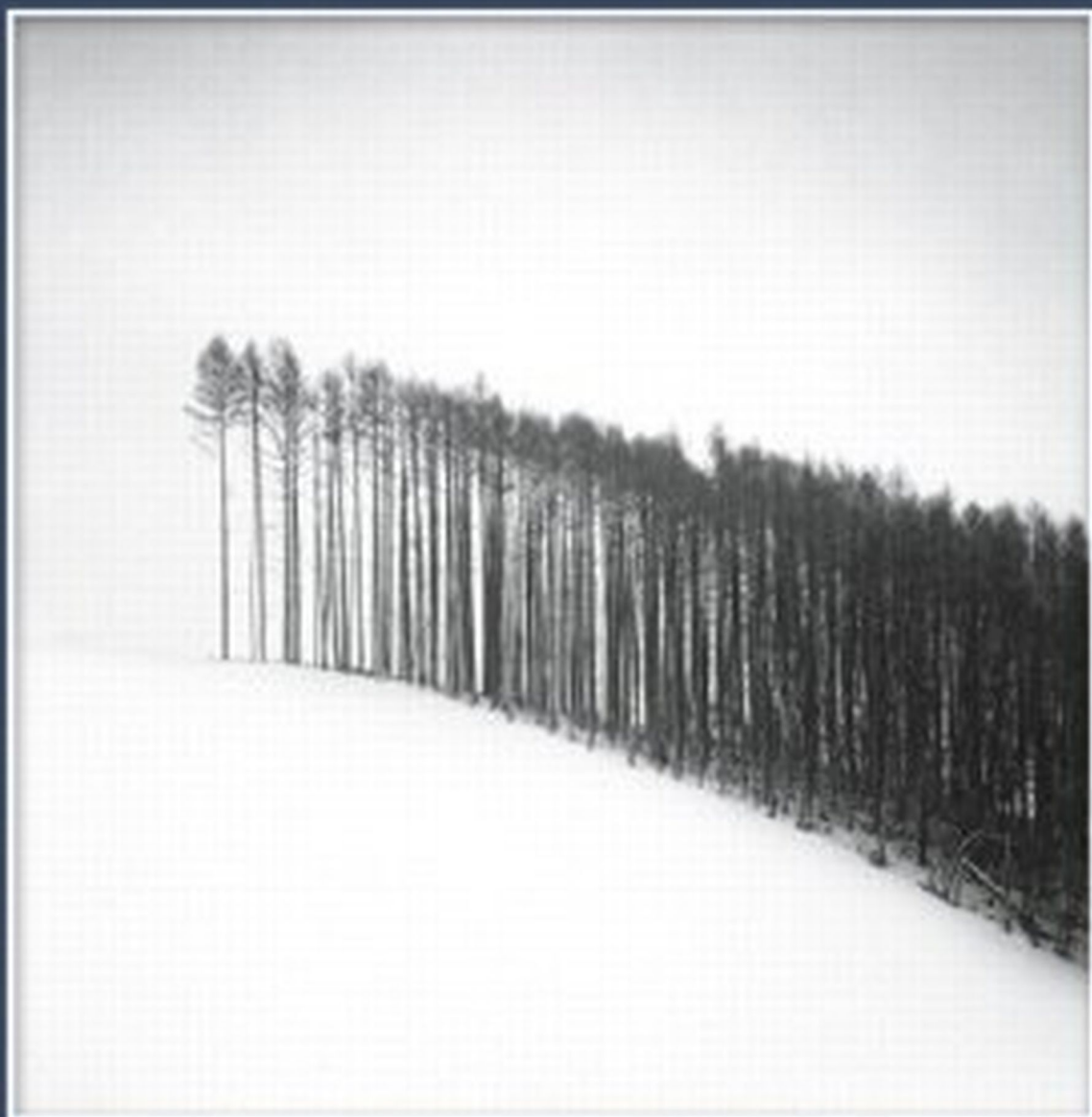


INSTRUCTOR'S SOLUTIONS MANUAL
MULTIVARIABLE



THOMAS'
CALCULUS

Twelfth Edition

Multivariable

INSTRUCTOR'S SOLUTIONS MANUAL MULTIVARIABLE

WILLIAM ARDIS
Collin County Community College

THOMAS' CALCULUS TWELFTH EDITION

BASED ON THE ORIGINAL WORK BY

George B. Thomas, Jr.
Massachusetts Institute of Technology

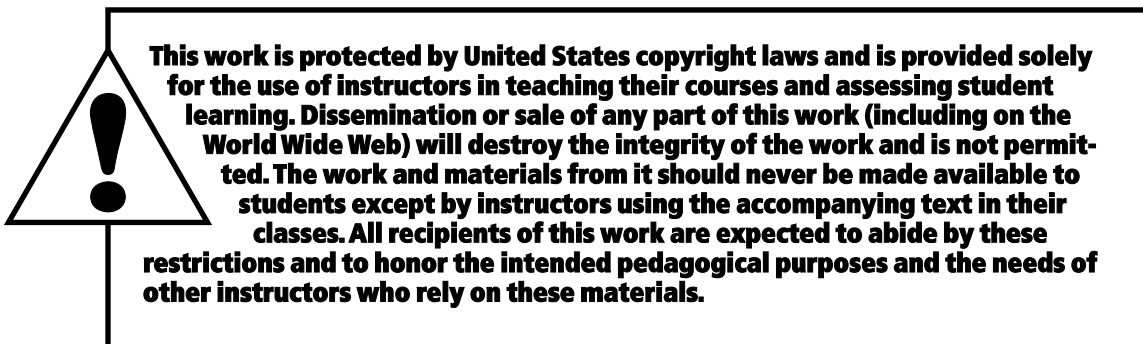
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PREFACE TO THE INSTRUCTOR

This Instructor's Solutions Manual contains the solutions to every exercise in the 12th Edition of THOMAS' CALCULUS by Maurice Weir and Joel Hass, including the Computer Algebra System (CAS) exercises. The corresponding Student's Solutions Manual omits the solutions to the even-numbered exercises as well as the solutions to the CAS exercises (because the CAS command templates would give them all away).

In addition to including the solutions to all of the new exercises in this edition of Thomas, we have carefully revised or rewritten every solution which appeared in previous solutions manuals to ensure that each solution

- conforms exactly to the methods, procedures and steps presented in the text
- is mathematically correct
- includes all of the steps necessary so a typical calculus student can follow the logical argument and algebra
- includes a graph or figure whenever called for by the exercise, or if needed to help with the explanation
- is formatted in an appropriate style to aid in its understanding

Every CAS exercise is solved in both the MAPLE and *MATHEMATICA* computer algebra systems. A template showing an example of the CAS commands needed to execute the solution is provided for each exercise type. Similar exercises within the text grouping require a change only in the input function or other numerical input parameters associated with the problem (such as the interval endpoints or the number of iterations).

For more information about other resources available with Thomas' Calculus, visit <http://pearsonhighered.com>.

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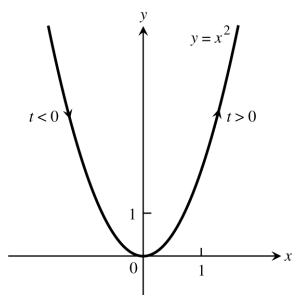
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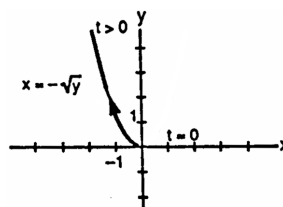
CHAPTER 11 PARAMETRIC EQUATIONS AND POLAR COORDINATES

11.1 PARAMETRIZATIONS OF PLANE CURVES

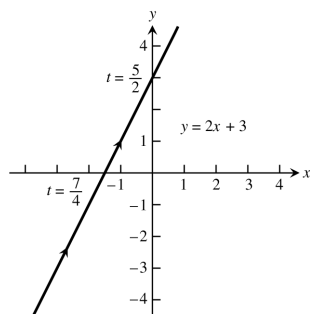
1. $x = 3t, y = 9t^2, -\infty < t < \infty \Rightarrow y = x^2$



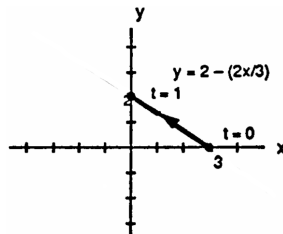
2. $x = -\sqrt{t}, y = t, t \geq 0 \Rightarrow x = -\sqrt{y}$
or $y = x^2, x \leq 0$



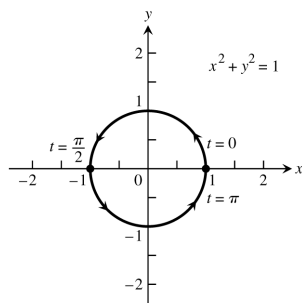
3. $x = 2t - 5, y = 4t - 7, -\infty < t < \infty$
 $\Rightarrow x + 5 = 2t \Rightarrow 2(x + 5) = 4t$
 $\Rightarrow y = 2(x + 5) - 7 \Rightarrow y = 2x + 3$



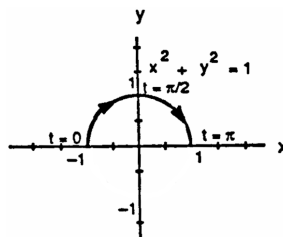
4. $x = 3 - 3t, y = 2t, 0 \leq t \leq 1 \Rightarrow \frac{y}{2} = t$
 $\Rightarrow x = 3 - 3\left(\frac{y}{2}\right) \Rightarrow 2x = 6 - 3y$
 $\Rightarrow y = 2 - \frac{2}{3}x, 0 \leq x \leq 3$



5. $x = \cos 2t, y = \sin 2t, 0 \leq t \leq \pi$
 $\Rightarrow \cos^2 2t + \sin^2 2t = 1 \Rightarrow x^2 + y^2 = 1$

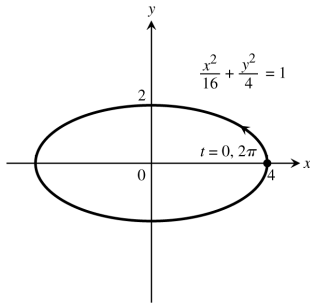


6. $x = \cos(\pi - t), y = \sin(\pi - t), 0 \leq t \leq \pi$
 $\Rightarrow \cos^2(\pi - t) + \sin^2(\pi - t) = 1$
 $\Rightarrow x^2 + y^2 = 1, y \geq 0$



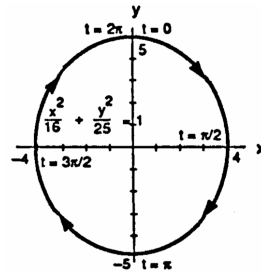
7. $x = 4 \cos t, y = 2 \sin t, 0 \leq t \leq 2\pi$

$$\Rightarrow \frac{16 \cos^2 t}{16} + \frac{4 \sin^2 t}{4} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$



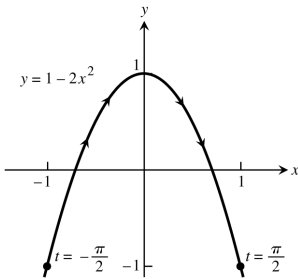
8. $x = 4 \sin t, y = 5 \cos t, 0 \leq t \leq 2\pi$

$$\Rightarrow \frac{16 \sin^2 t}{16} + \frac{25 \cos^2 t}{25} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{25} = 1$$



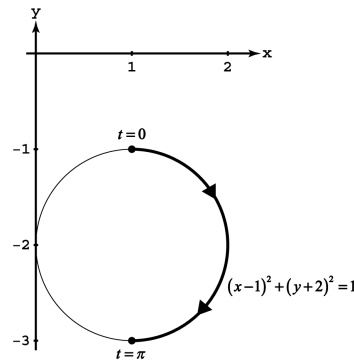
9. $x = \sin t, y = \cos 2t, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$$\Rightarrow y = \cos 2t = 1 - 2\sin^2 t \Rightarrow y = 1 - 2x^2$$



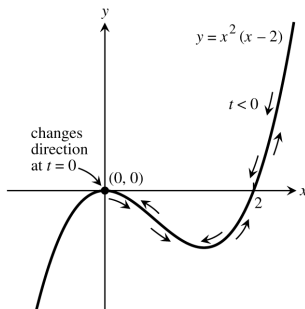
10. $x = 1 + \sin t, y = \cos t - 2, 0 \leq t \leq \pi$

$$\Rightarrow \sin^2 t + \cos^2 t = 1 \Rightarrow (x-1)^2 + (y+2)^2 = 1$$



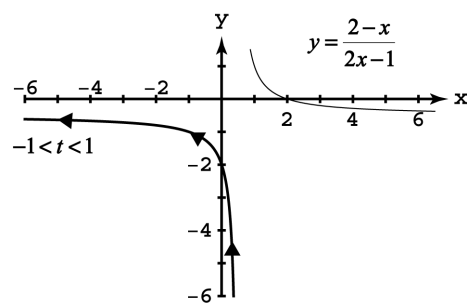
11. $x = t^2, y = t^6 - 2t^4, -\infty < t < \infty$

$$\Rightarrow y = (t^2)^3 - 2(t^2)^2 \Rightarrow y = x^3 - 2x^2$$



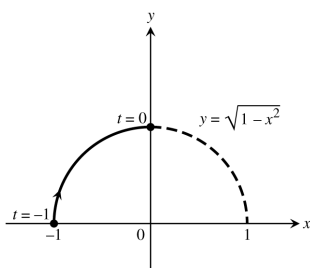
12. $x = \frac{t}{t-1}, y = \frac{t-2}{t+1}, -1 < t < 1$

$$\Rightarrow t = \frac{x}{x-1} \Rightarrow y = \frac{2-x}{2x-1}$$



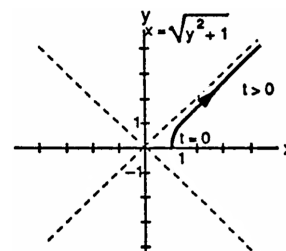
13. $x = t, y = \sqrt{1-t^2}, -1 \leq t \leq 0$

$$\Rightarrow y = \sqrt{1-x^2}$$

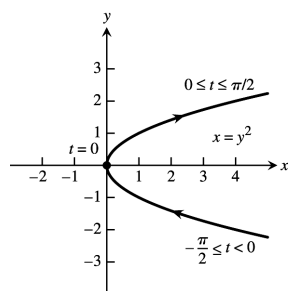


14. $x = \sqrt{t+1}, y = \sqrt{t}, t \geq 0$

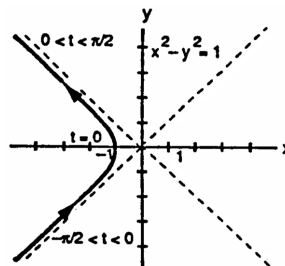
$$\Rightarrow y^2 = t \Rightarrow x = \sqrt{y^2+1}, y \geq 0$$



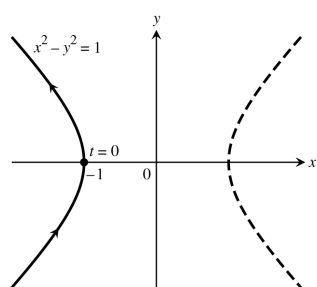
15. $x = \sec^2 t - 1, y = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}$
 $\Rightarrow \sec^2 t - 1 = \tan^2 t \Rightarrow x = y^2$



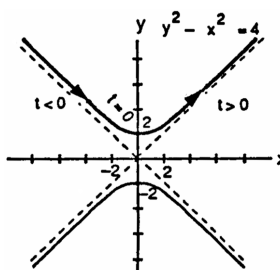
16. $x = -\sec t, y = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}$
 $\Rightarrow \sec^2 t - \tan^2 t = 1 \Rightarrow x^2 - y^2 = 1$



17. $x = -\cosh t, y = \sinh t, -\infty < t < \infty$
 $\Rightarrow \cosh^2 t - \sinh^2 t = 1 \Rightarrow x^2 - y^2 = 1$



18. $x = 2 \sinh t, y = 2 \cosh t, -\infty < t < \infty$
 $\Rightarrow 4 \cosh^2 t - 4 \sinh^2 t = 4 \Rightarrow y^2 - x^2 = 4$



19. (a) $x = a \cos t, y = -a \sin t, 0 \leq t \leq 2\pi$
 (b) $x = a \cos t, y = a \sin t, 0 \leq t \leq 2\pi$
 (c) $x = a \cos t, y = -a \sin t, 0 \leq t \leq 4\pi$
 (d) $x = a \cos t, y = a \sin t, 0 \leq t \leq 4\pi$

20. (a) $x = a \sin t, y = b \cos t, \frac{\pi}{2} \leq t \leq \frac{5\pi}{2}$
 (b) $x = a \cos t, y = b \sin t, 0 \leq t \leq 2\pi$
 (c) $x = a \sin t, y = b \cos t, \frac{\pi}{2} \leq t \leq \frac{9\pi}{2}$
 (d) $x = a \cos t, y = b \sin t, 0 \leq t \leq 4\pi$

21. Using $(-1, -3)$ we create the parametric equations $x = -1 + at$ and $y = -3 + bt$, representing a line which goes through $(-1, -3)$ at $t = 0$. We determine a and b so that the line goes through $(4, 1)$ when $t = 1$. Since $4 = -1 + a \Rightarrow a = 5$. Since $1 = -3 + b \Rightarrow b = 4$. Therefore, one possible parameterization is $x = -1 + 5t, y = -3 + 4t, 0 \leq t \leq 1$.
22. Using $(-1, 3)$ we create the parametric equations $x = -1 + at$ and $y = 3 + bt$, representing a line which goes through $(-1, 3)$ at $t = 0$. We determine a and b so that the line goes through $(3, -2)$ when $t = 1$. Since $3 = -1 + a \Rightarrow a = 4$. Since $-2 = 3 + b \Rightarrow b = -5$. Therefore, one possible parameterization is $x = -1 + 4t, y = 3 - 5t, 0 \leq t \leq 1$.
23. The lower half of the parabola is given by $x = y^2 + 1$ for $y \leq 0$. Substituting t for y , we obtain one possible parameterization $x = t^2 + 1, y = t, t \leq 0$.
24. The vertex of the parabola is at $(-1, -1)$, so the left half of the parabola is given by $y = x^2 + 2x$ for $x \leq -1$. Substituting t for x , we obtain one possible parameterization: $x = t, y = t^2 + 2t, t \leq -1$.
25. For simplicity, we assume that x and y are linear functions of t and that the point (x, y) starts at $(2, 3)$ for $t = 0$ and passes through $(-1, -1)$ at $t = 1$. Then $x = f(t)$, where $f(0) = 2$ and $f(1) = -1$. Since slope $= \frac{\Delta x}{\Delta t} = \frac{-1-2}{1-0} = -3, x = f(t) = -3t + 2 = 2 - 3t$. Also, $y = g(t)$, where $g(0) = 3$ and $g(1) = -1$. Since slope $= \frac{\Delta y}{\Delta t} = \frac{-1-3}{1-0} = -4, y = g(t) = -4t + 3 = 3 - 4t$. One possible parameterization is: $x = 2 - 3t, y = 3 - 4t, t \geq 0$.

26. For simplicity, we assume that x and y are linear functions of t and that the point (x, y) starts at $(-1, 2)$ for $t = 0$ and passes through $(0, 0)$ at $t = 1$. Then $x = f(t)$, where $f(0) = -1$ and $f(1) = 0$.
 Since slope $= \frac{\Delta x}{\Delta t} = \frac{0 - (-1)}{1 - 0} = 1$, $x = f(t) = 1t + (-1) = -1 + t$. Also, $y = g(t)$, where $g(0) = 2$ and $g(1) = 0$.
 Since slope $= \frac{\Delta y}{\Delta t} = \frac{0 - 2}{1 - 0} = -2$, $y = g(t) = -2t + 2 = 2 - 2t$.
 One possible parameterization is: $x = -1 + t$, $y = 2 - 2t$, $t \geq 0$.
27. Since we only want the top half of a circle, $y \geq 0$, so let $x = 2\cos t$, $y = 2|\sin t|$, $0 \leq t \leq 4\pi$
28. Since we want x to stay between -3 and 3 , let $x = 3 \sin t$, then $y = (3 \sin t)^2 = 9\sin^2 t$, thus $x = 3 \sin t$, $y = 9\sin^2 t$, $0 \leq t < \infty$
29. $x^2 + y^2 = a^2 \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$; let $t = \frac{dy}{dx} \Rightarrow -\frac{x}{y} = t \Rightarrow x = -yt$. Substitution yields
 $y^2 t^2 + y^2 = a^2 \Rightarrow y = \frac{a}{\sqrt{1+t^2}}$ and $x = \frac{-at}{\sqrt{1+t^2}}$, $-\infty < t < \infty$
30. In terms of θ , parametric equations for the circle are $x = a \cos \theta$, $y = a \sin \theta$, $0 \leq \theta < 2\pi$. Since $\theta = \frac{s}{a}$, the arc length parametrizations are: $x = a \cos \frac{s}{a}$, $y = a \sin \frac{s}{a}$, and $0 \leq \frac{s}{a} < 2\pi \Rightarrow 0 \leq s \leq 2\pi a$ is the interval for s .
31. Drop a vertical line from the point (x, y) to the x -axis, then θ is an angle in a right triangle, and from trigonometry we know that $\tan \theta = \frac{y}{x} \Rightarrow y = x \tan \theta$. The equation of the line through $(0, 2)$ and $(4, 0)$ is given by $y = -\frac{1}{2}x + 2$. Thus $x \tan \theta = -\frac{1}{2}x + 2 \Rightarrow x = \frac{4}{2 \tan \theta + 1}$ and $y = \frac{4 \tan \theta}{2 \tan \theta + 1}$ where $0 \leq \theta < \frac{\pi}{2}$.
32. Drop a vertical line from the point (x, y) to the x -axis, then θ is an angle in a right triangle, and from trigonometry we know that $\tan \theta = \frac{y}{x} \Rightarrow y = x \tan \theta$. Since $y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow (x \tan \theta)^2 = x \Rightarrow x = \cot^2 \theta \Rightarrow y = \cot \theta$ where $0 < \theta \leq \frac{\pi}{2}$.
33. The equation of the circle is given by $(x - 2)^2 + y^2 = 1$. Drop a vertical line from the point (x, y) on the circle to the x -axis, then θ is an angle in a right triangle. So that we can start at $(1, 0)$ and rotate in a clockwise direction, let $x = 2 - \cos \theta$, $y = \sin \theta$, $0 \leq \theta \leq 2\pi$.
34. Drop a vertical line from the point (x, y) to the x -axis, then θ is an angle in a right triangle, whose height is y and whose base is $x + 2$. By trigonometry we have $\tan \theta = \frac{y}{x+2} \Rightarrow y = (x + 2) \tan \theta$. The equation of the circle is given by $x^2 + y^2 = 1 \Rightarrow x^2 + ((x + 2) \tan \theta)^2 = 1 \Rightarrow x^2 \sec^2 \theta + 4x \tan^2 \theta + 4 \tan^2 \theta - 1 = 0$. Solving for x we obtain

$$x = \frac{-4 \tan^2 \theta \pm \sqrt{(4 \tan^2 \theta)^2 - 4 \sec^2 \theta (4 \tan^2 \theta - 1)}}{2 \sec^2 \theta} = \frac{-4 \tan^2 \theta \pm 2 \sqrt{1 - 3 \tan^2 \theta}}{2 \sec^2 \theta} = -2 \sin^2 \theta \pm \cos \theta \sqrt{\cos^2 \theta - 3 \sin^2 \theta}$$

$$= -2 + 2 \cos^2 \theta \pm \cos \theta \sqrt{4 \cos^2 \theta - 3} \text{ and } y = \left(-2 + 2 \cos^2 \theta \pm \cos \theta \sqrt{4 \cos^2 \theta - 3} + 2 \right) \tan \theta$$

$$= 2 \sin \theta \cos \theta \pm \sin \theta \sqrt{4 \cos^2 \theta - 3}. \text{ Since we only need to go from } (1, 0) \text{ to } (0, 1), \text{ let}$$

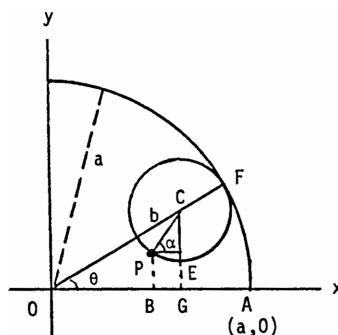
$$x = -2 + 2 \cos^2 \theta + \cos \theta \sqrt{4 \cos^2 \theta - 3}, y = 2 \sin \theta \cos \theta + \sin \theta \sqrt{4 \cos^2 \theta - 3}, 0 \leq \theta \leq \tan^{-1} \left(\frac{1}{2} \right).$$
 To obtain the upper limit for θ , note that $x = 0$ and $y = 1$, using $y = (x + 2) \tan \theta \Rightarrow 1 = 2 \tan \theta \Rightarrow \theta = \tan^{-1} \left(\frac{1}{2} \right)$.
35. Extend the vertical line through A to the x -axis and let C be the point of intersection. Then $OC = AQ = x$ and $\tan t = \frac{2}{OC} = \frac{2}{x} \Rightarrow x = \frac{2}{\tan t} = 2 \cot t$; $\sin t = \frac{2}{OA} \Rightarrow OA = \frac{2}{\sin t}$; and $(AB)(OA) = (AQ)^2 \Rightarrow AB \left(\frac{2}{\sin t} \right) = x^2$

$$\Rightarrow AB \left(\frac{2}{\sin t} \right) = \left(\frac{2}{\tan t} \right)^2 \Rightarrow AB = \frac{2 \sin t}{\tan^2 t}. \text{ Next } y = 2 - AB \sin t \Rightarrow y = 2 - \left(\frac{2 \sin t}{\tan^2 t} \right) \sin t =$$

$$2 - \frac{2 \sin^2 t}{\tan^2 t} = 2 - 2 \cos^2 t = 2 \sin^2 t. \text{ Therefore let } x = 2 \cot t \text{ and } y = 2 \sin^2 t, 0 < t < \pi.$$

36. Arc PF = Arc AF since each is the distance rolled and

$$\begin{aligned} \frac{\text{Arc PF}}{b} &= \angle FCP \Rightarrow \text{Arc PF} = b(\angle FCP); \frac{\text{Arc AF}}{a} = \theta \\ \Rightarrow \text{Arc AF} &= a\theta \Rightarrow a\theta = b(\angle FCP) \Rightarrow \angle FCP = \frac{a}{b}\theta; \\ \angle OCG &= \frac{\pi}{2} - \theta; \angle OCG = \angle OCP + \angle PCE \\ &= \angle OCP + \left(\frac{\pi}{2} - \alpha\right). \text{ Now } \angle OCP = \pi - \angle FCP \\ &= \pi - \frac{a}{b}\theta. \text{ Thus } \angle OCG = \pi - \frac{a}{b}\theta + \frac{\pi}{2} - \alpha \Rightarrow \frac{\pi}{2} - \theta \\ &= \pi - \frac{a}{b}\theta + \frac{\pi}{2} - \alpha \Rightarrow \alpha = \pi - \frac{a}{b}\theta + \theta = \pi - \left(\frac{a-b}{b}\right)\theta. \end{aligned}$$



$$\begin{aligned} \text{Then } x &= OG - BG = OG - PE = (a - b) \cos \theta - b \cos \alpha = (a - b) \cos \theta - b \cos \left(\pi - \frac{a-b}{b}\theta\right) \\ &= (a - b) \cos \theta + b \cos \left(\frac{a-b}{b}\theta\right). \text{ Also } y = EG = CG - CE = (a - b) \sin \theta - b \sin \alpha \\ &= (a - b) \sin \theta - b \sin \left(\pi - \frac{a-b}{b}\theta\right) = (a - b) \sin \theta - b \sin \left(\frac{a-b}{b}\theta\right). \text{ Therefore} \\ x &= (a - b) \cos \theta + b \cos \left(\frac{a-b}{b}\theta\right) \text{ and } y = (a - b) \sin \theta - b \sin \left(\frac{a-b}{b}\theta\right). \end{aligned}$$

$$\text{If } b = \frac{a}{4}, \text{ then } x = \left(a - \frac{a}{4}\right) \cos \theta + \frac{a}{4} \cos \left(\frac{a - (\frac{a}{4})}{(\frac{a}{4})}\theta\right)$$

$$\begin{aligned} &= \frac{3a}{4} \cos \theta + \frac{a}{4} \cos 3\theta = \frac{3a}{4} \cos \theta + \frac{a}{4} (\cos \theta \cos 2\theta - \sin \theta \sin 2\theta) \\ &= \frac{3a}{4} \cos \theta + \frac{a}{4} ((\cos \theta)(\cos^2 \theta - \sin^2 \theta) - (\sin \theta)(2 \sin \theta \cos \theta)) \\ &= \frac{3a}{4} \cos \theta + \frac{a}{4} \cos^3 \theta - \frac{a}{4} \cos \theta \sin^2 \theta - \frac{2a}{4} \sin^2 \theta \cos \theta \\ &= \frac{3a}{4} \cos \theta + \frac{a}{4} \cos^3 \theta - \frac{3a}{4} (\cos \theta)(1 - \cos^2 \theta) = a \cos^3 \theta; \end{aligned}$$

$$\begin{aligned} y &= \left(a - \frac{a}{4}\right) \sin \theta - \frac{a}{4} \sin \left(\frac{a - (\frac{a}{4})}{(\frac{a}{4})}\theta\right) = \frac{3a}{4} \sin \theta - \frac{a}{4} \sin 3\theta = \frac{3a}{4} \sin \theta - \frac{a}{4} (\sin \theta \cos 2\theta + \cos \theta \sin 2\theta) \\ &= \frac{3a}{4} \sin \theta - \frac{a}{4} ((\sin \theta)(\cos^2 \theta - \sin^2 \theta) + (\cos \theta)(2 \sin \theta \cos \theta)) \\ &= \frac{3a}{4} \sin \theta - \frac{a}{4} \sin \theta \cos^2 \theta + \frac{a}{4} \sin^3 \theta - \frac{2a}{4} \cos^2 \theta \sin \theta \\ &= \frac{3a}{4} \sin \theta - \frac{3a}{4} \sin \theta \cos^2 \theta + \frac{a}{4} \sin^3 \theta \\ &= \frac{3a}{4} \sin \theta - \frac{3a}{4} (\sin \theta)(1 - \sin^2 \theta) + \frac{a}{4} \sin^3 \theta = a \sin^3 \theta. \end{aligned}$$

37. Draw line AM in the figure and note that $\angle AMO$ is a right angle since it is an inscribed angle which spans the diameter of a circle. Then $AN^2 = MN^2 + AM^2$. Now, $OA = a$,

$$\frac{AN}{a} = \tan t, \text{ and } \frac{AM}{a} = \sin t. \text{ Next } MN = OP$$

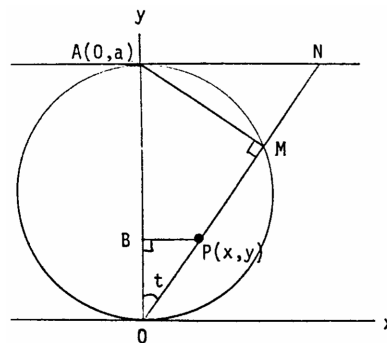
$$\Rightarrow OP^2 = AN^2 - AM^2 = a^2 \tan^2 t - a^2 \sin^2 t$$

$$\Rightarrow OP = \sqrt{a^2 \tan^2 t - a^2 \sin^2 t}$$

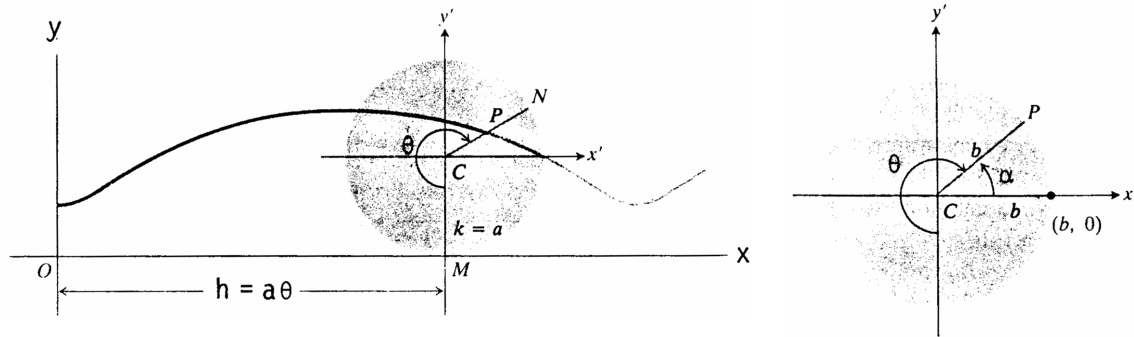
$$= (a \sin t) \sqrt{\sec^2 t - 1} = \frac{a \sin^3 t}{\cos t}. \text{ In triangle BPO,}$$

$$x = OP \sin t = \frac{a \sin^3 t}{\cos t} = a \sin^2 t \tan t \text{ and}$$

$$y = OP \cos t = a \sin^2 t \Rightarrow x = a \sin^2 t \tan t \text{ and } y = a \sin^2 t.$$



38. Let the x -axis be the line the wheel rolls along with the y -axis through a low point of the trochoid (see the accompanying figure).

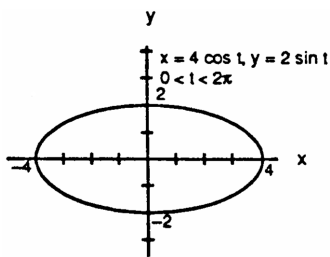


Let θ denote the angle through which the wheel turns. Then $h = a\theta$ and $k = a$. Next introduce $x'y'$ -axes parallel to the xy -axes and having their origin at the center C of the wheel. Then $x' = b \cos \alpha$ and $y' = b \sin \alpha$, where $\alpha = \frac{3\pi}{2} - \theta$. It follows that $x' = b \cos(\frac{3\pi}{2} - \theta) = -b \sin \theta$ and $y' = b \sin(\frac{3\pi}{2} - \theta) = -b \cos \theta \Rightarrow x = h + x' = a\theta - b \sin \theta$ and $y = k + y' = a - b \cos \theta$ are parametric equations of the trochoid.

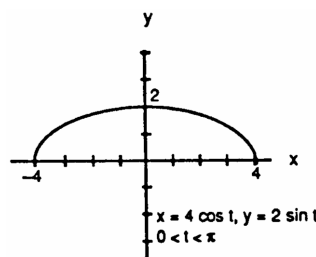
39. $D = \sqrt{(x-2)^2 + (y-\frac{1}{2})^2} \Rightarrow D^2 = (x-2)^2 + (y-\frac{1}{2})^2 = (t-2)^2 + (t^2 - \frac{1}{2})^2 \Rightarrow D^2 = t^4 - 4t + \frac{17}{4}$
 $\Rightarrow \frac{d(D^2)}{dt} = 4t^3 - 4 = 0 \Rightarrow t = 1$. The second derivative is always positive for $t \neq 0 \Rightarrow t = 1$ gives a local minimum for D^2 (and hence D) which is an absolute minimum since it is the only extremum $\Rightarrow$ the closest point on the parabola is $(1, 1)$.

40. $D = \sqrt{(2 \cos t - \frac{3}{4})^2 + (\sin t - 0)^2} \Rightarrow D^2 = (2 \cos t - \frac{3}{4})^2 + \sin^2 t \Rightarrow \frac{d(D^2)}{dt}$
 $= 2(2 \cos t - \frac{3}{4})(-2 \sin t) + 2 \sin t \cos t = (-2 \sin t)(3 \cos t - \frac{3}{2}) = 0 \Rightarrow -2 \sin t = 0$ or $3 \cos t - \frac{3}{2} = 0$
 $\Rightarrow t = 0, \pi$ or $t = \frac{\pi}{3}, \frac{5\pi}{3}$. Now $\frac{d^2(D^2)}{dt^2} = -6 \cos^2 t + 3 \cos t + 6 \sin^2 t$ so that $\frac{d^2(D^2)}{dt^2}(0) = -3 \Rightarrow$ relative maximum, $\frac{d^2(D^2)}{dt^2}(\pi) = -9 \Rightarrow$ relative maximum, $\frac{d^2(D^2)}{dt^2}(\frac{\pi}{3}) = \frac{9}{2} \Rightarrow$ relative minimum, and $\frac{d^2(D^2)}{dt^2}(\frac{5\pi}{3}) = \frac{9}{2} \Rightarrow$ relative minimum. Therefore both $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ give points on the ellipse closest to the point $(\frac{3}{4}, 0) \Rightarrow (1, \frac{\sqrt{3}}{2})$ and $(1, -\frac{\sqrt{3}}{2})$ are the desired points.

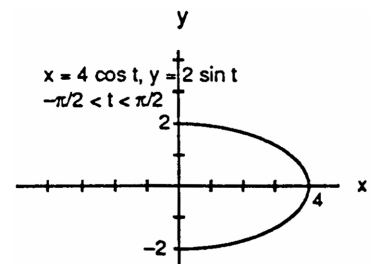
41. (a)



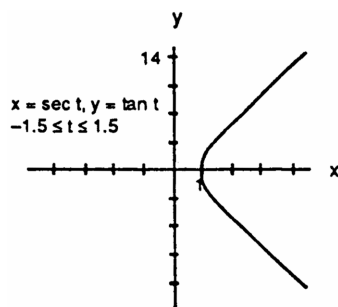
(b)



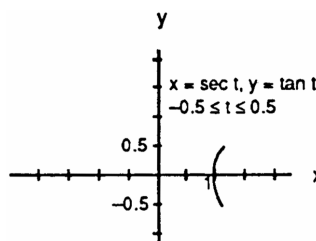
(c)



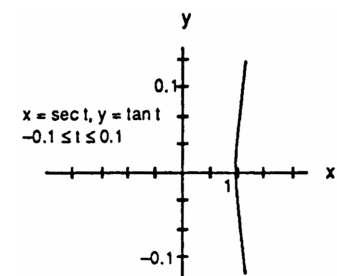
42. (a)



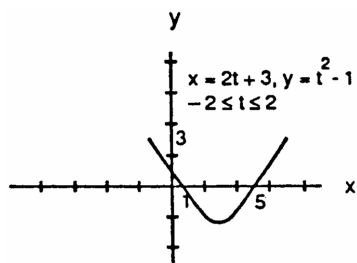
(b)



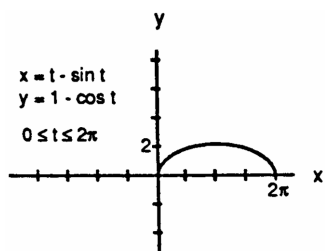
(c)



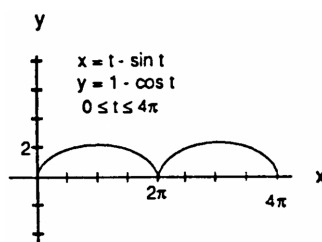
43.



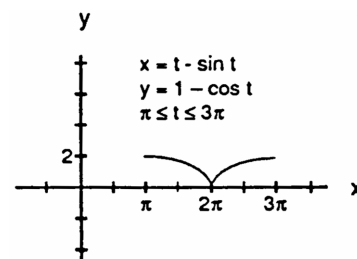
44. (a)



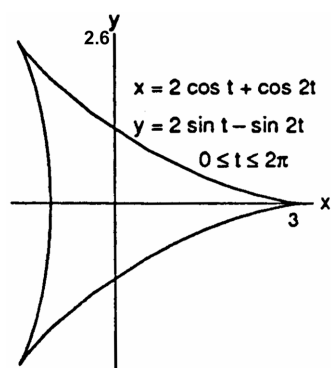
(b)



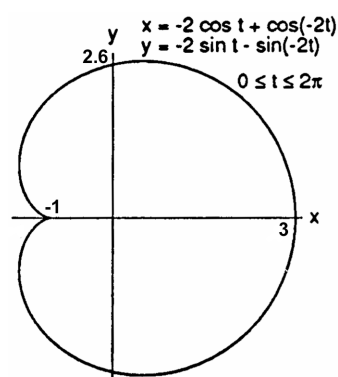
(c)



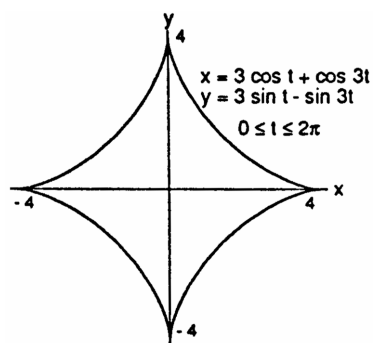
45. (a)



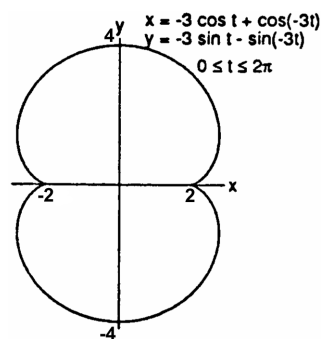
(b)



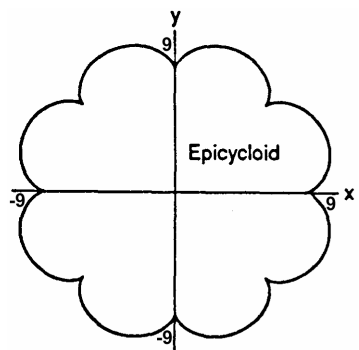
46. (a)



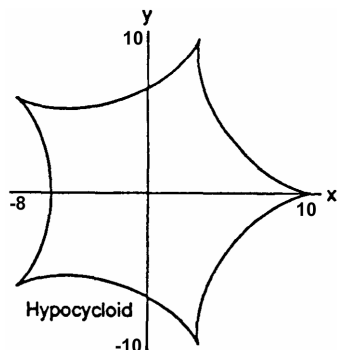
(b)



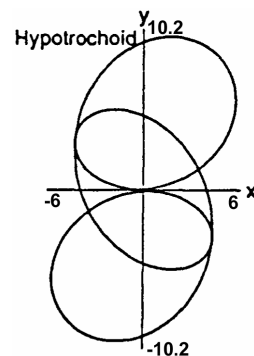
47. (a)



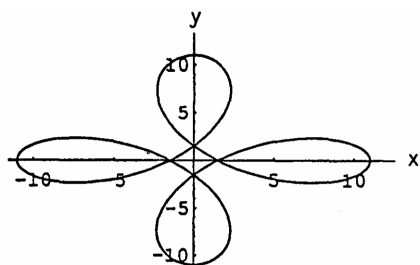
(b)



(c)

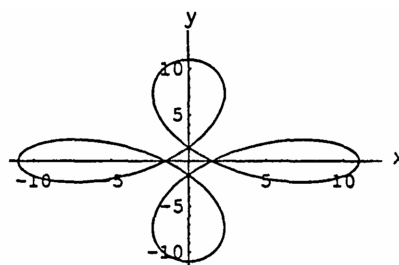


48. (a)



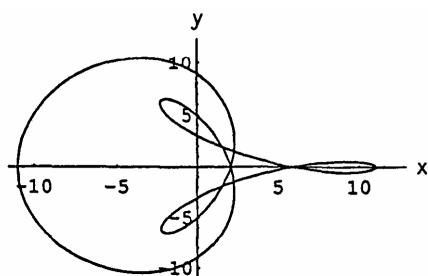
$$x = 6 \cos t + 5 \cos 3t, \quad y = 6 \sin t - 5 \sin 3t, \quad 0 \leq t \leq 2\pi$$

(b)



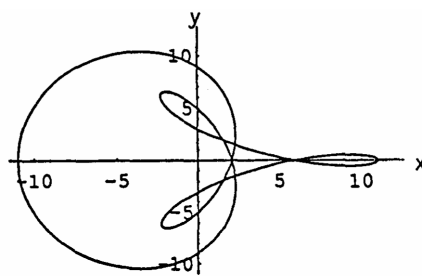
$$x = 6 \cos 2t + 5 \cos 6t, \quad y = 6 \sin 2t - 5 \sin 6t, \quad 0 \leq t \leq \pi$$

(c)



$$x = 6 \cos t + 5 \cos 3t, \quad y = 6 \sin 2t - 5 \sin 3t, \quad 0 \leq t \leq 2\pi$$

(d)



$$x = 6 \cos 2t + 5 \cos 6t, \quad y = 6 \sin 4t - 5 \sin 6t, \quad 0 \leq t \leq \pi$$

11.2 CALCULUS WITH PARAMETRIC CURVES

- $t = \frac{\pi}{4} \Rightarrow x = 2 \cos \frac{\pi}{4} = \sqrt{2}, y = 2 \sin \frac{\pi}{4} = \sqrt{2}; \frac{dx}{dt} = -2 \sin t, \frac{dy}{dt} = 2 \cos t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos t}{-2 \sin t} = -\cot t$
 $\Rightarrow \frac{dy}{dx} \Big|_{t=\frac{\pi}{4}} = -\cot \frac{\pi}{4} = -1; \text{ tangent line is } y - \sqrt{2} = -1(x - \sqrt{2}) \text{ or } y = -x + 2\sqrt{2}; \frac{dy'}{dt} = \csc^2 t$
 $\Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\csc^2 t}{-2 \sin t} = -\frac{1}{2 \sin^3 t} \Rightarrow \frac{d^2y}{dx^2} \Big|_{t=\frac{\pi}{4}} = -\sqrt{2}$
- $t = -\frac{1}{6} \Rightarrow x = \sin(2\pi(-\frac{1}{6})) = \sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}, y = \cos(2\pi(-\frac{1}{6})) = \cos(-\frac{\pi}{3}) = \frac{1}{2}; \frac{dx}{dt} = 2\pi \cos 2\pi t,$
 $\frac{dy}{dt} = -2\pi \sin 2\pi t \Rightarrow \frac{dy}{dx} = \frac{-2\pi \sin 2\pi t}{2\pi \cos 2\pi t} = -\tan 2\pi t \Rightarrow \frac{dy}{dx} \Big|_{t=-\frac{1}{6}} = -\tan(2\pi(-\frac{1}{6})) = -\tan(-\frac{\pi}{3}) = \sqrt{3};$
 tangent line is $y - \frac{1}{2} = \sqrt{3} \left[x - \left(-\frac{\sqrt{3}}{2}\right) \right]$ or $y = \sqrt{3}x + 2; \frac{dy'}{dt} = -2\pi \sec^2 2\pi t \Rightarrow \frac{d^2y}{dx^2} = \frac{-2\pi \sec^2 2\pi t}{2\pi \cos 2\pi t}$
 $= -\frac{1}{\cos^3 2\pi t} \Rightarrow \frac{d^2y}{dx^2} \Big|_{t=-\frac{1}{6}} = -8$

3. $t = \frac{\pi}{4} \Rightarrow x = 4 \sin \frac{\pi}{4} = 2\sqrt{2}, y = 2 \cos \frac{\pi}{4} = \sqrt{2}; \frac{dx}{dt} = 4 \cos t, \frac{dy}{dt} = -2 \sin t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \sin t}{4 \cos t}$
 $= -\frac{1}{2} \tan t \Rightarrow \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = -\frac{1}{2} \tan \frac{\pi}{4} = -\frac{1}{2};$ tangent line is $y - \sqrt{2} = -\frac{1}{2} \left(x - 2\sqrt{2} \right)$ or $y = -\frac{1}{2}x + 2\sqrt{2};$
 $\frac{dy'}{dt} = -\frac{1}{2} \sec^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\frac{1}{2} \sec^2 t}{4 \cos t} = -\frac{1}{8 \cos^3 t} \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=\frac{\pi}{4}} = -\frac{\sqrt{2}}{4}$
4. $t = \frac{2\pi}{3} \Rightarrow x = \cos \frac{2\pi}{3} = -\frac{1}{2}, y = \sqrt{3} \cos \frac{2\pi}{3} = -\frac{\sqrt{3}}{2}; \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = -\sqrt{3} \sin t \Rightarrow \frac{dy}{dx} = \frac{-\sqrt{3} \sin t}{-\sin t} = \sqrt{3}$
 $\Rightarrow \left. \frac{dy}{dx} \right|_{t=\frac{2\pi}{3}} = \sqrt{3};$ tangent line is $y - \left(-\frac{\sqrt{3}}{2} \right) = \sqrt{3} \left[x - \left(-\frac{1}{2} \right) \right]$ or $y = \sqrt{3}x; \frac{dy'}{dt} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{0}{-\sin t} = 0$
 $\Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=\frac{2\pi}{3}} = 0$
5. $t = \frac{1}{4} \Rightarrow x = \frac{1}{4}, y = \frac{1}{2}; \frac{dx}{dt} = 1, \frac{dy}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2\sqrt{t}} \Rightarrow \left. \frac{dy}{dx} \right|_{t=\frac{1}{4}} = \frac{1}{2\sqrt{\frac{1}{4}}} = 1;$ tangent line is
 $y - \frac{1}{2} = 1 \cdot \left(x - \frac{1}{4} \right)$ or $y = x + \frac{1}{4}; \frac{dy'}{dt} = -\frac{1}{4} t^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = -\frac{1}{4} t^{-3/2} \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=\frac{1}{4}} = -2$
6. $t = -\frac{\pi}{4} \Rightarrow x = \sec^2 \left(-\frac{\pi}{4} \right) - 1 = 1, y = \tan \left(-\frac{\pi}{4} \right) = -1; \frac{dx}{dt} = 2 \sec^2 t \tan t, \frac{dy}{dt} = \sec^2 t$
 $\Rightarrow \frac{dy}{dx} = \frac{\sec^2 t}{2 \sec^2 t \tan t} = \frac{1}{2 \tan t} = \frac{1}{2} \cot t \Rightarrow \left. \frac{dy}{dx} \right|_{t=-\frac{\pi}{4}} = \frac{1}{2} \cot \left(-\frac{\pi}{4} \right) = -\frac{1}{2};$ tangent line is
 $y - (-1) = -\frac{1}{2} (x - 1)$ or $y = -\frac{1}{2}x - \frac{1}{2}; \frac{dy'}{dt} = -\frac{1}{2} \csc^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{1}{2} \csc^2 t}{2 \sec^2 t \tan t} = -\frac{1}{4} \cot^3 t$
 $\Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=-\frac{\pi}{4}} = \frac{1}{4}$
7. $t = \frac{\pi}{6} \Rightarrow x = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}, y = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}; \frac{dx}{dt} = \sec t \tan t, \frac{dy}{dt} = \sec^2 t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
 $= \frac{\sec^2 t}{\sec t \tan t} = \csc t \Rightarrow \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{6}} = \csc \frac{\pi}{6} = 2;$ tangent line is $y - \frac{1}{\sqrt{3}} = 2 \left(x - \frac{2}{\sqrt{3}} \right)$ or $y = 2x - \sqrt{3};$
 $\frac{dy'}{dt} = -\csc t \cot t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\csc t \cot t}{\sec t \tan t} = -\cot^3 t \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=\frac{\pi}{6}} = -3\sqrt{3}$
8. $t = 3 \Rightarrow x = -\sqrt{3+1} = -2, y = \sqrt{3(3)} = 3; \frac{dx}{dt} = -\frac{1}{2} (t+1)^{-1/2}, \frac{dy}{dt} = \frac{3}{2} (3t)^{-1/2} \Rightarrow \frac{dy}{dx} = \frac{(\frac{3}{2})(3t)^{-1/2}}{(-\frac{1}{2})(t+1)^{-1/2}}$
 $= -\frac{3\sqrt{t+1}}{\sqrt{3t}} = \left. \frac{dy}{dx} \right|_{t=3} = \frac{-3\sqrt{3+1}}{\sqrt{3(3)}} = -2;$ tangent line is $y - 3 = -2[x - (-2)]$ or $y = -2x - 1;$
 $\frac{dy'}{dt} = \frac{\sqrt{3t}[-\frac{3}{2}(t+1)^{-1/2}] + 3\sqrt{t+1}[\frac{3}{2}(3t)^{-1/2}]}{3t} = \frac{3}{2t\sqrt{3t}\sqrt{t+1}} \Rightarrow \frac{d^2y}{dx^2} = \frac{(\frac{3}{2t\sqrt{3t}\sqrt{t+1}})}{(-\frac{1}{2\sqrt{t+1}})} = -\frac{3}{t\sqrt{3t}}$
 $\Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=3} = -\frac{1}{3}$
9. $t = -1 \Rightarrow x = 5, y = 1; \frac{dx}{dt} = 4t, \frac{dy}{dt} = 4t^3 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{4t} = t^2 \Rightarrow \left. \frac{dy}{dx} \right|_{t=-1} = (-1)^2 = 1;$ tangent line is
 $y - 1 = 1 \cdot (x - 5)$ or $y = x - 4; \frac{dy'}{dt} = 2t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2t}{4t} = \frac{1}{2} \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=-1} = \frac{1}{2}$
10. $t = 1 \Rightarrow x = 1, y = -2; \frac{dx}{dt} = -\frac{1}{t^2}, \frac{dy}{dt} = \frac{1}{t} \Rightarrow \frac{dy}{dx} = \frac{(\frac{1}{t})}{(-\frac{1}{t^2})} = -t \Rightarrow \left. \frac{dy}{dx} \right|_{t=1} = -1;$ tangent line is
 $y - (-2) = -1(x - 1)$ or $y = -x - 1; \frac{dy'}{dt} = -1 \Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{(-\frac{1}{t^2})} = t^2 \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=1} = 1$
11. $t = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3} - \sin \frac{\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}, y = 1 - \cos \frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2}; \frac{dx}{dt} = 1 - \cos t, \frac{dy}{dt} = \sin t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
 $= \frac{\sin t}{1 - \cos t} \Rightarrow \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{3}} = \frac{\sin(\frac{\pi}{3})}{1 - \cos(\frac{\pi}{3})} = \frac{(\frac{\sqrt{3}}{2})}{(\frac{1}{2})} = \sqrt{3};$ tangent line is $y - \frac{1}{2} = \sqrt{3} \left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2} \right)$

$$\Rightarrow y = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + 2; \frac{dy}{dt} = \frac{(1-\cos t)(\cos t) - (\sin t)(\sin t)}{(1-\cos t)^2} = \frac{-1}{1-\cos t} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(\frac{-1}{1-\cos t}\right)}{\frac{-1}{1-\cos t}} \\ = \frac{-1}{(1-\cos t)^2} \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=\frac{\pi}{3}} = -4$$

$$12. t = \frac{\pi}{2} \Rightarrow x = \cos \frac{\pi}{2} = 0, y = 1 + \sin \frac{\pi}{2} = 2; \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t \Rightarrow \frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t \\ \Rightarrow \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{2}} = -\cot \frac{\pi}{2} = 0; \text{tangent line is } y = 2; \frac{dy'}{dt} = \csc^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{\csc^2 t}{-\sin t} = -\csc^3 t \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=\frac{\pi}{2}} = -1$$

$$13. t = 2 \Rightarrow x = \frac{1}{2+1} = \frac{1}{3}, y = \frac{2}{2-1} = 2; \frac{dx}{dt} = \frac{-1}{(t+1)^2}, \frac{dy}{dt} = \frac{-1}{(t-1)^2} \Rightarrow \frac{dy}{dx} = \frac{(t+1)^2}{(t-1)^2} \Rightarrow \left. \frac{dy}{dx} \right|_{t=2} = \frac{(2+1)^2}{(2-1)^2} = 9; \\ \text{tangent line is } y = 9x - 1; \frac{dy'}{dt} = -\frac{4(t+1)}{(t-1)^3} \Rightarrow \frac{d^2y}{dx^2} = \frac{4(t+1)^3}{(t-1)^3} \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=2} = \frac{4(2+1)^3}{(2-1)^3} = 108$$

$$14. t = 0 \Rightarrow x = 0 + e^0 = 1, y = 1 - e^0 = 0; \frac{dx}{dt} = 1 + e^t, \frac{dy}{dt} = -e^t \Rightarrow \frac{dy}{dx} = \frac{-e^t}{1+e^t} \Rightarrow \left. \frac{dy}{dx} \right|_{t=0} = \frac{-e^0}{1+e^0} = -\frac{1}{2}; \\ \text{tangent line is } y = -\frac{1}{2}x + \frac{1}{2}; \frac{dy'}{dt} = \frac{-e^t}{(1+e^t)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{-e^t}{(1+e^t)^3} \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{t=0} = \frac{-e^0}{(1+e^0)^3} = -\frac{1}{8}$$

$$15. x^3 + 2t^2 = 9 \Rightarrow 3x^2 \frac{dx}{dt} + 4t = 0 \Rightarrow 3x^2 \frac{dx}{dt} = -4t \Rightarrow \frac{dx}{dt} = \frac{-4t}{3x^2}; \\ 2y^3 - 3t^2 = 4 \Rightarrow 6y^2 \frac{dy}{dt} - 6t = 0 \Rightarrow \frac{dy}{dt} = \frac{6t}{6y^2} = \frac{t}{y^2}; \text{thus } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{t}{y^2}\right)}{\left(\frac{-4t}{3x^2}\right)} = \frac{t(3x^2)}{y^2(-4t)} = \frac{3x^2}{-4y^2}; t = 2 \\ \Rightarrow x^3 + 2(2)^2 = 9 \Rightarrow x^3 + 8 = 9 \Rightarrow x^3 = 1 \Rightarrow x = 1; t = 2 \Rightarrow 2y^3 - 3(2)^2 = 4 \\ \Rightarrow 2y^3 = 16 \Rightarrow y^3 = 8 \Rightarrow y = 2; \text{therefore } \left. \frac{dy}{dx} \right|_{t=2} = \frac{3(1)^2}{-4(2)^2} = -\frac{3}{16}$$

$$16. x = \sqrt{5-\sqrt{t}} \Rightarrow \frac{dx}{dt} = \frac{1}{2} (5-\sqrt{t})^{-1/2} \left(-\frac{1}{2}t^{-1/2}\right) = -\frac{1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}; y(t-1) = \sqrt{t} \Rightarrow y + (t-1)\frac{dy}{dt} = \frac{1}{2}t^{-1/2} \\ \Rightarrow (t-1)\frac{dy}{dt} = \frac{1}{2\sqrt{t}} - y \Rightarrow \frac{dy}{dt} = \frac{\frac{1}{2\sqrt{t}} - y}{(t-1)} = \frac{1-2y\sqrt{t}}{2t\sqrt{t}-2\sqrt{t}}; \text{thus } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1-2y\sqrt{t}}{2t\sqrt{t}-2\sqrt{t}}}{\frac{-1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}} = \frac{1-2y\sqrt{t}}{2\sqrt{t}(t-1)} \cdot \frac{4\sqrt{t}\sqrt{5-\sqrt{t}}}{-1} \\ = \frac{2(1-2y\sqrt{t})\sqrt{5-\sqrt{t}}}{1-t}; t = 4 \Rightarrow x = \sqrt{5-\sqrt{4}} = \sqrt{3}; t = 4 \Rightarrow y \cdot 3 = \sqrt{4} \Rightarrow y = \frac{2}{3} \\ \text{therefore, } \left. \frac{dy}{dx} \right|_{t=4} = \frac{2\left(1-2\left(\frac{2}{3}\right)\sqrt{4}\right)\sqrt{5-\sqrt{4}}}{1-4} = \frac{10\sqrt{3}}{9}$$

$$17. x + 2x^{3/2} = t^2 + t \Rightarrow \frac{dx}{dt} + 3x^{1/2} \frac{dx}{dt} = 2t + 1 \Rightarrow (1 + 3x^{1/2}) \frac{dx}{dt} = 2t + 1 \Rightarrow \frac{dx}{dt} = \frac{2t+1}{1+3x^{1/2}}; y\sqrt{t+1} + 2t\sqrt{y} = 4 \\ \Rightarrow \frac{dy}{dt} \sqrt{t+1} + y \left(\frac{1}{2}\right)(t+1)^{-1/2} + 2\sqrt{y} + 2t \left(\frac{1}{2}y^{-1/2}\right) \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} \sqrt{t+1} + \frac{y}{2\sqrt{t+1}} + 2\sqrt{y} + \left(\frac{t}{\sqrt{y}}\right) \frac{dy}{dt} = 0 \\ \Rightarrow \left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right) \frac{dy}{dt} = \frac{-y}{2\sqrt{t+1}} - 2\sqrt{y} \Rightarrow \frac{dy}{dt} = \frac{\left(\frac{-y}{2\sqrt{t+1}} - 2\sqrt{y}\right)}{\left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)} = \frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}; \text{thus} \\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}\right)}{\left(\frac{2t+1}{1+3x^{1/2}}\right)}; t = 0 \Rightarrow x + 2x^{3/2} = 0 \Rightarrow x(1 + 2x^{1/2}) = 0 \Rightarrow x = 0; t = 0 \\ \Rightarrow y\sqrt{0+1} + 2(0)\sqrt{y} = 4 \Rightarrow y = 4; \text{therefore } \left. \frac{dy}{dx} \right|_{t=0} = \frac{\left(\frac{-4\sqrt{4} - 4(4)\sqrt{0+1}}{2\sqrt{4}(0+1) + 2(0)\sqrt{0+1}}\right)}{\left(\frac{2(0)+1}{1+3(0)^{1/2}}\right)} = -6$$

$$18. x \sin t + 2x = t \Rightarrow \frac{dx}{dt} \sin t + x \cos t + 2 \frac{dx}{dt} = 1 \Rightarrow (\sin t + 2) \frac{dx}{dt} = 1 - x \cos t \Rightarrow \frac{dx}{dt} = \frac{1-x \cos t}{\sin t + 2}; \\ t \sin t - 2t = y \Rightarrow \sin t + t \cos t - 2 = \frac{dy}{dt}; \text{thus } \frac{dy}{dx} = \frac{\sin t + t \cos t - 2}{\left(\frac{1-x \cos t}{\sin t + 2}\right)}; t = \pi \Rightarrow x \sin \pi + 2x = \pi \\ \Rightarrow x = \frac{\pi}{2}; \text{therefore } \left. \frac{dy}{dx} \right|_{t=\pi} = \frac{\sin \pi + \pi \cos \pi - 2}{\left[\frac{1-\left(\frac{\pi}{2}\right) \cos \pi}{\sin \pi + 2}\right]} = \frac{-4\pi - 8}{2 + \pi} = -4$$

$$19. \quad x = t^3 + t, y + 2t^3 = 2x + t^2 \Rightarrow \frac{dx}{dt} = 3t^2 + 1, \frac{dy}{dt} + 6t^2 = 2\frac{dx}{dt} + 2t \Rightarrow \frac{dy}{dt} = 2(3t^2 + 1) + 2t - 6t^2 = 2t + 2 \\ \Rightarrow \frac{dy}{dx} = \frac{2t+2}{3t^2+1} \Rightarrow \frac{dy}{dx} \Big|_{t=1} = \frac{2(1)+2}{3(1)^2+1} = 1$$

$$20. \quad t = \ln(x - t), y = te^t \Rightarrow 1 = \frac{1}{x-t} \left(\frac{dx}{dt} - 1 \right) \Rightarrow x - t = \frac{dx}{dt} - 1 \Rightarrow \frac{dx}{dt} = x - t + 1, \frac{dy}{dt} = te^t + e^t; \\ \Rightarrow \frac{dy}{dx} = \frac{te^t + e^t}{x - t + 1}; t = 0 \Rightarrow 0 = \ln(x - 0) \Rightarrow x = 1 \Rightarrow \frac{dy}{dx} \Big|_{t=0} = \frac{(0)e^0 + e^0}{1 - 0 + 1} = \frac{1}{2}$$

$$21. \quad A = \int_0^{2\pi} y \, dx = \int_0^{2\pi} a(1 - \cos t)a(1 - \cos t)dt = a^2 \int_0^{2\pi} (1 - \cos t)^2 dt = a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt \\ = a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right) dt = a^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2} \cos 2t\right) dt = a^2 \left[\frac{3}{2}t - 2\sin t + \frac{1}{4} \sin 2t \right]_0^{2\pi} \\ = a^2(3\pi - 0 + 0) - 0 = 3\pi a^2$$

$$22. \quad A = \int_0^1 x \, dy = \int_0^1 (t - t^2)(-e^{-t})dt \quad \left[u = t - t^2 \Rightarrow du = (1 - 2t)dt; dv = (-e^{-t})dt \Rightarrow v = e^{-t} \right] \\ = e^{-t}(t - t^2) \Big|_0^1 - \int_0^1 e^{-t}(1 - 2t)dt \quad \left[u = 1 - 2t \Rightarrow du = -2dt; dv = e^{-t}dt \Rightarrow v = -e^{-t} \right] \\ = e^{-t}(t - t^2) \Big|_0^1 - \left[-e^{-t}(1 - 2t) \Big|_0^1 - \int_0^1 2e^{-t}dt \right] = \left[e^{-t}(t - t^2) + e^{-t}(1 - 2t) - 2e^{-t} \right] \Big|_0^1 \\ = (e^{-1}(0) + e^{-1}(-1) - 2e^{-1}) - (e^0(0) + e^0(1) - 2e^0) = 1 - 3e^{-1} = 1 - \frac{3}{e}$$

$$23. \quad A = 2 \int_{\pi}^0 y \, dx = 2 \int_{\pi}^0 (b \sin t)(-a \sin t)dt = 2ab \int_0^{\pi} \sin^2 t \, dt = 2ab \int_0^{\pi} \frac{1 - \cos 2t}{2} dt = ab \int_0^{\pi} (1 - \cos 2t) dt \\ = ab \left[t - \frac{1}{2} \sin 2t \right]_0^{\pi} = ab((\pi - 0) - 0) = \pi ab$$

$$24. \quad (a) \quad x = t^2, y = t^6, 0 \leq t \leq 1 \Rightarrow A = \int_0^1 y \, dx = \int_0^1 (t^6)2t \, dt = \int_0^1 2t^7 \, dt = \left[\frac{1}{4}t^8 \right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4} \\ (b) \quad x = t^3, y = t^9, 0 \leq t \leq 1 \Rightarrow A = \int_0^1 y \, dx = \int_0^1 (t^9)3t^2 \, dt = \int_0^1 3t^{11} \, dt = \left[\frac{1}{4}t^{12} \right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$

$$25. \quad \frac{dx}{dt} = -\sin t \text{ and } \frac{dy}{dt} = 1 + \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1 + \cos t)^2} = \sqrt{2 + 2\cos t} \\ \Rightarrow \text{Length} = \int_0^{\pi} \sqrt{2 + 2\cos t} \, dt = \sqrt{2} \int_0^{\pi} \sqrt{\frac{1 + \cos t}{1 - \cos t}} (1 + \cos t) \, dt = \sqrt{2} \int_0^{\pi} \sqrt{\frac{\sin^2 t}{1 - \cos t}} \, dt \\ = \sqrt{2} \int_0^{\pi} \frac{\sin t}{\sqrt{1 - \cos t}} \, dt \text{ (since } \sin t \geq 0 \text{ on } [0, \pi]); [u = 1 - \cos t \Rightarrow du = \sin t \, dt; t = 0 \Rightarrow u = 0, \\ t = \pi \Rightarrow u = 2] \rightarrow \sqrt{2} \int_0^2 u^{-1/2} \, du = \sqrt{2} [2u^{1/2}]_0^2 = 4$$

$$26. \quad \frac{dx}{dt} = 3t^2 \text{ and } \frac{dy}{dt} = 3t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2)^2 + (3t)^2} = \sqrt{9t^4 + 9t^2} = 3t\sqrt{t^2 + 1} \text{ (since } t \geq 0 \text{ on } [0, \sqrt{3}]) \\ \Rightarrow \text{Length} = \int_0^{\sqrt{3}} 3t\sqrt{t^2 + 1} \, dt; [u = t^2 + 1 \Rightarrow \frac{3}{2} du = 3t \, dt; t = 0 \Rightarrow u = 1, t = \sqrt{3} \Rightarrow u = 4] \\ \rightarrow \int_1^4 \frac{3}{2} u^{1/2} \, du = \left[u^{3/2} \right]_1^4 = (8 - 1) = 7$$

$$27. \quad \frac{dx}{dt} = t \text{ and } \frac{dy}{dt} = (2t + 1)^{1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2 + (2t + 1)} = \sqrt{(t + 1)^2} = |t + 1| = t + 1 \text{ since } 0 \leq t \leq 4 \\ \Rightarrow \text{Length} = \int_0^4 (t + 1) \, dt = \left[\frac{t^2}{2} + t \right]_0^4 = (8 + 4) = 12$$

$$28. \frac{dx}{dt} = (2t+3)^{1/2} \text{ and } \frac{dy}{dt} = 1+t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t+3) + (1+t)^2} = \sqrt{t^2 + 4t + 4} = |t+2| = t+2$$

since $0 \leq t \leq 3 \Rightarrow \text{Length} = \int_0^3 (t+2) dt = \left[\frac{t^2}{2} + 2t\right]_0^3 = \frac{21}{2}$

$$29. \frac{dx}{dt} = 8t \cos t \text{ and } \frac{dy}{dt} = 8t \sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t \cos t)^2 + (8t \sin t)^2} = \sqrt{64t^2 \cos^2 t + 64t^2 \sin^2 t}$$

$$= |8t| = 8t \text{ since } 0 \leq t \leq \frac{\pi}{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 8t dt = [4t^2]_0^{\pi/2} = \pi^2$$

$$30. \frac{dx}{dt} = \left(\frac{1}{\sec t + \tan t}\right) (\sec t \tan t + \sec^2 t) - \cos t = \sec t - \cos t \text{ and } \frac{dy}{dt} = -\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2} = \sqrt{\sec^2 t - 1} = \sqrt{\tan^2 t} = |\tan t| = \tan t \text{ since } 0 \leq t \leq \frac{\pi}{3}$$

$$\Rightarrow \text{Length} = \int_0^{\pi/3} \tan t dt = \int_0^{\pi/3} \frac{\sin t}{\cos t} dt = [-\ln |\cos t|]_0^{\pi/3} = -\ln \frac{1}{2} + \ln 1 = \ln 2$$

$$31. \frac{dx}{dt} = -\sin t \text{ and } \frac{dy}{dt} = \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \text{Area} = \int 2\pi y ds$$

$$= \int_0^{2\pi} 2\pi(2 + \sin t)(1) dt = 2\pi [2t - \cos t]_0^{2\pi} = 2\pi[(4\pi - 1) - (0 - 1)] = 8\pi^2$$

$$32. \frac{dx}{dt} = t^{1/2} \text{ and } \frac{dy}{dt} = t^{-1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t + t^{-1}} = \sqrt{\frac{t^2+1}{t}} \Rightarrow \text{Area} = \int 2\pi x ds$$

$$= \int_0^{\sqrt{3}} 2\pi \left(\frac{2}{3} t^{3/2}\right) \sqrt{\frac{t^2+1}{t}} dt = \frac{4\pi}{3} \int_0^{\sqrt{3}} t \sqrt{t^2+1} dt; [u = t^2 + 1 \Rightarrow du = 2t dt; t = 0 \Rightarrow u = 1,$$

$$[t = \sqrt{3} \Rightarrow u = 4] \rightarrow \int_1^4 \frac{2\pi}{3} \sqrt{u} du = \left[\frac{4\pi}{9} u^{3/2}\right]_1^4 = \frac{28\pi}{9}$$

Note: $\int_0^{\sqrt{3}} 2\pi \left(\frac{2}{3} t^{3/2}\right) \sqrt{\frac{t^2+1}{t}} dt$ is an improper integral but $\lim_{t \rightarrow 0^+} f(t)$ exists and is equal to 0, where

$f(t) = 2\pi \left(\frac{2}{3} t^{3/2}\right) \sqrt{\frac{t^2+1}{t}}$. Thus the discontinuity is removable: define $F(t) = f(t)$ for $t > 0$ and $F(0) = 0$

$$\Rightarrow \int_0^{\sqrt{3}} F(t) dt = \frac{28\pi}{9}.$$

$$33. \frac{dx}{dt} = 1 \text{ and } \frac{dy}{dt} = t + \sqrt{2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1^2 + (t + \sqrt{2})^2} = \sqrt{t^2 + 2\sqrt{2}t + 3} \Rightarrow \text{Area} = \int 2\pi x ds$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} 2\pi (t + \sqrt{2}) \sqrt{t^2 + 2\sqrt{2}t + 3} dt; [u = t^2 + 2\sqrt{2}t + 3 \Rightarrow du = (2t + 2\sqrt{2}) dt; t = -\sqrt{2} \Rightarrow u = 1,$$

$$[t = \sqrt{2} \Rightarrow u = 9] \rightarrow \int_1^9 \pi \sqrt{u} du = \left[\frac{2}{3} \pi u^{3/2}\right]_1^9 = \frac{2\pi}{3} (27 - 1) = \frac{52\pi}{3}$$

$$34. \text{ From Exercise 30, } \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \tan t \Rightarrow \text{Area} = \int 2\pi y ds = \int_0^{\pi/3} 2\pi \cos t \tan t dt = 2\pi \int_0^{\pi/3} \sin t dt$$

$$= 2\pi [-\cos t]_0^{\pi/3} = 2\pi \left[-\frac{1}{2} - (-1)\right] = \pi$$

$$35. \frac{dx}{dt} = 2 \text{ and } \frac{dy}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2^2 + 1^2} = \sqrt{5} \Rightarrow \text{Area} = \int 2\pi y ds = \int_0^1 2\pi(t+1)\sqrt{5} dt$$

$$= 2\pi\sqrt{5} \left[\frac{t^2}{2} + t\right]_0^1 = 3\pi\sqrt{5}. \text{ Check: slant height is } \sqrt{5} \Rightarrow \text{Area is } \pi(1+2)\sqrt{5} = 3\pi\sqrt{5}.$$

36. $\frac{dx}{dt} = h$ and $\frac{dy}{dt} = r \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{h^2 + r^2} \Rightarrow \text{Area} = \int 2\pi y \, ds = \int_0^1 2\pi r t \sqrt{h^2 + r^2} \, dt$
 $= 2\pi r \sqrt{h^2 + r^2} \int_0^1 t \, dt = 2\pi r \sqrt{h^2 + r^2} \left[\frac{t^2}{2}\right]_0^1 = \pi r \sqrt{h^2 + r^2}$. Check: slant height is $\sqrt{h^2 + r^2} \Rightarrow \text{Area is}$
 $\pi r \sqrt{h^2 + r^2}$.
37. Let the density be $\delta = 1$. Then $x = \cos t + t \sin t \Rightarrow \frac{dx}{dt} = t \cos t$, and $y = \sin t - t \cos t \Rightarrow \frac{dy}{dt} = t \sin t$
 $\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(t \cos t)^2 + (t \sin t)^2} = |t| dt = t dt$ since $0 \leq t \leq \frac{\pi}{2}$. The curve's mass is
 $M = \int dm = \int_0^{\pi/2} t \, dt = \frac{\pi^2}{8}$. Also $M_x = \int \tilde{y} \, dm = \int_0^{\pi/2} (\sin t - t \cos t) t \, dt = \int_0^{\pi/2} t \sin t \, dt - \int_0^{\pi/2} t^2 \cos t \, dt$
 $= [\sin t - t \cos t]_0^{\pi/2} - [t^2 \sin t - 2 \sin t + 2t \cos t]_0^{\pi/2} = 3 - \frac{\pi^2}{4}$, where we integrated by parts. Therefore,
 $\bar{y} = \frac{M_x}{M} = \frac{(3 - \frac{\pi^2}{4})}{(\frac{\pi^2}{8})} = \frac{24}{\pi^2} - 2$. Next, $M_y = \int \tilde{x} \, dm = \int_0^{\pi/2} (\cos t + t \sin t) t \, dt = \int_0^{\pi/2} t \cos t \, dt + \int_0^{\pi/2} t^2 \sin t \, dt$
 $= [\cos t + t \sin t]_0^{\pi/2} + [-t^2 \cos t + 2 \cos t + 2t \sin t]_0^{\pi/2} = \frac{3\pi}{2} - 3$, again integrating by parts. Hence
 $\bar{x} = \frac{M_y}{M} = \frac{(\frac{3\pi}{2} - 3)}{(\frac{\pi^2}{8})} = \frac{12}{\pi} - \frac{24}{\pi^2}$. Therefore $(\bar{x}, \bar{y}) = (\frac{12}{\pi} - \frac{24}{\pi^2}, \frac{24}{\pi^2} - 2)$.
38. Let the density be $\delta = 1$. Then $x = e^t \cos t \Rightarrow \frac{dx}{dt} = e^t \cos t - e^t \sin t$, and $y = e^t \sin t \Rightarrow \frac{dy}{dt} = e^t \sin t + e^t \cos t$
 $\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} dt = \sqrt{2e^{2t}} dt = \sqrt{2} e^t dt$.
The curve's mass is $M = \int dm = \int_0^\pi \sqrt{2} e^t dt = \sqrt{2} e^\pi - \sqrt{2}$. Also $M_x = \int \tilde{y} \, dm = \int_0^\pi (e^t \sin t) (\sqrt{2} e^t) dt$
 $= \int_0^\pi \sqrt{2} e^{2t} \sin t \, dt = \sqrt{2} \left[\frac{e^{2t}}{5} (2 \sin t - \cos t)\right]_0^\pi = \sqrt{2} \left(\frac{e^{2\pi}}{5} + \frac{1}{5}\right) \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{\sqrt{2}(\frac{e^{2\pi}}{5} + \frac{1}{5})}{\sqrt{2}e^\pi - \sqrt{2}} = \frac{e^{2\pi} + 1}{5(e^\pi - 1)}$.
Next $M_y = \int \tilde{x} \, dm = \int_0^\pi (e^t \cos t) (\sqrt{2} e^t) dt = \int_0^\pi \sqrt{2} e^{2t} \cos t \, dt = \sqrt{2} \left[\frac{e^{2t}}{5} (2 \cos t + \sin t)\right]_0^\pi = -\sqrt{2} \left(\frac{2e^{2\pi}}{5} + \frac{2}{5}\right)$
 $\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{-\sqrt{2}(\frac{2e^{2\pi}}{5} + \frac{2}{5})}{\sqrt{2}e^\pi - \sqrt{2}} = -\frac{2e^{2\pi} + 2}{5(e^\pi - 1)}$. Therefore $(\bar{x}, \bar{y}) = \left(-\frac{2e^{2\pi} + 2}{5(e^\pi - 1)}, \frac{e^{2\pi} + 1}{5(e^\pi - 1)}\right)$.
39. Let the density be $\delta = 1$. Then $x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$, and $y = t + \sin t \Rightarrow \frac{dy}{dt} = 1 + \cos t$
 $\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(-\sin t)^2 + (1 + \cos t)^2} dt = \sqrt{2 + 2 \cos t} dt$. The curve's mass
is $M = \int dm = \int_0^\pi \sqrt{2 + 2 \cos t} dt = \sqrt{2} \int_0^\pi \sqrt{1 + \cos t} dt = \sqrt{2} \int_0^\pi \sqrt{2 \cos^2(\frac{t}{2})} dt = 2 \int_0^\pi |\cos(\frac{t}{2})| dt$
 $= 2 \int_0^\pi \cos(\frac{t}{2}) dt$ (since $0 \leq t \leq \pi \Rightarrow 0 \leq \frac{t}{2} \leq \frac{\pi}{2}$) $= 2 [2 \sin(\frac{t}{2})]_0^\pi = 4$. Also $M_x = \int \tilde{y} \, dm$
 $= \int_0^\pi (t + \sin t) (2 \cos \frac{t}{2}) dt = \int_0^\pi 2t \cos(\frac{t}{2}) dt + \int_0^\pi 2 \sin t \cos(\frac{t}{2}) dt$
 $= 2 [4 \cos(\frac{t}{2}) + 2t \sin(\frac{t}{2})]_0^\pi + 2 [-\frac{1}{3} \cos(\frac{3t}{2}) - \cos(\frac{1}{2}t)]_0^\pi = 4\pi - \frac{16}{3} \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{(4\pi - \frac{16}{3})}{4} = \pi - \frac{4}{3}$.
Next $M_y = \int \tilde{x} \, dm = \int_0^\pi (\cos t) (2 \cos \frac{t}{2}) dt = \int_0^\pi \cos t \cos(\frac{t}{2}) dt = 2 \left[\sin(\frac{t}{2}) + \frac{\sin(\frac{3t}{2})}{3}\right]_0^\pi = 2 - \frac{2}{3}$
 $= \frac{4}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{(\frac{4}{3})}{4} = \frac{1}{3}$. Therefore $(\bar{x}, \bar{y}) = (\frac{1}{3}, \pi - \frac{4}{3})$.
40. Let the density be $\delta = 1$. Then $x = t^3 \Rightarrow \frac{dx}{dt} = 3t^2$, and $y = \frac{3t^2}{2} \Rightarrow \frac{dy}{dt} = 3t \Rightarrow dm = 1 \cdot ds$
 $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(3t^2)^2 + (3t)^2} dt = 3|t| \sqrt{t^2 + 1} dt = 3t \sqrt{t^2 + 1} dt$ since $0 \leq t \leq \sqrt{3}$. The curve's mass
is $M = \int dm = \int_0^{\sqrt{3}} 3t \sqrt{t^2 + 1} dt = [(t^2 + 1)^{3/2}]_0^{\sqrt{3}} = 7$. Also $M_x = \int \tilde{y} \, dm = \int_0^{\sqrt{3}} \frac{3t^2}{2} (3t \sqrt{t^2 + 1}) dt$
 $= \frac{9}{2} \int_0^{\sqrt{3}} t^3 \sqrt{t^2 + 1} dt = \frac{87}{5} = 17.4$ (by computer) $\Rightarrow \bar{y} = \frac{M_x}{M} = \frac{17.4}{7} \approx 2.49$. Next $M_y = \int \tilde{x} \, dm$

$$= \int_0^{\sqrt{3}} t^3 \cdot 3t \sqrt{t^2 + 1} dt = 3 \int_0^{\sqrt{3}} t^4 \sqrt{t^2 + 1} dt \approx 16.4849 \text{ (by computer)} \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{16.4849}{7} \approx 2.35.$$

Therefore, $(\bar{x}, \bar{y}) \approx (2.35, 2.49)$.

$$41. (a) \frac{dx}{dt} = -2 \sin 2t \text{ and } \frac{dy}{dt} = 2 \cos 2t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-2 \sin 2t)^2 + (2 \cos 2t)^2} = 2$$

$$\Rightarrow \text{Length} = \int_0^{\pi/2} 2 dt = [2t]_0^{\pi/2} = \pi$$

$$(b) \frac{dx}{dt} = \pi \cos \pi t \text{ and } \frac{dy}{dt} = -\pi \sin \pi t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\pi \cos \pi t)^2 + (-\pi \sin \pi t)^2} = \pi$$

$$\Rightarrow \text{Length} = \int_{-1/2}^{1/2} \pi dt = [\pi t]_{-1/2}^{1/2} = \pi$$

$$42. (a) x = g(y) \text{ has the parametrization } x = g(y) \text{ and } y = y \text{ for } c \leq y \leq d \Rightarrow \frac{dx}{dy} = g'(y) \text{ and } \frac{dy}{dy} = 1; \text{ then}$$

$$\text{Length} = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2} dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d \sqrt{1 + [g'(y)]^2} dy$$

$$(b) x = y^{3/2}, 0 \leq y \leq \frac{4}{3} \Rightarrow \frac{dx}{dy} = \frac{3}{2}y^{1/2} \Rightarrow L = \int_0^{4/3} \sqrt{1 + \left(\frac{3}{2}y^{1/2}\right)^2} dy = \int_0^{4/3} \sqrt{1 + \frac{9}{4}y} dy = \left[\frac{4}{9} \cdot \frac{2}{3} \left(1 + \frac{9}{4}y\right)^{3/2} \right]_0^{4/3}$$

$$= \frac{8}{27}(4)^{3/2} - \frac{8}{27}(1)^{3/2} = \frac{56}{27}$$

$$(c) x = \frac{3}{2}y^{2/3}, 0 \leq y \leq 1 \Rightarrow \frac{dx}{dy} = y^{-1/3} \Rightarrow L = \int_0^1 \sqrt{1 + (y^{-1/3})^2} dy = \int_0^1 \sqrt{1 + \frac{1}{y^{2/3}}} dy = \lim_{a \rightarrow 0^+} \int_a^1 \sqrt{\frac{y^{2/3} + 1}{y^{2/3}}} dy$$

$$= \lim_{a \rightarrow 0^+} \frac{3}{2} \int_a^1 (y^{2/3} + 1)^{1/2} \left(\frac{2}{3}y^{-1/3}\right) dy = \lim_{a \rightarrow 0^+} \left[\frac{2}{3} \cdot \frac{2}{3} (y^{2/3} + 1)^{3/2} \right]_a^1 = \lim_{a \rightarrow 0^+} \left((2)^{3/2} - (a^{2/3} + 1)^{3/2} \right) = 2\sqrt{2} - 1$$

$$43. x = (1 + 2 \sin \theta) \cos \theta, y = (1 + 2 \sin \theta) \sin \theta \Rightarrow \frac{dx}{d\theta} = 2 \cos^2 \theta - \sin \theta (1 + 2 \sin \theta), \frac{dy}{d\theta} = 2 \cos \theta \sin \theta + \cos \theta (1 + 2 \sin \theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 \cos \theta \sin \theta + \cos \theta (1 + 2 \sin \theta)}{2 \cos^2 \theta - \sin \theta (1 + 2 \sin \theta)} = \frac{4 \cos \theta \sin \theta + \cos \theta}{2 \cos^2 \theta - 2 \sin^2 \theta - \sin \theta} = \frac{2 \sin 2\theta + \cos \theta}{2 \cos 2\theta - \sin \theta}$$

$$(a) x = (1 + 2 \sin(0)) \cos(0) = 1, y = (1 + 2 \sin(0)) \sin(0) = 0; \left. \frac{dy}{dx} \right|_{\theta=0} = \frac{2 \sin(2(0)) + \cos(0)}{2 \cos(2(0)) - \sin(0)} = \frac{0+1}{2-0} = \frac{1}{2}$$

$$(b) x = (1 + 2 \sin(\frac{\pi}{2})) \cos(\frac{\pi}{2}) = 0, y = (1 + 2 \sin(\frac{\pi}{2})) \sin(\frac{\pi}{2}) = 3; \left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \frac{2 \sin(2(\frac{\pi}{2})) + \cos(\frac{\pi}{2})}{2 \cos(2(\frac{\pi}{2})) - \sin(\frac{\pi}{2})} = \frac{0+0}{-2-1} = 0$$

$$(c) x = (1 + 2 \sin(\frac{4\pi}{3})) \cos(\frac{4\pi}{3}) = \frac{\sqrt{3}-1}{2}, y = (1 + 2 \sin(\frac{4\pi}{3})) \sin(\frac{4\pi}{3}) = \frac{3-\sqrt{3}}{2}; \left. \frac{dy}{dx} \right|_{\theta=4\pi/3} = \frac{2 \sin(2(\frac{4\pi}{3})) + \cos(\frac{4\pi}{3})}{2 \cos(2(\frac{4\pi}{3})) - \sin(\frac{4\pi}{3})}$$

$$= \frac{\sqrt{3}-\frac{1}{2}}{-1+\frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}-1}{\sqrt{3}-2} = -(4 + 3\sqrt{3})$$

$$44. x = t, y = 1 - \cos t, 0 \leq t \leq 2\pi \Rightarrow \frac{dx}{dt} = 1, \frac{dy}{dt} = \sin t \Rightarrow \frac{dy}{dx} = \frac{\sin t}{1} = \sin t \Rightarrow \frac{d}{dt} \left(\frac{dy}{dx} \right) = \cos t \Rightarrow \frac{d^2y}{dx^2} = \frac{\cos t}{1} = \cos t. \text{ The}$$

maximum and minimum slope will occur at points that maximize/minimize $\frac{dy}{dx}$, in other words, points where $\frac{d^2y}{dx^2} = 0$

$$\Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2} \text{ or } t = \frac{3\pi}{2} \Rightarrow \frac{d^2y}{dx^2} = \begin{array}{ccccc} +++ & & - & - & - \\ & \pi/2 & & 3\pi/2 & \end{array}$$

$$(a) \text{ the maximum slope is } \left. \frac{dy}{dx} \right|_{t=\pi/2} = \sin\left(\frac{\pi}{2}\right) = 1, \text{ which occurs at } x = \frac{\pi}{2}, y = 1 - \cos\left(\frac{\pi}{2}\right) = 1$$

$$(a) \text{ the minimum slope is } \left. \frac{dy}{dx} \right|_{t=3\pi/2} = \sin\left(\frac{3\pi}{2}\right) = -1, \text{ which occurs at } x = \frac{3\pi}{2}, y = 1 - \cos\left(\frac{3\pi}{2}\right) = 1$$

$$45. \frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = 2 \cos 2t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos 2t}{\cos t} = \frac{2(2 \cos^2 t - 1)}{\cos t}; \text{ then } \frac{dy}{dx} = 0 \Rightarrow \frac{2(2 \cos^2 t - 1)}{\cos t} = 0$$

$$\Rightarrow 2 \cos^2 t - 1 = 0 \Rightarrow \cos t = \pm \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \text{ In the 1st quadrant: } t = \frac{\pi}{4} \Rightarrow x = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \text{ and}$$

$$y = \sin 2\left(\frac{\pi}{4}\right) = 1 \Rightarrow \left(\frac{\sqrt{2}}{2}, 1\right) \text{ is the point where the tangent line is horizontal. At the origin: } x = 0 \text{ and } y = 0$$

$\Rightarrow \sin t = 0 \Rightarrow t = 0$ or $t = \pi$ and $\sin 2t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; thus $t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at origin: $\left. \frac{dy}{dx} \right|_{t=0} = 2 \Rightarrow y = 2x$ and $\left. \frac{dy}{dx} \right|_{t=\pi} = -2 \Rightarrow y = -2x$

46. $\frac{dx}{dt} = 2 \cos 2t$ and $\frac{dy}{dt} = 3 \cos 3t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3 \cos 3t}{2 \cos 2t} = \frac{3(\cos 2t \cos t - \sin 2t \sin t)}{2(2 \cos^2 t - 1)}$
 $= \frac{3[(2 \cos^2 t - 1)(\cos t) - 2 \sin t \cos t \sin t]}{2(2 \cos^2 t - 1)} = \frac{(3 \cos t)(2 \cos^2 t - 1 - 2 \sin^2 t)}{2(2 \cos^2 t - 1)} = \frac{(3 \cos t)(4 \cos^2 t - 3)}{2(2 \cos^2 t - 1)}$; then
 $\frac{dy}{dx} = 0 \Rightarrow \frac{(3 \cos t)(4 \cos^2 t - 3)}{2(2 \cos^2 t - 1)} = 0 \Rightarrow 3 \cos t = 0$ or $4 \cos^2 t - 3 = 0$: $3 \cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$ and
 $4 \cos^2 t - 3 = 0 \Rightarrow \cos t = \pm \frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. In the 1st quadrant: $t = \frac{\pi}{6} \Rightarrow x = \sin 2\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$
and $y = \sin 3\left(\frac{\pi}{6}\right) = 1 \Rightarrow \left(\frac{\sqrt{3}}{2}, 1\right)$ is the point where the graph has a horizontal tangent. At the origin: $x = 0$
and $y = 0 \Rightarrow \sin 2t = 0$ and $\sin 3t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \Rightarrow t = 0$ and $t = \pi$ give
the tangent lines at the origin. Tangents at the origin: $\left. \frac{dy}{dx} \right|_{t=0} = \frac{3 \cos 0}{2 \cos 0} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x$, and $\left. \frac{dy}{dx} \right|_{t=\pi} = \frac{3 \cos(3\pi)}{2 \cos(2\pi)} = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}x$

47. (a) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leq t \leq 2\pi \Rightarrow \frac{dx}{dt} = a(1 - \cos t)$, $\frac{dy}{dt} = a \sin t \Rightarrow \text{Length}$
 $= \int_0^{2\pi} \sqrt{(a(1 - \cos t))^2 + (a \sin t)^2} dt = \int_0^{2\pi} \sqrt{a^2 - 2a^2 \cos t + a^2 \cos^2 t + a^2 \sin^2 t} dt$
 $= a\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos t} dt = a\sqrt{2} \int_0^{2\pi} \sqrt{2 \sin^2\left(\frac{t}{2}\right)} dt = 2a \int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt = \left[-4a \cos\left(\frac{t}{2}\right)\right]_0^{2\pi}$
 $= -4a \cos \pi + 4a \cos(0) = 8a$
(b) $a = 1 \Rightarrow x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi \Rightarrow \frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t \Rightarrow \text{Surface area} =$
 $= \int_0^{2\pi} 2\pi(1 - \cos t) \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt = \int_0^{2\pi} 2\pi(1 - \cos t) \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt$
 $= 2\pi \int_0^{2\pi} (1 - \cos t) \sqrt{2 - 2 \cos t} dt = 2\sqrt{2}\pi \int_0^{2\pi} (1 - \cos t)^{3/2} dt = 2\sqrt{2}\pi \int_0^{2\pi} \left(1 - \cos\left(2 \cdot \frac{t}{2}\right)\right)^{3/2} dt$
 $= 2\sqrt{2}\pi \int_0^{2\pi} \left(2 \sin^2\left(\frac{t}{2}\right)\right)^{3/2} dt = 8\pi \int_0^{2\pi} \sin^3\left(\frac{t}{2}\right) dt$
 $\left[u = \frac{t}{2} \Rightarrow du = \frac{1}{2} dt \Rightarrow dt = 2 du; t = 0 \Rightarrow u = 0, t = 2\pi \Rightarrow u = \pi\right]$
 $= 16\pi \int_0^\pi \sin^3 u du = 16\pi \int_0^\pi \sin^2 u \sin u du = 16\pi \int_0^\pi (1 - \cos^2 u) \sin u du = 16\pi \int_0^\pi \sin u du - 16\pi \int_0^\pi \cos^2 u \sin u du$
 $= \left[-16\pi \cos u + \frac{16\pi}{3} \cos^3 u\right]_0^\pi = (16\pi - \frac{16\pi}{3}) - (-16\pi + \frac{16\pi}{3}) = \frac{64\pi}{3}$

48. $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$; Volume $= \int_0^{2\pi} \pi y^2 dx = \int_0^{2\pi} \pi(1 - \cos t)^2(1 - \cos t) dt$
 $= \pi \int_0^{2\pi} (1 - 3 \cos t + 3 \cos^2 t - \cos^3 t) dt = \pi \int_0^{2\pi} \left(1 - 3 \cos t + 3\left(\frac{1 + \cos 2t}{2}\right) - \cos^2 t \cos t\right) dt$
 $= \pi \int_0^{2\pi} \left(\frac{5}{2} - 3 \cos t + \frac{3}{2} \cos 2t - (1 - \sin^2 t) \cos t\right) dt = \pi \int_0^{2\pi} \left(\frac{5}{2} - 4 \cos t + \frac{3}{2} \cos 2t + \sin^2 t \cos t\right) dt$
 $= \pi \left[\frac{5}{2}t - 4 \sin t + \frac{3}{4} \sin 2t + \frac{1}{3} \sin^3 t\right]_0^{2\pi} = \pi(5\pi - 0 + 0 + 0) - 0 = 5\pi^2$

47-50. Example CAS commands:

Maple:

```
with(plots);
with(student);
x := t -> t^3/3;
y := t -> t^2/2;
a := 0;
b := 1;
N := [2, 4, 8];
for n in N do
```

```

tt := [seq( a+i*(b-a)/n, i=0..n )];
pts := [seq([x(t),y(t)],t=tt)];
L := simplify(add( student[distance](pts[i+1],pts[i]), i=1..n ));    # (b)
T := sprintf("#47(a) (Section 11.2)\nn=%3d L=%8.5f\n", n, L );
P[n] := plot( [[x(t),y(t),t=a..b],pts], title=T );                  # (a)
end do;
display( [seq(P[n],n=N)], insequence=true );
ds := t -> sqrt( simplify(D(x)(t)^2 + D(y)(t)^2) );                # (c)
L := Int( ds(t), t=a..b );
L = evalf(L);

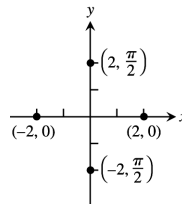
```

11.3 POLAR COORDINATES

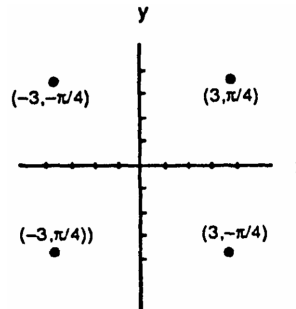
1. a, e; b, g; c, h; d, f

2. a, f; b, h; c, g; d, e

3. (a) $(2, \frac{\pi}{2} + 2n\pi)$ and $(-2, \frac{\pi}{2} + (2n+1)\pi)$, n an integer
 (b) $(2, 2n\pi)$ and $(-2, (2n+1)\pi)$, n an integer
 (c) $(2, \frac{3\pi}{2} + 2n\pi)$ and $(-2, \frac{3\pi}{2} + (2n+1)\pi)$, n an integer
 (d) $(2, (2n+1)\pi)$ and $(-2, 2n\pi)$, n an integer



4. (a) $(3, \frac{\pi}{4} + 2n\pi)$ and $(-3, \frac{5\pi}{4} + 2n\pi)$, n an integer
 (b) $(-3, \frac{\pi}{4} + 2n\pi)$ and $(3, \frac{5\pi}{4} + 2n\pi)$, n an integer
 (c) $(3, -\frac{\pi}{4} + 2n\pi)$ and $(-3, \frac{3\pi}{4} + 2n\pi)$, n an integer
 (d) $(-3, -\frac{\pi}{4} + 2n\pi)$ and $(3, \frac{3\pi}{4} + 2n\pi)$, n an integer



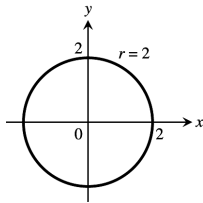
5. (a) $x = r \cos \theta = 3 \cos 0 = 3$, $y = r \sin \theta = 3 \sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(3, 0)$
 (b) $x = r \cos \theta = -3 \cos 0 = -3$, $y = r \sin \theta = -3 \sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(-3, 0)$
 (c) $x = r \cos \theta = 2 \cos \frac{2\pi}{3} = -1$, $y = r \sin \theta = 2 \sin \frac{2\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $(-1, \sqrt{3})$
 (d) $x = r \cos \theta = 2 \cos \frac{7\pi}{3} = 1$, $y = r \sin \theta = 2 \sin \frac{7\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $(1, \sqrt{3})$
 (e) $x = r \cos \theta = -3 \cos \pi = 3$, $y = r \sin \theta = -3 \sin \pi = 0 \Rightarrow$ Cartesian coordinates are $(3, 0)$
 (f) $x = r \cos \theta = 2 \cos \frac{\pi}{3} = 1$, $y = r \sin \theta = 2 \sin \frac{\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $(1, \sqrt{3})$
 (g) $x = r \cos \theta = -3 \cos 2\pi = -3$, $y = r \sin \theta = -3 \sin 2\pi = 0 \Rightarrow$ Cartesian coordinates are $(-3, 0)$
 (h) $x = r \cos \theta = -2 \cos(-\frac{\pi}{3}) = -1$, $y = r \sin \theta = -2 \sin(-\frac{\pi}{3}) = \sqrt{3} \Rightarrow$ Cartesian coordinates are $(-1, \sqrt{3})$
6. (a) $x = \sqrt{2} \cos \frac{\pi}{4} = 1$, $y = \sqrt{2} \sin \frac{\pi}{4} = 1 \Rightarrow$ Cartesian coordinates are $(1, 1)$
 (b) $x = 1 \cos 0 = 1$, $y = 1 \sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(1, 0)$
 (c) $x = 0 \cos \frac{\pi}{2} = 0$, $y = 0 \sin \frac{\pi}{2} = 0 \Rightarrow$ Cartesian coordinates are $(0, 0)$
 (d) $x = -\sqrt{2} \cos(\frac{\pi}{4}) = -1$, $y = -\sqrt{2} \sin(\frac{\pi}{4}) = -1 \Rightarrow$ Cartesian coordinates are $(-1, -1)$
 (e) $x = -3 \cos \frac{5\pi}{6} = \frac{3\sqrt{3}}{2}$, $y = -3 \sin \frac{5\pi}{6} = -\frac{3}{2} \Rightarrow$ Cartesian coordinates are $(\frac{3\sqrt{3}}{2}, -\frac{3}{2})$
 (f) $x = 5 \cos(\tan^{-1} \frac{4}{3}) = 3$, $y = 5 \sin(\tan^{-1} \frac{4}{3}) = 4 \Rightarrow$ Cartesian coordinates are $(3, 4)$

(g) $x = -1 \cos 7\pi = 1, y = -1 \sin 7\pi = 0 \Rightarrow$ Cartesian coordinates are $(1, 0)$

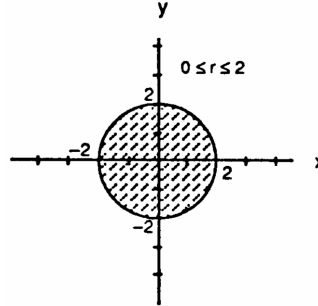
(h) $x = 2\sqrt{3} \cos \frac{2\pi}{3} = -\sqrt{3}, y = 2\sqrt{3} \sin \frac{2\pi}{3} = 3 \Rightarrow$ Cartesian coordinates are $(-\sqrt{3}, 3)$

7. (a) $(1, 1) \Rightarrow r = \sqrt{1^2 + 1^2} = \sqrt{2}, \sin \theta = \frac{1}{\sqrt{2}} \text{ and } \cos \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow$ Polar coordinates are $(\sqrt{2}, \frac{\pi}{4})$
- (b) $(-3, 0) \Rightarrow r = \sqrt{(-3)^2 + 0^2} = 3, \sin \theta = 0 \text{ and } \cos \theta = -1 \Rightarrow \theta = \pi \Rightarrow$ Polar coordinates are $(3, \pi)$
- (c) $(\sqrt{3}, -1) \Rightarrow r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2, \sin \theta = -\frac{1}{2} \text{ and } \cos \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{11\pi}{6} \Rightarrow$ Polar coordinates are $(2, \frac{11\pi}{6})$
- (d) $(-3, 4) \Rightarrow r = \sqrt{(-3)^2 + 4^2} = 5, \sin \theta = \frac{4}{5} \text{ and } \cos \theta = -\frac{3}{5} \Rightarrow \theta = \pi - \arctan(\frac{4}{3}) \Rightarrow$ Polar coordinates are $(5, \pi - \arctan(\frac{4}{3}))$
8. (a) $(-2, -2) \Rightarrow r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}, \sin \theta = -\frac{1}{\sqrt{2}} \text{ and } \cos \theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = -\frac{3\pi}{4} \Rightarrow$ Polar coordinates are $(2\sqrt{2}, -\frac{3\pi}{4})$
- (b) $(0, 3) \Rightarrow r = \sqrt{0^2 + 3^2} = 3, \sin \theta = 1 \text{ and } \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ Polar coordinates are $(3, \frac{\pi}{2})$
- (c) $(-\sqrt{3}, 1) \Rightarrow r = \sqrt{(-\sqrt{3})^2 + 1^2} = 2, \sin \theta = \frac{1}{2} \text{ and } \cos \theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{5\pi}{6} \Rightarrow$ Polar coordinates are $(2, \frac{5\pi}{6})$
- (d) $(5, -12) \Rightarrow r = \sqrt{5^2 + (-12)^2} = 13, \sin \theta = -\frac{12}{13} \text{ and } \cos \theta = \frac{5}{13} \Rightarrow \theta = -\arctan(\frac{12}{5}) \Rightarrow$ Polar coordinates are $(13, -\arctan(\frac{12}{5}))$
9. (a) $(3, 3) \Rightarrow r = -\sqrt{3^2 + 3^2} = -3\sqrt{2}, \sin \theta = -\frac{1}{\sqrt{2}} \text{ and } \cos \theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{5\pi}{4} \Rightarrow$ Polar coordinates are $(-3\sqrt{2}, \frac{5\pi}{4})$
- (b) $(-1, 0) \Rightarrow r = -\sqrt{(-1)^2 + 0^2} = -1, \sin \theta = 0 \text{ and } \cos \theta = 1 \Rightarrow \theta = 0 \Rightarrow$ Polar coordinates are $(-1, 0)$
- (c) $(-1, \sqrt{3}) \Rightarrow r = -\sqrt{(-1)^2 + (\sqrt{3})^2} = -2, \sin \theta = \frac{\sqrt{3}}{2} \text{ and } \cos \theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3} \Rightarrow$ Polar coordinates are $(-2, \frac{2\pi}{3})$
- (d) $(4, -3) \Rightarrow r = -\sqrt{4^2 + (-3)^2} = -5, \sin \theta = -\frac{3}{5} \text{ and } \cos \theta = \frac{4}{5} \Rightarrow \theta = \pi - \arctan(\frac{3}{4}) \Rightarrow$ Polar coordinates are $(-5, \pi - \arctan(\frac{3}{4}))$
10. (a) $(-2, 0) \Rightarrow r = -\sqrt{(-2)^2 + 0^2} = -2, \sin \theta = 0 \text{ and } \cos \theta = 1 \Rightarrow \theta = 0 \Rightarrow$ Polar coordinates are $(-2, 0)$
- (b) $(1, 0) \Rightarrow r = -\sqrt{1^2 + 0^2} = -1, \sin \theta = 0 \text{ and } \cos \theta = -1 \Rightarrow \theta = \pi \text{ or } \theta = -\pi \Rightarrow$ Polar coordinates are $(-1, \pi)$ or $(-1, -\pi)$
- (c) $(0, -3) \Rightarrow r = -\sqrt{0^2 + (-3)^2} = -3, \sin \theta = -1 \text{ and } \cos \theta = 0 \Rightarrow \theta = \frac{3\pi}{2} \Rightarrow$ Polar coordinates are $(-3, \frac{3\pi}{2})$
- (d) $(\frac{\sqrt{3}}{2}, \frac{1}{2}) \Rightarrow r = -\sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2} = -1, \sin \theta = \frac{1}{2} \text{ and } \cos \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \theta = -\frac{5\pi}{6} \Rightarrow$ Polar coordinates are $(-1, \frac{\pi}{6})$ or $(-1, -\frac{5\pi}{6})$

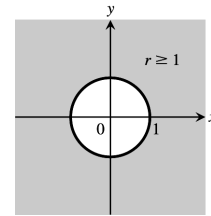
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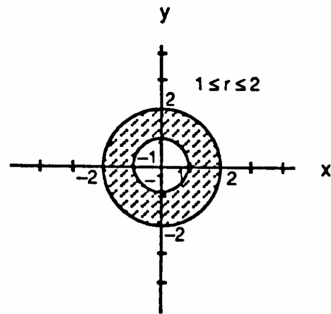
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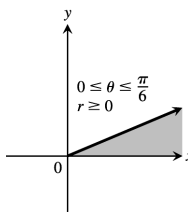
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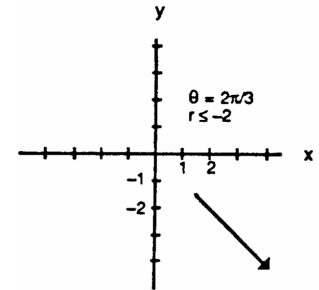
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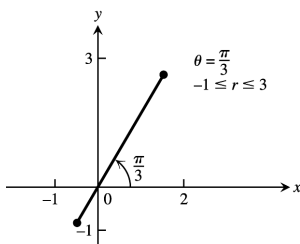
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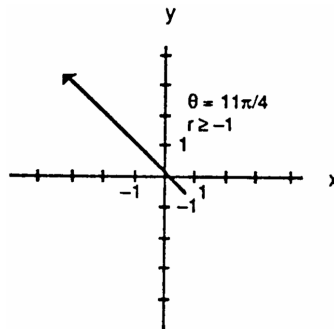
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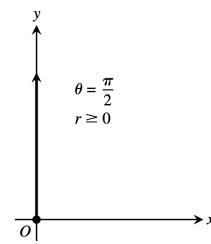
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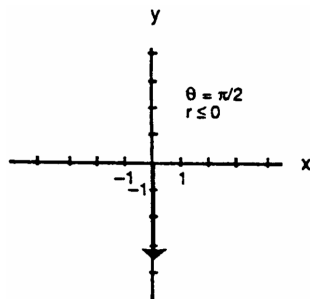
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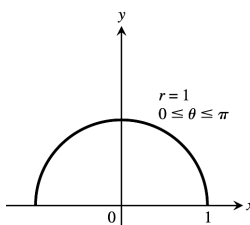
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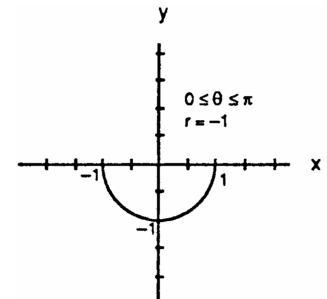
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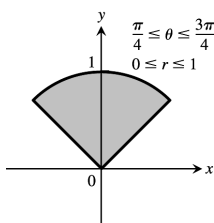
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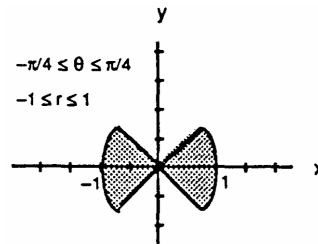
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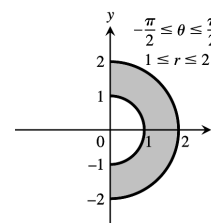
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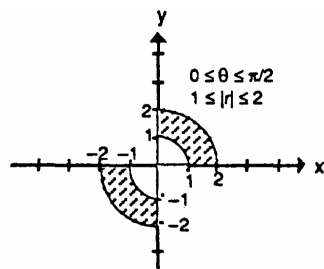
24.



