and assuming that the platform is released from rest when $\theta = 180°$, compute the speed of the person as a function θ for $0° \leq \theta \leq 180°$. In addition, find the speed of the person for $\theta = 0°$.

Solution to 8.22



Referring to the FBD shown, work is done only by gravity. We define ① as the position of the system at release. We define ② as the position of the system for a generic value of θ. Hence, we have

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, since the system is released from rest and due to our choice of datum, we have

$$T_1 = 0, \quad V_1 = 2mgH, \quad T_2 = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_{p2}^2, \quad V_2 = mgH(1 - \cos\theta), \tag{2}$$

where v_G is the speed of the mass center of the platform, ω_p is the angular speed of the platform, and where I_G, which is the mass moment of inertia of the platform about its own mass center.

Since the platform can only translate, we have that $\omega_{p2} = 0$ and the speed of the person moving along with the platform is equal t that of point G. Therefore, substituting Eq. (2) into Eq. (1) and solving for v_G, we have

$$\boxed{v_{\text{person}} = v_G = \sqrt{2gH(1 + \cos\theta)}.} \tag{3}$$

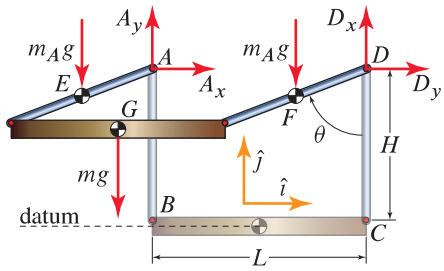
Evaluating Eq. (3) for $\theta = 0°$, we have

$$\boxed{v_{\text{person}}\big|_{\theta=0°} = 10.8\,\text{m/s},}$$

where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$ and $H = 3\,\text{m}$.

Solution to 8.23

Referring to the FBD shown, work is done only by gravity. We define ① as the position of the system at release. We define ② as the position of the system for a generic value of θ. Hence, we have



$$T_1 + V_1 = T_2 + V_2, \tag{4}$$

where, since the system is released from rest, due to our choice of datum, and given that the arms are identical and must always move parallel to one another, we have

$$T_1 = 0, \quad V_1 = 2mgH + 3m_A gH, \tag{5}$$

$$T_2 = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_{p2}^2 + \tfrac{1}{2}I_A\omega_{AB}^2 + \tfrac{1}{2}I_D\omega_{CD}^2 \quad V_2 = mgH(1-\cos\theta) + 2m_A gH[1 - \tfrac{1}{2}\cos\theta], \tag{6}$$

where v_G is the speed of the mass center of the platform, ω_p is the angular speed of the platform, I_G is the mass moment of inertia of the platform about its own mass center, I_A is the mass moment of inertia of bar AB about A, I_D is the mass moment of inertia of bar CD about D, ω_{AB} is the angular speed of bar AB, and ω_{CD} is the angular speed of bar CD.

The mass moments of inertia I_A and I_D are given by

$$I_A = I_D = \tfrac{1}{12}m_A H^2 + m_A(H/2)^2 = \tfrac{1}{3}m_A H^2. \tag{7}$$

We observe that the platform can only translate and that therefore the speed of the person is the same as that of point G. Also, notice that the arms must always remain parallel to one another and that points B and C must have the same speed as point G. Therefore, we have

$$\omega_{p2} = 0, \quad \omega_{AB2} = \omega_{CD2} = v_{G2}/H. \tag{8}$$

Substituting Eqs. (2)–(8) into Eq. (4) and solving for v_G, we have

$$\boxed{v_{\text{person}} = v_G = \sqrt{6gH\frac{(m+m_A)(1+\cos\theta)}{3m+2m_A}}.} \tag{9}$$

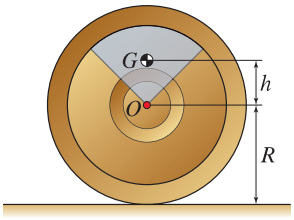
Evaluating Eq. (9) for $\theta = 0°$, we have

$$\boxed{v_{\text{person}}\big|_{\theta=0°} = 11.4\,\text{m/s},}$$

where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$ and $H = 3\,\text{m}$, $m = 400\,\text{kg}$, and $m_A = 150\,\text{kg}$.

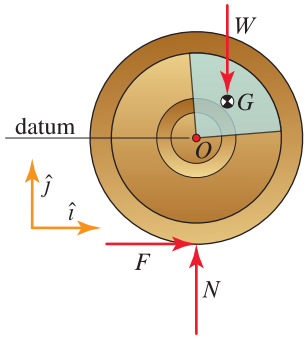
Problem 8.24

An eccentric wheel with weight $W = 250\,\text{lb}$, mass center G, and radius of gyration $k_G = 1.32\,\text{ft}$ is initially at rest in the position shown. Letting $R = 1.75\,\text{ft}$ and $h = 0.8\,\text{ft}$, and assuming that the wheel is gently nudged to the right and rolls without slip, determine the speed of O when G is closest to the ground.



Solution

Referring to the FBD shown, since the wheel rolls without slip, work is done only by gravity. We define ① as the position of the system at release. We define ② as the position of the system when G is closest to the ground. Hence, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, since the system is released from rest and due to our choice of datum, we have

$$T_1 = 0, \quad V_1 = Wh, \quad T_2 = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_{w2}^2, \quad V_2 = -Wh, \tag{2}$$

where v_G is the speed of the mass center of the wheel, ω_w is the angular speed of the wheel, and where I_G, which is the mass moment of inertia of the wheel about its own mass center. The mass moment of inertia I_G is given by

$$I_G = (W/g)k_G^2. \tag{3}$$

Since the wheel rolls without slip, we have

$$\omega_{w2} = \frac{v_O}{R} \quad \text{and} \quad v_{G2} = \frac{v_O}{R}(R-h). \tag{4}$$

Substituting Eqs. (2)–(4) into Eq. (1) and solving for v_O, we have

$$v_O = 2R\sqrt{\frac{gh}{(R-h)^2 + k_G^2}}. \tag{5}$$

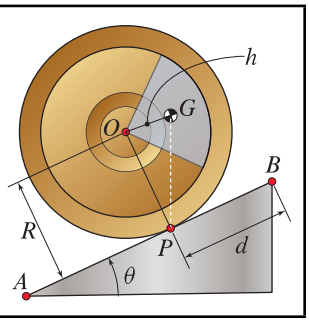
Evaluating Eq. (5), we have

$$\boxed{v_O = 10.9\,\text{ft/s},}$$

where we have used the following numerical data: $R = 1.75\,\text{ft}$, $g = 32.2\,\text{ft/s}^2$, $h = 0.8\,\text{ft}$, $k_G = 1.32\,\text{ft}$.

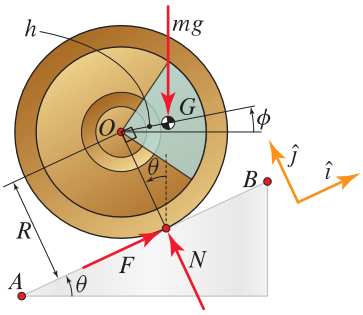
Problem 8.25

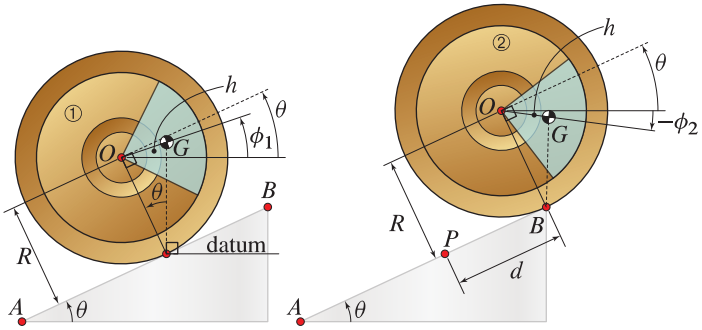
An eccentric wheel with mass $m = 150\,\text{kg}$, mass center G, and radius of gyration $k_G = 0.4\,\text{m}$ is placed on the incline shown such that the wheel's center of mass G is vertically aligned with P, which is the point of contact with the incline. If the wheel rolls without slip once it is gently nudged away from its initial placement, letting $R = 0.55\,\text{m}$, $h = 0.25\,\text{m}$, $\theta = 25°$, and $d = 0.5\,\text{m}$, determine whether the wheel arrives at B and, if yes, determine the corresponding speed of the center O. Note that the angle POG is not equal to 90° at release.



Solution

Referring to the FBD shown to the right, since the wheel rolls without slip, work is done only by gravity. Referring to the figure below, we define ① as the position of the system at release; and we define ② as the position of the system when the point of contact between the wheel and the incline is at B (see figure below).





Hence, we have

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, since the system is released from rest and due to our choice of datum, we have

$$T_1 = 0, \qquad V_1 = mg(R\cos\theta + h\sin\phi_1), \tag{2}$$

$$T_2 = \tfrac{1}{2}mv_{G2}^2 + \tfrac{1}{2}I_G\omega_{w2}^2, \qquad V_2 = mg(R\cos\theta + h\sin\phi_2 + d\sin\theta), \tag{3}$$

where v_G is the speed of the mass center of the wheel, ω_w is the angular speed of the wheel, and where I_G, which is the mass moment of inertia of the wheel about its own mass center. The mass moment of inertia I_G is given by

$$I_G = mk_G^2. \tag{4}$$

To evaluate V_1 and V_2 we need ϕ_1 and ϕ_2. At ① G is directly above P so that

$$h\cos\phi_1 = R\sin\theta \quad \Rightarrow \quad \phi_1 = \cos^{-1}[(R/h)\sin\theta] = 0.3770\,\text{rad}. \tag{5}$$

In going from ① to ②, the center of the wheel displaces a distance d up the incline while the wheel rolls without slip. Hence, we must have

$$\phi_2 = \phi_1 - d/R = -0.5321\,\text{rad}. \tag{6}$$

Again, due to rolling without slip we have

$$\omega_{w2} = v_{O2}/R \quad \text{and} \quad v_{G2} = (v_{O2}/R)d_{BG2}, \tag{7}$$

where d_{BG2} is the distance between B and G at ②, which, using the law of cosines, is given by

$$d_{BG2} = \sqrt{R^2 + h^2 - 2Rh\cos(90° - \theta + \phi_2)}. \tag{8}$$

Using Eqs. (7) and (8) into the expression for the kinetic energy, then we have

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}m\frac{v_{O2}^2}{R^2}[k_G^2 + R^2 + h^2 - 2Rh\cos(90° - \theta + \phi_2)]. \tag{9}$$

Substituting the expressions for the kinetic and potential energies into the work-energy principle and then solving for v_0, we obtain

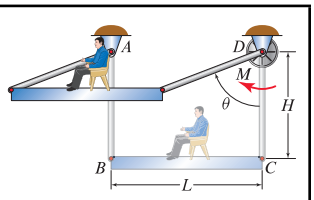
$$v_{O2} = R\sqrt{\frac{2g(\sin\phi_1 - \sin\phi_2) - d\,\sin\theta}{k_G^2 + R^2 + h^2 + 2Rh\sin(\phi_2 - \theta + \phi_2)}}, \tag{10}$$

which can be evaluated to obtain

$$\boxed{v_{O2} = 0.388\,\text{m/s},}$$

where we have used the values of ϕ_1 and ϕ_2 given in Eqs. (5) and (6), respectively, and where we have used the following numerical data: $R = 0.55\,\text{m}$, $g = 9.81\,\text{ft/s}^2$, $d = 0.5\,\text{m}$, $\theta = 25°$, $k_G = 0.4\,\text{m}$, and $h = 0.25\,\text{m}$. Since we have obtained an acceptable solution, we can conclude that the wheel does arrive at B if nudged away from equilibrium.

Problems 8.26 and 8.27

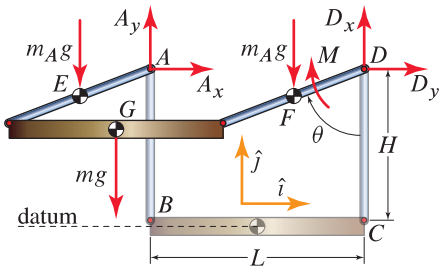


In a contraption built by a fraternity, a person sits at the center of a swinging platform with weight $W_p = 800\,\text{lb}$ and length $L = 12\,\text{ft}$ suspended via two identical arms each of length $H = 10\,\text{ft}$ and weight $W_a = 200\,\text{lb}$. The platform, which is at rest when $\theta = 0$, is put in motion by a motor that pumps the ride by exerting a constant moment M in the direction shown whenever $0 \le \theta \le \theta_p$ while exerting zero moment for any other value of θ.

Problem 8.26 Neglecting the mass of the person, neglecting friction, letting $M = 900\,\text{ft·lb}$, and letting $\theta_p = 25°$, find the minimum number of swings necessary to achieve $\theta > 90°$ and the ensuing speed achieved by the person at the lowest point in the swing. Model the arms AB and CD as uniform thin bars.

Problem 8.27 Neglecting the mass of the person, neglecting friction, and letting $\theta_p = 20°$, determine the value of M required to achieve a maximum value of θ equal to 90° in 6 full swings. Model the arms AB and CD as uniform thin bars.

Solution to 8.26



Referring to the FBD shown, work is done only by gravity and the moment due to the motor. Let ① be the initial position of the system. To determine the minimum number of pumping cycles to achieve $\theta = 90°$, we first compute the amount of energy needed for the system to barely achieve the position with $\theta = 90°$, which we choose as ②. Applying the work-energy principle, we have

$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2, \tag{1}$$

Due to our definitions of ① and ②, and due to our choice of datum, we have

$$T_1 = 0, \quad V_1 = 2W_a\frac{H}{2}, \quad T_2 = 0 \quad V_2 = 2W_a H + W_p H. \tag{2}$$

Since the moment provided by the motor is constant, the work done by the motor over a single pumping cycle is $M\theta_p$. The amount of work needed to go from ① to ② can be expressed as

$$(U_{1\text{-}2})_{\text{nc}} = (M\theta_p)\,f, \tag{3}$$

where f is a real valued factor measuring how much the "unit" amount of work $M\theta_p$ must be repeated to get the system to achieve ②. Substituting Eqs. (2) and (3) into Eq. (1) and solving for f, we obtain

$$f = \frac{H(W_a + W_p)}{M\theta_p} = 25.46, \tag{4}$$

where we have used the following numerical data: $H = 10\,\text{ft}$, $W_a = 200\,\text{lb}$, $W_p = 800\,\text{lb}$, $M = 900\,\text{ft·lb}$, and $\theta_p = 25°$. Hence, the minimum number of pumping cycles needed to achieve $\theta > 90°$ is the smallest integer that is larger then f, i.e.,

$$\boxed{n_{\text{min}} = 26.} \tag{5}$$

Speed of the person at $\theta = 0$ after 26 pumping cycles. Let ③ be the position with $\theta = 0°$ that is reached after 26 pumping cycles. Since the platform can only translate, and the arms AB and CD are in fixed axis rotations about A and D, respectively, we have

$$T_3 = \tfrac{1}{2} m_p v_{p3}^2 + \tfrac{1}{2} I_A \omega_{AB3}^2 + \tfrac{1}{2} I_D \omega_{CD3}^2, \tag{6}$$

where

$$I_A = I_D = \tfrac{1}{3} m_a H^2. \tag{7}$$

Then, applying the work-energy principle between ① and ③ and observing that $V_1 = V_3$ since $\theta_1 = \theta_3$, we have

$$M \theta_p n_{\min} = \tfrac{1}{2} m_p v_{p3}^2 + \tfrac{1}{3} m_a H^2 \left(\frac{v_{p3}}{H} \right)^2, \tag{8}$$

where we have used the fact that $\omega_{AB3} = \omega_{CD3} = v_{p3}/H$. Solving for v_{p3} we obtain

$$v_{p3} = \sqrt{\frac{6 M \theta_p n_{\min}}{3 m_p + 2 m_a}} = \sqrt{\frac{6 g M \theta_p n_{\min}}{3 W_p + 2 W_a}}, \tag{9}$$

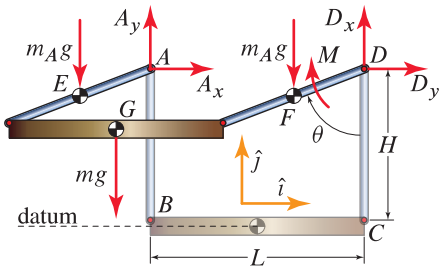
where g is the acceleration due to gravity. The above expression is also the speed person at ③ since the person translates with the platform. Hence, evaluating the above expression, we have,

$$\boxed{v_{\text{person}} = 26.5 \, \text{ft/s},}$$

where we have used the following numerical data: $g = 32.2 \, \text{ft/s}^2$, $H = 10 \, \text{ft}$, $W_a = 200 \, \text{lb}$, $W_p = 800 \, \text{lb}$, $M = 900 \, \text{ft·lb}$, and $\theta_p = 25°$.

Solution to 8.27

Referring to the FBD shown, work is done only by gravity and the moment due to the motor. Let ① be the initial position of the system and ② be the position with $\theta = \theta_{\max} = 90°$ that is achieved after 6 pumping cycles. Applying the work-energy principle, we have



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2, \tag{10}$$

Recalling that the system starts from rest and that the system is again at rest in ② because $\theta = \theta_{\max}$, we have

$$T_1 = 0, \quad V_1 = 2W_a \frac{H}{2}, \quad T_2 = 0 \quad V_2 = 2W_a H + W_p H, \quad (U_{1\text{-}2})_{\text{nc}} = 6M\theta_p. \tag{11}$$

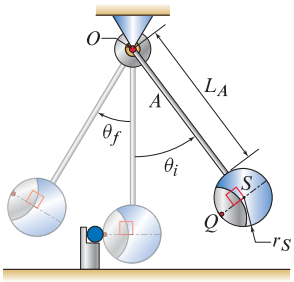
Substituting Eqs. (11) into Eq. (10) and solving for M, we obtain

$$\boxed{M = H\frac{W_p + W_a}{6\theta_p} = 4770\,\text{ft·lb},} \tag{12}$$

where we have used the following numerical data: $H = 10\,\text{ft}$, $W_a = 200\,\text{lb}$, $W_p = 800\,\text{lb}$, and $\theta_p = 20°$.

Problems 8.28 and 8.29

The *Charpy impact test* is one test that measures the resistance of a material to fracture. In this test, the fracture toughness is assessed by measuring the energy required to break a specimen of a given geometry. This is done by releasing a heavy pendulum from rest at an angle θ_i and then measuring the maximum swing angle θ_f reached by the pendulum after the specimen is broken.

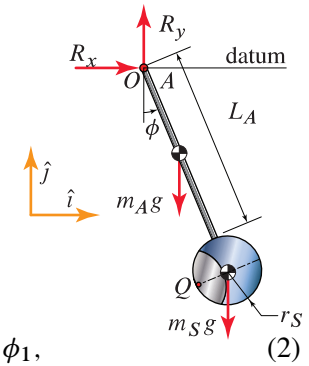


Problem 8.28 Consider a test rig in which the striker S (the pendulum's bob) can be modeled as a uniform disk of mass $m_S = 19.5\,\text{kg}$ and radius $r_S = 150\,\text{mm}$, and the arm can be modeled as a thin rod of mass $m_A = 2.5\,\text{kg}$ and length $L_A = 0.8\,\text{m}$. Neglecting friction and noting that the striker and the arm are rigidly connected, determine the fracture energy (i.e., the kinetic energy lost in breaking the specimen) in an experiment where $\theta_i = 158°$ and $\theta_f = 43°$.

Problem 8.29 Consider a test rig in which the striker S (the pendulum's bob) can be modeled as a uniform disk of weight $W_S = 40\,\text{lb}$ and radius $r_S = 6\,\text{in.}$, and the arm can be modeled as a thin rod of weight $W_A = 5.5\,\text{lb}$ and length $L_A = 2.75\,\text{ft}$. If the release angle of the striker is $\theta_i = 158°$ and if the striker impacts the specimen when the pendulum's arm is vertical, determine the speed of the point Q on the striker immediately before the striker impacts with the specimen. Neglect friction and observe that the striker and the arm are rigidly connected.

Solution to 8.28

When not in contact with the fracture specimen, the pendulum is acted upon by the forces in the FBD shown. Let ① be when $\phi = \phi_1 = \theta_i$; and ② be when $\phi = \phi_2 = -\theta_f$. In going from ① to ② the system is acted upon by gravity and by the contact forces between it and the specimen. Applying the work-energy principle, we have



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2, \tag{1}$$

Due to our definitions of ① and ②, and due to our choice of datum, we have

$$T_1 = 0, \qquad V_1 = -m_A g \frac{L_A}{2} \cos\phi_1 - m_S g (L_A + r_S) \cos\phi_1, \tag{2}$$

$$T_2 = 0 \qquad V_2 = -m_A g \frac{L_A}{2} \cos\phi_2 - m_S g (L_A + r_S) \cos\phi_2. \tag{3}$$

The work due to the nonconservative forces is all due to fracture, i.e.,

$$(U_{1\text{-}2})_{\text{nc}} = U_{\text{fracture}}. \tag{4}$$

Substituting Eqs. (2)–(4) into Eq. (1) and solving for U_{fracture}, we obtain

$$\boxed{U_{\text{fracture}} = -\tfrac{1}{2} g [L_A (m_A + 2m_S) + 2m_S r_S](\cos\theta_f - \cos\theta_i) = -318\,\text{J},} \tag{5}$$

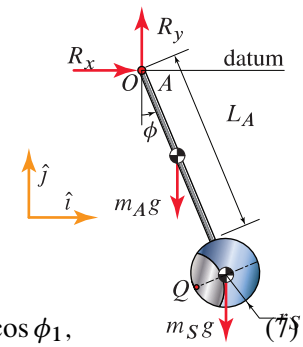
where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$, $L_A = 0.8\,\text{m}$, $m_A = 2.5\,\text{kg}$, $m_S = 19.5\,\text{kg}$, $r_S = 150\,\text{mm} = 150\times10^{-3}\,\text{m}$, $\theta_f = 43°$, and $\theta_i = 158°$.

Solution to 8.29

Let ① be when $\phi = \phi_1 = \theta_i$; and ② be when $\phi = \phi_2 = 0°$. In going from ① to ② the system is subject only to gravity. Applying conservation of eenergy, we have

$$T_1 + V_1 = T_2 + V_2, \tag{6}$$

Observe that the system is in a fixed axis rotation about O. Hence, due to our definitions of ① and ②, and due to our choice of datum, we have



$$T_1 = 0, \qquad V_1 = -m_A g \frac{L_A}{2} \cos\phi_1 - m_S g (L_A + r_S) \cos\phi_1, \tag{7}$$

$$T_2 = \tfrac{1}{2} I_O \omega_p^2 \qquad V_2 = -m_A g \frac{L_A}{2} \cos\phi_2 - m_S g (L_A + r_S) \cos\phi_2, \tag{8}$$

where ω_p is the angular speed of the system and where

$$I_O = \tfrac{1}{3}(W_A/g)L_A^2 + \tfrac{1}{2}(W_S/g)(r_S^2 + (W_S/g)(L_A + r_S)^2 = 13.71\,\text{slug}\cdot\text{ft}^2, \tag{9}$$

where we have used the following data: $g = 32.2\,\text{ft/s}^2$, $W_A = 5.5\,\text{lb}$, $W_S = 40\,\text{lb}$, $L_A = 2.75\,\text{ft}$, and $r_S = 6\,\text{in.} = 0.5000\,\text{ft}$.

Substituting Eqs. (7) and (8) into Eq. (1) and solving for ω_{p2}, we obtain

$$\omega_{p2} = \frac{1}{\sqrt{I_O}} \sqrt{[L_A W_A + 2(L_A + r_S)W_S](1 - \cos\theta_i)} \tag{10}$$

Since the system is in a fixed axis rotation about O, the speed of Q in ② will simply be given by $d_{OQ}\omega_{p2}$, where

$$d_{OQ} = \sqrt{(L_A + r_S)^2 + r_S^2} \tag{11}$$

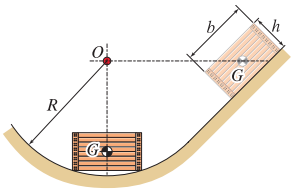
Hence, the solution of the problem is

$$\boxed{v_Q = \frac{\sqrt{(L_A + r_S)^2 + r_S^2}}{\sqrt{I_O}} \sqrt{[L_A W_A + 2(L_A + r_S)W_S](1 - \cos\theta_i)} = 20.4\,\text{ft/s},}$$

where we have used the following numerical data: $L_A = 2.75\,\text{ft}$, $r_S = 6\,\text{in.} = 0.5000\,\text{ft}$, $I_O = 13.71\,\text{slug}\cdot\text{ft}^2$ (see Eq. (9)), $W_A = 5.5\,\text{lb}$, $W_S = 40\,\text{lb}$, and $\theta_i = 158°$.

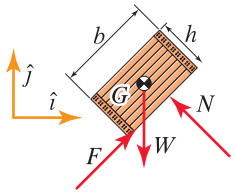
Problem 8.30

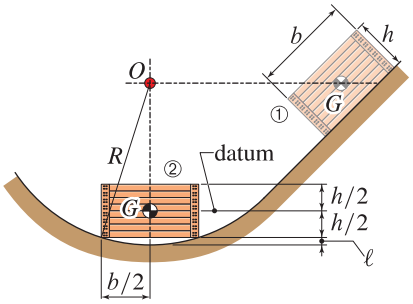
A crate, with weight $W = 155\,\text{lb}$ and mass center G, is placed on a slide and released from rest as shown. The lower part of the slide is circular with radius $R = 6\,\text{ft}$. Model the crate as a uniform body with $b = 3.6\,\text{ft}$ and $h = 2\,\text{ft}$, take into account the gap between the crate and the slide when the crate is in its lowest position, and assume that when the crate is in its lowest position on the slide, the crate's center of mass is moving to the left with a speed $v_G = 12\,\text{ft/s}$. Determine the work done by friction on the crate as the crate moves from the release point to the lowest point on the slide.



Solution

The crate is acted upon by the forces in the FBD shown to the right. Only gravity and friction do work since the point of application of the normal force does not move in the direction of that force. Referring to the figure given below, let ① be the position at release and ② be the lowest position achieved by the crate.





Applying the work-energy principle, we have

$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2, \tag{1}$$

where, due to our choice of datum and the definition of ① and ②, we have

$$T_1 = 0, \qquad V_1 = W(R - \ell - \tfrac{1}{2}h), \tag{2}$$

$$T_2 = \tfrac{1}{2}(W/g)v_{G2}^2 + \tfrac{1}{2}I_G\omega_{c2}^2, \qquad V_2 = 0. \tag{3}$$

where the mass moment of inertia about G is given by

$$I_G = \tfrac{1}{12}(W/g)(h^2 + b^2) = 6.803\,\text{slug}\cdot\text{ft}^2, \tag{4}$$

where we have used the following numerical data: $W = 155\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $h = 2\,\text{ft}$, and $b = 3.6\,\text{ft}$. Next, using trigonometry, we have

$$\ell = R - \sqrt{R^2 - (b/2)^2} = 0.2764\,\text{ft}, \tag{5}$$

where we have used the following numerical data: $R = 6\,\text{ft}$ and $b = 3.6\,\text{ft}$. Finally, notice that at ② the crate is in a fixed axis rotation about O. Hence, we have

$$\omega_{c2} = \frac{v_{G2}}{R - \ell - h/2} = 2.540\,\text{rad/s}, \tag{6}$$

where we have used the following numerical data: $R = 6\,\text{ft}$, $\ell = 0.2764\,\text{ft}$ (see Eq. (5)), and $v_{G2} = v_G = 12\,\text{ft/s}$.

Observing that the work of friction coincides with the term $(U_{1\text{-}2})_{\text{nc}}$, substituting Eqs. (2) and (3) in Eq. (1), and solving for $(U_{1\text{-}2})_{\text{nc}}$, we have

$$U_{\text{friction}} = (U_{1\text{-}2})_{\text{nc}} = \tfrac{1}{2}(W/g)v_{G2}^2 + \tfrac{1}{2}I_G\omega_{c2}^2 - W(R - \ell - \tfrac{1}{2}h), \tag{7}$$

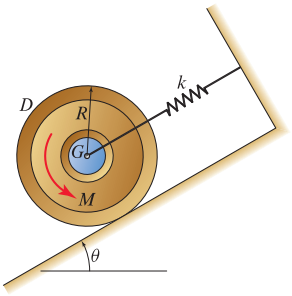
which can be evaluated to obtain

$$\boxed{U_{\text{friction}} = -364\,\text{ft}\cdot\text{lb},}$$

where we have used the following numerical data: $W = 155\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $v_{G2} = v_G = 12\,\text{ft/s}$, $\omega_{c2} = 2.540\,\text{rad/s}$ (see Eq. (6)), $R = 6\,\text{ft}$, $\ell = 0.2764\,\text{ft}$ (see Eq. (5)), and $h = 2\,\text{ft}$.

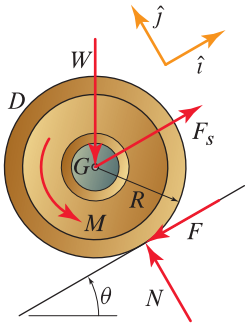
Problem 8.31

The disk D, which has weight W, mass center G coinciding with the disk's geometric center, and radius of gyration k_G, is at rest on an incline when the constant moment M is applied to it. The disk is attached at its center to a wall by a linear elastic spring of constant k. The spring is unstretched when the system is at rest. Assuming that the disk rolls without slipping and that it has not yet come to a stop, determine the angular velocity of the disk after its center G has moved a distance d down the incline. After doing so, using $k = 5\,\text{lb/ft}$, $R = 1.5\,\text{ft}$, $W = 10\,\text{lb}$, and $\theta = 30°$, determine the value of the moment M for the disk to stop after rolling $d_s = 5\,\text{ft}$ down the incline.



Solution

Given the FBD shown and recalling that the disk rolls without slip, work is done only by gravity, the spring force, and the moment M. Let ① be the position at release and ② be the position achieved by the disk after having rolled a distance d down the incline. Applying the work-energy principle between ① and ②, we have



$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2, \tag{1}$$

where, choosing the datum line to go through the mass center of the disk in ①, we have

$$T_1 = 0, \quad V_1 = 0, \quad T_2 = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_d^2, \quad V_2 = -Wd\sin\theta + \tfrac{1}{2}kd^2, \quad (U_{1\text{-}2})_{\text{nc}} = M\theta_D, \tag{2}$$

where v_G is the speed of G at ②, ω_d is the angular speed of the disk at ②, and θ_D is the angular displacement of the disk in the direction of M due to the rolling motion. Because of rolling without slip, we must have

$$v_G = R\omega_d \qquad \text{and} \qquad \theta_D = d/R. \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1) and solving for ω_d, we obtain

$$\boxed{\omega_d = \sqrt{\frac{g}{W(R^2 + k_G^2)}}\sqrt{d[(2M/R) + 2W\sin\theta - dk]}} \tag{4}$$

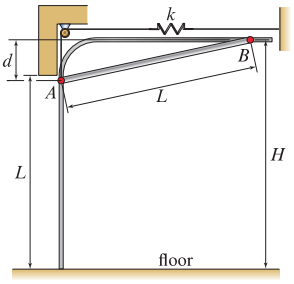
To determine the desired value of the moment M, we substitute d_s in place of d in Eq. (4), then we set ω_s equal to zero, and solve for M to obtain

$$\boxed{M = \tfrac{1}{2}R(d_s k - 2W\sin\theta) = 11.2\,\text{ft·lb},}$$

where we have used the following numerical data: $R = 1.5\,\text{ft}$, $d_s = 5\,\text{ft}$, $k = 5\,\text{lb/ft}$, $W = 10\,\text{lb}$, and $\theta = 30°$.

Problem 8.32

The figure shows the cross section of a garage door with length $L = 9\,\text{ft}$ and weight $W = 175\,\text{lb}$. At A and B there are rollers of negligible mass constrained to move in the guide whose horizontal portion is at a distance $H = 11\,\text{ft}$ from the floor. The door's motion is assisted by two springs, each with constant k (only one spring is shown). The door is released from rest when $d = 26\,\text{in.}$ and the spring is stretched 4 in. Neglecting friction, knowing that, when A touches the floor, B is in the vertical portion of the guide, and modeling the door as a uniform thin plate, determine the minimum value of k so that A will strike the ground with a speed no greater than $1\,\text{ft/s}$.



Solution

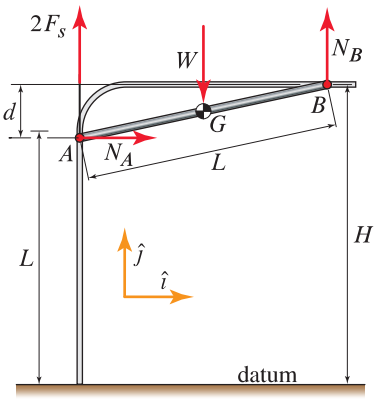
Referring to the FBD shown, work is done only by gravity and the spring force. Let ① be the position at release and ② be the position achieved by the door when A strikes the floor. Since energy is conserved in going from ① to ②, we have

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, recalling that the door is released from rest, given the choice of datum, and recalling that there are two springs, we have

$$T_1 = 0, \qquad V_1 = 2\tfrac{1}{2}k\delta_0^2 + W(H - \tfrac{1}{2}d), \tag{2}$$

$$T_2 = \tfrac{1}{2}mv_{G2}^2 + \tfrac{1}{2}I_G\omega_{d2}^2, \qquad V_2 = 2\tfrac{1}{2}k(\delta_0 + H - d)^2 + \tfrac{1}{2}LW. \tag{3}$$



In ② the door is undergoing a translation. Hence, we must have

$$v_{G2} = v_{A2} = v_{\max} \quad \text{and} \quad \omega_{d2} = 0, \tag{4}$$

where $v_{\max}$ is the maximum speed with which A strikes the floor.

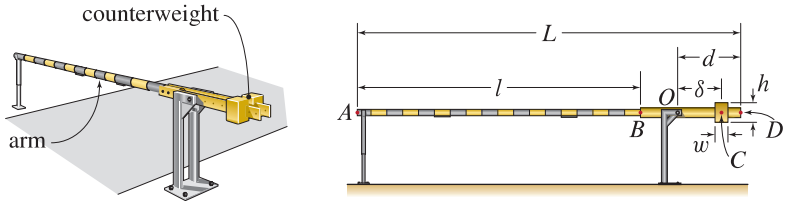
Substituting Eqs. (2)–(4) into Eq. (1), recalling that $m = W/g$, and solving for k, we have

$$k = -\frac{W}{2g}\frac{v_{\max}^2 + g(L + d - 2H)}{(H - d)(H + 2\delta_0 - d)} = 11.3\,\text{lb/ft},$$

where we have used the following numerical data: $W = 175\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $v_{\max} = 1\,\text{ft/s}$, $L = 9\,\text{ft}$, $d = 26\,\text{in.} = 2.167\,\text{ft}$, $H = 11\,\text{ft}$, and $\delta_0 = 4\,\text{in.} = 0.3333\,\text{ft}$.

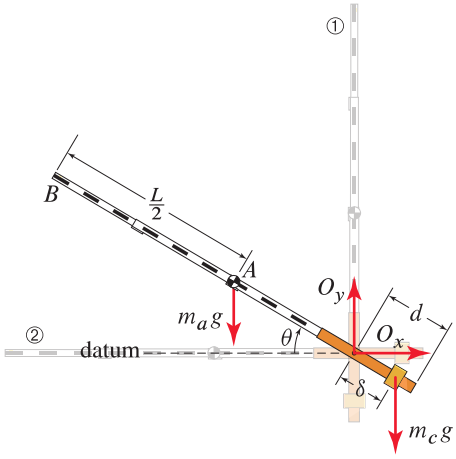
Problem 8.33

In Example 8.2 on p. 602 we ignored the rotational inertia of the counterweight. Let's revisit that example and remove that simplifying assumption. Assume that the arm AD is still a uniform thin bar of length $L = 15.7$ ft and weight 45 lb. The hinge O is still $d = 2.58$ ft from the right end of arm, and the 160 lb counterweight C is still $\delta = 1.4$ ft from the hinge. Now model the counterweight as a uniform block of height $h = 14$ in. and width $w = 9$ in. With this new assumption, solve for the angular velocity of the arm as it reaches the horizontal position after being nudged from the vertical position. Determine the percent change in angular velocity compared with that found in Example 8.2.



Solution

Given the FBD shown, work is done only by gravity. Let ① be the position at release when the gate is vertical, and ② be when the gate is horizontal. Energy is conserved between ① and ②, so that we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, given our choice of datum line and given that the system is in a fixed axis rotation about O, we have

$$T_1 = 0, \qquad V_1 = W_a(\tfrac{1}{2}L - d) - W_c\delta, \tag{2}$$

$$T_2 = \tfrac{1}{2} I_O \omega_{a2}^2 \qquad V_2 = 0, \tag{3}$$

where ω_a is the angular speed of the arm and I_O is the mass moment of inertia of the entire system about O and is therefore given by

$$I_O = \tfrac{1}{12}(W_a/g)L^2 + (W_a/g)(\tfrac{1}{2}L - d)^2 + \tfrac{1}{12}(W_c/g)(w^2 + h^2) + (W_c/g)\delta^2 = 78.05\ \text{slug}\cdot\text{ft}^2, \tag{4}$$

where we have used the following numerical data: $g = 32.2\ \text{ft/s}^2$, $W_a = 45$ lb, $L = 15.7$ ft, $d = 2.58$ ft, $W_c = 160$ lb, $w = 9$ in. $= 0.7500$ ft, $h = 14$ in. $= 1.167$ ft, and $\delta = 1.4$ ft.

Substituting Eqs. (2) and (3) into Eq. (1) and solving for ω_{a2} we have

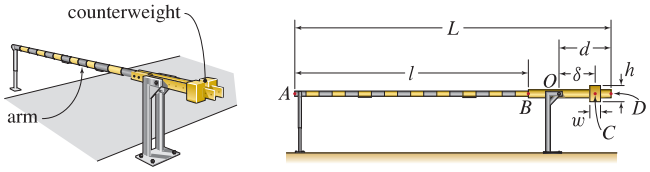
$$\omega_{a2} = \frac{1}{\sqrt{I_O}}\sqrt{LW_a - 2dW_a - 2W_c\delta} = 0.580\ \text{rad/s},$$

where we have used the result in Eq. (4) along with the following numerical data: $L = 15.7$ ft, $W_a = 45$ lb, $d = 2.58$ ft, $W_c = 160$ lb, and $\delta = 1.4$ ft. Hence, the percentage difference in angular speed relative to the result obtained in Example 8.2 is

$$\text{Percentage difference} = \frac{0.580 - 0.583}{0.583}100\% = -0.515\%.$$

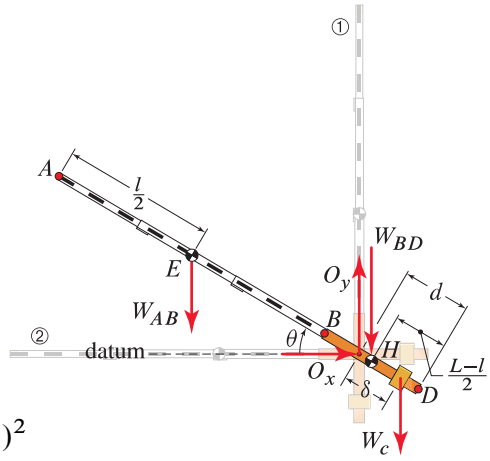
Problem 8.34

For the barrier gate shown, assume that the arm consists of a section of aluminum tubing from A to B of length $l = 11.6\,\text{ft}$ and weight $20\,\text{lb}$ and a steel support section from B to D of weight $40\,\text{lb}$. The overall length of the arm is $L = 15.7\,\text{ft}$. In addition, the $120\,\text{lb}$ counterweight C is placed a distance δ from the hinge at O, and the hinge is $d = 2.58\,\text{ft}$ from the right end of section BD. Model the two sections AB and BD as uniform thin bars, and model the counterweight as a uniform block of height $h = 14\,\text{in.}$ and width $w = 9\,\text{in.}$ Using these new assumptions, determine the distance δ so that the angular velocity of the arm is $0.25\,\text{rad/s}$ as it reaches the horizontal position after being nudged from the vertical position.



Solution

Given the FBD shown, work is done only by gravity. Let ① be when the gate is vertical, and ② be when the gate is horizontal. Energy is conserved between ① and ②, so that we have



$$T_1 + V_1 = T_2 + V_2 \tag{1}$$

Since the system is in a fixed axis rotation about O, we have

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2} I_O \omega_{a2}^2 \tag{2}$$

where ω_a is the angular speed of the arm and I_O is the mass moment of inertia of the entire system about O and is given by

$$I_O = \tfrac{1}{12}\frac{W_{AB}}{g} l^2 + \frac{W_{AB}}{g}(\tfrac{1}{2}l + L - l - d)^2 + \tfrac{1}{12}\frac{W_{BD}}{g}(L-l)^2 + \frac{W_{BD}}{g}[d - \tfrac{1}{2}(L-l)]^2 + \tfrac{1}{12}\frac{W_c}{g}(w^2 + h^2) + \frac{W_c}{g}\delta^2 = 42.93\,\text{slug}\cdot\text{ft}^2 - (3.727\,\text{slug})\delta^2, \tag{3}$$

where we have used the following numerical data: $W_{AB} = 20\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $l = 11.6\,\text{ft}$, $L = 15/7\,\text{ft}$, $d = 2.58\,\text{ft}$, $W_{BD} = 40\,\text{lb}$, $W_c = 120\,\text{lb}$, $w = 9\,\text{in.} = 0.7500\,\text{ft}$, and $h = 14\,\text{in.} = 1.167\,\text{ft}$. Given the choice of datum, the potential energies V_1 and V_2 are given by

$$V_1 = W_{AB}(\tfrac{1}{2}l + L - l - d) - W_{BD}[d - (L-l)/2] - W_c\delta = 125.2\,\text{ft}\cdot\text{lb} - (120.0\,\text{lb})\delta, \tag{4}$$

$$V_2 = 0, \tag{5}$$

where we have used the following numerical data: $W_{AB} = 20\,\text{lb}$, $l = 11.6\,\text{ft}$, $L = 15/7\,\text{ft}$, $d = 2.58\,\text{ft}$, $W_{BD} = 40\,\text{lb}$, and $W_c = 120\,\text{lb}$. Recalling that $\omega_{a2} = 0.25\,\text{rad/s}$, substituting Eqs. (2)–(5) into Eq. (1) yields:

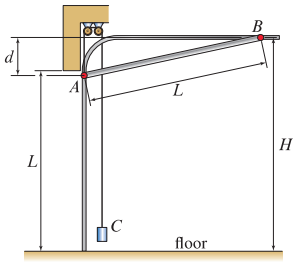
$$125.2\,\text{ft}\cdot\text{lb} - (120.0\,\text{lb})\delta = 1.342\,\text{ft}\cdot\text{lb} + (0.1165\,\text{lb/ft})\delta^2. \tag{6}$$

Equation (6) is a second order algebraic equation for δ with the two roots $\delta = -1031\,\text{ft}$ and $\delta = 1.031\,\text{ft}$. Since admissible values of δ must be positive, the solution to the problem is therefore

$$\boxed{\delta = 1.03\,\text{ft}.}$$

Problem 8.35

The figure shows the cross section of a garage door with length $L = 2.5\,\text{m}$ and mass $m = 90\,\text{kg}$. At the ends A and B there are rollers of negligible mass constrained to move in the guide whose horizontal portion is at a distance $H = 3\,\text{m}$ from the floor. The door's motion is assisted by two counterweights C, each of mass m_C (only one counterweight is shown). If the door is released from rest when $d = 53\,\text{cm}$, neglecting friction and modeling the door as a uniform thin plate, determine the minimum value of m_C so that A will strike the ground with a speed no greater than $0.25\,\text{m/s}$.

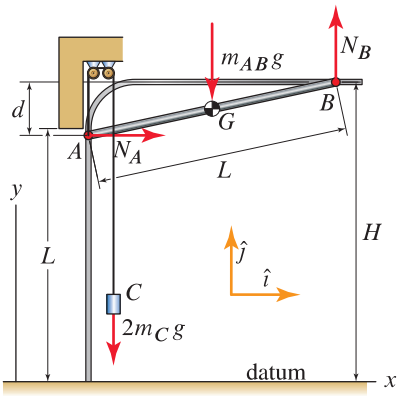


Solution

Referring to the FBD shown, work is done only by gravity. Let ① be the position at release and ② be the position achieved by the door when A strikes the floor. Since energy is conserved in going from ① to ②, we have

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, given the choice of datum, modeling the door as a rigid body and the counterweights as particles, and recalling that the door is released from rest, we have



$$T_1 = 0, \tag{2}$$

$$V_1 = 2m_C g y_{C1} + m_{AB} g(H - \tfrac{1}{2}d), \tag{3}$$

$$T_2 = \tfrac{1}{2}m_{AB} v_{G2}^2 + \tfrac{1}{2}I_G \omega_{d2}^2 + 2\left(\tfrac{1}{2}m_C v_{C2}^2\right), \tag{4}$$

$$V_2 = 2m_C g y_{C2} + \tfrac{1}{2}L m_{AB} g, \tag{5}$$

where m_{AB} is the mass of the door, y_{C1} and y_{C2} are the vertical coordinates of the counterweights at ① and ②, respectively, I_G is the mass moment of inertia of the door with respect to the door's center of mass G, v_G is the speed of G, and ω_d is the angular speed of the door.

Now observe that in ② the door is sliding within the vertical portion of the guide. This implies that at ② the door is translating so that the door's angular speed must be equal to zero. For the same reason, the speed of A must be equal to the speed of the center of mass G. In addition, notice that, in going from ① to ②, the counterweights C are lifted by the same amount that point A descends. Hence, recalling that the counterweights C must move with the same speed as A, we have

$$v_{G2} = v_{A2} = v_{\max}, \quad v_{C2} = v_{A2} = v_{\max}, \quad \omega_{d2} = 0, \quad \text{and} \quad y_{C2} = y_{C1} + (H - d), \tag{6}$$

where $v_{\max}$ is the maximum speed with which A strikes the floor.

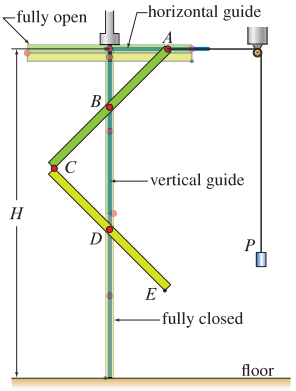
Substituting Eqs. (2)–(6) into Eq. (1), and solving for m_C, we have

$$m_C = \frac{m_{AB}}{2}\left[\frac{v_{\max}^2 + g(L + d - 2H)}{2dg - 2Hg - v_{\max}^2}\right] = 27.0\,\text{kg},$$

where we have used the following numerical data: $m_{AB} = 90\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $v_{\max} = 0.25\,\text{m/s}$, $L = 2.5\,\text{m}$, $d = 53\,\text{cm} = 0.5300\,\text{m}$, and $H = 3\,\text{m}$.

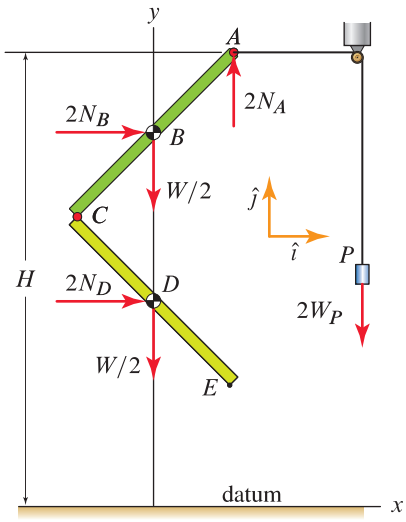
Problem 8.36

Revisit Example 8.5 on p. 608 and replace the two springs with a system of two counterweights P (only one counterweight is shown) each of weight W_P. Recalling that the door's weight is $W = 800\,\text{lb}$ and that the total height of the door is $H = 30\,\text{ft}$, if the door is released from rest in the fully open position and friction is negligible, determine the minimum value of W_P so that A will strike the left end of the horizontal guide with a speed no greater than $0.5\,\text{ft/s}$.



Solution

The FBD on the system is shown on the right. Point A can only move horizontally. Points B and D can only move vertically. Hence, the only forces that can do work are the weights of the system's components. As shown in the lower figure, let ① denote the position of the door when fully open, and ② the position of the door when fully closed. We then write the work-energy principle as follows:



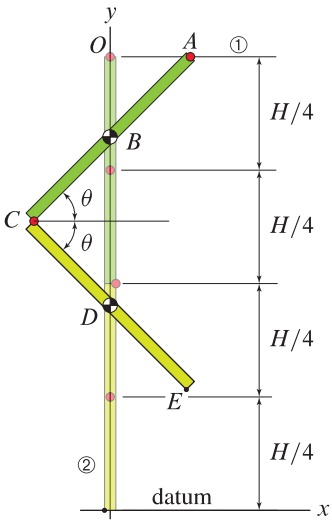
$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, T_1 and T_2 are the kinetic energies of the system at ① and ②, respectively, and V_1 and V_2 are the potential energies of the system at ① and ② respectively. Recalling that the system starts from rest, for the kinetic energies we have

$$T_1 = 0, \tag{2}$$

$$T_2 = \tfrac{1}{2}m_{AC}v_{B2}^2 + \tfrac{1}{2}I_B(\omega_{AC})_2^2 + \tfrac{1}{2}m_{CE}v_{D2}^2 + \tfrac{1}{2}I_D(\omega_{CE})_2^2 + 2\left(\tfrac{1}{2}m_P v_{P2}^2\right), \tag{3}$$

where, ω_{AC} and ω_{CE} are the angular speeds of AC and CE, respectively, v_B and v_D are the speeds of B and D, respectively, m_{AC} is the mass of AC, m_{CD} is the mass of CD, m_P is the mass of each of the counterweights, I_B is the mass moments of inertia of AC about B, and I_D is the mass moment of inertia of CE about D. Keeping in mind that W is the weight of the entire door and that the door consists of two identical parts, we have



$$m_{AC} = m_{CE} = \frac{W}{2g}, \quad m_P = \frac{W_P}{g}, \quad I_B = I_D = \frac{1}{12}\left(\frac{W}{2g}\right)\left(\frac{H}{2}\right)^2. \tag{4}$$

For the potential energies, we have

$$V_1 = W_{AC}y_{B1} + W_{CE}y_{D1} + 2W_P y_{P1}, \tag{5}$$

$$V_2 = W_{AC}y_{B2} + W_{CE}y_{D2} + 2W_P y_{P2}, \tag{6}$$

where

$$W_{AC} = \tfrac{1}{2}W, \quad \text{and} \quad W_{CE} = \tfrac{1}{2}W, \tag{7}$$

As discussed in Example 8.5 (on p. 608 of the textbook), the panels move in such a way that the angles formed by them with a horizontal line going through C remain always equal. This implies that

$$(\omega_{AC})_2 = (\omega_{CE})_2, \tag{8}$$

Furthermore, observe that at ② points B and D are the bottom of their respective trajectories. Hence, we must have

$$v_{B2} = v_{D2} = 0. \tag{9}$$

The above result implies that in ② point B is the instantaneous venter of rotation of AC. In turn, this implies that

$$(\omega_{AC})_2 \frac{H}{4} = v_{A2} \quad \Rightarrow \quad (\omega_{AC})_2 = \frac{4v_{A2}}{H}. \tag{10}$$

We now observe that the counterweights P move with the same speed as point A, i.e.,

$$v_{P2} = v_{A2}. \tag{11}$$

Finally we observe that

$$y_{B1} = H, \quad y_{D1} = H, \quad y_{B2} = \tfrac{3}{4}H, \quad y_{D2} = \tfrac{1}{4}H, \quad y_{P2} = y_{P1} + \tfrac{1}{4}H. \tag{12}$$

Substituting Eqs. (4), and Eqs. (8)–(11) into the expressions for the kinetic energies, after simplification, we have that Eqs. (2) and (3) can be rewritten as

$$T_1 = 0 \quad \text{and} \quad T_2 = \left(\frac{W}{6g} + \frac{W_P}{g}\right) v_{A2}^2. \tag{13}$$

Substituting Eqs. (7) and (12) into Eqs. (5) and (6), after simplification, we have

$$V_1 = WH + 2W_P y_{P1} \quad \text{and} \quad V_2 = \frac{WH}{2} + 2W_P \left(y_{P1} + \frac{H}{4}\right). \tag{14}$$

Now, substituting Eqs. (13) and (14) into Eq. (1), we have

$$WH + 2W_P y_{P1} = \left(\frac{W}{6g} + \frac{W_P}{g}\right) v_{A2}^2 + \frac{WH}{2} + 2W_P \left(y_{P1} + \frac{H}{4}\right). \tag{15}$$

Recalling that we are given the maximum admissible value of v_{A2}, setting v_{A2} equal to such a value, allows us to treat Eq. (15) as an equation in the single unknown W_P whose solution is

$$\boxed{W_P = \frac{3gH - v_{A2}^2}{3gH + 6v_{A2}^2} W = 800\,\text{lb},}$$

where we have used the following numerical data: $g = 32.2\,\text{ft/s}^2$, $H = 30\,\text{ft}$, $v_{A2} = 0.5\,\text{ft/s}$, and $W = 800\,\text{lb}$.

Problem 8.37

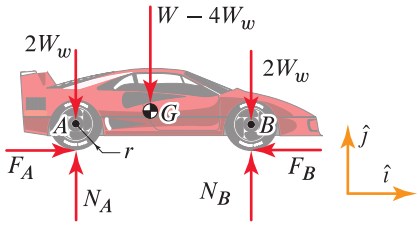
Torsional springs provide a simple propulsion mechanism for toy cars. When the rear wheels are rotated as if the car were moving backward, they cause a torsional spring (with one end attached to the axle and the other to the body of the car) to wind up and store energy. Therefore, a simple way to charge the spring is to place the car onto a surface and to pull it backward, making sure that the wheels roll without slipping. Note that the torsional spring can only be wound by pulling the car backward; that is, *the forward motion of the car unwinds the spring*.

Let the weight of the car (body and wheels) be $W = 5\,\text{oz}$, the weight of each of the wheels be $W_w = 0.15\,\text{oz}$, and the radius of the wheels be $r = 0.25\,\text{in.}$, where the wheels roll without slip and can be treated as uniform disks. Neglecting friction internal to the car and letting the car's torsional spring be linear with constant $k_t = 0.0002\,\text{ft·lb/rad}$, determine the maximum speed achieved by the car if it is released from rest after pulling it back a distance $L = 0.75\,\text{ft}$ from a position in which the spring is unwound.



Solution

Referring to the FBD shown, having assumed that the wheels roll without slip, we can conclude that no external force does work on the car due to the fact the car as a whole moves in a direction perpendicular to gravity. However, the internal torque provided by the (internal) torsional spring does work. This work term can be accounted for via the potential energy of the system. We define ① to be the position of the car at release (from rest) with the torsional spring wound as described in the problem statemen. Position 2 is the position corresponding to when the spring first becomes unwound. Hence, we can write



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, recalling that the system is released from rest and that the translational component of the velocity of the wheels is the same as the velocity of the car as a whole, we have

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}\frac{W}{g}v_{G2}^2 + 2\left(\tfrac{1}{2}I_A\omega_{A2}^2\right) + 2\left(\tfrac{1}{2}I_B\omega_{B2}^2\right), \tag{2}$$

v_G is the speed of the center of mass of the car and therefore of the car as a whole, ω_A is the angular speed of the rear wheels, ω_B is the angular speed of the front wheels, and where I_A and I_B are the mass moments of inertia of the rear and front wheel, respectively, with respect to their own center of mass and are given by

$$I_A = \tfrac{1}{2}\frac{W}{g}r^2 \quad \text{and} \quad I_B = \tfrac{1}{2}\frac{W}{g}r^2, \tag{3}$$

where we have accounted for the fact that all wheels are identical to one another.

Letting θ_1 describe the initial wounding angle, i.e., the angle by which the wheels rotate when the car is initially pulled backwards by the distance L, and recalling that the car's torsional spring is completely unwound at position two, we have

$$V_1 = \tfrac{1}{2}k_t\theta_1^2 \quad \text{and} \quad V_2 = 0. \tag{4}$$

We now deal with the kinematic equations of the problems. Since θ_1 describes the initial wounding angle, i.e., the angle by which the wheels rotate when the car is initially pulled backwards by the distance L, and since the wheels are identical to one another and roll without slip, then we must have

$$\theta_1 = \frac{L}{r}, \quad \omega_{A2} = \frac{v_{G2}}{r}, \quad \text{and} \quad \omega_{B2} = \frac{v_{G2}}{r}. \tag{5}$$

Substituting Eqs. (3) and the last two of Eqs (5) into Eqs. (2), we have that the kinetic energies can be rewritten as

$$T_1 = 0 \quad \text{and} \quad T_2 = \frac{W + 2W_w}{2g} v_{G2}^2. \tag{6}$$

Then substituting Eqs. (4) and (6) into Eq. (1), and accounting for the first of Eqs. (5) we obtain

$$\tfrac{1}{2} k_t \left(\frac{L}{r}\right)^2 = \frac{W + 2W_w}{2g} v_{G2}^2 \quad \Rightarrow \quad v_{G2} = \sqrt{\frac{k_t g L^2}{r^2 (W + 2W_w)}}. \tag{7}$$

Recalling that v_{G2} is the maximum speed of the car as a whole, evaluating the above expression we then have

$$\boxed{v_{\max} = 5.02\,\text{ft/s},}$$

where we have used the following numerical data: $k_t = 0.0002\,\text{ft·lb/rad}$, $g = 32.2\,\text{ft/s}^2$, $L = 0.75\,\text{ft}$, $r = 0.25\,\text{in.} = 0.2083\,\text{ft}$, $W = 5\,\text{oz} = 0.3125\,\text{lb}$, and $W_w = 0.15\,\text{oz} = 0.009375\,\text{lb}$.

Problem 8.38

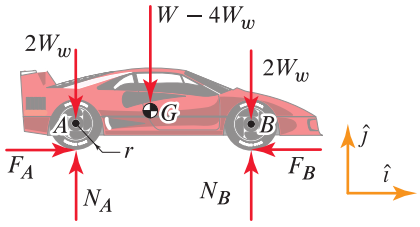
Torsional springs provide a simple propulsion mechanism for toy cars. When the rear wheels are rotated as if the car were moving backward, they cause a torsional spring (with one end attached to the axle and the other to the body of the car) to wind up and store energy. Therefore, a simple way to charge the spring is to place the car onto a surface and to pull it backward, making sure that the wheels roll without slipping. Note that the torsional spring can only be wound by pulling the car backward; that is, *the forward motion of the car unwinds the spring*.

Let the weight of the car (body and wheels) be $W = 5\,\text{oz}$, the weight of each of the wheels be $W_w = 0.15\,\text{oz}$, and the radius of the wheels be $r = 0.25\,\text{in.}$, where the wheels roll without slip and can be treated as uniform disks. In addition, let the torque M provided by the nonlinear torsional spring be given by $M = -\beta\theta^3$, where $\beta = 0.5\times10^{-6}\,\text{ft}\cdot\text{lb/rad}^3$, θ is the angular displacement of the rear axle, and the minus sign in front of β indicates that M acts opposite to the direction of θ. Neglecting any friction internal to the car, determine the maximum speed achieved by the car if it is released from rest after pulling it back a distance $L = 0.75\,\text{ft}$ from a position in which the spring is unwound.



Solution

Referring to the FBD shown, having assumed that the wheels roll without slip, we can conclude that no external force does work on the car due to the fact the car as a whole moves in a direction perpendicular to gravity. However, the internal torque provided by the (internal) torsional spring does work. This work term can be accounted for via the potential energy of the system. We define ① to be the position of the car at release (from rest) with the torsional spring wound as described in the problem statemen. Position 2 is the position corresponding to when the spring first becomes unwound. Hence, we can write



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, recalling that the system is released from rest and that the translational component of the velocity of the wheels is the same as the velocity of the car as a whole, we have

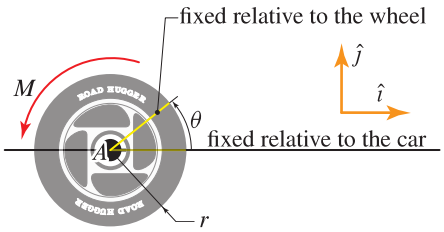
$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}\frac{W}{g}v_{G2}^2 + 2\left(\tfrac{1}{2}I_A\omega_{A2}^2\right) + 2\left(\tfrac{1}{2}I_B\omega_{B2}^2\right), \tag{2}$$

v_G is the speed of the center of mass of the car and therefore of the car as a whole, ω_A is the angular speed of the rear wheels, ω_B is the angular speed of the front wheels, and where I_A and I_B are the mass moments of inertia of the rear and front wheel, respectively, with respect to their own center of mass and are given by

$$I_A = \tfrac{1}{2}\frac{W}{g}r^2 \quad \text{and} \quad I_B = \tfrac{1}{2}\frac{W}{g}r^2, \tag{3}$$

where we have accounted for the fact that all wheels are identical to one another.

To evaluate the potential energy of the spring, we need to intergrate the expression of the moment with respect to θ. To do so, we refer to the figure on the right, and we recall that the potential energy of a force or a moment is equal to the negative of the work done by the force or moment in question. Hence, we have



$$V = -(\text{work of } M) = -\int_0^\theta M\, d\theta = -\left(\int_0^\theta -\beta\theta^3\, d\theta\right) = \tfrac{1}{4}\beta\theta^4.$$

Hence, using the above result, letting θ_1 describe the initial wounding angle, i.e., the angle by which the wheels rotate when the car is initially pulled backwards by the distance L, and recalling that the car's torsional spring is completely unwound at position two, we have

$$V_1 = \tfrac{1}{4}\beta\theta_1^4 \quad \text{and} \quad V_2 = 0. \tag{4}$$

We now deal with the kinematic equations of the problems. Since θ_1 describes the initial wounding angle, i.e., the angle by which the wheels rotate when the car is initially pulled backwards by the distance L, and since the wheels are identical to one another and roll without slip, then we must have

$$\theta_1 = \frac{L}{r}, \quad \omega_{A2} = \frac{v_{G2}}{r}, \quad \text{and} \quad \omega_{B2} = \frac{v_{G2}}{r}. \tag{5}$$

Substituting Eqs. (3) and the last two of Eqs (5) into Eqs. (2), we have that the kinetic energies can be rewritten as

$$T_1 = 0 \quad \text{and} \quad T_2 = \frac{W + 2W_w}{2g} v_{G2}^2. \tag{6}$$

Then substituting Eqs. (4) and (6) into Eq. (1), and accounting for the first of Eqs. (5) we obtain

$$\tfrac{1}{4}\beta\left(\frac{L}{r}\right)^4 = \frac{W + 2W_w}{2g} v_{G2}^2 \quad \Rightarrow \quad v_{G2} = \sqrt{\frac{\beta g L^4}{2r^4(W + 2W_w)}}. \tag{7}$$

Recalling that v_{G2} is the maximum speed of the car as a whole, evaluating the above expression we then have

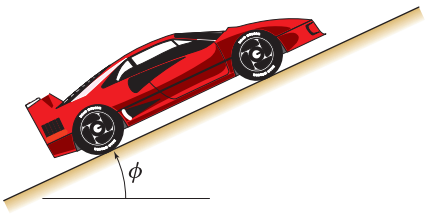
$$\boxed{v_{\max} = 6.39\,\text{ft/s},}$$

where we have used the following numerical data: $\beta = 0.5\times10^{-6}\,\text{ft·lb/rad}^3$, $g = 32.2\,\text{ft/s}^2$, $L = 0.75\,\text{ft}$, $r = 0.25\,\text{in.} = 0.2083\,\text{ft}$, $W = 5\,\text{oz} = 0.3125\,\text{lb}$, and $W_w = 0.15\,\text{oz} = 0.009375\,\text{lb}$.

Problem 8.39

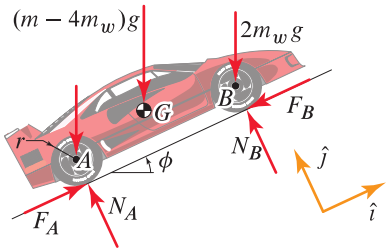
Torsional springs provide a simple propulsion mechanism for toy cars. When the rear wheels are rotated as if the car were moving backward, they cause a torsional spring (with one end attached to the axle and the other to the body of the car) to wind up and store energy. Therefore, a simple way to charge the spring is to place the car onto a surface and to pull it backward, making sure that the wheels roll without slipping. Note that the torsional spring can only be wound by pulling the car backward; that is, *the forward motion of the car unwinds the spring*.

Let the mass of the car (body and wheels) be $m = 120\,\text{g}$, the mass of each of the wheels be $m_w = 5\,\text{g}$, and the radius of the wheels be $r = 6\,\text{mm}$, where the wheels roll without slip and can be treated as uniform disks. In addition, let the car's torsional spring be linear with constant $k_t = 0.00025\,\text{N}\cdot\text{m/rad}$. Neglecting any friction internal to the car, if the angle of the incline is $\phi = 25°$ and the car is released from rest after pulling it back a distance $L = 25\,\text{cm}$ from a position in which the spring is unwound, determine the maximum distance $d_{\max}$ that the car will travel up the incline (from its release point), the maximum speed $v_{\max}$ achieved by the car, and the distance $d_{v\max}$ (from the release point) at which $v_{\max}$ is achieved.



Solution

Referring to the FBD shown, having assumed that the wheels roll without slip, we can conclude that the only external force that does work on the car is gravity. In addition, the internal torque provided by the (internal) torsional spring does work. This work term can be accounted for via the potential energy of the internal torsional spring. We define ① to be the position of the car at release (from rest) with the torsional spring wound as described in the problem statemen. Position 2 is the position corresponding corresponding to when the car having moved a distance d up the incline. Hence, we can write



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, recalling the translational component of the velocity of the wheels is the same as the velocity of the car as a whole, we have

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}mv_{G2}^2 + 2\left(\tfrac{1}{2}I_A\omega_{A2}^2\right) + 2\left(\tfrac{1}{2}I_B\omega_{B2}^2\right), \tag{2}$$

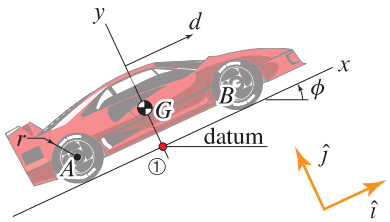
v_G is the speed of the center of mass of the car and therefore of the car as a whole, ω_A is the angular speed of the rear wheels, ω_B is the angular speed of the front wheels, and where I_A and I_B are the mass moments of inertia of the rear and front wheel, respectively, with respect to their own center of mass and are given by

$$I_A = \tfrac{1}{2}mr^2 \quad \text{and} \quad I_B = \tfrac{1}{2}mr^2, \tag{3}$$

where we have accounted for the fact that all wheels are identical to one another.

Before proceeding further, we observe that in determining the maximum distance achieved by the car, we will set $T_2 = 0$.

Referring to the figure on the right, the datum for gravity has been chosen to coincide with the mass center of the car in ①. Next let θ_1 describe the initial wounding angle, i.e., the angle by which the wheels rotate when the car is initially pulled backwards by the distance L. Then we have



$$V_1 = \tfrac{1}{2}k_t\theta_1^2 \quad \text{and} \quad V_2 = \begin{cases} \tfrac{1}{2}k_t\theta_2^2 + mgd\sin\phi & \text{for } d < L, \\ mgd\sin\phi & \text{for } d \geq L, \end{cases} \tag{4}$$

where for $d < L$ the spring has is not fully unwound whereas for $d > L$ no more potential energy in stored in the spring.

We now deal with the kinematic equations of the problems. Again we recall that θ_1 describes the initial wounding angle, i.e., the angle by which the wheels rotate when the car is initially pulled backwards by the distance L. Similarly, θ_2 is the angle corresponding to a wounding over a distance $L - d$. Hence, recalling that the wheels are identical to one another and roll without slip, then we must have

$$\theta_1 = \frac{L}{r}, \quad \theta_2 = \frac{L-d}{r}, \quad \omega_{A2} = \frac{v_{G2}}{r}, \quad \text{and} \quad \omega_{B2} = \frac{v_{G2}}{r}. \tag{5}$$

We are now ready to proceed to solve the problem. We start with the determination of the maximum distance traveled and then we proceed with the determination of the maximum speed achieved by the car.

Determination of maximum distance traveled. The maximum distance traveled $d_{\max}$ is achieved when the car comes to a stop for the first time and therefore $T_2 = 0$. We do not currently know if $d_{\max} \geq L$ or not. Hence, we derive a solution under the working assumption that indeed $d_{\max} \geq L$ and then we will verify whether or not our working assumption is satisfied. Hence, given that $T_1 = 0$ and $T_2 = 0$, substituting Eqs. (4) for the case with $d = d_{\max} \geq L$ into Eq. (1), and accounting for the first of Eqs. (5), we have

$$\tfrac{1}{2}k_t\left(\frac{L}{r}\right)^2 = mgd_{\max}\sin\phi. \tag{6}$$

which can be solved for $d_{\max}$ to obtain

$$\boxed{d_{\max} = \frac{k_t L^2}{2r^2 mg\sin\phi} = 0.436\,\text{m},}$$

where we have used the following numerical data: $k_t = 0.00025\,\text{N/m}$, $L = 25\,\text{cm} = 0.2500\,\text{m}$, $r = 6\,\text{mm} = 0.006000\,\text{m}$, $m = 120\,\text{g} = 0.1200\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $\phi = 25°$. Because the result we have obtained is consistent with the working assumption used to derive it, the result in question is indeed the answer to the first part of the problem.

Determination of the maximum speed. The maximum speed is necessarily achieved while the spring is actively propelling the car. Therefore, we proceed to determine the maximum speed using Eqs. (4) for the case with $d < L$. Clearly, this is a working assumption and we will need to verify that the result obtained under such an assumption is consistent with the assumption in question.

Substituting Eqs. (3) and the last two of Eqs. (5) in Eqs. (2) and simplifying, we have

$$T_1 = 0 \quad \text{and} \quad T_2 = \tfrac{1}{2}(m + 2m_w)v_{G2}^2. \tag{7}$$

Next, recalling that we are working under the assumption that $d < L$, substituting Eqs. (5) into Eqs. (4) and simplifying, we have

$$V_1 = \frac{k_t L^2}{2r^2} \quad \text{and} \quad V_2 = \frac{k_t(L-d)^2}{2r^2 + mgd\sin\phi}. \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (1), we have

$$\frac{k_t L^2}{2r^2} = \frac{k_t (L-d)^2}{2r^2 + mgd\sin\phi} + \tfrac{1}{2}(m+2m_w)v_{G2}^2 \quad \Rightarrow \quad v_{G2}^2 = \frac{k_t d(2L-d) - 2mgdr^2\sin\phi}{r^2(m+2m_w)}. \tag{9}$$

We need to find the value of d for which v_{G2} is maximum. For this purpose observe that v_{G2}^2 achieves its maximum for the same value of d for which v_{G2} is maximum, and maximizing v_{G2}^2 is simpler than maximizing v_{G2} (in this problem). Using this fact, we then have

$$\frac{dv_{G2}^2}{d(d)} = 0 \quad \Rightarrow \quad \frac{2k_t(L-d) - 2mgr^2\sin\phi}{r^2(m+2m_w)} = 0$$

$$\Rightarrow \quad d_{v\max} = \frac{k_t L - mgr^2\sin\phi}{k_t} = 0.1784\,\text{m}, \tag{10}$$

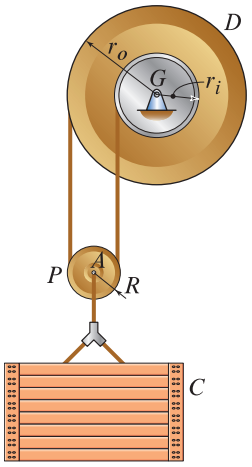
where we have used the following numerical data: $k_t = 0.00025\,\text{N/m}$, $L = 25\,\text{cm} = 0.2500\,\text{m}$, $m = 120\,\text{g} = 0.1200\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $r = 6\,\text{mm} = 0.006000\,\text{m}$, and $\phi = 25°$. We now observe that the result we have just obtained is consistent with our working assumption, i.e., $d < L$ and can therefore be considered acceptable. Then, recalling that v_G is also the speed of the car as a whole, setting d in the last of Eqs. (9) equal to $d_{v_{\max}}$ of Eq. (10), we can then compute the value of $v_{\max}$. This gives

$$\boxed{v_{\max} = 1.30\,\text{m/s} \quad \text{for} \quad d_{v\max} = 0.178\,\text{m},}$$

where we have used the following numerical data: $k_t = 0.00025\,\text{N/m}$, $L = 25\,\text{cm} = 0.2500\,\text{m}$, $m = 120\,\text{g} = 0.1200\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $r = 6\,\text{mm} = 0.006000\,\text{m}$, $\phi = 25°$, and $m_w = 5\,\text{g} = 0.005000\,\text{kg}$.

Problems 8.40 and 8.41

The double pulley D has mass of 15 kg, center of mass G coinciding with its geometric center, radius of gyration $k_G = 10$ cm, outer radius $r_o = 15$ cm, and inner radius $r_i = 7.5$ cm. It is connected to the pulley P with radius R via a cord of negligible mass that unwinds from the inner and outer spools of the double pulley D. The crate C, which has a mass of 20 kg, is released from rest.



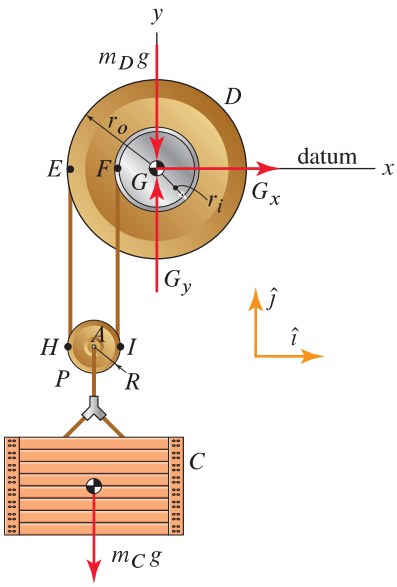
Problem 8.40 Neglecting the mass of the pulley P, determine the speed of the crate C and the angular velocity of the pulley D after the crate has dropped a distance $h = 2$ m.

Problem 8.41 Assuming that the pulley P has a mass of 1.5 kg and a radius of gyration $k_A = 3.5$ cm, determine the speed of the crate C and the angular velocity of the pulley D after the crate has dropped a distance $h = 2$ m.

Solution to 8.40

Referring to the FBD shown, work is done only by gravity. Let ① be the position at release and ② be the position achieved by the system when the crate C drops a distance h. Since the only force doing work in going from ① to ② is gravity, the energy of the system is conserved and we can write

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$



where, given the choice of datum, observing that point G is fixed, the mass of pulley P is being neglected, and recalling that the system is released from rest, we have

$$T_1 = 0, \tag{2}$$
$$V_1 = m_C g y_{C1}, \tag{3}$$
$$T_2 = \tfrac{1}{2} I_G \omega_{D2}^2 + \tfrac{1}{2} m_C v_{C2}^2, \tag{4}$$
$$V_2 = m_C g y_{C2}, \tag{5}$$

where m_D is the mass of pulley D, m_C is the mass of the crate, y_{C1} and y_{C2} are the vertical coordinates of the crate at ① and ②, respectively, I_G is the mass moment of inertia of pulley D relative to its won center of mass G, v_C is the speed of the crate, and ω_D is the component of the angular velocity of pulley D in the z direction.

For the mass moment of inertia I_G we can write

$$I_G = m_D k_G^2. \tag{6}$$

As far as the kinematics of the problem is concerned, let's observe that since G is a fixed we have

$$\vec{v}_E = -r_o \omega_D \, \hat{\jmath} \quad \text{and} \quad \vec{v}_F = -r_i \omega_D \, \hat{\jmath}. \tag{7}$$

Next, since the rope connecting the pulleys is assumed not to slip relative to any of the pulleys, we have

$$\vec{v}_H = \vec{v}_E \quad \text{and} \quad \vec{v}_I = \vec{v}_F. \tag{8}$$

Now observe that points H and I belong to the same rigid body, namely pulley P. Treating P as a rigid body, and letting R denote the radius of pulley P, we must have

$$\vec{v}_I = \vec{v}_H + \vec{\omega}_P \times \vec{r}_{I/H} \quad \Rightarrow \quad -r_i\omega_D\,\hat{j} = -r_o\omega_D\,\hat{j} + \omega_P\,\hat{k} \times (2R\,\hat{\imath}) \quad \Rightarrow \quad \omega_P = \frac{r_o - r_i}{2R}\omega_D, \quad (9)$$

where $\vec{\omega}_P$ is the angular velocity of pulley P. Observing that $R = \frac{1}{2}(r_o - r_i)$, we can rewrite ω_P simply as

$$\omega_P = \omega_D. \quad (10)$$

Using this result, we can the say that

$$\vec{v}_C = \vec{v}_A = \vec{v}_H + \vec{\omega}_P \times \vec{r}_{A/H} = -r_o\omega_D\,\hat{j} + \omega_D\,\hat{k} \times \tfrac{1}{2}(r_o - r_i)\,\hat{\imath} = -\tfrac{1}{2}(r_o + r_i)\omega_D\,\hat{j}. \quad (11)$$

The above equation implies that

$$v_{C2}^2 = \tfrac{1}{4}(r_o + r_i)^2\omega_{D2}^2. \quad (12)$$

Finally, we observe that

$$y_{C2} = y_{C1} - h. \quad (13)$$

Therefore, substituting Eqs. (2)–(5) into Eq. (1) and accounting for Eqs. (6), (12), and (13), we have

$$m_C g y_{C1} = \tfrac{1}{2}m_D k_G^2\omega_{D2}^2 + \tfrac{1}{2}m_C\left[\tfrac{1}{4}(r_o + r_i)^2\omega_{D2}^2\right] + m_C g(y_{C1} - h), \quad (14)$$

which, keeping in mind that we expect $\omega_{D2} > 0$, is an equation in ω_{D2} whose solution is

$$\omega_{D2} = 2\sqrt{2}\sqrt{\frac{ghm_C}{4k_G^2 m_D + m_C(r_i + r_o)^2}}. \quad (15)$$

Substituting this result in Eq. (12) and solving for v_{C2}, we have

$$v_{C2} = \sqrt{2}\sqrt{\frac{ghm_C(r_i + r_o)^2}{4k_G^2 m_D + m_C(r_i + r_o)^2}}. \quad (16)$$

Evaluating the above two results numerically, we have

$$\boxed{v_C = 4.96\,\text{m/s} \quad \text{and} \quad \vec{\omega}_D = (44.1\,\text{rad/s})\,\hat{k}}$$

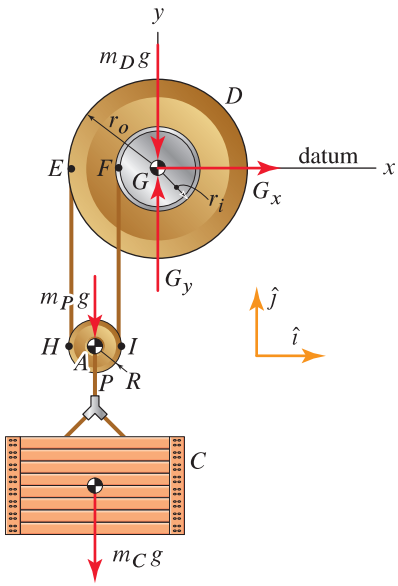


where we have dropped the subscript 2 for simplicity, and where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$, $h = 2\,\text{m}$, $m_C = 20\,\text{kg}$, $k_G = 10\,\text{cm} = 0.1000\,\text{m}$, $m_D = 15\,\text{kg}$, $r_i = 7.5\,\text{cm} = 0.07500\,\text{m}$, and $r_o = 15\,\text{cm} = 0.1500\,\text{m}$.

Solution to 8.41

Referring to the FBD shown, work is done only by gravity. Let ① be the position at release and ② be the position achieved by the system when the crate C drops a distance h. Since the only force doing work in going from ① to ② is gravity, the energy of the system is conserved and we can write

$$T_1 + V_1 = T_2 + V_2, \tag{17}$$



where, given the choice of datum, observing that point G is fixed, and recalling that the system is released from rest, we have

$$T_1 = 0, \tag{18}$$
$$V_1 = m_C g y_{C1} + m_P g y_{A1}, \tag{19}$$
$$T_2 = \tfrac{1}{2} I_G \omega_{D2}^2 + \tfrac{1}{2} m_C v_{C2}^2 + \tfrac{1}{2} m_P v_A^2 + \tfrac{1}{2} I_A \omega_P^2, \tag{20}$$
$$V_2 = m_C g y_{C2} + m_P g y_{A2}, \tag{21}$$

where m_D is the mass of pulley D, m_C is the mass of the crate, y_{C1} and y_{C2} are the vertical coordinates of the crate at ① and ②, respectively, I_G is the mass moment of inertia of pulley D relative to its own center of mass G, v_C is the speed of the crate, ω_D is the component of the angular velocity of pulley D in the z direction, y_{A1} and y_{A2} are the vertical coordinates of the mass center A of pulley P at ① and ②, respectively, v_A is the speed of A, ω_P is the component of the angular velocity of pulley P in the z direction, and I_A is the mass moment of inertia of pulley P with respect to its own center of mass.

For the mass moments of inertia I_G and I_A we can write

$$I_G = m_D k_G^2 \quad \text{and} \quad I_A = m_P k_A^2. \tag{22}$$

As far as the kinematics of the problem is concerned, let's observe that since G is a fixed we have

$$\vec{v}_E = -r_o \omega_D \, \hat{j} \quad \text{and} \quad \vec{v}_F = -r_i \omega_D \, \hat{j}. \tag{23}$$

Next, since the rope connecting the pulleys is assumed not to slip relative to any of the pulleys, we have

$$\vec{v}_H = \vec{v}_E \quad \text{and} \quad \vec{v}_I = \vec{v}_F. \tag{24}$$

Now observe that points H and I belong to the same rigid body, namely pulley P. Treating P as a rigid body, and letting R denote the radius of pulley P, we must have

$$\vec{v}_I = \vec{v}_H + \vec{\omega}_P \times \vec{r}_{I/H} \quad \Rightarrow \quad -r_i \omega_D \, \hat{j} = -r_o \omega_D \, \hat{j} + \omega_P \, \hat{k} \times (2R \, \hat{\imath}) \quad \Rightarrow \quad \omega_P = \frac{r_o - r_i}{2R} \omega_D, \tag{25}$$

where $\vec{\omega}_P$ is the angular velocity of pulley P. Observing that $R = \frac{1}{2}(r_o - r_i)$, we can rewrite ω_P simply as

$$\omega_P = \omega_D. \tag{26}$$

Using this result, we can the say that

$$\vec{v}_C = \vec{v}_A = \vec{v}_H + \vec{\omega}_P \times \vec{r}_{A/H} = -r_o \omega_D \, \hat{j} + \omega_D \, \hat{k} \times \tfrac{1}{2}(r_o - r_i) \, \hat{\imath} = -\tfrac{1}{2}(r_o + r_i) \omega_D \, \hat{j}. \tag{27}$$

The above equation implies that

$$v_{C2}^2 = v_{A2}^2 = \tfrac{1}{4}(r_o + r_i)^2 \omega_{D2}^2. \tag{28}$$

Finally, we observe that

$$y_{C2} = y_{C1} - h \quad \text{and} \quad y_{A2} = y_{A1} - h. \tag{29}$$

Therefore, substituting Eqs. (18)–(21) into Eq. (17) and accounting for Eqs. (22), (26), (28), and (29), we have

$$m_C g y_{C1} + m_P g y_{A1} = \tfrac{1}{2} m_D k_G^2 \omega_{D2}^2 + \tfrac{1}{2}(m_C + m_P)\left[\tfrac{1}{4}(r_o + r_i)^2 \omega_{D2}^2\right] \\ \tfrac{1}{2} m_P k_A^2 \omega_{D2}^2 + m_C g(y_{C1} - h) + m_P g(y_{A2} - h), \tag{30}$$

which, keeping in mind that we expect $\omega_{D2} > 0$, is an equation in ω_{D2} whose solution is

$$\omega_{D2} = 2\sqrt{2}\sqrt{\frac{gh(m_C + m_P)}{4k_G^2 m_D + 4k_A^2 m_P + (m_C + m_P)(r_i + r_o)^2}}. \tag{31}$$

Substituting this result in Eq. (28) and solving for v_{C2}, we have

$$v_{C2} = \sqrt{2}\sqrt{\frac{gh(m_C + m_P)(r_i + r_o)^2}{4k_G^2 m_D + 4k_A^2 m_P + (m_C + m_P)(r_i + r_o)^2}}. \tag{32}$$

Evaluating the above two results numerically, we have

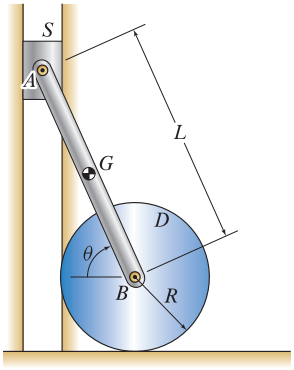
$$\boxed{v_C = 5.02\,\text{m/s} \quad \text{and} \quad \vec{\omega}_D = (44.6\,\text{rad/s})\,\hat{k}}$$



where we have dropped the subscript 2 for simplicity, and where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$, $h = 2\,\text{m}$, $m_C = 20\,\text{kg}$, $k_G = 10\,\text{cm} = 0.1000\,\text{m}$, $m_D = 15\,\text{kg}$, $r_i = 7.5\,\text{cm} = 0.07500\,\text{m}$, $r_o = 15\,\text{cm} = 0.1500\,\text{m}$, $m_P = 1.5\,\text{kg}$, and $k_A = 3.5\,\text{cm} = 0.03500\,\text{m}$.

Problems 8.42 through 8.44

The uniform thin rod AB is pin-connected to the slider S, which moves along the frictionless guide, and to the disk D, which rolls without slip over the horizontal surface. The pins at A and B are frictionless, and the system is released from rest. Neglect the vertical dimension of S.



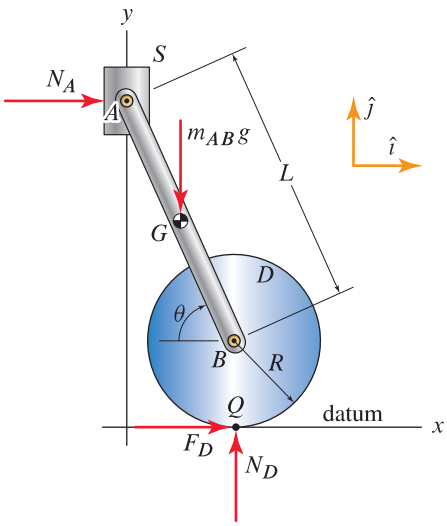
Problem 8.42 Letting $L = 1.75\,\text{m}$ and $R = 0.6\,\text{m}$, assuming that S and D are of negligible mass, that the mass of rod AB is $m_{AB} = 7\,\text{kg}$, and that the system is released from the angle $\theta_0 = 65°$, determine the speed of the slider S when it strikes the ground.

Problem 8.43 Letting $L = 4.5\,\text{ft}$ and $R = 1.2\,\text{ft}$, assuming that AB is of negligible mass, the weight of S is $W_S = 3\,\text{lb}$, D is a uniform disk of weight $W_D = 9\,\text{lb}$, and the system is released from the angle $\theta_0 = 67°$, determine the speed of the slider S when it strikes the ground.

Problem 8.44 Letting $L = 1.75\,\text{m}$ and $R = 0.6\,\text{m}$, assuming that the mass of S is $m_S = 4.2\,\text{kg}$, D is a uniform disk of mass $m_D = 12\,\text{kg}$, the mass of AB is $m_{AB} = 7\,\text{kg}$, and that the system is released from the angle $\theta = 69°$, determine the speed and the direction of motion of point B when the slider S strikes the ground.

Solution to 8.42

Referring to the FBD shown, work is done only by gravity. We define ① to be the position of the system at release and ② to be the position corresponding to S striking the ground. Since the only force doing work in going from ① to ② is gravity, the energy of the system is conserved and we can write



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, recalling that the system is released from rest and that the masses of the slider as well as of the disk are being neglected, we have

$$T_1 = 0, \tag{2}$$

$$T_2 = \tfrac{1}{2} m_{AB} v_{G2}^2 + \tfrac{1}{2} I_G (\omega_{AB})_2^2, \tag{3}$$

where v_G is the speed of G, which is the center of mass of bar AB, I_G is the mass moment of inertia of the bar AB with respect to G, and ω_{AB} is the component of the angular speed of AB.

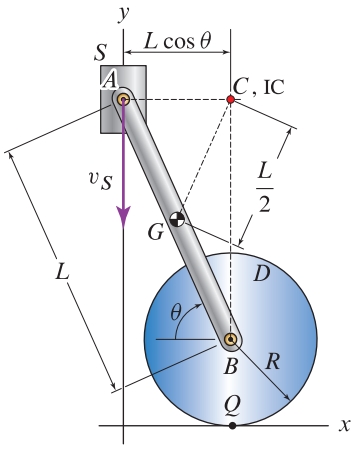
For the mass moment of inertia we have

$$I_G = \frac{1}{12} m_{AB} L^2. \tag{4}$$

For the potential energies, choosing the datum as shown in the figure, and recalling that the size of the slider is being ignored, we have

$$V_1 = m_{AB} g (R + \tfrac{1}{2} L \sin\theta_1) \quad \text{and} \quad V_2 = m_{AB} g \left(\tfrac{1}{2} R\right). \tag{5}$$

Referring to the figure on the right, we now consider the kinematic equations for this problem. We denote by C the instantaneous center of rotation of the bar. Because the distance between C and the slider is $L\cos\theta$, we must have that the angular speed of the bar is



$$\omega_{AB} = \frac{v_S}{L\cos\theta}. \tag{6}$$

Furthermore, since the distance between C and G is equal to $L/2$, we must also have

$$v_G = \omega_{AB}\frac{L}{2} = \frac{v_S}{2\cos\theta}. \tag{7}$$

where we have used the expression for ω_{AB} in Eq. (6). Next, we now observe that, in ② we have

$$\cos\theta_2 = \frac{\sqrt{L^2 - R^2}}{L}. \tag{8}$$

Using Eqs. (6)–(8), we can summarize the kinematics of the problem at ② as follows:

$$v_{G2} = \frac{v_{S2}L}{2\sqrt{L^2 - R^2}} \quad \text{and} \quad (\omega_{AB})_2^2 = \frac{v_{S2}^2}{\sqrt{L^2 - R^2}}. \tag{9}$$

Now, substituting Eqs. (4) and (9) into Eq. (3), and simplifying, we have

$$T_2 = \frac{m_{AB}L^2}{6(L^2 - R^2)}v_{S2}^2. \tag{10}$$

Then substituting Eqs. (2), (5), and (10) into Eq. (1), we have

$$m_{AB}g(R + \tfrac{1}{2}L\sin\theta_1) = \frac{m_{AB}L^2}{6(L^2 - R^2)}v_{S2}^2 + m_{AB}g\left(\tfrac{1}{2}R\right), \tag{11}$$

which is an equation in the only unknown v_{S2} whose solution is

$$v_{S2} = \frac{\sqrt{3}}{L}\sqrt{g(L^2 - R^2)(R + L\sin\theta_1)}$$

which can be evaluated to obtain

$$\boxed{v_S = 7.53\,\text{m/s},}$$

where we have dropped the subscript 2 for convenience, and where we have used the following numerical data: $L = 1.75\,\text{m}$, $g = 9.81\,\text{m/s}^2$, $R = 0.6\,\text{m}$, and $\theta_1 = 65°$.

Solution to 8.43

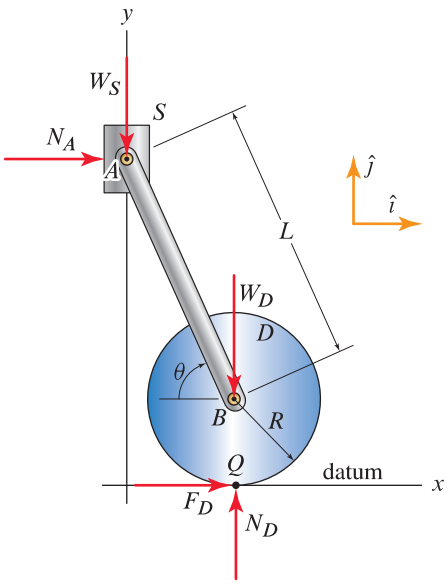
Referring to the FBD shown, work is done only by gravity. We define ① to be the position of the system at release and ② to be the position corresponding to S striking the ground. Since the only force doing work in going from ① to ② is gravity, the energy of the system is conserved and we can write

$$T_1 + V_1 = T_2 + V_2, \tag{12}$$

where, recalling that the system is released from rest,that the mass of the bar AB is negligible, and that point Q is the instantaneous center of rotation of the disk due to the rolling without slip condition, we have

$$T_1 = 0, \tag{13}$$

$$T_2 = \tfrac{1}{2}\frac{W_S}{g} v_{S2}^2 + \tfrac{1}{2} I_Q \omega_{D2}^2, \tag{14}$$



where v_S is the speed of the slider, I_Q is the mass moment of inertia of the disk D with respect to point Q, and ω_D is the component of the angular velocity of the disk D in the z direction.

For the mass moment of inertia we have

$$I_Q = \tfrac{1}{2}\frac{W_D}{g} R^2 + \frac{W_D}{g} R^2 = 0.6037\,\text{slug}\cdot\text{ft}^2, \tag{15}$$

where we have used the parallel axis theorem to express I_Q, and where we have used the following numerical data: $L = 4.5\,\text{ft}$, $W_D = 9\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, and $R = 1.2\,\text{ft}$.

For the potential energies, choosing the datum as shown in the figure, and recalling that the size of the slider is being ignored, we have

$$V_1 = W_S(R + L\sin\theta_1) \quad \text{and} \quad V_2 = 0. \tag{16}$$

We now consider the kinematic equations for this problem. Here we need to relate the speed of the slider S to the angular velocity of the disk. We start with observing that, S will only move in the vertical direction, i.e.,

$$\vec{v}_S = v_{Sy}\,\hat{\jmath}. \tag{17}$$

We now observe that because the physical dimensions of S are being neglected, S essentially coincides with point A. Hence, accounting for the fact that B can only move in the horizontal dierection, we have

$$\vec{v}_B = \vec{v}_S + \vec{\omega}_{AB} \times \vec{r}_{B/A} \quad \Rightarrow \quad v_{Bx}\,\hat{\imath} = v_{Sy}\,\hat{\jmath} + \omega_{AB}\,\hat{k} \times L(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$

$$\Rightarrow \quad v_{Bx} = -v_{Sy}\frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \omega_{AB} = -\frac{v_{Sy}}{L\cos\theta}, \tag{18}$$

where the last two results have been obtained by expanding the cross product and equating the left-hand side to the right-hand side on a component by component basis. Now that we have an expression for v_{Bx}, recalling that the disk D rolls without slip, we have

$$\omega_D = -\frac{v_{Bx}}{R} = \frac{v_{Sy}}{R}\frac{\sin\theta}{\cos\theta} \tag{19}$$

We now observe that, in ② we have

$$\frac{\sin\theta_2}{\cos\theta_2} = -\frac{R}{\sqrt{L^2 - R^2}}. \tag{20}$$

Hence, observing that $v_S^2 = v_{Sy}^2$, substituting Eq. (20) into Eq. (19), in ② we have

$$\omega_{D2}^2 = \frac{v_{S2}^2}{L^2 - R^2}. \tag{21}$$

Substituting Eqs. (21) into Eq. (14) and simplifying, we have

$$T_2 = \frac{1}{2}\left(\frac{I_Q}{L^2 - R^2} + \frac{W_S}{g}\right)v_{S2}^2, \tag{22}$$

Hence, recalling Eq. (13), we have that T_1 and T_2 can be given the following expressions:

$$T_1 = 0 \quad \text{and} \quad T_2 = (0.06263\,\text{slug})v_{S2}^2, \tag{23}$$

where we have used Eqs. (15) and the following numerical parameters: $L = 4.5\,\text{ft}$, $R = 1.2\,\text{ft}$, $W_S = 3\,\text{lb}$, and $g = 32.2\,\text{ft/s}^2$. Next, evaluating Eqs. (16) numerically, we have

$$V_1 = 16.03\,\text{ft·lb} \quad \text{and} \quad V_2 = 0, \tag{24}$$

where we have used the following numerical data: $W_S = 3\,\text{lb}$, $R = 1.2\,\text{ft}$, $L = 4.5\,\text{ft}$, and $\theta_1 = 67°$.

Next, substituting Eqs. (23) and (24) into Eq. (12), we have

$$16.03\,\text{ft·lb} = (0.06263\,\text{slug})v_{S2}^2, \tag{25}$$

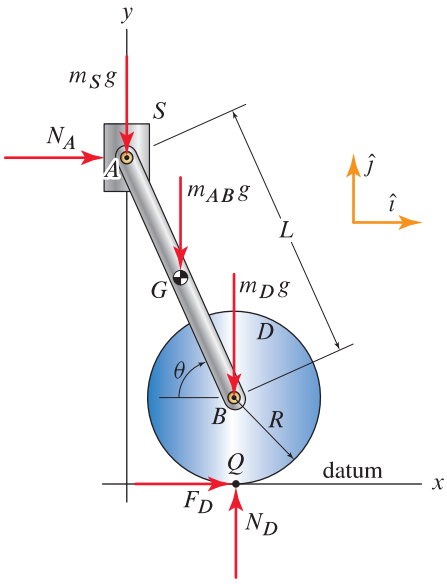
which is an equation in the only unknown v_{S2} whose solution is

$$\boxed{v_S = 16.0\,\text{ft/s},}$$

where we have dropped the subscript 2 for convenience.

Solution to 8.44

Referring to the FBD shown, work is done only by gravity. We define ① to be the position of the system at release and ② to be the position corresponding to S striking the ground. Since the only force doing work in going from ① to ② is gravity, the energy of the system is conserved and we can write



$$T_1 + V_1 = T_2 + V_2, \tag{26}$$

where, recalling that the system is released from rest and that point Q is the instantaneous center of rotation of the disk due to the rolling without slip condition, we have

$$T_1 = 0, \tag{27}$$

$$T_2 = \tfrac{1}{2} m_S v_{S2}^2 + \tfrac{1}{2} m_{AB} v_{G2}^2 + \tfrac{1}{2} I_G (\omega_{AB})_2^2 + \tfrac{1}{2} I_Q \omega_{D2}^2, \tag{28}$$

where v_S is the speed of the slider, v_G is the speed of G, which is the center of mass of bar AB, I_G is the mass moment of inertia of the bar AB with respect to G, ω_{AB} is the component of the angular velocity of AB in the z direction, I_Q is the mass moment of inertia of the disk D with respect to point Q, and ω_D is the component of the angular velocity of the disk D in the z direction.

For the mass moments of inertia we have

$$I_G = \frac{1}{12} m_{AB} L^2 = 1.786\,\text{kg}\cdot\text{m}^2 \quad \text{and} \quad I_Q = \tfrac{1}{2} m_D R^2 + m_D R^2 = 6.480\,\text{kg}\cdot\text{m}^2, \tag{29}$$

where we have used the parallel axis theorem to express I_Q, and where we have used the following numerical data: $m_{AB} = 7\,\text{kg}$, $L = 1.75\,\text{m}$, $m_D = 12\,\text{kg}$, and $R = 0.6\,\text{m}$.

For the potential energies, choosing the datum as shown in the figure, and recalling that the size of the slider is being ignored, we have

$$V_1 = m_S g(R + L\sin\theta_1) + m_{AB} g(R + \tfrac{1}{2} L\sin\theta_1) \quad \text{and} \quad V_2 = m_{AB} g\left(\tfrac{1}{2} R\right). \tag{30}$$

We now consider the kinematic equations for this problem. Here we need to relate the angular velocity of point B to the velocity of the slider S. We start with observing that, B will only move in the horizontal direction and that the disk D rolls without slip. Hence, we can write

$$\vec{v}_B = v_{Bx}\,\hat{\imath} \quad \text{and} \quad \omega_D = -\frac{v_{Bx}}{R}. \tag{31}$$

Next, observe that S can only move in the vertical direction and that it must move like point A. Hence, we have

$$\vec{v}_S = v_{Sy}\,\hat{\jmath} = \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{A/B} \quad \Rightarrow \quad v_{Sy}\,\hat{\jmath} = v_{Bx}\,\hat{\imath} = \omega_{AB}\,\hat{k} \times L(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$\Rightarrow \quad v_{Sy} = -v_{Bx}\frac{\cos\theta}{\sin\theta} \quad \text{and} \quad \omega_{AB} = \frac{v_{Bx}}{L\sin\theta}, \tag{32}$$

where the last two results have been obtained by expanding the cross product and equating the left-hand side to the right-hand side on a component by component basis. Next, using the the first of Eqs. (31) and the last of Eqs. (32), we then have

$$\vec{v}_G = \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{G/B} \quad \Rightarrow \quad \vec{v}_G = \tfrac{1}{2} v_{Bx}\,\hat{\imath} - \tfrac{1}{2} v_{Bx}\frac{\cos\theta}{\sin\theta}. \tag{33}$$

We now observe that, in ② we have

$$\frac{\cos\theta_2}{\sin\theta_2} = -\frac{\sqrt{L^2 - R^2}}{R} \quad \text{and} \quad \sin\theta_2 = -\frac{R}{L}. \tag{34}$$

Hence, observing that $v_S^2 = v_{Sy}^2$, $v_{Bx}^2 = v_B^2$, substituting Eqs. (34) into the last two relations in Eqs. (32), in ② we have

$$v_{S2}^2 = v_{B2}^2 \frac{L^2 - R^2}{R^2} \quad \text{and} \quad (\omega_{AB})_2^2 = \frac{v_{B2}^2}{R^2}. \tag{35}$$

Furthermore, substituting Eqs. (34) into the last of Eqs. (33) and then computing v_G^2, we have

$$v_{G2}^2 = \tfrac{1}{4}\left(1 + \frac{L^2 - R^2}{R^2}\right) v_{B2}^2. \tag{36}$$

Substituting Eqs. (35) and (36) into Eq. (28) and simplifying, we have

$$T_2 = \frac{4I_G + 4I_Q + (m_{AB} + 4m_S)L^2 - 4m_S R^2}{8R^2} v_{B2}^2. \tag{37}$$

Hence, recalling Eq. (27), we have that T_1 and T_2 can be given the following expressions:

$$T_1 = 0 \quad \text{and} \quad T_2 = (34.69\,\text{kg}) v_{B2}^2, \tag{38}$$

where we have used Eqs. (29) and the following numerical parameters: $L = 1.75\,\text{m}$, $m_{AB} = 7\,\text{kg}$, $m_S = 4.2\,\text{kg}$, and $R = 0.6\,\text{m}$. Next, evaluating Eqs. (30) numerically, we have

$$V_1 = 189.3\,\text{J} \quad \text{and} \quad V_2 = 20.60\,\text{J}, \tag{39}$$

where we have used the following numerical data: $m_S = 4.2\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $R = 0.6\,\text{m}$, $L = 1.75\,\text{m}$, $\theta_1 = 69°$, and $m_{AB} = 7\,\text{kg}$.

Next, substituting Eqs. (38) and (39) into Eq. (26), we have

$$189.3\,\text{J} = 20.60\,\text{J} + (34.69\,\text{kg}) v_{B2}^2, \tag{40}$$

which is an equation in the only unknown v_{B2} whose solution is

$$\boxed{v_B = 2.21\,\text{m/s},}$$

where we have dropped the subscript 2 for convenience.

As far as the direction of motion of point B in ② is concerned, substituting the first of Eqs. (34) into the second to the last of Eqs. (32) and solving for $(v_{Bx})_2$, we have

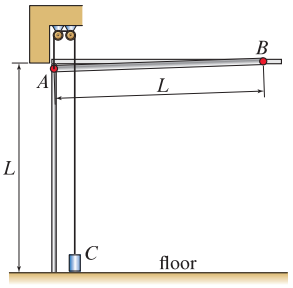
$$(v_{Bx})_2 = \frac{R}{\sqrt{L^2 - R^2}} (v_{Sy})_2. \tag{41}$$

Finally, we observe that in ② the slider strikes the ground and therefore is moving in the negative y direction, i.e., $(v_{Sy})_2 < 0$. This implies that in ② $(v_{Bx})_2 < 0$ and we conclude that

$$\boxed{\text{In ② } B \text{ is moving to the left.}}$$

Problem 8.45

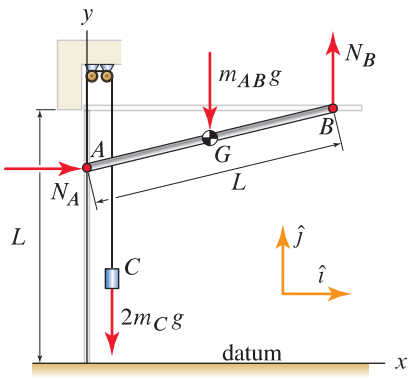
The figure shows the cross section of a garage door with length $L = 2.5\,\text{m}$ and mass $m = 90\,\text{kg}$. At the ends A and B there are rollers of negligible mass constrained to move in a vertical and a horizontal guide, respectively. The door's motion is assisted by two counterweights (only one counterweight is shown), each of mass $m_C = 42.5\,\text{kg}$. If the door is released from rest when horizontal, neglecting friction and modeling the door as a uniform thin plate, determine the speed with which B strikes the left end of the horizontal guide.



Solution

Note: The problem statement contains a typographical error. In this solution we set $m_C = 22\,\text{kg}$ instead of the (incorrect) value of 42.5 kg.

Referring to the FBD shown, work is done only by gravity. Let ① be the position at release and ② be the position achieved by the door when B strikes the left end of the horizontal guide. Since energy is conserved in going from ① to ②, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, given the choice of datum, modeling the door as a rigid body and the counterweights as particles, and recalling that the door is released from rest, we have

$$T_1 = 0, \tag{2}$$

$$V_1 = 2m_C g y_{C1} + m_{AB} g L, \tag{3}$$

$$T_2 = \tfrac{1}{2} m_{AB} v_{G2}^2 + \tfrac{1}{2} I_G \omega_{d2}^2 + 2\left(\tfrac{1}{2} m_C v_{C2}^2\right), \tag{4}$$

$$V_2 = 2m_C g y_{C2} + \tfrac{1}{2} L m_{AB} g, \tag{5}$$

where m_{AB} is the mass of the door, y_{C1} and y_{C2} are the vertical coordinates of the counterweights at ① and ②, respectively, I_G is the mass moment of inertia of the door with respect to the door's center of mass G, v_G is the speed of G, and ω_d is the angular speed of the door.

Now observe that in ② the door point A is at the very bottom of its trajectory. This implies that $v_{A2} = 0$ and that A is the instantaneous center of rotation of the door. In turn, this implies that the angular speed of the door in ② is $\omega_{d2} = v_{B2}/L$, and that $v_{G2} = \omega_{d2}(L/2) = v_{B2}/2$. Hence, observing that the counterweights C are directly attached to point A, we have

$$v_{G2} = \tfrac{1}{2} v_{B2}, \quad v_{C2} = 0, \quad \omega_{d2} = v_{B2}/L, \quad \text{and} \quad y_{C2} = y_{C1} + L. \tag{6}$$

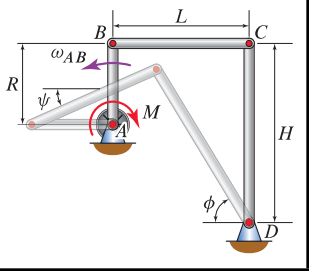
Substituting Eqs. (2)–(6) into Eq. (1), and solving for v_{B2}, we have

$$(v_B)_{\text{final}} = \sqrt{\frac{3}{m_{AB}}}\sqrt{gL(m_{AB} - 4m_C)} = 1.28\,\text{m/s},$$

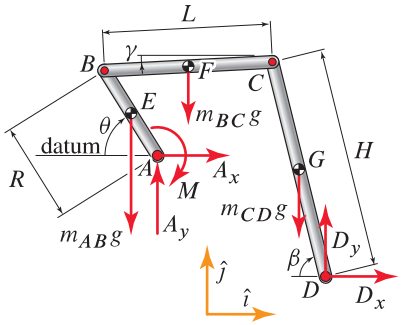
where we have written $(v_B)_{\text{final}}$ for v_{B2} and where we have used the following numerical data: $m_{AB} = 90\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $L = 2.5\,\text{m}$, and $m_C = 22\,\text{kg}$.

Problem 8.46

The uniform thin pin-connected bars AB, BC, and CD have masses $m_{AB} = 2.3\,\text{kg}$, $m_{BC} = 3.2\,\text{kg}$, and $m_{CD} = 5.0\,\text{kg}$, respectively. In addition, $R = 0.75\,\text{m}$, $L = 1.2\,\text{m}$, and $H = 1.55\,\text{m}$. When bars AB and CD are vertical, AB is rotating with angular speed $\omega_{AB} = 4\,\text{rad/s}$ in the direction shown. At this instant, the motor connected to AB starts to exert a constant torque M in the direction opposite to ω_{AB}. If the motor stops AB after AB has rotated 90° counterclockwise, determine M and the maximum power output of the motor during the stopping phase. In the final position, $\phi = 64.36°$ and $\psi = 29.85°$.



Solution



Referring to the FBD on the right, observe that the reaction forces at A and D do no work because points A and D are fixed. The only forces doing work are the weights of the bars in the system and the constant moment M. Because the calculation of the work of the force system in question is straightforward, it is indeed possible to solve this problem via the application of the work-energy principle.

Referring to the problem statement and to the FBD, observe that we have introduced angles θ, γ, and β to describe the orientation of bars AB, BC, and CD, respectively. In addition, we have labeled E, F, and G the mass centers of bars AB, BC, and CD, respectively. We define ① as the position at which $\theta = 90°$ and ② the position at which $\theta = 0$.

Then, the work-energy principle states that

$$T_1 + V_1 + (U_{1\text{-}2})_{\text{nc}} = T_2 + V_2, \tag{1}$$

where, observing that bars AB and CD are in fixed axis rotations about points A and D respectively, and observing that in ② the system comes to a stop, we have

$$T_1 = \tfrac{1}{2} I_A (\omega_{AB})_1^2 + \tfrac{1}{2} m_{BC} v_{F1}^2 + \tfrac{1}{2} I_F (\omega_{BC})_1^2 + \tfrac{1}{2} I_D (\omega_{CD})_1^2 \quad \text{and} \quad T_2 = 0, \tag{2}$$

where ω_{AB}, ω_{BC}, ω_{CD} are the angular speeds of the bars AB, BC, and CD, respectively, and where I_A is the mass moment of inertia of bar AB about A, I_F is the mass moment of inertia of bar BC about point F, and I_D is the mass moments of inertia of bar CD about D. The mass moments of inertia introduced are given by

$$I_A = \tfrac{1}{12} m_{AB} R^2 + m_{AB} (R/2)^2 = 0.4312\,\text{kg}\cdot\text{m}^2, \tag{3}$$

$$I_F = \tfrac{1}{12} m_{BC} L^2 = 0.3840\,\text{kg}\cdot\text{m}^2, \tag{4}$$

$$I_D = \tfrac{1}{12} m_{CD} H^2 + m_{CD} (H/2)^2 = 4.004\,\text{kg}\cdot\text{m}^2, \tag{5}$$

where we have used the following numerical data: $m_{AB} = 2.3\,\text{kg}$, $m_{BC} = 3.2\,\text{kg}$, $m_{CD} = 5.0\,\text{kg}$, $R = 0.75\,\text{m}$, $L = 1.2\,\text{m}$, and $H = 1.55\,\text{m}$.

As far as the potential energy terms in Eq. (1) are concerned, using the datum shown, we have

$$V_1 = m_{AB} g \frac{R}{2} + m_{BC} g R + m_{CD} g \left(R - \tfrac{1}{2} H\right), \tag{6}$$

$$V_2 = m_{BC} g \frac{L}{2} \sin\gamma_2 + m_{CD} g \left(L \sin\gamma_2 - \tfrac{1}{2} H \sin\beta_2\right). \tag{7}$$

Since the applied moment M is constant, and observing that both the moment and the angle θ have been defined positive if clockwise, then the work of M in going from ① to ② is

$$(U_{1\text{-}2})_{\text{nc}} = M(\theta_2 - \theta_1). \tag{8}$$

We now carry out the kinematic analysis of the system needed to compute the speeds that appear in the kinetic energy expression and the angles that appear in the computation of the potential energy as well as the work of the moment M. We being by observing that

$$\theta_1 = 90^\circ, \quad \theta_2 = 0, \quad \gamma_1 = 0, \quad \gamma_2 = \psi = 29.85^\circ, \quad \beta_1 = 90^\circ, \quad \text{and} \quad \beta_2 = \phi = 64.36^\circ, \tag{9}$$

where we have used the values of ϕ and ψ given in the problem statement. As far as the velocity analysis is concerned, we only need to determine the speeds of the system in ① because the system is at rest in ②. With this in mind, observe that in ① bars AB and CD are parallel to one another and that the angular speed of bar AB in ① is given and equal to 4 rad/s. Therefore, applying the concept of instantaneous center of rotation, we see that the IC of bar BC in ① is at infinity. In turn, this implies that bar BC is in translation, which also implies that the speed of points B and C must be the same and is equal to $\omega_{AB}R$. Therefore, we have

$$(\omega_{AB})_1 = \omega_{AB} = 4\,\text{rad/s}, \quad v_{F1} = \omega_{AB}R, \quad \text{and} \quad (\omega_{CD})_1 = \frac{R}{H}\omega_{AB}. \tag{10}$$

Now that the kinematic analysis is complete, substituting Eqs. (3)–(5) and Eqs. (10) into Eqs. (2), we have

$$T_1 = 25.35\,\text{J} \quad \text{and} \quad T_2 = 0, \tag{11}$$

where we have used the following numerical data: $\omega_{AB} = 4\,\text{rad/s}$, $R = 0.75\,\text{m}$, and $H = 1.55\,\text{m}$. Next, substituting Eqs. (9) into Eqs. (6)–(8), we have

$$V_1 = 30.78\,\text{J}, \quad V_2 = 4.401\,\text{J}, \quad \text{and} \quad (U_{1\text{-}2})_{\text{nc}} = -1.571M, \tag{12}$$

$m_{AB} = 2.3\,\text{kg}$, $m_{BC} = 3.2\,\text{kg}$, $m_{CD} = 5.0\,\text{kg}$, $R = 0.75\,\text{m}$, $L = 1.2\,\text{m}$, $H = 1.55\,\text{m}$, and $g = 9.81\,\text{m/s}^2$.

Substituting Eqs. (11) and (12) into Eq. (1) and solving for M, we obtain

$$M = 32.93\,\text{N·m}, \tag{13}$$

which, when expressed to three significant figures, gives

$$\boxed{M = 32.9\,\text{N·m}.}$$

As far as the maximum power output of the motor during the stopping phase is concerned, observe that, between ① and ②, the torque provided by the engine is equal and opposite to the torque that appears in the FBD shown. In addition, observe that the angular velocity of the motor must be equal to that of bar AB. This implies that the torque acting on the motor acts in the same direction as the angular velocity of the motor. In turn, this implies that the power output of the motor is given by the following expressions:

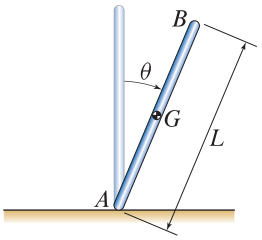
$$P = M\omega_{AB}. \tag{14}$$

Because M is constant, then the power is maximum when ω_{AB} is maximum, that is at ①. Hence, recalling that $(\omega_{AB})_1 = 4\,\text{rad/s}$ and using Eq. (13) to evaluate Eq. (14), we have

$$\boxed{P_{\text{max}} = 132\,\text{W}.}$$

Problem 8.47

A stick of length L and mass m is in equilibrium while standing on its end A when end B is gently nudged to the right, causing the stick to fall. Model the stick as a uniform slender bar, and assume that there is friction between the stick and the ground. Under these assumptions, there is a value of θ, let's call it $\theta_{\max}$, such that the stick *must* start slipping before reaching $\theta_{\max}$ for *any* value of the coefficient of static friction μ_s. To find the value of $\theta_{\max}$, follow the steps below.



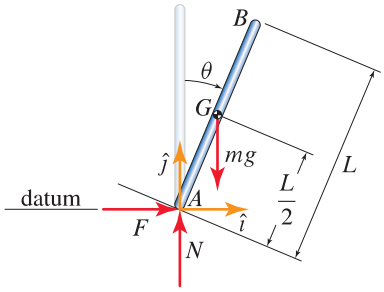
(a) Letting F and N be the friction and normal forces, respectively, between the stick and the ground, draw the FBD of the stick as it falls. Then set the sum of forces in the horizontal and vertical directions equal to the corresponding components of $m\vec{a}_G$. Express the components of $\vec{a}_G$ in terms of θ, $\dot{\theta}$, and $\ddot{\theta}$. Finally, express F and N as functions of θ, $\dot{\theta}$, and $\ddot{\theta}$.

(b) Use the work-energy principle to find an expression for $\dot{\theta}^2(\theta)$. Differentiate the expression for $\dot{\theta}^2(\theta)$ with respect to time, and find an expression for $\ddot{\theta}(\theta)$.

(c) Substitute the expressions for $\dot{\theta}^2(\theta)$ and $\ddot{\theta}(\theta)$ into the expressions for F and N to obtain F and N as functions of θ. For impending slip, $|F/N|$ must be equal to the coefficient of static friction. Use this fact to determine $\theta_{\max}$.

Solution

The FBD of the stick is shown on the right. Proceeding as indicated in Step (a), summing forces in the horizontal and vertical directions, we have



$$\sum F_x: \qquad F = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg = ma_{Gy}, \tag{2}$$

where G is the center of mass of the stick, and where F and N are the friction and normal forces at A, respectively. We will solve the problem under the working assumption that the stick does not slip at A. Consistent with this assumption, we then have that the position of G can be described as follows:

$$x_G = \tfrac{1}{2}L\sin\theta \quad \text{and} \quad y_G = \tfrac{1}{2}L\cos\theta. \tag{3}$$

Differentiating Eqs. (3) with respect to time, we have that the components of the velocity of G are

$$v_{Gx} = \tfrac{1}{2}L\dot{\theta}\cos\theta \quad \text{and} \quad v_{Gy} = -\tfrac{1}{2}L\dot{\theta}\sin\theta. \tag{4}$$

Differentiating Eqs. (4) with respect to time, we have that the components of the acceleration of G are

$$a_{Gx} = \tfrac{1}{2}L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \quad \text{and} \quad a_{Gy} = -\tfrac{1}{2}L(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta). \tag{5}$$

Substituting Eqs. (5) into Eqs. (1) and (2), and solving for F and N we have

$$F = \tfrac{1}{2}mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \quad \text{and} \quad N = mg\left[1 - \frac{L}{2g}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\right]. \tag{6}$$

We now proceed with Step (b) of the solution. We begin by defining ① to the the position of the stick at release and ② the position of the stick at a generic angle θ following ①. We observe that, as long as A does not slip, friction does no work and the system can be treated as being conservative. Hence, we can apply the work-energy principle as follows:

$$T_1 + V_1 = T_2 + V_2, \tag{7}$$

where, recalling that the system is released from rest and observing that while point A does not slip the stick is in a fixed axis rotation about A, we have

$$T_1 = 0, \quad V_1 = mg\frac{L}{2}, \quad T_2 = \tfrac{1}{2} I_A \dot{\theta}^2, \quad \text{and} \quad V_2 = mg\frac{L}{2}\cos\theta, \tag{8}$$

in which I_A is the mass moment of inertia of the stick about A and is therefore given by

$$I_A = \tfrac{1}{12} mL^2 + m\left(\frac{L}{2}\right)^2 = \tfrac{1}{3} mL^2, \tag{9}$$

where we have made use of the parallel axis theorem. Next, substituting Eq. (8) into Eq. (7) and solving for $\dot{\theta}^2$, we have

$$\dot{\theta}^2 = \frac{3g}{L}(1 - \cos\theta), \tag{10}$$

where we have also used the expression for I_A given in Eq. (9). Then, differentiating Eq. (10) with respect to time and solving for $\ddot{\theta}$, we have

$$2\dot{\theta}\ddot{\theta} = \frac{3g}{2L}(\sin\theta)\dot{\theta} \quad \Rightarrow \quad \ddot{\theta} = \frac{3g}{2L}\sin\theta. \tag{11}$$

We now proceed with Step (c) of the solution. Substituting the expressions for $\dot{\theta}^2$ and $\ddot{\theta}$ given in Eqs. (10) and (11) into the expressions for F and N in Eqs. (6), after simplification gives

$$F = \tfrac{3}{4} mg \sin\theta(3\cos\theta - 2) \quad \text{and} \quad N = \tfrac{1}{4} mg(1 - 3\cos\theta)^2. \tag{12}$$

We now consider the impending slip condition, which is the limiting case for the working assumption used thus far. In an impending slip condition we would have $|F/N| = \mu_s$, where μ_s is the coefficient of static friction between the stick and the ground. Using Eqs. (12) to enforce this condition gives

$$\left|\frac{F}{N}\right| = \left|\frac{3\sin\theta(3\cos\theta - 2)}{(1 - 3\cos\theta)^2}\right| = \mu_s. \tag{13}$$

Right now we are treating μ_s as a given of the problem and, in principle, μ_s could be a very large number. With this in mind, if we were to solve Eq. (13) for θ, we would find θ_{slip} the value of θ at which the stick would start slipping. Independently of θ_{slip}, let's denote by θ_0 the value of θ for which the denominator in Eq. (13) goes to zero. This value is as follows:

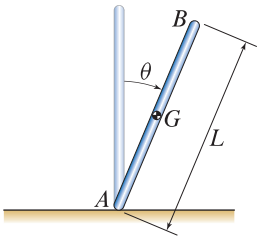
$$1 - 3\cos\theta_0 = 0 \quad \Rightarrow \quad \theta_0 = \cos^{-1}(1/3) = 70.53^\circ. \tag{14}$$

The importance of this result is in the fact that $|F/N| \to \infty$ as $\theta \to \theta_0$ and therefore, no matter how large μ_s is, we would find that $\theta_{\text{slip}} < \theta_0$. Based on this result, we can then say that the maximum value of θ what we were seeking is equal to θ_0, which, when expressed to three significant figures, is

$$\boxed{\theta_{\max} = 70.5^\circ.}$$

Problem 8.48

A stick of length L and mass m is in equilibrium while standing on its end A when the end B is gently nudged to the right, causing the stick to fall. Letting μ_s be the coefficient of static friction between the stick and the ground and modeling the stick as a uniform slender bar, find the largest value of μ_s for which the stick slides to the left as well as the corresponding value of θ at which sliding begins. To solve this problem, follow the steps below.



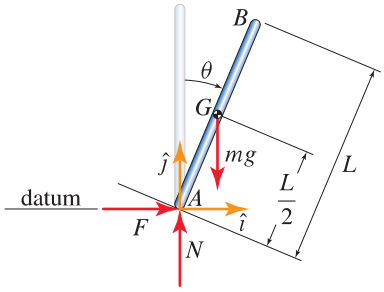
(a) Let F and N be the friction and normal forces, respectively, between the stick and the ground, and let F be positive to the right and N positive upward. Draw the FBD of the stick as it falls. Then set the sum of forces in the horizontal and vertical directions equal to the corresponding components of $m\vec{a}_G$. Express the components of $\vec{a}_G$ in terms of θ, $\dot{\theta}$, and $\ddot{\theta}$. Finally, express F and N as functions of θ, $\dot{\theta}$, and $\ddot{\theta}$.

(b) Use the work-energy principle to find an expression for $\dot{\theta}^2(\theta)$. Differentiate the expression for $\dot{\theta}^2(\theta)$ with respect to time, and find an expression for $\ddot{\theta}(\theta)$.

(c) Substitute the expressions for $\dot{\theta}^2(\theta)$ and $\ddot{\theta}(\theta)$ into the expressions for F and N to obtain F and N as functions of θ. When slip is impending (i.e., when $|F| = \mu_s|N|$), $|F/N|$ must be equal to the static coefficient of friction. Therefore, compute the maximum value of $|F/N|$ by differentiating it with respect to θ and setting the resulting derivative equal to zero.

Solution

The FBD of the stick is shown on the right. Proceeding as indicated in Step (a), summing forces in the horizontal and vertical directions, we have



$$\sum F_x: \qquad F = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg = ma_{Gy}, \tag{2}$$

where G is the center of mass of the stick, and where F and N are the friction and normal forces at A, respectively. We will solve the problem under the working assumption that the stick does not slip at A. Consistent with this assumption, we then have that the position of G can be described as follows:

$$x_G = \tfrac{1}{2}L\sin\theta \quad \text{and} \quad y_G = \tfrac{1}{2}L\cos\theta. \tag{3}$$

Differentiating Eqs. (3) with respect to time, we have that the components of the velocity of G are

$$v_{Gx} = \tfrac{1}{2}L\dot{\theta}\cos\theta \quad \text{and} \quad v_{Gy} = -\tfrac{1}{2}L\dot{\theta}\sin\theta. \tag{4}$$

Differentiating Eqs. (4) with respect to time, we have that the components of the acceleration of G are

$$a_{Gx} = \tfrac{1}{2}L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \quad \text{and} \quad a_{Gy} = -\tfrac{1}{2}L(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta). \tag{5}$$

Substituting Eqs. (5) into Eqs. (1) and (2), and solving for F and N we have

$$F = \tfrac{1}{2}mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \quad \text{and} \quad N = mg\left[1 - \frac{L}{2g}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\right]. \tag{6}$$

We now proceed with Step (b) of the solution. We begin by defining ① to the the position of the stick at release and ② the position of the stick at a generic angle θ following ①. We observe that, as long as A does not slip, friction does no work and the system can be treated as being conservative. Hence, we can apply the work-energy principle as follows:

$$T_1 + V_1 = T_2 + V_2, \tag{7}$$

where, recalling that the system is released from rest and observing that while point A does not slip the stick is in a fixed axis rotation about A, we have

$$T_1 = 0, \quad V_1 = mg\frac{L}{2}, \quad T_2 = \tfrac{1}{2} I_A \dot{\theta}^2, \quad \text{and} \quad V_2 = mg\frac{L}{2}\cos\theta, \tag{8}$$

in which I_A is the mass moment of inertia of the stick about A and is therefore given by

$$I_A = \tfrac{1}{12} mL^2 + m\left(\frac{L}{2}\right)^2 = \tfrac{1}{3} mL^2, \tag{9}$$

where we have made use of the parallel axis theorem. Next, substituting Eq. (8) into Eq. (7) and solving for $\dot{\theta}^2$, we have

$$\dot{\theta}^2 = \frac{3g}{L}(1 - \cos\theta), \tag{10}$$

where we have also used the expression for I_A given in Eq. (9). Then, differentiating Eq. (10) with respect to time and solving for $\ddot{\theta}$, we have

$$2\dot{\theta}\ddot{\theta} = \frac{3g}{2L}(\sin\theta)\dot{\theta} \quad \Rightarrow \quad \ddot{\theta} = \frac{3g}{2L}\sin\theta. \tag{11}$$

We now proceed with Step (c) of the solution. Substituting the expressions for $\dot{\theta}^2$ and $\ddot{\theta}$ given in Eqs. (10) and (11) into the expressions for F and N in Eqs. (6), after simplification gives

$$F = \tfrac{3}{4} mg \sin\theta(3\cos\theta - 2) \quad \text{and} \quad N = \tfrac{1}{4} mg(1 - 3\cos\theta)^2. \tag{12}$$

We now consider the impending slip condition, which is the limiting case for the working assumption used thus far. In an impending slip condition we would have $|F/N| = \mu_s$, where μ_s is the coefficient of static friction between the stick and the ground. Using Eqs. (12) to enforce this condition gives

$$\left|\frac{F}{N}\right| = \left|\frac{3\sin\theta(3\cos\theta - 2)}{(1 - 3\cos\theta)^2}\right| = \mu_s. \tag{13}$$

Let $f(\theta)$ denote the function of θ within absolute value signs, i.e.,

$$f(\theta) = \frac{3\sin\theta(3\cos\theta - 2)}{(1 - 3\cos\theta)^2}. \tag{14}$$

We observe that $f(\theta) = 0$ for $\theta = 0$. In addition, we observe that $f(\theta) \to \infty$ for $\theta \to \cos^{-1}(1/3)$, i.e., as the denominator $(1 - 3\cos\theta)^2 \to 0$. This indicates that the maximum value of $f(\theta)$ is ∞. However, this also indicates that the stick would slip no matter how large μ_s is. In other words, given a value of μ_s, the stick would start slipping before θ would ever become equation to $\cos^{-1}(1/3)$. In view of this observation, we then ask whether or not F/N has a *relative* maximum, that is a local maximum that is achieved before $\theta = \cos^{-1}(1/3)$. To answer this question, we proceed as indicated in the problem statement, i.e., by solving the equation $df(\theta)/d\theta = 0$. We begin by determining $df(\theta)/d\theta$, i.e.,

$$\frac{df(\theta)}{d\theta} = 3\frac{(3\cos^2\theta - 2\cos\theta - 3\sin^2\theta)(1 - 3\cos\theta)^2 - [\sin\theta(3\cos\theta - 2)]6(1 - 3\cos\theta)\sin\theta}{(1 - 3\cos\theta)^4}, \tag{15}$$

which, recalling that $\sin^{\theta} + \cos^2\theta = 1$, can be simplified to read

$$\frac{df(\theta)}{d\theta} = \frac{3(9 - 11\cos\theta)}{(1 - 3\cos\theta)^3}. \tag{16}$$

Setting the above expression to zero gives

$$\frac{df(\theta)}{d\theta} = 0 \quad \Rightarrow \quad \frac{3(9 - 11\cos\theta)}{(1 - 3\cos\theta)^3} = 0 \quad \Rightarrow \quad 9 - 11\cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}(9/11) = 35.10^\circ. \tag{17}$$

To understand the significance of the above result, we now compute the values of F, N and μ_s corresponding to the value $\theta = 53.10^\circ$. Substituting the value $\theta = 35.10^\circ$ into Eqs. (12), we have

$$F\big|_{\theta=35.10^\circ} = 0.1960mg \quad \text{and} \quad N\big|_{\theta=35.10^\circ} = 0.5289mg. \tag{18}$$

In addition, substituting the value $\theta = 35.10^\circ$ into Eq. (13), we have

$$\mu_s|_{\theta=35.10^\circ} = 0.3706. \tag{19}$$

Because $\theta = 35.10^\circ$ is the value of θ for which $df(\theta)/d\theta$ is equal to zero, then such a value of θ is corresponds to a relative maximum for μ_s. With this in mind, the result in the first of Eqs. (18) shows that, for $\theta = 35.10^\circ$, the friction force F is positive, i.e., points to the right. This means that if the given μ_s were indeed equal to that in Eq. (19), then the bar would start slipping at $\theta = 35.10^\circ$ and it would slip to the left (i.e., point A would move to the left).

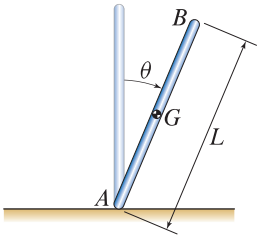
Now, what would happen if the given μ_s is larger than that in Eq. (19)? To answer this question, we first observe that the stick would slip for some value of θ larger than 35.10°. Second we observe that, for $\theta > 35.10^\circ$, the value of $|F/N|$ decreases (since $\theta = 35.10^\circ$ is a point of relative maximum for $|F/N|$). In fact, referring to the first of Eqs. (12), we have that $|F/N| = 0$ for $F = 0$, i.e., for $\theta = \cos^{-1}(2/3) = 48.19^\circ$. Furthermore, for $\theta > 48.19^\circ$ we have that $|F/N|$ increases again, but this time $F < 0$. Hence, if $\mu_s > 0.3706$, the stick will eventually slip, but it will do so with A moving to the right.

In conclusion, we can the say that the maximum value of μ_s for which the stick slips to the right is indeed that in Eq. (19) so that the answer to our problem is as follows:

$$\boxed{(\mu_s)_{\max} = 0.371 \quad \text{and} \quad \theta_{\text{slide}} = 35.1^\circ.}$$

Problem 8.49

A stick of length L and mass m is in equilibrium while standing on its end A when end B is gently nudged to the right, causing the stick to fall. Letting the coefficient of static friction between the stick and the ground be $\mu_s = 0.7$ and modeling the stick as a uniform slender bar, find the value of θ at which end A of the stick starts slipping and determine the corresponding direction of slip. As part of the solution, plot the absolute value of the ratio between the friction and normal force as a function of θ. To solve this problem, follow the steps below.



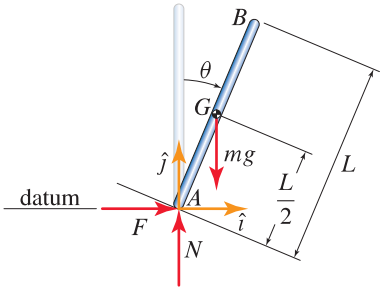
(a) Letting F and N be the friction and normal forces, respectively, between the stick and the ground, draw the FBD of the stick as it falls. Then set the sum of forces in the horizontal and vertical directions equal to the corresponding components of $m\vec{a}_G$. Express the components of $\vec{a}_G$ in terms of θ, $\dot{\theta}$, and $\ddot{\theta}$. Finally, express F and N as functions of θ, $\dot{\theta}$, and $\ddot{\theta}$.

(b) Use the work-energy principle to find an expression for $\dot{\theta}^2(\theta)$. Differentiate the expression for $\dot{\theta}^2(\theta)$ with respect to time, and find an expression for $\ddot{\theta}(\theta)$.

(c) After substituting the expressions for $\dot{\theta}^2(\theta)$ and $\ddot{\theta}(\theta)$ into the expressions for F and N, plot $|F/N|$ as a function of θ. For impending slip, $|F/N|$ must be equal to μ_s. Therefore, the desired value of θ corresponds to the intersection of the plot of $|F/N|$ with the horizontal line intercepting the vertical axis at the value 0.7. After determining the desired value of θ, the direction of slip can be found by determining the sign of F evaluated at the θ computed.

Solution

The FBD of the stick is shown on the right. Proceeding as indicated in Step (a), summing forces in the horizontal and vertical directions, we have



$$\sum F_x: \qquad F = ma_{Gx}, \tag{1}$$

$$\sum F_y: \quad N - mg = ma_{Gy}, \tag{2}$$

where G is the center of mass of the stick, and where F and N are the friction and normal forces at A, respectively. We will solve the problem under the working assumption that the stick does not slip at A. Consistent with this assumption, we then have that the position of G can be described as follows:

$$x_G = \tfrac{1}{2}L\sin\theta \quad \text{and} \quad y_G = \tfrac{1}{2}L\cos\theta. \tag{3}$$

Differentiating Eqs. (3) with respect to time, we have that the components of the velocity of G are

$$v_{Gx} = \tfrac{1}{2}L\dot{\theta}\cos\theta \quad \text{and} \quad v_{Gy} = -\tfrac{1}{2}L\dot{\theta}\sin\theta. \tag{4}$$

Differentiating Eqs. (4) with respect to time, we have that the components of the acceleration of G are

$$a_{Gx} = \tfrac{1}{2}L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \quad \text{and} \quad a_{Gy} = -\tfrac{1}{2}L(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta). \tag{5}$$

Substituting Eqs. (5) into Eqs. (1) and (2), and solving for F and N we have

$$F = \tfrac{1}{2}mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \quad \text{and} \quad N = mg\left[1 - \frac{L}{2g}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\right]. \tag{6}$$

We now proceed with Step (b) of the solution. We begin by defining ① to the the position of the stick at release and ② the position of the stick at a generic angle θ following ①. We observe that, as long as A does not slip, friction does no work and the system can be treated as being conservative. Hence, we can apply the work-energy principle as follows:

$$T_1 + V_1 = T_2 + V_2, \tag{7}$$

where, recalling that the system is released from rest and observing that while point A does not slip the stick is in a fixed axis rotation about A, we have

$$T_1 = 0, \quad V_1 = mg\frac{L}{2}, \quad T_2 = \tfrac{1}{2}I_A\dot{\theta}^2, \quad \text{and} \quad V_2 = mg\frac{L}{2}\cos\theta, \tag{8}$$

in which I_A is the mass moment of inertia of the stick about A and is therefore given by

$$I_A = \tfrac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \tfrac{1}{3}mL^2, \tag{9}$$

where we have made use of the parallel axis theorem. Next, substituting Eq. (8) into Eq. (7) and solving for $\dot{\theta}^2$, we have

$$\dot{\theta}^2 = \frac{3g}{L}(1 - \cos\theta), \tag{10}$$

where we have also used the expression for I_A given in Eq. (9). Then, differentiating Eq. (10) with respect to time and solving for $\ddot{\theta}$, we have

$$2\dot{\theta}\ddot{\theta} = \frac{3g}{2L}(\sin\theta)\dot{\theta} \quad \Rightarrow \quad \ddot{\theta} = \frac{3g}{2L}\sin\theta. \tag{11}$$

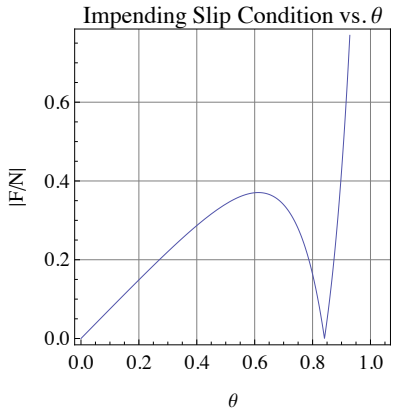
We now proceed with Step (c) of the solution. Substituting the expressions for $\dot{\theta}^2$ and $\ddot{\theta}$ given in Eqs. (10) and (11) into the expressions for F and N in Eqs. (6), after simplification gives

$$F = \tfrac{3}{4}mg\sin\theta(3\cos\theta - 2) \quad \text{and} \quad N = \tfrac{1}{4}mg(1 - 3\cos\theta)^2. \tag{12}$$

We now consider the impending slip condition, which is the limiting case for the working assumption used thus far. In an impending slip condition we would have $|F/N| = \mu_s$, where μ_s is the coefficient of static friction between the stick and the ground. Using Eqs. (12) to enforce this condition gives

$$\left|\frac{F}{N}\right| = \left|\frac{3\sin\theta(3\cos\theta - 2)}{(1 - 3\cos\theta)^2}\right| = \mu_s. \tag{13}$$

In order to solve this equation for the value of θ corresponding to the beginning of the slip motion we will need to use some appropriate mathematical software and we will need to provide a guess for the solution, that is, we will need to provide a value of θ which is close to the solution of the above equation. We will provide our guess by first plotting the expression for $|F/N|$, as we have done below:



The above plot was obtained using following Mathematica code:

```
μsθ = Abs[ 3 Sin[θ] (3 Cos[θ] - 2) / (1 - 3 Cos[θ])^2 ];

Plot[μsθ, {θ, 0, 60 Degree}, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"θ", "|F/N|"},
 PlotLabel → "Impending Slip Condition vs. θ"]
```

From the above plot, we see that the function $|F/N|$ achieves the value $\mu_s = 0.7$, near $\theta = 0.9\,\text{rad}$. Hence, we will find the numerical solution to our problem by providing the value 0.9 as the initial guess for θ. By doing so, we obtain the following solution

$$\theta_{\text{slip}} = 0.9237\,\text{rad} = 52.92^\circ, \tag{14}$$

which was obtained using the following code

```
ImpendingSlipSolution = FindRoot[μsθ == 0.7, {θ, 0.9}]

{θ → 0.923657}
```

Substituting the value of θ given in Eq.(14) into Eq. (12), we have

$$F\big|_{\theta=\theta_{\text{slip}}} = -0.1144mg. \tag{15}$$

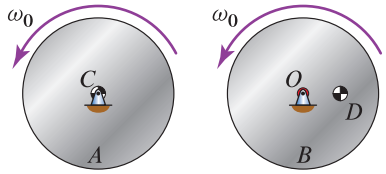
This result indicates that F points to the left and therefore point A will slide to the right.

In conclusion, we have

$\theta_{\text{slip}} = 52.9^\circ$ and A slips to the right.

Problem 8.50

Disks A and B have identical masses and mass moments of inertia about their respective mass centers. Point C is both the geometric center and center of mass of disk A. Points O and D are the geometric center and center of mass of disk B, respectively. If at the instant shown, the two disks are rotating about their centers with the same angular velocity ω_0, determine which of the following statements is true and why? (a) $|(\vec{h}_C)_A| < |(\vec{h}_O)_B|$, (b) $|(\vec{h}_C)_A| = |(\vec{h}_O)_B|$, (c) $|(\vec{h}_C)_A| > |(\vec{h}_O)_B|$.



Solution

Answer. The answer to the problem is (a), i.e.,

$$|(\vec{h}_C)_A| < |(\vec{h}_O)_B|.$$

Explanation. Both bodies are in fixes axes rotations. For body A, we have

$$|(\vec{h}_C)_A| = I_C|\vec{\omega}_O|.$$

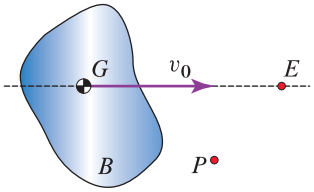
By contrast, for body B we have

$$|(\vec{h}_C)_O| = (I_D + m_B\overline{OD}^2)|\vec{\omega}_O|.$$

Since $I_C = I_D$, and since $m_B\overline{OD}^2 > 0$, then the magnitude of the angular momentum of B about is center is larger than the corresponding quantity for body C.

Problem 8.51

Body B has mass m and mass moment of inertia I_G, where G is the mass center of B. If B is translating as shown, determine which of the following statements is true and why: (a) $|(\vec{h}_E)_B| < |(\vec{h}_P)_B|$, (b) $|(\vec{h}_E)_B| = |(\vec{h}_P)_B|$, (c) $|(\vec{h}_E)_B| > |(\vec{h}_P)_B|$.



Solution

Answer. The answer to the problem is (a), i.e.,

$$|(\vec{h}_E)_B| < |(\vec{h}_P)_B|.$$

Explanation. Because the body is translating and point E is on a line parallel to the velocity vector of the center of mass we have that

$$(\vec{h}_E)_B = \vec{0}.$$

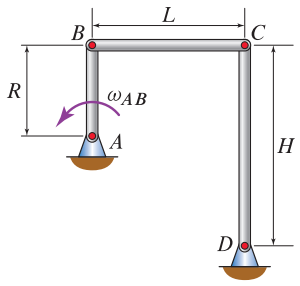
In addition, since P is not on a line parallel to the velocity vector of G and going through G, then we have

$$(\vec{h}_P)_B \neq \vec{0},$$

so that the magnitude of this vector must be greater than zero.

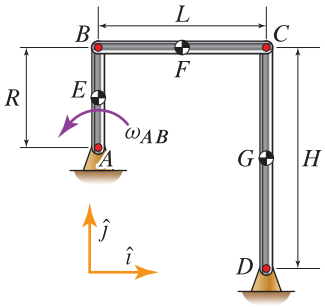
Problem 8.52

The uniform thin pin-connected bars AB, BC, and CD have masses $m_{AB} = 2.3\,\text{kg}$, $m_{BC} = 3.2\,\text{kg}$, and $m_{CD} = 5.0\,\text{kg}$, respectively. Letting $R = 0.75\,\text{m}$, $L = 1.2\,\text{m}$, and $H = 1.55\,\text{m}$, and knowing that bar AB rotates at a constant angular velocity $\omega_{AB} = 4\,\text{rad/s}$, compute the angular momentum of bar AB about A, of bar BC about A, and bar CD about D at the instant shown.



Solution

Angular momentum of bar AB about A. Using the component system shown, since AB is in a fixed axis rotation about A, using Eq. (8.46) on p. 628 of the textbook, we have



$$(\vec{h}_A)_{AB} = (I_A)_{AB}\,\vec{\omega}_{AB}, \tag{1}$$

where $(I_A)_{AB} = \frac{1}{3}m_{AB}R^2$ and $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$. Hence, we have

$$\boxed{(\vec{h}_A)_{AB} = \frac{1}{3}m_{AB}R^2\omega_{AB}\,\hat{k} = (1.72\,\text{kg}\cdot\text{m}^2/\text{s})\,\hat{k}} \tag{2}$$

where we have used the following numerical data $m_{AB} = 2.3\,\text{kg}$, $R = 0.75\,\text{m}$, and $\omega_{AB} = 4\,\text{rad/s}$.

Angular momentum of bar BC about A. At the instant shown bars AB and CD are parallel to one another. Therefore, we can conclude that the instantaneous center of rotation of bar BC is at infinity and consequently bar BC has zero angular velocity (at the instant shown). Let F denote the center of mass of bar BC. Because $\vec{\omega}_{BC} = \vec{0}$, then, using rigid body kinematics, we have $\vec{v}_F = \vec{v}_B = -\omega_{AB}R\,\hat{\imath}$. Then, applying Eq. (8.42) on p. 627 of the textbook, we have

$$(\vec{h}_A)_{BC} = \vec{r}_{F/A} \times m_{BC}\vec{v}_F = \left(\tfrac{1}{2}L\,\hat{\imath} + R\,\hat{\jmath}\right) \times m_{BC}(-\omega_{AB}R\,\hat{\imath}). \tag{3}$$

Carrying out the cross products, we have

$$\boxed{(\vec{h}_A)_{BC} = m_{BC}\omega_{AB}R^2\,\hat{k} = (7.20\,\text{kg}\cdot\text{m}^2/\text{s})\,\hat{k}} \tag{4}$$

where we have used the following numerical data: $m_{BC} = 3.2\,\text{kg}$, $\omega_{AB} = 4\,\text{rad/s}$, and $R = 0.75\,\text{m}$.

Angular momentum of bar CD about D. Bar BC is in a fixed axis rotation about D. Hence, using Eq. (8.46) on p. 628 of the textbook, we have

$$(\vec{h}_D)_{CD} = (I_D)_{CD}\,\vec{\omega}_{CD}, \tag{5}$$

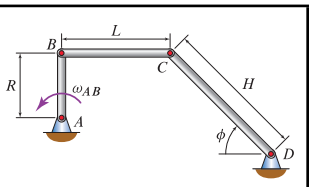
where $(I_D)_{CD} = \frac{1}{3}m_{CD}H^2$, is the mass moment of inertia of bar CD about D, and where, recalling that at the instant shown $\vec{v}_B = \vec{v}_C$, which implies that $\omega_{CD}H = \omega_{AB}R$ so that $\omega_{CD} = (R/H)\omega_{AB}$. In conclusion, we have

$$\boxed{(\vec{h}_D)_{CD} = \tfrac{1}{3}m_{CD}HR\omega_{AB}\,\hat{k} = (7.75\,\text{kg}\cdot\text{m}^2/\text{s})\,\hat{k}} \tag{6}$$

where we have used the following numerical data: $m_{CD} = 5.0\,\text{kg}$, $H = 1.55\,\text{m}$, $R = 0.75\,\text{m}$, and $\omega_{AB} = 4\,\text{rad/s}$.

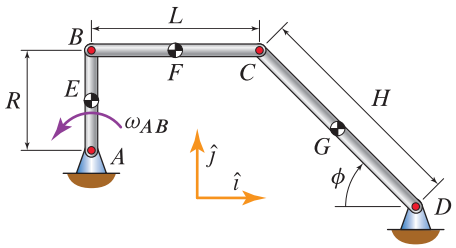
Problem 8.53

The weights of the uniform thin pin-connected bars AB, BC, and CD are $W_{AB} = 4\,\text{lb}$, $W_{BC} = 6.5\,\text{lb}$, and $W_{CD} = 10\,\text{lb}$, respectively. Letting $\phi = 47°$, $R = 2\,\text{ft}$, $L = 3.5\,\text{ft}$, and $H = 4.5\,\text{ft}$, and knowing that bar AB rotates at a constant angular velocity $\omega_{AB} = 4\,\text{rad/s}$, compute the magnitude of the linear momentum of the system at the instant shown.



Solution

The linear momentum of the system is the sum of the liner momenta of the individual components of the system. Hence, letting $\vec{p}$ denote the momentum of the system as a whole, we have



$$\vec{p} = m_{AB}\vec{v}_E + m_{BC}\vec{v}_F + m_{CD}\vec{v}_G = \frac{W_{AB}}{g}\vec{v}_E + \frac{W_{BC}}{g}\vec{v}_F + \frac{m_{CD}}{g}\vec{v}_G, \quad (1)$$

where, referring to the figure on the right, E, F, and G are the mass centers of bars AB, BC, and CD, respectively.

Given the above expression, the solution of the problem is obtained after we derive expressions for the velocities of points E, F, and G.

For the velocity of point E we can write

$$\vec{v}_E = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{E/A}, \quad (2)$$

where

$$\vec{v}_A = \vec{0}, \quad \vec{r}_{E/A} = \tfrac{1}{2} R\,\hat{\jmath}, \quad \text{and} \quad \vec{\omega}_{AB} = \omega_{AB}\,\hat{k}, \quad (3)$$

so that we have

$$\vec{v}_E = -\tfrac{1}{2}\omega_{AB} R\,\hat{\imath}. \quad (4)$$

Next, observing that $\vec{r}_{B/A} = R\,\hat{\jmath}$, we must also have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} = -\omega_{AB} R\,\hat{\imath}. \quad (5)$$

Now, notice that we can compute the velocity of C in the following two ways:

$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} \quad \text{and} \quad \vec{v}_C = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{C/D}, \quad (6)$$

where

$$\vec{\omega}_{BC} = \omega_{BC}\,\hat{k}, \quad \vec{r}_{C/B} = L\,\hat{\imath}, \quad \vec{v}_D = \vec{0}, \quad \vec{\omega}_{CD} = \omega_{CD}\,\hat{k}, \quad \vec{r}_{C/D} = H(-\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}). \quad (7)$$

Substituting the result in Eq. (5) along with Eqs. (7) into Eqs. (6), carrying out the cross products, and enforcing equality between the two expressions for $\vec{v}_C$, we have

$$-R\omega_{AB}\,\hat{\imath} + L\omega_{BC}\,\hat{\jmath} = -H\omega_{CD}\sin\phi\,\hat{\imath} - H\omega_{CD}\cos\phi\,\hat{\jmath}. \quad (8)$$

Equation (8) is a vector equation corresponding to two scalar equations in the two unknowns ω_{BC} and ω_{CD} whose solution is

$$\omega_{BC} = -\frac{R\omega_{AB}\cos\phi}{L\sin\phi} \quad \text{and} \quad \omega_{CD} = \frac{R\omega_{AB}}{H\sin\phi}. \quad (9)$$

We now observe that $\vec{r}_{F/B} = \frac{1}{2}L\,\hat{\imath}$ and that $\vec{r}_{G/D} = -\frac{1}{2}H(-\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})$. Using these expressions, we can then say

$$\vec{v}_F = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{F/B} = -R\omega_{AB}\,\hat{\imath} - \frac{R\omega_{AB}\cos\phi}{2\sin\phi}\,\hat{\jmath}, \tag{10}$$

$$\vec{v}_G = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{G/D} = -\tfrac{1}{2}R\omega_{AB}\,\hat{\imath} - \frac{R\omega_{AB}\cos\phi}{2\sin\phi}\,\hat{\jmath}, \tag{11}$$

where we have used the expressions of $\vec{v}_B$ in Eq. (5), of $\vec{v}_D$ in the third of Eqs. (7), as well as the expressions in Eqs. (9). Substituting Eqs. (4), (10), and (11) into Eq. (1) and simplifying we have

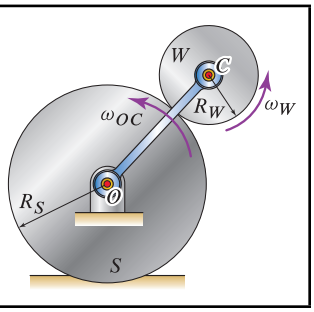
$$\vec{p} = \frac{R\omega_{AB}}{2g}\left[(W_{AB} + 2W_{BC} + W_{CD})\,\hat{\imath} - (W_{BC} + W_{CD})\frac{\cos\phi}{\sin\phi}\,\hat{\jmath}\right] = (-3.354\,\hat{\imath} - 1.911\,\hat{\jmath})\,\text{lb·s}, \tag{12}$$

where we have used the following numerical data: $R = 2\,\text{ft}$, $\omega_{AB} = 4\,\text{rad/s}$, $g = 32.2\,\text{ft/s}^2$, $W_{AB} = 4\,\text{lb}$, $W_{BC} = 6.5\,\text{lb}$, $W_{CD} = 10\,\text{lb}$, and $\phi = 47°$. Finally, computing the magnitude of the above expression, we have

$$\boxed{|\vec{p}| = 3.86\,\text{lb·s}.}$$

Problem 8.54

A uniform disk W of radius $R_W = 7\,\text{mm}$ and mass $m_W = 0.15\,\text{kg}$ is connected to point O via the rotating arm OC. Disk W also rolls without slip over the stationary cylinder S of radius $R_S = 15\,\text{mm}$. Assuming that $\omega_W = 25\,\text{rad/s}$, determine the angular momentum of W about its own center of mass C as well as about point O.



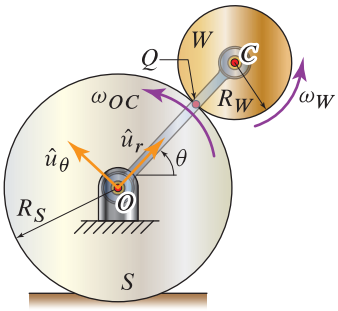
Solution

Recalling that in the component system shown $\hat{k} = \hat{u}_r \times \hat{u}_\theta$, the angular momentum of the the wheel W about its own mass center is

$$(\vec{h}_C) = I_C \omega_W \hat{k} = \tfrac{1}{2} m_W R_W^2 \omega_W \hat{k}, \tag{1}$$

where $I_C = \frac{1}{2} m_W R_W^2$ is the mass moment of inertia of the disk W about its own mass center C. The expression in Eq. (1) can be evaluated to obatin

$$\boxed{(\vec{h}_C)_W = (91.9\times10^{-6}\,\text{kg}\cdot\text{m}^2/\text{s})\,\hat{k} \quad \hat{u}_\theta \angle \hat{u}_r @ \theta,} \tag{2}$$



where we have used the following numerical data: $m_W = 0.15\,\text{kg}$, $R_W = 7\,\text{mm} = 0.007000\,\text{m}$, and $\omega_W = 25\,\text{rad/s}$.

To determine the angular momentum of the wheel W about O, observe that the wheel W rolls without slip on S, with instantaneous center of rotation at Q. This implies that the velocity of the center of W is $\vec{v}_C = \omega_W R_W \hat{u}_\theta$. Consequently, we have

$$\begin{aligned}
(\vec{h}_O)_W &= I_C \omega_W \hat{k} + \vec{r}_{C/O} \times m_W \vec{v}_C \\
&= I_C \omega_W \hat{k} + (R_S + R_W)\hat{u}_r \times (m_W \omega_W R_W)\,\hat{u}_\theta \\
&= [I_C + m_W (R_S + R_W) R_W] \omega_W \hat{k} \\
&= \left[\tfrac{1}{2} m_W R_W^2 + m_W (R_S + R_W) R_W\right] \omega_W \hat{k} \\
&= \tfrac{1}{2} m_W R_W \left(3R_W + 2R_S\right) \omega_W \hat{k},
\end{aligned} \tag{3}$$

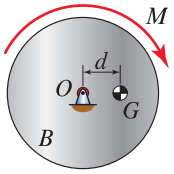
where we have used again the fact that $I_C = \frac{1}{2} m_W R_W^2$. The final expression in Eq. (3) can be evaluated to obtain

$$\boxed{(\vec{h}_O)_W = (669\times10^{-6}\,\text{kg}\cdot\text{m}^2/\text{s})\,\hat{k} \quad \hat{u}_\theta \angle \hat{u}_r @ \theta,}$$

where we have used the following numerical data: $m_W = 0.15\,\text{kg}$, $R_W = 7\,\text{mm} = 0.007000\,\text{m}$, $R_S = 15\,\text{mm} = 0.01500\,\text{m}$, and $\omega_W = 25\,\text{rad/s}$.

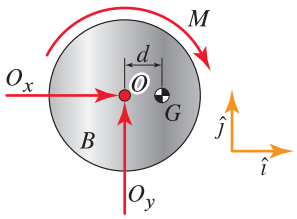
Problem 8.55

An eccentric wheel B weighing 150 lb has its mass center G at a distance $d = 4$ in. from the wheel's center O. The wheel is in the horizontal plane and is spun from rest by applying a constant torque $M = 32$ ft·lb. Determine the wheel's radius of gyration k_G if it takes 2 s to spin up the wheel to 140 rpm. Neglect all possible sources of friction.



Solution

We let the subscripts 1 and 2 denote the time instants at which the wheel is put in motion and when it achieves the angular velocity of 140 rad/s, respectively. Referring to the FBD on the right, we see that the only action contributing to a moment about the fixed point O is the constant moment M. Therefore, observing that M acts clockwise, we can apply the angular impulse–momentum principle in the z direction to obtain



$$I_O\omega_{B1} - \int_{t1}^{t2} M\,dt = I_O\omega_{B2} \quad \Rightarrow \quad I_O\omega_{B1} - M(t_2 - t_1) = I_O\omega_{B2} \tag{1}$$

where I_O is the mass moment of inertia of the wheel about O, $\vec{\omega}_B = \omega_B\,\hat{k}$ is the angular velocity of the wheel, and where we have accounted for the fact that M is constant.

Based on the data provided, for the mass moments of inertia we have

$$I_O = \frac{W_B}{g}k_G^2 + \frac{W_B}{g}d^2, \tag{2}$$

where g is the acceleration due to gravity, W_B is the weight of the wheel, k_G is the radius of gyration of the wheel about the wheel's center of mass G, and d is the distance between G and the center of rotation O.

There are no forces on the FBD for which force laws are needed.

In this problem, both the initial and final angular velocities are given. Hence, observing that the wheel will be rotating in clockwise, we can summarize the kinematics of the problem as follows:

$$\omega_{B1} = 0, \quad \omega_{B2} = -\omega_{\text{final}}, \tag{3}$$

where $\omega_{\text{final}} = 140$ rpm $= 14.66$ rad/s.

Letting $t_1 = 0$ and $t_2 = t_{\text{final}}$, and substituting Eqs. (2) and (3) into the last of Eqs. (1), we have

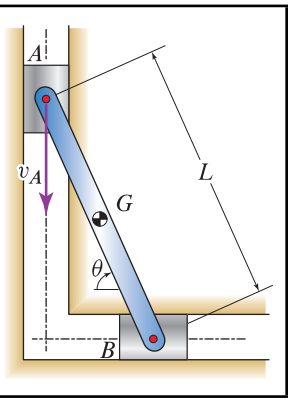
$$-Mt_{\text{final}} = -\frac{W_B}{g}k_G^2 - \omega_{\text{final}} - \frac{W_B}{g}d^2\omega_{\text{final}} \quad \Rightarrow \quad k_G = \sqrt{\frac{Mgt_{\text{final}}}{W_B\omega_{\text{final}}} - d^2}. \tag{4}$$

Recalling that $M = 32$ ft·lb, $g = 32.2$ ft/s^2, $t_{\text{final}} = 2$ s, $W_B = 150$ lb, $\omega_{\text{final}} = 140$ rpm $= 14.66$ rad/s, and $d = 4$ in. $= 0.3333$ ft, we have

$$\boxed{k_G = 0.909 \text{ ft.}}$$

Problem 8.56

The uniform bar AB has length $L = 4.5\,\text{ft}$ and weight $W_{AB} = 14\,\text{lb}$. At the instant shown, $\theta = 67°$ and $v_A = 5.8\,\text{ft/s}$. Determine the magnitude of the linear momentum of AB as well as the angular momentum of AB about its mass center G at the instant shown.

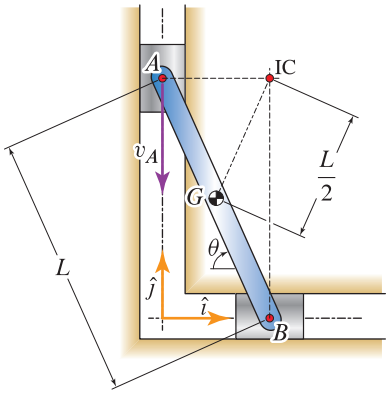


Solution

The linear momentum of the bar is $\vec{p}_{AB} = m_{AB}\vec{v}_G$ where G is the mass center of the bar, which is the midpoint of the bar. The magnitude of the linear momentum of the bar is therefore

$$|\vec{p}_{AB}| = m_{AB}v_G = \frac{W_{AB}}{g}v_G, \tag{1}$$

where v_G is the speed of the center of mass. Using the concept of instantaneous center of rotation, and observing that the distance between IC and G is $L/2$, we must have that $v_G = (L/2)|\omega_{AB}|$, where $|\omega_{AB}|$ is the angular speed of the bar. Using again the concept of instantaneous center of rotation, for point A we have



$$v_A = |\omega_{AB}|L\cos\theta \quad \Rightarrow \quad |\omega_{AB}| = \frac{v_A}{L\cos\theta} \quad \Rightarrow \quad v_G = \frac{v_A}{2\cos\theta}. \tag{2}$$

Hence we have

$$\boxed{|\vec{p}_{AB}| = \frac{W_{AB}v_A}{2g\cos\theta} = 3.23\,\text{lb·s},}$$

where we have used the following numerical data: $W_{AB} = 14\,\text{lb}$, $v_A = 5.8\,\text{ft/s}$, $g = 32.2\,\text{ft/s}^2$, and $\theta = 67°$.

The angular momentum of bar AB about G is given by

$$\vec{h}_G = I_G\vec{\omega}_{AB} = I_G\omega_{AB}\,\hat{k}, \tag{3}$$

where ω_{AB} represents the component of the angular velocity of the bar in the z direction, and where $I_G = \frac{1}{12}m_{AB}L^2 = \frac{1}{12}(W_{AB}/g)L^2$. Since A is moving downward, the the bar is rotating counterclockwise, so that, using the second of Eqs. (2), we have $\omega_A = v_A/(L\cos\theta)$ so that

$$\boxed{\vec{h}_G = \frac{W_{AB}Lv_A}{12g\cos\theta}\,\hat{k} = (2.42\,\text{ft·lb·s})\,\hat{k}}$$

where we have used the following numerical data: $W_{AB} = 14\,\text{lb}$, $L = 4.5\,\text{ft}$, $v_A = 5.8\,\text{ft/s}$, $g = 32.2\,\text{ft/s}^2$, and $\theta = 67°$.

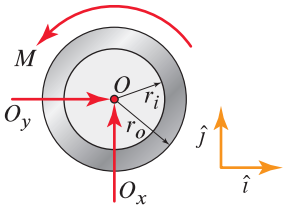
Problem 8.57

The top of the Space Needle in Seattle, Washington, hosts a revolving restaurant that goes through one full revolution every 47 min under the action of a motor with a power output of 1.5 hp. The portion of the restaurant that rotates is a ring-shaped turntable with internal and external radii $r_i = 33.3$ ft and $r_o = 47.3$ ft, respectively, and approximate weight $W = 125$ tons (1 ton $= 2000$ lb). Use the given values of power output and angular speed to estimate the torque M that the engine provides. Then, assuming that the motor can provide a constant torque equal to M, neglecting all friction, and modeling the turntable as a uniform body, determine the time t_s that it takes to spin up the revolving restaurant from rest to its working angular speed.



Solution

We let the subscripts 1 and 2 denote the time instants at which the system starts spinning and when it achieves its working angular velocity, respectively. The figure on the right is a top view of the FBD of the turntable. Choosing as our moment center the fixed point O, which is the center of rotation of the system as well as its center of mass, then the angular impulse–momentum principle gives



$$I_O\omega_{t1} + \int_{t_1}^{t_2} M\ dt = I_O\omega_{t2}, \tag{1}$$

where I_O is the system's mass moment of inertia about O and $\vec{\omega}_t = \omega_t\,\hat{k}$ is the angular velocity of the turntable, and M is the moment applied by the motor.

Modeling the turntable as a uniform body of mass W/g (with W the weight of the turntable and g the acceleration due to gravity) occupying a cicular crown of inner and ourter radii equal to r_i and r_o, respectively, we have

$$I_O = \tfrac{1}{2}(W/g)(r_i^2 + r_o^2) = 12.63\times10^6\ \text{slug}\cdot\text{ft}^2, \tag{2}$$

where we have used the following numerical data: $W = 125\,\text{ton} = 250.0\times10^3$ lb, $g = 32.2\,\text{ft/s}^2$, $r_i = 33.3$ ft, and $r_o = 46.3$ ft. Assuming that the torque provided by the motor is constant, letting P be the power output of the motor at steady state, i.e., for $t \geq t_s$, we must have

$$M = P/\omega_{t2}. \tag{3}$$

As far as the kinematic equations are concerned, observing that the system starts from rest and that we are given the number of revolutions per minute of the turntable, we have

$$\omega_{t1} = 0 \quad \text{and} \quad \omega_{t2} = \frac{1\,\text{rev}}{47\,\text{min}}\frac{2\pi\,\text{rad/rev}}{60\,\text{s/min}} = 0.002228\,\text{rad/s}, \tag{4}$$

Substituting Eq. (3) and the first of Eqs. (4) into Eq. (1), we have

$$(P/\omega_{t2})(t_2 - t_1) = I_O\omega_{t2}, \tag{5}$$

where we have used the fact that P and ω_{t2} are constant. Now, letting $t_s = t_2 - t_1$ and solving the above equation for t_s, we have

$$t_s = \frac{I_O\omega_{t2}^2}{P} \quad \Rightarrow \quad \boxed{t_s = 0.0760\,\text{s},} \tag{6}$$

where we have used the following numerical data: $I_O = 12.63\times10^6\,\text{slug}\cdot\text{ft}^2$ (see Eq. (2)), $\omega_{t2} = 0.002228\,\text{rad/s}$ (see the last of Eqs. (4)), and $P = 1.5\,\text{hp} = 825.0\,\text{ft}\cdot\text{lb/s}$.

Problem 8.58

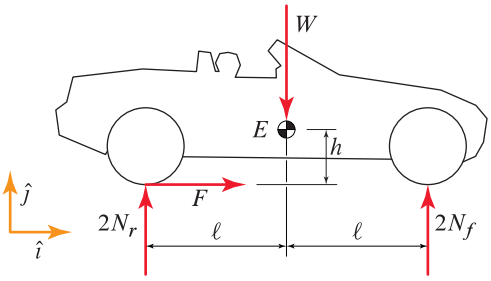
Moving on a straight and horizontal stretch of road, the rear-wheel-drive car shown can go from rest to 60 mph in $\Delta t = 8$ s. The car weighs 2570 lb (the weight includes the wheels). Each wheel has diameter $d = 24.3$ in., mass moment of inertia relative to its own center of mass $I_G = 0.989\,\text{slug}\cdot\text{ft}^2$, and the center of mass of each wheel coincides with its geometric center. Determine the average friction force F_{avg} acting on the car during Δt. In addition, if the wheels roll without slip, for each wheel, determine the average moment M_{avg}, computed relative to the wheel's center, that is applied to the wheel during Δt.



Solution

Applying the impulse-momentum principle to the car in the x direction, we have

$$m(v_{Ex})_1 + \int_{t_1}^{t_2} F_x\,dt = m(v_{Ex})_2, \qquad (1)$$



where m is the mass of the car, E is the center of mass of the car, and F_x is the x component of the total external force acting on the car. Since such a force consists of only the friction force, then, denoting the average friction force by F_{avg} and $t_2 - t_1 = \Delta t$, we have

$$\int_{t_1}^{t_2} F_x\,dt = F_{\text{avg}}\Delta t. \qquad (2)$$

Next, observe that the horizontal components of velocity in this problem are given. In particular, we have

$$(v_{Ex})_1 = 0 \quad \text{and} \quad (v_{Ex})_2 = v_{\text{final}}, \qquad (3)$$

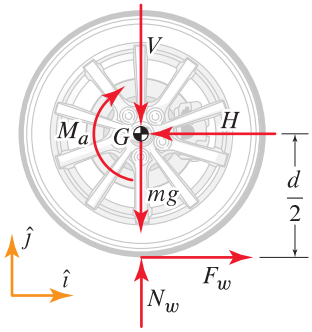
where $v_{\text{final}} = 60\,\text{mph} = 88.00\,\text{ft/s}$. Expressing the mass of the car as $m = W/g$, where W is the weight of the car and g is the acceleration due to gravity, substituting Eqs. (2) and (3) into Eq. (1), we have

$$F_{\text{avg}}\Delta t = \frac{W}{g}v_{\text{final}} \quad \Rightarrow \quad \boxed{\vec{F}_{\text{avg}} = \frac{W v_{\text{final}}}{g\Delta t}\,\hat{\imath} = (878\,\text{lb})\,\hat{\imath}}$$

where we have expressed the result in vector form and where we have used the following numerical data: $W = 2570$ lb, $v_{\text{final}} = 88.00\,\text{ft/s}$, $g = 32.2\,\text{ft/s}^2$, and $\Delta t = 8$ s.

Applying the angular impulse–momentum principle to a wheel in the z direction, and choosing the wheel's center of mass G as moment center, we have

$$(h_{Gz})_1 + \int_{t_1}^{t_2} M_{Gz}\,dt = (h_{Gz})_2, \qquad (4)$$



where h_{Gz} is the z component of the angular momentum of the wheel about G and M_{Gz} is the z component of the resultant moment about G of all of the forces and torques applied to the wheel.

By definition of average moment, we must have

$$M_{\text{avg}}\Delta t = \int_{t_1}^{t_2} M_{Gz}\,dt, \qquad (5)$$

where we have set $t_2 - t_1 = \Delta t$.

Now, recall that the wheel starts from rest. In addition, recall that $h_{Gz} = I_G \omega_w$, where I_G is the wheel's mass moment of inertia about G and $\vec{\omega}_w = \omega_w \hat{k}$ is the angular velocity of the wheel. Because the wheel rolls without slip, we must have

$$\omega_w = -\frac{v_{Gx}}{d/2} \quad \Rightarrow \quad \omega_{w2} = -\frac{2v_{\text{final}}}{d}, \tag{6}$$

where we have enforced the fact that the horizontal component of the velocity of G, namely v_{Gx}, is equal to the horizontal velocity of the car as a whole. Summarizing, we have

$$(h_{Gz})_1 = 0 \quad \text{and} \quad (h_{Gz})_2 = -\frac{2I_G v_{\text{final}}}{d}. \tag{7}$$

Substituting Eqs. (5) and (7) into Eq. (4), we have

$$M_{\text{avg}} \Delta_t = -\frac{2I_G v_{\text{final}}}{d} \quad \Rightarrow \quad \boxed{\vec{M}_{\text{avg}} = -\frac{2I_G v_{\text{final}}}{d\,\Delta t}\,\hat{k} = (-10.7\,\text{lb·ft})\,\hat{k}}$$



where we have expressed the result in vector form, and where we have used the following numerical data: $I_G = 0.989\,\text{slug·ft}^2$, $v_{\text{final}} = 88.00\,\text{ft/s}$, $d = 24.3\,\text{in.} = 2.025\,\text{ft}$, and $\Delta t = 8\,\text{s}$.

Problem 8.59

The rear-wheel-drive car can go from rest to 60 mph in $\Delta t = 8$ s. Assume that the wheels are all identical and that their geometric centers coincide with their mass centers. Let M_{rear} be the average moment applied to one of the rear wheels during Δt and computed relative to the wheel's center. Finally, let M_{front} be the average moment applied to one of the front wheels during Δt and computed relative to the wheel's center. Modeling the wheels as rigid bodies, determine which of the following statements is true and why. (a) $\left|M_{\text{rear}}\right| < \left|M_{\text{front}}\right|$, (b) $\left|M_{\text{rear}}\right| = \left|M_{\text{front}}\right|$, (c) $\left|M_{\text{rear}}\right| > \left|M_{\text{front}}\right|$.



Solution

The answer is (b). The reason for this is that, by the angular impulse-momentum principle the average moment about the mass center of a wheel is dictated only by the mass moment of inertia and the change angular velocity of the wheel. Since the motion and the rotational inertia of each wheel is the same, then the average moment on each wheel must also be the same.