25.



26.



27. $r \cos \theta = 2 \Rightarrow x = 2$, vertical line through $(2, 0)$ 28. $r \sin \theta = -1 \Rightarrow y = -1$, horizontal line through $(0, -1)$
29. $r \sin \theta = 0 \Rightarrow y = 0$, the x-axis 30. $r \cos \theta = 0 \Rightarrow x = 0$, the y-axis
31. $r = 4 \csc \theta \Rightarrow r = \frac{4}{\sin \theta} \Rightarrow r \sin \theta = 4 \Rightarrow y = 4$, a horizontal line through $(0, 4)$
32. $r = -3 \sec \theta \Rightarrow r = \frac{-3}{\cos \theta} \Rightarrow r \cos \theta = -3 \Rightarrow x = -3$, a vertical line through $(-3, 0)$
33. $r \cos \theta + r \sin \theta = 1 \Rightarrow x + y = 1$, line with slope $m = -1$ and intercept $b = 1$
34. $r \sin \theta = r \cos \theta \Rightarrow y = x$, line with slope $m = 1$ and intercept $b = 0$
35. $r^2 = 1 \Rightarrow x^2 + y^2 = 1$, circle with center $C = (0, 0)$ and radius 1
36. $r^2 = 4r \sin \theta \Rightarrow x^2 + y^2 = 4y \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + (y - 2)^2 = 4$, circle with center $C = (0, 2)$ and radius 2
37. $r = \frac{5}{\sin \theta - 2 \cos \theta} \Rightarrow r \sin \theta - 2r \cos \theta = 5 \Rightarrow y - 2x = 5$, line with slope $m = 2$ and intercept $b = 5$
38. $r^2 \sin 2\theta = 2 \Rightarrow 2r^2 \sin \theta \cos \theta = 2 \Rightarrow (r \sin \theta)(r \cos \theta) = 1 \Rightarrow xy = 1$, hyperbola with focal axis $y = x$
39. $r = \cot \theta \csc \theta = \left(\frac{\cos \theta}{\sin \theta}\right) \left(\frac{1}{\sin \theta}\right) \Rightarrow r \sin^2 \theta = \cos \theta \Rightarrow r^2 \sin^2 \theta = r \cos \theta \Rightarrow y^2 = x$, parabola with vertex $(0, 0)$ which opens to the right
40. $r = 4 \tan \theta \sec \theta \Rightarrow r = 4 \left(\frac{\sin \theta}{\cos^2 \theta}\right) \Rightarrow r \cos^2 \theta = 4 \sin \theta \Rightarrow r^2 \cos^2 \theta = 4r \sin \theta \Rightarrow x^2 = 4y$, parabola with vertex $(0, 0)$ which opens upward
41. $r = (\csc \theta) e^{r \cos \theta} \Rightarrow r \sin \theta = e^{r \cos \theta} \Rightarrow y = e^x$, graph of the natural exponential function
42. $r \sin \theta = \ln r + \ln \cos \theta = \ln(r \cos \theta) \Rightarrow y = \ln x$, graph of the natural logarithm function
43. $r^2 + 2r^2 \cos \theta \sin \theta = 1 \Rightarrow x^2 + y^2 + 2xy = 1 \Rightarrow x^2 + 2xy + y^2 = 1 \Rightarrow (x + y)^2 = 1 \Rightarrow x + y = \pm 1$, two parallel straight lines of slope -1 and y-intercepts $b = \pm 1$
44. $\cos^2 \theta = \sin^2 \theta \Rightarrow r^2 \cos^2 \theta = r^2 \sin^2 \theta \Rightarrow x^2 = y^2 \Rightarrow |x| = |y| \Rightarrow \pm x = y$, two perpendicular lines through the origin with slopes 1 and -1 , respectively.
45. $r^2 = -4r \cos \theta \Rightarrow x^2 + y^2 = -4x \Rightarrow x^2 + 4x + y^2 = 0 \Rightarrow x^2 + 4x + 4 + y^2 = 4 \Rightarrow (x + 2)^2 + y^2 = 4$, a circle with center $C(-2, 0)$ and radius 2

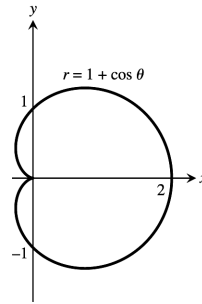
46. $r^2 = -6r \sin \theta \Rightarrow x^2 + y^2 = -6y \Rightarrow x^2 + y^2 + 6y = 0 \Rightarrow x^2 + y^2 + 6y + 9 = 9 \Rightarrow x^2 + (y + 3)^2 = 9$, a circle with center $C(0, -3)$ and radius 3
47. $r = 8 \sin \theta \Rightarrow r^2 = 8r \sin \theta \Rightarrow x^2 + y^2 = 8y \Rightarrow x^2 + y^2 - 8y = 0 \Rightarrow x^2 + y^2 - 8y + 16 = 16 \Rightarrow x^2 + (y - 4)^2 = 16$, a circle with center $C(0, 4)$ and radius 4
48. $r = 3 \cos \theta \Rightarrow r^2 = 3r \cos \theta \Rightarrow x^2 + y^2 = 3x \Rightarrow x^2 + y^2 - 3x = 0 \Rightarrow x^2 - 3x + \frac{9}{4} + y^2 = \frac{9}{4} \Rightarrow (x - \frac{3}{2})^2 + y^2 = \frac{9}{4}$, a circle with center $C(\frac{3}{2}, 0)$ and radius $\frac{3}{2}$
49. $r = 2 \cos \theta + 2 \sin \theta \Rightarrow r^2 = 2r \cos \theta + 2r \sin \theta \Rightarrow x^2 + y^2 = 2x + 2y \Rightarrow x^2 - 2x + y^2 - 2y = 0 \Rightarrow (x - 1)^2 + (y - 1)^2 = 2$, a circle with center $C(1, 1)$ and radius $\sqrt{2}$
50. $r = 2 \cos \theta - \sin \theta \Rightarrow r^2 = 2r \cos \theta - r \sin \theta \Rightarrow x^2 + y^2 = 2x - y \Rightarrow x^2 - 2x + y^2 + y = 0 \Rightarrow (x - 1)^2 + (y + \frac{1}{2})^2 = \frac{5}{4}$, a circle with center $C(1, -\frac{1}{2})$ and radius $\frac{\sqrt{5}}{2}$
51. $r \sin(\theta + \frac{\pi}{6}) = 2 \Rightarrow r(\sin \theta \cos \frac{\pi}{6} + \cos \theta \sin \frac{\pi}{6}) = 2 \Rightarrow \frac{\sqrt{3}}{2} r \sin \theta + \frac{1}{2} r \cos \theta = 2 \Rightarrow \frac{\sqrt{3}}{2} y + \frac{1}{2} x = 2 \Rightarrow \sqrt{3} y + x = 4$, line with slope $m = -\frac{1}{\sqrt{3}}$ and intercept $b = \frac{4}{\sqrt{3}}$
52. $r \sin(\frac{2\pi}{3} - \theta) = 5 \Rightarrow r(\sin \frac{2\pi}{3} \cos \theta - \cos \frac{2\pi}{3} \sin \theta) = 5 \Rightarrow \frac{\sqrt{3}}{2} r \cos \theta + \frac{1}{2} r \sin \theta = 5 \Rightarrow \frac{\sqrt{3}}{2} x + \frac{1}{2} y = 5 \Rightarrow \sqrt{3} x + y = 10$, line with slope $m = -\sqrt{3}$ and intercept $b = 10$
53. $x = 7 \Rightarrow r \cos \theta = 7$
54. $y = 1 \Rightarrow r \sin \theta = 1$
55. $x = y \Rightarrow r \cos \theta = r \sin \theta \Rightarrow \theta = \frac{\pi}{4}$
56. $x - y = 3 \Rightarrow r \cos \theta - r \sin \theta = 3$
57. $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$ or $r = -2$
58. $x^2 - y^2 = 1 \Rightarrow r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1 \Rightarrow r^2 (\cos^2 \theta - \sin^2 \theta) = 1 \Rightarrow r^2 \cos 2\theta = 1$
59. $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow 4x^2 + 9y^2 = 36 \Rightarrow 4r^2 \cos^2 \theta + 9r^2 \sin^2 \theta = 36$
60. $xy = 2 \Rightarrow (r \cos \theta)(r \sin \theta) = 2 \Rightarrow r^2 \cos \theta \sin \theta = 2 \Rightarrow 2r^2 \cos \theta \sin \theta = 4 \Rightarrow r^2 \sin 2\theta = 4$
61. $y^2 = 4x \Rightarrow r^2 \sin^2 \theta = 4r \cos \theta \Rightarrow r \sin^2 \theta = 4 \cos \theta$
62. $x^2 + xy + y^2 = 1 \Rightarrow x^2 + y^2 + xy = 1 \Rightarrow r^2 + r^2 \sin \theta \cos \theta = 1 \Rightarrow r^2 (1 + \sin \theta \cos \theta) = 1$
63. $x^2 + (y - 2)^2 = 4 \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + y^2 = 4y \Rightarrow r^2 = 4r \sin \theta \Rightarrow r = 4 \sin \theta$
64. $(x - 5)^2 + y^2 = 25 \Rightarrow x^2 - 10x + 25 + y^2 = 25 \Rightarrow x^2 + y^2 = 10x \Rightarrow r^2 = 10r \cos \theta \Rightarrow r = 10 \cos \theta$
65. $(x - 3)^2 + (y + 1)^2 = 4 \Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1 = 4 \Rightarrow x^2 + y^2 = 6x - 2y - 6 \Rightarrow r^2 = 6r \cos \theta - 2r \sin \theta - 6$
66. $(x + 2)^2 + (y - 5)^2 = 16 \Rightarrow x^2 + 4x + 4 + y^2 - 10y + 25 = 16 \Rightarrow x^2 + y^2 = -4x + 10y - 13 \Rightarrow r^2 = -4r \cos \theta + 10r \sin \theta - 13$

67. $(0, \theta)$ where θ is any angle

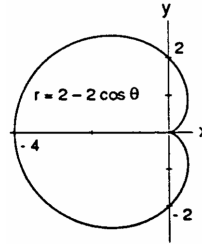
68. (a) $x = a \Rightarrow r \cos \theta = a \Rightarrow r = \frac{a}{\cos \theta} \Rightarrow r = a \sec \theta$
 (b) $y = b \Rightarrow r \sin \theta = b \Rightarrow r = \frac{b}{\sin \theta} \Rightarrow r = b \csc \theta$

11.4 GRAPHING IN POLAR COORDINATES

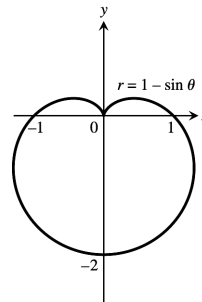
1. $1 + \cos(-\theta) = 1 + \cos \theta = r \Rightarrow$ symmetric about the x-axis; $1 + \cos(-\theta) \neq -r$ and $1 + \cos(\pi - \theta) = 1 - \cos \theta \neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



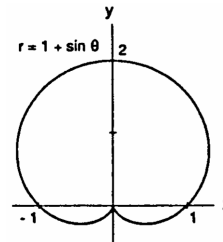
2. $2 - 2 \cos(-\theta) = 2 - 2 \cos \theta = r \Rightarrow$ symmetric about the x-axis; $2 - 2 \cos(-\theta) \neq -r$ and $2 - 2 \cos(\pi - \theta) = 2 + 2 \cos \theta \neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



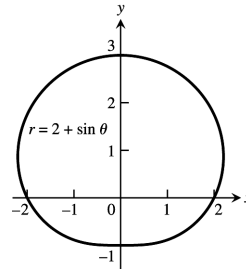
3. $1 - \sin(-\theta) = 1 + \sin \theta \neq r$ and $1 - \sin(\pi - \theta) = 1 - \sin \theta \neq -r \Rightarrow$ not symmetric about the x-axis; $1 - \sin(\pi - \theta) = 1 - \sin \theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



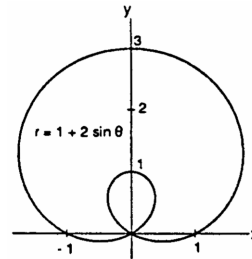
4. $1 + \sin(-\theta) = 1 - \sin \theta \neq r$ and $1 + \sin(\pi - \theta) = 1 + \sin \theta \neq -r \Rightarrow$ not symmetric about the x-axis; $1 + \sin(\pi - \theta) = 1 + \sin \theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



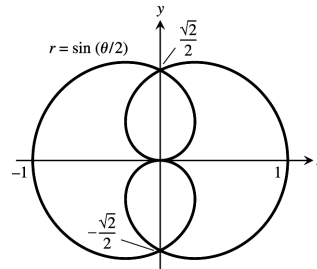
5. $2 + \sin(-\theta) = 2 - \sin \theta \neq r$ and $2 + \sin(\pi - \theta) = 2 + \sin \theta \neq -r \Rightarrow$ not symmetric about the x-axis;
 $2 + \sin(\pi - \theta) = 2 + \sin \theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



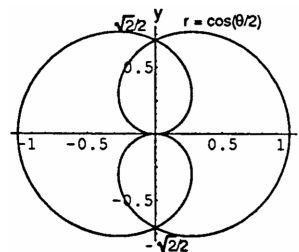
6. $1 + 2 \sin(-\theta) = 1 - 2 \sin \theta \neq r$ and $1 + 2 \sin(\pi - \theta) = 1 + 2 \sin \theta \neq -r \Rightarrow$ not symmetric about the x-axis;
 $1 + 2 \sin(\pi - \theta) = 1 + 2 \sin \theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



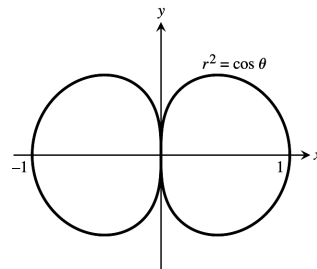
7. $\sin(-\frac{\theta}{2}) = -\sin(\frac{\theta}{2}) = -r \Rightarrow$ symmetric about the y-axis;
 $\sin(\frac{2\pi-\theta}{2}) = \sin(\frac{\theta}{2})$, so the graph is symmetric about the x-axis, and hence the origin.



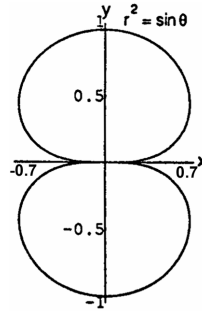
8. $\cos(-\frac{\theta}{2}) = \cos(\frac{\theta}{2}) = r \Rightarrow$ symmetric about the x-axis;
 $\cos(\frac{2\pi-\theta}{2}) = \cos(\frac{\theta}{2})$, so the graph is symmetric about the y-axis, and hence the origin.



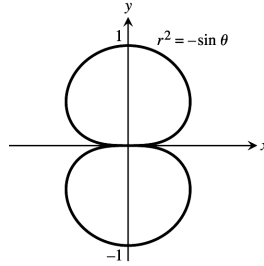
9. $\cos(-\theta) = \cos \theta = r^2 \Rightarrow (r, -\theta)$ and $(-r, -\theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the x-axis and the y-axis; therefore symmetric about the origin



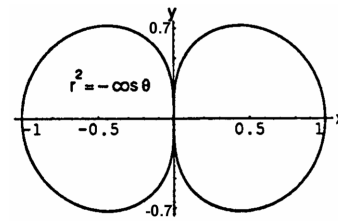
10. $\sin(\pi - \theta) = \sin \theta = r^2 \Rightarrow (r, \pi - \theta)$ and $(-r, \pi - \theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the y-axis and the x-axis; therefore symmetric about the origin



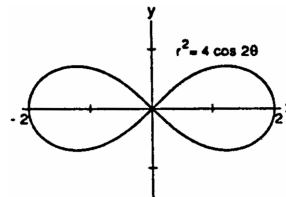
11. $-\sin(\pi - \theta) = -\sin \theta = r^2 \Rightarrow (r, \pi - \theta)$ and $(-r, \pi - \theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the y-axis and the x-axis; therefore symmetric about the origin



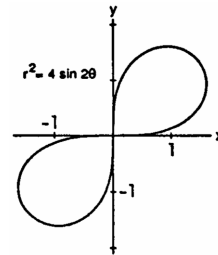
12. $-\cos(-\theta) = -\cos \theta = r^2 \Rightarrow (r, -\theta)$ and $(-r, -\theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the x-axis and the y-axis; therefore symmetric about the origin



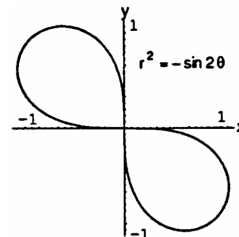
13. Since $(\pm r, -\theta)$ are on the graph when (r, θ) is on the graph $((\pm r)^2 = 4 \cos 2(-\theta) \Rightarrow r^2 = 4 \cos 2\theta)$, the graph is symmetric about the x-axis and the y-axis $\Rightarrow$ the graph is symmetric about the origin



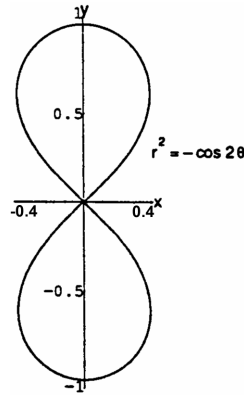
14. Since (r, θ) on the graph $\Rightarrow (-r, \theta)$ is on the graph $((\pm r)^2 = 4 \sin 2\theta \Rightarrow r^2 = 4 \sin 2\theta)$, the graph is symmetric about the origin. But $4 \sin 2(-\theta) = -4 \sin 2\theta \neq r^2$ and $4 \sin 2(\pi - \theta) = 4 \sin(2\pi - 2\theta) = 4 \sin(-2\theta) = -4 \sin 2\theta \neq r^2 \Rightarrow$ the graph is not symmetric about the x-axis; therefore the graph is not symmetric about the y-axis



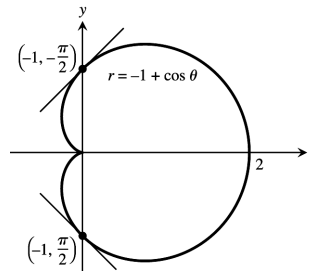
15. Since (r, θ) on the graph $\Rightarrow (-r, \theta)$ is on the graph $((\pm r)^2 = -\sin 2\theta \Rightarrow r^2 = -\sin 2\theta)$, the graph is symmetric about the origin. But $-\sin 2(-\theta) = -(-\sin 2\theta) = \sin 2\theta \neq r^2$ and $-\sin 2(\pi - \theta) = -\sin(2\pi - 2\theta) = -\sin(-2\theta) = \sin 2\theta \neq r^2 \Rightarrow$ the graph is not symmetric about the x-axis; therefore the graph is not symmetric about the y-axis



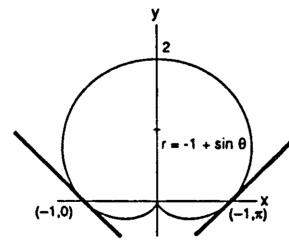
16. Since $(\pm r, -\theta)$ are on the graph when (r, θ) is on the graph $((\pm r)^2 = -\cos 2(-\theta) \Rightarrow r^2 = -\cos 2\theta)$, the graph is symmetric about the x-axis and the y-axis $\Rightarrow$ the graph is symmetric about the origin.



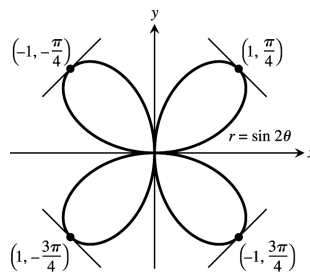
17. $\theta = \frac{\pi}{2} \Rightarrow r = -1 \Rightarrow (-1, \frac{\pi}{2})$, and $\theta = -\frac{\pi}{2} \Rightarrow r = -1 \Rightarrow (-1, -\frac{\pi}{2})$; $r' = \frac{dr}{d\theta} = -\sin \theta$; Slope $= \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$
 $= \frac{-\sin^2 \theta + r \cos \theta}{-\sin \theta \cos \theta - r \sin \theta} \Rightarrow$ Slope at $(-1, \frac{\pi}{2})$ is
 $\frac{-\sin^2(\frac{\pi}{2}) + (-1) \cos \frac{\pi}{2}}{-\sin \frac{\pi}{2} \cos \frac{\pi}{2} - (-1) \sin \frac{\pi}{2}} = -1$; Slope at $(-1, -\frac{\pi}{2})$ is
 $\frac{-\sin^2(-\frac{\pi}{2}) + (-1) \cos(-\frac{\pi}{2})}{-\sin(-\frac{\pi}{2}) \cos(-\frac{\pi}{2}) - (-1) \sin(-\frac{\pi}{2})} = 1$



18. $\theta = 0 \Rightarrow r = -1 \Rightarrow (-1, 0)$, and $\theta = \pi \Rightarrow r = -1 \Rightarrow (-1, \pi)$; $r' = \frac{dr}{d\theta} = \cos \theta$;
 Slope $= \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{\cos \theta \sin \theta + r \cos \theta}{\cos \theta \cos \theta - r \sin \theta}$
 $= \frac{\cos \theta \sin \theta + r \cos \theta}{\cos^2 \theta - r \sin \theta} \Rightarrow$ Slope at $(-1, 0)$ is $\frac{\cos 0 \sin 0 + (-1) \cos 0}{\cos^2 0 - (-1) \sin 0} = -1$;
 Slope at $(-1, \pi)$ is $\frac{\cos \pi \sin \pi + (-1) \cos \pi}{\cos^2 \pi - (-1) \sin \pi} = 1$



19. $\theta = \frac{\pi}{4} \Rightarrow r = 1 \Rightarrow (1, \frac{\pi}{4})$; $\theta = -\frac{\pi}{4} \Rightarrow r = -1 \Rightarrow (-1, -\frac{\pi}{4})$;
 $\theta = \frac{3\pi}{4} \Rightarrow r = -1 \Rightarrow (-1, \frac{3\pi}{4})$;
 $\theta = -\frac{3\pi}{4} \Rightarrow r = 1 \Rightarrow (1, -\frac{3\pi}{4})$;
 $r' = \frac{dr}{d\theta} = 2 \cos 2\theta$;
 Slope $= \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{2 \cos 2\theta \sin \theta + r \cos \theta}{2 \cos 2\theta \cos \theta - r \sin \theta}$
 $\Rightarrow$ Slope at $(1, \frac{\pi}{4})$ is $\frac{2 \cos(\frac{\pi}{2}) \sin(\frac{\pi}{4}) + (1) \cos(\frac{\pi}{4})}{2 \cos(\frac{\pi}{2}) \cos(\frac{\pi}{4}) - (1) \sin(\frac{\pi}{4})} = -1$;
 Slope at $(-1, -\frac{\pi}{4})$ is $\frac{2 \cos(-\frac{\pi}{2}) \sin(-\frac{\pi}{4}) + (-1) \cos(-\frac{\pi}{4})}{2 \cos(-\frac{\pi}{2}) \cos(-\frac{\pi}{4}) - (-1) \sin(-\frac{\pi}{4})} = 1$;
 Slope at $(-1, \frac{3\pi}{4})$ is $\frac{2 \cos(\frac{3\pi}{2}) \sin(\frac{3\pi}{4}) + (-1) \cos(\frac{3\pi}{4})}{2 \cos(\frac{3\pi}{2}) \cos(\frac{3\pi}{4}) - (-1) \sin(\frac{3\pi}{4})} = 1$;
 Slope at $(1, -\frac{3\pi}{4})$ is $\frac{2 \cos(-\frac{3\pi}{2}) \sin(-\frac{3\pi}{4}) + (1) \cos(-\frac{3\pi}{4})}{2 \cos(-\frac{3\pi}{2}) \cos(-\frac{3\pi}{4}) - (1) \sin(-\frac{3\pi}{4})} = -1$



$$\begin{aligned}
 20. \quad & \theta = 0 \Rightarrow r = 1 \Rightarrow (1, 0); \theta = \frac{\pi}{2} \Rightarrow r = -1 \Rightarrow (-1, \frac{\pi}{2}); \\
 & \theta = -\frac{\pi}{2} \Rightarrow r = -1 \Rightarrow (-1, -\frac{\pi}{2}); \theta = \pi \Rightarrow r = 1 \\
 & \Rightarrow (1, \pi); r' = \frac{dr}{d\theta} = -2 \sin 2\theta;
 \end{aligned}$$

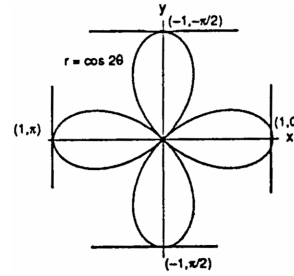
$$\text{Slope} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-2 \sin 2\theta \sin \theta + r \cos \theta}{-2 \sin 2\theta \cos \theta - r \sin \theta}$$

$$\Rightarrow \text{Slope at } (1, 0) \text{ is } \frac{-2 \sin 0 \sin 0 + \cos 0}{-2 \sin 0 \cos 0 - \sin 0} = \frac{1}{-1} = -1, \text{ which is undefined};$$

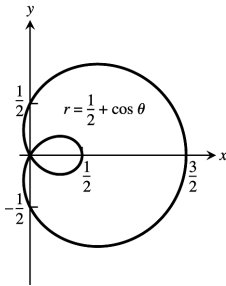
$$\text{Slope at } (-1, \frac{\pi}{2}) \text{ is } \frac{-2 \sin 2(\frac{\pi}{2}) \sin(\frac{\pi}{2}) + (-1) \cos(\frac{\pi}{2})}{-2 \sin 2(\frac{\pi}{2}) \cos(\frac{\pi}{2}) - (-1) \sin(\frac{\pi}{2})} = 0;$$

$$\text{Slope at } (-1, -\frac{\pi}{2}) \text{ is } \frac{-2 \sin 2(-\frac{\pi}{2}) \sin(-\frac{\pi}{2}) + (-1) \cos(-\frac{\pi}{2})}{-2 \sin 2(-\frac{\pi}{2}) \cos(-\frac{\pi}{2}) - (-1) \sin(-\frac{\pi}{2})} = 0;$$

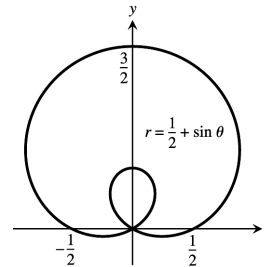
$$\text{Slope at } (1, \pi) \text{ is } \frac{-2 \sin 2\pi \sin \pi + \cos \pi}{-2 \sin 2\pi \cos \pi - \sin \pi} = \frac{-1}{-1} = 1, \text{ which is undefined}$$



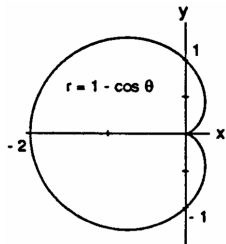
21. (a)



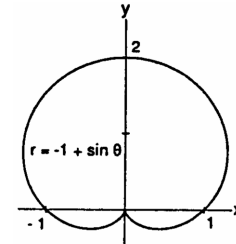
(b)



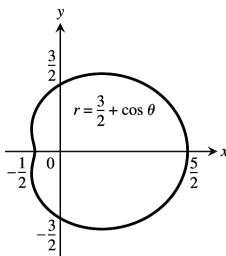
22. (a)



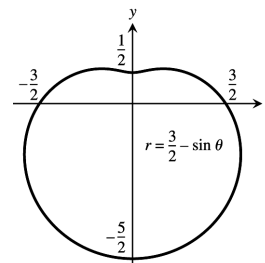
(b)



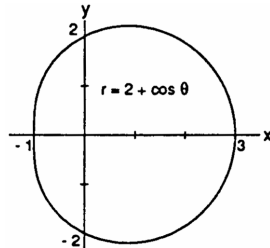
23. (a)



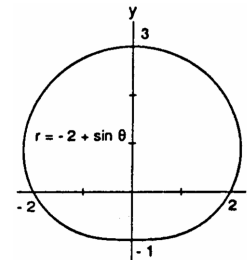
(b)



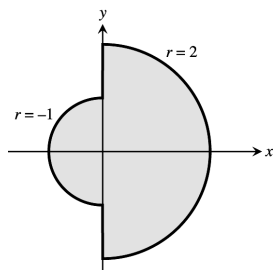
24. (a)



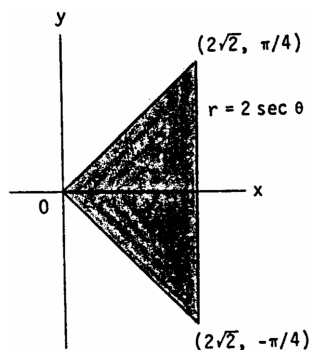
(b)



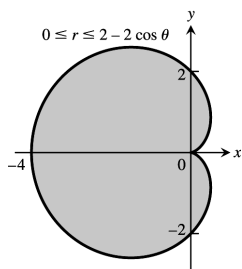
25.



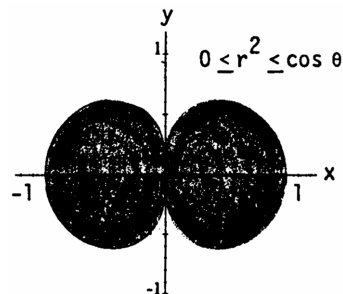
26. $r = 2 \sec \theta \Rightarrow r = \frac{2}{\cos \theta} \Rightarrow r \cos \theta = 2 \Rightarrow x = 2$



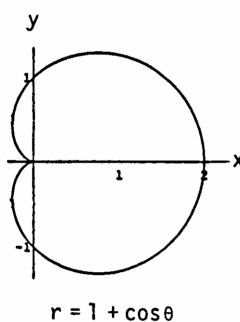
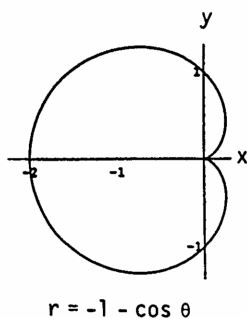
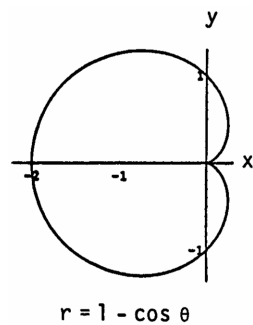
27.



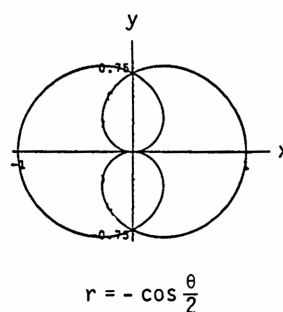
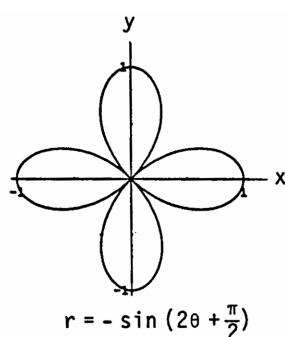
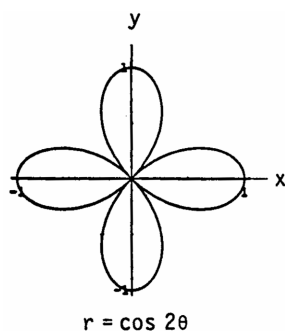
28.



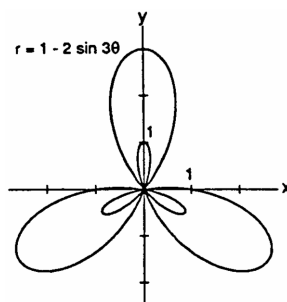
29. Note that (r, θ) and $(-r, \theta + \pi)$ describe the same point in the plane. Then $r = 1 - \cos \theta \Leftrightarrow -1 - \cos(\theta + \pi) = -1 - (\cos \theta \cos \pi - \sin \theta \sin \pi) = -1 + \cos \theta = -(1 - \cos \theta) = -r$; therefore (r, θ) is on the graph of $r = 1 - \cos \theta \Leftrightarrow (-r, \theta + \pi)$ is on the graph of $r = -1 - \cos \theta \Rightarrow$ the answer is (a).



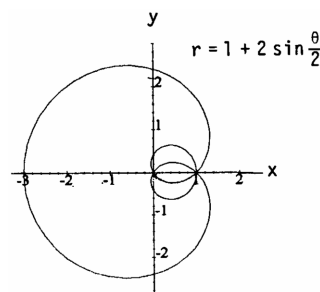
30. Note that (r, θ) and $(-r, \theta + \pi)$ describe the same point in the plane. Then $r = \cos 2\theta \Leftrightarrow -\sin(2(\theta + \pi) + \frac{\pi}{2}) = -\sin(2\theta + \frac{5\pi}{2}) = -\sin(2\theta) \cos(\frac{5\pi}{2}) - \cos(2\theta) \sin(\frac{5\pi}{2}) = -\cos 2\theta = -r$; therefore (r, θ) is on the graph of $r = -\sin(2\theta + \frac{\pi}{2}) \Rightarrow$ the answer is (a).



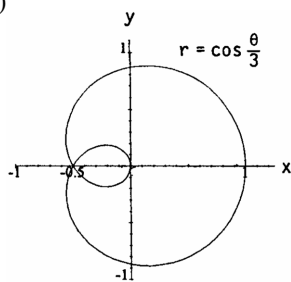
31.



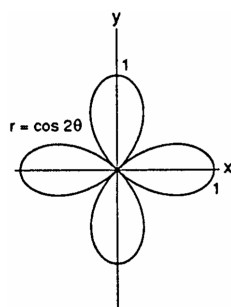
32.



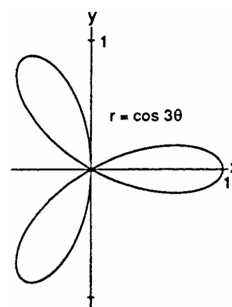
33. (a)



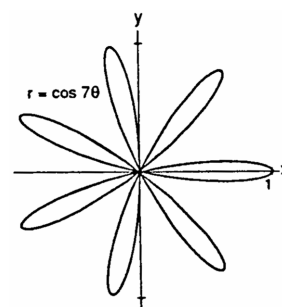
(b)



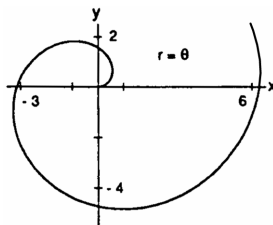
(c)



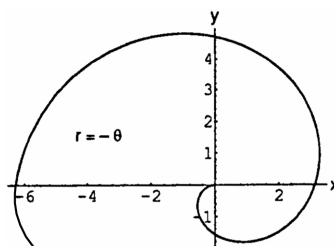
(d)



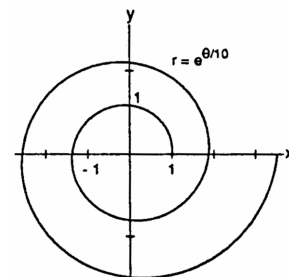
34. (a)



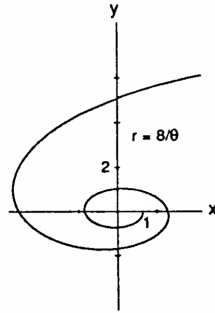
(b)



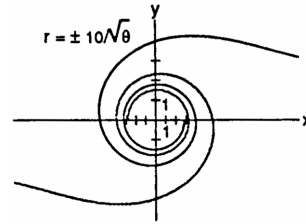
(c)



(d)



(e)



11.5 AREA AND LENGTHS IN POLAR COORDINATES

$$1. A = \int_0^{\pi} \frac{1}{2} \theta^2 d\theta = \left[\frac{1}{6} \theta^3 \right]_0^{\pi} = \frac{\pi^3}{6}$$

$$2. A = \int_{\pi/4}^{\pi/2} \frac{1}{2} (2 \sin \theta)^2 d\theta = 2 \int_{\pi/4}^{\pi/2} \sin^2 \theta d\theta = 2 \int_{\pi/4}^{\pi/2} \frac{1 - \cos 2\theta}{2} d\theta = \int_{\pi/4}^{\pi/2} (1 - \cos 2\theta) d\theta = \left[\theta - \frac{1}{2} \sin 2\theta \right]_{\pi/4}^{\pi/2} \\ = \left(\frac{\pi}{2} - 0 \right) - \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi}{4} + \frac{1}{2}$$

$$3. A = \int_0^{2\pi} \frac{1}{2} (4 + 2 \cos \theta)^2 d\theta = \int_0^{2\pi} \frac{1}{2} (16 + 16 \cos \theta + 4 \cos^2 \theta) d\theta = \int_0^{2\pi} \left[8 + 8 \cos \theta + 2 \left(\frac{1 + \cos 2\theta}{2} \right) \right] d\theta \\ = \int_0^{2\pi} (9 + 8 \cos \theta + \cos 2\theta) d\theta = \left[9\theta + 8 \sin \theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi} = 18\pi$$

$$4. A = \int_0^{2\pi} \frac{1}{2} [a(1 + \cos \theta)]^2 d\theta = \int_0^{2\pi} \frac{1}{2} a^2 (1 + 2 \cos \theta + \cos^2 \theta) d\theta = \frac{1}{2} a^2 \int_0^{2\pi} \left(1 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta \\ = \frac{1}{2} a^2 \int_0^{2\pi} \left(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta = \frac{1}{2} a^2 \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{3}{2} \pi a^2$$

$$5. A = 2 \int_0^{\pi/4} \frac{1}{2} \cos^2 2\theta d\theta = \int_0^{\pi/4} \frac{1 + \cos 4\theta}{2} d\theta = \frac{1}{2} \left[\theta + \frac{\sin 4\theta}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$

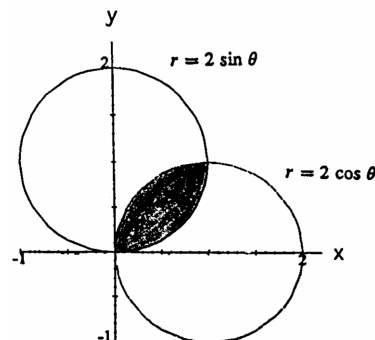
$$6. A = \int_{-\pi/6}^{\pi/6} \frac{1}{2} (\cos 3\theta)^2 d\theta = \frac{1}{2} \int_{-\pi/6}^{\pi/6} \cos^2 3\theta d\theta = \frac{1}{2} \int_{-\pi/6}^{\pi/6} \frac{1 + \cos 6\theta}{2} d\theta = \frac{1}{4} \int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta) d\theta \\ = \frac{1}{4} \left[\theta + \frac{1}{6} \sin 6\theta \right]_{-\pi/6}^{\pi/6} = \frac{1}{4} \left(\frac{\pi}{6} + 0 \right) - \frac{1}{4} \left(-\frac{\pi}{6} + 0 \right) = \frac{\pi}{12}$$

$$7. A = \int_0^{\pi/2} \frac{1}{2} (4 \sin 2\theta) d\theta = \int_0^{\pi/2} 2 \sin 2\theta d\theta = [-\cos 2\theta]_0^{\pi/2} = 2$$

$$8. A = (6)(2) \int_0^{\pi/6} \frac{1}{2} (2 \sin 3\theta) d\theta = 12 \int_0^{\pi/6} \sin 3\theta d\theta = 12 \left[-\frac{\cos 3\theta}{3} \right]_0^{\pi/6} = 4$$

$$9. r = 2 \cos \theta \text{ and } r = 2 \sin \theta \Rightarrow 2 \cos \theta = 2 \sin \theta \\ \Rightarrow \cos \theta = \sin \theta \Rightarrow \theta = \frac{\pi}{4}; \text{ therefore}$$

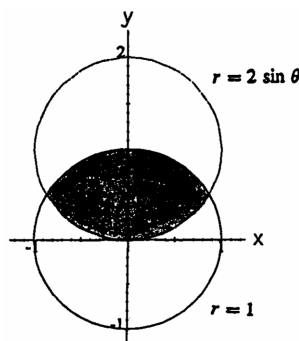
$$A = 2 \int_0^{\pi/4} \frac{1}{2} (2 \sin \theta)^2 d\theta = \int_0^{\pi/4} 4 \sin^2 \theta d\theta \\ = \int_0^{\pi/4} 4 \left(\frac{1 - \cos 2\theta}{2} \right) d\theta = \int_0^{\pi/4} (2 - 2 \cos 2\theta) d\theta \\ = [2\theta - \sin 2\theta]_0^{\pi/4} = \frac{\pi}{2} - 1$$



$$10. r = 1 \text{ and } r = 2 \sin \theta \Rightarrow 2 \sin \theta = 1 \Rightarrow \sin \theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}; \text{ therefore}$$

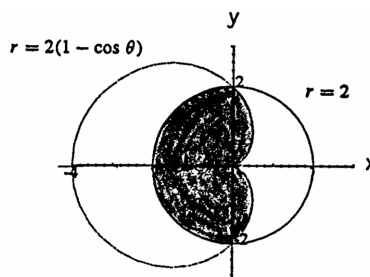
$$\begin{aligned} A &= \pi(1)^2 - \int_{\pi/6}^{5\pi/6} \frac{1}{2} [(2 \sin \theta)^2 - 1^2] d\theta \\ &= \pi - \int_{\pi/6}^{5\pi/6} (2 \sin^2 \theta - \frac{1}{2}) d\theta \\ &= \pi - \int_{\pi/6}^{5\pi/6} (1 - \cos 2\theta - \frac{1}{2}) d\theta \\ &= \pi - \int_{\pi/6}^{5\pi/6} (\frac{1}{2} - \cos 2\theta) d\theta = \pi - [\frac{1}{2}\theta - \frac{\sin 2\theta}{2}]_{\pi/6}^{5\pi/6} \\ &= \pi - (\frac{5\pi}{12} - \frac{1}{2} \sin \frac{5\pi}{3}) + (\frac{\pi}{12} - \frac{1}{2} \sin \frac{\pi}{3}) = \frac{4\pi - 3\sqrt{3}}{6} \end{aligned}$$



$$11. r = 2 \text{ and } r = 2(1 - \cos \theta) \Rightarrow 2 = 2(1 - \cos \theta)$$

$$\Rightarrow \cos \theta = 0 \Rightarrow \theta = \pm \frac{\pi}{2}; \text{ therefore}$$

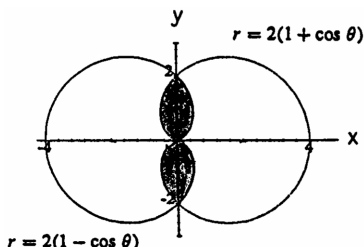
$$\begin{aligned} A &= 2 \int_0^{\pi/2} \frac{1}{2} [2(1 - \cos \theta)]^2 d\theta + \frac{1}{2} \text{area of the circle} \\ &= \int_0^{\pi/2} 4(1 - 2 \cos \theta + \cos^2 \theta) d\theta + (\frac{1}{2} \pi)(2)^2 \\ &= \int_0^{\pi/2} 4(1 - 2 \cos \theta + \frac{1 + \cos 2\theta}{2}) d\theta + 2\pi \\ &= \int_0^{\pi/2} (4 - 8 \cos \theta + 2 + 2 \cos 2\theta) d\theta + 2\pi \\ &= [6\theta - 8 \sin \theta + \sin 2\theta]_0^{\pi/2} + 2\pi = 5\pi - 8 \end{aligned}$$



$$12. r = 2(1 - \cos \theta) \text{ and } r = 2(1 + \cos \theta) \Rightarrow 1 - \cos \theta$$

$$= 1 + \cos \theta \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}; \text{ the graph also gives the point of intersection } (0, 0); \text{ therefore}$$

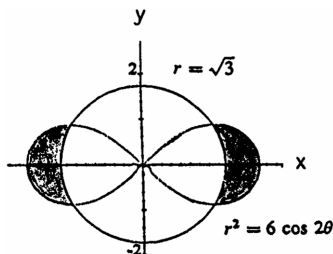
$$\begin{aligned} A &= 2 \int_0^{\pi/2} \frac{1}{2} [2(1 - \cos \theta)]^2 d\theta + 2 \int_{\pi/2}^{\pi} \frac{1}{2} [2(1 + \cos \theta)]^2 d\theta \\ &= \int_0^{\pi/2} 4(1 - 2 \cos \theta + \cos^2 \theta) d\theta \\ &\quad + \int_{\pi/2}^{\pi} 4(1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \int_0^{\pi/2} 4(1 - 2 \cos \theta + \frac{1 + \cos 2\theta}{2}) d\theta + \int_{\pi/2}^{\pi} 4(1 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2}) d\theta \\ &= \int_0^{\pi/2} (6 - 8 \cos \theta + 2 \cos 2\theta) d\theta + \int_{\pi/2}^{\pi} (6 + 8 \cos \theta + 2 \cos 2\theta) d\theta \\ &= [6\theta - 8 \sin \theta + \sin 2\theta]_0^{\pi/2} + [6\theta + 8 \sin \theta + \sin 2\theta]_{\pi/2}^{\pi} = 6\pi - 16 \end{aligned}$$



$$13. r = \sqrt{3} \text{ and } r^2 = 6 \cos 2\theta \Rightarrow 3 = 6 \cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6} \text{ (in the 1st quadrant); we use symmetry of the graph to find the area, so}$$

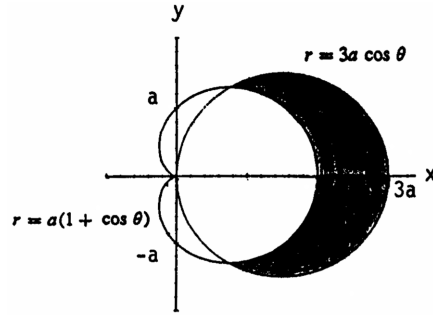
$$\begin{aligned} A &= 4 \int_0^{\pi/6} \left[\frac{1}{2} (6 \cos 2\theta) - \frac{1}{2} (\sqrt{3})^2 \right] d\theta \\ &= 2 \int_0^{\pi/6} (6 \cos 2\theta - 3) d\theta = 2 [3 \sin 2\theta - 3\theta]_0^{\pi/6} \\ &= 3\sqrt{3} - \pi \end{aligned}$$



$$14. \quad r = 3a \cos \theta \text{ and } r = a(1 + \cos \theta) \Rightarrow 3a \cos \theta = a(1 + \cos \theta) \\ \Rightarrow 3 \cos \theta = 1 + \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ or } -\frac{\pi}{3};$$

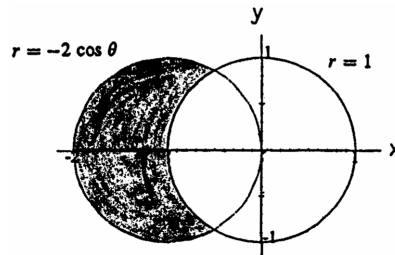
the graph also gives the point of intersection $(0, 0)$; therefore

$$\begin{aligned} A &= 2 \int_0^{\pi/3} \frac{1}{2} [(3a \cos \theta)^2 - a^2(1 + \cos \theta)^2] d\theta \\ &= \int_0^{\pi/3} (9a^2 \cos^2 \theta - a^2 - 2a^2 \cos \theta - a^2 \cos^2 \theta) d\theta \\ &= \int_0^{\pi/3} (8a^2 \cos^2 \theta - 2a^2 \cos \theta - a^2) d\theta \\ &= \int_0^{\pi/3} [4a^2(1 + \cos 2\theta) - 2a^2 \cos \theta - a^2] d\theta \\ &= \int_0^{\pi/3} (3a^2 + 4a^2 \cos 2\theta - 2a^2 \cos \theta) d\theta \\ &= [3a^2\theta + 2a^2 \sin 2\theta - 2a^2 \sin \theta]_0^{\pi/3} = \pi a^2 + 2a^2 \left(\frac{1}{2}\right) - 2a^2 \left(\frac{\sqrt{3}}{2}\right) = a^2 (\pi + 1 - \sqrt{3}) \end{aligned}$$

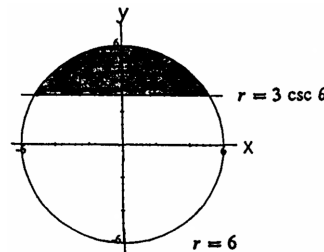


$$15. \quad r = 1 \text{ and } r = -2 \cos \theta \Rightarrow 1 = -2 \cos \theta \Rightarrow \cos \theta = -\frac{1}{2} \\ \Rightarrow \theta = \frac{2\pi}{3} \text{ in quadrant II; therefore}$$

$$\begin{aligned} A &= 2 \int_{2\pi/3}^{\pi} \frac{1}{2} [(-2 \cos \theta)^2 - 1^2] d\theta = \int_{2\pi/3}^{\pi} (4 \cos^2 \theta - 1) d\theta \\ &= \int_{2\pi/3}^{\pi} [2(1 + \cos 2\theta) - 1] d\theta = \int_{2\pi/3}^{\pi} (1 + 2 \cos 2\theta) d\theta \\ &= [\theta + \sin 2\theta]_{2\pi/3}^{\pi} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$

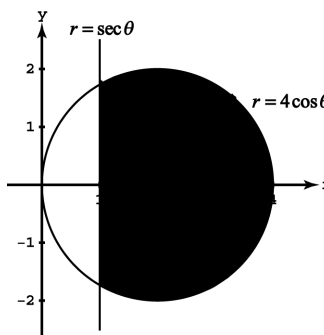


$$16. \quad r = 6 \text{ and } r = 3 \csc \theta \Rightarrow 6 \sin \theta = 3 \Rightarrow \sin \theta = \frac{1}{2} \\ \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}; \text{ therefore } A = \int_{\pi/6}^{5\pi/6} \frac{1}{2} (6^2 - 9 \csc^2 \theta) d\theta \\ = \int_{\pi/6}^{5\pi/6} (18 - \frac{9}{2} \csc^2 \theta) d\theta = [18\theta + \frac{9}{2} \cot \theta]_{\pi/6}^{5\pi/6} \\ = \left(15\pi - \frac{9}{2} \sqrt{3}\right) - \left(3\pi + \frac{9}{2} \sqrt{3}\right) = 12\pi - 9\sqrt{3}$$



$$17. \quad r = \sec \theta \text{ and } r = 4 \cos \theta \Rightarrow 4 \cos \theta = \sec \theta \Rightarrow \cos^2 \theta = \frac{1}{4} \\ \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}; \text{ therefore}$$

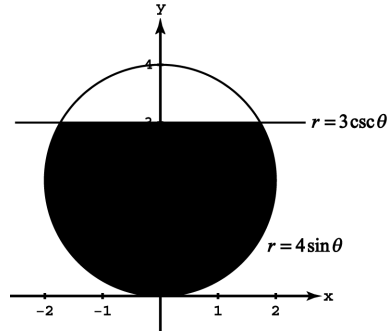
$$\begin{aligned} A &= 2 \int_0^{\pi/3} \frac{1}{2} (16 \cos^2 \theta - \sec^2 \theta) d\theta \\ &= \int_0^{\pi/3} (8 + 8 \cos 2\theta - \sec^2 \theta) d\theta \\ &= [8\theta + 4 \sin 2\theta - \tan \theta]_0^{\pi/3} \\ &= \left(\frac{8\pi}{3} + 2\sqrt{3} - \sqrt{3}\right) - (0 + 0 - 0) = \frac{8\pi}{3} + \sqrt{3} \end{aligned}$$



$$18. r = 3 \csc \theta \text{ and } r = 4 \sin \theta \Rightarrow 4 \sin \theta = 3 \csc \theta \Rightarrow \sin^2 \theta = \frac{3}{4}$$

$$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}; \text{ therefore}$$

$$\begin{aligned} A &= 4\pi - 2 \int_{\pi/3}^{\pi/2} \frac{1}{2} (16 \sin^2 \theta - 9 \csc^2 \theta) d\theta \\ &= 4\pi - \int_{\pi/3}^{\pi/2} (8 - 8 \cos 2\theta - 9 \csc^2 \theta) d\theta \\ &= 4\pi - [8\theta - 4 \sin 2\theta + 9 \cot \theta]_{\pi/3}^{\pi/2} \\ &= 4\pi - \left[(4\pi - 0 + 0) - \left(\frac{8\pi}{3} - 2\sqrt{3} + 3\sqrt{3} \right) \right] \\ &= \frac{8\pi}{3} + \sqrt{3} \end{aligned}$$



$$19. (a) r = \tan \theta \text{ and } r = \left(\frac{\sqrt{2}}{2} \right) \csc \theta \Rightarrow \tan \theta = \left(\frac{\sqrt{2}}{2} \right) \csc \theta$$

$$\Rightarrow \sin^2 \theta = \left(\frac{\sqrt{2}}{2} \right) \cos \theta \Rightarrow 1 - \cos^2 \theta = \left(\frac{\sqrt{2}}{2} \right) \cos \theta$$

$$\Rightarrow \cos^2 \theta + \left(\frac{\sqrt{2}}{2} \right) \cos \theta - 1 = 0 \Rightarrow \cos \theta = -\sqrt{2} \text{ or}$$

$$\frac{\sqrt{2}}{2} \text{ (use the quadratic formula)} \Rightarrow \theta = \frac{\pi}{4} \text{ (the solution}$$

in the first quadrant); therefore the area of R_1 is

$$A_1 = \int_0^{\pi/4} \frac{1}{2} \tan^2 \theta d\theta = \frac{1}{2} \int_0^{\pi/4} (\sec^2 \theta - 1) d\theta = \frac{1}{2} [\tan \theta - \theta]_0^{\pi/4} = \frac{1}{2} \left(\tan \frac{\pi}{4} - \frac{\pi}{4} \right) = \frac{1}{2} - \frac{\pi}{8}; AO = \left(\frac{\sqrt{2}}{2} \right) \csc \frac{\pi}{2}$$

$$= \frac{\sqrt{2}}{2} \text{ and } OB = \left(\frac{\sqrt{2}}{2} \right) \csc \frac{\pi}{4} = 1 \Rightarrow AB = \sqrt{1^2 - \left(\frac{\sqrt{2}}{2} \right)^2} = \frac{\sqrt{2}}{2} \Rightarrow \text{the area of } R_2 \text{ is } A_2 = \frac{1}{2} \left(\frac{\sqrt{2}}{2} \right) \left(\frac{\sqrt{2}}{2} \right) = \frac{1}{4};$$

therefore the area of the region shaded in the text is $2 \left(\frac{1}{2} - \frac{\pi}{8} + \frac{1}{4} \right) = \frac{3}{2} - \frac{\pi}{4}$. Note: The area must be found this way since no common interval generates the region. For example, the interval $0 \leq \theta \leq \frac{\pi}{4}$ generates the arc OB of $r = \tan \theta$

but does not generate the segment AB of the line $r = \frac{\sqrt{2}}{2} \csc \theta$. Instead the interval generates the half-line from B to $+\infty$ on the line $r = \frac{\sqrt{2}}{2} \csc \theta$.

$$(b) \lim_{\theta \rightarrow \pi/2^-} \tan \theta = \infty \text{ and the line } x = 1 \text{ is } r = \sec \theta \text{ in polar coordinates; then } \lim_{\theta \rightarrow \pi/2^-} (\tan \theta - \sec \theta)$$

$$= \lim_{\theta \rightarrow \pi/2^-} \left(\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta} \right) = \lim_{\theta \rightarrow \pi/2^-} \left(\frac{\sin \theta - 1}{\cos \theta} \right) = \lim_{\theta \rightarrow \pi/2^-} \left(\frac{\cos \theta}{-\sin \theta} \right) = 0 \Rightarrow r = \tan \theta \text{ approaches}$$

$r = \sec \theta$ as $\theta \rightarrow \frac{\pi}{2}^- \Rightarrow r = \sec \theta$ (or $x = 1$) is a vertical asymptote of $r = \tan \theta$. Similarly, $r = -\sec \theta$ (or $x = -1$) is a vertical asymptote of $r = \tan \theta$.

20. It is not because the circle is generated twice from $\theta = 0$ to 2π . The area of the cardioid is

$$A = 2 \int_0^\pi \frac{1}{2} (\cos \theta + 1)^2 d\theta = \int_0^\pi (\cos^2 \theta + 2 \cos \theta + 1) d\theta = \int_0^\pi \left(\frac{1 + \cos 2\theta}{2} + 2 \cos \theta + 1 \right) d\theta$$

$$= \left[\frac{3\theta}{2} + \frac{\sin 2\theta}{4} + 2 \sin \theta \right]_0^\pi = \frac{3\pi}{2}. \text{ The area of the circle is } A = \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{4} \Rightarrow \text{the area requested is actually } \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$$

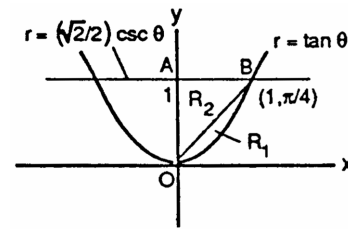
$$21. r = \theta^2, 0 \leq \theta \leq \sqrt{5} \Rightarrow \frac{dr}{d\theta} = 2\theta; \text{ therefore Length} = \int_0^{\sqrt{5}} \sqrt{(\theta^2)^2 + (2\theta)^2} d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2} d\theta$$

$$= \int_0^{\sqrt{5}} |\theta| \sqrt{\theta^2 + 4} d\theta = (\text{since } \theta \geq 0) \int_0^{\sqrt{5}} \theta \sqrt{\theta^2 + 4} d\theta; [u = \theta^2 + 4 \Rightarrow \frac{1}{2} du = \theta d\theta; \theta = 0 \Rightarrow u = 4,$$

$$\theta = \sqrt{5} \Rightarrow u = 9] \rightarrow \int_4^9 \frac{1}{2} \sqrt{u} du = \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_4^9 = \frac{19}{3}$$

$$22. r = \frac{e^\theta}{\sqrt{2}}, 0 \leq \theta \leq \pi \Rightarrow \frac{dr}{d\theta} = \frac{e^\theta}{\sqrt{2}}; \text{ therefore Length} = \int_0^\pi \sqrt{\left(\frac{e^\theta}{\sqrt{2}} \right)^2 + \left(\frac{e^\theta}{\sqrt{2}} \right)^2} d\theta = \int_0^\pi \sqrt{2 \left(\frac{e^{2\theta}}{2} \right)} d\theta$$

$$= \int_0^\pi e^\theta d\theta = [e^\theta]_0^\pi = e^\pi - 1$$



$$\begin{aligned}
 23. \quad r = 1 + \cos \theta &\Rightarrow \frac{dr}{d\theta} = -\sin \theta; \text{ therefore Length} = \int_0^{2\pi} \sqrt{(1 + \cos \theta)^2 + (-\sin \theta)^2} d\theta \\
 &= 2 \int_0^\pi \sqrt{2 + 2 \cos \theta} d\theta = 2 \int_0^\pi \sqrt{\frac{4(1 + \cos \theta)}{2}} d\theta = 4 \int_0^\pi \sqrt{\frac{1 + \cos \theta}{2}} d\theta = 4 \int_0^\pi \cos\left(\frac{\theta}{2}\right) d\theta = 4 \left[2 \sin \frac{\theta}{2}\right]_0^\pi = 8
 \end{aligned}$$

$$\begin{aligned}
 24. \quad r = a \sin^2 \frac{\theta}{2}, 0 \leq \theta \leq \pi, a > 0 &\Rightarrow \frac{dr}{d\theta} = a \sin \frac{\theta}{2} \cos \frac{\theta}{2}; \text{ therefore Length} = \int_0^\pi \sqrt{\left(a \sin^2 \frac{\theta}{2}\right)^2 + \left(a \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^2} d\theta \\
 &= \int_0^\pi \sqrt{a^2 \sin^4 \frac{\theta}{2} + a^2 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} d\theta = \int_0^\pi a \left|\sin \frac{\theta}{2}\right| \sqrt{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}} d\theta = (\text{since } 0 \leq \theta \leq \pi) \quad a \int_0^\pi \sin\left(\frac{\theta}{2}\right) d\theta \\
 &= \left[-2a \cos \frac{\theta}{2}\right]_0^\pi = 2a
 \end{aligned}$$

$$\begin{aligned}
 25. \quad r = \frac{6}{1 + \cos \theta}, 0 \leq \theta \leq \frac{\pi}{2} &\Rightarrow \frac{dr}{d\theta} = \frac{6 \sin \theta}{(1 + \cos \theta)^2}; \text{ therefore Length} = \int_0^{\pi/2} \sqrt{\left(\frac{6}{1 + \cos \theta}\right)^2 + \left(\frac{6 \sin \theta}{(1 + \cos \theta)^2}\right)^2} d\theta \\
 &= \int_0^{\pi/2} \sqrt{\frac{36}{(1 + \cos \theta)^2} + \frac{36 \sin^2 \theta}{(1 + \cos \theta)^4}} d\theta = 6 \int_0^{\pi/2} \left|\frac{1}{1 + \cos \theta}\right| \sqrt{1 + \frac{\sin^2 \theta}{(1 + \cos \theta)^2}} d\theta \\
 &= (\text{since } \frac{1}{1 + \cos \theta} > 0 \text{ on } 0 \leq \theta \leq \frac{\pi}{2}) \quad 6 \int_0^{\pi/2} \left(\frac{1}{1 + \cos \theta}\right) \sqrt{\frac{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 + \cos \theta)^2}} d\theta \\
 &= 6 \int_0^{\pi/2} \left(\frac{1}{1 + \cos \theta}\right) \sqrt{\frac{2 + 2 \cos \theta}{(1 + \cos \theta)^2}} d\theta = 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{(1 + \cos \theta)^{3/2}} = 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{(2 \cos^2 \frac{\theta}{2})^{3/2}} = 3 \int_0^{\pi/2} \left|\sec^3 \frac{\theta}{2}\right| d\theta \\
 &= 3 \int_0^{\pi/2} \sec^3 \frac{\theta}{2} d\theta = 6 \int_0^{\pi/4} \sec^3 u du = (\text{use tables}) \quad 6 \left(\left[\frac{\sec u \tan u}{2}\right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \sec u du \right) \\
 &= 6 \left(\frac{1}{\sqrt{2}} + \left[\frac{1}{2} \ln |\sec u + \tan u|\right]_0^{\pi/4} \right) = 3 \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right]
 \end{aligned}$$

$$\begin{aligned}
 26. \quad r = \frac{2}{1 - \cos \theta}, \frac{\pi}{2} \leq \theta \leq \pi &\Rightarrow \frac{dr}{d\theta} = \frac{-2 \sin \theta}{(1 - \cos \theta)^2}; \text{ therefore Length} = \int_{\pi/2}^\pi \sqrt{\left(\frac{2}{1 - \cos \theta}\right)^2 + \left(\frac{-2 \sin \theta}{(1 - \cos \theta)^2}\right)^2} d\theta \\
 &= \int_{\pi/2}^\pi \sqrt{\frac{4}{(1 - \cos \theta)^2} + \frac{4 \sin^2 \theta}{(1 - \cos \theta)^4}} d\theta = \int_{\pi/2}^\pi \left|\frac{2}{1 - \cos \theta}\right| \sqrt{\frac{(1 - \cos \theta)^2 + \sin^2 \theta}{(1 - \cos \theta)^2}} d\theta \\
 &= (\text{since } 1 - \cos \theta \geq 0 \text{ on } \frac{\pi}{2} \leq \theta \leq \pi) \quad 2 \int_{\pi/2}^\pi \left(\frac{1}{1 - \cos \theta}\right) \sqrt{\frac{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 - \cos \theta)^2}} d\theta \\
 &= 2 \int_{\pi/2}^\pi \left(\frac{1}{1 - \cos \theta}\right) \sqrt{\frac{2 - 2 \cos \theta}{(1 - \cos \theta)^2}} d\theta = 2\sqrt{2} \int_{\pi/2}^\pi \frac{d\theta}{(1 - \cos \theta)^{3/2}} = 2\sqrt{2} \int_{\pi/2}^\pi \frac{d\theta}{(2 \sin^2 \frac{\theta}{2})^{3/2}} = \int_{\pi/2}^\pi \left|\csc^3 \frac{\theta}{2}\right| d\theta \\
 &= \int_{\pi/2}^\pi \csc^3\left(\frac{\theta}{2}\right) d\theta = (\text{since } \csc \frac{\theta}{2} \geq 0 \text{ on } \frac{\pi}{2} \leq \theta \leq \pi) \quad 2 \int_{\pi/4}^{\pi/2} \csc^3 u du = (\text{use tables}) \\
 &2 \left(\left[-\frac{\csc u \cot u}{2}\right]_{\pi/4}^{\pi/2} + \frac{1}{2} \int_{\pi/4}^{\pi/2} \csc u du \right) = 2 \left(\frac{1}{\sqrt{2}} - \left[\frac{1}{2} \ln |\csc u + \cot u|\right]_{\pi/4}^{\pi/2} \right) = 2 \left[\frac{1}{\sqrt{2}} + \frac{1}{2} \ln(\sqrt{2} + 1) \right] \\
 &= \sqrt{2} + \ln(1 + \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 27. \quad r = \cos^3 \frac{\theta}{3} &\Rightarrow \frac{dr}{d\theta} = -\sin \frac{\theta}{3} \cos^2 \frac{\theta}{3}; \text{ therefore Length} = \int_0^{\pi/4} \sqrt{\left(\cos^3 \frac{\theta}{3}\right)^2 + \left(-\sin \frac{\theta}{3} \cos^2 \frac{\theta}{3}\right)^2} d\theta \\
 &= \int_0^{\pi/4} \sqrt{\cos^6 \left(\frac{\theta}{3}\right) + \sin^2 \left(\frac{\theta}{3}\right) \cos^4 \left(\frac{\theta}{3}\right)} d\theta = \int_0^{\pi/4} \left(\cos^2 \frac{\theta}{3}\right) \sqrt{\cos^2 \left(\frac{\theta}{3}\right) + \sin^2 \left(\frac{\theta}{3}\right)} d\theta = \int_0^{\pi/4} \cos^2 \left(\frac{\theta}{3}\right) d\theta \\
 &= \int_0^{\pi/4} \frac{1 + \cos\left(\frac{2\theta}{3}\right)}{2} d\theta = \frac{1}{2} \left[\theta + \frac{3}{2} \sin \frac{2\theta}{3} \right]_0^{\pi/4} = \frac{\pi}{8} + \frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad r = \sqrt{1 + \sin 2\theta}, 0 \leq \theta \leq \pi\sqrt{2} &\Rightarrow \frac{dr}{d\theta} = \frac{1}{2} (1 + \sin 2\theta)^{-1/2} (2 \cos 2\theta) = (\cos 2\theta)(1 + \sin 2\theta)^{-1/2}; \text{ therefore} \\
 \text{Length} &= \int_0^{\pi\sqrt{2}} \sqrt{(1 + \sin 2\theta) + \frac{\cos^2 2\theta}{(1 + \sin 2\theta)}} d\theta = \int_0^{\pi\sqrt{2}} \sqrt{\frac{1 + 2 \sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1 + \sin 2\theta}} d\theta \\
 &= \int_0^{\pi\sqrt{2}} \sqrt{\frac{2 + 2 \sin 2\theta}{1 + \sin 2\theta}} d\theta = \int_0^{\pi\sqrt{2}} \sqrt{2} d\theta = \left[\sqrt{2} \theta \right]_0^{\pi\sqrt{2}} = 2\pi
 \end{aligned}$$

29. Let $r = f(\theta)$. Then $x = f(\theta) \cos \theta \Rightarrow \frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \Rightarrow \left(\frac{dx}{d\theta}\right)^2 = [f'(\theta) \cos \theta - f(\theta) \sin \theta]^2$
 $= [f'(\theta)]^2 \cos^2 \theta - 2f'(\theta)f(\theta) \sin \theta \cos \theta + [f(\theta)]^2 \sin^2 \theta$; $y = f(\theta) \sin \theta \Rightarrow \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta$
 $\Rightarrow \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta) \sin \theta + f(\theta) \cos \theta]^2 = [f'(\theta)]^2 \sin^2 \theta + 2f'(\theta)f(\theta) \sin \theta \cos \theta + [f(\theta)]^2 \cos^2 \theta$. Therefore
 $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta)]^2 (\cos^2 \theta + \sin^2 \theta) + [f(\theta)]^2 (\cos^2 \theta + \sin^2 \theta) = [f'(\theta)]^2 + [f(\theta)]^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$.

Thus, $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$.

30. (a) $r = a \Rightarrow \frac{dr}{d\theta} = 0$; Length $= \int_0^{2\pi} \sqrt{a^2 + 0^2} d\theta = \int_0^{2\pi} |a| d\theta = [a\theta]_0^{2\pi} = 2\pi a$

(b) $r = a \cos \theta \Rightarrow \frac{dr}{d\theta} = -a \sin \theta$; Length $= \int_0^{\pi} \sqrt{(a \cos \theta)^2 + (-a \sin \theta)^2} d\theta = \int_0^{\pi} \sqrt{a^2 (\cos^2 \theta + \sin^2 \theta)} d\theta$
 $= \int_0^{\pi} |a| d\theta = [a\theta]_0^{\pi} = \pi a$

(c) $r = a \sin \theta \Rightarrow \frac{dr}{d\theta} = a \cos \theta$; Length $= \int_0^{\pi} \sqrt{(a \cos \theta)^2 + (a \sin \theta)^2} d\theta = \int_0^{\pi} \sqrt{a^2 (\cos^2 \theta + \sin^2 \theta)} d\theta$
 $= \int_0^{\pi} |a| d\theta = [a\theta]_0^{\pi} = \pi a$

31. (a) $r_{av} = \frac{1}{2\pi-0} \int_0^{2\pi} a(1 - \cos \theta) d\theta = \frac{a}{2\pi} [\theta - \sin \theta]_0^{2\pi} = a$

(b) $r_{av} = \frac{1}{2\pi-0} \int_0^{2\pi} a d\theta = \frac{1}{2\pi} [a\theta]_0^{2\pi} = a$

(c) $r_{av} = \frac{1}{(\frac{\pi}{2}) - (-\frac{\pi}{2})} \int_{-\pi/2}^{\pi/2} a \cos \theta d\theta = \frac{1}{\pi} [a \sin \theta]_{-\pi/2}^{\pi/2} = \frac{2a}{\pi}$

32. $r = 2f(\theta)$, $\alpha \leq \theta \leq \beta \Rightarrow \frac{dr}{d\theta} = 2f'(\theta) \Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = [2f(\theta)]^2 + [2f'(\theta)]^2 \Rightarrow$ Length $= \int_{\alpha}^{\beta} \sqrt{4[f(\theta)]^2 + 4[f'(\theta)]^2} d\theta$
 $= 2 \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$ which is twice the length of the curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$.

11.6 CONIC SECTIONS

1. $x = \frac{y^2}{8} \Rightarrow 4p = 8 \Rightarrow p = 2$; focus is $(2, 0)$, directrix is $x = -2$

2. $x = -\frac{y^2}{4} \Rightarrow 4p = 4 \Rightarrow p = 1$; focus is $(-1, 0)$, directrix is $x = 1$

3. $y = -\frac{x^2}{6} \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2}$; focus is $(0, -\frac{3}{2})$, directrix is $y = \frac{3}{2}$

4. $y = \frac{x^2}{2} \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}$; focus is $(0, \frac{1}{2})$, directrix is $y = -\frac{1}{2}$

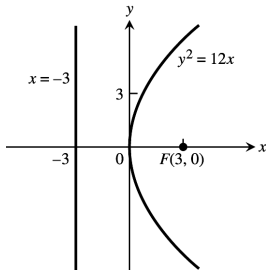
5. $\frac{x^2}{4} - \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{4+9} = \sqrt{13} \Rightarrow$ foci are $(\pm \sqrt{13}, 0)$; vertices are $(\pm 2, 0)$; asymptotes are $y = \pm \frac{3}{2}x$

6. $\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{9-4} = \sqrt{5} \Rightarrow$ foci are $(0, \pm \sqrt{5})$; vertices are $(0, \pm 3)$

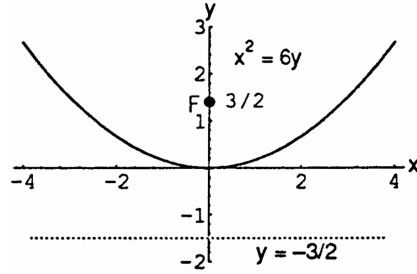
7. $\frac{x^2}{2} + y^2 = 1 \Rightarrow c = \sqrt{2-1} = 1 \Rightarrow$ foci are $(\pm 1, 0)$; vertices are $(\pm \sqrt{2}, 0)$

8. $\frac{y^2}{4} - x^2 = 1 \Rightarrow c = \sqrt{4+1} = \sqrt{5} \Rightarrow$ foci are $(0, \pm \sqrt{5})$; vertices are $(0, \pm 2)$; asymptotes are $y = \pm 2x$

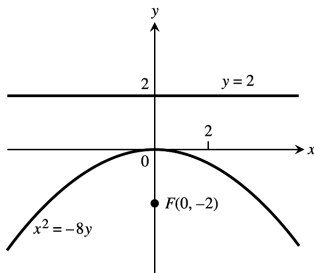
9. $y^2 = 12x \Rightarrow x = \frac{y^2}{12} \Rightarrow 4p = 12 \Rightarrow p = 3$;
focus is $(3, 0)$, directrix is $x = -3$



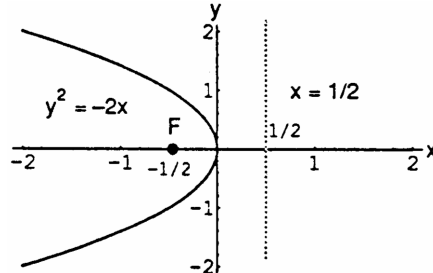
10. $x^2 = 6y \Rightarrow y = \frac{x^2}{6} \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2}$;
focus is $(0, \frac{3}{2})$, directrix is $y = -\frac{3}{2}$



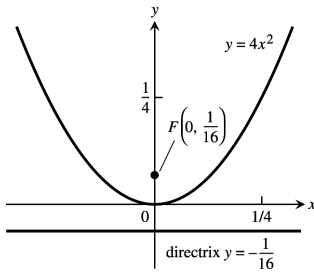
11. $x^2 = -8y \Rightarrow y = \frac{x^2}{-8} \Rightarrow 4p = 8 \Rightarrow p = 2$;
focus is $(0, -2)$, directrix is $y = 2$



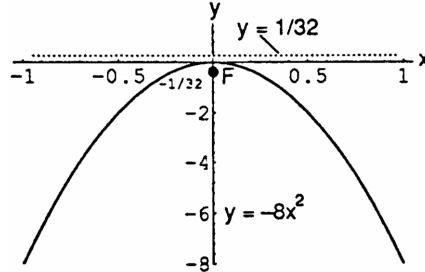
12. $y^2 = -2x \Rightarrow x = \frac{y^2}{-2} \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}$;
focus is $(-\frac{1}{2}, 0)$, directrix is $x = \frac{1}{2}$



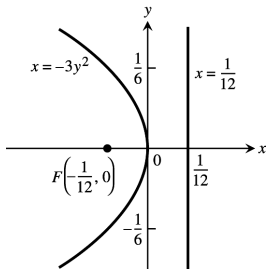
13. $y = 4x^2 \Rightarrow y = \frac{x^2}{(1/4)} \Rightarrow 4p = \frac{1}{4} \Rightarrow p = \frac{1}{16}$;
focus is $(0, \frac{1}{16})$, directrix is $y = -\frac{1}{16}$



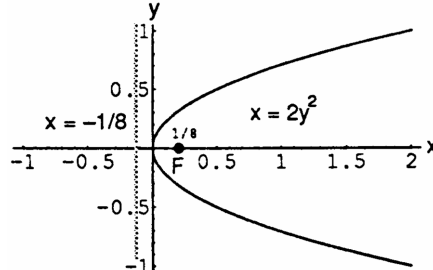
14. $y = -8x^2 \Rightarrow y = -\frac{x^2}{(1/8)} \Rightarrow 4p = \frac{1}{8} \Rightarrow p = \frac{1}{32}$;
focus is $(0, -\frac{1}{32})$, directrix is $y = \frac{1}{32}$



15. $x = -3y^2 \Rightarrow x = -\frac{y^2}{(1/3)} \Rightarrow 4p = \frac{1}{3} \Rightarrow p = \frac{1}{12}$;
focus is $(-\frac{1}{12}, 0)$, directrix is $x = \frac{1}{12}$

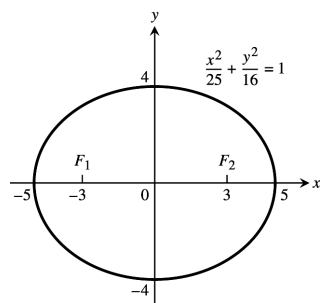


16. $x = 2y^2 \Rightarrow x = \frac{y^2}{(1/2)} \Rightarrow 4p = \frac{1}{2} \Rightarrow p = \frac{1}{8}$;
focus is $(\frac{1}{8}, 0)$, directrix is $x = -\frac{1}{8}$



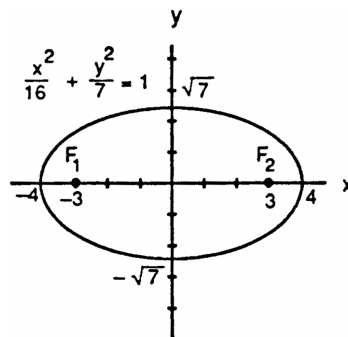
$$17. 16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = 3$$



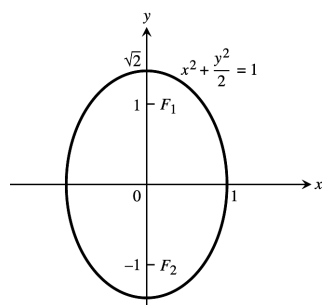
$$18. 7x^2 + 16y^2 = 112 \Rightarrow \frac{x^2}{16} + \frac{y^2}{7} = 1$$

$$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{16 - 7} = 3$$



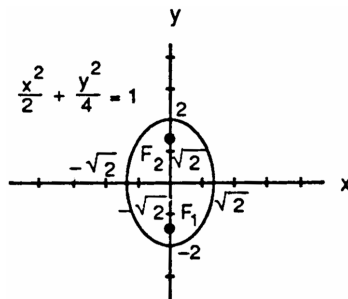
$$19. 2x^2 + y^2 = 2 \Rightarrow \frac{x^2}{1} + \frac{y^2}{2} = 1$$

$$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{2 - 1} = 1$$



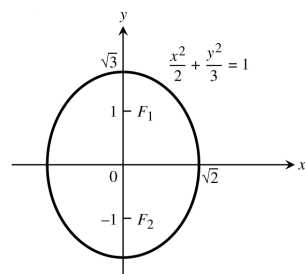
$$20. 2x^2 + y^2 = 4 \Rightarrow \frac{x^2}{2} + \frac{y^2}{4} = 1$$

$$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{4 - 2} = \sqrt{2}$$



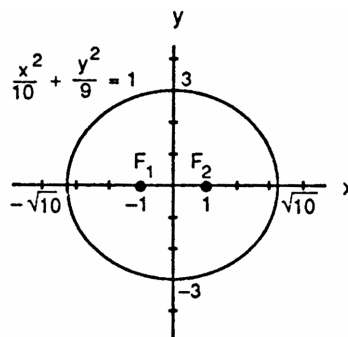
$$21. 3x^2 + 2y^2 = 6 \Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1$$

$$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{3 - 2} = 1$$

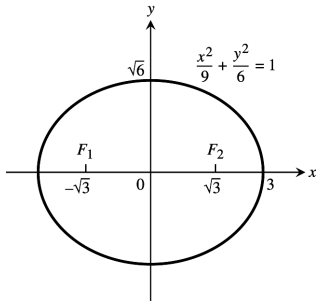


$$22. 9x^2 + 10y^2 = 90 \Rightarrow \frac{x^2}{10} + \frac{y^2}{9} = 1$$

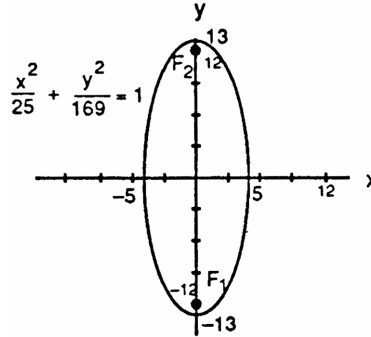
$$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{10 - 9} = 1$$



23. $6x^2 + 9y^2 = 54 \Rightarrow \frac{x^2}{9} + \frac{y^2}{6} = 1$
 $\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{9 - 6} = \sqrt{3}$



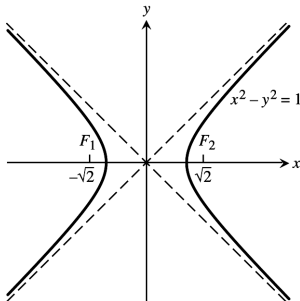
24. $169x^2 + 25y^2 = 4225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{169} = 1$
 $\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{169 - 25} = 12$



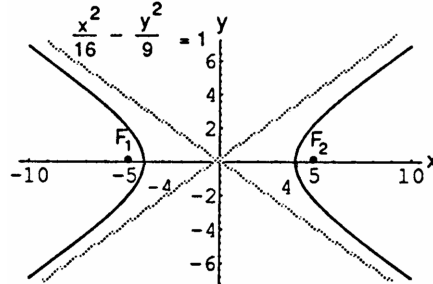
25. Foci: $(\pm \sqrt{2}, 0)$, Vertices: $(\pm 2, 0) \Rightarrow a = 2, c = \sqrt{2} \Rightarrow b^2 = a^2 - c^2 = 4 - (\sqrt{2})^2 = 2 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1$

26. Foci: $(0, \pm 4)$, Vertices: $(0, \pm 5) \Rightarrow a = 5, c = 4 \Rightarrow b^2 = 25 - 16 = 9 \Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$

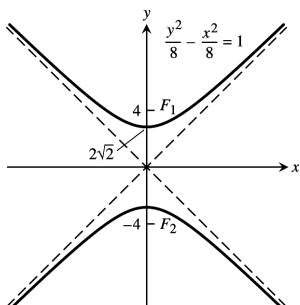
27. $x^2 - y^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2}$;
 asymptotes are $y = \pm x$



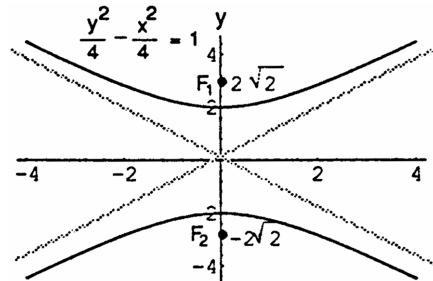
28. $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$
 $\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$;
 asymptotes are $y = \pm \frac{3}{4}x$



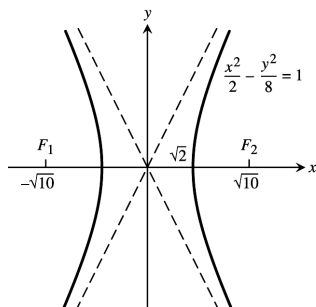
29. $y^2 - x^2 = 8 \Rightarrow \frac{y^2}{8} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{8 + 8} = 4$; asymptotes are $y = \pm x$



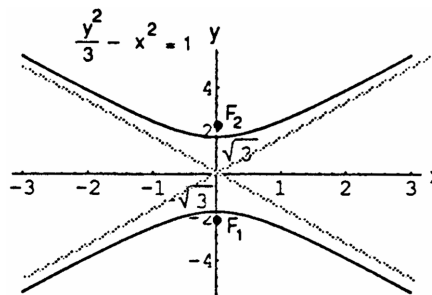
30. $y^2 - x^2 = 4 \Rightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{4 + 4} = 2\sqrt{2}$; asymptotes are $y = \pm x$



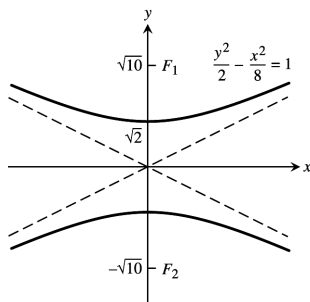
$$31. 8x^2 - 2y^2 = 16 \Rightarrow \frac{x^2}{2} - \frac{y^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{2 + 8} = \sqrt{10}; \text{ asymptotes are } y = \pm 2x$$



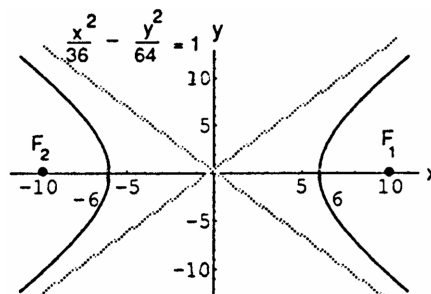
$$32. y^2 - 3x^2 = 3 \Rightarrow \frac{y^2}{3} - x^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{3 + 1} = 2; \text{ asymptotes are } y = \pm \sqrt{3}x$$



$$33. 8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{2 + 8} = \sqrt{10}; \text{ asymptotes are } y = \pm \frac{x}{2}$$



$$34. 64x^2 - 36y^2 = 2304 \Rightarrow \frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = 10; \text{ asymptotes are } y = \pm \frac{4}{3}x$$



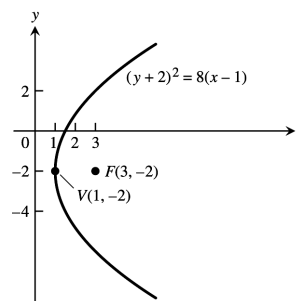
$$35. \text{ Foci: } (0, \pm \sqrt{2}), \text{ Asymptotes: } y = \pm x \Rightarrow c = \sqrt{2} \text{ and } \frac{a}{b} = 1 \Rightarrow a = b \Rightarrow c^2 = a^2 + b^2 = 2a^2 \Rightarrow 2 = 2a^2 \Rightarrow a = 1 \Rightarrow b = 1 \Rightarrow y^2 - x^2 = 1$$

$$36. \text{ Foci: } (\pm 2, 0), \text{ Asymptotes: } y = \pm \frac{1}{\sqrt{3}}x \Rightarrow c = 2 \text{ and } \frac{b}{a} = \frac{1}{\sqrt{3}} \Rightarrow b = \frac{a}{\sqrt{3}} \Rightarrow c^2 = a^2 + b^2 = a^2 + \frac{a^2}{3} = \frac{4a^2}{3} \Rightarrow 4 = \frac{4a^2}{3} \Rightarrow a^2 = 3 \Rightarrow a = \sqrt{3} \Rightarrow b = 1 \Rightarrow \frac{x^2}{3} - y^2 = 1$$

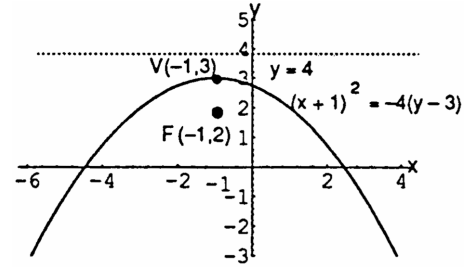
$$37. \text{ Vertices: } (\pm 3, 0), \text{ Asymptotes: } y = \pm \frac{4}{3}x \Rightarrow a = 3 \text{ and } \frac{b}{a} = \frac{4}{3} \Rightarrow b = \frac{4}{3}(3) = 4 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$$

$$38. \text{ Vertices: } (0, \pm 2), \text{ Asymptotes: } y = \pm \frac{1}{2}x \Rightarrow a = 2 \text{ and } \frac{b}{a} = \frac{1}{2} \Rightarrow b = 2(2) = 4 \Rightarrow \frac{y^2}{4} - \frac{x^2}{16} = 1$$

$$39. (a) y^2 = 8x \Rightarrow 4p = 8 \Rightarrow p = 2 \Rightarrow \text{directrix is } x = -2, \text{ focus is } (2, 0), \text{ and vertex is } (0, 0); \text{ therefore the new directrix is } x = -1, \text{ the new focus is } (3, -2), \text{ and the new vertex is } (1, -2)$$

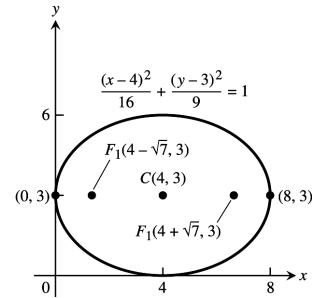


40. (a) $x^2 = -4y \Rightarrow 4p = 4 \Rightarrow p = 1 \Rightarrow$ directrix is $y = 1$, focus is $(0, -1)$, and vertex is $(0, 0)$; therefore the new directrix is $y = 4$, the new focus is $(-1, 2)$, and the new vertex is $(-1, 3)$



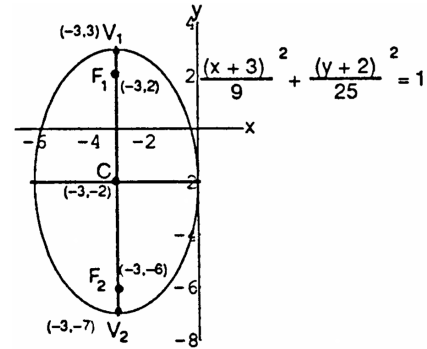
41. (a) $\frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(-4, 0)$ and $(4, 0)$; $c = \sqrt{a^2 - b^2} = \sqrt{7} \Rightarrow$ foci are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$; therefore the new center is $(4, 3)$, the new vertices are $(0, 3)$ and $(8, 3)$, and the new foci are $(4 \pm \sqrt{7}, 3)$

(b)