Problem 8.60

The rear-wheel-drive car can go from rest to 60 mph in $\Delta t = 8$ s. Assume that its wheels are identical, with their geometric centers coinciding with their mass centers. Let F_{avg} be the average friction force acting on the system during Δt due to contact with the ground. Modeling the car and the wheels as rigid bodies, does the value of F_{avg} change whether or not we account for the rotational inertia of the wheels? Why?

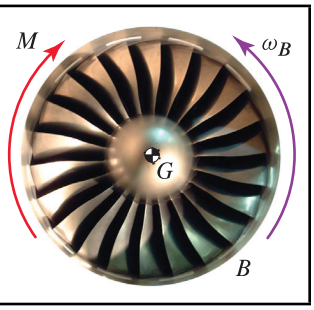


Solution

The answer is no. The reason for this is that the force of friction is dictated by the linear impulse-momentum principle and in this principle the only inertia property that matters is the total mass of the car. The rotational inertia of the wheels does not contribute to this principle.

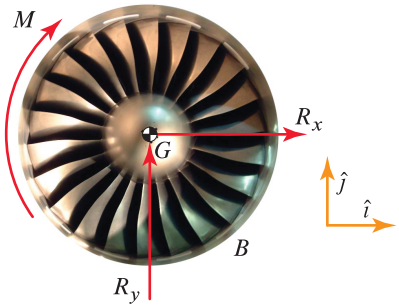
Problem 8.61

A rotor B with center of mass G, weight $W = 3000\,\text{lb}$, and radius of gyration $k_G = 15\,\text{ft}$ is spinning with an angular speed $\omega_B = 1200\,\text{rpm}$ when a braking system is applied to it, providing a time-dependent torque $M = M_0(1 + ct)$, with $M_0 = 3000\,\text{ft·lb}$ and $c = 0.01\,\text{s}^{-1}$. If G is also the geometric center of the rotor and is a fixed point, determine the time t_s that it takes to bring the rotor to a stop.



Solution

We let the subscripts 1 and 2 denote the time at which the braking moment is first applied ($t_1 = 0$) and the stopping time, respectively. Based on the FBD shown, choosing the fixed point G for the application of the angular impulse–momentum principle in the z direction gives



$$I_G\omega_{R1} - \int_0^{t_2} M\,dt = I_G\omega_{R2}, \tag{1}$$

where I_G is the mass moment of inertia of the rotor about G and $\vec{\omega}_R = \omega_R\,\hat{k}$ is the angular velocity of the rotor.

We can express the mass moment of inertia of the rotor as

$$I_G = \frac{W}{g}k_G^2, \tag{2}$$

where g is the acceleration due to gravity. Now recall that

$$M = M_0(1 + ct). \tag{3}$$

Next, we can summarize the kinematic equations for this problem as follows:

$$\omega_{R1} = \omega_B \quad \text{and} \quad \omega_{R2} = 0. \tag{4}$$

Then, substituting Eqs. (2)–(4) into Eq. (1), we obtain

$$\frac{W}{g}k_G^2\omega_B - \int_0^{t_2} M_0(1 + ct)\,dt = 0 \quad \Rightarrow \quad \frac{W}{g}k_G^2\omega_B - \frac{M_0}{2c}(1 + ct)^2\Big|_0^{t_2=t_s} = 0$$

$$\Rightarrow \quad \frac{2cWk_G^2\omega_B}{M_0 g} - (1 + ct)^2\Big|_0^{t_2=t_s} = 0 \quad \Rightarrow \quad \frac{2cWk_G^2\omega_B}{M_0 g} - (1 + ct_s)^2 + 1 = 0$$

$$\Rightarrow \quad t_s = \frac{1}{c}\left(\sqrt{1 + \frac{2cWk_G^2\omega_B}{M_0 g}} - 1\right), \tag{5}$$

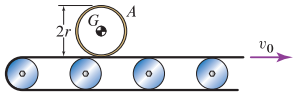
which can be evaluated to obtain

$$\boxed{t_s = 331\,\text{s},}$$

where we have used the following numerical data: $c = 0.01\,\text{s}^{-1}$, $W = 3000\,\text{lb}$, $k_G = 15\,\text{ft}$, $\omega_B = 1200\,\text{rpm} = 125.7\,\text{rad/s}$, $M_0 = 3000\,\text{ft·lb}$, and $g = 32.2\,\text{ft/s}^2$.

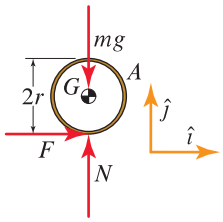
Problem 8.62

A uniform pipe section A of radius r, mass center G, and mass m is gently placed (i.e., with zero velocity) on a conveyor belt moving with a constant speed v_0 to the right. Friction between the belt and pipe causes the pipe to move to the right and eventually to roll without slip. If μ_k is the coefficient of kinetic friction between the pipe and the conveyor belt, find an expression for t_r, the time it takes for A to start rolling without slip. *Hint:* Using the methods of Chapter 7, we can show that the force between the pipe section and the belt is constant.



Solution

Using the FBD shown, letting t_r be the time at which the pipe section starts rolling without slip, the angular impulse-momentum principle for the pipe section written about the center of mass G gives



$$I_G\omega_{p1} + \int_0^{t_r} F\,dt = I_G\omega_{p2}, \tag{1}$$

where $\vec{\omega} = \omega_p\,\hat{k}$ is the angular velocity of the pipe section, I_G is the mass moment of inertia of the pipe section about its center of mass G, and F is the friction force between the pipe section and the conveyor belt. Since the pipe section is initially stationary, the pipe section will initially slip over the conveyor belt. The friction force s therefore given by $F = \mu_k N$, Because the pipe section does not move in the vertical direction, then the force N must balance the pipe's section weight mg, i.e., $N = mg$. Consequently, we have

$$F = \mu_k mgr = \text{constant}, \tag{2}$$

where g is the acceleration due to gravity. Substituting Eq. (2) into Eq. (1), and recalling that $\omega_{p1} = 0$ and that $I_G = mr^2$, we have

$$\mu_k mgrt_r = mr^2\omega_2 \quad \Rightarrow \quad t_r = \frac{r\omega_2}{\mu_k g}. \tag{3}$$

To find ω_2, we sum forces in the x direction to obtain

$$\mu_k mg = ma_{Gx} \quad \Rightarrow \quad a_{Gx} = \mu_k g \quad \Rightarrow \quad v_{Gx2} = \mu_k g t_r, \tag{4}$$

where, to obtain the x component of the velocity of the center of mass at time t_r, we have used the fact that the acceleration of the center of mass is constant between the initial time and time t_r. Finally, since at time t_r the pipe section achieves rolling without slip over a surface that moves with a horizontal velocity to the right equal to v_0, we must also have that

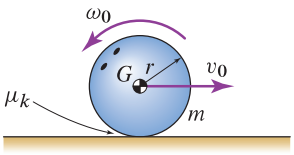
$$v_{Gx2} = v_0 - \omega_2 r \quad \Rightarrow \quad \omega_2 = \frac{v_0 - \mu_k g t_r}{r}. \tag{5}$$

Substituting this expression in the last of Eqs. (3) and solving for t_r, yields

$$\boxed{t_r = \frac{v_0}{2\mu_k g}.}$$

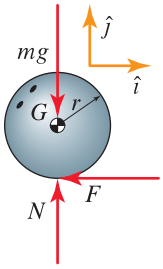
Problem 8.63

A 14 lb bowling ball is thrown onto a lane with a backspin $\omega_0 = 9\,\text{rad/s}$ and forward velocity $v_0 = 18\,\text{mph}$. Point G is both the geometric center and the mass center of the ball. After a few seconds, the ball starts rolling without slip. Let $r = 4.25\,\text{in.}$ and let the radius of gyration of the ball be $k_G = 2.6\,\text{in.}$ If the coefficient of kinetic friction between the ball and the floor is $\mu_k = 0.1$, determine the speed v_f that the ball will achieve when it starts rolling without slip. In addition, determine the time t_r the ball takes to achieve v_f. *Hint:* Using the methods of Chapter 7, we can show that the force between the ball and the floor is constant.



Solution

We will apply the both the linear and the angular impulse–momentum principles to solve this problem. We let the subscripts 1 and 2 denote the time instants at which the ball is thrown onto the lane and at which the ball starts rolling without slip, respectively. Referring to the FBD on the right, applying the linear impulse–momentum principle in the x direction, we have



$$m(v_{Gx})_1 - \int_{t_1}^{t_2} F\,dt = m(v_{Gx})_2, \tag{1}$$

where m is the mass of the ball, G is the center of mass of the ball, $\vec{v}_G$ is the velocity of the ball, and F is the total force acting in the horizontal direction, which, at this point is being treated as a function of time.

Choosing the center of mass as moment center, applying the angular impulse–momentum principle in the z direction, we have

$$I_G\omega_{b1} - \int_{t_1}^{t_2} Fr\,dt = I_G\omega_{b2}, \tag{2}$$

where I_G is the mass moment of inertia of the ball about G, ω_b is the component of the ball's angular velocity in the z direction, and r is the radius of the ball. Based on the given information, for m and I_G we can write

$$m = \frac{W}{g} \quad \text{and} \quad I_G = \frac{W}{g}k_G^2, \tag{3}$$

where W is the ball's weight, g is the acceleration due to gravity, and k_G is the radius of gyration of the ball.

Following the hint of the problem, summing forces in the y direction and observing that the ball does not move in the y direction, we have

$$\sum F_y: \quad N - mg = 0 \quad \text{and} \quad N = mg, \tag{4}$$

where N is the normal reaction between the ball and the lane.

Since the ball slides between t_1 and t_2, for the force laws we have

$$F = \mu_k N = \mu_k mg, \tag{5}$$

where we have accounted for the result in Eq. (4).

As far as the kinematic equations are concerned, observe that the horizontal component of the velocity of the ball at time t_1 is given, along with the initial angular velocity of the ball. In addition, we observe that at time t_2 the ball will still be moving in the positive x direction and so that $(v_{Gx})_2 = v_f$. Finally, we observe

that when the ball starts rolling without slip, the horizontal component of velocity and the angular velocity are no longer independent of one another, i.e., that we must have $(v_{Gx})_2 = -\omega_{b2} r$. Hence, we can summarize the kinematics of the problems as follows:

$$(v_{Gx})_1 = v_0, \quad \omega_{b1} = \omega_0, \quad (v_{Gx})_2 = v_f, \quad \omega_{b2} = -\frac{v_f}{r}. \tag{6}$$

Observe that the result in Eq. (5) implies that F is a constant. Using this fact to carry out the integrals in Eqs. (1) and (2), accounting for Eqs. (3) and (6), and letting $t_1 = 0$ and $t_2 = t_f$, after simplification Eqs. (1) and (2) become

$$v_0 - \mu_k g t_f = v_f, \tag{7}$$

$$k_G^2 \omega_0 - r t_f g \mu_k = -\frac{k_G^2 v_f}{r}. \tag{8}$$

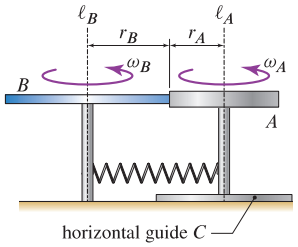
The above two equations form a system in the two unknowns v_f and t_f whose solution is

$$\boxed{v_f = \frac{r(r v_0 - k_G^2 \omega_0)}{k_G^2 + r^2} = 18.3\,\text{ft/s} \quad \text{and} \quad t_f = \frac{k_G^2 (v_0 + \omega_0 r)}{\mu_k g (k_G^2 + r^2)} = 2.50\,\text{s},}$$

where we have used the following numerical data: $r = 4.25\,\text{in.} = 0.3542\,\text{ft}$, $v_0 = 18\,\text{mph} = 26.40\,\text{ft/s}$, $k_G = 2.6\,\text{in.} = 0.2167\,\text{ft}$, $\omega_0 = 9\,\text{rad/s}$, $\mu_k = 0.1$, and $g = 32.2\,\text{ft/s}^2$.

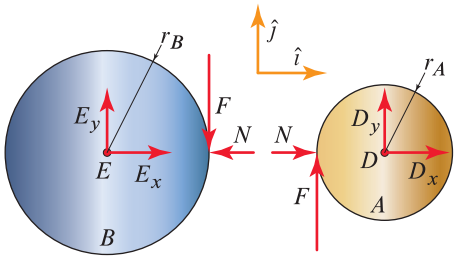
Problem 8.64

The uniform disk A, of mass $m_A = 1.2\,\text{kg}$ and radius $r_A = 0.25\,\text{m}$, is mounted on a vertical shaft that can translate along the horizontal guide C. The uniform disk B, of mass $m_B = 0.85\,\text{kg}$ and radius $r_B = 0.38\,\text{m}$, is mounted on a fixed vertical shaft. Both disks A and B can rotate about their own axes, namely, ℓ_A and ℓ_B, respectively. Disk A is initially spun with $\omega_A = 1000\,\text{rpm}$ and then brought into contact with B, which is initially stationary. The contact is maintained via a spring, and due to friction between A and B, disk B starts spinning and eventually A and B will stop slipping *relative to one another*. Neglecting any friction except at the contact between the two disks, determine the angular velocities of A and B when slipping stops.



Solution

A top view of the FBDs of the two disks after they come in contact is shown to the right. Points D and E are the centers of disks A and B, respectively. Points D and E do not move once the disks are in contact. Applying the angular impulse-momentum principle to each disk by choosing D as moment center for A and E as moment center for B, we have



$$I_D\omega_{A1} - \int_{t_1}^{t_2} F r_A\,dt = I_D\omega_{A2}, \tag{1}$$

$$I_E\omega_{B1} - \int_{t_1}^{t_2} F r_B\,dt = I_E\omega_{B2}, \tag{2}$$

where I_D and I_E are the mass moment of inertia of A and B, respectively, i.e.,

$$I_D = \tfrac{1}{2}m_A r_A^2 \quad \text{and} \quad I_E = \tfrac{1}{2}m_B r_B^2. \tag{3}$$

Since r_A and r_B are constant, and recalling that $\omega_{B1} = 0$, Eqs. (1) and (2) can be rewritten as

$$I_D\omega_{A1} - r_A\int_{t_1}^{t_2} F\,dt = I_D\omega_{A2}, \quad \text{and} \quad -r_B\int_{t_1}^{t_2} F\,dt = I_E\omega_{B2},$$

$$\Rightarrow \quad I_D\omega_{A1} + \frac{r_A}{r_B}I_E\omega_{B2} = I_D\omega_{A1}, \tag{4}$$

where we have obtained the last equation by eliminating the term $\int_{t_1}^{t_2} F\,dt$ from the first two equations. The last of Eqs. (4) and the fact that when rolling without slip is achieved we must have $-\omega_{A2}r_A = \omega_{B2}r_B$, give two equations in the two unknowns ω_{A2} and ω_{B2} whose solution is

$$\omega_{A2} = \frac{m_A}{m_A + m_B}\omega_{A1} \quad \text{and} \quad \omega_{B2} = -\frac{m_A r_A}{(m_A + m_B)r_B}\omega_{A1}, \tag{5}$$

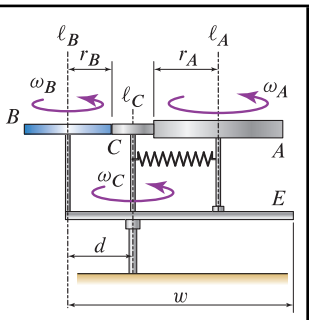
where we have also used the expressions for I_D and I_E in Eqs. (3). The results in Eqs. (5) can be evaluated to obtain

$$\boxed{\omega_A\big|_{\text{end of slip}} = 61.3\,\text{rad/s} \quad \text{and} \quad \omega_B\big|_{\text{end of slip}} = -40.3\,\text{rad/s},}$$

where we have used the following numerical data: $m_A = 1.2\,\text{kg}$, $m_B = 0.85\,\text{kg}$, $\omega_{A1} = 1000\,\text{rpm} = 104.7\,\text{rad/s}$, $r_A = 0.25\,\text{m}$, and $r_B = 0.38\,\text{m}$.

Problem 8.65

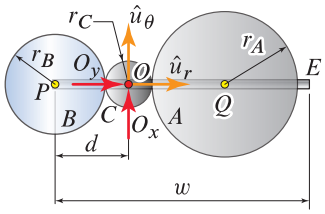
The uniform disk A, of mass $m_A = 1.2\,\text{kg}$ and radius $r_A = 0.25\,\text{m}$, is mounted on a vertical shaft that can translate along the horizontal arm E. The uniform disk B, of mass $m_B = 0.85\,\text{kg}$ and radius $r_B = 0.38\,\text{m}$, is mounted on a vertical shaft that is rigidly attached to arm E. Disk A can rotate about axis ℓ_A, disk B can rotate about axis ℓ_B, and the arm E, along with disk C, can rotate about the fixed axis ℓ_C. Disk C has negligible mass and is rigidly attached to E so that they rotate together. While keeping both B and C stationary, disk A is spun to $\omega_A = 1200\,\text{rpm}$. Disk A is then brought in contact with disk C (contact is maintained via a spring), and B and C (and the arm E) are then allowed to freely rotate. Due to friction between A and C, disks C (and arm E) and B start spinning. Eventually A and C stop slipping relative to one another. Disk B always rotates without slip over C. Let $d = 0.27\,\text{m}$ and $w = 0.95\,\text{m}$. If the only elements of the system that have mass are A and B, and if all friction in the system can be neglected except for that between A and C and between C and B, determine the angular speeds of A and C when they stop slipping relative to one another.



Solution

Note: The problem statement contains an error: the value of the quantity r_B is incorrect. In this solution we will use the following value: $r_B = 0.18\,\text{m}$.

We let the subscripts 1 and 2 denote the time instants at which the wheels are first brought into contact (and the system starts spinning) and when slip stops between the wheels. The figure above is a top view of the FBD of the system. Because the system is freely rotating, choosing as our moment center the fixed point O, we see that there are no external moments acting on the system and therefore the total angular momentum of the system about O must be conserved, i.e.,



$$\vec{h}_{O1} = \vec{h}_{O2}, \tag{1}$$

where $\vec{h}_O$ is the total angular momentum of the system about O.

We now proceed to compute the angular momentum of the system. We start with writing $\vec{h}_O$. Recalling that we are neglecting the mass of both disk C and the arm E, we have

$$\vec{h}_O = (\vec{h}_O)_A + (\vec{h}_O)_B, \tag{2}$$

where,

$$(\vec{h}_O)_A = I_Q\vec{\omega}_A + \vec{r}_{Q/O} \times m_A\vec{v}_Q \quad \text{and} \quad (\vec{h}_O)_B = I_P\vec{\omega}_B + \vec{r}_{P/O} \times m_B\vec{v}_P, \tag{3}$$

where I_Q is the mass moment of inertia of disk A about its mass center Q, and I_P is the mass moment of inertia of B about its mass center P:

$$I_Q = \tfrac{1}{2}m_A r_A^2 = 0.03750\,\text{kg}\cdot\text{m}^2 \quad \text{and} \quad I_P = \tfrac{1}{2}m_B r_B^2 = 0.01377\,\text{kg}\cdot\text{m}^2, \tag{4}$$

where we have used the following numerical data: $m_A = 1.2\,\text{kg}$, $r_A = 0.025\,\text{m}$, $m_B = 0.85\,\text{kg}$, and $r_B = 0.18\,\text{m}$. Going back to the solution of the problem, from Eqs. (3) we see that the key to the solution is expressing all the velocity terms at times t_1 and t_2.

At time t_1 B is at rest and A is spinning with $\vec{v}_{Q1} = \vec{0}$ (i.e., the arm E on which A is mounted is not rotating). Therefore, referring to Eqs. (3), at time t_1 we have

$$(\vec{h}_{O1})_A = I_Q \omega_{A1}\,\hat{k} \quad \text{and} \quad (\vec{h}_{O1})_B = \vec{0}, \tag{5}$$

where we have set $\vec{\omega}_A = \omega_{A1}\,\hat{k}$, and where $\omega_{A1} = 1200\,\text{rpm} = 125.7\,\text{rad/s}$.

We now observe that points P and Q move with the arm E. Recalling that E forms a single rigid body with disk C and that the center of rotation of such a body is the fixed point O, so that $\vec{\omega}_E = \vec{\omega}_C = \omega_C\,\hat{k}$, using rigid body kinematics at time t_2 we have

$$\vec{\omega}_{A2} = \omega_{A2}\,\hat{k}, \qquad \vec{v}_{Q2} = \vec{\omega}_{C2} \times \vec{r}_{Q/O} = \omega_{C2}\,\hat{k} \times (r_A + r_C)\,\hat{u}_r = \omega_{C2}(r_A + r_C)\,\hat{u}_\theta \tag{6}$$

$$\vec{\omega}_{B2} = \omega_{B2}\,\hat{k} \qquad \vec{v}_{P2} = \vec{\omega}_{C2} \times \vec{r}_{P/O} = \omega_{C2}\,\hat{k} \times [-(r_B + r_C)]\,\hat{u}_r = -\omega_{C2}(r_B + r_C)\,\hat{u}_\theta, \tag{7}$$

where we have used the fact that

$$\vec{r}_{Q/O} = (r_A + r_C)\,\hat{u}_r \quad \text{and} \quad \vec{r}_{P/O} = -(r_B + r_C)\,\hat{u}_r. \tag{8}$$

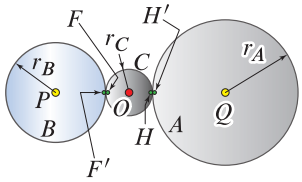
Substituting Eqs. (6) into Eqs. (3), and using Eqs. (8), at time t_2 we have

$$(\vec{h}_{O2})_A = \big[I_Q \omega_{A2} + m_A(r_A + r_C)^2 \omega_{C2}\big]\hat{k} \quad \text{and} \quad (\vec{h}_{O2})_B = \big[I_P \omega_{B2} + m_B(r_B + r_C)^2 \omega_{C2}\big]\hat{k}, \tag{9}$$

Then, using Eqs. (2), (5), and (9), Eq. (1) can be rewritten as

$$I_Q \omega_{A1} = I_Q \omega_{A2} + I_P \omega_{B2} + \big[m_A(r_A + r_C)^2 + m_B(r_B + r_C)^2\big]\omega_{C2}, \tag{10}$$

where we have written only the z component of the equation since it is the only nonzero component of the corresponding vector equation.



Observe that Eq. (10) is a single scalar equation in the three unknowns ω_{A2}, ω_{B2}, and ω_{C2}. To find the two additional equations we need, We recall that at t_2 the disks A and B no longer slip relative to C. Referring to the figure on the right, these two rolling without slip conditions imply $\vec{v}_{H2} = \vec{v}_{H'2}$ and $\vec{v}_{F2} = \vec{v}_{F'2}$, where points H and H' belong to C and A, respectively, and points F and F' belong to C and B respectively.

Applying rigid body kinematics, at time t_2 we have

$$\vec{v}_{H2} = \vec{v}_{H'2} \;\Rightarrow\; \vec{\omega}_{C2} \times \vec{r}_{H/O} = \vec{v}_{Q2} + \vec{\omega}_{A2} \times \vec{r}_{H'/Q} \;\Rightarrow\; \omega_{C2} r_C = (r_A + r_C)\omega_{C2} - r_A \omega_{A2}, \tag{11}$$

$$\vec{v}_{F2} = \vec{v}_{F'2} \;\Rightarrow\; \vec{\omega}_{C2} \times \vec{r}_{F/O} = \vec{v}_{P2} + \vec{\omega}_{B2} \times \vec{r}_{F'/P} \;\Rightarrow\; -\omega_{C2} r_C = -(r_B + r_C)\omega_{C2} + r_B \omega_{B2}, \tag{12}$$

where we have used Eqs. (6) and (7) and the fact that the only nonzero component of Eqs. (11) and (11) is the θ component.

We now observe that Eq. (10), (11), and (12) form a system of three equations in the three unknowns ω_{A2}, ω_{B2}, and ω_{C2}, whose solution is

$$\omega_{A2} = \omega_{B2} = \omega_{C2} = \frac{I_Q \omega_{A1}}{I_P + I_Q + m_A(r_A + r_C)^2 + m_B(r_B + r_C)^2}. \tag{13}$$

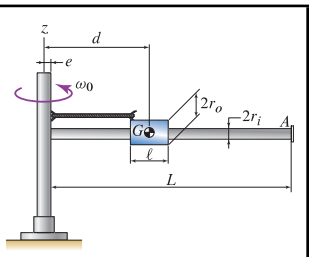
Therefore, the problem's answer is

$$\boxed{|\omega_A|_{\text{after slip stops}} = |\omega_B|_{\text{after slip stops}} = 18.7\,\text{rad/s},}$$

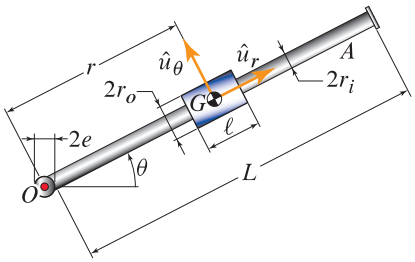
where we have used the following numerical data: $I_Q = 0.03750\,\text{kg}\cdot\text{m}^2$ (see Eqs. (4)), $\omega_{A1} = 1200\,\text{rpm} = 125.7\,\text{rad/s}$, $I_P = 0.01377\,\text{kg}\cdot\text{m}^2$ (see Eqs. (4)), $m_A = 1.2\,\text{kg}$, $r_A = 0.025\,\text{m}$, $r_B = 0.18\,\text{m}$, and $r_C = d - r_B = 0.09000\,\text{m}$ (given that $d = 0.27\,\text{m}$).

Problem 8.66

An 0.8 lb collar with center of mass at G and a uniform cylindrical horizontal arm A of length $L = 1$ ft, radius $r_i = 0.022$ ft, and weight $W_A = 1.5$ lb are rotating as shown with $\omega_0 = 1.5$ rad/s while the collar's mass center is at a distance $d = 0.44$ ft from the z axis. The vertical shaft has radius $e = 0.03$ ft and negligible mass. After the cord restraining the collar is cut, the collar slides with no friction relative to the arm. Assuming that no external forces and moments are applied to the system, determine the collar's impact speed with the end of A if (a) the collar is modeled as a particle coinciding with its own mass center (in this case neglect the collar's dimensions), and (b) the collar is modeled as a uniform hollow cylinder with length $\ell = 0.15$ ft, inner radius r_i, and outer radius $r_o = 0.048$ ft.



Solution



Let the subscripts 1 and 2 denote the time instants at which the collar is released and when when point G is at a generic distance r from the z axis, respectively. The figure to the right is a top view of the FBD of the system. The impact speed to be determined is the magnitude of the radial velocity of G (the mass center of the collar), i.e., $\dot{r}$. We first determine $\dot{\theta}$ using the angular impulse–momentum principle. Then we combine this result with the application of Newton's second law to the collar to obtain $\dot{r}$. Observe that the external moment about the z axis, identified by point O, is zero and therefore the angular momentum of the system about the z axis is conserved. Consequently, denoting the angular momentum of the system about O by $\vec{h}_O = I_O \omega_A \hat{k}$, we have

$$I_{O1}\omega_{A1} = I_{O2}\omega_{A2}, \tag{1}$$

where I_O is the system's mass moment of inertia about O and $\vec{\omega}_A = \omega_A \hat{k}$ is the angular velocity of the system (i.e., the common angular velocity of both the arm and the collar).

The quantity I_O varies as the collar slides down the arm and it consists of a contribution fue to the arm and a contribution do to the collar. The mass moment of inertia of just the arm A about O is

$$(I_O)_A = \tfrac{1}{12} m_A (3r_i^2 + L^2) + m_A \left(\tfrac{1}{2}L + e\right)^2 = 0.01697 \text{ slug}\cdot\text{ft}^2, \tag{2}$$

where we have used the following numerical data: $m_A = (1.5 \text{ lb})/(32.2 \text{ ft/s}^2) = 0.04658$ slug, $r_i = 0.022$ ft, $L = 1$ ft, and $e = 0.03$ ft. Observe that the term $\frac{1}{12} m_A (3r_i^2 + L^2)$ is the mass moment of inertia of the arm about its own mass center, the term $m_A \left(\frac{1}{2}L + e\right)^2$ is needed as per the parallel axis theorem and consist of the mass of the arm times the square of the distance between the arm's center of mass and the spin axis.

The mass moment of inertia of the collar about its own center of mass is

$$I_G = \begin{cases} 0, & \text{when the collar is modeled as a particle,} \\ \frac{m_C}{12}[3(r_i^2 + r_o^2) + \ell^2)] = 63.89\times10^{-6} \text{ slug}\cdot\text{ft}, & \text{when the collar is modeled as a rigid body,} \end{cases} \tag{3}$$

where we have used the following numerical data: $m_C = (0.8 \text{ lb})/(32.2 \text{ ft/s}^2) = 0.02484$ slug, $r_i = 0.022$ ft, $r_o = 0.048$ ft, and $\ell = 0.15$ ft.

Hence, overall, using the above definitions along with the parallel axis theorem applied to the collar, the total overall mass moment of inertia for the system about O is

$$I_O = (I_O)_A + I_G = I^* + m_C r^2, \tag{4}$$

where

$$I^* = (I_O)_A + I_G = \begin{cases} 0.01697\,\text{slug}\cdot\text{ft}^2 & \text{for the collar as a particle,} \\ 0.01703\,\text{slug}\cdot\text{ft}^2 & \text{for the collar as a rigid body.} \end{cases} \tag{5}$$

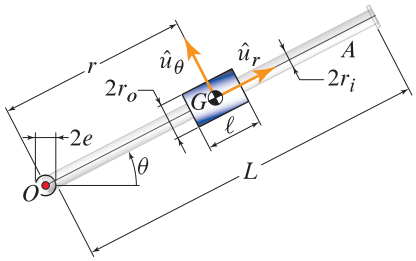
Going back to the statement of conservation of angular momentum, observing that $\omega_1 = \omega_0$ and letting $\omega_2 = \dot{\theta}$, and using the definition in Eq. (4), Eq. (1) gives

$$\dot{\theta} = \frac{I^* + m_C d^2}{I^* + m_C r^2}\omega_0, \tag{6}$$

where we have set $r_1 = d$.

Referring to the FBD on the right, we observe that no force acts on the collar in the r direction so that

$$\sum F_r = m_C a_{Gr}: \quad 0 = m_C(\ddot{r} - r\dot{\theta}^2) \quad \Rightarrow \quad \ddot{r} = r\dot{\theta}^2. \tag{7}$$



Recalling that $\ddot{r} = \dot{r}(d\dot{r}/dr)$ and using Eq. (6) we then have

$$\dot{r}\,d\dot{r} = r\frac{(I^* + m_C d^2)^2}{(I^* + m_C r^2)^2}\omega_0^2\,dr$$

$$\Rightarrow \quad \int_{\dot{r}_1}^{\dot{r}} \dot{r}\,d\dot{r} = \int_d^r r\frac{(I^* + m_C d^2)^2}{(I^* + m_C r^2)^2}\omega_0^2\,dr. \tag{8}$$

Recalling that $\dot{r}_1 = 0$ and $r_1 = d$, the above expression yields

$$\tfrac{1}{2}\dot{r}^2 = -\frac{(I^* + m_C d^2)^2\omega_0^2}{2m_C}\left.\frac{1}{(I^* + m_C r^2)}\right|_d^r = \frac{(I^* + m_C d^2)^2\omega_0^2}{2m_C}\left(\frac{1}{I^* + m_C d^2} - \frac{1}{I^* + m_C r^2}\right). \tag{9}$$

Solving for $\dot{r}$ and simplifying, we obtain

$$\dot{r} = \omega_0\sqrt{\frac{I^* + m_C d^2}{I^* + m_C r^2}(r^2 - d^2)}. \tag{10}$$

Answer to part (a). If the collar is modeled as a particle, then $I^* = 0.01697\,\text{slug}\cdot\text{ft}^2$ (see Eq. (5)) and the impact with the end of the arm occurs for $r = L + e = 1.030\,\text{ft}$ (given that $L = 1\,\text{ft}$ and $e = 0.03\,\text{ft}$). Using these considerations and recalling that $\dot{r}$ does represent the impact speed, Eq. (10) yields

Collar modeled as a particle: $v_{\text{impact}} = 0.990\,\text{ft/s}$,

where we also have used the following numerical data: $\omega_0 = 1.5\,\text{rad/s}$, $m_C = (0.8\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.02484\,\text{slug}$, and $d = 0.44\,\text{ft}$.

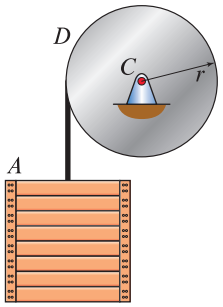
Answer to part (b). If the collar is modeled as a rigid body, then $I^* = 0.01703\,\text{slug}\cdot\text{ft}^2$ (see Eq. (5)) and the impact with the end of the arm occurs for $r = L + e - \frac{1}{2}\ell = 0.9550\,\text{ft}$ (given that $L = 1\,\text{ft}$, $e = 0.03\,\text{ft}$, and $\ell = 0.15\,\text{ft}$). Using these considerations along with Eqs. (3)–(4), and recalling that $\dot{r}$ does represent the impact speed, Eq. (10) yields

Collar modeled as a rigid body: $v_{\text{impact}} = 0.943\,\text{ft/s}$,

where we have again used the following numerical data: $\omega_0 = 1.5\,\text{rad/s}$, $m_C = (0.8\,\text{lb})/(32.2\,\text{ft/s}^2) = 0.02484\,\text{slug}$, and $d = 0.44\,\text{ft}$.

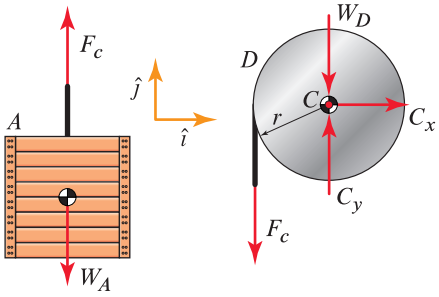
Problem 8.67

A crate A with weight $W_A = 250\,\text{lb}$ is hanging from a rope wound around a uniform drum D of radius $r = 1.2\,\text{ft}$, weight $W_D = 125\,\text{lb}$, and center C. The system is initially at rest when the restraining system holding the drum stationary fails, thus causing the drum to rotate, the rope to unwind, and, consequently, the crate to fall. Assuming that the rope does not stretch or slip relative to the drum and neglecting the inertia of the rope, determine the speed of the crate 1.5 s after the system starts to move.



Solution

We let the subscripts 1 and 2 denote the time instants at which the system is released and 1.5 s after release, respectively. Referring to the FBDs on the right, we apply the linear impulse–momentum principle to the crate along the vertical direction and the angular impulse–momentum principle to the drum. Specifically, we have



$$m_A(v_{Ay})_1 + \int_{t_1}^{t_2} (F_c - W_A)\,dt = m_A(v_{Ay})_2, \qquad (1)$$

where m_A is the mass of the crate, $\vec{v}_A = v_{Ay}\,\hat{\jmath}$ is the velocity of the crate, F_c is the tension in the cord assumed to be some general function of time, and W_A is the weight of the crate. For the drum, choosing the fixed point C as the moment center, we have

$$(h_{Cz})_1 + \int_{t_1}^{t_2} rF_c\,dt = (h_{Cz})_2, \qquad (2)$$

where $\vec{h}_C = h_{Cz}\,\hat{k}$ is the angular momentum of the drum about C, C coincides with the center of mass of the drum, and r is the radius of the drum. Since C is the mass center of the drum, we can express the angular momenta of the drum as follows:

$$(h_{Cz})_1 = I_C\omega_{D1} \quad \text{and} \quad (h_{Cz})_2 = I_C\omega_{D2}, \qquad (3)$$

where $\vec{\omega}_D = \omega_D\,\hat{k}$ is the angular velocity of the drum, and where I_C is the mass moment of the drum about C.

Based on the information provided in the problem statement, we can express the mass of the crate and the mass moment of inertia of the drum as follows:

$$m_A = \frac{W_A}{g} \quad \text{and} \quad I_C = \frac{1}{2}\frac{W_D}{g}r^2, \qquad (4)$$

where g is the acceleration due to gravity and W_D is the weight of the drum. Next, because the cord does not slip relative to the drum, we observe that we must have $v_{Ay} = -\omega_D r$. Therefore, recalling that the system is released from rest, we can summarize the kinematic equations for the problem as follows:

$$(v_{Ay})_1 = 0, \quad \omega_{D1} = 0, \quad \text{and} \quad \omega_{D2} = -\frac{(v_{Ay})_2}{r}. \qquad (5)$$

Letting $t_1 = 0$ and $t_2 = 1.5\,\text{s}$, substituting the first of Eqs. (4) and the first of Eqs. (5) into Eq. (1), and rearranging terms, we have

$$\int_0^{t_2} F_c\,dt = W_A t_2 + \frac{W_A}{g}(v_{Ay})_2. \tag{6}$$

Substituting the last of Eqs. (4) and the last two of Eqs. (5) into Eqs. (3), we obtain

$$(h_{Cz})_1 = 0 \quad \text{and} \quad (h_{Cz})_2 = -\frac{1}{2}\frac{W_D}{g} r (v_{Ay})_2. \tag{7}$$

Then substituting Eqs. (7) into Eq. (2), again letting $t_1 = 0$ and $t_2 = 1.5\,\text{s}$, and simplifying, we have

$$\int_0^{t_2} F_c\,dt = -\frac{1}{2}\frac{W_D}{g}(v_{Ay})_2. \tag{8}$$

Eliminating the integral from Eqs. (6) and (8) we have

$$-\frac{1}{2}\frac{W_D}{g}(v_{Ay})_2 = W_A t_2 + \frac{W_A}{g}(v_{Ay})_2, \tag{9}$$

which is an equation in the single unknown $(v_{Ay})_2$ whose solution is

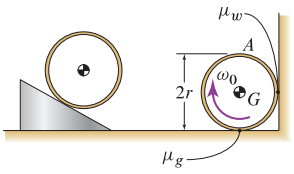
$$(v_{Ay})_2 = -\frac{2g t_2 W_A}{W_D + 2W_A} = -38.64\,\text{ft/s}, \tag{10}$$

where we have used the following numerical data: $g = 32.2\,\text{ft/s}^2$, $t_2 = 1.5\,\text{s}$, $W_A = 250\,\text{lb}$, and $W_D = 125\,\text{lb}$. Finally, taking the absolute value of the result here above and expressing the final answer to three significant figures, we have

$$\boxed{v_A\big|_{\text{after } 1.5\,\text{s}} = 38.6\,\text{ft/s}.}$$

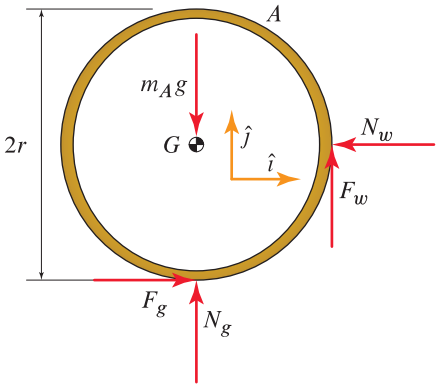
Problem 8.68

Some pipe sections of radius r and mass m are being unloaded and placed in a row against a wall. The first of these pipe sections, A, is made to roll without slipping into a corner with an angular velocity ω_0 as shown. Upon touching the wall, A does not rebound but slips against the ground and against the wall. Modeling A as a uniform thin ring with center at G and letting μ_g and μ_w be the coefficients of kinetic friction of the contacts between A and the ground and between A and the wall, respectively, determine an expression for the angular velocity of A as a function of time from the moment A touches the wall until it stops. *Hint:* Using the methods learned in Chapter 7, we can show that the friction forces at the ground and at the wall are constant.



Solution

Using the FBD shown we have



$$\sum F_x: \quad F_g - N_w = m_A a_{Gx}, \tag{1}$$

$$\sum F_y: \quad N_g + F_w - m_A g = m_A a_{Gy}, \tag{2}$$

where F_g and N_g are the friction and normal forces between the pipe and the ground, whereas F_w and N_w are the friction and normal forces between the pipe and the wall. Since $a_{Gx} = 0$ and $a_{Gy} = 0$, and since $F_g = \mu_g N_g$ and $F_w = \mu_w N_w$, we can solve Eqs. (1) and (2) for F_g and F_w to obtain

$$F_g = \frac{\mu_g}{1 + \mu_g \mu_w} m_A g \quad \text{and} \quad F_w = \frac{\mu_g \mu_w}{1 + \mu_g \mu_w} m_A g. \tag{3}$$

Next, writing the angular impulse–momentum principle for the pipe about G, we have

$$(h_{Gz})_1 + \int_0^t (F_g + F_w) r \, dt = (h_{Gz})_2, \tag{4}$$

where

$$(h_{Gz})_1 = -I_G \omega_0 \quad \text{and} \quad (h_{Gz})_2 = I_G \omega(t), \tag{5}$$

where $I_G = m_A r^2$ and $\vec{\omega}(t) = \omega(t)\,\hat{k}$ is the angular velocity of the pipe. Substituting Eqs. (3) and (5) into Eq. (4), and using the expression for I_G, we have

$$-m_A r^2 \omega_0 + \frac{\mu_g (1 + \mu_w) m_A g r}{1 + \mu_g \mu_w} t = m_A r^2 \omega(t), \tag{6}$$

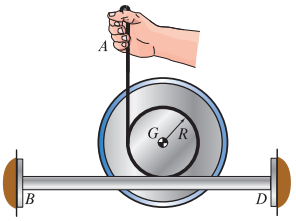
which can be solved for $\omega(t)$ to obtain

$$\boxed{\vec{\omega}(t) = \left[\frac{\mu_g (1 + \mu_w) g}{(1 + \mu_g \mu_w) r} t - \omega_0 \right] \hat{k}}$$

where we have expressed the final answer in vector form.

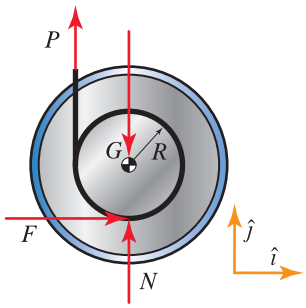
Problem 8.69

A cord, which is wrapped around the inner radius of the spool of mass $m = 35\,\text{kg}$, is pulled vertically at A by a constant force $P = 120\,\text{N}$ (the cord is pulled in such a way that it remains vertical), causing the spool to roll over the horizontal bar BD. The inner radius of the spool is $R = 0.3\,\text{m}$, and the center of mass of the spool is at G, which also coincides with the geometric center of the spool. The spool's radius of gyration is $k_G = 0.18\,\text{m}$. Assuming that the spool starts from rest, that the cord's inertia and extensibility can be neglected, and that the spool rolls without slip, determine the speed of the spool's center 3 s after the application of the force. In addition, determine the minimum static friction coefficient for rolling without slip to be maintained during the time interval in question.



Solution

We let the subscripts 1 and 2 denote the initial time at which the force is applied ($t_1 = 0$) and the instant 3 seconds after the force is applied begins ($t_2 = 3\,\text{s}$), respectively. Based on the FBD shown, the application of the linear impulse–mometum principle in the x direction gives



$$m(v_{Gx})_1 + \int_0^{t_2} F\,dt = m(v_{Gx})_2, \tag{1}$$

where $\vec{v}_G = v_{Gx}\,\hat{\imath}$ is the velocity of the center of mass G of the spool, and where the friction force F is being treated as a function of time.

The application of the angular impulse–momentum principle about G in the z direction yields

$$I_G\omega_{S1} + \int_0^{t_2} (FR - PR)\,dt = I_G\omega_{S2}, \tag{2}$$

where I_G is the mass moment of inertia of the spool about G and $\vec{\omega}_S = \omega_S\,\hat{k}$ is the angular velocity of the spool. We can express the mass moment of inertia of the spool as

$$I_G = mk_G^2. \tag{3}$$

Recalling that starts from rest and that it rolls without slip, we can summarize the kinematic equations of the problem as follows:

$$(v_{Gx})_1 = 0, \quad \omega_{S1} = 0, \quad \omega_{S2} = -\frac{(v_{Gx})_2}{R}. \tag{4}$$

Substituting the first of Eqs. (4) into Eq. (1), we have

$$\int_0^{t_2} F\,dt = m(v_{Gx})_2. \tag{5}$$

Substituting Eq. (3) and the last two of Eqs. (4) into Eq. (2) and recalling that P is constant, we have

$$R\int_0^{t_2} F\,dt - PRt_2 = -mk_G^2\frac{(v_{Gx})_2}{R}. \tag{6}$$

Substituting Eq. (5) into Eq. (6) and solving for $(v_{Gx})_2$, we have

$$(v_{Gx})_2 = \frac{PR^2 t_2}{m(R^2 + k_G^2)}. \tag{7}$$

Observing that the speed of the spool is $v_G = |v_{Gx}|$, evaluating the result in Eq. (7), we have

$$\boxed{v_G\big|_{t=3\,\mathrm{s}} = 7.56\,\mathrm{m/s},}$$

where we have used the following numerical data: $P = 120\,\mathrm{N}$, $R = 0.3\,\mathrm{m}$, $t_2 = 3\,\mathrm{s}$, $m = 35\,\mathrm{kg}$, and $k_G = 0.18\,\mathrm{m}$.

Since the spool does not move in the y direction, the equilibrium of the forces in the y direction implies that $N + P = mg$, which implies that $N = mg - P$. Letting μ_s denote the coefficient of static friction between the spool and the rail, for the spool to roll without slip, at the very minimum we must have $F_{\min} = (\mu_s)_{\min}(mg - P)$, which indicates that $F_{\min}$ is a constant force. Substituting $F_{\min}$ for F in Eq. (5) and using the result in Eq. (7), we have

$$\int_0^{t_2} (\mu_s)_{\min}(mg - P)\,dt = m\frac{PR^2 t_2}{m(R^2 + k_G^2)} \quad \Rightarrow \quad (\mu_s)_{\min}(mg - P)t_2 = \frac{PR^2 t_2}{R^2 + k_G^2}, \tag{8}$$

which can be solved for μ_s to obtain

$$(\mu_s)_{\min} = \frac{PR^2}{(mg - P)(R^2 + k_G^2)}, \tag{9}$$

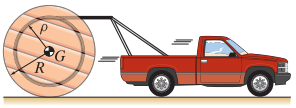
which can be evaluated to obtain

$$\boxed{(\mu_s)_{\min} = 0.395,}$$

where we have used the following numerical data: $P = 120\,\mathrm{N}$, $R = 0.3\,\mathrm{m}$, $m = 35\,\mathrm{kg}$, $g = 9.81\,\mathrm{m/s^2}$, and $k_G = 0.18\,\mathrm{m}$.

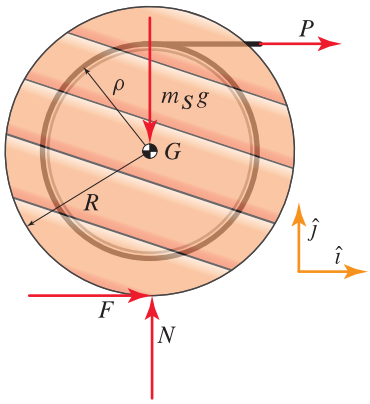
Problem 8.70

A spool has weight $W = 450\,\text{lb}$, outer and inner radii $R = 6\,\text{ft}$ and $\rho = 4.5\,\text{ft}$, respectively, center of mass G coinciding with its geometric center, and radius of gyration $k_G = 4.0\,\text{ft}$. The spool is being pulled to the right as shown, and the cable wrapped around the spool can be modeled as being inextensible and of negligible mass. Assume that the spool rolls without slip relative to both the cable and the ground. If the cable is pulled with a force $P = 125\,\text{lb}$, determine the speed of the center of spool after 2 s and the minimum value of the static friction coefficient between the spool and the ground necessary to guarantee rolling without slip.



Solution

We let the subscripts 1 and 2 denote the initial time ($t_1 = 0$) and the instant 2 seconds after the motion begins ($t_2 = 2\,\text{s}$), respectively. Based on the FBD shown, the application of the linear impulse–mometum principle in the x direction gives



$$m_S(v_{Gx})_1 + \int_0^{t_2} (P + F)\,dt = m_S(v_{Gx})_2, \tag{1}$$

where m_S is the mass of the spool, $\vec{v}_G = v_{Gx}\,\hat{\imath}$ is the velocity of the center of mass G of the spool, and where the friction force F is being treated as a function of time. The application of the angular impulse–momentum principle about G in the z direction yields

$$I_G\omega_{S1} + \int_0^{t_2} (FR - P\rho)\,dt = I_G\omega_{S2}, \tag{2}$$

where I_G is the mass moment of inertia of the spool about G and $\vec{\omega}_S = \omega_S\,\hat{k}$ is the angular velocity of the spool.

Since P is constant and recalling that the system starts from rest so that $(v_{Gx})_1 = 0$, from Eq. (1) we have

$$\int_0^{t_2} F\,dt = m_S(v_{Gx})_2 - Pt_2. \tag{3}$$

Recalling again that P is constant and that the spool starts from rest so that $\omega_{S1} = 0$, we see that Eq. (2) can be rewritten as

$$R\int_0^{t_2} F\,dt - P\rho t_2 = I_G\omega_{S2}. \tag{4}$$

Then, substituting Eq. (3) into Eq. (4), we have

$$R\left[\frac{W}{g}(v_{Gx})_2 - Pt_2\right] - P\rho t_2 = \frac{W}{g}k_G^2\omega_{S2}, \tag{5}$$

where we have set $m_S = W/g$ and $I_G = m_S k_G^2 = (W/g)k_G^2$ (g is the acceleration due to gravity).

The spool rolls without slip so that we must have $\omega_{S2} = -(v_{Gx})_2/R$. Substituting this result in Eq. (5) and solving for $(v_{Gx})_2$, we have

$$(v_{Gx})_2 = \frac{gR(R+\rho)}{W(R^2 + k_G^2)}Pt_2, \tag{6}$$

Observing that the speed of the spool is $v_G = |v_{Gx}|$, evaluating the result in Eq. (6), we have

$$\boxed{v_G\big|_{t=2\,\mathrm{s}} = 21.7\,\mathrm{ft/s},}$$

where we have used the following numerical data: $g = 32.2\,\mathrm{ft/s^2}$, $R = 6\,\mathrm{ft}$, $\rho = 4.5\,\mathrm{ft}$, $W = 450\,\mathrm{lb}$, $k_G = 4\,\mathrm{ft}$, $P = 125\,\mathrm{lb}$, and $t_2 = 2\,\mathrm{s}$.

Since the spool does not move in the y direction, the equilibrium of the forces in the y direction implies that $N = m_S g = W$. Letting μ_s denote the coefficient of static friction between the spool and the ground, for the spool to roll without slip, at the very minimum we must have $F_{\min} = (\mu_s)_{\min} N = (\mu_s)_{\min} W$, which indicates that $F_{\min}$ is a constant force. Substituting $F_{\min}$ for F in Eq. (3) and using the result in Eq. (6), we have

$$\int_0^{t_2} (\mu_s)_{\min} W\, dt = \frac{W}{g}\frac{gR(R+\rho)}{W(R^2+k_G^2)} P t_2 - P t_2 \quad \Rightarrow \quad (\mu_s)_{\min} W t_2 = \left[\frac{R(R+\rho)}{(R^2+k_G^2)} - 1\right] P t_2, \tag{7}$$

which can be solved for μ_s to obtain

$$(\mu_s)_{\min} = \frac{P(R\rho - k_G^2)}{W(R^2+k_G^2)}, \tag{8}$$

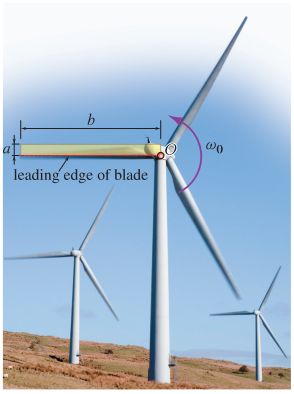
which can be evaluated to obtain

$$\boxed{(\mu_s)_{\min} = 0.0588,}$$

where we have used the following numerical data: $P = 125\,\mathrm{lb}$, $W = 450\,\mathrm{lb}$, $R = 6\,\mathrm{ft}$, $\rho = 4.5\,\mathrm{ft}$, and $k_G = 4\,\mathrm{ft}$.

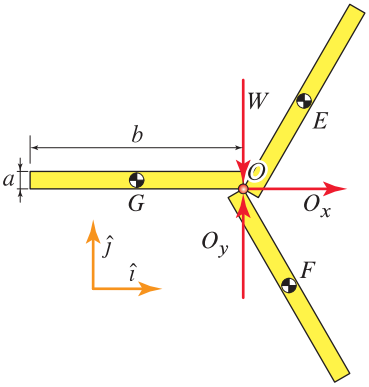
Problem 8.71

The wind turbine in the figure consists of three equally spaced blades that are rotating as shown about the fixed point O with an angular velocity $\omega_0 = 30$ rpm. Suppose that each 38,000 lb blade can be modeled as a narrow uniform rectangle of length $b = 182$ ft, width $a = 12$ ft, and negligible thickness, with one of its corners coinciding with the center of rotation O. The orientation of each blade can be controlled by rotating the blade about an axis going through the center O and coinciding with the blade's leading edge. Neglecting aerodynamic forces and any source of friction, and assuming that turbine is freely rotating, determine the turbine's angular velocity ω_f after each blade has been rotated 90° about its own leading edge.



Solution

We let the subscripts 1 and 2 denote the time at which the blades are oriented as shown and after being rotated 90°, respectively. Based on the assumptions stated, the FBD of the turbine is as shown. Therefore, the angular momentum of the turbine about the fixed point O, is conserved, i.e.,



$$(h_{Oz})_1 = (h_{Oz})_2, \tag{1}$$

where h_{Oz} is the z component of the angular momentum about O, and where, due to the symmetry of the system, the angular momentum of the rotor consists of the three equal contributions, one for every blade. Therefore we have

$$(h_{Oz})_1 = 3I_{O1}\omega_{r1} \quad \text{and} \quad (h_{Oz})_2 = 3I_{O2}\omega_{r2}, \tag{2}$$

where $\vec{\omega}_r = \omega_r \hat{k}$ is the angular velocity of the rotor, and where I_{O1} and I_{O2} are the mass moments of inertia of each blade about O at times t_1 and t_2, respectively. Using the parallel axis theorem, these mass moments of inertia are given by

$$I_{O1} = \tfrac{1}{12}(W_b/g)(a^2 + b^2) + (W_b/g)\big[(a/2) + (b/2)^2\big] = 13.09\times10^6 \text{ slug}\cdot\text{ft}^2, \tag{3}$$

$$I_{O2} = \tfrac{1}{12}(W_b/g)b^2 + (W_b/g)(b/2)^2 = 13.03\times10^6 \text{ slug}\cdot\text{ft}^2, \tag{4}$$

where W_b is the weight of each individual blade, g is the acceleration due to gravity, and where we have used the following numerical data: $W_b = 38{,}000$ lb, $g = 32.2$ ft/s^2, $a = 12$ ft, and $b = 182$ ft.

Substituting Eqs. (2) into Eq. (1) and solving for ω_{r2}, we have

$$\omega_f = \frac{I_{O1}}{I_{O2}}\omega_0, \tag{5}$$

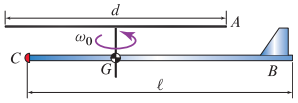
where we have used the fact that $\omega_{r1} = \omega_0$ and $\omega_{r2} = \omega_f$. The above expression can be evaluated to obatin

$$\boxed{\omega_f = 3.16 \text{ rad/s} = 30.1 \text{ rpm},}$$

where we have used the following numerical data: $I_{O1} = 13.09\times10^6$ slug·ft^2 (see Eq. (3)), $I_{O2} = 13.03\times10^6$ slug·ft^2 (see Eq. (4)), and $\omega_0 = 30$ rpm $= 3.142$ rad/s.

Problem 8.72

A toy helicopter consists of a rotor A, a body B, and a small ballast C. The axis of rotation of the rotor goes through G, which is the center of mass of the body B and ballast C. While holding the body (and ballast) fixed, the rotor is spun as shown with a given angular velocity ω_0. If there is *no friction* between the helicopter's body and the rotor's shaft, will the body of the helicopter start spinning once the toy is released?

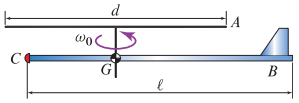


Solution

The answer is no. The reason for this is that without friction no force can be transmitted between the rotor and the body of the airplane. Therefore, the state of motion of the airplane would have no cause to change based on the fact that the rotor is spinning.

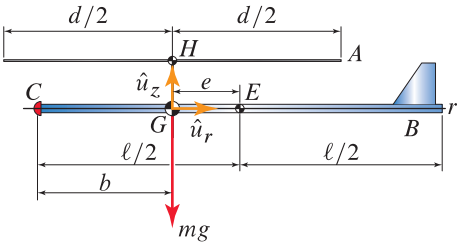
Problem 8.73

A toy helicopter consists of a rotor A with diameter $d = 10\,\text{in.}$ and weight $W_r = 0.09\times10^{-3}\,\text{oz}$, a thin body B of length $\ell = 12\,\text{in.}$ and weight $W_B = 0.144\times10^{-3}\,\text{oz}$, and a small ballast C placed at the front end of the body and with weight $W_C = 0.0723\times10^{-3}\,\text{oz}$. The ballast's weight is such that the axis of the rotation of the rotor goes through G, which is the center of mass of the body B and ballast C. While holding the body (and ballast) fixed, the rotor is spun as shown with $\omega_0 = 150\,\text{rpm}$. Neglecting aerodynamic effects, the weights of the rotor's shaft and the body's tail, and assuming there is friction between the helicopter's body and the rotor's shaft, determine the angular velocity of the body once the toy is released and the angular velocity of the rotor decreases to 120 rpm. Model the body as a uniform thin rod and the ballast as a particle. Assume that the rotor and the body remain horizontal after release.



Solution

We are told that G is the mass center of the ballast-C-and-body-B system. The center of mass of the rotor (point H) is vertically aligned with G. Therefore the overall weight of the toy mg, which the only external force on the system contributed no moment about G. Therefore the angular momentum of the helicopter about G must be conserved. Since the vertical motion of the toy does not play a role in this problem, we will treat G as a fixed point.



We let the subscripts 1 and 2 denote the time at which the helicopter is first released and when the angular speed of the rotor becomes 120 rpm, respectively. Hence, we have

$$\vec{h}_{G1} = \vec{h}_{G2}, \tag{1}$$

where $\vec{h}_G$ is the overall momentum of the system about G and is given by

$$\vec{h}_G = (\vec{h}_G)_r + (\vec{h}_G)_C + (\vec{h}_G)_B, \tag{2}$$

where the three terms on the right-hand side of Eq. (1) are the angular momenta about G due to the rotor, ballast C, and body B, respectively. We now proceed to determine these contributions to the angular momentum one at a time.

Angular Momentum of the Rotor. Modeling the rotor as a uniform thin rod of length d with its mass center vertically aligned with G, and recalling that the initial angular velocity of the rotor is $\omega_0\,\hat{u}_z$, we have

$$(\vec{h}_{G1})_r = (I_G)_r\omega_0\,\hat{u}_z \quad \text{and} \quad (\vec{h}_{G2})_r = (I_G)_r\omega_{r2}\,\hat{u}_z, \tag{3}$$

where $(I_G)_r$ is the mass moment of inertia of the rotor about a vertical axis going through G, i.e., a vertical axis going through the center of mass of the rotor itself so that

$$(I_G)_r = \tfrac{1}{12}(W_r/g)d^2 = 1.011\times10^{-8}\,\text{slug}\cdot\text{ft}^2, \tag{4}$$

where g is the acceleration due to gravity and (W_r/g) is the mass of the rotor, and where we have used the following numerical data: $W_r = 0.09\times10^{-3}\,\text{oz} = 5.625\times10^{-6}\,\text{lb}$ $d = 10\,\text{in.} = 0.8333\,\text{ft}$, and $g = 32.2\,\text{ft/s}^2$.

Angular Momentum of the Ballast. Modeling the ballast C as a particle we have

$$(\vec{h}_G)_C = \vec{r}_{C/G} \times m_C \vec{v}_C, \tag{5}$$

where, referring to the FBD, $\vec{r}_{C/G} = -b\,\hat{u}_r$, $m_C = W_C/g$ is the mass of the ballast, and $\vec{v}_C$ is the velocity of point C. Because the ballast moves with B, which, in turn, rotates about the z, we have

$$\vec{v}_C = \vec{\omega}_B \times \vec{r}_{C/G}, \tag{6}$$

where $\vec{\omega}_B = \omega_B\,\hat{u}_z$ is the angular velocity of the body. Substituting Eq. (6) into Eq. (5) and observing that the vectors $\vec{r}_{C/G}$ and $\vec{\omega}_B$ are perpendicular to one another, we have

$$(\vec{h}_G)_C = m_C \vec{r}_{C/G} \times \vec{\omega}_B \times \vec{r}_{C/G} = (W_C/g)\left|\vec{r}_{C/G}\right|^2 \vec{\omega}_B = (W_C/g) b^2 \omega_B\,\hat{u}_z, \tag{7}$$

where we have used the vector identity $\vec{a} \times \vec{c} \times \vec{a} = |\vec{a}|^2 \vec{c}$ when $\vec{a}$ and $\vec{c}$ are mutually perpendicular. To determine b, recalling that G is the center of mass of the ballast-body system, we must have

$$r_G = 0 = \frac{m_C r_C + m_B r_E}{m_C + m_B} \quad \Rightarrow \quad \frac{W_C}{g}(-b) + \frac{W_B}{g}(\tfrac{1}{2}\ell - b) = 0 \quad \Rightarrow \quad b = \frac{W_B \ell}{2(W_B + W_C)} = 0.3329\,\text{ft}, \tag{8}$$

where we have used the following numerical data $W_B = 0.144 \times 10^{-3}\,\text{oz} = 9.000 \times 10^{-6}\,\text{lb}$, $\ell = 12\,\text{in.} = 1.000\,\text{ft}$, and $W_C = 0.0723 \times 10^{-3}\,\text{oz} = 4.519 \times 10^{-6}\,\text{lb}$. Since C starts from rest, using Eq. (7), we have

$$(\vec{h}_{G1})_C = \vec{0} \quad \text{and} \quad (\vec{h}_{G2})_C = (W_C/g) b^2 \omega_{B2}\,\hat{u}_z. \tag{9}$$

Angular Momentum of the Body B. The body B is rotates about the (fixed) z axis with angular velocity $\vec{\omega}_B = \omega_B\,\hat{u}_z$. Therefore, recalling that B starts from rest, we have

$$(\vec{h}_{G1})_B = \vec{0} \quad \text{and} \quad (\vec{h}_{G2})_B = (I_G)_B \omega_{B2}\,\hat{u}_z, \tag{10}$$

where $(I_G)_B$ is the mass moment of inertia of the body about G. To obtain $(I_G)_B$ we model the body as a thin bar and, referring to the FBD, we apply the parallel axis theorem to obtain

$$(I_G)_B = \tfrac{1}{12}(W_B/g)\ell^2 + (W_B/g)e^2 = \tfrac{1}{12}(W_B/g)\ell^2 + (W_B/g)[(\ell/2) - b]^2 = 3.110 \times 10^{-8}\,\text{slug}\cdot\text{ft}^2, \tag{11}$$

where we have used the following numerical data: $W_B = 0.144 \times 10^{-3}\,\text{oz} = 9.000 \times 10^{-6}\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $\ell = 12\,\text{in.} = 1.000\,\text{ft}$, and $b = 0.3329\,\text{ft}$ (see last of Eqs. (8)).

Final Computation Substituting Eq. (2) into Eq. (1), we have

$$(\vec{h}_{G1})_r + (\vec{h}_{G1})_C + (\vec{h}_{G2})_B = (\vec{h}_{G2})_r + (\vec{h}_{G2})_C + (\vec{h}_{G2})_B. \tag{12}$$

Then substituting Eqs. (3), (9), and (10) into Eq. (12), we have

$$(I_G)_r \omega_0 = (I_G)_r \omega_{r2} + (W_C/g) b^2 \omega_{B2} + (I_G)_B \omega_{B2}, \tag{13}$$

where we have written the equation in scalar form (all of the nonzero terms are the z direction). Equation (13) is a single scalar equation in the only unknown ω_{B2} whose solution is

$$\omega_{B2} = \frac{(I_G)_r(\omega_0 - \omega_{r2})}{(W_C/g)b^2 + (I_G)_B}, \tag{14}$$

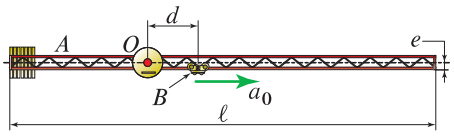
which can be evaluated to obtain

$$\boxed{\omega_B\big|_{\omega_r = 120\,\text{rpm}} = 0.680\,\text{rad/s} = 6.50\,\text{rpm},}$$

where we have used the following numerical data: $(I_G)_r = 1.011 \times 10^{-8}\,\text{slug}\cdot\text{ft}^2$ (see Eq. (4)), $\omega_0 = 150\,\text{rpm} = 15.71\,\text{rad/s}$, $\omega_{r2} = 120\,\text{rpm} = 12.57\,\text{rad/s}$, $W_C = 0.0723 \times 10^{-3}\,\text{oz} = 4.519 \times 10^{-6}\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $b = 0.3329\,\text{ft}$ (see the last of Eqs. (8)), and $(I_G)_B = 3.110 \times 10^{-8}\,\text{slug}\cdot\text{ft}^2$ (see Eq. (11)).

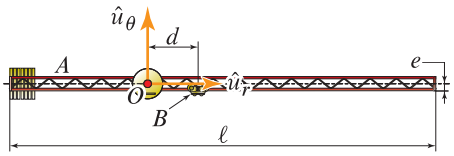
Problem 8.74

A crane has a boom A of mass m_A and length ℓ that can rotate in the horizontal plane about a fixed point O. A trolley B of mass m_B is mounted on one side of A such that the mass center of B is always at a distance e from the longitudinal axis of A. The position of B is controlled via a cable and a system of pulleys. Both A and B are initially at rest in the position shown, where d is the initial distance of B from O measured along the longitudinal axis of A. The boom A is free to rotate about O and, for a short time interval $0 \le t \le t_f$, B moves with constant acceleration a_0 without reaching the end of A. Letting I_O be the mass moment of inertia of A, modeling B as a particle, and accounting only for the inertia of A and B, determine the direction of rotation of A and the angle θ swept by A from $t = 0$ to $t = t_f$. Neglect the mass of the cable and of the pulleys.



Solution

Referring to the FBD below,



the stated assumptions imply that the boom and the trolley form an isolated system from the viewpoint of the rotational motion about O in the $r\theta$ plane. Therefore the system's angular momentum about O is conserved, i.e.,

$$(h_{Oz})_1 = (h_{Oz})_2, \tag{1}$$

where the subscripts 1 and 2 correspond to the initial time and a generic time t such that $0 < t < t_f$, respectively, and where the z direction is oriented by the unit vector $\hat{k} = \hat{u}_r \times \hat{u}_\theta$. Since the system starts from rest we have $(h_{Oz})_1 = 0$. For $(h_{Oz})_2$ we have

$$(h_{Oz})_2 = 0 = I_O \omega_A + m_B v_{Br} e, \tag{2}$$

where $\vec{\omega}_A = \omega_A \hat{k}$ is the angular velocity of the boom, and where $v_{Br} = a_0 t$ is the component of the velocity of B that is parallel to the radial direction shown in the figure. Hence we must have

$$\omega_A = -\frac{m_B a_0 e}{I_O} t, \tag{3}$$

Therefore we have that

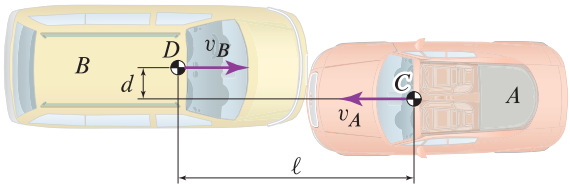
The system rotates clockwise.

The angle $\Delta\theta$ swept by the boom is obtained by integrating the angular velocity of the boom Eq. (3) with respect to time from time $t = 0$ to $t = t_f$. This yields

$$\Delta\theta = \frac{m_B a_0 e}{2 I_O} t_f^2, \text{ (clockwise).}$$

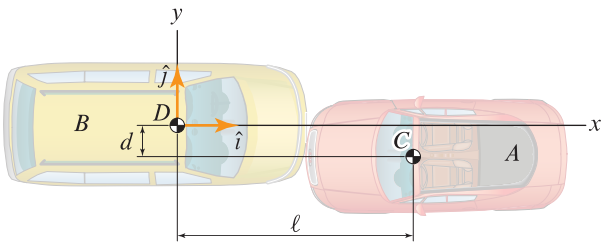
Problem 8.75

Cars A and B collide as shown. Neglecting the effect of friction, what would be the angular velocity of A and B immediately after impact if A and B were to form a single rigid body as a result of the collision? In solving the problem, let C and D be the mass centers of A and B, respectively, and use the following data: the weight of A is $W_A = 3130\,\text{lb}$, the radius of gyration of A is $k_C = 34.5\,\text{in.}$, the speed of A right before impact is $v_A = 12\,\text{mph}$, the weight of B is $W_B = 3520\,\text{lb}$, the radius of gyration of B is $k_D = 39.3\,\text{in.}$, the speed of B right before impact is $v_B = 15\,\text{mph}$, $d = 19\,\text{in.}$, and $\ell = 144\,\text{in.}$ Finally, assume that while A and B form a single rigid body right after impact, the mass center of the rigid body formed by A and B coincides with the mass center of the A-B system right before impact.



Solution

We let the subscripts 1 and 2 denote the time instants right before and right after impact, respectively.



Referring to the FBD above, we see that, as far as the motion in the xy plane is concerned, the system is isolated. Therefore, both the linear and the angular momentum of the system are conserved.

The conservation of the system's linear momentum is expressed by the following two equations:

$$m_A(v_{Cx})_1 + m_B(v_{Dx})_1 = m_A(v_{Cx})_2 + m_B(v_{Dx})_2, \tag{1}$$

$$m_A(v_{Cy})_1 + m_B(v_{Dy})_1 = m_A(v_{Cy})_2 + m_B(v_{Dy})_2, \tag{2}$$

where m_A and m_B are the masses of A and B, respectively, and where $\vec{v}_A = v_{Ax}\,\hat{\imath} + v_{Ay}\,\hat{\jmath}$ is the velocity of A, whereas $\vec{v}_B = v_{Bx}\,\hat{\imath} + v_{By}\,\hat{\jmath}$ is the velocity of B. Choosing point D, which is the origin of the chosen coordinate system, as moment center, and recalling that A and B do not move significantly right after impact, the conservation of the system's angular momentum is expressed as follows:

$$I_D\vec{\omega}_{B1} + I_C\vec{\omega}_{A1} + \vec{r}_{C/D} \times m_A\vec{v}_{C1} = I_D\vec{\omega}_{B2} + I_C\vec{\omega}_{A2} + \vec{r}_{C/D} \times m_A\vec{v}_{C2}, \tag{3}$$

where I_D and I_C are the mass moments of inertia of A and B about their respective mass centers, $\vec{\omega}_A = \omega_A\,\hat{k}$ and $\vec{\omega}_B = \omega_B\,\hat{k}$ are the angular velocities of A and B, respectively, and where $\vec{r}_{C/D}$, given by

$$\vec{r}_{C/D} = \ell\,\hat{\imath} - d\,\hat{\jmath} = (12.00\,\text{ft})\,\hat{\imath} - (1.583\,\text{ft})\,\hat{\jmath}, \tag{4}$$

is the position of C relative to D both right before and right after impact; notice that we have used the following numerical data: $d = 19\,\text{in.} = 1.583\,\text{ft}$ and $\ell = 144\,\text{in.} = 12.00\,\text{ft}$.

Based on the data provided, for the masses and the mass moments of inertia, we have

$$m_A = \frac{W_A}{g} = 97.20\,\text{slug}, \qquad I_C = \frac{W_A}{g}k_C^2 = 803.5\,\text{slug}\cdot\text{ft}^2, \tag{5}$$

$$m_B = \frac{W_B}{g} = 109.3\,\text{slug}, \qquad I_D = \frac{W_B}{g}k_D^2 = 1172\,\text{slug}\cdot\text{ft}^2, \tag{6}$$

where $g = 32.2\,\text{ft/s}^2$ is the acceleration due to gravity, $W_A = 3130\,\text{lb}$ and $k_C = 34.5\,\text{in.} = 2.875\,\text{ft}$ are the weight of A and the radius of gyration of A about C, respectively, and where $W_B = 3520\,\text{lb}$ and $k_D = 39.3\,\text{in.} = 3.275\,\text{ft}$ are the weight of B and the radius of gyration of B about D, respectively.

Observing that there are no forces on the FBD for which force laws are needed, we now proceed to the kinematic analysis. We begin by observing that the velocities before right impact are given and can be summarized as follows:

$$(v_{Cx})_1 = -v_A = -17.60\,\text{ft/s}, \qquad (v_{Cy})_1 = 0, \qquad \omega_{A1} = 0, \tag{7}$$

$$(v_{Dx})_1 = v_B = 22.00\,\text{ft/s}, \qquad (v_{Dy})_1 = 0, \qquad \omega_{B1} = 0, \tag{8}$$

where we have used the following numerical data: $v_A = 12\,\text{mph} = 17.60\,\text{ft/s}$ and $v_B = 15\,\text{mph} = 22.00\,\text{ft/s}$. Next, since A and B form a single rigid body after impact, we must have

$$\omega_{A2} = \omega_{B2} = \omega_2, \tag{9}$$

where $\vec{\omega}_2 = \omega_2\,\hat{k}$ is the common value of the angular velocity of A and B after impact. The fact that A and B form a single rigid body after impact (with the geometry described in the problem statement) also implies that we must have

$$\begin{aligned}\vec{v}_{C2} = \vec{v}_{D2} + \vec{\omega}_2 \times \vec{r}_{C/D} \quad &\Rightarrow \quad (v_{Cx})_2 = (v_{Dx})_2 + d\omega_2 \quad \text{and} \quad (v_{Cy})_2 = (v_{Dy})_2 + \ell\omega_2, \\ &\Rightarrow \quad (v_{Cx})_2 = (v_{Dx})_2 + (1.583\,\text{ft})\omega_2 \quad \text{and} \quad (v_{Cy})_2 = (v_{Dy})_2 + (12.00\,\text{ft})\omega_2,\end{aligned} \tag{10}$$

where $\vec{r}_{C/D}$ is given in Eq. (4).

Substituting the values of the mass properties (see Eq. (5) and (6)) and the preimpact velocities (see Eqs. (7) and (8)) into Eqs. (1)–(3), we have

$$693.9\,\text{lb}\cdot\text{s} = (97.20\,\text{slug})(v_{Cx})_2 = (109.3\,\text{slug})(v_{Dx})_2, \tag{11}$$

$$0 = (97.20\,\text{slug})(v_{Cy})_2 = (109.3\,\text{slug})(v_{Dy})_2, \tag{12}$$

$$-2708\,\text{ft}\cdot\text{lb}\cdot\text{s} = (153.9\,\text{lb}\cdot\text{s}^2)(v_{Cx})_2 + (1166\,\text{lb}\cdot\text{s}^2)(v_{Cy})_2 + (1976\,\text{ft}\cdot\text{lb}\cdot\text{s})\omega_2, \tag{13}$$

where we have used Eq. (4) again, and where we considered only the z component of Eq. (3) since it is the only nonzero component of the conservation of angular momentum equation. Now we observe that the last two of Eqs. (10) along with Eqs. (11)–(13) form a system of 5 equations in the five unknowns $(v_{Cx})_2$, $(v_{Cy})_2$, $(v_{Dx})_2$, $(v_{Dy})_2$, and ω_2 whose solution is

$$(v_{Cx})_2 = 3.076\,\text{ft/s}, \quad (v_{Cy})_2 = -2.154\,\text{ft/s}, \tag{14}$$

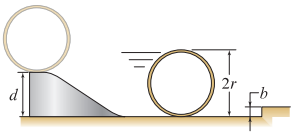
$$(v_{Dx})_2 = 3.613\,\text{ft/s}, \quad (v_{Dy})_2 = 1.915\,\text{ft/s}, \quad \text{and} \quad \omega_2 = -0.3391\,\text{rad/s}. \tag{15}$$

Hence, expressing the angular velocity of the system right after impact in vector form and to three significant digits, we have

$$\boxed{\vec{\omega}_{\text{right after impact}} = (-0.339\,\text{rad/s})\,\hat{k}}$$

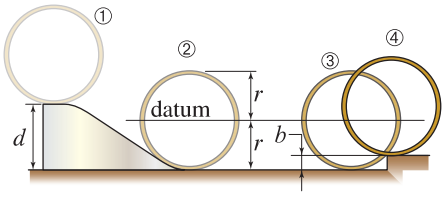
Problem 8.76

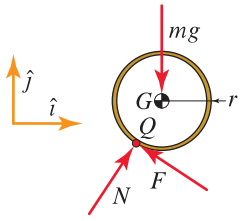
Some pipe sections are gently nudged from rest down an incline and roll without slipping all the way to a step of height b. Assume that each pipe section does not slide or rebound against the step, so that the pipes move as if hinged at the corner of the step. Modeling a pipe as a uniform thin ring of mass m and radius r, and letting d be the height from which the pipes are released, determine the minimum value of d so that the pipes can roll over the step. *Hint:* When a pipe hits the corner of the step, its motion changes almost instantaneously from rolling without slip on the ground to a fixed axis rotation about the corner of the step. Model this transition, using the ideas presented in Section 5.2 on p. 356. That is, assume that there is an infinitesimal time interval right after the impact between a pipe and the corner of the step, in which the pipe does not change its position significantly, the pipe loses contact with the ground, and its weight is negligible relative to the contact forces between the pipe and the step.



Solution

Referring to the figure below (left), let ① denote the position at release, ② when the pipe reaches the bottom of the incline, ③ when the incline impacts against point D of the step, and ④ when the pipe reaches the top of the step.





Referring to the FBD above (right), because of rolling without slip the friction forces F does not do any work so that energy is conserved between ① and ②, i.e.,

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where $T_1 = 0$ because the pipe is released from rest. Because of rolling without slip, denoting by Q the point of contact of the pipe with the flor at ②,T_2 can be written as

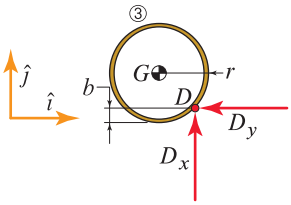
$$T_2 = \tfrac{1}{2} I_Q \omega_p^2, \tag{2}$$

where $\vec{\omega}_p = \omega_p \hat{k}$ is the angular velocity of the pipe, and where, modeling the pipe as a thin ring, $I_Q = 2mr^2$ is the mass moment of inertia of the pipe relative to Q. Choosing the datum at a distance r from the ground, we have $V_1 = mgd$ and $V_2 = 0$. Hence, using these considerations along with Eq. (1) gives

$$mgd = mr^2\omega_p^2 \quad \Rightarrow \quad \omega_p = -\frac{\sqrt{gd}}{r}, \tag{3}$$

where we have used the fact that at ② the pipe rotates clockwise. Between ② and ③ the pipe will continue rolling without slip and thefore move with a constant angular velocity.

At time t_3 we have the collision of the pipe against point D. From the impact-relevant FBD (i.e., neglecting any force that is not impulsive) is shown below. From this FBD we see that the impulsive forces acting on the pipe only act at point D. Since D is a fixed point, then we must have



$$h_{Dz}^{-} = h_{Dz}^{+}. \tag{4}$$

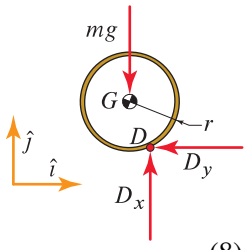
where the superscripts $-$ and $+$ denote the instants right before and right after impact. By assumption, right after impact the pipe is in a fixed axis rotation about D. Hence, Eq. (4) yields

$$I_G\omega_{p3}^{-} - m(r-b)v_{Gx3}^{-} = I_D\omega_{p3}^{+}, \tag{5}$$

where $I_G = mr^2$, $I_D = 2mr^2$, $\omega_{p3}^{-} = \omega_2$, and $v_{Gx3}^{-} = -\omega_{p3}^{-}$. Hence, Eq. (5) can be rewritten as

$$-mr^2\frac{\sqrt{gd}}{r} - m(r-b)\sqrt{gd} = 2mr^2\omega_{p3}^{+} \quad \Rightarrow \quad \omega_3^{+} = -\frac{\sqrt{gd}}{2r^2}(2r-b). \tag{6}$$

Finally, using work energy to study the motion from right after impact to when the pipe barely makes it to the top of the step, we have



$$T_3^{+} + V_3^{+} = T_4 + V_4, \tag{7}$$

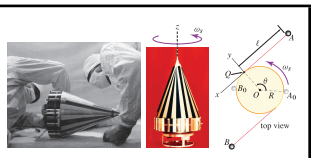
where, since after impact the pipe is in a fixed axis rotation about D, we have

$$T_3^{+} = \tfrac{1}{2}I_D(\omega_{P3}^{+})^2, \quad V_3^{+} = 0, \quad T_4 = 0, \quad V_4 = mgb. \tag{8}$$

Substituting the expressions in Eq. (8) into Eq. (7) and solving for d, we have

$$\boxed{d = \frac{4r^2 d}{(2r-b)^2}.}$$

Problem 8.77



Following up on parts (b) and (c) of the Pioneer 3 despin in Prob. 7.65, it turns out that it is possible to analytically determine the length of the unwound wire needed to achieve *any* value of ω_s by making use of conservation of energy and conservation of angular momentum. In doing so, let the masses of A and B each be m, and the mass moment of inertia of the spacecraft body be I_O. Let the initial conditions of the system be $\omega_s(0) = \omega_0$, $\ell(0) = 0$, and $\dot{\ell}(0) = 0$, and neglect gravity and the mass of each wire.

(a) Find the velocity of each of the masses A and B as a function of the wire length $\ell(t)$ and the angular velocity of the spacecraft body $\omega_s(t)$ (and the radius of the spacecraft R). *Hint:* This part of the problem involves just kinematics — refer to Prob. 6.117 if you need help with the kinematics.

(b) Apply the work-energy principle to the spacecraft system between the time just before the masses start to unwind and any arbitrary later time. You should obtain an expression relating ℓ, $\dot{\ell}$, ω_s, and constants. *Hint:* No external work is done on the system.

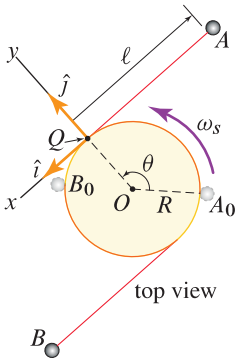
(c) Since no external forces act on the system, its total angular momentum must be conserved about point O. Relate the angular momentum for this system between the time just before the masses start unwinding and any arbitrary later time. As with part (b), you should obtain an expression relating ℓ, $\dot{\ell}$, ω_s, and constants.

(d) Solve the energy and angular momentum equations obtained in parts (b) and (c), respectively, for $\dot{\ell}$ and ω_s. Now, letting $\omega_s = 0$, show that the length of the unwound wire when the angular velocity of the spacecraft body is zero is given by $\ell_{\omega_s=0} = \sqrt{(I_O + 2mR^2)/(2m)}$.

(e) From your solutions for $\dot{\ell}$ and ω_s in part (d), find the equations for $\ell(t)$ and $\omega_s(t)$. These are the general solutions to the nonlinear equations of motion found in Prob. 7.64.

Solution

Part (a). We attach a rotating xyz reference frame such that the x axis is always aligned with the wire attached to A and such that the origin of the frame always coincides with the point on the wire that is about to unwind (see point Q in the figure on the right). Then, treating O as a fixed point, and using concepts from rotating reference frames, we have



$$\vec{v}_A = \vec{v}_Q + \vec{v}_{A\text{rel}} + \vec{\Omega} \times \vec{r}_{A/Q}, \tag{1}$$

where, observing that the unwind angle θ is related to the unwound cord length ℓ via the relation $\theta = \ell/R$,

$$\vec{v}_Q = R(\omega_s + \dot{\theta})\,\hat{\imath} = (R\omega_s + \dot{\ell})\,\hat{\imath}, \tag{2}$$

and where, recalling that $\vec{v}_{A\text{rel}}$ is the velocity of A as perceived in the rotating frame and $\vec{\Omega}$ is the (absolute) angular velocity of the rotating frame, we also have that

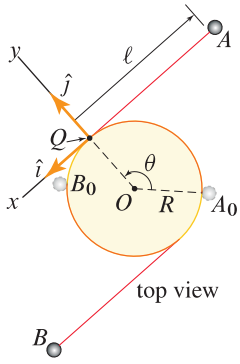
$$\vec{v}_{A\text{rel}} = -\dot{\ell}\,\hat{\imath}, \quad \vec{\Omega} = -(\omega_s + \dot{\theta})\,\hat{k} = -\left(\omega_s + \frac{\dot{\ell}}{R}\right)\hat{k}, \quad \vec{r}_{A/Q} = -\ell\,\hat{\imath}. \tag{3}$$

Then, substituting Eqs. (2) and (3) into Eq. (1) and simplifying, we have

$$\boxed{\vec{v}_A = R\omega_s\,\hat{\imath} + \ell\left(\omega_s + \frac{\dot{\ell}}{R}\right)\hat{\jmath},} \tag{4}$$

Part (b). We define ① to be the position of the system right before the masses start to unwind; and ② is the position corresponding to the cords having unwound an amount ℓ.

The FBD shown on the right indicates that there are no external forces act on the system, i.e., the system is isolated. Consequently, no work is being done on the satellite. In addition, we assume the cords are inextensible and that any source if internal friction is negligible. This then implies that that no internal work is done on the system. Based on these assumptions, the work-energy principle requires the kinetic energy of the system be conserved, i.e.,



$$T_1 = T_2, \tag{5}$$

where, recalling that the A and B move with identical speeds, and denoting by I_O the mass moment of inertia of the satellite, which is assumed not to change appreciably as the cords unwind, we have

$$T_1 = 2\left(\tfrac{1}{2}mv_{A1}^2\right) + \tfrac{1}{2}I_O\omega_{s1}^2 \quad \text{and} \quad T_2 = 2\left(\tfrac{1}{2}mv_{A2}^2\right) + \tfrac{1}{2}I_O\omega_{s2}^2. \tag{6}$$

Recalling that $\omega_s(0) = \omega_0$, $\ell(0) = 0$, and $\dot{\ell}(0) = 0$, using the expression for the velocity of A in Eq. (4), we have

$$v_{A1}^2 = \omega_0^2 R^2, \quad \omega_{s1}^2 = \omega_0^2, \quad v_{A2}^2 = \vec{v}_A \cdot \vec{v}_A = R^2\omega_s^2 + \ell^2\left(\omega_s + \frac{\dot{\ell}}{R}\right)^2, \quad \text{and} \quad \omega_{s2}^2 = \omega_s^2. \tag{7}$$

Then, substituting Eqs. (6) and (7) into Eq. (5) and simplifying, we obtain the following equation:

$$\boxed{\tfrac{1}{2}(I_O + 2mR^2)\omega_0^2 = m\left[R^2\omega_s^2 + \ell^2\left(\omega_s + \frac{\dot{\ell}}{R}\right)^2\right] + \tfrac{1}{2}I_O\omega_s^2.} \tag{8}$$

Part (c). We define the time instants 1 and 2 to be the instants in time corresponding to the positions 1 and 2, respectively, defined for the application of the work-energy principle in Part (b).

As remarked earlier, there are no external forces acting on the system. Therefore the angular momentum of the system must be conserved. Therefore, choosing the fixed point O as our moment center, we have

$$\vec{h}_{O1} = \vec{h}_{O2} \tag{9}$$

where $\vec{h}_O$ is the angular momentum of the system about O and it is given by the sum of two contributions, one due to the body of the satellite and the other due to A and B. Specifically, observing that A and B contribute to the angular momentum in question in identical ways, we have

$$\vec{h}_{O1} = I_O\vec{\omega}_{s1} + 2(\vec{r}_{A/O})_1 \times m\vec{v}_{A1} \quad \text{and} \quad \vec{h}_{O2} = I_O\vec{\omega}_{s2} + 2(\vec{r}_{A/O})_2 \times m\vec{v}_{A1}, \tag{10}$$

where, recalling that $\omega_s(0) = \omega_0$, $\ell(0) = 0$, and $\dot{\ell}(0) = 0$, using the expression for the velocity of A in Eq. (4)

$$\vec{\omega}_{s1} = -\omega_0 \hat{k}, \qquad (\vec{r}_{A/O})_1 = R\,\hat{\jmath}, \qquad \vec{v}_{A1} = R\omega_0 \hat{\imath}, \tag{11}$$

$$\vec{\omega}_{s2} = -\omega_s \hat{k}, \qquad (\vec{r}_{A/O})_2 = -\ell\,\hat{\imath} + R\,\hat{\jmath}, \qquad \vec{v}_{A2} = \vec{v}_A = R\omega_s \hat{\imath} + \ell\left(\omega_s + \frac{\dot{\ell}}{R}\right)\hat{\jmath}. \tag{12}$$

Substituting Eqs. (11) and (12) into Eqs. (10), and then substituting the result in Eq. (9) and simplifying, we obtain

$$\boxed{(I_O + 2mR^2)\omega_0 = \left[I_O + 2m(\ell^2 + R^2)\right]\omega_s + 2m\frac{\ell^2}{R}\dot{\ell},} \tag{13}$$

where we have written only the z component of the conservation of angular momentum equation because this component is the only nonzero component of the corresponding vector equation.

Part (d). Equation (8) and (13) form a system of two equations in the two unknowns $\dot{\ell}$ and ω_s whose solution is

$$\boxed{\dot{\ell} = R\omega_0 \quad \text{and} \quad \omega_s = \frac{I_O + 2m(R^2 - \ell^2)}{I_O + 2m(R^2 + \ell^2)}\omega_0.} \tag{14}$$

From the second of Eqs. (14), we see that for ω_s to be equal to zero it is sufficient to set the numerator of the expression for ω_s be equal to zero. Doing so and solving for ℓ, we have

$$\boxed{\ell\big|_{\omega_s=0} = \sqrt{\frac{I_O}{2m} + R^2},} \tag{15}$$

as expected.

Part (e). The solution for $\dot{\ell}$ Eq. (14) shows that $\dot{\ell}$ is a constant. Hence, integrating $\dot{\ell}$ with respect to time and recalling that $\ell(0) = 0$, we have

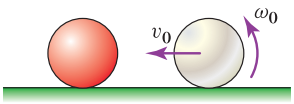
$$\boxed{\ell(t) = R\omega_0 t.} \tag{16}$$

Substituting this result in the second of Eq. (14) we obtain

$$\boxed{\omega_s(t) = \frac{I_O + 2mR^2(1 - \omega_0^2 t^2)}{I_O + 2mR^2(1 + \omega_0^2 t^2)}\omega_0.} \tag{17}$$

Problem 8.78

A *stop shot* is a pool shot in which the cue ball (white) stops upon striking the object ball (red). Modeling the collision between the two balls as a perfectly elastic collision of two rigid bodies with frictionless contact, determine which condition must be true for the preimpact angular velocity of the cue ball in order to properly execute a stop shot: (a) $\omega_0 < 0$; (b) $\omega_0 = 0$; (c) $\omega_0 > 0$.

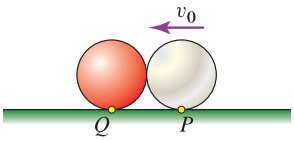


Solution

To execute a stop shot under the stated assumptions, the cue ball must not have any angular velocity when it hits the object ball. The reason is that the collision in question is a direct central impact with frictionless contact. This implies that the angular velocity of the cue ball (as well as that of the object ball) is conserved through the impact. Hence, if the angular velocity of the cue ball were not equal to zero, the ball would continue rolling after the impact.

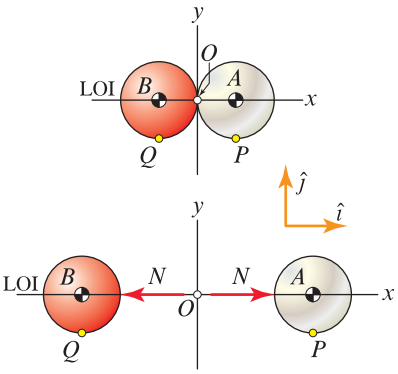
Problem 8.79

The cue ball (white) is rolling without slip to the left, and its center is moving with a speed $v_0 = 6$ ft while the object ball (red) is stationary. The diameter d of the two balls is the same and is equal to 2.25 in. The coefficient of restitution of the impact is $e = 0.98$. Let $W_c = 6$ oz and $W_o = 5.5$ oz be the weights of the cue ball and object ball, respectively. Let P and Q be the points on the cue ball and on the object ball, respectively, that are in contact with the table at the time of impact. Assuming that the contact between the two balls is frictionless and modeling the balls as uniform spheres, determine the postimpact velocities of P and Q.



Solution

We model the impact as an unconstrainted direct central impact. Point O is the fixed point in space that happens to coincide with the points on the two balls that come into contact at the time of impact. The chosen x axis is aligned with the LOI. Points A and B are both the geometric and mass centers of the cue and object balls, respectively. We will denote the angular velocities of the cue and object balls as $\vec{\omega}_c = \omega_c \hat{k}$ and $\vec{\omega}_o = \omega_o \hat{k}$, respectively. We will first determine the postimpact angular velocities of the balls and the postimpact velocities of the balls' centers and then we will use rigid body kinematics to determine the velocities of points P and Q.



The impact in question is governed by the following equations:

$$m_c v_{Ax}^- + m_o v_{Bx}^- = m_c v_{Ax}^+ + m_o v_{Bx}^+, \tag{1}$$

$$\omega_c^- = \omega_c^+, \tag{2}$$

$$\omega_o^- = \omega_o^+, \tag{3}$$

$$v_{Ax}^+ - v_{Bx}^+ = e(v_{Bx}^- - v_{Ax}^-), \tag{4}$$

which, in order, represent the conservation of linear momentum for the system along the LOI, the conservation of the angular momentum of the cue ball about O, the conservation of the angular momentum of the object ball about O, and the coefficient of restitution equation. The simplicity of these equations is due to the particular nature of the impact in this problem (see discussion beginning on p. 649 of the textbook).

The preimpact conditions are given and can be summarized as follows:

$$v_{Ax}^- = -v_0, \quad v_{Bx}^- = 0, \quad \omega_c^- = \frac{2v_0}{d}, \quad \omega_o^- = 0, \tag{5}$$

where the third of Eqs. (5) expresses the fact that the cue ball is rolling without slip before impact. Substituting the last two of Eqs. (5) into Eqs. (2) and (3) we obtain the postimpact angular velocities of the balls. Substituting the first two of Eqs. (5) into Eqs. (1) and (4) we obtain a system of two equations in the two unknowns v_{Ax}^+ and v_{Bx}^+. Solving this system of equations and including the results from Eqs. (2) and (3), we can summarize the postimpact kinematics of the balls as follows:

$$v_{Ax}^+ = \frac{(eW_o - W_c)v_0}{W_c + W_o} = -0.3178\,\text{ft/s}, \qquad v_{Bx}^+ = -\frac{(1+e)W_c v_0}{W_c + W_o} = -6.198\,\text{ft/s} \tag{6}$$

$$\omega_c^+ = \frac{2v_0}{d} = 64.00\,\text{rad/s}, \qquad \omega_o^+ = 0, \tag{7}$$

where we have set $m_c = W_c/g$ and $m_o = W_o/g$ ($g = 32.2\,\text{ft/s}^2$ is the acceleration due to gravity), and where we have used the following numerical data: $e = 0.98$, $W_o = 5.5\,\text{oz} = 0.3438\,\text{lb}$, $W_c = 6\,\text{oz} = 0.3750\,\text{lb}$, $v_0 = 6\,\text{ft/s}$, and $d = 2.25\,\text{in.} = 0.1875\,\text{ft}$.

We are now in a position to determine the velocities of points P and Q. For point P we have

$$\vec{v}_P^+ = \vec{v}_A^+ + \vec{\omega}_c^+ \times \vec{r}_{P/A}. \tag{8}$$

Observing that we have $\vec{r}_{P/A} = -(d/2)\,\hat{\jmath}$ and that A moves only in the x direction, we have

$$\vec{v}_P^+ = v_{Ax}^+\,\hat{\imath} + \omega_c^+\,\hat{k} \times -\tfrac{1}{2}d\,\hat{\jmath} = \left(v_{Ax}^+ + \tfrac{1}{2}\omega_c^+ d\right)\hat{\imath} = (5.682\,\text{ft/s})\,\hat{\imath}, \tag{9}$$

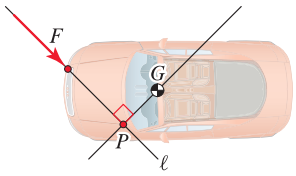
where we have used the results in the first of Eqs. (6) and (7), and where we have used the fact that $d = 2.25\,\text{in.} = 0.1875\,\text{ft}$.

In the case of point Q, we observe that because $\omega_o^+ = 0$ (see the second of Eqs. (7)), then the object ball is translating so that $\vec{v}_Q^+ = \vec{v}_B^+$, where $\vec{v}_B^+ = v_{Bx}^+\,\hat{\imath}$ and where v_{Bx}^+ is given by the second of Eqs. (6). In conclusion, we have

$$\boxed{\vec{v}_P^+ = (5.68\,\text{ft/s})\,\hat{\imath} \quad \text{and} \quad \vec{v}_Q^+ = (-6.20\,\text{ft/s})\,\hat{\imath}}\ \hat{\jmath}\ \hat{\imath}.$$

Problem 8.80

Consider the impact-relevant FBD of a car involved in a collision. Assume that, at the time of impact, the car was stationary. In addition, assume that the impulsive force F, with line of action ℓ, is the only impulsive force acting on the car at the time of impact. The point P at the intersection of ℓ and the line perpendicular to ℓ and passing through G, the center of mass of the car, is sometimes referred to as the *center of percussion* (for an alternative definition of center of percussion see the Mini-Example on p. 552). Is it true that, at the time of impact, the instantaneous center of rotation of the car lies on the same line as P and G?

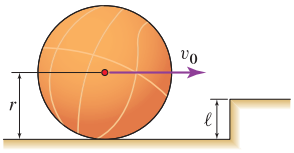


Solution

Yes, it is true that the instantaneous center of rotation will lie on the line connecting points P and G. The reason for this is that, by the linear impulse momentum principle the post impact velocity of the center of mass will have to be parallel to the force F. In turn, using the concept of instantaneous center of rotation, this means that the instantaneous center of rotation will have to lie on a line per perpendicular to the velocity of G and going through G. But this line is the line that also contains the point P.

Problem 8.81

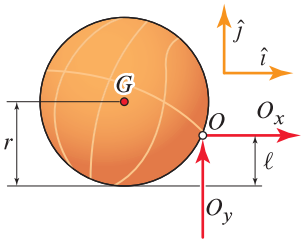
A basketball with mass $m = 0.6\,\text{kg}$ is rolling without slipping as shown when it hits a small step with $\ell = 7\,\text{cm}$. Letting the ball's diameter be $r = 12.0\,\text{cm}$, modeling the ball as a thin spherical shell (the mass moment of inertia of a spherical shell about its mass center is $\frac{2}{3}mr^2$), and assuming that the ball does not rebound off the step or slip relative to it, determine v_0 such that the ball barely makes it over the step.



Solution

The solution is organized in two parts. In the first, we consider the collision between the ball and the step. In the second, we apply the work energy principle to relate the postimpact speed of the ball to the ball's change in elevation in going over the step.

The impact-relevant FBD of the ball is shown on the right. This FBD indicates that the impact is a constrained impact in which the ball behaves as if hinged at the fixed point O right after impact. Since point O is fixed, we can conclude that the impact is governed by the conservation of angular momentum about O. Keeping in mind that the vertical component of the velocity of G is equal to zero before impact, we have



$$I_G\omega_b^- - m_b(r-\ell)v_{Gx}^- = I_O\omega_b^+, \tag{1}$$

where $\vec{\omega}_b = \omega_b\,\hat{k}$ is the angular velocity of the ball, and where I_G and I_O are the mass moments of inertia of the ball relative to points G and O, respectively. Specifically, we have

$$I_G = \tfrac{2}{3}m_b r^2 \quad \text{and} \quad I_O = I_G + m_b r^2 = \tfrac{5}{3}m_b r^2. \tag{2}$$

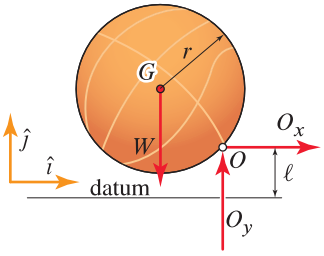
Since G is moving with speed v_0 to the right and the ball rolls without slip before impact, we must have

$$v_{Gx}^- = v_0 \quad \text{and} \quad \omega_b^- = -v_0/r. \tag{3}$$

Hence, substituting Eqs. (2) and (3) into Eq. (1) and solving for ω_b^+, we have

$$\omega_b^+ = -\frac{5r - 3\ell}{5r^2}v_0. \tag{4}$$

We now consider the second part of the problem. We define ① to coincide with the postimpact position of the system, and we define ② to be the position of the ball after it barely makes it to the top of the step. The FBD for this part of the problem is shown on the right. Since the the ball moves as if hinged at O, the only force doing work is gravity and therefore we can write



$$T_1 + V_1 = T_2 + V_2, \tag{5}$$

where, given our choice of datum, that the ball is in a state of fixed axis rotation about O after impact, and that the ball barely makes it to the top of the step, we have

$$T_1 = \tfrac{1}{2}I_O\omega_{b1}^2 = \tfrac{5}{6}m_b r^2(\omega_b^+)^2, \quad V_1 = 0, \quad T_2 = 0, \quad V_2 = m_b g\ell. \tag{6}$$

Recalling that ω_b^+ is given in Eq. (4), substituting Eqs. (6) into Eq. (5), and solving for v_0 we have

$$v_0 = \sqrt{\frac{30g\ell r^2}{(5r - 3\ell)^2}},$$

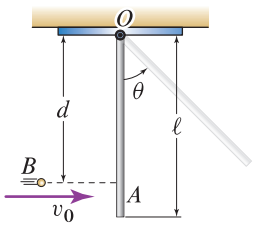
which can be evaluated to obtain

$$\boxed{v_0 = 1.40\,\text{ft/s},}$$

where we have used the following numerical data: $g = 9.81\,\text{m/s}^2$, $\ell = 7\,\text{cm} = 0.07000\,\text{m}$, and $r = 12\,\text{cm} = 0.1200\,\text{m}$.

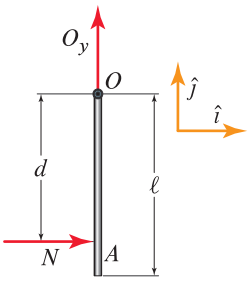
Problem 8.82

A bullet B of mass m_b is fired with a speed v_0 as shown against a uniform thin rod A of length ℓ, mass m_r, and that is pinned at O. Determine the distance d such that no horizontal reaction is felt at the pin when the bullet strikes the rod.



Solution

Under the assumption that no horizontal reaction is felt at the pint during the impact, the impact-relevant FBD of the bar is as shown on the right. Using this FBD and applying the linear impulse–momentum principle in the horizontal direction, we have



$$mv_{Gx}^{-} + \int_{t^-}^{t^+} N\,dt = mv_{Gx}^{+} \quad \Rightarrow \quad \int_{t^-}^{t^+} N\,dt = mv_{Gx}^{+}, \tag{1}$$

where G is the center of mass of the bar, $\vec{v}_G = v_{Gx}\,\hat{\imath} + v_{Gy}\,\hat{\jmath}$ is the velocity of G, N is the force exerted by the bullet on the bar during the collision, t^- and t^+ are the time instants immediately before and after the collision, respectively, and where we have accounted for the fact that the bar starts from rest and therefore $v_{Gx}^{-} = 0$. Choosing the fixed point O as moment center, and applying the angular–impulse momentum principle, we have

$$I_O\omega_A^{-} + \int_{t^-}^{t^+} Nd\,dt = I_O\omega_A^{+}, \quad \Rightarrow \quad d\int_{t^-}^{t^+} N\,dt = I_O\omega_A^{+}, \tag{2}$$

where $\vec{\omega}_A = \omega_A\,\hat{k}$ is the angular velocity of the bar, $\omega_A^{-} = 0$ since the bar starts from rest, and where I_O is the mass moment of inertia of the bar about O and is therefore given by

$$I_O = \tfrac{1}{12}m\ell^2 + m(\ell/2)^2 = \tfrac{1}{3}m\ell^2. \tag{3}$$

Substituting the last of Eqs. (1) into the last of Eqs. (2), and using the expression for I_O in Eq. (3), we have

$$dmv_{Gx}^{+} = \tfrac{1}{3}m\ell^2\omega_A^{+}. \tag{4}$$

Now, observe that rigid body kinematics requires that $\vec{v}_G = \vec{v}_O + \vec{\omega}_A \times \vec{r}_{G/O}$. Hence, accounting for the fact that O is a fixed point, we must have

$$\vec{v}_G^{+} = \omega_A^{+}\,\hat{k} \times (-\ell/2)\,\hat{\jmath} \quad \Rightarrow \quad \vec{v}_G^{+} = \tfrac{1}{2}\ell\omega_A^{+}\,\hat{\imath} \quad \Rightarrow \quad v_{Gx}^{+} = \tfrac{1}{2}\ell\omega_A^{+}. \tag{5}$$

Substituting the last of Eq. (5) into Eq. (4), we have

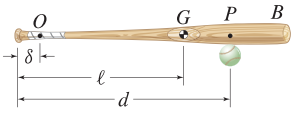
$$dm\tfrac{1}{2}\ell\omega_A^{+} = \tfrac{1}{3}m\ell^2\omega_A^{+}. \tag{6}$$

Solving the above equation for d, we have

$$\boxed{d = \tfrac{2}{3}\ell.}$$

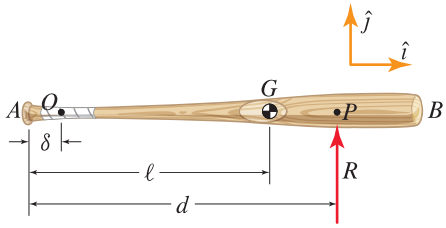
Problem 8.83

Solve the problem in the Mini-Example on p. 552 using momentum methods and the concept of impulsive force. Specifically, consider a ball hitting a bat at a distance d from the handle when the batter has "choked up" a distance δ. Find the "sweet spot" P (more properly called the center of percussion) of the bat B by determining the distance d at which the ball should be hit so that the lateral force (i.e., perpendicular to the bat) at O is zero. Assume that the bat is pinned at O, it has mass m, the mass center is at G, and the mass moment of inertia is I_G.



Solution

The impact-relevant FBD of the bat assuming no impulsive forces at the grip is shown on the right, where R is the impulsive force exerted by the ball on the bat. Letting t^- and t^+ denote the time instants right before and right after impact, respectively, the linear impulse–momentum principle in the y direction, yields



$$mv_{Gy}^- + \int_{t^-}^{t^+} R\,dt = mv_{Gy}^+, \tag{1}$$

where v_{Gy} is the y component of the velocity of G. If the bat rotates about the fixed point O, then rigid body kinematics requires that

$$v_{Gy}^- = \omega_b^-(\ell - d) \quad \text{and} \quad v_{Gy}^+ = \omega_b^+(\ell - d), \tag{2}$$

where $\vec{\omega}_b = \omega_b \hat{k}$ is the angular velocity of the bat. Substituting Eqs. (2) into Eq. (1) we write

$$\int_{t^-}^{t^+} R\,dt = m(\ell - d)(\omega_b^+ - \omega_b^-). \tag{3}$$

Next, choosing the fixed point O as the moment center, the angular impulse–momentum principle yields

$$I_O\omega_b^- + \int_{t^-}^{t^+} R(d-\delta)\,dt = I_O\omega_b^+ \quad \Rightarrow \quad I_O\omega_b^- + (d-\delta)\int_{t^-}^{t^+} R\,dt = I_O\omega_b^+ \tag{4}$$

where I_O is the mass moment of inertia of the bat relative to O and therefore is given by

$$I_O = I_G + m(\ell - \delta)^2, \tag{5}$$

where I_G is the mass moment of inertia of the bat relative to its own mass center. Substituting Eqs. (3) and (5) into the last of Eqs. (4), and simplifying, we have

$$\begin{aligned}\left[I_G + m(\ell-\delta)^2\right]\omega_b^- + m(\ell-d)(d-\delta)(\omega_b^+ - \omega_b^-) &= \left[I_G + m(\ell-\delta)^2\right]\omega_b^+ \\ \Rightarrow \quad (\omega_b^+ - \omega_b^-)[I_G + m(\ell-\delta)^2 - m(\ell-\delta)(d-\delta)] &= 0. \end{aligned} \tag{6}$$

Since we expect that $(\omega_b^+ - \omega_b^-) \neq 0$, then, in order for the above equation to be satisfied, we must have

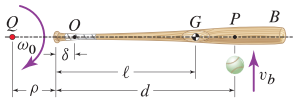
$$I_G + m(\ell-\delta)^2 - m(\ell-\delta)(d-\delta) = 0, \tag{7}$$

which can be solved for d to obtain the following expected result:

$$\boxed{d = \frac{I_G + m\ell(\ell-\delta)}{m(\ell-\delta)}.}$$

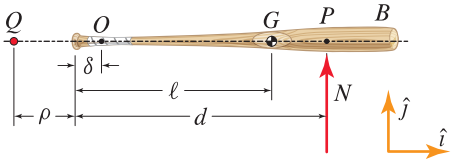
Problem 8.84

A batter is swinging a 34 in. long bat with weight $W_B = 32$ oz, mass center G, and mass moment of inertia $I_G = 0.0413$ slug·ft^2. The center of rotation of the bat is point Q. Compute the distance d identifying the position of point P, the bat's "sweet spot" or center of percussion, such that the batter will not feel any impulsive forces at O where he is grasping the bat. In addition, knowing that the ball, weighing 5 oz, is traveling at a speed $v_b = 90$ mph and that the batter is swinging the bat with an angular velocity $\omega_0 = 45$ rad/s, determine the speed of the ball and the angular velocity of the bat immediately after impact. To solve the problem, use the following data: $\delta = 6$ in., $\rho = 14$ in., $\ell = 22.5$ in., and COR $e = 0.5$.



Solution

If there is no impulsive force at the grip, then the impact-relevant FBD of the system is that shown on the right. Let t^- and t^+ are the time instants right before and right after the impact, respectively, and where ω_B^+ denotes the component of the angular velocity of the bat in the z direction right after impact, whereas $\omega_B^- = -\omega_0$ is the angular velocity of the bat right before impact. Then, using the FBD on the right, the linear impulse–momentum principle for the bat in the y direction gives



$$\underbrace{-m_B\omega_0(\rho+\ell)}_{m_B v_{Gy}^-} + \int_{t_-}^{t^+} N\,dt = \underbrace{m_B\omega_B^+(\rho+\ell)}_{m_B v_{Gy}^+} \quad \Rightarrow \quad \int_{t_-}^{t^+} N\,dt = m_B(\rho+\ell)(\omega_B^+ + \omega_0) \qquad (1)$$

where m_B is the mass of the bat, and where we have used rigid body kinematics to express the y component of the velocity of G, i.e. v_{Gy}, before and after impact.

Similarly, choosing point Q as the moment center, and since the bat is in a fixed axis rotation about Q, the application of the angular impulse–momentum principle gives,

$$-I_Q\omega_0 + \int_{t_-}^{t^+} N(\rho+d)\,dt = I_Q\omega_B^+ \quad \Rightarrow \quad -I_Q\omega_0 + (\rho+d)\int_{t_-}^{t^+} N\,dt = I_Q\omega_B^+, \qquad (2)$$

where we have accounted for the fact that ρ and d are constants, and where I_Q is the mass moment of inertia of the bat about Q and is therefore given by

$$I_Q = I_G + m_B(\rho+\ell)^2 = 0.6161\ \text{slug·ft}^2, \qquad (3)$$

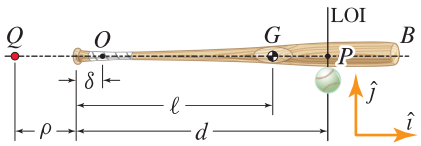
where we have used the following data: $I_G = 0.0413$ slug·ft^2, $m_B = 32\,\text{oz}(16\,\text{oz/lb})^{-1}/(32.2\,\text{ft/s}^2) = 0.06211$ slug, $\rho = 14$ in. $= 1.167$ ft, and $\ell = 22.5$ in. $= 1.875$ ft. Substituting the expression for the impulse of N from Eq. (1) and the expression for I_Q from the last of Eqs. (3) into the last of Eqs. (2), and solving for d we have

$$d = \frac{I_G}{m_B(\rho+\ell)} + \ell = 2.094\ \text{ft}, \qquad (4)$$

where we have used the following data: $I_G = 0.0413$ slug·ft^2, $m_B = 32\,\text{oz}(16\,\text{oz/lb})^{-1}/(32.2\,\text{ft/s}^2) = 0.06211$ slug, $\rho = 14$ in. $= 1.167$ ft, and $\ell = 22.5$ in. $= 1.875$ ft. Expressing the above result to three significant figures, we have

$$\boxed{d = 2.09\ \text{ft}.}$$

The impact-relevant FBD of the bat-ball system is shown to the right. We observe that the LOI is parallel to the y direction and that system's angular momentum about the fixed point Q must be conserved since there are no external impulsive forces acting of the system when the ball hits the bat right on the "sweet spot." Therefore, we have



$$-I_Q\omega_0 + m_b v_{by}^-(\rho + d) = I_Q\omega_B^+(\rho + \ell) + m_b v_{by}^+(\rho + d), \tag{5}$$

where m_b is the mass of the ball, and where v_{by}^- and v_{by}^+ are the components of the velocity of the ball in the y direction right before and right after impact. Neglecting the thickness of the bat, the point on the bat that is hit by the ball moves just like point P. Therefore the COR equation for this problem is

$$v_{by}^+ - v_{Py}^+ = e(v_{Py}^- - v_{by}^-). \tag{6}$$

Because the bat is in a fixed axis rotation about point Q, we have that

$$v_{Py}^- = -\omega_0(\rho + d) \quad \text{and} \quad v_{Py}^+ = \omega_B^+(\rho + d). \tag{7}$$

Observing that the ball moves parallel to the y direction before impact that using the standard assumption that the contact between the two colliding bodies is frictionless, we must have

$$v_{by}^- = v_b, \quad v_{bx}^- = 0, \quad \text{and} \quad v_{bx}^+ = 0. \tag{8}$$

Substituting Eqs. (7) and the first of Eqs. (8) into Eq. (6) we have

$$v_{by}^+ - \omega_B^+(\rho + d) = e[-\omega_0(\rho + d) - v_{by}^-]. \tag{9}$$

After substituting the first of Eqs. (8) into Eq. (5), we see that Eqs. (5) and (9) form a system of two equations in the two unknowns ω_B^+ and v_{by}^+ whose solution is

$$\omega_B^+ = \frac{m_b(\rho + d)[e\omega_0(\rho + d) + (1 + e)v_b] - I_Q\omega_0}{m_b(\rho + d)^2 + I_Q}, \tag{10}$$

$$v_{by}^+ = \frac{v_b[m_b(\rho + d)^2 - eI_Q] - (1 + e)I_Q\omega_0(\rho + d)}{m_b(\rho + d)^2 + I_Q}. \tag{11}$$

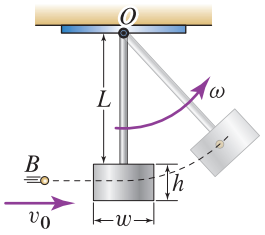
Finally, recalling that the post impact speed of the ball is $v_b^+ = |v_{by}^+|$ and expressing the angular velocity of the bat in vector form, we have

$$\boxed{\vec{\omega}_B = (-26.6\,\text{rad/s})\,\hat{k} \quad \text{and} \quad v_b^+ = 226\,\text{ft/s}}$$

where we have used the following numerical data: $m_b = 5\,\text{oz}(16\,\text{oz/lb})^{-1}/(32.2\,\text{ft/s}^2) = 0.009705\,\text{slug}$, $\rho = 14\,\text{in.} = 1.167\,\text{ft}$, $d = 2.094\,\text{ft}$ (see Eq. (4)), $e = 0.5$, $\omega_0 = 45\,\text{rad/s}$, $v_b = 90\,\text{mph} = 132.0\,\text{ft/s}$, and $I_Q = 0.6161\,\text{slug}\cdot\text{ft}^2$ (see Eq. (3)).

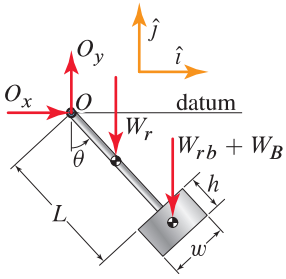
Problem 8.85

A bullet B weighing 147 gr (1 lb = 7000 gr) is fired with a speed v_0 as shown and becomes embedded in the center of a rubber block of dimensions $h = 4.5$ in. and $w = 6$ in. weighing $W_{\text{rb}} = 2$ lb. The rubber block is attached to the end of a uniform thin rod A of length $L = 1.5$ ft, weight $W_r = 5$ lb, and that is pinned at O. After the impact, the rod (with the block and the bullet embedded in it) swings upward to an angle of 60°. Determine the speed of the bullet right before impact.



Solution

The solution of this problem is organized in two parts. The first is the determination of the postimpact angular velocity of the ballistic pendulum based on the knowledge of the maximum swing angle. The second is the determination of the preimpact speed of the bullet based on the result of the first part.



We will relate the maximum swing angle to the post-impact angular velocity of the ballistic pendulum using the work-energy principle. We therefore begin by defining the ① and ② needed to apply the work-energy principle. We define ① to be the position with $\theta = 0$ immediately after impact and ② will be that corresponding to $\theta = \theta_{\max} = 60°$.

Referring to the FBD above on the right, observe that the system rotates about the fixed point O and that the only forces doing work are the weights of the bar, of the block, and of the bullet. Since the forces doing work are conservative, we can write

$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where observing that the ballistic pendulum is in a fixed axis rotation about O and that it comes to a stop when at ②, we have

$$T_1 = \tfrac{1}{2} I_O \omega_{p1}^2 \quad \text{and} \quad T_2 = 0, \tag{2}$$

where I_O is the mass moment of inertia of the entire system relative to O and where $\vec{\omega}_p = \omega_p \hat{k}$ is the angular velocity of the ballistic pendulum. Observing that by assumption the bullet ends up occupying the same position as the center of mass of the block, and applying the parallel axis theorem, I_O is given by

$$I_O = \tfrac{1}{12}(W_r/g)L^2 + (W_r/g)\left(\tfrac{1}{2}L\right)^2 + \tfrac{1}{12}(W_{\text{rb}}/g)(w^2 + h^2) + (W_{\text{rb}}/g)\left(L + \tfrac{1}{2}h\right)^2 + (W_B/g)\left(L + \tfrac{1}{2}h\right)^2 = 0.2972\ \text{slug}\cdot\text{ft}^2, \tag{3}$$

where g is the acceleration due to gravity, W_r/g is the mass of the rod, W_{rb}/g is the mass of the rubber block, W_B/g is the mass of the bullet, and where we have used the following numerical data: $W_r = 5$ lb, $g = 32.2\ \text{ft/s}^2$, $L = 1.5$ ft, $W_{\text{rb}} = 2$ lb, $w = 6$ in. $= 0.5000$ ft, $h = 4.5$ in. $= 0.3750$ ft, and $W_B = 147$ gr $= 0.02100$ lb.

As far as the potential energies are concerned, due to our choice of datum, we have

$$V_1 = -W_r(L/2) - (W_{\text{rb}} + W_B)[L + (h/2)] = -7.160\ \text{ft}\cdot\text{lb}, \tag{4}$$

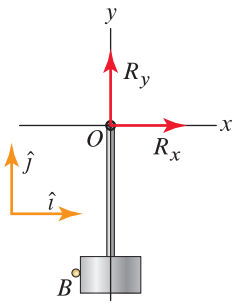
$$V_2 = -W_r(L/2)\cos\theta_{\max} - (W_{\text{rb}} + W_B)[L + (h/2)]\cos\theta_{\max} = -3.580\ \text{ft}\cdot\text{lb}, \tag{5}$$

Substituting Eqs. (2) into Eq. (1) and solving for ω_{p1} we have

$$\omega_{p1} = \sqrt{\frac{2(V_2 - V_1)}{I_O}} = 4.908\ \text{rad/s}, \tag{6}$$

where we have let ω_{p1} be positive since the pendulum will move counterclockwise after impact, and where we have used the following numerical data: $V_2 = -3.580\,\text{ft·lb}$ (see Eq. (5)), $V_1 = -7.160\,\text{ft·lb}$ (see Eq. (4)), and $I_O = 0.2972\,\text{slug·ft}^2$ (see Eq. (3)).

Now, we deal with the impact part of the problem. The impact at hand is a constrained eccentric impact. The impact-relevat FBD is shown on the right and it implies that the angular momentum of the system about the fixed point O must be conserved. This is expressed by the following equation:



$$\vec{h}_O^- = \vec{h}_O^+, \tag{7}$$

where the angular momentum of the system about O consists of a contribution due to the ballistic pendulum and one due to the bullet. To compute these contributions and recalling the pre-imapct conditions of the system, we define the following kinematic quantities:

$$\vec{\omega}_p^- = \vec{0}, \quad \vec{\omega}_p^+ = \omega_p^+, \quad \vec{r}_{B/O} = -\left(L + \tfrac{1}{2}h\right)\hat{\jmath}, \quad \vec{v}_B^- = v_0\,\hat{\imath}, \quad \vec{v}_B^+ = \vec{\omega}_p^+ \times \vec{r}_{B/O} = \omega_p^+\left(L + \tfrac{1}{2}h\right)\hat{\imath}, \tag{8}$$

where the last of Eqs. (8) expresses the fact that right after impact the bullet moves as part of the ballistic pendulum, where the latter is modeled as a rigid body.

Next, using Eqs. (8), we have

$$\vec{h}_O^- = I_O^-\vec{\omega}_p^- + \vec{r}_{B/O} \times (W_B/g)\vec{v}_B^- = \frac{W_B v_0}{2g}(h + 2L)\,\hat{k} \quad \text{and} \quad \vec{h}_O^+ = I_O^+\omega_p^+\,\hat{k}, \tag{9}$$

where I_O^- and I_O^+ are the mass moment of inertia of the ballistic pendulum about O before and after the bullet becomes embedded into the rubber block, respectively.

Substituting Eqs. (9) into Eq. (7) and solving for v_0 we obtain

$$v_0 = \frac{2g I_O^+ \omega_p^+}{W_B(h + 2L)}. \tag{10}$$

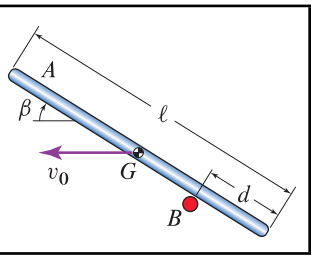
Now we observe that I_O^+ is precisely the mass moment of inertia given in Eq. (3) and that ω_p^+ is the angular velocity computed in Eq. (6). Therefore, Eq. (10) can be evaluated to obtain

$$\boxed{v_0 = 1330\,\text{ft/s},}$$

where we have used the following numerical data: $g = 32.2\,\text{ft/s}^2$, $I_O^+ = I_O = 0.2972\,\text{slug·ft}^2$ (see Eq. (3)), $\omega_p^+ = \omega_{p1} = 4.908\,\text{rad/s}$ (see Eq. (6)), $h = 4.5\,\text{in.} = 0.3750\,\text{ft}$, $L = 1.5\,\text{ft}$, and $W_B = 147\,\text{gr} = 0.02100\,\text{lb}$.

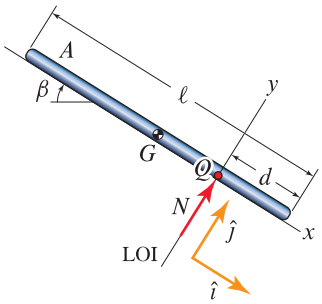
Problem 8.86

A thin homogeneous bar A of length $\ell = 1.75\,\text{m}$ and mass $m = 23\,\text{kg}$ is translating as shown with a speed $v_0 = 12\,\text{m/s}$ when it collides with the fixed obstacle B. Modeling the contact between the bar and obstacle as frictionless, letting $\beta = 32°$, and letting the distance $d = 0.46\,\text{m}$, determine the angular velocity of the bar immediately after the collision, knowing that the COR for the impact is $e = 0.74$.



Solution

The impact-relevant FBD of the bar under the stated assumptions is shown on the right. Based on this FBD, and recalling that the superscripts $-$ and $+$ denote the time instants right before and right after the collision, we see that the linear momentum of the bar in the direction perpendicular the the LOI is conserved, i.e.,



$$mv_{Gx}^{-} = mv_{Gx}^{+}, \tag{1}$$

where m is the mass of the bar, G denotes the bar's center of mass, and $\vec{v}_G = v_{Gx}\,\hat{\imath} + v_{Gy}\,\hat{\jmath}$ is the velocity of the mass center of the bar. In addition, observe that the impulsive force acting on the bar contributes no moment about point Q, which is the point on the bar that comes into contact with the obstacle B. Therefore, the angular momentum of the bar about the fixed point that happens to coincide with Q at the instant of impact must be conserved through the impact and this conservation statement can be written in scalar form as follows:

$$I_G\omega_b^{-} - m[(\ell/2) - d]v_{Gy}^{-} = I_G\omega_b^{+} - m[(\ell/2) - d]v_{Gy}^{+}, \tag{2}$$

where $I_G = \frac{1}{12}m\ell^2$ is the mass moment of inertia of the bar, and where $\vec{\omega}_b = \omega_b\,\hat{k}$ is the angular velocity of the bar.

The COR equation for the problem is

$$v_{Qy}^{+} = -ev_{Qy}^{-}, \tag{3}$$

Given the motion of the bar preceding the impact, we have

$$\omega_b^{-} = 0, \quad v_{Gx}^{-} = -v_0\cos\beta, \quad v_{Gy}^{-} = -v_0\sin\beta, \quad v_{Qy}^{-} = -v_0\sin\beta. \tag{4}$$

In addition, using rigid body kinematics, right after impact we must have

$$v_{Qy}^{+} = v_{Gy}^{+} + \omega_b^{+}[(\ell/2) - d]. \tag{5}$$

After substituting Eqs. (5) and (4) into Eq. (3), Eqs. (1)–(3) form a system of three equations in the three unknowns v_{Gx}^{+}, v_{Gy}^{+} and ω_b^{+} whose solution is

$$v_{Gx}^{+} = -v_0\cos\beta, \quad v_{Gy}^{+} = -\frac{(3-e)\ell^2 - 12d\ell + 12d^2}{4(\ell^2 - 3d\ell + 3d^2)}v_0\sin\beta, \quad \omega_b^{+} = \frac{3(1+e)(\ell-2d)v_0\sin\beta}{2(\ell^2 - 3d\ell + 3d^2)},$$

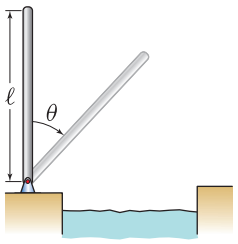
so that the last of the above results can be evaluated to obatin

$$\boxed{\vec{\omega}_b^{+} = (10.7\,\text{rad/s})\,\hat{k}}$$

where we have used the following numerical data: $e = 0.74$, $\ell = 1.75\,\text{m}$, $d = 0.46\,\text{m}$, $v_0 = 12\,\text{m/s}$, and $\beta = 32°$.

Problem 8.87

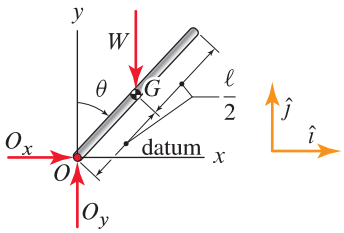
A drawbridge of length $\ell = 30\,\text{ft}$ and weight $W = 600\,\text{lb}$ is released in the position shown and freely pivots clockwise until it strikes the right end of the moat. If the COR for the collision between the bridge and the ground is $e = 0.45$ and if the contact point between the bridge and the ground is effectively ℓ away from the bridge's pivot point, determine the angle to which the bridge rebounds after the collision. Neglect any possible source of friction.



Solution

The solution is organized in three parts. The first part will deal with the determination of the preimpact conditions between the bridge and the ground. This part will be solved applying the work-energy principle. The second part will concern the solution of the impact problem between the bridge and the ground. The third part will concern the determination of the rebound angle and will again be solved using the work-energy principle.

We define ① and ② to be the positions at released and when the bridge first strijes the ground respectively. From the above FBD we see that the only force doing work is gravity. Hence, we can apply the work-energy principle in the form of the conservation of energy, i.e.,



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, given the fact that the system is released from rest, our choice of datum, and that the system is in a fixed axis rotation about O once released, we have

$$T_1 = 0 \quad V_1 = \tfrac{1}{2}\ell W, \quad T_2 = \tfrac{1}{2} I_O \omega_{b2}^2, \quad V_2 = 0, \tag{2}$$

and where $\vec{\omega}_b = \omega_b \hat{k}$ is the angular velocity of the bridge and, letting g denote the acceleration due to gravity,

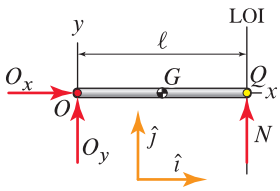
$$I_O = \frac{1}{12}\frac{W}{g}\ell + \frac{W}{g}\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}\frac{W}{g}\ell^2. \tag{3}$$

Substituting Eqs. (2) into Eq. (1), using the expression of I_O in Eq. (3), and solving for ω_{b2}, we have

$$\omega_{b2} = -\sqrt{\frac{3g}{\ell}}, \tag{4}$$

where the minus sign is due to the fact that we know that the bridge is rotating clockwise in going from ① to ②.

Here we consider the impact of the bridge and the ground. The impact-relevant FBD of the bridge illustrates the fact that this is a constrained impact in which no momentum conservation principle can be invoked. The only governing equation that can be written for this impact is the coefficient of restitution equation long the LOI. Let Q be the point on the bridge that comes into contact with the ground. Then, accounting for the fact that the ground does not move, we have



$$v_{Qy}^{+} = -e v_{Qy}^{-}. \tag{5}$$

Now observe that ② defined earlier coincides with the preimapact configuration of the body. Hence, using rigid body kinematics, and observing that the body can only rigidly rotate about O, we must have

$$v_{Qy}^{-} = \omega_{b2}\ell = -\sqrt{eg\ell} \quad \text{and} \quad v_{Qy}^{+} = \omega_b^{+}\ell. \tag{6}$$

Substituting Eq. (6) into Eq. (5) and solving for ω_b^+, we have

$$\omega_b^{+} = e\sqrt{\frac{3g}{\ell}}. \tag{7}$$

We now apply the work-energy principle again to determine the rebound angle. Let ③ and ④ be the postimpact and max swing angle positions, respectively, The FBD for this part of the problem is identical to that given at the beginning of the problem. Hence, we have

$$T_3 + V_3 = T_4 + V_4, \tag{8}$$

where, using Eq. (3) and the result in eq. (7), as well as the datum defined earlier and the fact that the angular velocity of the bridge at ④ is equal to zero, we have

$$T_3 = \frac{1}{2}\frac{1}{3}\frac{W}{g}\ell^2\left(e\sqrt{\frac{3g}{\ell}}\right)^2 = \tfrac{1}{2}e^2\ell W, \quad V_3 = 0, \quad T_2 = 0, \quad V_2 = \tfrac{1}{2}\ell\cos\theta_4 W. \tag{9}$$

Substituting Eqs. (9) into Eq. (8) and solving for θ_4, we have

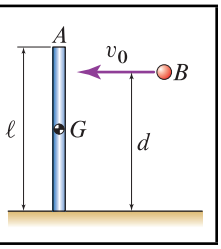
$$\theta_4 = \cos^{-1}(e^2), \tag{10}$$

which, recalling that θ_4 is the rebound angle we are looking for and that $e = 0.45$, we have

$$\boxed{\theta_{\text{rebound}} = 78.3^\circ.}$$

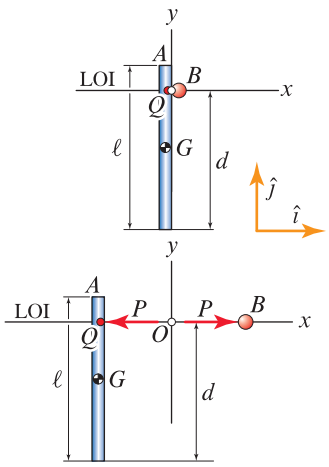
Problem 8.88

A stick A with length $\ell = 1.55\,\text{m}$ and mass $m_A = 6\,\text{kg}$ is in static equilibrium as shown when a ball B with mass $m_B = 0.15\,\text{kg}$ traveling at a speed $v_0 = 30\,\text{m/s}$ strikes the stick at distance $d = 1.3\,\text{m}$ from the lower end of the stick. If the COR for the impact is $e = 0.85$, determine the velocity of the mass center G of the stick as well as the stick's angular velocity right after the impact.



Solution

The point on the bar A that first comes into contact with the ball B is point Q. As indicated in the FBDs on the right, we will use a Cartesian coordinate system with origin at the fixed point O that happens to coincide with the points Q and B at the time of impact. The LOI coincides with the chosen x axis. The impact is an unconstrained direct eccentric impact between the rigid body A and the particle B. This impact is therefore governed by the following equations:



$$m_A v_{Gx}^- + m_B v_{Bx}^- = m_A v_{Gx}^+ + m_B v_{Bx}^+, \tag{1}$$

$$v_{Gy}^- = v_{Gy}^+, \tag{2}$$

$$v_{By}^- = v_{By}^+, \tag{3}$$

$$I_G \vec{\omega}_A^- + \vec{r}_{G/O} \times \vec{v}_G^- = I_G \vec{\omega}_A^+ + \vec{r}_{G/O} \times \vec{v}_G^+, \tag{4}$$

$$v_{Qx}^+ - v_{Bx}^+ = e(v_{Bx}^- - v_{Qx}^-), \tag{5}$$

where I_G is the mass oment of inertia of bar A about its own mass center. Equation (1) expresses the conservation of the linear momentum of the system along the LOI. Equation (2) expresses the conservation of the linear momentum of A along the line perpendicular to the LOI. Equation (3) expresses the conservation of the linear momentum of B along the line perpendicular to the LOI. Equation (4) expresses the conservation of the angular linear momentum of A about the fixed point O. Finally, Eq. (5) is the COR equation along the line of impact expressed in terms of the velocities of the points that actually come into contact.

We now consider the kinematic equations of the problem. The preimpact velocities are specified and can be summarized as follows:

$$\omega_A^- = 0, \quad v_{Gx}^- = 0, \quad v_{Gy}^- = 0, \quad v_{Qx}^- = 0, \quad v_{Bx}^- = -v_0, \quad \text{and} \quad v_{By}^- = 0. \tag{6}$$

We now need to express the postimpact velocity of Q in terms of post-mpact the velocity of G and of the postimpact angular velocity of A. Using rigid body kinematics we have

$$\vec{v}_Q^+ = \vec{v}_G^+ + \vec{\omega}_A^+ \times \vec{r}_{Q/G} \quad \Rightarrow \quad \vec{v}_Q^+ = v_{Gx}^+ \hat{\imath} + v_{Gy}^+ \hat{\jmath} + \omega_A^+ \hat{k} \times \left(d - \tfrac{1}{2}\ell\right) \hat{\jmath}$$

$$\Rightarrow \quad v_{Qx}^+ = v_{Gx}^+ - \left(d - \tfrac{1}{2}\ell\right)\omega_A^+. \tag{7}$$

Substituting Eqs. (6) and the last of Eqs. (7) into Eqs. (1)–(5) we have

$$-m_B v_0 = m_A v_{Gx}^+ + m_B v_{Bx}^+, \tag{8}$$

$$0 = v_{Gy}^+, \tag{9}$$

$$0 = v_{By}^+, \tag{10}$$

$$0 = \tfrac{1}{12} m_A \ell^2 \omega_A^+ + \left(d - \tfrac{1}{2}\ell\right) v_{Gx}^+, \tag{11}$$

$$v_{Gx}^+ - \left(d - \tfrac{1}{2}\ell\right)\omega_A^+ - v_{Bx}^+ = -e v_0, \tag{12}$$

where we have used the fact that $I_G = \frac{1}{12} m_A \ell^2$ and $\vec{r}_{G/O} = -\left(d - \frac{1}{2}\ell\right) \hat{\jmath}$, and where we have written Eq. (11) in scalar form since the corresponding vector equation has only one nonzero component, namely that in the z direction.

Equations (9) and (10) simply indicate that neither G nor B move in the y direction right after impact. Equations (8), (11), and (12) form a system of three equations in the three unknowns ω_A^+, v_{Gx}^+, and v_{Bx}^+ whose solution is

$$\omega_A^+ = \frac{6 m_B v_0 (1+e)(2d-\ell)}{12 d m_B (d-\ell) + (m_A + 4m_B)\ell^2}, \tag{13}$$

$$v_{Gx}^+ = -\frac{m_B v_0 (1+e)\ell^2}{12 d m_B (d-\ell) + (m_A + 4m_B)\ell^2}, \tag{14}$$

$$v_{Bx}^+ = \frac{m_A v_0 (1+e)\ell^2}{12 d m_B (d-\ell) + (m_A + 4m_B)\ell^2} - v_0. \tag{15}$$

Hence, using Eqs. (14), (9), and (13), we have

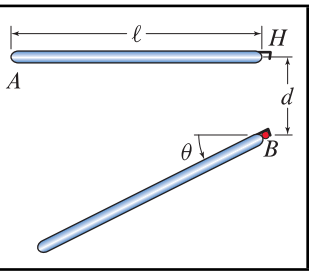
$$\boxed{\vec{v}_G^+ = (-1.31\,\text{m/s})\,\hat{\imath} \quad \text{and} \quad \vec{\omega}_A^+ = (3.43\,\text{rad/s})\,\hat{k}}$$



where we have used the following numerical data: $m_B = 0.15\,\text{kg}$, $v_0 = 30\,\text{m/s}$, $e = 0.85$, $d = 1.3\,\text{m}$, $\ell = 1.55\,\text{m}$, and $m_A = 6\,\text{kg}$.

Problem 8.89

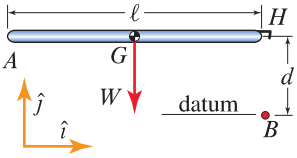
A uniform bar A with a hook H at the end is dropped as shown from a height $d = 3$ ft over a fixed pin B. Letting the weight and length of A be $W = 100$ lb and $\ell = 7$ ft, respectively, determine the angle θ that the bar will sweep through if the bar becomes hooked with B and does not rebound. Although bar A becomes hooked with B, assume that there is no friction between the hook and the pin.



Solution

The solution of this problem is organized in three parts. The first is the determination of the impact speed between the bar and the pin B. The second is the determination of the postimpact angular velocity of the bar using impact theory. The third is the determination of the angle swept by the bar using the work-energy principle.

We define ① to be the position of the bar at release and ② to be the position of the bar right before impact. Since there is no force that would cause the bar to rotate between ① and ②, the bar will translate between the two positions. The only force doing work is gravity. Hence, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

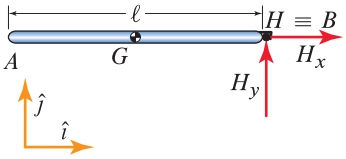
where, accounting for the fact that the system is only translating after being released from rest and due to our choice of datum, we have

$$T_1 = 0, \quad T_2 = \tfrac{1}{2}(W/g)v_{G2}^2, \quad V_1 = Wd, \quad V_2 = 0, \tag{2}$$

where g is the acceleration due to gravity, W/g is the mass of the bar, G is the mass center of the bar, and v_G is the speed of G. Substituting Eqs. (2) into Eq. (1) and solving for v_{G2}, we have

$$v_{G2} = \sqrt{2gd}. \tag{3}$$

Now, we deal with the impact part of the problem with the understanding that the preimpact conditions coincide with the conditions in ② just computed. The impact-relevat FBD is shown on the right, where we observe that the hook H now is in contact (and therefore coincides) with the fixed point B. We classify the impact as a constrained eccentric impact and its behavior of governed by the conservation of angular momentum of the bar about B, which, in scalar form reads



$$h_{Bz}^- = h_{Bz}^+, \tag{4}$$

where, recalling that before impact the bar is simply translating in the vertical direction and that after impact the bar is in a fixed axis rotation about B, we have

$$h_{Bz}^- = -\tfrac{1}{2}\ell(W/g)v_{Gy}^- \quad \text{and} \quad h_{Bz}^+ = I_B\omega_b^+, \tag{5}$$

where v_{Gy} is the y component of the velocity of the center of mass G, I_B is the mass moment of inertia of the bar about B, and $\vec{\omega}_b = \omega_b\,\hat{k}$ is the angular velocity of the bar.

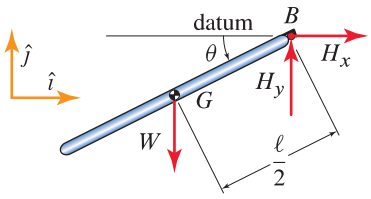
Using the result from the work-energy part of the solution and using the parallel axis theorem, we have

$$v_{\bar{G}y} = -v_{G2} = -\sqrt{2gd} \quad \text{and} \quad I_B = \tfrac{1}{12}(W/g)\ell^2 + (W/g)(\ell/2)^2 = \tfrac{1}{3}(W/g)\ell^2. \tag{6}$$

Then substituting Eqs. (6) into Eqs. (5) and then substituting the result in Eq. (4), we have

$$\tfrac{1}{2}\ell(W/g)\sqrt{2gd} = \tfrac{1}{3}(W/g)\ell^2\omega_b^+ \quad \Rightarrow \quad \omega_b^+ = \frac{3\sqrt{gd}}{\sqrt{2}\ell}. \tag{7}$$

We now apply the work-energy principle again. We define ③ to be the position of the bar immediately after impact and ④ to be the position of the bar as soon as it reaches the maximum swing angle θ_{swept}. The FBD of the bar in a generic position between ③ and ④ is shown on the right. The reactions H_x and H_y are applied to the fixed point B, the only force doing work is gravity, so that the work-energy principle can be written in the following form:



$$T_3 + V_3 = T_4 + V_4, \tag{8}$$

where, accounting for the fact that the system is in a fixed axis rotation about B and due to our choice of datum, we have

$$T_3 = \tfrac{1}{2}I_B\omega_{b3}^2, \quad T_4 = 0, \quad V_3 = 0, \quad V_4 = -W\tfrac{1}{2}\ell\sin\theta_{\text{swept}}. \tag{9}$$

Recalling that ③ coincides with the position of the system right after impact and repeating the second of Eqs. (6), we have

$$\omega_{b3} = \omega_{b3}^+ = \frac{3\sqrt{gd}}{\sqrt{2}\ell} \quad \text{and} \quad I_B = \tfrac{1}{3}(W/g)\ell^2. \tag{10}$$

Substituting Eqs. (10) into Eqs. (9) and then substituting the result in Eq. (8), we obtain

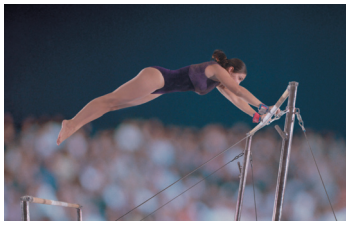
$$\tfrac{1}{6}(W/g)\ell^2\frac{9gd}{2\ell^2} = -W\tfrac{1}{2}\ell\sin\theta_{\text{swept}} \quad \Rightarrow \quad \theta_{\text{swept}} = -\sin^{-1}\left(\frac{3d}{2\ell}\right). \tag{11}$$

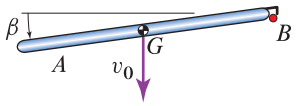
Now, recalling that $d = 3$ ft and $\ell = 7$ ft, observe that $-\sin^{-1}[3d/(2\ell)] = -40.01°$, and therefore, based on the definition of the angle θ, this corresponds to a solution for θ_{swept} equal to $180° + 40.01°$, i.e.,

$$\boxed{\theta_{\text{swept}} = 220°.}$$

Problem 8.90

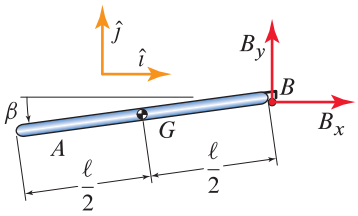
A gymnast on the uneven parallel bars has a vertical speed v_0 and no angular speed when she grasps the upper bar. Model the gymnast as a single uniform rigid bar A of weight $W = 92\,\text{lb}$ and length $\ell = 6\,\text{ft}$. Neglecting all friction, letting $\beta = 12°$, and assuming that the upper bar B does not move after the gymnast grasps it, determine v_0 if the gymnast is to swing (counterclockwise) so as to become horizontal. Assume that, during the motion, the friction between the gymnast's hands and the upper bar is negligible.





Solution

The problem can be divided into two parts. First we relate v_0 to the angular velocity of the gymnast right after she grasps the upper bar. Then we use the work energy principle to relate the angular velocity in question to the final position of the gymnast. The impact-relevant FBD of the system is shown to the right. This FBD implies that the angular momentum of the gymnast about B is conserved, i.e.,



$$I_G\omega_g^- - m_g v_{Gy}^-(\ell/2)\cos\beta = I_B\omega_g^+, \tag{1}$$

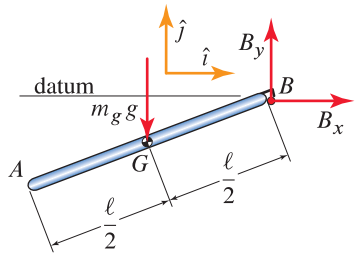
where m_g is the mass of the gymnast, I_G and I_B are the mass moments of inertia of the gymnast about her own mass center and point B, respectively, and where we have used the fact that, after she grasps the upper bar, the gymnast will be in a fixed axis rotation about B. For I_G and I_B we have

$$I_G = \tfrac{1}{12}m_g\ell^2 \quad \text{and} \quad I_B = I_G + m_g(\ell/2)^2 = \tfrac{1}{3}m_g\ell^2. \tag{2}$$

Recalling that $\omega_g^- = 0$ and that $v_{Gy}^- = -v_0$, substituting Eqs. (2) into Eq. (1) and solving for ω_g^+ we have

$$\omega_g^+ = \frac{3v_0\cos\beta}{2\ell}. \tag{3}$$

Now let ① and ② be the positions of the gymnast right after she grasps the upper bar and when she becomes horizontal. The FBD of the gymnast between ① and ② is shown to the right. Since the only forces acting on the system are conservative, we have



$$T_1 + V_1 = T_2 + V_2. \tag{4}$$

Since the gymnast is in a fixed axis rotation about B, and because of the choice of datum line, we have

$$T_1 = \tfrac{1}{3}I_B(\omega_g^+)^2, \quad V_1 = -m_g g(\ell/2)\sin\beta, \quad T_2 = 0, \quad V_2 = 0. \tag{5}$$

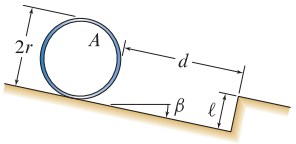
Substituting Eqs. (5) into Eq. (4) and solving for v_0, we have

$$v_0 = \sqrt{\frac{4g\ell \sin\beta}{3\cos^2\beta}} = 7.48\,\text{ft/s},$$

where we have used the following numerical data: $g = 32.2\,\text{ft/s}^2$, $\ell = 6\,\text{ft}$, and $\beta = 12°$.

Problem 8.91

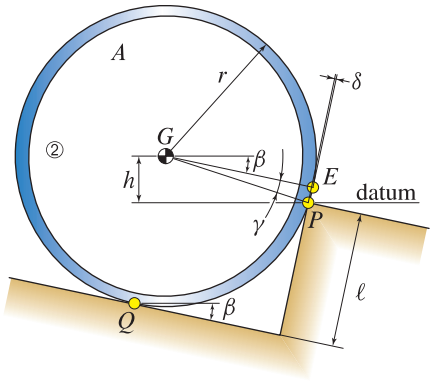
A uniform thin ring A of mass $m = 7\,\text{kg}$ and radius $r = 0.5\,\text{m}$ is released from rest as shown and rolls without slip until it meets a step of height $\ell = 0.45\,\text{m}$. Letting $\beta = 12°$ and assuming that the ring does not rebound off the step or slip relative to it, determine the distance d such that the ring barely makes it over the step.



Solution

The solution is organized in three parts. In the first part we relate the preimpact angular velocity of the ring to the distance d. This part will be solved applying the work-energy principle. The second part will concern the solution of the impact problem between the ring and the step. The solution of this second part will allow us to express the postimpact velocity of the ring as a function of d. In the third part we will establish a relation between the postimpact angular velocity of the ring, viewed as a function of d, and the size of the step such that the ring barely makes it over the step. This last part will be solved via the work-energy principle.

We denote by ① the position of the ring at release. We then denote by ② the position of the ring when it first strikes the step. To make the calculations easier to follow, referring to the figure on the right, we now consider some geometrical details of the ring at ②. Point P is the edge of the step. Point G is both the center and the center of mass of the ring. Point E is the point on the ring such that the segment GE is parallel to the incline. While this may appear a small detail, point E hangs a distance δ over the step. The length h denotes the vertical distance in ② between G and P. Using trigonometry we have



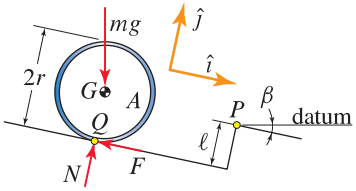
$$r\sin\gamma = r - \ell \quad \Rightarrow \quad \gamma = \sin^{-1}\left(\frac{r-\ell}{r}\right) = 5.739°, \qquad (1)$$

where we have used the following numerical data: $r = 0.5\,\text{m}$ and $\ell = 0.45\,\text{m}$. Furthermore, we have

$$\delta = r(1-\cos\gamma) = 0.002506\,\text{m} \quad \text{and} \quad h = r\sin(\beta+\gamma) = 0.1523\,\text{m}, \qquad (2)$$

where we have used the value of γ in Eq. (1) and the fact that $r = 0.5\,\text{m}$ and $\beta = 12°$.

We now apply the work-energy principle between ① and ②. Given the FBD on the right and recalling that the ring rolls without slip between ① and ②, we have



$$T_1 + V_1 = T_2 + V_2, \qquad (3)$$

where, given our choice of datum and again recalling that the system is released from rest and then rolls without slip, we have

$$T_1 = 0, \quad V_1 = mg[(d+\delta)\sin\beta + h], \quad T_2 = \tfrac{1}{2} I_Q \omega_{r2}^2, \quad V_2 = mgh, \qquad (4)$$

where $\vec{\omega}_r = \omega_r\,\hat{k}$ is the angular velocity of the ring, the length $d + \delta$ is the length parallel to the incline by which the center of mass G moves between ① and ②, and where I_Q is the mass moment of inertia with respect to Q and, applying the parallel axis theorem, is given by

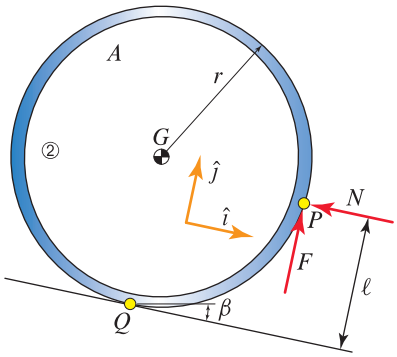
$$I_Q = mr^2 + mr^2 = 2mr^2. \qquad (5)$$

Substituting Eqs. (4) into Eq. (3) and using Eq. (5), we have

$$mg[(d+\delta)\sin\beta + h] = mr^2\omega_{r2}^2 + mgh \quad \Rightarrow \quad \omega_{r2} = -\frac{\sqrt{(d+\delta)g\sin\beta}}{r}, \tag{6}$$

where the minus sign expresses the fact that in ② the ring is rotating clockwise.

We now consider the collision of the ring with the step. The impact-relevant FBD is shown on the right and it indicates that this is a constrained impact. The only conservation principle we can invoke is the conservation of angular momentum about P. Keeping in mind that before impact G does not have a y component of velocity and that right after impact the ring moves as though in a fixed axis rotation about P, we have



$$I_G\omega_r^- - mv_{Gx}^-(r-\ell) = I_P\omega_r^+, \tag{7}$$

where

$$I_G = mr^2, \qquad I_P = mr^2 + mr^2 = 2mr^2, \tag{8}$$

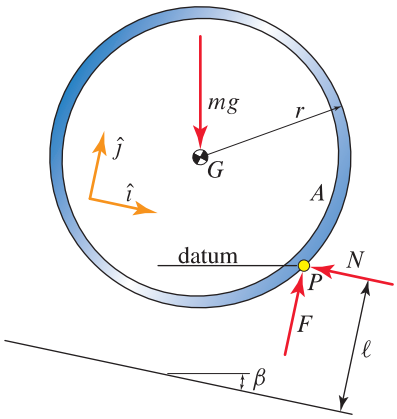
$$\omega_r^- = \omega_{r2} = -\frac{\sqrt{(d+\delta)g\sin\beta}}{r}, \qquad v_{Gx}^- = -r\omega_r^- = \sqrt{(d+\delta)g\sin\beta}, \tag{9}$$

where we have used the parallel axis theorem in the last of Eqs. (8), and where the last of Eqs. (9) expresses the fact that right before impact the ring is rolling without slip.

Substituting Eqs. (8) and (9) into Eq. (7) and solving for ω_r^+, we obtain

$$\omega_r^+ = -\frac{2r-\ell}{2r^2}\sqrt{(d+\delta)g\sin\beta}. \tag{10}$$

We now define ③ to coincide with the position of the ring right after impact and ④ with the position in which point G is vertically aligned with point P. To say that the ring barely makes it over the step implies that the ring comes to a stop in ④. The FBD of the ring in a generic position between ③ and ④ is shown on the right. Under the assumption that the ring does not slip over the edge of the step, we have



$$T_3 + V_3 = T_4 = V_4, \tag{11}$$

where, recalling that between ③ and ④ the ring moves as though in a fixed axis rotation about P, we have

$$T_3 = \tfrac{1}{2}I_P\omega_{r3}^2, \quad V_3 = mgh, \quad T_4 = 0, \quad V_4 = mgr, \tag{12}$$

where the length h is shown in the figure at the beginning of this solution.

Observing that I_P is given in the second of Eqs. (8) and that $\omega_{r3} = \omega_r^+$, which is given in Eq. (10), substituting Eqs. (12) into Eq. (11) we have

$$\tfrac{1}{2}2mr^2\left(\frac{2r-\ell}{2r^2}\sqrt{(d+\delta)g\sin\beta}\right)^2 + mgh = mgr \quad \Rightarrow \quad d = \frac{4r^2(r-h)}{(2r-\ell)^2\sin\beta} - \delta, \tag{13}$$

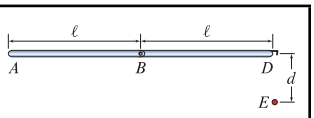
which can be evaluated to obtain

$$\boxed{d = 5.53\,\text{m},}$$

where we have used the following numerical data: $r = 0.5\,\text{m}$, $h = 0.1523\,\text{m}$ (see the second of Eq. (2)), $\ell = 0.45\,\text{m}$, $\beta = 12°$, and $\delta = 0.002506\,\text{m}$ (see the first of Eqs. (2)).

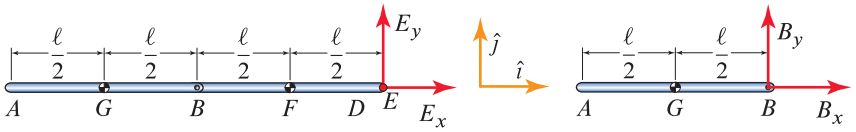
Problem 8.92

Two identical uniform bars AB and BD are pin-connected at B, and bar BD has a hook at the free end. The two bars are dropped as shown from a height $d = 3\,\text{ft}$ over a fixed pin E (shown in cross section). Letting the weight and length of each bar be $W = 100\,\text{lb}$ and $\ell = 7\,\text{ft}$, respectively, determine the angular velocities of AB and BD immediately after bar BD becomes hooked on E and does not rebound. *Hint:* The angular momentum of bar AB is conserved about B during impact.



Solution

The figure below shows the impact-relevant FBDs of the system and of bar AB.



FBD of system FBD of AB

These FBDs imply that the angular momentum of the system about E and the angular momentum of bar AB about B are conserved through the impact. Hence, letting $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$ and $\vec{\omega}_{BD} = \omega_{BD}\,\hat{k}$ denote the angular velocities of AB and BD, respectively, we have

$$I_G\omega_{AB}^- - mv_{Gy}^-\tfrac{3}{2}\ell - I_F\omega_{BD}^- - mv_{Fy}^-\tfrac{1}{2}\ell = I_G\omega_{AB}^+ - mv_{Gy}^+\tfrac{3}{2}\ell - I_F\omega_{BD}^+ - mv_{Fy}^+\tfrac{1}{2}\ell, \tag{1}$$

$$I_G\omega_{AB}^- - mv_{Gy}^-\tfrac{1}{2}\ell = I_G\omega_{AB}^+ - mv_{Gy}^+\tfrac{1}{2}\ell, \tag{2}$$

where, m is the common value of the masses of the bars AB and BD, and where I_G and I_F are the mass moments of inertia of bars AB and BD about their respective mass centers. Since these bars are identical, we have

$$I_G = I_F = \tfrac{1}{12}m\ell^2. \tag{3}$$

Since both bars fall from rest over a distance d while remaining horizontal, we have

$$v_{Gy}^- = -\sqrt{2gd}, \quad \omega_{AB}^- = 0, \quad v_{Fy}^- = -\sqrt{2gd}, \quad \omega_{BD}^- = 0. \tag{4}$$

Once bar BD is hooked at E, bar BD will be in a fixed axis rotation about E. Using this fact, and recalling that we must have $\vec{v}_G = \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{G/B}$, right after impact (i.e., while the bars are still in the same horizontal position they had right before impact), we have

$$v_{Fy}^+ = -\omega_{BD}^+(\ell/2), \quad v_{By}^+ = -\omega_{BD}^+\ell \quad \Rightarrow \quad v_{Gy}^+ = -\omega_{BD}^+\ell - \omega_{AB}^+(\ell/2). \tag{5}$$

Substituting Eqs. (3)–(5) into Eqs. (1) and (2), we obtain a system of two equations in the two unknowns ω_{AB}^+ and ω_{BD}^+ whose solution is

$$\omega_{AB}^+ = -\frac{3\sqrt{2dg}}{7\ell} \quad \text{and} \quad \omega_{BD}^+ = \frac{9\sqrt{2dg}}{7\ell}, \tag{6}$$

which can be evaluated to obtain

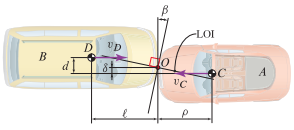
$$\vec{\omega}_{AB}^+ = (-0.851\,\text{rad/s})\,\hat{k} \quad \text{and} \quad \vec{\omega}_{BD}^+ = (2.55\,\text{rad/s})\,\hat{k}$$



where we have used the following numerical data: $d = 3\,\text{ft}$, $\ell = 7\,\text{ft}$, and $g = 32.2\,\text{ft/s}^2$.

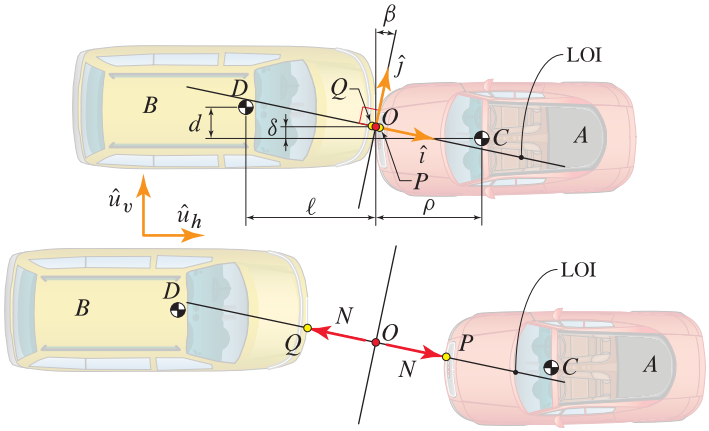
Problem 8.93

Cars A and B collide as shown. Determine the angular velocities of A and B immediately after impact if the COR is $e = 0.35$. In solving the problem, let C and D be the mass centers of A and B, respectively. In addition, enforce assumption 3 on p. 649 and use the following data: $W_A = 3130\,\text{lb}$ (weight of A), $k_C = 34.5\,\text{in.}$ (radius of gyration of A), $v_C = 12\,\text{mph}$ (speed of the mass center of A), $W_B = 3520\,\text{lb}$ (weight of B), $k_D = 39.3\,\text{in.}$ (radius of gyration of B), $v_D = 15\,\text{mph}$ (speed of the mass center of B), $d = 19\,\text{in.}$, $\ell = 79\,\text{in.}$, $\delta = 7.1\,\text{in.}$, $\rho = 65\,\text{in.}$, and $\beta = 12°$.



Solution

We model the impact as an unconstrainted oblique eccentric impact of two rigid bodies. The figure below shows the FBD of the system as a whole (top) and the FBDs of cars A and B separately (bottom).



The contact between the two bodies occurs at point P on car A and at point Q on car B. Point O is the fixed point in space that happens to coincide with points P and Q at the time of impact. We define a Cartesian coordinate system with origin at O and the x axis aligned with the LOI. For convenience, we also introduce unit vectors $\hat{u}_h$ and $\hat{u}_v$ (the subscripts h and v stand for horizontal and vertical, respectively) as shown on the FBDs. Based on the assumptions underlying the impact theory in the textbook, the impact is governed by the following relations:

$$m_A v_{Cx}^- + m_B v_{Dx}^- = m_A v_{Cx}^+ + m_B v_{Dx}^+, \tag{1}$$

$$v_{Cy}^- = v_{Cy}^+, \tag{2}$$

$$v_{Dy}^- = v_{Dy}^+, \tag{3}$$

$$(\vec{h}_O^-)_A = (\vec{h}_O^+)_A, \tag{4}$$

$$(\vec{h}_O^-)_B = (\vec{h}_O^+)_B, \tag{5}$$

$$v_{Qx}^+ - v_{Px}^+ = e(v_{Px}^- - v_{Qx}^-), \tag{6}$$

which, in order, express the conservation of linear momentum for the system along the LOI, the conservation of linear momentum for car A in the direction perpendicular to the LOI, the conservation of linear momentum for car B in the direction perpendicular to the LOI, the conservation of the angular momentum of A about the fixed point O, the conservation of the angular momentum of B about the fixed point O, and the COR equation along the LOI written in terms of the velocities of the points on A and B that actually come into contact.

We now need to express the angular momentum conservation laws and the COR equation in terms of the the velocities of the cars' mass centers as well as the cars' angular velocities. For this purpose, we introduce the following unit vectors:

$$\hat{u}_h = \cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_v = -\sin\beta\,\hat{\imath} = \cos\beta\,\hat{\jmath}. \tag{7}$$

We also introduce the position vectors necessary to compute the angular momenta of the cars as well as the velocities of points P and Q:

$$\vec{r}_{C/O} = \rho\,\hat{u}_h - \delta\,\hat{u}_v = (\rho\cos\beta + \delta\sin\beta)\,\hat{\imath} + (\rho\sin\beta - \delta\cos\beta)\,\hat{\jmath}, \tag{8}$$
$$\vec{r}_{D/O} = -\ell\,\hat{u}_h + d\,\hat{u}_v = -(\ell\cos\beta + d\sin\beta)\,\hat{\imath} + (-\ell\sin\beta + d\cos\beta)\,\hat{\jmath}, \tag{9}$$
$$\vec{r}_{P/C} = -\vec{r}_{C/O} = -(\rho\cos\beta + \delta\sin\beta)\,\hat{\imath} - (\rho\sin\beta - \delta\cos\beta)\,\hat{\jmath}, \tag{10}$$
$$\vec{r}_{Q/D} = -\vec{r}_{D/O} = (\ell\cos\beta + d\sin\beta)\,\hat{\imath} + (\ell\sin\beta - d\cos\beta)\,\hat{\jmath}. \tag{11}$$

Next we describe the velocity vectors needed to evaluate the angular momenta and the COR equation. Specifically, accounting for the preimpact conditions, for car A, we write

$$\vec{v}_C^{\,-} = -v_C\,\hat{u}_h = -v_C(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}), \qquad \vec{v}_C^{\,+} = v_{Cx}^+\,\hat{\imath} + v_{Cy}^+\,\hat{\jmath}, \tag{12}$$
$$\vec{\omega}_A^{\,-} = \vec{0}, \qquad \vec{\omega}_A^{\,+} = \omega_A^+\,\hat{k}, \tag{13}$$
$$v_{Px}^- = -v_C\cos\beta, \qquad v_{Px}^+ = v_{Cx}^+ + \omega_A^+(\rho\sin\beta - \delta\cos\beta), \tag{14}$$

where, to evaluate v_{Px}^- and v_{Px}^+, we have used rigid body kinematics according to which we must have $\vec{v}_P^{\,\pm} = \vec{v}_C^{\,\pm} + \vec{\omega}_A^{\,\pm} \times \vec{r}_{P/C}$ (which requires the use of Eq. (10) above). Similarly, for car B we have

$$\vec{v}_D^{\,-} = v_D\,\hat{u}_h = v_D(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}), \qquad \vec{v}_D^{\,+} = v_{Dx}^+\,\hat{\imath} + v_{Dy}^+\,\hat{\jmath}, \tag{15}$$
$$\vec{\omega}_B^{\,-} = \vec{0}, \qquad \vec{\omega}_B^{\,+} = \omega_B^+\,\hat{k}, \tag{16}$$
$$v_{Qx}^- = v_D\cos\beta, \qquad v_{Qx}^+ = v_{Dx}^+ + \omega_B^+(d\cos\beta - \ell\sin\beta), \tag{17}$$

where, to evaluate v_{Qx}^- and v_{Qx}^+, we have used rigid body kinematics according to which we must have $\vec{v}_Q^{\,\pm} = \vec{v}_D^{\,\pm} + \vec{\omega}_B^{\,\pm} \times \vec{r}_{Q/D}$ (which requires the use of Eq. (11) above).

We are now in a position to compute the expressions of the angular momenta of A and B about point O that we will need to substitute in Eqs. (4) and (5). Specifically, we have

$$(\vec{h}_O^{\,-})_A = I_C\vec{\omega}_A^{\,-} + \vec{r}_{C/O} \times m_A\vec{v}_C^{\,-} = -m_A v_C\,\delta\,\hat{k}, \tag{18}$$
$$\begin{aligned}(\vec{h}_O^{\,+})_A &= I_C\vec{\omega}_A^{\,+} + \vec{r}_{C/O} \times m_A\vec{v}_C^{\,+} \\ &= \left[I_C\omega_A^+ + m_A v_{Cx}^+(\delta\cos\beta - \rho\sin\beta) + m_A v_{Cy}^+(\delta\sin\beta + \rho\cos\beta)\right]\hat{k},\end{aligned} \tag{19}$$
$$(\vec{h}_O^{\,-})_B = I_D\vec{\omega}_B^{\,-} + \vec{r}_{D/O} \times m_B\vec{v}_D^{\,-} = -m_B v_D\,d\,\hat{k}, \tag{20}$$
$$\begin{aligned}(\vec{h}_O^{\,+})_B &= I_D\vec{\omega}_A^{\,+} + \vec{r}_{C/O} \times m_B\vec{v}_D^{\,+} \\ &= \left[I_D\omega_B^+ + m_B v_{Dx}^+(-d\cos\beta + \ell\sin\beta) - m_B v_{Dy}^+(d\sin\beta + \ell\cos\beta)\right]\hat{k},\end{aligned} \tag{21}$$

where we have made use of Eqs. (8) and (9) and well as of Eqs. (12), (13), (15), and (16).

We are now ready to rewrite Eqs. (1)–(6). Specifically, using the appropriate kinematic relations developed above we have

$$-m_A v_C\cos\beta + m_B v_D\cos\beta = m_A v_{Cx}^+ + m_B v_{Dx}^+, \tag{22}$$
$$-v_C\sin\beta = v_{Cy}^+, \tag{23}$$

$$v_D \sin\beta = v_{Dy}^+, \tag{24}$$

$$-m_A v_C \delta = I_C \omega_A^+ + m_A v_{Cx}^+ (\delta \cos\beta - \rho \sin\beta) + m_A v_{Cy}^+ (\delta \sin\beta + \rho \cos\beta), \tag{25}$$

$$-m_B v_D d = I_D \omega_B^+ + m_B v_{Dx}^+ (-d \cos\beta + \ell \sin\beta) - m_B v_{Dy}^+ (d \sin\beta + \ell \cos\beta), \tag{26}$$

$$v_{Dx}^+ + \omega_B^+ (d \cos\beta - \ell \sin\beta) - v_{Cx}^+ + \omega_A^+ (\rho \sin\beta - \delta \cos\beta) = -e(v_C \cos\beta + v_D \cos\beta), \tag{27}$$

where we have written Eqs. (25) and (26) in scalar form because the corresponding vector equations had only one nonzero component, namely, that in the z direction.