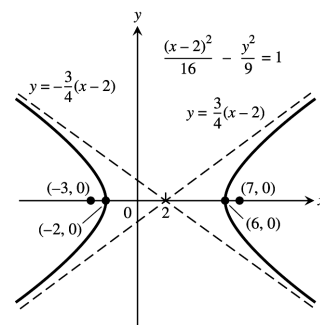
42. (a) $\frac{x^2}{9} + \frac{y^2}{25} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 5)$ and $(0, -5)$; $c = \sqrt{a^2 - b^2} = \sqrt{16} = 4 \Rightarrow$ foci are $(0, 4)$ and $(0, -4)$; therefore the new center is $(-3, -2)$, the new vertices are $(-3, 3)$ and $(-3, -7)$, and the new foci are $(-3, 2)$ and $(-3, -6)$

(b)

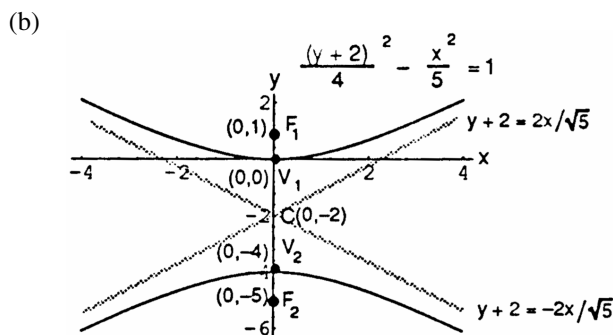


43. (a) $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(-4, 0)$ and $(4, 0)$, and the asymptotes are $\frac{x}{4} = \pm \frac{y}{3}$ or $y = \pm \frac{3x}{4}$; $c = \sqrt{a^2 + b^2} = \sqrt{25} = 5 \Rightarrow$ foci are $(-5, 0)$ and $(5, 0)$; therefore the new center is $(2, 0)$, the new vertices are $(-2, 0)$ and $(6, 0)$, the new foci are $(-3, 0)$ and $(7, 0)$, and the new asymptotes are $y = \pm \frac{3(x - 2)}{4}$

(b)



44. (a) $\frac{y^2}{4} - \frac{x^2}{5} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, -2)$ and $(0, 2)$, and the asymptotes are $\frac{y}{2} = \pm \frac{x}{\sqrt{5}}$ or $y = \pm \frac{2x}{\sqrt{5}}$; $c = \sqrt{a^2 + b^2} = \sqrt{9} = 3 \Rightarrow$ foci are $(0, 3)$ and $(0, -3)$; therefore the new center is $(0, -2)$, the new vertices are $(0, -4)$ and $(0, 0)$, the new foci are $(0, 1)$ and $(0, -5)$, and the new asymptotes are $y + 2 = \pm \frac{2x}{\sqrt{5}}$



45. $y^2 = 4x \Rightarrow 4p = 4 \Rightarrow p = 1 \Rightarrow$ focus is $(1, 0)$, directrix is $x = -1$, and vertex is $(0, 0)$; therefore the new vertex is $(-2, -3)$, the new focus is $(-1, -3)$, and the new directrix is $x = -3$; the new equation is $(y + 3)^2 = 4(x + 2)$
46. $y^2 = -12x \Rightarrow 4p = 12 \Rightarrow p = 3 \Rightarrow$ focus is $(-3, 0)$, directrix is $x = 3$, and vertex is $(0, 0)$; therefore the new vertex is $(4, 3)$, the new focus is $(1, 3)$, and the new directrix is $x = 7$; the new equation is $(y - 3)^2 = -12(x - 4)$
47. $x^2 = 8y \Rightarrow 4p = 8 \Rightarrow p = 2 \Rightarrow$ focus is $(0, 2)$, directrix is $y = -2$, and vertex is $(0, 0)$; therefore the new vertex is $(1, -7)$, the new focus is $(1, -5)$, and the new directrix is $y = -9$; the new equation is $(x - 1)^2 = 8(y + 7)$
48. $x^2 = 6y \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2} \Rightarrow$ focus is $(0, \frac{3}{2})$, directrix is $y = -\frac{3}{2}$, and vertex is $(0, 0)$; therefore the new vertex is $(-3, -2)$, the new focus is $(-3, -\frac{1}{2})$, and the new directrix is $y = -\frac{7}{2}$; the new equation is $(x + 3)^2 = 6(y + 2)$
49. $\frac{x^2}{6} + \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 3)$ and $(0, -3)$; $c = \sqrt{a^2 - b^2} = \sqrt{9 - 6} = \sqrt{3} \Rightarrow$ foci are $(0, \sqrt{3})$ and $(0, -\sqrt{3})$; therefore the new center is $(-2, -1)$, the new vertices are $(-2, 2)$ and $(-2, -4)$, and the new foci are $(-2, -1 \pm \sqrt{3})$; the new equation is $\frac{(x+2)^2}{6} + \frac{(y+1)^2}{9} = 1$
50. $\frac{x^2}{2} + y^2 = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$; $c = \sqrt{a^2 - b^2} = \sqrt{2 - 1} = 1 \Rightarrow$ foci are $(-1, 0)$ and $(1, 0)$; therefore the new center is $(3, 4)$, the new vertices are $(3 \pm \sqrt{2}, 4)$, and the new foci are $(2, 4)$ and $(4, 4)$; the new equation is $\frac{(x-3)^2}{2} + (y - 4)^2 = 1$
51. $\frac{x^2}{3} + \frac{y^2}{2} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$; $c = \sqrt{a^2 - b^2} = \sqrt{3 - 2} = 1 \Rightarrow$ foci are $(-1, 0)$ and $(1, 0)$; therefore the new center is $(2, 3)$, the new vertices are $(2 \pm \sqrt{3}, 3)$, and the new foci are $(1, 3)$ and $(3, 3)$; the new equation is $\frac{(x-2)^2}{3} + \frac{(y-3)^2}{2} = 1$
52. $\frac{x^2}{16} + \frac{y^2}{25} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 5)$ and $(0, -5)$; $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = 3 \Rightarrow$ foci are $(0, 3)$ and $(0, -3)$; therefore the new center is $(-4, -5)$, the new vertices are $(-4, 0)$ and $(-4, -10)$, and the new foci are $(-4, -2)$ and $(-4, -8)$; the new equation is $\frac{(x+4)^2}{16} + \frac{(y+5)^2}{25} = 1$
53. $\frac{x^2}{4} - \frac{y^2}{5} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(2, 0)$ and $(-2, 0)$; $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3 \Rightarrow$ foci are $(3, 0)$ and $(-3, 0)$; the asymptotes are $\pm \frac{x}{2} = \frac{y}{\sqrt{5}} \Rightarrow y = \pm \frac{\sqrt{5}x}{2}$; therefore the new center is $(2, 2)$, the new vertices are

(4, 2) and (0, 2), and the new foci are (5, 2) and (-1, 2); the new asymptotes are $y - 2 = \pm \frac{\sqrt{5}(x-2)}{2}$; the new equation is $\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1$

54. $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow$ center is (0, 0), vertices are (4, 0) and (-4, 0); $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5 \Rightarrow$ foci are (-5, 0) and (5, 0); the asymptotes are $\pm \frac{x}{4} = \frac{y}{3} \Rightarrow y = \pm \frac{3x}{4}$; therefore the new center is (-5, -1), the new vertices are (-1, -1) and (-9, -1), and the new foci are (-10, -1) and (0, -1); the new asymptotes are $y + 1 = \pm \frac{3(x+5)}{4}$; the new equation is $\frac{(x+5)^2}{16} - \frac{(y+1)^2}{9} = 1$
55. $y^2 - x^2 = 1 \Rightarrow$ center is (0, 0), vertices are (0, 1) and (0, -1); $c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2} \Rightarrow$ foci are $(0, \pm \sqrt{2})$; the asymptotes are $y = \pm x$; therefore the new center is (-1, -1), the new vertices are (-1, 0) and (-1, -2), and the new foci are $(-1, -1 \pm \sqrt{2})$; the new asymptotes are $y + 1 = \pm (x + 1)$; the new equation is $(y + 1)^2 - (x + 1)^2 = 1$
56. $\frac{y^2}{3} - x^2 = 1 \Rightarrow$ center is (0, 0), vertices are $(0, \sqrt{3})$ and $(0, -\sqrt{3})$; $c = \sqrt{a^2 + b^2} = \sqrt{3 + 1} = 2 \Rightarrow$ foci are (0, 2) and (0, -2); the asymptotes are $\pm x = \frac{y}{\sqrt{3}} \Rightarrow y = \pm \sqrt{3}x$; therefore the new center is (1, 3), the new vertices are $(1, 3 \pm \sqrt{3})$, and the new foci are (1, 5) and (1, 1); the new asymptotes are $y - 3 = \pm \sqrt{3}(x - 1)$; the new equation is $\frac{(y-3)^2}{3} - (x - 1)^2 = 1$
57. $x^2 + 4x + y^2 = 12 \Rightarrow x^2 + 4x + 4 + y^2 = 12 + 4 \Rightarrow (x + 2)^2 + y^2 = 16$; this is a circle: center at C(-2, 0), $a = 4$
58. $2x^2 + 2y^2 - 28x + 12y + 114 = 0 \Rightarrow x^2 - 14x + 49 + y^2 + 6y + 9 = -57 + 49 + 9 \Rightarrow (x - 7)^2 + (y + 3)^2 = 1$; this is a circle: center at C(7, -3), $a = 1$
59. $x^2 + 2x + 4y - 3 = 0 \Rightarrow x^2 + 2x + 1 = -4y + 3 + 1 \Rightarrow (x + 1)^2 = -4(y - 1)$; this is a parabola: V(-1, 1), F(-1, 0)
60. $y^2 - 4y - 8x - 12 = 0 \Rightarrow y^2 - 4y + 4 = 8x + 12 + 4 \Rightarrow (y - 2)^2 = 8(x + 2)$; this is a parabola: V(-2, 2), F(0, 2)
61. $x^2 + 5y^2 + 4x = 1 \Rightarrow x^2 + 4x + 4 + 5y^2 = 5 \Rightarrow (x + 2)^2 + 5y^2 = 5 \Rightarrow \frac{(x+2)^2}{5} + y^2 = 1$; this is an ellipse: the center is (-2, 0), the vertices are $(-2 \pm \sqrt{5}, 0)$; $c = \sqrt{a^2 - b^2} = \sqrt{5 - 1} = 2 \Rightarrow$ the foci are (-4, 0) and (0, 0)
62. $9x^2 + 6y^2 + 36y = 0 \Rightarrow 9x^2 + 6(y^2 + 6y + 9) = 54 \Rightarrow 9x^2 + 6(y + 3)^2 = 54 \Rightarrow \frac{x^2}{6} + \frac{(y+3)^2}{9} = 1$; this is an ellipse: the center is (0, -3), the vertices are (0, 0) and (0, -6); $c = \sqrt{a^2 - b^2} = \sqrt{9 - 6} = \sqrt{3} \Rightarrow$ the foci are $(0, -3 \pm \sqrt{3})$
63. $x^2 + 2y^2 - 2x - 4y = -1 \Rightarrow x^2 - 2x + 1 + 2(y^2 - 2y + 1) = 2 \Rightarrow (x - 1)^2 + 2(y - 1)^2 = 2 \Rightarrow \frac{(x-1)^2}{2} + (y - 1)^2 = 1$; this is an ellipse: the center is (1, 1), the vertices are $(1 \pm \sqrt{2}, 1)$; $c = \sqrt{a^2 - b^2} = \sqrt{2 - 1} = 1 \Rightarrow$ the foci are (2, 1) and (0, 1)
64. $4x^2 + y^2 + 8x - 2y = -1 \Rightarrow 4(x^2 + 2x + 1) + y^2 - 2y + 1 = 4 \Rightarrow 4(x + 1)^2 + (y - 1)^2 = 4 \Rightarrow (x + 1)^2 + \frac{(y-1)^2}{4} = 1$; this is an ellipse: the center is (-1, 1), the vertices are (-1, 3) and (-1, -1); $c = \sqrt{a^2 - b^2} = \sqrt{4 - 1} = \sqrt{3} \Rightarrow$ the foci are $(-1, 1 \pm \sqrt{3})$

65. $x^2 - y^2 - 2x + 4y = 4 \Rightarrow x^2 - 2x + 1 - (y^2 - 4y + 4) = 1 \Rightarrow (x - 1)^2 - (y - 2)^2 = 1$; this is a hyperbola: the center is $(1, 2)$, the vertices are $(2, 2)$ and $(0, 2)$; $c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2} \Rightarrow$ the foci are $(1 \pm \sqrt{2}, 2)$; the asymptotes are $y - 2 = \pm (x - 1)$

66. $x^2 - y^2 + 4x - 6y = 6 \Rightarrow x^2 + 4x + 4 - (y^2 + 6y + 9) = 1 \Rightarrow (x + 2)^2 - (y + 3)^2 = 1$; this is a hyperbola: the center is $(-2, -3)$, the vertices are $(-1, -3)$ and $(-3, -3)$; $c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2} \Rightarrow$ the foci are $(-2 \pm \sqrt{2}, -3)$; the asymptotes are $y + 3 = \pm (x + 2)$

67. $2x^2 - y^2 + 6y = 3 \Rightarrow 2x^2 - (y^2 - 6y + 9) = -6 \Rightarrow \frac{(y-3)^2}{6} - \frac{x^2}{3} = 1$; this is a hyperbola: the center is $(0, 3)$, the vertices are $(0, 3 \pm \sqrt{6})$; $c = \sqrt{a^2 + b^2} = \sqrt{6 + 3} = 3 \Rightarrow$ the foci are $(0, 6)$ and $(0, 0)$; the asymptotes are $\frac{y-3}{\sqrt{6}} = \pm \frac{x}{\sqrt{3}} \Rightarrow y = \pm \sqrt{2}x + 3$

68. $y^2 - 4x^2 + 16x = 24 \Rightarrow y^2 - 4(x^2 - 4x + 4) = 8 \Rightarrow \frac{y^2}{8} - \frac{(x-2)^2}{2} = 1$; this is a hyperbola: the center is $(2, 0)$, the vertices are $(2, \pm \sqrt{8})$; $c = \sqrt{a^2 + b^2} = \sqrt{8 + 2} = \sqrt{10} \Rightarrow$ the foci are $(2, \pm \sqrt{10})$; the asymptotes are $\frac{y}{\sqrt{8}} = \pm \frac{x-2}{\sqrt{2}} \Rightarrow y = \pm 2(x - 2)$

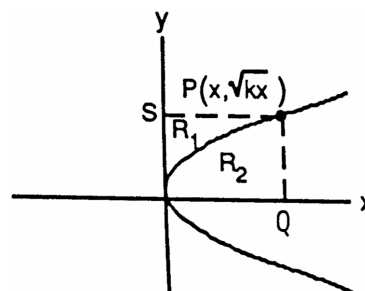
69. (a) $y^2 = kx \Rightarrow x = \frac{y^2}{k}$; the volume of the solid formed by

revolving R_1 about the y -axis is $V_1 = \int_0^{\sqrt{kx}} \pi \left(\frac{y^2}{k} \right)^2 dy$

$= \frac{\pi}{k^2} \int_0^{\sqrt{kx}} y^4 dy = \frac{\pi x^2 \sqrt{kx}}{5}$; the volume of the right

circular cylinder formed by revolving PQ about the y -axis is $V_2 = \pi x^2 \sqrt{kx} \Rightarrow$ the volume of the solid formed by revolving R_2 about the y -axis is

$V_3 = V_2 - V_1 = \frac{4\pi x^2 \sqrt{kx}}{5}$. Therefore we can see the ratio of V_3 to V_1 is 4:1.



(b) The volume of the solid formed by revolving R_2 about the x -axis is $V_1 = \int_0^x \pi \left(\sqrt{kt} \right)^2 dt = \pi k \int_0^x t dt$
 $= \frac{\pi k x^2}{2}$. The volume of the right circular cylinder formed by revolving PS about the x -axis is

$V_2 = \pi \left(\sqrt{kx} \right)^2 x = \pi k x^2 \Rightarrow$ the volume of the solid formed by revolving R_1 about the x -axis is

$V_3 = V_2 - V_1 = \pi k x^2 - \frac{\pi k x^2}{2} = \frac{\pi k x^2}{2}$. Therefore the ratio of V_3 to V_1 is 1:1.

70. $y = \int \frac{w}{H} x dx = \frac{w}{H} \left(\frac{x^2}{2} \right) + C = \frac{wx^2}{2H} + C$; $y = 0$ when $x = 0 \Rightarrow 0 = \frac{w(0)^2}{2H} + C \Rightarrow C = 0$; therefore $y = \frac{wx^2}{2H}$ is the equation of the cable's curve

71. $x^2 = 4py$ and $y = p \Rightarrow x^2 = 4p^2 \Rightarrow x = \pm 2p$. Therefore the line $y = p$ cuts the parabola at points $(-2p, p)$ and $(2p, p)$, and these points are $\sqrt{[2p - (-2p)]^2 + (p - p)^2} = 4p$ units apart.

72. $\lim_{x \rightarrow \infty} \left(\frac{b}{a} x - \frac{b}{a} \sqrt{x^2 - a^2} \right) = \frac{b}{a} \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 - a^2} \right) = \frac{b}{a} \lim_{x \rightarrow \infty} \left[\frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} \right]$
 $= \frac{b}{a} \lim_{x \rightarrow \infty} \left[\frac{x^2 - (x^2 - a^2)}{x + \sqrt{x^2 - a^2}} \right] = \frac{b}{a} \lim_{x \rightarrow \infty} \left[\frac{a^2}{x + \sqrt{x^2 - a^2}} \right] = 0$

73. Let $y = \sqrt{1 - \frac{x^2}{4}}$ on the interval $0 \leq x \leq 2$. The area of the inscribed rectangle is given by

$$A(x) = 2x \left(2\sqrt{1 - \frac{x^2}{4}} \right) = 4x\sqrt{1 - \frac{x^2}{4}} \text{ (since the length is } 2x \text{ and the height is } 2y)$$

$$\Rightarrow A'(x) = 4\sqrt{1 - \frac{x^2}{4}} - \frac{x^2}{\sqrt{1 - \frac{x^2}{4}}}. \text{ Thus } A'(x) = 0 \Rightarrow 4\sqrt{1 - \frac{x^2}{4}} - \frac{x^2}{\sqrt{1 - \frac{x^2}{4}}} = 0 \Rightarrow 4\left(1 - \frac{x^2}{4}\right) - x^2 = 0 \Rightarrow x^2 = 2$$

$\Rightarrow x = \sqrt{2}$ (only the positive square root lies in the interval). Since $A(0) = A(2) = 0$ we have that $A(\sqrt{2}) = 4$ is the maximum area when the length is $2\sqrt{2}$ and the height is $\sqrt{2}$.

74. (a) Around the x-axis: $9x^2 + 4y^2 = 36 \Rightarrow y^2 = 9 - \frac{9}{4}x^2 \Rightarrow y = \pm \sqrt{9 - \frac{9}{4}x^2}$ and we use the positive root

$$\Rightarrow V = 2 \int_0^2 \pi \left(\sqrt{9 - \frac{9}{4}x^2} \right)^2 dx = 2 \int_0^2 \pi \left(9 - \frac{9}{4}x^2 \right) dx = 2\pi \left[9x - \frac{3}{4}x^3 \right]_0^2 = 24\pi$$

(b) Around the y-axis: $9x^2 + 4y^2 = 36 \Rightarrow x^2 = 4 - \frac{4}{9}y^2 \Rightarrow x = \pm \sqrt{4 - \frac{4}{9}y^2}$ and we use the positive root

$$\Rightarrow V = 2 \int_0^3 \pi \left(\sqrt{4 - \frac{4}{9}y^2} \right)^2 dy = 2 \int_0^3 \pi \left(4 - \frac{4}{9}y^2 \right) dy = 2\pi \left[4y - \frac{4}{27}y^3 \right]_0^3 = 16\pi$$

75. $9x^2 - 4y^2 = 36 \Rightarrow y^2 = \frac{9x^2 - 36}{4} \Rightarrow y = \pm \frac{3}{2}\sqrt{x^2 - 4}$ on the interval $2 \leq x \leq 4 \Rightarrow V = \int_2^4 \pi \left(\frac{3}{2}\sqrt{x^2 - 4} \right)^2 dx$

$$= \frac{9\pi}{4} \int_2^4 (x^2 - 4) dx = \frac{9\pi}{4} \left[\frac{x^3}{3} - 4x \right]_2^4 = \frac{9\pi}{4} \left[\left(\frac{64}{3} - 16 \right) - \left(\frac{8}{3} - 8 \right) \right] = \frac{9\pi}{4} \left(\frac{56}{3} - 8 \right) = \frac{3\pi}{4} (56 - 24) = 24\pi$$

76. Let $P_1(-p, y_1)$ be any point on $x = -p$, and let $P(x, y)$ be a point where a tangent intersects $y^2 = 4px$. Now

$$y^2 = 4px \Rightarrow 2y \frac{dy}{dx} = 4p \Rightarrow \frac{dy}{dx} = \frac{2p}{y}; \text{ then the slope of a tangent line from } P_1 \text{ is } \frac{y - y_1}{x - (-p)} = \frac{dy}{dx} = \frac{2p}{y}$$

$$\Rightarrow y^2 - yy_1 = 2px + 2p^2. \text{ Since } x = \frac{y^2}{4p}, \text{ we have } y^2 - yy_1 = 2p \left(\frac{y^2}{4p} \right) + 2p^2 \Rightarrow y^2 - yy_1 = \frac{1}{2}y^2 + 2p^2$$

$$\Rightarrow \frac{1}{2}y^2 - yy_1 - 2p^2 = 0 \Rightarrow y = \frac{2y_1 \pm \sqrt{4y_1^2 + 16p^2}}{2} = y_1 \pm \sqrt{y_1^2 + 4p^2}. \text{ Therefore the slopes of the two}$$

$$\text{tangents from } P_1 \text{ are } m_1 = \frac{2p}{y_1 + \sqrt{y_1^2 + 4p^2}} \text{ and } m_2 = \frac{2p}{y_1 - \sqrt{y_1^2 + 4p^2}} \Rightarrow m_1 m_2 = \frac{4p^2}{y_1^2 - (y_1^2 + 4p^2)} = -1$$

$\Rightarrow$ the lines are perpendicular

77. $(x - 2)^2 + (y - 1)^2 = 5 \Rightarrow 2(x - 2) + 2(y - 1) \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x-2}{y-1}; y = 0 \Rightarrow (x - 2)^2 + (0 - 1)^2 = 5$

$$\Rightarrow (x - 2)^2 = 4 \Rightarrow x = 4 \text{ or } x = 0 \Rightarrow \text{the circle crosses the x-axis at } (4, 0) \text{ and } (0, 0); x = 0$$

$$\Rightarrow (0 - 2)^2 + (y - 1)^2 = 5 \Rightarrow (y - 1)^2 = 1 \Rightarrow y = 2 \text{ or } y = 0 \Rightarrow \text{the circle crosses the y-axis at } (0, 2) \text{ and } (0, 0).$$

$$\text{At } (4, 0): \frac{dy}{dx} = -\frac{4-2}{0-1} = 2 \Rightarrow \text{the tangent line is } y = 2(x - 4) \text{ or } y = 2x - 8$$

$$\text{At } (0, 0): \frac{dy}{dx} = -\frac{0-2}{0-1} = -2 \Rightarrow \text{the tangent line is } y = -2x$$

$$\text{At } (0, 2): \frac{dy}{dx} = -\frac{0-2}{2-1} = 2 \Rightarrow \text{the tangent line is } y - 2 = 2x \text{ or } y = 2x + 2$$

78. $x^2 - y^2 = 1 \Rightarrow x = \pm \sqrt{1 + y^2}$ on the interval $-3 \leq y \leq 3 \Rightarrow V = \int_{-3}^3 \pi (\sqrt{1 + y^2})^2 dy = 2 \int_0^3 \pi (\sqrt{1 + y^2})^2 dy$

$$= 2\pi \int_0^3 (1 + y^2) dy = 2\pi \left[y + \frac{y^3}{3} \right]_0^3 = 24\pi$$

79. Let $y = \sqrt{16 - \frac{16}{9}x^2}$ on the interval $-3 \leq x \leq 3$. Since the plate is symmetric about the y-axis, $\bar{x} = 0$. For a

$$\text{vertical strip: } (\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{16 - \frac{16}{9}x^2}}{2} \right), \text{ length} = \sqrt{16 - \frac{16}{9}x^2}, \text{ width} = dx \Rightarrow \text{area} = dA = \sqrt{16 - \frac{16}{9}x^2} dx$$

$$\Rightarrow \text{mass} = dm = \delta dA = \delta \sqrt{16 - \frac{16}{9}x^2} dx. \text{ Moment of the strip about the x-axis:}$$

$$\tilde{y} dm = \frac{\sqrt{16 - \frac{16}{9}x^2}}{2} \left(\delta \sqrt{16 - \frac{16}{9}x^2} \right) dx = \delta \left(8 - \frac{8}{9}x^2 \right) dx \text{ so the moment of the plate about the x-axis is}$$

$$M_x = \int \tilde{y} \, dm = \int_{-3}^3 \delta \left(8 - \frac{8}{9} x^2 \right) dx = \delta \left[8x - \frac{8}{27} x^3 \right]_{-3}^3 = 32\delta; \text{ also the mass of the plate is}$$

$$M = \int_{-3}^3 \delta \sqrt{16 - \frac{16}{9} x^2} dx = \int_{-3}^3 4\delta \sqrt{1 - \left(\frac{1}{3} x\right)^2} dx = 4\delta \int_{-1}^1 3\sqrt{1 - u^2} du \text{ where } u = \frac{x}{3} \Rightarrow 3 du = dx; x = -3 \\ \Rightarrow u = -1 \text{ and } x = 3 \Rightarrow u = 1. \text{ Hence, } 4\delta \int_{-1}^1 3\sqrt{1 - u^2} du = 12\delta \int_{-1}^1 \sqrt{1 - u^2} du \\ = 12\delta \left[\frac{1}{2} \left(u\sqrt{1 - u^2} + \sin^{-1} u \right) \right]_{-1}^1 = 6\pi\delta \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{32\delta}{6\pi\delta} = \frac{16}{3\pi}. \text{ Therefore the center of mass is } \left(0, \frac{16}{3\pi} \right).$$

$$80. y = \sqrt{x^2 + 1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} (x^2 + 1)^{-1/2} (2x) = \frac{x}{\sqrt{x^2 + 1}} \Rightarrow \left(\frac{dy}{dx} \right)^2 = \frac{x^2}{x^2 + 1} \Rightarrow \sqrt{1 + \left(\frac{dy}{dx} \right)^2} = \sqrt{1 + \frac{x^2}{x^2 + 1}} \\ = \sqrt{\frac{2x^2 + 1}{x^2 + 1}} \Rightarrow S = \int_0^{\sqrt{2}} 2\pi y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx = \int_0^{\sqrt{2}} 2\pi \sqrt{x^2 + 1} \sqrt{\frac{2x^2 + 1}{x^2 + 1}} dx = \int_0^{\sqrt{2}} 2\pi \sqrt{2x^2 + 1} dx; \\ \left[\begin{array}{l} u = \sqrt{2}x \\ du = \sqrt{2} dx \end{array} \right] \rightarrow \frac{2\pi}{\sqrt{2}} \int_0^2 \sqrt{u^2 + 1} du = \frac{2\pi}{\sqrt{2}} \left[\frac{1}{2} \left(u\sqrt{u^2 + 1} + \ln(u + \sqrt{u^2 + 1}) \right) \right]_0^2 = \frac{\pi}{\sqrt{2}} \left[2\sqrt{5} + \ln(2 + \sqrt{5}) \right]$$

$$81. (a) \tan \beta = m_L \Rightarrow \tan \beta = f'(x_0) \text{ where } f(x) = \sqrt{4px};$$

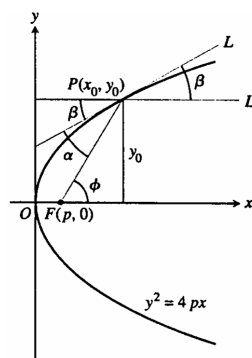
$$f'(x) = \frac{1}{2} (4px)^{-1/2} (4p) = \frac{2p}{\sqrt{4px}} \Rightarrow f'(x_0) = \frac{2p}{\sqrt{4px_0}}$$

$$= \frac{2p}{y_0} \Rightarrow \tan \beta = \frac{2p}{y_0}.$$

$$(b) \tan \phi = m_{FP} = \frac{y_0 - 0}{x_0 - p} = \frac{y_0}{x_0 - p}$$

$$(c) \tan \alpha = \frac{\tan \phi - \tan \beta}{1 + \tan \phi \tan \beta} = \frac{\left(\frac{y_0}{x_0 - p} - \frac{2p}{y_0} \right)}{1 + \left(\frac{y_0}{x_0 - p} \right) \left(\frac{2p}{y_0} \right)}$$

$$= \frac{\frac{y_0^2 - 2p(x_0 - p)}{y_0(x_0 - p)}}{\frac{y_0(x_0 - p) + 2p}{y_0(x_0 - p)}} = \frac{4px_0 - 2px_0 + 2p^2}{y_0(x_0 + p)} = \frac{2p(x_0 + p)}{y_0(x_0 + p)} = \frac{2p}{y_0}$$

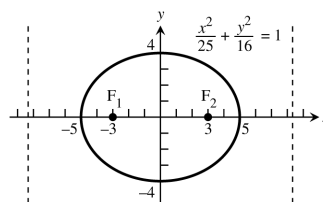


11.7 CONICS IN POLAR COORDINATES

$$1. 16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1 \Rightarrow c = \sqrt{a^2 - b^2}$$

$$= \sqrt{25 - 16} = 3 \Rightarrow e = \frac{c}{a} = \frac{3}{5}; F(\pm 3, 0);$$

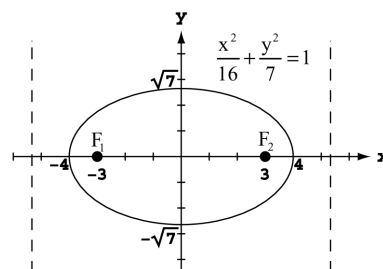
$$\text{directrices are } x = 0 \pm \frac{a}{e} = \pm \frac{5}{(3/5)} = \pm \frac{25}{3}$$



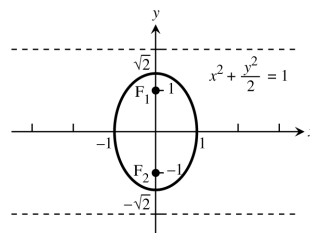
$$2. 7x^2 + 16y^2 = 112 \Rightarrow \frac{x^2}{16} + \frac{y^2}{7} = 1 \Rightarrow c = \sqrt{a^2 - b^2}$$

$$= \sqrt{16 - 7} = 3 \Rightarrow e = \frac{c}{a} = \frac{3}{4}; F(\pm 3, 0);$$

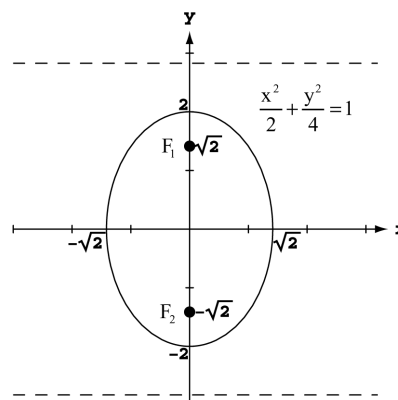
$$\text{directrices are } x = 0 \pm \frac{a}{e} = \pm \frac{4}{(3/4)} = \pm \frac{16}{3}$$



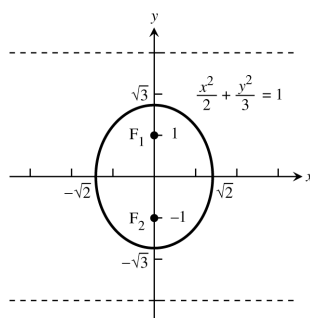
$$\begin{aligned}
 3. \quad 2x^2 + y^2 = 2 &\Rightarrow x^2 + \frac{y^2}{2} = 1 \Rightarrow c = \sqrt{a^2 - b^2} \\
 &= \sqrt{2 - 1} = 1 \Rightarrow e = \frac{c}{a} = \frac{1}{\sqrt{2}}; F(0, \pm 1); \\
 \text{directrices are } y &= 0 \pm \frac{a}{e} = \pm \frac{\sqrt{2}}{(\frac{1}{\sqrt{2}})} = \pm 2
 \end{aligned}$$



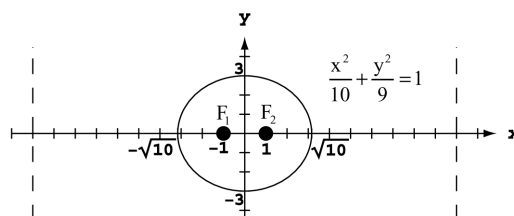
$$\begin{aligned}
 4. \quad 2x^2 + y^2 = 4 &\Rightarrow \frac{x^2}{2} + \frac{y^2}{4} = 1 \Rightarrow c = \sqrt{a^2 - b^2} \\
 &= \sqrt{4 - 2} = \sqrt{2} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{2}}{2}; F(0, \pm \sqrt{2}); \\
 \text{directrices are } y &= 0 \pm \frac{a}{e} = \pm \frac{2}{(\frac{\sqrt{2}}{2})} = \pm 2\sqrt{2}
 \end{aligned}$$



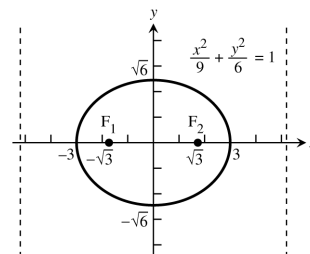
$$\begin{aligned}
 5. \quad 3x^2 + 2y^2 = 6 &\Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1 \Rightarrow c = \sqrt{a^2 - b^2} \\
 &= \sqrt{3 - 2} = 1 \Rightarrow e = \frac{c}{a} = \frac{1}{\sqrt{3}}; F(0, \pm 1); \\
 \text{directrices are } y &= 0 \pm \frac{a}{e} = \pm \frac{\sqrt{3}}{(\frac{1}{\sqrt{3}})} = \pm 3
 \end{aligned}$$



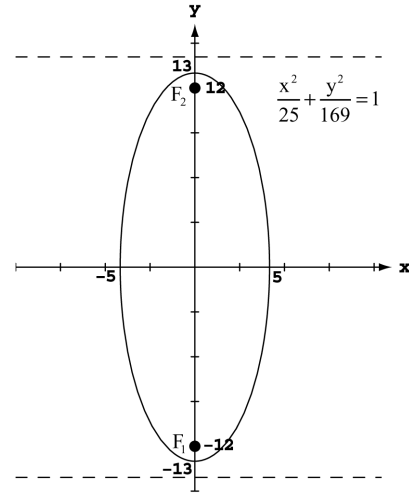
$$\begin{aligned}
 6. \quad 9x^2 + 10y^2 = 90 &\Rightarrow \frac{x^2}{10} + \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{a^2 - b^2} \\
 &= \sqrt{10 - 9} = 1 \Rightarrow e = \frac{c}{a} = \frac{1}{\sqrt{10}}; F(\pm 1, 0); \\
 \text{directrices are } x &= 0 \pm \frac{a}{e} = \pm \frac{\sqrt{10}}{(\frac{1}{\sqrt{10}})} = \pm 10
 \end{aligned}$$



$$\begin{aligned}
 7. \quad 6x^2 + 9y^2 = 54 &\Rightarrow \frac{x^2}{9} + \frac{y^2}{6} = 1 \Rightarrow c = \sqrt{a^2 - b^2} \\
 &= \sqrt{9 - 6} = \sqrt{3} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{3}}{3}; F(\pm \sqrt{3}, 0); \\
 \text{directrices are } x &= 0 \pm \frac{a}{e} = \pm \frac{3}{(\frac{\sqrt{3}}{3})} = \pm 3\sqrt{3}
 \end{aligned}$$



8. $169x^2 + 25y^2 = 4225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{169} = 1 \Rightarrow c = \sqrt{a^2 - b^2}$
 $= \sqrt{169 - 25} = 12 \Rightarrow e = \frac{c}{a} = \frac{12}{13}; F(0, \pm 12);$
 directrices are $y = 0 \pm \frac{a}{e} = \pm \frac{13}{(\frac{12}{13})} = \pm \frac{169}{12}$



9. Foci: $(0, \pm 3)$, $e = 0.5 \Rightarrow c = 3$ and $a = \frac{c}{e} = \frac{3}{0.5} = 6 \Rightarrow b^2 = 36 - 9 = 27 \Rightarrow \frac{x^2}{27} + \frac{y^2}{36} = 1$

10. Foci: $(\pm 8, 0)$, $e = 0.2 \Rightarrow c = 8$ and $a = \frac{c}{e} = \frac{8}{0.2} = 40 \Rightarrow b^2 = 1600 - 64 = 1536 \Rightarrow \frac{x^2}{1600} + \frac{y^2}{1536} = 1$

11. Vertices: $(0, \pm 70)$, $e = 0.1 \Rightarrow a = 70$ and $c = ae = 70(0.1) = 7 \Rightarrow b^2 = 4900 - 49 = 4851 \Rightarrow \frac{x^2}{4851} + \frac{y^2}{4900} = 1$

12. Vertices: $(\pm 10, 0)$, $e = 0.24 \Rightarrow a = 10$ and $c = ae = 10(0.24) = 2.4 \Rightarrow b^2 = 100 - 5.76 = 94.24 \Rightarrow \frac{x^2}{100} + \frac{y^2}{94.24} = 1$

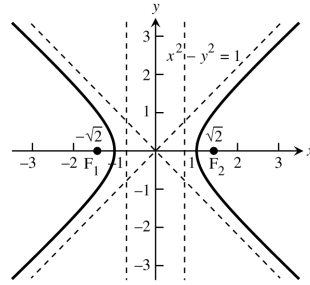
13. Focus: $(\sqrt{5}, 0)$, Directrix: $x = \frac{9}{\sqrt{5}} \Rightarrow c = ae = \sqrt{5}$ and $\frac{a}{e} = \frac{9}{\sqrt{5}} \Rightarrow \frac{ae}{e^2} = \frac{9}{\sqrt{5}} \Rightarrow \frac{\sqrt{5}}{e^2} = \frac{9}{\sqrt{5}} \Rightarrow e^2 = \frac{5}{9}$
 $\Rightarrow e = \frac{\sqrt{5}}{3}$. Then $PF = \frac{\sqrt{5}}{3} PD \Rightarrow \sqrt{(x - \sqrt{5})^2 + (y - 0)^2} = \frac{\sqrt{5}}{3} \left| x - \frac{9}{\sqrt{5}} \right| \Rightarrow (x - \sqrt{5})^2 + y^2 = \frac{5}{9} \left(x - \frac{9}{\sqrt{5}} \right)^2$
 $\Rightarrow x^2 - 2\sqrt{5}x + 5 + y^2 = \frac{5}{9} \left(x^2 - \frac{18}{\sqrt{5}}x + \frac{81}{5} \right) \Rightarrow \frac{4}{9}x^2 + y^2 = 4 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$

14. Focus: $(4, 0)$, Directrix: $x = \frac{16}{3} \Rightarrow c = ae = 4$ and $\frac{a}{e} = \frac{16}{3} \Rightarrow \frac{ae}{e^2} = \frac{16}{3} \Rightarrow \frac{4}{e^2} = \frac{16}{3} \Rightarrow e^2 = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$. Then
 $PF = \frac{\sqrt{3}}{2} PD \Rightarrow \sqrt{(x - 4)^2 + (y - 0)^2} = \frac{\sqrt{3}}{2} \left| x - \frac{16}{3} \right| \Rightarrow (x - 4)^2 + y^2 = \frac{3}{4} \left(x - \frac{16}{3} \right)^2 \Rightarrow x^2 - 8x + 16 + y^2$
 $= \frac{3}{4} \left(x^2 - \frac{32}{3}x + \frac{256}{9} \right) \Rightarrow \frac{1}{4}x^2 + y^2 = \frac{16}{3} \Rightarrow \frac{x^2}{64} + \frac{y^2}{16} = 1$

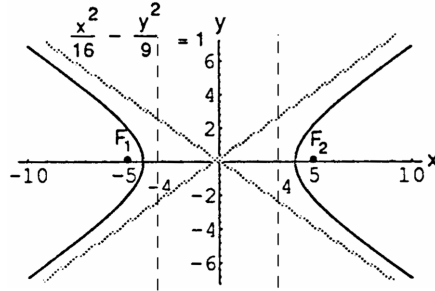
15. Focus: $(-4, 0)$, Directrix: $x = -16 \Rightarrow c = ae = 4$ and $\frac{a}{e} = 16 \Rightarrow \frac{ae}{e^2} = 16 \Rightarrow \frac{4}{e^2} = 16 \Rightarrow e^2 = \frac{1}{4} \Rightarrow e = \frac{1}{2}$. Then
 $PF = \frac{1}{2} PD \Rightarrow \sqrt{(x + 4)^2 + (y - 0)^2} = \frac{1}{2} |x + 16| \Rightarrow (x + 4)^2 + y^2 = \frac{1}{4} (x + 16)^2 \Rightarrow x^2 + 8x + 16 + y^2$
 $= \frac{1}{4} (x^2 + 32x + 256) \Rightarrow \frac{3}{4}x^2 + y^2 = 48 \Rightarrow \frac{x^2}{64} + \frac{y^2}{48} = 1$

16. Focus: $(-\sqrt{2}, 0)$, Directrix: $x = -2\sqrt{2} \Rightarrow c = ae = \sqrt{2}$ and $\frac{a}{e} = 2\sqrt{2} \Rightarrow \frac{ae}{e^2} = 2\sqrt{2} \Rightarrow \frac{\sqrt{2}}{e^2} = 2\sqrt{2} \Rightarrow e^2 = \frac{1}{2}$
 $\Rightarrow e = \frac{1}{\sqrt{2}}$. Then $PF = \frac{1}{\sqrt{2}} PD \Rightarrow \sqrt{(x + \sqrt{2})^2 + (y - 0)^2} = \frac{1}{\sqrt{2}} |x + 2\sqrt{2}| \Rightarrow (x + \sqrt{2})^2 + y^2$
 $= \frac{1}{2} (x + 2\sqrt{2})^2 \Rightarrow x^2 + 2\sqrt{2}x + 2 + y^2 = \frac{1}{2} (x^2 + 4\sqrt{2}x + 8) \Rightarrow \frac{1}{2}x^2 + y^2 = 2 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1$

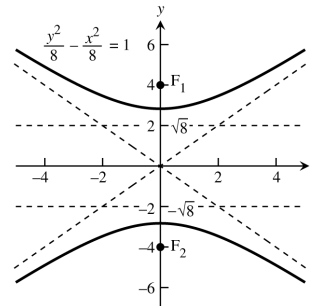
$$\begin{aligned}
 17. \quad x^2 - y^2 = 1 &\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2} \Rightarrow e = \frac{c}{a} \\
 &= \frac{\sqrt{2}}{1} = \sqrt{2}; \text{ asymptotes are } y = \pm x; F(\pm\sqrt{2}, 0); \\
 &\text{directrices are } x = 0 \pm \frac{a}{e} = \pm \frac{1}{\sqrt{2}}
 \end{aligned}$$



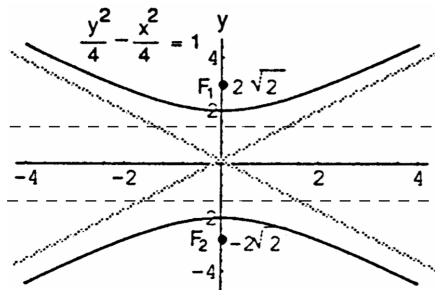
$$\begin{aligned}
 18. \quad 9x^2 - 16y^2 = 144 &\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{a^2 + b^2} \\
 &= \sqrt{16 + 9} = 5 \Rightarrow e = \frac{c}{a} = \frac{5}{4}; \text{ asymptotes are } \\
 &y = \pm \frac{3}{4}x; F(\pm 5, 0); \text{ directrices are } x = 0 \pm \frac{a}{e} \\
 &= \pm \frac{16}{5}
 \end{aligned}$$



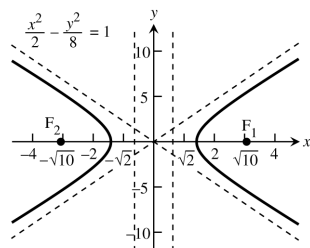
$$\begin{aligned}
 19. \quad y^2 - x^2 = 8 &\Rightarrow \frac{y^2}{8} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2} \\
 &= \sqrt{8 + 8} = 4 \Rightarrow e = \frac{c}{a} = \frac{4}{\sqrt{8}} = \sqrt{2}; \text{ asymptotes are } \\
 &y = \pm x; F(0, \pm 4); \text{ directrices are } y = 0 \pm \frac{a}{e} \\
 &= \pm \frac{\sqrt{8}}{\sqrt{2}} = \pm 2
 \end{aligned}$$



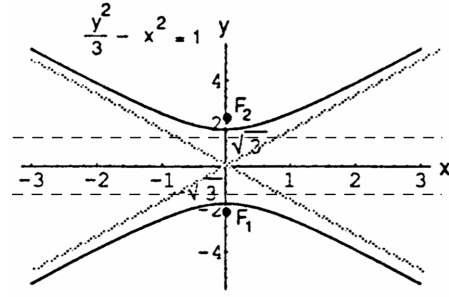
$$\begin{aligned}
 20. \quad y^2 - x^2 = 4 &\Rightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow c = \sqrt{a^2 + b^2} \\
 &= \sqrt{4 + 4} = 2\sqrt{2} \Rightarrow e = \frac{c}{a} = \frac{2\sqrt{2}}{2} = \sqrt{2}; \text{ asymptotes } \\
 &\text{are } y = \pm x; F(0, \pm 2\sqrt{2}); \text{ directrices are } y = 0 \pm \frac{a}{e} \\
 &= \pm \frac{2}{\sqrt{2}} = \pm \sqrt{2}
 \end{aligned}$$



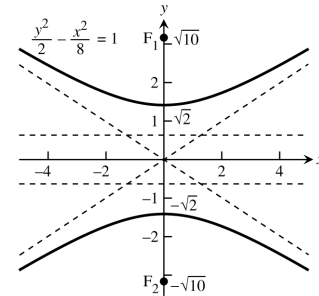
$$\begin{aligned}
 21. \quad 8x^2 - 2y^2 = 16 &\Rightarrow \frac{x^2}{2} - \frac{y^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2} \\
 &= \sqrt{2 + 8} = \sqrt{10} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}; \text{ asymptotes } \\
 &\text{are } y = \pm 2x; F(\pm\sqrt{10}, 0); \text{ directrices are } x = 0 \pm \frac{a}{e} \\
 &= \pm \frac{\sqrt{2}}{\sqrt{5}} = \pm \frac{2}{\sqrt{10}}
 \end{aligned}$$



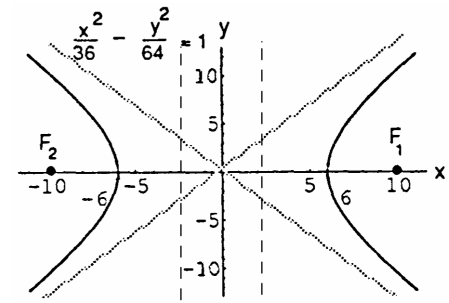
$$\begin{aligned}
 22. \quad y^2 - 3x^2 &= 3 \Rightarrow \frac{y^2}{3} - x^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} \\
 &= \sqrt{3 + 1} = 2 \Rightarrow e = \frac{c}{a} = \frac{2}{\sqrt{3}}; \text{ asymptotes are} \\
 y &= \pm \sqrt{3}x; F(0, \pm 2); \text{ directrices are } y = 0 \pm \frac{a}{e} \\
 &= \pm \frac{\sqrt{3}}{\left(\frac{2}{\sqrt{3}}\right)} = \pm \frac{3}{2}
 \end{aligned}$$



$$\begin{aligned}
 23. \quad 8y^2 - 2x^2 &= 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2} \\
 &= \sqrt{2 + 8} = \sqrt{10} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}; \text{ asymptotes} \\
 \text{are } y &= \pm \frac{1}{2}x; F(0, \pm \sqrt{10}); \text{ directrices are } y = 0 \pm \frac{a}{e} \\
 &= \pm \frac{\sqrt{2}}{\sqrt{5}} = \pm \frac{2}{\sqrt{10}}
 \end{aligned}$$



$$\begin{aligned}
 24. \quad 64x^2 - 36y^2 &= 2304 \Rightarrow \frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow c = \sqrt{a^2 + b^2} \\
 &= \sqrt{36 + 64} = 10 \Rightarrow e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}; \text{ asymptotes are} \\
 y &= \pm \frac{4}{3}x; F(\pm 10, 0); \text{ directrices are } x = 0 \pm \frac{a}{e} \\
 &= \pm \frac{6}{\left(\frac{5}{3}\right)} = \pm \frac{18}{5}
 \end{aligned}$$



$$25. \quad \text{Vertices } (0, \pm 1) \text{ and } e = 3 \Rightarrow a = 1 \text{ and } e = \frac{c}{a} = 3 \Rightarrow c = 3a = 3 \Rightarrow b^2 = c^2 - a^2 = 9 - 1 = 8 \Rightarrow y^2 - \frac{x^2}{8} = 1$$

$$26. \quad \text{Vertices } (\pm 2, 0) \text{ and } e = 2 \Rightarrow a = 2 \text{ and } e = \frac{c}{a} = 2 \Rightarrow c = 2a = 4 \Rightarrow b^2 = c^2 - a^2 = 16 - 4 = 12 \Rightarrow \frac{x^2}{4} - \frac{y^2}{12} = 1$$

$$27. \quad \text{Foci } (\pm 3, 0) \text{ and } e = 3 \Rightarrow c = 3 \text{ and } e = \frac{c}{a} = 3 \Rightarrow c = 3a \Rightarrow a = 1 \Rightarrow b^2 = c^2 - a^2 = 9 - 1 = 8 \Rightarrow x^2 - \frac{y^2}{8} = 1$$

$$\begin{aligned}
 28. \quad \text{Foci } (0, \pm 5) \text{ and } e = 1.25 \Rightarrow c = 5 \text{ and } e = \frac{c}{a} = 1.25 = \frac{5}{4} \Rightarrow c = \frac{5}{4}a \Rightarrow 5 = \frac{5}{4}a \Rightarrow a = 4 \Rightarrow b^2 = c^2 - a^2 \\
 = 25 - 16 = 9 \Rightarrow \frac{y^2}{16} - \frac{x^2}{9} = 1
 \end{aligned}$$

$$29. \quad e = 1, x = 2 \Rightarrow k = 2 \Rightarrow r = \frac{2(1)}{1 + (1)\cos\theta} = \frac{2}{1 + \cos\theta}$$

$$30. \quad e = 1, y = 2 \Rightarrow k = 2 \Rightarrow r = \frac{2(1)}{1 + (1)\sin\theta} = \frac{2}{1 + \sin\theta}$$

$$31. \quad e = 5, y = -6 \Rightarrow k = 6 \Rightarrow r = \frac{6(5)}{1 - 5\sin\theta} = \frac{30}{1 - 5\sin\theta}$$

$$32. \quad e = 2, x = 4 \Rightarrow k = 4 \Rightarrow r = \frac{4(2)}{1 + 2\cos\theta} = \frac{8}{1 + 2\cos\theta}$$

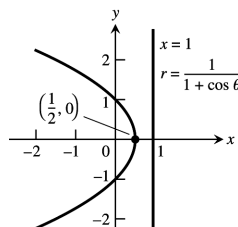
$$33. \quad e = \frac{1}{2}, x = 1 \Rightarrow k = 1 \Rightarrow r = \frac{\left(\frac{1}{2}\right)(1)}{1 + \left(\frac{1}{2}\right)\cos\theta} = \frac{1}{2 + \cos\theta}$$

$$34. e = \frac{1}{4}, x = -2 \Rightarrow k = 2 \Rightarrow r = \frac{\left(\frac{1}{4}\right)(2)}{1 - \left(\frac{1}{4}\right)\cos\theta} = \frac{2}{4 - \cos\theta}$$

$$35. e = \frac{1}{5}, y = -10 \Rightarrow k = 10 \Rightarrow r = \frac{\left(\frac{1}{5}\right)(10)}{1 - \left(\frac{1}{5}\right)\sin\theta} = \frac{10}{5 - \sin\theta}$$

$$36. e = \frac{1}{3}, y = 6 \Rightarrow k = 6 \Rightarrow r = \frac{\left(\frac{1}{3}\right)(6)}{1 + \left(\frac{1}{3}\right)\sin\theta} = \frac{6}{3 + \sin\theta}$$

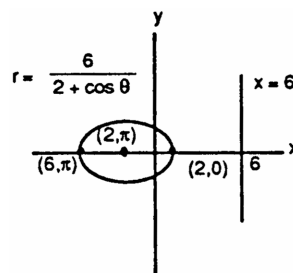
$$37. r = \frac{1}{1 + \cos\theta} \Rightarrow e = 1, k = 1 \Rightarrow x = 1$$



$$38. r = \frac{6}{2 + \cos\theta} = \frac{3}{1 + \left(\frac{1}{2}\right)\cos\theta} \Rightarrow e = \frac{1}{2}, k = 6 \Rightarrow x = 6;$$

$$a(1 - e^2) = ke \Rightarrow a\left[1 - \left(\frac{1}{2}\right)^2\right] = 3 \Rightarrow \frac{3}{4}a = 3$$

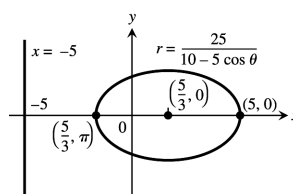
$$\Rightarrow a = 4 \Rightarrow ea = 2$$



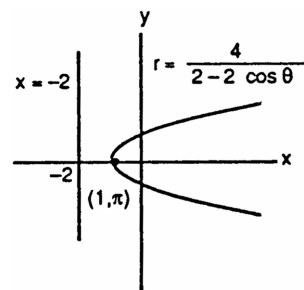
$$39. r = \frac{25}{10 - 5\cos\theta} \Rightarrow r = \frac{\left(\frac{25}{10}\right)}{1 - \left(\frac{5}{10}\right)\cos\theta} = \frac{\left(\frac{5}{2}\right)}{1 - \left(\frac{1}{2}\right)\cos\theta}$$

$$\Rightarrow e = \frac{1}{2}, k = 5 \Rightarrow x = -5; a(1 - e^2) = ke$$

$$\Rightarrow a\left[1 - \left(\frac{1}{2}\right)^2\right] = \frac{5}{2} \Rightarrow \frac{3}{4}a = \frac{5}{2} \Rightarrow a = \frac{10}{3} \Rightarrow ea = \frac{5}{3}$$



$$40. r = \frac{4}{2 - 2\cos\theta} \Rightarrow r = \frac{2}{1 - \cos\theta} \Rightarrow e = 1, k = 2 \Rightarrow x = -2$$

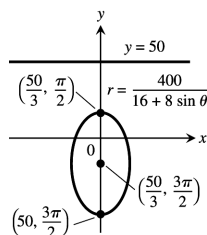


$$41. r = \frac{400}{16 + 8\sin\theta} \Rightarrow r = \frac{\left(\frac{400}{16}\right)}{1 + \left(\frac{8}{16}\right)\sin\theta} \Rightarrow r = \frac{25}{1 + \left(\frac{1}{2}\right)\sin\theta}$$

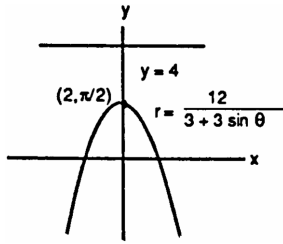
$$e = \frac{1}{2}, k = 50 \Rightarrow y = 50; a(1 - e^2) = ke$$

$$\Rightarrow a\left[1 - \left(\frac{1}{2}\right)^2\right] = 25 \Rightarrow \frac{3}{4}a = 25 \Rightarrow a = \frac{100}{3}$$

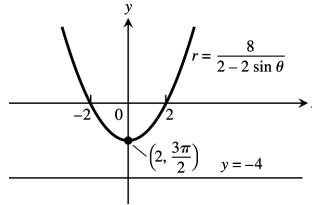
$$\Rightarrow ea = \frac{50}{3}$$



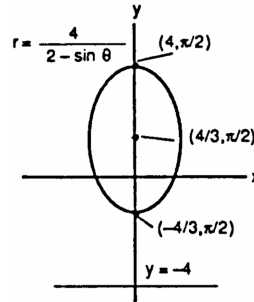
$$42. r = \frac{12}{3+3\sin\theta} \Rightarrow r = \frac{4}{1+\sin\theta} \Rightarrow e = 1, \\ k = 4 \Rightarrow y = 4$$



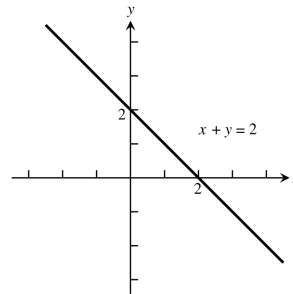
$$43. r = \frac{8}{2-2\sin\theta} \Rightarrow r = \frac{4}{1-\sin\theta} \Rightarrow e = 1, \\ k = 4 \Rightarrow y = -4$$



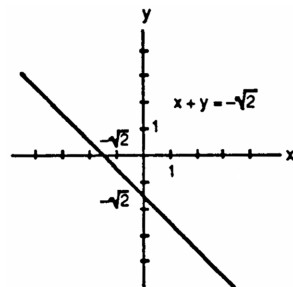
$$44. r = \frac{4}{2-\sin\theta} \Rightarrow r = \frac{2}{1-(\frac{1}{2})\sin\theta} \Rightarrow e = \frac{1}{2}, k = 4 \\ \Rightarrow y = -4; a(1-e^2) = ke \Rightarrow a\left[1 - \left(\frac{1}{2}\right)^2\right] = 2 \\ \Rightarrow \frac{3}{4}a = 2 \Rightarrow a = \frac{8}{3} \Rightarrow ea = \frac{4}{3}$$



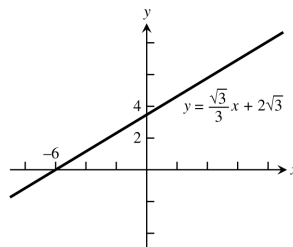
$$45. r \cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2} \Rightarrow r\left(\cos\theta \cos\frac{\pi}{4} + \sin\theta \sin\frac{\pi}{4}\right) \\ = \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}r \cos\theta + \frac{1}{\sqrt{2}}r \sin\theta = \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ = \sqrt{2} \Rightarrow x + y = 2 \Rightarrow y = 2 - x$$



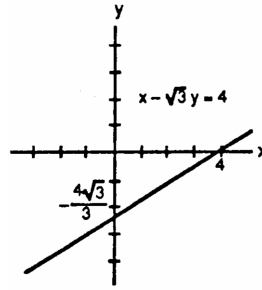
$$46. r \cos\left(\theta + \frac{3\pi}{4}\right) = 1 \Rightarrow r\left(\cos\theta \cos\frac{3\pi}{4} - \sin\theta \sin\frac{3\pi}{4}\right) = 1 \\ \Rightarrow -\frac{\sqrt{2}}{2}r \cos\theta - \frac{\sqrt{2}}{2}r \sin\theta = 1 \Rightarrow x + y = -\sqrt{2} \\ \Rightarrow y = -x - \sqrt{2}$$



$$47. r \cos\left(\theta - \frac{2\pi}{3}\right) = 3 \Rightarrow r\left(\cos\theta \cos\frac{2\pi}{3} + \sin\theta \sin\frac{2\pi}{3}\right) = 3 \\ \Rightarrow -\frac{1}{2}r \cos\theta + \frac{\sqrt{3}}{2}r \sin\theta = 3 \Rightarrow -\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3 \\ \Rightarrow -x + \sqrt{3}y = 6 \Rightarrow y = \frac{\sqrt{3}}{3}x + 2\sqrt{3}$$



$$\begin{aligned}
 48. \quad r \cos \left(\theta + \frac{\pi}{3} \right) &= 2 \Rightarrow r \left(\cos \theta \cos \frac{\pi}{3} - \sin \theta \sin \frac{\pi}{3} \right) = 2 \\
 &\Rightarrow \frac{1}{2} r \cos \theta - \frac{\sqrt{3}}{2} r \sin \theta = 2 \Rightarrow \frac{1}{2} x - \frac{\sqrt{3}}{2} y = 2 \\
 &\Rightarrow x - \sqrt{3} y = 4 \Rightarrow y = \frac{\sqrt{3}}{3} x - \frac{4\sqrt{3}}{3}
 \end{aligned}$$



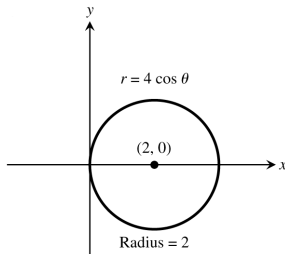
$$\begin{aligned}
 49. \quad \sqrt{2}x + \sqrt{2}y &= 6 \Rightarrow \sqrt{2}r \cos \theta + \sqrt{2}r \sin \theta = 6 \Rightarrow r \left(\frac{\sqrt{2}}{2} \cos \theta + \frac{\sqrt{2}}{2} \sin \theta \right) = 3 \Rightarrow r \left(\cos \frac{\pi}{4} \cos \theta + \sin \frac{\pi}{4} \sin \theta \right) \\
 &= 3 \Rightarrow r \cos \left(\theta - \frac{\pi}{4} \right) = 3
 \end{aligned}$$

$$\begin{aligned}
 50. \quad \sqrt{3}x - y &= 1 \Rightarrow \sqrt{3}r \cos \theta - r \sin \theta = 1 \Rightarrow r \left(\frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta \right) = \frac{1}{2} \Rightarrow r \left(\cos \frac{\pi}{6} \cos \theta - \sin \frac{\pi}{6} \sin \theta \right) \\
 &= \frac{1}{2} \Rightarrow r \cos \left(\theta + \frac{\pi}{6} \right) = \frac{1}{2}
 \end{aligned}$$

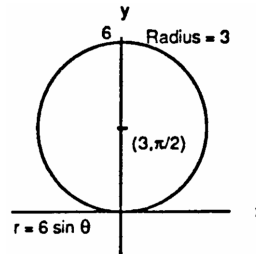
$$51. \quad y = -5 \Rightarrow r \sin \theta = -5 \Rightarrow -r \sin \theta = 5 \Rightarrow r \sin(-\theta) = 5 \Rightarrow r \cos \left(\frac{\pi}{2} - (-\theta) \right) = 5 \Rightarrow r \cos \left(\theta + \frac{\pi}{2} \right) = 5$$

$$52. \quad x = -4 \Rightarrow r \cos \theta = -4 \Rightarrow -r \cos \theta = 4 \Rightarrow r \cos(\theta - \pi) = 4$$

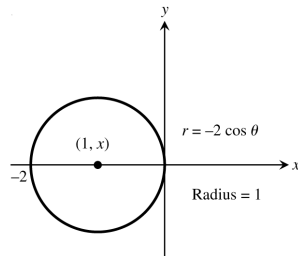
53.



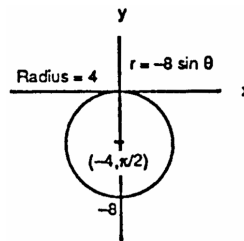
54.



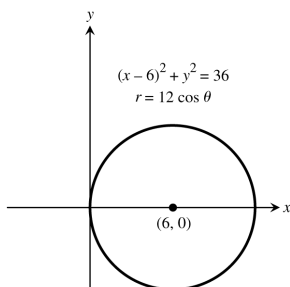
55.



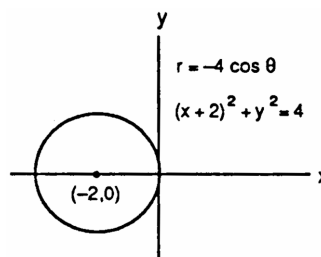
56.



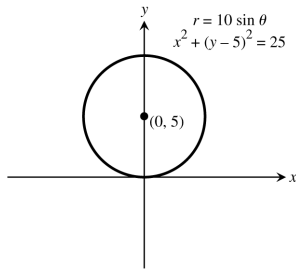
$$\begin{aligned}
 57. \quad (x - 6)^2 + y^2 &= 36 \Rightarrow C = (6, 0), a = 6 \\
 &\Rightarrow r = 12 \cos \theta \text{ is the polar equation}
 \end{aligned}$$



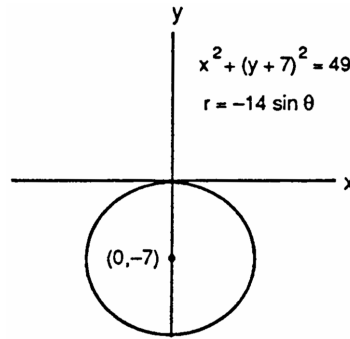
$$\begin{aligned}
 58. \quad (x + 2)^2 + y^2 &= 4 \Rightarrow C = (-2, 0), a = 2 \\
 &\Rightarrow r = -4 \cos \theta \text{ is the polar equation}
 \end{aligned}$$



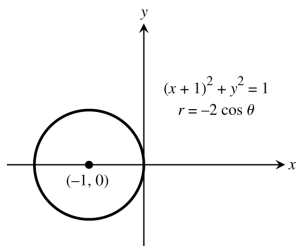
59. $x^2 + (y - 5)^2 = 25 \Rightarrow C = (0, 5), a = 5$
 $\Rightarrow r = 10 \sin \theta$ is the polar equation



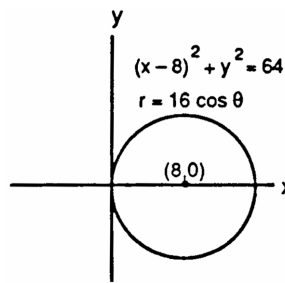
60. $x^2 + (y + 7)^2 = 49 \Rightarrow C = (0, -7), a = 7$
 $\Rightarrow r = -14 \sin \theta$ is the polar equation



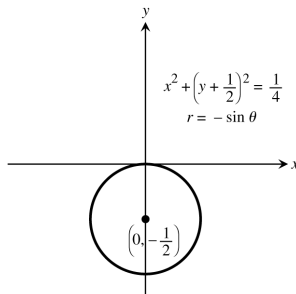
61. $x^2 + 2x + y^2 = 0 \Rightarrow (x + 1)^2 + y^2 = 1$
 $\Rightarrow C = (-1, 0), a = 1 \Rightarrow r = -2 \cos \theta$ is the polar equation



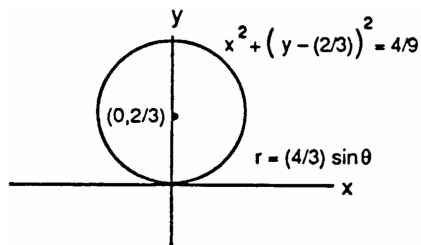
62. $x^2 - 16x + y^2 = 0 \Rightarrow (x - 8)^2 + y^2 = 64$
 $\Rightarrow C = (8, 0), a = 8 \Rightarrow r = 16 \cos \theta$ is the polar equation



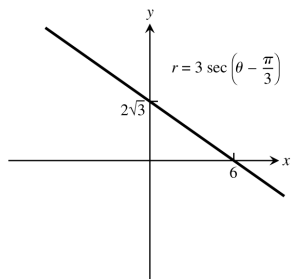
63. $x^2 + y^2 + y = 0 \Rightarrow x^2 + (y + \frac{1}{2})^2 = \frac{1}{4}$
 $\Rightarrow C = (0, -\frac{1}{2}), a = \frac{1}{2} \Rightarrow r = -\sin \theta$ is the polar equation



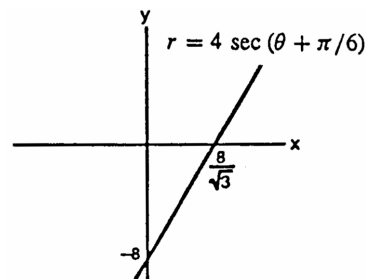
64. $x^2 + y^2 - \frac{4}{3}y = 0 \Rightarrow x^2 + (y - \frac{2}{3})^2 = \frac{4}{9}$
 $\Rightarrow C = (0, \frac{2}{3}), a = \frac{2}{3} \Rightarrow r = \frac{4}{3} \sin \theta$ is the polar equation



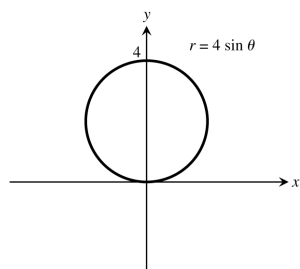
65.



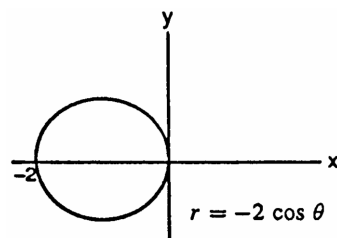
66.



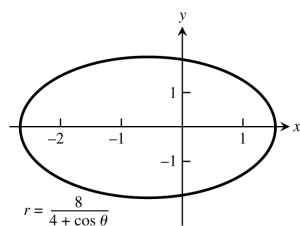
67.



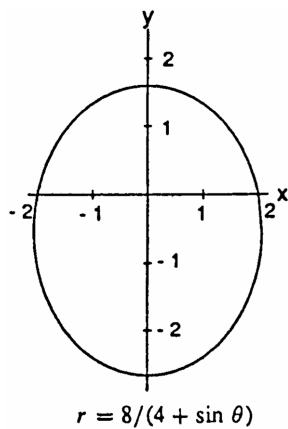
68.



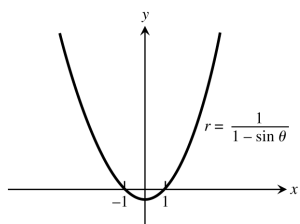
69.



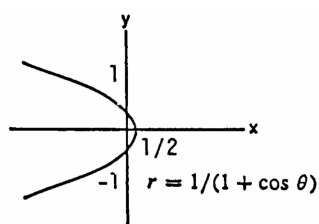
70.



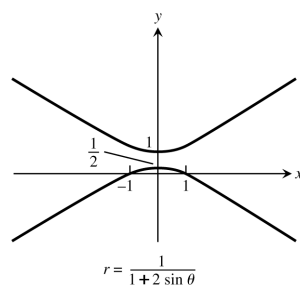
71.



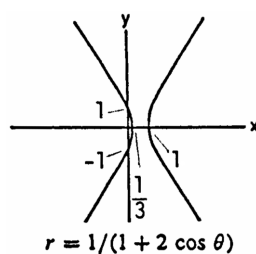
72.



73.



74.



75. (a) Perihelion = $a - ae = a(1 - e)$, Aphelion = $ea + a = a(1 + e)$

(b)

Planet	Perihelion	Aphelion
Mercury	0.3075 AU	0.4667 AU
Venus	0.7184 AU	0.7282 AU
Earth	0.9833 AU	1.0167 AU
Mars	1.3817 AU	1.6663 AU
Jupiter	4.9512 AU	5.4548 AU
Saturn	9.0210 AU	10.0570 AU
Uranus	18.2977 AU	20.0623 AU
Neptune	29.8135 AU	30.3065 AU

76. Mercury: $r = \frac{(0.3871)(1 - 0.2056^2)}{1 + 0.2056 \cos \theta} = \frac{0.3707}{1 + 0.2056 \cos \theta}$

Venus: $r = \frac{(0.7233)(1 - 0.0068^2)}{1 + 0.0068 \cos \theta} = \frac{0.7233}{1 + 0.0068 \cos \theta}$

Earth: $r = \frac{1(1 - 0.0167^2)}{1 + 0.0167 \cos \theta} = \frac{0.9997}{1 + 0.0167 \cos \theta}$

Mars: $r = \frac{(1.524)(1 - 0.0934^2)}{1 + 0.0934 \cos \theta} = \frac{1.511}{1 + 0.0934 \cos \theta}$

Jupiter: $r = \frac{(5.203)(1 - 0.0484^2)}{1 + 0.0484 \cos \theta} = \frac{5.191}{1 + 0.0484 \cos \theta}$

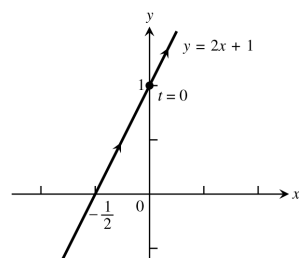
Saturn: $r = \frac{(9.539)(1 - 0.0543^2)}{1 + 0.0543 \cos \theta} = \frac{9.511}{1 + 0.0543 \cos \theta}$

Uranus: $r = \frac{(19.18)(1 - 0.0460^2)}{1 + 0.0460 \cos \theta} = \frac{19.14}{1 + 0.0460 \cos \theta}$

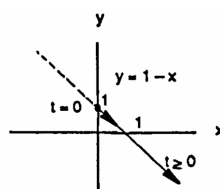
Neptune: $r = \frac{(30.06)(1 - 0.0082^2)}{1 + 0.0082 \cos \theta} = \frac{30.06}{1 + 0.0082 \cos \theta}$

CHAPTER 11 PRACTICE EXERCISES

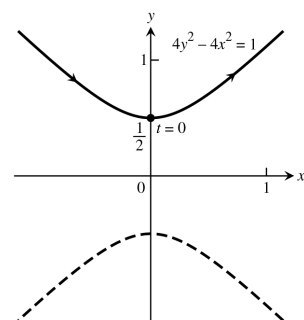
1. $x = \frac{1}{2}t$ and $y = t + 1 \Rightarrow 2x = t \Rightarrow y = 2x + 1$



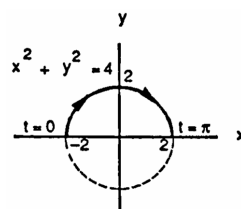
2. $x = \sqrt{t}$ and $y = 1 - \sqrt{t} \Rightarrow y = 1 - x$



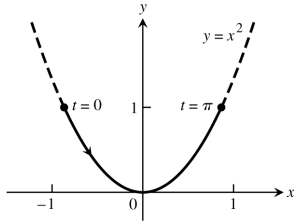
3. $x = \frac{1}{2} \tan t$ and $y = \frac{1}{2} \sec t \Rightarrow x^2 = \frac{1}{4} \tan^2 t$
and $y^2 = \frac{1}{4} \sec^2 t \Rightarrow 4x^2 = \tan^2 t$ and $4y^2 = \sec^2 t \Rightarrow 4x^2 + 1 = 4y^2 \Rightarrow 4y^2 - 4x^2 = 1$



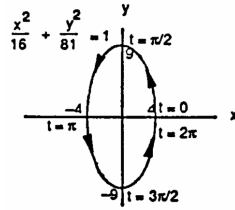
4. $x = -2 \cos t$ and $y = 2 \sin t \Rightarrow x^2 = 4 \cos^2 t$ and $y^2 = 4 \sin^2 t \Rightarrow x^2 + y^2 = 4$



5. $x = -\cos t$ and $y = \cos^2 t \Rightarrow y = (-x)^2 = x^2$



6. $x = 4 \cos t$ and $y = 9 \sin t \Rightarrow x^2 = 16 \cos^2 t$ and $y^2 = 81 \sin^2 t \Rightarrow \frac{x^2}{16} + \frac{y^2}{81} = 1$



7. $16x^2 + 9y^2 = 144 \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1 \Rightarrow a = 3$ and $b = 4 \Rightarrow x = 3 \cos t$ and $y = 4 \sin t, 0 \leq t \leq 2\pi$

8. $x^2 + y^2 = 4 \Rightarrow x = -2 \cos t$ and $y = 2 \sin t, 0 \leq t \leq 6\pi$

9. $x = \frac{1}{2} \tan t, y = \frac{1}{2} \sec t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2} \sec t \tan t}{\frac{1}{2} \sec^2 t} = \frac{\tan t}{\sec t} = \sin t \Rightarrow \frac{dy}{dx} \Big|_{t=\pi/3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}; t = \frac{\pi}{3}$
 $\Rightarrow x = \frac{1}{2} \tan \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2} \sec \frac{\pi}{3} = 1 \Rightarrow y = \frac{\sqrt{3}}{2} x + \frac{1}{4}; \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\cos t}{\frac{1}{2} \sec^2 t} = 2 \cos^3 t \Rightarrow \frac{d^2y}{dx^2} \Big|_{t=\pi/3} = 2 \cos^3 \left(\frac{\pi}{3}\right) = \frac{1}{4}$

10. $x = 1 + \frac{1}{t^2}, y = 1 - \frac{3}{t} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{3}{t^3}\right)}{\left(-\frac{2}{t^3}\right)} = -\frac{3}{2} t \Rightarrow \frac{dy}{dx} \Big|_{t=2} = -\frac{3}{2} (2) = -3; t = 2 \Rightarrow x = 1 + \frac{1}{2^2} = \frac{5}{4}$ and $y = 1 - \frac{3}{2} = -\frac{1}{2} \Rightarrow y = -3x + \frac{13}{4}; \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(-\frac{3}{t^3}\right)}{\left(-\frac{2}{t^3}\right)} = \frac{3}{2} t^3 \Rightarrow \frac{d^2y}{dx^2} \Big|_{t=2} = \frac{3}{2} (2)^3 = 6$

11. (a) $x = 4t^2, y = t^3 - 1 \Rightarrow t = \pm \frac{\sqrt{x}}{2} \Rightarrow y = \left(\pm \frac{\sqrt{x}}{2}\right)^3 - 1 = \pm \frac{x^{3/2}}{8} - 1$

(b) $x = \cos t, y = \tan t \Rightarrow \sec t = \frac{1}{x} \Rightarrow \tan^2 t + 1 = \sec^2 t \Rightarrow y^2 = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2} \Rightarrow y = \pm \frac{\sqrt{1-x^2}}{x}$

12. (a) The line through $(1, -2)$ with slope 3 is $y = 3x - 5 \Rightarrow x = t, y = 3t - 5, -\infty < t < \infty$

(b) $(x-1)^2 + (y+2)^2 = 9 \Rightarrow x-1 = 3 \cos t, y+2 = 3 \sin t \Rightarrow x = 1 + 3 \cos t, y = -2 + 3 \sin t, 0 \leq t \leq 2\pi$

(c) $y = 4x^2 - x \Rightarrow x = t, y = 4t^2 - t, -\infty < t < \infty$

(d) $9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow x = 2 \cos t, y = 3 \sin t, 0 \leq t \leq 2\pi$

13. $y = x^{1/2} - \frac{x^{3/2}}{3} \Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-1/2} - \frac{1}{2} x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4} \left(\frac{1}{x} - 2 + x\right) \Rightarrow L = \int_1^4 \sqrt{1 + \frac{1}{4} \left(\frac{1}{x} - 2 + x\right)} dx$
 $\Rightarrow L = \int_1^4 \sqrt{\frac{1}{4} \left(\frac{1}{x} + 2 + x\right)} dx = \int_1^4 \sqrt{\frac{1}{4} (x^{-1/2} + x^{1/2})^2} dx = \int_1^4 \frac{1}{2} (x^{-1/2} + x^{1/2}) dx = \frac{1}{2} \left[2x^{1/2} + \frac{2}{3} x^{3/2}\right]_1^4$
 $= \frac{1}{2} \left[\left(4 + \frac{2}{3} \cdot 8\right) - \left(2 + \frac{2}{3}\right)\right] = \frac{1}{2} \left(2 + \frac{14}{3}\right) = \frac{10}{3}$

14. $x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3} x^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4x^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}} dy$
 $= \int_1^8 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}} dx = \frac{1}{3} \int_1^8 \sqrt{9x^{2/3} + 4} (x^{-1/3}) dx; [u = 9x^{2/3} + 4 \Rightarrow du = 6y^{-1/3} dy; x = 1 \Rightarrow u = 13,$
 $x = 8 \Rightarrow u = 40] \rightarrow L = \frac{1}{18} \int_{13}^{40} u^{1/2} du = \frac{1}{18} \left[\frac{2}{3} u^{3/2}\right]_{13}^{40} = \frac{1}{27} [40^{3/2} - 13^{3/2}] \approx 7.634$

15. $y = \frac{5}{12} x^{6/5} - \frac{5}{8} x^{4/5} \Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{1/5} - \frac{1}{2} x^{-1/5} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4} (x^{2/5} - 2 + x^{-2/5})$
 $\Rightarrow L = \int_1^{32} \sqrt{1 + \frac{1}{4} (x^{2/5} - 2 + x^{-2/5})} dx \Rightarrow L = \int_1^{32} \sqrt{\frac{1}{4} (x^{2/5} + 2 + x^{-2/5})} dx = \int_1^{32} \sqrt{\frac{1}{4} (x^{1/5} + x^{-1/5})^2} dx$

$$= \int_1^{32} \frac{1}{2} (x^{1/5} + x^{-1/5}) dx = \frac{1}{2} \left[\frac{5}{6} x^{6/5} + \frac{5}{4} x^{4/5} \right]_1^{32} = \frac{1}{2} \left[\left(\frac{5}{6} \cdot 2^6 + \frac{5}{4} \cdot 2^4 \right) - \left(\frac{5}{6} + \frac{5}{4} \right) \right] = \frac{1}{2} \left(\frac{315}{6} + \frac{75}{4} \right) \\ = \frac{1}{48} (1260 + 450) = \frac{1710}{48} = \frac{285}{8}$$

$$16. x = \frac{1}{12} y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4} y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy} \right)^2 = \frac{1}{16} y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16} y^4 - \frac{1}{2} + \frac{1}{y^4} \right)} dy \\ = \int_1^2 \sqrt{\frac{1}{16} y^4 + \frac{1}{2} + \frac{1}{y^4}} dy = \int_1^2 \sqrt{\left(\frac{1}{4} y^2 + \frac{1}{y^2} \right)^2} dy = \int_1^2 \left(\frac{1}{4} y^2 + \frac{1}{y^2} \right) dy = \left[\frac{1}{12} y^3 - \frac{1}{y} \right]_1^2 \\ = \left(\frac{8}{12} - \frac{1}{2} \right) - \left(\frac{1}{12} - 1 \right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$$

$$17. \frac{dx}{dt} = -5 \sin t + 5 \sin 5t \text{ and } \frac{dy}{dt} = 5 \cos t - 5 \cos 5t \Rightarrow \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \\ = \sqrt{(-5 \sin t + 5 \sin 5t)^2 + (5 \cos t - 5 \cos 5t)^2} \\ = 5 \sqrt{\sin^2 5t - 2 \sin t \sin 5t + \sin^2 t + \cos^2 t - 2 \cos t \cos 5t + \cos^2 5t} = 5 \sqrt{2 - 2(\sin t \sin 5t + \cos t \cos 5t)} \\ = 5 \sqrt{2(1 - \cos 4t)} = 5 \sqrt{4 \left(\frac{1}{2} \right) (1 - \cos 4t)} = 10 \sqrt{\sin^2 2t} = 10 |\sin 2t| = 10 \sin 2t \text{ (since } 0 \leq t \leq \frac{\pi}{2} \text{)} \\ \Rightarrow \text{Length} = \int_0^{\pi/2} 10 \sin 2t dt = [-5 \cos 2t]_0^{\pi/2} = (-5)(-1) - (-5)(1) = 10$$

$$18. \frac{dx}{dt} = 3t^2 - 12t \text{ and } \frac{dy}{dt} = 3t^2 + 12t \Rightarrow \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} = \sqrt{(3t^2 - 12t)^2 + (3t^2 + 12t)^2} = \sqrt{288t^2 + 18t^4} \\ = 3\sqrt{2} |t| \sqrt{16 + t^2} \Rightarrow \text{Length} = \int_0^1 3\sqrt{2} |t| \sqrt{16 + t^2} dt = 3\sqrt{2} \int_0^1 t \sqrt{16 + t^2} dt; \left[u = 16 + t^2 \Rightarrow du = 2t dt \right. \\ \Rightarrow \frac{1}{2} du = t dt; t = 0 \Rightarrow u = 16; t = 1 \Rightarrow u = 17 \left. \right]; \frac{3\sqrt{2}}{2} \int_{16}^{17} \sqrt{u} du = \frac{3\sqrt{2}}{2} \left[\frac{2}{3} u^{3/2} \right]_{16}^{17} = \frac{3\sqrt{2}}{2} \left(\frac{2}{3} (17)^{3/2} - \frac{2}{3} (16)^{3/2} \right) \\ = \frac{3\sqrt{2}}{2} \cdot \frac{2}{3} \left((17)^{3/2} - 64 \right) = \sqrt{2} \left((17)^{3/2} - 64 \right) \approx 8.617.$$

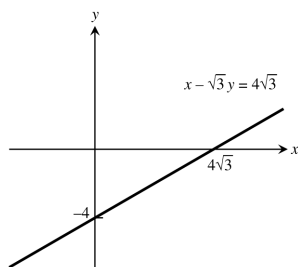
$$19. \frac{dx}{d\theta} = -3 \sin \theta \text{ and } \frac{dy}{d\theta} = 3 \cos \theta \Rightarrow \sqrt{\left(\frac{dx}{d\theta} \right)^2 + \left(\frac{dy}{d\theta} \right)^2} = \sqrt{(-3 \sin \theta)^2 + (3 \cos \theta)^2} = \sqrt{3(\sin^2 \theta + \cos^2 \theta)} = 3 \\ \Rightarrow \text{Length} = \int_0^{3\pi/2} 3 d\theta = 3 \int_0^{3\pi/2} d\theta = 3 \left(\frac{3\pi}{2} - 0 \right) = \frac{9\pi}{2}$$

$$20. x = t^2 \text{ and } y = \frac{t^3}{3} - t, -\sqrt{3} \leq t \leq \sqrt{3} \Rightarrow \frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = t^2 - 1 \Rightarrow \text{Length} = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (t^2 - 1)^2} dt \\ = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1} dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1} dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(t^2 + 1)^2} dt = \int_{-\sqrt{3}}^{\sqrt{3}} (t^2 + 1) dt = \left[\frac{t^3}{3} + t \right]_{-\sqrt{3}}^{\sqrt{3}} \\ = 4\sqrt{3}$$

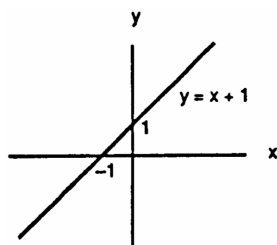
$$21. x = \frac{t^2}{2} \text{ and } y = 2t, 0 \leq t \leq \sqrt{5} \Rightarrow \frac{dx}{dt} = t \text{ and } \frac{dy}{dt} = 2 \Rightarrow \text{Surface Area} = \int_0^{\sqrt{5}} 2\pi(2t)\sqrt{t^2 + 4} dt = \int_4^9 2\pi u^{1/2} du \\ = 2\pi \left[\frac{2}{3} u^{3/2} \right]_4^9 = \frac{76\pi}{3}, \text{ where } u = t^2 + 4 \Rightarrow du = 2t dt; t = 0 \Rightarrow u = 4, t = \sqrt{5} \Rightarrow u = 9$$

$$22. x = t^2 + \frac{1}{2t} \text{ and } y = 4\sqrt{t}, \frac{1}{\sqrt{2}} \leq t \leq 1 \Rightarrow \frac{dx}{dt} = 2t - \frac{1}{2t^2} \text{ and } \frac{dy}{dt} = \frac{2}{\sqrt{t}} \\ \Rightarrow \text{Surface Area} = \int_{1/\sqrt{2}}^1 2\pi \left(t^2 + \frac{1}{2t} \right) \sqrt{\left(2t - \frac{1}{2t^2} \right)^2 + \left(\frac{2}{\sqrt{t}} \right)^2} dt = 2\pi \int_{1/\sqrt{2}}^1 \left(t^2 + \frac{1}{2t} \right) \sqrt{\left(2t + \frac{1}{2t^2} \right)^2} dt \\ = 2\pi \int_{1/\sqrt{2}}^1 \left(t^2 + \frac{1}{2t} \right) \left(2t + \frac{1}{2t^2} \right) dt = 2\pi \int_{1/\sqrt{2}}^1 \left(2t^3 + \frac{3}{2} + \frac{1}{4} t^{-3} \right) dt = 2\pi \left[\frac{1}{2} t^4 + \frac{3}{2} t - \frac{1}{8} t^{-2} \right]_{1/\sqrt{2}}^1 \\ = 2\pi \left(2 - \frac{3\sqrt{2}}{4} \right)$$

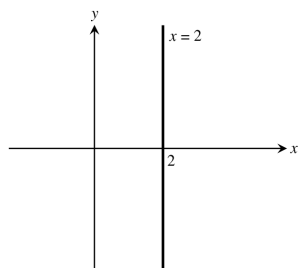
$$\begin{aligned}
 23. \quad r \cos \left(\theta + \frac{\pi}{3} \right) &= 2\sqrt{3} \Rightarrow r \left(\cos \theta \cos \frac{\pi}{3} - \sin \theta \sin \frac{\pi}{3} \right) \\
 &= 2\sqrt{3} \Rightarrow \frac{1}{2} r \cos \theta - \frac{\sqrt{3}}{2} r \sin \theta = 2\sqrt{3} \\
 &\Rightarrow r \cos \theta - \sqrt{3} r \sin \theta = 4\sqrt{3} \Rightarrow x - \sqrt{3} y = 4\sqrt{3} \\
 &\Rightarrow y = \frac{\sqrt{3}}{3} x - 4
 \end{aligned}$$



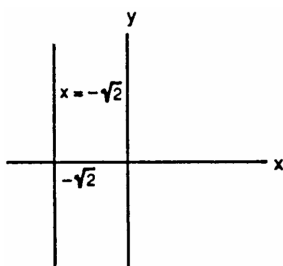
$$\begin{aligned}
 24. \quad r \cos \left(\theta - \frac{3\pi}{4} \right) &= \frac{\sqrt{2}}{2} \Rightarrow r \left(\cos \theta \cos \frac{3\pi}{4} + \sin \theta \sin \frac{3\pi}{4} \right) \\
 &= \frac{\sqrt{2}}{2} \Rightarrow -\frac{\sqrt{2}}{2} r \cos \theta + \frac{\sqrt{2}}{2} r \sin \theta = \frac{\sqrt{2}}{2} \Rightarrow -x + y = 1 \\
 &\Rightarrow y = x + 1
 \end{aligned}$$



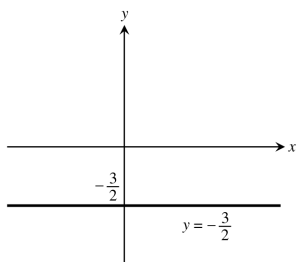
$$25. \quad r = 2 \sec \theta \Rightarrow r = \frac{2}{\cos \theta} \Rightarrow r \cos \theta = 2 \Rightarrow x = 2$$



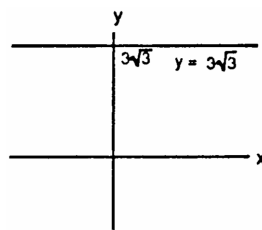
$$26. \quad r = -\sqrt{2} \sec \theta \Rightarrow r \cos \theta = -\sqrt{2} \Rightarrow x = -\sqrt{2}$$



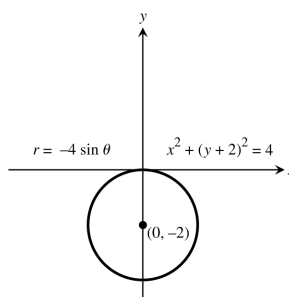
$$27. \quad r = -\frac{3}{2} \csc \theta \Rightarrow r \sin \theta = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}$$



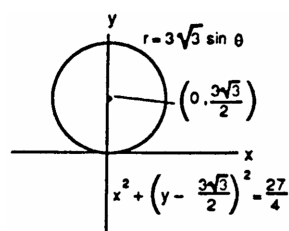
$$28. r = 3\sqrt{3} \csc \theta \Rightarrow r \sin \theta = 3\sqrt{3} \Rightarrow y = 3\sqrt{3}$$



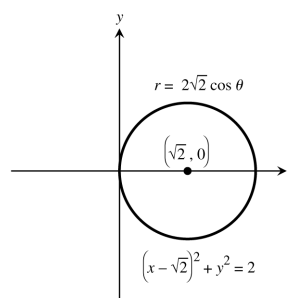
$$29. r = -4 \sin \theta \Rightarrow r^2 = -4r \sin \theta \Rightarrow x^2 + y^2 + 4y = 0 \\ \Rightarrow x^2 + (y + 2)^2 = 4; \text{ circle with center } (0, -2) \text{ and radius } 2.$$



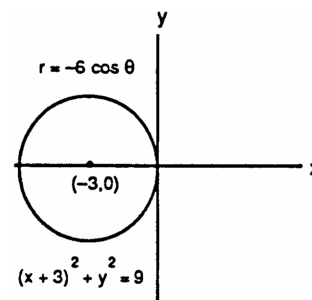
$$30. r = 3\sqrt{3} \sin \theta \Rightarrow r^2 = 3\sqrt{3} r \sin \theta \\ \Rightarrow x^2 + y^2 - 3\sqrt{3} y = 0 \Rightarrow x^2 + \left(y - \frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4}; \\ \text{circle with center } \left(0, \frac{3\sqrt{3}}{2}\right) \text{ and radius } \frac{3\sqrt{3}}{2}$$



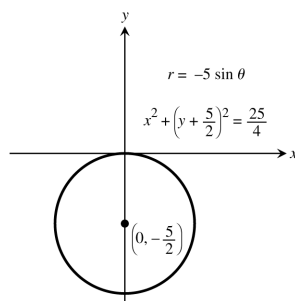
$$31. r = 2\sqrt{2} \cos \theta \Rightarrow r^2 = 2\sqrt{2} r \cos \theta \\ \Rightarrow x^2 + y^2 - 2\sqrt{2} x = 0 \Rightarrow (x - \sqrt{2})^2 + y^2 = 2; \\ \text{circle with center } (\sqrt{2}, 0) \text{ and radius } \sqrt{2}$$



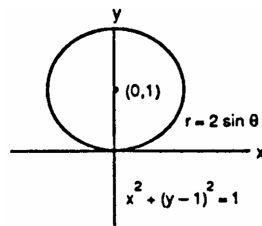
$$32. r = -6 \cos \theta \Rightarrow r^2 = -6r \cos \theta \Rightarrow x^2 + y^2 + 6x = 0 \\ \Rightarrow (x + 3)^2 + y^2 = 9; \text{ circle with center } (-3, 0) \text{ and radius } 3$$



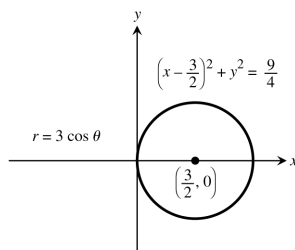
33. $x^2 + y^2 + 5y = 0 \Rightarrow x^2 + \left(y + \frac{5}{2}\right)^2 = \frac{25}{4} \Rightarrow C = \left(0, -\frac{5}{2}\right)$
 and $a = \frac{5}{2}; r^2 + 5r \sin \theta = 0 \Rightarrow r = -5 \sin \theta$



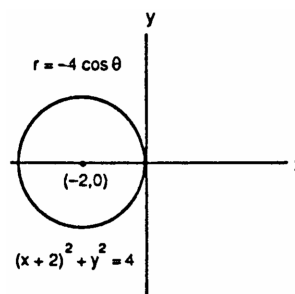
34. $x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y - 1)^2 = 1 \Rightarrow C = (0, 1)$ and
 $a = 1; r^2 - 2r \sin \theta = 0 \Rightarrow r = 2 \sin \theta$



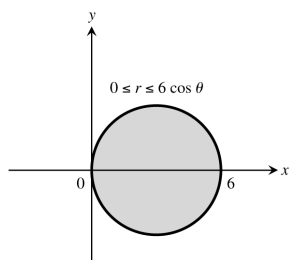
35. $x^2 + y^2 - 3x = 0 \Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4} \Rightarrow C = \left(\frac{3}{2}, 0\right)$
 and $a = \frac{3}{2}; r^2 - 3r \cos \theta = 0 \Rightarrow r = 3 \cos \theta$



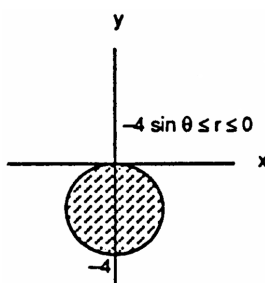
36. $x^2 + y^2 + 4x = 0 \Rightarrow (x + 2)^2 + y^2 = 4 \Rightarrow C = (-2, 0)$
 and $a = 2; r^2 + 4r \cos \theta = 0 \Rightarrow r = -4 \cos \theta$



37.



38.