Equations (22)–(27) form a system of six equations in the six unknowns v_{Cx}^+, v_{Cy}^+, ω_A^+, v_{Dx}^+, v_{Dy}^+, ω_B^+. Notice that Eqs. (23) and (24), allow us to directly evaluate v_{Cy}^+ and v_{Dy}^+, wfor which we obtain

$$v_{Cy}^+ = -3.659\,\text{ft/s} \quad \text{and} \quad v_{Dy}^+ = 4.574\,\text{ft/s}, \tag{28}$$

where we have used the following numerical data: $v_C = 12.0\,\text{mph} = 17.60\,\text{ft/s}$, $\beta = 12°$, and $v_D = 15\,\text{mph} = 22.00\,\text{ft/s}$. This result allows us to treat the quantities in Eqs. (28) as known, and therefore to rewrite Eqs. (22) and (25)–(27) numerically as the following system of 4 equations in the 4 unknowns v_{Cx}^+, ω_A^+, v_{Dx}^+, ω_B^+:

$$(97.20\,\text{slug})v_{Cx}^+ + (109.3\,\text{slug})v_{Dx}^+ = 678.7\,\text{lb·s}, \tag{29}$$

$$(-53.22\,\text{slug·ft})v_{Cx}^+ + (803.5\,\text{slug·ft}^2)\omega_A^+ = 916.0\,\text{ft·lb·s}, \tag{30}$$

$$(-19.64\,\text{slug·ft})v_{Dx}^+ + (1172\,\text{slug·ft}^2)\omega_B^+ = -422.0\,\text{ft·lb·s}, \tag{31}$$

$$-v_{Cx}^+ + v_{Dx}^+ + (-0.5475\,\text{ft})\omega_A^+ + (0.1797\,\text{ft})\omega_B^+ = -13.56\,\text{ft/s}, \tag{32}$$

where we have placed all of the unknowns on the left-hand sides of the above equaitions and where, in addition to the results in Eqs. (28), we have used the following numerical data:

$$\delta = 7.1\,\text{in.} = 0.5917\,\text{ft}, \quad \rho = 65\,\text{in.} = 5.417\,\text{ft}, \tag{33}$$

$$d = 19\,\text{in.} = 1.583\,\text{ft}, \quad \ell = 79\,\text{in.} = 6.583\,\text{ft}, \quad e = 0.35, \tag{34}$$

and

$$m_A = (W_A/g) = 97.20\,\text{slug}, \qquad I_C = (W_A/g)k_C^2 = 803.5\,\text{slug·ft}^2, \tag{35}$$

$$m_B = (W_B/g) = 109.3\,\text{slug}, \qquad I_D = (W_B/g)k_D^2 = 1172\,\text{slug·ft}^2, \tag{36}$$

where g is the acceleration due to gravity and $W_A = 3130\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $k_C = 34.5\,\text{in.} = 2.875\,\text{ft}$, $W_B = 3520\,\text{lb}$, and $k_D = 39.3\,\text{in.} = 1.583\,\text{ft}$. The solution of Eqs. (29)–(32) is as follows:

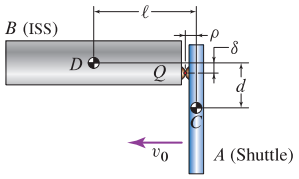
$$v_{Cx}^+ = 9.905\,\text{ft/s}, \quad \omega_A^+ = 1.796\,\text{rad/s}, \quad v_{Dx}^+ = -2.599\,\text{ft/s}, \quad \omega_B^+ = -0.4036\,\text{rad/s}. \tag{37}$$

Hence, expressing the answer in vector form and using three significant figures, we have

$$\boxed{\vec{\omega}_A^+ = (1.80\,\text{rad/s})\,\hat{k} \quad \text{and} \quad \vec{\omega}_B^+ = (-0.404\,\text{rad/s})\,\hat{k} \quad \hat{\imath}\,\hat{\jmath}\,@ -12°.}$$

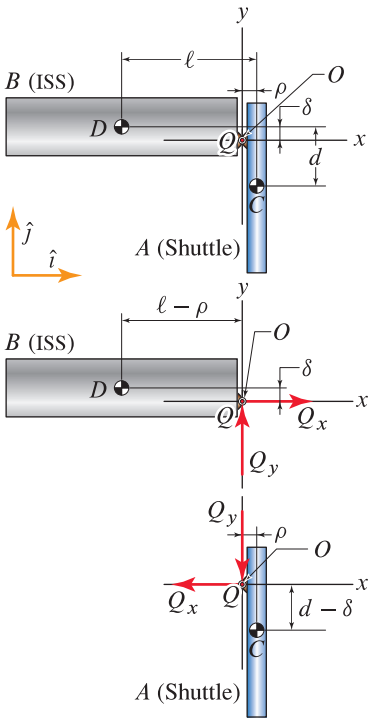
Problem 8.94

Consider the collision of two rigid bodies A and B, which, referring to Example 8.10 on p. 635, models the docking of the Space Shuttle (body A) to the International Space Station (body B). As in Example 8.10, we assume that B is stationary relative to an inertial frame of reference and that A translates as shown. In contrast to Example 8.10, here we assume that A and B join at point Q but, due to the flexibility of the docking system, can rotate relative to one another. Determine the angular velocities of A and B right after docking if $v_0 = 0.03\,\text{m/s}$. In solving the problem, let C and D be the centers of mass of A and B, respectively. In addition, let the mass and mass moment of inertia of A be $m_A = 120{\times}10^3\,\text{kg}$ and $I_C = 14{\times}10^6\,\text{kg}{\cdot}\text{m}^2$, respectively, and the mass and mass moment of inertia of B be $m_B = 180{\times}10^3\,\text{kg}$ and $I_D = 34{\times}10^6\,\text{kg}{\cdot}\text{m}^2$, respectively. Finally, use the following dimensions: $\ell = 24\,\text{m}$, $d = 8\,\text{m}$, $\rho = 2.6\,\text{m}$, and $\delta = 2.4\,\text{m}$.



Solution

On the right we show the impact-relevant FBD of the system as a whole (top) and of A and B individually (two bottom diagrams). Point O is the fixed point in space which happens to coincide with point Q when A and B first come into contact. We model the impact as a constrained eccentric impact. Notice that we cannot apply a coefficient of restitution equation in this problem since the impacting bodies become connected after impact. The governing equations for this impact are therefore as follows:



$$m_A v_{Cx}^- + m_B v_{Dx}^- = m_A v_{Cx}^+ + m_B v_{Dx}^+, \tag{1}$$

$$m_A v_{Cy}^- + m_B v_{Dy}^- = m_A v_{Cy}^+ + m_B v_{Dy}^+, \tag{2}$$

$$I_C \omega_A^- + m_A[v_{Cx}^-(d-\delta) + v_{Cy}^-\rho] = I_C \omega_A^+ + m_A[v_{Cx}^+(d-\delta) + v_{Cy}^+\rho], \tag{3}$$

$$I_D \omega_B^- - m_B[v_{Dx}^-\delta + v_{Dy}^-(\ell-\rho)] = I_D \omega_B^+ - m_B[v_{Dx}^+\delta + v_{Dy}^+(\ell-\rho)], \tag{4}$$

which, in order, express the conservation of linear momentum for the system in the x direction, the conservation of linear momentum for the system in the y direction, the conservation of the angular momentum for A about O, and the conservation of angular momentum about O for B. The kinematic equations for this problem must enforce the fact that the two bodies become connected at Q after impact. For this purpose we introduce the following vectors:

$$\vec{\omega}_A^+ = \omega_A^+ \hat{k}, \qquad \vec{\omega}_B^+ = \omega_B^+ \hat{k}, \tag{5}$$

$$\vec{v}_C^+ = v_{Cx}^+ \hat{\imath} + v_{Cy}^+ \hat{\jmath}, \qquad \vec{v}_D^+ = v_{Dx}^+ \hat{\imath} + v_{Dy}^+ \hat{\jmath}, \tag{6}$$

$$\vec{r}_{Q/C} = -\rho\hat{\imath} + (d-\delta)\hat{\jmath}, \qquad \vec{r}_{Q/D} = (\ell-\rho)\hat{\imath} - \delta\hat{\jmath}. \tag{7}$$

Then, the fact that A and B become connected at Q, i.e., that Q ia a common point of the two rigid bodies A and B, can be given the following vector form: $\vec{v}_Q^+ = \vec{v}_C^+ + \vec{\omega}_A^+ \times \vec{r}_{Q/C} = \vec{v}_+^D \vec{\omega}_B^+ \times \vec{r}_{Q/D}$. This

relationship, using Eqs. (5)–(7), carrying out the needed cross-products, and symplifying, yields the following two scalar relations:

$$v_{Cx}^{+} - (d-\delta)\omega_A^{+} = v_{Dx}^{+} + \delta\omega_B^{+}, \tag{8}$$

$$v_{Cy}^{+} - \rho\omega_A^{+} = v_{Dy}^{+} + (\ell-\rho)\omega_B^{+}. \tag{9}$$

Finally, we observe that the problem statement specifies the preimpact conditions, which are as follows:

$$v_{Cx}^{-} = -v_0, \quad v_{Cy}^{-} = 0, \quad \omega_A^{-} = 0, \quad v_{Dx}^{-} = 0, \quad v_{Dy}^{-} = 0, \quad \omega_B^{-} = 0. \tag{10}$$

Using Eqs. (10) and taking advantage of the given data, i.e., recalling that $v_0 = 0.03\,\text{m/s}$, $m_A = 120\times10^3\,\text{kg}$, $I_C = 14\times10^6\,\text{kg·m}^2$, $m_B = 180\times10^3\,\text{kg}$, $I_D = 34\times10^6\,\text{kg·m}^2$, $\ell = 24\,\text{m}$, $d = 8\,\text{m}$, $\rho = 2.6\,\text{m}$, and $\delta = 2.4\,\text{m}$, we can rewrite Eqs. (1)–(4) and Eqs. (8) and (9) as follows:

$$\begin{aligned}
(1.200\times10^5\,\text{kg})v_{Cx}^{+} + (1.800\times10^5\,\text{kg})v_{Dx}^{+} &= -3600\,\text{kg·m/s},\\
(1.200\times10^5\,\text{kg})v_{Cy}^{+} + (1.800\times10^5\,\text{kg})v_{Dy}^{+} &= 0,\\
(6.720\times10^5\,\text{kg·m})v_{Cx}^{+} + (3.120\times10^5\,\text{kg·m})v_{Cy}^{+} + (1.400\times10^7\,\text{kg·m}^2)\omega_A^{+} &= -2.016\times10^4\,\text{kg·m}^2/\text{s},\\
(-4.320\times10^5\,\text{kg·m})v_{Dx}^{+} + (-3.852\times10^6\,\text{kg·m})v_{Dy}^{+} + (3.400\times10^7\,\text{kg·m}^2)\omega_B^{+} &= 0,\\
v_{Cx}^{+} + (-5.600\,\text{m})\omega_A^{+} - v_{Dx}^{+} - (2.400\,\text{m})\omega_B^{+} &= 0,\\
v_{Cy}^{+} - (2.600\,\text{m})\omega_A^{+} - v_{Dy}^{+} - (21.40\,\text{m})\omega_B^{+} &= 0,
\end{aligned}$$

where we have placed all unknowns on the left-had sides of the equations and all known terms on the right-hand side. The above system of equations is a system of 6 equations in the 6 unknowns v_{Cx}^{+}, v_{Cy}^{+}, ω_A^{+}, v_{Dx}^{+}, v_{Dy}^{+}, and ω_B^{+} whose solution is

$$\begin{aligned}
v_{Cx}^{+} &= -0.01444\,\text{m/s}, & v_{Cy}^{+} &= -0.001426\,\text{m/s}, & \omega_A^{+} &= -0.0007152\,\text{rad/s},\\
v_{Dx}^{+} &= -0.01037\,\text{m/s}, & v_{Dy}^{+} &= 0.0009505\,\text{m/s}, & \omega_B^{+} &= -0.00002414\,\text{rad/s}.
\end{aligned}$$

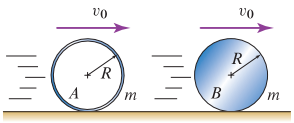
Hence, expressing our final result in vector form and to three significant figures, we have

$$\vec{\omega}_A^{+} = (-0.000715\,\text{rad/s})\,\hat{k} \quad \text{and} \quad \vec{\omega}_B^{+} = (-0.0000241\,\text{rad/s})\,\hat{k}$$



Problem 8.95

A uniform thin ring A and a uniform disk B roll without slip as shown. Letting T_A and T_B be the kinetic energies of A and B, respectively, if the two objects have the same mass and radius and if their centers are moving with the same speed v_0, state which of the following statements is true and why: (a) $T_A < T_B$; (b) $T_A = T_B$; (c) $T_A > T_B$.



Solution

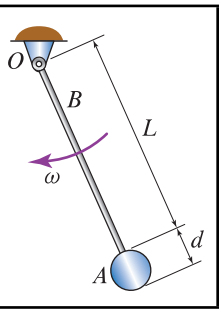
The kinetic energies of A and B are different. In particular, the kinetic energy of A is greater then that of B. The reason for this is that the kinetic energy of a rigid body in planar motion is given by

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_b^2,$$

where, m is the mass of the body, G is the center of mass of the body, v_G is the speed of the center of mass, I_G is the mass moment of inertia of the body relative to the mass center, and ω_b represents the angular speed of the body. In the case of A and B, the quantities $m_A = m_B$, $v_{G_A} = v_{G_B}$, and $\omega_A = \omega_B$. However, the mass moments of inertia of A and B relative to the respective mass centers are different. Specifically, the rotational inertia of A is larger than that of B and this therefore causes the kinetic energy of A to be larger than that of B.

Problem 8.96

A pendulum consists of a uniform disk A of diameter $d = 5\,\text{in.}$ and weight $W_A = 0.25\,\text{lb}$ attached at the end of a uniform bar B of length $L = 2.75\,\text{ft}$ and weight $W_B = 1.3\,\text{lb}$. At the instant shown, the pendulum is swinging with an angular velocity $\omega = 0.55\,\text{rad/s}$ clockwise. Determine the kinetic energy of the pendulum at this instant, using Eq. (8.13) on p. 594.



Solution

Observe that the system is in a fixed axis rotation about point O. Therefore the kinetic energy of the system is given by

$$T = \tfrac{1}{2} I_O \omega^2, \tag{1}$$

where I_O is the mass moment of the system about O. Using the parallel axis theorem, we have

$$I_O = \frac{1}{2}\frac{W_A}{g}\left(\frac{d}{2}\right)^2 + \frac{W_A}{g}\left(L + \frac{d}{2}\right)^2 + \frac{1}{12}\frac{W_B}{g}L^2 + \frac{W_B}{g}\left(\frac{L}{2}\right)^2 = 0.1699\,\text{slug}\cdot\text{ft}^2, \tag{2}$$

where the terms W_A/g and W_B/g are the masses of A and B, respectively, g is the acceleration due to gravity, and where we have used the following numerical data: $g = 32.2\,\text{ft/s}^2$, $W_A = 0.25\,\text{lb}$, $d = 5\,\text{in.} = 0.4167\,\text{ft}$, $L = 2.75\,\text{ft}$, and $W_B = 1.3\,\text{lb}$.

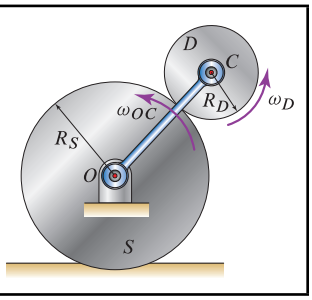
Substituting the value for I_O given in Eq. (2) into Eq. (1), we have

$$\boxed{T = 0.0257\,\text{ft}\cdot\text{lb},}$$

where we have also used the fact that $\omega = 0.55\,\text{rad/s}$.

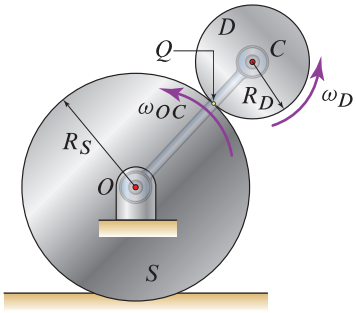
Problem 8.97

A uniform disk D of radius $R_D = 7\,\text{mm}$ and mass $m_D = 0.15\,\text{kg}$ is connected to point O via the rotating arm OC and rolls without slip over the stationary cylinder S of radius $R_S = 15\,\text{mm}$. Assuming that $\omega_D = 25\,\text{rad/s}$ and treating the arm OC as a uniform slender bar of length $L = R_D + R_S$ and mass $m_{OC} = 0.08\,\text{kg}$, determine the kinetic energy of the system.



Solution

The kinetic energy of the system us equal to the sum of the kinetic energy of each part of the system. In this problem the system consists of the disk D and the arm OC. Referring to the figure on the right, we label Q the point on the disk D that is in contact with the stationary cylinder S. Observing that the arm is in a fixed axis rotation about O and that the disk D is rolling without slip and that therefore the instantaneous center of rotation of the disk D is point Q, we have



$$T = \tfrac{1}{2} I_O \omega_{OC}^2 + \tfrac{1}{2} I_Q \omega_D^2, \tag{1}$$

where I_O is the mass moment of inertia of the arm OC with respect to O, ω_{OC} is the angular speed of the arm OC, I_Q is the mass moment of inertia of the disk D relative to its on center of instantaneous rotation Q, and ω_D is the angular speed of the disk D.

Using the parallel axis theorem and the modeling assumptions described in the problem statement, we have

$$I_O = \tfrac{1}{12} m_{OC} (R_D + R_S)^2 + m_{OC} \left[\tfrac{1}{2}(R_D + R_S)\right]^2 = \tfrac{1}{3} m_{OC} (R_D + R_S)^2 = 12.91 \times 10^{-6}\,\text{kg}\cdot\text{m}^2, \tag{2}$$

$$I_Q = \tfrac{1}{2} m_D R_D^2 + m_D R_D^2 = \tfrac{3}{2} m_D R_D^2 = 11.02 \times 10^{-6}\,\text{kg}\cdot\text{m}^2, \tag{3}$$

where we have used the following numerical data: $m_{OC} = 0.08\,\text{kg}$, $R_D = 7\,\text{mm} = 0.007000\,\text{m}$, $R_S = 15\,\text{mm} = 0.01500\,\text{m}$, and $m_D = 0.15\,\text{kg}$.

Observe that ω_D is given. Hence, the only missing piece of information is the angular speed of the arm OC. Observing that the point C is shared by both the arm OC and the disk D, and recalling that the disk D is rolling without slip over a stationary surface, we must have

$$v_C = \omega_D R_D = \omega_{OC} (R_S + R_D) \quad \Rightarrow \quad \omega_{OC} = \frac{R_D}{R_S + R_D} \omega_D. \tag{4}$$

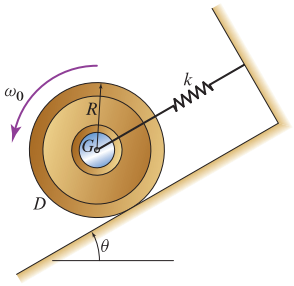
Substituting this relation into Eqs. (1), we have

$$\boxed{T = \frac{1}{2}\left[\frac{R_D^2}{(R_S + R_D)^2} I_O + I_Q\right] \omega_D^2 = 0.00385\,\text{J},}$$

where we have used the values of I_O and I_Q in Eqs. (2) and (3) as well as the following numerical data: $R_D = 7\,\text{mm} = 0.007000\,\text{m}$, $R_S = 15\,\text{mm} = 0.01500\,\text{m}$, and $\omega_D = 25\,\text{rad/s}$.

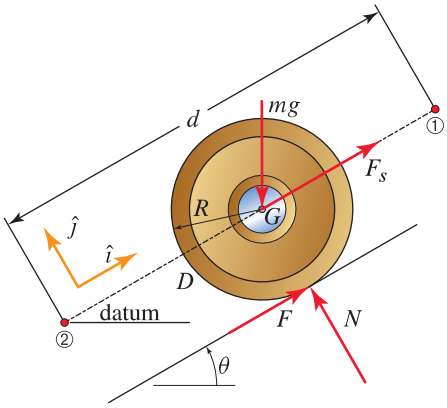
Problem 8.98

At the instant shown, the disk D, which has mass m and radius of gyration k_G, is rolling without slip down the flat incline with angular velocity ω_0. The disk is attached at its center to a wall by a linear elastic spring of constant k. If, at the instant shown, the spring is unstretched, determine the distance d down the incline that the disk rolls before coming to a stop. Use $k = 65\,\text{N/m}$, $R = 0.3\,\text{m}$, $m = 10\,\text{kg}$, $k_G = 0.25\,\text{m}$, $\omega_0 = 60\,\text{rpm}$, and $\theta = 30°$.



Solution

Referring to the FBD to the right and recalling that the disk rolls without slip, we see that the only forces doing work on the system are conservative. Hence, we can apply the work-energy principle in the form of conservation of energy. Let ① and ② be the positions of the systems at release and when the disk stops for the first time, respectively. Then we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, taking the datum for the potential energy of gravity at ②, and accounting for the fact that at ① the spring is unstretched and that at ② the disk comes to a stop, we have

$$T_1 = \tfrac{1}{2}mv_{G1}^2 + \tfrac{1}{2}I_G\omega_{D1}^2, \quad V_1 = mgd\sin\theta, \quad T_2 = 0, \quad V_2 = \tfrac{1}{2}kd^2, \tag{2}$$

where m is the mass of the disk, ω_D is the angular speed of the disk, and where $I_G = mk_G^2$ is the mass moment of inertia of the disk about its mass center. Because the disk is rolling without slip, at ① we must have

$$v_{G1} = R\omega_{D1} = R\omega_0. \tag{3}$$

Substituting Eqs. (2) into Eq. (1) and then substituting Eq. (3) into the resulting expression yields a second order algebraic equation with the only known d whose roots are

$$d = \frac{1}{k}\left(mg\sin\theta \pm \sqrt{m^2g^2\sin^2\theta + mkk_G^2\omega_0^2 + mkR^2\omega_0^2}\right). \tag{4}$$

Since the square root term is greater than the term $mg\sin\theta$, the only acceptable root is the one given by

$$d = \frac{1}{k}\left(mg\sin\theta + \sqrt{m^2g^2\sin^2\theta + mkk_G^2\omega_0^2 + mkR^2\omega_0^2}\right), \tag{5}$$

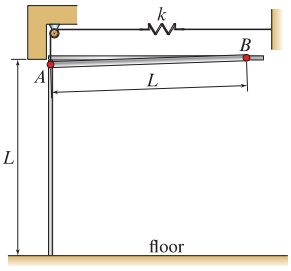
which can be evaluated to obtain

$$\boxed{d = 1.98\,\text{m},}$$

where we have used the following data: $m = 10\,\text{kg}$, $g = 9.81\,\text{ft/s}^2$, $\theta = 30°$, $k = 65\,\text{N/m}$, $k_G = 0.25\,\text{m}$, $R = 0.3\,\text{m}$, and $\omega_0 = 60\,\text{rpm} = 6.283\,\text{rad/s}$.

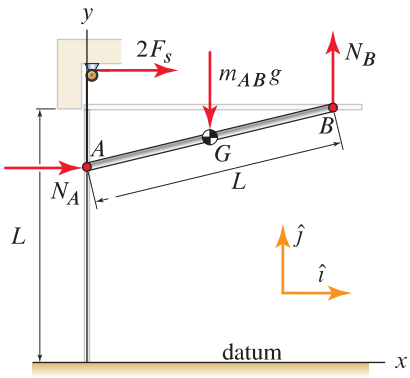
Problem 8.99

The figure shows the cross section of a garage door with length $L = 9\,\text{ft}$ and weight $W = 175\,\text{lb}$. At the ends A and B there are rollers of negligible mass constrained to move in a vertical and a horizontal guide, respectively. The door's motion is assisted by two springs (only one spring is shown), each with constant $k = 9.05\,\text{lb/ft}$. If the door is released from rest when horizontal and the spring is stretched 4 in., neglecting friction, and modeling the door as a uniform thin plate, determine the speed with which B strikes the left end of the horizontal guide.



Solution

Referring to the FBD shown, work is done only by gravity and the spring force. Let ① be the position at release and ② be the position achieved by the door when B strikes the left end of the horizontal guide. Since energy is conserved in going from ① to ②, we have



$$T_1 + V_1 = T_2 + V_2, \tag{1}$$

where, given the choice of datum, modeling the door as a rigid body, and recalling that the door is released from rest, we have

$$T_1 = 0, \tag{2}$$

$$V_1 = 2\tfrac{1}{2}\delta_i^2 + m_{AB}gL, \tag{3}$$

$$T_2 = \tfrac{1}{2}m_{AB}v_{G2}^2 + \tfrac{1}{2}I_G\omega_{d2}^2, \tag{4}$$

$$V_2 = 2\tfrac{1}{2}k(L+\delta_i)^2 + \tfrac{1}{2}Lm_{AB}g, \tag{5}$$

where m_{AB} is the mass of the door, δ_i is the stretch of the spring at ①, I_G is the mass moment of inertia of the door with respect to its own mass center G, v_G is the speed of G, and ω_d is the angular speed of the door.

Recalling that the weight of the door is denoted by W, we have

$$m_{AB} = \frac{W}{g} \quad \text{and} \quad I_G = \frac{1}{12}\frac{W}{g}L^2, \tag{6}$$

where g is the acceleration due to gravity. Now observe that in ② the door point A is at the very bottom of its trajectory. This implies that $v_{A2} = 0$ and that A is the instantaneous center of rotation of the door. In turn, this implies that the angular speed of the door in ② is $\omega_{d2} = v_{B2}/L$, and that $v_{G2} = \omega_{d2}(L/2) = v_{B2}/2$. Hence, we have

$$v_{G2} = \tfrac{1}{2}v_{B2}, \quad \text{and} \quad \omega_{d2} = v_{B2}/L, \tag{7}$$

Substituting Eqs. (2)–(5) into Eq. (1), then substituting in the resulting relation Eqs. (6) and (7), simplifying, and solving for v_{B2}, we have

$$v_{B2} = \sqrt{\frac{3}{W}}\sqrt{gL[W - 2k(L+2\delta_i)]}, \tag{8}$$

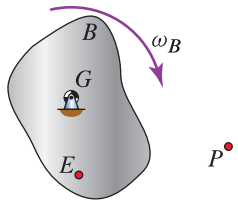
which can be evaluated to obtain

$$\boxed{(v_B)_{\text{final}} = 0.414\,\text{ft/s},}$$

where we have written $(v_B)_{\text{final}}$ for v_{B2} and where we have used the following numerical data: $W = 175\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $L = 9\,\text{ft}$, $k = 9.05\,\text{lb/ft}$, and $\delta_i = 4\,\text{in.} = 0.3333\,\text{ft}$.

Problem 8.100

Body B has mass m and mass moment of inertia I_G, where G is the mass center of B. If B is in fixed axis rotation about its center of mass G, determine which of the following statements is true and why: (a) $\left|(\vec{h}_E)_B\right| < \left|(\vec{h}_P)_B\right|$, (b) $\left|(\vec{h}_E)_B\right| = \left|(\vec{h}_P)_B\right|$, (c) $\left|(\vec{h}_E)_B\right| > \left|(\vec{h}_P)_B\right|$.

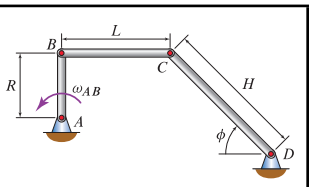


Solution

The answer is (b). The reason for this is that the angular momentum about a generic moment center is equal to the angular momentum of the body computed with respect to the mass center plus the moment of the linear momentum of the body viewed as a vector applied at the mass center relative to the moment center in question. The body shown is in a fixed axis rotation about the mass center. This implies that the linear momentum of the body is equal to zero. In turn, this means that the angular momentum of the body about any moment center is simply equal to the angular momentum computed about the mass center. In other words, in a fixed axis rotation about the mass center, the angular momentum of the body is the same no matter the choice of moment center.

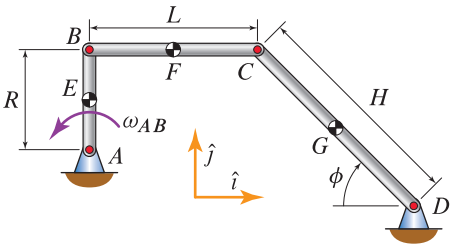
Problem 8.101

The weights of the uniform thin pin-connected bars AB, BC, and CD are $W_{AB} = 4\,\text{lb}$, $W_{BC} = 6.5\,\text{lb}$, and $W_{CD} = 10\,\text{lb}$, respectively. Letting $\phi = 47°$, $R = 2\,\text{ft}$, $L = 3.5\,\text{ft}$, and $H = 4.5\,\text{ft}$, and knowing that bar AB rotates at a constant angular velocity $\omega_{AB} = 4\,\text{rad/s}$, compute the angular momentum of the system about D at the instant shown.



Solution

The angular momentum of the system about D is the sum of the angular momenta about D of the individual components of the system. Hence, letting $\vec{h}_D$ denote the angular momentum about D of the system as a whole, we have



$$\vec{h}_D = (\vec{h}_D)_{AB} + (\vec{h}_D)_{BC} + (\vec{h}_D)_{CD}, \tag{1}$$

where, letting E and F denote the mass centers of AB and BC, respectively, and observing that bar CD is in a fixed axis rotation about D, we have

$$(\vec{h}_D)_{AB} = I_E\vec{\omega}_{AB} + \vec{r}_{E/D}\times m_{AB}\vec{v}_E,\ (\vec{h}_D)_{BC} = I_F\vec{\omega}_{BC} + \vec{r}_{F/D}\times m_{BC}\vec{v}_F,\ (\vec{h}_D)_{CD} = I_D\vec{\omega}_{CD}, \tag{2}$$

where I_E is the mass moment of inertia of bar AB with respect to its own mass center, $\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$ is the angular velocity of bar AB, $\vec{r}_{E/D}$ is the position of E relative to D, $\vec{v}_E$ is the velocity of E; I_F is the mass moment of inertia of bar BC with respect to its own mass center, $\vec{\omega}_{BC} = \omega_{BC}\,\hat{k}$ is the angular velocity of bar BC, $\vec{r}_{F/D}$ is the position of F relative to D, $\vec{v}_F$ is the velocity of F; I_D is the mass moment of inertia of bar BC relative to its own rotation center D, and $\vec{\omega}_{CD} = \omega_{CD}\,\hat{k}$ is the angular velocity of bar CD.

We now proceed to determine all the contributions to the terms in Eqs. (2), beginning with the relative position vectors:

$$\vec{r}_{E/D} = -(H\cos\phi + L)\,\hat{\imath} + (H\sin\phi - \tfrac{1}{2}R)\,\hat{\jmath} \quad \text{and} \quad \vec{r}_{F/D} = -(H\cos\phi + \tfrac{1}{2}L)\,\hat{\imath} + H\sin\phi\,\hat{\jmath}. \tag{3}$$

For the velocity of point E we can write

$$\vec{v}_E = \vec{v}_A + \vec{\omega}_{AB}\times\vec{r}_{E/A}, \tag{4}$$

where

$$\vec{v}_A = \vec{0}, \quad \vec{r}_{E/A} = \tfrac{1}{2}R\,\hat{\jmath}, \quad \text{and} \quad \vec{\omega}_{AB} = \omega_{AB}\,\hat{k}, \tag{5}$$

so that we have

$$\vec{v}_E = -\tfrac{1}{2}\omega_{AB}R\,\hat{\imath}. \tag{6}$$

Next, observing that $\vec{r}_{B/A} = R\,\hat{\jmath}$, we must also have

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB}\times\vec{r}_{B/A} = -\omega_{AB}R\,\hat{\imath}. \tag{7}$$

Now, notice that we can compute the velocity of C in the following two ways:

$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC}\times\vec{r}_{C/B} \quad \text{and} \quad \vec{v}_C = \vec{v}_D + \vec{\omega}_{CD}\times\vec{r}_{C/D}, \tag{8}$$

where

$$\vec{\omega}_{BC} = \omega_{BC}\,\hat{k}, \quad \vec{r}_{C/B} = L\,\hat{\imath}, \quad \vec{v}_D = \vec{0}, \quad \vec{\omega}_{CD} = \omega_{CD}\,\hat{k}, \quad \vec{r}_{C/D} = H(-\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}). \tag{9}$$

Substituting the result in Eq. (7) along with Eqs. (9) into Eqs. (8), carrying out the cross products, and enforcing equality between the two expressions for $\vec{v}_C$, we have

$$-R\omega_{AB}\,\hat{\imath} + L\omega_{BC}\,\hat{\jmath} = -H\omega_{CD}\sin\phi\,\hat{\imath} - H\omega_{CD}\cos\phi\,\hat{\jmath}. \tag{10}$$

Equation (10) is a vector equation corresponding to two scalar equations in the two unknowns ω_{BC} and ω_{CD} whose solution is

$$\omega_{BC} = -\frac{R\omega_{AB}\cos\phi}{L\sin\phi} \quad \text{and} \quad \omega_{CD} = \frac{R\omega_{AB}}{H\sin\phi}. \tag{11}$$

We now observe that $\vec{r}_{F/B} = \frac{1}{2}L\,\hat{\imath}$ and that $\vec{r}_{G/D} = -\frac{1}{2}H(-\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})$. Using these expressions, we can then say

$$\vec{v}_F = \vec{v}_B + \vec{\omega}_{BC}\times\vec{r}_{F/B} = -R\omega_{AB}\,\hat{\imath} - \frac{R\omega_{AB}\cos\phi}{2\sin\phi}\,\hat{\jmath}, \tag{12}$$

where we have used the expressions of $\vec{v}_B$ in Eq. (7), of $\vec{v}_D$ in the third of Eqs. (9), as well as the first of Eqs. (11).

Letting g denote the acceleration due to gravity, the mass properties of each bar can be summarized as follows:

$$m_{AB} = \frac{W_{AB}}{g} = 0.1242\,\text{slug}, \quad I_E = \frac{1}{12}\frac{W_{AB}}{g}R^2 = 0.04141\,\text{slug}\cdot\text{ft}^2, \tag{13}$$

$$m_{BC} = \frac{W_{BC}}{g} = 0.2019\,\text{slug}, \quad I_F = \frac{1}{12}\frac{W_{BC}}{g}L^2 = 0.2061\,\text{slug}\cdot\text{ft}^2, \tag{14}$$

$$I_D = \frac{1}{12}\frac{W_{CD}}{g}H^2 + \frac{W_{CD}}{g}(H/2)^2 = 2.096\,\text{slug}\cdot\text{ft}^2, \tag{15}$$

where we have used the following numerical data: $W_{AB} = 4\,\text{lb}$, $g = 32.2\,\text{ft/s}^2$, $R = 2\,\text{ft}$, $W_{BC} = 6.5\,\text{lb}$, $L = 3.5\,\text{ft}$, $W_{CD} = 10\,\text{lb}$, and $H = 4.5\,\text{ft}$.

We are now ready to evaluate the angular momenta about D due to each bar. Substituting into the first of Eqs. (2) the last of Eq. (5), the first of Eqs. (3), and Eq. (6), carrying out the cross product and simplifying, we have

$$(\vec{h}_D)_{AB} = \tfrac{1}{4}\omega_{AB}\big[4I_E + m_{AB}R(2H\sin\phi - R)\big]\hat{k} = (1.304\,\text{ft}\cdot\text{lb}\cdot\text{s})\,\hat{k}, \tag{16}$$

where we have used the fact that $\omega_{AB} = 4\,\text{rad/s}$, m_{AB} and I_E are given by Eqs. (13), $R = 2\,\text{ft}$, $H = 4.5\,\text{ft}$, and $\phi = 47^\circ$. Similarly, substituting into the second of Eqs. (2) all the required expressions, we have

$$(\vec{h}_D)_{AB} = \frac{R\omega_{AB}\big[m_{BC}L(2H\cos\phi + L) - 4I_F\big]\cos\phi}{4L\sin\phi}\,\hat{k} = (8.506\,\text{ft}\cdot\text{lb}\cdot\text{s})\,\hat{k}, \tag{17}$$

where we have used the values in Eqs. (14) and $R = 2\,\text{ft}$, $\omega_{AB} = 4\,\text{rad/s}$, $L = 3.5\,\text{ft}$, $H = 4.5\,\text{ft}$, and $\phi = 47^\circ$. Finally, substituting into the last of Eqs. (2) all the required expressions, we have

$$(\vec{h}_D)_{AB} = \frac{I_D R\omega_{AB}}{H\sin\phi}\,\hat{k} = (5.095\,\text{ft}\cdot\text{lb}\cdot\text{s})\,\hat{k}, \tag{18}$$

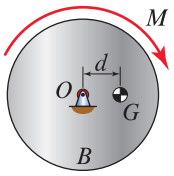
where we have used the value of I_D in Eq. (15) and $R = 2\,\text{ft}$, $\omega_{AB} = 4\,\text{rad/s}$, $H = 4.5\,\text{ft}$, and $\phi = 47^\circ$. Hence, summing the contributions computed in the last three equations, we have

$$\boxed{\vec{h}_D = (14.9\,\text{ft}\cdot\text{lb}\cdot\text{s})\,\hat{k}}$$



Problem 8.102

Consider Prob. 8.55 on p. 637 in which an eccentric wheel B is spun from rest under the action of a known torque M. In that problem, it was said that the wheel was in the *horizontal* plane. Is it possible to solve Prob. 8.55 by just applying Eq. (8.44) on p. 627 if the wheel is in the *vertical* plane? Why?

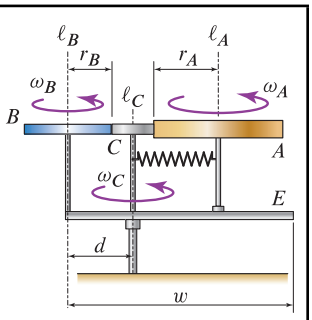


Solution

The answer is no. In fact, if point O is chosen as moment center, the contribution to the time integral of the moment of the external forces about O due to the reactions at O would be equal to zero. However, not enough information is provided for calculating the contribution due to gravity. If G is chosen as moment center, the moment due to gravity about G is equal to zero. However, not enough information is provided to compute the time integral of the moment about O due to the reactions at O. Any other choice of moment center would combine the two cases already discussed.

Problem 8.103

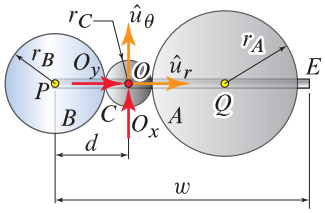
The uniform disk A, of mass $m_A = 1.2\,\text{kg}$ and radius $r_A = 0.25\,\text{m}$, is mounted on a vertical shaft that can translate along the horizontal rod E. The uniform disk B, of mass $m_B = 0.85\,\text{kg}$ and radius $r_B = 0.38\,\text{m}$, is mounted on a vertical shaft that is rigidly attached to E. Disk C has a negligible mass and is rigidly attached to E; i.e., C and E form a single rigid body. Disk A can rotate about the axis ℓ_A, disk B can rotate about the axis ℓ_B, and the arm E along with C can rotate about the fixed axis ℓ_C. While keeping both B and C stationary, disk A is initially spun with $\omega_A = 1200\,\text{rpm}$. Disk A is then brought in contact with C (contact is maintained via a spring), and at the same time, both B and C (and the arm E) are free to rotate. Due to friction between A and C, C along with E and disk B start spinning. Eventually A and C will stop slipping relative to one another. Disk B always rotates without slip over C. Let $d = 0.27\,\text{m}$ and $w = 0.95\,\text{m}$. Assuming that the only elements of the system that have mass are A, B, and E and that $m_E = 0.3\,\text{kg}$, and assuming that all friction in the system can be neglected except for that between A and C and between C and B, determine the angular speeds of A, B, and C (the angular velocity of C is the same as that of E since they form a single rigid body), when A and C stop slipping relative to one another.



Solution

Note: The value of the r_B in the problem statement is incorrect. Here we will set $r_B = 0.18\,\text{m}$.

We let the subscripts 1 and 2 denote the time instants at which the wheels are first brought into contact and when slip between the wheels stops, respectively. The figure on the right is a top view of the FBD of the system. Because the system is freely rotating, choosing as our moment center the fixed point O, we see that there are no external moments acting on the system and therefore the total angular momentum of the system about O, $\vec{h}_O$, must be conserved, i.e.,



$$\vec{h}_{O1} = \vec{h}_{O2} \tag{1}$$

To compute $\vec{h}_O$, we write it as the sum of three contributions due A, B, and E (disk C does not contribute because its mass is being neglected), respectively,

$$\vec{h}_O = (\vec{h}_O)_A + (\vec{h}_O)_B + (\vec{h}_O)_E, \tag{2}$$

where, recalling that E is in a fixed axis rotation about O with angular velocity $\vec{\omega}_C$ (since E and C form a single rigid body), we have

$$(\vec{h}_O)_A = I_Q\vec{\omega}_A + \vec{r}_{Q/O}\times m_A\vec{v}_Q, \quad (\vec{h}_O)_B = I_P\vec{\omega}_B + \vec{r}_{P/O}\times m_B\vec{v}_P, \quad (\vec{h}_O)_E = I_O\vec{\omega}_C, \tag{3}$$

where I_Q is the mass moment of inertia A about its mass center Q, and I_P is the mass moment of inertia of B about its mass center P, and I_O is the mass moment of inertia of E about O:

$$I_Q = \tfrac{1}{2}m_A r_A^2 = 0.03750\,\text{kg}\cdot\text{m}^2, \quad I_P = \tfrac{1}{2}m_B r_B^2 = 0.01377\,\text{kg}\cdot\text{m}^2,$$
$$I_O = \tfrac{1}{12}m_E w^2 + m_E\left(\tfrac{1}{2}w - d\right)^2 = 0.03517\,\text{kg}\cdot\text{m}^2 \tag{4}$$

where we have used the following numerical data: $m_A = 1.2\,\text{kg}$, $r_A = 0.025\,\text{m}$, $m_B = 0.85\,\text{kg}$, $r_B = 0.18\,\text{m}$, $m_E = 0.3\,\text{kg}$, $w = 0.95\,\text{m}$, and $d = 0.27\,\text{m}$. Going back to the solution of the problem, from Eqs. (3) we see that the key to the solution is expressing all the velocity terms at times t_1 and t_2.

At time t_1 B and E are at rest while A is spinning with $\vec{v}_{Q1} = \vec{0}$. Therefore, referring to Eqs. (3), at time t_1 we have

$$(\vec{h}_{O1})_A = I_Q \omega_{A1}\,\hat{k}, \quad (\vec{h}_{O1})_B = \vec{0}, \quad \text{and} \quad (\vec{h}_{O1})_E = \vec{0}, \tag{5}$$

where we have set $\vec{\omega}_A = \omega_{A1}\,\hat{k}$, and where $\omega_{A1} = 1200\,\text{rpm} = 125.7\,\text{rad/s}$.

We now observe that points P and Q move with the arm E. Recalling that E forms a single rigid body with disk C and that the center of rotation of such a body is the fixed point O, so that $\vec{\omega}_E = \vec{\omega}_C = \omega_C\,\hat{k}$, using rigid body kinematics at time t_2 we have

$$\vec{\omega}_{A2} = \omega_{A2}\,\hat{k}, \qquad \vec{v}_{Q2} = \vec{\omega}_{C2} \times \vec{r}_{Q/O} = \omega_{C2}\,\hat{k} \times (r_A + r_C)\,\hat{u}_r = \omega_{C2}(r_A + r_C)\,\hat{u}_\theta \tag{6}$$

$$\vec{\omega}_{B2} = \omega_{B2}\,\hat{k} \qquad \vec{v}_{P2} = \vec{\omega}_{C2} \times \vec{r}_{P/O} = \omega_{C2}\,\hat{k} \times [-(r_B + r_C)]\,\hat{u}_r = -\omega_{C2}(r_B + r_C)\,\hat{u}_\theta, \tag{7}$$

where we have used the fact that

$$\vec{r}_{Q/O} = (r_A + r_C)\,\hat{u}_r \quad \text{and} \quad \vec{r}_{P/O} = -(r_B + r_C)\,\hat{u}_r. \tag{8}$$

Substituting Eqs. (6) into Eqs. (3), and using Eqs. (8), at time t_2 we have

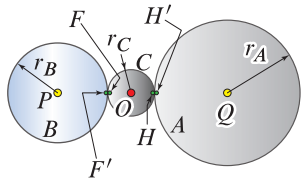
$$(\vec{h}_{O2})_A = \left[I_Q \omega_{A2} + m_A(r_A + r_C)^2 \omega_{C2}\right]\hat{k} \quad \text{and} \quad (\vec{h}_{O2})_B = \left[I_P \omega_{B2} + m_B(r_B + r_C)^2 \omega_{C2}\right]\hat{k}. \tag{9}$$

Then, using Eqs. (2), (5), and (9), and recalling that $(\vec{h}_{O2})_E = I_O \omega_{C2}\,\hat{k}$, Eq. (1) can be rewritten as

$$I_Q \omega_{A1} = I_Q \omega_{A2} + I_P \omega_{B2} + \left[I_O + m_A(r_A + r_C)^2 + m_B(r_B + r_C)^2\right]\omega_{C2}, \tag{10}$$

where we have written only the z component of the equation since it is the only nonzero component of the corresponding vector equation.

Observe that Eq. (10) is a single scalar equation in the three unknowns ω_{A2}, ω_{B2}, and ω_{C2}. To find the two additional equations we need, We recall that at t_2 the disks A and B no longer slip relative to C. Referring to the figure on the right, these two rolling without slip conditions imply $\vec{v}_{H2} = \vec{v}_{H'2}$ and $\vec{v}_{F2} = \vec{v}_{F'2}$, where points H and H' belong to C and A, respectively, and points F and F' belong to C and B respectively.



Applying rigid body kinematics, at time t_2 we have

$$\vec{v}_{H2} = \vec{v}_{H'2} \;\Rightarrow\; \vec{\omega}_{C2} \times \vec{r}_{H/O} = \vec{v}_{Q2} + \vec{\omega}_{A2} \times \vec{r}_{H'/Q} \;\Rightarrow\; \omega_{C2} r_C = (r_A + r_C)\omega_{C2} - r_A \omega_{A2}, \tag{11}$$

$$\vec{v}_{F2} = \vec{v}_{F'2} \;\Rightarrow\; \vec{\omega}_{C2} \times \vec{r}_{F/O} = \vec{v}_{P2} + \vec{\omega}_{B2} \times \vec{r}_{F'/P} \;\Rightarrow\; -\omega_{C2} r_C = -(r_B + r_C)\omega_{C2} + r_B \omega_{B2}, \tag{12}$$

where we have used Eqs. (6) and (7) and the fact that the only nonzero component of Eqs. (11) and (11) is the θ component.

We now observe that Eq. (10), (11), and (12) form a system of three equations in the three unknowns ω_{A2}, ω_{B2}, and ω_{C2}, whose solution is

$$\omega_{A2} = \omega_{B2} = \omega_{C2} = \frac{I_Q \omega_{A1}}{I_O + I_P + I_Q + m_A(r_A + r_C)^2 + m_B(r_B + r_C)^2}. \tag{13}$$

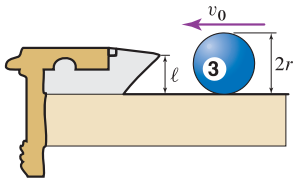
Therefore, the problem's answer is

$$\boxed{|\omega_A|_{\text{after slip stops}} = |\omega_B|_{\text{after slip stops}} = |\omega_C|_{\text{after slip stops}} = 16.4\,\text{rad/s},}$$

where we have used the values of I_O, I_P, and I_Q in Eqs. (4), and where we have used the following numerical data: $\omega_{A1} = 1200\,\text{rpm} = 125.7\,\text{rad/s}$, $m_A = 1.2\,\text{kg}$, $r_A = 0.025\,\text{m}$, $r_B = 0.18\,\text{m}$, and $r_C = d - r_B = 0.09000\,\text{m}$ (given that $d = 0.27\,\text{m}$).

Problem 8.104

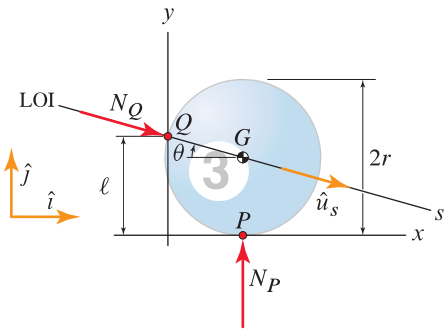
A billiard ball is rolling without slipping with a speed $v_0 = 6\,\text{ft/s}$ as shown when it hits the rail. According to regulations, the nose of the rail is at a height from the table bed of 63.5% of the ball's diameter (i.e., $\ell/(2r) = 0.635$). Model the impact with the rail as perfectly elastic, neglect friction between the ball and the rail as well as between the ball and the table, and neglect any vertical motion of the ball. Based on the stated assumptions, determine the velocity of the point of contact between the ball and the table right after impact. The diameter of the ball is $2r = 2.25\,\text{in.}$, and the weight of the ball is $W = 5.5\,\text{oz}$.



Solution

The impact-relevant FBD is shown to the right. Based on the stated assumptions, the angular momentum about the center of the ball is conserved, so that we have

$$I_G\omega_b^- = I_G\omega_b^+ \quad \Rightarrow \quad \omega_b^+ = \omega_b^- = \frac{v_0}{r}, \tag{1}$$



where $\vec{\omega}_B = \omega_b\,\hat{k}$ is the angular velocity of the ball and where we have accounted for the fact that the ball is rolling without slip before impact. Observe that the LOI for this impact is oriented by unit vector $\hat{u}_s$, which is defined via the angle θ.

The angle θ is found by observing that

$$\sin\theta = \frac{\ell - r}{r} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{\ell - r}{r}\right). \tag{2}$$

The COR equation for this problem is

$$v_{Qs}^+ = -e\,v_{Qs}^-, \tag{3}$$

where Q is the point on the ball that comes into contact with the nose of the table's rail and where we have assumed that the table does not move. Due to rigid body kinematics, we have that $\vec{v}_Q = \vec{v}_G + \vec{\omega}_b \times \vec{r}_{Q/G}$. Using the xy coordinate system shown, this kinematic relation is expressed as

$$\vec{v}_Q = v_{Gx}\,\hat{\imath} + \omega_b\,\hat{k} \times (-r\,\hat{u}_s), \tag{4}$$

where we have accounted for the fact that G can only move in the x direction, and where

$$\hat{u}_s = \cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}. \tag{5}$$

Using Eqs. (4) and (5) together, for we can then state that

$$\vec{v}_{Qs}^- = -v_0\cos\theta \quad \text{and} \quad \vec{v}_{Qs}^+ = v_{Gx}^+\cos\theta. \tag{6}$$

Substituting Eq. (6) into Eq. (3) and solving for v_{Gx}^+, we obtain that

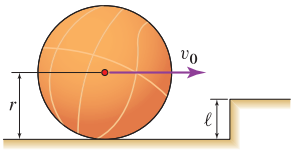
$$v_{Gx}^+ = v_0. \tag{7}$$

Letting P be the point of contact between the table and the ball, recalling that $v_0 = 6\,\text{ft/s}$, and recalling that we must have $\vec{v}_P = \vec{v}_G + \vec{\omega}_b \times \vec{r}_{P/G}$, with $\vec{r}_{P/G} = -r\,\hat{\jmath}$, we then have

$$\boxed{\vec{v}_P^+ = 2v_0\,\hat{\imath} = (12\,\text{ft/s})\,\hat{\imath}}$$

Problem 8.105

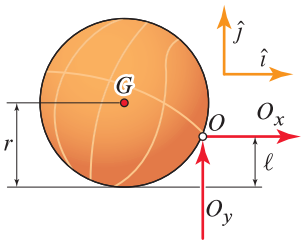
A basketball with mass $m = 0.6\,\text{kg}$ is rolling without slipping as shown when it hits a small step with $\ell = 7\,\text{cm}$. Letting the ball's diameter be $r = 12.0\,\text{cm}$, modeling the ball as a thin spherical shell (the mass moment of inertia of a spherical shell about its mass center is $\frac{2}{3}mr^2$), and assuming that the ball does not rebound off the step or slip relative to it, determine the maximum value of v_0 for which the ball will roll over the step without losing contact with it.



Solution

The solution is organized in three parts. In the first, we consider the collision between the ball and the step. In the second, we apply the work-energy principle to relate the postimpact angular speed of the ball to the ball's change in elevation in reaching the top of the step. In the third part we will relate the value of the angular speed in question to the condition that would be met if the ball were to lose contact with the top of the step.

The impact-relevant FBD of the ball is shown on the right. This FBD indicates that the impact is a constrained impact in which the ball behaves as if hinged at the fixed point O right after impact. Since point O is fixed, we can conclude that the impact is governed by the conservation of angular momentum about O. Keeping in mind that the vertical component of the velocity of G is equal to zero before impact, we have



$$I_G\omega_b^- - m_b(r-\ell)v_{Gx}^- = I_O\omega_b^+, \tag{1}$$

where $\vec{\omega}_b = \omega_b\,\hat{k}$ is the angular velocity of the ball, and where I_G and I_O are the mass moments of inertia of the ball relative to points G and O, respectively. Specifically, we have

$$I_G = \tfrac{2}{3}m_b r^2 \quad \text{and} \quad I_O = I_G + m_b r^2 = \tfrac{5}{3}m_b r^2. \tag{2}$$

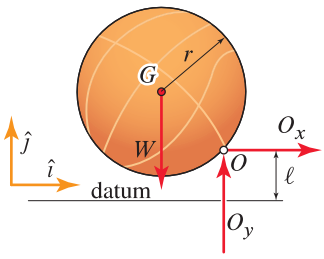
Since G is moving with speed v_0 to the right and the ball rolls without slip before impact, we must have

$$v_{Gx}^- = v_0 \quad \text{and} \quad \omega_b^- = -v_0/r. \tag{3}$$

Hence, substituting Eqs. (2) and (3) into Eq. (1) and solving for ω_b^+, we have

$$\omega_b^+ = -\frac{5r - 3\ell}{5r^2}v_0. \tag{4}$$

We now consider the second part of the problem. We define ① to coincide with the postimpact position of the system, and we define ② to be the position of the ball after it barely makes it to the top of the step. The FBD for this part of the problem is shown on the right. Since the the ball moves as if hinged at O, the only force doing work is gravity and therefore we can write



$$T_1 + V_1 = T_2 + V_2, \tag{5}$$

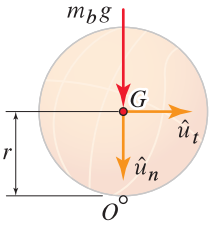
where, given our choice of datum, that the ball is in a state of fixed axis rotation about O after impact, we have

$$T_1 = \tfrac{1}{2}I_O\omega_{b1}^2 = \tfrac{5}{6}m_b r^2(\omega_b^+)^2, \quad V_1 = 0, \quad T_2 = \tfrac{1}{2}I_O\omega_{b2}^2 = \tfrac{5}{6}m_b r^2\omega_{b2}^2, \quad V_2 = m_b g\ell. \tag{6}$$

Recalling that ω_b^+ is given in Eq. (4), substituting Eqs. (6) into Eq. (5), and solving for ω_{b2} we have

$$\omega_{b2} = \sqrt{\frac{(5r-3\ell)^2 v_0^2}{25r^4} - \frac{6g\ell}{5r^2}}. \tag{7}$$

The expression of the angular speed in Eq. (7) is computed under the assumption that the ball makes it to the top of the step by rolling over the edge of the step, i.e., by not losing contact with the edge of the step. We now consider the limiting case where the contact force between the edge of the step and the ball becomes equal to zero as the ball reaches the top of the step. The FBD describing this condition is shown on the right, where we have chosen to use a normal-tangential component system attached the the ball's mass center G. Notice that it is assumed that the ball is still rotating about the fixed point O, which, at the instant considered, is vertically aligned with G.



Using the FBD in question and summing forces in the normal direction, we have

$$\sum F_n : \quad m_b g = m_b (a_{Gn})_2, \tag{8}$$

where, as long at the ball is still in a rotation about O, we have

$$(a_{Gn})_2 = \frac{v_{G2}^2}{r} = r\omega_{b2}^2 = \frac{(5r-3\ell)^2 v_0^2}{25r^3} - \frac{6g\ell}{5r}, \tag{9}$$

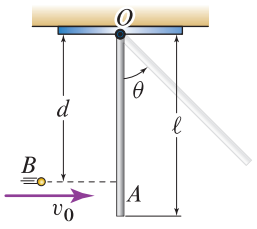
where we have used teh expression for ω_{b2} in Eq. (7). Substituting Eq. (9) into Eq. (8) and solving for v_0, we obtain the maximum value of v_0 such that the ball will not lose contact with the step. Letting such value of v_0 be denoted by $(v_0)_{\max}$, we have

$$\boxed{(v_0)_{\max} = \sqrt{\frac{5r^2(5r+6\ell)g}{(5r-3\ell)^2}} = 2.18\,\text{m/s},}$$

where we have used the following numerical data: $r = 12\,\text{cm} = 0.1200\,\text{m}$, $\ell = 7\,\text{cm} = 0.07000\,\text{m}$, and $g = 9.81\,\text{m/s}^2$.

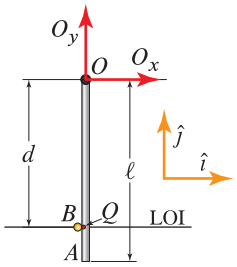
Problem 8.106

A bullet B weighing 147 gr (1 lb = 7000 gr) is fired with a speed $v_0 = 2750\,\text{ft/s}$ as shown against a thin uniform rod A of length $\ell = 3\,\text{ft}$, weight $W_r = 35\,\text{lb}$, and pinned at O. If $d = 1.5\,\text{ft}$ and the COR for the impact is $e = 0.25$, determine the bar's angular velocity immediately after the impact. In addition, determine the maximum value of the angle θ to which the bar swings after impact.



Solution

From the impact-relevant FBD of the system we see that the angular momentum of the system about O is conserved. Since the bar A is initially at rest and since its motion must be a fixed axis rotation about O, the conservation of angular momentum of the system about O can be written as follows:



$$m_B v_0 d = m_B v_{Bx}^{+} d + I_O \omega_A^{+}, \tag{1}$$

where m_B is the mass of B, we have accounted for the fact that $v_{Bx}^{-} = v_0$, $\vec{\omega}_A = \omega_A \hat{k}$ is the angular velocity of the bar, and I_O is the mass moment of inertia of A relative to O, i.e.,

$$I_O = \tfrac{1}{12} m_r \ell^2 + m_r \left(\frac{\ell}{2}\right)^2 = \tfrac{1}{3} m_r \ell^2. \tag{2}$$

Observing that the LOI is parallel to the horizontal direction, the COR equation for this problem is

$$v_{Bx}^{+} - v_{Qx}^{+} = -e v_0, \tag{3}$$

where point Q is the point on the bar that his hit by the bullet, and where, again, we have taken into account the fact that A is initially at rest (i.e., $v_{Qx}^{-} = 0$). Since A is constrained to rotate about O, we must have that

$$v_{Qx}^{+} = \omega_A^{+} d. \tag{4}$$

The equation that results from substituting Eq. (4) into Eq. (3), together with Eq. (1) form a system of two equations in the two unknowns v_{Bx}^{+} and ω_A^{+} whose solution is

$$v_{Qx}^{+} = \frac{3d^2 m_B - e\ell^2 m_r}{3d^2 m_B + \ell^2 m_r} v_0 = \frac{3d^2 W_B - e\ell^2 W_r}{3d^2 W_B + \ell^2 W_r} v_0 = -686.0\,\text{ft/s} \tag{5}$$

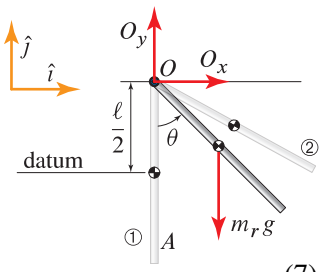
and

$$\omega_A^{+} = \frac{3d(m_B + e m_r)}{3d^2 m_B + \ell^2 m_r} v_0 = \frac{3d(W_B + e W_r)}{3d^2 W_B + \ell^2 W_r} v_0 = 1.031\,\text{rad/s}, \tag{6}$$

where we have first replaced the masses with the corresponding weights by multiplying both the numerators and the denominators by g, namely the acceleration due to gravity, and where obtained numerical values by using the following data: $d = 1.5\,\text{ft}$, $W_B = 147\,\text{gr} = 0.02100\,\text{lb}$, $e = 0.25$, $\ell = 3\,\text{ft}$, $W_r = 35\,\text{lb}$, and $v_0 = 2750\,\text{ft/s}$. Expressing the result in Eq. (6) in vector form and to 3 significant figures, we have

$$\vec{\omega}_A^{+} = (1.03\,\text{rad/s})\,\hat{k}.$$

Now, to determine the maximum swing angle of the bar A, we can apply the work energy principle. An FBD of the bar between the post-impact position and the position with the maximum swing angle is shown to the right. The post-impact position is denoted by ① whereas the position of the bar corresponding to the maximum angle θ is denoted as ②. Observe that only gravity does work on the bar in going from ① to ②. Hence, choosing the datum for gravity at ①, we have



$$T_1 + V_1 = T_2 + V_2, \tag{7}$$

where

$$T_1 = \tfrac{1}{2} I_O (\omega_A^+)^2, \quad V_1 = 0, \quad T_2 = 0, \quad V_2 = m_r g(\ell/2)(1 - \cos\theta_{\max}), \tag{8}$$

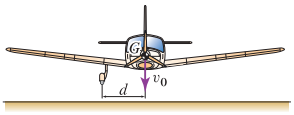
where the bar achieves $\theta_{\max}$ at ②. Substituting Eq. (8) into Eq. (7) yields an equation for $\theta_{\max}$ whose solution is

$$\theta_{\max} = \cos^{-1}\left[1 - \frac{I_O(\omega_A^+)^2}{m_r g \ell}\right] = \cos^{-1}\left[1 - \frac{\ell(\omega_A^+)^2}{3g}\right], \tag{9}$$

where, to obtain the last expression, we have used the result in Eq. (2). Using the value of ω_A^+ in Eq. (6) and recalling that $\ell = 3\,\text{ft}$ and $g = 32.2\,\text{ft/s}^2$, we have

$$\boxed{\theta_{\max} = 14.8°.}$$

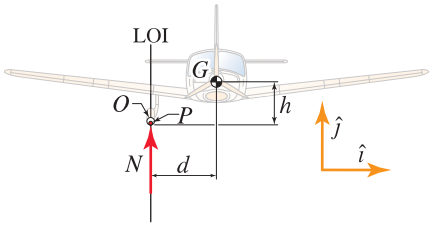
Problem 8.107



An airplane is about to crash-land with a vertical component of speed $v_0 = 2\,\text{ft/s}$ and zero roll, pitch, and yaw. Determine the vertical component of velocity of the center of mass of the airplane G as well as the airplane's angular velocity immediately after touching down, assuming that (1) the only available landing gear is rigid and rigidly attached to the airplane, (2) the coefficient of restitution between the landing gear and the ground is $e = 0.1$, (3) the airplane can be modeled as a rigid body, (4) the mass center G and the point of first contact between the landing gear and the ground are in the same plane perpendicular to the longitudinal axis of the airplane, and (5) friction between the landing gear and the ground is negligible. In solving the problem use the following data: $W = 2500\,\text{lb}$ (weight of the airplane), G is the mass center of the airplane, $k_G = 3\,\text{ft}$ is the radius of gyration of the airplane, and $d = 5.08\,\text{ft}$.

Solution

We model the airplane's touchdown as a constrained impact. the impact-relevant FBD is shown on the right where P is the point on the airplane that comes into contact with the ground and point O is the (fixed) point on the ground that is first struck by the plane. Based on the FBD, we can invoke conservation of angular momentum of the airplane about O. This gives



$$I_G\omega_p^- + md v_{Gy}^- = I_G\omega_p^+ + md v_{Gy}^+, \qquad (1)$$

where I_G is the mass moment of inertia of the airplane relative to its own mass center, $\vec{\omega}_p = \omega_p\,\hat{k}$ is the angular velocity of the airplane, $\vec{v}_G = v_{Gx}\,\hat{\imath} + v_{Gy}\,\hat{\jmath}$ is the velocity of G. The mass moment of inertia I_G is given by

$$I_G = mk_G^2. \qquad (2)$$

Next, we write the coefficient of restitution equation for point P. Keeping in mind that the ground is stationary and that the LOI is parallel to the vertical direction, this equation takes on the form

$$v_{Py}^+ = -e\,v_{Py}^-. \qquad (3)$$

Next, we recall that due to the stated conditions before impact, we must have

$$\omega_p^- = 0, \quad v_{Gy}^- = -v_0, \quad v_{Py}^- = -v_0, \qquad (4)$$

where the last of Eqs. (4) is due to the fact that before impact the airplane is simply translating. Since P and G belong to the same rigid body, we must also have

$$\vec{v}_P^+ = \vec{v}_G^+ + \vec{\omega}_p^+ \times \vec{r}_{P/G} \quad \Rightarrow \quad \vec{v}_P^+ = v_{Gx}^+\,\hat{\imath} + v_{Gy}^+\,\hat{\jmath} + \omega_p^+\,\hat{k} \times \underbrace{(-d\,\hat{\imath} - h\,\hat{\jmath})}_{\vec{r}_{P/G}}$$

$$\Rightarrow \quad \vec{v}_P^+ = (v_{Gx}^+ + h\omega_p^+)\,\hat{\imath} + (v_{Gy}^+ - d\omega_p^+)\,\hat{\jmath} \quad \Rightarrow \quad v_{Py}^+ = v_{Gy}^+ - d\omega_p^+. \qquad (5)$$

Substituting Eq. (2) and the first two of Eqs. (4) into Eq. (1), and symplifying we have

$$-dv_0 = k_G^2\omega_p^+ + dv_{Gy}^+. \qquad (6)$$

Substituting the last of Eqs. (4) and the last of Eqs. (5) into Eq. (3), we have

$$v_{Gy}^{+} - d\omega_p^{+} = ev_0. \tag{7}$$

Equations (6) and (7) for a system of to equations in two unknowns whose solution is

$$v_{Gy}^{+} = -\frac{d^2 - ek_G^2}{d^2 + k_G^2} v_0 \quad \text{and} \quad \omega_p^{+} = -\frac{(1+e)dv_0}{d^2 + k_G^2}, \tag{8}$$

which can be evaluated to obtain

$$\boxed{v_{Gy}^{+} = -1.43\,\text{ft/s} \quad \text{and} \quad \vec{\omega}_p^{+} = (-0.321\,\text{rad/s})\,\hat{k}}$$



where we have used the following numerical data: $d = 5.08\,\text{ft}$, $e = 0.1$, $k_G = 3\,\text{ft}$, and $v_0 = 2\,\text{ft/s}$.

Solutions Manual

Engineering Mechanics: Dynamics

1st Edition

Gary L. Gray
The Pennsylvania State University

Francesco Costanzo
The Pennsylvania State University

Michael E. Plesha
University of Wisconsin–Madison

With the assistance of:
Chris Punshon
Andrew J. Miller
Justin High
Chris O'Brien
Chandan Kumar
Joseph Wyne

Version: August 10, 2009

The McGraw-Hill Companies, Inc.

Copyright © 2002–2010
Gary L. Gray, Francesco Costanzo, and Michael E. Plesha

This solutions manual, in any print or electronic form, remains the property of McGraw-Hill, Inc. It may be used and/or possessed only by permission of McGraw-Hill, and must be surrendered upon request of McGraw-Hill. Any duplication or distribution, either in print or electronic form, without the permission of McGraw-Hill, is prohibited.

Important Information about this Solutions Manual

Even though this solutions manual is nearly complete, we encourage you to visit

`http://www.mhhe.com/pgc`

often to obtain the most up-to-date version. In particular, as of July 30, 2009, please note the following:

- The solutions for Chapters 1 and 2 have been accuracy checked and have been edited by us. They are in their final form.
- The solutions for Chapters 4 and 7 have been accuracy checked and should be error free. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapters 3, 6, 8, and 9 are being accuracy checked and the accuracy checked versions should be available by the end of August 2009. We will be adding some additional detail to these solutions in the coming weeks.
- The solutions for Chapter 10 should be available in their entirety by the end of August 2009.

All of the figures in Chapters 6–10 are in color. Color will be added to the figures in Chapters 1–5 over the coming weeks.

Contact the Authors

If you find any errors and/or have questions concerning a solution, please do not hesitate to contact the authors and editors via email at:

`dyn_solns@email.esm.psu.edu`

We welcome your input.

Accuracy of Numbers in Calculations

Throughout this solutions manual, we will generally assume that the data given for problems is accurate to 3 significant digits. When calculations are performed, all intermediate numerical results are reported to 4 significant digits. Final answers are usually reported with 3 significant digits. If you verify the calculations in this solutions manual using the rounded intermediate numerical results that are reported, you should obtain the final answers that are reported to 3 significant digits.

Chapter 9 Solutions

Problem 9.1

Show that Eq. (9.15) is equivalent to Eq. (9.3) if $C = \sqrt{A^2 + B^2}$ and $\tan\phi = A/B$.

Solution

Start with Eq. (9.15), which is given by

$$x(t) = x_i \cos\omega_n t + \frac{v_i}{\omega_n}\sin\omega_n t = A\cos\omega_n t + B\sin\omega_n t,$$

where we have let $x_i = A$ and $v_i/\omega_n = B$. Factoring out B, we obtain

$$x(t) = B\left(\frac{A}{B}\cos\omega_n t + \sin\omega_n t\right).$$

Letting $A/B = \tan\phi$, this becomes

$$x(t) = B(\tan\phi\cos\omega_n t + \sin\omega_n t) = \frac{B}{\cos\phi}(\sin\phi\cos\omega_n t + \cos\phi\sin\omega_n t).$$

Since $\sin\phi\cos\omega_n t + \cos\phi\sin\omega_n t = \sin(\omega_n t + \phi)$, this equation becomes

$$x(t) = \frac{B}{\cos\phi}\sin(\omega_n t + \phi).$$

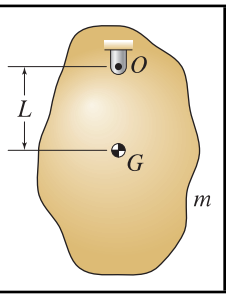
This is Eq. (9.3) as long as we can show that $B/\cos\phi = \sqrt{A^2 + B^2}$. To show this, we write

$$\frac{B}{\cos\phi} = \frac{B}{\frac{B}{\sqrt{A^2+B^2}}} = \sqrt{A^2 + B^2} = C,$$

where we have use the identity that if $\tan\phi = A/B$, then $\cos\phi = B/\sqrt{A^2 + B^2}$. We have thus shown that Eq. (9.15) is equivalent to Eq. (9.3) for the proper definition of C and ϕ.

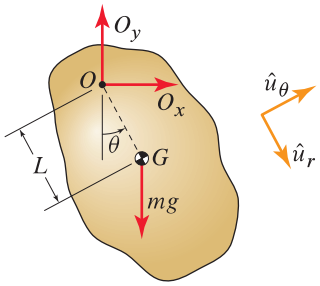
Problem 9.2

Derive the formula for the mass moment of inertia of an arbitrarily shaped rigid body about its mass center based on the body's period of oscillation τ when suspended as a pendulum. Assume that the mass of the body m is known and that the location of the mass center G is known relative to the pivot point O.



Solution

Using the FBD shown at the right, we can sum moments about the pivot O to obtain



$$\sum M_O: \quad -mgL\sin\theta = I_O\ddot{\theta}.$$

Rearranging this equation and assuming small oscillations, we obtain

$$\ddot{\theta} + \frac{mgL}{I_O}\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{mgL}{I_O}\theta = 0.$$

This is the standard form of the harmonic oscillator, so the natural frequency can be written as

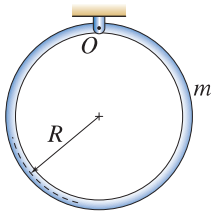
$$\omega_n^2 = \frac{mgL}{I_O} \quad \Rightarrow \quad I_O = \frac{mgL}{\omega_n^2}.$$

Since $\omega_n = 2\pi/\tau$, we have that

$$\boxed{I_O = \frac{mgL\tau^2}{4\pi^2}.}$$

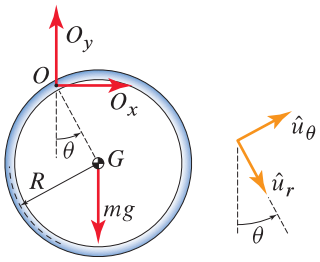
Problem 9.3

The thin ring of radius R and mass m is suspended by the pin at O. Determine its period of vibration if it is displaced a small amount and released.



Solution

Referring to the FBD at the right, we can sum moments about point O to obtain



$$\sum M_O: \quad -mgR\sin\theta = I_O\alpha_G, \tag{1}$$

where $\alpha = \ddot{\theta}$ is the angular acceleration of the thin ring. Using the parallel axis theorem, we find that $I_O = mR^2 + mR^2 = 2mR^2$. Substituting this into Eq. (1), we find the equation of motion to be

$$-mgR\sin\theta = 2mR^2\ddot{\theta} \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{2R}\sin\theta = 0.$$

Assuming small angles θ, we can use the approximation $\sin\theta \approx \theta$ so that the equation of motion becomes

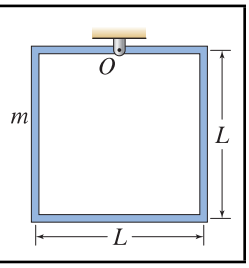
$$\ddot{\theta} + \frac{g}{2R}\theta = 0.$$

Therefore, the natural frequency and period are found to be

$$\omega_n = \sqrt{\frac{g}{2R}} \quad \Rightarrow \quad \boxed{\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2R}{g}}.}$$

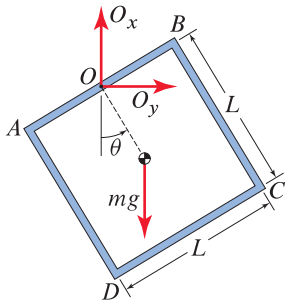
Problem 9.4

The thin square hoop has mass m and is suspended by the pin at O. Determine its period of vibration if it is displaced a small amount and released.



Solution

Using the FBD shown at the right, we can sum moments about the pivot O to obtain



$$\sum M_O: \quad -mg\frac{L}{2}\sin\theta = I_O\ddot{\theta}.$$

Rearranging this equation and assuming small oscillations, we obtain

$$\ddot{\theta} + \frac{mgL}{2I_O}\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{mgL}{2I_O}\theta = 0.$$

This is the standard form of the harmonic oscillator, so the natural frequency and period can be written as

$$\omega_n^2 = \frac{mgL}{2I_O} \quad \text{and} \quad \tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2I_O}{mgL}}, \tag{1}$$

respectively. Noting that each of the four segments of the thin square hoop has mass $m/4$, computing I_O we obtain

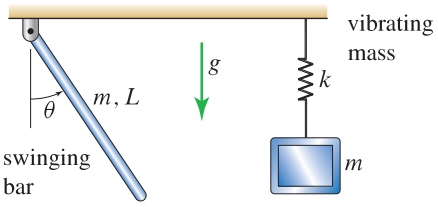
$$I_O = \underbrace{\frac{1}{12}\left(\frac{m}{4}\right)L^2}_{\text{segment } AB} + \underbrace{2\left[\frac{1}{12}\left(\frac{m}{4}\right)L^2 + \frac{m}{4}\left(\frac{L^2}{4}+\frac{L^2}{4}\right)\right]}_{\text{segments } BC \text{ and } AD} + \underbrace{\left[\frac{1}{12}\left(\frac{m}{4}\right)L^2 + \frac{m}{4}L^2\right]}_{\text{segment } CD} = \frac{7}{12}mL^2.$$

Substituting this expression for I_O into the expression for the period given in Eq. (1), we obtain

$$\boxed{\tau = 2\pi\sqrt{\frac{7L}{6g}}.}$$

Problem 9.5

The swinging bar and the vibrating mass are made to vibrate on Earth, and their respective natural frequencies are measured. The two systems are then taken to the Moon and are again allowed to vibrate at their respective natural frequencies. How will the natural frequency of each system change when compared with that on the Earth, and which of the two systems will experience the larger change in natural frequency?

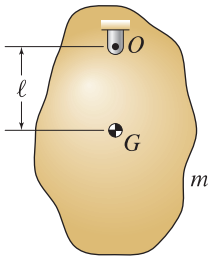


Solution

As can be seen in Example 9.1, the natural frequency of a rigid body that swings like a pendulum under the action of gravity depends on the value of the acceleration due to gravity g such that $\omega_n \propto \sqrt{g}$. On the other hand, as can be seen in the mini-example on p. 675, the natural frequency of a mass at the end of a spring that oscillates under the action of gravity does *not* depend on g. Therefore, on the Moon, where the acceleration due to gravity g_{moon} is less than g on Earth, the natural frequency of the swinging bar will decrease and the natural frequency of the vibrating mass will stay the same.

Problems 9.6 and 9.7

A rigid body of mass m, mass center at G, and mass moment of inertia I_G is pinned at an arbitrary point O and allowed to oscillate as a pendulum.

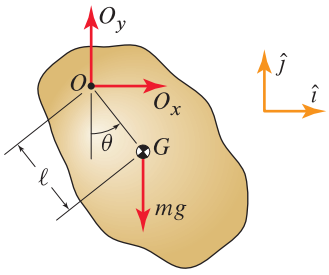


Problem 9.6 By writing the Newton-Euler equations, determine the distance ℓ from G to the pivot point O so that the pendulum has the highest possible natural frequency of oscillation.

Problem 9.7 Using the energy method, determine the distance ℓ from G to the pivot point O so that the pendulum has the highest possible natural frequency of oscillation.

Solution to 9.6

We will write the equation of motion using the Newton-Euler equations, find ω_n as a function of ℓ, and then find $(\omega_n)_{\max}$ by letting $d\omega_n/d\ell = 0$.



Referring to the FBD at the right and summing moments about O, we obtain

$$\sum M_O: \quad -mg\ell\sin\theta = I_O\ddot{\theta}, \tag{1}$$

where, using the parallel axis theorem, $I_O = I_G + m\ell^2$. Substituting I_O into Eq. (1), we obtain

$$\left(I_G + m\ell^2\right)\ddot{\theta} + mg\ell\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{mg\ell}{I_G + m\ell^2}\theta = 0,$$

where the second equation was obtained from the first by dividing through by the coefficient of $\ddot{\theta}$ and assuming small θ. Therefore, the natural frequency is given by

$$\omega_n = \sqrt{\frac{mg\ell}{I_G + m\ell^2}}.$$

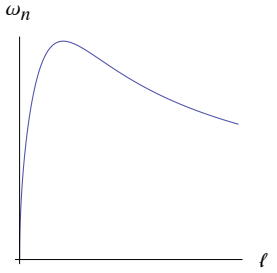
Differentiating ω_n with respect to ℓ and setting it equal to zero, we find that

$$\frac{d\omega_n}{d\ell} = \frac{d}{d\ell}\left(\frac{mg\ell}{I_G + m\ell^2}\right)^{1/2} = \frac{1}{2}\left(\frac{mg\ell}{I_G + m\ell^2}\right)^{-1/2}\left[\frac{mg\left(I_G + m\ell^2\right) - 2m\ell(mg\ell)}{\left(I_G + m\ell^2\right)^2}\right] = 0,$$

which implies that

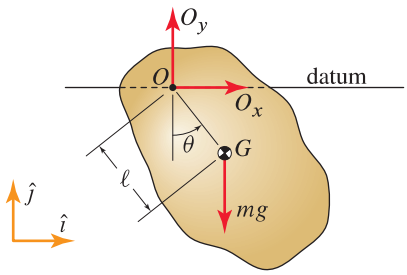
$$mg\left(I_G + m\ell^2\right) - 2m\ell(mg\ell) = 0 \quad \Rightarrow \quad I_G - m\ell^2 = 0 \quad \Rightarrow \quad \boxed{\ell = \sqrt{\frac{I_G}{m}}}$$

where we have used the fact that ℓ needs to be positive and real. Since ω_n as a function of ℓ has the shape shown below, we can see that this expression for ℓ gives a maximum value of ω_n and not a minimum.



Solution to 9.7

We will write the equation of motion using the energy method, find ω_n as a function of ℓ, and then find $(\omega_n)_{\max}$ by letting $d\omega_n/d\ell = 0$.



Referring to the FBD at the right we see that the only force that does work as the body swings is the weight force mg. Since this system is conservative, we can use the energy method to determine the equation of motion.

The kinetic energy can be written as

$$T = \tfrac{1}{2} I_O \omega_G^2 = \tfrac{1}{2}\left(I_G + m\ell^2\right)\dot{\theta}^2,$$

where ω_G is the angular velocity of the rigid body and we have used the parallel axis theorem to find I_O. In addition, the potential energy can be written as

$$V = -mg\ell\cos\theta \approx -mg\ell\left(1 - \frac{\theta^2}{2}\right),$$

where we have assumed small θ in approximating the cosine function. Applying the energy method, we obtain

$$\frac{d}{dt}(T+V) = \left(I_G + m\ell^2\right)\dot{\theta}\ddot{\theta} + mg\ell\theta\dot{\theta} = 0 \quad \Rightarrow \quad \left(I_G + m\ell^2\right)\ddot{\theta} + mg\ell\theta = 0,$$

which implies that

$$\omega_n = \sqrt{\frac{mg\ell}{I_G + m\ell^2}}.$$

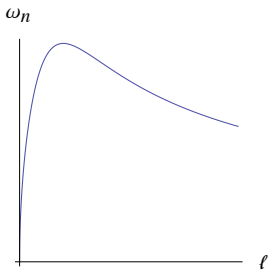
Differentiating ω_n with respect to ℓ and setting it equal to zero, we find that

$$\frac{d\omega_n}{d\ell} = \frac{d}{d\ell}\left(\frac{mg\ell}{I_G + m\ell^2}\right)^{1/2} = \frac{1}{2}\left(\frac{mg\ell}{I_G + m\ell^2}\right)^{-1/2}\left[\frac{mg\left(I_G + m\ell^2\right) - 2m\ell(mg\ell)}{\left(I_G + m\ell^2\right)^2}\right] = 0,$$

which implies that

$$mg\left(I_G + m\ell^2\right) - 2m\ell(mg\ell) = 0 \quad \Rightarrow \quad I_G - m\ell^2 = 0 \quad \Rightarrow \quad \boxed{\ell = \sqrt{\frac{I_G}{m}}}$$

where we have used the fact that ℓ needs to be positive and real. Since ω_n as a function of ℓ has the shape shown below, we can see that this expression for ℓ gives a maximum value of ω_n and not a minimum.

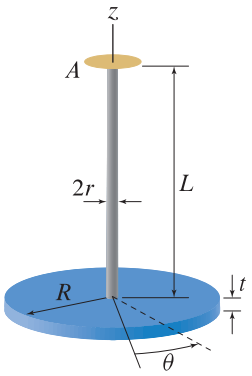


Problem 9.8

The uniform disk of radius R and thickness t is attached to the thin shaft of radius r, length L, and negligible mass. The end A of the shaft is fixed. From mechanics of materials, it can be shown that if a torque M_z is applied to the free end of the shaft, then it can be related to the twist angle θ via

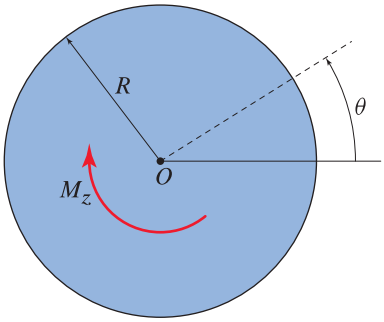
$$\theta = \frac{M_z L}{GJ},$$

where G is the shear modulus of elasticity of the shaft and $J = \frac{\pi}{2} r^4$ is the polar moment of inertia of the cross-sectional area of the shaft. Letting ρ be the mass density of the disk and using the given relationship between M_z and θ, determine the natural frequency of vibration of the disk in terms of the given dimensions and material properties when it is given a small angular displacement θ in the plane of the disk.



Solution

If the disk is displaced through a positive angle θ then the FBD (as viewed down the z axis) is as shown on the right. The moment M_z is the restoring moment due to the thin shaft of radius r and length L. Summing moments about O gives



$$\sum M_O: \quad -M_z = I_O \ddot{\theta}. \tag{1}$$

From the given relationship between M_z and θ, we can find that

$$M_z = \frac{GJ}{L}\theta \quad \Rightarrow \quad -\frac{GJ}{L}\theta = I_O \ddot{\theta}. \tag{2}$$

where we have made use of Eq. (1). Since

$$I_O = \tfrac{1}{2} m R^2, \quad m = \rho \pi R^2 t, \quad \text{and} \quad J = \tfrac{\pi}{2} r^4,$$

Eq. (2) becomes

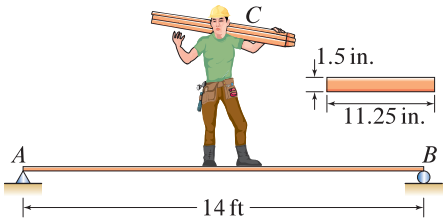
$$-\frac{G\pi r^4/2}{L} = \frac{1}{2}\pi \rho R^4 t \ddot{\theta} \quad \Rightarrow \quad \ddot{\theta} + \left(\frac{Gr^4}{\rho L R^4 t}\right)\theta = 0.$$

Therefore, the natural frequency of oscillation is

$$\boxed{\omega_n = \sqrt{\frac{Gr^4}{\rho L R^4 t}}.}$$

Problem 9.9

A construction worker C is standing at the midpoint of a 14 ft long pine board that is simply supported. The board is a standard 2×12, so its cross-sectional dimensions are as shown. Assuming the worker weighs 180 lb and he flexes his knees once to get the board oscillating, determine his vibration frequency. Neglect the weight of the beam and use the fact that a load P applied to a simply supported beam will deflect the center of the beam $PL^3/(48EI_{cs})$, where L is the length of the beam, E is its modulus of elasticity, and I_{cs} is the area moment of inertia of the cross section of the beam. The elastic modulus of pine is 1.8×10^6 psi.



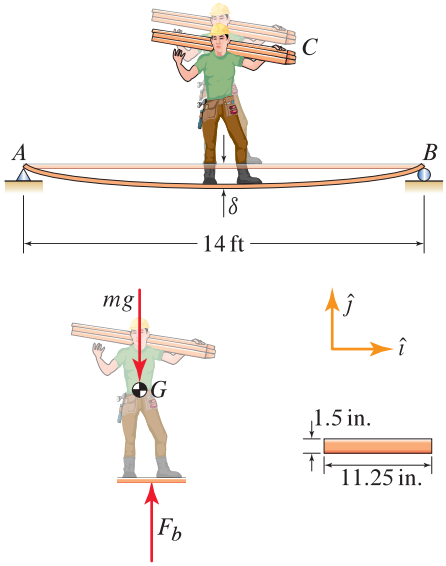
Solution

Referring to the FBD of the construction worker shown at the right, we see that the elastic restoring force due to the board F_b as well as the weight of the worker mg both do work on the worker. Since the force due to the board is assumed to be linear elastic with the force law

$$F_b = \frac{48EI_{cs}}{L^3}\,\delta = k_{eq}\delta, \quad \text{where} \quad k_{eq} = \frac{48EI_{cs}}{L^3}$$

and where δ is the deflection of the center of the board due to the force F_b as shown on the right and k_{eq} is the equivalent linear spring constant of the deflected board. Since both forces doing work are conservative, we can apply the energy method to determine the equation of motion of the construction worker on the board. The kinetic energy of worker is

$$T = \tfrac{1}{2}mv_G^2 = \tfrac{1}{2}m\dot{y}^2.$$



The potential energy is given by

$$V = -mg(y+\delta_{st}) + \tfrac{1}{2}k_{eq}(y+\delta_{st})^2,$$

where y is measured from the static equilibrium position of the worker standing on the board. Applying the energy method, we obtain

$$\frac{d}{dt}(T+V) = m\dot{y}\ddot{y} - mg\dot{y} + k_{eq}(y+\delta_{st})\dot{y} = 0 \quad \Rightarrow \quad m\ddot{y} - mg + k_{eq}(y+\delta_{st}) = 0.$$

Using the fact that $mg = k_{eq}\delta_{st}$, this equation becomes the standard harmonic oscillator equation

$$\ddot{y} + \frac{48EI_{cs}}{mL^3}\,y = 0 \quad \Rightarrow \quad f = \frac{\omega_n}{2\pi} = \frac{2}{\pi}\sqrt{\frac{3EI_{cs}}{mL^3}},$$

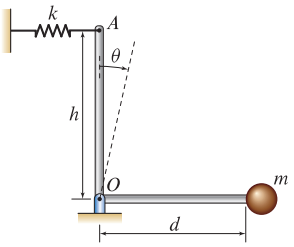
where f is the vibration frequency. Substituting in given values

$$f = \frac{2}{\pi}\left[\frac{3\left(1.8\times10^6\ \frac{\text{lb}}{\text{in.}^2}\right)\left(3.164\ \text{in.}^4\right)}{\left(0.4658\ \frac{\text{lb}\cdot\text{s}^2}{\text{in.}}\right)\left(168\ \text{in.}\right)^3}\right]^{1/2} \quad \Rightarrow \quad \boxed{f = 1.77\ \text{Hz},}$$

where $I_{cs} = \frac{1}{12}bh^3$ in which $b = 11.25$ in. and $h = 1.5$ in., $L = 14$ ft $= 168$ in., and $m = 5.590$ slug $= 0.4658\ \text{lb}\cdot\text{s}^2/\text{in}$.