39. d

40. e

41. l

42. f

43. k

44. h

45. i

46. j

$$47. A = 2 \int_0^\pi \frac{1}{2} r^2 d\theta = \int_0^\pi (2 - \cos \theta)^2 d\theta = \int_0^\pi (4 - 4 \cos \theta + \cos^2 \theta) d\theta = \int_0^\pi (4 - 4 \cos \theta + \frac{1 + \cos 2\theta}{2}) d\theta \\ = \int_0^\pi (\frac{9}{2} - 4 \cos \theta + \frac{\cos 2\theta}{2}) d\theta = [\frac{9}{2} \theta - 4 \sin \theta + \frac{\sin 2\theta}{4}]_0^\pi = \frac{9}{2} \pi$$

$$48. A = \int_0^{\pi/3} \frac{1}{2} (\sin^2 3\theta) d\theta = \int_0^{\pi/3} (\frac{1 - \cos 6\theta}{2}) d\theta = \frac{1}{4} [\theta - \frac{1}{6} \sin 6\theta]_0^{\pi/3} = \frac{\pi}{12}$$

$$49. r = 1 + \cos 2\theta \text{ and } r = 1 \Rightarrow 1 = 1 + \cos 2\theta \Rightarrow 0 = \cos 2\theta \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}; \text{ therefore}$$

$$A = 4 \int_0^{\pi/4} \frac{1}{2} [(1 + \cos 2\theta)^2 - 1^2] d\theta = 2 \int_0^{\pi/4} (1 + 2 \cos 2\theta + \cos^2 2\theta - 1) d\theta \\ = 2 \int_0^{\pi/4} (2 \cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}) d\theta = 2 [\sin 2\theta + \frac{1}{2} \theta + \frac{\sin 4\theta}{8}]_0^{\pi/4} = 2 (1 + \frac{\pi}{8} + 0) = 2 + \frac{\pi}{4}$$

50. The circle lies interior to the cardioid. Thus,

$$A = 2 \int_{-\pi/2}^{\pi/2} \frac{1}{2} [2(1 + \sin \theta)]^2 d\theta - \pi \text{ (the integral is the area of the cardioid minus the area of the circle)} \\ = \int_{-\pi/2}^{\pi/2} 4(1 + 2 \sin \theta + \sin^2 \theta) d\theta - \pi = \int_{-\pi/2}^{\pi/2} (6 + 8 \sin \theta - 2 \cos 2\theta) d\theta - \pi = [6\theta - 8 \cos \theta - \sin 2\theta]_{-\pi/2}^{\pi/2} - \pi \\ = [3\pi - (-3\pi)] - \pi = 5\pi$$

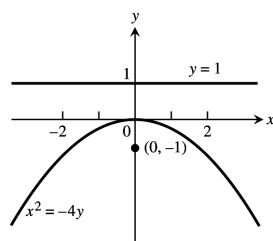
$$51. r = -1 + \cos \theta \Rightarrow \frac{dr}{d\theta} = -\sin \theta; \text{ Length} = \int_0^{2\pi} \sqrt{(-1 + \cos \theta)^2 + (-\sin \theta)^2} d\theta = \int_0^{2\pi} \sqrt{2 - 2 \cos \theta} d\theta \\ = \int_0^{2\pi} \sqrt{\frac{4(1 - \cos \theta)}{2}} d\theta = \int_0^{2\pi} 2 \sin \frac{\theta}{2} d\theta = [-4 \cos \frac{\theta}{2}]_0^{2\pi} = (-4)(-1) - (-4)(1) = 8$$

$$52. r = 2 \sin \theta + 2 \cos \theta, 0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = 2 \cos \theta - 2 \sin \theta; r^2 + (\frac{dr}{d\theta})^2 = (2 \sin \theta + 2 \cos \theta)^2 + (2 \cos \theta - 2 \sin \theta)^2 \\ = 8(\sin^2 \theta + \cos^2 \theta) = 8 \Rightarrow L = \int_0^{\pi/2} \sqrt{8} d\theta = [2\sqrt{2} \theta]_0^{\pi/2} = 2\sqrt{2} (\frac{\pi}{2}) = \pi\sqrt{2}$$

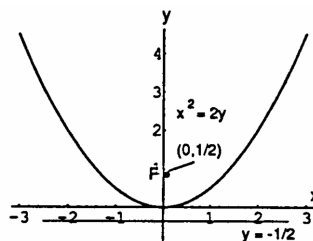
$$53. r = 8 \sin^3 (\frac{\theta}{3}), 0 \leq \theta \leq \frac{\pi}{4} \Rightarrow \frac{dr}{d\theta} = 8 \sin^2 (\frac{\theta}{3}) \cos (\frac{\theta}{3}); r^2 + (\frac{dr}{d\theta})^2 = [8 \sin^3 (\frac{\theta}{3})]^2 + [8 \sin^2 (\frac{\theta}{3}) \cos (\frac{\theta}{3})]^2 \\ = 64 \sin^4 (\frac{\theta}{3}) \Rightarrow L = \int_0^{\pi/4} \sqrt{64 \sin^4 (\frac{\theta}{3})} d\theta = \int_0^{\pi/4} 8 \sin^2 (\frac{\theta}{3}) d\theta = \int_0^{\pi/4} 8 [\frac{1 - \cos (\frac{2\theta}{3})}{2}] d\theta \\ = \int_0^{\pi/4} [4 - 4 \cos (\frac{2\theta}{3})] d\theta = [4\theta - 6 \sin (\frac{2\theta}{3})]_0^{\pi/4} = 4(\frac{\pi}{4}) - 6 \sin (\frac{\pi}{6}) - 0 = \pi - 3$$

$$54. r = \sqrt{1 + \cos 2\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2} (1 + \cos 2\theta)^{-1/2} (-2 \sin 2\theta) = \frac{-\sin 2\theta}{\sqrt{1 + \cos 2\theta}} \Rightarrow (\frac{dr}{d\theta})^2 = \frac{\sin^2 2\theta}{1 + \cos 2\theta} \\ \Rightarrow r^2 + (\frac{dr}{d\theta})^2 = 1 + \cos 2\theta + \frac{\sin^2 2\theta}{1 + \cos 2\theta} = \frac{(1 + \cos 2\theta)^2 + \sin^2 2\theta}{1 + \cos 2\theta} = \frac{1 + 2 \cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}{1 + \cos 2\theta} \\ = \frac{2 + 2 \cos 2\theta}{1 + \cos 2\theta} = 2 \Rightarrow L = \int_{-\pi/2}^{\pi/2} \sqrt{2} d\theta = \sqrt{2} [\frac{\pi}{2} - (-\frac{\pi}{2})] = \sqrt{2} \pi$$

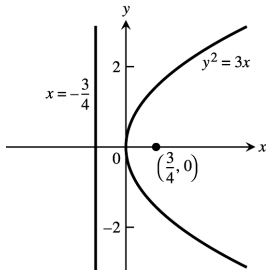
$$55. x^2 = -4y \Rightarrow y = -\frac{x^2}{4} \Rightarrow 4p = 4 \Rightarrow p = 1; \\ \text{therefore Focus is } (0, -1), \text{ Directrix is } y = 1$$



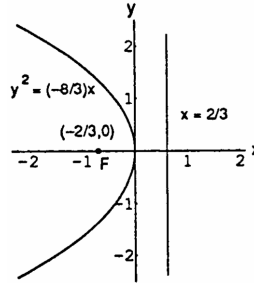
$$56. x^2 = 2y \Rightarrow \frac{x^2}{2} = y \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}; \\ \text{therefore Focus is } (0, \frac{1}{2}), \text{ Directrix is } y = -\frac{1}{2}$$



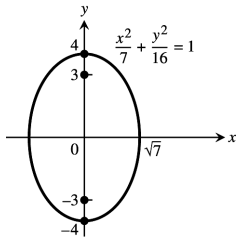
57. $y^2 = 3x \Rightarrow x = \frac{y^2}{3} \Rightarrow 4p = 3 \Rightarrow p = \frac{3}{4}$;
therefore Focus is $(\frac{3}{4}, 0)$, Directrix is $x = -\frac{3}{4}$



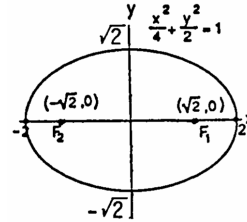
58. $y^2 = -\frac{8}{3}x \Rightarrow x = -\frac{y^2}{(8/3)} \Rightarrow 4p = \frac{8}{3} \Rightarrow p = \frac{2}{3}$;
therefore Focus is $(-\frac{2}{3}, 0)$, Directrix is $x = \frac{2}{3}$



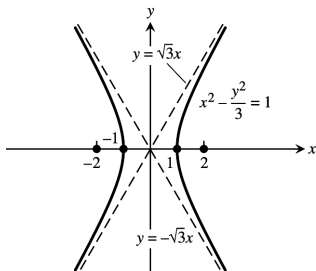
59. $16x^2 + 7y^2 = 112 \Rightarrow \frac{x^2}{7} + \frac{y^2}{16} = 1$
 $\Rightarrow c^2 = 16 - 7 = 9 \Rightarrow c = 3$; $e = \frac{c}{a} = \frac{3}{4}$



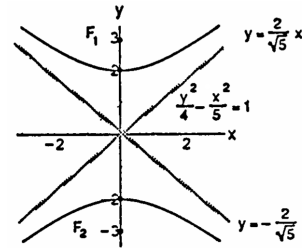
60. $x^2 + 2y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1 \Rightarrow c^2 = 4 - 2 = 2$
 $\Rightarrow c = \sqrt{2}$; $e = \frac{c}{a} = \frac{\sqrt{2}}{2}$



61. $3x^2 - y^2 = 3 \Rightarrow x^2 - \frac{y^2}{3} = 1 \Rightarrow c^2 = 1 + 3 = 4$
 $\Rightarrow c = 2$; $e = \frac{c}{a} = \frac{2}{1} = 2$; the asymptotes are
 $y = \pm \sqrt{3}x$



62. $5y^2 - 4x^2 = 20 \Rightarrow \frac{y^2}{4} - \frac{x^2}{5} = 1 \Rightarrow c^2 = 4 + 5 = 9$
 $\Rightarrow c = 3$, $e = \frac{c}{a} = \frac{3}{2}$; the asymptotes are $y = \pm \frac{2}{\sqrt{5}}x$



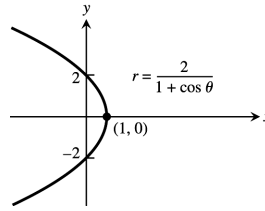
63. $x^2 = -12y \Rightarrow -\frac{x^2}{12} = y \Rightarrow 4p = 12 \Rightarrow p = 3 \Rightarrow$ focus is $(0, -3)$, directrix is $y = 3$, vertex is $(0, 0)$; therefore new vertex is $(2, 3)$, new focus is $(2, 0)$, new directrix is $y = 6$, and the new equation is $(x - 2)^2 = -12(y - 3)$

64. $y^2 = 10x \Rightarrow \frac{y^2}{10} = x \Rightarrow 4p = 10 \Rightarrow p = \frac{5}{2} \Rightarrow$ focus is $(\frac{5}{2}, 0)$, directrix is $x = -\frac{5}{2}$, vertex is $(0, 0)$; therefore new vertex is $(-\frac{1}{2}, -1)$, new focus is $(2, -1)$, new directrix is $x = -3$, and the new equation is $(y + 1)^2 = 10(x + \frac{1}{2})$

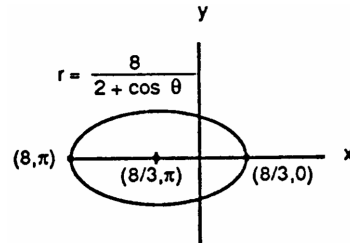
65. $\frac{x^2}{9} + \frac{y^2}{25} = 1 \Rightarrow a = 5$ and $b = 3 \Rightarrow c = \sqrt{25 - 9} = 4 \Rightarrow$ foci are $(0, \pm 4)$, vertices are $(0, \pm 5)$, center is $(0, 0)$; therefore the new center is $(-3, -5)$, new foci are $(-3, -1)$ and $(-3, -9)$, new vertices are $(-3, -10)$ and $(-3, 0)$, and the new equation is $\frac{(x+3)^2}{9} + \frac{(y+5)^2}{25} = 1$

66. $\frac{x^2}{169} + \frac{y^2}{144} = 1 \Rightarrow a = 13$ and $b = 12 \Rightarrow c = \sqrt{169 - 144} = 5 \Rightarrow$ foci are $(\pm 5, 0)$, vertices are $(\pm 13, 0)$, center is $(0, 0)$; therefore the new center is $(5, 12)$, new foci are $(10, 12)$ and $(0, 12)$, new vertices are $(18, 12)$ and $(-8, 12)$, and the new equation is $\frac{(x-5)^2}{169} + \frac{(y-12)^2}{144} = 1$
67. $\frac{y^2}{8} - \frac{x^2}{2} = 1 \Rightarrow a = 2\sqrt{2}$ and $b = \sqrt{2} \Rightarrow c = \sqrt{8 + 2} = \sqrt{10} \Rightarrow$ foci are $(0, \pm \sqrt{10})$, vertices are $(0, \pm 2\sqrt{2})$, center is $(0, 0)$, and the asymptotes are $y = \pm 2x$; therefore the new center is $(2, 2\sqrt{2})$, new foci are $(2, 2\sqrt{2} \pm \sqrt{10})$, new vertices are $(2, 4\sqrt{2})$ and $(2, 0)$, the new asymptotes are $y = 2x - 4 + 2\sqrt{2}$ and $y = -2x + 4 + 2\sqrt{2}$; the new equation is $\frac{(y-2\sqrt{2})^2}{8} - \frac{(x-2)^2}{2} = 1$
68. $\frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow a = 6$ and $b = 8 \Rightarrow c = \sqrt{36 + 64} = 10 \Rightarrow$ foci are $(\pm 10, 0)$, vertices are $(\pm 6, 0)$, the center is $(0, 0)$ and the asymptotes are $\frac{y}{8} = \pm \frac{x}{6}$ or $y = \pm \frac{4}{3}x$; therefore the new center is $(-10, -3)$, the new foci are $(-20, -3)$ and $(0, -3)$, the new vertices are $(-16, -3)$ and $(-4, -3)$, the new asymptotes are $y = \frac{4}{3}x + \frac{31}{3}$ and $y = -\frac{4}{3}x - \frac{49}{3}$; the new equation is $\frac{(x+10)^2}{36} - \frac{(y+3)^2}{64} = 1$
69. $x^2 - 4x - 4y^2 = 0 \Rightarrow x^2 - 4x + 4 - 4y^2 = 4 \Rightarrow (x-2)^2 - 4y^2 = 4 \Rightarrow \frac{(x-2)^2}{4} - y^2 = 1$, a hyperbola; $a = 2$ and $b = 1 \Rightarrow c = \sqrt{1 + 4} = \sqrt{5}$; the center is $(2, 0)$, the vertices are $(0, 0)$ and $(4, 0)$; the foci are $(2 \pm \sqrt{5}, 0)$ and the asymptotes are $y = \pm \frac{x-2}{2}$
70. $4x^2 - y^2 + 4y = 8 \Rightarrow 4x^2 - y^2 + 4y - 4 = 4 \Rightarrow 4x^2 - (y-2)^2 = 4 \Rightarrow x^2 - \frac{(y-2)^2}{4} = 1$, a hyperbola; $a = 1$ and $b = 2 \Rightarrow c = \sqrt{1 + 4} = \sqrt{5}$; the center is $(0, 2)$, the vertices are $(1, 2)$ and $(-1, 2)$, the foci are $(\pm \sqrt{5}, 2)$ and the asymptotes are $y = \pm 2x + 2$
71. $y^2 - 2y + 16x = -49 \Rightarrow y^2 - 2y + 1 = -16x - 48 \Rightarrow (y-1)^2 = -16(x+3)$, a parabola; the vertex is $(-3, 1)$; $4p = 16 \Rightarrow p = 4 \Rightarrow$ the focus is $(-7, 1)$ and the directrix is $x = 1$
72. $x^2 - 2x + 8y = -17 \Rightarrow x^2 - 2x + 1 = -8y - 16 \Rightarrow (x-1)^2 = -8(y+2)$, a parabola; the vertex is $(1, -2)$; $4p = 8 \Rightarrow p = 2 \Rightarrow$ the focus is $(1, -4)$ and the directrix is $y = 0$
73. $9x^2 + 16y^2 + 54x - 64y = -1 \Rightarrow 9(x^2 + 6x) + 16(y^2 - 4y) = -1 \Rightarrow 9(x^2 + 6x + 9) + 16(y^2 - 4y + 4) = 144 \Rightarrow 9(x+3)^2 + 16(y-2)^2 = 144 \Rightarrow \frac{(x+3)^2}{16} + \frac{(y-2)^2}{9} = 1$, an ellipse; the center is $(-3, 2)$; $a = 4$ and $b = 3 \Rightarrow c = \sqrt{16 - 9} = \sqrt{7}$; the foci are $(-3 \pm \sqrt{7}, 2)$; the vertices are $(1, 2)$ and $(-7, 2)$
74. $25x^2 + 9y^2 - 100x + 54y = 44 \Rightarrow 25(x^2 - 4x) + 9(y^2 + 6y) = 44 \Rightarrow 25(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = 225 \Rightarrow \frac{(x-2)^2}{9} + \frac{(y+3)^2}{25} = 1$, an ellipse; the center is $(2, -3)$; $a = 5$ and $b = 3 \Rightarrow c = \sqrt{25 - 9} = 4$; the foci are $(2, 1)$ and $(2, -7)$; the vertices are $(2, 2)$ and $(2, -8)$
75. $x^2 + y^2 - 2x - 2y = 0 \Rightarrow x^2 - 2x + 1 + y^2 - 2y + 1 = 2 \Rightarrow (x-1)^2 + (y-1)^2 = 2$, a circle with center $(1, 1)$ and radius $= \sqrt{2}$
76. $x^2 + y^2 + 4x + 2y = 1 \Rightarrow x^2 + 4x + 4 + y^2 + 2y + 1 = 6 \Rightarrow (x+2)^2 + (y+1)^2 = 6$, a circle with center $(-2, -1)$ and radius $= \sqrt{6}$

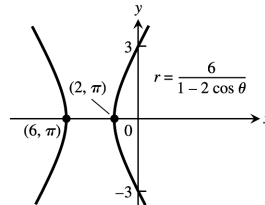
77. $r = \frac{2}{1 + \cos \theta} \Rightarrow e = 1 \Rightarrow$ parabola with vertex at $(1, 0)$



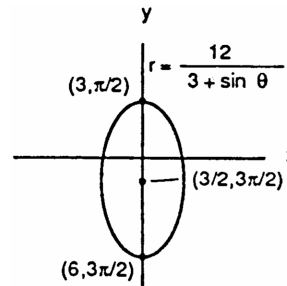
78. $r = \frac{8}{2 + \cos \theta} \Rightarrow r = \frac{4}{1 + (\frac{1}{2}) \cos \theta} \Rightarrow e = \frac{1}{2} \Rightarrow$ ellipse;
 $ke = 4 \Rightarrow \frac{1}{2}k = 4 \Rightarrow k = 8; k = \frac{a}{e} - ea \Rightarrow 8 = \frac{a}{(\frac{1}{2})} - \frac{1}{2}a$
 $\Rightarrow a = \frac{16}{3} \Rightarrow ea = (\frac{1}{2}) (\frac{16}{3}) = \frac{8}{3};$ therefore the center is $(\frac{8}{3}, \pi);$ vertices are $(8, \pi)$ and $(\frac{8}{3}, 0)$



79. $r = \frac{6}{1 - 2 \cos \theta} \Rightarrow e = 2 \Rightarrow$ hyperbola; $ke = 6 \Rightarrow 2k = 6$
 $\Rightarrow k = 3 \Rightarrow$ vertices are $(2, \pi)$ and $(6, \pi)$



80. $r = \frac{12}{3 + \sin \theta} \Rightarrow r = \frac{4}{1 + (\frac{1}{3}) \sin \theta} \Rightarrow e = \frac{1}{3}; ke = 4$
 $\Rightarrow \frac{1}{3}k = 4 \Rightarrow k = 12; a(1 - e^2) = 4 \Rightarrow a[1 - (\frac{1}{3})^2]$
 $= 4 \Rightarrow a = \frac{9}{2} \Rightarrow ea = (\frac{1}{3}) (\frac{9}{2}) = \frac{3}{2};$ therefore the center is $(\frac{3}{2}, \frac{3\pi}{2});$ vertices are $(3, \frac{\pi}{2})$ and $(6, \frac{3\pi}{2})$



81. $e = 2$ and $r \cos \theta = 2 \Rightarrow x = 2$ is directrix $\Rightarrow k = 2;$ the conic is a hyperbola; $r = \frac{ke}{1 + e \cos \theta} \Rightarrow r = \frac{(2)(2)}{1 + 2 \cos \theta}$
 $\Rightarrow r = \frac{4}{1 + 2 \cos \theta}$

82. $e = 1$ and $r \cos \theta = -4 \Rightarrow x = -4$ is directrix $\Rightarrow k = 4;$ the conic is a parabola; $r = \frac{ke}{1 - e \cos \theta} \Rightarrow r = \frac{(4)(1)}{1 - \cos \theta}$
 $\Rightarrow r = \frac{4}{1 - \cos \theta}$

83. $e = \frac{1}{2}$ and $r \sin \theta = 2 \Rightarrow y = 2$ is directrix $\Rightarrow k = 2;$ the conic is an ellipse; $r = \frac{ke}{1 + e \sin \theta} \Rightarrow r = \frac{(2)(\frac{1}{2})}{1 + (\frac{1}{2}) \sin \theta}$
 $\Rightarrow r = \frac{2}{2 + \sin \theta}$

84. $e = \frac{1}{3}$ and $r \sin \theta = -6 \Rightarrow y = -6$ is directrix $\Rightarrow k = 6;$ the conic is an ellipse; $r = \frac{ke}{1 - e \sin \theta} \Rightarrow r = \frac{(6)(\frac{1}{3})}{1 - (\frac{1}{3}) \sin \theta}$
 $\Rightarrow r = \frac{6}{3 - \sin \theta}$

85. (a) Around the x-axis: $9x^2 + 4y^2 = 36 \Rightarrow y^2 = 9 - \frac{9}{4}x^2 \Rightarrow y = \pm \sqrt{9 - \frac{9}{4}x^2}$ and we use the positive root:

$$V = 2 \int_0^2 \pi \left(\sqrt{9 - \frac{9}{4}x^2} \right)^2 dx = 2 \int_0^2 \pi \left(9 - \frac{9}{4}x^2 \right) dx = 2\pi \left[9x - \frac{3}{4}x^3 \right]_0^2 = 24\pi$$

(b) Around the y-axis: $9x^2 + 4y^2 = 36 \Rightarrow x^2 = 4 - \frac{4}{9}y^2 \Rightarrow x = \pm \sqrt{4 - \frac{4}{9}y^2}$ and we use the positive root:

$$V = 2 \int_0^3 \pi \left(\sqrt{4 - \frac{4}{9}y^2} \right)^2 dy = 2 \int_0^3 \pi \left(4 - \frac{4}{9}y^2 \right) dy = 2\pi \left[4y - \frac{4}{27}y^3 \right]_0^3 = 16\pi$$

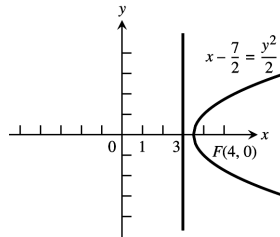
$$86. \quad 9x^2 - 4y^2 = 36, x = 4 \Rightarrow y^2 = \frac{9x^2 - 36}{4} \Rightarrow y = \frac{3}{2} \sqrt{x^2 - 4}; V = \int_2^4 \pi \left(\frac{3}{2} \sqrt{x^2 - 4} \right)^2 dx = \frac{9\pi}{4} \int_2^4 (x^2 - 4) dx \\ = \frac{9\pi}{4} \left[\frac{x^3}{3} - 4x \right]_2^4 = \frac{9\pi}{4} \left[\left(\frac{64}{3} - 16 \right) - \left(\frac{8}{3} - 8 \right) \right] = \frac{9\pi}{4} \left(\frac{56}{3} - \frac{24}{3} \right) = \frac{3\pi}{4} (32) = 24\pi$$

87. (a) $r = \frac{k}{1 + e \cos \theta} \Rightarrow r + e r \cos \theta = k \Rightarrow \sqrt{x^2 + y^2} + ex = k \Rightarrow \sqrt{x^2 + y^2} = k - ex \Rightarrow x^2 + y^2 = k^2 - 2kex + e^2x^2 \Rightarrow x^2 - e^2x^2 + y^2 + 2kex - k^2 = 0 \Rightarrow (1 - e^2)x^2 + y^2 + 2kex - k^2 = 0$
 (b) $e = 0 \Rightarrow x^2 + y^2 - k^2 = 0 \Rightarrow x^2 + y^2 = k^2 \Rightarrow$ circle;
 $0 < e < 1 \Rightarrow e^2 < 1 \Rightarrow e^2 - 1 < 0 \Rightarrow B^2 - 4AC = 0^2 - 4(1 - e^2)(1) = 4(e^2 - 1) < 0 \Rightarrow$ ellipse;
 $e = 1 \Rightarrow B^2 - 4AC = 0^2 - 4(0)(1) = 0 \Rightarrow$ parabola;
 $e > 1 \Rightarrow e^2 > 1 \Rightarrow B^2 - 4AC = 0^2 - 4(1 - e^2)(1) = 4e^2 - 4 > 0 \Rightarrow$ hyperbola

88. Let (r_1, θ_1) be a point on the graph where $r_1 = a\theta_1$. Let (r_2, θ_2) be on the graph where $r_2 = a\theta_2$ and $\theta_2 = \theta_1 + 2\pi$. Then r_1 and r_2 lie on the same ray on consecutive turns of the spiral and the distance between the two points is $r_2 - r_1 = a\theta_2 - a\theta_1 = a(\theta_2 - \theta_1) = 2\pi a$, which is constant.

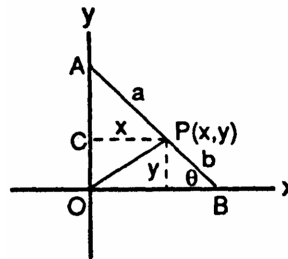
CHAPTER 11 ADDITIONAL AND ADVANCED EXERCISES

1. Directrix $x = 3$ and focus $(4, 0) \Rightarrow$ vertex is $\left(\frac{7}{2}, 0\right)$
 $\Rightarrow p = \frac{1}{2} \Rightarrow$ the equation is $x - \frac{7}{2} = \frac{y^2}{2}$



2. $x^2 - 6x - 12y + 9 = 0 \Rightarrow x^2 - 6x + 9 = 12y \Rightarrow \frac{(x-3)^2}{12} = y \Rightarrow$ vertex is $(3, 0)$ and $p = 3 \Rightarrow$ focus is $(3, 3)$ and the directrix is $y = -3$
3. $x^2 = 4y \Rightarrow$ vertex is $(0, 0)$ and $p = 1 \Rightarrow$ focus is $(0, 1)$; thus the distance from $P(x, y)$ to the vertex is $\sqrt{x^2 + y^2}$ and the distance from P to the focus is $\sqrt{x^2 + (y - 1)^2} \Rightarrow \sqrt{x^2 + y^2} = 2\sqrt{x^2 + (y - 1)^2}$
 $\Rightarrow x^2 + y^2 = 4[x^2 + (y - 1)^2] \Rightarrow x^2 + y^2 = 4x^2 + 4y^2 - 8y + 4 \Rightarrow 3x^2 + 3y^2 - 8y + 4 = 0$, which is a circle

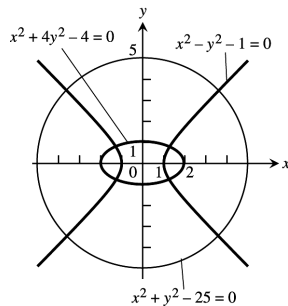
4. Let the segment $a + b$ intersect the y-axis in point A and intersect the x-axis in point B so that $PB = b$ and $PA = a$ (see figure). Draw the horizontal line through P and let it intersect the y-axis in point C. Let $\angle PBO = \theta$
 $\Rightarrow \angle APC = \theta$. Then $\sin \theta = \frac{y}{b}$ and $\cos \theta = \frac{x}{a}$
 $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \theta + \sin^2 \theta = 1$.



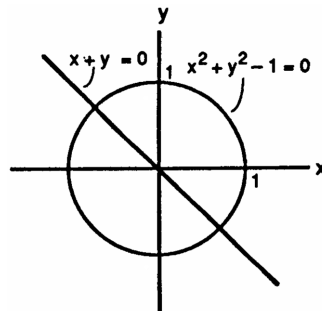
5. Vertices are $(0, \pm 2) \Rightarrow a = 2; e = \frac{c}{a} \Rightarrow 0.5 = \frac{c}{2} \Rightarrow c = 1 \Rightarrow$ foci are $(0, \pm 1)$

6. Let the center of the ellipse be $(x, 0)$; directrix $x = 2$, focus $(4, 0)$, and $e = \frac{2}{3} \Rightarrow \frac{a}{c} - c = 2 \Rightarrow \frac{a}{c} = 2 + c$
 $\Rightarrow a = \frac{2}{3}(2 + c)$. Also $c = ae = \frac{2}{3}a \Rightarrow a = \frac{2}{3}(2 + \frac{2}{3}a) \Rightarrow a = \frac{4}{3} + \frac{4}{9}a \Rightarrow \frac{5}{9}a = \frac{4}{3} \Rightarrow a = \frac{12}{5}$; $x - 2 = \frac{a}{c}$
 $\Rightarrow x - 2 = (\frac{12}{5})(\frac{3}{2}) = \frac{18}{5} \Rightarrow x = \frac{28}{5} \Rightarrow$ the center is $(\frac{28}{5}, 0)$; $x - 4 = c \Rightarrow c = \frac{28}{5} - 4 = \frac{8}{5}$ so that $c^2 = a^2 - b^2$
 $= (\frac{12}{5})^2 - (\frac{8}{5})^2 = \frac{80}{25}$; therefore the equation is $\frac{(x - \frac{28}{5})^2}{(\frac{144}{25})} + \frac{y^2}{(\frac{80}{25})} = 1$ or $\frac{25(x - \frac{28}{5})^2}{144} + \frac{5y^2}{16} = 1$
7. Let the center of the hyperbola be $(0, y)$.
 (a) Directrix $y = -1$, focus $(0, -7)$ and $e = 2 \Rightarrow c - \frac{a}{e} = 6 \Rightarrow \frac{a}{c} = c - 6 \Rightarrow a = 2c - 12$. Also $c = ae = 2a$
 $\Rightarrow a = 2(2a) - 12 \Rightarrow a = 4 \Rightarrow c = 8$; $y - (-1) = \frac{a}{e} = \frac{4}{2} = 2 \Rightarrow y = 1 \Rightarrow$ the center is $(0, 1)$; $c^2 = a^2 + b^2$
 $\Rightarrow b^2 = c^2 - a^2 = 64 - 16 = 48$; therefore the equation is $\frac{(y-1)^2}{16} - \frac{x^2}{48} = 1$
 (b) $e = 5 \Rightarrow c - \frac{a}{e} = 6 \Rightarrow \frac{a}{c} = c - 6 \Rightarrow a = 5c - 30$. Also, $c = ae = 5a \Rightarrow a = 5(5a) - 30 \Rightarrow 24a = 30 \Rightarrow a = \frac{5}{4}$
 $\Rightarrow c = \frac{25}{4}$; $y - (-1) = \frac{a}{e} = \frac{(\frac{5}{4})}{5} = \frac{1}{4} \Rightarrow y = -\frac{3}{4} \Rightarrow$ the center is $(0, -\frac{3}{4})$; $c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$
 $= \frac{625}{16} - \frac{25}{16} = \frac{75}{2}$; therefore the equation is $\frac{(y + \frac{3}{4})^2}{(\frac{25}{16})} - \frac{x^2}{(\frac{75}{2})} = 1$ or $\frac{16(y + \frac{3}{4})^2}{25} - \frac{2x^2}{75} = 1$
8. The center is $(0, 0)$ and $c = 2 \Rightarrow 4 = a^2 + b^2 \Rightarrow b^2 = 4 - a^2$. The equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{49}{a^2} - \frac{144}{b^2} = 1$
 $\Rightarrow \frac{49}{a^2} - \frac{144}{(4-a^2)} = 1 \Rightarrow 49(4-a^2) - 144a^2 = a^2(4-a^2) \Rightarrow 196 - 49a^2 - 144a^2 = 4a^2 - a^4 \Rightarrow a^4 - 197a^2 + 196$
 $= 0 \Rightarrow (a^2 - 196)(a^2 - 1) = 0 \Rightarrow a = 14$ or $a = 1$; $a = 14 \Rightarrow b^2 = 4 - (14)^2 < 0$ which is impossible; $a = 1$
 $\Rightarrow b^2 = 4 - 1 = 3$; therefore the equation is $y^2 - \frac{x^2}{3} = 1$
9. $b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y}$; at (x_1, y_1) the tangent line is $y - y_1 = \left(-\frac{b^2x_1}{a^2y_1}\right)(x - x_1)$
 $\Rightarrow a^2yy_1 + b^2xx_1 = b^2x_1^2 + a^2y_1^2 = a^2b^2 \Rightarrow b^2xx_1 + a^2yy_1 - a^2b^2 = 0$
10. $b^2x^2 - a^2y^2 = a^2b^2 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$; at (x_1, y_1) the tangent line is $y - y_1 = \left(\frac{b^2x_1}{a^2y_1}\right)(x - x_1)$
 $\Rightarrow b^2xx_1 - a^2yy_1 = b^2x_1^2 - a^2y_1^2 = a^2b^2 \Rightarrow b^2xx_1 - a^2yy_1 - a^2b^2 = 0$

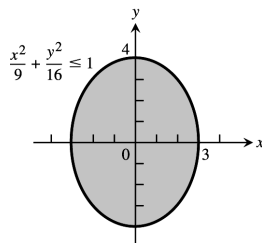
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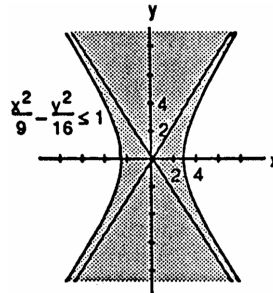
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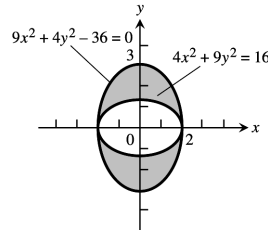
13.



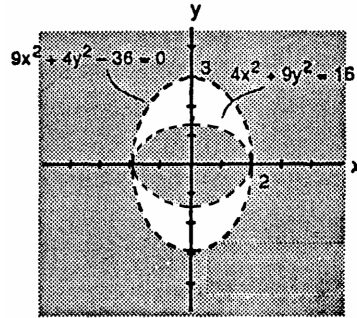
14.



15. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) \leq 0$
 $\Rightarrow 9x^2 + 4y^2 - 36 \leq 0$ and $4x^2 + 9y^2 - 16 \geq 0$
 or $9x^2 + 4y^2 - 36 \geq 0$ and $4x^2 + 9y^2 - 16 \leq 0$

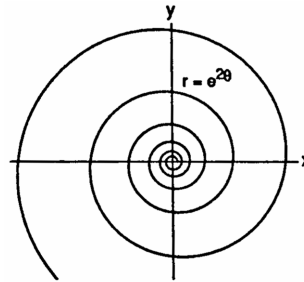


16. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) > 0$, which is the complement of the set in Exercise 15



17. (a) $x = e^{2t} \cos t$ and $y = e^{2t} \sin t \Rightarrow x^2 + y^2 = e^{4t} \cos^2 t + e^{4t} \sin^2 t = e^{4t}$. Also $\frac{y}{x} = \frac{e^{2t} \sin t}{e^{2t} \cos t} = \tan t$
 $\Rightarrow t = \tan^{-1} \left(\frac{y}{x} \right) \Rightarrow x^2 + y^2 = e^{4 \tan^{-1}(y/x)}$ is the Cartesian equation. Since $r^2 = x^2 + y^2$ and $\theta = \tan^{-1} \left(\frac{y}{x} \right)$, the polar equation is $r^2 = e^{4\theta}$ or $r = e^{2\theta}$ for $r > 0$

- (b) $ds^2 = r^2 d\theta^2 + dr^2$; $r = e^{2\theta} \Rightarrow dr = 2e^{2\theta} d\theta$
 $\Rightarrow ds^2 = r^2 d\theta^2 + (2e^{2\theta} d\theta)^2 = (e^{2\theta})^2 d\theta^2 + 4e^{4\theta} d\theta^2$
 $= 5e^{4\theta} d\theta^2 \Rightarrow ds = \sqrt{5} e^{2\theta} d\theta \Rightarrow L = \int_0^{2\pi} \sqrt{5} e^{2\theta} d\theta$
 $= \left[\frac{\sqrt{5} e^{2\theta}}{2} \right]_0^{2\pi} = \frac{\sqrt{5}}{2} (e^{4\pi} - 1)$



18. $r = 2 \sin^3 \left(\frac{\theta}{3} \right) \Rightarrow dr = 2 \sin^2 \left(\frac{\theta}{3} \right) \cos \left(\frac{\theta}{3} \right) d\theta \Rightarrow ds^2 = r^2 d\theta^2 + dr^2 = [2 \sin^3 \left(\frac{\theta}{3} \right)]^2 d\theta^2 + [2 \sin^2 \left(\frac{\theta}{3} \right) \cos \left(\frac{\theta}{3} \right) d\theta]^2$
 $= 4 \sin^6 \left(\frac{\theta}{3} \right) d\theta^2 + 4 \sin^4 \left(\frac{\theta}{3} \right) \cos^2 \left(\frac{\theta}{3} \right) d\theta^2 = [4 \sin^4 \left(\frac{\theta}{3} \right)] [\sin^2 \left(\frac{\theta}{3} \right) + \cos^2 \left(\frac{\theta}{3} \right)] d\theta^2 = 4 \sin^4 \left(\frac{\theta}{3} \right) d\theta^2$
 $\Rightarrow ds = 2 \sin^2 \left(\frac{\theta}{3} \right) d\theta$. Then $L = \int_0^{3\pi} 2 \sin^2 \left(\frac{\theta}{3} \right) d\theta = \int_0^{3\pi} [1 - \cos \left(\frac{2\theta}{3} \right)] d\theta = [\theta - \frac{3}{2} \sin \left(\frac{2\theta}{3} \right)]_0^{3\pi} = 3\pi$

19. $e = 2$ and $r \cos \theta = 2 \Rightarrow x = 2$ is the directrix $\Rightarrow k = 2$; the conic is a hyperbola with $r = \frac{ke}{1 + e \cos \theta}$
 $\Rightarrow r = \frac{(2)(2)}{1 + 2 \cos \theta} = \frac{4}{1 + 2 \cos \theta}$

20. $e = 1$ and $r \cos \theta = -4 \Rightarrow x = -4$ is the directrix $\Rightarrow k = 4$; the conic is a parabola with $r = \frac{ke}{1 - e \cos \theta}$
 $\Rightarrow r = \frac{(4)(1)}{1 - \cos \theta} = \frac{4}{1 - \cos \theta}$

21. $e = \frac{1}{2}$ and $r \sin \theta = 2 \Rightarrow y = 2$ is the directrix $\Rightarrow k = 2$; the conic is an ellipse with $r = \frac{ke}{1 + e \sin \theta}$
 $\Rightarrow r = \frac{2(\frac{1}{2})}{1 + (\frac{1}{2}) \sin \theta} = \frac{2}{2 + \sin \theta}$

22. $e = \frac{1}{3}$ and $r \sin \theta = -6 \Rightarrow y = -6$ is the directrix $\Rightarrow k = 6$; the conic is an ellipse with $r = \frac{ke}{1 - e \sin \theta}$
 $\Rightarrow r = \frac{6(\frac{1}{3})}{1 - (\frac{1}{3}) \sin \theta} = \frac{6}{3 - \sin \theta}$

23. Arc PF = Arc AF since each is the distance rolled;

$$\angle PCF = \frac{\text{Arc PF}}{b} \Rightarrow \text{Arc PF} = b(\angle PCF); \theta = \frac{\text{Arc AF}}{a}$$

$$\Rightarrow \text{Arc AF} = a\theta \Rightarrow a\theta = b(\angle PCF) \Rightarrow \angle PCF = \left(\frac{a}{b}\right)\theta;$$

$$\angle OCB = \frac{\pi}{2} - \theta \text{ and } \angle OCB = \angle PCF - \angle PCE$$

$$= \angle PCF - \left(\frac{\pi}{2} - \alpha\right) = \left(\frac{a}{b}\right)\theta - \left(\frac{\pi}{2} - \alpha\right) \Rightarrow \frac{\pi}{2} - \theta$$

$$= \left(\frac{a}{b}\right)\theta - \left(\frac{\pi}{2} - \alpha\right) \Rightarrow \frac{\pi}{2} - \theta = \left(\frac{a}{b}\right)\theta - \frac{\pi}{2} + \alpha$$

$$\Rightarrow \alpha = \pi - \theta - \left(\frac{a}{b}\right)\theta \Rightarrow \alpha = \pi - \left(\frac{a+b}{b}\right)\theta.$$

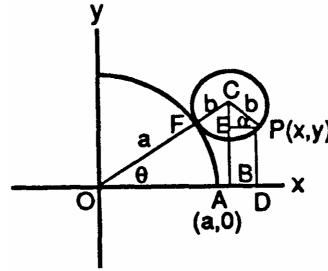
$$\text{Now } x = OB + BD = OB + EP = (a+b)\cos\theta + b\cos\left(\pi - \left(\frac{a+b}{b}\right)\theta\right)$$

$$= (a+b)\cos\theta + b\cos\pi\cos\left(\left(\frac{a+b}{b}\right)\theta\right) + b\sin\pi\sin\left(\left(\frac{a+b}{b}\right)\theta\right) = (a+b)\cos\theta - b\cos\left(\left(\frac{a+b}{b}\right)\theta\right) \text{ and}$$

$$y = PD = CB - CE = (a+b)\sin\theta - b\sin\alpha = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$$

$$= (a+b)\sin\theta - b\sin\pi\cos\left(\left(\frac{a+b}{b}\right)\theta\right) + b\cos\pi\sin\left(\left(\frac{a+b}{b}\right)\theta\right) = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right);$$

$$\text{therefore } x = (a+b)\cos\theta - b\cos\left(\left(\frac{a+b}{b}\right)\theta\right) \text{ and } y = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$$



- 24.
- $x = a(t - \sin t) \Rightarrow \frac{dx}{dt} = a(1 - \cos t)$
- and let
- $\delta = 1 \Rightarrow dm = dA = y dx = y \left(\frac{dx}{dt}\right) dt$

$$= a(1 - \cos t) a(1 - \cos t) dt = a^2(1 - \cos t)^2 dt; \text{ then } A = \int_0^{2\pi} a^2(1 - \cos t)^2 dt$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) dt = a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\right) dt = a^2 \left[\frac{3}{2}t - 2\sin t + \frac{\sin 2t}{4}\right]_0^{2\pi}$$

$$= 3\pi a^2; \bar{x} = x = a(t - \sin t) \text{ and } \bar{y} = \frac{1}{2}y = \frac{1}{2}a(1 - \cos t) \Rightarrow M_x = \int \bar{y} dm = \int \bar{y} \delta dA$$

$$= \int_0^{2\pi} \frac{1}{2} a(1 - \cos t) a^2(1 - \cos t)^2 dt = \frac{1}{2} a^3 \int_0^{2\pi} (1 - \cos t)^3 dt = \frac{a^3}{2} \int_0^{2\pi} (1 - 3\cos t + 3\cos^2 t - \cos^3 t) dt$$

$$= \frac{a^3}{2} \int_0^{2\pi} \left[1 - 3\cos t + \frac{3}{2} + \frac{3\cos 2t}{2} - (1 - \sin^2 t)(\cos t)\right] dt = \frac{a^3}{2} \left[\frac{5}{2}t - 3\sin t + \frac{3\sin 2t}{4} - \sin t + \frac{\sin^3 t}{3}\right]_0^{2\pi}$$

$$= \frac{5\pi a^3}{2}. \text{ Therefore } \bar{y} = \frac{M_x}{M} = \frac{\left(\frac{5\pi a^3}{2}\right)}{3\pi a^2} = \frac{5}{6}a. \text{ Also, } M_y = \int \bar{x} dm = \int \bar{x} \delta dA$$

$$= \int_0^{2\pi} a(t - \sin t) a^2(1 - \cos t)^2 dt = a^3 \int_0^{2\pi} (t - 2t\cos t + t\cos^2 t - \sin t + 2\sin t\cos t - \sin t\cos^2 t) dt$$

$$= a^3 \left[\frac{t^2}{2} - 2\cos t - 2t\sin t + \frac{1}{4}t^2 + \frac{1}{8}\cos 2t + \frac{1}{4}\sin 2t + \cos t + \sin^2 t + \frac{\cos^3 t}{3}\right]_0^{2\pi} = 3\pi^2 a^3. \text{ Thus}$$

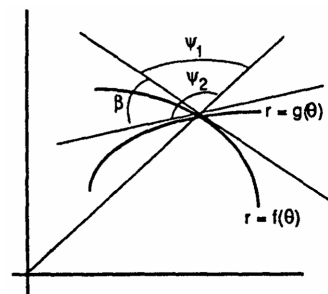
$$\bar{x} = \frac{M_y}{M} = \frac{3\pi^2 a^3}{3\pi a^2} = \pi a \Rightarrow \left(\pi a, \frac{5}{6}a\right) \text{ is the center of mass.}$$

- 25.
- $\beta = \psi_2 - \psi_1 \Rightarrow \tan \beta = \tan(\psi_2 - \psi_1) = \frac{\tan \psi_2 - \tan \psi_1}{1 + \tan \psi_2 \tan \psi_1};$

the curves will be orthogonal when $\tan \beta$ is undefined, or

$$\text{when } \tan \psi_2 = \frac{-1}{\tan \psi_1} \Rightarrow \frac{r}{g'(\theta)} = \frac{-1}{\left[\frac{r}{f'(\theta)}\right]}$$

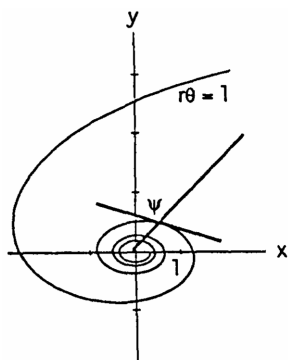
$$\Rightarrow r^2 = -f'(\theta)g'(\theta)$$



- 26.
- $r = \sin^4\left(\frac{\theta}{4}\right) \Rightarrow \frac{dr}{d\theta} = \sin^3\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right) \Rightarrow \tan \psi = \frac{\sin^4\left(\frac{\theta}{4}\right)}{\sin^3\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right)} = \tan\left(\frac{\theta}{4}\right)$

- 27.
- $r = 2a \sin 3\theta \Rightarrow \frac{dr}{d\theta} = 6a \cos 3\theta \Rightarrow \tan \psi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{2a \sin 3\theta}{6a \cos 3\theta} = \frac{1}{3} \tan 3\theta; \text{ when } \theta = \frac{\pi}{6}, \tan \psi = \frac{1}{3} \tan \frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{2}$

28. (a)



(b) $r\theta = 1 \Rightarrow r = \theta^{-1} \Rightarrow \frac{dr}{d\theta} = -\theta^{-2} \Rightarrow \tan \psi|_{\theta=1}$
 $= \frac{\theta^{-1}}{-\theta^{-2}} = -\theta \Rightarrow \lim_{\theta \rightarrow \infty} \tan \psi = -\infty$
 $\Rightarrow \psi \rightarrow \frac{\pi}{2}$ from the right as the spiral winds in around the origin.

29. $\tan \psi_1 = \frac{\sqrt{3} \cos \theta}{-\sqrt{3} \sin \theta} = -\cot \theta$ is $-\frac{1}{\sqrt{3}}$ at $\theta = \frac{\pi}{3}$; $\tan \psi_2 = \frac{\sin \theta}{\cos \theta} = \tan \theta$ is $\sqrt{3}$ at $\theta = \frac{\pi}{3}$; since the product of these slopes is -1 , the tangents are perpendicular

30. $\tan \psi = \frac{r}{(\frac{dr}{d\theta})} = \frac{a(1 - \cos \theta)}{a \sin \theta}$ is 1 at $\theta = \frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{4}$

NOTES:

CHAPTER 12 VECTORS AND THE GEOMETRY OF SPACE

12.1 THREE-DIMENSIONAL COORDINATE SYSTEMS

1. The line through the point $(2, 3, 0)$ parallel to the z -axis
2. The line through the point $(-1, 0, 0)$ parallel to the y -axis
3. The x -axis
4. The line through the point $(1, 0, 0)$ parallel to the z -axis
5. The circle $x^2 + y^2 = 4$ in the xy -plane
6. The circle $x^2 + y^2 = 4$ in the plane $z = -2$
7. The circle $x^2 + z^2 = 4$ in the xz -plane
8. The circle $y^2 + z^2 = 1$ in the yz -plane
9. The circle $y^2 + z^2 = 1$ in the yz -plane
10. The circle $x^2 + z^2 = 9$ in the plane $y = -4$
11. The circle $x^2 + y^2 = 16$ in the xy -plane
12. The circle $x^2 + z^2 = 3$ in the xz -plane
13. The ellipse formed by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = y$.
14. The circle formed by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $y = x$.
15. The parabola $y = x^2$ in the the xy -plane.
16. The parabola $z = y^2$ in the the plane $x = 1$.
17. (a) The first quadrant of the xy -plane (b) The fourth quadrant of the xy -plane
18. (a) The slab bounded by the planes $x = 0$ and $x = 1$
(b) The square column bounded by the planes $x = 0, x = 1, y = 0, y = 1$
(c) The unit cube in the first octant having one vertex at the origin
19. (a) The solid ball of radius 1 centered at the origin
(b) The exterior of the sphere of radius 1 centered at the origin
20. (a) The circumference and interior of the circle $x^2 + y^2 = 1$ in the xy -plane
(b) The circumference and interior of the circle $x^2 + y^2 = 1$ in the plane $z = 3$

- (c) A solid cylindrical column of radius 1 whose axis is the z -axis
21. (a) The solid enclosed between the sphere of radius 1 and radius 2 centered at the origin
(b) The solid upper hemisphere of radius 1 centered at the origin
22. (a) The line $y = x$ in the xy -plane
(b) The plane $y = x$ consisting of all points of the form (x, x, z)
23. (a) The region on or inside the parabola $y = x^2$ in the xy -plane and all points above this region.
(b) The region on or to the left of the parabola $x = y^2$ in the xy -plane and all points above it that are 2 units or less away from the xy -plane.
24. (a) All the points that lie on the plane $z = 1 - y$.
(b) All points that lie on the curve $z = y^3$ in the plane $x = -2$.
25. (a) $x = 3$ (b) $y = -1$ (c) $z = -2$
26. (a) $x = 3$ (b) $y = -1$ (c) $z = 2$
27. (a) $z = 1$ (b) $x = 3$ (c) $y = -1$
28. (a) $x^2 + y^2 = 4, z = 0$ (b) $y^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 0$
29. (a) $x^2 + (y - 2)^2 = 4, z = 0$ (b) $(y - 2)^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 2$
30. (a) $(x + 3)^2 + (y - 4)^2 = 1, z = 1$ (b) $(y - 4)^2 + (z - 1)^2 = 1, x = -3$
(c) $(x + 3)^2 + (z - 1)^2 = 1, y = 4$
31. (a) $y = 3, z = -1$ (b) $x = 1, z = -1$ (c) $x = 1, y = 3$
32. $\sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + (y - 2)^2 + z^2} \Rightarrow x^2 + y^2 + z^2 = x^2 + (y - 2)^2 + z^2 \Rightarrow y^2 = y^2 - 4y + 4 \Rightarrow y = 1$
33. $x^2 + y^2 + z^2 = 25, z = 3 \Rightarrow x^2 + y^2 = 16$ in the plane $z = 3$
34. $x^2 + y^2 + (z - 1)^2 = 4$ and $x^2 + y^2 + (z + 1)^2 = 4 \Rightarrow x^2 + y^2 + (z - 1)^2 = x^2 + y^2 + (z + 1)^2 \Rightarrow z = 0, x^2 + y^2 = 3$
35. $0 \leq z \leq 1$ 36. $0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2$
37. $z \leq 0$ 38. $z = \sqrt{1 - x^2 - y^2}$
39. (a) $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 < 1$ (b) $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 > 1$
40. $1 \leq x^2 + y^2 + z^2 \leq 4$
41. $|P_1P_2| = \sqrt{(3 - 1)^2 + (3 - 1)^2 + (0 - 1)^2} = \sqrt{9} = 3$
42. $|P_1P_2| = \sqrt{(2 + 1)^2 + (5 - 1)^2 + (0 - 5)^2} = \sqrt{50} = 5\sqrt{2}$

$$43. |P_1P_2| = \sqrt{(4-1)^2 + (-2-4)^2 + (7-5)^2} = \sqrt{49} = 7$$

$$44. |P_1P_2| = \sqrt{(2-3)^2 + (3-4)^2 + (4-5)^2} = \sqrt{3}$$

$$45. |P_1P_2| = \sqrt{(2-0)^2 + (-2-0)^2 + (-2-0)^2} = \sqrt{3 \cdot 4} = 2\sqrt{3}$$

$$46. |P_1P_2| = \sqrt{(0-5)^2 + (0-3)^2 + (0+2)^2} = \sqrt{38}$$

$$47. \text{ center } (-2, 0, 2), \text{ radius } 2\sqrt{2}$$

$$48. \text{ center } (1, -\frac{1}{2}, -3), \text{ radius } 5$$

$$49. \text{ center } (\sqrt{2}, \sqrt{2}, -\sqrt{2}), \text{ radius } \sqrt{2}$$

$$50. \text{ center } (0, -\frac{1}{3}, \frac{1}{3}), \text{ radius } \frac{4}{3}$$

$$51. (x-1)^2 + (y-2)^2 + (z-3)^2 = 14$$

$$52. x^2 + (y+1)^2 + (z-5)^2 = 4$$

$$53. (x+1)^2 + (y-\frac{1}{2})^2 + (z+\frac{2}{3})^2 = \frac{16}{81}$$

$$54. x^2 + (y+7)^2 + z^2 = 49$$

$$55. x^2 + y^2 + z^2 + 4x - 4z = 0 \Rightarrow (x^2 + 4x + 4) + y^2 + (z^2 - 4z + 4) = 4 + 4 \\ \Rightarrow (x+2)^2 + (y-0)^2 + (z-2)^2 = (\sqrt{8})^2 \Rightarrow \text{the center is at } (-2, 0, 2) \text{ and the radius is } \sqrt{8}$$

$$56. x^2 + y^2 + z^2 - 6y + 8z = 0 \Rightarrow x^2 + (y^2 - 6y + 9) + (z^2 + 8z + 16) = 9 + 16 \Rightarrow (x-0)^2 + (y-3)^2 + (z+4)^2 = 5^2 \\ \Rightarrow \text{the center is at } (0, 3, -4) \text{ and the radius is } 5$$

$$57. 2x^2 + 2y^2 + 2z^2 + x + y + z = 9 \Rightarrow x^2 + \frac{1}{2}x + y^2 + \frac{1}{2}y + z^2 + \frac{1}{2}z = \frac{9}{2} \\ \Rightarrow (x^2 + \frac{1}{2}x + \frac{1}{16}) + (y^2 + \frac{1}{2}y + \frac{1}{16}) + (z^2 + \frac{1}{2}z + \frac{1}{16}) = \frac{9}{2} + \frac{3}{16} \Rightarrow (x + \frac{1}{4})^2 + (y + \frac{1}{4})^2 + (z + \frac{1}{4})^2 = (\frac{5\sqrt{3}}{4})^2 \\ \Rightarrow \text{the center is at } (-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}) \text{ and the radius is } \frac{5\sqrt{3}}{4}$$

$$58. 3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9 \Rightarrow x^2 + y^2 + \frac{2}{3}y + z^2 - \frac{2}{3}z = 3 \Rightarrow x^2 + (y^2 + \frac{2}{3}y + \frac{1}{9}) + (z^2 - \frac{2}{3}z + \frac{1}{9}) = 3 + \frac{2}{9} \\ \Rightarrow (x-0)^2 + (y + \frac{1}{3})^2 + (z - \frac{1}{3})^2 = (\frac{\sqrt{29}}{3})^2 \Rightarrow \text{the center is at } (0, -\frac{1}{3}, \frac{1}{3}) \text{ and the radius is } \frac{\sqrt{29}}{3}$$

$$59. (a) \text{ the distance between } (x, y, z) \text{ and } (x, 0, 0) \text{ is } \sqrt{y^2 + z^2}$$

$$(b) \text{ the distance between } (x, y, z) \text{ and } (0, y, 0) \text{ is } \sqrt{x^2 + z^2}$$

$$(c) \text{ the distance between } (x, y, z) \text{ and } (0, 0, z) \text{ is } \sqrt{x^2 + y^2}$$

$$60. (a) \text{ the distance between } (x, y, z) \text{ and } (x, y, 0) \text{ is } z$$

$$(b) \text{ the distance between } (x, y, z) \text{ and } (0, y, z) \text{ is } x$$

$$(c) \text{ the distance between } (x, y, z) \text{ and } (x, 0, z) \text{ is } y$$

$$61. |AB| = \sqrt{(1-(-1))^2 + (-1-2)^2 + (3-1)^2} = \sqrt{4+9+4} = \sqrt{17}$$

$$|BC| = \sqrt{(3-1)^2 + (4-(-1))^2 + (5-3)^2} = \sqrt{4+25+4} = \sqrt{33}$$

$$|CA| = \sqrt{(-1-3)^2 + (2-4)^2 + (1-5)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$$

Thus the perimeter of triangle ABC is $\sqrt{17} + \sqrt{33} + 6$.

$$62. |\mathbf{PA}| = \sqrt{(2-3)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$|\mathbf{PB}| = \sqrt{(4-3)^2 + (3-1)^2 + (1-2)^2} = \sqrt{1+4+1} = \sqrt{6}$$

Thus P is equidistant from A and B.

$$63. \sqrt{(x-x)^2 + (y-(-1))^2 + (z-z)^2} = \sqrt{(x-x)^2 + (y-3)^2 + (z-z)^2} \Rightarrow (y+1)^2 = (y-3)^2 \Rightarrow 2y+1 = -6y+9 \\ \Rightarrow y = 1$$

$$64. \sqrt{(x-0)^2 + (y-0)^2 + (z-2)^2} = \sqrt{(x-x)^2 + (y-y)^2 + (z-0)^2} \Rightarrow x^2 + y^2 + (z-2)^2 = z^2 \\ \Rightarrow x^2 + y^2 - 4z + 4 = 0 \Rightarrow z = \frac{x^2}{4} + \frac{y^2}{4} + 1$$

65. (a) Since the entire sphere is below the xy -plane, the point on the sphere closest to the xy -plane is the point at the top of the sphere, which occurs when $x = 0$ and $y = 3 \Rightarrow 0^2 + (3-3)^2 + (z+5)^2 = 4 \Rightarrow z = -5 \pm 2 \Rightarrow z = -3 \\ \Rightarrow (0, 3, -3).$

(b) Both the center $(0, 3, -5)$ and the point $(0, 7, -5)$ lie in the plane $z = -5$, so the point on the sphere closest to $(0, 7, -5)$ should also be in the same plane. In fact it should lie on the line segment between $(0, 3, -5)$ and $(0, 7, -5)$, thus the point occurs when $x = 0$ and $z = -5 \Rightarrow 0^2 + (y-3)^2 + (-5+5)^2 = 4 \Rightarrow y = 3 \pm 2 \Rightarrow y = 5 \\ \Rightarrow (0, 5, -5).$

$$66. \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{(x-0)^2 + (y-4)^2 + (z-0)^2} = \sqrt{(x-3)^2 + (y-0)^2 + (z-0)^2} \\ = \sqrt{(x-2)^2 + (y-2)^2 + (z+3)^2} \\ \Rightarrow x^2 + y^2 + z^2 = x^2 + y^2 - 8y + 16 + z^2 = x^2 - 6x + 9 + y^2 + z^2 = x^2 - 4x + y^2 - 4y + z^2 + 6z + 17 \\ \text{Solve: } x^2 + y^2 + z^2 = x^2 + y^2 - 8y + 16 + z^2 \Rightarrow 0 = -8y + 16 \Rightarrow y = 2 \\ \text{Solve: } x^2 + y^2 + z^2 = x^2 - 6x + 9 + y^2 + z^2 \Rightarrow 0 = -6x + 9 \Rightarrow x = \frac{3}{2} \\ \text{Solve: } x^2 + y^2 + z^2 = x^2 - 4x + y^2 - 4y + z^2 + 6z + 17 \Rightarrow 0 = -4x - 4y + 6z + 17 \Rightarrow 0 = -4\left(\frac{3}{2}\right) - 4(2) + 6z + 17 \\ \Rightarrow z = -\frac{1}{2} \Rightarrow \left(\frac{3}{2}, 2, -\frac{1}{2}\right)$$

12.2 VECTORS

$$1. (a) \langle 3(3), 3(-2) \rangle = \langle 9, -6 \rangle$$

$$(b) \sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$$

$$2. (a) \langle -2(-2), -2(5) \rangle = \langle 4, -10 \rangle$$

$$(b) \sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$$

$$3. (a) \langle 3 + (-2), -2 + 5 \rangle = \langle 1, 3 \rangle$$

$$(b) \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$4. (a) \langle 3 - (-2), -2 - 5 \rangle = \langle 5, -7 \rangle$$

$$(b) \sqrt{5^2 + (-7)^2} = \sqrt{74}$$

$$5. (a) 2\mathbf{u} = \langle 2(3), 2(-2) \rangle = \langle 6, -4 \rangle$$

$$3\mathbf{v} = \langle 3(-2), 3(5) \rangle = \langle -6, 15 \rangle$$

$$2\mathbf{u} - 3\mathbf{v} = \langle 6 - (-6), -4 - 15 \rangle = \langle 12, -19 \rangle$$

$$(b) \sqrt{12^2 + (-19)^2} = \sqrt{505}$$

$$6. (a) -2\mathbf{u} = \langle -2(3), -2(-2) \rangle = \langle -6, 4 \rangle$$

$$5\mathbf{v} = \langle 5(-2), 5(5) \rangle = \langle -10, 25 \rangle$$

$$-2\mathbf{u} + 5\mathbf{v} = \langle -6 + (-10), 4 + 25 \rangle = \langle -16, 29 \rangle$$

$$(b) \sqrt{(-16)^2 + 29^2} = \sqrt{1097}$$

$$7. \quad (a) \quad \frac{3}{5}\mathbf{u} = \left\langle \frac{3}{5}(3), \frac{3}{5}(-2) \right\rangle = \left\langle \frac{9}{5}, -\frac{6}{5} \right\rangle$$

$$\frac{4}{5}\mathbf{v} = \left\langle \frac{4}{5}(-2), \frac{4}{5}(5) \right\rangle = \left\langle -\frac{8}{5}, 4 \right\rangle$$

$$\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v} = \left\langle \frac{9}{5} + \left(-\frac{8}{5}\right), -\frac{6}{5} + 4 \right\rangle = \left\langle \frac{1}{5}, \frac{14}{5} \right\rangle$$

$$(b) \quad \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{14}{5}\right)^2} = \frac{\sqrt{197}}{5}$$

$$8. \quad (a) \quad -\frac{5}{13}\mathbf{u} = \left\langle -\frac{5}{13}(3), -\frac{5}{13}(-2) \right\rangle = \left\langle -\frac{15}{13}, \frac{10}{13} \right\rangle$$

$$\frac{12}{13}\mathbf{v} = \left\langle \frac{12}{13}(-2), \frac{12}{13}(5) \right\rangle = \left\langle -\frac{24}{13}, \frac{60}{13} \right\rangle$$

$$-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v} = \left\langle -\frac{15}{13} + \left(-\frac{24}{13}\right), \frac{10}{13} + \frac{60}{13} \right\rangle = \left\langle -3, \frac{70}{13} \right\rangle$$

$$(b) \quad \sqrt{(-3)^2 + \left(\frac{70}{13}\right)^2} = \frac{\sqrt{6421}}{13}$$

$$9. \quad \langle 2 - 1, -1 - 3 \rangle = \langle 1, -4 \rangle$$

$$10. \quad \left\langle \frac{2+(-4)}{2} - 0, \frac{-1+3}{2} - 0 \right\rangle = \langle -1, 1 \rangle$$

$$11. \quad \langle 0 - 2, 0 - 3 \rangle = \langle -2, -3 \rangle$$

$$12. \quad \vec{AB} = \langle 2 - 1, 0 - (-1) \rangle = \langle 1, 1 \rangle, \quad \vec{CD} = \langle -2 - (-1), 2 - 3 \rangle = \langle -1, -1 \rangle, \quad \vec{AB} + \vec{CD} = \langle 0, 0 \rangle$$

$$13. \quad \left\langle \cos \frac{2\pi}{3}, \sin \frac{2\pi}{3} \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$14. \quad \left\langle \cos \left(-\frac{3\pi}{4}\right), \sin \left(-\frac{3\pi}{4}\right) \right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

15. This is the unit vector which makes an angle of $120^\circ + 90^\circ = 210^\circ$ with the positive x-axis;

$$\langle \cos 210^\circ, \sin 210^\circ \rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$

$$16. \quad \langle \cos 135^\circ, \sin 135^\circ \rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$17. \quad \vec{P_1P_2} = (2 - 5)\mathbf{i} + (9 - 7)\mathbf{j} + (-2 - (-1))\mathbf{k} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$18. \quad \vec{P_1P_2} = (-3 - 1)\mathbf{i} + (0 - 2)\mathbf{j} + (5 - 0)\mathbf{k} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

$$19. \quad \vec{AB} = (-10 - (-7))\mathbf{i} + (8 - (-8))\mathbf{j} + (1 - 1)\mathbf{k} = -3\mathbf{i} + 16\mathbf{j}$$

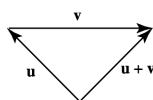
$$20. \quad \vec{AB} = (-1 - 1)\mathbf{i} + (4 - 0)\mathbf{j} + (5 - 3)\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

$$21. \quad 5\mathbf{u} - \mathbf{v} = 5\langle 1, 1, -1 \rangle - \langle 2, 0, 3 \rangle = \langle 5, 5, -5 \rangle - \langle 2, 0, 3 \rangle = \langle 5 - 2, 5 - 0, -5 - 3 \rangle = \langle 3, 5, -8 \rangle = 3\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$$

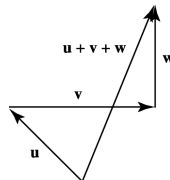
$$22. \quad -2\mathbf{u} + 3\mathbf{v} = -2\langle -1, 0, 2 \rangle + 3\langle 1, 1, 1 \rangle = \langle 2, 0, -4 \rangle + \langle 3, 3, 3 \rangle = \langle 5, 3, -1 \rangle = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

23. The vector $\mathbf{v}$ is horizontal and 1 in. long. The vectors $\mathbf{u}$ and $\mathbf{w}$ are $\frac{11}{16}$ in. long. $\mathbf{w}$ is vertical and $\mathbf{u}$ makes a 45° angle with the horizontal. All vectors must be drawn to scale.

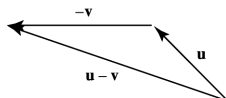
(a)



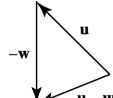
(b)



(c)

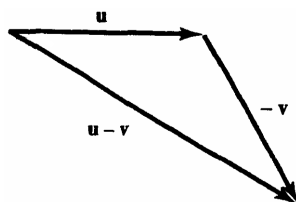


(d)

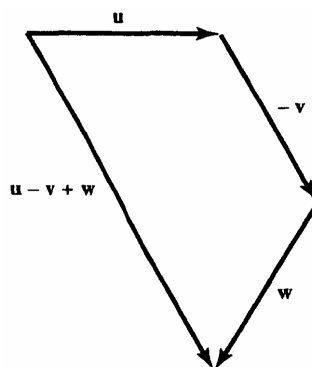


24. The angle between the vectors is 120° and vector $\mathbf{u}$ is horizontal. They are all 1 in. long. Draw to scale.

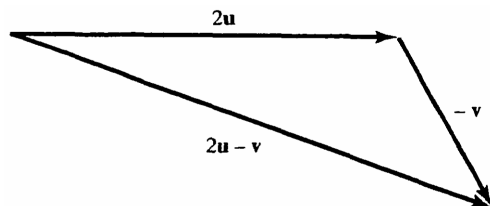
(a)



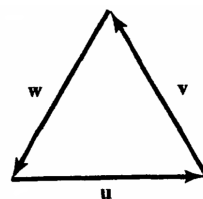
(b)



(c)



(d)



$$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$$

25. length $= |\mathbf{2i} + \mathbf{j} - \mathbf{2k}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, the direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{2i} + \mathbf{j} - \mathbf{2k} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$

26. length $= |\mathbf{9i} - \mathbf{2j} + \mathbf{6k}| = \sqrt{81 + 4 + 36} = 11$, the direction is $\frac{9}{11}\mathbf{i} - \frac{2}{11}\mathbf{j} + \frac{6}{11}\mathbf{k} \Rightarrow \mathbf{9i} - \mathbf{2j} + \mathbf{6k} = 11\left(\frac{9}{11}\mathbf{i} - \frac{2}{11}\mathbf{j} + \frac{6}{11}\mathbf{k}\right)$

27. length $= |\mathbf{5k}| = \sqrt{25} = 5$, the direction is $\mathbf{k} \Rightarrow \mathbf{5k} = 5(\mathbf{k})$

28. length $= \left|\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}\right| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$, the direction is $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k} = 1\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}\right)$

29. length $= \left|\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{2}}$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
 $\Rightarrow \frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} = \sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$

30. length $= \left|\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{3}}\right)^2} = 1$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
 $\Rightarrow \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} = 1\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$

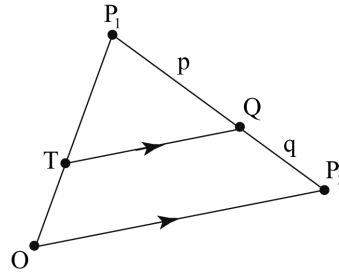
31. (a) $\mathbf{2i}$ (b) $-\sqrt{3}\mathbf{k}$ (c) $\frac{3}{10}\mathbf{j} + \frac{2}{5}\mathbf{k}$ (d) $\mathbf{6i} - \mathbf{2j} + \mathbf{3k}$

32. (a) $-\mathbf{7j}$ (b) $-\frac{3\sqrt{2}}{5}\mathbf{i} - \frac{4\sqrt{2}}{5}\mathbf{k}$ (c) $\frac{1}{4}\mathbf{i} - \frac{1}{3}\mathbf{j} - \mathbf{k}$ (d) $\frac{a}{\sqrt{2}}\mathbf{i} + \frac{a}{\sqrt{3}}\mathbf{j} - \frac{a}{\sqrt{6}}\mathbf{k}$

33. $|\mathbf{v}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{13}\mathbf{v} = \frac{1}{13}(12\mathbf{i} - 5\mathbf{k}) \Rightarrow$ the desired vector is $\frac{7}{13}(12\mathbf{i} - 5\mathbf{k})$

34. $|\mathbf{v}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow$ the desired vector is $-3\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) = -\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}$
35. (a) $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} = 5\sqrt{2}\left(\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$
 (b) the midpoint is $\left(\frac{1}{2}, 3, \frac{5}{2}\right)$
36. (a) $3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} = 7\left(\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$
 (b) the midpoint is $\left(\frac{5}{2}, 1, 6\right)$
37. (a) $-\mathbf{i} - \mathbf{j} - \mathbf{k} = \sqrt{3}\left(-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
 (b) the midpoint is $\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$
38. (a) $2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
 (b) the midpoint is $(1, -1, -1)$
39. $\vec{AB} = (5-a)\mathbf{i} + (1-b)\mathbf{j} + (3-c)\mathbf{k} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow 5-a=1, 1-b=4, \text{ and } 3-c=-2 \Rightarrow a=4, b=-3, \text{ and } c=5 \Rightarrow A$ is the point $(4, -3, 5)$
40. $\vec{AB} = (a+2)\mathbf{i} + (b+3)\mathbf{j} + (c-6)\mathbf{k} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k} \Rightarrow a+2=-7, b+3=3, \text{ and } c-6=8 \Rightarrow a=-9, b=0, \text{ and } c=14 \Rightarrow B$ is the point $(-9, 0, 14)$
41. $2\mathbf{i} + \mathbf{j} = a(\mathbf{i} + \mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a+b)\mathbf{i} + (a-b)\mathbf{j} \Rightarrow a+b=2 \text{ and } a-b=1 \Rightarrow 2a=3 \Rightarrow a=\frac{3}{2} \text{ and } b=a-1=\frac{1}{2}$
42. $\mathbf{i} - 2\mathbf{j} = a(2\mathbf{i} + 3\mathbf{j}) + b(\mathbf{i} + \mathbf{j}) = (2a+b)\mathbf{i} + (3a+b)\mathbf{j} \Rightarrow 2a+b=1 \text{ and } 3a+b=-2 \Rightarrow a=-3 \text{ and } b=1-2a=7 \Rightarrow \mathbf{u}_1 = a(2\mathbf{i} + 3\mathbf{j}) = -6\mathbf{i} - 9\mathbf{j} \text{ and } \mathbf{u}_2 = b(\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 7\mathbf{j}$
43. 25° west of north is $90^\circ + 25^\circ = 115^\circ$ north of east. $800\langle \cos 115^\circ, \sin 115^\circ \rangle \approx \langle -338.095, 725.046 \rangle$
44. Let $\mathbf{u} = \langle x, y \rangle$ be represent the velocity of the plane alone, $\mathbf{v} = \langle 70 \cos 60^\circ, 70 \sin 60^\circ \rangle = \langle 35, 35\sqrt{3} \rangle$, and let the resultant $\mathbf{u} + \mathbf{v} = \langle 500, 0 \rangle$. Then $\langle x, y \rangle + \langle 35, 35\sqrt{3} \rangle = \langle 500, 0 \rangle \Rightarrow \langle x+35, y+35\sqrt{3} \rangle = \langle 500, 0 \rangle \Rightarrow x+35=500 \text{ and } y+35\sqrt{3}=0 \Rightarrow x=465 \text{ and } y=-35\sqrt{3} \Rightarrow \mathbf{u} = \langle 465, -35\sqrt{3} \rangle \Rightarrow |\mathbf{u}| = \sqrt{465^2 + (-35\sqrt{3})^2} \approx 468.9 \text{ mph, and } \tan \theta = \frac{-35\sqrt{3}}{465} \Rightarrow \theta \approx -7.4^\circ \Rightarrow 7.4^\circ \text{ south of east.}$
45. $\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 30^\circ, |\mathbf{F}_1|\sin 30^\circ \rangle = \langle -\frac{\sqrt{3}}{2}|\mathbf{F}_1|, \frac{1}{2}|\mathbf{F}_1| \rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 45^\circ, |\mathbf{F}_2|\sin 45^\circ \rangle = \langle \frac{1}{\sqrt{2}}|\mathbf{F}_2|, \frac{1}{\sqrt{2}}|\mathbf{F}_2| \rangle$, and $\mathbf{w} = \langle 0, -100 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 100 \rangle \Rightarrow \langle -\frac{\sqrt{3}}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2|, \frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2| \rangle = \langle 0, 100 \rangle \Rightarrow -\frac{\sqrt{3}}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2| = 0 \text{ and } \frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2| = 100$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{\sqrt{6}}{2}|\mathbf{F}_1|$. Substituting this result into the second equation gives us: $\frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{6}}{2}|\mathbf{F}_1|\right) = 100 \Rightarrow |\mathbf{F}_1| = \frac{200}{1+\sqrt{3}} \approx 73.205 \text{ N} \Rightarrow |\mathbf{F}_2| = \frac{100\sqrt{6}}{1+\sqrt{3}} \approx 89.658 \text{ N} \Rightarrow \mathbf{F}_1 \approx \langle -63.397, 36.603 \rangle \text{ and } \mathbf{F}_2 \approx \langle 63.397, 63.397 \rangle$

46. $\mathbf{F}_1 = \langle -35 \cos \alpha, 35 \sin \alpha \rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2| \cos 60^\circ, |\mathbf{F}_2| \sin 60^\circ \rangle = \langle \frac{1}{2}|\mathbf{F}_2|, \frac{\sqrt{3}}{2}|\mathbf{F}_2| \rangle$, and $\mathbf{w} = \langle 0, -50 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 50 \rangle \Rightarrow \langle -35 \cos \alpha + \frac{1}{2}|\mathbf{F}_2|, 35 \sin \alpha + \frac{\sqrt{3}}{2}|\mathbf{F}_2| \rangle = \langle 0, 50 \rangle \Rightarrow -35 \cos \alpha + \frac{1}{2}|\mathbf{F}_2| = 0$ and $35 \sin \alpha + \frac{\sqrt{3}}{2}|\mathbf{F}_2| = 50$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = 70 \cos \alpha$. Substituting this result into the second equation gives us: $35 \sin \alpha + 35\sqrt{3} \cos \alpha = 50 \Rightarrow \sqrt{3} \cos \alpha = \frac{10}{7} - \sin \alpha \Rightarrow 3 \cos^2 \alpha = \frac{100}{49} - \frac{20}{7} \sin \alpha + \sin^2 \alpha \Rightarrow 3(1 - \sin^2 \alpha) = \frac{100}{49} - \frac{20}{7} \sin \alpha + \sin^2 \alpha \Rightarrow 196 \sin^2 \alpha - 140 \sin \alpha - 47 = 0 \Rightarrow \sin \alpha = \frac{5 \pm 6\sqrt{2}}{14}$. Since $\alpha > 0 \Rightarrow \sin \alpha > 0 \Rightarrow \sin \alpha = \frac{5+6\sqrt{2}}{14} \Rightarrow \alpha \approx 74.42^\circ$, and $|\mathbf{F}_2| = 70 \cos \alpha \approx 18.81$ N.
47. $\mathbf{F}_1 = \langle -|\mathbf{F}_1| \cos 40^\circ, |\mathbf{F}_1| \sin 40^\circ \rangle$, $\mathbf{F}_2 = \langle 100 \cos 35^\circ, 100 \sin 35^\circ \rangle$, and $\mathbf{w} = \langle 0, -w \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, w \rangle \Rightarrow \langle -|\mathbf{F}_1| \cos 40^\circ + 100 \cos 35^\circ, |\mathbf{F}_1| \sin 40^\circ + 100 \sin 35^\circ \rangle = \langle 0, w \rangle \Rightarrow -|\mathbf{F}_1| \cos 40^\circ + 100 \cos 35^\circ = 0$ and $|\mathbf{F}_1| \sin 40^\circ + 100 \sin 35^\circ = w$. Solving the first equation for $|\mathbf{F}_1|$ results in: $|\mathbf{F}_1| = \frac{100 \cos 35^\circ}{\cos 40^\circ} \approx 106.933$ N. Substituting this result into the second equation gives us: $w \approx 126.093$ N.
48. $\mathbf{F}_1 = \langle -|\mathbf{F}_1| \cos \alpha, |\mathbf{F}_1| \sin \alpha \rangle = \langle -75 \cos \alpha, 75 \sin \alpha \rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2| \cos \beta, |\mathbf{F}_2| \sin \beta \rangle = \langle 75 \cos \alpha, 75 \sin \alpha \rangle$, and $\mathbf{w} = \langle 0, -25 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 25 \rangle \Rightarrow \langle -75 \cos \alpha + 75 \cos \alpha, 75 \sin \alpha + 75 \sin \alpha \rangle = \langle 0, 25 \rangle \Rightarrow 150 \sin \alpha = 25 \Rightarrow \alpha \approx 9.59^\circ$.
49. (a) The tree is located at the tip of the vector $\vec{OP} = (5 \cos 60^\circ)\mathbf{i} + (5 \sin 60^\circ)\mathbf{j} = \frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} \Rightarrow P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$
- (b) The telephone pole is located at the point Q, which is the tip of the vector $\vec{OP} + \vec{PQ}$
 $= \left(\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}\right) + (10 \cos 315^\circ)\mathbf{i} + (10 \sin 315^\circ)\mathbf{j} = \left(\frac{5}{2} + \frac{10\sqrt{2}}{2}\right)\mathbf{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\mathbf{j}$
 $\Rightarrow Q = \left(\frac{5+10\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$
50. Let $t = \frac{q}{p+q}$ and $s = \frac{p}{p+q}$. Choose T on $\vec{OP}_1$ so that $\vec{TQ}$ is parallel to $\vec{OP}_2$, so that $\triangle TP_1Q$ is similar to $\triangle OP_1P_2$. Then $\frac{|\vec{OT}|}{|\vec{OP}_1|} = t \Rightarrow \vec{OT} = t\vec{OP}_1$ so that $T = (tx_1, ty_1, tz_1)$. Also, $\frac{|\vec{TQ}|}{|\vec{OP}_2|} = s \Rightarrow \vec{TQ} = s\vec{OP}_2 = s\langle x_2, y_2, z_2 \rangle$. Letting $Q = (x, y, z)$, we have that $\vec{TQ} = \langle x - tx_1, y - ty_1, z - tz_1 \rangle = s\langle x_2, y_2, z_2 \rangle$. Thus $x = tx_1 + sx_2, y = ty_1 + sy_2, z = tz_1 + sz_2$. (Note that if Q is the midpoint, then $\frac{p}{q} = 1$ and $t = s = \frac{1}{2}$ so that $x = \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{x_1+x_2}{2}, y = \frac{y_1+y_2}{2}, z = \frac{z_1+z_2}{2}$ so that this result agrees with the midpoint formula.)



51. (a) the midpoint of AB is $M\left(\frac{5}{2}, \frac{5}{2}, 0\right)$ and $\vec{CM} = \left(\frac{5}{2} - 1\right)\mathbf{i} + \left(\frac{5}{2} - 1\right)\mathbf{j} + (0 - 3)\mathbf{k} = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$
- (b) the desired vector is $\left(\frac{2}{3}\right)\vec{CM} = \frac{2}{3}\left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
- (c) the vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass $\Rightarrow$ the terminal point of $(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is the point $(2, 2, 1)$, which is the location of the center of mass
52. The midpoint of AB is $M\left(\frac{3}{2}, 0, \frac{5}{2}\right)$ and $\left(\frac{2}{3}\right)\vec{CM} = \frac{2}{3}\left[\left(\frac{3}{2} + 1\right)\mathbf{i} + (0 - 2)\mathbf{j} + \left(\frac{5}{2} + 1\right)\mathbf{k}\right] = \frac{2}{3}\left(\frac{5}{2}\mathbf{i} - 2\mathbf{j} + \frac{7}{2}\mathbf{k}\right) = \frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$. The vector from the origin to the point of intersection of the medians is $\left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + \vec{OC} = \left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + (-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}$.

53. Without loss of generality we identify the vertices of the quadrilateral such that $A(0, 0, 0)$, $B(x_b, 0, 0)$, $C(x_c, y_c, 0)$ and $D(x_d, y_d, z_d) \Rightarrow$ the midpoint of AB is $M_{AB}(\frac{x_b}{2}, 0, 0)$, the midpoint of BC is $M_{BC}(\frac{x_b+x_c}{2}, \frac{y_c}{2}, 0)$, the midpoint of CD is $M_{CD}(\frac{x_c+x_d}{2}, \frac{y_c+y_d}{2}, \frac{z_d}{2})$ and the midpoint of AD is $M_{AD}(\frac{x_d}{2}, \frac{y_d}{2}, \frac{z_d}{2}) \Rightarrow$ the midpoint of $M_{AB}M_{CD}$ is $(\frac{\frac{x_b}{2} + \frac{x_c+x_d}{2}}{2}, \frac{\frac{y_c}{2} + \frac{y_c+y_d}{2}}{2}, \frac{\frac{z_d}{2}}{2})$ which is the same as the midpoint of $M_{AD}M_{BC} = (\frac{\frac{x_b+x_c}{2} + \frac{x_d}{2}}{2}, \frac{y_c+y_d}{4}, \frac{z_d}{4})$.
54. Let $V_1, V_2, V_3, \dots, V_n$ be the vertices of a regular n -sided polygon and $\mathbf{v}_i$ denote the vector from the center to V_i for $i = 1, 2, 3, \dots, n$. If $\mathbf{S} = \sum_{i=1}^n \mathbf{v}_i$ and the polygon is rotated through an angle of $\frac{i(2\pi)}{n}$ where $i = 1, 2, 3, \dots, n$, then $\mathbf{S}$ would remain the same. Since the vector $\mathbf{S}$ does not change with these rotations we conclude that $\mathbf{S} = \mathbf{0}$.
55. Without loss of generality we can coordinatize the vertices of the triangle such that $A(0, 0)$, $B(b, 0)$ and $C(x_c, y_c) \Rightarrow$ a is located at $(\frac{b+x_c}{2}, \frac{y_c}{2})$, b is at $(\frac{x_c}{2}, \frac{y_c}{2})$ and c is at $(\frac{b}{2}, 0)$. Therefore, $\vec{Aa} = (\frac{b}{2} + \frac{x_c}{2})\mathbf{i} + (\frac{y_c}{2})\mathbf{j}$, $\vec{Bb} = (\frac{x_c}{2} - b)\mathbf{i} + (\frac{y_c}{2})\mathbf{j}$, and $\vec{Cc} = (\frac{b}{2} - x_c)\mathbf{i} + (-y_c)\mathbf{j} \Rightarrow \vec{Aa} + \vec{Bb} + \vec{Cc} = \mathbf{0}$.
56. Let $\mathbf{u}$ be any unit vector in the plane. If $\mathbf{u}$ is positioned so that its initial point is at the origin and terminal point is at (x, y) , then $\mathbf{u}$ makes an angle θ with $\mathbf{i}$, measured in the counter-clockwise direction. Since $|\mathbf{u}| = 1$, we have that $x = \cos \theta$ and $y = \sin \theta$. Thus $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$. Since $\mathbf{u}$ was assumed to be any unit vector in the plane, this holds for every unit vector in the plane.

12.3 THE DOT PRODUCT

NOTE: In Exercises 1-8 below we calculate $\text{proj}_{\mathbf{v}} \mathbf{u}$ as the vector $(\frac{|\mathbf{u}| \cos \theta}{|\mathbf{v}|}) \mathbf{v}$, so the scalar multiplier of $\mathbf{v}$ is the number in column 5 divided by the number in column 2.

	$\mathbf{v} \cdot \mathbf{u}$	$ \mathbf{v} $	$ \mathbf{u} $	$\cos \theta$	$ \mathbf{u} \cos \theta$	$\text{proj}_{\mathbf{v}} \mathbf{u}$
1.	-25	5	5	-1	-5	$-2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
2.	3	1	13	$\frac{3}{13}$	3	$3(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k})$
3.	25	15	5	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{9}(10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k})$
4.	13	15	3	$\frac{13}{45}$	$\frac{13}{15}$	$\frac{13}{225}(2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k})$
5.	2	$\sqrt{34}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}\sqrt{34}}$	$\frac{2}{\sqrt{34}}$	$\frac{1}{17}(5\mathbf{j} - 3\mathbf{k})$
6.	$\sqrt{3} - \sqrt{2}$	$\sqrt{2}$	3	$\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$
7.	$10 + \sqrt{17}$	$\sqrt{26}$	$\sqrt{21}$	$\frac{10+\sqrt{17}}{\sqrt{546}}$	$\frac{10+\sqrt{17}}{\sqrt{26}}$	$\frac{10+\sqrt{17}}{26}(5\mathbf{i} + \mathbf{j})$
8.	$\frac{1}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{5}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$

$$9. \theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{(2)(1) + (1)(2) + (0)(-1)}{\sqrt{2^2 + 1^2 + 0^2} \sqrt{1^2 + 2^2 + (-1)^2}} \right) = \cos^{-1} \left(\frac{4}{\sqrt{5} \sqrt{6}} \right) = \cos^{-1} \left(\frac{4}{\sqrt{30}} \right) \approx 0.75 \text{ rad}$$

$$10. \theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{(2)(3) + (-2)(0) + (1)(4)}{\sqrt{2^2 + (-2)^2 + 1^2} \sqrt{3^2 + 0^2 + 4^2}} \right) = \cos^{-1} \left(\frac{10}{\sqrt{9} \sqrt{25}} \right) = \cos^{-1} \left(\frac{2}{3} \right) \approx 0.84 \text{ rad}$$

$$11. \theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{(\sqrt{3})(\sqrt{3}) + (-7)(1) + (0)(-2)}{\sqrt{(\sqrt{3})^2 + (-7)^2 + 0^2} \sqrt{(\sqrt{3})^2 + (1)^2 + (-2)^2}} \right) = \cos^{-1} \left(\frac{3-7}{\sqrt{52} \sqrt{8}} \right) \\ = \cos^{-1} \left(\frac{-1}{\sqrt{26}} \right) \approx 1.77 \text{ rad}$$

$$12. \theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{(1)(-1) + (\sqrt{2})(1) + (-\sqrt{2})(1)}{\sqrt{(1)^2 + (\sqrt{2})^2 + (-\sqrt{2})^2} \sqrt{(-1)^2 + (1)^2 + (1)^2}} \right) = \cos^{-1} \left(\frac{-1}{\sqrt{5} \sqrt{3}} \right) \\ = \cos^{-1} \left(\frac{-1}{\sqrt{15}} \right) \approx 1.83 \text{ rad}$$

$$13. \vec{AB} = \langle 3, 1 \rangle, \vec{BC} = \langle -1, -3 \rangle, \text{ and } \vec{AC} = \langle 2, -2 \rangle. \vec{BA} = \langle -3, -1 \rangle, \vec{CB} = \langle 1, 3 \rangle, \vec{CA} = \langle -2, 2 \rangle.$$

$$|\vec{AB}| = |\vec{BA}| = \sqrt{10}, |\vec{BC}| = |\vec{CB}| = \sqrt{10}, |\vec{AC}| = |\vec{CA}| = 2\sqrt{2},$$

$$\text{Angle at A} = \cos^{-1} \left(\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} \right) = \cos^{-1} \left(\frac{3(2) + 1(-2)}{(\sqrt{10})(2\sqrt{2})} \right) = \cos^{-1} \left(\frac{1}{\sqrt{5}} \right) \approx 63.435^\circ$$

$$\text{Angle at B} = \cos^{-1} \left(\frac{\vec{BC} \cdot \vec{BA}}{|\vec{BC}| |\vec{BA}|} \right) = \cos^{-1} \left(\frac{(-1)(-3) + (-3)(-1)}{(\sqrt{10})(\sqrt{10})} \right) = \cos^{-1} \left(\frac{3}{5} \right) \approx 53.130^\circ, \text{ and}$$

$$\text{Angle at C} = \cos^{-1} \left(\frac{\vec{CB} \cdot \vec{CA}}{|\vec{CB}| |\vec{CA}|} \right) = \cos^{-1} \left(\frac{1(-2) + 3(2)}{(\sqrt{10})(2\sqrt{2})} \right) = \cos^{-1} \left(\frac{1}{\sqrt{5}} \right) \approx 63.435^\circ$$

$$14. \vec{AC} = \langle 2, 4 \rangle \text{ and } \vec{BD} = \langle 4, -2 \rangle. \vec{AC} \cdot \vec{BD} = 2(4) + 4(-2) = 0, \text{ so the angle measures are all } 90^\circ.$$

$$15. (a) \cos \alpha = \frac{\mathbf{i} \cdot \mathbf{v}}{|\mathbf{i}| |\mathbf{v}|} = \frac{a}{|\mathbf{v}|}, \cos \beta = \frac{\mathbf{j} \cdot \mathbf{v}}{|\mathbf{j}| |\mathbf{v}|} = \frac{b}{|\mathbf{v}|}, \cos \gamma = \frac{\mathbf{k} \cdot \mathbf{v}}{|\mathbf{k}| |\mathbf{v}|} = \frac{c}{|\mathbf{v}|} \text{ and}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \left(\frac{a}{|\mathbf{v}|} \right)^2 + \left(\frac{b}{|\mathbf{v}|} \right)^2 + \left(\frac{c}{|\mathbf{v}|} \right)^2 = \frac{a^2 + b^2 + c^2}{|\mathbf{v}|^2} = \frac{|\mathbf{v}|^2}{|\mathbf{v}|^2} = 1$$

$$(b) |\mathbf{v}| = 1 \Rightarrow \cos \alpha = \frac{a}{|\mathbf{v}|} = a, \cos \beta = \frac{b}{|\mathbf{v}|} = b \text{ and } \cos \gamma = \frac{c}{|\mathbf{v}|} = c \text{ are the direction cosines of } \mathbf{v}$$

$$16. \mathbf{u} = 10\mathbf{i} + 2\mathbf{k} \text{ is parallel to the pipe in the north direction and } \mathbf{v} = 10\mathbf{j} + \mathbf{k} \text{ is parallel to the pipe in the east direction. The angle between the two pipes is } \theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{2}{\sqrt{104} \sqrt{101}} \right) \approx 1.55 \text{ rad} \approx 88.88^\circ.$$

17. The sum of two vectors of equal length is *always* orthogonal to their difference, as we can see from the equation

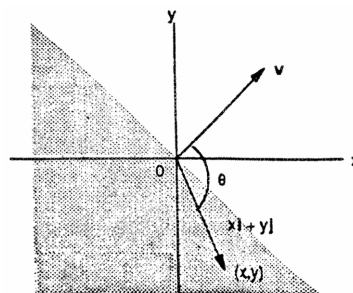
$$(\mathbf{v}_1 + \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_1 - \mathbf{v}_1 \cdot \mathbf{v}_2 - \mathbf{v}_2 \cdot \mathbf{v}_2 = |\mathbf{v}_1|^2 - |\mathbf{v}_2|^2 = 0$$

18. $\vec{CA} \cdot \vec{CB} = (-\mathbf{v} + (-\mathbf{u})) \cdot (-\mathbf{v} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$ since both equal the radius of the circle. Therefore, $\vec{CA}$ and $\vec{CB}$ are orthogonal.

19. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rhombus $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u} + \mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u} + \mathbf{v}$

$\Rightarrow \mathbf{d}_1 \cdot \mathbf{d}_2 = (\mathbf{u} + \mathbf{v}) \cdot (-\mathbf{u} + \mathbf{v}) = -\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$, since a rhombus has equal sides.

20. Suppose the diagonals of a rectangle are perpendicular, and let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rectangle $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u} + \mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u} + \mathbf{v}$. Since the diagonals are perpendicular we have $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$
 $\Leftrightarrow (\mathbf{u} + \mathbf{v}) \cdot (-\mathbf{u} + \mathbf{v}) = -\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = 0 \Leftrightarrow |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0 \Leftrightarrow (|\mathbf{v}| + |\mathbf{u}|)(|\mathbf{v}| - |\mathbf{u}|) = 0$
 $\Leftrightarrow (|\mathbf{v}| + |\mathbf{u}|) = 0$ which is not possible, or $(|\mathbf{v}| - |\mathbf{u}|) = 0$ which is equivalent to $|\mathbf{v}| = |\mathbf{u}| \Rightarrow$ the rectangle is a square.
21. Clearly the diagonals of a rectangle are equal in length. What is not as obvious is the statement that equal diagonals happen only in a rectangle. We show this is true by letting the adjacent sides of a parallelogram be the vectors $(v_1\mathbf{i} + v_2\mathbf{j})$ and $(u_1\mathbf{i} + u_2\mathbf{j})$. The equal diagonals of the parallelogram are $\mathbf{d}_1 = (v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j})$ and $\mathbf{d}_2 = (v_1\mathbf{i} + v_2\mathbf{j}) - (u_1\mathbf{i} + u_2\mathbf{j})$. Hence $|\mathbf{d}_1| = |\mathbf{d}_2| = |(v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j})| = |(v_1\mathbf{i} + v_2\mathbf{j}) - (u_1\mathbf{i} + u_2\mathbf{j})|$
 $\Rightarrow |(v_1 + u_1)\mathbf{i} + (v_2 + u_2)\mathbf{j}| = |(v_1 - u_1)\mathbf{i} + (v_2 - u_2)\mathbf{j}| \Rightarrow \sqrt{(v_1 + u_1)^2 + (v_2 + u_2)^2} = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$
 $\Rightarrow v_1^2 + 2v_1u_1 + u_1^2 + v_2^2 + 2v_2u_2 + u_2^2 = v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 \Rightarrow 2(v_1u_1 + v_2u_2)$
 $= -2(v_1u_1 + v_2u_2) \Rightarrow v_1u_1 + v_2u_2 = 0 \Rightarrow (v_1\mathbf{i} + v_2\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) = 0 \Rightarrow$ the vectors $(v_1\mathbf{i} + v_2\mathbf{j})$ and $(u_1\mathbf{i} + u_2\mathbf{j})$ are perpendicular and the parallelogram must be a rectangle.
22. If $|\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{u} + \mathbf{v}$ is the indicated diagonal, then $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} = |\mathbf{u}|^2 + \mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$
 $= \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} \Rightarrow$ the angle $\cos^{-1} \left(\frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u}}{|\mathbf{u} + \mathbf{v}| |\mathbf{u}|} \right)$ between the diagonal and $\mathbf{u}$ and the angle $\cos^{-1} \left(\frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}}{|\mathbf{u} + \mathbf{v}| |\mathbf{v}|} \right)$ between the diagonal and $\mathbf{v}$ are equal because the inverse cosine function is one-to-one.
 Therefore, the diagonal bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.
23. horizontal component: $1200 \cos(8^\circ) \approx 1188$ ft/s; vertical component: $1200 \sin(8^\circ) \approx 167$ ft/s
24. $|\mathbf{w}| \cos(33^\circ - 15^\circ) = 2.5$ lb, so $|\mathbf{w}| = \frac{2.5 \text{ lb}}{\cos 18^\circ}$. Then $\mathbf{w} = \frac{2.5 \text{ lb}}{\cos 18^\circ} \langle \cos 33^\circ, \sin 33^\circ \rangle \approx \langle 2.205, 1.432 \rangle$
25. (a) Since $|\cos \theta| \leq 1$, we have $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\cos \theta| \leq |\mathbf{u}| |\mathbf{v}| (1) = |\mathbf{u}| |\mathbf{v}|$.
 (b) We have equality precisely when $|\cos \theta| = 1$ or when one or both of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π , i.e., when the vectors are parallel.
26. $(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{v} = |x\mathbf{i} + y\mathbf{j}| |\mathbf{v}| \cos \theta \leq 0$ when $\frac{\pi}{2} \leq \theta \leq \pi$. This means (x, y) has to be a point whose position vector makes an angle with $\mathbf{v}$ that is a right angle or bigger.



27. $\mathbf{v} \cdot \mathbf{u}_1 = (a\mathbf{u}_1 + b\mathbf{u}_2) \cdot \mathbf{u}_1 = a\mathbf{u}_1 \cdot \mathbf{u}_1 + b\mathbf{u}_2 \cdot \mathbf{u}_1 = a|\mathbf{u}_1|^2 + b(\mathbf{u}_2 \cdot \mathbf{u}_1) = a(1)^2 + b(0) = a$
28. No, $\mathbf{v}_1$ need not equal $\mathbf{v}_2$. For example, $\mathbf{i} + \mathbf{j} \neq \mathbf{i} + 2\mathbf{j}$ but $\mathbf{i} \cdot (\mathbf{i} + \mathbf{j}) = \mathbf{i} \cdot \mathbf{i} + \mathbf{i} \cdot \mathbf{j} = 1 + 0 = 1$ and $\mathbf{i} \cdot (\mathbf{i} + 2\mathbf{j}) = \mathbf{i} \cdot \mathbf{i} + 2\mathbf{i} \cdot \mathbf{j} = 1 + 2 \cdot 0 = 1$.
29. $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \Rightarrow \left(\mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right) \cdot \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right) = \mathbf{u} \cdot \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right) - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right) \cdot \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) (\mathbf{u} \cdot \mathbf{v}) - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right)^2 (\mathbf{v} \cdot \mathbf{v})$
 $= \frac{(\mathbf{u} \cdot \mathbf{v})^2}{|\mathbf{v}|^2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{|\mathbf{v}|^4} |\mathbf{v}|^2 = 0$

30. $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - \mathbf{j} \Rightarrow \text{proj}_{\mathbf{v}} \mathbf{F} = \frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{5}{(\sqrt{10})^2} (3\mathbf{i} - \mathbf{j}) = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$, is the vector parallel to $\mathbf{v}$.

$\mathbf{F} - \text{proj}_{\mathbf{v}} \mathbf{F} = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) - (\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}) = \frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$ is the vector orthogonal to $\mathbf{v}$.

31. $P(x_1, y_1) = P(x_1, \frac{c}{b} - \frac{a}{b}x_1)$ and $Q(x_2, y_2) = Q(x_2, \frac{c}{b} - \frac{a}{b}x_2)$ are any two points P and Q on the line with $b \neq 0$

$$\Rightarrow \vec{PQ} = (x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_1 - x_2)\mathbf{j} \Rightarrow \vec{PQ} \cdot \mathbf{v} = [(x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_1 - x_2)\mathbf{j}] \cdot (a\mathbf{i} + b\mathbf{j}) = a(x_2 - x_1) + b(\frac{a}{b})(x_1 - x_2)$$

$$= 0 \Rightarrow \mathbf{v} \text{ is perpendicular to } \vec{PQ} \text{ for } b \neq 0. \text{ If } b = 0, \text{ then } \mathbf{v} = a\mathbf{i} \text{ is perpendicular to the vertical line } ax = c.$$

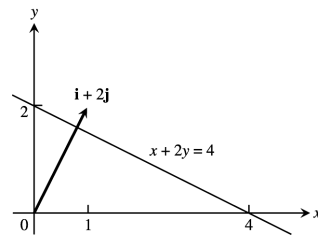
Alternatively, the slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of the line $ax + by = c$ is $-\frac{a}{b}$, so the slopes are negative reciprocals

$\Rightarrow$ the vector $\mathbf{v}$ and the line are perpendicular.

32. The slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of $bx - ay = c$ is $\frac{b}{a}$, provided that $a \neq 0$. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is parallel to the vertical line $bx = c$. In either case, the vector $\mathbf{v}$ is parallel to the line $bx - ay = c$.

33. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ is perpendicular to the line $x + 2y = c$;

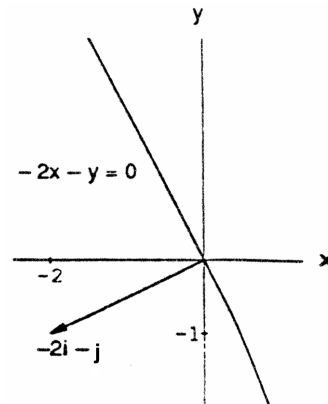
$$P(2, 1) \text{ on the line} \Rightarrow 2 + 2 = c \Rightarrow x + 2y = 4$$



34. $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$ is perpendicular to the line $-2x - y = c$;

$$P(-1, 2) \text{ on the line} \Rightarrow (-2)(-1) - 2 = c$$

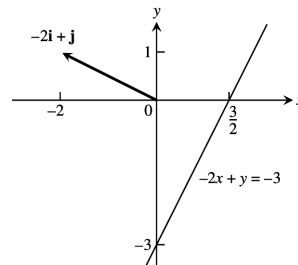
$$\Rightarrow -2x - y = 0$$



35. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ is perpendicular to the line $-2x + y = c$;

$$P(-2, -7) \text{ on the line} \Rightarrow (-2)(-2) - 7 = c$$

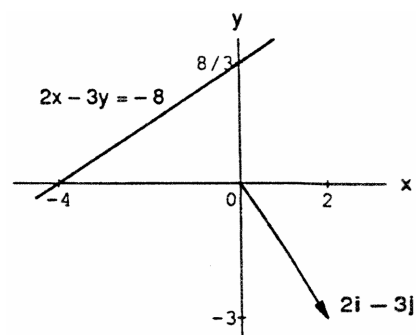
$$\Rightarrow -2x + y = -3$$



36. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ is perpendicular to the line $2x - 3y = c$;

$$P(11, 10) \text{ on the line} \Rightarrow (2)(11) - (3)(10) = c$$

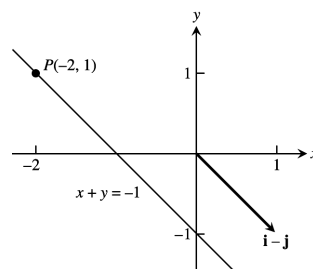
$$\Rightarrow 2x - 3y = -8$$



37. $\mathbf{v} = \mathbf{i} - \mathbf{j}$ is parallel to the line $-x - y = c$;

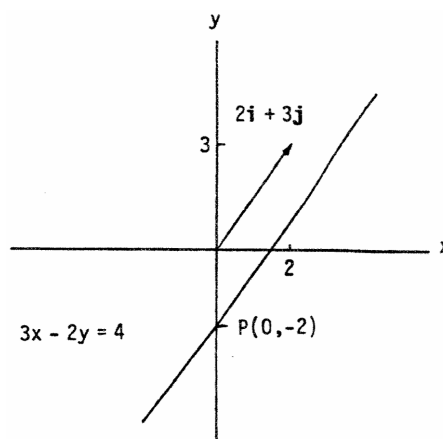
$$P(-2, 1) \text{ on the line} \Rightarrow -(-2) - 1 = c \Rightarrow -x - y = 1$$

$$\text{or } x + y = -1.$$



38. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ is parallel to the line $3x - 2y = c$;

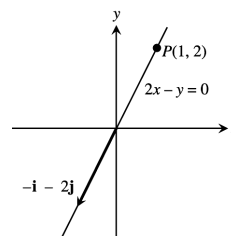
$$P(0, -2) \text{ on the line} \Rightarrow 0 - 2(-2) = c \Rightarrow 3x - 2y = 4$$



39. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x + y = c$;

$$P(1, 2) \text{ on the line} \Rightarrow -2(1) + 2 = c \Rightarrow -2x + y = 0$$

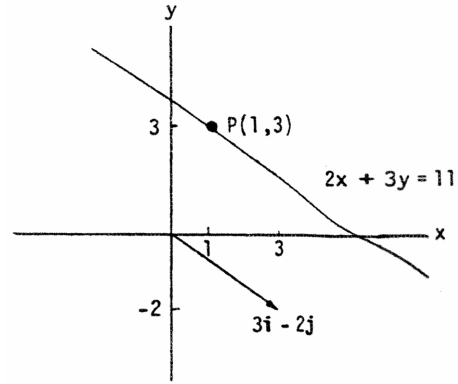
$$\text{or } 2x - y = 0.$$



- 40.
- $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$
- is parallel to the line
- $-2x - 3y = c$
- ;

$$P(1, 3) \text{ on the line} \Rightarrow (-2)(1) - (3)(3) = c$$

$$\Rightarrow -2x - 3y = -11 \text{ or } 2x + 3y = 11$$



- 41.
- $P(0, 0)$
- ,
- $Q(1, 1)$
- and
- $\mathbf{F} = 5\mathbf{j} \Rightarrow \vec{PQ} = \mathbf{i} + \mathbf{j}$
- and
- $\mathbf{W} = \mathbf{F} \cdot \vec{PQ} = (5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) = 5 \text{ N} \cdot \text{m} = 5 \text{ J}$

- 42.
- $\mathbf{W} = |\mathbf{F}| (\text{distance}) \cos \theta = (602,148 \text{ N})(605 \text{ km})(\cos 0) = 364,299,540 \text{ N} \cdot \text{km} = (364,299,540)(1000) \text{ N} \cdot \text{m}$
-
- $= 3.6429954 \times 10^{11} \text{ J}$

- 43.
- $\mathbf{W} = |\mathbf{F}| |\vec{PQ}| \cos \theta = (200)(20)(\cos 30^\circ) = 2000\sqrt{3} = 3464.10 \text{ N} \cdot \text{m} = 3464.10 \text{ J}$

- 44.
- $\mathbf{W} = |\mathbf{F}| |\vec{PQ}| \cos \theta = (1000)(5280)(\cos 60^\circ) = 2,640,000 \text{ ft} \cdot \text{lb}$

In Exercises 45-50 we use the fact that $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line $ax + by = c$.

- 45.
- $\mathbf{n}_1 = 3\mathbf{i} + \mathbf{j}$
- and
- $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{6-1}{\sqrt{10}\sqrt{5}} \right) = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) = \frac{\pi}{4}$

- 46.
- $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$
- and
- $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{-3+1}{\sqrt{4}\sqrt{4}} \right) = \cos^{-1} \left(-\frac{1}{2} \right) = \frac{2\pi}{3}$

- 47.
- $\mathbf{n}_1 = \sqrt{3}\mathbf{i} - \mathbf{j}$
- and
- $\mathbf{n}_2 = \mathbf{i} - \sqrt{3}\mathbf{j} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{4}\sqrt{4}} \right) = \cos^{-1} \left(\frac{\sqrt{3}}{2} \right) = \frac{\pi}{6}$

- 48.
- $\mathbf{n}_1 = \mathbf{i} + \sqrt{3}\mathbf{j}$
- and
- $\mathbf{n}_2 = (1 - \sqrt{3})\mathbf{i} + (1 + \sqrt{3})\mathbf{j} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right)$
-
- $= \cos^{-1} \left(\frac{1 - \sqrt{3} + \sqrt{3} + 3}{\sqrt{1+3}\sqrt{1-2\sqrt{3}+3+1+2\sqrt{3}+3}} \right) = \cos^{-1} \left(\frac{4}{2\sqrt{8}} \right) = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) = \frac{\pi}{4}$

- 49.
- $\mathbf{n}_1 = 3\mathbf{i} - 4\mathbf{j}$
- and
- $\mathbf{n}_2 = \mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{3-4}{\sqrt{25}\sqrt{2}} \right) = \cos^{-1} \left(\frac{7}{5\sqrt{2}} \right) \approx 0.14 \text{ rad}$

- 50.
- $\mathbf{n}_1 = 12\mathbf{i} + 5\mathbf{j}$
- and
- $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{24-10}{\sqrt{169}\sqrt{8}} \right) = \cos^{-1} \left(\frac{14}{26\sqrt{2}} \right) \approx 1.18 \text{ rad}$

12.4 THE CROSS PRODUCT

- 1.
- $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 3 \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \right) \Rightarrow \text{length} = 3 \text{ and the direction is } \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k};$
-
- $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -3 \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \right) \Rightarrow \text{length} = 3 \text{ and the direction is } -\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

$$2. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 5(\mathbf{k}) \Rightarrow \text{length} = 5 \text{ and the direction is } \mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -5(\mathbf{k}) \Rightarrow \text{length} = 5 \text{ and the direction is } -\mathbf{k}$$

$$3. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \mathbf{0} \Rightarrow \text{length} = 0 \text{ and has no direction}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \mathbf{0} \Rightarrow \text{length} = 0 \text{ and has no direction}$$

$$4. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \Rightarrow \text{length} = 0 \text{ and has no direction}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \mathbf{0} \Rightarrow \text{length} = 0 \text{ and has no direction}$$

$$5. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = -6(\mathbf{k}) \Rightarrow \text{length} = 6 \text{ and the direction is } -\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 6(\mathbf{k}) \Rightarrow \text{length} = 6 \text{ and the direction is } \mathbf{k}$$

$$6. \mathbf{u} \times \mathbf{v} = (\mathbf{i} \times \mathbf{j}) \times (\mathbf{j} \times \mathbf{k}) = \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \Rightarrow \text{length} = 1 \text{ and the direction is } \mathbf{j}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -\mathbf{j} \Rightarrow \text{length} = 1 \text{ and the direction is } -\mathbf{j}$$

$$7. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i} - 12\mathbf{k} \Rightarrow \text{length} = 6\sqrt{5} \text{ and the direction is } \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$$

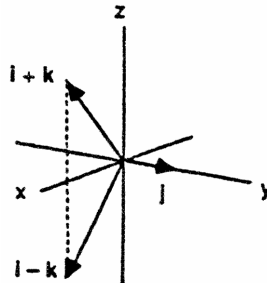
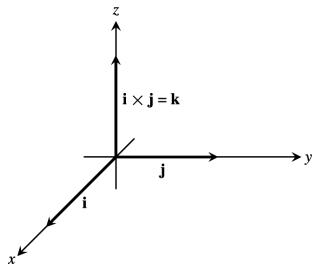
$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(6\mathbf{i} - 12\mathbf{k}) \Rightarrow \text{length} = 6\sqrt{5} \text{ and the direction is } -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$$

$$8. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow \text{length} = 2\sqrt{3} \text{ and the direction is } -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

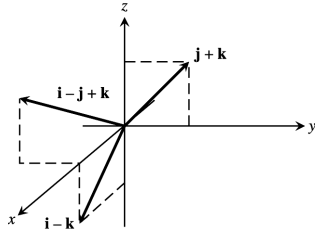
$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \Rightarrow \text{length} = 2\sqrt{3} \text{ and the direction is } \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$$

$$9. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$$

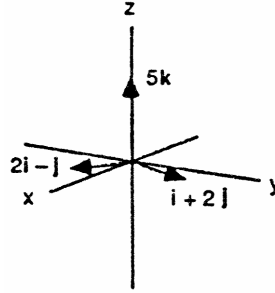
$$10. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{k}$$



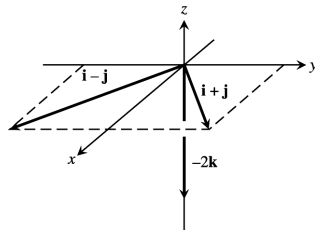
$$11. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$



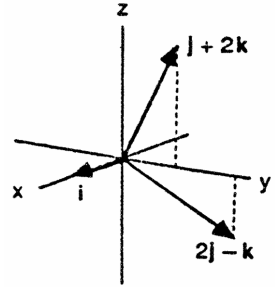
$$12. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 5\mathbf{k}$$



$$13. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2\mathbf{k}$$



$$14. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - \mathbf{k}$$



$$15. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{64 + 16 + 16} = 2\sqrt{6}$$

$$(b) \mathbf{u} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{1}{\sqrt{6}} (2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$16. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{16 + 16 + 4} = 3$$

$$(b) \mathbf{u} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{1}{3} (2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

$$17. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} \Rightarrow \text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{1 + 1} = \frac{\sqrt{2}}{2}$$

$$(b) \mathbf{u} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{1}{\sqrt{2}} (-\mathbf{i} + \mathbf{j}) = -\frac{1}{\sqrt{2}} (\mathbf{i} - \mathbf{j})$$

$$18. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{4 + 9 + 1} = \frac{\sqrt{14}}{2}$$

$$(b) \mathbf{u} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{1}{\sqrt{14}} (2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

19. If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

$(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ and $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ which all have the same absolute value, since the

interchanging of two rows in a determinant does not change its absolute value $\Rightarrow$ the volume is

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

20. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 4$ (for details about verification, see Exercise 19)

21. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = |-7| = 7$ (for details about verification, see Exercise 19)

22. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 8$ (for details about verification, see Exercise 19)

23. (a) $\mathbf{u} \cdot \mathbf{v} = -6$, $\mathbf{u} \cdot \mathbf{w} = -81$, $\mathbf{v} \cdot \mathbf{w} = 18 \Rightarrow$ none are perpendicular

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = \mathbf{0}$, $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix} \neq \mathbf{0}$
 $\Rightarrow \mathbf{u}$ and $\mathbf{w}$ are parallel

24. (a) $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u} \times \mathbf{w} = \mathbf{0}$, $\mathbf{u} \cdot \mathbf{r} = -3\pi$, $\mathbf{v} \cdot \mathbf{w} = 0$, $\mathbf{v} \cdot \mathbf{r} = 0$, $\mathbf{w} \cdot \mathbf{r} = 0 \Rightarrow \mathbf{u} \perp \mathbf{v}$, $\mathbf{u} \perp \mathbf{w}$, $\mathbf{v} \perp \mathbf{w}$, $\mathbf{v} \perp \mathbf{r}$ and $\mathbf{w} \perp \mathbf{r}$

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} = \mathbf{0}$
 $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{v} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} \neq \mathbf{0}$, $\mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} \neq \mathbf{0}$
 $\Rightarrow \mathbf{u}$ and $\mathbf{r}$ are parallel

25. $|\vec{\mathbf{PQ}} \times \mathbf{F}| = |\vec{\mathbf{PQ}}| |\mathbf{F}| \sin(60^\circ) = \frac{2}{3} \cdot 30 \cdot \frac{\sqrt{3}}{2} \text{ ft} \cdot \text{lb} = 10\sqrt{3} \text{ ft} \cdot \text{lb}$

26. $|\vec{\mathbf{PQ}} \times \mathbf{F}| = |\vec{\mathbf{PQ}}| |\mathbf{F}| \sin(135^\circ) = \frac{2}{3} \cdot 30 \cdot \frac{\sqrt{2}}{2} \text{ ft} \cdot \text{lb} = 10\sqrt{2} \text{ ft} \cdot \text{lb}$

27. (a) true, $|\mathbf{u}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

(b) not always true, $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

(c) true, $\mathbf{u} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ 0 & 0 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$ and $\mathbf{0} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

$$(d) \text{ true, } \mathbf{u} \times (-\mathbf{u}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ -u_1 & -u_2 & -u_3 \end{vmatrix} = (-u_2u_3 + u_2u_3)\mathbf{i} - (-u_1u_3 + u_1u_3)\mathbf{j} + (-u_1u_2 + u_1u_2)\mathbf{k} = \mathbf{0}$$

(e) not always true, $\mathbf{i} \times \mathbf{j} = \mathbf{k} \neq -\mathbf{k} = \mathbf{j} \times \mathbf{i}$ for example

(f) true, distributive property of the cross product

(g) true, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{v}) = \mathbf{u} \cdot \mathbf{0} = 0$

(h) true, the volume of a parallelepiped with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ along the three edges is the same whether the plane containing $\mathbf{u}$ and $\mathbf{v}$ or the plane containing $\mathbf{v}$ and $\mathbf{w}$ is used as the base plane, and the dot product is commutative.

$$28. (a) \text{ true, } \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v} \cdot \mathbf{u}$$

$$(b) \text{ true, } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -(\mathbf{v} \times \mathbf{u})$$

$$(c) \text{ true, } (-\mathbf{u}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -u_1 & -u_2 & -u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -(\mathbf{u} \times \mathbf{v})$$

$$(d) \text{ true, } (c\mathbf{u}) \cdot \mathbf{v} = (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 = u_1(cv_1) + u_2(cv_2) + u_3(cv_3) = \mathbf{u} \cdot (c\mathbf{v}) = c(u_1v_1 + u_2v_2 + u_3v_3) = c(\mathbf{u} \cdot \mathbf{v})$$

$$(e) \text{ true, } c(\mathbf{u} \times \mathbf{v}) = c \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (c\mathbf{u}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ cv_1 & cv_2 & cv_3 \end{vmatrix} = \mathbf{u} \times (c\mathbf{v})$$

$$(f) \text{ true, } \mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = (\sqrt{u_1^2 + u_2^2 + u_3^2})^2 = |\mathbf{u}|^2$$

$$(g) \text{ true, } (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = \mathbf{0} \cdot \mathbf{u} = 0$$

$$(h) \text{ true, } \mathbf{u} \times \mathbf{v} \perp \mathbf{u} \text{ and } \mathbf{u} \times \mathbf{v} \perp \mathbf{v} \Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$$

$$29. (a) \text{ proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \quad (b) (\mathbf{u} \times \mathbf{v}) \quad (c) ((\mathbf{u} \times \mathbf{v}) \times \mathbf{w}) \quad (d) |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

$$(e) (\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) \quad (f) |\mathbf{u}| \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$30. (\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}; \mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}. \text{ The cross product is not associative.}$$

31. (a) yes, $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w}$ are both vectors

(b) no, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is a scalar

(c) yes, $\mathbf{u}$ and $\mathbf{u} \times \mathbf{w}$ are both vectors

(d) no, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is a scalar

32. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is parallel to a vector in the plane of $\mathbf{u}$ and $\mathbf{v}$ which means it lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$.

The situation is degenerate if $\mathbf{u}$ and $\mathbf{v}$ are parallel so $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and the vectors do not determine a plane.

Similar reasoning shows that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$ provided $\mathbf{v}$ and $\mathbf{w}$ are nonparallel.

33. No, $\mathbf{v}$ need not equal $\mathbf{w}$. For example, $\mathbf{i} + \mathbf{j} \neq -\mathbf{i} + \mathbf{j}$, but $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$ and $\mathbf{i} \times (-\mathbf{i} + \mathbf{j}) = \mathbf{i} \times (-\mathbf{i}) + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$.

34. Yes. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$ and $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$. Suppose now that $\mathbf{v} \neq \mathbf{w}$.

Then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$ implies that $\mathbf{v} - \mathbf{w} = k\mathbf{u}$ for some real number $k \neq 0$. This in turn implies that

$\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot (k\mathbf{u}) = k|\mathbf{u}|^2 = 0$, which implies that $\mathbf{u} = \mathbf{0}$. Since $\mathbf{u} \neq \mathbf{0}$, it cannot be true that $\mathbf{v} \neq \mathbf{w}$, so $\mathbf{v} = \mathbf{w}$.

$$35. \vec{AB} = -\mathbf{i} + \mathbf{j} \text{ and } \vec{AD} = -\mathbf{i} - \mathbf{j} \Rightarrow \vec{AB} \times \vec{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 2\mathbf{k} \Rightarrow \text{area} = |\vec{AB} \times \vec{AD}| = 2$$

$$36. \vec{AB} = 7\mathbf{i} + 3\mathbf{j} \text{ and } \vec{AD} = 2\mathbf{i} + 5\mathbf{j} \Rightarrow \vec{AB} \times \vec{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 29\mathbf{k} \Rightarrow \text{area} = |\vec{AB} \times \vec{AD}| = 29$$

$$37. \vec{AB} = 3\mathbf{i} - 2\mathbf{j} \text{ and } \vec{AD} = 5\mathbf{i} + \mathbf{j} \Rightarrow \vec{AB} \times \vec{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 5 & 1 & 0 \end{vmatrix} = 13\mathbf{k} \Rightarrow \text{area} = |\vec{AB} \times \vec{AD}| = 13$$

$$38. \vec{AB} = 7\mathbf{i} - 4\mathbf{j} \text{ and } \vec{AD} = 2\mathbf{i} + 5\mathbf{j} \Rightarrow \vec{AB} \times \vec{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 43\mathbf{k} \Rightarrow \text{area} = |\vec{AB} \times \vec{AD}| = 43$$

$$39. \vec{AB} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \text{ and } \vec{DC} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \Rightarrow \vec{AB} \text{ is parallel to } \vec{DC}; \vec{BC} = 2\mathbf{i} - \mathbf{j} \text{ and } \vec{AD} = 2\mathbf{i} - \mathbf{j} \Rightarrow \vec{BC} \text{ is parallel to } \vec{AD}. \vec{AB} \times \vec{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 2 & -1 & 0 \end{vmatrix} = 4\mathbf{i} + 8\mathbf{j} - 7\mathbf{k} \Rightarrow \text{area} = |\vec{AB} \times \vec{BC}| = \sqrt{129}$$

$$40. \vec{AC} = \mathbf{i} + 4\mathbf{j} \text{ and } \vec{DB} = \mathbf{i} + 4\mathbf{j} \Rightarrow \vec{AC} \text{ is parallel to } \vec{DB}; \vec{AD} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \text{ and } \vec{CB} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \Rightarrow \vec{AD} \text{ is parallel to } \vec{CB}. \vec{AC} \times \vec{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ -1 & 3 & 3 \end{vmatrix} = 12\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} \Rightarrow \text{area} = |\vec{AC} \times \vec{AD}| = \sqrt{202}$$

$$41. \vec{AB} = -2\mathbf{i} + 3\mathbf{j} \text{ and } \vec{AC} = 3\mathbf{i} + \mathbf{j} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} = -11\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{11}{2}$$

$$42. \vec{AB} = 4\mathbf{i} + 4\mathbf{j} \text{ and } \vec{AC} = 3\mathbf{i} + 2\mathbf{j} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} = -4\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = 2$$

$$43. \vec{AB} = 6\mathbf{i} - 5\mathbf{j} \text{ and } \vec{AC} = 11\mathbf{i} - 5\mathbf{j} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 0 \\ 11 & -5 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{25}{2}$$

$$44. \vec{AB} = 16\mathbf{i} - 5\mathbf{j} \text{ and } \vec{AC} = 4\mathbf{i} + 4\mathbf{j} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & -5 & 0 \\ 4 & 4 & 0 \end{vmatrix} = 84\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = 42$$

$$45. \vec{AB} = -\mathbf{i} + 2\mathbf{j} \text{ and } \vec{AC} = -\mathbf{i} - \mathbf{k} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{3}{2}$$

$$46. \vec{AB} = -\mathbf{i} + \mathbf{j} - \mathbf{k} \text{ and } \vec{AC} = 3\mathbf{i} + 3\mathbf{k} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 3 & 0 & 3 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{3\sqrt{2}}{2}$$

$$47. \vec{AB} = -\mathbf{i} + 2\mathbf{j} \text{ and } \vec{AC} = \mathbf{j} - 2\mathbf{k} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & 1 & -2 \end{vmatrix} = -4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{\sqrt{21}}{2}$$

$$48. \vec{AB} = \mathbf{i} + 2\mathbf{j}, \vec{AC} = -3\mathbf{j} + 2\mathbf{k} \text{ and } \vec{AD} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} \Rightarrow (\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix} = 5$$

$$\Rightarrow \text{volume} = \left| (\vec{AB} \times \vec{AC}) \cdot \vec{AD} \right| = 5$$

$$49. \text{ If } \mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} \text{ and } \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}, \text{ then } \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \text{ and the triangle's area is}$$

$$\frac{1}{2} |\mathbf{A} \times \mathbf{B}| = \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \text{ The applicable sign is (+) if the acute angle from } \mathbf{A} \text{ to } \mathbf{B} \text{ runs counterclockwise}$$

in the xy -plane, and $(-)$ if it runs clockwise, because the area must be a nonnegative number.

$$50. \text{ If } \mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}, \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}, \text{ and } \mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}, \text{ then the area of the triangle is } \frac{1}{2} |\vec{AB} \times \vec{AC}|. \text{ Now,}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix} \mathbf{k} \Rightarrow \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2} |(b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)| = \frac{1}{2} |a_1(b_2 - c_2) + a_2(c_1 - b_1) + (b_1c_2 - c_1b_2)|$$

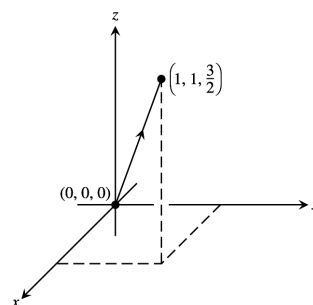
$$= \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}. \text{ The applicable sign ensures the area formula gives a nonnegative number.}$$

12.5 LINES AND PLANES IN SPACE

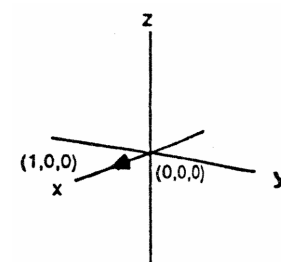
1. The direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $P(3, -4, -1) \Rightarrow x = 3 + t, y = -4 + t, z = -1 + t$
2. The direction $\vec{PQ} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $P(1, 2, -1) \Rightarrow x = 1 - 2t, y = 2 - 2t, z = -1 + 2t$
3. The direction $\vec{PQ} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$ and $P(-2, 0, 3) \Rightarrow x = -2 + 5t, y = 5t, z = 3 - 5t$
4. The direction $\vec{PQ} = -\mathbf{j} - \mathbf{k}$ and $P(1, 2, 0) \Rightarrow x = 1, y = 2 - t, z = -t$
5. The direction $2\mathbf{j} + \mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = 0, y = 2t, z = t$
6. The direction $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $P(3, -2, 1) \Rightarrow x = 3 + 2t, y = -2 - t, z = 1 + 3t$
7. The direction $\mathbf{k}$ and $P(1, 1, 1) \Rightarrow x = 1, y = 1, z = 1 + t$
8. The direction $3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$ and $P(2, 4, 5) \Rightarrow x = 2 + 3t, y = 4 + 7t, z = 5 - 5t$
9. The direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $P(0, -7, 0) \Rightarrow x = t, y = -7 + 2t, z = 2t$
10. The direction is $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $P(2, 3, 0) \Rightarrow x = 2 - 2t, y = 3 + 4t, z = -2t$
11. The direction $\mathbf{i}$ and $P(0, 0, 0) \Rightarrow x = t, y = 0, z = 0$

12. The direction $\mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = 0, y = 0, z = t$

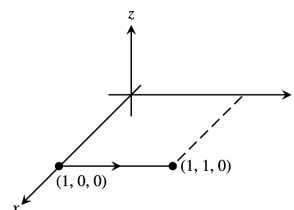
13. The direction $\vec{PQ} = \mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = t, y = t, z = \frac{3}{2}t$, where $0 \leq t \leq 1$



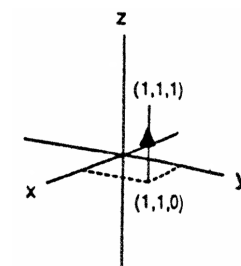
14. The direction $\vec{PQ} = \mathbf{i}$ and $P(0, 0, 0) \Rightarrow x = t, y = 0, z = 0$, where $0 \leq t \leq 1$



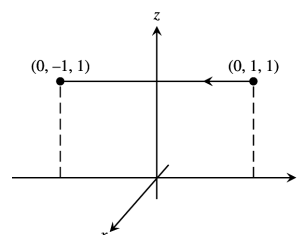
15. The direction $\vec{PQ} = \mathbf{j}$ and $P(1, 1, 0) \Rightarrow x = 1, y = 1 + t, z = 0$, where $-1 \leq t \leq 0$



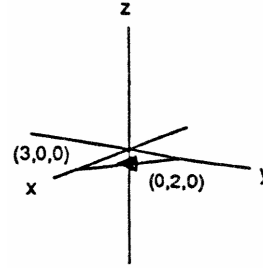
16. The direction $\vec{PQ} = \mathbf{k}$ and $P(1, 1, 0) \Rightarrow x = 1, y = 1, z = t$, where $0 \leq t \leq 1$



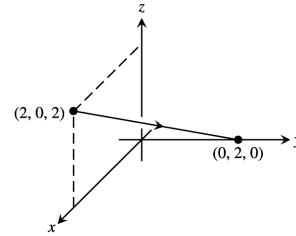
17. The direction $\vec{PQ} = -2\mathbf{j}$ and $P(0, 1, 1) \Rightarrow x = 0, y = 1 - 2t, z = 1$, where $0 \leq t \leq 1$



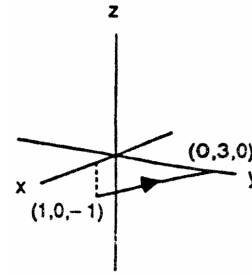
18. The direction $\vec{PQ} = 3\mathbf{i} - 2\mathbf{j}$ and $P(0, 2, 0) \Rightarrow x = 3t$,
 $y = 2 - 2t$, $z = 0$, where $0 \leq t \leq 1$



19. The direction $\vec{PQ} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $P(2, 0, 2)$
 $\Rightarrow x = 2 - 2t$, $y = 2t$, $z = 2 - 2t$, where $0 \leq t \leq 1$



20. The direction $\vec{PQ} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $P(1, 0, -1)$
 $\Rightarrow x = 1 - t$, $y = 3t$, $z = -1 + t$, where $0 \leq t \leq 1$



21. $3(x - 0) + (-2)(y - 2) + (-1)(z + 1) = 0 \Rightarrow 3x - 2y - z = -3$

22. $3(x - 1) + (1)(y + 1) + (1)(z - 3) = 0 \Rightarrow 3x + y + z = 5$

23. $\vec{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\vec{PS} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \Rightarrow \vec{PQ} \times \vec{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ is normal to the plane
 $\Rightarrow 7(x - 2) + (-5)(y - 0) + (-4)(z - 2) = 0 \Rightarrow 7x - 5y - 4z = 6$

24. $\vec{PQ} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\vec{PS} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \vec{PQ} \times \vec{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to the plane
 $\Rightarrow (-1)(x - 1) + (-3)(y - 5) + (1)(z - 7) = 0 \Rightarrow x + 3y - z = 9$

25. $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $P(2, 4, 5) \Rightarrow (1)(x - 2) + (3)(y - 4) + (4)(z - 5) = 0 \Rightarrow x + 3y + 4z = 34$

26. $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $P(1, -2, 1) \Rightarrow (1)(x - 1) + (-2)(y + 2) + (1)(z - 1) = 0 \Rightarrow x - 2y + z = 6$

27. $\begin{cases} x = 2t + 1 = s + 2 \\ y = 3t + 2 = 2s + 4 \end{cases} \Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 4t - 2s = 2 \\ 3t - 2s = 2 \end{cases} \Rightarrow t = 0 \text{ and } s = -1; \text{ then } z = 4t + 3 = -4s - 1$
 $\Rightarrow 4(0) + 3 = (-4)(-1) - 1$ is satisfied $\Rightarrow$ the lines intersect when $t = 0$ and $s = -1 \Rightarrow$ the point of intersection is $x = 1$, $y = 2$, and $z = 3$ or $P(1, 2, 3)$. A vector normal to the plane determined by these lines is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\mathbf{i} + 12\mathbf{j} + \mathbf{k}, \text{ where } \mathbf{n}_1 \text{ and } \mathbf{n}_2 \text{ are directions of the lines} \Rightarrow \text{the plane}$$

containing the lines is represented by $(-20)(x-1) + (12)(y-2) + (1)(z-3) = 0 \Rightarrow -20x + 12y + z = 7$.

$$28. \begin{cases} x = t = 2s + 2 \\ y = -t + 2 = s + 3 \end{cases} \Rightarrow \begin{cases} t - 2s = 2 \\ -t - s = 1 \end{cases} \Rightarrow s = -1 \text{ and } t = 0; \text{ then } z = t + 1 = 5s + 6 \Rightarrow 0 + 1 = 5(-1) + 6$$

is satisfied $\Rightarrow$ the lines do intersect when $s = -1$ and $t = 0 \Rightarrow$ the point of intersection is $x = 0, y = 2$ and $z = 1$

$$\text{or } P(0, 2, 1). \text{ A vector normal to the plane determined by these lines is } \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k},$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by

$$(-6)(x-0) + (-3)(y-2) + (3)(z-1) = 0 \Rightarrow 6x + 3y - 3z = 3.$$

29. The cross product of $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ has the same direction as the normal to the plane

$$\Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\mathbf{j} + 6\mathbf{k}. \text{ Select a point on either line, such as } P(-1, 2, 1). \text{ Since the lines are given}$$

to intersect, the desired plane is $0(x+1) + 6(y-2) + 6(z-1) = 0 \Rightarrow 6y + 6z = 18 \Rightarrow y + z = 3$.

30. The cross product of $\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$ has the same direction as the normal to the plane

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}. \text{ Select a point on either line, such as } P(0, 3, -2). \text{ Since the lines are}$$

given to intersect, the desired plane is $(-2)(x-0) + (-2)(y-3) + (4)(z+2) = 0 \Rightarrow -2x - 2y + 4z = -14$
 $\Rightarrow x + y - 2z = 7$.

$$31. \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \text{ is a vector in the direction of the line of intersection of the planes}$$

$\Rightarrow 3(x-2) + (-3)(y-1) + 3(z+1) = 0 \Rightarrow 3x - 3y + 3z = 0 \Rightarrow x - y + z = 0$ is the desired plane containing $P_0(2, 1, -1)$

$$32. \text{ A vector normal to the desired plane is } \vec{P_1P_2} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}; \text{ choosing } P_1(1, 2, 3) \text{ as a point on}$$

the plane $\Rightarrow (-2)(x-1) + (-12)(y-2) + (-2)(z-3) = 0 \Rightarrow -2x - 12y - 2z = -32 \Rightarrow x + 6y + z = 16$ is the desired plane

$$33. S(0, 0, 12), P(0, 0, 0) \text{ and } \mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow \vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix} = 24\mathbf{i} + 48\mathbf{j} = 24(\mathbf{i} + 2\mathbf{j})$$

$$\Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{24\sqrt{1+4}}{\sqrt{16+4+4}} = \frac{24\sqrt{5}}{\sqrt{24}} = \sqrt{5 \cdot 24} = 2\sqrt{30} \text{ is the distance from } S \text{ to the line}$$

$$34. S(0, 0, 0), P(5, 5, -3) \text{ and } \mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \Rightarrow \vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix} = 13\mathbf{i} - 16\mathbf{j} - 5\mathbf{k}$$

$$\Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{169+256+25}}{\sqrt{9+16+25}} = \frac{\sqrt{450}}{\sqrt{50}} = \sqrt{9} = 3 \text{ is the distance from } S \text{ to the line}$$

35. $S(2, 1, 3)$, $P(2, 1, 3)$ and $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} \Rightarrow \vec{PS} \times \mathbf{v} = \mathbf{0} \Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{0}{\sqrt{40}} = 0$ is the distance from S to the line (i.e., the point S lies on the line)

36. $S(2, 1, -1)$, $P(0, 1, 0)$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$
 $\Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{4+36+16}}{\sqrt{4+4+4}} = \frac{\sqrt{56}}{\sqrt{12}} = \sqrt{\frac{14}{3}}$ is the distance from S to the line

37. $S(3, -1, 4)$, $P(4, 3, -5)$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & 9 \\ -1 & 2 & 3 \end{vmatrix} = -30\mathbf{i} - 6\mathbf{j} - 6\mathbf{k}$
 $\Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{900+36+36}}{\sqrt{1+4+9}} = \frac{\sqrt{972}}{\sqrt{14}} = \frac{\sqrt{486}}{\sqrt{7}} = \frac{\sqrt{81 \cdot 6}}{\sqrt{7}} = \frac{9\sqrt{42}}{7}$ is the distance from S to the line

38. $S(-1, 4, 3)$, $P(10, -3, 0)$ and $\mathbf{v} = 4\mathbf{i} + 4\mathbf{k} \Rightarrow \vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -11 & 7 & 3 \\ 4 & 0 & 4 \end{vmatrix} = 28\mathbf{i} + 56\mathbf{j} - 28\mathbf{k} = 28(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
 $\Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{28\sqrt{1+4+1}}{4\sqrt{1+1}} = 7\sqrt{3}$ is the distance from S to the line

39. $S(2, -3, 4)$, $x + 2y + 2z = 13$ and $P(13, 0, 0)$ is on the plane $\Rightarrow \vec{PS} = -11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
 $\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-11-6+8}{\sqrt{1+4+4}} \right| = \left| \frac{-9}{\sqrt{9}} \right| = 3$

40. $S(0, 0, 0)$, $3x + 2y + 6z = 6$ and $P(2, 0, 0)$ is on the plane $\Rightarrow \vec{PS} = -2\mathbf{i}$ and $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$
 $\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-6}{\sqrt{9+4+36}} \right| = \frac{6}{\sqrt{49}} = \frac{6}{7}$

41. $S(0, 1, 1)$, $4y + 3z = -12$ and $P(0, -3, 0)$ is on the plane $\Rightarrow \vec{PS} = 4\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{j} + 3\mathbf{k}$
 $\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{16+3}{\sqrt{16+9}} \right| = \frac{19}{5}$

42. $S(2, 2, 3)$, $2x + y + 2z = 4$ and $P(2, 0, 0)$ is on the plane $\Rightarrow \vec{PS} = 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
 $\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{2+6}{\sqrt{4+1+4}} \right| = \frac{8}{3}$

43. $S(0, -1, 0)$, $2x + y + 2z = 4$ and $P(2, 0, 0)$ is on the plane $\Rightarrow \vec{PS} = -2\mathbf{i} - \mathbf{j}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
 $\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-4-1+0}{\sqrt{4+1+4}} \right| = \frac{5}{3}$

44. $S(1, 0, -1)$, $-4x + y + z = 4$ and $P(-1, 0, 0)$ is on the plane $\Rightarrow \vec{PS} = 2\mathbf{i} - \mathbf{k}$ and $\mathbf{n} = -4\mathbf{i} + \mathbf{j} + \mathbf{k}$
 $\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-8-1}{\sqrt{16+1+1}} \right| = \frac{9}{\sqrt{18}} = \frac{3\sqrt{2}}{2}$

45. The point $P(1, 0, 0)$ is on the first plane and $S(10, 0, 0)$ is a point on the second plane $\Rightarrow \vec{PS} = 9\mathbf{i}$, and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the first plane $\Rightarrow$ the distance from S to the first plane is $d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$
 $= \left| \frac{9}{\sqrt{1+4+36}} \right| = \frac{9}{\sqrt{41}}$, which is also the distance between the planes.

46. The line is parallel to the plane since $\mathbf{v} \cdot \mathbf{n} = (\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) = 1 + 2 - 3 = 0$. Also the point $S(1, 0, 0)$ when $t = -1$ lies on the line, and the point $P(10, 0, 0)$ lies on the plane $\Rightarrow \vec{PS} = -9\mathbf{i}$. The distance from S to the plane is $d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-9}{\sqrt{1+4+36}} \right| = \frac{9}{\sqrt{41}}$, which is also the distance from the line to the plane.
47. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{2+1}{\sqrt{2}\sqrt{9}} \right) = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) = \frac{\pi}{4}$
48. $\mathbf{n}_1 = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{5-2-3}{\sqrt{27}\sqrt{14}} \right) = \cos^{-1}(0) = \frac{\pi}{2}$
49. $\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{4-4-2}{\sqrt{12}\sqrt{9}} \right) = \cos^{-1} \left(\frac{-1}{3\sqrt{3}} \right) \approx 1.76 \text{ rad}$
50. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{1}{\sqrt{3}\sqrt{1}} \right) \approx 0.96 \text{ rad}$
51. $\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{2+4-1}{\sqrt{9}\sqrt{6}} \right) = \cos^{-1} \left(\frac{5}{3\sqrt{6}} \right) \approx 0.82 \text{ rad}$
52. $\mathbf{n}_1 = 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n}_2 = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{8+18}{\sqrt{25}\sqrt{49}} \right) = \cos^{-1} \left(\frac{26}{35} \right) \approx 0.73 \text{ rad}$
53. $2x - y + 3z = 6 \Rightarrow 2(1-t) - (3t) + 3(1+t) = 6 \Rightarrow -2t + 5 = 6 \Rightarrow t = -\frac{1}{2} \Rightarrow x = \frac{3}{2}, y = -\frac{3}{2}$ and $z = \frac{1}{2} \Rightarrow \left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2} \right)$ is the point
54. $6x + 3y - 4z = -12 \Rightarrow 6(2) + 3(3+2t) - 4(-2-2t) = -12 \Rightarrow 14t + 29 = -12 \Rightarrow t = -\frac{41}{14} \Rightarrow x = 2, y = 3 - \frac{41}{7}$, and $z = -2 + \frac{41}{7} \Rightarrow \left(2, -\frac{20}{7}, \frac{27}{7} \right)$ is the point
55. $x + y + z = 2 \Rightarrow (1+2t) + (1+5t) + (3t) = 2 \Rightarrow 10t + 2 = 2 \Rightarrow t = 0 \Rightarrow x = 1, y = 1$ and $z = 0 \Rightarrow (1, 1, 0)$ is the point
56. $2x - 3z = 7 \Rightarrow 2(-1+3t) - 3(5t) = 7 \Rightarrow -9t - 2 = 7 \Rightarrow t = -1 \Rightarrow x = -1 - 3, y = -2$ and $z = -5 \Rightarrow (-4, -2, -5)$ is the point
57. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j}$, the direction of the desired line; $(1, 1, -1)$ is on both planes $\Rightarrow$ the desired line is $x = 1 - t, y = 1 + t, z = -1$
58. $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$, the direction of the desired line; $(1, 0, 0)$ is on both planes $\Rightarrow$ the desired line is $x = 1 + 14t, y = 2t, z = 15t$
59. $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 6\mathbf{j} + 3\mathbf{k}$, the direction of the desired line; $(4, 3, 1)$ is on both planes $\Rightarrow$ the desired line is $x = 4, y = 3 + 6t, z = 1 + 3t$

60. $\mathbf{n}_1 = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{n}_2 = 4\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -2 & 0 \\ 0 & 4 & -5 \end{vmatrix} = 10\mathbf{i} + 25\mathbf{j} + 20\mathbf{k}$, the direction of the desired line; $(1, -3, 1)$ is on both planes $\Rightarrow$ the desired line is $x = 1 + 10t$, $y = -3 + 25t$, $z = 1 + 20t$

61. L1 & L2: $x = 3 + 2t = 1 + 4s$ and $y = -1 + 4t = 1 + 2s \Rightarrow \begin{cases} 2t - 4s = -2 \\ 4t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 2t - 4s = -2 \\ 2t - s = 1 \end{cases}$
 $\Rightarrow -3s = -3 \Rightarrow s = 1$ and $t = 1 \Rightarrow$ on L1, $z = 1$ and on L2, $z = 1 \Rightarrow$ L1 and L2 intersect at $(5, 3, 1)$.

L2 & L3: The direction of L2 is $\frac{1}{6}(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ which is the same as the direction $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ of L3; hence L2 and L3 are parallel.

L1 & L3: $x = 3 + 2t = 3 + 2r$ and $y = -1 + 4t = 2 + r \Rightarrow \begin{cases} 2t - 2r = 0 \\ 4t - r = 3 \end{cases} \Rightarrow \begin{cases} t - r = 0 \\ 4t - r = 3 \end{cases} \Rightarrow 3t = 3$
 $\Rightarrow t = 1$ and $r = 1 \Rightarrow$ on L1, $z = 2$ while on L3, $z = 0 \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{21}}(2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ while the direction of L3 is $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and neither is a multiple of the other; hence L1 and L3 are skew.

62. L1 & L2: $x = 1 + 2t = 2 - s$ and $y = -1 - t = 3s \Rightarrow \begin{cases} 2t + s = 1 \\ -t - 3s = 1 \end{cases} \Rightarrow -5s = 3 \Rightarrow s = -\frac{3}{5}$ and $t = \frac{4}{5} \Rightarrow$ on L1, $z = \frac{12}{5}$ while on L2, $z = 1 - \frac{3}{5} = \frac{2}{5} \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ while the direction of L2 is $\frac{1}{\sqrt{11}}(-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ and neither is a multiple of the other; hence, L1 and L2 are skew.

L2 & L3: $x = 2 - s = 5 + 2r$ and $y = 3s = 1 - r \Rightarrow \begin{cases} -s - 2r = 3 \\ 3s + r = 1 \end{cases} \Rightarrow 5s = 5 \Rightarrow s = 1$ and $r = -2 \Rightarrow$ on L2, $z = 2$ and on L3, $z = 2 \Rightarrow$ L2 and L3 intersect at $(1, 3, 2)$.

L1 & L3: L1 and L3 have the same direction $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$; hence L1 and L3 are parallel.

63. $x = 2 + 2t$, $y = -4 - t$, $z = 7 + 3t$; $x = -2 - t$, $y = -2 + \frac{1}{2}t$, $z = 1 - \frac{3}{2}t$

64. $1(x - 4) - 2(y - 1) + 1(z - 5) = 0 \Rightarrow x - 4 - 2y + 2 + z - 5 = 0 \Rightarrow x - 2y + z = 7$;
 $-\sqrt{2}(x - 3) + 2\sqrt{2}(y + 2) - \sqrt{2}(z - 0) = 0 \Rightarrow -\sqrt{2}x + 2\sqrt{2}y - \sqrt{2}z = -7\sqrt{2}$

65. $x = 0 \Rightarrow t = -\frac{1}{2}$, $y = -\frac{1}{2}$, $z = -\frac{3}{2} \Rightarrow (0, -\frac{1}{2}, -\frac{3}{2})$; $y = 0 \Rightarrow t = -1$, $x = -1$, $z = -3 \Rightarrow (-1, 0, -3)$; $z = 0 \Rightarrow t = 0$, $x = 1$, $y = -1 \Rightarrow (1, -1, 0)$

66. The line contains $(0, 0, 3)$ and $(\sqrt{3}, 1, 3)$ because the projection of the line onto the xy -plane contains the origin and intersects the positive x -axis at a 30° angle. The direction of the line is $\sqrt{3}\mathbf{i} + \mathbf{j} + 0\mathbf{k} \Rightarrow$ the line in question is $x = \sqrt{3}t$, $y = t$, $z = 3$.

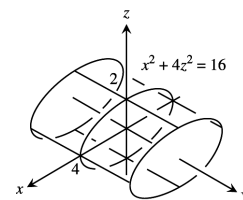
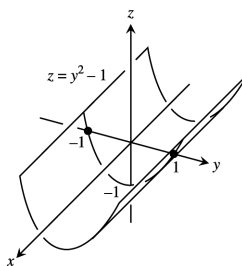
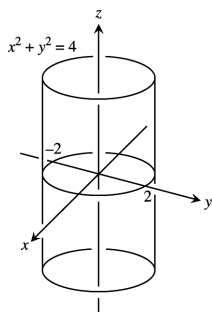
67. With substitution of the line into the plane we have $2(1 - 2t) + (2 + 5t) - (-3t) = 8 \Rightarrow 2 - 4t + 2 + 5t + 3t = 8 \Rightarrow 4t + 4 = 8 \Rightarrow t = 1 \Rightarrow$ the point $(-1, 7, -3)$ is contained in both the line and plane, so they are not parallel.

68. The planes are parallel when either vector $A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}$ or $A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}$ is a multiple of the other or when $(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \times (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}) = \mathbf{0}$. The planes are perpendicular when their normals are perpendicular, or $(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \cdot (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}) = 0$.

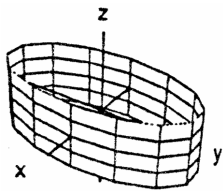
69. There are many possible answers. One is found as follows: eliminate t to get $t = x - 1 = 2 - y = \frac{z-3}{2}$
 $\Rightarrow x - 1 = 2 - y$ and $2 - y = \frac{z-3}{2} \Rightarrow x + y = 3$ and $2y + z = 7$ are two such planes.
70. Since the plane passes through the origin, its general equation is of the form $Ax + By + Cz = 0$. Since it meets the plane M at a right angle, their normal vectors are perpendicular $\Rightarrow 2A + 3B + C = 0$. One choice satisfying this equation is $A = 1$, $B = -1$ and $C = 1 \Rightarrow x - y + z = 0$. Any plane $Ax + By + Cz = 0$ with $2A + 3B + C = 0$ will pass through the origin and be perpendicular to M .
71. The points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ are the x , y , and z intercepts of the plane. Since a , b , and c are all nonzero, the plane must intersect all three coordinate axes and cannot pass through the origin. Thus,
 $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ describes all planes except those through the origin or parallel to a coordinate axis.
72. Yes. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors parallel to the lines, then $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ is perpendicular to the lines.
73. (a) $\vec{EP} = c\vec{EP}_1 \Rightarrow -x_0\mathbf{i} + y\mathbf{j} + z\mathbf{k} = c[(x_1 - x_0)\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}] \Rightarrow -x_0 = c(x_1 - x_0)$, $y = cy_1$ and $z = cz_1$,
 where c is a positive real number
 (b) At $x_1 = 0 \Rightarrow c = 1 \Rightarrow y = y_1$ and $z = z_1$; at $x_1 = x_0 \Rightarrow x_0 = 0$, $y = 0$, $z = 0$; $\lim_{x_0 \rightarrow \infty} c = \lim_{x_0 \rightarrow \infty} \frac{-x_0}{x_1 - x_0}$
 $= \lim_{x_0 \rightarrow \infty} \frac{-1}{-1} = 1 \Rightarrow c \rightarrow 1$ so that $y \rightarrow y_1$ and $z \rightarrow z_1$
74. The plane which contains the triangular plane is $x + y + z = 2$. The line containing the endpoints of the line segment is $x = 1 - t$, $y = 2t$, $z = 2t$. The plane and the line intersect at $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The visible section of the line segment is $\sqrt{(\frac{1}{3})^2 + (\frac{2}{3})^2 + (\frac{2}{3})^2} = 1$ unit in length. The length of the line segment is $\sqrt{1^2 + 2^2 + 2^2} = 3 \Rightarrow \frac{2}{3}$ of the line segment is hidden from view.

12.6 CYLINDERS AND QUADRIC SURFACES

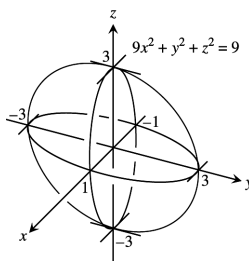
1. d, ellipsoid	2. i, hyperboloid	3. a, cylinder
4. g, cone	5. l, hyperbolic paraboloid	6. e, paraboloid
7. b, cylinder	8. j, hyperboloid	9. k, hyperbolic paraboloid
10. f, paraboloid	11. h, cone	12. c, ellipsoid
13. $x^2 + y^2 = 4$	14. $z = y^2 - 1$	15. $x^2 + 4z^2 = 16$



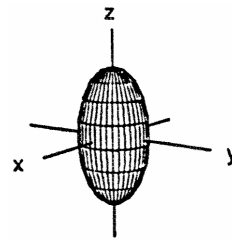
16. $4x^2 + y^2 = 36$



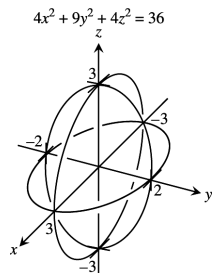
17. $9x^2 + y^2 + z^2 = 9$



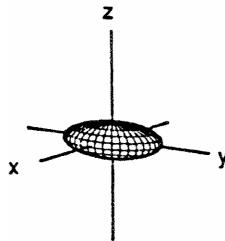
18. $4x^2 + 4y^2 + z^2 = 16$



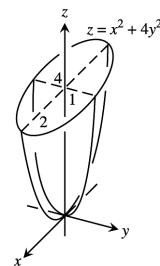
19. $4x^2 + 9y^2 + 4z^2 = 36$



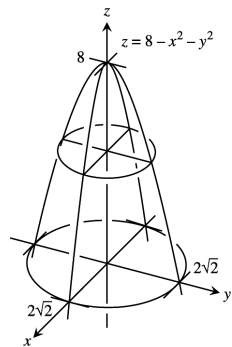
20. $9x^2 + 4y^2 + 36z^2 = 36$



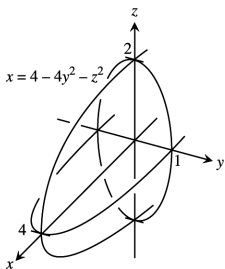
21. $x^2 + 4y^2 = z$



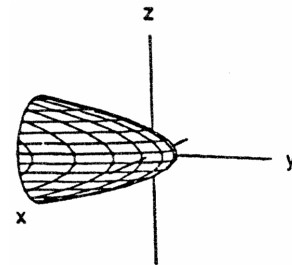
22. $z = 8 - x^2 - y^2$



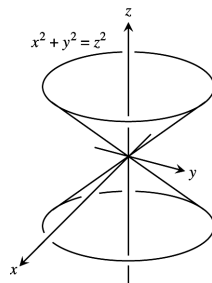
23. $x = 4 - 4y^2 - z^2$



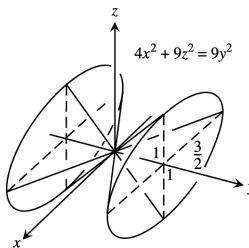
24. $y = 1 - x^2 - z^2$



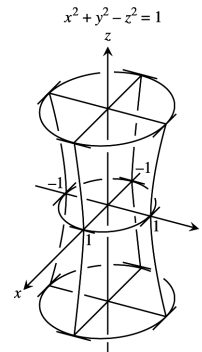
25. $x^2 + y^2 = z^2$



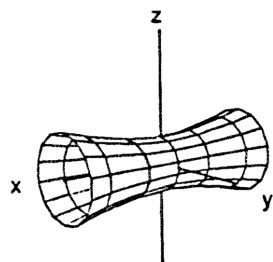
26. $4x^2 + 9z^2 = 9y^2$



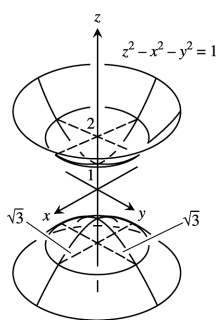
27. $x^2 + y^2 - z^2 = 1$



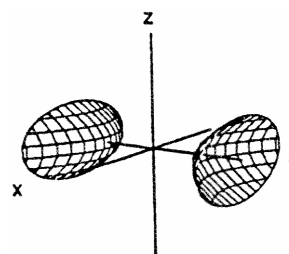
28. $y^2 + z^2 - x^2 = 1$



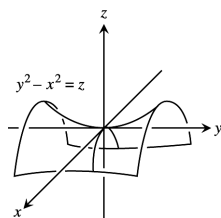
29. $z^2 - x^2 - y^2 = 1$



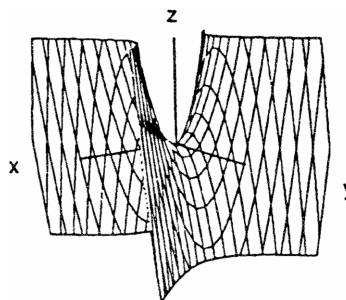
30. $\frac{y^2}{4} - \frac{x^2}{4} - z^2 = 1$



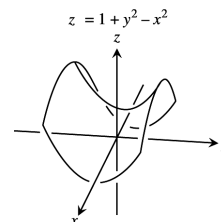
31. $y^2 - x^2 = z$



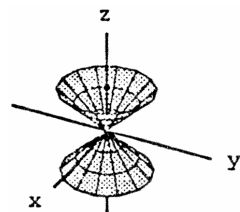
32. $x^2 - y^2 = z$



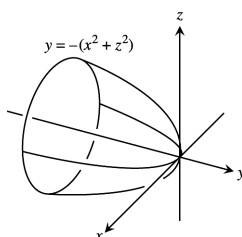
33. $z = 1 + y^2 - x^2$



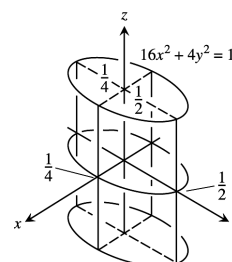
34. $4x^2 + 4y^2 = z^2$



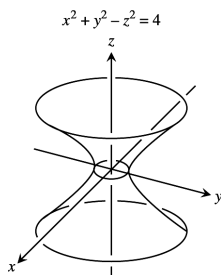
35. $y = -(x^2 + z^2)$



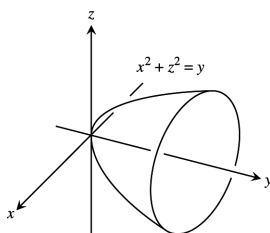
36. $16x^2 + 4y^2 = 1$



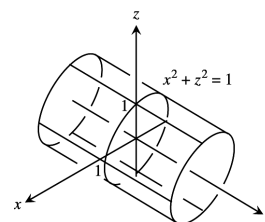
37. $x^2 + y^2 - z^2 = 4$



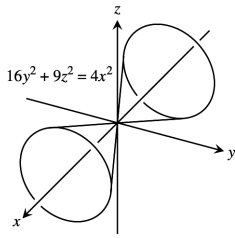
38. $x^2 + z^2 = y$



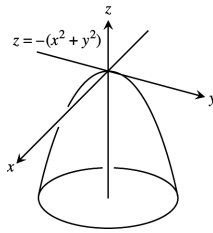
39. $x^2 + z^2 = 1$



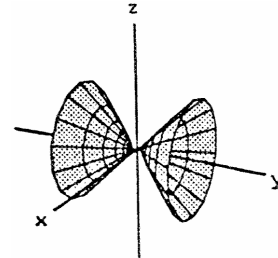
40. $16y^2 + 9z^2 = 4x^2$



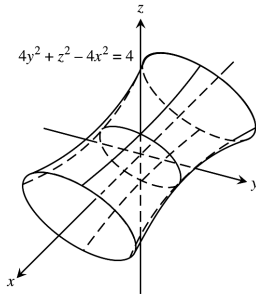
41. $z = -(x^2 + y^2)$



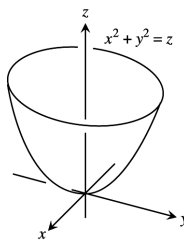
42. $y^2 - x^2 - z^2 = 1$



43. $4y^2 + z^2 - 4x^2 = 4$



44. $x^2 + y^2 = z$



45. (a) If $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ and $z = c$, then $x^2 + \frac{y^2}{4} = \frac{9-c^2}{9} \Rightarrow \frac{x^2}{\left(\frac{9-c^2}{9}\right)} + \frac{y^2}{\left[\frac{4(9-c^2)}{9}\right]} = 1 \Rightarrow A = ab\pi$

$$= \pi \left(\frac{\sqrt{9-c^2}}{3} \right) \left(\frac{2\sqrt{9-c^2}}{3} \right) = \frac{2\pi(9-c^2)}{9}$$

(b) From part (a), each slice has the area $\frac{2\pi(9-z^2)}{9}$, where $-3 \leq z \leq 3$. Thus $V = 2 \int_0^3 \frac{2\pi}{9} (9 - z^2) dz$

$$= \frac{4\pi}{9} \int_0^3 (9 - z^2) dz = \frac{4\pi}{9} \left[9z - \frac{z^3}{3} \right]_0^3 = \frac{4\pi}{9} (27 - 9) = 8\pi$$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{x^2}{\left[\frac{a^2(c^2-z^2)}{c^2}\right]} + \frac{y^2}{\left[\frac{b^2(c^2-z^2)}{c^2}\right]} = 1 \Rightarrow A = \pi \left(\frac{a\sqrt{c^2-z^2}}{c} \right) \left(\frac{b\sqrt{c^2-z^2}}{c} \right)$

$$\Rightarrow V = 2 \int_0^c \frac{\pi ab}{c^2} (c^2 - z^2) dz = \frac{2\pi ab}{c^2} \left[c^2 z - \frac{z^3}{3} \right]_0^c = \frac{2\pi ab}{c^2} \left(\frac{2}{3} c^3 \right) = \frac{4\pi abc}{3}. \text{ Note that if } r = a = b = c, \text{ then } V = \frac{4\pi r^3}{3}, \text{ which is the volume of a sphere.}$$

46. The ellipsoid has the form $\frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{c^2} = 1$. To determine c^2 we note that the point $(0, r, h)$ lies on the surface of the barrel. Thus, $\frac{r^2}{R^2} + \frac{h^2}{c^2} = 1 \Rightarrow c^2 = \frac{h^2 R^2}{R^2 - r^2}$. We calculate the volume by the disk method:

$$\begin{aligned} V &= \pi \int_{-h}^h y^2 dz. \text{ Now, } \frac{y^2}{R^2} + \frac{z^2}{c^2} = 1 \Rightarrow y^2 = R^2 \left(1 - \frac{z^2}{c^2} \right) = R^2 \left[1 - \frac{z^2 (R^2 - r^2)}{h^2 R^2} \right] = R^2 - \left(\frac{R^2 - r^2}{h^2} \right) z^2 \\ \Rightarrow V &= \pi \int_{-h}^h \left[R^2 - \left(\frac{R^2 - r^2}{h^2} \right) z^2 \right] dz = \pi \left[R^2 z - \frac{1}{3} \left(\frac{R^2 - r^2}{h^2} \right) z^3 \right]_{-h}^h = 2\pi \left[R^2 h - \frac{1}{3} (R^2 - r^2) h \right] = 2\pi \left(\frac{2R^2 h}{3} + \frac{r^2 h}{3} \right) \\ &= \frac{4}{3} \pi R^2 h + \frac{2}{3} \pi r^2 h, \text{ the volume of the barrel. If } r = R, \text{ then } V = 2\pi R^2 h \text{ which is the volume of a cylinder of radius } R \text{ and height } 2h. \text{ If } r = 0 \text{ and } h = R, \text{ then } V = \frac{4}{3} \pi R^3 \text{ which is the volume of a sphere.} \end{aligned}$$

47. We calculate the volume by the slicing method, taking slices parallel to the xy -plane. For fixed z , $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ gives the ellipse $\frac{x^2}{\left(\frac{za^2}{c}\right)} + \frac{y^2}{\left(\frac{zb^2}{c}\right)} = 1$. The area of this ellipse is $\pi \left(a\sqrt{\frac{z}{c}} \right) \left(b\sqrt{\frac{z}{c}} \right) = \frac{\pi abz}{c}$ (see Exercise 45a). Hence

$$\text{the volume is given by } V = \int_0^h \frac{\pi abz}{c} dz = \left[\frac{\pi abz^2}{2c} \right]_0^h = \frac{\pi abh^2}{c}.$$

$$\text{Now the area of the elliptic base when } z = h \text{ is } A = \frac{\pi abh}{c}, \text{ as determined previously. Thus, } V = \frac{\pi abh^2}{c} = \frac{1}{2} \left(\frac{\pi abh}{c} \right) h = \frac{1}{2} (\text{base})(\text{altitude}), \text{ as claimed.}$$

48. (a) For each fixed value of z , the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ results in a cross-sectional ellipse

$$\left[\frac{x^2}{a^2(c^2 + z^2)} \right] + \left[\frac{y^2}{b^2(c^2 + z^2)} \right] = 1. \text{ The area of the cross-sectional ellipse (see Exercise 45a) is}$$

$$A(z) = \pi \left(\frac{a}{c} \sqrt{c^2 + z^2} \right) \left(\frac{b}{c} \sqrt{c^2 + z^2} \right) = \frac{\pi ab}{c^2} (c^2 + z^2). \text{ The volume of the solid by the method of slices is}$$

$$V = \int_0^h A(z) dz = \int_0^h \frac{\pi ab}{c^2} (c^2 + z^2) dz = \frac{\pi ab}{c^2} \left[c^2 z + \frac{1}{3} z^3 \right]_0^h = \frac{\pi ab}{c^2} \left(c^2 h + \frac{1}{3} h^3 \right) = \frac{\pi abh}{3c^2} (3c^2 + h^2)$$

$$(b) A_0 = A(0) = \pi ab \text{ and } A_h = A(h) = \frac{\pi ab}{c^2} (c^2 + h^2), \text{ from part (a)} \Rightarrow V = \frac{\pi abh}{3c^2} (3c^2 + h^2)$$

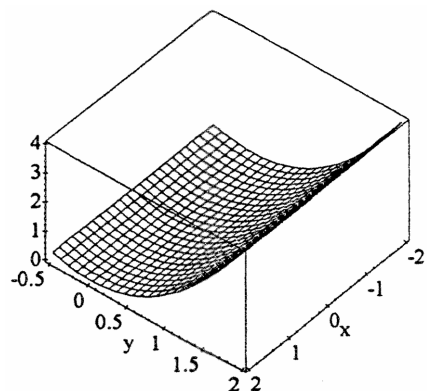
$$= \frac{\pi abh}{3} \left(2 + 1 + \frac{h^2}{c^2} \right) = \frac{\pi abh}{3} \left(2 + \frac{c^2 + h^2}{c^2} \right) = \frac{h}{3} \left[2\pi ab + \frac{\pi ab}{c^2} (c^2 + h^2) \right] = \frac{h}{3} (2A_0 + A_h)$$

$$(c) A_m = A\left(\frac{h}{2}\right) = \frac{\pi ab}{c^2} \left(c^2 + \frac{h^2}{4} \right) = \frac{\pi ab}{4c^2} (4c^2 + h^2) \Rightarrow \frac{h}{6} (A_0 + 4A_m + A_h)$$

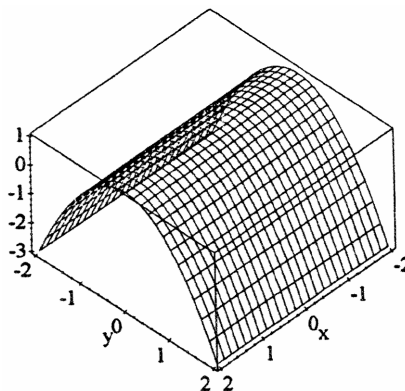
$$= \frac{h}{6} \left[\pi ab + \frac{\pi ab}{c^2} (4c^2 + h^2) + \frac{\pi ab}{c^2} (c^2 + h^2) \right] = \frac{\pi abh}{6c^2} (c^2 + 4c^2 + h^2 + c^2 + h^2) = \frac{\pi abh}{6c^2} (6c^2 + 2h^2)$$

$$= \frac{\pi abh}{3c^2} (3c^2 + h^2) = V \text{ from part (a)}$$

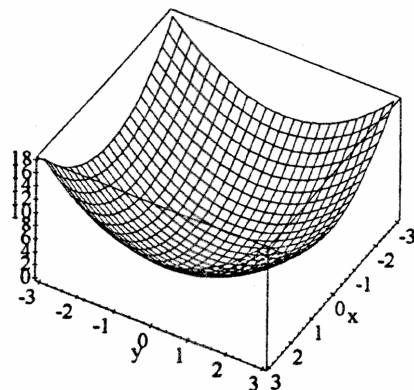
49. $z = y^2$



50. $z = 1 - y^2$

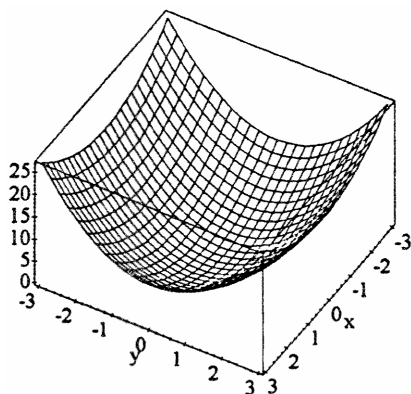


51. $z = x^2 + y^2$

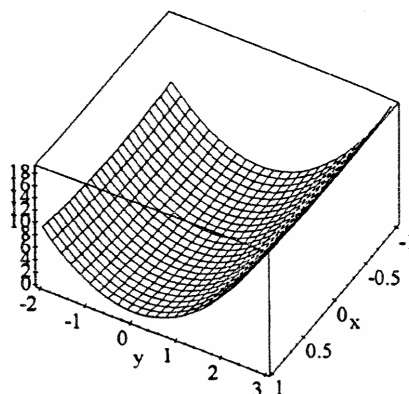


52. $z = x^2 + 2y^2$

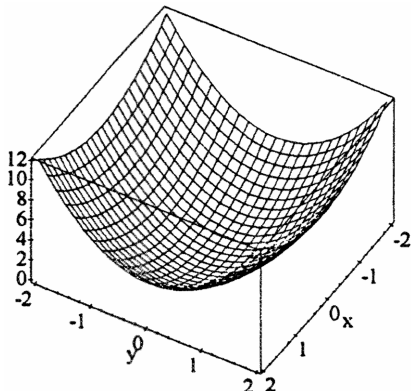
(a)



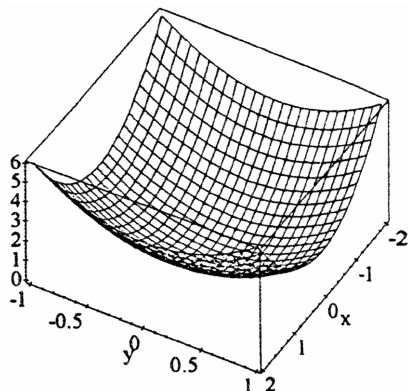
(b)



(c)



(d)



53-58. Example CAS commands:

Maple:

```
with( plots );
eq := x^2/9 + y^2/36 = 1 - z^2/25;
implicitplot3d( eq, x=-3..3, y=-6..6, z=-5..5, scaling=constrained,
                shading=zhue, axes=boxed, title="#89 (Section 11.6)" );
```

Mathematica: (functions and domains may vary):

In the following chapter, you will consider contours or level curves for surfaces in three dimensions. For the purposes of plotting the functions of two variables expressed implicitly in this section, we will call upon the function **ContourPlot3D**. To insert the stated function, write all terms on the same side of the equal sign and the default contour equating that expression to zero will be plotted.

This built-in function requires the loading of a special graphics package.

```
<<Graphics`ContourPlot3D`
Clear[x, y, z]
ContourPlot3D[x^2/9 - y^2/16 - z^2/2 - 1, {x, -9, 9}, {y, -12, 12}, {z, -5, 5},
  Axes -> True, AxesLabel -> {x, y, z}, Boxed -> False,
  PlotLabel -> "Elliptic Hyperboloid of Two Sheets"]
```

Your identification of the plot may or may not be able to be done without considering the graph.

CHAPTER 12 PRACTICE EXERCISES

- $3\langle -3, 4 \rangle - 4\langle 2, -5 \rangle = \langle -9 - 8, 12 + 20 \rangle = \langle -17, 32 \rangle$
 - $\sqrt{17^2 + 32^2} = \sqrt{1313}$

2. (a) $\langle -3 + 2, 4 - 5 \rangle = \langle -1, -1 \rangle$

(b) $\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

3. (a) $\langle -2(-3), -2(4) \rangle = \langle 6, -8 \rangle$

(b) $\sqrt{6^2 + (-8)^2} = 10$

4. (a) $\langle 5(2), 5(-5) \rangle = \langle 10, -25 \rangle$

(b) $\sqrt{10^2 + (-25)^2} = \sqrt{725} = 5\sqrt{29}$

5. $\frac{\pi}{6}$ radians below the negative x-axis: $\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle$ [assuming counterclockwise].

6. $\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

7. $2\left(\frac{1}{\sqrt{4^2+1^2}}\right)(4\mathbf{i} - \mathbf{j}) = \left(\frac{8}{\sqrt{17}}\mathbf{i} - \frac{2}{\sqrt{17}}\mathbf{j}\right)$

8. $-5\left(\frac{1}{\sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2}}\right)\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = -3\mathbf{i} - 4\mathbf{j}$

9. length = $|\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}| = \sqrt{2+2} = 2$, $\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} = 2\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

10. length = $|\mathbf{i} - \mathbf{j}| = \sqrt{1+1} = \sqrt{2}$, $\mathbf{i} - \mathbf{j} = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$

11. $t = \frac{\pi}{2} \Rightarrow \mathbf{v} = (-2 \sin \frac{\pi}{2})\mathbf{i} + (2 \cos \frac{\pi}{2})\mathbf{j} = -2\mathbf{i}$; length = $|-2\mathbf{i}| = \sqrt{4+0} = 2$; $-2\mathbf{i} = 2(-\mathbf{i}) \Rightarrow$ the direction is $-\mathbf{i}$

12. $t = \ln 2 \Rightarrow \mathbf{v} = (e^{\ln 2} \cos(\ln 2) - e^{\ln 2} \sin(\ln 2))\mathbf{i} + (e^{\ln 2} \sin(\ln 2) + e^{\ln 2} \cos(\ln 2))\mathbf{j}$
 $= (2 \cos(\ln 2) - 2 \sin(\ln 2))\mathbf{i} + (2 \sin(\ln 2) + 2 \cos(\ln 2))\mathbf{j} = 2[(\cos(\ln 2) - \sin(\ln 2))\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\mathbf{j}]$

length = $|2[(\cos(\ln 2) - \sin(\ln 2))\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\mathbf{j}]| = 2\sqrt{(\cos(\ln 2) - \sin(\ln 2))^2 + (\cos(\ln 2) + \sin(\ln 2))^2}$
 $= 2\sqrt{2\cos^2(\ln 2) + 2\sin^2(\ln 2)} = 2\sqrt{2}$;

$2[(\cos(\ln 2) - \sin(\ln 2))\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\mathbf{j}] = 2\sqrt{2}\left(\frac{(\cos(\ln 2) - \sin(\ln 2))\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\mathbf{j}}{\sqrt{2}}\right)$

$\Rightarrow$ direction = $\frac{(\cos(\ln 2) - \sin(\ln 2))}{\sqrt{2}}\mathbf{i} + \frac{(\sin(\ln 2) + \cos(\ln 2))}{\sqrt{2}}\mathbf{j}$

13. length = $|2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}| = \sqrt{4+9+36} = 7$, $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} = 7\left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$

14. length = $|\mathbf{i} + 2\mathbf{j} - \mathbf{k}| = \sqrt{1+4+1} = \sqrt{6}$, $\mathbf{i} + 2\mathbf{j} - \mathbf{k} = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right) \Rightarrow$ the direction is

$\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$

15. $2 \frac{\mathbf{v}}{|\mathbf{v}|} = 2 \cdot \frac{4\mathbf{i} - \mathbf{j} + 4\mathbf{k}}{\sqrt{4^2 + (-1)^2 + 4^2}} = 2 \cdot \frac{4\mathbf{i} - \mathbf{j} + 4\mathbf{k}}{\sqrt{33}} = \frac{8}{\sqrt{33}}\mathbf{i} - \frac{2}{\sqrt{33}}\mathbf{j} + \frac{8}{\sqrt{33}}\mathbf{k}$

16. $-5 \frac{\mathbf{v}}{|\mathbf{v}|} = -5 \cdot \frac{(\frac{3}{5})\mathbf{i} + (\frac{4}{5})\mathbf{k}}{\sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2}} = -5 \cdot \frac{(\frac{3}{5})\mathbf{i} + (\frac{4}{5})\mathbf{k}}{\sqrt{\frac{9}{25} + \frac{16}{25}}} = -3\mathbf{i} - 4\mathbf{k}$

17. $|\mathbf{v}| = \sqrt{1+1} = \sqrt{2}$, $|\mathbf{u}| = \sqrt{4+1+4} = 3$, $\mathbf{v} \cdot \mathbf{u} = 3$, $\mathbf{u} \cdot \mathbf{v} = 3$, $\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 1 & -2 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$,

$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $|\mathbf{v} \times \mathbf{u}| = \sqrt{4+4+1} = 3$, $\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$,

$|\mathbf{u}| \cos \theta = \frac{3}{\sqrt{2}}$, $\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right) \mathbf{v} = \frac{3}{2}(\mathbf{i} + \mathbf{j})$

$$18. |\mathbf{v}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}, |\mathbf{u}| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}, \mathbf{v} \cdot \mathbf{u} = (1)(-1) + (1)(0) + (2)(-1) = -3,$$

$$\mathbf{u} \cdot \mathbf{v} = -3, \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) = \mathbf{i} + \mathbf{j} - \mathbf{k},$$

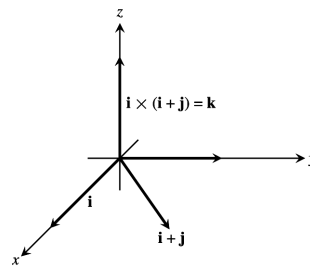
$$|\mathbf{v} \times \mathbf{u}| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{3}, \theta = \cos^{-1} \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}| |\mathbf{u}|} \right) = \cos^{-1} \left(\frac{-3}{\sqrt{6} \sqrt{2}} \right) = \cos^{-1} \left(\frac{-3}{\sqrt{12}} \right)$$

$$= \cos^{-1} \left(-\frac{\sqrt{3}}{2} \right) = \frac{5\pi}{6}, |\mathbf{u}| \cos \theta = \sqrt{2} \cdot \left(-\frac{\sqrt{3}}{2} \right) = -\frac{\sqrt{6}}{2}, \text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}| |\mathbf{v}|} \right) \mathbf{v} = \frac{-3}{6} (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = -\frac{1}{2} (\mathbf{i} + \mathbf{j} + \mathbf{k})$$

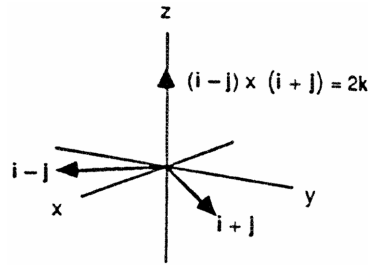
$$19. \text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}| |\mathbf{v}|} \right) \mathbf{v} = \frac{4}{3} (2\mathbf{i} + \mathbf{j} - \mathbf{k}) \text{ where } \mathbf{v} \cdot \mathbf{u} = 8 \text{ and } \mathbf{v} \cdot \mathbf{v} = 6$$

$$20. \text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}| |\mathbf{v}|} \right) \mathbf{v} = -\frac{1}{3} (\mathbf{i} - 2\mathbf{j}) \text{ where } \mathbf{v} \cdot \mathbf{u} = -1 \text{ and } \mathbf{v} \cdot \mathbf{v} = 3$$

$$21. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{k}$$



$$22. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{k}$$



$$\begin{aligned} 23. \text{ Let } \mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \text{ and } \mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}. \text{ Then } |\mathbf{v} - 2\mathbf{w}|^2 = |(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) - 2(w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k})|^2 \\ &= |(v_1 - 2w_1)\mathbf{i} + (v_2 - 2w_2)\mathbf{j} + (v_3 - 2w_3)\mathbf{k}|^2 = (\sqrt{(v_1 - 2w_1)^2 + (v_2 - 2w_2)^2 + (v_3 - 2w_3)^2})^2 \\ &= (v_1^2 + v_2^2 + v_3^2) - 4(v_1w_1 + v_2w_2 + v_3w_3) + 4(w_1^2 + w_2^2 + w_3^2) = |\mathbf{v}|^2 - 4\mathbf{v} \cdot \mathbf{w} + 4|\mathbf{w}|^2 \\ &= |\mathbf{v}|^2 - 4|\mathbf{v}||\mathbf{w}|\cos\theta + 4|\mathbf{w}|^2 = 4 - 4(2)(3)\left(\cos\frac{\pi}{3}\right) + 36 = 40 - 24\left(\frac{1}{2}\right) = 40 - 12 = 28 \Rightarrow |\mathbf{v} - 2\mathbf{w}| = \sqrt{28} \\ &= 2\sqrt{7} \end{aligned}$$

$$\begin{aligned} 24. \mathbf{u} \text{ and } \mathbf{v} \text{ are parallel when } \mathbf{u} \times \mathbf{v} &= \mathbf{0} \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -4 & -8 & a \end{vmatrix} = \mathbf{0} \Rightarrow (4a - 40)\mathbf{i} + (20 - 2a)\mathbf{j} + (0)\mathbf{k} = \mathbf{0} \\ &\Rightarrow 4a - 40 = 0 \text{ and } 20 - 2a = 0 \Rightarrow a = 10 \end{aligned}$$

$$25. (a) \text{ area} = |\mathbf{u} \times \mathbf{v}| = \text{abs} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = |2\mathbf{i} - 3\mathbf{j} - \mathbf{k}| = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$(b) \text{ volume} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix} = 1(3 + 2) - 1(6 - (-1)) - 1(-4 + 1) = 1$$

$$26. (a) \text{ area} = |\mathbf{u} \times \mathbf{v}| = \text{abs} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = |\mathbf{k}| = 1$$

$$(b) \text{ volume} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-0) - 1(0-0) + 0 = 1$$

27. The desired vector is $\mathbf{n} \times \mathbf{v}$ or $\mathbf{v} \times \mathbf{n}$ since $\mathbf{n} \times \mathbf{v}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{v}$ and, therefore, also parallel to the plane.

28. If $a = 0$ and $b \neq 0$, then the line $by = c$ and $\mathbf{i}$ are parallel. If $a \neq 0$ and $b = 0$, then the line $ax = c$ and $\mathbf{j}$ are parallel. If a and b are both $\neq 0$, then $ax + by = c$ contains the points $(\frac{c}{a}, 0)$ and $(0, \frac{c}{b}) \Rightarrow$ the vector $ab(\frac{c}{a}\mathbf{i} - \frac{c}{b}\mathbf{j}) = c(b\mathbf{i} - a\mathbf{j})$ and the line are parallel. Therefore, the vector $b\mathbf{i} - a\mathbf{j}$ is parallel to the line $ax + by = c$ in every case.

29. The line L passes through the point $P(0, 0, -1)$ parallel to $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} - (2+1)\mathbf{j} + (2+2)\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}, \text{ we find the distance}$$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}.$$

30. The line L passes through the point $P(2, 2, 0)$ parallel to $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} - (-2-1)\mathbf{j} + (-2-2)\mathbf{k} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \text{ we find the distance}$$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}.$$

31. Parametric equations for the line are $x = 1 - 3t$, $y = 2$, $z = 3 + 7t$.

32. The line is parallel to $\overrightarrow{PQ} = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$ and contains the point $P(1, 2, 0) \Rightarrow$ parametric equations are $x = 1$, $y = 2 + t$, $z = -t$ for $0 \leq t \leq 1$.

33. The point $P(4, 0, 0)$ lies on the plane $x - y = 4$, and $\overrightarrow{PS} = (6-4)\mathbf{i} + 0\mathbf{j} + (-6+0)\mathbf{k} = 2\mathbf{i} - 6\mathbf{k}$ with $\mathbf{n} = \mathbf{i} - \mathbf{j}$

$$\Rightarrow d = \frac{|\mathbf{n} \cdot \overrightarrow{PS}|}{|\mathbf{n}|} = \left| \frac{2+0+0}{\sqrt{1+1+0}} \right| = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

34. The point $P(0, 0, 2)$ lies on the plane $2x + 3y + z = 2$, and $\overrightarrow{PS} = (3-0)\mathbf{i} + (0-0)\mathbf{j} + (10+2)\mathbf{k} = 3\mathbf{i} + 8\mathbf{k}$ with

$$\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow d = \frac{|\mathbf{n} \cdot \overrightarrow{PS}|}{|\mathbf{n}|} = \left| \frac{6+0+8}{\sqrt{4+9+1}} \right| = \frac{14}{\sqrt{14}} = \sqrt{14}.$$

35. $P(3, -2, 1)$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow (2)(x-3) + (1)(y-(-2)) + (1)(z-1) = 0 \Rightarrow 2x + y + z = 5$

36. $P(-1, 6, 0)$ and $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow (1)(x-(-1)) + (-2)(y-6) + (3)(z-0) = 0 \Rightarrow x - 2y + 3z = -13$

37. $P(1, -1, 2)$, $Q(2, 1, 3)$ and $R(-1, 2, -1) \Rightarrow \vec{PQ} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\vec{PR} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and $\vec{PQ} \times \vec{PR}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ -2 & 3 & -3 \end{vmatrix} = -9\mathbf{i} + \mathbf{j} + 7\mathbf{k} \text{ is normal to the plane } \Rightarrow (-9)(x-1) + (1)(y+1) + (7)(z-2) = 0$$

$$\Rightarrow -9x + y + 7z = 4$$

38. $P(1, 0, 0)$, $Q(0, 1, 0)$ and $R(0, 0, 1) \Rightarrow \vec{PQ} = -\mathbf{i} + \mathbf{j}$, $\vec{PR} = -\mathbf{i} + \mathbf{k}$ and $\vec{PQ} \times \vec{PR}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ is normal to the plane } \Rightarrow (1)(x-1) + (1)(y-0) + (1)(z-0) = 0$$

$$\Rightarrow x + y + z = 1$$

39. $(0, -\frac{1}{2}, -\frac{3}{2})$, since $t = -\frac{1}{2}$, $y = -\frac{1}{2}$ and $z = -\frac{3}{2}$ when $x = 0$; $(-1, 0, -3)$, since $t = -1$, $x = -1$ and $z = -3$ when $y = 0$; $(1, -1, 0)$, since $t = 0$, $x = 1$ and $y = -1$ when $z = 0$

40. $x = 2t$, $y = -t$, $z = -t$ represents a line containing the origin and perpendicular to the plane $2x - y - z = 4$; this line intersects the plane $3x - 5y + 2z = 6$ when t is the solution of $3(2t) - 5(-t) + 2(-t) = 6$

$$\Rightarrow t = \frac{2}{3} \Rightarrow \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right) \text{ is the point of intersection}$$

41. $\mathbf{n}_1 = \mathbf{i}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow$ the desired angle is $\cos^{-1} \left(\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{3}$

42. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{j} + \mathbf{k} \Rightarrow$ the desired angle is $\cos^{-1} \left(\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{3}$

43. The direction of the line is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. Since the point $(-5, 3, 0)$ is on both planes, the desired line is $x = -5 + 5t$, $y = 3 - t$, $z = -3t$.

44. The direction of the intersection is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 5 & -2 & -1 \end{vmatrix} = -6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k} = -3(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$ and is the same as the direction of the given line.

45. (a) The corresponding normals are $\mathbf{n}_1 = 3\mathbf{i} + 6\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and since $\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(2) + (0)(2) + (6)(-1) = 6 + 0 - 6 = 0$, we have that the planes are orthogonal

(b) The line of intersection is parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{vmatrix} = -12\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$. Now to find a point in the intersection, solve $\begin{cases} 3x + 6z = 1 \\ 2x + 2y - z = 3 \end{cases} \Rightarrow \begin{cases} 3x + 6z = 1 \\ 12x + 12y - 6z = 18 \end{cases} \Rightarrow 15x + 12y = 19 \Rightarrow x = 0 \text{ and } y = \frac{19}{12}$

$$\Rightarrow \left(0, \frac{19}{12}, \frac{1}{6}\right) \text{ is a point on the line we seek. Therefore, the line is } x = -12t, y = \frac{19}{12} + 15t \text{ and } z = \frac{1}{6} + 6t.$$

46. A vector in the direction of the plane's normal is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $P(1, 2, 3)$ on the plane $\Rightarrow 7(x-1) - 3(y-2) - 5(z-3) = 0 \Rightarrow 7x - 3y - 5z = -14$.

47. Yes; $\mathbf{v} \cdot \mathbf{n} = (2\mathbf{i} - 4\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 0\mathbf{k}) = 2 \cdot 2 - 4 \cdot 1 + 1 \cdot 0 = 0 \Rightarrow$ the vector is orthogonal to the plane's normal
 $\Rightarrow \mathbf{v}$ is parallel to the plane

48. $\mathbf{n} \cdot \vec{PP_0} > 0$ represents the half-space of points lying on one side of the plane in the direction which the normal $\mathbf{n}$ points

49. A normal to the plane is $\mathbf{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance is $d = \left| \frac{\vec{AP} \cdot \mathbf{n}}{\mathbf{n}} \right|$
 $= \left| \frac{(\mathbf{i} + 4\mathbf{j}) \cdot (-\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})}{\sqrt{1+4+4}} \right| = \left| \frac{-1-8+0}{3} \right| = 3$

50. $P(0, 0, 0)$ lies on the plane $2x + 3y + 5z = 0$, and $\vec{PS} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} \Rightarrow$
 $d = \left| \frac{\vec{PS} \cdot \mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{4+6+15}{\sqrt{4+9+25}} \right| = \frac{25}{\sqrt{38}}$

51. $\mathbf{n} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ is normal to the plane $\Rightarrow \mathbf{n} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} = -3\mathbf{j} + 3\mathbf{k}$ is orthogonal
to $\mathbf{v}$ and parallel to the plane

52. The vector $\mathbf{B} \times \mathbf{C}$ is normal to the plane of $\mathbf{B}$ and $\mathbf{C} \Rightarrow \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is orthogonal to $\mathbf{A}$ and parallel to the plane of $\mathbf{B}$ and $\mathbf{C}$:

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} - \mathbf{k} \text{ and } \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -5 & 3 & -1 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

$$\Rightarrow |\mathbf{A} \times (\mathbf{B} \times \mathbf{C})| = \sqrt{4+9+1} = \sqrt{14} \text{ and } \mathbf{u} = \frac{1}{\sqrt{14}}(-2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \text{ is the desired unit vector.}$$

53. A vector parallel to the line of intersection is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{25+1+9} = \sqrt{35} \Rightarrow 2 \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = \frac{2}{\sqrt{35}}(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ is the desired vector.

54. The line containing $(0, 0, 0)$ normal to the plane is represented by $x = 2t$, $y = -t$, and $z = -t$. This line intersects the plane $3x - 5y + 2z = 6$ when $3(2t) - 5(-t) + 2(-t) = 6 \Rightarrow t = \frac{2}{3} \Rightarrow$ the point is $(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3})$.

55. The line is represented by $x = 3 + 2t$, $y = 2 - t$, and $z = 1 + 2t$. It meets the plane $2x - y + 2z = -2$ when $2(3 + 2t) - (2 - t) + 2(1 + 2t) = -2 \Rightarrow t = -\frac{8}{9} \Rightarrow$ the point is $(\frac{11}{9}, \frac{26}{9}, -\frac{7}{9})$.

56. The direction of the intersection is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{v} \cdot \mathbf{i}}{|\mathbf{v}|} \right)$
 $= \cos^{-1} \left(\frac{3}{\sqrt{35}} \right) \approx 59.5^\circ$

57. The intersection occurs when $(3 + 2t) + 3(2t) - t = -4 \Rightarrow t = -1 \Rightarrow$ the point is $(1, -2, -1)$. The required line must be perpendicular to both the given line and to the normal, and hence is parallel to $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix}$
 $= -5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \Rightarrow$ the line is represented by $x = 1 - 5t$, $y = -2 + 3t$, and $z = -1 + 4t$.

58. If $P(a, b, c)$ is a point on the line of intersection, then P lies in both planes $\Rightarrow a - 2b + c + 3 = 0$ and $2a - b - c + 1 = 0 \Rightarrow (a - 2b + c + 3) + k(2a - b - c + 1) = 0$ for all k .

59. The vector $\vec{AB} \times \vec{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ \frac{26}{5} & 0 & -\frac{26}{5} \end{vmatrix} = \frac{26}{5}(2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k})$ is normal to the plane and $A(-2, 0, -3)$ lies on the plane $\Rightarrow 2(x + 2) + 7(y - 0) + 2(z - (-3)) = 0 \Rightarrow 2x + 7y + 2z + 10 = 0$ is an equation of the plane.

60. Yes; the line's direction vector is $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ which is parallel to the line and also parallel to the normal $-4\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$ to the plane $\Rightarrow$ the line is orthogonal to the plane.

61. The vector $\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix} = -\mathbf{i} - 11\mathbf{j} - 3\mathbf{k}$ is normal to the plane.

- (a) No, the plane is not orthogonal to $\vec{PQ} \times \vec{PR}$.
 (b) No, these equations represent a line, not a plane.
 (c) No, the plane $(x + 2) + 11(y - 1) - 3z = 0$ has normal $\mathbf{i} + 11\mathbf{j} - 3\mathbf{k}$ which is not parallel to $\vec{PQ} \times \vec{PR}$.
 (d) No, this vector equation is equivalent to the equations $3y + 3z = 3$, $3x - 2z = -6$, and $3x + 2y = -4 \Rightarrow x = -\frac{4}{3} - \frac{2}{3}t$, $y = t$, $z = 1 - t$, which represents a line, not a plane.
 (e) Yes, this is a plane containing the point $R(-2, 1, 0)$ with normal $\vec{PQ} \times \vec{PR}$.
62. (a) The line through A and B is $x = 1 + t$, $y = -t$, $z = -1 + 5t$; the line through C and D must be parallel and is $L_1: x = 1 + t$, $y = 2 - t$, $z = 3 + 5t$. The line through B and C is $x = 1$, $y = 2 + 2s$, $z = 3 + 4s$; the line through A and D must be parallel and is $L_2: x = 2$, $y = -1 + 2s$, $z = 4 + 4s$. The lines L_1 and L_2 intersect at $D(2, 1, 8)$ where $t = 1$ and $s = 1$.
 (b) $\cos \theta = \frac{(2\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + 5\mathbf{k})}{\sqrt{20}\sqrt{27}} = \frac{3}{\sqrt{15}}$
 (c) $\left(\frac{\vec{BA} \cdot \vec{BC}}{\vec{BC} \cdot \vec{BC}} \right) \vec{BC} = \frac{18}{20} \vec{BC} = \frac{9}{5}(\mathbf{j} + 2\mathbf{k})$ where $\vec{BA} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\vec{BC} = 2\mathbf{j} + 4\mathbf{k}$
 (d) $\text{area} = |(2\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} - \mathbf{j} + 5\mathbf{k})| = |14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = 6\sqrt{6}$
 (e) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is normal to the plane $\Rightarrow 14(x - 1) + 4(y - 0) - 2(z + 1) = 0 \Rightarrow 7x + 2y - z = 8$.
 (f) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow$ the area of the projection on the yz -plane is $|\mathbf{n} \cdot \mathbf{i}| = 14$; the area of the projection on the xy -plane is $|\mathbf{n} \cdot \mathbf{j}| = 4$; and the area of the projection on the xz -plane is $|\mathbf{n} \cdot \mathbf{k}| = 2$.