Problems 9.10 through 9.13

The L-shaped bar lies in the vertical plane and is pinned at O. One end of the bar has a linear elastic spring with constant k attached to it, and attached at the other end is a mass m of negligible size. The angle θ is measured from the equilibrium position of the system and it is assumed to be small.



Problem 9.10 Assuming that the L-shaped bar has negligible mass, determine the natural period of vibration of the system by writing the Newton-Euler equations.

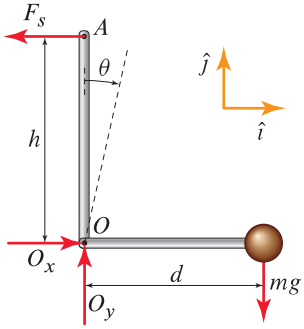
Problem 9.11 Assuming that the L-shaped bar has negligible mass, determine the natural period of vibration of the system via the energy method.

Problem 9.12 Assuming that the L-shaped bar has mass per unit length ρ, determine the natural period of vibration of the system by writing the Newton-Euler equations.

Problem 9.13 Assuming that the L-shaped bar has mass per unit length ρ, determine the natural period of vibration of the system via the energy method.

Solution to 9.10

Ignoring the mass of the L-shaped bar, the FBD of the system is as shown on the right. Summing moments about point O, we find that



$$\sum M_O: \quad F_s h - mgd = -I_O \ddot{\theta}, \tag{1}$$

where the minus sign on the right side is due to the fact that θ is positive in the clockwise direction and positive z is in the counterclockwise direction. The force in the spring F_s is due to the static deflection δ_{st} at equilibrium as well as any additional deflection during vibration. Therefore the spring force is

$$F_s = k(\delta_{\text{st}} + h\theta).$$

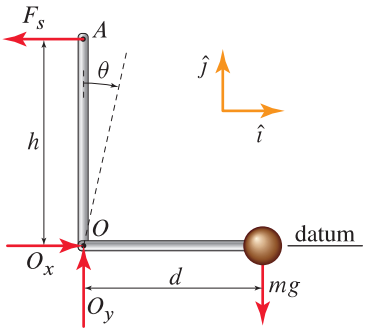
In addition, if we neglect the mass of the bar, then $I_O = md^2$, which means that Eq. (1) becomes

$$k(\delta_{\text{st}} + h\theta)h - mgd = -md^2\ddot{\theta},$$

Since $k\delta_{\text{st}}h = mgd$, this equation becomes $md^2\ddot{\theta} + kh^2\theta = 0$, which means that the natural frequency and period of vibration are given by

$$\omega_n = \sqrt{\frac{kh^2}{md^2}} = \frac{h}{d}\sqrt{\frac{k}{m}} \quad \Rightarrow \quad \boxed{\tau = \frac{2\pi d}{h}\sqrt{\frac{m}{k}}.}$$

Solution to 9.11



Ignoring the mass of the L-shaped bar, the FBD of the system is as shown on the right. From this FBD we can see that the spring force F_s and the weight force mg both do work and thus the system is conservative. Applying the energy method, the kinetic energy of the system is

$$T = \tfrac{1}{2} I_O \dot{\theta}^2 = \tfrac{1}{2} m d^2 \dot{\theta}^2,$$

where we have used the parallel axis theorem to obtain $I_O = md^2$. Changes in the potential energy of the system are due to the deflection of the spring as well as changes in height of the mass m. The deflection of the spring is due to the static deflection δ_{st} at equilibrium as well as any additional deflection during vibration. Therefore, the potential energy of the system can be written as

$$V = \tfrac{1}{2} k(\delta_{\text{st}} + h\theta)^2 - mgd \sin\theta = \tfrac{1}{2} k(\delta_{\text{st}} + h\theta)^2 - mgd\theta,$$

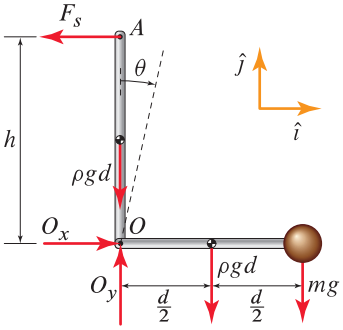
where we have approximated $\sin\theta$ as θ. Applying the energy method, we obtain

$$\frac{d}{dt}(T + V) = md^2 \dot{\theta}\ddot{\theta} + k(\delta_{\text{st}} + h\theta)h\dot{\theta} - mgd\dot{\theta} = 0 \quad \Rightarrow \quad md^2\ddot{\theta} + k(\delta_{\text{st}} + h\theta)h - mgd = 0.$$

Since $k\delta_{\text{st}}h = mgd$, this equation becomes $md^2\ddot{\theta} + kh^2\theta = 0$, which means that the natural frequency and period of vibration are given by

$$\omega_n = \sqrt{\frac{kh^2}{md^2}} = \frac{h}{d}\sqrt{\frac{k}{m}} \quad \Rightarrow \quad \boxed{\tau = \frac{2\pi d}{h}\sqrt{\frac{m}{k}}.}$$

Solution to 9.12



Including the mass of the L-shaped bar, the FBD of the system is as shown on the right. Summing moments about point O, we find that

$$\sum M_O: \quad F_s h - mgd - \rho dg\tfrac{d}{2} = -I_O\ddot{\theta}, \tag{2}$$

where the minus sign on the right side is due to the fact that θ is positive in the clockwise direction and positive z is in the counterclockwise direction. The force in the spring F_s is due to the static deflection δ_{st} at equilibrium as well as any additional deflection during vibration. Therefore the spring force is

$$F_s = k(\delta_{\text{st}} + h\theta).$$

In addition, if we include the mass of the bar, then

$$I_O = md^2 + \tfrac{1}{3}(\rho h)h^2 + \tfrac{1}{3}(\rho d)d^2,$$

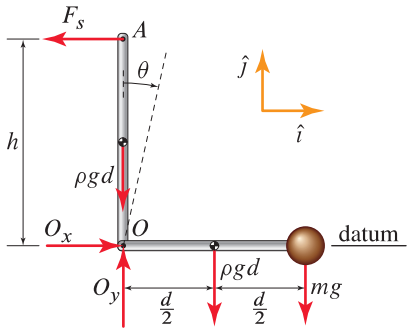
which means that Eq. (2) becomes

$$k(\delta_{\text{st}} + h\theta)h - mgd - \rho dg\tfrac{d}{2} = -\left[md^2 + \tfrac{1}{3}(\rho h)h^2 + \tfrac{1}{3}(\rho d)d^2\right]\ddot{\theta},$$

Since $k\delta_{\text{st}}h = mgd + \frac{1}{2}\rho g d^2$, this equation becomes $\left[\frac{1}{3}\rho(h^3 + d^3) + md^2\right]\ddot{\theta} + kh^2\theta = 0$, which means that the natural frequency and period of vibration are given by

$$\omega_n = \sqrt{\frac{kh^2}{\frac{1}{3}\rho(h^3 + d^3) + md^2}} \quad \Rightarrow \quad \boxed{\tau = 2\pi\sqrt{\frac{\frac{1}{3}\rho(h^3 + d^3) + md^2}{kh^2}}.}$$

Solution to 9.13



Including the mass of the L-shaped bar, the FBD of the system is as shown on the right. From this FBD we can see that the spring force F_s and the weight forces mg and ρgd do work and thus the system is conservative. Applying the energy method, the kinetic energy of the system is

$$T = \tfrac{1}{2} I_O \dot{\theta}^2 = \tfrac{1}{2}\left[\tfrac{1}{3}\rho\left(h^3 + d^3\right) + md^2\right]\dot{\theta}^2,$$

where we have used the parallel axis theorem to find $I_O = md^2 + \frac{1}{3}(\rho h)h^2 + \frac{1}{3}(\rho d)d^2 = \frac{1}{3}\rho\left(h^3 + d^3\right) + md^2$. Changes in the potential energy of the system are due to the deflection of the spring as well as changes in height of the mass m and the centers of mass of each of the two segments of the L-shaped bar. The deflection of the spring is due to the static deflection δ_{st} at equilibrium as well as any additional deflection during vibration. Therefore, the potential energy of the system can be written as

$$\begin{aligned} V &= \tfrac{1}{2}k(\delta_{\text{st}} + h\theta)^2 - mgd\sin\theta - \rho dg\tfrac{d}{2}\sin\theta + \rho hg\tfrac{h}{2}\cos\theta \\ &= \tfrac{1}{2}k(\delta_{\text{st}} + h\theta)^2 - mgd\theta - \rho dg\tfrac{d}{2}\theta + \rho hg\tfrac{h}{2}\left(1 - \tfrac{\theta^2}{2}\right), \end{aligned}$$

where we have approximated $\sin\theta$ as θ and $\cos\theta$ as $1 - \theta^2/2$. Applying the energy method, we obtain

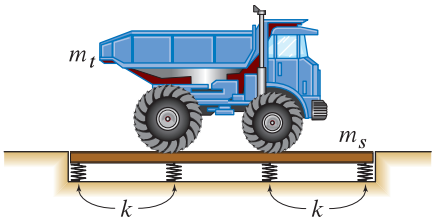
$$\begin{aligned} \frac{d}{dt}(T+V) &= \left[\tfrac{1}{3}\rho\left(h^3 + d^3\right) + md^2\right]\dot{\theta}\ddot{\theta} + k(\delta_{\text{st}} + h\theta)h\dot{\theta} - mgd\dot{\theta} - \tfrac{1}{2}\rho d^2 g\dot{\theta} - \tfrac{1}{2}\rho h^2 g\theta\dot{\theta} = 0 \\ &\Rightarrow \left[\tfrac{1}{3}\rho\left(h^3 + d^3\right) + md^2\right]\ddot{\theta} + k(\delta_{\text{st}} + h\theta)h - mgd - \tfrac{1}{2}\rho g\left(d^2 - h^2\theta\right) = 0. \end{aligned}$$

Since $k\delta_{\text{st}}h = mgd + \frac{1}{2}\rho g\left(d^2 + h^2\theta\right)$, this equation becomes $\left[\frac{1}{3}\rho(h^3 + d^3) + md^2\right]\ddot{\theta} + kh^2\theta = 0$, which means that the natural frequency and period of vibration are given by

$$\omega_n = \sqrt{\frac{kh^2}{\frac{1}{3}\rho(h^3 + d^3) + md^2}} \quad \Rightarrow \quad \boxed{\tau = 2\pi\sqrt{\frac{\frac{1}{3}\rho(h^3 + d^3) + md^2}{kh^2}}.}$$

Problem 9.14

An off-highway truck drives onto a concrete deck scale to be weighed, thus causing the truck and scale to vibrate vertically at the natural frequency of the system. The empty truck weighs 74,000 lb, the scale platform weighs 51,000 lb, and the platform is supported by eight identical springs (four of which are shown), each with constant $k = 3.6\times10^5$ lb/ft. Modeling the truck, its contents, and the concrete deck as a single particle, if a vibration frequency of 3.3 Hz is measured, what is the weight of the payload being carried by the truck?



Solution

Since the frequency of vibration f is 3.3 Hz, the natural frequency ω_n is given by

$$\omega_n = 2\pi f \quad \Rightarrow \quad \sqrt{\frac{k_{\text{eq}}}{m_{\text{tot}}}} = 2\pi f, \tag{1}$$

where k_{eq} is the equivalent spring constant of the eight springs supporting the platform and m_{tot} is the total mass that is vibrating (i.e., the mass of the truck, the scale platform, and the payload of the truck). The equivalent spring constant is equal to the sum of the eight individual spring constants so that

$$k_{\text{eq}} = 8k = 2.88\times10^6 \text{ lb/ft},$$

and the total vibrating mass is given by

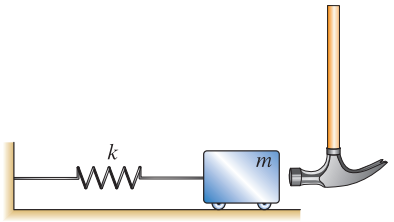
$$m_{\text{tot}} = \frac{W_{\text{tot}}}{g} = \frac{W_{\text{scale}} + W_{\text{truck}} + W_{\text{payload}}}{g} = \frac{51{,}000 \text{ lb} + 74{,}000 \text{ lb} + W_{\text{payload}}}{32.2 \text{ ft/s}^2},$$

where W_{tot} is the total vibrating weight. Substituting $f = 3.3$ Hz, k_{eq}, and m_{tot} into Eq. (1), we obtain

$$\boxed{W_{\text{payload}} = 90{,}700 \text{ lb}.}$$

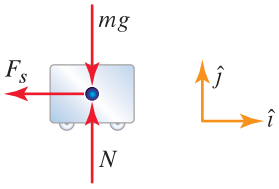
Problem 9.15

A mass m of 3 kg is in equilibrium when a hammer hits it, imparting a velocity v_0 of 2 m/s to it. If k is 120 N/m, determine the amplitude of the ensuing vibration and find the maximum acceleration experienced by the mass.



Solution

Using the FBD at the right and summing forces in the x direction, we obtain



$$\sum F_x: \quad -F_s = ma_x = m\ddot{x} \quad \Rightarrow \quad m\ddot{x} + kx = 0, \qquad (1)$$

where $F_s = kx$ is the force in the spring and x is measured from the undeformed position of the spring. Since the initial velocity of the mass is $\dot{x}(0) = v_0 = 2\,\text{m/s}$ and its initial position is $x(0) = x_0 = 0$ and since $\omega_n^2 = k/m$, the amplitude of vibration is

$$C = \sqrt{\frac{v_0^2}{\omega_n^2} + x_0^2} = \sqrt{\frac{v_0^2}{k/m}} = \sqrt{\frac{mv_0^2}{k}} \quad \Rightarrow \quad \boxed{C = 0.316\,\text{m}.}$$

To find the maximum acceleration of the mass, we write the position as a function of time in the form

$$x(t) = C\sin(\omega_n t + \phi),$$

and then differentiate it twice to obtain

$$\ddot{x}(t) = -C\omega_n^2 \sin(\omega_n t + \phi), \quad \Rightarrow \quad \ddot{x}_{\max} = C\omega_n^2 \quad \Rightarrow \quad \boxed{\ddot{x}_{\max} = 12.6\,\text{m/s}^2.}$$

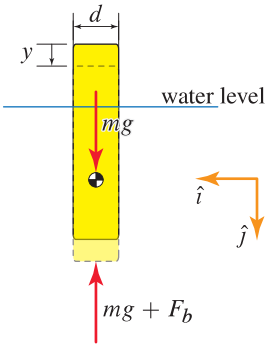
Problem 9.16

The buoy in the photograph can be modeled as a circular cylinder of diameter d and mass m. If the buoy is pushed down in the water, which has density ρ, it will oscillate vertically. Determine the frequency of oscillation. Evaluate your result for $d = 1.2\,\text{m}$, $m = 900\,\text{kg}$, and surface seawater, which has a density of $\rho = 1027\,\text{kg/m}^3$. *Hint:* Use Archimedes' principle, which states that a body wholly or partially submerged in a fluid is buoyed up by a force equal to the weight of the displaced fluid.



Solution

Referring to the FBD of the buoy at the right, mg is it's weight and F_b is the added buoyancy force due to the displacement of the buoy by an amount y from it's static equilibrium position. Summing forces in the y direction, we obtain



$$\sum F_y: \quad mg - (mg + F_b) = m\ddot{y} \quad \Rightarrow \quad m\ddot{y} + F_b = 0. \tag{1}$$

The added buoyancy force is equal to the weight of the fluid displaced when the buoy is deflected by the distance y. Therefore, it must be

$$F_b = \rho \Omega g = \rho \left[\pi \left(\frac{d}{2} \right)^2 y \right] g,$$

where Ω is the volume of the fluid displaced. Therefore, Eq. (1) becomes

$$m\ddot{y} + \tfrac{\pi}{4}\rho g d^2 y = 0 \quad \Rightarrow \quad \ddot{y} + \frac{\pi \rho g d^2}{4m} y = 0.$$

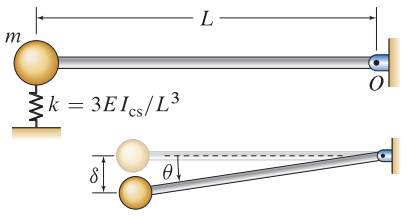
This equation is in standard form, so we can write the natural frequency ω_n and the frequency f as

$$\omega_n = \frac{d}{2}\sqrt{\frac{\pi \rho g}{m}} \quad \Rightarrow \quad \boxed{f = \frac{\omega_n}{2\pi} = \frac{d}{4}\sqrt{\frac{\rho g}{\pi m}} = 0.566\,\text{Hz},}$$

where, to obtain the numerical result, we have used $d = 1.2\,\text{m}$, $\rho = 1027\,\text{kg/m}^3$, $m = 900\,\text{kg}$, and $g = 9.81\,\text{m/s}^2$.

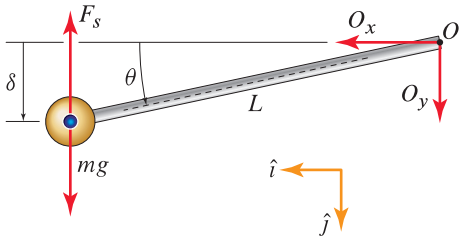
Problem 9.17

For the silicon nanowire in Example 9.2, use the lumped mass model shown, in which a point mass m is connected to a rod of negligible mass and length L that is pinned at O, to determine the natural frequency ω_n and frequency f of the nanowire. Use the values given in Example 9.2 for the mass of the lumped mass, the length of the massless rod, and the parameters used to determine the spring constant $k = 3EI_{cs}/L^3$. You may use either δ or θ as the position variable in your solution. Assume that the displacement of m is small so that it moves vertically.



Solution

Referring to the FBD of the nanowire at the right, which shows the nanowire displaced from its static equilibrium position, we let F_s be the elastic force in the spring. Assuming that both δ and θ are small and summing moments about point O, we obtain



$$\sum M_O: \quad (mg - F_s)L = I_O\alpha_n = I_O\ddot{\theta}, \qquad (1)$$

where $\alpha_n = \ddot{\theta}$ from the angular acceleration of the nanowire, $I_O = mL^2$ by the parallel axis theorem, and θ and δ are measured from the static equilibrium position of the nanowire. Noting that the force in the spring is given by $F_s = k(\delta + \delta_{st}) = kL(\theta + \theta_{st})$, we can write Eq. (1) as

$$mL^2\ddot{\theta} + kL^2(\theta + \theta_{st}) - mgL = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{k}{m}\theta = 0,$$

where we have used the fact that $kL\theta_{st} = mg$.

From Example 9.2, we know that the cross section of the Si nanowire is circular and that it has the following properties:

$$L = 9.8\times10^{-6}\,\text{m}, \quad d = 2r = 330\times10^{-9}\,\text{m}, \quad \rho = 2330\,\text{kg/m}^3, \quad \text{and} \quad E = 152\,\text{GPa},$$

where d is the diameter of the wire, r is its radius, ρ is its density, and E is its modulus of elasticity. Since the centroidal area moment of inertia of the cross section is $I_{cs} = \frac{1}{4}\pi r^4$, we can compute k as

$$k = \frac{3(152\times10^9\,\text{N/m}^2)\left[\frac{1}{4}\pi(165\times10^{-9}\,\text{m})^4\right]}{(9.8\times10^{-6}\,\text{m})^3} = 0.2820\,\text{N/m}.$$

The mass m of the lumped mass at the end of the wire is simply the mass of the wire, which is the density of Si multiplied by the volume of the wire, that is,

$$m = \rho\pi r^2 L = 1.953\times10^{-15}\,\text{kg}.$$

Using the values of k and m found above, we find that the natural frequency ω_n and frequency f are

$$\omega_n = \sqrt{\frac{k}{m}} = 1.20\times10^7\,\text{rad/s} \quad \text{and} \quad f = \frac{\omega_n}{2\pi} = 1.91\times10^6\,\text{Hz}. \qquad (2)$$

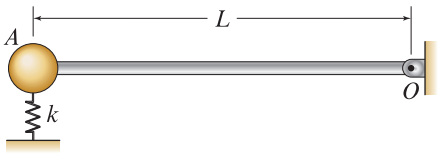
The frequency of vibration goes down when all the mass is lumped at the end of the wire since it now has a larger mass moment of inertia with respect to point O when compared with rigid body model used in Example 9.2.

Problem 9.18

The small sphere A has mass m and is fixed at the end of the arm OA of negligible mass, which is pinned at O. If the linear elastic spring has stiffness k, determine the equation of motion for small oscillations, using

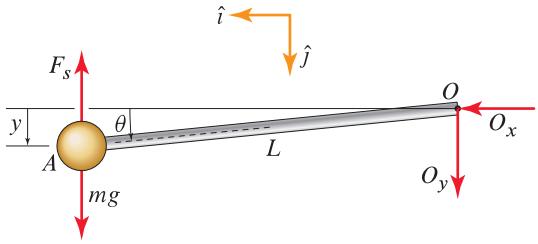
(a) the vertical position of the mass A as the position coordinate,

(b) the angle formed by the arm OA with the horizontal as the position coordinate.



Solution

When the nanowire is displaced from its equilibrium position, the FBD is as shown on the right, where F_s is the elastic restoring force due to the spring. Summing moments about point O and using the fact that both y and θ are small, we obtain



$$\sum M_O: \quad mgL - F_s L = I_O \alpha_{OA}, \qquad (1)$$

where $I_O = mL^2$ by the parallel axis theorem and α_{OA} is the angular acceleration of the nanowire.

Part (a). With y as the vertical position of m measured from the static equibrium position of the nanowire, the force in the spring can be written as

$$F_s = k(\delta_{\text{st}} + y),$$

where δ_{st} is the deflection of the spring in the static equilibrium position. Using this force law, along with the kinematics relation $\alpha_{OA} = \ddot{y}/L$, the equation of motion (Eq. (1)) in terms of y becomes

$$mgL - k(\delta_{\text{st}} + y)L = mL^2\ddot{y}/L.$$

Since $mg = k\delta_{\text{st}}$, we have

$$\boxed{m\ddot{y} + ky = 0.}$$

Part (b). Using θ as the coordinate measuring the position of the nanowire from its static equibrium position, the force in the spring can be written as

$$F_s = k(\delta_{\text{st}} + L\theta),$$

where δ_{st} is the deflection of the spring in the static equilibrium position. Using this force law, along with the kinematics relation $\alpha_{OA} = \ddot{\theta}$, the equation of motion (Eq. (1)) in terms of y becomes

$$mgL - k(\delta_{\text{st}} + L\theta)L = mL^2\ddot{\theta}.$$

Since $mg = k\delta_{\text{st}}$, we have

$$\boxed{m\ddot{\theta} + k\theta = 0.}$$

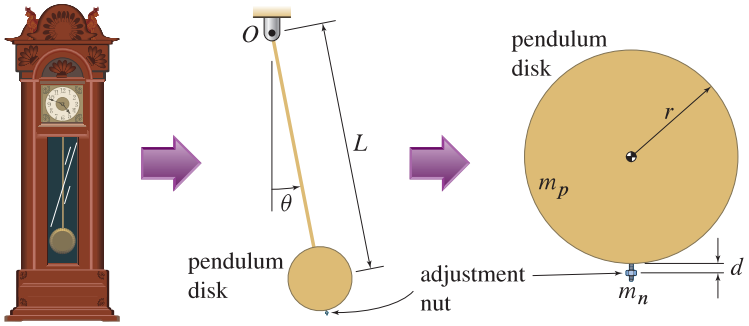
Of course both equations of motion give the same vibration frequency.

Problems 9.19 and 9.20

Grandfather clocks keep time by advancing the hands a set amount per oscillation of the pendulum. Therefore, the pendulum needs to have a very accurate period for the clock to keep time accurately. As a fine adjustment of the pendulum's period, many grandfather clocks have an adjustment nut on a bolt at the bottom of the pendulum disk. By screwing this nut inward or outward, the mass distribution of the pendulum can be changed and its period adjusted. In the following problems, model the pendulum as a uniform disk of radius r and mass m_p, which is at the end of a rod of negligible mass and length $L - r$. Model the adjustment nut as a particle of mass m_n, and let the distance between the bottom of the pendulum disk and the nut be d.

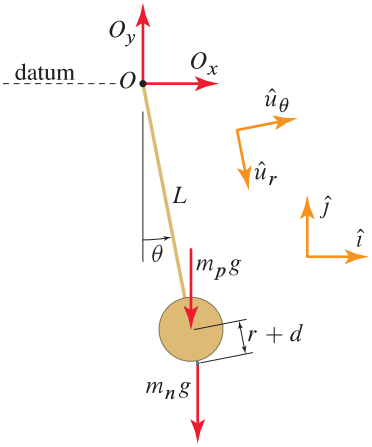
Problem 9.19 If the adjustment is initially at a distance $d = 9\,\text{mm}$ from the bottom of the pendulum disk, how much would the period of the pendulum change if the nut were screwed 4 mm closer to the disk? In addition, how much time would the clock gain or lose in a 24 h period if this were done? Let $m_p = 0.7\,\text{kg}$, $r = 0.1\,\text{m}$, $m_n = 8\,\text{g}$, and $L = 0.85\,\text{m}$.

Problem 9.20 The clock is running slow so that it is losing 1 minute every 24 hours (i.e., the clock takes 1441 minutes to complete a 1440 minute day). If the adjustment nut is at $d = 2\,\text{cm}$, what would its mass need to have to correct the pendulum's period if the nut is moved to $d = 0\,\text{cm}$? Let $m_p = 0.7\,\text{kg}$, $r = 0.1\,\text{m}$, and $L = 0.85\,\text{m}$.



Solution to 9.19

Referring to the FBD at the right, we see that as the pendulum swings, on the weight of the pendulum disk and the adjustment nut do work. Since the system is conservative, we apply the energy method to find the equation of motion and thus the natural frequency of oscillation. The kinetic energy of the system can be written as



$$T = \tfrac{1}{2}(I_O)_d \omega_p^2 + \tfrac{1}{2}(I_O)_n \omega_p^2 = \tfrac{1}{2}\left[(I_O)_d + I_O)_n\right]\omega_p^2,$$

where $(I_O)_d$ is the mass moment of inertia of the pendulum disk with respect to point O, $(I_O)_n$ is the mass moment of inertia of the adjustment nut with respect to point O, and $\omega_p = \dot{\theta}$ is the angular velocity of the pendulum. The moments of inertia are given by

$$(I_O)_p = \tfrac{1}{2}m_p r^2 + m_p L^2 \quad \text{and} \quad (I_O)_n = m_n(L + r + d)^2$$
$$= m_p(\tfrac{1}{2}r^2 + L^2).$$

The potential energy of the system can be written as

$$\begin{aligned} V &= -m_p g L\cos\theta - m_n g(L+r+d)\cos\theta \\ &= -g\left[m_p L + m_n(L+r+d)\right]\cos\theta \\ &\approx -g\left[m_p L + m_n(L+r+d)\right]\left(1-\frac{\theta^2}{2}\right), \end{aligned}$$

where we have used the approximation $\cos\theta \approx 1-\theta^2/2$. Applying the energy method, we find

$$\begin{aligned} \frac{d}{dt}(T+V) = \left[m_p\left(\tfrac{1}{2}r^2+L^2\right)+m_n(L+r+d)^2\right]\dot\theta\ddot\theta + g\left[m_p L + m_n(L+r+d)\right]\theta\dot\theta = 0 \\ \Rightarrow \quad \left[m_p\left(\tfrac{1}{2}r^2+L^2\right)+m_n(L+r+d)^2\right]\ddot\theta + g\left[m_p L + m_n(L+r+d)\right]\theta = 0, \end{aligned}$$

and therefore the natural frequency is

$$\omega_n^2 = \frac{g\left[m_p L + m_n(L+r+d)\right]}{m_p\left(\tfrac{1}{2}r^2+L^2\right)+m_n(L+r+d)^2}.$$

Using $g = 9.81\,\text{m/s}^2$, $m_p = 0.7\,\text{kg}$, $L = 0.85\,\text{m}$, $m_n = 0.008\,\text{kg}$, $r = 0.1\,\text{m}$, and the fact that the period is $\tau = 2\pi/\omega_n$, with the initial position of the adjustment nut at $d = 9\,\text{mm} = 0.009\,\text{m}$, the period τ_9 is (note that we keep more than four significant figures since the difference in period is very small)

$$\tau_9 = \left.\frac{2\pi}{\omega_n}\right|_{d=0.009} = 1.85731339\,\text{s},$$

and with the nut 4 mm closer at $d = 5\,\text{mm} = 0.005\,\text{m}$, the period τ_5 is

$$\tau_5 = \left.\frac{2\pi}{\omega_n}\right|_{d=0.005} = 1.85725259\,\text{s},$$

so that the change in period is

$$\boxed{\tau_9 - \tau_5 = 0.00006080\,\text{s}.}$$

To determine how much time the clock would gain or lose in a 24 hr period, we write

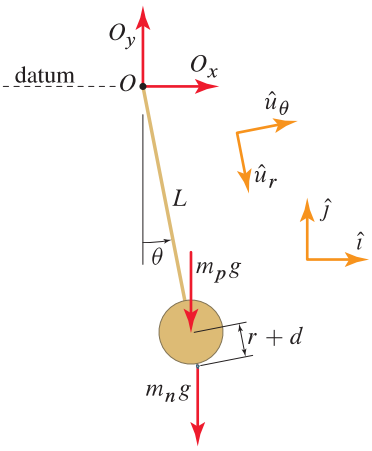
$$\frac{\tau_9}{\tau_5} = 1.00003274 = \frac{1440\,\text{min}}{t_{\text{new}}} \quad \Rightarrow \quad t_{\text{new}} = 1439.953\,\text{min}.$$

so that the day would be *shorter* by

$$1440 - 1439.953 = 0.047\,\text{min} \quad \Rightarrow \quad \boxed{0.047\,\text{min}\left(60\,\tfrac{\text{s}}{\text{min}}\right) = 2.82\,\text{s}}$$

Solution to 9.20

Referring to the FBD at the right, we see that as the pendulum swings, on the weight of the pendulum disk and the adjustment nut do work. Since the system is conservative, we apply the energy method to find the equation of motion and thus the natural frequency of oscillation. The kinetic energy of the system can be written as



$$T = \tfrac{1}{2}(I_O)_d \omega_p^2 + \tfrac{1}{2}(I_O)_n \omega_p^2 = \tfrac{1}{2}\left[(I_O)_d + I_O)_n\right]\omega_p^2,$$

where $(I_O)_d$ is the mass moment of inertia of the pendulum disk with respect to point O, $(I_O)_n$ is the mass moment of inertia of the adjustment nut with respect to point O, and $\omega_p = \dot{\theta}$ is the angular velocity of the pendulum. The moments of inertia are given by

$$\begin{aligned}(I_O)_p &= \tfrac{1}{2}m_p r^2 + m_p L^2 \quad \text{and} \quad (I_O)_n = m_n(L+r+d)^2\\ &= m_p(\tfrac{1}{2}r^2 + L^2).\end{aligned}$$

The potential energy of the system can be written as

$$\begin{aligned}V &= -m_p g L\cos\theta - m_n g(L+r+d)\cos\theta\\ &= -g\left[m_p L + m_n(L+r+d)\right]\cos\theta\\ &\approx -g\left[m_p L + m_n(L+r+d)\right]\left(1-\frac{\theta^2}{2}\right),\end{aligned}$$

where we have used the approximation $\cos\theta \approx 1 - \theta^2/2$. Applying the energy method, we find

$$\begin{aligned}\frac{d}{dt}(T+V) &= \left[m_p\left(\tfrac{1}{2}r^2+L^2\right) + m_n(L+r+d)^2\right]\dot{\theta}\ddot{\theta} + g\left[m_p L + m_n(L+r+d)\right]\theta\dot{\theta} = 0\\ \Rightarrow \quad &\left[m_p\left(\tfrac{1}{2}r^2+L^2\right) + m_n(L+r+d)^2\right]\ddot{\theta} + g\left[m_p L + m_n(L+r+d)\right]\theta = 0,\end{aligned}$$

and therefore the natural frequency is

$$\omega_n^2 = \frac{g\left[m_p L + m_n(L+r+d)\right]}{m_p\left(\tfrac{1}{2}r^2+L^2\right) + m_n(L+r+d)^2}.$$

Now that we have the natural frequency, the period can be calculated using $\tau = 2\pi/\omega_n$. Since the clock is running slow and loses one minute every 24 hours, it takes 1441 minutes to complete a 1440 minute day. That is, $\tau_{\text{correct}} = 1440\,\text{min}$ and $\tau_{\text{actual}} = 1441\,\text{min}$. Therefore

$$\frac{\tau_{\text{correct}}}{\tau_{\text{actual}}} = \frac{1440}{1441} \quad \Rightarrow \quad \frac{\tau_{\text{correct}}(d = 0\,\text{m})}{\tau_{\text{actual}}(d = 0.02\,\text{m})} = \frac{1440}{1441}, \tag{1}$$

that is, the *actual* period is when the nut is at $d = 0.02\,\text{m}$ and the *correct* period is when the nut is at $d = 0\,\text{m}$. Substituting in the expression for the period in terms of the natural frequency into Eq. (1), we obtain

$$\frac{\frac{2\pi}{(\omega_n)_{\text{correct}}}}{\frac{2\pi}{(\omega_n)_{\text{actual}}}} = \frac{\sqrt{\frac{g\left[m_p L + m_n(L+r+0.02)\right]}{m_p\left(\frac{1}{2}r^2+L^2\right)+m_n(L+r+0.02)^2}}}{\sqrt{\frac{g\left[m_p L + m_n(L+r)\right]}{m_p\left(\frac{1}{2}r^2+L^2\right)+m_n(L+r)^2}}} = \frac{1440}{1441}, \tag{2}$$

where we have substituted $d = 0\,\text{m}$ into $(\omega_n)_{\text{correct}}$ and $d = 0.02\,\text{m}$ into $(\omega_n)_{\text{actual}}$. When we substitute $g = 9.81\,\text{m/s}^2$, $m_p = 0.7\,\text{kg}$, $L = 0.85\,\text{m}$, and $r = 0.1\,\text{m}$ into Eq. (2), both sides are squared, and it is simplified, it becomes the following quadratic equation in the unknown m_n

$$m_n^2 + 0.6524 m_n - 0.02446 = 0. \tag{3}$$

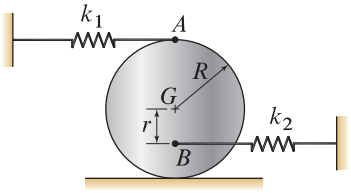
Solving Eq. (3) for m_n, we obtain

$$m_n = -0.688\,\text{kg} \quad \text{and} \quad m_n = 0.0356\,\text{kg} \quad \Rightarrow \quad \boxed{m_n = 35.6\,\text{g},}$$

where the positive root has been chosen for the mass.

Problems 9.21 and 9.22

The uniform cylinder rolls without slipping on a flat surface. Let $k_1 = k_2 = k$ and $r = R/2$. Assume that the horizontal motion of G is small.

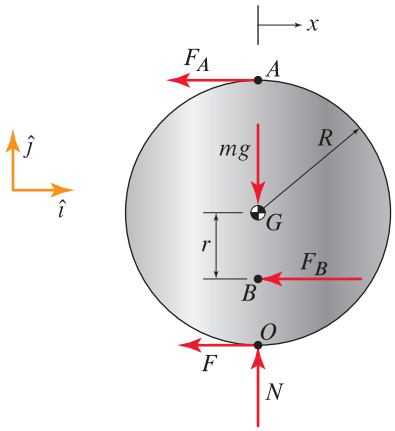


Problem 9.21 Determine the equation of motion for the cylinder by writing its Newton-Euler equations. Use the horizontal position of the mass center G as the degree of freedom.

Problem 9.22 Determine the equation of motion for the cylinder using the energy method. Use the horizontal position of the mass center G as the degree of freedom.

Solution to 9.21

Referring to the FBD on the right, we can sum moments about point O to obtain



$$\sum M_O: \quad 2RF_A + (R-r)F_B = I_O\alpha_G, \qquad (1)$$

where F_A is the force in the spring of constant k_1, F_B is the force in the spring of constant k_2, α_G is the angular acceleration of the cylinder, and where

$$I_O = I_G + mR^2 = \tfrac{1}{2}mR^2 + mR^2 = \tfrac{3}{2}mR^2.$$

For the spring forces, we can write

$$F_A = k_1 x_A \quad \text{and} \quad F_B = k_2 x_B,$$

where x_A and x_B are the horizontal deflection of the springs at A and B, respectively. Since the cylinder rolls without slip at O, we can write the velocity of points A and B as

$$\vec{v}_A = \vec{v}_O + \omega_G \times \vec{r}_{A/O} = \omega_G \hat{k} \times 2R\,\hat{\jmath} = -2R\omega_G\,\hat{\imath},$$
$$\vec{v}_B = \vec{v}_O + \omega_G \times \vec{r}_{B/G} = \omega_G \hat{k} \times (R-r)\,\hat{\jmath} = -(R-r)\omega_G\,\hat{\imath} = -\tfrac{1}{2}R\omega_G\,\hat{\imath},$$

where we have used the fact that $r = R/2$. Since we are assuming that the horizontal motions are small, these expressions for velocity imply that

$$x_A = -2R\theta_G \quad \text{and} \quad x_B = -\tfrac{1}{2}R\theta_G,$$

where θ_G is the rotation of the cylinder. Since $x_G = -R\theta_G$, we can write the above two expressions for the motion of points A and B as

$$x_A = 2x_G \quad \text{and} \quad x_B = \tfrac{1}{2}x_G,$$

and so the spring force equations become (also using $k_1 = k_2 = k$)

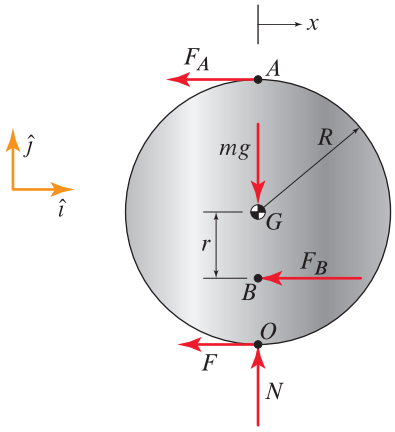
$$F_A = 2kx_G \quad \text{and} \quad F_B = \tfrac{1}{2}kx_G.$$

Substituting everything into Eq. (1) and noting that $\alpha_G = -\ddot{x}_G/R$, we obtain

$$4kRx_G + \left(\tfrac{1}{2}R\right)\tfrac{1}{2}kx_G = \tfrac{3}{2}mR^2\left(-\frac{\ddot{x}_G}{R}\right) \quad \Rightarrow \quad \boxed{\ddot{x}_G + \frac{17k}{6m}x_G = 0.}$$

Solution to 9.22

Referring to the FBD on the right, we that only F_A and F_B do work as the wheel undergoes small horizontal motion, where F_A is the force in the spring of constant k_1, F_B is the force in the spring of constant k_2. Since both of those spring forces are conservative, we can apply the energy method to determine the equation of motion of the cylinder. With this in mind, the kinetic energy of the cylinder is



$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_G^2,$$

here ω_G is the angular speed of the cylinder as it rolls without slip at point O. Since $I_G = \frac{1}{2}mR^2$, $v_G = \dot{x}_G$, and $\omega_G = -\dot{x}_G/R$, the kinetic energy can be written as

$$T = \tfrac{1}{2}m\dot{x}_G^2 + \tfrac{1}{2}\left(\tfrac{1}{2}mR^2\right)\left(-\frac{\dot{x}_G}{R}\right)^2 = \tfrac{3}{4}m\dot{x}_G^2.$$

For the potential energy of the springs, we can write

$$V = \tfrac{1}{2}k_1x_A^2 + \tfrac{1}{2}k_2x_B^2 = \tfrac{1}{2}kx_A^2 + \tfrac{1}{2}kx_B^2,$$

where x_A and x_B are the horizontal deflection of the springs at A and B, respectively, and we have used the fact that $k_1 = k_2 = k$. Since the cylinder rolls without slip at O, we can write the velocity of points A and B as

$$\vec{v}_A = \vec{v}_O + \omega_G \times \vec{r}_{A/O} = \omega_G\,\hat{k} \times 2R\,\hat{\jmath} = -2R\omega_G\,\hat{\imath},$$
$$\vec{v}_B = \vec{v}_O + \omega_G \times \vec{r}_{B/G} = \omega_G\,\hat{k} \times (R-r)\,\hat{\jmath} = -(R-r)\omega_G\,\hat{\imath} = -\tfrac{1}{2}R\omega_G\,\hat{\imath},$$

where we have used the fact that $r = R/2$. Since we are assuming that the horizontal motions are small, these expressions for velocity imply that

$$x_A = -2R\theta_G \quad \text{and} \quad x_B = -\tfrac{1}{2}R\theta_G,$$

where θ_G is the rotation of the cylinder. Since $x_G = -R\theta_G$, we can write the above two expressions for the motion of points A and B as

$$x_A = 2x_G \quad \text{and} \quad x_B = \tfrac{1}{2}x_G,$$

and so the potential energy becomes

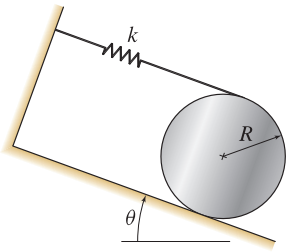
$$V = \tfrac{1}{2}k(2x_G)^2 + \tfrac{1}{2}k\left(\tfrac{1}{2}x_G\right)^2 = \tfrac{17}{8}kx_G^2.$$

Substituting T and V into $\frac{d}{dt}(T+V) = 0$, we obtain

$$\tfrac{3}{2}m\dot{x}_G\ddot{x}_G + \tfrac{17}{4}kx_G\dot{x}_G = 0 \quad \Rightarrow \quad \boxed{\ddot{x}_G + \frac{17k}{6m}x_G = 0.}$$

Problems 9.23 and 9.24

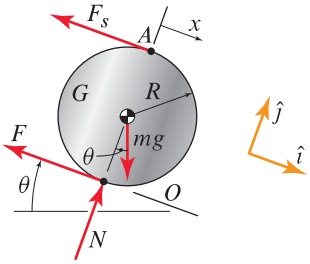
The uniform cylinder of mass m and radius R rolls without slipping on the inclined surface. The spring with constant k wraps around the cylinder as it rolls.



Problem 9.23 Determine the equation of motion for the cylinder by writing its Newton-Euler equations. Determine the numerical value of the period of oscillation of the cylinder using $k = 30\,\text{N/m}$, $m = 10\,\text{kg}$, $R = 30\,\text{cm}$, and $\theta = 20°$.

Problem 9.24 Determine the equation of motion for the cylinder using the energy method. Determine the numerical value of the period of oscillation of the cylinder using $k = 30\,\text{N/m}$, $m = 10\,\text{kg}$, $R = 30\,\text{cm}$, and $\theta = 20°$.

Solution to 9.23



Referring to the FBD at the right and summing moments about point O, we obtain

$$\sum M_O: \quad -(mg\sin\theta)R + 2RF_s = I_G\alpha_G + mRa_{Gx}, \tag{1}$$

where F_s is the force in the spring, α_G is the angular acceleration of the cylinder, F is the force of friction between the cylinder and the ground, and a_{Gx} is the x component of the acceleration of the mass center of the cylinder. If the center of the cylinder at G moves a distance x_G down the incline, then point A must move a distance $2x_G$ and so the total deflection of the spring must consist of its deflection when the system is in static equilibrium δ_{st} plus its deflection away from static equilibrium $2x_G$, which implies that

$$F_s = k(\delta_{\text{st}} + 2x_G),$$

In addition, since the cylinder rolls without slip at point O, we can write the angular acceleration of the cylinder as $\alpha_G = -\ddot{x}_G/R$. Substituting this kinematics equation and the force law shown above into Eq. (1) and using $I_G = \frac{1}{2}mR^2$, we obtain

$$-mgR\sin\theta + 2kR(\delta_{\text{st}} + 2x_G) = \tfrac{1}{2}mR^2\left(-\frac{\ddot{x}_G}{R}\right) - mR\ddot{x}_G.$$

Using the fact that $mgR\sin\theta = 2kR\delta_{\text{st}}$ and then simplifying this equation, we obtain the equation of motion as

$$\boxed{\ddot{x}_G + \frac{8k}{3m}x_G = 0.} \tag{2}$$

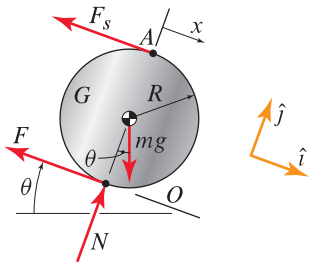
From Eq. (2), we can see that the natural frequency and period are given by

$$\omega_n = \sqrt{\frac{8k}{3m}} \quad \Rightarrow \quad \boxed{\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3m}{8k}} = 2.22\,\text{s},}$$

where we have substituted in $m = 10\,\text{kg}$ and $k = 30\,\text{N/m}$.

Solution to 9.24

Referring to the FBD at the right we see that F_s and mg are the only two forces that do work (since the cylinder rolls without slip on a stationary surface, the friction force F does no work). Since both of these forces are conservative, we can apply the energy method to obtain the equation of motion for the system. The kinetic energy of the system is



$$T = \tfrac{1}{2} I_O \omega_G^2 = \tfrac{1}{2}\left(\tfrac{1}{2}mR^2 + mR^2\right)\omega_G^2 = \tfrac{3}{4}mR^2\omega_G^2 = \tfrac{3}{4}mR\dot{x}_G^2,$$

where ω_G is the angular velocity of the cylinder, we have used the parallel axis theorem to find I_O, and we have used the kinematic relation that $\dot{x}_G = -R\omega_G$ to write ω_G since the cylinder rolls without slip at point O. If the center of the cylinder at G moves a distance x_G down the incline, then point A must move a distance $2x_G$ and so the total deflection of the spring must consist of its deflection when the system is in static equilibrium δ_{st} plus its deflection away from static equilibrium $2x_G$, which implies that

$$V = \tfrac{1}{2}k(\delta_{\text{st}} + 2x_G)^2 - mgx_G \sin\theta,$$

Substituting the kinetic and potential energies into the energy method, we obtain

$$\frac{d}{dt}(T + V) = \tfrac{3}{2}m\dot{x}_G\ddot{x}_G + k(\delta_{\text{st}} + 2x_G)(2\dot{x}_G) - mg\dot{x}_G \sin\theta = 0,$$

Canceling $\dot{x}_G$, using the fact that $mg \sin\theta = 2k\delta_{\text{st}}$, and then simplifying this equation, we obtain the equation of motion as

$$\boxed{\ddot{x}_G + \frac{8k}{3m}x_G = 0.} \tag{3}$$

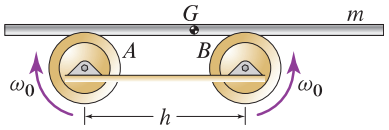
From Eq. (3), we can see that the natural frequency and period are given by

$$\omega_n = \sqrt{\frac{8k}{3m}} \quad \Rightarrow \quad \boxed{\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3m}{8k}} = 2.22\,\text{s},}$$

where we have substituted in $m = 10\,\text{kg}$ and $k = 30\,\text{N/m}$.

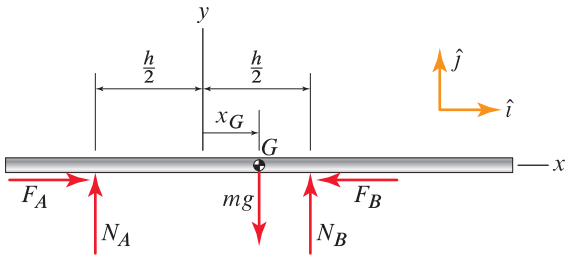
Problem 9.25

A uniform bar of mass m is placed off-center on two counter-rotating drums A and B. Each drum is driven with constant angular speed ω_0, and the coefficient of kinetic friction between the drums and the bar is μ_k. Determine the natural frequency of oscillation of the bar on the rollers. *Hint:* Measure the horizontal position of G relative to the midpoint between the two drums, and assume that the drums rotate sufficiently fast so that the drums are always slipping relative to the bar.



Solution

Referring to the FBD at the right, notice that the coordinate system we are using measures the horizontal position of the mass center of the bar x_G relative to the midpoint between the two drums. Since drum A rotates clockwise, the friction force it exerts on the bar F_A is to the right and since drum B rotates counterclockwise, the friction force it exerts on the bar F_B is to the left. With this in mind, the balance laws for the bar shown can be written as



$$\sum F_x: \qquad F_A - F_B = ma_{Gx}, \tag{1}$$

$$\sum F_y: \qquad N_A + N_B - mg = ma_{Gy}, \tag{2}$$

$$\sum M_G: \quad N_B(h/2 - x_G) - N_A(h/2 + x_G) = I_G\alpha_{\text{bar}}. \tag{3}$$

Since the counter-rotating drums must be slipping relative to the bar, the force laws are given by

$$F_A = \mu_k N_A \quad \text{and} \quad F_B = \mu_k N_B.$$

Since the bar does not rotate or move vertically, the kinematics equations are given by

$$a_{Gx} = \ddot{x}_G, \quad a_{Gy} = 0, \quad \text{and} \quad \alpha_{\text{bar}} = 0.$$

Substituting the force laws and kinematics into Eq. (1), we obtain

$$\mu_k N_A - \mu_k N_B = m\ddot{x}_G. \tag{4}$$

Substituting the kinematics into Eqs. (2) and (3) and then solving the resulting two equations for N_A and N_B gives

$$N_A = mg\left(\frac{1}{2} - \frac{x_G}{h}\right) \quad \text{and} \quad N_B = mg\left(\frac{1}{2} + \frac{x_G}{h}\right).$$

Substituting the two normal forces into Eq. (4), we obtain the equation of motion as

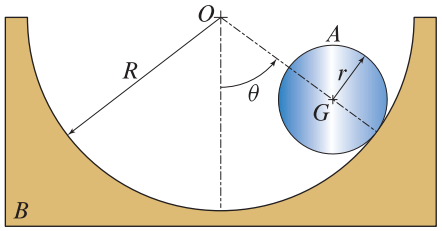
$$\mu_k mg\left(\frac{1}{2} - \frac{x_G}{h}\right) - \mu_k mg\left(\frac{1}{2} + \frac{x_G}{h}\right) = m\ddot{x}_G \quad \Rightarrow \quad \ddot{x}_G + \frac{2\mu_k g}{h}x_G = 0.$$

Therefore, the natural frequency of oscillation is

$$\boxed{\omega_n = \sqrt{\frac{2\mu_k g}{h}}.}$$

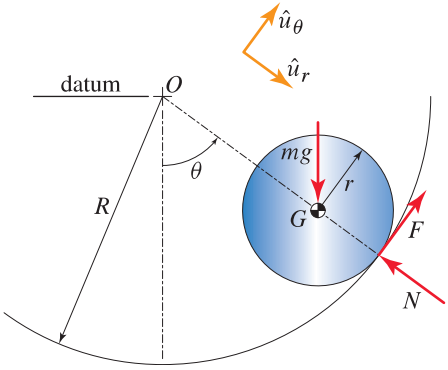
Problem 9.26

The uniform cylinder A of radius r and mass m is released from a small angle θ inside the large cylinder of radius R. Assuming that it rolls without slipping, determine the natural frequency and period of oscillation of A.



Solution

Referring to the FBD shown at the right, as long as the cylinder rolls without slipping, only the weight force mg will do work on it. Since the weight force is conservative, we can apply the energy method to find the equation of motion. With this in mind, the kinetic energy can be written as



$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_G^2,$$

where v_G is the speed of the mass center of the cylinder, ω_G is the angular speed of the cylinder, and I_G is its mass moment of inertia. Since point G is moving in a circle centered at O and since the cylinder A of radius r is rolling without slip inside the larger cylinder B, we can write the speed of G as

$$v_G = (R-r)\dot{\theta} = r\omega_G \quad \Rightarrow \quad \omega_G = \frac{R-r}{r}\dot{\theta},$$

where ω_G is the angular speed of the cylinder A. Using these results, the kinetic energy becomes

$$T = \tfrac{1}{2}m(R-r)^2\dot{\theta}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}mr^2\right)\left(\frac{R-r}{r}\right)^2\dot{\theta}^2 = \tfrac{3}{4}m(R-r)^2\dot{\theta}^2, \tag{1}$$

where we have used $I_G = \frac{1}{2}mr^2$ for a uniform cylinder. In addition, using the datum line shown in the figure above, the potential energy of the cylinder can be written as

$$V = -mg(R-r)\cos\theta.$$

Adding the potential energy to the kinetic energy in Eq. (1) and applying the energy method, we obtain

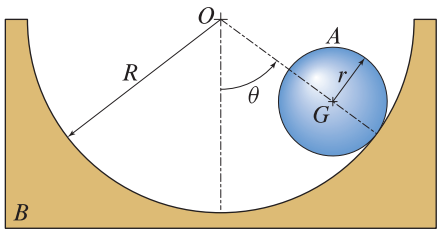
$$\frac{d}{dt}(T+V) = \tfrac{3}{2}m(R-r)^2\dot{\theta}\ddot{\theta} + mg(R-r)\dot{\theta}\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{2g}{3(R-r)}\sin\theta = 0.$$

For small values of θ, we can approximate $\sin\theta$ by θ and we obtain

$$\ddot{\theta} + \frac{2g}{3(R-r)}\theta = 0 \quad \Rightarrow \quad \boxed{\omega_n = \sqrt{\frac{2g}{3(R-r)}}} \quad \text{and} \quad \boxed{\tau = 2\pi\sqrt{\frac{3(R-r)}{2g}}}.$$

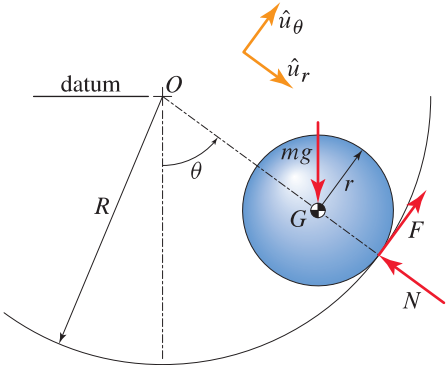
Problem 9.27

The uniform sphere A of radius r and mass m is released from a small angle θ inside the large cylinder of radius R. Assuming that it rolls without slipping, determine the natural frequency and period of oscillation of the sphere.



Solution

Referring to the FBD shown at the right, as long as the sphere rolls without slipping, only the weight force mg will do work on it. Since the weight force is conservative, we can apply the energy method to find the equation of motion. With this in mind, the kinetic energy can be written as



$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega_G^2,$$

where v_G is the speed of the mass center of the sphere, ω_G is the angular speed of the sphere, and I_G is its mass moment of inertia. Since point G is moving in a circle centered at O and since the sphere A of radius r is rolling without slip inside the larger cylinder B, we can write the speed of G as

$$v_G = (R-r)\dot{\theta} = r\omega_G \quad \Rightarrow \quad \omega_G = \frac{R-r}{r}\dot{\theta},$$

where ω_G is the angular speed of the sphere A. Using these results, the kinetic energy becomes

$$T = \tfrac{1}{2}m(R-r)^2\dot{\theta}^2 + \tfrac{1}{2}\left(\tfrac{2}{5}mr^2\right)\left(\frac{R-r}{r}\right)^2\dot{\theta}^2 = \tfrac{7}{10}m(R-r)^2\dot{\theta}^2, \tag{1}$$

where we have used $I_G = \frac{2}{5}mr^2$ for a uniform cylinder. In addition, using the datum line shown in the figure above, the potential energy of the sphere can be written as

$$V = -mg(R-r)\cos\theta.$$

Adding the potential energy to the kinetic energy in Eq. (1) and applying the energy method, we obtain

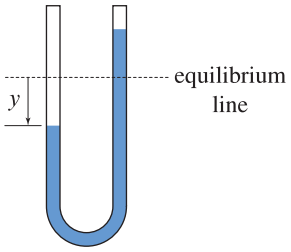
$$\frac{d}{dt}(T+V) = \tfrac{7}{5}m(R-r)^2\dot{\theta}\ddot{\theta} + mg(R-r)\dot{\theta}\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{5g}{7(R-r)}\sin\theta = 0.$$

For small values of θ, we can approximate $\sin\theta$ by θ and we obtain

$$\ddot{\theta} + \frac{5g}{7(R-r)}\theta = 0 \quad \Rightarrow \quad \boxed{\omega_n = \sqrt{\frac{5g}{7(R-r)}}} \quad \text{and} \quad \boxed{\tau = 2\pi\sqrt{\frac{7(R-r)}{5g}}}.$$

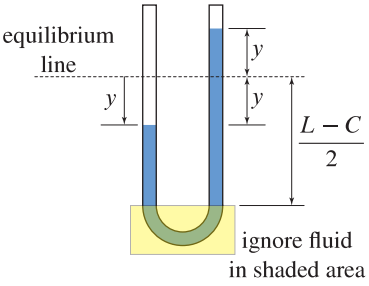
Problem 9.28

The U-tube manometer lies in the vertical plane and contains a fluid of density ρ that has been displaced a distance y and oscillates in the tube. If the cross-sectional area of the tube is A and the total length of the fluid in the tube is L, determine the natural period of oscillation of the fluid, using the energy method. *Hint:* As long as the curved portion of the tube is always filled with liquid (i.e., the oscillations don't get large enough to empty part of it), the contribution of the liquid in the curved portion to the potential energy is *constant*.



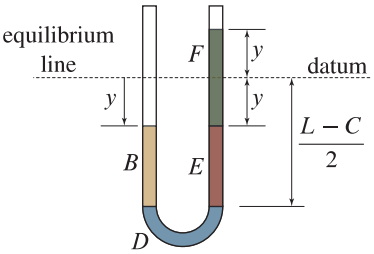
Solution

To apply the energy method to find the equation of motion of the oscillating fluid, we need to write the kinetic and potential energy of the fluid in the tube. Referring to the top figure on the right and assuming the all the fluid moves as a unit, we can write the kinetic energy of the fluid as



$$T = \tfrac{1}{2}m\dot{y}^2 = \tfrac{1}{2}\rho AL\dot{y}^2,$$

where we have computed the mass of the fluid in the tube as $m = \rho AL$, $\dot{y}$ is the speed of each element of the fluid, and the coordinate y is measure downward from the equilibrium level of the surface of the fluid. Referring the bottom figure on the right, the total potential energy of the fluid can be computed as the sum of the individual potential energies of the four segments of fluid B, D, E, and F, that is,



$$\begin{aligned} V &= V_B + V_D + V_E + V_F \\ &= -\rho Ag \underbrace{\left(\frac{L-C}{2} - y\right)}_{\text{length of } B} \underbrace{\left[y + \frac{1}{2}\left(\frac{L-C}{2} - y\right)\right]}_{\text{dist. from datum to center of } B} + V_D \\ &\quad - \rho Ag \underbrace{\left(\frac{L-C}{2} - y\right)}_{\text{length of } E} \underbrace{\left[y + \frac{1}{2}\left(\frac{L-C}{2} - y\right)\right]}_{\text{dist. from datum to center of } E} + V_F, \end{aligned}$$

where C is the *constant* length of the fluid in the curved portion of the tube and V_F is always equal to zero. Simplifying this expression for V, we obtain

$$V = -\tfrac{1}{4}\rho Ag\left[(C-L)^2 - 4y^2\right].$$

Applying the energy method to this system, we obtain

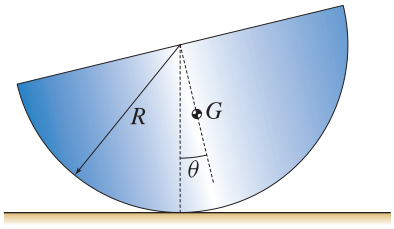
$$\frac{d}{dt}(T+V) = 0 \quad \Rightarrow \quad \rho AL\dot{y}\ddot{y} + 2\rho Agy\dot{y} = 0 \quad \Rightarrow \quad \ddot{y} + \frac{2g}{L}y = 0 \quad \Rightarrow \quad \omega_n = \sqrt{\frac{2g}{L}},$$

where we have used the fact that V_D, C, and L are all constant. Therefore, the period of oscillation of the fluid is

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{L}{2g}}.$$

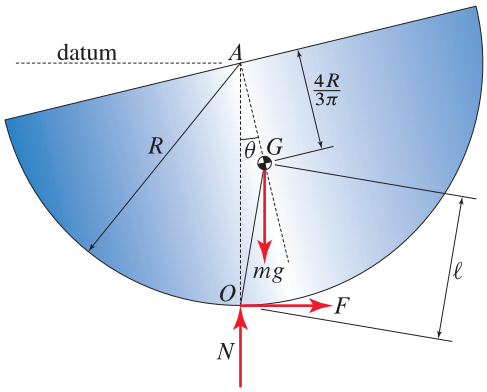
Problem 9.29

The uniform semicylinder of radius R and mass m rolls without slip on the horizontal surface. Using the energy method, determine the period of oscillation for small θ.



Solution

Referring to the FBD on the right and noting the semicylinder rolls with slip on the surface at point O, we see that the only force that does work on the semicylinder as it rocks back and forth is the weight force mg. Since that force is conservative, we can apply the energy method to find the equation of motion as instructed. Using the datum indicated, the potential energy of the semicylinder is



$$V = -mg\left(\frac{4R}{3\pi}\right)\cos\theta \approx -mg\left(\frac{4R}{3\pi}\right)(1-\theta^2/2), \quad (1)$$

where we have used the small angle approximation for $\cos\theta$. Since the semicylinder is rolling without slip at point O, that point must be the IC of the semicylinder. Therefore, we can write the kinetic energy as

$$T = \tfrac{1}{2} I_O \dot{\theta}^2, \quad (2)$$

where $\dot{\theta}$ is the angular velocity of the semicylinder. To find I_O we first note that

$$I_G = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)mR^2 \quad \text{and then} \quad I_O = I_G + m\ell^2 \approx I_G + m\left(R - \frac{4R}{3\pi}\right)^2,$$

where ℓ is the distance from O to G as seen in the figure above and we have approximated ℓ by $R - \frac{4R}{3\pi}$ for small θ. Simplifying I_O, we obtain

$$I_O = \frac{9\pi - 16}{6\pi} mR^2. \quad (3)$$

Substituting Eq. (3) into Eq. (2) and then using Eqs. (1) and (2) to apply the energy method to the semicylinder, we obtain the following equation of motion:

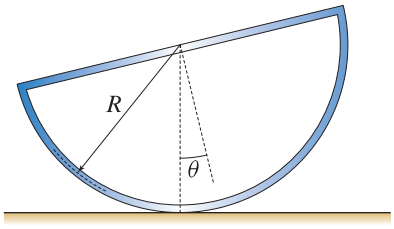
$$\frac{d}{dt}(T+V) = 0 \quad \Rightarrow \quad \left(\frac{9\pi-16}{6\pi}\right)mR^2\dot{\theta}\ddot{\theta} + mg\left(\frac{4R}{3\pi}\right)\theta\dot{\theta} = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{8g}{(9\pi-16)R}\theta = 0.$$

Therefore, the natural frequency and period of oscillation are

$$\omega_n = \sqrt{\frac{8g}{(9\pi-16)R}} \quad \Rightarrow \quad \boxed{\tau = \pi\sqrt{\frac{(9\pi-16)R}{2g}}.}$$

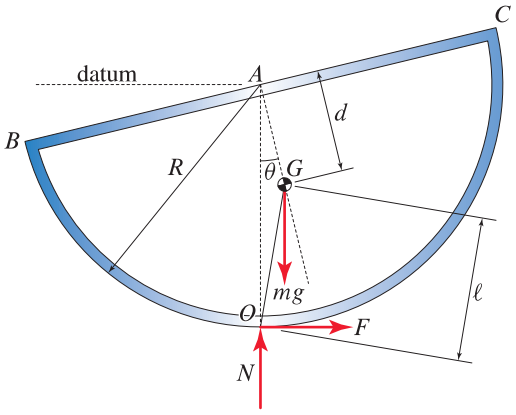
Problem 9.30

The thin shell semicylinder of radius R and mass m rolls without slip on the horizontal surface. Using the energy method, determine the period of oscillation for small θ.



Solution

Referring to the FBD on the right and noting the thin shell semicylinder rolls with slip on the surface at point O, we see that the only force that does work on the thin shell semicylinder as it rocks back and forth is the weight force mg. Since that force is conservative, we can apply the energy method to find the equation of motion as instructed. Before doing so, we need to find the distance d from point A to the center of mass of the thin shell semicylinder at G. Since the total mass of the thin shell semicylinder is m and that the length of the shell is $2R + \pi R$, its mass per unit length is $\rho = m/(2R + \pi R)$. Therefore, applying the definition of the center of mass, the distance d is given by



$$md = 0 + \pi R\rho\left(\frac{2R}{\pi}\right) = \pi R\left(\frac{m}{2R + \pi R}\right)\left(\frac{2R}{\pi}\right) \quad \Rightarrow \quad d = \frac{2R}{2 + \pi}.$$

Now that we have d, using the datum indicated, the potential energy of the thin shell semicylinder is

$$V = -mg\left(\frac{2R}{2 + \pi}\right)\cos\theta \approx -mg\left(\frac{2R}{2 + \pi}\right)(1 - \theta^2/2), \tag{1}$$

where we have used the small angle approximation for $\cos\theta$. Since the thin shell semicylinder is rolling without slip at point O, that point must be the IC of the thin shell semicylinder. Therefore, we can write the kinetic energy as

$$T = \tfrac{1}{2} I_O \dot{\theta}^2, \tag{2}$$

where $\dot{\theta}$ is the angular velocity of the thin shell semicylinder. To find I_O we must first find I_G and then apply the parallel axis theorem to obtain as $I_O = I_G + m\ell^2$, where ℓ is the distance from G to O. Since I_G for the body we are considering does not appear in the inside back cover of the books, we will first find I_A by treating the thin shell semicylinder as a composite body made up of the straight segment BC and semicircular segment BOC. Therefore

$$I_A = (I_A)_{BC} + (I_A)_{BOC} = \tfrac{1}{12} m_{BC}(2R)^2 + m_{BOC} R^2.$$

Since the linear density of the thin shell semicylinder is ρ as given above, we can write the mass of each segment as

$$m_{BC} = \rho(2R) = \frac{2m}{2 + \pi} \quad \text{and} \quad m_{BOC} = \rho(\pi R) = \frac{\pi m}{2 + \pi},$$

and therefore I_A becomes

$$I_A = \frac{1}{12}\left(\frac{2m}{2+\pi}\right)(2R)^2 + \frac{\pi m}{2+\pi}R^2 = \frac{2+3\pi}{3(2+\pi)}mR^2.$$

We can now find I_G using the parallel axis theorem as

$$I_A = I_G + md^2 \quad \Rightarrow \quad I_G = I_A - md^2 = \frac{2+3\pi}{3(2+\pi)}mR^2 - m\left(\frac{2R}{2+\pi}\right)^2 = \frac{3\pi^2+8\pi-8}{3(2+\pi)^2}mR^2.$$

Finally, we can find I_O by again using the parallel axis theorem as

$$I_O = I_G + m\ell^2 \approx I_G + m(R-d)^2 = \frac{3\pi^2+8\pi-8}{3(2+\pi)^2}mR^2 + m\left(R - \frac{2R}{2+\pi}\right)^2 = \frac{2(3\pi-2)}{3(2+\pi)}mR^2,$$

where we have approximated ℓ by $R-d$ for small θ. Substituting I_O into Eq. (2) and then using Eqs. (1) and (2) to apply the energy method to the thin shell semicylinder, we obtain the following equation of motion:

$$\frac{d}{dt}(T+V) = 0 \quad \Rightarrow \quad \left[\frac{2(3\pi-2)}{3(2+\pi)}\right]mR^2\dot{\theta}\ddot{\theta} + mg\left(\frac{2R}{2+\pi}\right)\theta\dot{\theta} = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{3g}{(3\pi-2)R}\theta = 0.$$

Therefore, the natural frequency and period of oscillation are

$$\omega_n = \sqrt{\frac{3g}{(3\pi-2)R}} \quad \Rightarrow \quad \boxed{\tau = 2\pi\sqrt{\frac{(3\pi-2)R}{3g}}.}$$

Problem 9.31

The magnification factor for a forced (undamped) harmonic oscillator is measured to be equal to 5. Determine the driving frequency of the forcing if the natural frequency of the system is 100 rad/s.

Solution

The MF for a harmonic oscillator depends on the forcing frequency and the natural frequency of a system according to

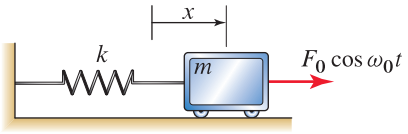
$$\text{MF} = \frac{1}{1-(\omega_0/\omega_n)^2}.$$

Since we know that MF $= 5$ and $\omega_n = 100\,\text{rad/s}$, we can substitute these values in to the above equation and solve for the forcing frequency ω_0. Doing so, we obtain

$$5 = \frac{1}{1-(\omega_0/100)^2} \quad \Rightarrow \quad \boxed{\omega_0 = 89.4\,\text{rad/s}.}$$

Problem 9.32

Suppose that equation of motion of a forced harmonic oscillator is given by $\ddot{x} + \omega_n^2 x = (F_0/m)\cos\omega_0 t$. Obtain the expression for the response of the oscillator, and compare it to the response presented in Eq. (9.42) (which is for a forced harmonic oscillator with the equation of motion given in Eq. (9.37)).



Solution

The response is given by general solution to the equation of motion, that is, it is given by

$$x(t) = x_c + x_p,$$

where x_c is the complementary solution and x_p is the particular solution. The complementary solution for the given equation of motion is still given by

$$x_c = A\sin\omega_n t + B\cos\omega_n t,$$

where A and B are constants determined by the initial conditions. To find a particular solution, we try a solution of the form $x_p = D\cos\omega_0 t$, where D is a constant to be determined. Substituting this assumed solution into the equation of motion, we obtain

$$-D\omega_0^2\cos\omega_0 t + \omega_n^2 D\cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t \quad \Rightarrow \quad D(\omega_n^2 - \omega_0^2) = \frac{F_0}{m} \quad \Rightarrow \quad D = \frac{F_0/m}{\omega_n^2 - \omega_0^2}$$

$$\Rightarrow \quad D = \frac{\frac{F_0}{m}\left(\frac{1}{\omega_n^2}\right)}{1-(\omega_0/\omega_n)^2} = \frac{\frac{F_0}{m}\left(\frac{m}{k}\right)}{1-(\omega_0/\omega_n)^2} = \frac{F_0/k}{1-(\omega_0/\omega_n)^2}$$

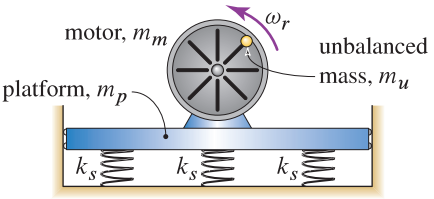
Therefore, the response is given by

$$\boxed{x(t) = A\sin\omega_n t + B\cos\omega_n t + \frac{F_0/k}{1-(\omega_0/\omega_n)^2}\cos\omega_0 t.}$$

Notice that we can find the response due to $\cos\omega_0 t$ forcing by replacing the "sin" with a "cos" in the response due to $\sin\omega_0 t$ forcing.

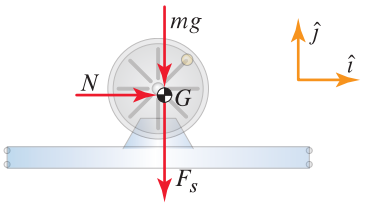
Problem 9.33

Derive the equations of motion for the unbalanced motor introduced in this section by applying Newton's second law to the center of mass of the system shown in Fig. 9.12.



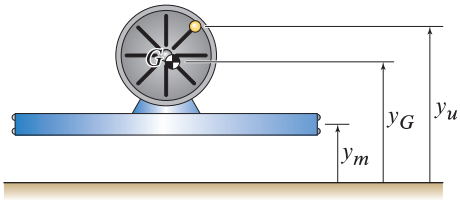
Solution

The FBD of the mass center G of the system is shown in the top figure on the right, where N is the net horizontal force acting on the system, m is the total mass of the system (i.e., $m = m_m + m_p$), and F_s is the force on the platform due to the springs. Applying Newton's second law to the center of mass and summing forces in the vertical direction, we find



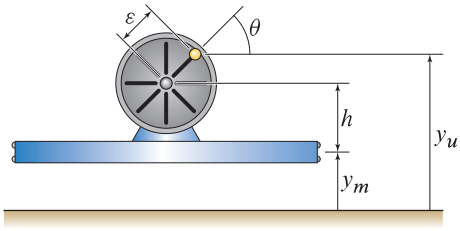
$$\sum F_y: \quad -mg - F_s = m\ddot{y}_G, \tag{1}$$

where, referring to the middle figure on the right, y_G measures the vertical position of the mass center. The force law for the springs is given by



$$F_s = k_{\text{eq}}(y_m - \delta_{\text{st}}),$$

where, referring to the middle figure on the right, y_m is the vertical position of the motor and platforms (they move together), k_{eq} is the equivalent spring constant of the six springs supporting the platform, and δ_{st} is the static deflection of the spring when the system is in static equilibrium. Substituting the force law into the balance law in Eq. (1) we find



$$-mg - k_{\text{eq}}(y_m - \delta_{\text{st}}) = m\ddot{y}_G \quad \Rightarrow \quad m\ddot{y}_G + k_{\text{eq}}y_m = 0, \tag{2}$$

where we have used the fact that $mg = k_{\text{eq}}\delta_{\text{st}}$. Notice that Eq. (2) contains two dependent variables y_m and y_G and so we need to eliminate one of them. To show that this equation is equivalent to Eq. (9.36) in the textbook, we will need to eliminate y_G by using the definition of the mass center. Referring to the bottom figure above, we find

$$my_G = m_u y_u + (m - m_u)y_m \quad \Rightarrow \quad my_G = m_u(y_m + h + \varepsilon \sin\theta) + (m - m_u)y_m.$$

Canceling $m_u y_m$ and noting that $\sin\theta = \sin\omega_r t$, this equation becomes

$$my_G = m_u(h + \varepsilon \sin\omega_r t) + my_m \quad \Rightarrow \quad m\ddot{y}_G = -m_u\varepsilon\omega_r^2 \sin\omega_r t + m\ddot{y}_m,$$

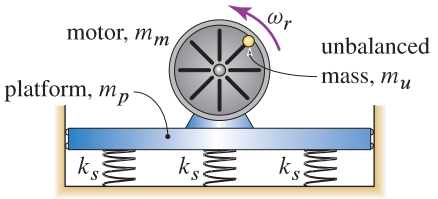
where we have used the fact that h is constant in taking the second derivative. Substituting $m\ddot{y}_G$ from this expression into Eq. (2), we obtain the equation of motion as

$$(m_m + m_p)\ddot{y}_m + k_{\text{eq}}y_m = m_u\varepsilon\omega_r^2 \sin\omega_r t \quad \Rightarrow \quad \boxed{\ddot{y}_m + \frac{k_{\text{eq}}}{m_m + m_p}y_m = \frac{m_u\varepsilon\omega_r^2}{m_m + m_p}\sin\omega_r t,}$$

where we have made the substitution $m = m_m + m_p$.

Problem 9.34

Determine the amplitude of vibration of the unbalanced motor we studied in Example 9.4 if the forcing frequency of the motor is $0.95\omega_n$.

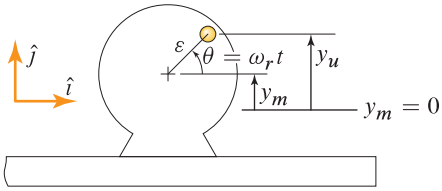


Solution

Referring to Example 9.4, the response if the unbalance motor was found to be

$$y_m = \left[\frac{v_{m0}}{\omega_n} - \frac{\omega_r}{\omega_n}\frac{m_u\varepsilon\omega_r^2/k_{\text{eq}}}{1-(\omega_r/\omega_n)^2}\right]\sin\omega_n t + \frac{m_u\varepsilon\omega_r^2/k_{\text{eq}}}{1-(\omega_r/\omega_n)^2}\sin\omega_r t, \tag{1}$$

where y_m in defined in the figure on the right, $v_{m0} = \dot{y}_m(0)$, and the natural frequency was found to be



$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m_m + m_p}}.$$

The goal then is to find the amplitude of y_m in Eq. (1). We begin by writing Equation (1) as

$$y_m = P\sin\omega_n t + Q\sin\omega_r t,$$

where

$$P = \frac{v_{m0}}{\omega_n} - \frac{\omega_r}{\omega_n}\frac{m_u\varepsilon\omega_r^2/k_{\text{eq}}}{1-(\omega_r/\omega_n)^2} \quad \text{and} \quad Q = \frac{m_u\varepsilon\omega_r^2/k_{\text{eq}}}{1-(\omega_r/\omega_n)^2}.$$

We now let $\Delta\omega = \omega_r - \omega_n$, which means that y_m can be written as

$$\begin{aligned} y_m &= P\sin\omega_n t + Q\sin(\Delta\omega + \omega_n)t = P\sin\omega_n t + Q(\sin\Delta\omega t\cos\omega_n t + \cos\Delta\omega t\sin\omega_n t) \\ &= \underbrace{Q\sin\Delta\omega t}_{A}\cos\omega_n t + \underbrace{(P + Q\cos\Delta\omega t)}_{B}\sin\omega_n t \\ &= \sqrt{A^2+B^2}\,\sin(\omega_n t + \phi), \end{aligned}$$

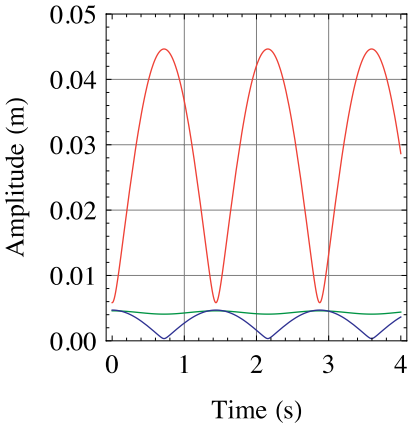
where the angle ϕ can be determined, but is not needed to find the amplitude of vibration. Thus we see that the amplitude is

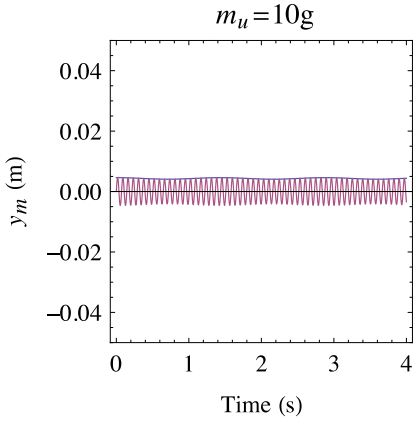
$$|y_m| = \sqrt{A^2+B^2} = \sqrt{(Q\sin\Delta\omega t)^2 + (P + Q\cos\Delta\omega t)^2} = \sqrt{P^2+Q^2+2PQ\cos(\omega_r-\omega_n)t},$$

where we have substituted $\omega_r - \omega_n$ for $\Delta\omega$. Notice that the amplitude of vibration is itself a function of time. We can now apply the known quantities from Example 9.4, which were $m_m = 40\,\text{kg}$, $m_p = 15\,\text{kg}$, $k_{\text{eq}} = 6k_s = 420{,}000\,\text{N/m}$, $\varepsilon = 15\,\text{cm}$, $\omega_r = 0.95\omega_n$, $v_m(0) = 0.4\,\text{m/s}$, and three different values of m_u: 10 g, 100 g, and 1000 g. Using these, we find that the amplitude of vibration for each of the three unbalanced masses is

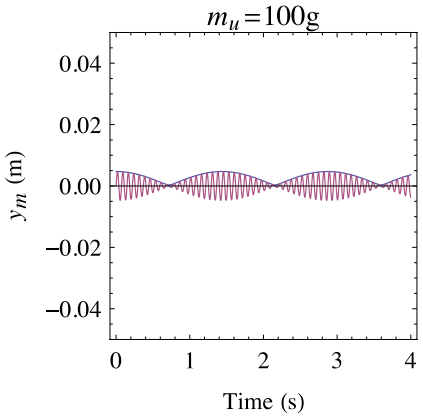
$$\begin{aligned} m_u &= 0.01\,\text{kg} \quad \Rightarrow \quad |y_m| = \sqrt{2.19\times10^{-6}\cos(4.37t) + 1.89\times10^{-5}}\,\text{m}, \\ m_u &= 0.1\,\text{kg} \quad \Rightarrow \quad |y_m| = \sqrt{1.10\times10^{-5}\cos(4.37t) + 1.11\times10^{-5}}\,\text{m}, \\ m_u &= 1\,\text{kg} \quad \Rightarrow \quad |y_m| = \sqrt{9.80\times10^{-4}\cos(4.37t) + 1.01\times10^{-3}}\,\text{m}, \end{aligned}$$

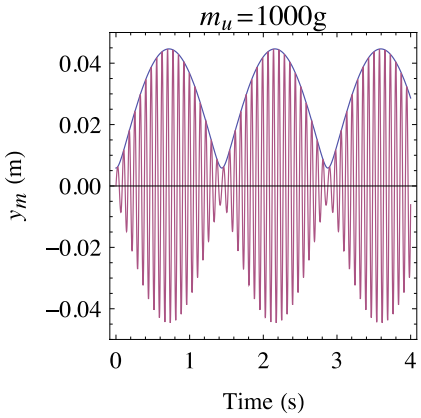
A Closer Look. A plot showing each of these three amplitudes as a function of time is shown at the right. The green curve corresponds to $m_u = 10\,\text{g}$, the blue curve corresponds to $m_u = 100\,\text{g}$, and the red curve corresponds to $m_u = 1000\,\text{g}$. Note that these curves represent the *amplitude* of the response at any time t. These curves envelope the *actual* response y_m as can be seen in the three figures below. The first figure shows the response as well as the amplitude of the response for $m_u = 10\,\text{g}$.





The second plot shows the response as well as the amplitude of the response for $m_u = 100\,\text{g}$. The third plot shows the response as well as the amplitude of the response for $m_u = 1000\,\text{g}$.

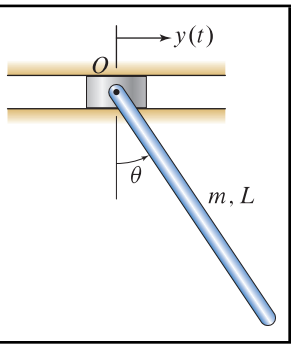




Notice that in each case the amplitude of the response $|y_m|$ that we found forms an envelope over the response y_m. When the forcing frequency is very close to the natural frequency as it is in this problem, the amplitude of the response increases and decreases as we have seen in this problem. If the vibrations were sound waves, the volume would alternately increase and decrease; this is referred to as the *beat phenomenon*.

Problem 9.35

A uniform bar of mass m and length L is pinned to a slider at O. The slider is forced to oscillate horizontally according to $y(t) = Y \sin \omega_s t$. The system lies in the vertical plane.

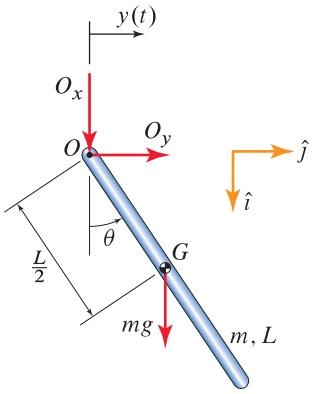


(a) Derive the equation of motion of the bar for small angles θ.

(b) Determine the amplitude of steady-state vibration of the bar.

Solution

Referring to the FBD on the right, to obtain the equation of motion of the bar, we can sum moments about point O using the equation



$$\vec{M}_O = I_O \vec{\alpha}_{\text{bar}} + \vec{r}_{G/O} \times m\vec{a}_O.$$

Doing so, we obtain

$$\sum M_O: \quad -mg\frac{L}{2}\sin\theta\,\hat{k} = I_O\ddot{\theta}\,\hat{k} + \frac{L}{2}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \times m\ddot{y}_O\,\hat{\jmath},$$

where $\ddot{y}_O\,\hat{\jmath}$ is the acceleration of point O. Carrying out the cross product, noting that $I_O = \frac{1}{3}mL^2$, and that $\ddot{y}_O = -Y\omega_s^2 \sin\omega_s t$, the equation of motion becomes

$$\frac{1}{3}mL^2\ddot{\theta} + mg\frac{L}{2}\sin\theta - m\frac{L}{2}Y\omega_s^2 \sin\omega_s t\cos\theta = 0,$$

which, for small θ can be written as

$$\frac{1}{3}mL^2\ddot{\theta} + mg\frac{L}{2}\theta = m\frac{L}{2}Y\omega_s^2\sin\omega_s t \quad \Rightarrow \quad \boxed{\ddot{\theta} + \frac{3g}{2L}\theta = \frac{3Y}{2L}\omega_s^2\sin\omega_s t.}$$

To find the amplitude of steady-state vibration of the bar, we need to find a particular solution and then the amplitude of that solution will be what we seek. We assume a particular solution of the form

$$\theta_p = D\sin\omega_s t,$$

where D is a constant to be determined. Substituting this particular solution into the equation of motion, we obtain

$$-D\omega_s^2\sin\omega_s t + \frac{3g}{2L}D\sin\omega_s t = \frac{3Y}{2L}\omega_s^2\sin\omega_s t \quad \Rightarrow \quad D\left(\omega_n^2 - \omega_s^2\right) = \frac{3Y}{2L}\omega_s^2$$

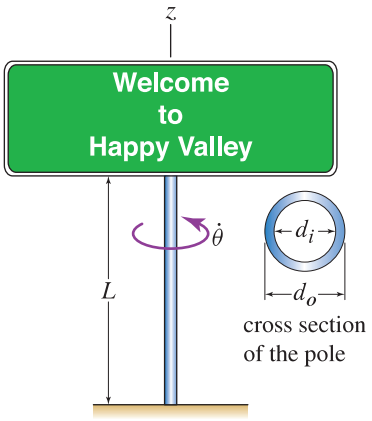
$$\Rightarrow \quad D = \frac{3Y/(2L)(\omega_s/\omega_n)^2}{1-(\omega_s/\omega_n)^2},$$

where we have used $\omega_n^2 = 3g/(2L)$ from the equation of motion. Therefore, the amplitude of steady-state vibration is

$$\boxed{\theta_{\text{amp}} = \frac{3Y/(2L)(\omega_s/\omega_n)^2}{1-(\omega_s/\omega_n)^2}.}$$

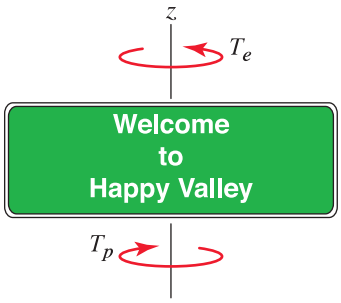
Problem 9.36

Consider a sign mounted on a circular hollow steel pole of length $L = 5\,\text{m}$, outer diameter $d_o = 5\,\text{cm}$, and inner diameter $d_i = 4\,\text{cm}$. Aerodynamic forces due to wind provide a harmonic torsional excitation with frequency $f_0 = 3\,\text{Hz}$ and amplitude $M_0 = 10\,\text{N}\cdot\text{m}$ about the z axis. The mass center of the sign lies on the central axis z of the pole. The mass moment of inertia of the sign is $I_z = 0.1\,\text{kg}\cdot\text{m}^2$. The torsional stiffness of the pole can be estimated as $k_t = \pi G_{\text{st}}(d_o^4 - d_i^4)/(32L)$, where G_{st} is the shear modulus of steel, which is 79 GPa. Neglecting the inertia of the pole, calculate the amplitude of vibration of the sign.