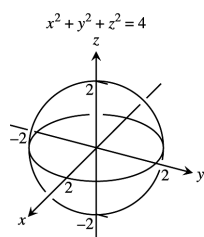
63. $\vec{AB} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\vec{CD} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$, and $\vec{AC} = 2\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 4 & -1 \end{vmatrix} = -5\mathbf{i} - \mathbf{j} - 9\mathbf{k} \Rightarrow$ the distance is

$$d = \left| \frac{(2\mathbf{i} + \mathbf{j}) \cdot (-5\mathbf{i} - \mathbf{j} - 9\mathbf{k})}{\sqrt{25 + 1 + 81}} \right| = \frac{11}{\sqrt{107}}$$

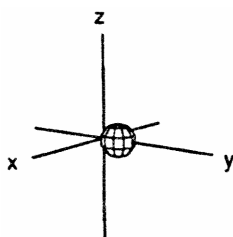
64. $\vec{AB} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\vec{CD} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $\vec{AC} = -3\mathbf{i} + 3\mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance

$$\text{is } d = \left| \frac{(-3\mathbf{i} + 3\mathbf{j}) \cdot (7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})}{\sqrt{49 + 9 + 4}} \right| = \frac{12}{\sqrt{62}}$$

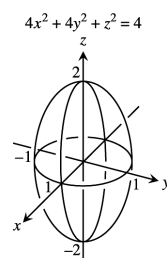
65. $x^2 + y^2 + z^2 = 4$



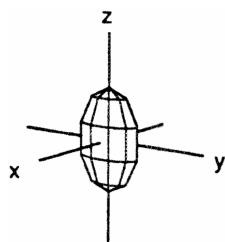
66. $x^2 + (y - 1)^2 + z^2 = 1$



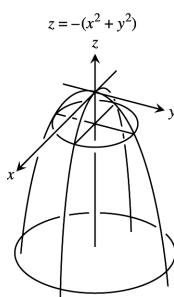
67. $4x^2 + 4y^2 + z^2 = 4$



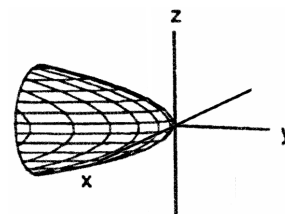
68. $36x^2 + 9y^2 + 4z^2 = 36$



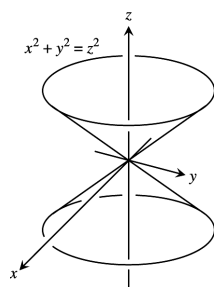
69. $z = -(x^2 + y^2)$



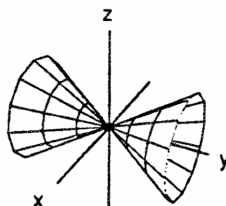
70. $y = -(x^2 + z^2)$



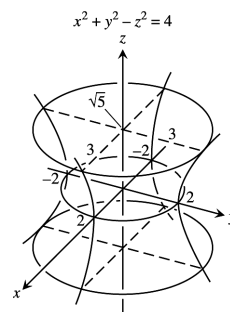
71. $x^2 + y^2 = z^2$



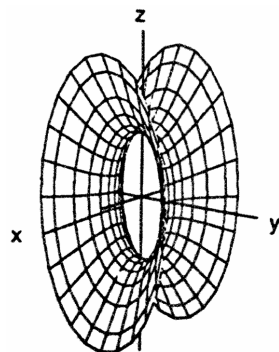
72. $x^2 + z^2 = y^2$



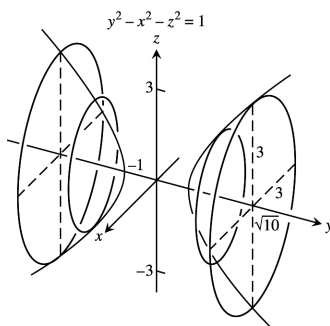
73. $x^2 + y^2 - z^2 = 4$



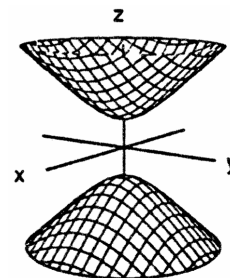
74. $4y^2 + z^2 - 4x^2 = 4$



75. $y^2 - x^2 - z^2 = 1$



76. $z^2 - x^2 - y^2 = 1$



CHAPTER 12 ADDITIONAL AND ADVANCED EXERCISES

- Information from ship A indicates the submarine is now on the line $L_1: x = 4 + 2t, y = 3t, z = -\frac{1}{3}t$; information from ship B indicates the submarine is now on the line $L_2: x = 18s, y = 5 - 6s, z = -s$. The current position of the sub is $(6, 3, -\frac{1}{3})$ and occurs when the lines intersect at $t = 1$ and $s = \frac{1}{3}$. The straight line path of the submarine contains both points $P(2, -1, -\frac{1}{3})$ and $Q(6, 3, -\frac{1}{3})$; the line representing this path is $L: x = 2 + 4t, y = -1 + 4t, z = -\frac{1}{3}$. The submarine traveled the distance between P and Q in 4 minutes $\Rightarrow$ a speed of $\frac{|\overrightarrow{PQ}|}{4} = \frac{\sqrt{32}}{4} = \sqrt{2}$ thousand ft/min. In 20 minutes the submarine will move $20\sqrt{2}$ thousand ft from Q along the line L
 $\Rightarrow 20\sqrt{2} = \sqrt{(2+4t-6)^2 + (-1+4t-3)^2 + 0^2} \Rightarrow 800 = 16(t-1)^2 + 16(t-1)^2 = 32(t-1)^2 \Rightarrow (t-1)^2 = \frac{800}{32}$
 $= 25 \Rightarrow t = 6 \Rightarrow$ the submarine will be located at $(26, 23, -\frac{1}{3})$ in 20 minutes.
- H_2 stops its flight when $6 + 110t = 446 \Rightarrow t = 4$ hours. After 6 hours, H_1 is at $P(246, 57, 9)$ while H_2 is at $(446, 13, 0)$. The distance between P and Q is $\sqrt{(246-446)^2 + (57-13)^2 + (9-0)^2} \approx 204.98$ miles. At 150 mph, it would take about 1.37 hours for H_1 to reach H_2 .
- Torque $= |\overrightarrow{PQ} \times \mathbf{F}| \Rightarrow 15 \text{ ft}\cdot\text{lb} = |\overrightarrow{PQ}| |\mathbf{F}| \sin \frac{\pi}{2} = \frac{3}{4} \text{ ft} \cdot |\mathbf{F}| \Rightarrow |\mathbf{F}| = 20 \text{ lb}$
- Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ be the vector from O to A and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ be the vector from O to B. The vector $\mathbf{v}$ orthogonal to $\mathbf{a}$ and $\mathbf{b} \Rightarrow \mathbf{v}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (since the rotation is clockwise). Now $\mathbf{b} \times \mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
 $\Rightarrow (2, 2, 2)$ is the center of the circular path $(1, 3, 2)$ takes $\Rightarrow \text{radius} = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2} \Rightarrow$ arc length per second covered by the point is $\frac{3}{2}\sqrt{2}$ units/sec $= |\mathbf{v}|$ (velocity is constant). A unit vector in the direction of $\mathbf{v}$ is $\frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b} \times \mathbf{a}|}$
 $= \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v} = |\mathbf{v}| \left(\frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b} \times \mathbf{a}|}\right) = \frac{3}{2}\sqrt{2} \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \sqrt{3}\mathbf{k}$
- (a) By the Law of Cosines we have $\cos \alpha = \frac{3^2 + 5^2 - 4^2}{2(3)(5)} = \frac{3}{5}$ and $\cos \beta = \frac{4^2 + 5^2 - 3^2}{2(4)(5)} = \frac{4}{5} \Rightarrow \sin \alpha = \frac{4}{5}$ and $\sin \beta = \frac{3}{5}$
 $\Rightarrow \mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos \alpha, |\mathbf{F}_1|\sin \alpha \rangle = \langle -\frac{3}{5}|\mathbf{F}_1|, \frac{4}{5}|\mathbf{F}_1| \rangle, \mathbf{F}_2 = \langle |\mathbf{F}_2|\cos \beta, |\mathbf{F}_2|\sin \beta \rangle = \langle \frac{4}{5}|\mathbf{F}_2|, \frac{3}{5}|\mathbf{F}_2| \rangle$, and
 $\mathbf{w} = \langle 0, -100 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 100 \rangle \Rightarrow \langle -\frac{3}{5}|\mathbf{F}_1| + \frac{4}{5}|\mathbf{F}_2|, \frac{4}{5}|\mathbf{F}_1| + \frac{3}{5}|\mathbf{F}_2| \rangle = \langle 0, 100 \rangle \Rightarrow -\frac{3}{5}|\mathbf{F}_1| + \frac{4}{5}|\mathbf{F}_2| = 0$
and $\frac{4}{5}|\mathbf{F}_1| + \frac{3}{5}|\mathbf{F}_2| = 100$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{3}{4}|\mathbf{F}_1|$. Substituting this result into the second equation gives us: $\frac{4}{5}|\mathbf{F}_1| + \frac{9}{20}|\mathbf{F}_1| = 100 \Rightarrow |\mathbf{F}_1| = 80 \text{ lb} \Rightarrow |\mathbf{F}_2| = 60 \text{ lb} \Rightarrow \mathbf{F}_1 = \langle -48, 64 \rangle$ and
 $\mathbf{F}_2 = \langle 48, 36 \rangle$, and $\alpha = \tan^{-1}(\frac{4}{3})$ and $\beta = \tan^{-1}(\frac{3}{4})$
(b) By the Law of Cosines we have $\cos \alpha = \frac{5^2 + 13^2 - 12^2}{2(5)(13)} = \frac{5}{13}$ and $\cos \beta = \frac{12^2 + 13^2 - 5^2}{2(12)(13)} = \frac{12}{13} \Rightarrow \sin \alpha = \frac{12}{13}$ and $\sin \beta = \frac{5}{13}$
 $\Rightarrow \mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos \alpha, |\mathbf{F}_1|\sin \alpha \rangle = \langle -\frac{5}{13}|\mathbf{F}_1|, \frac{12}{13}|\mathbf{F}_1| \rangle, \mathbf{F}_2 = \langle |\mathbf{F}_2|\cos \beta, |\mathbf{F}_2|\sin \beta \rangle = \langle \frac{12}{13}|\mathbf{F}_2|, \frac{5}{13}|\mathbf{F}_2| \rangle$, and
 $\mathbf{w} = \langle 0, -200 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 200 \rangle \Rightarrow \langle -\frac{5}{13}|\mathbf{F}_1| + \frac{12}{13}|\mathbf{F}_2|, \frac{12}{13}|\mathbf{F}_1| + \frac{5}{13}|\mathbf{F}_2| \rangle = \langle 0, 200 \rangle$
 $\Rightarrow -\frac{5}{13}|\mathbf{F}_1| + \frac{12}{13}|\mathbf{F}_2| = 0$ and $\frac{12}{13}|\mathbf{F}_1| + \frac{5}{13}|\mathbf{F}_2| = 200$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{5}{12}|\mathbf{F}_1|$.
Substituting this result into the second equation gives us: $\frac{12}{13}|\mathbf{F}_1| + \frac{25}{156}|\mathbf{F}_1| = 200 \Rightarrow |\mathbf{F}_1| = \frac{2400}{13} \approx 184.615 \text{ lb}$.
 $\Rightarrow |\mathbf{F}_2| = \frac{1000}{13} \approx 76.923 \text{ lb} \Rightarrow \mathbf{F}_1 = \langle -\frac{12000}{1169}, \frac{28800}{1169} \rangle \approx \langle -71.006, 170.414 \rangle$ and $\mathbf{F}_2 = \langle \frac{12000}{1169}, \frac{5000}{1169} \rangle$
 $\approx \langle 71.006, 29.586 \rangle$.
- (a) $\mathbf{T}_1 = \langle -|\mathbf{T}_1|\cos \alpha, |\mathbf{T}_1|\sin \alpha \rangle, \mathbf{T}_2 = \langle |\mathbf{T}_2|\cos \beta, |\mathbf{T}_2|\sin \beta \rangle$, and $\mathbf{w} = \langle 0, -w \rangle$. Since $\mathbf{T}_1 + \mathbf{T}_2 = \langle 0, w \rangle \Rightarrow$
 $\langle -|\mathbf{T}_1|\cos \alpha + |\mathbf{T}_2|\cos \beta, |\mathbf{T}_1|\sin \alpha + |\mathbf{T}_2|\sin \beta \rangle = \langle 0, w \rangle \Rightarrow -|\mathbf{T}_1|\cos \alpha + |\mathbf{T}_2|\cos \beta = 0$ and
 $|\mathbf{T}_1|\sin \alpha + |\mathbf{T}_2|\sin \beta = w$. Solving the first equation for $|\mathbf{T}_2|$ results in: $|\mathbf{T}_2| = \frac{\cos \alpha}{\cos \beta} |\mathbf{T}_1|$. Substituting this result into

the second equation gives us: $|\mathbf{T}_1| \sin \alpha + \frac{\cos \alpha \sin \beta}{\cos \beta} |\mathbf{T}_1| = w \Rightarrow |\mathbf{T}_1| = \frac{w \cos \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta} = \frac{w \cos \beta}{\sin(\alpha + \beta)}$ and $|\mathbf{T}_2| = \frac{w \cos \alpha}{\sin(\alpha + \beta)}$

$$(b) \frac{d}{d\alpha}(|\mathbf{T}_1|) = \frac{d}{d\alpha} \left(\frac{w \cos \beta}{\sin(\alpha + \beta)} \right) = \frac{-w \cos \beta \cos(\alpha + \beta)}{\sin^2(\alpha + \beta)}; \frac{d}{d\alpha}(|\mathbf{T}_1|) = 0 \Rightarrow -w \cos \beta \cos(\alpha + \beta) = 0 \Rightarrow \cos(\alpha + \beta) = 0$$

$$\Rightarrow \alpha + \beta = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{2} - \beta; \frac{d^2}{d\alpha^2}(|\mathbf{T}_1|) = \frac{d}{d\alpha} \left(\frac{-w \cos \beta \cos(\alpha + \beta)}{\sin^2(\alpha + \beta)} \right) = \frac{w \cos \beta (\cos^2(\alpha + \beta) + 1)}{\sin^3(\alpha + \beta)};$$

$$\frac{d^2}{d\alpha^2}(|\mathbf{T}_1|) \bigg|_{\alpha=\frac{\pi}{2}-\beta} = w \cos \beta > 0 \Rightarrow \text{local minimum when } \alpha = \frac{\pi}{2} - \beta$$

$$(c) \frac{d}{d\beta}(|\mathbf{T}_2|) = \frac{d}{d\beta} \left(\frac{w \cos \alpha}{\sin(\alpha + \beta)} \right) = \frac{-w \cos \alpha \cos(\alpha + \beta)}{\sin^2(\alpha + \beta)}; \frac{d}{d\beta}(|\mathbf{T}_2|) = 0 \Rightarrow -w \cos \alpha \cos(\alpha + \beta) = 0 \Rightarrow \cos(\alpha + \beta) = 0$$

$$\Rightarrow \alpha + \beta = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - \alpha; \frac{d^2}{d\beta^2}(|\mathbf{T}_2|) = \frac{d}{d\beta} \left(\frac{-w \cos \alpha \cos(\alpha + \beta)}{\sin^2(\alpha + \beta)} \right) = \frac{w \cos \alpha (\cos^2(\alpha + \beta) + 1)}{\sin^3(\alpha + \beta)};$$

$$\frac{d^2}{d\beta^2}(|\mathbf{T}_2|) \bigg|_{\beta=\frac{\pi}{2}-\alpha} = w \cos \alpha > 0 \Rightarrow \text{local minimum when } \beta = \frac{\pi}{2} - \alpha$$

7. (a) If $P(x, y, z)$ is a point in the plane determined by the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$, then the vectors $\vec{PP}_1$, $\vec{PP}_2$ and $\vec{PP}_3$ all lie in the plane. Thus $\vec{PP}_1 \cdot (\vec{PP}_2 \times \vec{PP}_3) = 0$
- $$\Rightarrow \begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0 \text{ by the determinant formula for the triple scalar product in Section 12.4.}$$
- (b) Subtract row 1 from rows 2, 3, and 4 and evaluate the resulting determinant (which has the same value as the given determinant) by cofactor expansion about column 4. This expansion is exactly the determinant in part (a) so we have all points $P(x, y, z)$ in the plane determined by $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

8. Let $L_1: x = a_1s + b_1, y = a_2s + b_2, z = a_3s + b_3$ and $L_2: x = c_1t + d_1, y = c_2t + d_2, z = c_3t + d_3$. If $L_1 \parallel L_2$,

$$\text{then for some } k, a_i = kc_i, i = 1, 2, 3 \text{ and the determinant } \begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = \begin{vmatrix} kc_1 & c_1 & b_1 - d_1 \\ kc_2 & c_2 & b_2 - d_2 \\ kc_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0,$$

since the first column is a multiple of the second column. The lines L_1 and L_2 intersect if and only if the

$$\text{system } \begin{cases} a_1s - c_1t + (b_1 - d_1) = 0 \\ a_2s - c_2t + (b_2 - d_2) = 0 \\ a_3s - c_3t + (b_3 - d_3) = 0 \end{cases} \text{ has a nontrivial solution } \Leftrightarrow \text{the determinant of the coefficients is zero.}$$

9. (a) Place the tetrahedron so that A is at $(0, 0, 0)$, the point P is on the y -axis, and $\triangle ABC$ lies in the xy -plane. Since $\triangle ABC$ is an equilateral triangle, all the angles in the triangle are 60° and since AP bisects $BC \Rightarrow \triangle ABP$ is a 30° - 60° - 90° triangle. Thus the coordinates of P are $(0, \sqrt{3}, 0)$, the coordinates of B are $(1, \sqrt{3}, 0)$, and the coordinates of C are $(-1, \sqrt{3}, 0)$. Let the coordinates of D be given by (a, b, c) . Since all of the faces are equilateral triangles $\Rightarrow$ all the angles in each of the triangles are $60^\circ \Rightarrow \cos(\angle DAB) = \cos(60^\circ) = \frac{\vec{AD} \cdot \vec{AB}}{|\vec{AD}||\vec{AB}|} = \frac{a+b\sqrt{3}}{(2)(2)} = \frac{1}{2}$
- $$\Rightarrow a + b\sqrt{3} = 2 \text{ and } \cos(\angle DAC) = \cos(60^\circ) = \frac{\vec{AD} \cdot \vec{AC}}{|\vec{AD}||\vec{AC}|} = \frac{-a+b\sqrt{3}}{(2)(2)} = \frac{1}{2} \Rightarrow -a + b\sqrt{3} = 2. \text{ Add the two equations}$$
- to obtain: $2b\sqrt{3} = 4 \Rightarrow b = \frac{2}{\sqrt{3}}$. Substituting this value for b in the first equation gives us: $a + \left(\frac{2}{\sqrt{3}}\right)\sqrt{3} = 2$
- $$\Rightarrow a = 0. \text{ Since } |\vec{AD}| = \sqrt{a^2 + b^2 + c^2} = 2 \Rightarrow 0^2 + \left(\frac{2}{\sqrt{3}}\right)^2 + c^2 = 4 \Rightarrow c = \frac{2\sqrt{2}}{\sqrt{3}}. \text{ Thus the coordinates of } D \text{ are}$$
- $$\left(0, \frac{2}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right). \cos \theta = \cos(\angle DAP) = \frac{\vec{AD} \cdot \vec{AP}}{|\vec{AD}||\vec{AP}|} = \frac{2}{2\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow 57.74^\circ.$$
- (b) Since $\triangle ABC$ lies in the xy -plane $\Rightarrow$ the normal to the face given by $\triangle ABC$ is $\mathbf{n}_1 = \mathbf{k}$. The face given by $\triangle BCD$ is an adjacent face. The vectors $\vec{DB} = \mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$ and $\vec{DC} = -\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$ both lie in the plane containing

$\triangle BCD$. The normal to this plane is given by $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{1}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} \\ -1 & \frac{1}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} \end{vmatrix} = \frac{4\sqrt{2}}{\sqrt{3}}\mathbf{j} + \frac{2}{\sqrt{3}}\mathbf{k}$. The angle θ between two

adjacent faces is given by $\cos \theta = \cos(\angle DAP) = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{2/\sqrt{3}}{(1)(6/\sqrt{3})} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53^\circ$.

10. Extend $\vec{CD}$ to $\vec{CG}$ so that $\vec{CD} = \vec{DG}$. Then $\vec{CG} = t\vec{CF} = \vec{CB} + \vec{BG}$ and $t\vec{CF} = 3\vec{CE} + \vec{CA}$, since ACBG is a parallelogram. If $t\vec{CF} - 3\vec{CE} - \vec{CA} = \mathbf{0}$, then $t - 3 - 1 = 0 \Rightarrow t = 4$, since F, E, and A are collinear. Therefore, $\vec{CG} = 4\vec{CF} \Rightarrow \vec{CD} = 2\vec{CF} \Rightarrow F$ is the midpoint of $\vec{CD}$.

11. If $Q(x, y)$ is a point on the line $ax + by = c$, then $\vec{P_1Q} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j}$, and $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line. The distance is $|\text{proj}_{\mathbf{n}} \vec{P_1Q}| = \left| \frac{[(x - x_1)\mathbf{i} + (y - y_1)\mathbf{j}] \cdot (a\mathbf{i} + b\mathbf{j})}{\sqrt{a^2 + b^2}} \right| = \frac{|a(x - x_1) + b(y - y_1)|}{\sqrt{a^2 + b^2}} = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$, since $c = ax + by$.

12. (a) Let $Q(x, y, z)$ be any point on $Ax + By + Cz - D = 0$. Let $\vec{QP_1} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$, and $\mathbf{n} = \frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}$. The distance is $|\text{proj}_{\mathbf{n}} \vec{QP_1}| = \left| ((x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}) \cdot \left(\frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}} \right) \right| = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}$.

- (b) Since both tangent planes are parallel, one-half of the distance between them is equal to the radius of the sphere, i.e., $r = \frac{1}{2} \frac{|3 - 9|}{\sqrt{1 + 1 + 1}} = \sqrt{3}$ (see also Exercise 12a). Clearly, the points $(1, 2, 3)$ and $(-1, -2, -3)$ are on the line containing the sphere's center. Hence, the line containing the center is $x = 1 + 2t$, $y = 2 + 4t$, $z = 3 + 6t$. The distance from the plane $x + y + z - 3 = 0$ to the center is $\sqrt{3} \Rightarrow \frac{|(1 + 2t) + (2 + 4t) + (3 + 6t) - 3|}{\sqrt{1 + 1 + 1}} = \sqrt{3}$ from part (a) $\Rightarrow t = 0 \Rightarrow$ the center is at $(1, 2, 3)$. Therefore an equation of the sphere is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 3$.

13. (a) If (x_1, y_1, z_1) is on the plane $Ax + By + Cz = D_1$, then the distance d between the planes is

$$d = \frac{|Ax_1 + By_1 + Cz_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D_1 - D_2|}{|A\mathbf{i} + B\mathbf{j} + C\mathbf{k}|}, \text{ since } Ax_1 + By_1 + Cz_1 = D_1, \text{ by Exercise 12(a).}$$

(b) $d = \frac{|12 - 6|}{\sqrt{4 + 9 + 1}} = \frac{6}{\sqrt{14}}$

(c) $\frac{|2(3) + (-1)(2) + 2(-1) + 4|}{\sqrt{14}} = \frac{|2(3) + (-1)(2) + 2(-1) - D|}{\sqrt{14}} \Rightarrow D = 8 \text{ or } -4 \Rightarrow$ the desired plane is

$$2x - y + 2z = 8$$

(d) Choose the point $(2, 0, 1)$ on the plane. Then $\frac{|3 - D|}{\sqrt{6}} = 5 \Rightarrow D = 3 \pm 5\sqrt{6} \Rightarrow$ the desired planes are

$$x - 2y + z = 3 + 5\sqrt{6} \text{ and } x - 2y + z = 3 - 5\sqrt{6}.$$

14. Let $\mathbf{n} = \vec{AB} \times \vec{BC}$ and $D(x, y, z)$ be any point in the plane determined by A, B and C. Then the point D lies in this plane if and only if $\vec{AD} \cdot \mathbf{n} = 0 \Leftrightarrow \vec{AD} \cdot (\vec{AB} \times \vec{BC}) = 0$.

15. $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the plane $x + 2y + 6z = 6$; $\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ is parallel to the

plane and perpendicular to the plane of $\mathbf{v}$ and $\mathbf{n} \Rightarrow \mathbf{w} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 6 \\ 4 & -5 & 1 \end{vmatrix} = 32\mathbf{i} + 23\mathbf{j} - 13\mathbf{k}$ is a

vector parallel to the plane $x + 2y + 6z = 6$ in the direction of the projection vector $\text{proj}_p \mathbf{v}$. Therefore,

$$\text{proj}_p \mathbf{v} = \text{proj}_w \mathbf{v} = \left(\mathbf{v} \cdot \frac{\mathbf{w}}{|\mathbf{w}|} \right) \frac{\mathbf{w}}{|\mathbf{w}|} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \right) \mathbf{w} = \left(\frac{32 + 23 - 13}{32^2 + 23^2 + 13^2} \right) \mathbf{w} = \frac{42}{1722} \mathbf{w} = \frac{1}{41} \mathbf{w} = \frac{32}{41} \mathbf{i} + \frac{23}{41} \mathbf{j} - \frac{13}{41} \mathbf{k}$$

$$16. \text{proj}_z \mathbf{w} = -\text{proj}_z \mathbf{v} \text{ and } \mathbf{w} - \text{proj}_z \mathbf{w} = \mathbf{v} - \text{proj}_z \mathbf{v} \Rightarrow \mathbf{w} = (\mathbf{w} - \text{proj}_z \mathbf{w}) + \text{proj}_z \mathbf{w} = (\mathbf{v} - \text{proj}_z \mathbf{v}) + \text{proj}_z \mathbf{w} \\ = \mathbf{v} - 2 \text{proj}_z \mathbf{v} = \mathbf{v} - 2 \left(\frac{\mathbf{v} \cdot \mathbf{z}}{|\mathbf{z}|^2} \right) \mathbf{z}$$

$$17. (a) \mathbf{u} \times \mathbf{v} = 2\mathbf{i} \times 2\mathbf{j} = 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}; (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 0\mathbf{v} - 0\mathbf{u} = \mathbf{0}; \mathbf{v} \times \mathbf{w} = 4\mathbf{i} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}; \\ (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 0\mathbf{v} - 0\mathbf{w} = \mathbf{0}$$

$$(b) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ -1 & 2 & -1 \end{vmatrix} = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 2(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k};$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 4 & 5 \end{vmatrix} = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-1)(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$$

$$(c) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -4 \\ 1 & 0 & 2 \end{vmatrix} = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 4(2\mathbf{i} + \mathbf{j}) = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k};$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -2 & -3 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 3(\mathbf{i} + 2\mathbf{k}) = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$$

$$(d) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 2 & 4 & -2 \end{vmatrix} = -10\mathbf{i} - 10\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 10(-\mathbf{i} - \mathbf{k}) - 0(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -10\mathbf{i} - 10\mathbf{k};$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 4 & -4 & -4 \end{vmatrix} = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 10(-\mathbf{i} - \mathbf{k}) - 1(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$$

$$18. (a) \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} + (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} = \mathbf{0}$$

$$(b) [\mathbf{u} \cdot (\mathbf{v} \times \mathbf{i})]\mathbf{i} + [(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{j}))\mathbf{j}] + [(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{k}))\mathbf{k}] = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{i}]\mathbf{i} + [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{j}]\mathbf{j} + [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{k}]\mathbf{k} = \mathbf{u} \times \mathbf{v}$$

$$(c) (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{r}) = \mathbf{u} \cdot [\mathbf{v} \times (\mathbf{w} \times \mathbf{r})] = \mathbf{u} \cdot [(\mathbf{v} \cdot \mathbf{r})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{r}] = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{r}) - (\mathbf{u} \cdot \mathbf{r})(\mathbf{v} \cdot \mathbf{w}) = \begin{vmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{u} \cdot \mathbf{r} & \mathbf{v} \cdot \mathbf{r} \end{vmatrix}$$

$$19. \text{The formula is always true; } \mathbf{u} \times [\mathbf{u} \times (\mathbf{u} \times \mathbf{v})] \cdot \mathbf{w} = \mathbf{u} \times [(\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{v}] \cdot \mathbf{w}$$

$$= [(\mathbf{u} \cdot \mathbf{v})\mathbf{u} \times \mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u} \times \mathbf{v}] \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$$

$$20. \text{If } \mathbf{u} = (\cos B)\mathbf{i} + (\sin B)\mathbf{j} \text{ and } \mathbf{v} = (\cos A)\mathbf{i} + (\sin A)\mathbf{j}, \text{ where } A > B, \text{ then } \mathbf{u} \times \mathbf{v} = [|\mathbf{u}| |\mathbf{v}| \sin(A - B)] \mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos B & \sin B & 0 \\ \cos A & \sin A & 0 \end{vmatrix} = (\cos B \sin A - \sin B \cos A)\mathbf{k} \Rightarrow \sin(A - B) = \cos B \sin A - \sin B \cos A, \text{ since}$$

$$|\mathbf{u}| = 1 \text{ and } |\mathbf{v}| = 1.$$

21. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$, then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta \Rightarrow ac + bd = \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} \cos \theta$
 $\Rightarrow (ac + bd)^2 = (a^2 + b^2)(c^2 + d^2) \cos^2 \theta \Rightarrow (ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$, since $\cos^2 \theta \leq 1$.

22. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $\mathbf{u} \cdot \mathbf{u} = a^2 + b^2 + c^2 \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ iff $a = b = c = 0$.

23. $|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \leq |\mathbf{u}|^2 + 2|\mathbf{u}| |\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2 \Rightarrow |\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$

24. Let α denote the angle between $\mathbf{w}$ and $\mathbf{u}$, and β the angle between $\mathbf{w}$ and $\mathbf{v}$. Let $a = |\mathbf{u}|$ and $b = |\mathbf{v}|$. Then

$$\cos \alpha = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{w}| |\mathbf{u}|} = \frac{(a\mathbf{v} + b\mathbf{u}) \cdot \mathbf{u}}{|\mathbf{w}| |\mathbf{u}|} = \frac{(a\mathbf{v} \cdot \mathbf{u} + b\mathbf{u} \cdot \mathbf{u})}{|\mathbf{w}| |\mathbf{u}|} = \frac{(a\mathbf{v} \cdot \mathbf{u} + b\mathbf{u} \cdot \mathbf{u})}{|\mathbf{w}| |\mathbf{u}|} = \frac{(a\mathbf{v} \cdot \mathbf{u} + ba^2)}{|\mathbf{w}| a} = \frac{\mathbf{v} \cdot \mathbf{u} + ba}{|\mathbf{w}|}, \text{ and likewise, } \cos \beta = \frac{\mathbf{u} \cdot \mathbf{v} + ba}{|\mathbf{w}|}.$$

Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is always $\leq \frac{\pi}{2}$ and $\cos \alpha = \cos \beta$, we have that $\alpha = \beta \Rightarrow \mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

25. $(|\mathbf{u}| |\mathbf{v}| + |\mathbf{v}| |\mathbf{u}|) \cdot (|\mathbf{v}| |\mathbf{u}| - |\mathbf{u}| |\mathbf{v}|) = |\mathbf{u}| |\mathbf{v}| \cdot |\mathbf{v}| |\mathbf{u}| + |\mathbf{v}| |\mathbf{u}| \cdot |\mathbf{v}| |\mathbf{u}| - |\mathbf{u}| |\mathbf{v}| \cdot |\mathbf{u}| |\mathbf{v}| - |\mathbf{v}| |\mathbf{u}| \cdot |\mathbf{u}| |\mathbf{v}|$
 $= |\mathbf{v}| |\mathbf{u}| \cdot |\mathbf{u}| |\mathbf{v}| + |\mathbf{v}|^2 |\mathbf{u}| \cdot |\mathbf{u}| - |\mathbf{u}|^2 |\mathbf{v}| \cdot |\mathbf{v}| - |\mathbf{v}| |\mathbf{u}| \cdot |\mathbf{u}| |\mathbf{v}| = |\mathbf{v}|^2 |\mathbf{u}|^2 - |\mathbf{u}|^2 |\mathbf{v}|^2 = 0$

CHAPTER 13 VECTOR-VALUED FUNCTIONS AND MOTION IN SPACE

13.1 CURVES IN SPACE AND THEIR TANGENTS

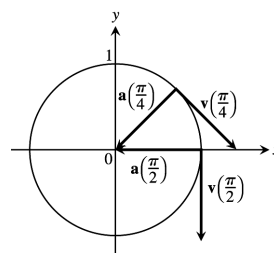
1. $x = t + 1$ and $y = t^2 - 1 \Rightarrow y = (x - 1)^2 - 1 = x^2 - 2x$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a} = 2\mathbf{j}$ at $t = 1$

2. $x = \frac{t}{t+1}$ and $y = \frac{1}{t} \Rightarrow x = \frac{\frac{1}{y}}{\frac{1}{y}+1} = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{1}{(t+1)^2}\mathbf{i} - \frac{1}{t^2}\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = -\frac{2}{(t+1)^3}\mathbf{i} + \frac{2}{t^3}\mathbf{j}$
 $\Rightarrow \mathbf{v} = 4\mathbf{i} - 4\mathbf{j}$ and $\mathbf{a} = -16\mathbf{i} - 16\mathbf{j}$ at $t = -\frac{1}{2}$

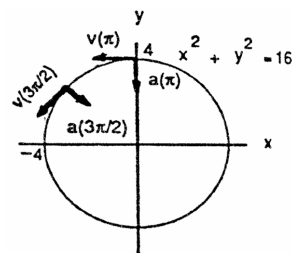
3. $x = e^t$ and $y = \frac{2}{9}e^{2t} \Rightarrow y = \frac{2}{9}x^2$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = e^t\mathbf{i} + \frac{4}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{a} = e^t\mathbf{i} + \frac{8}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 3\mathbf{i} + 8\mathbf{j}$ at $t = \ln 3$

4. $x = \cos 2t$ and $y = 3 \sin 2t \Rightarrow x^2 + \frac{1}{9}y^2 = 1$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = (-4 \cos 2t)\mathbf{i} + (-12 \sin 2t)\mathbf{j} \Rightarrow \mathbf{v} = 6\mathbf{j}$ and $\mathbf{a} = -4\mathbf{i}$ at $t = 0$

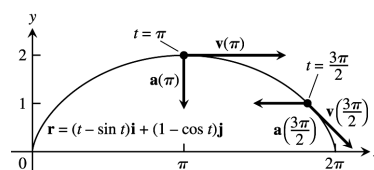
5. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j}$
 $\Rightarrow$ for $t = \frac{\pi}{4}$, $\mathbf{v}(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and
 $\mathbf{a}(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$; for $t = \frac{\pi}{2}$, $\mathbf{v}(\frac{\pi}{2}) = -\mathbf{j}$ and
 $\mathbf{a}(\frac{\pi}{2}) = -\mathbf{i}$



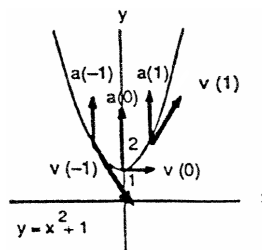
6. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2 \sin \frac{t}{2})\mathbf{i} + (2 \cos \frac{t}{2})\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-\cos \frac{t}{2})\mathbf{i} + (-\sin \frac{t}{2})\mathbf{j} \Rightarrow$ for $t = \pi$, $\mathbf{v}(\pi) = -2\mathbf{i}$ and
 $\mathbf{a}(\pi) = -\mathbf{j}$; for $t = \frac{3\pi}{2}$, $\mathbf{v}(\frac{3\pi}{2}) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$ and
 $\mathbf{a}(\frac{3\pi}{2}) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$



7. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow$ for $t = \pi$, $\mathbf{v}(\pi) = 2\mathbf{i}$ and $\mathbf{a}(\pi) = -\mathbf{j}$;
for $t = \frac{3\pi}{2}$, $\mathbf{v}(\frac{3\pi}{2}) = \mathbf{i} - \mathbf{j}$ and $\mathbf{a}(\frac{3\pi}{2}) = -\mathbf{i}$



8. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow$ for $t = -1$,
 $\mathbf{v}(-1) = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{a}(-1) = 2\mathbf{j}$; for $t = 0$, $\mathbf{v}(0) = \mathbf{i}$ and
 $\mathbf{a}(0) = 2\mathbf{j}$; for $t = 1$, $\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a}(1) = 2\mathbf{j}$

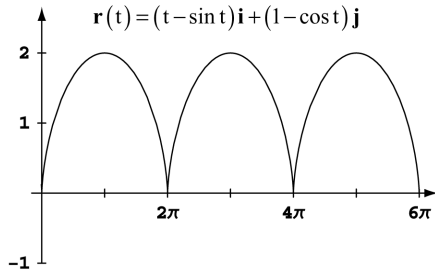


9. $\mathbf{r} = (t+1)\mathbf{i} + (t^2-1)\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + (2(1))^2 + 2^2} = 3$;
Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + 2(1)\mathbf{j} + 2\mathbf{k}}{3} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}(1) = 3\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$
10. $\mathbf{r} = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{2t}{\sqrt{2}}\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{2}{\sqrt{2}}\mathbf{j} + 2t\mathbf{k}$; Speed: $|\mathbf{v}(1)|$
 $= \sqrt{1^2 + \left(\frac{2(1)}{\sqrt{2}}\right)^2 + (1^2)^2} = 2$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + \frac{2(1)}{\sqrt{2}}\mathbf{j} + (1^2)\mathbf{k}}{2} = \frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k} \Rightarrow \mathbf{v}(1)$
 $= 2\left(\frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$
11. $\mathbf{r} = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-2\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$;
Speed: $|\mathbf{v}(\frac{\pi}{2})| = \sqrt{(-2\sin \frac{\pi}{2})^2 + (3\cos \frac{\pi}{2})^2 + 4^2} = 2\sqrt{5}$; Direction: $\frac{\mathbf{v}(\frac{\pi}{2})}{|\mathbf{v}(\frac{\pi}{2})|}$
 $= \left(-\frac{2}{2\sqrt{5}}\sin \frac{\pi}{2}\right)\mathbf{i} + \left(\frac{3}{2\sqrt{5}}\cos \frac{\pi}{2}\right)\mathbf{j} + \frac{4}{2\sqrt{5}}\mathbf{k} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k} \Rightarrow \mathbf{v}(\frac{\pi}{2}) = 2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}\right)$
12. $\mathbf{r} = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (\sec t \tan t)\mathbf{i} + (\sec^2 t)\mathbf{j} + \frac{4}{3}\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$
 $= (\sec t \tan^2 t + \sec^3 t)\mathbf{i} + (2\sec^2 t \tan t)\mathbf{j}$; Speed: $|\mathbf{v}(\frac{\pi}{6})| = \sqrt{(\sec \frac{\pi}{6} \tan \frac{\pi}{6})^2 + (\sec^2 \frac{\pi}{6})^2 + (\frac{4}{3})^2} = 2$;
Direction: $\frac{\mathbf{v}(\frac{\pi}{6})}{|\mathbf{v}(\frac{\pi}{6})|} = \frac{(\sec \frac{\pi}{6} \tan \frac{\pi}{6})\mathbf{i} + (\sec^2 \frac{\pi}{6})\mathbf{j} + \frac{4}{3}\mathbf{k}}{2} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}(\frac{\pi}{6}) = 2\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$
13. $\mathbf{r} = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t+1}\right)\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{-2}{(t+1)^2}\right]\mathbf{i} + 2\mathbf{j} + \mathbf{k}$;
Speed: $|\mathbf{v}(1)| = \sqrt{\left(\frac{2}{1+1}\right)^2 + (2(1))^2 + 1^2} = \sqrt{6}$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\left(\frac{2}{1+1}\right)\mathbf{i} + 2(1)\mathbf{j} + (1)\mathbf{k}}{\sqrt{6}}$
 $= \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}\right)$
14. $\mathbf{r} = (e^{-t})\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-e^{-t})\mathbf{i} - (6\sin 3t)\mathbf{j} + (6\cos 3t)\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$
 $= (e^{-t})\mathbf{i} - (18\cos 3t)\mathbf{j} - (18\sin 3t)\mathbf{k}$; Speed: $|\mathbf{v}(0)| = \sqrt{(-e^0)^2 + [-6\sin 3(0)]^2 + [6\cos 3(0)]^2} = \sqrt{37}$;
Direction: $\frac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \frac{(-e^0)\mathbf{i} - 6\sin 3(0)\mathbf{j} + 6\cos 3(0)\mathbf{k}}{\sqrt{37}} = -\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k} \Rightarrow \mathbf{v}(0) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}\right)$
15. $\mathbf{v} = 3\mathbf{i} + \sqrt{3}\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{v}(0) = 3\mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{a}(0) = 2\mathbf{k} \Rightarrow |\mathbf{v}(0)| = \sqrt{3^2 + (\sqrt{3})^2 + 0^2} = \sqrt{12}$ and
 $|\mathbf{a}(0)| = \sqrt{2^2} = 2$; $\mathbf{v}(0) \cdot \mathbf{a}(0) = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$
16. $\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} + \left(\frac{\sqrt{2}}{2} - 32t\right)\mathbf{j}$ and $\mathbf{a} = -32\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$ and $\mathbf{a}(0) = -32\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$
 $= 1$ and $|\mathbf{a}(0)| = \sqrt{(-32)^2} = 32$; $\mathbf{v}(0) \cdot \mathbf{a}(0) = \left(\frac{\sqrt{2}}{2}\right)(-32) = -16\sqrt{2} \Rightarrow \cos \theta = \frac{-16\sqrt{2}}{1(32)} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{3\pi}{4}$
17. $\mathbf{v} = \left(\frac{2t}{t^2+1}\right)\mathbf{i} + \left(\frac{1}{t^2+1}\right)\mathbf{j} + t(t^2+1)^{-1/2}\mathbf{k}$ and $\mathbf{a} = \left[\frac{-2t^2+2}{(t^2+1)^2}\right]\mathbf{i} - \left[\frac{2t}{(t^2+1)^2}\right]\mathbf{j} + \left[\frac{1}{(t^2+1)^{3/2}}\right]\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{j}$ and
 $\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k} \Rightarrow |\mathbf{v}(0)| = 1$ and $|\mathbf{a}(0)| = \sqrt{2^2 + 1^2} = \sqrt{5}$; $\mathbf{v}(0) \cdot \mathbf{a}(0) = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$
18. $\mathbf{v} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and $\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and
 $\mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1$ and $|\mathbf{a}(0)| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{2}}{3}$; $\mathbf{v}(0) \cdot \mathbf{a}(0) = \frac{2}{9} - \frac{2}{9}$
 $= 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

19. $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k}; t_0 = 0 \Rightarrow \mathbf{v}(t_0) = \mathbf{i} + \mathbf{k}$ and $\mathbf{r}(t_0) = \mathbf{P}_0 = (0, -1, 1) \Rightarrow x = 0 + t = t, y = -1$, and $z = 1 + t$ are parametric equations of the tangent line
20. $\mathbf{r}(t) = t^2\mathbf{i} + (2t - 1)\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j} + 3t^2\mathbf{k}; t_0 = 2 \Rightarrow \mathbf{v}(2) = 4\mathbf{i} + 2\mathbf{j} + 12\mathbf{k}$ and $\mathbf{r}(t_0) = \mathbf{P}_0 = (4, 3, 8) \Rightarrow x = 4 + 4t, y = 3 + 2t$, and $z = 8 + 12t$ are parametric equations of the tangent line
21. $\mathbf{r}(t) = (\ln t)\mathbf{i} + \frac{t-1}{t+2}\mathbf{j} + (t \ln t)\mathbf{k} \Rightarrow \mathbf{v}(t) = \frac{1}{t}\mathbf{i} + \frac{3}{(t+2)^2}\mathbf{j} + (\ln t + 1)\mathbf{k}; t_0 = 1 \Rightarrow \mathbf{v}(1) = \mathbf{i} + \frac{1}{3}\mathbf{j} + \mathbf{k}$ and $\mathbf{r}(t_0) = \mathbf{P}_0 = (0, 0, 0) \Rightarrow x = 0 + t = t, y = 0 + \frac{1}{3}t = \frac{1}{3}t$, and $z = 0 + t = t$ are parametric equations of the tangent line
22. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k} \Rightarrow \mathbf{v}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (2 \cos 2t)\mathbf{k}; t_0 = \frac{\pi}{2} \Rightarrow \mathbf{v}(t_0) = -\mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}(t_0) = \mathbf{P}_0 = (0, 1, 0) \Rightarrow x = 0 - t = -t, y = 1$, and $z = 0 - 2t = -2t$ are parametric equations of the tangent line
23. (a) $\mathbf{v}(t) = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j};$
 (i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow$ constant speed;
 (ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;
 (iii) counterclockwise movement;
 (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$
- (b) $\mathbf{v}(t) = -(2 \sin 2t)\mathbf{i} + (2 \cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4 \cos 2t)\mathbf{i} - (4 \sin 2t)\mathbf{j};$
 (i) $|\mathbf{v}(t)| = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = 2 \Rightarrow$ constant speed;
 (ii) $\mathbf{v} \cdot \mathbf{a} = 8 \sin 2t \cos 2t - 8 \cos 2t \sin 2t = 0 \Rightarrow$ yes, orthogonal;
 (iii) counterclockwise movement;
 (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$
- (c) $\mathbf{v}(t) = -\sin(t - \frac{\pi}{2})\mathbf{i} + \cos(t - \frac{\pi}{2})\mathbf{j} \Rightarrow \mathbf{a}(t) = -\cos(t - \frac{\pi}{2})\mathbf{i} - \sin(t - \frac{\pi}{2})\mathbf{j};$
 (i) $|\mathbf{v}(t)| = \sqrt{\sin^2(t - \frac{\pi}{2}) + \cos^2(t - \frac{\pi}{2})} = 1 \Rightarrow$ constant speed;
 (ii) $\mathbf{v} \cdot \mathbf{a} = \sin(t - \frac{\pi}{2})\cos(t - \frac{\pi}{2}) - \cos(t - \frac{\pi}{2})\sin(t - \frac{\pi}{2}) = 0 \Rightarrow$ yes, orthogonal;
 (iii) counterclockwise movement;
 (iv) no, $\mathbf{r}(0) = 0\mathbf{i} - \mathbf{j}$ instead of $\mathbf{i} + 0\mathbf{j}$
- (d) $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j};$
 (i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow$ constant speed;
 (ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;
 (iii) clockwise movement;
 (iv) yes, $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$
- (e) $\mathbf{v}(t) = -(2t \sin t)\mathbf{i} + (2t \cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(2 \sin t + 2t \cos t)\mathbf{i} + (2 \cos t - 2t \sin t)\mathbf{j};$
 (i) $|\mathbf{v}(t)| = \sqrt{[-(2t \sin t)]^2 + (2t \cos t)^2} = \sqrt{4t^2(\sin^2 t + \cos^2 t)} = 2|t| = 2t, t \geq 0 \Rightarrow$ variable speed;
 (ii) $\mathbf{v} \cdot \mathbf{a} = 4(t \sin^2 t + t^2 \sin t \cos t) + 4(t \cos^2 t - t^2 \cos t \sin t) = 4t \neq 0$ in general $\Rightarrow$ not orthogonal in general;
 (iii) counterclockwise movement;
 (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$
24. Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ denote the position vector of the point $(2, 2, 1)$ and let, $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. Then $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$. Note that $(2, 2, 1)$ is a point on the plane and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the plane. Moreover, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors with $\mathbf{u} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. Therefore, $\mathbf{r}(t)$ identifies a point that lies in the plane for each t . Also, for each t , $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ is a unit vector. Starting at the point $(2 + \frac{1}{\sqrt{2}}, 2 - \frac{1}{\sqrt{2}}, 1)$ the vector $\mathbf{r}(t)$ traces out a circle of radius 1 and center $(2, 2, 1)$ in the plane $x + y - 2z = 2$.

25. The velocity vector is tangent to the graph of $y^2 = 2x$ at the point $(2, 2)$, has length 5, and a positive $\mathbf{i}$ component. Now, $y^2 = 2x \Rightarrow 2y \frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} \Big|_{(2,2)} = \frac{2}{2 \cdot 2} = \frac{1}{2} \Rightarrow$ the tangent vector lies in the direction of the vector $\mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow$ the velocity vector is $\mathbf{v} = \frac{5}{\sqrt{1+\frac{1}{4}}} (\mathbf{i} + \frac{1}{2}\mathbf{j}) = \frac{5}{(\frac{\sqrt{5}}{2})} (\mathbf{i} + \frac{1}{2}\mathbf{j}) = 2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j}$

26. (a)



- (b) $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $|\mathbf{v}|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t \Rightarrow |\mathbf{v}|^2$ is at a max when $\cos t = -1 \Rightarrow t = \pi, 3\pi, 5\pi$, etc., and at these values of t , $|\mathbf{v}|^2 = 4 \Rightarrow \max |\mathbf{v}| = \sqrt{4} = 2$; $|\mathbf{v}|^2$ is at a min when $\cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi$, etc., and at these values of t , $|\mathbf{v}|^2 = 0 \Rightarrow \min |\mathbf{v}| = 0$; $|\mathbf{a}|^2 = \sin^2 t + \cos^2 t = 1$ for every $t \Rightarrow \max |\mathbf{a}| = \min |\mathbf{a}| = \sqrt{1} = 1$
27. $\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 2 \cdot 0 = 0 \Rightarrow \mathbf{r} \cdot \mathbf{r}$ is a constant $\Rightarrow |\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}}$ is constant
28. (a) $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \left(\frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{v} \times \frac{d\mathbf{w}}{dt} \right)$
 $= \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \times \frac{d\mathbf{w}}{dt}$
- (b) $\frac{d}{dt} \left[\mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right) \right] = \frac{d\mathbf{r}}{dt} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right) + \mathbf{r} \cdot \left(\frac{d^2\mathbf{r}}{dt^2} \times \frac{d^2\mathbf{r}}{dt^2} \right) + \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3} \right) = \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3} \right)$, since $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$ and $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{B}) = 0$ for any vectors $\mathbf{A}$ and $\mathbf{B}$
29. (a) $\mathbf{u} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow c\mathbf{u} = cf(t)\mathbf{i} + cg(t)\mathbf{j} + ch(t)\mathbf{k} \Rightarrow \frac{d}{dt}(c\mathbf{u}) = c \frac{df}{dt}\mathbf{i} + c \frac{dg}{dt}\mathbf{j} + c \frac{dh}{dt}\mathbf{k}$
 $= c \left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k} \right) = c \frac{d\mathbf{u}}{dt}$
- (b) $f\mathbf{u} = f[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] \Rightarrow \frac{d}{dt}(f\mathbf{u}) = \left[\frac{df}{dt}f(t) + f \frac{df}{dt} \right] \mathbf{i} + \left[\frac{df}{dt}g(t) + f \frac{dg}{dt} \right] \mathbf{j} + \left[\frac{df}{dt}h(t) + f \frac{dh}{dt} \right] \mathbf{k}$
 $= \frac{df}{dt} [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] + f \left[\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k} \right] = \frac{df}{dt}\mathbf{u} + f \frac{d\mathbf{u}}{dt}$
30. Let $\mathbf{u} = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ and $\mathbf{v} = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$. Then
 $\mathbf{u} + \mathbf{v} = [f_1(t) + g_1(t)]\mathbf{i} + [f_2(t) + g_2(t)]\mathbf{j} + [f_3(t) + g_3(t)]\mathbf{k}$
 $\Rightarrow \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = [f'_1(t) + g'_1(t)]\mathbf{i} + [f'_2(t) + g'_2(t)]\mathbf{j} + [f'_3(t) + g'_3(t)]\mathbf{k}$
 $= [f'_1(t)\mathbf{i} + f'_2(t)\mathbf{j} + f'_3(t)\mathbf{k}] + [g'_1(t)\mathbf{i} + g'_2(t)\mathbf{j} + g'_3(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$;
 $\mathbf{u} - \mathbf{v} = [f_1(t) - g_1(t)]\mathbf{i} + [f_2(t) - g_2(t)]\mathbf{j} + [f_3(t) - g_3(t)]\mathbf{k}$
 $\Rightarrow \frac{d}{dt}(\mathbf{u} - \mathbf{v}) = [f'_1(t) - g'_1(t)]\mathbf{i} + [f'_2(t) - g'_2(t)]\mathbf{j} + [f'_3(t) - g'_3(t)]\mathbf{k}$
 $= [f'_1(t)\mathbf{i} + f'_2(t)\mathbf{j} + f'_3(t)\mathbf{k}] - [g'_1(t)\mathbf{i} + g'_2(t)\mathbf{j} + g'_3(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$
31. Suppose $\mathbf{r}$ is continuous at $t = t_0$. Then $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \Leftrightarrow \lim_{t \rightarrow t_0} [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]$
 $= f(t_0)\mathbf{i} + g(t_0)\mathbf{j} + h(t_0)\mathbf{k} \Leftrightarrow \lim_{t \rightarrow t_0} f(t) = f(t_0), \lim_{t \rightarrow t_0} g(t) = g(t_0), \text{ and } \lim_{t \rightarrow t_0} h(t) = h(t_0) \Leftrightarrow f, g, \text{ and } h \text{ are}$
 continuous at $t = t_0$.

$$\begin{aligned}
 32. \lim_{t \rightarrow t_0} [\mathbf{r}_1(t) \times \mathbf{r}_2(t)] &= \lim_{t \rightarrow t_0} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix} = \begin{vmatrix} \lim_{t \rightarrow t_0} f_1(t) & \lim_{t \rightarrow t_0} f_2(t) & \lim_{t \rightarrow t_0} f_3(t) \\ \lim_{t \rightarrow t_0} g_1(t) & \lim_{t \rightarrow t_0} g_2(t) & \lim_{t \rightarrow t_0} g_3(t) \end{vmatrix} \\
 &= \lim_{t \rightarrow t_0} \mathbf{r}_1(t) \times \lim_{t \rightarrow t_0} \mathbf{r}_2(t) = \mathbf{A} \times \mathbf{B}
 \end{aligned}$$

33. $\mathbf{r}'(t_0)$ exists $\Rightarrow f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$ exists $\Rightarrow f'(t_0), g'(t_0), h'(t_0)$ all exist $\Rightarrow f, g,$ and h are continuous at $t = t_0 \Rightarrow \mathbf{r}(t)$ is continuous at $t = t_0$

34. $\mathbf{u} = \mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with a, b, c real constants $\Rightarrow \frac{d\mathbf{u}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

35-38. Example CAS commands:

Maple:

```

> with( plots );
r := t -> [sin(t)-t*cos(t),cos(t)+t*sin(t),t^2];
t0 := 3*Pi/2;
lo := 0;
hi := 6*Pi;
P1 := spacecurve( r(t), t=lo..hi, axes=boxed, thickness=3 );
display( P1, title="#35(a) (Section 13.1)" );
Dr := unapply( diff(r(t),t), t );           # (b)
Dr(t0);                                     # (c)
q1 := expand( r(t0) + Dr(t0)*(t-t0) );
T := unapply( q1, t );
P2 := spacecurve( T(t), t=lo..hi, axes=boxed, thickness=3, color=black );
display( [P1,P2], title="#35(d) (Section 13.1)" );

```

39-40. Example CAS commands:

Maple:

```

a := 'a'; b := 'b';
r := (a,b,t) -> [cos(a*t),sin(a*t),b*t];
Dr := unapply( diff(r(a,b,t),t), (a,b,t) );
t0 := 3*Pi/2;
q1 := expand( r(a,b,t0) + Dr(a,b,t0)*(t-t0) );
T := unapply( q1, (a,b,t) );
lo := 0;
hi := 4*Pi;
P := NULL;
for a in [ 1, 2, 4, 6 ] do
    P1 := spacecurve( r(a,1,t), t=lo..hi, thickness=3 );
    P2 := spacecurve( T(a,1,t), t=lo..hi, thickness=3, color=black );
    P := P, display( [P1,P2], axes=boxed, title=sprintf("#39 (Section 13.1)\n a=%a",a) );
end do;
display( [P], insequence=true );

```

35-40. Example CAS commands:

Mathematica: (assigned functions, parameters, and intervals will vary)

The x-y-z components for the curve are entered as a list of functions of t. The unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are not inserted.

If a graph is too small, highlight it and drag out a corner or side to make it larger.

Only the components of $\mathbf{r}[t]$ and values for t_0 , $t_{\min}$, and $t_{\max}$ require alteration for each problem.

```
Clear[r, v, t, x, y, z]
r[t_]:= { Sin[t] - t Cos[t], Cos[t] + t Sin[t], t^2}
t0= 3π / 2; tmin= 0; tmax= 6π;
ParametricPlot3D[Evaluate[r[t]], {t, tmin, tmax}, AxesLabel -> {x, y, z}];
v[t_]= r'[t]
tanline[t_]= v[t0] t + r[t0]
ParametricPlot3D[Evaluate[{r[t], tanline[t]}], {t, tmin, tmax}, AxesLabel -> {x, y, z}];
```

For 39 and 40, the curve can be defined as a function of t , a , and b . Leave a space between a and t and b and t .

```
Clear[r, v, t, x, y, z, a, b]
r[t_,a_,b_]:= { Cos[a t], Sin[a t], b t}
t0= 3π / 2; tmin= 0; tmax= 4π;
v[t_,a_,b_]= D[r[t, a, b], t]
tanline[t_,a_,b_]= v[t0, a, b] t + r[t0, a, b]
pa1=ParametricPlot3D[Evaluate[{r[t, 1, 1], tanline[t, 1, 1]}], {t,tmin, tmax}, AxesLabel -> {x, y, z}];
pa2=ParametricPlot3D[Evaluate[{r[t, 2, 1], tanline[t, 2, 1]}], {t,tmin, tmax}, AxesLabel -> {x, y, z}];
pa4=ParametricPlot3D[Evaluate[{r[t, 4, 1], tanline[t, 4, 1]}], {t,tmin, tmax}, AxesLabel -> {x, y, z}];
pa6=ParametricPlot3D[Evaluate[{r[t, 6, 1], tanline[t, 6, 1]}], {t,tmin, tmax}, AxesLabel -> {x, y, z}];
Show[GraphicsRow[{pa1, pa2, pa4, pa6}]]
```

13.2 INTEGRALS OF VECTOR FUNCTIONS; PROJECTILE MOTION

- $\int_0^1 [t^3 \mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}] dt = \left[\frac{t^4}{4} \right]_0^1 \mathbf{i} + [7t]_0^1 \mathbf{j} + \left[\frac{t^2}{2} + t \right]_0^1 \mathbf{k} = \frac{1}{4} \mathbf{i} + 7\mathbf{j} + \frac{3}{2} \mathbf{k}$
- $\int_1^2 [(6-6t)\mathbf{i} + 3\sqrt{t}\mathbf{j} + (\frac{4}{t^2})\mathbf{k}] dt = [6t - 3t^2]_1^2 \mathbf{i} + [2t^{3/2}]_1^2 \mathbf{j} + [-4t^{-1}]_1^2 \mathbf{k} = -3\mathbf{i} + (4\sqrt{2} - 2)\mathbf{j} + 2\mathbf{k}$
- $\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1 + \cos t)\mathbf{j} + (\sec^2 t)\mathbf{k}] dt = [-\cos t]_{-\pi/4}^{\pi/4} \mathbf{i} + [t + \sin t]_{-\pi/4}^{\pi/4} \mathbf{j} + [\tan t]_{-\pi/4}^{\pi/4} \mathbf{k} = \left(\frac{\pi+2\sqrt{2}}{2} \right) \mathbf{j} + 2\mathbf{k}$
- $\int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\mathbf{k}] dt = \int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (\sin 2t)\mathbf{k}] dt$
 $= [\sec t]_0^{\pi/3} \mathbf{i} + [-\ln(\cos t)]_0^{\pi/3} \mathbf{j} + [-\frac{1}{2} \cos 2t]_0^{\pi/3} \mathbf{k} = \mathbf{i} + (\ln 2)\mathbf{j} + \frac{3}{4} \mathbf{k}$
- $\int_1^4 (\frac{1}{t} \mathbf{i} + \frac{1}{5-t} \mathbf{j} + \frac{1}{2t} \mathbf{k}) dt = [\ln t]_1^4 \mathbf{i} + [-\ln(5-t)]_1^4 \mathbf{j} + [\frac{1}{2} \ln t]_1^4 \mathbf{k} = (\ln 4)\mathbf{i} + (\ln 4)\mathbf{j} + (\ln 2)\mathbf{k}$
- $\int_0^1 \left(\frac{2}{\sqrt{1-t^2}} \mathbf{i} + \frac{\sqrt{3}}{1+t^2} \mathbf{k} \right) dt = [2 \sin^{-1} t]_0^1 \mathbf{i} + \left[\sqrt{3} \tan^{-1} t \right]_0^1 \mathbf{k} = \pi \mathbf{i} + \frac{\pi\sqrt{3}}{4} \mathbf{k}$
- $\int_0^1 (te^{t^2} \mathbf{i} + e^{-t} \mathbf{j} + \mathbf{k}) dt = \left[\frac{1}{2} e^{t^2} \right]_0^1 \mathbf{i} - [e^{-t}]_0^1 \mathbf{j} + [t]_0^1 \mathbf{k} = \frac{e-1}{2} \mathbf{i} + \frac{e-1}{e} \mathbf{j} + \mathbf{k}$
- $\int_1^{\ln 3} (te^t \mathbf{i} + e^t \mathbf{j} + \ln t \mathbf{k}) dt = [te^t - e^t]_1^{\ln 3} \mathbf{i} - [e^t]_1^{\ln 3} \mathbf{j} + [t \ln t - t]_1^{\ln 3} \mathbf{k}$
 $= 3(\ln 3 - 1)\mathbf{i} + (3 - e)\mathbf{j} + (\ln 3(\ln(\ln 3) - 1) + 1)\mathbf{k}$
- $\int_0^{\pi/2} [(\cos t)\mathbf{i} - (\sin 2t)\mathbf{j} + (\sin^2 t)\mathbf{k}] dt = \int_0^{\pi/2} [(\cos t)\mathbf{i} - (\sin 2t)\mathbf{j} + (\frac{1}{2} - \frac{1}{2} \cos 2t)\mathbf{k}] dt =$
 $= [\sin t]_0^{\pi/2} \mathbf{i} + [\frac{1}{2} \cos t]_0^{\pi/2} \mathbf{j} + [\frac{1}{2} t - \frac{1}{4} \sin 2t]_0^{\pi/2} \mathbf{k} = \mathbf{i} - \mathbf{j} + \frac{\pi}{4} \mathbf{k}$

10. $\int_0^{\pi/4} [(\sec t)\mathbf{i} + (\tan^2 t)\mathbf{j} - (t \sin t)\mathbf{k}] dt = \int_0^{\pi/4} [(\sec t)\mathbf{i} + (\sec^2 t - 1)\mathbf{j} - (t \sin t)\mathbf{k}] dt$
 $= [\ln(\sec t + \tan t)]_0^{\pi/4} \mathbf{i} + [\tan t - t]_0^{\pi/4} \mathbf{j} + [t \cos t - \sin t]_0^{\pi/4} \mathbf{k} = \ln(1 + \sqrt{2})\mathbf{i} + (1 - \frac{\pi}{4})\mathbf{j} + (\frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}})\mathbf{k}$
11. $\mathbf{r} = \int (-t\mathbf{i} - t\mathbf{j} - t\mathbf{k}) dt = -\frac{t^2}{2}\mathbf{i} - \frac{t^2}{2}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}; \mathbf{r}(0) = 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} + \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
 $\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 1\right)\mathbf{i} + \left(-\frac{t^2}{2} + 2\right)\mathbf{j} + \left(-\frac{t^2}{2} + 3\right)\mathbf{k}$
12. $\mathbf{r} = \int [(180t)\mathbf{i} + (180t - 16t^2)\mathbf{j}] dt = 90t^2\mathbf{i} + (90t^2 - \frac{16}{3}t^3)\mathbf{j} + \mathbf{C}; \mathbf{r}(0) = 90(0)^2\mathbf{i} + [90(0)^2 - \frac{16}{3}(0)^3]\mathbf{j} + \mathbf{C}$
 $= 100\mathbf{j} \Rightarrow \mathbf{C} = 100\mathbf{j} \Rightarrow \mathbf{r} = 90t^2\mathbf{i} + (90t^2 - \frac{16}{3}t^3 + 100)\mathbf{j}$
13. $\mathbf{r} = \int \left[\left(\frac{3}{2}(t+1)^{1/2}\right)\mathbf{i} + e^{-t}\mathbf{j} + \left(\frac{1}{t+1}\right)\mathbf{k}\right] dt = (t+1)^{3/2}\mathbf{i} - e^{-t}\mathbf{j} + \ln(t+1)\mathbf{k} + \mathbf{C};$
 $\mathbf{r}(0) = (0+1)^{3/2}\mathbf{i} - e^{-0}\mathbf{j} + \ln(0+1)\mathbf{k} + \mathbf{C} = \mathbf{k} \Rightarrow \mathbf{C} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$
 $\Rightarrow \mathbf{r} = [(t+1)^{3/2} - 1]\mathbf{i} + (1 - e^{-t})\mathbf{j} + [1 + \ln(t+1)]\mathbf{k}$
14. $\mathbf{r} = \int [(t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}] dt = \left(\frac{t^4}{4} + 2t^2\right)\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{2t^3}{3}\mathbf{k} + \mathbf{C}; \mathbf{r}(0) = \left[\frac{0^4}{4} + 2(0)^2\right]\mathbf{i} + \frac{0^2}{2}\mathbf{j} + \frac{2(0)^3}{3}\mathbf{k} + \mathbf{C}$
 $= \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{C} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{r} = \left(\frac{t^4}{4} + 2t^2 + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + 1\right)\mathbf{j} + \frac{2t^3}{3}\mathbf{k}$
15. $\frac{d\mathbf{r}}{dt} = \int (-32\mathbf{k}) dt = -32t\mathbf{k} + \mathbf{C}_1; \frac{d\mathbf{r}}{dt}(0) = 8\mathbf{i} + 8\mathbf{j} \Rightarrow -32(0)\mathbf{k} + \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j} \Rightarrow \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$
 $\Rightarrow \frac{d\mathbf{r}}{dt} = 8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}; \mathbf{r} = \int (8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}) dt = 8t\mathbf{i} + 8t\mathbf{j} - 16t^2\mathbf{k} + \mathbf{C}_2; \mathbf{r}(0) = 100\mathbf{k}$
 $\Rightarrow 8(0)\mathbf{i} + 8(0)\mathbf{j} - 16(0)^2\mathbf{k} + \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{r} = 8t\mathbf{i} + 8t\mathbf{j} + (100 - 16t^2)\mathbf{k}$
16. $\frac{d\mathbf{r}}{dt} = \int -(\mathbf{i} + \mathbf{j} + \mathbf{k}) dt = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) + \mathbf{C}_1; \frac{d\mathbf{r}}{dt}(0) = \mathbf{0} \Rightarrow -(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + \mathbf{C}_1 = \mathbf{0} \Rightarrow \mathbf{C}_1 = \mathbf{0}$
 $\Rightarrow \frac{d\mathbf{r}}{dt} = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}); \mathbf{r} = \int -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) dt = -\left(\frac{t^2}{2}\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}\right) + \mathbf{C}_2; \mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$
 $\Rightarrow -\left(\frac{0^2}{2}\mathbf{i} + \frac{0^2}{2}\mathbf{j} + \frac{0^2}{2}\mathbf{k}\right) + \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k} \Rightarrow \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$
 $\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j} + \left(-\frac{t^2}{2} + 10\right)\mathbf{k}$
17. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3t\mathbf{i} - t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(4-1)\mathbf{i} + (1-2)\mathbf{j} + (4-3)\mathbf{k} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed 2 $\Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{9+1+1}}(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(3t + \frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{k}$
 $\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{k} + \mathbf{C}_2; \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{C}_2$
 $\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\mathbf{k}$
 $= \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
18. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(3-1)\mathbf{i} + (0-(-1))\mathbf{j} + (3-2)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed 2 $\Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{4+1+1}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{k}$
 $\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{k} + \mathbf{C}_2; \mathbf{r}(0) = \mathbf{i} - \mathbf{j} + 2\mathbf{k} = \mathbf{C}_2$
 $\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t + 1\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t - 1\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + 2\right)\mathbf{k} = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

19. $x = (v_0 \cos \alpha)t \Rightarrow (21 \text{ km})\left(\frac{1000 \text{ m}}{1 \text{ km}}\right) = (840 \text{ m/s})(\cos 60^\circ)t \Rightarrow t = \frac{21,000 \text{ m}}{(840 \text{ m/s})(\cos 60^\circ)} = 50 \text{ seconds}$

20. $R = \frac{v_0^2}{g} \sin 2\alpha$ and maximum R occurs when $\alpha = 45^\circ \Rightarrow 24.5 \text{ km} = \left(\frac{v_0^2}{9.8 \text{ m/s}^2}\right)(\sin 90^\circ)$
 $\Rightarrow v_0 = \sqrt{(9.8)(24,500) \text{ m}^2/\text{s}^2} = 490 \text{ m/s}$

21. (a) $t = \frac{2v_0 \sin \alpha}{g} = \frac{2(500 \text{ m/s})(\sin 45^\circ)}{9.8 \text{ m/s}^2} \approx 72.2 \text{ seconds}$; $R = \frac{v_0^2}{g} \sin 2\alpha = \frac{(500 \text{ m/s})^2}{9.8 \text{ m/s}^2} (\sin 90^\circ) \approx 25,510.2 \text{ m}$

(b) $x = (v_0 \cos \alpha)t \Rightarrow 5000 \text{ m} = (500 \text{ m/s})(\cos 45^\circ)t \Rightarrow t = \frac{5000 \text{ m}}{(500 \text{ m/s})(\cos 45^\circ)} \approx 14.14 \text{ s}$; thus,

$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y \approx (500 \text{ m/s})(\sin 45^\circ)(14.14 \text{ s}) - \frac{1}{2}(9.8 \text{ m/s}^2)(14.14 \text{ s})^2 \approx 4020 \text{ m}$

(c) $y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} = \frac{((500 \text{ m/s})(\sin 45^\circ))^2}{2(9.8 \text{ m/s}^2)} \approx 6378 \text{ m}$

22. $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 32 \text{ ft} + (32 \text{ ft/sec})(\sin 30^\circ)t - \frac{1}{2}(32 \text{ ft/sec}^2)t^2 \Rightarrow y = 32 + 16t - 16t^2$;
the ball hits the ground when $y = 0 \Rightarrow 0 = 32 + 16t - 16t^2 \Rightarrow t = -1$ or $t = 2 \Rightarrow t = 2 \text{ sec}$ since $t > 0$; thus,
 $x = (v_0 \cos \alpha)t \Rightarrow x = (32 \text{ ft/sec})(\cos 30^\circ)t = 32\left(\frac{\sqrt{3}}{2}\right)(2) \approx 55.4 \text{ ft}$

23. (a) $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 10 \text{ m} = \left(\frac{v_0^2}{9.8 \text{ m/s}^2}\right)(\sin 90^\circ) \Rightarrow v_0^2 = 98 \text{ m}^2/\text{s}^2 \Rightarrow v_0 \approx 9.9 \text{ m/s}$;

(b) $6 \text{ m} \approx \frac{(9.9 \text{ m/s})^2}{9.8 \text{ m/s}^2} (\sin 2\alpha) \Rightarrow \sin 2\alpha \approx 0.59999 \Rightarrow 2\alpha \approx 36.87^\circ$ or $143.12^\circ \Rightarrow \alpha \approx 18.4^\circ$ or 71.6°

24. $v_0 = 5 \times 10^6 \text{ m/s}$ and $x = 40 \text{ cm} = 0.4 \text{ m}$; thus $x = (v_0 \cos \alpha)t \Rightarrow 0.4 \text{ m} = (5 \times 10^6 \text{ m/s})(\cos 0^\circ)t$
 $\Rightarrow t = 0.08 \times 10^{-6} \text{ s} = 8 \times 10^{-8} \text{ s}$; also, $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
 $\Rightarrow y = (5 \times 10^6 \text{ m/s})(\sin 0^\circ)(8 \times 10^{-8} \text{ s}) - \frac{1}{2}(9.8 \text{ m/s}^2)(8 \times 10^{-8} \text{ s})^2 = -3.136 \times 10^{-14} \text{ m}$ or
 $-3.136 \times 10^{-12} \text{ cm}$. Therefore, it drops $3.136 \times 10^{-12} \text{ cm}$.

25. $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 16,000 \text{ m} = \frac{(400 \text{ m/s})^2}{9.8 \text{ m/s}^2} \sin 2\alpha \Rightarrow \sin 2\alpha = 0.98 \Rightarrow 2\alpha \approx 78.5^\circ$ or $2\alpha \approx 101.5^\circ \Rightarrow \alpha \approx 39.3^\circ$
or 50.7°

26. (a) $R = \frac{(2v_0)^2}{g} \sin 2\alpha = \frac{4v_0^2}{g} \sin 2\alpha = 4\left(\frac{v_0^2}{g} \sin \alpha\right)$ or 4 times the original range.

(b) Now, let the initial range be $R = \frac{v_0^2}{g} \sin 2\alpha$. Then we want the factor p so that pv_0 will double the range

$\Rightarrow \frac{(pv_0)^2}{g} \sin 2\alpha = 2\left(\frac{v_0^2}{g} \sin 2\alpha\right) \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$ or about 141%. The same percentage will approximately

double the height: $\frac{(pv_0 \sin \alpha)^2}{2g} = \frac{2(v_0 \sin \alpha)^2}{2g} \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$.

27. The projectile reaches its maximum height when its vertical component of velocity is zero $\Rightarrow \frac{dy}{dt} = v_0 \sin \alpha - gt = 0$

$\Rightarrow t = \frac{v_0 \sin \alpha}{g} \Rightarrow y_{\max} = (v_0 \sin \alpha)\left(\frac{v_0 \sin \alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin \alpha}{g}\right)^2 = \frac{(v_0 \sin \alpha)^2}{g} - \frac{(v_0 \sin \alpha)^2}{2g} = \frac{(v_0 \sin \alpha)^2}{2g}$. To find the flight time

we find the time when the projectile lands: $(v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 0 \Rightarrow t(v_0 \sin \alpha - \frac{1}{2}gt) = 0 \Rightarrow t = 0$ or $t = \frac{2v_0 \sin \alpha}{g}$.

$t = 0$ is the time when the projectile is fired, so $t = \frac{2v_0 \sin \alpha}{g}$ is the time when the projectile strikes the ground. The range is

the value of the horizontal component when $t = \frac{2v_0 \sin \alpha}{g} \Rightarrow R = x = (v_0 \cos \alpha)\left(\frac{2v_0 \sin \alpha}{g}\right) = \frac{v_0^2}{g}(2 \sin \alpha \cos \alpha) = \frac{v_0^2}{g} \sin 2\alpha$.

The range is largest when $\sin 2\alpha = 1 \Rightarrow \alpha = 45^\circ$.

28. When marble A is located R units downrange, we have $x = (v_0 \cos \alpha)t \Rightarrow R = (v_0 \cos \alpha)t \Rightarrow t = \frac{R}{v_0 \cos \alpha}$. At

that time the height of marble A is $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = (v_0 \sin \alpha)\left(\frac{R}{v_0 \cos \alpha}\right) - \frac{1}{2}g\left(\frac{R}{v_0 \cos \alpha}\right)^2$

$\Rightarrow y = R \tan \alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2 \cos^2 \alpha}\right)$. The height of marble B at the same time $t = \frac{R}{v_0 \cos \alpha}$ seconds is

$h = R \tan \alpha - \frac{1}{2} g t^2 = R \tan \alpha - \frac{1}{2} g \left(\frac{R^2}{v_0^2 \cos^2 \alpha} \right)$. Since the heights are the same, the marbles collide regardless of the initial velocity v_0 .

$$\begin{aligned} 29. \quad \frac{d\mathbf{r}}{dt} &= \int (-g\mathbf{j}) dt = -gt\mathbf{j} + \mathbf{C}_1 \text{ and } \frac{d\mathbf{r}}{dt}(0) = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow -g(0)\mathbf{j} + \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \\ &\Rightarrow \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}; \mathbf{r} = \int [(v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}] dt \\ &= (v_0 t \cos \alpha)\mathbf{i} + \left(v_0 t \sin \alpha - \frac{1}{2} g t^2 \right) \mathbf{j} + \mathbf{C}_2 \text{ and } \mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow [v_0(0) \cos \alpha]\mathbf{i} + [v_0(0) \sin \alpha - \frac{1}{2} g(0)^2] \mathbf{j} + \mathbf{C}_2 \\ &= x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{C}_2 = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{r} = (x_0 + v_0 t \cos \alpha)\mathbf{i} + \left(y_0 + v_0 t \sin \alpha - \frac{1}{2} g t^2 \right) \mathbf{j} \Rightarrow x = x_0 + v_0 t \cos \alpha \text{ and} \\ &y = y_0 + v_0 t \sin \alpha - \frac{1}{2} g t^2 \end{aligned}$$

$$\begin{aligned} 30. \quad \text{The maximum height is } y &= \frac{(v_0 \sin \alpha)^2}{2g} \text{ and this occurs for } x = \frac{v_0^2}{2g} \sin 2\alpha = \frac{v_0^2 \sin \alpha \cos \alpha}{g}. \text{ These equations describe} \\ \text{parametrically the points on a curve in the } xy\text{-plane associated with the maximum heights on the parabolic trajectories in} \\ \text{terms of the parameter (launch angle) } \alpha. \text{ Eliminating the parameter } \alpha, \text{ we have } x^2 &= \frac{v_0^4 \sin^2 \alpha \cos^2 \alpha}{g^2} = \frac{(v_0^4 \sin^2 \alpha)(1 - \sin^2 \alpha)}{g^2} \\ &= \frac{v_0^4 \sin^2 \alpha}{g^2} - \frac{v_0^4 \sin^4 \alpha}{g^2} = \frac{v_0^2}{g} (2y) - (2y)^2 \Rightarrow x^2 + 4y^2 - \left(\frac{2v_0^2}{g} \right) y = 0 \Rightarrow x^2 + 4 \left[y^2 - \left(\frac{v_0^2}{2g} \right) y + \frac{v_0^4}{16g^2} \right] = \frac{v_0^4}{4g^2} \\ &\Rightarrow x^2 + 4 \left(y - \frac{v_0^2}{4g} \right)^2 = \frac{v_0^4}{4g^2}, \text{ where } x \geq 0. \end{aligned}$$

31. (a) At the time t when the projectile hits the line OR we

$$\text{have } \tan \beta = \frac{y}{x}; x = [v_0 \cos(\alpha - \beta)]t \text{ and}$$

$$y = [v_0 \sin(\alpha - \beta)]t - \frac{1}{2} g t^2 < 0 \text{ since R is}$$

below level ground. Therefore let

$$|y| = \frac{1}{2} g t^2 - [v_0 \sin(\alpha - \beta)]t > 0$$

$$\text{so that } \tan \beta = \frac{[\frac{1}{2} g t^2 - v_0 \sin(\alpha - \beta)]t}{[v_0 \cos(\alpha - \beta)]t} = \frac{[\frac{1}{2} g t - v_0 \sin(\alpha - \beta)]}{v_0 \cos(\alpha - \beta)}$$

$$\Rightarrow v_0 \cos(\alpha - \beta) \tan \beta = \frac{1}{2} g t - v_0 \sin(\alpha - \beta)$$

$$\Rightarrow t = \frac{2v_0 \sin(\alpha - \beta) + 2v_0 \cos(\alpha - \beta) \tan \beta}{g}, \text{ which is the time}$$

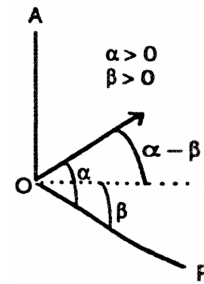
when the projectile hits the downhill slope. Therefore,

$$x = [v_0 \cos(\alpha - \beta)] \left[\frac{2v_0 \sin(\alpha - \beta) + 2v_0 \cos(\alpha - \beta) \tan \beta}{g} \right] = \frac{2v_0^2}{g} [\cos^2(\alpha - \beta) \tan \beta + \sin(\alpha - \beta) \cos(\alpha - \beta)]. \text{ If } x \text{ is}$$

$$\text{maximized, then OR is maximized: } \frac{dx}{d\alpha} = \frac{2v_0^2}{g} [-\sin 2(\alpha - \beta) \tan \beta + \cos 2(\alpha - \beta)] = 0$$

$$\Rightarrow -\sin 2(\alpha - \beta) \tan \beta + \cos 2(\alpha - \beta) = 0 \Rightarrow \tan \beta = \cot 2(\alpha - \beta) \Rightarrow 2(\alpha - \beta) = 90^\circ - \beta$$

$$\Rightarrow \alpha - \beta = \frac{1}{2} (90^\circ - \beta) \Rightarrow \alpha = \frac{1}{2} (90^\circ + \beta) = \frac{1}{2} \text{ of } \angle \text{AOR.}$$



- (b) At the time t when the projectile hits OR we have

$$\tan \beta = \frac{y}{x}; x = [v_0 \cos(\alpha + \beta)]t \text{ and}$$

$$y = [v_0 \sin(\alpha + \beta)]t - \frac{1}{2} g t^2$$

$$\Rightarrow \tan \beta = \frac{[v_0 \sin(\alpha + \beta)]t - \frac{1}{2} g t^2}{[v_0 \cos(\alpha + \beta)]t} = \frac{[v_0 \sin(\alpha + \beta) - \frac{1}{2} g t]}{v_0 \cos(\alpha + \beta)}$$

$$\Rightarrow v_0 \cos(\alpha + \beta) \tan \beta = v_0 \sin(\alpha + \beta) - \frac{1}{2} g t$$

$$\Rightarrow t = \frac{2v_0 \sin(\alpha + \beta) - 2v_0 \cos(\alpha + \beta) \tan \beta}{g}, \text{ which is the time}$$

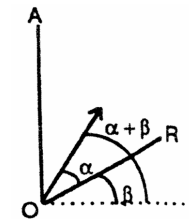
when the projectile hits the uphill slope. Therefore,

$$x = [v_0 \cos(\alpha + \beta)] \left[\frac{2v_0 \sin(\alpha + \beta) - 2v_0 \cos(\alpha + \beta) \tan \beta}{g} \right] = \frac{2v_0^2}{g} [\sin(\alpha + \beta) \cos(\alpha + \beta) - \cos^2(\alpha + \beta) \tan \beta]. \text{ If } x \text{ is}$$

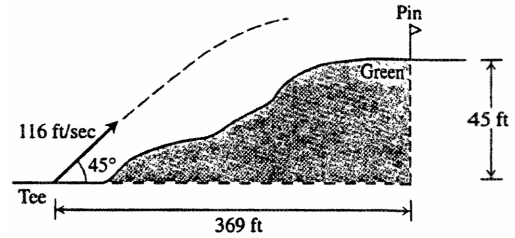
$$\text{maximized, then OR is maximized: } \frac{dx}{d\alpha} = \frac{2v_0^2}{g} [\cos 2(\alpha + \beta) + \sin 2(\alpha + \beta) \tan \beta] = 0$$

$$\Rightarrow \cos 2(\alpha + \beta) + \sin 2(\alpha + \beta) \tan \beta = 0 \Rightarrow \cot 2(\alpha + \beta) + \tan \beta = 0 \Rightarrow \cot 2(\alpha + \beta) = -\tan \beta$$

$$= \tan(-\beta) \Rightarrow 2(\alpha + \beta) = 90^\circ - (-\beta) = 90^\circ + \beta \Rightarrow \alpha = \frac{1}{2} (90^\circ - \beta) = \frac{1}{2} \text{ of } \angle \text{AOR. Therefore } v_0 \text{ would bisect } \angle \text{AOR for maximum range uphill.}$$



32. $v_0 = 116 \text{ ft/sec}$, $\alpha = 45^\circ$, and $x = (v_0 \cos \alpha)t$
 $\Rightarrow 369 = (116 \cos 45^\circ)t \Rightarrow t \approx 4.50 \text{ sec}$;
 also $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
 $\Rightarrow y = (116 \sin 45^\circ)(4.50) - \frac{1}{2}(32)(4.50)^2$
 $\approx 45.11 \text{ ft}$. It will take the ball 4.50 sec to travel
 369 ft. At that time the ball will be 45.11 ft in
 the air and will hit the green past the pin.



33. (a) (Assuming that "x" is zero at the point of impact:)
 $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = (35 \cos 27^\circ)t$ and $y(t) = 4 + (35 \sin 27^\circ)t - 16t^2$.
 (b) $y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} + 4 = \frac{(35 \sin 27^\circ)^2}{64} + 4 \approx 7.945$ feet, which is reached at $t = \frac{v_0 \sin \alpha}{g} = \frac{35 \sin 27^\circ}{32} \approx 0.497$ seconds.
 (c) For the time, solve $y = 4 + (35 \sin 27^\circ)t - 16t^2 = 0$ for t , using the quadratic formula
 $t = \frac{35 \sin 27^\circ + \sqrt{(-35 \sin 27^\circ)^2 + 256}}{32} \approx 1.201$ sec. Then the range is about $x(1.201) = (35 \cos 27^\circ)(1.201) \approx 37.453$ feet.
 (d) For the time, solve $y = 4 + (35 \sin 27^\circ)t - 16t^2 = 7$ for t , using the quadratic formula
 $t = \frac{35 \sin 27^\circ + \sqrt{(-35 \sin 27^\circ)^2 - 192}}{32} \approx 0.254$ and 0.740 seconds. At those times the ball is about
 $x(0.254) = (35 \cos 27^\circ)(0.254) \approx 7.921$ feet and $x(0.740) = (35 \cos 27^\circ)(0.740) \approx 23.077$ feet the impact point,
 or about $37.453 - 7.921 \approx 29.532$ feet and $37.453 - 23.077 \approx 14.376$ feet from the landing spot.
 (e) Yes. It changes things because the ball won't clear the net ($y_{\max} \approx 7.945$).
34. $x = x_0 + (v_0 \cos \alpha)t = 0 + (v_0 \cos 40^\circ)t \approx 0.766 v_0 t$ and $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 6.5 + (v_0 \sin 40^\circ)t - 16t^2$
 $\approx 6.5 + 0.643 v_0 t - 16t^2$; now the shot went 73.833 ft $\Rightarrow 73.833 = 0.766 v_0 t \Rightarrow t \approx \frac{96.383}{v_0}$ sec; the shot lands when $y = 0$
 $\Rightarrow 0 = 6.5 + (0.643)(96.383) - 16 \left(\frac{96.383}{v_0} \right)^2 \Rightarrow 0 \approx 68.474 - \frac{148.635}{v_0^2} \Rightarrow v_0 \approx \sqrt{\frac{148.635}{68.474}} \approx 46.6 \text{ ft/sec}$, the shot's initial
 speed
35. Flight time = 1 sec and the measure of the angle of elevation is about 64° (using a protractor) so that $t = \frac{2v_0 \sin \alpha}{g}$
 $\Rightarrow 1 = \frac{2v_0 \sin 64^\circ}{32} \Rightarrow v_0 \approx 17.80 \text{ ft/sec}$. Then $y_{\max} = \frac{(17.80 \sin 64^\circ)^2}{2(32)} \approx 4.00$ ft and $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow R = \frac{(17.80)^2}{32} \sin 128^\circ$
 $\approx 7.80 \text{ ft} \Rightarrow$ the engine traveled about 7.80 ft in 1 sec $\Rightarrow$ the engine velocity was about 7.80 ft/sec
36. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = (145 \cos 23^\circ - 14)t$ and $y(t) = 2.5 + (145 \sin 23^\circ)t - 16t^2$.
 (b) $y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} + 2.5 = \frac{(145 \sin 23^\circ)^2}{64} + 2.5 \approx 52.655$ feet, which is reached at $t = \frac{v_0 \sin \alpha}{g} = \frac{145 \sin 23^\circ}{32} \approx 1.771$ seconds.
 (c) For the time, solve $y = 2.5 + (145 \sin 23^\circ)t - 16t^2 = 0$ for t , using the quadratic formula
 $t = \frac{145 \sin 23^\circ + \sqrt{(145 \sin 23^\circ)^2 + 160}}{32} \approx 3.585$ sec. Then the range at $t \approx 3.585$ is about $x = (145 \cos 23^\circ - 14)(3.585)$
 ≈ 428.311 feet.
 (d) For the time, solve $y = 2.5 + (145 \sin 23^\circ)t - 16t^2 = 20$ for t , using the quadratic formula
 $t = \frac{145 \sin 23^\circ + \sqrt{(145 \sin 23^\circ)^2 - 1120}}{32} \approx 0.342$ and 3.199 seconds. At those times the ball is about
 $x(0.342) = (145 \cos 23^\circ - 14)(0.342) \approx 40.860$ feet from home plate and $x(3.199) = (145 \cos 23^\circ - 14)(3.199)$
 ≈ 382.195 feet from home plate.
 (e) Yes. According to part (d), the ball is still 20 feet above the ground when it is 382 feet from home plate.
37. $\frac{d^2 \mathbf{r}}{dt^2} + k \frac{d\mathbf{r}}{dt} = -g\mathbf{j} \Rightarrow P(t) = k$ and $Q(t) = -g\mathbf{j} \Rightarrow \int P(t) dt = kt \Rightarrow v(t) = e^{\int P(t) dt} = e^{kt} \Rightarrow \frac{d\mathbf{r}}{dt} = \frac{1}{v(t)} \int v(t) Q(t) dt$
 $= -ge^{-kt} \int e^{kt} \mathbf{j} dt = -ge^{-kt} \left[\frac{e^{kt}}{k} \mathbf{j} + \mathbf{C}_1 \right] = -\frac{g}{k} \mathbf{j} + \mathbf{C} e^{-kt}$, where $\mathbf{C} = -g\mathbf{C}_1$; apply the initial condition:
 $\left. \frac{d\mathbf{r}}{dt} \right|_{t=0} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} = -\frac{g}{k} \mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = (v_0 \cos \alpha)\mathbf{i} + \left(\frac{g}{k} + v_0 \sin \alpha \right) \mathbf{j}$
 $\Rightarrow \frac{d\mathbf{r}}{dt} = (v_0 e^{-kt} \cos \alpha)\mathbf{i} + \left(-\frac{g}{k} + e^{-kt} \left(\frac{g}{k} + v_0 \sin \alpha \right) \right) \mathbf{j}$, $\mathbf{r} = \int \left[(v_0 e^{-kt} \cos \alpha)\mathbf{i} + \left(-\frac{g}{k} + e^{-kt} \left(\frac{g}{k} + v_0 \sin \alpha \right) \right) \mathbf{j} \right] dt$

$$= \left(-\frac{v_0}{k} e^{-kt} \cos \alpha\right) \mathbf{i} + \left(-\frac{gt}{k} - \frac{e^{-kt}}{k} \left(\frac{g}{k} + v_0 \sin \alpha\right)\right) \mathbf{j} + \mathbf{C}_2; \text{ apply the initial condition:}$$

$$\mathbf{r}(0) = \mathbf{0} = \left(-\frac{v_0}{k} \cos \alpha\right) \mathbf{i} + \left(-\frac{g}{k^2} - \frac{v_0 \sin \alpha}{k}\right) \mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \left(\frac{v_0}{k} \cos \alpha\right) \mathbf{i} + \left(\frac{g}{k^2} + \frac{v_0 \sin \alpha}{k}\right) \mathbf{j}$$

$$\Rightarrow \mathbf{r}(t) = \left(\frac{v_0}{k} (1 - e^{-kt}) \cos \alpha\right) \mathbf{i} + \left(\frac{v_0}{k} (1 - e^{-kt}) \sin \alpha + \frac{g}{k^2} (1 - kt - e^{-kt})\right) \mathbf{j}$$

38. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\cos 20^\circ)$ and $y(t) = 3 + \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\sin 20^\circ) + \left(\frac{32}{0.12^2}\right)(1 - 0.12t - e^{-0.12t})$
- (b) Solve graphically using a calculator or CAS: At $t \approx 1.484$ seconds the ball reaches a maximum height of about 40.435 feet.
- (c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.126 seconds. The range is about $x(3.126) = \left(\frac{152}{0.12}\right)(1 - e^{-0.12(3.126)})(\cos 20^\circ) \approx 372.311$ feet.
- (d) Use a graphing calculator or CAS to find that $y = 30$ for $t \approx 0.689$ and 2.305 seconds, at which times the ball is about $x(0.689) \approx 94.454$ feet and $x(2.305) \approx 287.621$ feet from home plate.
- (e) Yes, the batter has hit a home run since a graph of the trajectory shows that the ball is more than 14 feet above the ground when it passes over the fence.

39. (a) $\int_a^b k\mathbf{r}(t) dt = \int_a^b [kf(t)\mathbf{i} + kg(t)\mathbf{j} + kh(t)\mathbf{k}] dt = \int_a^b [kf(t)] dt \mathbf{i} + \int_a^b [kg(t)] dt \mathbf{j} + \int_a^b [kh(t)] dt \mathbf{k}$

$$= k \left(\int_a^b f(t) dt \mathbf{i} + \int_a^b g(t) dt \mathbf{j} + \int_a^b h(t) dt \mathbf{k} \right) = k \int_a^b \mathbf{r}(t) dt$$

(b) $\int_a^b [\mathbf{r}_1(t) \pm \mathbf{r}_2(t)] dt = \int_a^b ([f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] \pm [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}]) dt$

$$= \int_a^b ([f_1(t) \pm f_2(t)] \mathbf{i} + [g_1(t) \pm g_2(t)] \mathbf{j} + [h_1(t) \pm h_2(t)] \mathbf{k}) dt$$

$$= \int_a^b [f_1(t) \pm f_2(t)] dt \mathbf{i} + \int_a^b [g_1(t) \pm g_2(t)] dt \mathbf{j} + \int_a^b [h_1(t) \pm h_2(t)] dt \mathbf{k}$$

$$= \left[\int_a^b f_1(t) dt \mathbf{i} \pm \int_a^b f_2(t) dt \mathbf{i} \right] + \left[\int_a^b g_1(t) dt \mathbf{j} \pm \int_a^b g_2(t) dt \mathbf{j} \right] + \left[\int_a^b h_1(t) dt \mathbf{k} \pm \int_a^b h_2(t) dt \mathbf{k} \right]$$

$$= \int_a^b \mathbf{r}_1(t) dt \pm \int_a^b \mathbf{r}_2(t) dt$$

(c) Let $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Then $\int_a^b \mathbf{C} \cdot \mathbf{r}(t) dt = \int_a^b [c_1f(t) + c_2g(t) + c_3h(t)] dt$

$$= c_1 \int_a^b f(t) dt + c_2 \int_a^b g(t) dt + c_3 \int_a^b h(t) dt = \mathbf{C} \cdot \int_a^b \mathbf{r}(t) dt;$$

$$\int_a^b \mathbf{C} \times \mathbf{r}(t) dt = \int_a^b [c_2h(t) - c_3g(t)] \mathbf{i} + [c_3f(t) - c_1h(t)] \mathbf{j} + [c_1g(t) - c_2f(t)] \mathbf{k} dt$$

$$= \left[c_2 \int_a^b h(t) dt - c_3 \int_a^b g(t) dt \right] \mathbf{i} + \left[c_3 \int_a^b f(t) dt - c_1 \int_a^b h(t) dt \right] \mathbf{j} + \left[c_1 \int_a^b g(t) dt - c_2 \int_a^b f(t) dt \right] \mathbf{k}$$

$$= \mathbf{C} \times \int_a^b \mathbf{r}(t) dt$$

40. (a) Let u and $\mathbf{r}$ be continuous on $[a, b]$. Then $\lim_{t \rightarrow t_0} u(t)\mathbf{r}(t) = \lim_{t \rightarrow t_0} [u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$

$$= u(t_0)f(t_0)\mathbf{i} + u(t_0)g(t_0)\mathbf{j} + u(t_0)h(t_0)\mathbf{k} = u(t_0)\mathbf{r}(t_0) \Rightarrow u\mathbf{r} \text{ is continuous for every } t_0 \text{ in } [a, b].$$

(b) Let u and $\mathbf{r}$ be differentiable. Then $\frac{d}{dt}(u\mathbf{r}) = \frac{d}{dt}[u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$

$$= \left(\frac{du}{dt}f(t) + u(t)\frac{df}{dt}\right)\mathbf{i} + \left(\frac{du}{dt}g(t) + u(t)\frac{dg}{dt}\right)\mathbf{j} + \left(\frac{du}{dt}h(t) + u(t)\frac{dh}{dt}\right)\mathbf{k}$$

$$= [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] \frac{du}{dt} + u(t) \left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = \mathbf{r} \frac{du}{dt} + u \frac{d\mathbf{r}}{dt}$$

41. (a) If $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on I , then $\frac{d\mathbf{R}_1}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{dg_1}{dt}\mathbf{j} + \frac{dh_1}{dt}\mathbf{k} = \frac{df_2}{dt}\mathbf{i} + \frac{dg_2}{dt}\mathbf{j} + \frac{dh_2}{dt}\mathbf{k}$

$$= \frac{d\mathbf{R}_2}{dt} \Rightarrow \frac{df_1}{dt} = \frac{df_2}{dt}, \frac{dg_1}{dt} = \frac{dg_2}{dt}, \frac{dh_1}{dt} = \frac{dh_2}{dt} \Rightarrow f_1(t) = f_2(t) + c_1, g_1(t) = g_2(t) + c_2, h_1(t) = h_2(t) + c_3$$

$$\Rightarrow f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k} = [f_2(t) + c_1]\mathbf{i} + [g_2(t) + c_2]\mathbf{j} + [h_2(t) + c_3]\mathbf{k} \Rightarrow \mathbf{R}_1(t) = \mathbf{R}_2(t) + \mathbf{C}, \text{ where}$$

$$\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

- (b) Let $\mathbf{R}(t)$ be an antiderivative of $\mathbf{r}(t)$ on I . Then $\mathbf{R}'(t) = \mathbf{r}(t)$. If $\mathbf{U}(t)$ is an antiderivative of $\mathbf{r}(t)$ on I , then $\mathbf{U}'(t) = \mathbf{r}(t)$. Thus $\mathbf{U}'(t) = \mathbf{R}'(t)$ on $I \Rightarrow \mathbf{U}(t) = \mathbf{R}(t) + \mathbf{C}$.

$$42. \frac{d}{dt} \int_a^t \mathbf{r}(\tau) d\tau = \frac{d}{dt} \int_a^t [f(\tau)\mathbf{i} + g(\tau)\mathbf{j} + h(\tau)\mathbf{k}] d\tau = \frac{d}{dt} \int_a^t f(\tau) d\tau \mathbf{i} + \frac{d}{dt} \int_a^t g(\tau) d\tau \mathbf{j} + \frac{d}{dt} \int_a^t h(\tau) d\tau \mathbf{k} \\ = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \mathbf{r}(t). \text{ Since } \frac{d}{dt} \int_a^t \mathbf{r}(\tau) d\tau = \mathbf{r}(t), \text{ we have that } \int_a^t \mathbf{r}(\tau) d\tau \text{ is an antiderivative of } \mathbf{r}. \\ \text{If } \mathbf{R} \text{ is any antiderivative of } \mathbf{r}, \text{ then } \mathbf{R}(t) = \int_a^t \mathbf{r}(\tau) d\tau + \mathbf{C} \text{ by Exercise 41(b). Then } \mathbf{R}(a) = \int_a^a \mathbf{r}(\tau) d\tau + \mathbf{C} \\ = \mathbf{0} + \mathbf{C} \Rightarrow \mathbf{C} = \mathbf{R}(a) \Rightarrow \int_a^t \mathbf{r}(\tau) d\tau = \mathbf{R}(t) - \mathbf{C} = \mathbf{R}(t) - \mathbf{R}(a) \Rightarrow \int_a^b \mathbf{r}(\tau) d\tau = \mathbf{R}(b) - \mathbf{R}(a).$$

43. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08t})(152 \cos 20^\circ - 17.6)$ and $y(t) = 3 + \left(\frac{152}{0.08}\right)(1 - e^{-0.08t})(\sin 20^\circ) + \left(\frac{32}{0.08^2}\right)(1 - 0.08t - e^{-0.08t})$
 (b) Solve graphically using a calculator or CAS: At $t \approx 1.527$ seconds the ball reaches a maximum height of about 41.893 feet.
 (c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.181 seconds. The range is about $x(3.181) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08(3.181)})(152 \cos 20^\circ - 17.6) \approx 351.734$ feet.
 (d) Use a graphing calculator or CAS to find that $y = 35$ for $t \approx 0.877$ and 2.190 seconds, at which times the ball is about $x(0.877) \approx 106.028$ feet and $x(2.190) \approx 251.530$ feet from home plate.
 (e) No; the range is less than 380 feet. To find the wind needed for a home run, first use the method of part (d) to find that $y = 20$ at $t \approx 0.376$ and 2.716 seconds. Then define $x(w) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08(2.716)})(152 \cos 20^\circ + w)$, and solve $x(w) = 380$ to find $w \approx 12.846$ ft/sec.

$$44. y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} \Rightarrow \frac{3}{4} y_{\max} = \frac{3(v_0 \sin \alpha)^2}{8g} \text{ and } y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow \frac{3(v_0 \sin \alpha)^2}{8g} = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \\ \Rightarrow 3(v_0 \sin \alpha)^2 = (8gv_0 \sin \alpha)t - 4g^2t^2 \Rightarrow 4g^2t^2 - (8gv_0 \sin \alpha)t + 3(v_0 \sin \alpha)^2 = 0 \Rightarrow 2gt - 3v_0 \sin \alpha = 0 \text{ or } \\ 2gt - v_0 \sin \alpha = 0 \Rightarrow t = \frac{3v_0 \sin \alpha}{2g} \text{ or } t = \frac{v_0 \sin \alpha}{2g}. \text{ Since the time it takes to reach } y_{\max} \text{ is } t_{\max} = \frac{v_0 \sin \alpha}{g}, \\ \text{then the time it takes the projectile to reach } \frac{3}{4} \text{ of } y_{\max} \text{ is the shorter time } t = \frac{v_0 \sin \alpha}{2g} \text{ or half the time it takes} \\ \text{to reach the maximum height.}$$

13.3 ARC LENGTH IN SPACE

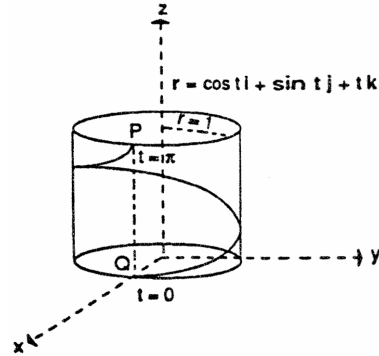
- $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + \sqrt{5}t\mathbf{k} \Rightarrow \mathbf{v} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + \sqrt{5}\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (\sqrt{5})^2} = \sqrt{4 \sin^2 t + 4 \cos^2 t + 5} = 3; \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \left(-\frac{2}{3} \sin t\right)\mathbf{i} + \left(\frac{2}{3} \cos t\right)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k} \text{ and Length} = \int_0^\pi |\mathbf{v}| dt = \int_0^\pi 3 dt = [3t]_0^\pi = 3\pi$
- $\mathbf{r} = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12 \cos 2t)\mathbf{i} + (-12 \sin 2t)\mathbf{j} + 5\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{(12 \cos 2t)^2 + (-12 \sin 2t)^2 + 5^2} = \sqrt{144 \cos^2 2t + 144 \sin^2 2t + 25} = 13; \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \left(\frac{12}{13} \cos 2t\right)\mathbf{i} - \left(\frac{12}{13} \sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k} \text{ and Length} = \int_0^\pi |\mathbf{v}| dt = \int_0^\pi 13 dt = [13t]_0^\pi = 13\pi$
- $\mathbf{r} = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (t^{1/2})^2} = \sqrt{1+t}; \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k}$
 $\text{and Length} = \int_0^8 \sqrt{1+t} dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^8 = \frac{52}{3}$
- $\mathbf{r} = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}; \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
 $\text{and Length} = \int_0^3 \sqrt{3} dt = \left[\sqrt{3}t\right]_0^3 = 3\sqrt{3}$

5. $\mathbf{r} = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k} \Rightarrow \mathbf{v} = (-3 \cos^2 t \sin t)\mathbf{j} + (3 \sin^2 t \cos t)\mathbf{k} \Rightarrow |\mathbf{v}|$
 $= \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} = \sqrt{(9 \cos^2 t \sin^2 t)(\cos^2 t + \sin^2 t)} = 3 |\cos t \sin t|;$
 $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-3 \cos^2 t \sin t}{3 |\cos t \sin t|} \mathbf{j} + \frac{3 \sin^2 t \cos t}{3 |\cos t \sin t|} \mathbf{k} = (-\cos t)\mathbf{j} + (\sin t)\mathbf{k}, \text{ if } 0 \leq t \leq \frac{\pi}{2}, \text{ and}$
 $\text{Length} = \int_0^{\pi/2} 3 |\cos t \sin t| dt = \int_0^{\pi/2} 3 \cos t \sin t dt = \int_0^{\pi/2} \frac{3}{2} \sin 2t dt = \left[-\frac{3}{4} \cos 2t\right]_0^{\pi/2} = \frac{3}{2}$
6. $\mathbf{r} = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k} \Rightarrow \mathbf{v} = 18t^2\mathbf{i} - 6t^2\mathbf{j} - 9t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2} = \sqrt{441t^4} = 21t^2;$
 $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{18t^2}{21t^2} \mathbf{i} - \frac{6t^2}{21t^2} \mathbf{j} - \frac{9t^2}{21t^2} \mathbf{k} = \frac{6}{7} \mathbf{i} - \frac{2}{7} \mathbf{j} - \frac{3}{7} \mathbf{k} \text{ and } \text{Length} = \int_1^2 21t^2 dt = [7t^3]_1^2 = 49$
7. $\mathbf{r} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + \frac{2\sqrt{2}}{3} t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} + \left(\sqrt{2} t^{1/2}\right)\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + \left(\sqrt{2} t\right)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1, \text{ if } t \geq 0;$
 $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - t \sin t}{t+1}\right)\mathbf{i} + \left(\frac{\sin t + t \cos t}{t+1}\right)\mathbf{j} + \left(\frac{\sqrt{2} t^{1/2}}{t+1}\right)\mathbf{k} \text{ and } \text{Length} = \int_0^{\pi} (t+1) dt = \left[\frac{t^2}{2} + t\right]_0^{\pi} = \frac{\pi^2}{2} + \pi$
8. $\mathbf{r} = (t \sin t + \cos t)\mathbf{i} + (t \cos t - \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (\sin t + t \cos t - \sin t)\mathbf{i} + (\cos t - t \sin t - \cos t)\mathbf{j}$
 $= (t \cos t)\mathbf{i} - (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (-t \sin t)^2} = \sqrt{t^2} = |t| = t \text{ if } \sqrt{2} \leq t \leq 2; \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \left(\frac{t \cos t}{t}\right)\mathbf{i} - \left(\frac{t \sin t}{t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} \text{ and } \text{Length} = \int_{\sqrt{2}}^2 t dt = \left[\frac{t^2}{2}\right]_{\sqrt{2}}^2 = 1$
9. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (5 \cos t)\mathbf{i} - (5 \sin t)\mathbf{j} + 12\mathbf{k}$ and $26\pi = \int_0^{t_0} \sqrt{25 \cos^2 t + 25 \sin^2 t + 144} dt$
 $= \int_0^{t_0} 13 dt = 13t_0 \Rightarrow t_0 = 2\pi, \text{ and the point is } P(2\pi) = (5 \sin 2\pi, 5 \cos 2\pi, 24\pi) = (0, 5, 24\pi)$
10. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (12 \cos t)\mathbf{i} + (12 \sin t)\mathbf{j} + 5\mathbf{k}$ and
 $-13\pi = \int_0^{t_0} \sqrt{144 \cos^2 t + 144 \sin^2 t + 25} dt = \int_0^{t_0} 13 dt = 13t_0 \Rightarrow t_0 = -\pi, \text{ and the point is}$
 $P(-\pi) = (12 \sin(-\pi), -12 \cos(-\pi), -5\pi) = (0, 12, -5\pi)$
11. $\mathbf{r} = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v} = (-4 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4 \sin t)^2 + (4 \cos t)^2 + 3^2}$
 $= \sqrt{25} = 5 \Rightarrow s(t) = \int_0^t 5 d\tau = 5t \Rightarrow \text{Length} = s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$
12. $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + \sin t + t \cos t)\mathbf{i} + (\cos t - \cos t + t \sin t)\mathbf{j}$
 $= (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} = t, \text{ since } \frac{\pi}{2} \leq t \leq \pi \Rightarrow s(t) = \int_0^t \tau d\tau = \frac{t^2}{2}$
 $\Rightarrow \text{Length} = s(\pi) - s\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{3\pi^2}{8}$
13. $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2} = \sqrt{3e^{2t}} = \sqrt{3} e^t \Rightarrow s(t) = \int_0^t \sqrt{3} e^{\tau} d\tau$
 $= \sqrt{3} e^t - \sqrt{3} \Rightarrow \text{Length} = s(0) - s(-\ln 4) = 0 - \left(\sqrt{3} e^{-\ln 4} - \sqrt{3}\right) = \frac{3\sqrt{3}}{4}$
14. $\mathbf{r} = (1 + 2t)\mathbf{i} + (1 + 3t)\mathbf{j} + (6 - 6t)\mathbf{k} \Rightarrow \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + 3^2 + (-6)^2} = 7 \Rightarrow s(t) = \int_0^t 7 d\tau = 7t$
 $\Rightarrow \text{Length} = s(0) - s(-1) = 0 - (-7) = 7$

$$15. \mathbf{r} = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1 - t^2)\mathbf{k} \Rightarrow \mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + (-2t)^2} = \sqrt{4 + 4t^2} \\ = 2\sqrt{1 + t^2} \Rightarrow \text{Length} = \int_0^1 2\sqrt{1 + t^2} dt = \left[2\left(\frac{1}{2}\sqrt{1 + t^2} + \frac{1}{2}\ln\left(t + \sqrt{1 + t^2}\right)\right) \right]_0^1 = \sqrt{2} + \ln(1 + \sqrt{2})$$

16. Let the helix make one complete turn from $t = 0$ to $t = 2\pi$.

Note that the radius of the cylinder is 1 $\Rightarrow$ the circumference of the base is 2π . When $t = 2\pi$, the point P is $(\cos 2\pi, \sin 2\pi, 2\pi) = (1, 0, 2\pi) \Rightarrow$ the cylinder is 2π units high. Cut the cylinder along PQ and flatten. The resulting rectangle has a width equal to the circumference of the cylinder $= 2\pi$ and a height equal to 2π , the height of the cylinder. Therefore, the rectangle is a square and the portion of the helix from $t = 0$ to $t = 2\pi$ is its diagonal.



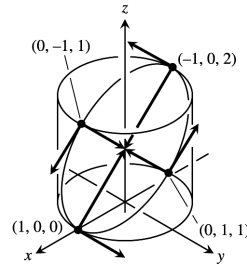
17. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1 - \cos t)\mathbf{k}$, $0 \leq t \leq 2\pi \Rightarrow x = \cos t$, $y = \sin t$, $z = 1 - \cos t \Rightarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, a right circular cylinder with the z -axis as the axis and radius $= 1$. Therefore $P(\cos t, \sin t, 1 - \cos t)$ lies on the cylinder $x^2 + y^2 = 1$; $t = 0 \Rightarrow P(1, 0, 0)$ is on the curve; $t = \frac{\pi}{2} \Rightarrow Q(0, 1, 1)$ is on the curve; $t = \pi \Rightarrow R(-1, 0, 2)$ is on the curve. Then $\vec{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\vec{PR} = -2\mathbf{i} + 2\mathbf{k}$

$$\Rightarrow \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{k} \text{ is a vector normal to the plane of } P, Q, \text{ and } R. \text{ Then the}$$

plane containing P, Q, and R has an equation $2x + 2z = 2(1) + 2(0)$ or $x + z = 1$. Any point on the curve will satisfy this equation since $x + z = \cos t + (1 - \cos t) = 1$. Therefore, any point on the curve lies on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1 \Rightarrow$ the curve is an ellipse.

- (b) $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} = \sqrt{1 + \sin^2 t} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \\ = \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k}}{\sqrt{1 + \sin^2 t}} \Rightarrow \mathbf{T}(0) = \mathbf{j}$, $\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}$, $\mathbf{T}(\pi) = -\mathbf{j}$, $\mathbf{T}\left(\frac{3\pi}{2}\right) = \frac{\mathbf{i} - \mathbf{k}}{\sqrt{2}}$

- (c) $\mathbf{a} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$; $\mathbf{n} = \mathbf{i} + \mathbf{k}$ is normal to the plane $x + z = 1 \Rightarrow \mathbf{n} \cdot \mathbf{a} = -\cos t + \cos t = 0 \Rightarrow \mathbf{a}$ is orthogonal to $\mathbf{n} \Rightarrow \mathbf{a}$ is parallel to the plane; $\mathbf{a}(0) = -\mathbf{i} + \mathbf{k}$, $\mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{j}$, $\mathbf{a}(\pi) = \mathbf{i} - \mathbf{k}$, $\mathbf{a}\left(\frac{3\pi}{2}\right) = \mathbf{j}$



(d) $|\mathbf{v}| = \sqrt{1 + \sin^2 t}$ (See part (b)) $\Rightarrow L = \int_0^{2\pi} \sqrt{1 + \sin^2 t} dt$

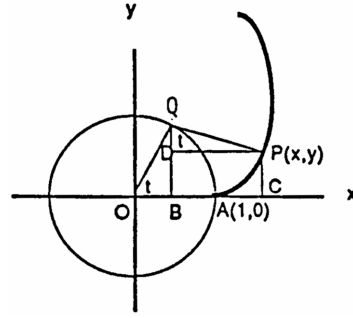
(e) $L \approx 7.64$ (by *Mathematica*)

18. (a) $\mathbf{r} = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin 4t)\mathbf{i} + (4\cos 4t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4\sin 4t)^2 + (4\cos 4t)^2 + 4^2} \\ = \sqrt{32} = 4\sqrt{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 4\sqrt{2} dt = \left[4\sqrt{2}t \right]_0^{\pi/2} = 2\pi\sqrt{2}$

(b) $\mathbf{r} = \left(\cos \frac{t}{2}\right)\mathbf{i} + \left(\sin \frac{t}{2}\right)\mathbf{j} + \frac{t}{2}\mathbf{k} \Rightarrow \mathbf{v} = \left(-\frac{1}{2}\sin \frac{t}{2}\right)\mathbf{i} + \left(\frac{1}{2}\cos \frac{t}{2}\right)\mathbf{j} + \frac{1}{2}\mathbf{k} \\ \Rightarrow |\mathbf{v}| = \sqrt{\left(-\frac{1}{2}\sin \frac{t}{2}\right)^2 + \left(\frac{1}{2}\cos \frac{t}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2} \Rightarrow \text{Length} = \int_0^{4\pi} \frac{\sqrt{2}}{2} dt = \left[\frac{\sqrt{2}}{2}t \right]_0^{4\pi} = 2\pi\sqrt{2}$

(c) $\mathbf{r} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + (-1)^2} = \sqrt{1 + 1} \\ = \sqrt{2} \Rightarrow \text{Length} = \int_{-2\pi}^0 \sqrt{2} dt = \left[\sqrt{2}t \right]_{-2\pi}^0 = 2\pi\sqrt{2}$

19. $\angle PQB = \angle QOB = t$ and $PQ = \text{arc}(AQ) = t$ since
 $PQ = \text{length of the unwound string} = \text{length of arc}(AQ)$;
 thus $x = OB + BC = OB + DP = \cos t + t \sin t$, and
 $y = PC = QB - QD = \sin t - t \cos t$



20. $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + t \cos t + \sin t)\mathbf{i} + (\cos t - (t(-\sin t) + \cos t))\mathbf{j}$
 $= (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} = |t| = t, t \geq 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{t \cos t}{t}\mathbf{i} + \frac{t \sin t}{t}\mathbf{j}$
 $= \cos t \mathbf{i} + \sin t \mathbf{j}$
21. $\mathbf{v} = \frac{d}{dt}(x_0 + t u_1)\mathbf{i} + \frac{d}{dt}(y_0 + t u_2)\mathbf{j} + \frac{d}{dt}(z_0 + t u_3)\mathbf{k} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \mathbf{u}$, so $s(t) = \int_0^t |\mathbf{v}| dt = \int_0^t |\mathbf{u}| d\tau = \int_0^t 1 d\tau = t$
22. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}(t)| = \sqrt{(1)^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$. $(0, 0, 0) \Rightarrow t = 0$
 and $(2, 4, 8) \Rightarrow t = 2$. Thus $L = \int_0^2 |\mathbf{v}(t)| dt = \int_0^2 \sqrt{1 + 4t^2 + 9t^4} dt$. Using Simpson's rule with $n = 10$ and
 $\Delta x = \frac{2-0}{10} = 0.2 \Rightarrow L \approx \frac{0.2}{3} \left(|\mathbf{v}(0)| + 4|\mathbf{v}(0.2)| + 2|\mathbf{v}(0.4)| + 4|\mathbf{v}(0.6)| + 2|\mathbf{v}(0.8)| + 4|\mathbf{v}(1)| + 2|\mathbf{v}(1.2)| + 4|\mathbf{v}(1.4)| \right.$
 $\left. + 2|\mathbf{v}(1.6)| + 4|\mathbf{v}(1.8)| + |\mathbf{v}(2)| \right) \approx \frac{0.2}{3} \left(1 + 4(1.0837) + 2(1.3676) + 4(1.8991) + 2(2.6919) + 4(3.7417) \right.$
 $\left. + 2(5.0421) + 4(6.5890) + 2(8.3800) + 4(10.4134) + 12.6886 \right) = \frac{0.2}{3}(143.5594) \approx 9.5706$

13.4 CURVATURE AND NORMAL VECTORS OF A CURVE

1. $\mathbf{r} = t\mathbf{i} + \ln(\cos t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + \left(\frac{-\sin t}{\cos t}\right)\mathbf{j} = \mathbf{i} - (\tan t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-\tan t)^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$, since
 $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sec t}\right)\mathbf{i} - \left(\frac{\tan t}{\sec t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$
 $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t} \cdot 1 = \cos t$.
2. $\mathbf{r} = \ln(\sec t)\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{\sec t \tan t}{\sec t}\right)\mathbf{i} + \mathbf{j} = (\tan t)\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(\tan t)^2 + 1^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$,
 since $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\tan t}{\sec t}\right)\mathbf{i} - \left(\frac{1}{\sec t}\right)\mathbf{j} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \frac{d\mathbf{T}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$
 $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t} \cdot 1 = \cos t$.
3. $\mathbf{r} = (2t + 3)\mathbf{i} + (5 - t^2)\mathbf{j} \Rightarrow \mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-2t)^2} = 2\sqrt{1 + t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{2\sqrt{1+t^2}}\mathbf{i} + \frac{-2t}{2\sqrt{1+t^2}}\mathbf{j}$
 $= \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j}; \frac{d\mathbf{T}}{dt} = \frac{-t}{(\sqrt{1+t^2})^3}\mathbf{i} - \frac{1}{(\sqrt{1+t^2})^3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-t}{(\sqrt{1+t^2})^3}\right)^2 + \left(\frac{-1}{(\sqrt{1+t^2})^3}\right)^2}$
 $= \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j}; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2\sqrt{1+t^2}} \cdot \frac{1}{1+t^2} = \frac{1}{2(1+t^2)^{3/2}}$
4. $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow \mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} = |t| = t$, since
 $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(t \cos t)\mathbf{i} + (t \sin t)\mathbf{j}}{t} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}; \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2}$
 $= 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t} \cdot 1 = \frac{1}{t}$

5. (a) $\kappa(x) = \frac{1}{|\mathbf{v}(x)|} \cdot \left| \frac{d\mathbf{T}(x)}{dt} \right|$. Now, $\mathbf{v} = \mathbf{i} + f'(x)\mathbf{j} \Rightarrow |\mathbf{v}(x)| = \sqrt{1 + [f'(x)]^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \left(1 + [f'(x)]^2\right)^{-1/2} \mathbf{i} + f'(x) \left(1 + [f'(x)]^2\right)^{-1/2} \mathbf{j}$. Thus $\frac{d\mathbf{T}}{dt}(x) = \frac{-f'(x)f''(x)}{(1 + [f'(x)]^2)^{3/2}} \mathbf{i} + \frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}} \mathbf{j}$
 $\Rightarrow \left| \frac{d\mathbf{T}(x)}{dt} \right| = \sqrt{\left[\frac{-f'(x)f''(x)}{(1 + [f'(x)]^2)^{3/2}} \right]^2 + \left[\frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}} \right]^2} = \sqrt{\frac{[f''(x)]^2(1 + [f'(x)]^2)}{(1 + [f'(x)]^2)^3}} = \frac{|f''(x)|}{1 + [f'(x)]^2}$
Thus $\kappa(x) = \frac{1}{(1 + [f'(x)]^2)^{1/2}} \cdot \frac{|f''(x)|}{1 + [f'(x)]^2} = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$
- (b) $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)(-\sin x) = -\tan x \Rightarrow \frac{d^2y}{dx^2} = -\sec^2 x \Rightarrow \kappa = \frac{|-\sec^2 x|}{[1 + (-\tan x)^2]^{3/2}} = \frac{\sec^2 x}{|\sec^3 x|}$
 $= \frac{1}{\sec x} = \cos x$, since $-\frac{\pi}{2} < x < \frac{\pi}{2}$
- (c) Note that $f''(x) = 0$ at an inflection point.
6. (a) $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\mathbf{i} + \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\mathbf{j}$
 $\frac{d\mathbf{T}}{dt} = \frac{\dot{y}(\dot{y}\dot{x} - \dot{x}\dot{y})}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\mathbf{i} + \frac{\dot{x}(\dot{x}\dot{y} - \dot{y}\dot{x})}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\left[\frac{\dot{y}(\dot{y}\dot{x} - \dot{x}\dot{y})}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \right]^2 + \left[\frac{\dot{x}(\dot{x}\dot{y} - \dot{y}\dot{x})}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \right]^2} = \sqrt{\frac{(\dot{y}^2 + \dot{x}^2)(\dot{y}\dot{x} - \dot{x}\dot{y})^2}{(\dot{x}^2 + \dot{y}^2)^3}}$
 $= \frac{|\dot{y}\dot{x} - \dot{x}\dot{y}|}{|\dot{x}^2 + \dot{y}^2|}; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \cdot \frac{|\dot{y}\dot{x} - \dot{x}\dot{y}|}{|\dot{x}^2 + \dot{y}^2|} = \frac{|\dot{x}\dot{y} - \dot{y}\dot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$
- (b) $\mathbf{r}(t) = t\mathbf{i} + \ln(\sin t)\mathbf{j}, 0 < t < \pi \Rightarrow x = t$ and $y = \ln(\sin t) \Rightarrow \dot{x} = 1, \ddot{x} = 0; \dot{y} = \frac{\cos t}{\sin t} = \cot t, \ddot{y} = -\csc^2 t$
 $\Rightarrow \kappa = \frac{|-\csc^2 t - 0|}{(1 + \cot^2 t)^{3/2}} = \frac{\csc^2 t}{\csc^3 t} = \sin t$
- (c) $\mathbf{r}(t) = \tan^{-1}(\sinh t)\mathbf{i} + \ln(\cosh t)\mathbf{j} \Rightarrow x = \tan^{-1}(\sinh t)$ and $y = \ln(\cosh t) \Rightarrow \dot{x} = \frac{\cosh t}{1 + \sinh^2 t} = \frac{1}{\cosh t}$
 $= \operatorname{sech} t, \ddot{x} = -\operatorname{sech} t \tanh t; \dot{y} = \frac{\sinh t}{\cosh t} = \tanh t, \ddot{y} = \operatorname{sech}^2 t \Rightarrow \kappa = \frac{|\operatorname{sech}^3 t + \operatorname{sech} t \tanh^2 t|}{(\operatorname{sech}^2 t + \tanh^2 t)} = |\operatorname{sech} t| = \operatorname{sech} t$
7. (a) $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ is tangent to the curve at the point $(f(t), g(t))$;
 $\mathbf{n} \cdot \mathbf{v} = [-g'(t)\mathbf{i} + f'(t)\mathbf{j}] \cdot [f'(t)\mathbf{i} + g'(t)\mathbf{j}] = -g'(t)f'(t) + f'(t)g'(t) = 0; -\mathbf{n} \cdot \mathbf{v} = -(\mathbf{n} \cdot \mathbf{v}) = 0$; thus, $\mathbf{n}$ and $-\mathbf{n}$ are both normal to the curve at the point
- (b) $\mathbf{r}(t) = t\mathbf{i} + e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2e^{2t}\mathbf{j} \Rightarrow \mathbf{n} = -2e^{2t}\mathbf{i} + \mathbf{j}$ points toward the concave side of the curve; $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{4e^{4t} + 1} \Rightarrow \mathbf{N} = \frac{-2e^{2t}}{\sqrt{1 + 4e^{4t}}}\mathbf{i} + \frac{1}{\sqrt{1 + 4e^{4t}}}\mathbf{j}$
- (c) $\mathbf{r}(t) = \sqrt{4 - t^2}\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \frac{-t}{\sqrt{4 - t^2}}\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = -\mathbf{i} - \frac{t}{\sqrt{4 - t^2}}\mathbf{j}$ points toward the concave side of the curve;
 $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{1 + \frac{t^2}{4 - t^2}} = \frac{2}{\sqrt{4 - t^2}} \Rightarrow \mathbf{N} = -\frac{1}{2} \left(\sqrt{4 - t^2}\mathbf{i} + t\mathbf{j} \right)$
8. (a) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{n} = t^2\mathbf{i} - \mathbf{j}$ points toward the concave side of the curve when $t < 0$ and $-\mathbf{n} = -t^2\mathbf{i} + \mathbf{j}$ points toward the concave side when $t > 0 \Rightarrow \mathbf{N} = \frac{1}{\sqrt{1 + t^4}}(t^2\mathbf{i} - \mathbf{j})$ for $t < 0$ and $\mathbf{N} = \frac{1}{\sqrt{1 + t^4}}(-t^2\mathbf{i} + \mathbf{j})$ for $t > 0$
- (b) From part (a), $|\mathbf{v}| = \sqrt{1 + t^4} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1 + t^4}}\mathbf{i} + \frac{t^2}{\sqrt{1 + t^4}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{-2t^3}{(1 + t^4)^{3/2}}\mathbf{i} + \frac{2t}{(1 + t^4)^{3/2}}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\frac{4t^6 + 4t^2}{(1 + t^4)^3}}$
 $= \frac{2|t|}{1 + t^4}; \mathbf{N} = \frac{(\frac{d\mathbf{T}}{dt})}{|\frac{d\mathbf{T}}{dt}|} = \frac{1 + t^4}{2|t|} \left(\frac{-2t^3}{(1 + t^4)^{3/2}}\mathbf{i} + \frac{2t}{(1 + t^4)^{3/2}}\mathbf{j} \right) = \frac{-t^3}{|t|\sqrt{1 + t^4}}\mathbf{i} + \frac{t}{|t|\sqrt{1 + t^4}}\mathbf{j}; t \neq 0. \mathbf{N}$ does not exist at $t = 0$, where the curve has a point of inflection; $\frac{d\mathbf{T}}{dt} \Big|_{t=0} = 0$ so the curvature $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds} \right| = 0$ at $t = 0 \Rightarrow \mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}$ is undefined. Since $x = t$ and $y = \frac{1}{3}t^3 \Rightarrow y = \frac{1}{3}x^3$, the curve is the cubic power curve which is concave down for $x = t < 0$ and concave up for $x = t > 0$.
9. $\mathbf{r} = (3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (3 \cos t)\mathbf{i} + (-3 \sin t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(3 \cos t)^2 + (-3 \sin t)^2 + 4^2} = \sqrt{25} = 5 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{3}{5} \cos t\right)\mathbf{i} - \left(\frac{3}{5} \sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{3}{5} \sin t\right)\mathbf{i} - \left(\frac{3}{5} \cos t\right)\mathbf{j}$

$$\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\left(-\frac{3}{5} \sin t\right)^2 + \left(-\frac{3}{5} \cos t\right)^2} = \frac{3}{5} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}; \kappa = \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}$$

$$\begin{aligned} 10. \mathbf{r} &= (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} \\ &= |t| = t, \text{ if } t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}, t > 0 \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \\ &\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \kappa = \frac{1}{t} \cdot 1 = \frac{1}{t} \end{aligned}$$

$$\begin{aligned} 11. \mathbf{r} &= (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} \Rightarrow \\ |\mathbf{v}| &= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2e^{2t}} = e^t \sqrt{2}; \\ \mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{j} \\ &\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}; \\ \kappa &= \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{e^t \sqrt{2}} \cdot 1 = \frac{1}{e^t \sqrt{2}} \end{aligned}$$

$$\begin{aligned} 12. \mathbf{r} &= (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12 \cos 2t)\mathbf{i} - (12 \sin 2t)\mathbf{j} + 5\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(12 \cos 2t)^2 + (-12 \sin 2t)^2 + 5^2} \\ &= \sqrt{169} = 13 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{12}{13} \cos 2t\right)\mathbf{i} - \left(\frac{12}{13} \sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{24}{13} \sin 2t\right)\mathbf{i} - \left(\frac{24}{13} \cos 2t\right)\mathbf{j} \\ &\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\left(-\frac{24}{13} \sin 2t\right)^2 + \left(-\frac{24}{13} \cos 2t\right)^2} = \frac{24}{13} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j}; \\ \kappa &= \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{13} \cdot \frac{24}{13} = \frac{24}{169}. \end{aligned}$$

$$\begin{aligned} 13. \mathbf{r} &= \left(\frac{t^3}{3}\right)\mathbf{i} + \left(\frac{t^2}{2}\right)\mathbf{j}, t > 0 \Rightarrow \mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{t^4 + t^2} = t\sqrt{t^2 + 1}, \text{ since } t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{t}{\sqrt{t^2 + 1}}\mathbf{i} + \frac{1}{\sqrt{t^2 + 1}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{1}{(t^2 + 1)^{3/2}}\mathbf{i} - \frac{t}{(t^2 + 1)^{3/2}}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\left(\frac{1}{(t^2 + 1)^{3/2}}\right)^2 + \left(\frac{-t}{(t^2 + 1)^{3/2}}\right)^2} \\ &= \sqrt{\frac{1 + t^2}{(t^2 + 1)^3}} = \frac{1}{t^2 + 1} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1}{\sqrt{t^2 + 1}}\mathbf{i} - \frac{t}{\sqrt{t^2 + 1}}\mathbf{j}; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{t\sqrt{t^2 + 1}} \cdot \frac{1}{t^2 + 1} = \frac{1}{t(t^2 + 1)^{3/2}}. \end{aligned}$$

$$\begin{aligned} 14. \mathbf{r} &= (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, 0 < t < \frac{\pi}{2} \Rightarrow \mathbf{v} = (-3 \cos^2 t \sin t)\mathbf{i} + (3 \sin^2 t \cos t)\mathbf{j} \\ &\Rightarrow |\mathbf{v}| = \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} = \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} = 3 \cos t \sin t, \text{ since } 0 < t < \frac{\pi}{2} \\ &\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} \\ &= (\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{3 \cos t \sin t} \cdot 1 = \frac{1}{3 \cos t \sin t}. \end{aligned}$$

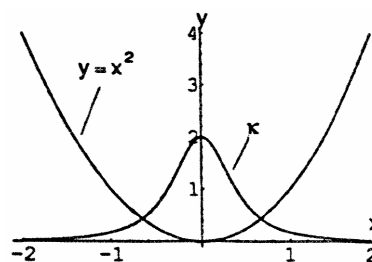
$$\begin{aligned} 15. \mathbf{r} &= t\mathbf{i} + \left(a \cosh \frac{t}{a}\right)\mathbf{j}, a > 0 \Rightarrow \mathbf{v} = \mathbf{i} + \left(\sinh \frac{t}{a}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + \sinh^2 \left(\frac{t}{a}\right)} = \sqrt{\cosh^2 \left(\frac{t}{a}\right)} = \cosh \frac{t}{a} \\ &\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{i} + \left(\tanh \frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{a} \operatorname{sech} \frac{t}{a} \tanh \frac{t}{a}\right)\mathbf{i} + \left(\frac{1}{a} \operatorname{sech}^2 \frac{t}{a}\right)\mathbf{j} \\ &\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\frac{1}{a^2} \operatorname{sech}^2 \left(\frac{t}{a}\right) \tanh^2 \left(\frac{t}{a}\right) + \frac{1}{a^2} \operatorname{sech}^4 \left(\frac{t}{a}\right)} = \frac{1}{a} \operatorname{sech} \left(\frac{t}{a}\right) \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\tanh \frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{j}; \\ \kappa &= \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\cosh \frac{t}{a}} \cdot \frac{1}{a} \operatorname{sech} \left(\frac{t}{a}\right) = \frac{1}{a} \operatorname{sech}^2 \left(\frac{t}{a}\right). \end{aligned}$$

$$\begin{aligned} 16. \mathbf{r} &= (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sinh^2 t + (-\cosh t)^2 + 1} = \sqrt{2} \cosh t \\ &\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{1}{\sqrt{2}} \operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}} \operatorname{sech} t \tanh t\right)\mathbf{k} \\ &\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\frac{1}{2} \operatorname{sech}^4 t + \frac{1}{2} \operatorname{sech}^2 t \tanh^2 t} = \frac{1}{\sqrt{2}} \operatorname{sech} t \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k}; \\ \kappa &= \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\sqrt{2} \cosh t} \cdot \frac{1}{\sqrt{2}} \operatorname{sech} t = \frac{1}{2} \operatorname{sech}^2 t. \end{aligned}$$

17. $y = ax^2 \Rightarrow y' = 2ax \Rightarrow y'' = 2a$; from Exercise 5(a), $\kappa(x) = \frac{|2a|}{(1+4a^2x^2)^{3/2}} = |2a| (1+4a^2x^2)^{-3/2}$
 $\Rightarrow \kappa'(x) = -\frac{3}{2} |2a| (1+4a^2x^2)^{-5/2} (8a^2x)$; thus, $\kappa'(x) = 0 \Rightarrow x = 0$. Now, $\kappa'(x) > 0$ for $x < 0$ and $\kappa'(x) < 0$ for $x > 0$ so that $\kappa(x)$ has an absolute maximum at $x = 0$ which is the vertex of the parabola. Since $x = 0$ is the only critical point for $\kappa(x)$, the curvature has no minimum value.
18. $\mathbf{r} = (a \cos t)\mathbf{i} + (b \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (-a \sin t)\mathbf{i} + (b \cos t)\mathbf{j} \Rightarrow \mathbf{a} = (-a \cos t)\mathbf{i} - (b \sin t)\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a}$
 $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & b \cos t & 0 \\ -a \cos t & -b \sin t & 0 \end{vmatrix} = ab\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = |ab| = ab$, since $a > b > 0$; $\kappa(t) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
 $= ab(a^2 \sin^2 t + b^2 \cos^2 t)^{-3/2}$; $\kappa'(t) = -\frac{3}{2} (ab) (a^2 \sin^2 t + b^2 \cos^2 t)^{-5/2} (2a^2 \sin t \cos t - 2b^2 \sin t \cos t)$
 $= -\frac{3}{2} (ab) (a^2 - b^2) (\sin 2t) (a^2 \sin^2 t + b^2 \cos^2 t)^{-5/2}$; thus, $\kappa'(t) = 0 \Rightarrow \sin 2t = 0 \Rightarrow t = 0, \pi$ identifying points on the major axis, or $t = \frac{\pi}{2}, \frac{3\pi}{2}$ identifying points on the minor axis. Furthermore, $\kappa'(t) < 0$ for $0 < t < \frac{\pi}{2}$ and for $\pi < t < \frac{3\pi}{2}$; $\kappa'(t) > 0$ for $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$. Therefore, the points associated with $t = 0$ and $t = \pi$ on the major axis give absolute maximum curvature and the points associated with $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ on the minor axis give absolute minimum curvature.
19. $\kappa = \frac{a}{a^2+b^2} \Rightarrow \frac{d\kappa}{da} = \frac{-a^2+b^2}{(a^2+b^2)^2}$; $\frac{d\kappa}{da} = 0 \Rightarrow -a^2+b^2 = 0 \Rightarrow a = \pm b \Rightarrow a = b$ since $a, b \geq 0$. Now, $\frac{d\kappa}{da} > 0$ if $a < b$ and $\frac{d\kappa}{da} < 0$ if $a > b \Rightarrow \kappa$ is at a maximum for $a = b$ and $\kappa(b) = \frac{b}{b^2+b^2} = \frac{1}{2b}$ is the maximum value of κ .
20. (a) From Example 5, the curvature of the helix $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + b\mathbf{k}$, $a, b \geq 0$ is $\kappa = \frac{a}{a^2+b^2}$; also $|\mathbf{v}| = \sqrt{a^2+b^2}$. For the helix $\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 4\pi$, $a = 3$ and $b = 1 \Rightarrow \kappa = \frac{3}{3^2+1^2} = \frac{3}{10}$ and $|\mathbf{v}| = \sqrt{10} \Rightarrow K = \int_0^{4\pi} \frac{3}{10} \sqrt{10} dt = \left[\frac{3}{\sqrt{10}} t \right]_0^{4\pi} = \frac{12\pi}{\sqrt{10}}$
- (b) $y = x^2 \Rightarrow x = t$ and $y = t^2$, $-\infty < t < \infty \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2}$;
 $\mathbf{T} = \frac{1}{\sqrt{1+4t^2}}\mathbf{i} + \frac{2t}{\sqrt{1+4t^2}}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = \frac{-4t}{(1+4t^2)^{3/2}}\mathbf{i} + \frac{2}{(1+4t^2)^{3/2}}\mathbf{j}$; $\left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\frac{16t^2+4}{(1+4t^2)^3}} = \frac{2}{1+4t^2}$. Thus
 $\kappa = \frac{1}{\sqrt{1+4t^2}} \cdot \frac{2}{1+4t^2} = \frac{2}{(\sqrt{1+4t^2})^3}$. Then $K = \int_{-\infty}^{\infty} \frac{2}{(\sqrt{1+4t^2})^3} (\sqrt{1+4t^2}) dt = \int_{-\infty}^{\infty} \frac{2}{1+4t^2} dt$
 $= \lim_{a \rightarrow -\infty} \int_a^0 \frac{2}{1+4t^2} dt + \lim_{b \rightarrow \infty} \int_0^b \frac{2}{1+4t^2} dt = \lim_{a \rightarrow -\infty} [\tan^{-1} 2t]_a^0 + \lim_{b \rightarrow \infty} [\tan^{-1} 2t]_0^b$
 $= \lim_{a \rightarrow -\infty} (-\tan^{-1} 2a) + \lim_{b \rightarrow \infty} (\tan^{-1} 2b) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$
21. $\mathbf{r} = t\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (\cos t)^2} = \sqrt{1 + \cos^2 t} \Rightarrow |\mathbf{v}(\frac{\pi}{2})| = \sqrt{1 + \cos^2(\frac{\pi}{2})} = 1$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \frac{\mathbf{i} + \cos t \mathbf{j}}{\sqrt{1 + \cos^2 t}} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{\sin t \cos t}{(1 + \cos^2 t)^{3/2}}\mathbf{i} + \frac{-\sin t}{(1 + \cos^2 t)^{3/2}}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\sin t|}{1 + \cos^2 t}$; $\left| \frac{d\mathbf{T}}{dt} \right|_{t=\frac{\pi}{2}} = \frac{|\sin \frac{\pi}{2}|}{1 + \cos^2(\frac{\pi}{2})} = \frac{1}{1} = 1$. Thus $\kappa(\frac{\pi}{2}) = \frac{1}{1} \cdot 1 = 1$
 $\Rightarrow \rho = \frac{1}{1} = 1$ and the center is $(\frac{\pi}{2}, 0) \Rightarrow (x - \frac{\pi}{2})^2 + y^2 = 1$
22. $\mathbf{r} = (2 \ln t)\mathbf{i} - (t + \frac{1}{t})\mathbf{j} \Rightarrow \mathbf{v} = (\frac{2}{t})\mathbf{i} - (1 - \frac{1}{t^2})\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\frac{4}{t^2} + (1 - \frac{1}{t^2})^2} = \frac{t^2+1}{t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2t}{t^2+1}\mathbf{i} - \frac{t^2-1}{t^2+1}\mathbf{j}$;
 $\frac{d\mathbf{T}}{dt} = \frac{-2(t^2-1)}{(t^2+1)^2}\mathbf{i} - \frac{4t}{(t^2+1)^2}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\frac{4(t^2-1)^2 + 16t^2}{(t^2+1)^4}} = \frac{2}{t^2+1}$. Thus $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{t^2}{t^2+1} \cdot \frac{2}{t^2+1} = \frac{2t^2}{(t^2+1)^2} \Rightarrow \kappa(1) = \frac{2}{2^2}$
 $= \frac{1}{2} \Rightarrow \rho = \frac{1}{\kappa} = 2$. The circle of curvature is tangent to the curve at $P(0, -2) \Rightarrow$ circle has same tangent as the curve
 $\Rightarrow \mathbf{v}(1) = 2\mathbf{i}$ is tangent to the circle $\Rightarrow$ the center lies on the y-axis. If $t \neq 1$ ($t > 0$), then $(t-1)^2 > 0$
 $\Rightarrow t^2 - 2t + 1 > 0 \Rightarrow t^2 + 1 > 2t \Rightarrow \frac{t^2+1}{t} > 2$ since $t > 0 \Rightarrow t + \frac{1}{t} > 2 \Rightarrow -(t + \frac{1}{t}) < -2 \Rightarrow y < -2$ on both sides of $(0, -2) \Rightarrow$ the curve is concave down $\Rightarrow$ center of circle of curvature is $(0, -4) \Rightarrow x^2 + (y+4)^2 = 4$
 is an equation of the circle of curvature

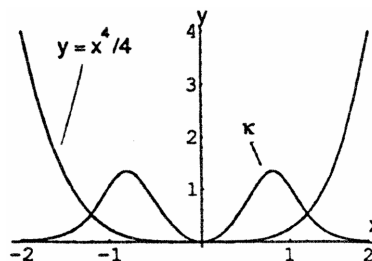
$$23. y = x^2 \Rightarrow f'(x) = 2x \text{ and } f''(x) = 2$$

$$\Rightarrow \kappa = \frac{|2|}{(1 + (2x)^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$



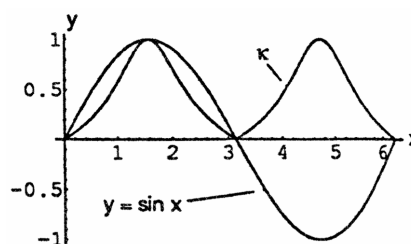
$$24. y = \frac{x^4}{4} \Rightarrow f'(x) = x^3 \text{ and } f''(x) = 3x^2$$

$$\Rightarrow \kappa = \frac{|3x^2|}{(1 + (x^3)^2)^{3/2}} = \frac{3x^2}{(1 + x^6)^{3/2}}$$



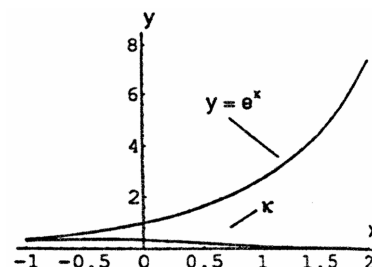
$$25. y = \sin x \Rightarrow f'(x) = \cos x \text{ and } f''(x) = -\sin x$$

$$\Rightarrow \kappa = \frac{|-\sin x|}{(1 + \cos^2 x)^{3/2}} = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}$$



$$26. y = e^x \Rightarrow f'(x) = e^x \text{ and } f''(x) = e^x$$

$$\Rightarrow \kappa = \frac{|e^x|}{(1 + (e^x)^2)^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}$$



27-34. Example CAS commands:

Maple:

```
with( plots );
r := t -> [3*cos(t), 5*sin(t)];
lo := 0;
hi := 2*Pi;
t0 := Pi/4;
P1 := plot( [r(t)[], t=lo..hi] );
display( P1, scaling=constrained, title="#27(a) (Section 13.4)" );
CURVATURE := (x,y,t) -> simplify(abs(diff(x,t)*diff(y,t,t)-diff(y,t)*diff(x,t,t))/(diff(x,t)^2+diff(y,t)^2)^(3/2));
kappa := eval(CURVATURE(r(t)[],t),t=t0);
UnitNormal := (x,y,t) -> expand( [-diff(y,t),diff(x,t)]/sqrt(diff(x,t)^2+diff(y,t)^2) );
N := eval( UnitNormal(r(t)[],t), t=t0 );
C := expand( r(t0) + N/kappa );
OscCircle := (x-C[1])^2+(y-C[2])^2 = 1/kappa^2;
evalf( OscCircle );
```

```
P2 := implicitplot( (x-C[1])^2+(y-C[2])^2 = 1/kappa^2, x=-7..4, y=-4..6, color=blue );
display( [P1,P2], scaling=constrained, title="#27(e) (Section 13.4)" );
```

Mathematica: (assigned functions and parameters may vary)

In Mathematica, the dot product can be applied either with a period "." or with the word, "Dot".

Similarly, the cross product can be applied either with a very small "x" (in the palette next to the arrow) or with the word, "Cross". However, the Cross command assumes the vectors are in three dimensions

For the purposes of applying the cross product command, we will define the position vector $\mathbf{r}$ as a three dimensional vector with zero for its z-component. For graphing, we will use only the first two components.

```
Clear[r, t, x, y]
r[t_]= {3 Cos[t], 5 Sin[t] }
t0= π /4; tmin= 0; tmax= 2π;
r2[t_]= {r[t][[1]], r[t][[2]]}
pp=ParametricPlot[r2[t], {t, tmin, tmax}];
mag[v_]=Sqrt[v.v]
vel[t_]= r'[t]
speed[t_]=mag[vel[t]]
acc[t_]= vel'[t]
curv[t_]= mag[Cross[vel[t],acc[t]]]/speed[t]^3//Simplify
unittan[t_]= vel[t]/speed[t]//Simplify
unitnorm[t_]= unittan'[t] / mag[unittan'[t]]
ctr= r[t0] + (1 / curv[t0]) unitnorm[t0] //Simplify
{a,b}= {ctr[[1]], ctr[[2]]}
```

To plot the osculating circle, load a graphics package and then plot it, and show it together with the original curve.

```
<<Graphics`ImplicitPlot`
pc=ImplicitPlot[(x - a)^2 + (y - b)^2 == 1/curv[t0]^2, {x, -8, 8},{y, -8, 8}]
radius=Graphics[Line[{ {a, b}, r2[t0]}]]
Show[pp, pc, radius, AspectRatio -> 1]
```

13.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

- $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + b\mathbf{k} \Rightarrow \mathbf{v} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2 + b^2}$
 $= \sqrt{a^2 + b^2} \Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = 0; \mathbf{a} = (-a \cos t)\mathbf{i} + (-a \sin t)\mathbf{j} \Rightarrow |\mathbf{a}| = \sqrt{(-a \cos t)^2 + (-a \sin t)^2} = \sqrt{a^2} = |a|$
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{|\mathbf{a}|^2 - 0^2} = |\mathbf{a}| = |a| \Rightarrow \mathbf{a} = (0)\mathbf{T} + |a|\mathbf{N} = |a|\mathbf{N}$
- $\mathbf{r} = (1 + 3t)\mathbf{i} + (t - 2)\mathbf{j} - 3t\mathbf{k} \Rightarrow \mathbf{v} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19} \Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = 0; \mathbf{a} = \mathbf{0}$
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = 0 \Rightarrow \mathbf{a} = (0)\mathbf{T} + (0)\mathbf{N} = \mathbf{0}$
- $\mathbf{r} = (t + 1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2} \Rightarrow a_T = \frac{1}{2} (5 + 4t^2)^{-1/2} (8t)$
 $= 4t (5 + 4t^2)^{-1/2} \Rightarrow a_T(1) = \frac{4}{\sqrt{9}} = \frac{4}{3}; \mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{k} \Rightarrow |\mathbf{a}(1)| = 2 \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - \left(\frac{4}{3}\right)^2}$
 $= \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3} \Rightarrow \mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}$
- $\mathbf{r} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} + 2t\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (2t)^2} = \sqrt{5t^2 + 1} \Rightarrow a_T = \frac{1}{2} (5t^2 + 1)^{-1/2} (10t)$

$$= \frac{5t}{\sqrt{5t^2+1}} \Rightarrow a_T(0) = 0; \mathbf{a} = (-2 \sin t - t \cos t)\mathbf{i} + (2 \cos t - t \sin t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{a}(0)|$$

$$= \sqrt{2^2 + 2^2} = 2\sqrt{2} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{(2\sqrt{2})^2 - 0^2} = 2\sqrt{2} \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\sqrt{2}\mathbf{N} = 2\sqrt{2}\mathbf{N}$$

5. $\mathbf{r} = t^2\mathbf{i} + (t + \frac{1}{3}t^3)\mathbf{j} + (t - \frac{1}{3}t^3)\mathbf{k} \Rightarrow \mathbf{v} = 2t\mathbf{i} + (1 + t^2)\mathbf{j} + (1 - t^2)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(2t)^2 + (1 + t^2)^2 + (1 - t^2)^2}$
 $= \sqrt{2(t^4 + 2t^2 + 1)} = \sqrt{2}(1 + t^2) \Rightarrow a_T = 2t\sqrt{2} \Rightarrow a_T(0) = 0; \mathbf{a} = 2\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{i} \Rightarrow |\mathbf{a}(0)| = 2$
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - 0^2} = 2 \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\mathbf{N} = 2\mathbf{N}$

6. $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (\sqrt{2}e^t)^2} = \sqrt{4e^{2t}} = 2e^t \Rightarrow a_T = 2e^t \Rightarrow a_T(0) = 2;$
 $\mathbf{a} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
 $= (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{a}(0)| = \sqrt{2^2 + (\sqrt{2})^2} = \sqrt{6}$
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{(\sqrt{6})^2 - 2^2} = \sqrt{2} \Rightarrow \mathbf{a}(0) = 2\mathbf{T} + \sqrt{2}\mathbf{N}$

7. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{T}(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}; \frac{d\mathbf{T}}{dt} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{(-\cos t)^2 + (-\sin t)^2}$
 $= 1 \Rightarrow \mathbf{N} = \frac{(\frac{d\mathbf{T}}{dt})}{|\frac{d\mathbf{T}}{dt}|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \mathbf{N}(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}; \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{k}$
 $\Rightarrow \mathbf{B}(\frac{\pi}{4}) = \mathbf{k}$, the normal to the osculating plane; $\mathbf{r}(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{P} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1)$ lies on the
 osculating plane $\Rightarrow 0(x - \frac{\sqrt{2}}{2}) + 0(y - \frac{\sqrt{2}}{2}) + (z - (-1)) = 0 \Rightarrow z = -1$ is the osculating plane; $\mathbf{T}$ is normal
 to the normal plane $\Rightarrow (-\frac{\sqrt{2}}{2})(x - \frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(y - \frac{\sqrt{2}}{2}) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 0$
 $\Rightarrow -x + y = 0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane
 $\Rightarrow (-\frac{\sqrt{2}}{2})(x - \frac{\sqrt{2}}{2}) + (-\frac{\sqrt{2}}{2})(y - \frac{\sqrt{2}}{2}) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = -1 \Rightarrow x + y = \sqrt{2}$ is the
 rectifying plane

8. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= (-\frac{1}{\sqrt{2}}\sin t)\mathbf{i} + (\frac{1}{\sqrt{2}}\cos t)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = (-\frac{1}{\sqrt{2}}\cos t)\mathbf{i} + (-\frac{1}{\sqrt{2}}\sin t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right|$
 $= \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \frac{1}{\sqrt{2}} \Rightarrow \mathbf{N} = \frac{(\frac{d\mathbf{T}}{dt})}{|\frac{d\mathbf{T}}{dt}|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j};$ thus $\mathbf{T}(0) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ and $\mathbf{N}(0) = -\mathbf{i}$
 $\Rightarrow \mathbf{B}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$, the normal to the osculating plane; $\mathbf{r}(0) = \mathbf{i} \Rightarrow \mathbf{P}(1, 0, 0)$ lies on
 the osculating plane $\Rightarrow 0(x - 1) - \frac{1}{\sqrt{2}}(y - 0) + \frac{1}{\sqrt{2}}(z - 0) = 0 \Rightarrow y - z = 0$ is the osculating plane; $\mathbf{T}$ is normal
 to the normal plane $\Rightarrow 0(x - 1) + \frac{1}{\sqrt{2}}(y - 0) + \frac{1}{\sqrt{2}}(z - 0) = 0 \Rightarrow y + z = 0$ is the normal plane; $\mathbf{N}$ is normal to
 the rectifying plane $\Rightarrow -1(x - 1) + 0(y - 0) + 0(z - 0) = 0 \Rightarrow x = 1$ is the rectifying plane.

9. By Exercise 9 in Section 13.4, $\mathbf{T} = \left(\frac{3}{5} \cos t\right) \mathbf{i} + \left(-\frac{3}{5} \sin t\right) \mathbf{j} + \frac{4}{5} \mathbf{k}$ and $\mathbf{N} = (-\sin t) \mathbf{i} - (\cos t) \mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5} \cos t & -\frac{3}{5} \sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} = \left(\frac{4}{5} \cos t\right) \mathbf{i} - \left(\frac{4}{5} \sin t\right) \mathbf{j} - \frac{3}{5} \mathbf{k}. \text{ Also } \mathbf{v} = (3 \cos t) \mathbf{i} + (-3 \sin t) \mathbf{j} + 4 \mathbf{k}$$

$$\Rightarrow \mathbf{a} = (-3 \sin t) \mathbf{i} + (-3 \cos t) \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (-3 \cos t) \mathbf{i} + (3 \sin t) \mathbf{j} \text{ and } \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 \cos t & -3 \sin t & 4 \\ -3 \sin t & -3 \cos t & 0 \end{vmatrix}$$

$$= (12 \cos t) \mathbf{i} - (12 \sin t) \mathbf{j} - 9 \mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (12 \cos t)^2 + (-12 \sin t)^2 + (-9)^2 = 225. \text{ Thus}$$

$$\tau = \frac{\begin{vmatrix} 3 \cos t & -3 \sin t & 4 \\ -3 \sin t & -3 \cos t & 0 \\ -3 \cos t & 3 \sin t & 0 \end{vmatrix}}{225} = \frac{4(-9 \sin^2 t - 9 \cos^2 t)}{225} = \frac{-36}{225} = -\frac{4}{25}$$

10. By Exercise 10 in Section 13.4, $\mathbf{T} = (\cos t) \mathbf{i} + (\sin t) \mathbf{j}$ and $\mathbf{N} = (-\sin t) \mathbf{i} + (\cos t) \mathbf{j}$; thus $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \end{vmatrix} = (\cos^2 t + \sin^2 t) \mathbf{k} = \mathbf{k}. \text{ Also } \mathbf{v} = (t \cos t) \mathbf{i} + (t \sin t) \mathbf{j}$$

$$\Rightarrow \mathbf{a} = (t(-\sin t) + \cos t) \mathbf{i} + (t \cos t + \sin t) \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (-t \cos t - \sin t - \sin t) \mathbf{i} + (-t \sin t + \cos t + \cos t) \mathbf{j}$$

$$= (-t \cos t - 2 \sin t) \mathbf{i} + (2 \cos t - t \sin t) \mathbf{j}. \text{ Thus } \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t \cos t & t \sin t & 0 \\ (-t \sin t + \cos t) & (t \cos t + \sin t) & 0 \end{vmatrix}$$

$$= [(t \cos t)(t \cos t + \sin t) - (t \sin t)(-t \sin t + \cos t)] \mathbf{k} = t^2 \mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (t^2)^2 = t^4. \text{ Thus}$$

$$\tau = \frac{\begin{vmatrix} t \cos t & t \sin t & 0 \\ \cos t - t \sin t & \sin t + t \cos t & 0 \\ -2 \sin t - t \cos t & 2 \cos t - t \sin t & 0 \end{vmatrix}}{t^4} = \frac{0}{t^4} = 0$$

11. By Exercise 11 in Section 13.4, $\mathbf{T} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right) \mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right) \mathbf{j}$ and $\mathbf{N} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right) \mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right) \mathbf{j}$; Thus

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos t - \sin t}{\sqrt{2}} & \frac{\sin t + \cos t}{\sqrt{2}} & 0 \\ \frac{-\cos t - \sin t}{\sqrt{2}} & \frac{-\sin t + \cos t}{\sqrt{2}} & 0 \end{vmatrix} = \left[\left(\frac{\cos^2 t - 2 \cos t \sin t + \sin^2 t}{2} \right) + \left(\frac{\sin^2 t + 2 \sin t \cos t + \cos^2 t}{2} \right) \right] \mathbf{k}$$

$$= \left[\left(\frac{1 - \sin(2t)}{2} \right) + \left(\frac{1 + \sin(2t)}{2} \right) \right] \mathbf{k} = \mathbf{k}. \text{ Also, } \mathbf{v} = (e^t \cos t - e^t \sin t) \mathbf{i} + (e^t \sin t + e^t \cos t) \mathbf{j}$$

$$\Rightarrow \mathbf{a} = [e^t(-\sin t - \cos t) + e^t(\cos t - \sin t)] \mathbf{i} + [e^t(\cos t - \sin t) + e^t(\sin t + \cos t)] \mathbf{j} = (-2e^t \sin t) \mathbf{i} + (2e^t \cos t) \mathbf{j}$$

$$\Rightarrow \frac{d\mathbf{a}}{dt} = -2e^t(\cos t + \sin t) \mathbf{i} + 2e^t(-\sin t + \cos t) \mathbf{j}. \text{ Thus } \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \end{vmatrix} = 2e^{2t} \mathbf{k}$$

$$\Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (2e^{2t})^2 = 4e^{4t}. \text{ Thus } \tau = \frac{\begin{vmatrix} e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \\ -2e^t(\cos t + \sin t) & 2e^t(-\sin t + \cos t) & 0 \end{vmatrix}}{4e^{4t}} = 0$$

12. By Exercise 12 in Section 13.4, $\mathbf{T} = \left(\frac{12}{13} \cos 2t\right) \mathbf{i} - \left(\frac{12}{13} \sin 2t\right) \mathbf{j} + \frac{5}{13} \mathbf{k}$ and $\mathbf{N} = (-\sin 2t) \mathbf{i} - (\cos 2t) \mathbf{j}$ so

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{12}{13} \cos 2t & -\frac{12}{13} \sin 2t & \frac{5}{13} \\ (-\sin 2t) & (-\cos 2t) & 0 \end{vmatrix} = \left(\frac{5}{13} \cos 2t\right) \mathbf{i} - \left(\frac{5}{13} \sin 2t\right) \mathbf{j} - \frac{12}{13} \mathbf{k}. \text{ Also,}$$

$$\mathbf{v} = (12 \cos 2t) \mathbf{i} - (12 \sin 2t) \mathbf{j} + 5 \mathbf{k} \Rightarrow \mathbf{a} = (-24 \sin 2t) \mathbf{i} - (24 \cos 2t) \mathbf{j} \text{ and } \frac{d\mathbf{a}}{dt} = (-48 \cos 2t) \mathbf{i} + (48 \sin 2t) \mathbf{j}$$

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 \cos 2t & -12 \sin 2t & 5 \\ -24 \sin 2t & -24 \cos 2t & 0 \end{vmatrix} = (120 \cos 2t) \mathbf{i} - (120 \sin 2t) \mathbf{j} - 288 \mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2$$

$$= (120 \cos 2t)^2 + (-120 \sin 2t)^2 + (-288)^2 = 120^2(\cos^2 2t + \sin^2 2t) + 288^2 = 97344. \text{ Thus}$$

$$\tau = \frac{\begin{vmatrix} 12 \cos 2t & -12 \sin 2t & 5 \\ -24 \sin 2t & -24 \cos 2t & 0 \\ -48 \cos 2t & 48 \sin 2t & 0 \end{vmatrix}}{97344} = \frac{5(-24 \cdot 48)}{97344} = -\frac{10}{169}$$

13. By Exercise 13 in Section 13.4, $\mathbf{T} = \frac{t}{(t^2+1)^{1/2}} \mathbf{i} + \frac{1}{(t^2+1)^{1/2}} \mathbf{j}$ and $\mathbf{N} = \frac{1}{\sqrt{t^2+1}} \mathbf{i} - \frac{t}{\sqrt{t^2+1}} \mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{t}{\sqrt{t^2+1}} & \frac{1}{\sqrt{t^2+1}} & 0 \\ \frac{1}{\sqrt{t^2+1}} & -\frac{t}{\sqrt{t^2+1}} & 0 \end{vmatrix} = -\mathbf{k}. \text{ Also, } \mathbf{v} = t^2 \mathbf{i} + t \mathbf{j} \Rightarrow \mathbf{a} = 2t \mathbf{i} + \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = 2 \mathbf{i} \text{ so that } \begin{vmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$$

14. By Exercise 14 in Section 13.4, $\mathbf{T} = (-\cos t) \mathbf{i} + (\sin t) \mathbf{j}$ and $\mathbf{N} = (\sin t) \mathbf{i} + (\cos t) \mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{vmatrix} = -\mathbf{k}. \text{ Also, } \mathbf{v} = (-3 \cos^2 t \sin t) \mathbf{i} + (3 \sin^2 t \cos t) \mathbf{j}$$

$$\Rightarrow \mathbf{a} = \frac{d}{dt}(-3 \cos^2 t \sin t) \mathbf{i} + \frac{d}{dt}(3 \sin^2 t \cos t) \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = \frac{d}{dt}\left(\frac{d}{dt}(-3 \cos^2 t \sin t)\right) \mathbf{i} + \frac{d}{dt}\left(\frac{d}{dt}(3 \sin^2 t \cos t)\right) \mathbf{j}$$

$$\Rightarrow \begin{vmatrix} -3 \cos^2 t \sin t & 3 \sin^2 t \cos t & 0 \\ \frac{d}{dt}(-3 \cos^2 t \sin t) & \frac{d}{dt}(3 \sin^2 t \cos t) & 0 \\ \frac{d}{dt}\left(\frac{d}{dt}(-3 \cos^2 t \sin t)\right) & \frac{d}{dt}\left(\frac{d}{dt}(3 \sin^2 t \cos t)\right) & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$$

15. By Exercise 15 in Section 13.4, $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (\operatorname{sech} \frac{t}{a}) \mathbf{i} + (\tanh \frac{t}{a}) \mathbf{j}$ and $\mathbf{N} = (-\tanh \frac{t}{a}) \mathbf{i} + (\operatorname{sech} \frac{t}{a}) \mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \operatorname{sech}(\frac{t}{a}) & \tanh(\frac{t}{a}) & 0 \\ -\tanh(\frac{t}{a}) & \operatorname{sech}(\frac{t}{a}) & 0 \end{vmatrix} = \mathbf{k}. \text{ Also, } \mathbf{v} = \mathbf{i} + (\sinh \frac{t}{a}) \mathbf{j} \Rightarrow \mathbf{a} = (\frac{1}{a} \cosh \frac{t}{a}) \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = \frac{1}{a^2} \sinh(\frac{t}{a}) \mathbf{j} \text{ so that}$$

$$\begin{vmatrix} 1 & \sinh(\frac{t}{a}) & 0 \\ 0 & \frac{1}{a} \cosh(\frac{t}{a}) & 0 \\ 0 & \frac{1}{a^2} \sinh(\frac{t}{a}) & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$$

16. By Exercise 16 in Section 13.4, $\mathbf{T} = (\frac{1}{\sqrt{2}} \tanh t) \mathbf{i} - \frac{1}{\sqrt{2}} \mathbf{j} + (\frac{1}{\sqrt{2}} \operatorname{sech} t) \mathbf{k}$ and $\mathbf{N} = (\operatorname{sech} t) \mathbf{i} - (\tanh t) \mathbf{k}$ so that

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} \tanh t & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = (\frac{1}{\sqrt{2}} \tanh t) \mathbf{i} + \frac{1}{\sqrt{2}} \mathbf{j} + (\frac{1}{\sqrt{2}} \operatorname{sech} t) \mathbf{k}. \text{ Also, } \mathbf{v} = (\sinh t) \mathbf{i} - (\cosh t) \mathbf{j} + \mathbf{k}$$

$$\mathbf{a} = (\cosh t) \mathbf{i} - (\sinh t) \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (\sinh t) \mathbf{i} - (\cosh t) \mathbf{j} \text{ and } \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix}$$

$$= (\sinh t) \mathbf{i} + (\cosh t) \mathbf{j} + (\cosh^2 t - \sinh^2 t) \mathbf{k} = (\sinh t) \mathbf{i} + (\cosh t) \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = \sinh^2 t + \cosh^2 t + 1. \text{ Thus}$$

$$\tau = \frac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{2 \cosh^2 t}.$$

17. Yes. If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N = \kappa |\mathbf{v}|^2 \neq 0 \Rightarrow \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} \neq \mathbf{0}$.

18. $|\mathbf{v}|$ constant $\Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = 0 \Rightarrow \mathbf{a} = a_N \mathbf{N}$ is orthogonal to $\mathbf{T} \Rightarrow$ the acceleration is normal to the path

19. $\mathbf{a} \perp \mathbf{v} \Rightarrow \mathbf{a} \perp \mathbf{T} \Rightarrow a_T = 0 \Rightarrow \frac{d}{dt} |\mathbf{v}| = 0 \Rightarrow |\mathbf{v}|$ is constant

20. $\mathbf{a}(t) = a_T \mathbf{T} + a_N \mathbf{N}$, where $a_T = \frac{d}{dt} |\mathbf{v}| = \frac{d}{dt} (10) = 0$ and $a_N = \kappa |\mathbf{v}|^2 = 100\kappa \Rightarrow \mathbf{a} = 0\mathbf{T} + 100\kappa \mathbf{N}$. Now, from Exercise 5(a) Section 12.4, we find for $y = f(x) = x^2$ that $\kappa = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{2}{[1 + (2x)^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$; also,

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$ is the position vector of the moving mass $\Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + 4t^2}$
 $\Rightarrow \mathbf{T} = \frac{1}{\sqrt{1 + 4t^2}}(\mathbf{i} + 2t\mathbf{j})$. At $(0, 0)$: $\mathbf{T}(0) = \mathbf{i}$, $\mathbf{N}(0) = \mathbf{j}$ and $\kappa(0) = 2 \Rightarrow \mathbf{F} = m\mathbf{a} = m(100\kappa)\mathbf{N} = 200m\mathbf{j}$;
 At $(\sqrt{2}, 2)$: $\mathbf{T}(\sqrt{2}) = \frac{1}{3}(\mathbf{i} + 2\sqrt{2}\mathbf{j}) = \frac{1}{3}\mathbf{i} + \frac{2\sqrt{2}}{3}\mathbf{j}$, $\mathbf{N}(\sqrt{2}) = -\frac{2\sqrt{2}}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$, and $\kappa(\sqrt{2}) = \frac{2}{27} \Rightarrow \mathbf{F} = m\mathbf{a}$
 $= m(100\kappa)\mathbf{N} = \left(\frac{200}{27}m\right)\left(-\frac{2\sqrt{2}}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}\right) = -\frac{400\sqrt{2}}{81}m\mathbf{i} + \frac{200}{81}m\mathbf{j}$

21. By $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ we have $\mathbf{v} \times \mathbf{a} = \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\right] = \left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T} \times \mathbf{T}) + \kappa\left(\frac{ds}{dt}\right)^3(\mathbf{T} \times \mathbf{N})$
 $= \kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}$. It follows that $|\mathbf{v} \times \mathbf{a}| = \kappa\left|\frac{ds}{dt}\right|^3|\mathbf{B}| = \kappa|\mathbf{v}|^3 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

22. $a_N = 0 \Rightarrow \kappa|\mathbf{v}|^2 = 0 \Rightarrow \kappa = 0$ (since the particle is moving, we cannot have zero speed) $\Rightarrow$ the curvature is zero so the particle is moving along a straight line

23. From Example 1, $|\mathbf{v}| = t$ and $a_N = t$ so that $a_N = \kappa|\mathbf{v}|^2 \Rightarrow \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{t}{t^2} = \frac{1}{t}$, $t \neq 0 \Rightarrow \rho = \frac{1}{\kappa} = t$

24. $\mathbf{r} = (x_0 + At)\mathbf{i} + (y_0 + Bt)\mathbf{j} + (z_0 + Ct)\mathbf{k} \Rightarrow \mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \Rightarrow \mathbf{a} = \mathbf{0} \Rightarrow \mathbf{v} \times \mathbf{a} = \mathbf{0} \Rightarrow \kappa = 0$. Since the curve is a plane curve, $\tau = 0$.

25. If a plane curve is sufficiently differentiable the torsion is zero as the following argument shows:

$\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} \Rightarrow \mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = f'''(t)\mathbf{i} + g'''(t)\mathbf{j}$
 $\Rightarrow \tau = \frac{\begin{vmatrix} f'(t) & g'(t) & 0 \\ f''(t) & g''(t) & 0 \\ f'''(t) & g'''(t) & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0$

26. $\mathbf{v} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}$ and $\mathbf{a} = -(a \cos t)\mathbf{i} - (a \sin t)\mathbf{j}$

To find the torsion: $\tau = \frac{\begin{vmatrix} -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \\ a \sin t & -a \cos t & 0 \end{vmatrix}}{(a\sqrt{a^2 + b^2})^2} = \frac{b(a^2 \cos^2 t + a^2 \sin^2 t)}{a^2(a^2 + b^2)} = \frac{a^2 b(\cos^2 t + \sin^2 t)}{a^2(a^2 + b^2)} = \frac{b}{a^2 + b^2} \Rightarrow \tau'(b) = \frac{a^2 - b^2}{(a^2 + b^2)^2}$;

$\tau'(b) = 0 \Rightarrow \frac{a^2 - b^2}{(a^2 + b^2)^2} = 0 \Rightarrow a^2 - b^2 = 0 \Rightarrow b = \pm a \Rightarrow b = a$ since $a, b > 0$. Also $b < a \Rightarrow \tau' > 0$ and $b > a \Rightarrow \tau' < 0$ so $\tau_{\max}$ occurs when $b = a \Rightarrow \tau_{\max} = \frac{a}{a^2 + a^2} = \frac{1}{2a}$

27. $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$; $\mathbf{v} \cdot \mathbf{k} = 0 \Rightarrow h'(t) = 0 \Rightarrow h(t) = C$
 $\Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + C\mathbf{k}$ and $\mathbf{r}(a) = f(a)\mathbf{i} + g(a)\mathbf{j} + C\mathbf{k} = \mathbf{0} \Rightarrow f(a) = 0, g(a) = 0$ and $C = 0 \Rightarrow h(t) = 0$.

28. From Exercise 26, $\mathbf{v} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{a^2 + b^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \frac{1}{\sqrt{a^2 + b^2}}[-(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}]$; $\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2 + b^2}}[-(a \cos t)\mathbf{i} - (a \sin t)\mathbf{j}] \Rightarrow \mathbf{N} = \frac{(\frac{d\mathbf{T}}{dt})}{|\frac{d\mathbf{T}}{dt}|}$
 $= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$; $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a \sin t}{\sqrt{a^2 + b^2}} & \frac{a \cos t}{\sqrt{a^2 + b^2}} & \frac{b}{\sqrt{a^2 + b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$
 $= \frac{b \sin t}{\sqrt{a^2 + b^2}}\mathbf{i} - \frac{b \cos t}{\sqrt{a^2 + b^2}}\mathbf{j} + \frac{a}{\sqrt{a^2 + b^2}}\mathbf{k} \Rightarrow \frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{a^2 + b^2}}[(b \cos t)\mathbf{i} + (b \sin t)\mathbf{j}] \Rightarrow \frac{d\mathbf{B}}{dt} \cdot \mathbf{N} = -\frac{b}{\sqrt{a^2 + b^2}}$
 $\Rightarrow \tau = -\frac{1}{|\mathbf{v}|} \left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right) = \left(-\frac{1}{\sqrt{a^2 + b^2}}\right)\left(-\frac{b}{\sqrt{a^2 + b^2}}\right) = \frac{b}{a^2 + b^2}$, which is consistent with the result in Exercise 26.

29-32. Example CAS commands:

Maple:

```
with( LinearAlgebra );
r := < t*cos(t) | t*sin(t) | t >;
t0 := sqrt(3);
rr := eval( r, t=t0 );
v := map( diff, r, t );
vv := eval( v, t=t0 );
a := map( diff, v, t );
aa := eval( a, t=t0 );
s := simplify(Norm( v, 2 )) assuming t::real;
ss := eval( s, t=t0 );
T := v/s;
TT := vv/ss ;
q1 := map( diff, simplify(T), t );
NN := simplify(eval( q1/Norm(q1,2), t=t0 ));
BB := CrossProduct( TT, NN );
kappa := Norm(CrossProduct(vv,aa),2)/ss^3;
tau := simplify( Determinant(< vv, aa, eval(map(diff,a,t),t=t0) >)/Norm(CrossProduct(vv,aa),2)^3 );
a_t := eval( diff( s, t ), t=t0 );
a_n := evalf[4]( kappa*ss^2 );
```

Mathematica: (assigned functions and value for t0 will vary)

```
Clear[t, v, a, t]
mag[vector_]:=Sqrt[vector.vector]
Print["The position vector is ", r[t_]={t Cos[t], t Sin[t], t}]
Print["The velocity vector is ", v[t_]=r'[t]]
Print["The acceleration vector is ", a[t_]=v'[t]]
Print["The speed is ", speed[t_]=mag[v[t]]//Simplify]
Print["The unit tangent vector is ", utan[t_]=v[t]/speed[t] //Simplify]
Print["The curvature is ", curv[t_]=mag[Cross[v[t],a[t]]] / speed[t]^3 //Simplify]
Print["The torsion is ", torsion[t_]=Det[{v[t], a[t], a'[t]}] / mag[Cross[v[t],a[t]]]^2 //Simplify]
Print["The unit normal vector is ", unorm[t_]=utan'[t] / mag[utan'[t]] //Simplify]
Print["The unit binormal vector is ", ubinorm[t_]=Cross[utan[t],unorm[t]] //Simplify]
Print["The tangential component of the acceleration is ", at[t_]=a[t].utan[t] //Simplify]
Print["The normal component of the acceleration is ", an[t_]=a[t].unorm[t] //Simplify]
```

You can evaluate any of these functions at a specified value of t.

```
t0= Sqrt[3]
{utan[t0], unorm[t0], ubinorm[t0]}
N[{utan[t0], unorm[t0], ubinorm[t0]}]
{curv[t0], torsion[t0]}
N[{curv[t0], torsion[t0]}]
{at[t0], an[t0]}
N[{at[t0], an[t0]}]
```

To verify that the tangential and normal components of the acceleration agree with the formulas in the book:

```
at[t]== speed'[t] //Simplify
an[t]==curv [t] speed[t]^2 //Simplify
```

13.6 VELOCITY AND ACCELERATION IN POLAR COORDINATES

- $\frac{d\theta}{dt} = 3 = \dot{\theta} \Rightarrow \ddot{\theta} = 0, r = a(1 - \cos \theta) \Rightarrow \dot{r} = a \sin \theta \frac{d\theta}{dt} = 3a \sin \theta \Rightarrow \ddot{r} = 3a \cos \theta \frac{d\theta}{dt} = 9a \cos \theta$
 $\mathbf{v} = (3a \sin \theta)\mathbf{u}_r + (a(1 - \cos \theta))(3)\mathbf{u}_\theta = (3a \sin \theta)\mathbf{u}_r + 3a(1 - \cos \theta)\mathbf{u}_\theta$
 $\mathbf{a} = \left(9a \cos \theta - a(1 - \cos \theta)(3)^2\right)\mathbf{u}_r + (a(1 - \cos \theta) \cdot 0 + 2(3a \sin \theta)(3))\mathbf{u}_\theta$
 $= (9a \cos \theta - 9a + 9a \cos \theta)\mathbf{u}_r + (18a \sin \theta)\mathbf{u}_\theta = 9a(2 \cos \theta - 1)\mathbf{u}_r + (18a \sin \theta)\mathbf{u}_\theta$
- $\frac{d\theta}{dt} = 2t = \dot{\theta} \Rightarrow \ddot{\theta} = 2, r = a \sin 2\theta \Rightarrow \dot{r} = a \cos 2\theta \cdot 2 \frac{d\theta}{dt} = 4ta \cos 2\theta \Rightarrow \ddot{r} = 4ta(-\sin 2\theta \cdot 2 \frac{d\theta}{dt}) + 4a \cos 2\theta$
 $= -16t^2 a \sin 2\theta + 4a \cos 2\theta$
 $\mathbf{v} = (4ta \cos 2\theta)\mathbf{u}_r + (a \sin 2\theta)(2t)\mathbf{u}_\theta = (4ta \cos 2\theta)\mathbf{u}_r + (2ta \sin 2\theta)\mathbf{u}_\theta$
 $\mathbf{a} = \left[(-16t^2 a \sin 2\theta + 4a \cos 2\theta) - (a \sin 2\theta)(2t)^2\right]\mathbf{u}_r + [(a \sin 2\theta)(2) + 2(4ta \cos 2\theta)(2t)]\mathbf{u}_\theta$
 $= \left[-16t^2 a \sin 2\theta + 4a \cos 2\theta - 4t^2 a \sin 2\theta\right]\mathbf{u}_r + [2a \sin 2\theta + 16t^2 a \cos 2\theta]\mathbf{u}_\theta$
 $= \left[-20t^2 a \sin 2\theta + 4a \cos 2\theta\right]\mathbf{u}_r + [2a \sin 2\theta + 16t^2 a \cos 2\theta]\mathbf{u}_\theta = 4a(\cos 2\theta - 5t^2 \sin 2\theta)\mathbf{u}_r + 2a(\sin 2\theta + 8t^2 \cos 2\theta)\mathbf{u}_\theta$
- $\frac{d\theta}{dt} = 2 = \dot{\theta} \Rightarrow \ddot{\theta} = 0, r = e^{a\theta} \Rightarrow \dot{r} = e^{a\theta} \cdot a \frac{d\theta}{dt} = 2a e^{a\theta} \Rightarrow \ddot{r} = 2a e^{a\theta} \cdot a \frac{d\theta}{dt} = 4a^2 e^{a\theta}$
 $\mathbf{v} = (2a e^{a\theta})\mathbf{u}_r + (e^{a\theta})(2)\mathbf{u}_\theta = (2a e^{a\theta})\mathbf{u}_r + (2e^{a\theta})\mathbf{u}_\theta$
 $\mathbf{a} = \left[(4a^2 e^{a\theta}) - (e^{a\theta})(2)^2\right]\mathbf{u}_r + \left[(e^{a\theta})(0) + 2(2a e^{a\theta})(2)\right]\mathbf{u}_\theta = \left[4a^2 e^{a\theta} - 4e^{a\theta}\right]\mathbf{u}_r + \left[0 + 8a e^{a\theta}\right]\mathbf{u}_\theta$
 $= 4e^{a\theta}(a^2 - 1)\mathbf{u}_r + (8a e^{a\theta})\mathbf{u}_\theta$
- $\theta = 1 - e^{-t} \Rightarrow \dot{\theta} = e^{-t} \Rightarrow \ddot{\theta} = -e^{-t}, r = a(1 + \sin t) \Rightarrow \dot{r} = a \cos t \Rightarrow \ddot{r} = -a \sin t$
 $\mathbf{v} = (a \cos t)\mathbf{u}_r + (a(1 + \sin t))(e^{-t})\mathbf{u}_\theta = (a \cos t)\mathbf{u}_r + a e^{-t}(1 + \sin t)\mathbf{u}_\theta$
 $\mathbf{a} = \left[(-a \sin t) - (a(1 + \sin t))(e^{-t})^2\right]\mathbf{u}_r + \left[(a(1 + \sin t))(-e^{-t}) + 2(a \cos t)(e^{-t})\right]\mathbf{u}_\theta$
 $= \left[-a \sin t - a e^{-2t}(1 + \sin t)\right]\mathbf{u}_r + \left[-a e^{-t}(1 + \sin t) + 2a e^{-t} \cos t\right]\mathbf{u}_\theta$
 $= -a(\sin t + e^{-2t}(1 + \sin t))\mathbf{u}_r + a e^{-t}(-(1 + \sin t) + 2 \cos t)\mathbf{u}_\theta$
 $= -a(\sin t + e^{-2t}(1 + \sin t))\mathbf{u}_r + a e^{-t}(2 \cos t - 1 - \sin t)\mathbf{u}_\theta$
- $\theta = 2t \Rightarrow \dot{\theta} = 2 \Rightarrow \ddot{\theta} = 0, r = 2 \cos 4t \Rightarrow \dot{r} = -8 \sin 4t \Rightarrow \ddot{r} = -32 \cos 4t$
 $\mathbf{v} = (-8 \sin 4t)\mathbf{u}_r + (2 \cos 4t)(2)\mathbf{u}_\theta = -8(\sin 4t)\mathbf{u}_r + 4(\cos 4t)\mathbf{u}_\theta$
 $\mathbf{a} = \left[(-32 \cos 4t) - (2 \cos 4t)(2)^2\right]\mathbf{u}_r + ((2 \cos 4t) \cdot 0 + 2(-8 \sin 4t)(2))\mathbf{u}_\theta$
 $= (-32 \cos 4t - 8 \cos 4t)\mathbf{u}_r + (0 - 32 \sin 4t)\mathbf{u}_\theta = -40(\cos 4t)\mathbf{u}_r - 32(\sin 4t)\mathbf{u}_\theta$
- $e = \frac{r_0 v_0^2}{GM} - 1 \Rightarrow v_0^2 = \frac{GM(e+1)}{r_0} \Rightarrow v_0 = \sqrt{\frac{GM(e+1)}{r_0}};$
 Circle: $e = 0 \Rightarrow v_0 = \sqrt{\frac{GM}{r_0}}$
 Ellipse: $0 < e < 1 \Rightarrow \sqrt{\frac{GM}{r_0}} < v_0 < \sqrt{\frac{2GM}{r_0}}$
 Parabola: $e = 1 \Rightarrow v_0 = \sqrt{\frac{2GM}{r_0}}$
 Hyperbola: $e > 1 \Rightarrow v_0 > \sqrt{\frac{2GM}{r_0}}$
- $r = \frac{GM}{v^2} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$ which is constant since $G, M,$ and r (the radius of orbit) are constant

$$\begin{aligned}
 8. \quad \Delta A &= \frac{1}{2} |\mathbf{r}(t + \Delta t) \times \mathbf{r}(t)| \Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t) + \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| \\
 &= \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) + \frac{1}{\Delta t} \mathbf{r}(t) \times \mathbf{r}(t) \right| = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| \Rightarrow \frac{dA}{dt} = \lim_{\Delta t \rightarrow 0} \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| \\
 &= \frac{1}{2} \left| \frac{d\mathbf{r}}{dt} \times \mathbf{r}(t) \right| = \frac{1}{2} |\mathbf{r}(t) \times \frac{d\mathbf{r}}{dt}| = \frac{1}{2} |\mathbf{r} \times \dot{\mathbf{r}}|
 \end{aligned}$$

$$\begin{aligned}
 9. \quad T &= \left(\frac{2\pi a^2}{r_0 v_0} \right) \sqrt{1 - e^2} \Rightarrow T^2 = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) (1 - e^2) = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[1 - \left(\frac{r_0 v_0^2}{GM} - 1 \right)^2 \right] \text{ (from Equation 5)} \\
 &= \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[-\frac{r_0^2 v_0^4}{G^2 M^2} + 2 \left(\frac{r_0 v_0^2}{GM} \right) \right] = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[\frac{2GM r_0 v_0^2 - r_0^2 v_0^4}{G^2 M^2} \right] = \frac{(4\pi^2 a^4) (2GM - r_0 v_0^2)}{r_0 G^2 M^2} \\
 &= (4\pi^2 a^4) \left(\frac{2GM - r_0 v_0^2}{2r_0 GM} \right) \left(\frac{2}{GM} \right) = (4\pi^2 a^4) \left(\frac{1}{2a} \right) \left(\frac{2}{GM} \right) \text{ (from Equation 10)} \Rightarrow T^2 = \frac{4\pi^2 a^3}{GM} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad r &= 365.256 \text{ days} = 365.256 \text{ days} \times 24 \frac{\text{hours}}{\text{day}} \times 60 \frac{\text{minutes}}{\text{hour}} \times 60 \frac{\text{seconds}}{\text{minute}} = 31,558,118.4 \text{ seconds} \approx 3.16 \times 10^7, \\
 G &= 6.6726 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}, \text{ and the mass of the sun } M = 1.99 \times 10^{30} \text{ kg. } \frac{T^2}{a^3} = \frac{4\pi^2}{GM} \Rightarrow a^3 = T^2 \frac{GM}{4\pi^2} \\
 \Rightarrow a^3 &= (3.16 \times 10^7)^2 \frac{(6.6726 \times 10^{-11})(1.99 \times 10^{30})}{4\pi^2} \approx 3.35863335 \times 10^{33} \Rightarrow a = \sqrt[3]{3.35863335 \times 10^{33}} \\
 &\approx 149757138111 \text{ m} \approx 149.757 \text{ billion km}
 \end{aligned}$$

CHAPTER 13 PRACTICE EXERCISES

$$1. \quad \mathbf{r}(t) = (4 \cos t)\mathbf{i} + (\sqrt{2} \sin t)\mathbf{j} \Rightarrow x = 4 \cos t$$

$$\text{and } y = \sqrt{2} \sin t \Rightarrow \frac{x^2}{16} + \frac{y^2}{2} = 1;$$

$$\mathbf{v} = (-4 \sin t)\mathbf{i} + (\sqrt{2} \cos t)\mathbf{j} \text{ and}$$

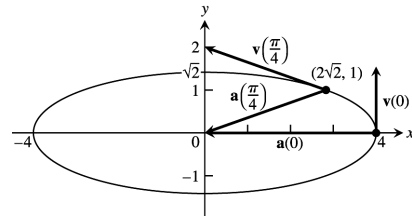
$$\mathbf{a} = (-4 \cos t)\mathbf{i} - (\sqrt{2} \sin t)\mathbf{j}; \mathbf{r}(0) = 4\mathbf{i}, \mathbf{v}(0) = \sqrt{2}\mathbf{j},$$

$$\mathbf{a}(0) = -4\mathbf{i}; \mathbf{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\mathbf{i} + \mathbf{j}, \mathbf{v}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} + \mathbf{j},$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} - \mathbf{j}; |\mathbf{v}| = \sqrt{16 \sin^2 t + 2 \cos^2 t}$$

$$\Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = \frac{14 \sin t \cos t}{\sqrt{16 \sin^2 t + 2 \cos^2 t}}; \text{ at } t = 0: a_T = 0, a_N = \sqrt{|\mathbf{a}|^2 - 0} = 4, \mathbf{a} = 0\mathbf{T} + 4\mathbf{N} = 4\mathbf{N}, \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4}{2} = 2;$$

$$\text{at } t = \frac{\pi}{4}: a_T = \frac{7}{\sqrt{8+1}} = \frac{7}{3}, a_N = \sqrt{9 - \frac{49}{9}} = \frac{4\sqrt{2}}{3}, \mathbf{a} = \frac{7}{3}\mathbf{T} + \frac{4\sqrt{2}}{3}\mathbf{N}, \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4\sqrt{2}}{27}$$



$$2. \quad \mathbf{r}(t) = (\sqrt{3} \sec t)\mathbf{i} + (\sqrt{3} \tan t)\mathbf{j} \Rightarrow x = \sqrt{3} \sec t \text{ and } y = \sqrt{3} \tan t \Rightarrow \frac{x^2}{3} - \frac{y^2}{3} = \sec^2 t - \tan^2 t = 1;$$

$$\Rightarrow x^2 - y^2 = 3; \mathbf{v} = (\sqrt{3} \sec t \tan t)\mathbf{i} + (\sqrt{3} \sec^2 t)\mathbf{j}$$

and

$$\mathbf{a} = (\sqrt{3} \sec t \tan^2 t + \sqrt{3} \sec^3 t)\mathbf{i} - (2\sqrt{3} \sec^2 t \tan t)\mathbf{j};$$

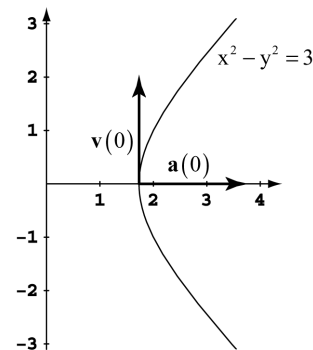
$$\mathbf{r}(0) = \sqrt{3}\mathbf{i}, \mathbf{v}(0) = \sqrt{3}\mathbf{j}, \mathbf{a}(0) = \sqrt{3}\mathbf{i};$$

$$|\mathbf{v}| = \sqrt{3 \sec^2 t \tan^2 t + 3 \sec^4 t}$$

$$\Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = \frac{6 \sec^2 t \tan^3 t + 18 \sec^4 t \tan t}{2\sqrt{3 \sec^2 t \tan^2 t + 3 \sec^4 t}};$$

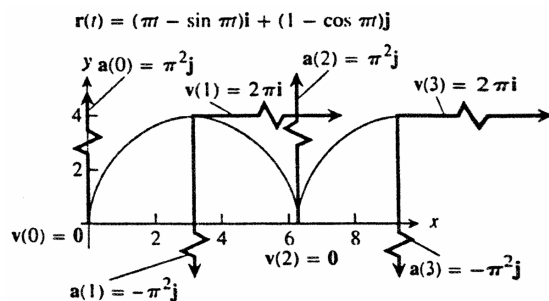
$$\text{at } t = 0: a_T = 0, a_N = \sqrt{|\mathbf{a}|^2 - 0} = \sqrt{3},$$

$$\mathbf{a} = 0\mathbf{T} + \sqrt{3}\mathbf{N} = \sqrt{3}\mathbf{N}, \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$



3. $\mathbf{r} = \frac{1}{\sqrt{1+t^2}} \mathbf{i} + \frac{t}{\sqrt{1+t^2}} \mathbf{j} \Rightarrow \mathbf{v} = -t(1+t^2)^{-3/2} \mathbf{i} + (1+t^2)^{-3/2} \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{[-t(1+t^2)^{-3/2}]^2 + [(1+t^2)^{-3/2}]^2}$
 $= \frac{1}{1+t^2}$. We want to maximize $|\mathbf{v}|$: $\frac{d|\mathbf{v}|}{dt} = \frac{-2t}{(1+t^2)^2}$ and $\frac{d|\mathbf{v}|}{dt} = 0 \Rightarrow \frac{-2t}{(1+t^2)^2} = 0 \Rightarrow t = 0$. For $t < 0$, $\frac{-2t}{(1+t^2)^2} > 0$; for $t > 0$, $\frac{-2t}{(1+t^2)^2} < 0 \Rightarrow |\mathbf{v}|_{\max}$ occurs when $t = 0 \Rightarrow |\mathbf{v}|_{\max} = 1$
4. $\mathbf{r} = (e^t \cos t) \mathbf{i} + (e^t \sin t) \mathbf{j} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t) \mathbf{i} + (e^t \sin t + e^t \cos t) \mathbf{j}$
 $\Rightarrow \mathbf{a} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t) \mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t) \mathbf{j}$
 $= (-2e^t \sin t) \mathbf{i} + (2e^t \cos t) \mathbf{j}$. Let θ be the angle between $\mathbf{r}$ and $\mathbf{a}$. Then $\theta = \cos^{-1} \left(\frac{\mathbf{r} \cdot \mathbf{a}}{|\mathbf{r}| |\mathbf{a}|} \right)$
 $= \cos^{-1} \left(\frac{-2e^{2t} \sin t \cos t + 2e^{2t} \sin t \cos t}{\sqrt{(e^t \cos t)^2 + (e^t \sin t)^2} \sqrt{(-2e^t \sin t)^2 + (2e^t \cos t)^2}} \right) = \cos^{-1} \left(\frac{0}{2e^{2t}} \right) = \cos^{-1} 0 = \frac{\pi}{2}$ for all t
5. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 5\mathbf{i} + 15\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 5 & 15 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 25$; $|\mathbf{v}| = \sqrt{3^2 + 4^2} = 5$
 $\Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{25}{5^3} = \frac{1}{5}$
6. $\kappa = \frac{|y''|}{[1+(y')^2]^{3/2}} = e^x (1+e^{2x})^{-3/2} \Rightarrow \frac{d\kappa}{dx} = e^x (1+e^{2x})^{-3/2} + e^x \left[-\frac{3}{2} (1+e^{2x})^{-5/2} (2e^{2x}) \right]$
 $= e^x (1+e^{2x})^{-3/2} - 3e^{3x} (1+e^{2x})^{-5/2} = e^x (1+e^{2x})^{-5/2} [(1+e^{2x}) - 3e^{2x}] = e^x (1+e^{2x})^{-5/2} (1-2e^{2x})$;
 $\frac{d\kappa}{dx} = 0 \Rightarrow (1-2e^{2x}) = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow 2x = -\ln 2 \Rightarrow x = -\frac{1}{2} \ln 2 = -\ln \sqrt{2} \Rightarrow y = \frac{1}{\sqrt{2}}$; therefore κ is at a maximum at the point $\left(-\ln \sqrt{2}, \frac{1}{\sqrt{2}} \right)$
7. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j}$ and $\mathbf{v} \cdot \mathbf{i} = y \Rightarrow \frac{dx}{dt} = y$. Since the particle moves around the unit circle $x^2 + y^2 = 1$, $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y} (y) = -x$. Since $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$, we have $\mathbf{v} = y\mathbf{i} - x\mathbf{j} \Rightarrow$ at $(1, 0)$, $\mathbf{v} = -\mathbf{j}$ and the motion is clockwise.
8. $9y = x^3 \Rightarrow 9 \frac{dy}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{3} x^2 \frac{dx}{dt}$. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t , then $\mathbf{v} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j}$. Hence $\mathbf{v} \cdot \mathbf{i} = 4 \Rightarrow \frac{dx}{dt} = 4$ and $\mathbf{v} \cdot \mathbf{j} = \frac{dy}{dt} = \frac{1}{3} x^2 \frac{dx}{dt} = \frac{1}{3} (3)^2 (4) = 12$ at $(3, 3)$. Also, $\mathbf{a} = \frac{d^2x}{dt^2} \mathbf{i} + \frac{d^2y}{dt^2} \mathbf{j}$ and $\frac{d^2y}{dt^2} = \left(\frac{2}{3} x \right) \left(\frac{dx}{dt} \right)^2 + \left(\frac{1}{3} x^2 \right) \frac{d^2x}{dt^2}$. Hence $\mathbf{a} \cdot \mathbf{i} = -2 \Rightarrow \frac{d^2x}{dt^2} = -2$ and $\mathbf{a} \cdot \mathbf{j} = \frac{d^2y}{dt^2} = \frac{2}{3} (3)(4)^2 + \frac{1}{3} (3)^2 (-2) = 26$ at the point $(x, y) = (3, 3)$.
9. $\frac{d\mathbf{r}}{dt}$ orthogonal to $\mathbf{r} \Rightarrow 0 = \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = \frac{1}{2} \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} + \frac{1}{2} \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{2} \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) \Rightarrow \mathbf{r} \cdot \mathbf{r} = K$, a constant. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t , then $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 \Rightarrow x^2 + y^2 = K$, which is the equation of a circle centered at the origin.

10. (a)

(b) $\mathbf{v} = (\pi - \pi \cos \pi t) \mathbf{i} + (\pi \sin \pi t) \mathbf{j}$

$$\Rightarrow \mathbf{a} = (\pi^2 \sin \pi t) \mathbf{i} + (\pi^2 \cos \pi t) \mathbf{j};$$

$$\mathbf{v}(0) = \mathbf{0} \text{ and } \mathbf{a}(0) = \pi^2 \mathbf{j};$$

$$\mathbf{v}(1) = 2\pi \mathbf{i} \text{ and } \mathbf{a}(1) = -\pi^2 \mathbf{j};$$

$$\mathbf{v}(2) = \mathbf{0} \text{ and } \mathbf{a}(2) = \pi^2 \mathbf{j};$$

$$\mathbf{v}(3) = 2\pi \mathbf{i} \text{ and } \mathbf{a}(3) = -\pi^2 \mathbf{j}$$

- (c) Forward speed at the topmost point is $|\mathbf{v}(1)| = |\mathbf{v}(3)| = 2\pi$ ft/sec; since the circle makes $\frac{1}{2}$ revolution per second, the center moves π ft parallel to the x-axis each second $\Rightarrow$ the forward speed of C is π ft/sec.

11. $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 6.5 + (44 \text{ ft/sec})(\sin 45^\circ)(3 \text{ sec}) - \frac{1}{2}(32 \text{ ft/sec}^2)(3 \text{ sec})^2 = 6.5 + 66\sqrt{2} - 144 \approx -44.16 \text{ ft} \Rightarrow$ the shot put is on the ground. Now, $y = 0 \Rightarrow 6.5 + 22\sqrt{2}t - 16t^2 = 0 \Rightarrow t \approx 2.13 \text{ sec}$ (the positive root) $\Rightarrow x \approx (44 \text{ ft/sec})(\cos 45^\circ)(2.13 \text{ sec}) \approx 66.27 \text{ ft}$ or about 66 ft, 3 in. from the stopboard

12. $y_{\max} = y_0 + \frac{(v_0 \sin \alpha)^2}{2g} = 7 \text{ ft} + \frac{[(80 \text{ ft/sec})(\sin 45^\circ)]^2}{(2)(32 \text{ ft/sec}^2)} \approx 57 \text{ ft}$

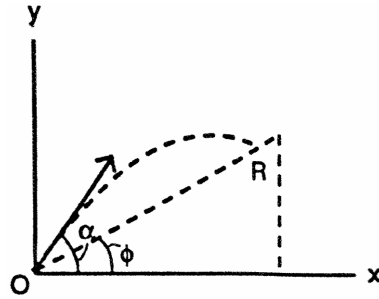
13. $x = (v_0 \cos \alpha)t$ and $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow \tan \phi = \frac{y}{x} = \frac{(v_0 \sin \alpha)t - \frac{1}{2}gt^2}{(v_0 \cos \alpha)t} = \frac{(v_0 \sin \alpha) - \frac{1}{2}gt}{v_0 \cos \alpha}$
 $\Rightarrow v_0 \cos \alpha \tan \phi = v_0 \sin \alpha - \frac{1}{2}gt \Rightarrow t = \frac{2v_0 \sin \alpha - 2v_0 \cos \alpha \tan \phi}{g}$, which is the time when the golf ball hits the upward slope. At this time $x = (v_0 \cos \alpha) \left(\frac{2v_0 \sin \alpha - 2v_0 \cos \alpha \tan \phi}{g} \right) = \left(\frac{2}{g} \right) (v_0^2 \sin \alpha \cos \alpha - v_0^2 \cos^2 \alpha \tan \phi)$.