Solution

Referring to the FBD of the sign on the right, T_e is the exciting torque on the sign due to the wind and T_p is the restoring torque due to the steel pole of length L. Summing moments about the z axis, we obtain



$$\sum M_z: \quad T_e - T_p = I_z\ddot{\theta}, \tag{1}$$

where θ is the angle of rotation of the sign about the z axis from its static equilibrium position. The force law for the excitation torque and restoring torque are given by, respectively,

$$T_e = M_0 \sin(2\pi f_0 t) \quad \text{and} \quad T_p = k_t\theta = \left[\frac{\pi G_{\text{st}}(d_o^4 - d_i^4)}{32L}\right]\theta,$$

where k_t is the torsional stiffness of the pole and we have substituted in the given relationship for k_t. Substituting the force laws in to Eq. (1), we obtain

$$M_0 \sin(2\pi f_0 t) - \left[\frac{\pi G_{\text{st}}(d_o^4 - d_i^4)}{32L}\right]\theta = I_z\ddot{\theta} \quad \Rightarrow \quad I_z\ddot{\theta} + \left[\frac{\pi G_{\text{st}}(d_o^4 - d_i^4)}{32L}\right]\theta = M_0 \sin(2\pi f_0 t),$$

so we see that using $d_o = 0.05\,\text{m}$, $d_i = 0.04\,\text{m}$, $L = 5\,\text{m}$, $G_{\text{st}} = 79{\times}10^9\,\text{Pa}$, and $I_z = 0.1\,\text{kg}\cdot\text{m}^2$, we obtain

$$\omega_n = \sqrt{\frac{\pi G_{\text{st}}(d_o^4 - d_i^4)}{32LI_z}} = 239.2\,\text{rad/s}, \qquad \omega_0 = 2\pi f_0 = 18.85\,\text{rad/s}$$

$$k_{\text{eff}} = \frac{\pi G_{\text{st}}(d_o^4 - d_i^4)}{32L} = 5724\,\text{N}\cdot\text{m}, \qquad (F_0)_{\text{eff}} = M_0 = 10\,\text{N}\cdot\text{m},$$

where ω_0 is the forcing frequency, k_{eff} is the effective spring constant, and $(F_0)_{\text{eff}}$ is the effective forcing amplitude. Applying Eq. (9.43) from the text, we find the following amplitude

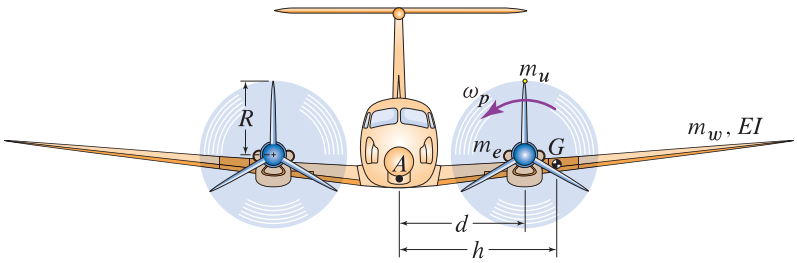
$$\theta_{\text{amp}} = \frac{(F_0)_{\text{eff}}/k_{\text{eff}}}{1 - (\omega_0/\omega_n)^2} \quad \Rightarrow \quad \boxed{\theta_{\text{amp}} = 0.00176\,\text{rad/s}.}$$

Problems 9.37 and 9.38

One of the propellers on the Beech King Air 200 is unbalanced such that the eccentric mass m_u is a distance R from the spin axis of the propeller. The propellers spin at a constant rate ω_r, and the mass of each engine is m_e (this includes the mass of the propeller). Assume that the wing is a uniform beam that is cantilevered at A, has mass m_w and bending stiffness EI, and whose mass center is at G. For each problem, evaluate your answers for $m_u = 3\,\text{oz}$, $m_e = 450\,\text{lb}$, $R = 5.1\,\text{ft}$, $\omega_r = 2000\,\text{rpm}$, $EI = 1.13\times10^{11}\,\text{lb·in.}^2$, $d = 8.7\,\text{ft}$, and $h = 10.9\,\text{ft}$.

Problem 9.37 Neglect the mass of the wing and model the wing as done in Example 9.2. Determine the resonance frequency of the system, and find the MF for the given parameters.

Problem 9.38 Let the mass of the wing be $m_w = 350\,\text{lb}$, and model the wing as done in Example 9.2. Determine the resonance frequency of the system and find the MF for the given parameters.



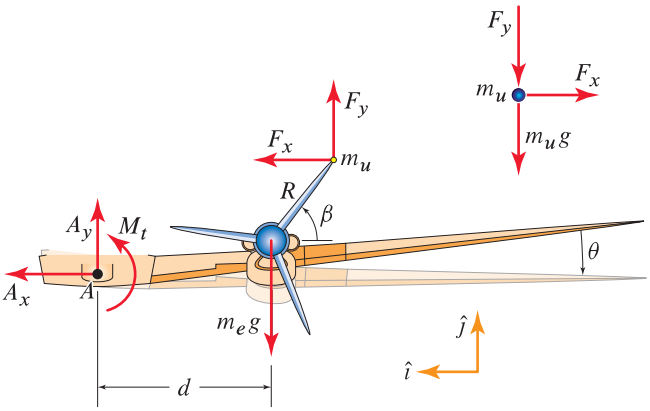
Solution to 9.37

The figure at the right shows an FBD of the wing and engine with the unbalanced mass removed as well as an FBD of just the unbalanced mass m_u. Summing forces in the x and y directions on the unbalanced mass m_u gives

$$\sum F_x: \qquad -F_x = m_u a_{ux}, \qquad (1)$$

$$\sum F_y: \quad -F_y - m_u g = m_u a_{uy}, \qquad (2)$$

where a_{ux} and a_{uy} are the x and y components, respectively, of the acceleration of the unbalanced mass. Next, summing moments about point A on the wing, we obtain



$$\sum M_A: \quad m_e g d - F_x R \sin\beta - F_y(d + R\cos\beta) - M_t = I_A \alpha_{\text{wing}}. \qquad (3)$$

where M_t is the lumped torsional stiffness of the wing at point A, $I_A = m_e d^2$, and $\alpha_{\text{wing}} = \ddot{\theta}$. Note that we have treated the engine as a point mass. The force law for the torsional stiffness is given by

$$M_t = k_t(\theta + \theta_{\text{st}}) = \frac{3EI}{d}(\theta + \theta_{\text{st}}),$$

where k_t is the lumped torsional spring constant, θ_{st} is the rotation of the wing in its static equilibrium position, and k_t was determined using the result from Example 9.2. The last thing we need to do is determine the acceleration of the unbalanced mass. Noting that the propeller is rotating with a constant angular velocity

and neglecting the *rotation* of the wing when computing the angular velocity and acceleration of the propeller, we find the acceleration of the unbalanced mass to be

$$\vec{a}_u = \vec{a}_e + \vec{\alpha}_{\text{prop}} \times \vec{r}_{u/e} - \omega_{\text{prop}}^2 \vec{r}_{u/e} = -d\ddot{\theta}\,\hat{\jmath} - \omega_p^2(-R\cos\beta\,\hat{\imath} + R\sin\beta\,\hat{\jmath}),$$

so that

$$a_{ux} = R\omega_p^2\cos\beta \quad \text{and} \quad a_{uy} = -d\ddot{\theta} - R\omega_p^2\sin\beta.$$

Substituting the kinematics equations into Eqs. (1) and (2) gives

$$F_x = -m_u R\omega_p^2\cos\beta \quad \text{and} \quad F_y = m_u d\ddot{\theta} + m_u R\omega_p^2\sin\beta - m_u g.$$

Substituting F_x and F_y into Eq. (3), we obtain the following equation of motion

$$m_e g d + m_u R^2\omega_p^2\sin\beta\cos\beta - m_u(d + R\cos\beta)\big(d\ddot{\theta} + R\omega_p^2\sin\beta - g\big) - \frac{3EI}{d}(\theta + \theta_{\text{st}}) = m_e d^2\ddot{\theta}.$$

Canceling terms and rearranging, we obtain

$$(m_u + m_e)d^2\ddot{\theta} + m_u Rd\cos\beta\ddot{\theta} - (m_u + m_e)dg + m_u dR\omega_p^2\sin\beta - m_u gR\cos\beta + \frac{3EI}{d}(\theta + \theta_{\text{st}}) = 0.$$

Since $(3EI/d)\theta_{\text{st}} = m_e g d + m_u g(d + R\cos\beta)$, this becomes

$$\underbrace{\big[(m_u + m_e)d^2 + m_u Rd\cos\omega_p t\big]}_{\text{time-dependent inertia}}\ddot{\theta} + \frac{3EI}{d}\theta = -m_u dR\omega_p^2\sin\omega_p t,$$

where we have substituted $\omega_p t$ for β. Notice that the inertia term (i.e., the coefficient of $\ddot{\theta}$) is time-dependent since it contains $\cos\omega_p t$, which means that this equation of motion is not in our standard form, which required that the coefficients be constant. On the other hand, also notice that

$$\big|(m_u + m_e)d^2\big| = 1058\,\text{slug}\cdot\text{ft}^2 \quad \text{and} \quad |m_u Rd\cos\omega_p t| \le 0.2584\,\text{slug}\cdot\text{ft}^2,$$

and therefore, the inertia term is dominated by the constant part, which means that we can approximate the equation of motion as

$$(m_u + m_e)d^2\ddot{\theta} + \frac{3EI}{d}\theta = -m_u dR\omega_p^2\sin\omega_p t, \tag{4}$$

and so using

$$\begin{aligned}
EI &= 1.13\times10^{11}\,\text{lb}\cdot\text{in.}^2 = 7.847\times10^{8}\,\text{lb}\cdot\text{ft}^2,\\
m_u &= 3\,\text{oz}\left(\frac{1\,\text{lb}}{16\,\text{oz}}\right)\left(\frac{1}{32.2\,\text{ft/s}^2}\right) = 5.823\times10^{-3}\,\text{slug},\\
m_e &= 450\,\text{lb}\left(\frac{1}{32.2\,\text{ft/s}^2}\right) = 13.98\,\text{slug}, \quad \text{and}\\
d &= 8.7\,\text{ft},
\end{aligned}$$

we find that the resonance frequency of the wing is

$$\boxed{\omega_n = \sqrt{\frac{3EI}{(m_u + m_e)d^3}} = 506\,\text{rad/s}} \quad \text{or} \quad \boxed{f = \frac{\omega_n}{2\pi} = 80.5\,\text{Hz}.}$$

As for the MF, we need to find a particular solution θ_p to the equation of motion in Eq. (4), and then use its amplitude to find determine the MF. The most direct way to find a particular solution is to assume a solution of the form $\theta_p = D \sin \omega_p t$ and then substitute it into the equation of motion. Doing so, we obtain

$$-D(m_u + m_e)d^2\omega_p^2 \sin \omega_p t + \frac{3EI}{d} D \sin \omega_p t = -m_u dR\omega_p^2 \sin \omega_p t,$$

which, upon canceling $\sin \omega_p t$ and solving for D gives

$$\begin{aligned} D &= \frac{-m_u dR\omega_p^2}{\frac{3EI}{d} - (m_u + m_e)d^2\omega_p^2} = \frac{-m_u d^2 R\omega_p^2/(3EI)}{1 - (m_u + m_e)d^3\omega_p^2/(3EI)} = \frac{-m_u d^2 R\omega_p^2/(3EI)}{1 - (\omega_p/\omega_n)^2} \\ &= -\frac{m_u R}{(m_u + m_e)d} \frac{(\omega_p/\omega_n)^2}{1 - (\omega_p/\omega_n)^2}. \end{aligned}$$

Therefore,

$$|\theta_p| = \frac{m_u R}{(m_u + m_e)d} \frac{(\omega_p/\omega_n)^2}{1 - (\omega_p/\omega_n)^2}$$

$$\Rightarrow \quad \boxed{\text{MF} = \frac{|\theta_p|(m_u + m_e)d}{m_u R} = \frac{(\omega_p/\omega_n)^2}{1 - (\omega_p/\omega_n)^2} = 0.207,}$$

where we have used $\omega_n = 505.7\,\text{rad/s}$ and $\omega_p = 2000\,\text{rpm} = 209.4\,\text{rad/s}$.

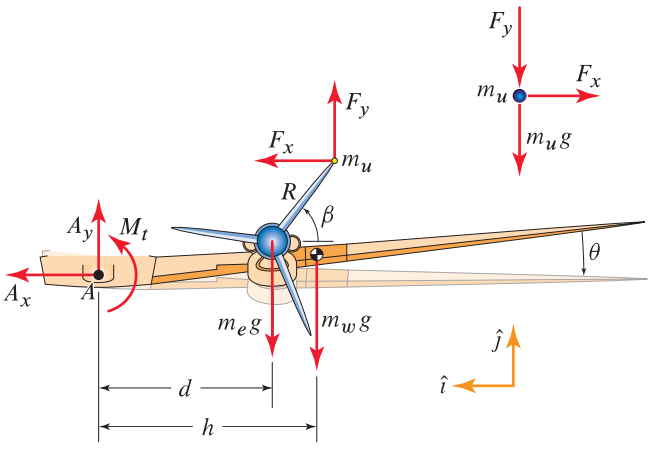
Solution to 9.38

The figure at the right shows an FBD of the wing and engine with the unbalanced mass removed as well as an FBD of just the unbalanced mass m_u. Summing forces in the x and y directions on the unbalanced mass m_u gives

$$\sum F_x: \qquad -F_x = m_u a_{ux}, \qquad (5)$$

$$\sum F_y: \quad -F_y - m_u g = m_u a_{uy}, \qquad (6)$$

where a_{ux} and a_{uy} are the x and y components, respectively, of the acceleration of the unbalanced mass. Next, summing moments about point A on the wing, we obtain



$$\sum M_A: \quad m_w g h + m_e g d - F_x R \sin\beta - F_y(d + R\cos\beta) - M_t = I_A \alpha_{\text{wing}}. \qquad (7)$$

where M_t is the lumped torsional stiffness of the wing at point A and $\alpha_{\text{wing}} = \ddot{\theta}$. Note that we have treated the engine as a point mass. The mass moment of inertia with respect to point A is

$$I_A = m_e d^2 + \tfrac{1}{12} m_w (2h)^2 + m_w h^2 = m_e d^2 + \tfrac{4}{3} m_w h^2,$$

where we have assumed that the wing is a uniform beam of length $2h$. The force law for the torsional stiffness is given by

$$M_t = k_t(\theta + \theta_{\text{st}}) = \frac{3EI}{d}(\theta + \theta_{\text{st}}),$$

where k_t is the lumped torsional spring constant, θ_{st} is the rotation of the wing in its static equilibrium position, and k_t was determined using the result from Example 9.2. The last thing we need to do is determine the acceleration of the unbalanced mass. Noting that the propeller is rotating with a constant angular velocity and neglecting the *rotation* of the wing when computing the angular velocity and acceleration of the propeller, we find the acceleration of the unbalanced mass to be

$$\vec{a}_u = \vec{a}_e + \vec{\alpha}_{\text{prop}} \times \vec{r}_{u/e} - \omega_{\text{prop}}^2 \vec{r}_{u/e} = -d\ddot{\theta}\,\hat{\jmath} - \omega_p^2(-R\cos\beta\,\hat{\imath} + R\sin\beta\,\hat{\jmath}),$$

so that

$$a_{ux} = R\omega_p^2 \cos\beta \quad \text{and} \quad a_{uy} = -d\ddot{\theta} - R\omega_p^2 \sin\beta.$$

Substituting the kinematics equations into Eqs. (5) and (6) gives

$$F_x = -m_u R\omega_p^2 \cos\beta \quad \text{and} \quad F_y = m_u d\ddot{\theta} + m_u R\omega_p^2 \sin\beta - m_u g.$$

Substituting F_x and F_y into Eq. (7), we obtain the following equation of motion

$$m_w g h + m_e g d + m_u R^2 \omega_p^2 \sin\beta\cos\beta - m_u(d + R\cos\beta)\big(d\ddot{\theta} + R\omega_p^2 \sin\beta - g\big) - \frac{3EI}{d}(\theta + \theta_{\text{st}}) = \big(m_e d^2 + \tfrac{4}{3} m_w h^2\big)\ddot{\theta}.$$

Canceling terms and rearranging, we obtain

$$\big[(m_u + m_e)d^2 + \tfrac{4}{3} m_w h^2 + m_u R d\cos\beta\big]\ddot{\theta} - (m_u + m_e)dg - m_w g h + m_u d R\omega_p^2 \sin\beta$$

$$-m_u gR\cos\beta + \frac{3EI}{d}(\theta + \theta_{\text{st}}) = 0,$$

or, since $(3EI/d)\theta_{\text{st}} = m_w gh + m_e gd + m_u g(d + R\cos\beta)$, this becomes

$$\underbrace{\left[(m_u + m_e)d^2 + \tfrac{4}{3}m_w h^2 + m_u Rd\cos\omega_p t\right]}_{\text{time-dependent inertia}}\ddot{\theta} + \frac{3EI}{d}\theta = -m_u dR\omega_p^2 \sin\omega_p t,$$

where we have substituted $\omega_p t$ for β. Notice that the inertia term (i.e., the coefficient of $\ddot{\theta}$) is time-dependent since it contains $\cos\omega_p t$, which means that this equation of motion is not in our standard form, which required that the coefficients be constant. On the other hand, also notice that

$$\left|(m_u + m_e)d^2 + \tfrac{4}{3}m_w h^2\right| = 2780\,\text{slug}\cdot\text{ft}^2 \quad \text{and} \quad |m_u Rd\cos\omega_p t| \le 0.2584\,\text{slug}\cdot\text{ft}^2,$$

and therefore, the inertia term is dominated by the constant part, which means that we can approximate the equation of motion as

$$\left[(m_u + m_e)d^2 + \tfrac{4}{3}m_w h^2\right]\ddot{\theta} + \frac{3EI}{d}\theta = -m_u dR\omega_p^2 \sin\omega_p t. \tag{8}$$

Therefore, the resonance frequency of the wing is

$$\boxed{\omega_n = \sqrt{\frac{3EI}{(m_u + m_e)d^3 + \frac{4}{3}m_w h^2 d}} = 312\,\text{rad/s}} \quad \text{or} \quad \boxed{f = \frac{\omega_n}{2\pi} = 49.7\,\text{Hz},}$$

where we have used

$$\begin{aligned}
EI &= 1.13\times10^{11}\,\text{lb}\cdot\text{in.}^2 = 7.847\times10^{8}\,\text{lb}\cdot\text{ft}^2,\\
m_u &= 3\,\text{oz}\left(\frac{1\,\text{lb}}{16\,\text{oz}}\right)\left(\frac{1}{32.2\,\text{ft/s}^2}\right) = 5.823\times10^{-3}\,\text{slug},\\
m_e &= 450\,\text{lb}\left(\frac{1}{32.2\,\text{ft/s}^2}\right) = 13.98\,\text{slug},\\
m_w &= 350\,\text{lb}\left(\frac{1}{32.2\,\text{ft/s}^2}\right) = 10.87\,\text{slug}\\
d &= 8.7\,\text{ft}, \quad \text{and}\\
h &= 10.9\,\text{ft}.
\end{aligned}$$

As for the MF, we need to find a particular solution θ_p to the equation of motion in Eq. (8), and then use its amplitude to find determine the MF. The most direct way to find a particular solution is to assume a solution of the form $\theta_p = D\sin\omega_p t$ and then substitute it into the equation of motion. Doing so, we obtain

$$-D\left[(m_u + m_e)d^2 + \tfrac{4}{3}m_w h^2\right]\omega_p^2 \sin\omega_p t + \frac{3EI}{d}D\sin\omega_p t = -m_u dR\omega_p^2 \sin\omega_p t,$$

which, upon canceling $\sin\omega_p t$ and solving for D gives

$$\begin{aligned}
D &= \frac{-m_u dR\omega_p^2}{\frac{3EI}{d} - \left[(m_u + m_e)d^2 + \frac{4}{3}m_w h^2\right]\omega_p^2} = \frac{-m_u d^2 R\omega_p^2/(3EI)}{1 - \left[(m_u + m_e)d^3 + \frac{4}{3}m_w h^2 d\right]\omega_p^2/(3EI)}\\
&= \frac{-m_u d^2 R\omega_p^2/(3EI)}{1 - (\omega_p/\omega_n)^2} = -\frac{m_u R}{(m_u + m_e)d}\,\frac{(\omega_p/\omega_n)^2}{1 - (\omega_p/\omega_n)^2}.
\end{aligned}$$

Therefore,

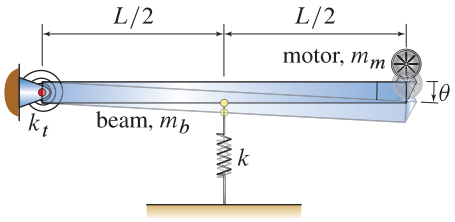
$$|\theta_p| = \frac{m_u R}{(m_u + m_e)d} \frac{(\omega_p/\omega_n)^2}{1-(\omega_p/\omega_n)^2}$$

$$\Rightarrow \quad \boxed{\text{MF} = \frac{|\theta_p|(m_u + m_e)d}{m_u R} = \frac{(\omega_p/\omega_n)^2}{1-(\omega_p/\omega_n)^2} = 0.820,}$$

where we have used $\omega_n = 312.0\,\text{rad/s}$ and $\omega_p = 2000\,\text{rpm} = 209.4\,\text{rad/s}$.

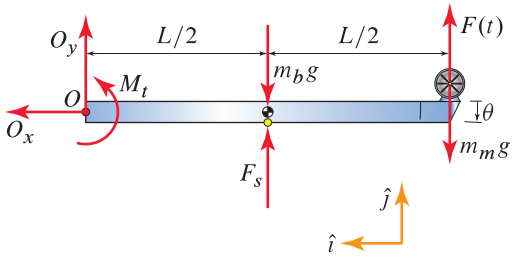
Problem 9.39

An unbalanced motor is mounted at the tip of a rigid beam of mass m_b and length L. The beam is restrained by a torsional spring of stiffness k_t and an additional support of stiffness k located at the half length of the beam. In the static equilibrium position, the beam is horizontal and the torsional spring does not exert any moment on the beam. The mass of the motor is m_m, and the unbalance results in a harmonic excitation $F(t) = F \sin \omega_0 t$ in the vertical direction. Derive the equation of motion for the system assuming that θ is small.



Solution

Referring to the FBD at the right, we can sum moments about point O to obtain the equation of motion of the system. Doing so, we obtain



$$\sum M_O: \quad -M_t + (m_b g - F_s)\tfrac{L}{2} + [m_m g - F(t)]L = I_O \alpha_{\text{beam}}. \qquad (1)$$

The force laws for the torsional spring and the spring of stiffness k are given by

$$M_t = k_t \theta \quad \text{and} \quad F_s = k\left(\tfrac{L}{2}\theta + \delta_{\text{st}}\right),$$

where θ is measured from the static equilibrium position of the beam and δ_{st} is the deflection of the midpoint of the beam when it is in static equilibrium. Substituting these force laws into Eq. (1), noting that $\alpha_{\text{beam}} = \ddot{\theta}$, that $F(t) = F \sin \omega_0 t$, and that $I_O = \frac{1}{3} m_b L^2 + m_m L^2$, the equation of motion becomes

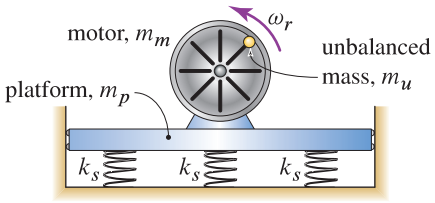
$$-k_t \theta + \left[m_b g - k\left(\tfrac{L}{2}\theta + \delta_{\text{st}}\right)\right]\tfrac{L}{2} + (m_m g - F \sin \omega_0 t)L = \left(\tfrac{1}{3} m L^2 + m_m L^2\right)\ddot{\theta}.$$

Now, if we realize that $m_b g L/2 + m_m g L = k \delta_{\text{st}} L/2$, we are left with the following equation of motion

$$\boxed{\left(\tfrac{1}{3} m_b + m_m\right) L^2 \ddot{\theta} + \left(k_t + \tfrac{1}{4} k L^2\right)\theta = FL \sin \omega_0 t.}$$

Problem 9.40

Revisit Example 9.4 and discuss whether it is possible to obtain the equation of motion of the system via the energy method.

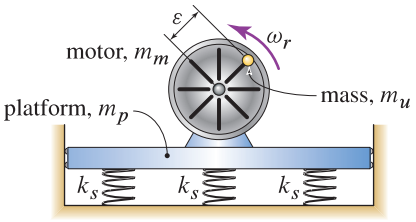


Solution

No, it is not possible since the system is not conservative. The system is not conservative since the amount of energy the motor adds or takes away from the system is not known and the efficiency of the motor is not known.

Problem 9.41

A fatigue-testing machine for electronic components consists of a platform with an unbalanced motor. Assume that the rotor in the motor spins at $\omega_0 = 3000$ rpm, the mass of the platform is $m_p = 20$ kg, the mass of the motor is $m_m = 15$ kg, the eccentric mass is $m_u = 0.5$ kg, and the equivalent stiffness of the platform suspension is $k = 5\times10^6$ N/m. For the testing machine, the distance ε between the spin axis of the rotor and the location at which m_u is placed can be varied to obtain the desired vibration level. Calculate the range of values of ε that would provide amplitudes of the particular solution ranging from 0.1 mm to 2 mm.



Solution

This system is identical to that found in Example 9.4. Therefore, using $k = 5\times10^6$ N/m, $m_m = 15$ kg, and $m_p = 20$ kg, we know that the natural frequency of the system is given by

$$\omega_n^2 = \frac{k}{m_m + m_p} = 1.429\times10^5 \text{ rad}^2/\text{s}^2,$$

The amplitude of the particular or steady-state solution is then given by

$$|y_{mp}| = \frac{m_u \varepsilon \omega_r^2 / k}{1 - (\omega_r/\omega_n)^2} = (0.03191\varepsilon) \text{ m},$$

in which we used $\omega_r = 3000$ rpm $= 314.2$ rad/s is the angular velocity of the rotor and we have substituted in $m_u = 0.5$ kg, $k = 5\times10^6$ N/m, and $\omega_n^2 = 1.429\times10^5$ rad^2/s^2. Therefore,

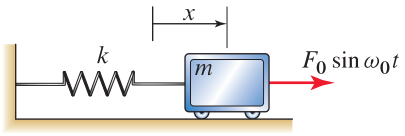
$$\text{for } |y_{mp}| = 0.0001 \text{ m} \quad \Rightarrow \quad \varepsilon = 0.00313 \text{ m} = 3.13 \text{ mm},$$
$$\text{for } |y_{mp}| = 0.002 \text{ m} \quad \Rightarrow \quad \varepsilon = 0.0627 \text{ m} = 62.7 \text{ mm},$$

and therefore the range of ε is

$$3.13 \text{ mm} < \varepsilon < 62.7 \text{ mm}.$$

Problem 9.42

At time $t = 0$, a forced harmonic oscillator occupies position $x(0) = 0.1\,\text{m}$ and has a velocity $\dot{x}(0) = 0$. The mass of the oscillator is $m = 10\,\text{kg}$, and the stiffness of the spring is $k = 1000\,\text{N/m}$. Calculate the motion of the system if the forcing function is $F(t) = F_0 \sin\omega_0 t$, with $F_0 = 10\,\text{N}$ and $\omega_0 = 200\,\text{rad/s}$.



Solution

As we have seen, the equation of motion for this system is given by

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega_0 t,$$

where ω_n is the natural frequency of the system, which is given by $\omega_n = \sqrt{k/m} = 10.00\,\text{rad/s}$. We found the response or motion of this system to be

$$x(t) = A\sin\omega_n t + B\cos\omega_n t + \frac{F_0/k}{1-(\omega_0/\omega_n)^2}\sin\omega_0 t, \tag{1}$$

where A and B are constants that depend on the given initial conditions. Evaluating the response at $t = 0$, we obtain

$$x(0) = B = 0.1 \quad \Rightarrow \quad B = 0.1\,\text{m},$$

and

$$\dot{x} = A\omega_n\cos\omega_n t - B\omega_n\sin\omega_n t + \frac{F_0/k}{1-(\omega_0/\omega_n)^2}\omega_0\cos\omega_0 t$$

$$\Rightarrow \quad \dot{x}(0) = A\omega_n + \frac{F_0/k}{1-(\omega_0/\omega_n)^2}\omega_0 \quad \Rightarrow \quad A = -\frac{F_0/k}{1-(\omega_0/\omega_n)^2}\frac{\omega_0}{\omega_n} = 0.0005013\,\text{m},$$

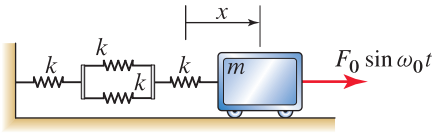
where we have used $F_0 = 10\,\text{N}$, $k = 1000\,\text{N/m}$, $\omega_0 = 200\,\text{rad/s}$, and $\omega_n = 10.00\,\text{rad/s}$. Substituting A and B into Eq. (1), we find that

$$\boxed{x(t) = \left(0.000501\sin 10t + 0.100\cos 10t - 2.51\times10^{-5}\sin 200t\right)\text{m},}$$

where we have used $F_0 = 10\,\text{N}$, $k = 1000\,\text{N/m}$, $\omega_0 = 200\,\text{rad/s}$, and $\omega_n = 10.00\,\text{rad/s}$ to obtain the coefficient of the $\sin 200t$ term.

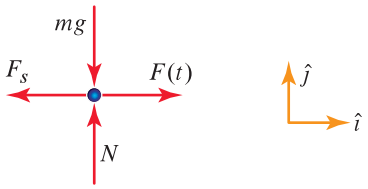
Problem 9.43

The forced harmonic oscillator shown has a mass $m = 10\,\text{kg}$. In addition the harmonic excitation is such that $F_0 = 150\,\text{N}$ and $\omega_0 = 200\,\text{rad/s}$. If all sources of friction can be neglected, determine the spring constant k such that the magnification factor MF = 5.



Solution

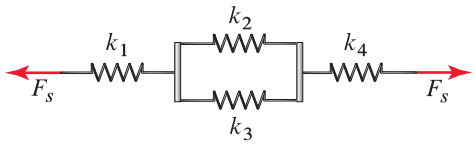
Treating m as a point mass, we obtain the FBD shown on the right. Summing forces in the x direction gives



$$\sum F_x: \quad F(t) - F_s = ma_x = m\ddot{x}, \tag{1}$$

where F_s is the force on m due to the springs and x is measured from the location of m when the springs are undeformed. The force law for the springs is

$$F_s = k_{\text{eq}} x,$$



where k_{eq} is the single spring constant that is equivalent to the four springs shown. To find k_{eq} we refer to the bottom figure shown above, where we have assigned to each spring a different constant to illustrate how to find the equivalent spring constant. We note that the total deformation of the springs is given by

$$\delta_{\text{tot}} = \delta_1 + \delta_{23} + \delta_4,$$

where δ_1 is the deformation of spring k_1, δ_{23} is the deformation of springs k_2 and k_3, and δ_4 is the deformation of spring k_4. Since the force in each of the three segments is equal to F_s, the displacements can be written in terms of F_s as

$$\frac{F_s}{k_{\text{eq}}} = \frac{F_s}{k_1} + \frac{F_s}{k_2 + k_3} + \frac{F_s}{k_4} \quad \Rightarrow \quad \frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2 + k_3} + \frac{1}{k_4}, \tag{2}$$

where the expression for δ_{23} was found using

$$F_s = F_2 + F_3 = k_2\delta_{23} + k_3\delta_{23} \quad \Rightarrow \quad \delta_{23} = \frac{F_s}{k_2 + k_3},$$

and where F_2 is the force in k_2 and F_3 is the force in k_3. Solving Eq. (2) for k_{eq}, we obtain

$$k_{\text{eq}} = \frac{k_1 k_4 (k_2 + k_3)}{k_1k_2 + k_1k_3 + k_1k_4 + k_2k_4 + k_3k_4} = \tfrac{2}{5}k,$$

where the last result was obtained by letting $k_1 = k_2 = k_3 = k_4 = k$. Substituting F_s and $F(t) = F_0 \sin \omega_0 t$ into Eq. (1), we obtain the equation of motion as

$$m\ddot{x} + \tfrac{2}{5}kx = F_0 \sin \omega_0 t.$$

The MF for this system was found to be (see Eq. (9.44))

$$\text{MF} = \frac{1}{1 - (\omega_0/\omega_n)^2},$$

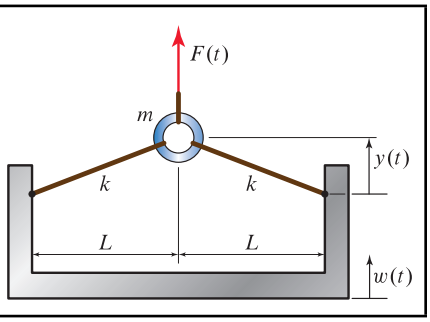
where $\omega_n^2 = \frac{2k}{5m}$ from the equation of motion. Substituting ω_n^2 into the equation above for the MF and solving for k gives

$$k = \frac{5m\omega_0^2\,\text{MF}}{2(\text{MF} - 1)} \quad \Rightarrow \quad \boxed{k = 1.25\times10^6\,\text{N/m},}$$

where we have used MF = 5, $m = 10\,\text{kg}$, and $\omega_0 = 200\,\text{rad/s}$ to obtain the numerical result.

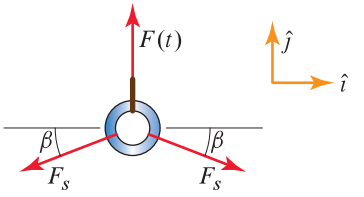
Problem 9.44

A ring of mass m is attached by two linear elastic cords with elastic constant k and unstretched length $L_0 < L$ to a support, as shown. Assuming that the pretension in the cords is large, so that the cords' deflection due to the ring's weight can be neglected, find the linearized equation of motion for the case where $F(t) = F_0 \sin \omega_0 t$ and $w(t) = 0$ (i.e., the support is stationary). In addition, find the response of the system for $y(0) = 0$ and $\dot{y} = 0$.



Solution

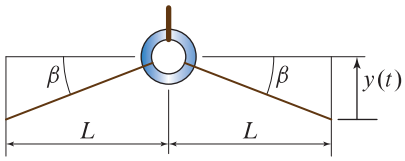
Referring to the FBD on the right, we can sum forces in the y direction to obtain



$$\sum F_y : \quad F(t) - 2F_s \sin\beta = ma_y = m\ddot{y}, \tag{1}$$

where F_s is the force in each of the elastic cords. The force law for each cord is given by

$$F_s = k\left(\sqrt{y^2 + L^2} - L_0\right).$$



Kinematically, we can write β in terms of y if we note that

$$\sin\beta = \frac{y}{\sqrt{y^2 + L^2}}.$$

Substituting the kinematics and the force law into the balance law in Eq. (1), we obtain

$$m\ddot{y} + 2k\left(\sqrt{y^2 + L^2} - L_0\right)\frac{y}{\sqrt{y^2 + L^2}} = F_0 \sin\omega_0 t \quad \Rightarrow \quad m\ddot{y} + 2k\left(1 - \frac{L_0}{\sqrt{y^2 + L^2}}\right)y = F_0 \sin\omega_0 t.$$

Looking at the second term in parentheses, we note that using a Taylor series expansion about $y = 0$ (this is sometimes called a binomial expansion) we can write it as

$$\frac{L_0}{\sqrt{y^2 + L^2}} = \frac{L_0}{L}\frac{1}{\sqrt{1 + (y/L)^2}} = \frac{L_0}{L}\left[1 - \frac{1}{2}\left(\frac{y}{L}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} + 1\right)}{2!}\left(\frac{y}{L}\right)^4 - \ldots\right] \approx \frac{L_0}{L},$$

where we obtained the last approximation by noting that $y \ll L$ and so $(y/L)^2$ and higher powers can be ignored. Therefore the linearized equation of motion becomes

$$\boxed{m\ddot{y} + 2k\left(1 - \frac{L_0}{L}\right)y = F_0 \sin\omega_0 t.}$$

We have already seen that the response is given by Eq. (9.42), which is

$$y = A \sin\omega_n t + B \cos\omega_n t + \frac{F_0/k_{\text{eq}}}{1 - (\omega_0/\omega_n)^2} \sin\omega_0 t, \tag{2}$$

where A and B are constants determined by enforcing the initial conditions, k_{eq} is the equivalent spring constant, and ω_n is the natural frequency, which are given by

$$k_{\text{eq}} = 2k\left(1 - \frac{L_0}{L}\right) \quad \text{and} \quad \omega_n = \sqrt{\frac{2k}{m}\left(1 - \frac{L_0}{L}\right)}. \tag{3}$$

Evaluating Eq. (2) at $y(0) = 0$, we obtain

$$y(0) = 0 = B \quad \Rightarrow \quad B = 0.$$

Evaluating Eq. (2) at $\dot{y}(0) = 0$, we find

$$\dot{y}(0) = A\omega_n + \frac{F_0/k_{\text{eq}}}{1 - (\omega_0/\omega_n)^2}\omega_0 = 0 \quad \Rightarrow \quad A = -\frac{F_0/k_{\text{eq}}}{1 - (\omega_0/\omega_n)^2}\frac{\omega_0}{\omega_n},$$

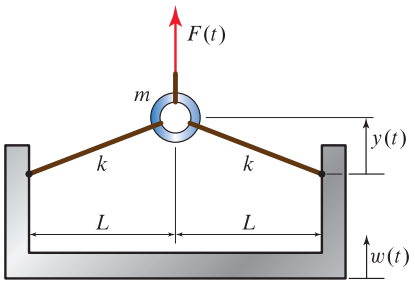
so that the response of the system for the given initial conditions is

$$y(t) = -\frac{F_0/k_{\text{eq}}}{1 - (\omega_0/\omega_n)^2}\left[\frac{\omega_0}{\omega_n}\sin\omega_n t + \sin\omega_0 t\right],$$

where $k_{\text{eq}} = 2k\left(1 - \frac{L_0}{L}\right)$ and $\omega_n = \sqrt{\frac{2k}{m}\left(1 - \frac{L_0}{L}\right)}$.

Problem 9.45

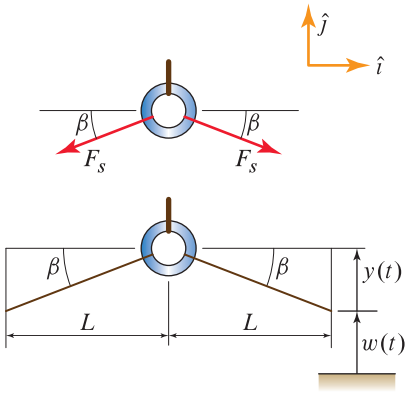
A ring of mass m is attached by two linear elastic cords with elastic constant k and unstretched length $L_0 < L$ to a support, as shown. Assuming that the pretension in the cords is large, so that the cords' deflection due to the ring's weight can be neglected, find the linearized equation of motion for the case where $F(t) = 0$ and $w(t) = w_0 \sin \omega t$. In addition, find the response of the system for $y(0) = 0$ and $\dot{y}(0) = 0$.



Solution

Referring to the FBD at the right, we can sum forces in the y direction to obtain the following balance law

$$\sum F_y : \quad -2F_s \sin\beta = ma_y, \tag{1}$$



where F_s is the force in each of the elastic cords, y is measured from the point of attachment of the elastic cords to the support, and a_y is the *inertial* acceleration of the ring in the y direction. The force law for each cord is given by

$$F_s = k\left(\sqrt{y^2 + L^2} - L_0\right).$$

Kinematically, we can write β in terms of y if we note that

$$\sin\beta = \frac{y}{\sqrt{y^2 + L^2}}.$$

In addition, the acceleration of the ring can be found using

$$a_y = \frac{d^2}{dt^2}[w(t) + y(t)] = -\omega^2 w_0 \sin\omega t + \ddot{y},$$

where we note that $\ddot{y}$ is *not* the inertial acceleration of the ring. Substituting the kinematics and the force law into the balance law in Eq. (1), we obtain

$$m\ddot{y} + 2k\left(\sqrt{y^2 + L^2} - L_0\right)\frac{y}{\sqrt{y^2 + L^2}} = m\omega^2 w_0 \sin\omega t$$

$$\Rightarrow \quad m\ddot{y} + 2k\left(1 - \frac{L_0}{\sqrt{y^2 + L^2}}\right)y = m\omega^2 w_0 \sin\omega t.$$

Looking at the second term in parentheses, we note that using a Taylor series expansion about $y = 0$ (this is sometimes called a binomial expansion) we can write it as

$$\frac{L_0}{\sqrt{y^2 + L^2}} = \frac{L_0}{L}\frac{1}{\sqrt{1 + (y/L)^2}} = \frac{L_0}{L}\left[1 - \frac{1}{2}\left(\frac{y}{L}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} + 1\right)}{2!}\left(\frac{y}{L}\right)^4 - \ldots\right] \approx \frac{L_0}{L},$$

where we obtained the last approximation by noting that $y \ll L$ and so $(y/L)^2$ and higher powers can be ignored. Therefore the linearized equation of motion becomes

$$\boxed{m\ddot{y} + 2k\left(1 - \frac{L_0}{L}\right)y = m\omega^2 w_0 \sin\omega t.}$$

We have already seen that the general solution to this equation is given by Eq. (9.42), which is

$$y = A\sin\omega_n t + B\cos\omega_n t + \frac{F_0/k_{\text{eq}}}{1-(\omega/\omega_n)^2}\sin\omega t, \tag{2}$$

where A and B are constants determined by enforcing the initial conditions, k_{eq} is the equivalent spring constant, ω_n is the natural frequency, and F_0 is the amplitude of the forcing, which are given by

$$k_{\text{eq}} = 2k\left(1 - \frac{L_0}{L}\right), \quad \omega_n = \sqrt{\frac{2k}{m}\left(1 - \frac{L_0}{L}\right)}, \quad \text{and} \quad F_0 = m\omega^2 w_0. \tag{3}$$

Evaluating Eq. (2) at $y(0) = 0$, we obtain

$$y(0) = 0 = B \quad \Rightarrow \quad B = 0.$$

Evaluating Eq. (2) at $\dot{y}(0) = 0$, we find

$$\dot{y}(0) = A\omega_n + \frac{F_0/k_{\text{eq}}}{1-(\omega/\omega_n)^2}\omega = 0 \quad \Rightarrow \quad A = -\frac{F_0/k_{\text{eq}}}{1-(\omega/\omega_n)^2}\frac{\omega}{\omega_n},$$

so that the response of the system, which is given by $w(t) + y(t)$, for the given initial conditions is

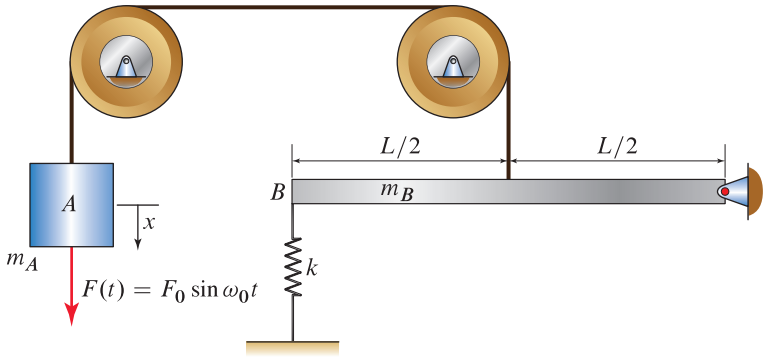
$$w(t) + y(t) = w_0\sin\omega t - \frac{F_0/k_{\text{eq}}}{1-(\omega/\omega_n)^2}\left[\frac{\omega}{\omega_n}\sin\omega_n t + \sin\omega t\right],$$

or

$$\boxed{\begin{aligned} w(t) + y(t) &= -\frac{F_0/k_{\text{eq}}}{1-(\omega/\omega_n)^2}\frac{\omega}{\omega_n}\sin\omega_n t + \left[w_0 - \frac{F_0/k_{\text{eq}}}{1-(\omega/\omega_n)^2}\right]\sin\omega t, \\ \text{where} \quad k_{\text{eq}} &= 2k\left(1 - \frac{L_0}{L}\right), \quad \omega_n = \sqrt{\frac{2k}{m}\left(1 - \frac{L_0}{L}\right)}, \quad \text{and} \quad F_0 = m\omega^2 w_0. \end{aligned}}$$

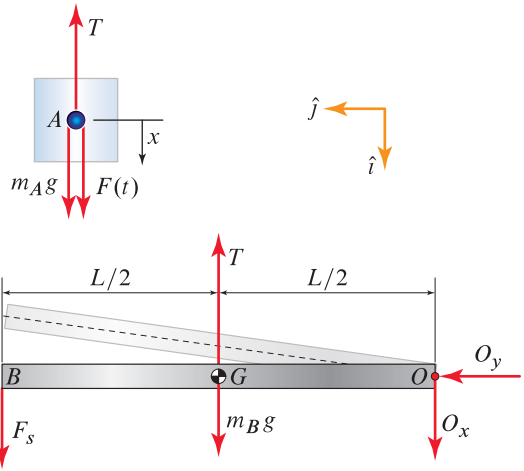
Problem 9.46

Modeling the beam as a rigid uniform thin bar, ignoring the inertia of the pulleys, assuming that the system is in static equilibrium when the bar is horizontal, and assuming that the cord is inextensible and does not go slack, determine the linearized equation of motion of the system in terms of x, which is the position of A. Finally, determine the amplitude of the steady-state vibration of block A.



Solution

An FBD of the block A, an FBD of the bar B, and the positive coordinate directions are shown at the right. Summing forces in the x direction on the block, we obtain the following balance law



$$\sum F_y: \quad m_A g + F(t) - T = m_A a_{Ax}$$
$$= m_A \ddot{x}. \qquad (1)$$

where x measures the vertical position of the block A. Summing moments about point O for bar B gives

$$\sum M_O: \quad T\tfrac{L}{2} - F_s L - m_B g \tfrac{L}{2} = I_O \alpha_B, \quad (2)$$

where T is the cord tension, α_B is the angular acceleration of the bar, F_s is the force in the spring, and $I_O = \frac{1}{3}mL^2$ is the mass moment of inertia of the bar B about point O. The force laws are

$$F(t) = F_0 \sin \omega_0 t \quad \text{and} \quad F_s k\left[\delta_B - (\delta_B)_{\text{st}}\right],$$

where $(\delta_B)_{\text{st}}$ is the deflection of the spring when the system is in static equilibrium. For the kinematics relations, we need to relate the angular acceleration of the bar (α_B) to the acceleration of the mass A ($\ddot{x}$) and we need to relate the deflection of point B (δ_B) to the position of mass A (x). Assuming small displacements and noting that the vertical displacement of A is equal and opposite to that of G, we can write

$$\delta_B = L\theta_B \quad \text{and} \quad x = \tfrac{L}{2}\theta_B \quad \Rightarrow \quad \delta_B = 2x \quad \text{and} \quad \ddot{\theta}_B = \alpha_B = \tfrac{2}{L}\ddot{x}.$$

Solving Eq. (1) for T and substituting the result into Eq. (2), we obtain

$$\tfrac{L}{2}\left[m_A g + F(t) - m_A \ddot{x}\right] - kL\left[\delta_B - (\delta_B)_{\text{st}}\right] - m_B g \tfrac{L}{2} = \tfrac{1}{3}mL^2\ddot{\theta}_B.$$

Now substituting in the force laws and kinematics relations, this becomes

$$\tfrac{L}{2}\left[m_A g + F_0 \sin \omega_0 t - m_A \ddot{x}\right] - kL\left[2x - (\delta_B)_{\text{st}}\right] - m_B g \tfrac{L}{2} = \tfrac{1}{3}mL^2\left(\tfrac{2}{L}\ddot{x}\right)$$

Canceling an L and noting that $\frac{1}{2}g(m_A - m_B) = k(\delta_B)_{\text{st}}$, we find the equation of motion to be

$$\boxed{\left(\tfrac{2}{3}m_B + \tfrac{1}{2}m_A\right)\ddot{x} + 2kx = \tfrac{1}{2}F_0 \sin \omega_0 t.}$$

In Eq. (9.43) we found that the amplitude of steady-state vibration is

$$x_{\text{amp}} = \frac{(F_0)_{\text{eq}}/k_{\text{eq}}}{1 - (\omega_0/\omega_n)^2},$$

where $(F_0)_{\text{eq}}$ is the amplitude of the forcing, k_{eq} is the equivalent spring constant, and ω_n is the natural frequency of vibration. Referring to the above equation of motion, these three quantities are given by

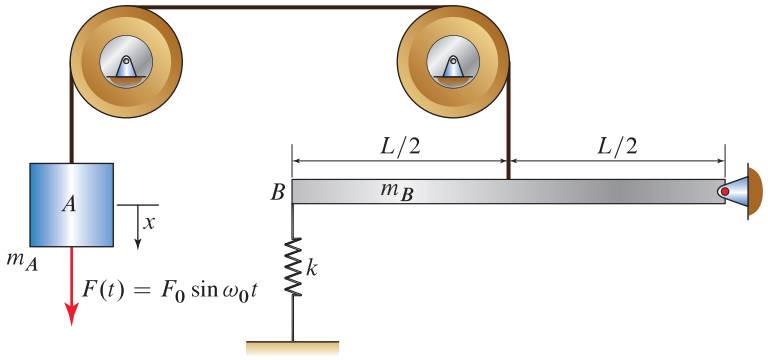
$$(F_0)_{\text{eq}} = \tfrac{1}{2}F_0, \quad k_{\text{eq}} = 2k, \quad \text{and} \quad \omega_n^2 = \frac{2k}{\frac{2}{3}m_B + \frac{1}{2}m_A}.$$

Substituting in these quantities, x_{amp} becomes

$$\boxed{x_{\text{amp}} = \frac{F_0}{4k - \left(\frac{4}{3}m_B + m_A\right)\omega_0^2}.}$$

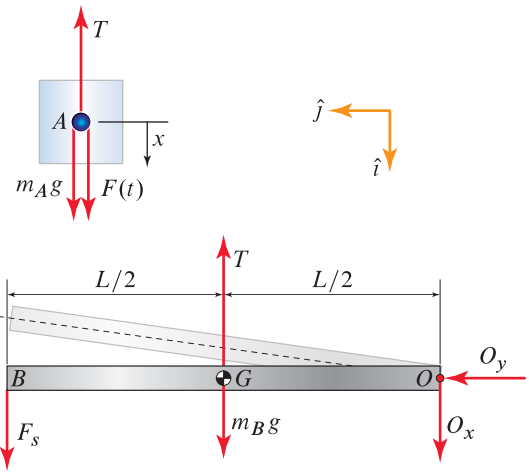
Problem 9.47

For the system in Prob. 9.46 determine the maximum forcing frequency ω_0 for steady state motion such that the cord does not go slack.



Solution

An FBD of the block A, an FBD of the bar B, and the positive coordinate directions are shown at the right. Summing forces in the x direction on the block, we obtain the following balance law



$$\sum F_y: \quad m_A g + F(t) - T = m_A a_{Ax} = m_A \ddot{x}. \qquad (1)$$

where x measures the vertical position of the block A and T is the tension in the cord. To determine the maximum ω_0 such that the cord does not go slack, we need to make sure that $T \geq 0$ in Eq. (1). Summing moments about point O for bar B gives

$$\sum M_O: \quad T\tfrac{L}{2} - F_s L - m_B g \tfrac{L}{2} = I_O \alpha_B, \qquad (2)$$

where α_B is the angular acceleration of the bar, F_s is the force in the spring, and $I_O = \frac{1}{3} m L^2$ is the mass moment of inertia of the bar B about point O. The force laws are

$$F(t) = F_0 \sin \omega_0 t \quad \text{and} \quad F_s k\left[\delta_B - (\delta_B)_{\text{st}}\right],$$

where $(\delta_B)_{\text{st}}$ is the deflection of the spring when the system is in static equilibrium. For the kinematics relations, we need to relate the angular acceleration of the bar (α_B) to the acceleration of the mass A ($\ddot{x}$) and we need to relate the deflection of point B (δ_B) to the position of mass A (x). Assuming small displacements and noting that the vertical displacement of A is equal and opposite to that of G, we can write

$$\delta_B = L\theta_B \quad \text{and} \quad x = \tfrac{L}{2}\theta_B \quad \Rightarrow \quad \delta_B = 2x \quad \text{and} \quad \ddot{\theta}_B = \alpha_B = \tfrac{2}{L}\ddot{x}.$$

Solving Eq. (1) for T and substituting the result into Eq. (2), we obtain

$$\tfrac{L}{2}\left[m_A g + F(t) - m_A \ddot{x}\right] - kL\left[\delta_B - (\delta_B)_{\text{st}}\right] - m_B g \tfrac{L}{2} = \tfrac{1}{3} m L^2 \ddot{\theta}_B.$$

Now substituting in the force laws and kinematics relations, this becomes

$$\tfrac{L}{2}\left[m_A g + F_0 \sin \omega_0 t - m_A \ddot{x}\right] - kL\left[2x - (\delta_B)_{\text{st}}\right] - m_B g \tfrac{L}{2} = \tfrac{1}{3} m L^2 \left(\tfrac{2}{L}\ddot{x}\right)$$

Canceling an L and noting that $\frac{1}{2}g(m_A - m_B) = k(\delta_B)_{\text{st}}$, we find the equation of motion to be

$$\left(\tfrac{2}{3}m_B + \tfrac{1}{2}m_A\right)\ddot{x} + 2kx = \tfrac{1}{2}F_0 \sin\omega_0 t.$$

In Eq. (9.43) we found that the amplitude of steady-state vibration is

$$x_{\text{amp}} = \frac{(F_0)_{\text{eq}}/k_{\text{eq}}}{1 - (\omega_0/\omega_n)^2},$$

where $(F_0)_{\text{eq}}$ is the amplitude of the forcing, k_{eq} is the equivalent spring constant, and ω_n is the natural frequency of vibration. Referring to the above equation of motion, these three quantities are given by

$$(F_0)_{\text{eq}} = \tfrac{1}{2}F_0, \quad k_{\text{eq}} = 2k, \quad \text{and} \quad \omega_n^2 = \frac{2k}{\frac{2}{3}m_B + \frac{1}{2}m_A}.$$

Substituting in these quantities, x_{amp} becomes

$$x_{\text{amp}} = \frac{F_0}{4k - \left(\frac{4}{3}m_B + m_A\right)\omega_0^2}.$$

and the steady-state vibration response is

$$x(t) = x_{\text{amp}} \sin\omega_0 t = \frac{F_0}{4k - \left(\frac{4}{3}m_B + m_A\right)\omega_0^2} \sin\omega_0 t.$$

Taking two time derivatives of x, substituting the result into Eq. (1), and then solving for T, we obtain

$$T = m_A g + F_0 \sin\omega_0 t - \frac{m_A F_0 \omega_0^2}{4k - \left(\frac{4}{3}m_B + m_A\right)\omega_0^2} \sin\omega_0 t.$$

Since we want $T \geq 0$, we can write

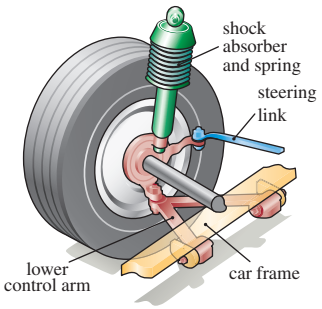
$$m_A g + \left[F_0 - \frac{m_A F_0 \omega_0^2}{4k - \left(\frac{4}{3}m_B + m_A\right)\omega_0^2}\right] \sin\omega_0 t \geq 0.$$

Since $\sin\omega_0 t$ oscillates between -1 and 1, for this expression to be true, we must satisfy

$$m_A g \geq F_0 - \frac{m_A F_0 \omega_0^2}{4k - \left(\frac{4}{3}m_B + m_A\right)\omega_0^2} \quad \text{or} \quad \boxed{\omega_0^2 \leq \frac{4k(F_0 - m_A g)}{\frac{4}{3}m_B(F_0 - m_A g) - m_A^2 g}.}$$

Problem 9.48

In the design of a MacPherson strut suspension, what would you choose for the damping ratio ζ? Explain your answer in terms of automotive ride and comfort.



Solution

You would want to choose a damping ratio as close to 1 as possible. The reason is that this provides the level of damping that returns the system to equilibrium in the minimum time and without any vibration.

Problem 9.49

For identical systems, one with damping and the other without, would you expect the period of damped vibration to be greater, less than, or equal to the period of undamped vibration? Explain your answer.

Solution

Equation (9.68) tells us that frequency of damped vibration is always less than the frequency of undamped vibration. Since the period of vibration is inversely proportional to the frequency, the period of vibration of a damped system should always be greater than the period of an identical undamped system.

Problem 9.50

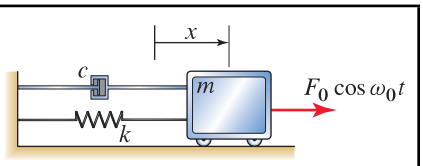
A vibration test is performed on a structure, in which both the magnification factor MF and the phase angle ϕ are recorded as a function of excitation frequency ω_0. After the test, it is discovered that, for some unfortunate reason, the recording of the magnification factor data is corrupted so that only the phase angle data is available for analysis. Is it possible to determine the resonant frequency from the available data? What can be inferred about the amount of damping in the system from the phase data?

Solution

Yes, the resonance frequency can be closely approximated from the phase angle data. As can be seen in Fig. 9.32, the phase angle ϕ is approximately equal to $\pi/2$ at resonance, so knowing the forcing frequency when $\phi = \pi/2$ will tell us the resonance frequency. As for the amount of damping, Fig. 9.32 also demonstrates that the steeper the slope at resonance (i.e., at $\phi = \pi/2$), the smaller the amount of damping. Therefore, we can infer something about the amount of damping by the slope of the phase angle plot at resonance.

Problem 9.51

Suppose that equation of motion of a damped forced harmonic oscillator is given by $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = (F_0/m)\cos\omega_0 t$, where x is measured from the equilibrium position of the system. Obtain the expression for the amplitude of the steady-state response of the oscillator, and compare it with the expression presented in Eq. (9.78) (which is for a system with equation of motion $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = (F_0/m)\sin\omega_0 t$).



Solution

To find the steady-state response for $\cos\omega_0 t$ forcing, we will follow the same procedure we used to find the steady-state response for $\sin\omega_0 t$ forcing.

We will assume a particular solution of either of the form

$$x_p = D\cos(\omega_0 t - \phi), \tag{1}$$

where D and ϕ are constants to be determined and D is assumed to be a positive quantity. Substituting this expression for x_p into the given equation of motion, $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = (F_0/m)\cos\omega_0 t$, we obtain

$$-D\omega_0^2\cos(\omega_0 t - \phi) - 2D\zeta\omega_n\omega_0\sin(\omega_0 t - \phi) + D\omega_n^2\cos(\omega_0 t - \phi) = \frac{F_0}{m}\cos\omega_0 t. \tag{2}$$

Using the trigonometric identities $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ and $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\cos\beta$ and then collecting terms, we obtain

$$D\left[\left(\omega_n^2 - \omega_0^2\right)\sin\phi - 2\zeta\omega_0\omega_n\cos\phi\right]\sin\omega_0 t + D\left[\left(\omega_n^2 - \omega_0^2\right)\cos\phi + 2\zeta\omega_0\omega_n\sin\phi\right]\cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t.$$

Since this equation must be true for all time, we can equate the coefficients of $\sin\omega_0 t$ and $\cos\omega_0 t$ to obtain two equations for the unknowns D and ϕ. Doing this for $\sin\omega_0 t$ allows us to solve for $\tan\phi$ as

$$\tan\phi = \frac{2\zeta\omega_0\omega_n}{\omega_n^2 - \omega_0^2} = \frac{2\zeta\omega_0/\omega_n}{1 - (\omega_0/\omega_n)^2}. \tag{3}$$

Equating the coefficients of $\cos\omega_0 t$, we obtain

$$D = \frac{F_0/m}{\left(\omega_n^2 - \omega_0^2\right)\cos\phi + 2\zeta\omega_0\omega_n\sin\phi} = \frac{F_0/(m\omega_n^2\cos\phi)}{1 - (\omega_0/\omega_n)^2 + 2\zeta(\omega_0/\omega_n)\tan\phi},$$

where we have divided the numerator and denominator by $\omega_n^2\cos\phi$. Now noting that $\omega_n^2 = k/m$, that we can write $1/\cos\phi = \sqrt{1 + \tan^2\phi}$, and that $\tan\phi$ is given by Eq. (3), D becomes

$$D = \frac{\frac{F_0}{k}\sqrt{1 + \left[\frac{2\zeta(\omega_0/\omega_n)}{1-(\omega_0/\omega_n)^2}\right]^2}}{1 - (\omega_0/\omega_n)^2 + 2\zeta(\omega_0/\omega_n)\left[\frac{2\zeta(\omega_0/\omega_n)}{1-(\omega_0/\omega_n)^2}\right]} = \frac{\frac{F_0}{k}\sqrt{\left[1 - (\omega_0/\omega_n)^2\right]^2 + \left[2\zeta(\omega_0/\omega_n)\right]^2}}{\left[1 - (\omega_0/\omega_n)^2\right]^2 + \left[2\zeta(\omega_0/\omega_n)\right]^2}$$

This is easily seen to simplify to

$$\boxed{D = \frac{F_0/k}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}}.}$$

Thus, we see that both the amplitude and the phase of the steady-state response are the same for $\cos\omega_0 t$ forcing as they are for $\sin\omega_0 t$ forcing.

Problem 9.52

Differentiate Eq. (9.79) with respect to ω_0/ω_n and set the result equal to zero to determine the frequency ω_0 at which peaks in the MF curve occur as a function of ζ and ω_n. Use this result to show that the peak always occurs at $\omega_0/\omega_n \leq 1$. Finally, determine the value of ζ for which the MF has no peak.

Solution

Equation (9.79) is given by

$$\text{MF} = \frac{1}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}},$$

where ω_0 is the forcing frequency, ω_n is the natural frequency, and ζ is the damping ratio. Letting $x = \omega_0/\omega_n$ and then differentiating this equation with respect to x, we obtain

$$\frac{d\,\text{MF}}{dx} = \frac{-\frac{1}{2}\left[\left(1-x^2\right)^2 + (2\zeta x)^2\right]^{-1/2}\left[2\left(1-x^2\right)(-2x) + 8\zeta^2 x\right]}{\left(1-x^2\right)^2 + (2\zeta x)^2} = 0$$

$$\Rightarrow \quad -\frac{8\zeta^2 x - 4x\left(1-x^2\right)}{2\left[\left(1-x^2\right)^2 + (2\zeta x)^2\right]^{3/2}} = 0 \quad \Rightarrow \quad 8\zeta^2 x - 4x\left(1-x^2\right) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \sqrt{1-2\zeta^2} \quad \Rightarrow \quad \frac{\omega_0}{\omega_n} = \sqrt{1-2\zeta^2}.$$

Therefore, the peak in the MF curve occurs at $\omega_0/\omega_n = \sqrt{1-2\zeta^2}$. Since $\zeta \geq 0$, we see that $1-2\zeta^2 \leq 1$ and so $\omega_0/\omega_n \leq 1$.

For the MF to have no peak, the quantity under the square root in the equation above must be negative. This occurs when

no peak in MF for $\zeta \geq \sqrt{1/2}$.

Problem 9.53

Calculate the response described by the equations listed below, in which x is measured in feet and time is measured in seconds.

(a) $5\ddot{x} + 10\dot{x} + 100x = 0$, with $x(0) = 0.1$ and $\dot{x}(0) = -0.1$

(b) $3\ddot{x} + 15\dot{x} + 12x = 0$, with $x(0) = 0$ and $\dot{x}(0) = 0.5$

(c) $\ddot{x} + 10\dot{x} + 25x = 0$, with $x(0) = 0.15$ and $\dot{x}(0) = 0$

(d) $25\ddot{x} + 200\dot{x} + 1500x = 0$, with $x(0) = 0.01$ and $\dot{x}(0) = 0$

Solution

In each case, before finding the response, we will first need to determine the character of the damping. That is, is the system overdamped, critically damped, or underdamped.

Part (a). For this system, $m = 5$, $c = 10$, and $k = 100$. Therefore the damping ratio is

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = 0.2236 < 1 \quad \Rightarrow \quad \text{underdamped.}$$

Therefore the response is given by

$$x(t) = De^{-\zeta\omega_n t}\sin(\omega_d t + \phi),$$

where D and ϕ are constants determined by the initial conditions and ω_n and ω_d are the natural frequency and damped natural frequency, respectively, which are given by

$$\omega_n = \sqrt{k/m} = 4.472\,\text{rad/s} \quad \text{and} \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 4.359\,\text{rad/s}.$$

Therefore, we can write the response as

$$x(t) = De^{-t}\sin(4.359t + \phi).$$

Applying $x(0) = 0.1$ gives

$$0.1 = D\sin\phi. \tag{1}$$

Since $\dot{x} = D[-e^{-t}\sin(4.359t + \phi) + 4.359e^{-t}\cos(4.359t + \phi)]$, we can apply $\dot{x}(0) = -0.1$ to obtain

$$-0.1 = -D\sin\phi + 4.359D\cos\phi. \tag{2}$$

Substituting Eq. (1) into Eq. (2), we find that

$$-0.1 = -0.1 + 4.359D\cos\phi \quad \Rightarrow \quad \cos\phi = 0 \quad \Rightarrow \quad \phi = \pi/2,$$

where we have used the fact that D cannot be zero. Since $\phi = \pi/2$, Eq. (1) gives $D = 0.1$ and so the response is

$$\boxed{x(t) = 0.1e^{-t}\sin(4.36t + \pi/2).}$$

Part (b). For this system, $m = 3$, $c = 15$, and $k = 12$. Therefore the damping ratio is

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = 1.25 > 1 \quad \Rightarrow \quad \text{overdamped.}$$

Therefore the response is given by

$$x(t) = e^{-\zeta\omega_n t}\left(Ae^{\sqrt{\zeta^2-1}\,\omega_n t} + Be^{-\sqrt{\zeta^2-1}\,\omega_n t}\right)$$

where A and B are constants determined by the initial conditions and where ω_n is the natural frequency, which is given by

$$\omega_n = \sqrt{k/m} = 2.000\,\text{rad/s}.$$

Therefore, we can write the response as

$$x(t) = e^{-2.5t}\left(Ae^{1.5t} + Be^{-1.5t}\right) = Ae^{-t} + Be^{-4t}.$$

Applying $x(0) = 0$ gives

$$A + B = 0 \quad \Rightarrow \quad A = -B. \tag{3}$$

Since $\dot{x} = -Ae^{-t} - 4Be^{-4t}$, we can apply $\dot{x}(0) = 0.5$ to obtain

$$-A - 4B = 0.5 \quad \Rightarrow \quad B - 4B = 0.5 \quad \Rightarrow \quad B = -0.1667 \quad \Rightarrow \quad A = 0.1667. \tag{4}$$

Therefore, the response is

$$\boxed{x(t) = 0.167e^{-t} - 0.167e^{-4t}.}$$

Part (c). For this system, $m = 1$, $c = 10$, and $k = 25$. Therefore the damping ratio is

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = 1 \quad \Rightarrow \quad \text{critically damped.}$$

Therefore the response is given by

$$x(t) = (A + Bt)e^{-\omega_n t} = (A + Bt)e^{-5t},$$

where A and B are constants determined from the initial conditions and $\omega_n = \sqrt{k/m} = 5\,\text{rad/s}$ is the natural frequency. Applying $x(0) = 0.15$, we obtain

$$A = 0.15.$$

Since $\dot{x} = Be^{-5t} - 5(A + Bt)e^{-5t}$, then applying $\dot{x}(0) = 0$ gives

$$B - 5A = 0 \quad \Rightarrow \quad B = 0.75,$$

which means that the response is

$$\boxed{x(t) = (0.15 + 0.75t)e^{-5t}.}$$

Part (d). For this system, $m = 25$, $c = 200$, and $k = 1500$. Therefore the damping ratio is

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = 0.5164 < 1 \quad \Rightarrow \quad \text{underdamped.}$$

Therefore the response is given by

$$x(t) = De^{-\zeta\omega_n t}\sin(\omega_d t + \phi),$$

where D and ϕ are constants determined by the initial conditions and ω_n and ω_d are the natural frequency and damped natural frequency, respectively, which are given by

$$\omega_n = \sqrt{k/m} = 7.746\,\text{rad/s} \quad \text{and} \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 6.633\,\text{rad/s}.$$

Therefore, we can write the response as

$$x(t) = De^{-4t}\sin(6.633t + \phi).$$

Applying $x(0) = 0.01$ gives

$$D\sin\phi = 0.01. \tag{5}$$

Since $\dot{x} = D[-4e^{-t}\sin(6.633t + \phi) + 6.633e^{-4t}\cos(6.633t + \phi)]$, we can apply $\dot{x}(0) = 0$ to obtain

$$-4D\sin\phi + 6.633D\cos\phi = 0. \tag{6}$$

Substituting Eq. (5) into Eq. (6), we find that

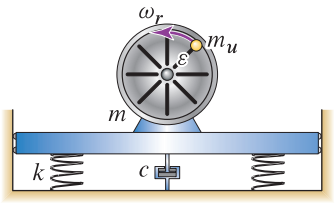
$$-0.04 + 6.633D\cos\phi = 0 \quad \Rightarrow \quad 6.633D\cos\phi = 0.04 \quad \Rightarrow \quad \frac{D\sin\phi}{6.633D\cos\phi} = \frac{0.01}{0.04}$$
$$\Rightarrow \quad \tan\phi = 1.658 \quad \Rightarrow \quad \phi = 58.91^\circ = 1.028\,\text{rad}.$$

Since $\phi = 58.91^\circ = 1.028$ rad, Eq. (5) gives $D = 0.01168$ and so the response is

$$\boxed{x(t) = 0.0117e^{-4t}\sin(6.63t + 1.03).}$$

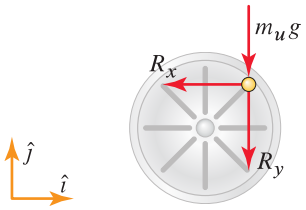
Problem 9.54

Derive the equation of motion given in Eq. (1) of Example 9.6 for the system in that example. The independent variable y is measured from the equilibrium position of the system, m is the mass of the motor and platform, c is the total damping coefficient of the dashpots, k is the total constant of the linear elastic springs, ω_r is the angular velocity of the unbalanced rotor, ε is the distance of the eccentric mass from the rotor axis, and m_u is the eccentric mass. Note that m *includes* the eccentric mass so that the nonrotating mass is equal to $m - m_u$.



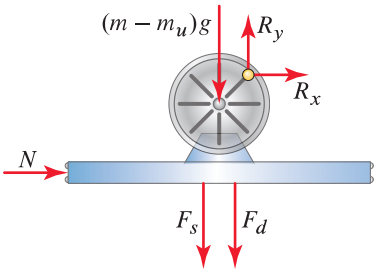
Solution

Referring to the figure at the right, we have drawn an FBD of the unbalanced mass m_u and an FBD of the system with the unbalanced mass removed. Summing forces in the y direction on the unbalanced mass (top figure) we obtain



$$\sum F_y: \quad -R_y - m_u g = m_u a_{uy} = m_u \ddot{y}_u, \tag{1}$$

where y_u is defined in the bottom figure on the right. Summing forces in the y direction on the motor and platform (middle figure), we obtain

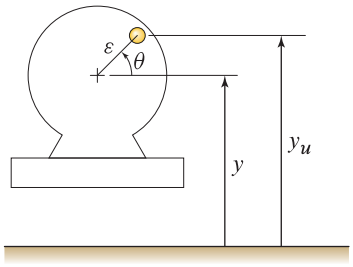


$$\sum F_y: \quad R_y - (m - m_u)g - F_s - F_d = (m - m_u)a_{my} = (m - m_u)\ddot{y}, \tag{2}$$

where F_s is the force in the spring, F_d is the force in the dashpot, and y is the vertical position of the motor and platform as shown on the right. The force laws for the spring and dashpot forces are given by

$$F_s = k(y - \delta_{\text{st}}) \quad \text{and} \quad F_d = c\dot{y},$$

where δ_{st} is the deflection of the spring when the system is in static equilibrium. Kinematically, we need to write $\ddot{y}_u$ in terms of y and/or its time derivatives. We can do this using the relation



$$y_u = y + \varepsilon \sin\theta \quad \Rightarrow \quad \ddot{y}_u = \ddot{y} - \varepsilon\dot{\theta}^2 \sin\theta = \ddot{y} - \varepsilon\omega_r^2 \sin\omega_r t,$$

where we have used the fact that $\dot{\theta} = \omega_r$, ω_r is constant, and $\theta = \omega_r t$. Eliminating R_y from Eqs. (1) and (2) and then wubstituting the kinematics equation and the force laws into the result, we obtain

$$-m_u g - m_u(\ddot{y} - \varepsilon\omega_r^2 \sin\omega_r t) - (m - m_u)g - k(y - \delta_{\text{st}}) - c\dot{y} = (m - m_u)\ddot{y}.$$

Since $mg = k\delta_{\text{st}}$, the equation of motion becomes

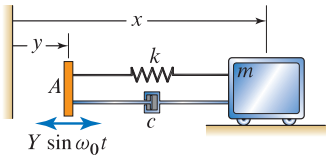
$$m\ddot{y} + c\dot{y} + ky = m_u\varepsilon\omega_r^2 \sin\omega_r t \quad \Rightarrow \quad \ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = \frac{m_u\varepsilon\omega_r^2}{m} \sin\omega_r t,$$

or, using $c/m = 2\zeta\omega_n$ and $k/m = \omega_n^2$, this becomes the equation of motion in Example 9.6, that is

$$\boxed{\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{m_u\varepsilon\omega_r^2}{m} \sin\omega_r t.}$$

Problem 9.55

The mass m is coupled to the support A, which is displacing harmonically according to $y = Y \sin \omega_0 t$, by the linear elastic spring of constant k and the dashpot with constant c.



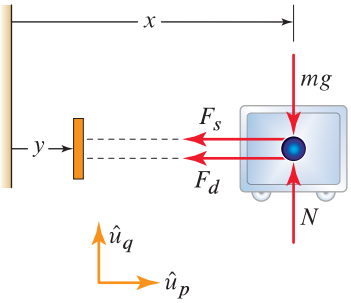
(a) Derive its equation of motion, using x as the independent variable, and explain in what way the resulting equation of motion is not in the form of Eq. (9.71).

(b) Next, let $z = x - y$ and substitute it into the equation of motion found in part (a). After doing so, show that you obtain an equation of motion in z that is of the same form as Eq. (9.71).

(c) Find the steady-state solution to the equation of motion found in part (b) and then using that, determine the steady-state solution for x.

Solution

Part (a). Referring to the FBD on the right, if we sum forces in the p direction, we obtain the following balance law



$$\sum F_p: \quad -F_s - F_d = m\ddot{x}.$$

The force laws for the spring and dashpot forces are, respectively,

$$F_s = k(x - y) \quad \text{and} \quad F_d = c(\dot{x} - \dot{y}).$$

Substituting the force laws into the balance law, we obtain the equation of motion as

$$-k(x - y) - c(\dot{x} - \dot{y}) = m\ddot{x} \quad \Rightarrow \quad m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

$$\Rightarrow \quad \boxed{m\ddot{x} + c\dot{x} + kx = c\omega_0 Y \cos \omega_0 t + kY \sin \omega_0 t,}$$

where we have used $y = Y \sin \omega_0 t$ and $\dot{y} = \omega_0 \cos \omega_0 t$. This equation is not in form of Eq. (9.71) since the forcing term on the right hand side of the form $R \cos \omega_0 t + S \sin \omega_0 t$ and not just $R \cos \omega_0 t$ *or* $S \sin \omega_0 t$. Therefore, we cannot solve this equation with what we have learned in this section.*

Part (b). Letting $z = x - y$ and substituting it into the first form of the equation of motion from Part (a), we obtain

$$-kz - c\dot{z} = m(\ddot{z} + \ddot{y}) \quad \Rightarrow \quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega_0^2 Y \sin \omega_0 t,$$

which means that the equation of motion in terms of z is

$$\boxed{m\ddot{z} + c\dot{z} + kz = m\omega_0^2 Y \sin \omega_0 t,}$$

which is of the same form as Eq. (9.71).

*Since the system is linear, it turns out that it is actually easy to solve as the superposition of a particular solution for $R \cos \omega_0 t$ and one for $S \sin \omega_0 t$.

Part (c). To determine the steady-state solution for z, that is to the equation of motion found in Part (b), we can either apply the results derived in Section 9.3, or we can assume a particular solution of the form

$$z_p = A\sin\omega_0 t + B\cos\omega_0 t \quad \text{or} \quad z_p = D\sin(\omega_0 t - \phi).$$

Choosing the former and substituting it into the equation of motion in z, we obtain

$$-m\omega_0^2(A\sin\omega_0 t + B\cos\omega_0 t) + c\omega_0(A\cos\omega_0 t - B\sin\omega_0 t) + k(A\sin\omega_0 t + B\cos\omega_0 t) = m\omega_0^2 Y\sin\omega_0 t,$$

Since this expression must be true for all time, we can equate the coefficients of $\sin\omega_0 t$ and $\cos\omega_0 t$ to obtain two equations whose solution gives A and B. These equations are:

$$\begin{aligned} -m\omega_0^2 A - c\omega_0 B + kA = m\omega_0^2 Y \quad &\Rightarrow \quad \left(k - m\omega_0^2\right)A - c\omega_0 B = m\omega_0^2 Y, \\ -m\omega_0^2 B + c\omega_0 A + kB = 0 \quad &\Rightarrow \quad c\omega_0 A + \left(k - m\omega_0^2\right)B = 0. \end{aligned}$$

Solving, we obtain the following for B

$$-\frac{\left(k - m\omega_0^2\right)^2}{c\omega_0}B - c\omega_0 B = m\omega_0^2 Y \quad \Rightarrow \quad B = -\frac{m\omega_0^2 Y}{\left(k - m\omega_0^2\right)^2/c\omega_0 + c\omega_0} = \frac{-mc\omega_0^3 Y}{\left(k - m\omega_0^2\right)^2 + c^2\omega_0^2},$$

and for A, we obtain

$$A = \frac{mc\omega_0^3 Y}{\left(k - m\omega_0^2\right)^2 + c^2\omega_0^2}\left(\frac{k - m\omega_0^2}{c\omega_0}\right) = \frac{m\omega_0^2 Y\left(k - m\omega_0^2\right)}{\left(k - m\omega_0^2\right)^2 + c^2\omega_0^2}.$$

Substituting A and B into the assumed form of the particular solution, we obtain the steady-state response for z as

$$z_p = \frac{m\omega_0^2 Y\left(k - m\omega_0^2\right)}{\left(k - m\omega_0^2\right)^2 + c^2\omega_0^2}\sin\omega_0 t - \frac{mc\omega_0^3 Y}{\left(k - m\omega_0^2\right)^2 + c^2\omega_0^2}\cos\omega_0 t$$

or

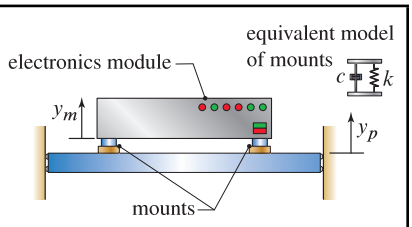
$$\boxed{z_p = \frac{m\omega_0^2 Y}{\left(k - m\omega_0^2\right)^2 + c^2\omega_0^2}\left[\left(k - m\omega_0^2\right)\sin\omega_0 t + c\omega_0\cos\omega_0 t\right].}$$

Therefore, since $x = z + y$, the steady-state solution for x must be

$$\boxed{x(t) = \frac{m\omega_0^2 Y}{\left(k - m\omega_0^2\right)^2 + c^2\omega_0^2}\left[\left(k - m\omega_0^2\right)\sin\omega_0 t + c\omega_0\cos\omega_0 t\right] + Y\sin\omega_0 t.}$$

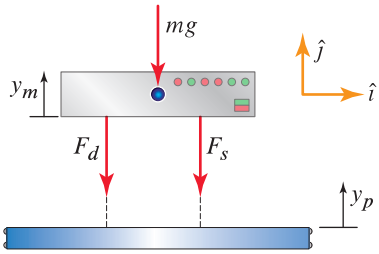
Problem 9.56

A module with sensitive electronics is mounted on a panel that vibrates due to excitation from a nearby diesel generator. To prevent fatigue failure, the module is placed on vibration-absorbing mounts. The displacement of the panel is measured to be $y_p(t) = y_0 \sin \omega_0 t$, where $y_0 = 0.001$ m, $\omega_0 = 300$ rad/s, and the time t is measured in seconds. Letting the mass of the electronic module be $m = 0.5$ kg, calculate the amplitude of the vibration of the module if the equivalent stiffness and damping coefficients for all the mounts combined are $k = 10{,}000$ N/m and $c = 40$ N·s/m, respectively.



Solution

The figure at the right shows an FBD of just the electronics module. Summing forces in the y direction, we find that



$$\sum F_y: \quad -mg - F_d - F_s = m\ddot{y}_m, \tag{1}$$

where F_s is the force in the equivalent spring and F_d is the force in the equivalent damper. The forces laws for the spring and damper forces are, respectively,

$$F_s = k(y_m - y_p - \delta_{\text{st}}) \quad \text{and} \quad F_d = c(\dot{y}_m - \dot{y}_p),$$

where δ_{st} is the deflection of the spring in the static equilibrium position of the system. Substituting the force laws into the equation of motion, we obtain

$$m\ddot{y}_m + c(\dot{y}_m - \dot{y}_p) + k(y_m - y_p - \delta_{\text{st}}) + mg = 0 \quad \Rightarrow \quad m\ddot{y}_m + c\dot{y}_m + ky_m = c\dot{y}_p + ky_p, \tag{2}$$

where we have used the fact that $mg = k\delta_{\text{st}}$. Substituting in $y_p = y_0 \sin \omega_0 t$ and $\dot{y}_p = \omega_0 y_0 \cos \omega_0 t$, we obtain the equation of motion as

$$m\ddot{y}_m + c\dot{y}_m + ky_m = c\omega_0 y_0 \cos \omega_0 t + ky_0 \sin \omega_0 t,$$

which is *not* in the form of Eq. (9.69). Letting $z = y_m - y_p$, we can write Eq. (2) as

$$m(\ddot{z} + \ddot{y}_p) + c(\dot{z} + \dot{y}_p) + k(z + y_p) = c\dot{y}_p + ky_p \quad \Rightarrow \quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y}_p = my_0\omega_0^2 \sin \omega_0 t,$$

which *is* in the form of Eq. (9.69). Assuming a particular solution of the form

$$z = D \sin(\omega_0 t - \phi),$$

and then substituting it into the equation of motion in z, we obtain

$$-mD\omega_0^2 \sin(\omega_0 t - \phi) + cD\omega_0 \cos(\omega_0 t - \phi) + kD \sin(\omega_0 t - \phi) = my_0\omega_0^2 \sin \omega_0 t,$$

Applying the trigonometric identities $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$ and $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \cos\beta$ and then collecting terms, we obtain

$$D(-m\omega_0^2 \cos\phi + c\omega_0 \sin\phi + k\cos\phi) \sin\omega_0 t + D(m\omega_0^2 \sin\phi + c\omega_0 \cos\phi - k\sin\phi) \cos\omega_0 t = my_0\omega_0^2 \sin\omega_0 t.$$

Since this equation must be true for all time, we can equate the coefficients of $\sin\omega_0 t$ and $\cos\omega_0 t$ to obtain two equations for the unknowns D and ϕ. Doing this for $\cos\omega_0 t$ allows us to solve for $\tan\phi$ as

$$\tan\phi = \frac{c\omega_0}{k - m\omega_0^2} \quad \Rightarrow \quad \phi = \tan^{-1}\left(\frac{c\omega_0}{k - m\omega_0^2}\right) = -18.92^\circ = -0.3303\,\text{rad}, \tag{3}$$

where we have used $c = 40\,\text{N}\cdot\text{s/m}$, $\omega_0 = 300\,\text{rad/s}$, $k = 10{,}000\,\text{N/m}$, and $m = 0.5\,\text{kg}$. Equating the coefficients of $\sin\omega_0 t$, we obtain

$$D = \frac{my_0\omega_0^2}{(k - m\omega_0^2)\cos\phi + c\omega_0\sin\phi} = \frac{my_0\omega_0^2/\cos\phi}{k - m\omega_0^2 + c\omega_0\tan\phi}.$$

Since $\tan\phi$ is given by Eq. (3), we can write $\cos\phi$ as

$$\cos\phi = \frac{1}{\sqrt{1 + \left(\frac{c\omega_0}{k - m\omega_0^2}\right)^2}},$$

and so the expression for D becomes

$$D = \frac{my_0\omega_0^2\sqrt{1 + \left(\frac{c\omega_0}{k - m\omega_0^2}\right)^2}}{k - m\omega_0^2 + c\omega_0\left(\frac{c\omega_0}{k - m\omega_0^2}\right)} = \frac{my_0\omega_0^2\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}}{(k - m\omega_0^2)^2 + c^2\omega_0^2}$$

$$= \frac{my_0\omega_0^2}{\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}} = 0.001216\,\text{m}, \tag{4}$$

where we have used $c = 40\,\text{N}\cdot\text{s/m}$, $\omega_0 = 300\,\text{rad/s}$, $k = 10{,}000\,\text{N/m}$, $y_0 = 0.001\,\text{m}$, and $m = 0.5\,\text{kg}$. Now that we have D and ϕ, a particular solution for z is

$$z = 0.001216\sin(300t + 0.3303),$$

and so the solution for y_m is

$$y_m = z + y_p = 0.001216\sin(300t + 0.3303) + 0.001\sin 300t.$$

To calculate the amplitude of this vibration, we will write y_m as

$$y_m = y_0\sin\omega_0 t + D\sin(\omega_0 t - \phi) = y_0\sin\omega_0 t + D\sin\omega_0 t\cos\phi - D\cos\omega_0 t\sin\phi$$

$$= (y_0 + D\cos\phi)\sin\omega_0 t - D\sin\phi\cos\omega_0 t = (y_0 + D\cos\phi)\left[\sin\omega_0 t - \underbrace{\frac{D\sin\phi}{y_0 + D\cos\phi}}_{\tan\psi}\cos\omega_0 t\right]$$

$$= (y_0 + D\cos\phi)(\sin\omega_0 t - \tan\psi\cos\omega_0 t) = \frac{y_0 + D\cos\phi}{\cos\psi}(\sin\omega_0 t\cos\psi - \cos\omega_0 t\sin\psi)$$

$$= \frac{y_0 + D\cos\phi}{\cos\psi}\sin(\omega_0 t - \psi),$$

where we have let $\tan\psi = D\sin\phi/(y_0 + D\cos\phi)$. Therefore, the amplitude of vibration is given by

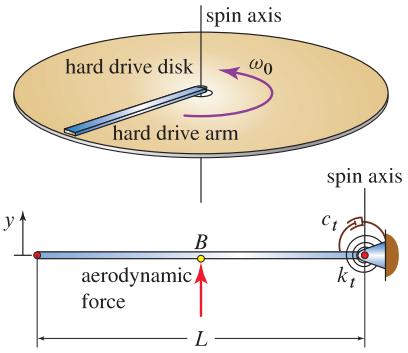
$$|y_m| = \frac{y_0 + D\cos\phi}{\cos\psi} = (y_0 + D\cos\phi)\sqrt{1 + \left(\frac{D\sin\phi}{y_0 + D\cos\phi}\right)^2} \quad \Rightarrow \quad \boxed{|y_m| = 0.00219\,\text{m},}$$

where we have used the fact that if $\tan\beta = q$ then $\cos\beta = 1/\sqrt{1 + q^2}$.

We could have saved a little time with this solution by noting that Eq. (3) above could have been directly obtained from Eq. (9.75) in the textbook and Eq. (4) above could have been obtained from Eq. (9.76).

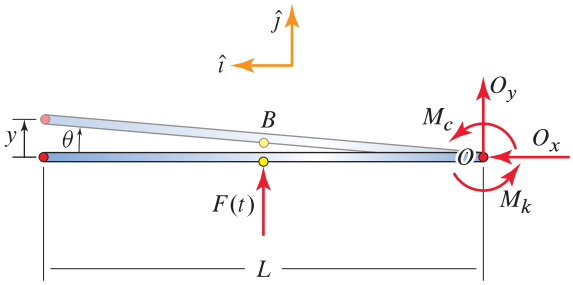
Problem 9.57

A hard drive arm undergoes flow-induced vibration caused by the vortices of air produced by a platter that rotates at $\omega_0 = 10{,}000$ rpm. The arm has length $L = 0.037$ m and mass $m = 0.00075$ kg, and it is made from aluminum with a modulus of elasticity $E = 70$ GPa. In addition, assume that the cross section of the arm has an area moment of inertia $I_{cs} = 8.5\times10^{-14}$ m^4. Following the steps in Example 9.2 on p. 682, the arm can be modeled as a rigid rod that is pinned at one end and is restrained by a torsional spring with equivalent spring constant $k_t = 3EI_{cs}/L$. In addition to the torsional spring, assume that the arm's motion is affected by a torsional damper with torsional damping coefficient c_t. Assuming that the damping ratio is $\zeta = 0.02$ and that the vortices produce an aerodynamic force with the same frequency as the rotation of the platter, determine the amplitude of the aerodynamic force needed to cause a steady-state vibration amplitude of 0.0001 m at the tip of the arm. Assume that the aerodynamic force is applied at the midpoint B of the hard drive. What vibration amplitude will result if the same excitation is applied to a hard drive arm assembly with the damping ratio of 0.05?



Solution

Referring to the FBD on the right and summing moments about point O, we find that



$$\sum M_O: \quad -M_k - M_c + F(t)\frac{L}{2} = I_O\alpha_{\text{arm}}, \qquad (1)$$

where M_k is the moment due to the torsional spring, M_c is the moment due to the torsional damper, $F(t)$ is the aerodynamic force on the arm, and α_{arm} is the angular acceleration of the arm. The force laws are given by

$$M_k = k_t\theta = k_t\frac{y}{L}, \quad M_c = c_t\dot{\theta} = c_t\frac{\dot{y}}{L}, \quad \text{and} \quad F(t) = F_0\sin\omega_0 t,$$

where F_0 is the desired amplitude of the aerodynamic forcing and θ and y are defined as shown on the FBD. Since $\alpha_{\text{arm}} = \ddot{\theta} = \ddot{y}/L$ and $I_O = \frac{1}{3}mL^2$, we can write the equation of motion in Eq. (1) as

$$-k_t\frac{y}{L} - c_t\frac{\dot{y}}{L} + \frac{L}{2}F_0\sin\omega_0 t = \frac{1}{3}mL^2\frac{\ddot{y}}{L} \quad \Rightarrow \quad \tfrac{1}{3}mL^2\ddot{y} + c_t\dot{y} + k_t y = \tfrac{1}{2}L^2F_0\sin\omega_0 t. \qquad (2)$$

Defining the following "equivalent" quantities

$$m_{\text{eq}} = \tfrac{1}{3}mL^2 \quad \text{and} \quad (F_0)_{\text{eq}} = \tfrac{1}{2}L^2F_0,$$

we see that Eq. (2) becomes

$$m_{\text{eq}}\ddot{y} + c_t\dot{y} + k_t y = (F_0)_{\text{eq}}\sin\omega_0 t.$$

Therefore the amplitude of steady-state vibration is given by

$$|y| = \frac{(F_0)_{\text{eq}}/k_t}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}}, \qquad (3)$$

where $\omega_n = \sqrt{k_t/m_{\text{eq}}}$ is the natural frequency, ω_0 is the forcing frequency, and $\zeta = c_t/\left(2\sqrt{k_t m_{\text{eq}}}\right)$ is the damping ratio.* Computing the numerical values of the quantities in Eq. (3), we find that

$$\begin{aligned}
(F_0)_{\text{eq}} &= \tfrac{1}{2}L^2 F_0 = (0.0006845 F_0)\,\text{N}, \\
k_t &= \frac{3EI_{\text{cs}}}{L} = 0.4824\,\text{N·m/rad}, \\
\omega_0 &= 10{,}000\,\text{rpm} = 1047\,\text{rad/s}, \\
m_{\text{eq}} &= \tfrac{1}{3}mL^2 = 3.422\times10^{-7}\,\text{kg}, \\
\omega_n &= \sqrt{\frac{k_t}{m_{\text{eq}}}} = 1187\,\text{rad/s}.
\end{aligned}$$

Substituting $|y| = 0.0001$ m as well as the values above into Eq. (3) and then solving for F_0, we obtain

$$0.0001 = \frac{0.0006845F_0/0.4824}{\sqrt{[1-(1047/1187)^2]^2 + [2(0.02)1047/1187]^2}} \quad \Rightarrow \quad F_0 = 0.01582\,\text{N}$$

$$\Rightarrow \quad \boxed{F_0 = 0.0158\,\text{N},}$$

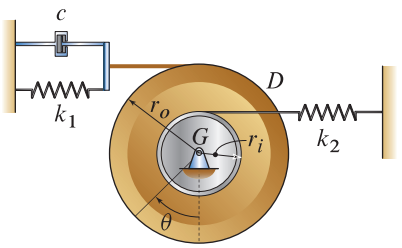
where we have reported the result to both 3 and 4 significant figures. Applying this level of excitation to a drive arm assembly with damping ratio $\zeta = 0.05$, we obtain

$$|y| = \frac{0.0006845(0.01582)/0.4824}{\sqrt{[1-(1047/1187)^2]^2 + [2(0.05)1047/1187]^2}} \quad \Rightarrow \quad \boxed{|y| = 0.0000941\,\text{m}.}$$

*Note that this definition for ζ would allow us to find c_t if we desired, but c_t is not explicitly needed.

Problem 9.58

The mechanism consists of a disk D pinned at G, which is both the geometric center of the disk and its mass center. The outer circumference of the disk has radius $r_o = 0.1\,\text{m}$ and is connected to an element consisting of a linear spring with stiffness $k_1 = 100\,\text{N/m}$ in parallel with a dashpot with damping coefficient $c = 50\,\text{N·s/m}$. The disk has a hub of radius $r_i = 0.05\,\text{m}$ that is connected to a linear spring with constant $k_2 = 350\,\text{N/m}$. Knowing that for $\theta = 0$ the disk is in static equilibrium and that the mass moment of inertia of the disk is $I_G = 0.001\,\text{kg·m}^2$, derive the linearized equation of motion of the disk in terms of θ. In addition, calculate the resulting vibrational motion if the system is released from rest with an initial angular displacement $\theta_i = 0.05\,\text{rad}$.



Solution

Referring to the FBD at the right, we can obtain the equation of motion by summing moments about the mass center G to obtain

$$\sum M_G: \quad -(F_c + F_1)r_o - F_2 r_i = I_G \alpha_D, \tag{1}$$

where F_c is the for due to the dashpot, F_1 is the force due to spring k_1, F_2 is the force due to spring k_2, and the reactions due to the pin at G are designated by R_x and R_y. Using θ as the displacement variable and assuming small displacements, the force laws can be written as

$$F_1 = k_1 r_o \theta, \quad F_2 = k_2 r_i \theta, \quad \text{and} \quad F_c = c r_o \dot{\theta}.$$

Using these force laws, along with the fact that $\alpha_D = \ddot{\theta}$, Eq. (1) becomes

$$\boxed{I_G \ddot{\theta} + c r_o^2 \dot{\theta} + (k_1 r_o^2 + k_2 r_i^2)\theta = 0.}$$

We now compute the motion of the system when it is released with the initial conditions $\theta(0) = 0.05\,\text{rad}$ and $\dot{\theta}(0) = 0\,\text{rad/s}$. Before we can determine the response, we need to determine character of the damping. Computing the critical damping for this system, we find

$$c_c = 2\sqrt{k_{\text{eq}} m_{\text{eq}}} = 2\sqrt{(k_1 r_o^2 + k_2 r_i^2) I_G} = 0.08660\,\text{N·s/m},$$

where we are using the definitions

$$m_{\text{eq}} = I_G = 0.001\,\text{kg·m}^2, \quad c_{\text{eq}} = c r_0^2 = 0.5000\,\text{N·m·s}, \quad \text{and} \quad k_{\text{eq}} = k_1 r_o^2 + k_2 r_i^2 = 1.875\,\text{N·m}.$$

Since $c > c_c$, the system is overdamped, $\zeta = c/c_c = 5.774$, and the solution is given by

$$\theta = e^{-\zeta \omega_n t}\left(A e^{\sqrt{\zeta^2 - 1}\,\omega_n t} + B e^{-\sqrt{\zeta^2 - 1}\,\omega_n t}\right),$$

where ω_n is the natural frequency and is given by

$$\omega_n = \sqrt{k_{\text{eq}}/m_{\text{eq}}} = 43.30\,\text{rad/s},$$

and A and B are constants determined from the initial conditions. Substituting the computed quantities into the solution for θ, we obtain

$$\theta = e^{-250.0t}\left(Ae^{246.2t} + Be^{-246.2t}\right).$$

Applying the initial condition $\theta(0) = 0.05\,\text{rad}$, we obtain

$$0.05 = A + B. \tag{2}$$

Finding $\dot{\theta}$, we obtain

$$\begin{aligned}\dot{\theta} = -250.0e^{-250.0t}\left(Ae^{246.2t} + Be^{-246.2t}\right) + e^{-250.0t}\left(246.2Ae^{246.2t} - 246.2Be^{-246.2t}\right)\\ = -3.800Ae^{-3.800t} - 496.2Be^{-496.2t},\end{aligned}$$

and then applying the initial condition $\dot{\theta}(0) = 0$, we get

$$0 = -3.800A - 496.2B. \tag{3}$$

Solving Eqs. (2) and (3) for A and B, we obtain

$$A = 0.05039\,\text{rad} \quad \text{and} \quad B = -0.0003859\,\text{rad},$$

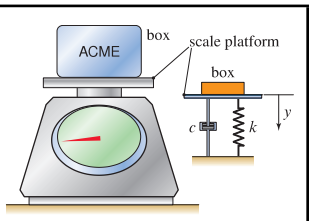
which means that the vibrational motion is given by

$$\theta(t) = e^{-250.0t}\left(0.05039e^{246.2t} - 0.0003859e^{-246.2t}\right)$$

$$\Rightarrow \quad \boxed{\theta(t) = 0.0504e^{-3.80t} - 0.000386e^{-496t}.}$$

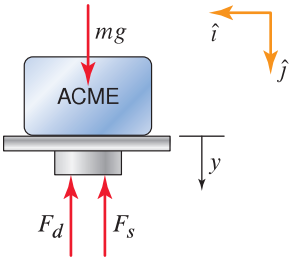
Problem 9.59

A box of mass 0.75 kg is thrown on a scale, causing both the scale and the box to move vertically downward with an initial speed of 0.5 m/s. Before the box lands on the scale, the scale is in equilibrium. The total mass of the scale's moving platform and the box is $m = 1.25$ kg. Modeling the platform's support as a spring and dashpot with stiffness $k = 1000$ N/m and damping coefficient $c = 70.7$ N·s/m, find the response of the scale. *Hint:* Place the origin of the y axis at the position of the platform corresponding to the equilibrium configuration of the platform and box together.



Solution

Referring to the FBD of the box and the platform on the right, F_s is the force due to the spring and F_d is the force due to the dashpot. Summing forces in the y direction gives



$$\sum F_y: \quad mg - F_d - F_s = m\ddot{y}, \tag{1}$$

Writing the force laws for the spring and dashpot forces, we obtain

$$F_s = k(y + y_{\text{st}}) \quad \text{and} \quad F_d = c\dot{y},$$

where y_{st} is the static deflection of the spring when the box is on the platform. Substituting the force laws into Eq. (1) gives

$$mg - c\dot{y} - k(y + y_{\text{st}}) = m\ddot{y} \quad \Rightarrow \quad m\ddot{y} + c\dot{y} + ky = 0,$$

where we have used the fact that $mg = ky_{\text{st}}$. To determine the nature of the response, we need to determine the nature of the damping. Doing so, we find

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = 0.9998 \approx 1,$$

where we have used $c = 70.7$ N·s/m, $k = 1000$ N/m, and $m = 1.25$ kg. Since the damping ratio ζ is so close to 1, can consider the system to be critically damped. The response of a critically damped system is given by

$$y = (A + Bt)e^{-\omega_n t},$$

where A and B are constants determined from the initial conditions and ω_n is the natural frequency, which is given by

$$\omega_n = \sqrt{\frac{k}{m}} = 28.28 \text{ rad/s} \quad \Rightarrow \quad y = (A + Bt)e^{-28.28t},$$

where we have used $k = 1000$ N/m and $m = 1.25$ kg and we have substituted ω_n into the response. Evaluating the response at the initial condition $y(0) = 0$, we obtain

$$y(0) = A \quad \Rightarrow \quad A = 0.$$

Differentiating the response with respect to time and then evaluating it at the initial condition $\dot{y}(0) = 0.5$ m/s, we obtain

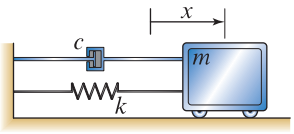
$$\dot{y} = Be^{-28.28t} - 28.28(A + Bt)e^{-28.28t} \quad \Rightarrow \quad \dot{y}(0) = B - 28.28A = 0.5 \quad \Rightarrow \quad B = 0.5 \text{ m/s},$$

where we have used the fact that $A = 0$. Putting everything together, the response is

$$\boxed{y(t) = 0.5te^{-28.3t}.}$$

Problem 9.60

Consider a simple viscously damped harmonic oscillator governed by Eq. (9.51), and analyze the case in which the damping coefficient c is negative. Calculate the general expression for the response (without taking into account specific initial conditions), using $m = 1\,\text{kg}$, $c = -1\,\text{N·s/m}$, and $k = 10\,\text{N/m}$. Comment on the system's response.



Solution

We are told that the equation of motion for this system is given by Eq. (9.51), which is given by

$$m\ddot{x} + c\dot{x} + kx = 0.$$

The solution of this equation was found by assuming a solution of the form

$$x = e^{\lambda t} \quad \Rightarrow \quad m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + k e^{\lambda t} = 0 \quad \Rightarrow \quad m\lambda^2 + c\lambda + k = 0.$$

The roots of this characteristic equation we found to be

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}, \qquad \lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}.$$

For $m = 1\,\text{kg}$, $c = -1\,\text{N·s/m}$, and $k = 10\,\text{N/m}$, the term under the square root has the value -9.75, which is less than zero and so the response is given by

$$x = e^{-(c/2m)t}(A \sin \omega_d t + B \cos \omega_d t),$$

where ω_d is the damped natural frequency given by

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 3.122\,\text{rad/s}.$$

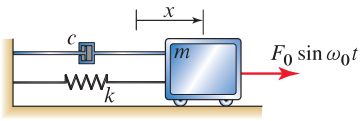
Substituting in $m = 1\,\text{kg}$, $c = -1\,\text{N·s/m}$, and $k = 10\,\text{N/m}$, the equation of motion and response become, respectively,

$$\ddot{x} - \dot{x} + 10x = 0 \quad \text{and} \quad \boxed{x(t) = e^{0.5t}(A \sin 3.12t + B \cos 3.12t).}$$

Since the exponent on the exponential is positive, the response will grow without limit and so the system is unstable.

Problem 9.61

The MF for a harmonically excited spring-mass-damper system at $\omega_0/\omega_n \approx 1$ is equal to 5. Calculate the damping ratio of the system. What would the damping ratio be if the MF were equal to 10? Sketch the magnification factor at $\omega_0/\omega_n \approx 1$ as a function of the damping ratio.



Solution

The equation for the magnification factor for this system is given by Eq. (9.79), which is

$$\text{MF} = \frac{1}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}}$$

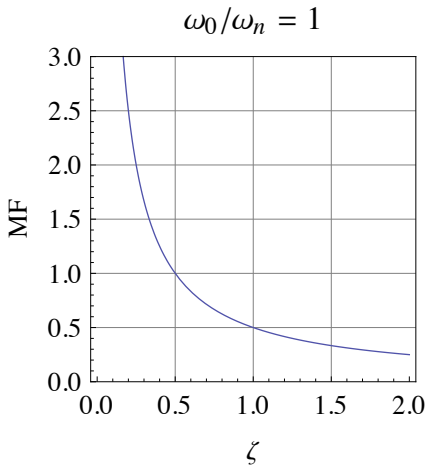
where ζ is the damping ratio, ω_n is the natural frequency, and ω_0 is the forcing frequency. Noting that MF $= 5$, when $\omega_0/\omega_n = 1$, we have

$$5 = \frac{1}{\sqrt{[1-(1)^2]^2 + (2\zeta)^2}} = \frac{1}{\sqrt{(2\zeta)^2}} = \frac{1}{2\zeta} \quad \Rightarrow \quad \boxed{\zeta = \frac{1}{10} = 0.1 \quad \text{for MF} = 5.}$$

For MF $= 10$, the damping ratio would be

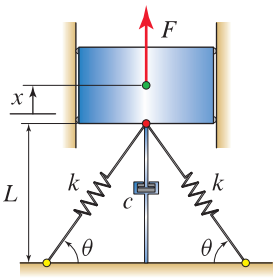
$$10 = \frac{1}{2\zeta} \quad \Rightarrow \quad \boxed{\zeta = \frac{1}{20} = 0.05 \quad \text{for MF} = 10.}$$

A plot of MF as a function of ζ for $\omega_0/\omega_n = 1$ is shown below.



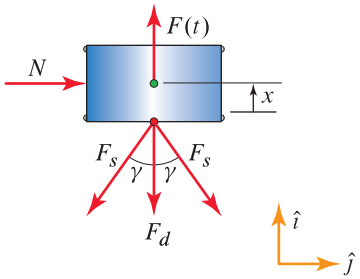
Problem 9.62

A slider moves in the horizontal plane under the action of the harmonic forcing $F(t) = F_0 \sin \omega_0 t$. The slider is connected to two identical linear springs, each of which has constant k. When $t = 0$, $x(0) = 0$, the springs are unstretched, $\theta = 45°$, and $L = L_0$. The slider is also connected to a damper with damping coefficient c. Treating F_0, k, c, and L_0 as known quantities, neglecting friction, and letting $\dot{x}(0) = v_i$, (a) derive the equations of motion of the system, (b) derive the linearized equations of motion about the initial position, and (c) determine the amplitude of the steady-state vibrations for the linearized equations of motion.



Solution

Referring to the FBD at the right and summing forces in the x direction, we find that

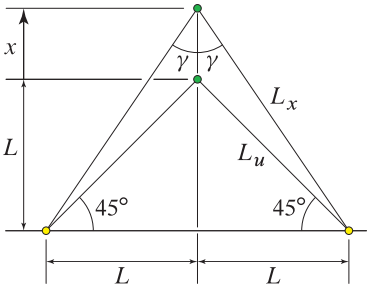


$$\sum F_y: \quad F(t) - 2F_s \cos\gamma - F_d = m\ddot{x}, \tag{1}$$

where F_s is the force in each spring and F_d is the force in the dashpot. The forces in the spring and dashpot can be written as, respectively,

$$F_s = k(L_x - L_u) \quad \text{and} \quad F_d = c\dot{x},$$

where L_x is the length of the spring in its deformed configuration (i.e., for arbitrary x) and L_u is the undeformed length of the spring. Referring to the bottom figure on the right, we find L_x and L_u to be, respectively,



$$L_u = \sqrt{2}L \quad \text{and} \quad L_x = \sqrt{L^2 + (L+x)^2}.$$

We can write $\cos\gamma$ as

$$\cos\gamma = \frac{L+x}{\sqrt{L^2 + (L+x)^2}}.$$

Substituting the force laws and the geometric relations into Eq. (1), we obtain the equation of motion

$$F(t) - 2k\left[\sqrt{L^2 + (L+x)^2} - \sqrt{2}L\right]\frac{L+x}{\sqrt{L^2 + (L+x)^2}} - c\dot{x} = m\ddot{x}$$

$$\Rightarrow \quad \boxed{m\ddot{x} + c\dot{x} + 2k\left[\sqrt{L^2 + (L+x)^2} - \sqrt{2}L\right]\frac{L+x}{\sqrt{L^2 + (L+x)^2}} = F_0 \sin\omega_0 t,}$$

where we have replaced $F(t)$ with $F_0 \sin\omega_0 t$. To linearize the equation of motion, we only need to linearize the term containing k since it is the only term that is nonlinear. Copying that term and then rearranging gives

$$2k\left[\sqrt{L^2 + (L+x)^2} - \sqrt{2}L\right]\frac{L+x}{\sqrt{L^2 + (L+x)^2}} = 2k(L+x)\left[1 - \underbrace{\frac{\sqrt{2}L}{\sqrt{L^2 + (L+x)^2}}}_{f(x)}\right]$$

$$= 2k(L+x)[1 - f(x)]. \tag{2}$$

We can now use the Taylor series for $f(x)$ about $x = 0$ to linearize the stiffness term and thus the equation of motion. Writing the Taylor series, we have

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + \cdots \\ &= \frac{\sqrt{2}L}{\sqrt{L^2+(L+x)^2}}\bigg|_{x=0} - \frac{\sqrt{2}L(L+x)}{\left[L^2+(L+x)^2\right]^{3/2}}\bigg|_{x=0} x + \cdots \\ &= 1 - \frac{\sqrt{2}L^2}{(2L^2)^{3/2}}\,x + \cdots = 1 - \frac{x}{2L} + \ldots, \end{aligned} \tag{3}$$

where the prime ($'$) indicates differentiation with respect to x and we have omitted the x^2 term in the Taylor series expansion since it is nonlinear. Substituting Eq. (3) into Eq. (2) and then substituting the result into the equation of motion, we obtain

$$m\ddot{x} + c\dot{x} + 2k(L+x)\left[1 - \left(1 - \frac{x}{2L}\right)\right] = F_0 \sin\omega_0 t \quad \Rightarrow \quad m\ddot{x} + c\dot{x} + kx + \frac{k}{L}x^2 = F_0 \sin\omega_0 t$$

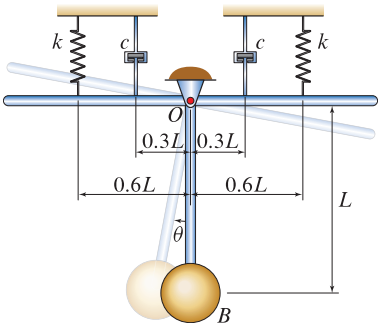
$$\Rightarrow \quad \boxed{m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega_0 t,}$$

where we have ignored terms of the order x^2 and higher. Since the linearized equation of motion for this system is identical to Eq. (9.69), the amplitude of steady-state vibration is given by Eq. (9.78), which is

$$\boxed{D = \frac{F_0/k}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}}, \quad \text{where} \quad \omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \zeta = \frac{c}{2\sqrt{km}}.}$$

Problem 9.63

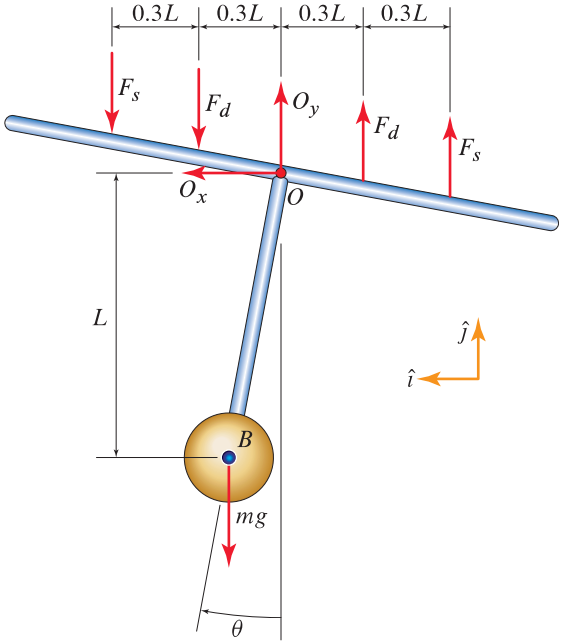
The mechanism shown is a pendulum consisting of a pendulum bob B with mass m and a T bar, which is pinned at O and has negligible mass. The horizontal portion of the T bar is connected to two supports, each of which has an identical spring and dashpot system, each with spring constant k and damping coefficient c. The springs are unstretched when B is vertically aligned with the pin at O. Modeling B as a particle, derive the linearized equations of motion of the system. In addition, assuming that the system is underdamped, derive the expression for the damped natural frequency of vibration of the system.



Solution

Referring to the FBD on the right and assuming that the angle θ is small (so the horizontal and vertical dimensions are as shown on the FBD), we can sum moments about point O to obtain the equation of motion

$$\sum M_O: \quad -mgL\theta - 2F_d(0.3L) - 2F_s(0.6L) = I_O\ddot{\theta}, \quad (1)$$



where F_s is the force in each of the two springs and F_d is the force in each of the two dashpots. The mass moment of inertia $I_O = mL^2$ and the forces F_s and F_d can be written as

$$F_s = k(0.6L\theta) \quad \text{and} \quad F_d = c\left(0.3L\dot{\theta}\right).$$

Substituting the force laws and the expression for the moment of inertia into Eq. (1), we obtain

$$-mgL\theta - 0.72kL^2\theta - 0.18cL^2\dot{\theta} = mL^2\ddot{\theta},$$

or, rearranging and canceling an L, we get the linear equation of motion as

$$\boxed{mL\ddot{\theta} + 0.18cL\dot{\theta} + (mg + 0.72kL)\theta = 0.}$$

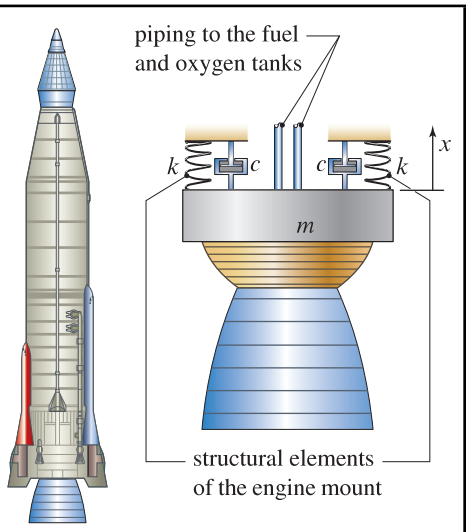
Assuming the system is underdamped, the damped natural frequency, which is given by Eq. (9.61), is

$$\omega_d = \sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}} - \left(\frac{c_{\text{eq}}}{2m_{\text{eq}}}\right)^2} = \sqrt{\frac{mg + 0.72kL}{mL} - \left(\frac{0.18cL}{2mL}\right)^2}$$

$$\Rightarrow \quad \boxed{\omega_d = \sqrt{\frac{g}{L} + 0.72\frac{k}{m} - 0.0081\frac{c^2}{m^2}}.}$$

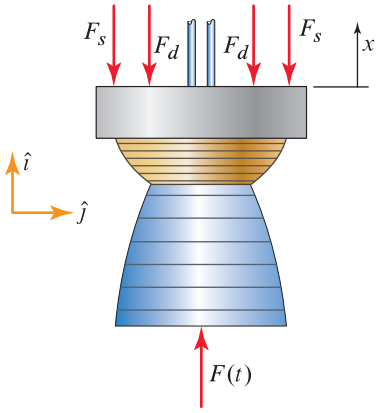
Problem 9.64

The engine in the rocket shown is supposed to provide a constant thrust of 5000 kN. The turbopump unit in the engine nominally operates at 7000 rpm, and as a result of a design issue, the actual thrust provided by the engine oscillates harmonically with an amplitude of 10 kN at the same rotational frequency of the turbopump unit. The mass of the engine is $m = 5000$ kg. The rest of the rocket is much heavier than the engine and can be treated as being fixed. The engine is mounted to the rocket via two structural members, each of which can be modeled as consisting of a linear spring of stiffness k in parallel with a dashpot with linear viscous damping coefficient c. Determine the smallest values of k and c such that the static deflection due to the constant component of the thrust is less than 0.01 m and so that the vibration amplitude in the nominal operating regime is less than 0.001 m. Ignore the stiffness and damping due to the piping. *Hint:* If x is measured from the equilibrium position of the engine that results from the combined effect of the thrust and gravity, then the engine is subject to an externally applied forcing equal to (10 kN) $\sin \omega_0 t$, where ω_0 is the rotational frequency of the turbopump.



Solution

Referring to the figure at the right and summing forces in the x direction, we obtain



$$\sum F_x: \quad -2F_s - 2F_d + F(t) = m\ddot{x}, \tag{1}$$

or, noting that $F_s = kx$, $F_d = c\dot{x}$, and $F(t) = F_0 \sin \omega_0 t$, the balance law becomes the following equation of motion

$$m\ddot{x} + 2c\dot{x} + 2kx = F_0 \sin \omega_0 t,$$

where $F_0 = 10{,}000$ N and $\omega_0 = 7000$ rpm $= 733.0$ rad/s. Writing the static deflection as δ_{st}, we note that we can write it as

$$\delta_{\text{st}} = \frac{F_{\text{st}}}{k_{\text{eq}}} = \frac{F_{\text{st}}}{2k} < 0.01 \text{ m} \quad \Rightarrow \quad k > 50F_{\text{st}} \quad \Rightarrow \quad k > 2.5\times10^8 \text{ N/m},$$

where $F_{\text{st}} = 5000\times10^3$ N is the constant thrust static force. Note that we also want the amplitude of the deflection during vibration to not exceed 0.001 m. The steady-state amplitude for damped system is

$$D = \frac{F_0/k_{\text{eq}}}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}} < 0.001 \text{ m},$$

where

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m}} = \sqrt{\frac{2k}{m}} \quad \text{and} \quad \zeta = \frac{c_{\text{eq}}}{\sqrt{2k_{\text{eq}}m}} = \frac{2c}{2\sqrt{2km}} = \frac{c}{\sqrt{2km}}.$$

As a limiting case, we can ask whether or not we can satisfy this limit on D with $c = 0$? Let's check. Since $c = 0$ implies that $\zeta = 0$, the steady-state amplitude becomes

$$D = \frac{F_0/k_{\text{eq}}}{1-(\omega_0/\omega_n)^2} = \frac{F_0/(2k)}{1-\omega_0^2 m/(2k)} = \frac{F_0}{2k-m\omega_0^2} < 0.001\,\text{m} \quad \Rightarrow \quad k > 1.348\times 10^9\,\text{N/m},$$

where we have used $F_0 = 10{,}000\,\text{N}$, $m = 5000\,\text{kg}$, and $\omega_0 = 733.0\,\text{rad/s}$. Therefore, we can satisfy the given constraints with

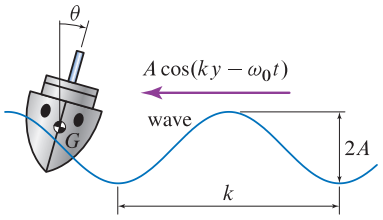
$$\boxed{k > 1.35\times 10^9\,\text{N/m} \quad \text{and} \quad c = 0.}$$

Of course, with $c = 0$, the amplitude would be infinite at resonance, which is generally undesirable.

Problem 9.65

A simple model for a ship rolling on waves treats the waves as *sinusoids*. Using this model, it can be shown that a linear model for the roll angle θ is given by

$$I_G\ddot{\theta} + c\dot{\theta} + mg\theta = -I_G k\omega_0^2 \sin\omega_0 t,$$



where G denotes the mass center of the ship, I_G is the ship's mass moment of inertia, c is a rotational viscous damping constant coefficient, m is the mass of the ship, A is the wave amplitude, and k is the wavelength of the waves.

(a) What is the natural frequency of the system?

(b) Find the magnification factor for the system.

(c) Assuming that the damping is negligible (i.e., $c \approx 0$), if the maximum amplitude of oscillation that the ship can undergo without capsizing is $\theta_{\max} = 1$ rad, find the maximum A so that the crew remains safe.

Solution

Note: The given equation of motion is missing an A in the forcing term so that it should be written as

$$I_G\ddot{\theta} + c\dot{\theta} + mg\theta = -I_G Ak\omega_0^2 \sin\omega_0 t.$$

This does not affect parts (a) and (b) of the problem, but it does impact part (c).

Part (a). The natural frequency is

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}}} \quad \Rightarrow \quad \boxed{\omega_n = \sqrt{\frac{mg}{I_G}}.}$$

Part (b). For an equation of motion of the given form, the MF is given by

$$\text{MF} = \frac{1}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}},$$

where ω_n is given above and ζ is given by

$$\zeta = \frac{c}{2\sqrt{k_{\text{eq}}m_{\text{eq}}}} = \frac{c}{2\sqrt{mgI_G}}.$$

Substituting ω_n and ζ into the MF, we obtain

$$\text{MF} = \frac{1}{\sqrt{\left(1-\frac{\omega_0^2 I_G}{mg}\right)^2 + \frac{4\omega_0^2 c^2 I_G}{4mgI_G mg}}} = \frac{1}{\sqrt{\left(1-\frac{\omega_0^2 I_G}{mg}\right)^2 + \frac{\omega_0^2 c^2}{m^2 g^2}}}$$

$$\Rightarrow \quad \boxed{\text{MF} = \frac{mg}{\sqrt{\left(mg - I_G\omega_0^2\right)^2 + c^2\omega_0^2}}.}$$

Part (c). If $c = 0$, then the amplitude of the response is given by

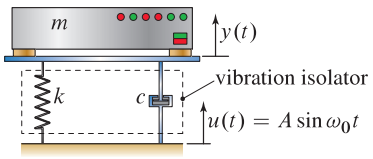
$$\theta_{\text{amp}} = \frac{(F_0)_{\text{eq}}/k_{\text{eq}}}{1-(\omega_0/\omega_n)^2} = \frac{I_G A k\omega_0^2/(mg)}{1-\frac{I_G\omega_0^2}{mg}} = \frac{I_G A k\omega_0^2}{mg - I_G\omega_0^2} < 1 \text{ rad},$$

where we have used the fact that we want the amplitude of vibration to be less than 1 rad. Solving for A, we obtain

$$I_G A k\omega_0^2 < mg - I_G\omega_0^2 \quad \Rightarrow \quad \boxed{A < \frac{mg - I_G\omega_0^2}{I_G k\omega_0^2}.}$$

Problem 9.66

A delicate instrument of mass m must be isolated from excessive vibration of the ground, which is described by the function $u(t) = A \sin \omega_0 t$. To do so, we need to design a *vibration isolating mount*, modeled by the spring and dashpot system shown.



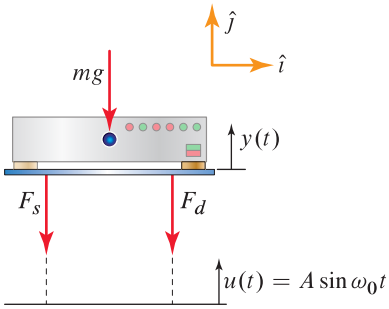
(a) Find the equation of motion of the instrument and reduce it to standard form.

(b) Find the steady-state response $y(t)$.

(c) Find the *displacement transmissibility*, i.e., the response amplitude D divided by A, where A is the amplitude of the ground's vibration.

Solution

Part (a). Referring to the FBD on the right and summing forces in the y direction, we obtain



$$\sum F_y: \quad -mg - F_s - F_d = m\ddot{y}, \tag{1}$$

where F_s is the force in the spring and F_d is the force in the dashpot. The force laws for the spring and dashpot are, respectively,

$$F_s = k[y(t) - u(t) - \delta_{\text{st}}] = k[y(t) - A \sin \omega_0 t - \delta_{\text{st}}],$$
$$F_d = c[\dot{y}(t) - \dot{u}(t)] = c[\dot{y}(t) - A\omega_0 \cos \omega_0 t].$$

Substituting F_s and F_d into Eq. (1), we obtain the equation of motion as

$$-mg - k[y(t) - A \sin \omega_0 t - \delta_{\text{st}}] - c[\dot{y}(t) - A\omega_0 \cos \omega_0 t] = m\ddot{y}$$
$$\Rightarrow \quad m\ddot{y} + c\dot{y} + ky = kA \sin \omega_0 t + cA\omega_0 \cos \omega_0 t, \tag{2}$$

where we have used the fact that $mg = k\delta_{\text{st}}$. Equation (2) has a forcing function that is *not* in standard form. We can put it in standard form if we let $s(t) = y(t) - u(t)$. Doing so, the force laws become

$$F_s = k(s - \delta_{\text{st}}) \quad \text{and} \quad F_d = c\dot{s}.$$

Noting that $\ddot{y} = \ddot{s} + \ddot{u} = \ddot{s} - A\omega_0^2 \sin \omega_0 t$, Eq. (1) becomes

$$-mg - k(s - \delta_{\text{st}}) - c\dot{s} = m(\ddot{s} - A\omega_0^2 \sin \omega_0 t) \quad \Rightarrow \quad \boxed{m\ddot{s} + c\dot{s} + ks = mA\omega_0^2 \sin \omega_0 t,}$$

where we still have $mg = k\delta_{\text{st}}$.

Part (b). The steady-state response for this system is given by

$$s = B \sin \omega_0 t + C \cos \omega_0 t,$$

where B and C are constants to be determined. Substituting this assumed solution into the equation of motion, we obtain

$$-m\omega_0^2(B\sin\omega_0 t + C\cos\omega_0 t) + c\omega_0(B\cos\omega_0 t - C\sin\omega_0 t) + k(B\sin\omega_0 t + C\cos\omega_0 t) = mA\omega_0^2\sin\omega_0 t.$$

Because of the orthogonality of $\sin\omega_0 t$ and $\cos\omega_0 t$, we can equate the coefficients of each to obtain the following two equations for the unknowns B and C

$$\begin{aligned} -m\omega_0^2 B - c\omega_0 C + kB = mA\omega_0^2 \quad &\Rightarrow \quad (k - m\omega_0^2)B - c\omega_0 C = mA\omega_0^2, \\ -m\omega_0^2 C + c\omega_0 B + kC = 0 \quad &\Rightarrow \quad c\omega_0 B + (k - m\omega_0^2)C = 0. \end{aligned}$$

Solving these two equations for B and C, we obtain

$$B = \frac{mA\omega_0^2(k - m\omega_0^2)}{(k - m\omega_0^2)^2 + c^2\omega_0^2} \quad \text{and} \quad C = \frac{-mcA\omega_0^3}{(k - m\omega_0^2)^2 + c^2\omega_0^2}.$$

Therefore, the solution for $s(t)$ is given by

$$s(t) = \left[\frac{mA\omega_0^2(k - m\omega_0^2)}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\right]\sin\omega_0 t - \left[\frac{mcA\omega_0^3}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\right]\cos\omega_0 t,$$

and therefore the solution for the steady-state response $y(t)$ is

$$y(t) = s(t) + u(t) = \left[\frac{mA\omega_0^2(k - m\omega_0^2)}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\right]\sin\omega_0 t - \left[\frac{mcA\omega_0^3}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\right]\cos\omega_0 t + A\sin\omega_0 t,$$

or,

$$\boxed{y(t) = \left[\frac{mA\omega_0^2(k - m\omega_0^2)}{(k - m\omega_0^2)^2 + c^2\omega_0^2} + A\right]\sin\omega_0 t - \left[\frac{mcA\omega_0^3}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\right]\cos\omega_0 t.}$$

Part (c). To find the displacement transmissibility, we divide the amplitude of the response by the amplitude of the ground's vibration. To find the amplitude of the response, we first define E and F as follows

$$y(t) = \underbrace{\left[\frac{mA\omega_0^2(k - m\omega_0^2)}{(k - m\omega_0^2)^2 + c^2\omega_0^2} + A\right]}_{E}\sin\omega_0 t - \underbrace{\left[\frac{mcA\omega_0^3}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\right]}_{F}\cos\omega_0 t,$$

and then, since $\sin\omega_0 t$ and $\cos\omega_0 t$ are orthogonal, the response amplitude D is given by

$$\begin{aligned} D = \sqrt{E^2 + F^2} &= \sqrt{\left[\frac{mA\omega_0^2(k - m\omega_0^2)}{(k - m\omega_0^2)^2 + c^2\omega_0^2} + A\right]^2 + \left[\frac{mcA\omega_0^3}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\right]^2} \\ &= \frac{A}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\sqrt{\left[m\omega_0^2(k - m\omega_0^2) + (k - m\omega_0^2)^2 + c^2\omega_0^2\right]^2 + (mc\omega_0^3)^2} \\ &= \frac{A}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\sqrt{\left[(k - m\omega_0^2)(m\omega_0^2 + k - m\omega_0^2) + c^2\omega_0^2\right]^2 + (mc\omega_0^3)^2} \\ &= \frac{A}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\sqrt{\left[k(k - m\omega_0^2) + c^2\omega_0^2\right]^2 + (mc\omega_0^3)^2}. \end{aligned}$$

Calling the term under the square root Q and then expanding it, we obtain

$$\begin{aligned}
Q &= k^2\left(k-m\omega_0^2\right)^2 + c^2\omega_0^2\left[c^2\omega_0^2 + 2k\left(k-m\omega_0^2\right) + m^2\omega_0^4\right] \\
&= k^2\left(k-m\omega_0^2\right)^2 + c^2\omega_0^2\Big[c^2\omega_0^2 + k^2 + \underbrace{k^2 - 2mk\omega_0^2 + m^2\omega_0^4}_{\left(k-m\omega_0^2\right)^2}\Big] \\
&= k^2\left(k-m\omega_0^2\right)^2 + \left(c^2\omega_0^2\right)^2 + c^2\omega_0^2k^2 + c^2\omega_0^2\left(k-m\omega_0^2\right)^2 \\
&= \left(k-m\omega_0^2\right)^2\left(k^2+c^2\omega_0^2\right) + c^2\omega_0^2\left(k^2+c^2\omega_0^2\right) \\
&= \left(k^2+c^2\omega_0^2\right)\left[\left(k-m\omega_0^2\right)^2 + c^2\omega_0^2\right].
\end{aligned}$$

Substituting Q back into the expression for D, we obtain

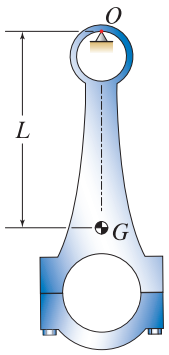
$$D = A\sqrt{\frac{k^2+c^2\omega_0^2}{\left(k-m\omega_0^2\right)^2+c^2\omega_0^2}}.$$

Finally, since the displacement transmissibility, DT, becomes

$$\mathrm{DT} = \frac{D}{A} \quad \Rightarrow \quad \boxed{\mathrm{DT} = \sqrt{\frac{k^2+c^2\omega_0^2}{\left(k-m\omega_0^2\right)^2+c^2\omega_0^2}}.}$$

Problem 9.67

When the connecting rod shown is suspended from the knife-edge at point O and displaced slightly so that it oscillates as a pendulum, its period of oscillation is 0.77 s. In addition, it is known that the mass center G is located a distance $L = 110\,\text{mm}$ from O and that the mass of the connecting rod is 661 g. Using the energy method, determine the mass moment of inertia of the connecting rod I_G.



Solution

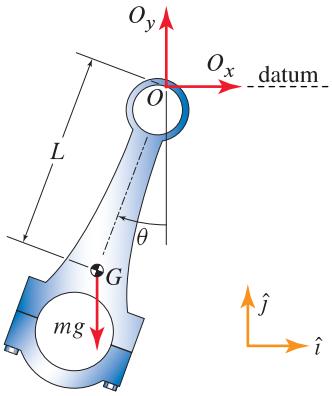
Referring to the FBD on the right, the kinetic energy of the connecting rod can be written as

$$T = \frac{1}{2} I_O \dot{\theta}^2 = \frac{1}{2}(I_G + mL^2)\dot{\theta}^2.$$

Using the datum shown, the potential energy of the connecting rod can be written as

$$V = -mgL\cos\theta \approx -mgL\left(1 - \frac{\theta^2}{2}\right),$$

where we have approximated the cosine function using Eq. (9.28). Applying the energy method to the kinetic and potentials energies shown above, we obtain



$$\frac{d}{dt}(T+V) = \frac{d}{dt}\left[\frac{1}{2}(I_G + mL^2)\dot{\theta}^2 - mgL\left(1 - \frac{\theta^2}{2}\right)\right] = 0$$
$$= (I_G + mL^2)\dot{\theta}\ddot{\theta} + mgL\theta\dot{\theta} = 0.$$

Canceling $\dot{\theta}$, we obtain

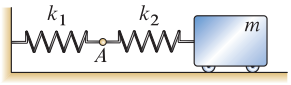
$$(I_G + mL^2)\ddot{\theta} + mgL\theta = 0 \quad \Rightarrow \quad \omega_n = \sqrt{\frac{mgL}{I_G + mL^2}} \quad \Rightarrow \quad \tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I_G + mL^2}{mgL}}.$$

Solving for I_G

$$\frac{\tau^2}{4\pi^2} = \frac{I_G + mL^2}{mgL} \quad \Rightarrow \quad I_G = mL\left(\frac{g\tau^2}{4\pi^2} - L\right) \quad \Rightarrow \quad \boxed{I_G = 0.00271\,\text{kg}\cdot\text{m}^2.}$$

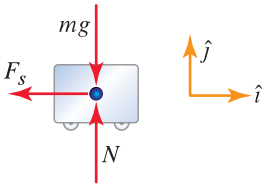
Problem 9.68

Derive the equation of motion for the system, in which the springs with constants k_1 and k_2 connecting m to the wall are joined in series. Neglect the mass of the small wheels, and assume that the attachment point A between the two springs has negligible mass. *Hint:* The force in the two springs must be the same; use this fact, along with the fact that the total deflection of the mass must equal the sum of the deflections of the springs to find an equivalent spring constant k_{eq}.



Solution

Referring to the FBD at the right and summing forces in the x direction, we obtain the following balance law for the mass m



$$\sum F_x: \quad -F_s = m\ddot{x}. \tag{1}$$

To find the force law for F_s, we begin by noting that the total displacement of the mass m, which is given by x, is the sum of the deflection of k_1 and the deflection of k_2, that is

$$x = \delta_1 + \delta_2,$$

where δ_1 is the deflection of k_1 and δ_2 is the deflection of k_2. Since we are neglecting the mass of the springs, the force in each spring must equal F_s so that

$$F_s = k_1\delta_1 = k_2\delta_2 = k_{\text{eq}}x,$$

where k_{eq} is the equivalent spring constant of the two springs in series. Therefore

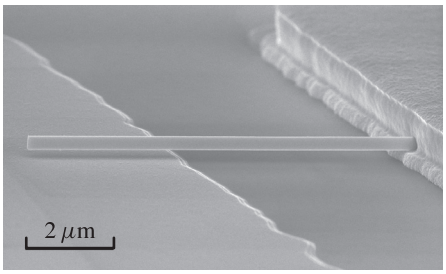
$$x = \frac{F_s}{k_1} + \frac{F_s}{k_2} = \frac{F_s}{k_{\text{eq}}} \quad \Rightarrow \quad \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{k_{\text{eq}}} \quad \Rightarrow \quad k_{\text{eq}} = \frac{k_1k_2}{k_1 + k_2}.$$

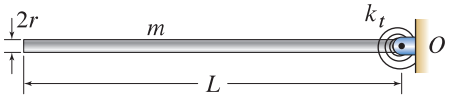
Therefore, since $F_s = k_{\text{eq}}x$, the equation of motion becomes

$$\boxed{m\ddot{x} + \frac{k_1k_2}{k_1 + k_2}x = 0.}$$

Problem 9.69

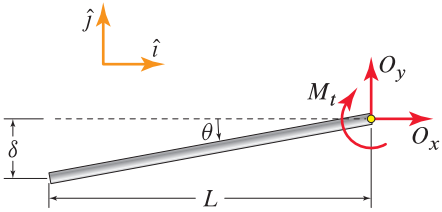
Revisit Example 9.2 and compute the natural frequency of the silicon nanowire, using the energy method. Use a uniform Si nanowire with a circular cross section that is 9.8 μm long and 330 nm in diameter and with all its flexibility lumped in a torsional spring at the base of the wire. In addition, use $\rho = 2330\,\text{kg/m}^3$ for the density of silicon and $E = 152\,\text{GPa}$ for its modulus of elasticity.





Solution

Referring to the FBD at the right, we see that only the moment due to the torsional spring does work. With this in mind, we can write the kinetic energy as



$$T = \frac{1}{2} I_O \dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{3} m L^2\right)\dot{\theta}^2 = \frac{1}{6} m L^2 \dot{\theta}^2,$$

where I_O is the mass moment of inertia of the bar with respect to point O. The potential energy of the torsional spring can be written as

$$V = \frac{1}{2} k_t \theta^2 = \frac{1}{2}\left(\frac{3EI_{cs}}{L}\right)\theta^2,$$

where we have used the fact that $k_t = 3EI_{cs}/L$ from Eq. (3) in Example 9.2. Applying the energy method, we obtain

$$\frac{d}{dt}(T+V) = \frac{d}{dt}\left[\frac{1}{6} m L^2 \dot{\theta}^2 + \frac{1}{2}\left(\frac{3EI_{cs}}{L}\right)\theta^2\right] = \frac{1}{3} m L^2 \dot{\theta}\ddot{\theta} + \left(\frac{3EI_{cs}}{L}\right)\theta\dot{\theta} = 0.$$

Canceling $\dot{\theta}$ and we obtain the equation of motion and the natural frequency as

$$\frac{1}{3} m L^2 \ddot{\theta} + \left(\frac{3EI_{cs}}{L}\right)\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \left(\frac{9EI_{cs}}{mL^3}\right)\theta = 0 \quad \Rightarrow \quad \omega_n = 3\sqrt{\frac{EI_{cs}}{mL^3}}.$$

To obtain a numerical value for ω_n, we find that the volume of the nanowire is $\pi r^2 L$, where $r = 165\,\text{nm} = 165\times10^{-9}\,\text{m}$ and $L = 9.8\times10^{-6}\,\text{m}$. Therefore, the mass of the wire is $m = \rho\pi r^2 L = 1.953\times10^{-15}\,\text{kg}$, where $\rho = 2330\,\text{kg/m}^3$. Referring back to Example 9.2, the area moment of inertia is $I_{cs} = \frac{1}{4}\pi r^4 = 5.821\times10^{-28}\,\text{m}^4$. Using these results, along with $E = 152\times10^9\,\text{N/m}^2$, we find that

$$\boxed{\omega_n = 2.08\times10^7\,\text{rad/s}.}$$

Problem 9.70

Structural health monitoring technology detects damage in civil, aerospace, and other structures. Structural damage is usually comprised of cracking, delaminations, or loose fasteners, which result in the reduction of stiffness. Many structural health monitoring methods are based on tracking changes in natural frequencies. Modeling a structure as a one DOF harmonic oscillator, calculate the change in stiffness needed to cause a 3% reduction in the natural frequency of the structure being monitored.

Solution

With a 3% reduction in frequency, the ratio of the new ω_1 frequency to the old ω_0 can be written as

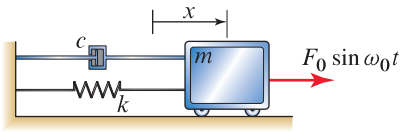
$$\frac{\omega_1}{\omega_0} = 0.97 \quad \Rightarrow \quad \omega_1 = 0.97\omega_0 \quad \Rightarrow \quad \sqrt{\frac{k_1}{m}} = 0.97\sqrt{\frac{k_0}{m}} \quad \Rightarrow \quad \frac{k_1}{m} = 0.9409\,\frac{k_0}{m}.$$

Therefore, since $k_1 = 0.9409k_0$, the percent change in stiffness is

$$\boxed{\Delta k = -5.91\%.}$$

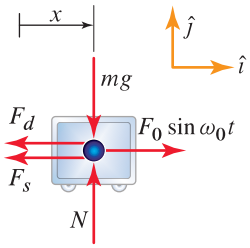
Problem 9.71

The harmonic oscillator shown has a mass $m = 5\,\text{kg}$, a spring with constant $k = 4000\,\text{N/m}$, and a dashpot with a damping coefficient $c = 20\,\text{N·s/m}$. Calculate the amplitude F_0 of the sinusoidal excitation force that is necessary to produce a steady-state vibration with a velocity amplitude of 10 m/s at resonance. What is the corresponding amplitude of the acceleration?



Solution

Referring to the FBD at the right and summing forces in the x direction, we obtain the following balance law



$$\sum F_x: \quad -F_d - F_s + F_0 \sin \omega_0 t = m\ddot{x}, \tag{1}$$

where F_d is the force in the dashpot and F_s is the force in the spring. Writing the force laws as follows,

$$F_d = c\dot{x} \quad \text{and} \quad F_s = kx,$$

Eq. (1) becomes

$$-c\dot{x} - kx + F_0 \sin \omega_0 t = m\ddot{x} \quad \Rightarrow \quad m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_0 t.$$

Now that we have the equation of motion, we know that we can write its steady-state response and its time derivative as

$$x = D \sin(\omega_0 t - \phi) \quad \Rightarrow \quad \dot{x} = \omega_0 D \cos(\omega_0 t - \phi) \quad \Rightarrow \quad |\dot{x}| = \omega_0 D = 10\,\text{m/s}, \tag{2}$$

where we have noted that we want the velocity $|\dot{x}|$ to be 10 m/s (at resonance). The position amplitude D can be written as

$$D = \frac{F_0/k}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}}, \tag{3}$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency. Combining Eqs. (2) and (3) and solving for the amplitude of the forcing F_0, we obtain

$$\frac{\omega_0 F_0/k}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}} = |\dot{x}| \quad \Rightarrow \quad F_0 = \frac{|\dot{x}|k}{\omega_0}\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}.$$

Since we are interested in finding F_0 at resonance, we note that

$$\frac{\omega_0}{\omega_n} = 1 \quad \Rightarrow \quad \frac{\omega_0}{\sqrt{k/m}} = 1 \quad \Rightarrow \quad \omega_0 = \sqrt{k/m} = 28.28\,\text{rad/s},$$

where we have used $k = 4000\,\text{N/m}$ and $m = 5\,\text{kg}$. In addition, calculating the damping ratio ζ, we find

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = 0.07071,$$

where we used $c = 20\,\text{N·s/m}$, $k = 4000\,\text{N/m}$, and $m = 5\,\text{kg}$. Substituting into the F_0 we found above, we obtain

$$F_0 = \frac{|\dot{x}|k}{\omega_0}\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2} = \frac{10(4000)}{28.28}\sqrt{\left[1-(1)^2\right]^2 + [2(0.07071)(1)]^2}$$

$$\Rightarrow \quad \boxed{F_0 = 200\,\text{N}.}$$

To find the corresponding amplitude of the acceleration, we differentiate x in Eq. (2) a second time to obtain

$$\ddot{x} = -\omega_0^2 D\sin(\omega_0 t - \phi) \quad \Rightarrow \quad |\ddot{x}| = \omega_0^2 D = \frac{\omega_0^2 F_0/k}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + (2\zeta\omega_0/\omega_n)^2}}$$

$$\Rightarrow \quad \boxed{|\ddot{x}| = 283\,\text{m/s}^2,}$$

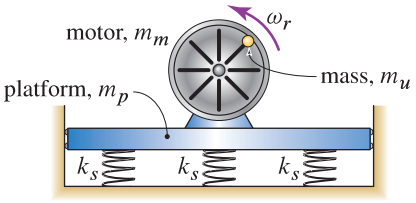
where, to obtain the final numerical result, we used $\omega_0 = 28.28\,\text{rad/s}$, $F_0 = 200.0\,\text{N}$, $k = 4000\,\text{N/m}$, $\omega_0/\omega_n = 1$, and $\zeta = 0.07071$.

Problem 9.72

Revisit Example 9.4 and derive the equations of motion of the motor, using the following equation, called *Lagrange's equation*,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_m}\right) - \frac{\partial T}{\partial y_m} + \frac{\partial V}{\partial y_m} = 0,$$

where T and V are the kinetic and potential energies of the system, respectively.

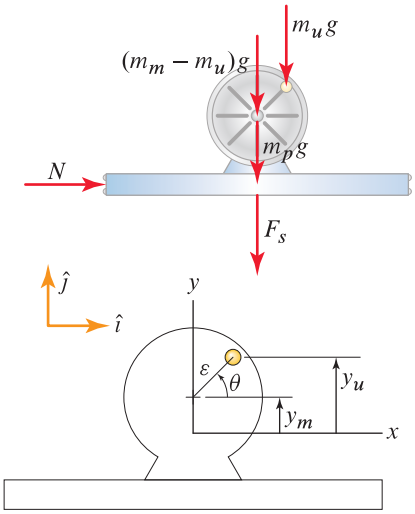


Solution

Referring to the FBD at the right, we see that the spring force F_s as well as all three weight forces do work as the system oscillates vertically. On the other hand, we have also seen that as long as we measure the position of the system y_m from its static equilibrium position, we do *not* have to include the potential energy of gravity. Doing so, we find that the potential energy can be written as simply

$$V = \tfrac{1}{2}k_{\text{eq}}y_m^2,$$

where, as we saw in Example 9.4, k_{eq} is the single spring constant equivalent to all the springs supporting the motor and platform. To find the kinetic energy T, we note that there are two contributions: one from motor and platform, which move together, and one from the rotating mass. Therefore, we can write T as



$$T = \tfrac{1}{2}(m_m - m_u + m_p)\dot{y}_m^2 + \tfrac{1}{2}m_u v_u^2,$$

where v_u is the speed of the unbalanced mass. To determine v_u, we can write its position as

$$\vec{r}_u = \varepsilon\cos\theta\,\hat{\imath} + (y_m + \varepsilon\sin\theta)\,\hat{\jmath} = \varepsilon\cos\omega_r t\,\hat{\imath} + (y_m + \varepsilon\sin\omega_r t)\,\hat{\jmath},$$

where we have let $\theta = \omega_r t$ since the unbalanced mass spins at a constant angular velocity ω_r. Differentiating $\vec{r}_u$ with respect to time and noting that $\hat{\imath}$ and $\hat{\jmath}$ do not rotate, we have that

$$\vec{v}_u = -\varepsilon\omega_r\sin\omega_r t\,\hat{\imath} + (\dot{y}_m + \varepsilon\omega_r\cos\omega_r t)\,\hat{\jmath}.$$

Since we need v_u^2, we sum the square of the components of $\vec{v}_u$ to obtain

$$v_u^2 = \varepsilon^2\omega_r^2\sin^2\omega_r t + \dot{y}_m^2 + 2\dot{y}_m\varepsilon\omega_r\cos\omega_r t + \varepsilon^2\omega_r^2\cos^2\omega_r t = \varepsilon^2\omega_r^2 + \dot{y}_m^2 + 2\dot{y}_m\varepsilon\omega_r\cos\omega_r t.$$

Therefore, the kinetic energy becomes

$$T = \tfrac{1}{2}(m_m - m_u + m_p)\dot{y}_m^2 + \tfrac{1}{2}m_u\left(\varepsilon^2\omega_r^2 + \dot{y}_m^2 + 2\dot{y}_m\varepsilon\omega_r\cos\omega_r t\right).$$

Carrying out the derivatives in the given Lagrange's equation, we obtain

$$\frac{\partial T}{\partial \dot{y}_m} = (m_m - m_u + m_p)\dot{y}_m + m_u\dot{y}_m + m_u\varepsilon\omega_r\cos\omega_r t = (m_m + m_p)\dot{y}_m + m_u\varepsilon\omega_r\cos\omega_r t,$$

and then

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_m}\right) = (m_m + m_p)\ddot{y}_m - m_u \varepsilon \omega_r^2 \sin \omega_r t,$$

where we have used the fact that ω_r is constant. Taking the remaining two derivatives, we obtain

$$\frac{\partial T}{\partial y_m} = 0 \quad \text{and} \quad \frac{\partial V}{\partial y_m} = k_{\text{eq}} y_m.$$

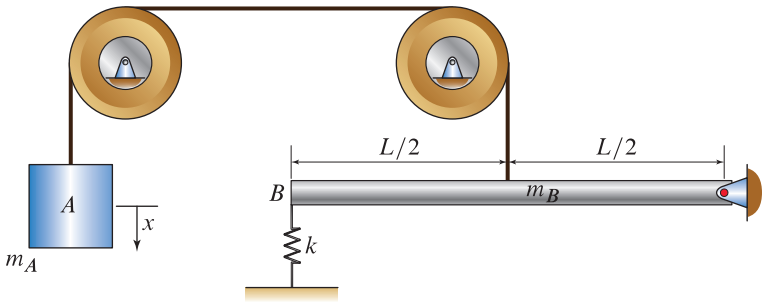
Substituting these derivatives into the given Lagrange's equation, we obtain

$$(m_m + m_p)\ddot{y}_m - m_u \varepsilon \omega_r^2 \sin \omega_r t + k_{\text{eq}} y_m = 0 \quad \Rightarrow \quad \boxed{(m_m + m_p)\ddot{y}_m + k_{\text{eq}} y_m = m_u \varepsilon \omega_r^2 \sin \omega_r t,}$$

which agrees with the equation of motion used in Example 9.4.

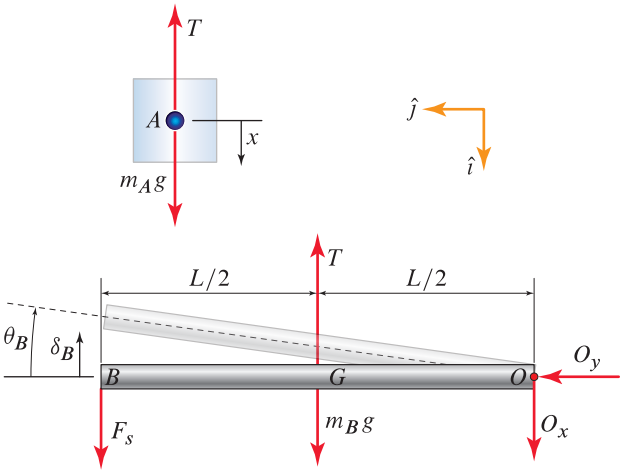
Problem 9.73

Modeling the beam as a uniform thin bar, ignoring the inertia of the pulleys, assuming that the system is in static equilibrium when the bar is horizontal, and assuming that the cord is inextensible and does not go slack, determine the linearized equation of motion of the system. In addition, determine the system's natural frequency of vibration. Treat the parameters shown in the figure as known.



Solution

An FBD of the block A, an FBD of the bar B, and the positive coordinate directions are shown at the right. Summing forces in the x direction on the block, we obtain the following balance law



$$\sum F_y: \quad m_A g - T = m_A a_{Ax} = m_A \ddot{x}. \qquad (1)$$

where x measures the vertical position of the block A. Summing moments about point O on bar B gives

$$\sum M_O: \quad T\tfrac{L}{2} - F_s L - m_B g \tfrac{L}{2} = I_O \alpha_B, \qquad (2)$$

where T is the cord tension, α_B is the angular acceleration of the bar, F_s is the force in the spring, and $I_O = \frac{1}{3} m_B L^2$ is the mass moment of inertia of the bar B about point O. The only force law is for F_s and it can be written as

$$F_s = k\left[\delta_B - (\delta_B)_{\text{st}}\right],$$

where $(\delta_B)_{\text{st}}$ is the deflection of the spring when the system is in static equilibrium. For the kinematics relations, we need to relate the angular acceleration of the bar (α_B) to the acceleration of the mass A ($\ddot{x}$) and we need to relate the deflection of point B (δ_B) to the position of mass A (x). Assuming small displacements and noting that the vertical displacement of A is equal and opposite to that of G, we can write

$$\delta_B = L\theta_B \quad \text{and} \quad x = \tfrac{L}{2}\theta_B \quad \Rightarrow \quad \delta_B = 2x \quad \text{and} \quad \ddot{\theta}_B = \alpha_B = \tfrac{2}{L}\ddot{x}.$$

Solving Eq. (1) for T and substituting the result into Eq. (2), we obtain

$$\tfrac{L}{2}\left(m_A g - m_A \ddot{x}\right) - kL\left[\delta_B - (\delta_B)_{\text{st}}\right] - m_B g \tfrac{L}{2} = \tfrac{1}{3} m_B L^2 \ddot{\theta}_B.$$

Now substituting in the force laws and kinematics relations, this becomes

$$\tfrac{L}{2} m_A \left(g - \ddot{x}\right) - kL\left[2x - (\delta_B)_{\text{st}}\right] - m_B g \tfrac{L}{2} = \tfrac{1}{3} m_B L^2 \left(\tfrac{2}{L}\ddot{x}\right)$$

Canceling an L and noting that $\frac{1}{2}g(m_A - m_B) = k(\delta_B)_{\text{st}}$, we find the equation of motion to be

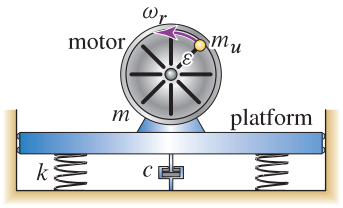
$$\boxed{\left(\tfrac{2}{3}m_B + \tfrac{1}{2}m_A\right)\ddot{x} + 2kx = 0.}$$

Therefore, the natural frequency of vibration is

$$\boxed{\omega_n = \sqrt{\frac{2k}{\frac{2}{3}m_B + \frac{1}{2}m_A}}.}$$

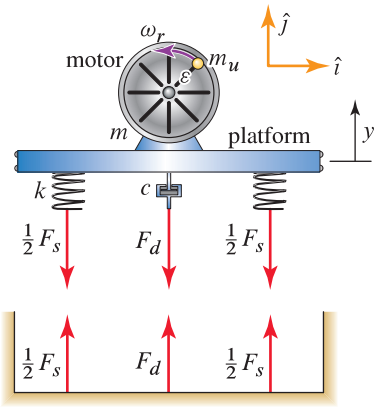
Problem 9.74

Revisit Example 9.6 and obtain the expression for the force transmitted to the floor, using the expression for the steady-state response of the unbalanced motor.



Solution

Referring to the diagram at the right, we see that the force transmitted to the floor F_t is equal to the force in the springs F_s plus the force in the dashpots F_d, i.e.,



$$F_t = F_s + F_d = k y_{ss} + c \dot{y}_{ss},$$

where y_{ss} is the steady-state response and is measured from the static equilibrium position of the system and we have ignored the static force transmitted to the floor. In Example 9.6 we found the steady-state response to be

$$y_{ss} = 0.00352 m_u \sin(125.7t + 0.842)\ \text{m},$$

in which the following values were used for k and c

$$k = 420{,}000\ \text{N/m} \quad \text{and} \quad c = 4000\ \text{N·s/m}.$$

Differentiating y_{ss} with respect to time and then substituting y_{ss} and $\dot{y}_{ss}$ into the equation for F_t, we obtain

$$\dot{y}_{ss} = 0.442 m_u \cos(125.7t + 0.842)\ \text{m/s}$$

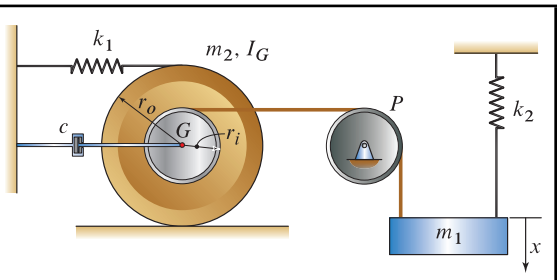
$$\Rightarrow \quad F_t = 1480 m_u \sin(125.7t + 0.842) + 1770 m_u \cos(125.7t + 0.842)\ \text{N}.$$

Therefore, for each of the values of the unbalanced mass $m_u = 0.01$ kg, $m_u = 0.1$ kg, and $m_u = 1$ kg, we obtain

$$\begin{aligned}
m_u = 0.01\ \text{kg}: &\quad F_t = 14.8 m_u \sin(125.7t + 0.842) + 17.7 m_u \cos(125.7t + 0.842)\ \text{N},\\
m_u = 0.1\ \text{kg}: &\quad F_t = 148 m_u \sin(125.7t + 0.842) + 177 m_u \cos(125.7t + 0.842)\ \text{N},\\
m_u = 1\ \text{kg}: &\quad F_t = 1480 m_u \sin(125.7t + 0.842) + 1770 m_u \cos(125.7t + 0.842)\ \text{N}.
\end{aligned}$$

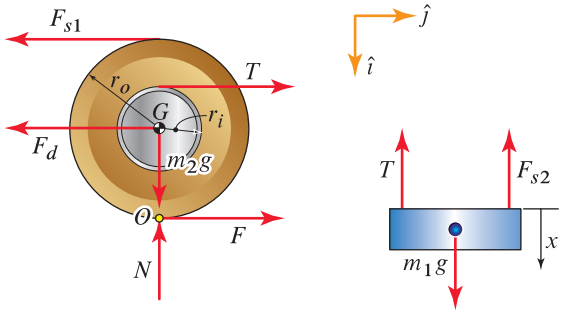
Problem 9.75

The system shown is released from rest when both springs are unstretched and $x = 0$. Neglecting the inertia of the pulley P and assuming that the disk rolls without slip, derive the equation of motion of the system in terms of x. Assume that point G is both the mass center of the disk and its geometric center. Treat the quantities k_1, k_2, c, m_1, m_2, and I_G as known, where I_G is the mass moment of inertia of the disk. Finally, assuming that the system is underdamped, derive an expression for the damped natural frequency of the system.



Solution

The figure at the right shows an FBD of the mass m_1 as well as an FBD of the disk of mass m_2, where T is the tension in the connecting cable, F is the friction force between the disk and the ground, F_{s1} is the force in spring k_1, F_{s2} is the force in spring k_2, and F_d is the force in the dashpot. Summing forces in the x direction on mass m_1, we obtain



$$\sum F_x: \quad m_1 g - T - F_{s2} = m_1 \ddot{x}. \qquad (1)$$

Summing moments about point O on the disk of mass m_2, we obtain

$$\sum M_O: \quad 2r_o F_{s1} + r_o F_d - (r_o + r_i)T = I_O \alpha_{\text{disk}}, \qquad (2)$$

where α_{disk} is the angular acceleration of the disk and $I_O = I_G + m_2 r_o^2$ via the parallel axis theorem. For the force laws, we have that

$$F_{s2} = k_2 x, \quad F_{s1} = k_1 \delta_{k1}, \quad \text{and} \quad F_d = c\dot{y}_G,$$

where δ_{k1} is the deflection of spring k_1 and $\dot{y}_G$ is velocity of point G. In addition, we need to relate the position of mass m_1, that is x, to δ_{k1} and $\dot{y}_G$. For small displacements, this is done via the relations

$$\begin{aligned}
\frac{\delta_{k1}}{2r_o} &= \frac{x}{r_o + r_i} \quad \Rightarrow \quad \delta_{k1} = \frac{2r_o}{r_o + r_i} x, \\
\frac{y_G}{r_o} &= \frac{x}{r_o + r_i} \quad \Rightarrow \quad \dot{y}_G = \frac{r_o}{r_o + r_i} \dot{x}, \\
\frac{y_G}{r_o} &= \frac{x}{r_o + r_i} \quad \Rightarrow \quad \ddot{y}_G = -r_o \alpha_{\text{disk}} = \frac{r_o}{r_o + r_i} \ddot{x} \quad \Rightarrow \quad \alpha_{\text{disk}} = -\frac{\ddot{x}}{r_o + r_i},
\end{aligned}$$

where we have used the fact that point O is the instantaneous center of rotation for the disk. Substituting the kinematics relations into the force laws and then substituting the force laws into Eqs. (1) and (2), we obtain

$$m_1 g - T - k_2 x = m_1 \ddot{x} \quad \text{and} \quad \frac{4r_o^2 k_1}{r_o + r_i} x + \frac{c r_o^2}{r_o + r_i} \dot{x} - (r_o + r_i)T = -\left(I_G + m_2 r_o^2\right) \frac{\ddot{x}}{r_o + r_i}.$$

Solving the first equation for T and substituting the result into the second, we obtain the equation of motion as

$$\frac{4r_o^2k_1}{r_o+r_i}x+\frac{cr_o^2}{r_o+r_i}\dot{x}-(r_o+r_i)(m_1g-k_2x-m_1\ddot{x})=-\left(I_G+m_2r_o^2\right)\frac{\ddot{x}}{r_o+r_i}$$

or, rearranging

$$\left[\frac{I_G+m_2r_o^2}{r_o+r_i}+m_1(r_o+r_i)\right]\ddot{x}+\left(\frac{cr_o^2}{r_o+r_i}\right)\dot{x}+\left[\frac{4r_o^2k_1}{r_o+r_i}+k_2(r_o+r_i)\right]x=(r_o+r_i)m_1g$$

$$\Rightarrow \quad \boxed{\left[I_G+m_2r_o^2+m_1(r_o+r_i)^2\right]\ddot{x}+cr_o^2\dot{x}+\left[4r_o^2k_1+k_2(r_o+r_i)^2\right]x=(r_o+r_i)^2m_1g.}$$

Notice that because we measured the position of mass m_1 from the *unstretched* state of the springs, we have the constant forcing term $(r_o+r_i)^2m_1g$ on the right-hand side of the equation of motion. Notice that the static solution (i.e., for $\ddot{x}=0$ and $\dot{x}=0$ is given by

$$x_{\text{st}}=\frac{(r_o+r_i)^2m_1g}{4r_o^2k_1+k_2(r_o+r_i)^2},$$

which is clearly a constant. As we have seen many times, if we were to measure x from this position instead, we would add this constant to x to obtain $x_{\text{new}}=x+x_{\text{st}}$, then the right-hand side would disappear, and the coefficients of $\ddot{x}$ and $\dot{x}$ would not change since $\dot{x}_{\text{new}}=\dot{x}$ and $\ddot{x}_{\text{new}}=\ddot{x}$.

The damped natural frequency is given by $\omega_d=\omega_n\sqrt{1-\zeta^2}$, where

$$\omega_n=\sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}}}=\sqrt{\frac{4r_o^2k_1+k_2(r_o+r_i)^2}{I_G+m_2r_o^2+m_1(r_o+r_i)^2}},$$

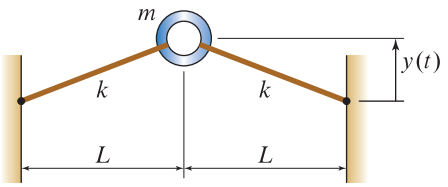
$$\zeta=\frac{c_{\text{eq}}}{2\sqrt{k_{\text{eq}}m_{\text{eq}}}}=\frac{cr_o^2}{2\sqrt{\left[4r_o^2k_1+k_2(r_o+r_i)^2\right]\left[I_G+m_2r_o^2+m_1(r_o+r_i)^2\right]}}.$$

Therefore, we can write the damped natural frequency as

$$\boxed{\omega_d=\sqrt{\frac{4r_o^2k_1+k_2(r_o+r_i)^2}{I_G+m_2r_o^2+m_1(r_o+r_i)^2}}\left[1-\frac{c^2r_o^4}{4\left[4r_o^2k_1+k_2(r_o+r_i)^2\right]\left[I_G+m_2r_o^2+m_1(r_o+r_i)^2\right]}\right]^{1/2}}$$

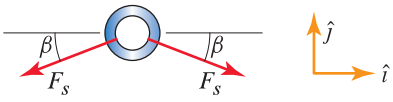
Problem 9.76

A ring of mass m is attached by two linear elastic cords to the vertical supports as shown. The cords have elastic constant k and unstretched length $L_0 < L$. Assuming that the pretension in the cords is large enough that the deflection of the cords due to the ring's weight can be neglected, find the nonlinear equation of motion for the mass m.



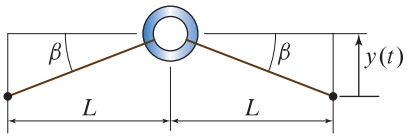
Solution

Referring to the FBD on the right, we can sum forces in the y direction to obtain



$$\sum F_y : \quad -2F_s \sin\beta = ma_y = m\ddot{y}, \tag{1}$$

where F_s is the force in each of the elastic cords. The force law for each cord is given by



$$F_s = k\left(\sqrt{y^2 + L^2} - L_0\right).$$

Kinematically, we can write β in terms of y if we note that

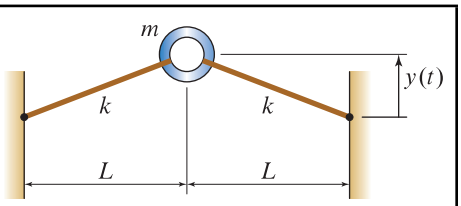
$$\sin\beta = \frac{y}{\sqrt{y^2 + L^2}}.$$

Substituting the kinematics and the force law into the balance law in Eq. (1), we obtain

$$m\ddot{y} + 2k\left(\sqrt{y^2 + L^2} - L_0\right)\frac{y}{\sqrt{y^2 + L^2}} = 0 \quad \Rightarrow \quad \boxed{m\ddot{y} + 2k\left(1 - \frac{L_0}{\sqrt{y^2 + L^2}}\right)y = 0.}$$

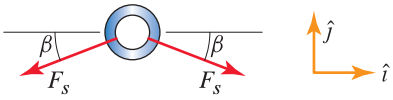
Problem 9.77

A ring of mass m is attached by two linear elastic cords to the vertical supports as shown. The cords have elastic constant k and unstretched length $L_0 < L$. Assuming that the pretension in the cords is large enough that the deflection of the cords due to the ring's weight can be neglected, use Newton's second law to find the linearized equation of motion about $y = 0$ for the mass m. In addition, determine the natural frequency of the ring's vibration.



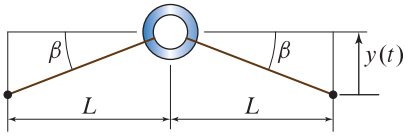
Solution

Referring to the FBD on the right, we can sum forces in the y direction to obtain



$$\sum F_y : \quad -2F_s \sin\beta = ma_y = m\ddot{y}, \tag{1}$$

where F_s is the force in each of the elastic cords. The force law for each cord is given by



$$F_s = k\left(\sqrt{y^2 + L^2} - L_0\right).$$

Kinematically, we can write β in terms of y if we note that

$$\sin\beta = \frac{y}{\sqrt{y^2 + L^2}}.$$

Substituting the kinematics and the force law into the balance law in Eq. (1), we obtain

$$m\ddot{y} + 2k\left(\sqrt{y^2 + L^2} - L_0\right)\frac{y}{\sqrt{y^2 + L^2}} = 0 \quad \Rightarrow \quad m\ddot{y} + 2k\left(1 - \frac{L_0}{\sqrt{y^2 + L^2}}\right)y = 0.$$

Looking at the second term in parentheses, we note that using a Taylor series expansion about $y = 0$ (this is sometimes called a binomial expansion) we can write it as

$$\frac{L_0}{\sqrt{y^2 + L^2}} = \frac{L_0}{L}\frac{1}{\sqrt{1 + (y/L)^2}} = \frac{L_0}{L}\left[1 - \frac{1}{2}\left(\frac{y}{L}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} + 1\right)}{2!}\left(\frac{y}{L}\right)^4 - \ldots\right] \approx \frac{L_0}{L},$$

where we obtained the last approximation by noting that $y \ll L$ and so $(y/L)^2$ and higher powers can be ignored. Therefore the linearized equation of motion becomes

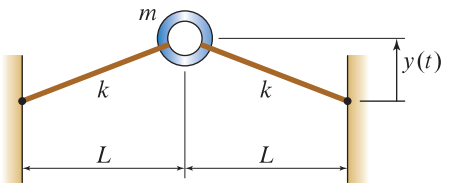
$$\boxed{m\ddot{y} + 2k\left(1 - \frac{L_0}{L}\right)y = 0.}$$

The natural frequency of the ring's vibration is given by $\omega_n = \sqrt{k_{\text{eq}}/m_{\text{eq}}}$, and so we can write

$$k_{\text{eq}} = 2k\left(1 - \frac{L_0}{L}\right), \quad m_{\text{eq}} = m, \quad \text{and} \quad \boxed{\omega_n = \sqrt{\frac{2k}{m}\left(1 - \frac{L_0}{L}\right)}.}$$

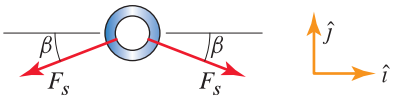
Problem 9.78

Solve Prob. 9.77 by finding the linearized equations of motion via the energy method.



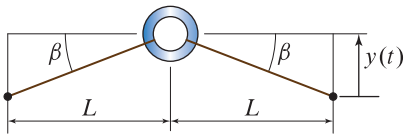
Solution

Referring to the FBD at the right, we see that only the forces due to the elastic cords do work on the ring. Since they are linear elastic, the system is conservative and we can apply the energy method to find the equation of motion. The kinetic energy can be written as



$$T = \tfrac{1}{2} m \dot{y}^2.$$

In addition, the potential energy of the two elastic can be written as



$$V = 2\left[\tfrac{1}{2}k\left(\sqrt{y^2+L^2}-L_0\right)^2\right] = k\left(y^2+L^2-2L_0\sqrt{y^2+L^2}+L_0^2\right).$$

Since V is nonlinear, to apply the energy method, we need to expand it in a Taylor series about $y = 0$ and keep up through quadratic terms (see Section 9.1). Rewriting V and expanding in a Taylor series, we obtain

$$\begin{aligned}
V &= k\left(L^2+L_0^2\right) + ky^2 - 2kL_0\sqrt{y^2+L^2} \\
&= k\left(L^2+L_0^2\right) + ky^2 \\
&\quad - 2kL_0\left\{L + \frac{y}{\sqrt{y^2+L^2}}\bigg|_{y=0} y + \frac{1}{2}\left[\frac{1}{\sqrt{y^2+L^2}} - \frac{y^2}{\left(y^2+L^2\right)^{3/2}}\right]\bigg|_{y=0} y^2 + \cdots\right\} \\
&= k\left(L^2+L_0^2\right) + ky^2 - 2kL_0\left(L + \frac{y^2}{2L}\right) + \cdots \\
&\approx \underbrace{k\left(L^2+L_0^2-2L_0L\right)}_{\text{constant}} + k\left(1-\frac{L_0}{L}\right)y^2
\end{aligned}$$

where we have used the Taylor series expansion about $y = 0$, which is given by $f(y) = f(0) + f'(0)y + \frac{1}{2}f''(0)y^2 + \ldots$ and where $f(y) = \sqrt{y^2+L^2}$. Notice that the first term in V is a constant and so we can ignore it when applying the energy method. Doing so, we obtain

$$\frac{d}{dt}(T+V) = m\dot{y}\ddot{y} + 2k\left(1-\frac{L_0}{L}\right)y\dot{y} = 0 \quad \Rightarrow \quad \boxed{m\ddot{y} + 2k\left(1-\frac{L_0}{L}\right)y = 0.}$$