Now $OR = \frac{x}{\cos \phi} \Rightarrow OR = \left(\frac{2}{g} \right) \left(\frac{v_0^2 \sin \alpha \cos \alpha - v_0^2 \cos^2 \alpha \tan \phi}{\cos \phi} \right)$
 $= \left(\frac{2v_0^2 \cos \alpha}{g} \right) \left(\frac{\sin \alpha}{\cos \phi} - \frac{\cos \alpha \tan \phi}{\cos \phi} \right)$
 $= \left(\frac{2v_0^2 \cos \alpha}{g} \right) \left(\frac{\sin \alpha \cos \phi - \cos \alpha \sin \phi}{\cos^2 \phi} \right)$
 $= \left(\frac{2v_0^2 \cos \alpha}{g \cos^2 \phi} \right) [\sin(\alpha - \phi)]$. The distance OR is maximized

when x is maximized:

$$\frac{dx}{d\alpha} = \left(\frac{2v_0^2}{g} \right) (\cos 2\alpha + \sin 2\alpha \tan \phi) = 0$$

$$\Rightarrow (\cos 2\alpha + \sin 2\alpha \tan \phi) = 0 \Rightarrow \cot 2\alpha + \tan \phi = 0 \Rightarrow \cot 2\alpha = \tan(-\phi) \Rightarrow 2\alpha = \frac{\pi}{2} + \phi \Rightarrow \alpha = \frac{\phi}{2} + \frac{\pi}{4}$$



14. (a) $x = v_0(\cos 40^\circ)t$ and $y = 6.5 + v_0(\sin 40^\circ)t - \frac{1}{2}gt^2 = 6.5 + v_0(\sin 40^\circ)t - 16t^2$; $x = 262 \frac{5}{12} \text{ ft}$ and $y = 0 \text{ ft}$
 $\Rightarrow 262 \frac{5}{12} = v_0(\cos 40^\circ)t$ or $v_0 = \frac{262.4167}{(\cos 40^\circ)t}$ and $0 = 6.5 + \left[\frac{262.4167}{(\cos 40^\circ)t} \right] (\sin 40^\circ)t - 16t^2 \Rightarrow t^2 = 14.1684$
 $\Rightarrow t \approx 3.764 \text{ sec}$. Therefore, $262.4167 \approx v_0(\cos 40^\circ)(3.764 \text{ sec}) \Rightarrow v_0 \approx \frac{262.4167}{(\cos 40^\circ)(3.764 \text{ sec})} \Rightarrow v_0 \approx 91 \text{ ft/sec}$

(b) $y_{\max} = y_0 + \frac{(v_0 \sin \alpha)^2}{2g} \approx 6.5 + \frac{((91)(\sin 40^\circ))^2}{(2)(32)} \approx 60 \text{ ft}$

15. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (2t)^2}$
 $= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^{\pi/4} 2\sqrt{1+t^2} dt = \left[t\sqrt{1+t^2} + \ln \left| t + \sqrt{1+t^2} \right| \right]_0^{\pi/4} = \frac{\pi}{4} \sqrt{1 + \frac{\pi^2}{16}} + \ln \left(\frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}} \right)$

16. $\mathbf{r} = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + 2t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (-3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 3t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (3t^{1/2})^2}$
 $= \sqrt{9+9t} = 3\sqrt{1+t} \Rightarrow \text{Length} = \int_0^3 3\sqrt{1+t} dt = [2(1+t)^{3/2}]_0^3 = 14$

17. $\mathbf{r} = \frac{4}{9}(1+t)^{3/2}\mathbf{i} + \frac{4}{9}(1-t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{\left[\frac{2}{3}(1+t)^{1/2} \right]^2 + \left[-\frac{2}{3}(1-t)^{1/2} \right]^2 + \left(\frac{1}{3} \right)^2} = 1 \Rightarrow \mathbf{T} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$
 $\Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$; $\frac{d\mathbf{T}}{dt} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt}(0) \right| = \frac{\sqrt{2}}{3}$
 $\Rightarrow \mathbf{N}(0) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} = -\frac{1}{3\sqrt{2}}\mathbf{i} + \frac{1}{3\sqrt{2}}\mathbf{j} + \frac{4}{3\sqrt{2}}\mathbf{k}$;
 $\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$ and $\mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \Rightarrow \mathbf{v}(0) \times \mathbf{a}(0)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix} = -\frac{1}{9}\mathbf{i} + \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \frac{\sqrt{2}}{3} \Rightarrow \kappa(0) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\left(\frac{\sqrt{2}}{3}\right)}{1^3} = \frac{\sqrt{2}}{3};$$

$$\dot{\mathbf{a}} = -\frac{1}{6}(1+t)^{-3/2}\mathbf{i} + \frac{1}{6}(1-t)^{-3/2}\mathbf{j} \Rightarrow \dot{\mathbf{a}}(0) = -\frac{1}{6}\mathbf{i} + \frac{1}{6}\mathbf{j} \Rightarrow \tau(0) = \frac{\begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{\left(\frac{1}{3}\right)\left(\frac{2}{18}\right)}{\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{6}$$

18. $\mathbf{r} = (e^t \sin 2t)\mathbf{i} + (e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \sin 2t + 2e^t \cos 2t)\mathbf{i} + (e^t \cos 2t - 2e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{(e^t \sin 2t + 2e^t \cos 2t)^2 + (e^t \cos 2t - 2e^t \sin 2t)^2 + (2e^t)^2} = 3e^t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \left(\frac{1}{3} \sin 2t + \frac{2}{3} \cos 2t\right)\mathbf{i} + \left(\frac{1}{3} \cos 2t - \frac{2}{3} \sin 2t\right)\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k};$
 $\frac{d\mathbf{T}}{dt} = \left(\frac{2}{3} \cos 2t - \frac{4}{3} \sin 2t\right)\mathbf{i} + \left(-\frac{2}{3} \sin 2t - \frac{4}{3} \cos 2t\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{2}{3}\sqrt{5}$
 $\Rightarrow \mathbf{N}(0) = \frac{\left(\frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j}\right)}{\left(\frac{2\sqrt{5}}{3}\right)} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}; \mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix} = \frac{4}{3\sqrt{5}}\mathbf{i} + \frac{2}{3\sqrt{5}}\mathbf{j} - \frac{5}{3\sqrt{5}}\mathbf{k};$
 $\mathbf{a} = (4e^t \cos 2t - 3e^t \sin 2t)\mathbf{i} + (-3e^t \cos 2t - 4e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
 $\Rightarrow \mathbf{v}(0) \times \mathbf{a}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} - 10\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{64 + 16 + 100} = 6\sqrt{5} \text{ and } |\mathbf{v}(0)| = 3$
 $\Rightarrow \kappa(0) = \frac{6\sqrt{5}}{3^3} = \frac{2\sqrt{5}}{9};$
 $\dot{\mathbf{a}} = (4e^t \cos 2t - 8e^t \sin 2t - 3e^t \sin 2t - 6e^t \cos 2t)\mathbf{i} + (-3e^t \cos 2t + 6e^t \sin 2t - 4e^t \sin 2t - 8e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k}$
 $= (-2e^t \cos 2t - 11e^t \sin 2t)\mathbf{i} + (-11e^t \cos 2t + 2e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \dot{\mathbf{a}}(0) = -2\mathbf{i} - 11\mathbf{j} + 2\mathbf{k}$
 $\Rightarrow \tau(0) = \frac{\begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ -2 & -11 & 2 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-80}{180} = -\frac{4}{9}$

19. $\mathbf{r} = t\mathbf{i} + \frac{1}{2}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + e^{2t}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + e^{4t}} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1 + e^{4t}}}\mathbf{i} + \frac{e^{2t}}{\sqrt{1 + e^{4t}}}\mathbf{j} \Rightarrow \mathbf{T}(\ln 2) = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j};$
 $\frac{d\mathbf{T}}{dt} = \frac{-2e^{4t}}{(1 + e^{4t})^{3/2}}\mathbf{i} + \frac{2e^{2t}}{(1 + e^{4t})^{3/2}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \frac{-32}{17\sqrt{17}}\mathbf{i} + \frac{8}{17\sqrt{17}}\mathbf{j} \Rightarrow \mathbf{N}(\ln 2) = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j};$
 $\mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} = \mathbf{k}; \mathbf{a} = 2e^{2t}\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 8\mathbf{j} \text{ and } \mathbf{v}(\ln 2) = \mathbf{i} + 4\mathbf{j}$
 $\Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix} = 8\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 8 \text{ and } |\mathbf{v}(\ln 2)| = \sqrt{17} \Rightarrow \kappa(\ln 2) = \frac{8}{17\sqrt{17}}; \dot{\mathbf{a}} = 4e^{2t}\mathbf{j}$
 $\Rightarrow \dot{\mathbf{a}}(\ln 2) = 16\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0$

20. $\mathbf{r} = (3 \cosh 2t)\mathbf{i} + (3 \sinh 2t)\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{v} = (6 \sinh 2t)\mathbf{i} + (6 \cosh 2t)\mathbf{j} + 6\mathbf{k}$
 $\Rightarrow |\mathbf{v}| = \sqrt{36 \sinh^2 2t + 36 \cosh^2 2t + 36} = 6\sqrt{2} \cosh 2t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \tanh 2t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} 2t\right)\mathbf{k}$
 $\Rightarrow \mathbf{T}(\ln 2) = \frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{8}{17\sqrt{2}}\mathbf{k}; \frac{d\mathbf{T}}{dt} = \left(\frac{2}{\sqrt{2}} \operatorname{sech}^2 2t\right)\mathbf{i} - \left(\frac{2}{\sqrt{2}} \operatorname{sech} 2t \tanh 2t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2)$
 $= \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)^2\mathbf{i} - \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)\mathbf{k} = \frac{128}{289\sqrt{2}}\mathbf{i} - \frac{240}{289\sqrt{2}}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(\ln 2)\right| = \sqrt{\left(\frac{128}{289\sqrt{2}}\right)^2 + \left(-\frac{240}{289\sqrt{2}}\right)^2} = \frac{8\sqrt{2}}{17}$

$$\Rightarrow \mathbf{N}(\ln 2) = \frac{8}{17}\mathbf{i} - \frac{15}{17}\mathbf{k}; \mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{15}{17\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{8}{17\sqrt{2}} \\ \frac{8}{17} & 0 & -\frac{15}{17} \end{vmatrix} = -\frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \frac{8}{17\sqrt{2}}\mathbf{k};$$

$$\mathbf{a} = (12 \cosh 2t)\mathbf{i} + (12 \sinh 2t)\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 12\left(\frac{17}{8}\right)\mathbf{i} + 12\left(\frac{15}{8}\right)\mathbf{j} = \frac{51}{2}\mathbf{i} + \frac{45}{2}\mathbf{j} \text{ and}$$

$$\mathbf{v}(\ln 2) = 6\left(\frac{15}{8}\right)\mathbf{i} + 6\left(\frac{17}{8}\right)\mathbf{j} + 6\mathbf{k} = \frac{45}{4}\mathbf{i} + \frac{51}{4}\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \end{vmatrix}$$

$$= -135\mathbf{i} + 153\mathbf{j} - 72\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 153\sqrt{2} \text{ and } |\mathbf{v}(\ln 2)| = \frac{51}{4}\sqrt{2} \Rightarrow \kappa(\ln 2) = \frac{153\sqrt{2}}{\left(\frac{51}{4}\sqrt{2}\right)^3} = \frac{32}{867};$$

$$\dot{\mathbf{a}} = (24 \sinh 2t)\mathbf{i} + (24 \cosh 2t)\mathbf{j} \Rightarrow \dot{\mathbf{a}}(\ln 2) = 45\mathbf{i} + 51\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \\ 45 & 51 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{32}{867}$$

$$21. \mathbf{r} = (2 + 3t + 3t^2)\mathbf{i} + (4t + 4t^2)\mathbf{j} - (6 \cos t)\mathbf{k} \Rightarrow \mathbf{v} = (3 + 6t)\mathbf{i} + (4 + 8t)\mathbf{j} + (6 \sin t)\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(3 + 6t)^2 + (4 + 8t)^2 + (6 \sin t)^2} = \sqrt{25 + 100t + 100t^2 + 36 \sin^2 t}$$

$$\Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}(25 + 100t + 100t^2 + 36 \sin^2 t)^{-1/2}(100 + 200t + 72 \sin t \cos t) \Rightarrow a_T(0) = \frac{d|\mathbf{v}|}{dt}(0) = 10;$$

$$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j} + (6 \cos t)\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{6^2 + 8^2 + (6 \cos t)^2} = \sqrt{100 + 36 \cos^2 t} \Rightarrow |\mathbf{a}(0)| = \sqrt{136}$$

$$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{136 - 10^2} = \sqrt{36} = 6 \Rightarrow \mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$$

$$22. \mathbf{r} = (2 + t)\mathbf{i} + (t + 2t^2)\mathbf{j} + (1 + t^2)\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + (1 + 4t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (1 + 4t)^2 + (2t)^2}$$

$$= \sqrt{2 + 8t + 20t^2} \Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}(2 + 8t + 20t^2)^{-1/2}(8 + 40t) \Rightarrow a_T = \frac{d|\mathbf{v}|}{dt}(0) = 2\sqrt{2}; \mathbf{a} = 4\mathbf{j} + 2\mathbf{k}$$

$$\Rightarrow |\mathbf{a}| = \sqrt{4^2 + 2^2} = \sqrt{20} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{20 - (2\sqrt{2})^2} = \sqrt{12} = 2\sqrt{3} \Rightarrow \mathbf{a}(0) = 2\sqrt{2}\mathbf{T} + 2\sqrt{3}\mathbf{N}$$

$$23. \mathbf{r} = (\sin t)\mathbf{i} + (\sqrt{2} \cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (\cos t)\mathbf{i} - (\sqrt{2} \sin t)\mathbf{j} + (\cos t)\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t)^2 + (-\sqrt{2} \sin t)^2 + (\cos t)^2} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{i} - (\sin t)\mathbf{j} + \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{k};$$

$$\frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t\right)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{1}{\sqrt{2}} \sin t\right)^2 + (-\cos t)^2 + \left(-\frac{1}{\sqrt{2}} \sin t\right)^2} = 1$$

$$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t\right)\mathbf{k}; \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} \cos t & -\sin t & \frac{1}{\sqrt{2}} \cos t \\ -\frac{1}{\sqrt{2}} \sin t & -\cos t & -\frac{1}{\sqrt{2}} \sin t \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}; \mathbf{a} = (-\sin t)\mathbf{i} - (\sqrt{2} \cos t)\mathbf{j} - (\sin t)\mathbf{k} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \end{vmatrix}$$

$$= \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{4} = 2 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}}; \dot{\mathbf{a}} = (-\cos t)\mathbf{i} + (\sqrt{2} \sin t)\mathbf{j} - (\cos t)\mathbf{k}$$

$$\Rightarrow \tau = \frac{\begin{vmatrix} \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \\ -\cos t & \sqrt{2} \sin t & -\cos t \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{(\cos t)(\sqrt{2}) - (\sqrt{2} \sin t)(0) + (\cos t)(-\sqrt{2})}{4} = 0$$

$$24. \mathbf{r} = \mathbf{i} + (5 \cos t)\mathbf{j} + (3 \sin t)\mathbf{k} \Rightarrow \mathbf{v} = (-5 \sin t)\mathbf{j} + (3 \cos t)\mathbf{k} \Rightarrow \mathbf{a} = (-5 \cos t)\mathbf{j} - (3 \sin t)\mathbf{k}$$

$$\Rightarrow \mathbf{v} \cdot \mathbf{a} = 25 \sin t \cos t - 9 \sin t \cos t = 16 \sin t \cos t; \mathbf{v} \cdot \mathbf{a} = 0 \Rightarrow 16 \sin t \cos t = 0 \Rightarrow \sin t = 0 \text{ or } \cos t = 0$$

$$\Rightarrow t = 0, \frac{\pi}{2} \text{ or } \pi$$

$$25. \mathbf{r} = 2\mathbf{i} + \left(4 \sin \frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k} \Rightarrow 0 = \mathbf{r} \cdot (\mathbf{i} - \mathbf{j}) = 2(1) + \left(4 \sin \frac{t}{2}\right)(-1) \Rightarrow 0 = 2 - 4 \sin \frac{t}{2} \Rightarrow \sin \frac{t}{2} = \frac{1}{2} \Rightarrow \frac{t}{2} = \frac{\pi}{6} \\ \Rightarrow t = \frac{\pi}{3} \text{ (for the first time)}$$

$$26. \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1 + 4t^2 + 9t^4} \Rightarrow |\mathbf{v}(1)| = \sqrt{14} \\ \Rightarrow \mathbf{T}(1) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}, \text{ which is normal to the normal plane} \\ \Rightarrow \frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0 \text{ or } x + 2y + 3z = 6 \text{ is an equation of the normal plane. Next we} \\ \text{calculate } \mathbf{N}(1) \text{ which is normal to the rectifying plane. Now, } \mathbf{a} = 2\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(1) \times \mathbf{a}(1) \\ = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{v}(1) \times \mathbf{a}(1)| = \sqrt{76} \Rightarrow \kappa(1) = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{\sqrt{19}}{7\sqrt{14}}; \frac{ds}{dt} = |\mathbf{v}(t)| \Rightarrow \left. \frac{d^2s}{dt^2} \right|_{t=1} \\ = \frac{1}{2}(1 + 4t^2 + 9t^4)^{-1/2}(8t + 36t^3) \Big|_{t=1} = \frac{22}{\sqrt{14}}, \text{ so } \mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} \Rightarrow 2\mathbf{j} + 6\mathbf{k} \\ = \frac{22}{\sqrt{14}}\left(\frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}}\right) + \frac{\sqrt{19}}{7\sqrt{14}}(\sqrt{14})^2\mathbf{N} \Rightarrow \mathbf{N} = \frac{\sqrt{14}}{2\sqrt{19}}\left(-\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k}\right) \Rightarrow -\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1) \\ = 0 \text{ or } 11x + 8y - 9z = 10 \text{ is an equation of the rectifying plane. Finally, } \mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) \\ = \left(\frac{\sqrt{14}}{2\sqrt{19}}\right)\left(\frac{1}{\sqrt{14}}\right)\left(\frac{1}{7}\right) \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}}(3\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \Rightarrow 3(x-1) - 3(y-1) + (z-1) = 0 \text{ or } 3x - 3y + z \\ = 1 \text{ is an equation of the osculating plane.}$$

$$27. \mathbf{r} = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k} \Rightarrow \mathbf{v} = e^t\mathbf{i} + (\cos t)\mathbf{j} - \left(\frac{1}{1-t}\right)\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}; \mathbf{r}(0) = \mathbf{i} \Rightarrow (1, 0, 0) \text{ is on the line} \\ \Rightarrow x = 1 + t, y = t, \text{ and } z = -t \text{ are parametric equations of the line}$$

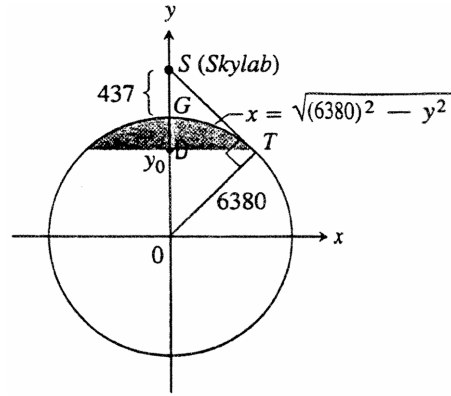
$$28. \mathbf{r} = (\sqrt{2} \cos t)\mathbf{i} + (\sqrt{2} \sin t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (-\sqrt{2} \sin t)\mathbf{i} + (\sqrt{2} \cos t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{4}\right) \\ = (-\sqrt{2} \sin \frac{\pi}{4})\mathbf{i} + (\sqrt{2} \cos \frac{\pi}{4})\mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k} \text{ is a vector tangent to the helix when } t = \frac{\pi}{4} \Rightarrow \text{the tangent line} \\ \text{is parallel to } \mathbf{v}\left(\frac{\pi}{4}\right); \text{ also } \mathbf{r}\left(\frac{\pi}{4}\right) = (\sqrt{2} \cos \frac{\pi}{4})\mathbf{i} + (\sqrt{2} \sin \frac{\pi}{4})\mathbf{j} + \frac{\pi}{4}\mathbf{k} \Rightarrow \text{the point } (1, 1, \frac{\pi}{4}) \text{ is on the line} \\ \Rightarrow x = 1 - t, y = 1 + t, \text{ and } z = \frac{\pi}{4} + t \text{ are parametric equations of the line}$$

$$29. x^2 = (v_0^2 \cos^2 \alpha) t^2 \text{ and } \left(y + \frac{1}{2}gt^2\right)^2 = (v_0^2 \sin^2 \alpha) t^2 \Rightarrow x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = v_0^2 t^2$$

$$30. \dot{s} = \frac{d}{dt} \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow \dot{x}^2 + \dot{y}^2 - \dot{s}^2 = \dot{x}^2 + \dot{y}^2 - \frac{(\dot{x}\dot{x} + \dot{y}\dot{y})^2}{\dot{x}^2 + \dot{y}^2} \\ = \frac{(\dot{x}^2 + \dot{y}^2)(\dot{x}^2 + \dot{y}^2) - (\dot{x}\dot{x} + \dot{y}\dot{y})^2}{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}^2\dot{y}^2 + \dot{y}^2\dot{x}^2 - 2\dot{x}\dot{x}\dot{y}\dot{y}}{\dot{x}^2 + \dot{y}^2} = \frac{(\dot{x}\dot{y} - \dot{y}\dot{x})^2}{\dot{x}^2 + \dot{y}^2} \\ \Rightarrow \sqrt{\dot{x}^2 + \dot{y}^2 - \dot{s}^2} = \frac{|\dot{x}\dot{y} - \dot{y}\dot{x}|}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow \frac{\dot{x}^2 + \dot{y}^2}{\sqrt{\dot{x}^2 + \dot{y}^2 - \dot{s}^2}} = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\dot{x}\dot{y} - \dot{y}\dot{x}|} = \frac{1}{\kappa} = \rho$$

$$31. s = a\theta \Rightarrow \theta = \frac{s}{a} \Rightarrow \phi = \frac{s}{a} + \frac{\pi}{2} \Rightarrow \frac{d\phi}{ds} = \frac{1}{a} \Rightarrow \kappa = \left|\frac{1}{a}\right| = \frac{1}{a} \text{ since } a > 0$$

32. (1) $\Delta SOT \approx \Delta TOD \Rightarrow \frac{DO}{OT} = \frac{OT}{SO} \Rightarrow \frac{y_0}{6380} = \frac{6380}{6380+437}$
 $\Rightarrow y_0 = \frac{6380^2}{6817} \Rightarrow y_0 \approx 5971 \text{ km};$
- (2) $VA = \int_{5971}^{6380} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$
 $= 2\pi \int_{5971}^{6380} \sqrt{6380^2 - y^2} \left(\frac{6380}{\sqrt{6380^2 - y^2}}\right) dy$
 $= 2\pi \int_{5971}^{6380} 6380 dy = 2\pi [6380y]_{5971}^{6380}$
 $= 16,395,469 \text{ km}^2 \approx 1.639 \times 10^7 \text{ km}^2;$
- (3) percentage visible $\approx \frac{16,395,469 \text{ km}^2}{4\pi(6380 \text{ km})^2} \approx 3.21\%$



CHAPTER 13 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $\mathbf{r}(\theta) = (a \cos \theta)\mathbf{i} + (a \sin \theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + \mathbf{b}\mathbf{k}] \frac{d\theta}{dt}; |\mathbf{v}| = \sqrt{2gz} = \left|\frac{d\mathbf{r}}{dt}\right|$
 $= \sqrt{a^2 + b^2} \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \sqrt{\frac{2gz}{a^2 + b^2}} = \sqrt{\frac{2gb\theta}{a^2 + b^2}} \Rightarrow \frac{d\theta}{dt} \Big|_{\theta=2\pi} = \sqrt{\frac{4\pi gb}{a^2 + b^2}} = 2\sqrt{\frac{\pi gb}{a^2 + b^2}}$
- (b) $\frac{d\theta}{dt} = \sqrt{\frac{2gb\theta}{a^2 + b^2}} \Rightarrow \frac{d\theta}{\sqrt{\theta}} = \sqrt{\frac{2gb}{a^2 + b^2}} dt \Rightarrow 2\theta^{1/2} = \sqrt{\frac{2gb}{a^2 + b^2}} t + C; t = 0 \Rightarrow \theta = 0 \Rightarrow C = 0$
 $\Rightarrow 2\theta^{1/2} = \sqrt{\frac{2gb}{a^2 + b^2}} t \Rightarrow \theta = \frac{gbt^2}{2(a^2 + b^2)}; z = b\theta \Rightarrow z = \frac{gb^2t^2}{2(a^2 + b^2)}$
- (c) $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + \mathbf{b}\mathbf{k}] \frac{d\theta}{dt} = [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + \mathbf{b}\mathbf{k}] \left(\frac{gbt}{a^2 + b^2}\right)$, from part (b)
 $\Rightarrow \mathbf{v}(t) = \left[\frac{(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + \mathbf{b}\mathbf{k}}{\sqrt{a^2 + b^2}}\right] \left(\frac{gbt}{\sqrt{a^2 + b^2}}\right) = \frac{gbt}{\sqrt{a^2 + b^2}} \mathbf{T};$
 $\frac{d^2\mathbf{r}}{dt^2} = [(-a \cos \theta)\mathbf{i} - (a \sin \theta)\mathbf{j}] \left(\frac{d\theta}{dt}\right)^2 + [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + \mathbf{b}\mathbf{k}] \frac{d^2\theta}{dt^2}$
 $= \left(\frac{gbt}{a^2 + b^2}\right)^2 [(-a \cos \theta)\mathbf{i} - (a \sin \theta)\mathbf{j}] + [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + \mathbf{b}\mathbf{k}] \left(\frac{gb}{a^2 + b^2}\right)$
 $= \left[\frac{(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + \mathbf{b}\mathbf{k}}{\sqrt{a^2 + b^2}}\right] \left(\frac{gb}{\sqrt{a^2 + b^2}}\right) + a \left(\frac{gbt}{a^2 + b^2}\right)^2 [(-\cos \theta)\mathbf{i} - (\sin \theta)\mathbf{j}]$
 $= \frac{gb}{\sqrt{a^2 + b^2}} \mathbf{T} + a \left(\frac{gbt}{a^2 + b^2}\right)^2 \mathbf{N}$ (there is no component in the direction of $\mathbf{B}$).
2. (a) $\mathbf{r}(\theta) = (a\theta \cos \theta)\mathbf{i} + (a\theta \sin \theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = [(a \cos \theta - a\theta \sin \theta)\mathbf{i} + (a \sin \theta + a\theta \cos \theta)\mathbf{j} + \mathbf{b}\mathbf{k}] \frac{d\theta}{dt};$
 $|\mathbf{v}| = \sqrt{2gz} = \left|\frac{d\mathbf{r}}{dt}\right| = (a^2 + a^2\theta^2 + b^2)^{1/2} \left(\frac{d\theta}{dt}\right) \Rightarrow \frac{d\theta}{dt} = \frac{\sqrt{2gb\theta}}{\sqrt{a^2 + a^2\theta^2 + b^2}}$
- (b) $s = \int_0^t |\mathbf{v}| dt = \int_0^t (a^2 + a^2\theta^2 + b^2)^{1/2} \frac{d\theta}{dt} dt = \int_0^t (a^2 + a^2\theta^2 + b^2)^{1/2} d\theta = \int_0^\theta (a^2 + a^2u^2 + b^2)^{1/2} du$
 $= \int_0^\theta a \sqrt{\frac{a^2 + b^2}{a^2} + u^2} du = a \int_0^\theta \sqrt{c^2 + u^2} du$, where $c = \frac{\sqrt{a^2 + b^2}}{|a|}$
 $\Rightarrow s = a \left[\frac{u}{2} \sqrt{c^2 + u^2} + \frac{c^2}{2} \ln |u + \sqrt{c^2 + u^2}| \right]_0^\theta = \frac{a}{2} \left(\theta \sqrt{c^2 + \theta^2} + c^2 \ln |\theta + \sqrt{c^2 + \theta^2}| - c^2 \ln c \right)$
3. $\mathbf{r} = \frac{(1+e)r_0}{1+e \cos \theta} \Rightarrow \frac{d\mathbf{r}}{d\theta} = \frac{(1+e)r_0(e \sin \theta)}{(1+e \cos \theta)^2}; \frac{d\mathbf{r}}{dt} = 0 \Rightarrow \frac{(1+e)r_0(e \sin \theta)}{(1+e \cos \theta)^2} = 0 \Rightarrow (1+e)r_0(e \sin \theta) = 0$
 $\Rightarrow \sin \theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$. Note that $\frac{d\mathbf{r}}{d\theta} > 0$ when $\sin \theta > 0$ and $\frac{d\mathbf{r}}{d\theta} < 0$ when $\sin \theta < 0$. Since $\sin \theta < 0$ on $-\pi < \theta < 0$ and $\sin \theta > 0$ on $0 < \theta < \pi$, r is a minimum when $\theta = 0$ and $r(0) = \frac{(1+e)r_0}{1+e \cos 0} = r_0$
4. (a) $f(x) = x - 1 - \frac{1}{2} \sin x = 0 \Rightarrow f(0) = -1$ and $f(2) = 2 - 1 - \frac{1}{2} \sin 2 \geq \frac{1}{2}$ since $|\sin 2| \leq 1$; since f is continuous on $[0, 2]$, the Intermediate Value Theorem implies there is a root between 0 and 2
- (b) Root ≈ 1.4987011335179

5. (a) $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ and $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta = (\dot{r})[(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}] + (r\dot{\theta})[(-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}] \Rightarrow \mathbf{v} \cdot \mathbf{i} = \dot{x}$ and $\mathbf{v} \cdot \mathbf{j} = \dot{y}$
 $\mathbf{v} \cdot \mathbf{i} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta \Rightarrow \dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta$; $\mathbf{v} \cdot \mathbf{j} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta$
 $\Rightarrow \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta$

(b) $\mathbf{u}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} \Rightarrow \mathbf{v} \cdot \mathbf{u}_r = \dot{x} \cos \theta + \dot{y} \sin \theta$
 $= (\dot{r} \cos \theta - r \dot{\theta} \sin \theta)(\cos \theta) + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta)(\sin \theta)$ by part (a),
 $\Rightarrow \mathbf{v} \cdot \mathbf{u}_r = \dot{r}$; therefore, $\dot{r} = \dot{x} \cos \theta + \dot{y} \sin \theta$;
 $\mathbf{u}_\theta = -(\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j} \Rightarrow \mathbf{v} \cdot \mathbf{u}_\theta = -\dot{x} \sin \theta + \dot{y} \cos \theta$
 $= (\dot{r} \cos \theta - r \dot{\theta} \sin \theta)(-\sin \theta) + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta)(\cos \theta)$ by part (a) $\Rightarrow \mathbf{v} \cdot \mathbf{u}_\theta = r \dot{\theta}$;
therefore, $r \dot{\theta} = -\dot{x} \sin \theta + \dot{y} \cos \theta$

6. $\mathbf{r} = f(\theta) \Rightarrow \frac{dr}{dt} = f'(\theta) \frac{d\theta}{dt} \Rightarrow \frac{d^2r}{dt^2} = f''(\theta) \left(\frac{d\theta}{dt}\right)^2 + f'(\theta) \frac{d^2\theta}{dt^2}$; $\mathbf{v} = \frac{dr}{dt} \mathbf{u}_r + r \frac{d\theta}{dt} \mathbf{u}_\theta$
 $= (\cos \theta \frac{dr}{dt} - r \sin \theta \frac{d\theta}{dt}) \mathbf{i} + (\sin \theta \frac{dr}{dt} + r \cos \theta \frac{d\theta}{dt}) \mathbf{j} \Rightarrow |\mathbf{v}| = \left[\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \right]^{1/2} = \left[(f')^2 + f^2 \right]^{1/2} \left(\frac{d\theta}{dt}\right)$;
 $|\mathbf{v} \times \mathbf{a}| = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$, where $x = r \cos \theta$ and $y = r \sin \theta$. Then $\frac{dx}{dt} = (-r \sin \theta) \frac{d\theta}{dt} + (\cos \theta) \frac{dr}{dt}$
 $\Rightarrow \frac{d^2x}{dt^2} = (-2 \sin \theta) \frac{d\theta}{dt} \frac{dr}{dt} - (r \cos \theta) \left(\frac{d\theta}{dt}\right)^2 - (r \sin \theta) \frac{d^2\theta}{dt^2} + (\cos \theta) \frac{d^2r}{dt^2}$; $\frac{dy}{dt} = (r \cos \theta) \frac{d\theta}{dt} + (\sin \theta) \frac{dr}{dt}$
 $\Rightarrow \frac{d^2y}{dt^2} = (2 \cos \theta) \frac{d\theta}{dt} \frac{dr}{dt} - (r \sin \theta) \left(\frac{d\theta}{dt}\right)^2 + (r \cos \theta) \frac{d^2\theta}{dt^2} + (\sin \theta) \frac{d^2r}{dt^2}$. Then $|\mathbf{v} \times \mathbf{a}|$
 $= (\text{after much algebra}) r^2 \left(\frac{d\theta}{dt}\right)^3 + r \frac{d^2\theta}{dt^2} \frac{dr}{dt} - r \frac{d\theta}{dt} \frac{d^2r}{dt^2} + 2 \frac{d\theta}{dt} \left(\frac{dr}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^3 (f^2 - f \cdot f'' + 2(f')^2)$
 $\Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{f^2 - f \cdot f'' + 2(f')^2}{[(f')^2 + f^2]^{3/2}}$

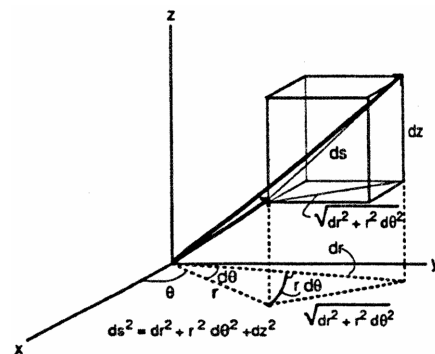
7. (a) Let $r = 2 - t$ and $\theta = 3t \Rightarrow \frac{dr}{dt} = -1$ and $\frac{d\theta}{dt} = 3 \Rightarrow \frac{d^2r}{dt^2} = \frac{d^2\theta}{dt^2} = 0$. The halfway point is $(1, 3) \Rightarrow t = 1$;
 $\mathbf{v} = \frac{dr}{dt} \mathbf{u}_r + r \frac{d\theta}{dt} \mathbf{u}_\theta \Rightarrow \mathbf{v}(1) = -\mathbf{u}_r + 3\mathbf{u}_\theta$; $\mathbf{a} = \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2 \right] \mathbf{u}_r + \left[r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right] \mathbf{u}_\theta \Rightarrow \mathbf{a}(1) = -9\mathbf{u}_r - 6\mathbf{u}_\theta$

(b) It takes the beetle 2 min to crawl to the origin $\Rightarrow$ the rod has revolved 6 radians
 $\Rightarrow L = \int_0^6 \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_0^6 \sqrt{(2 - \frac{\theta}{3})^2 + (-\frac{1}{3})^2} d\theta = \int_0^6 \sqrt{4 - \frac{4\theta}{3} + \frac{\theta^2}{9} + \frac{1}{9}} d\theta$
 $= \int_0^6 \sqrt{\frac{37 - 12\theta + \theta^2}{9}} d\theta = \frac{1}{3} \int_0^6 \sqrt{(\theta - 6)^2 + 1} d\theta = \frac{1}{3} \left[\frac{(\theta - 6)}{2} \sqrt{(\theta - 6)^2 + 1} + \frac{1}{2} \ln |\theta - 6 + \sqrt{(\theta - 6)^2 + 1}| \right]_0^6$
 $= \sqrt{37} - \frac{1}{6} \ln(\sqrt{37} - 6) \approx 6.5$ in.

8. (a) $x = r \cos \theta \Rightarrow dx = \cos \theta dr - r \sin \theta d\theta$; $y = r \sin \theta \Rightarrow dy = \sin \theta dr + r \cos \theta d\theta$; thus
 $dx^2 = \cos^2 \theta dr^2 - 2r \sin \theta \cos \theta dr d\theta + r^2 \sin^2 \theta d\theta^2$ and
 $dy^2 = \sin^2 \theta dr^2 + 2r \sin \theta \cos \theta dr d\theta + r^2 \cos^2 \theta d\theta^2 \Rightarrow ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2$

(c) $r = e^\theta \Rightarrow dr = e^\theta d\theta$ (b)

$\Rightarrow L = \int_0^{\ln 8} \sqrt{dr^2 + r^2 d\theta^2 + dz^2}$
 $= \int_0^{\ln 8} \sqrt{e^{2\theta} + e^{2\theta} + e^{2\theta}} d\theta$
 $= \int_0^{\ln 8} \sqrt{3} e^\theta d\theta = [\sqrt{3} e^\theta]_0^{\ln 8}$
 $= 8\sqrt{3} - \sqrt{3} = 7\sqrt{3}$



$$9. (a) \mathbf{u}_r \times \mathbf{u}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \end{vmatrix} = \mathbf{k} \Rightarrow \text{a right-handed frame of unit vectors}$$

$$(b) \frac{d\mathbf{u}_r}{d\theta} = (-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j} = \mathbf{u}_\theta \text{ and } \frac{d\mathbf{u}_\theta}{d\theta} = (-\cos \theta)\mathbf{i} - (\sin \theta)\mathbf{j} = -\mathbf{u}_r$$

$$(c) \text{ From Eq. (7), } \mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \Rightarrow \mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r) + (\dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta) + \ddot{z}\mathbf{k} \\ = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$$

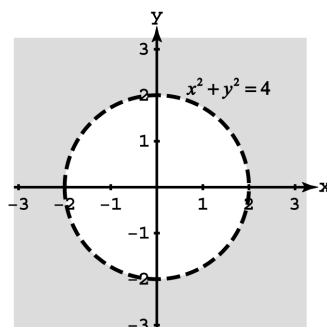
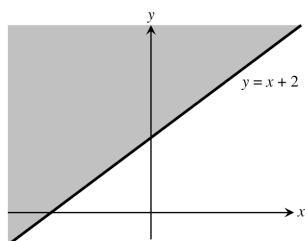
$$10. \mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t) \Rightarrow \frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) + \left(\mathbf{r} \times m \frac{d^2\mathbf{r}}{dt^2}\right) \Rightarrow \frac{d\mathbf{L}}{dt} = (\mathbf{v} \times m\mathbf{v}) + (\mathbf{r} \times m\mathbf{a}) = \mathbf{r} \times m\mathbf{a}; \mathbf{F} = m\mathbf{a} \Rightarrow -\frac{c}{|\mathbf{r}|^3} \mathbf{r} \\ = m\mathbf{a} \Rightarrow \frac{d\mathbf{L}}{dt} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \left(-\frac{c}{|\mathbf{r}|^3} \mathbf{r}\right) = -\frac{c}{|\mathbf{r}|^3} (\mathbf{r} \times \mathbf{r}) = \mathbf{0} \Rightarrow \mathbf{L} = \text{constant vector}$$

NOTES:

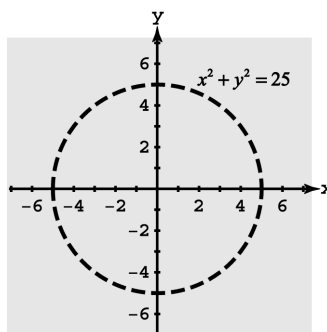
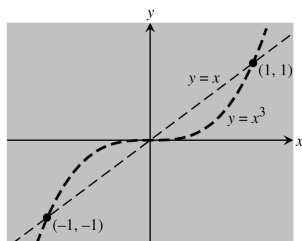
CHAPTER 14 PARTIAL DERIVATIVES

14.1 FUNCTIONS OF SEVERAL VARIABLES

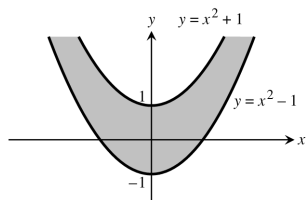
- $f(0, 0) = 0$
 - $f(-3, -2) = 33$
- $f(2, \frac{\pi}{6}) = \frac{\sqrt{3}}{2}$
 - $f(-\frac{\pi}{2}, -7) = -1$
- $f(3, -1, 2) = \frac{4}{5}$
 - $f(2, 2, 100) = 0$
- $f(0, 0, 0) = 7$
 - $f(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}) = \sqrt{\frac{21}{2}}$
- Domain: all points (x, y) on or above the line $y = x + 2$
- Domain: all points (x, y) outside the circle $x^2 + y^2 = 4$



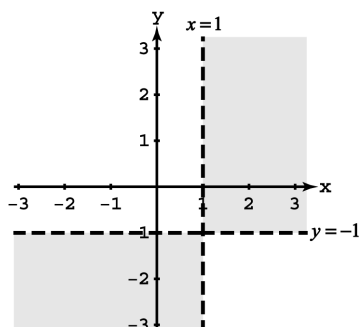
- Domain: all points (x, y) not lying on the graph of $y = x$ or $y = x^3$
- Domain: all points (x, y) not lying on the graph of $x^2 + y^2 = 25$



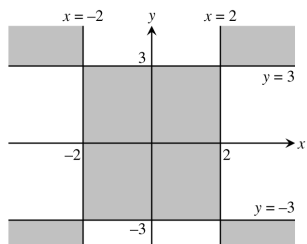
9. Domain: all points (x, y) satisfying
 $x^2 - 1 \leq y \leq x^2 + 1$



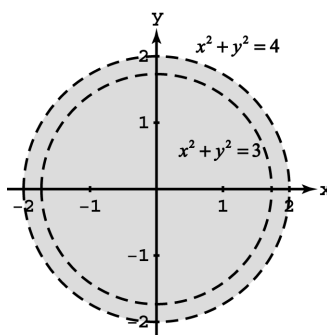
10. Domain: all points (x, y) satisfying
 $(x - 1)(y + 1) > 0$



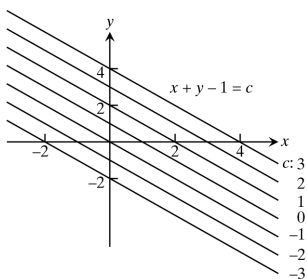
11. Domain: all points (x, y) satisfying
 $(x - 2)(x + 2)(y - 3)(y + 3) \geq 0$



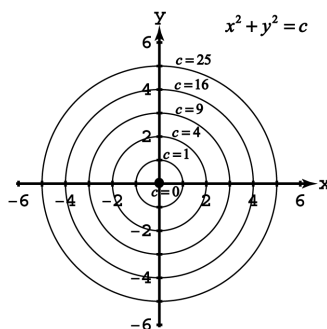
12. Domain: all points (x, y) inside the circle
 $x^2 + y^2 = 4$ such that $x^2 + y^2 \neq 3$



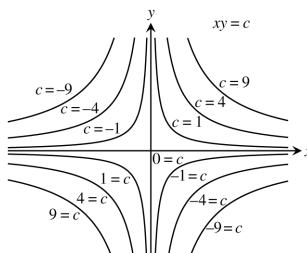
13.



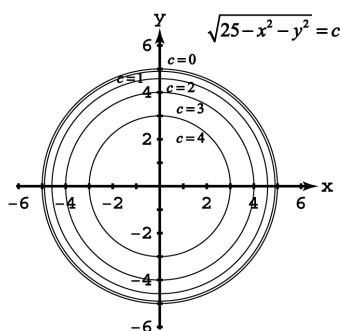
14.



15.



16.



17. (a) Domain: all points in the xy -plane
 (b) Range: all real numbers

- (c) level curves are straight lines $y - x = c$ parallel to the line $y = x$
 - (d) no boundary points
 - (e) both open and closed
 - (f) unbounded
18. (a) Domain: set of all (x, y) so that $y - x \geq 0 \Rightarrow y \geq x$
- (b) Range: $z \geq 0$
 - (c) level curves are straight lines of the form $y - x = c$ where $c \geq 0$
 - (d) boundary is $\sqrt{y - x} = 0 \Rightarrow y = x$, a straight line
 - (e) closed
 - (f) unbounded
19. (a) Domain: all points in the xy -plane
- (b) Range: $z \geq 0$
 - (c) level curves: for $f(x, y) = 0$, the origin; for $f(x, y) = c > 0$, ellipses with center $(0, 0)$ and major and minor axes along the x - and y -axes, respectively
 - (d) no boundary points
 - (e) both open and closed
 - (f) unbounded
20. (a) Domain: all points in the xy -plane
- (b) Range: all real numbers
 - (c) level curves: for $f(x, y) = 0$, the union of the lines $y = \pm x$; for $f(x, y) = c \neq 0$, hyperbolas centered at $(0, 0)$ with foci on the x -axis if $c > 0$ and on the y -axis if $c < 0$
 - (d) no boundary points
 - (e) both open and closed
 - (f) unbounded
21. (a) Domain: all points in the xy -plane
- (b) Range: all real numbers
 - (c) level curves are hyperbolas with the x - and y -axes as asymptotes when $f(x, y) \neq 0$, and the x - and y -axes when $f(x, y) = 0$
 - (d) no boundary points
 - (e) both open and closed
 - (f) unbounded
22. (a) Domain: all $(x, y) \neq (0, 0)$
- (b) Range: all real numbers
 - (c) level curves: for $f(x, y) = 0$, the x -axis minus the origin; for $f(x, y) = c \neq 0$, the parabolas $y = cx^2$ minus the origin
 - (d) boundary is the line $x = 0$
 - (e) open
 - (f) unbounded
23. (a) Domain: all (x, y) satisfying $x^2 + y^2 < 16$
- (b) Range: $z \geq \frac{1}{4}$
 - (c) level curves are circles centered at the origin with radii $r < 4$
 - (d) boundary is the circle $x^2 + y^2 = 16$

- (e) open
 - (f) bounded
24. (a) Domain: all (x, y) satisfying $x^2 + y^2 \leq 9$
 (b) Range: $0 \leq z \leq 3$
 (c) level curves are circles centered at the origin with radii $r \leq 3$
 (d) boundary is the circle $x^2 + y^2 = 9$
 (e) closed
 (f) bounded
25. (a) Domain: $(x, y) \neq (0, 0)$
 (b) Range: all real numbers
 (c) level curves are circles with center $(0, 0)$ and radii $r > 0$
 (d) boundary is the single point $(0, 0)$
 (e) open
 (f) unbounded
26. (a) Domain: all points in the xy -plane
 (b) Range: $0 < z \leq 1$
 (c) level curves are the origin itself and the circles with center $(0, 0)$ and radii $r > 0$
 (d) no boundary points
 (e) both open and closed
 (f) unbounded
27. (a) Domain: all (x, y) satisfying $-1 \leq y - x \leq 1$
 (b) Range: $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$
 (c) level curves are straight lines of the form $y - x = c$ where $-1 \leq c \leq 1$
 (d) boundary is the two straight lines $y = 1 + x$ and $y = -1 + x$
 (e) closed
 (f) unbounded
28. (a) Domain: all (x, y) , $x \neq 0$
 (b) Range: $-\frac{\pi}{2} < z < \frac{\pi}{2}$
 (c) level curves are the straight lines of the form $y = cx$, c any real number and $x \neq 0$
 (d) boundary is the line $x = 0$
 (e) open
 (f) unbounded
29. (a) Domain: all points (x, y) outside the circle $x^2 + y^2 = 1$
 (b) Range: all reals
 (c) Circles centered at the origin with radii $r > 1$
 (d) Boundary: the circle $x^2 + y^2 = 1$
 (e) open
 (f) unbounded
30. (a) Domain: all points (x, y) inside the circle $x^2 + y^2 = 9$
 (b) Range: $z < \ln 9$
 (c) Circles centered at the origin with radii $r < 3$
 (d) Boundary: the circle $x^2 + y^2 = 9$

- (e) open
(f) bounded

31. f

32. e

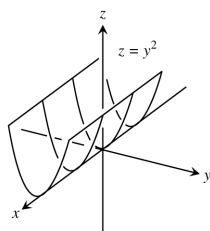
33. a

34. c

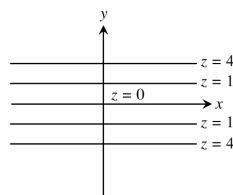
35. d

36. b

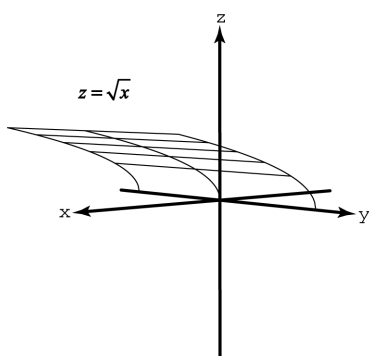
37. (a)



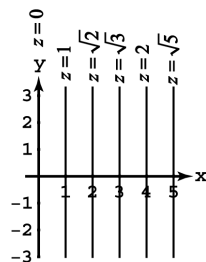
(b)



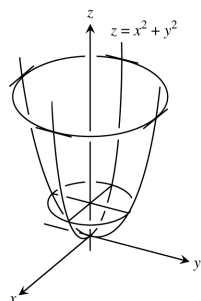
38. (a)



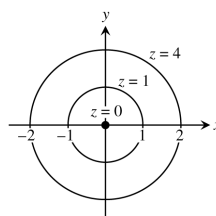
(b)



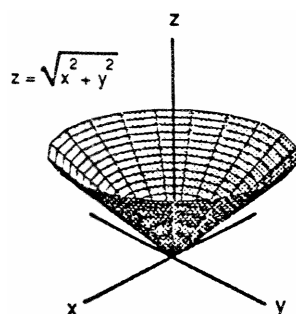
39. (a)



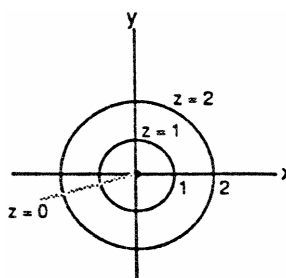
(b)



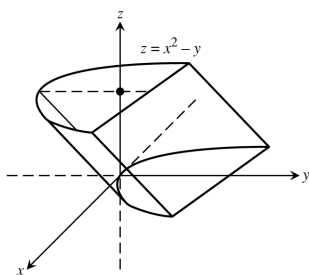
40. (a)



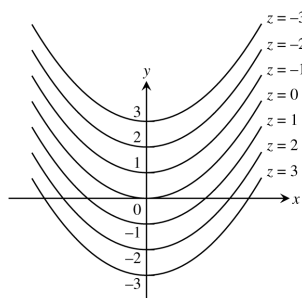
(b)



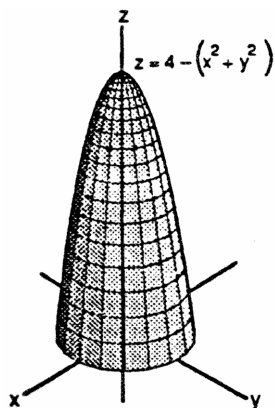
41. (a)



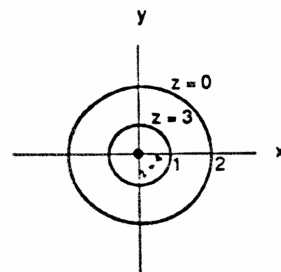
(b)



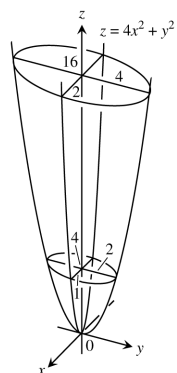
42. (a)



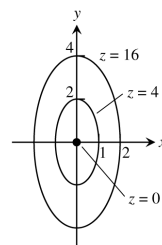
(b)



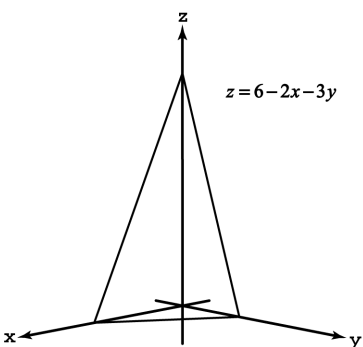
43. (a)



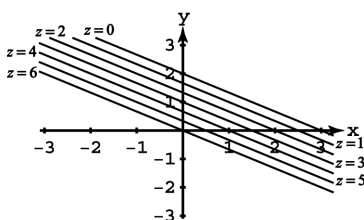
(b)



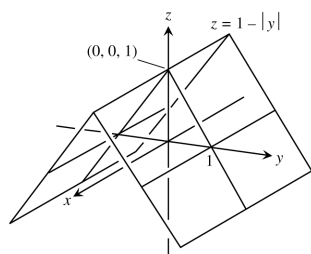
44. (a)



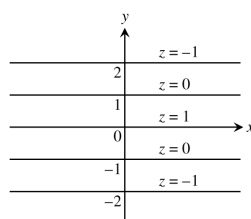
(b)



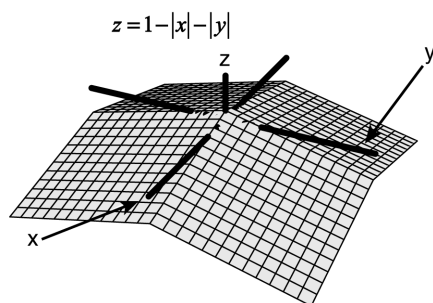
45. (a)



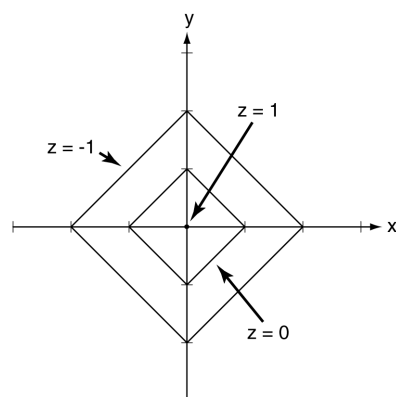
(b)



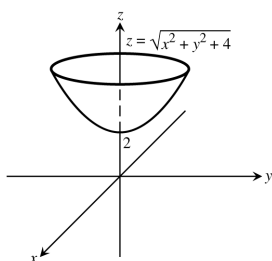
46. (a)



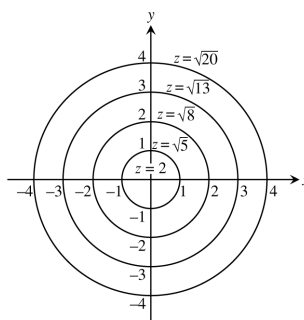
(b)



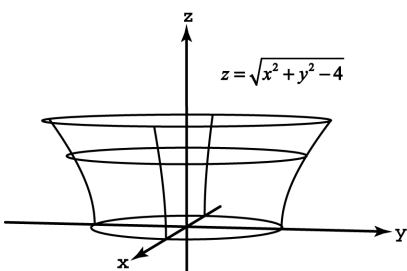
47. (a)



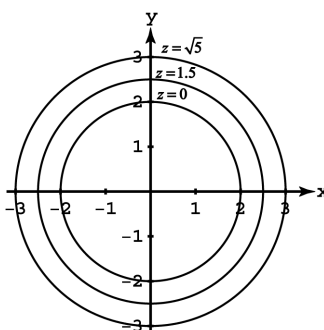
(b)



48. (a)



(b)



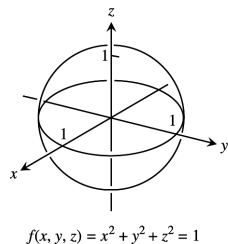
$$49. f(x, y) = 16 - x^2 - y^2 \text{ and } (2\sqrt{2}, \sqrt{2}) \Rightarrow z = 16 - (2\sqrt{2})^2 - (\sqrt{2})^2 = 6 \Rightarrow 6 = 16 - x^2 - y^2 \Rightarrow x^2 + y^2 = 10$$

$$50. f(x, y) = \sqrt{x^2 - 1} \text{ and } (1, 0) \Rightarrow z = \sqrt{1^2 - 1} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1 \text{ or } x = -1$$

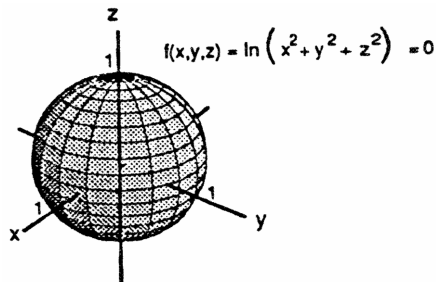
51. $f(x, y) = \sqrt{x + y^2 - 3}$ and $(3, -1) \Rightarrow z = \sqrt{3 + (-1)^2 - 3} = 1 \Rightarrow x + y^2 - 3 = 1 \Rightarrow x + y^2 = 4$

52. $f(x, y) = \frac{2y-x}{x+y+1}$ and $(-1, 1) \Rightarrow z = \frac{2(1)-(-1)}{(-1)+1+1} = 3 \Rightarrow 3 = \frac{2y-x}{x+y+1} \Rightarrow y = -4x - 3$

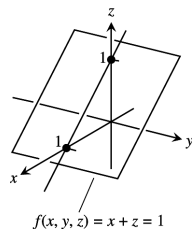
53.



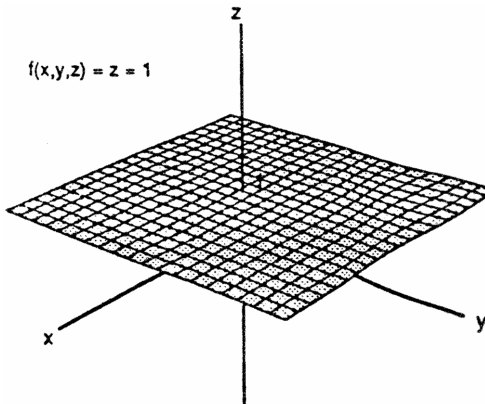
54.



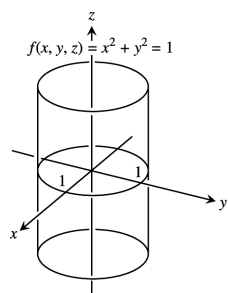
55.



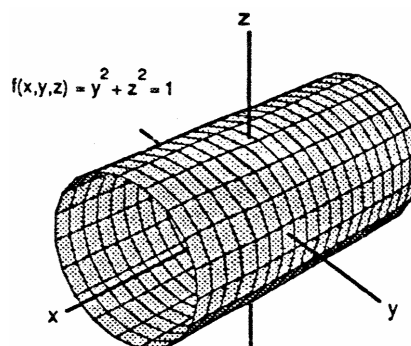
56.



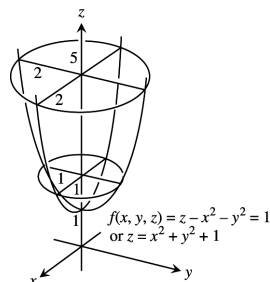
57.



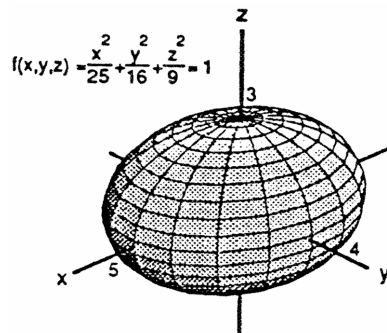
58.



59.



60.



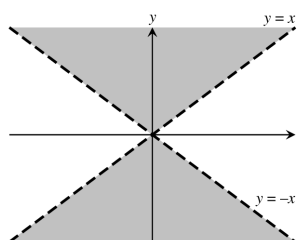
$$61. f(x, y, z) = \sqrt{x-y} - \ln z \text{ at } (3, -1, 1) \Rightarrow w = \sqrt{x-y} - \ln z; \text{ at } (3, -1, 1) \Rightarrow w = \sqrt{3 - (-1)} - \ln 1 = 2 \\ \Rightarrow \sqrt{x-y} - \ln z = 2$$

$$62. f(x, y, z) = \ln(x^2 + y + z^2) \text{ at } (-1, 2, 1) \Rightarrow w = \ln(x^2 + y + z^2); \text{ at } (-1, 2, 1) \Rightarrow w = \ln(1 + 2 + 1) = \ln 4 \\ \Rightarrow \ln 4 = \ln(x^2 + y + z^2) \Rightarrow x^2 + y + z^2 = 4$$

$$63. g(x, y, z) = \sqrt{x^2 + y^2 + z^2} \text{ at } (1, -1, \sqrt{2}) \Rightarrow w = \sqrt{x^2 + y^2 + z^2}; \text{ at } (1, -1, \sqrt{2}) \Rightarrow w = \sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} \\ = 2 \Rightarrow 2 = \sqrt{x^2 + y^2 + z^2} \Rightarrow x^2 + y^2 + z^2 = 4$$

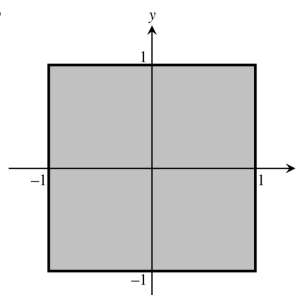
$$64. g(x, y, z) = \frac{x-y+z}{2x+y-z} \text{ at } (1, 0, -2) \Rightarrow w = \frac{x-y+z}{2x+y-z}; \text{ at } (1, 0, -2) \Rightarrow w = \frac{1-0+(-2)}{2(1)+0-(-2)} = -\frac{1}{4} \Rightarrow -\frac{1}{4} = \frac{x-y+z}{2x+y-z} \\ \Rightarrow 2x - y + z = 0$$

$$65. f(x, y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n = \frac{1}{1 - \left(\frac{x}{y}\right)} = \frac{y}{y-x} \text{ for} \\ \left|\frac{x}{y}\right| < 1 \Rightarrow \text{Domain: all points } (x, y) \text{ satisfying } |x| < |y|; \\ \text{at } (1, 2) \Rightarrow \text{since } \left|\frac{1}{2}\right| < 1 \Rightarrow z = \frac{2}{2-1} = 2 \\ \Rightarrow \frac{y}{y-x} = 2 \Rightarrow y = 2x$$



$$66. g(x, y, z) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n! z^n} = e^{(x+y)/z} \Rightarrow \text{Domain: all points } (x, y, z) \text{ satisfying } z \neq 0; \text{ at } (\ln 4, \ln 9, 2) \\ \Rightarrow w = e^{(\ln 4 + \ln 9)/2} = e^{(\ln 36)/2} = e^{\ln 6} = 6 \Rightarrow 6 = e^{(x+y)/z} \Rightarrow \frac{x+y}{z} = \ln 6$$

$$67. f(x, y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} = \sin^{-1}y - \sin^{-1}x \Rightarrow \text{Domain: all points} \\ (x, y) \text{ satisfying } -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1; \\ \text{at } (0, 1) \Rightarrow \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2} \Rightarrow \sin^{-1}y - \sin^{-1}x \\ = \frac{\pi}{2}. \text{ Since } -\frac{\pi}{2} \leq \sin^{-1}y \leq \frac{\pi}{2} \text{ and } -\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}, \text{ in} \\ \text{order for } \sin^{-1}y - \sin^{-1}x \text{ to equal } \frac{\pi}{2}, 0 \leq \sin^{-1}y \leq \frac{\pi}{2} \text{ and} \\ -\frac{\pi}{2} \leq \sin^{-1}x \leq 0; \text{ that is } 0 \leq y \leq 1 \text{ and } -1 \leq x \leq 0. \text{ Thus} \\ y = \sin\left(\frac{\pi}{2} + \sin^{-1}x\right) = \sqrt{1-x^2}, x \leq 0$$



$$68. g(x, y, z) = \int_x^y \frac{dt}{1+t^2} + \int_0^z \frac{d\theta}{\sqrt{4-\theta^2}} = \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right) \Rightarrow \text{Domain: all points } (x, y, z) \text{ satisfying } -2 \leq z \leq 2; \\ \text{at } (0, 1, \sqrt{3}) \Rightarrow \tan^{-1}1 - \tan^{-1}0 + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{7\pi}{12} \Rightarrow \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right) = \frac{7\pi}{12}. \text{ Since } -\frac{\pi}{2} \leq \sin^{-1}\left(\frac{z}{2}\right) \leq \frac{\pi}{2}, \\ \frac{\pi}{12} \leq \tan^{-1}y - \tan^{-1}x \leq \frac{13\pi}{12} \Rightarrow z = 2 \sin\left(\frac{7\pi}{12} - \tan^{-1}y + \tan^{-1}x\right), \frac{\pi}{12} \leq \tan^{-1}y - \tan^{-1}x \leq \frac{13\pi}{12}$$

69-72. Example CAS commands:

Maple:

```
with(plots);
f := (x,y) -> x*sin(y/2) + y*sin(2*x);
xdomain := x=0..5*Pi;
ydomain := y=0..5*Pi;
x0,y0 := 3*Pi,3*Pi;
plot3d( f(x,y), xdomain, ydomain, axes=boxed, style=patch, shading=zhue, title="#69(a) (Section 14.1)");
```



```

plot3d( f(x,y), xdomain, ydomain, grid=[50,50], axes=boxed, shading=zhue, style=patchcontour, orientation=[-90,0],
        title="#69(b) (Section 14.1)" ); # (b)
L := evalf( f(x0,y0) ); # (c)
plot3d( f(x,y), xdomain, ydomain, grid=[50,50], axes=boxed, shading=zhue, style=patchcontour, contours=[L],
        orientation=[-90,0], title="#45(c) (Section 13.1)" );

```

73-76. Example CAS commands:

Maple:

```

eq := 4*ln(x^2+y^2+z^2)=1;
implicitplot3d( eq, x=-2..2, y=-2..2, z=-2..2, grid=[30,30,30], axes=boxed, title="#73 (Section 14.1)" );

```

77-80. Example CAS commands:

Maple:

```

x := (u,v) -> u*cos(v);
y := (u,v) -> u*sin(v);
z := (u,v) -> u;
plot3d( [x(u,v),y(u,v),z(u,v)], u=0..2, v=0..2*Pi, axes=boxed, style=patchcontour, contours=[($0..4)/2], shading=zhue,
        title="#77 (Section 14.1)" );

```

69-60. Example CAS commands:

Mathematica: (assigned functions and bounds will vary)

For 69 - 72, the command **ContourPlot** draws 2-dimensional contours that are z-level curves of surfaces $z = f(x,y)$.

```

Clear[x, y, f]
f[x_, y_] := x Sin[y/2] + y Sin[2x]
xmin= 0; xmax= 5π; ymin= 0; ymax= 5π; {x0, y0}={3π, 3π};
cp= ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, ContourShading -> False];
cp0= ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, Contours -> {f[x0,y0]}, ContourShading -> False,
        PlotStyle -> {RGBColor[1,0,0]}];
Show[cp, cp0]

```

For 73 - 76, the command **ContourPlot3D** will be used. Write the function $f[x, y, z]$ so that when it is equated to zero, it represents the level surface given.

For 73, the problem associated with $\text{Log}[0]$ can be avoided by rewriting the function as $x^2 + y^2 + z^2 - e^{1/4}$

```

Clear[x, y, z, f]
f[x_, y_, z_] := x^2 + y^2 + z^2 - Exp[1/4]
ContourPlot3D[f[x,y,z], {x, -5, 5}, {y, -5, 5}, {z, -5, 5}, PlotPoints -> {7, 7}];

```

For 77 - 80, the command **ParametricPlot3D** will be used. To get the z-level curves here, we solve x and y in terms of z and either u or v (v here), create a table of level curves, then plot that table.

```

Clear[x, y, z, u, v]
ParametricPlot3D[{u Cos[v], u Sin[v], u}, {u, 0, 2}, {v, 0, 2π}];
zlevel= Table[{z Cos[v], z Sin[v]}, {z, 0, 2, .1}];
ParametricPlot[Evaluate[zlevel], {v, 0, 2π}];

```

14.2 LIMITS AND CONTINUITY IN HIGHER DIMENSIONS

$$1. \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3(0)^2 - 0^2 + 5}{0^2 + 0^2 + 2} = \frac{5}{2}$$

$$2. \lim_{(x,y) \rightarrow (0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$$

3. $\lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2 - 1} = \sqrt{3^2 + 4^2 - 1} = \sqrt{24} = 2\sqrt{6}$
4. $\lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2 = \left[\frac{1}{2} + \left(\frac{1}{-3}\right)\right]^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$
5. $\lim_{(x,y) \rightarrow (0, \frac{\pi}{4})} \sec x \tan y = (\sec 0) \left(\tan \frac{\pi}{4}\right) = (1)(1) = 1$
6. $\lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^2 + y^3}{x + y + 1}\right) = \cos\left(\frac{0^2 + 0^3}{0 + 0 + 1}\right) = \cos 0 = 1$
7. $\lim_{(x,y) \rightarrow (0, \ln 2)} e^{x-y} = e^{0 - \ln 2} = e^{\ln(\frac{1}{2})} = \frac{1}{2}$
8. $\lim_{(x,y) \rightarrow (1,1)} \ln |1 + x^2 y^2| = \ln |1 + (1)^2 (1)^2| = \ln 2$
9. $\lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x} = \lim_{(x,y) \rightarrow (0,0)} (e^y) \left(\frac{\sin x}{x}\right) = e^0 \cdot \lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right) = 1 \cdot 1 = 1$
10. $\lim_{(x,y) \rightarrow (1/27, \pi^3)} \cos \sqrt[3]{xy} = \cos \sqrt[3]{\left(\frac{1}{27}\right)\pi^3} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
11. $\lim_{(x,y) \rightarrow (1, \pi/6)} \frac{x \sin y}{x^2 + 1} = \frac{1 \cdot \sin(\frac{\pi}{6})}{1^2 + 1} = \frac{1/2}{2} = \frac{1}{4}$
12. $\lim_{(x,y) \rightarrow (\frac{\pi}{2}, 0)} \frac{\cos y + 1}{y - \sin x} = \frac{(\cos 0) + 1}{0 - \sin(\frac{\pi}{2})} = \frac{1 + 1}{-1} = -2$
13. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x - y)^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} (x - y) = (1 - 1) = 0$
14. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x + y)(x - y)}{x - y} = \lim_{(x,y) \rightarrow (1,1)} (x + y) = (1 + 1) = 2$
15. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x - 1} = \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{(x - 1)(y - 2)}{x - 1} = \lim_{(x,y) \rightarrow (1,1)} (y - 2) = (1 - 2) = -1$
16. $\lim_{\substack{(x,y) \rightarrow (2,-4) \\ y \neq -4, x \neq x^2}} \frac{y + 4}{x^2 y - xy + 4x^2 - 4x} = \lim_{\substack{(x,y) \rightarrow (2,-4) \\ y \neq -4, x \neq x^2}} \frac{y + 4}{x(x - 1)(y + 4)} = \lim_{\substack{(x,y) \rightarrow (2,-4) \\ x \neq x^2}} \frac{1}{x(x - 1)} = \frac{1}{2(2 - 1)} = \frac{1}{2}$
17. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ x \neq y}} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}} = \lim_{\substack{(x,y) \rightarrow (0,0) \\ x \neq y}} \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y} + 2)}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} (\sqrt{x} + \sqrt{y} + 2)$
 $= (\sqrt{0} + \sqrt{0} + 2) = 2$
 Note: (x, y) must approach $(0, 0)$ through the first quadrant only with $x \neq y$.
18. $\lim_{\substack{(x,y) \rightarrow (2,2) \\ x + y \neq 4}} \frac{x + y - 4}{\sqrt{x + y} - 2} = \lim_{\substack{(x,y) \rightarrow (2,2) \\ x + y \neq 4}} \frac{(\sqrt{x + y} + 2)(\sqrt{x + y} - 2)}{\sqrt{x + y} - 2} = \lim_{\substack{(x,y) \rightarrow (2,2) \\ x + y \neq 4}} (\sqrt{x + y} + 2)$
 $= (\sqrt{2 + 2} + 2) = 2 + 2 = 4$

19. $\lim_{\substack{(x,y) \rightarrow (2,0) \\ 2x-y \neq 4}} \frac{\sqrt{2x-y}-2}{2x-y-4} = \lim_{\substack{(x,y) \rightarrow (2,0) \\ 2x-y \neq 4}} \frac{\sqrt{2x-y}-2}{(\sqrt{2x-y}+2)(\sqrt{2x-y}-2)} = \lim_{(x,y) \rightarrow (2,0)} \frac{1}{\sqrt{2x-y}+2}$
 $= \frac{1}{\sqrt{(2)(2)-0}+2} = \frac{1}{2+2} = \frac{1}{4}$
20. $\lim_{\substack{(x,y) \rightarrow (4,3) \\ x-y \neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{\substack{(x,y) \rightarrow (4,3) \\ x-y \neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{(\sqrt{x}+\sqrt{y+1})(\sqrt{x}-\sqrt{y+1})} = \lim_{(x,y) \rightarrow (4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}}$
 $= \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2} = \frac{1}{4}$
21. $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r \rightarrow 0} \frac{\sin(r^2)}{r^2} = \lim_{r \rightarrow 0} \frac{2r \cos(r^2)}{2r} = \lim_{r \rightarrow 0} \cos(r^2) = 1$
22. $\lim_{(x,y) \rightarrow (0,0)} \frac{1-\cos(xy)}{xy} = \lim_{u \rightarrow 0} \frac{1-\cos u}{u} = \lim_{u \rightarrow 0} \frac{\sin u}{1} = 0$
23. $\lim_{(x,y) \rightarrow (1,-1)} \frac{x^3+y^3}{x+y} = \lim_{(x,y) \rightarrow (1,-1)} \frac{(x+y)(x^2-xy+y^2)}{x+y} = \lim_{(x,y) \rightarrow (1,-1)} (x^2-xy+y^2) = (1^2 - (1)(-1) + (-1)^2) = 3$
24. $\lim_{(x,y) \rightarrow (2,2)} \frac{x-y}{x^4-y^4} = \lim_{(x,y) \rightarrow (2,2)} \frac{x-y}{(x+y)(x-y)(x^2+y^2)} = \lim_{(x,y) \rightarrow (2,2)} \frac{1}{(x+y)(x^2+y^2)} = \frac{1}{(2+2)(2^2+2^2)} = \frac{1}{32}$
25. $\lim_{P \rightarrow (1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = \frac{1}{1} + \frac{1}{3} + \frac{1}{4} = \frac{12+4+3}{12} = \frac{19}{12}$
26. $\lim_{P \rightarrow (1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{2(1)(-1)+(-1)(-1)}{1^2+(-1)^2} = \frac{-2+1}{1+1} = -\frac{1}{2}$
27. $\lim_{P \rightarrow (3,3,0)} (\sin^2 x + \cos^2 y + \sec^2 z) = (\sin^2 3 + \cos^2 3) + \sec^2 0 = 1 + 1^2 = 2$
28. $\lim_{P \rightarrow (-\frac{1}{4}, \frac{\pi}{2}, 2)} \tan^{-1}(xyz) = \tan^{-1} \left(-\frac{1}{4} \cdot \frac{\pi}{2} \cdot 2 \right) = \tan^{-1} \left(-\frac{\pi}{4} \right)$
29. $\lim_{P \rightarrow (\pi, 0, 3)} ze^{-2y} \cos 2x = 3e^{-2(0)} \cos 2\pi = (3)(1)(1) = 3$
30. $\lim_{P \rightarrow (2, -3, 6)} \ln \sqrt{x^2 + y^2 + z^2} = \ln \sqrt{2^2 + (-3)^2 + 6^2} = \ln \sqrt{49} = \ln 7$
31. (a) All (x, y) (b) All (x, y) except $(0, 0)$
32. (a) All (x, y) so that $x \neq y$ (b) All (x, y)
33. (a) All (x, y) except where $x = 0$ or $y = 0$ (b) All (x, y)
34. (a) All (x, y) so that $x^2 - 3x + 2 \neq 0 \Rightarrow (x-2)(x-1) \neq 0 \Rightarrow x \neq 2$ and $x \neq 1$
 (b) All (x, y) so that $y \neq x^2$
35. (a) All (x, y, z) (b) All (x, y, z) except the interior of the cylinder $x^2 + y^2 = 1$
36. (a) All (x, y, z) so that $xyz > 0$ (b) All (x, y, z)
37. (a) All (x, y, z) with $z \neq 0$ (b) All (x, y, z) with $x^2 + z^2 \neq 1$

38. (a) All (x, y, z) except $(x, 0, 0)$ (b) All (x, y, z) except $(0, y, 0)$ or $(x, 0, 0)$
39. (a) All (x, y, z) such that $z > x^2 + y^2 + 1$ (b) All (x, y, z) such that $z \neq \sqrt{x^2 + y^2}$
40. (a) All (x, y, z) such that $x^2 + y^2 + z^2 \leq 4$
 (b) All (x, y, z) such that $x^2 + y^2 + z^2 \geq 9$ except when $x^2 + y^2 + z^2 = 25$
41. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x \\ x > 0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim_{x \rightarrow 0^+} -\frac{x}{\sqrt{x^2+x^2}} = \lim_{x \rightarrow 0^+} -\frac{x}{\sqrt{2}|x|} = \lim_{x \rightarrow 0^+} -\frac{x}{\sqrt{2}x} = \lim_{x \rightarrow 0^+} -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}};$
 $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x \\ x < 0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim_{x \rightarrow 0^-} -\frac{x}{\sqrt{2}|x|} = \lim_{x \rightarrow 0^-} -\frac{x}{\sqrt{2}(-x)} = \lim_{x \rightarrow 0^-} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$
42. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=0}} \frac{x^4}{x^4+y^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^4+0^2} = 1; \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x^2}} \frac{x^4}{x^4+y^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^4+(x^2)^2} = \lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \frac{1}{2}$
43. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx^2}} \frac{x^4-y^2}{x^4+y^2} = \lim_{x \rightarrow 0} \frac{x^4-(kx^2)^2}{x^4+(kx^2)^2} = \lim_{x \rightarrow 0} \frac{x^4-k^2x^4}{x^4+k^2x^4} = \frac{1-k^2}{1+k^2} \Rightarrow \text{different limits for different values of } k$
44. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx \\ k \neq 0}} \frac{xy}{|xy|} = \lim_{x \rightarrow 0} \frac{x(kx)}{|x(kx)|} = \lim_{x \rightarrow 0} \frac{kx^2}{|kx^2|} = \lim_{x \rightarrow 0} \frac{k}{|k|}; \text{ if } k > 0, \text{ the limit is } 1; \text{ but if } k < 0, \text{ the limit is } -1$
45. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx \\ k \neq -1}} \frac{x-y}{x+y} = \lim_{x \rightarrow 0} \frac{x-kx}{x+kx} = \frac{1-k}{1+k} \Rightarrow \text{different limits for different values of } k, k \neq -1$
46. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx \\ k \neq 1}} \frac{x^2-y}{x-y} = \lim_{x \rightarrow 0} \frac{x^2-kx}{x-kx} = \lim_{x \rightarrow 0} \frac{x-k}{1-k} = \frac{-k}{1-k} \Rightarrow \text{different limits for different values of } k, k \neq 1$
47. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx^2 \\ k \neq 0}} \frac{x^2+y}{y} = \lim_{x \rightarrow 0} \frac{x^2+kx^2}{kx^2} = \frac{1+k}{k} \Rightarrow \text{different limits for different values of } k, k \neq 0$
48. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx^2}} \frac{x^2y}{x^4+y^2} = \lim_{x \rightarrow 0} \frac{kx^4}{x^4+k^2x^4} = \frac{k}{1+k^2} \Rightarrow \text{different limits for different values of } k$
49. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ \text{along } x=1}} \frac{xy^2-1}{y-1} = \lim_{y \rightarrow 1} \frac{y^2-1}{y-1} = \lim_{y \rightarrow 1} (y+1) = 2; \lim_{\substack{(x,y) \rightarrow (1,1) \\ \text{along } y=x}} \frac{xy^2-1}{y-1} = \lim_{y \rightarrow 1} \frac{y^3-1}{y-1} = \lim_{y \rightarrow 1} (y^2+y+1) = 3$
50. $\lim_{\substack{(x,y) \rightarrow (1,-1) \\ \text{along } y=-1}} \frac{xy+1}{x^2-y^2} = \lim_{x \rightarrow 1} \frac{-x+1}{x^2-1} = \lim_{x \rightarrow 1} \frac{-1}{x+1} = -\frac{1}{2}; \lim_{\substack{(x,y) \rightarrow (1,-1) \\ \text{along } y=-x^2}} \frac{xy+1}{x^2-y^2} = \lim_{x \rightarrow 1} \frac{-x^3+1}{x^2-x^4} = \lim_{x \rightarrow 1} \frac{x^2+x+1}{(x+1)(x^2+1)} = \frac{3}{2}$

$$51. f(x, y) = \begin{cases} 1 & \text{if } y \geq x^4 \\ 1 & \text{if } y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

- (a) $\lim_{(x,y) \rightarrow (0,1)} f(x, y) = 1$ since any path through $(0, 1)$ that is close to $(0, 1)$ satisfies $y \geq x^4$
- (b) $\lim_{(x,y) \rightarrow (2,3)} f(x, y) = 0$ since any path through $(2, 3)$ that is close to $(2, 3)$ does not satisfy either $y \geq x^4$ or $y \leq 0$
- (c) $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } x=0}} f(x, y) = 1$ and $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x^2}} f(x, y) = 0 \Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist

$$52. f(x, y) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$$

- (a) $\lim_{(x,y) \rightarrow (3,-2)} f(x, y) = 3^2 = 9$ since any path through $(3, -2)$ that is close to $(3, -2)$ satisfies $x \geq 0$
- (b) $\lim_{(x,y) \rightarrow (-2,1)} f(x, y) = (-2)^3 = -8$ since any path through $(-2, 1)$ that is close to $(-2, 1)$ satisfies $x < 0$
- (c) $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ since the limit is 0 along any path through $(0, 0)$ with $x < 0$ and the limit is also zero along any path through $(0, 0)$ with $x \geq 0$

53. First consider the vertical line $x = 0 \Rightarrow \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } x=0}} \frac{2x^2y}{x^4+y^2} = \lim_{y \rightarrow 0} \frac{2(0)^2y}{(0)^4+y^2} = \lim_{y \rightarrow 0} 0 = 0$. Now consider any nonvertical

through $(0, 0)$. The equation of any line through $(0, 0)$ is of the form $y = mx \Rightarrow \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=mx}} f(x, y) = \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=mx}} \frac{2x^2y}{x^4+y^2}$

$$= \lim_{x \rightarrow 0} \frac{2x^2(mx)}{x^4+(mx)^2} = \lim_{x \rightarrow 0} \frac{2mx^3}{x^4+m^2x^2} = \lim_{x \rightarrow 0} \frac{2mx^3}{x^2(x^2+m^2)} = \lim_{x \rightarrow 0} \frac{2mx}{(x^2+m^2)} = 0. \text{ Thus } \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{any line through } (0,0)}} \frac{2x^2y}{x^4+y^2} = 0.$$

54. If f is continuous at (x_0, y_0) , then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ must equal $f(x_0, y_0) = 3$. If f is not continuous at (x_0, y_0) , the limit could have any value different from 3, and need not even exist.

55. $\lim_{(x,y) \rightarrow (0,0)} \left(1 - \frac{x^2y^2}{3}\right) = 1$ and $\lim_{(x,y) \rightarrow (0,0)} 1 = 1 \Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{\tan^{-1}xy}{xy} = 1$, by the Sandwich Theorem

56. If $xy > 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{2|xy| - \left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} \frac{2xy - \left(\frac{x^2y^2}{6}\right)}{xy} = \lim_{(x,y) \rightarrow (0,0)} \left(2 - \frac{xy}{6}\right) = 2$ and

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2|xy|}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} 2 = 2; \text{ if } xy < 0, \lim_{(x,y) \rightarrow (0,0)} \frac{2|xy| - \left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} \frac{-2xy - \left(\frac{x^2y^2}{6}\right)}{-xy} \\ = \lim_{(x,y) \rightarrow (0,0)} \left(2 + \frac{xy}{6}\right) = 2 \text{ and } \lim_{(x,y) \rightarrow (0,0)} \frac{2|xy|}{|xy|} = 2 \Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|} = 2, \text{ by the Sandwich Theorem}$$

57. The limit is 0 since $|\sin(\frac{1}{x})| \leq 1 \Rightarrow -1 \leq \sin(\frac{1}{x}) \leq 1 \Rightarrow -y \leq y \sin(\frac{1}{x}) \leq y$ for $y \geq 0$, and $-y \geq y \sin(\frac{1}{x}) \geq y$ for $y \leq 0$. Thus as $(x, y) \rightarrow (0, 0)$, both $-y$ and y approach 0 $\Rightarrow y \sin(\frac{1}{x}) \rightarrow 0$, by the Sandwich Theorem.

58. The limit is 0 since $|\cos(\frac{1}{y})| \leq 1 \Rightarrow -1 \leq \cos(\frac{1}{y}) \leq 1 \Rightarrow -x \leq x \cos(\frac{1}{y}) \leq x$ for $x \geq 0$, and $-x \geq x \cos(\frac{1}{y}) \geq x$ for $x \leq 0$. Thus as $(x, y) \rightarrow (0, 0)$, both $-x$ and x approach 0 $\Rightarrow x \cos(\frac{1}{y}) \rightarrow 0$, by the Sandwich Theorem.

59. (a) $f(x, y)|_{y=mx} = \frac{2m}{1+m^2} = \frac{2 \tan \theta}{1+\tan^2 \theta} = \sin 2\theta$. The value of $f(x, y) = \sin 2\theta$ varies with θ , which is the line's angle of inclination.

- (b) Since $f(x, y)|_{y=mx} = \sin 2\theta$ and since $-1 \leq \sin 2\theta \leq 1$ for every θ , $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ varies from -1 to 1 along $y = mx$.

$$\begin{aligned} 60. \quad |xy(x^2 - y^2)| &= |xy| |x^2 - y^2| \leq |x| |y| |x^2 + y^2| = \sqrt{x^2} \sqrt{y^2} |x^2 + y^2| \leq \sqrt{x^2 + y^2} \sqrt{x^2 + y^2} |x^2 + y^2| \\ &= (x^2 + y^2)^2 \Rightarrow \left| \frac{xy(x^2 - y^2)}{x^2 + y^2} \right| \leq \frac{(x^2 + y^2)^2}{x^2 + y^2} = x^2 + y^2 \Rightarrow -(x^2 + y^2) \leq \frac{xy(x^2 - y^2)}{x^2 + y^2} \leq (x^2 + y^2) \\ &\Rightarrow \lim_{(x, y) \rightarrow (0, 0)} \left(xy \frac{x^2 - y^2}{x^2 + y^2} \right) = 0 \text{ by the Sandwich Theorem, since } \lim_{(x, y) \rightarrow (0, 0)} \pm (x^2 + y^2) = 0; \text{ thus, define } f(0, 0) = 0 \end{aligned}$$

$$61. \quad \lim_{(x, y) \rightarrow (0, 0)} \frac{x^3 - xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta - (r \cos \theta)(r^2 \sin^2 \theta)}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = \lim_{r \rightarrow 0} \frac{r(\cos^3 \theta - \cos \theta \sin^2 \theta)}{1} = 0$$

$$62. \quad \lim_{(x, y) \rightarrow (0, 0)} \cos \left(\frac{x^3 - y^3}{x^2 + y^2} \right) = \lim_{r \rightarrow 0} \cos \left(\frac{r^3 \cos^3 \theta - r^3 \sin^3 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \right) = \lim_{r \rightarrow 0} \cos \left[\frac{r(\cos^3 \theta - \sin^3 \theta)}{1} \right] = \cos 0 = 1$$

$$63. \quad \lim_{(x, y) \rightarrow (0, 0)} \frac{y^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^2 \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} (\sin^2 \theta) = \sin^2 \theta; \text{ the limit does not exist since } \sin^2 \theta \text{ is between } 0 \text{ and } 1 \text{ depending on } \theta$$

$$64. \quad \lim_{(x, y) \rightarrow (0, 0)} \frac{2x}{x^2 + x + y^2} = \lim_{r \rightarrow 0} \frac{2r \cos \theta}{r^2 + r \cos \theta} = \lim_{r \rightarrow 0} \frac{2 \cos \theta}{r + \cos \theta} = \frac{2 \cos \theta}{\cos \theta}; \text{ the limit does not exist for } \cos \theta = 0$$

$$\begin{aligned} 65. \quad \lim_{(x, y) \rightarrow (0, 0)} \tan^{-1} \left[\frac{|x| + |y|}{x^2 + y^2} \right] &= \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|r \cos \theta| + |r \sin \theta|}{r^2} \right] = \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|r|(|\cos \theta| + |\sin \theta|)}{r^2} \right]; \\ \text{if } r \rightarrow 0^+, \text{ then } \lim_{r \rightarrow 0^+} \tan^{-1} \left[\frac{|r|(|\cos \theta| + |\sin \theta|)}{r^2} \right] &= \lim_{r \rightarrow 0^+} \tan^{-1} \left[\frac{|\cos \theta| + |\sin \theta|}{r} \right] = \frac{\pi}{2}; \text{ if } r \rightarrow 0^-, \text{ then} \\ \lim_{r \rightarrow 0^-} \tan^{-1} \left[\frac{|r|(|\cos \theta| + |\sin \theta|)}{r^2} \right] &= \lim_{r \rightarrow 0^-} \tan^{-1} \left(\frac{|\cos \theta| + |\sin \theta|}{-r} \right) = \frac{\pi}{2} \Rightarrow \text{the limit is } \frac{\pi}{2} \end{aligned}$$

$$66. \quad \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} (\cos^2 \theta - \sin^2 \theta) = \lim_{r \rightarrow 0} (\cos 2\theta) \text{ which ranges between } -1 \text{ and } 1 \text{ depending on } \theta \Rightarrow \text{the limit does not exist}$$

$$\begin{aligned} 67. \quad \lim_{(x, y) \rightarrow (0, 0)} \ln \left(\frac{3x^2 - x^2 y^2 + 3y^2}{x^2 + y^2} \right) &= \lim_{r \rightarrow 0} \ln \left(\frac{3r^2 \cos^2 \theta - r^4 \cos^2 \theta \sin^2 \theta + 3r^2 \sin^2 \theta}{r^2} \right) \\ &= \lim_{r \rightarrow 0} \ln (3 - r^2 \cos^2 \theta \sin^2 \theta) = \ln 3 \Rightarrow \text{define } f(0, 0) = \ln 3 \end{aligned}$$

$$68. \quad \lim_{(x, y) \rightarrow (0, 0)} \frac{3xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{(3r \cos \theta)(r^2 \sin^2 \theta)}{r^2} = \lim_{r \rightarrow 0} 3r \cos \theta \sin^2 \theta = 0 \Rightarrow \text{define } f(0, 0) = 0$$

$$69. \quad \text{Let } \delta = 0.1. \text{ Then } \sqrt{x^2 + y^2} < \delta \Rightarrow \sqrt{x^2 + y^2} < 0.1 \Rightarrow x^2 + y^2 < 0.01 \Rightarrow |x^2 + y^2 - 0| < 0.01 \\ \Rightarrow |f(x, y) - f(0, 0)| < 0.01 = \epsilon.$$

$$70. \quad \text{Let } \delta = 0.05. \text{ Then } |x| < \delta \text{ and } |y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left| \frac{y}{x^2 + 1} - 0 \right| = \left| \frac{y}{x^2 + 1} \right| \leq |y| < 0.05 = \epsilon.$$

$$71. \quad \text{Let } \delta = 0.005. \text{ Then } |x| < \delta \text{ and } |y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left| \frac{x+y}{x^2+1} - 0 \right| = \left| \frac{x+y}{x^2+1} \right| \leq |x+y| < |x| + |y| < 0.005 + 0.005 = 0.01 = \epsilon.$$

$$\begin{aligned} 72. \quad \text{Let } \delta = 0.01. \text{ Since } -1 \leq \cos x \leq 1 \Rightarrow 1 \leq 2 + \cos x \leq 3 \Rightarrow \frac{1}{3} \leq \frac{1}{2 + \cos x} \leq 1 \Rightarrow \frac{|x+y|}{3} \leq \left| \frac{x+y}{2 + \cos x} \right| \leq |x+y| \\ \leq |x| + |y|. \text{ Then } |x| < \delta \text{ and } |y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left| \frac{x+y}{2 + \cos x} - 0 \right| = \left| \frac{x+y}{2 + \cos x} \right| \leq |x| + |y| < 0.01 + 0.01 \\ = 0.02 = \epsilon. \end{aligned}$$

73. Let $\delta = 0.04$. Since $y^2 \leq x^2 + y^2 \Rightarrow \frac{y^2}{x^2 + y^2} \leq 1 \Rightarrow \frac{|x|y^2}{x^2 + y^2} \leq |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} < \delta \Rightarrow |f(x, y) - f(0, 0)|$
 $= \left| \frac{xy^2}{x^2 + y^2} - 0 \right| < 0.04 = \epsilon.$

74. Let $\delta = 0.01$. If $|y| \leq 1$, then $y^2 \leq |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2}$, so $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} \Rightarrow |x| + y^2 \leq 2\sqrt{x^2 + y^2}$. Since
 $x^2 \leq x^2 + y^2 \Rightarrow \frac{x^2}{x^2 + y^2} \leq 1$ and $y^2 \leq x^2 + y^2 \Rightarrow \frac{y^2}{x^2 + y^2} \leq 1$. Then $\frac{|x^3 + y^4|}{x^2 + y^2} \leq \frac{x^2}{x^2 + y^2}|x| + \frac{y^2}{x^2 + y^2}y^2 \leq |x| + y^2 < 2\delta$
 $\Rightarrow |f(x, y) - f(0, 0)| = \left| \frac{x^3 + y^4}{x^2 + y^2} - 0 \right| < 2(0.01) = 0.002 = \epsilon.$

75. Let $\delta = \sqrt{0.015}$. Then $\sqrt{x^2 + y^2 + z^2} < \delta \Rightarrow |f(x, y, z) - f(0, 0, 0)| = |x^2 + y^2 + z^2 - 0| = |x^2 + y^2 + z^2|$
 $= \left(\sqrt{x^2 + y^2 + z^2} \right)^2 < \left(\sqrt{0.015} \right)^2 = 0.015 = \epsilon.$

76. Let $\delta = 0.2$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x, y, z) - f(0, 0, 0)| = |xyz - 0| = |xyz| = |x| |y| |z| < (0.2)^3$
 $= 0.008 = \epsilon.$

77. Let $\delta = 0.005$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x, y, z) - f(0, 0, 0)| = \left| \frac{x + y + z}{x^2 + y^2 + z^2 + 1} - 0 \right|$
 $= \left| \frac{x + y + z}{x^2 + y^2 + z^2 + 1} \right| \leq |x + y + z| \leq |x| + |y| + |z| < 0.005 + 0.005 + 0.005 = 0.015 = \epsilon.$

78. Let $\delta = \tan^{-1}(0.1)$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x, y, z) - f(0, 0, 0)| = |\tan^2 x + \tan^2 y + \tan^2 z|$
 $\leq |\tan^2 x| + |\tan^2 y| + |\tan^2 z| = \tan^2 x + \tan^2 y + \tan^2 z < \tan^2 \delta + \tan^2 \delta + \tan^2 \delta = 0.01 + 0.01 + 0.01 = 0.03 = \epsilon.$

79. $\lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} f(x, y, z) = \lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} (x + y - z) = x_0 + y_0 - z_0 = f(x_0, y_0, z_0) \Rightarrow f$ is continuous at
every (x_0, y_0, z_0)

80. $\lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} f(x, y, z) = \lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} (x^2 + y^2 + z^2) = x_0^2 + y_0^2 + z_0^2 = f(x_0, y_0, z_0) \Rightarrow f$ is continuous at
every point (x_0, y_0, z_0)

14.3 PARTIAL DERIVATIVES

- $\frac{\partial f}{\partial x} = 4x, \frac{\partial f}{\partial y} = -3$
- $\frac{\partial f}{\partial x} = 2x - y, \frac{\partial f}{\partial y} = -x + 2y$
- $\frac{\partial f}{\partial x} = 2x(y + 2), \frac{\partial f}{\partial y} = x^2 - 1$
- $\frac{\partial f}{\partial x} = 5y - 14x + 3, \frac{\partial f}{\partial y} = 5x - 2y - 6$
- $\frac{\partial f}{\partial x} = 2y(xy - 1), \frac{\partial f}{\partial y} = 2x(xy - 1)$
- $\frac{\partial f}{\partial x} = 6(2x - 3y)^2, \frac{\partial f}{\partial y} = -9(2x - 3y)^2$
- $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$
- $\frac{\partial f}{\partial x} = \frac{2x^2}{\sqrt[3]{x^3 + (\frac{y}{2})}}, \frac{\partial f}{\partial y} = \frac{1}{3\sqrt[3]{x^3 + (\frac{y}{2})}}$
- $\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial x}(x+y) = -\frac{1}{(x+y)^2}, \frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial y}(x+y) = -\frac{1}{(x+y)^2}$
- $\frac{\partial f}{\partial x} = \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{\partial f}{\partial y} = \frac{(x^2 + y^2)(0) - x(2y)}{(x^2 + y^2)^2} = -\frac{2xy}{(x^2 + y^2)^2}$
- $\frac{\partial f}{\partial x} = \frac{(xy - 1)(1) - (x+y)(y)}{(xy - 1)^2} = \frac{-y^2 - 1}{(xy - 1)^2}, \frac{\partial f}{\partial y} = \frac{(xy - 1)(1) - (x+y)(x)}{(xy - 1)^2} = \frac{-x^2 - 1}{(xy - 1)^2}$

12. $\frac{\partial f}{\partial x} = \frac{1}{1 + (\frac{y}{x})^2} \cdot \frac{\partial}{\partial x} \left(\frac{y}{x} \right) = -\frac{y}{x^2 \left[1 + (\frac{y}{x})^2 \right]} = -\frac{y}{x^2 + y^2}, \frac{\partial f}{\partial y} = \frac{1}{1 + (\frac{y}{x})^2} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x} \right) = \frac{1}{x \left[1 + (\frac{y}{x})^2 \right]} = \frac{x}{x^2 + y^2}$
13. $\frac{\partial f}{\partial x} = e^{(x+y+1)} \cdot \frac{\partial}{\partial x} (x + y + 1) = e^{(x+y+1)}, \frac{\partial f}{\partial y} = e^{(x+y+1)} \cdot \frac{\partial}{\partial y} (x + y + 1) = e^{(x+y+1)}$
14. $\frac{\partial f}{\partial x} = -e^{-x} \sin(x + y) + e^{-x} \cos(x + y), \frac{\partial f}{\partial y} = e^{-x} \cos(x + y)$
15. $\frac{\partial f}{\partial x} = \frac{1}{x+y} \cdot \frac{\partial}{\partial x} (x + y) = \frac{1}{x+y}, \frac{\partial f}{\partial y} = \frac{1}{x+y} \cdot \frac{\partial}{\partial y} (x + y) = \frac{1}{x+y}$
16. $\frac{\partial f}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x} (xy) \cdot \ln y = ye^{xy} \ln y, \frac{\partial f}{\partial y} = e^{xy} \cdot \frac{\partial}{\partial y} (xy) \cdot \ln y + e^{xy} \cdot \frac{1}{y} = xe^{xy} \ln y + \frac{e^{xy}}{y}$
17. $\frac{\partial f}{\partial x} = 2 \sin(x - 3y) \cdot \frac{\partial}{\partial x} \sin(x - 3y) = 2 \sin(x - 3y) \cos(x - 3y) \cdot \frac{\partial}{\partial x} (x - 3y) = 2 \sin(x - 3y) \cos(x - 3y),$
 $\frac{\partial f}{\partial y} = 2 \sin(x - 3y) \cdot \frac{\partial}{\partial y} \sin(x - 3y) = 2 \sin(x - 3y) \cos(x - 3y) \cdot \frac{\partial}{\partial y} (x - 3y) = -6 \sin(x - 3y) \cos(x - 3y)$
18. $\frac{\partial f}{\partial x} = 2 \cos(3x - y^2) \cdot \frac{\partial}{\partial x} \cos(3x - y^2) = -2 \cos(3x - y^2) \sin(3x - y^2) \cdot \frac{\partial}{\partial x} (3x - y^2)$
 $= -6 \cos(3x - y^2) \sin(3x - y^2),$
 $\frac{\partial f}{\partial y} = 2 \cos(3x - y^2) \cdot \frac{\partial}{\partial y} \cos(3x - y^2) = -2 \cos(3x - y^2) \sin(3x - y^2) \cdot \frac{\partial}{\partial y} (3x - y^2)$
 $= -4y \cos(3x - y^2) \sin(3x - y^2)$
19. $\frac{\partial f}{\partial x} = yx^{y-1}, \frac{\partial f}{\partial y} = x^y \ln x$
20. $f(x, y) = \frac{\ln x}{\ln y} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{x \ln y} \text{ and } \frac{\partial f}{\partial y} = \frac{-\ln x}{y(\ln y)^2}$
21. $\frac{\partial f}{\partial x} = -g(x), \frac{\partial f}{\partial y} = g(y)$
22. $f(x, y) = \sum_{n=0}^{\infty} (xy)^n, |xy| < 1 \Rightarrow f(x, y) = \frac{1}{1-xy} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial x} (1 - xy) = \frac{y}{(1-xy)^2} \text{ and}$
 $\frac{\partial f}{\partial y} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial y} (1 - xy) = \frac{x}{(1-xy)^2}$
23. $f_x = y^2, f_y = 2xy, f_z = -4z$
24. $f_x = y + z, f_y = x + z, f_z = y + x$
25. $f_x = 1, f_y = -\frac{y}{\sqrt{y^2 + z^2}}, f_z = -\frac{z}{\sqrt{y^2 + z^2}}$
26. $f_x = -x(x^2 + y^2 + z^2)^{-3/2}, f_y = -y(x^2 + y^2 + z^2)^{-3/2}, f_z = -z(x^2 + y^2 + z^2)^{-3/2}$
27. $f_x = \frac{yz}{\sqrt{1-x^2y^2z^2}}, f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}}, f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$
28. $f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}, f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}, f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$
29. $f_x = \frac{1}{x+2y+3z}, f_y = \frac{2}{x+2y+3z}, f_z = \frac{3}{x+2y+3z}$
30. $f_x = yz \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial x} (xy) = \frac{(yz)(y)}{xy} = \frac{yz}{x}, f_y = z \ln(xy) + yz \cdot \frac{\partial}{\partial y} \ln(xy) = z \ln(xy) + \frac{yz}{xy} \cdot \frac{\partial}{\partial y} (xy) = z \ln(xy) + z,$
 $f_z = y \ln(xy) + yz \cdot \frac{\partial}{\partial z} \ln(xy) = y \ln(xy)$
31. $f_x = -2xe^{-(x^2+y^2+z^2)}, f_y = -2ye^{-(x^2+y^2+z^2)}, f_z = -2ze^{-(x^2+y^2+z^2)}$
32. $f_x = -yze^{-xyz}, f_y = -xze^{-xyz}, f_z = -xye^{-xyz}$

$$33. f_x = \operatorname{sech}^2(x + 2y + 3z), f_y = 2 \operatorname{sech}^2(x + 2y + 3z), f_z = 3 \operatorname{sech}^2(x + 2y + 3z)$$

$$34. f_x = y \cosh(xy - z^2), f_y = x \cosh(xy - z^2), f_z = -2z \cosh(xy - z^2)$$

$$35. \frac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha), \frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$$

$$36. \frac{\partial g}{\partial u} = v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial u} \left(\frac{2u}{v} \right) = 2ve^{(2u/v)}, \frac{\partial g}{\partial v} = 2ve^{(2u/v)} + v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial v} \left(\frac{2u}{v} \right) = 2ve^{(2u/v)} - 2ue^{(2u/v)}$$

$$37. \frac{\partial h}{\partial \rho} = \sin \phi \cos \theta, \frac{\partial h}{\partial \phi} = \rho \cos \phi \cos \theta, \frac{\partial h}{\partial \theta} = -\rho \sin \phi \sin \theta$$

$$38. \frac{\partial g}{\partial r} = 1 - \cos \theta, \frac{\partial g}{\partial \theta} = r \sin \theta, \frac{\partial g}{\partial z} = -1$$

$$39. W_p = V, W_v = P + \frac{\delta v^2}{2g}, W_\delta = \frac{Vv^2}{2g}, W_v = \frac{2V\delta v}{2g} = \frac{V\delta v}{g}, W_g = -\frac{V\delta v^2}{2g^2}$$

$$40. \frac{\partial A}{\partial c} = m, \frac{\partial A}{\partial h} = \frac{q}{2}, \frac{\partial A}{\partial k} = \frac{m}{q}, \frac{\partial A}{\partial m} = \frac{k}{q} + c, \frac{\partial A}{\partial q} = -\frac{km}{q^2} + \frac{h}{2}$$

$$41. \frac{\partial f}{\partial x} = 1 + y, \frac{\partial f}{\partial y} = 1 + x, \frac{\partial^2 f}{\partial x^2} = 0, \frac{\partial^2 f}{\partial y^2} = 0, \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$$

$$42. \frac{\partial f}{\partial x} = y \cos xy, \frac{\partial f}{\partial y} = x \cos xy, \frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy, \frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy, \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy$$

$$43. \frac{\partial g}{\partial x} = 2xy + y \cos x, \frac{\partial g}{\partial y} = x^2 - \sin y + \sin x, \frac{\partial^2 g}{\partial x^2} = 2y - y \sin x, \frac{\partial^2 g}{\partial y^2} = -\cos y, \frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x$$

$$44. \frac{\partial h}{\partial x} = e^y, \frac{\partial h}{\partial y} = xe^y + 1, \frac{\partial^2 h}{\partial x^2} = 0, \frac{\partial^2 h}{\partial y^2} = xe^y, \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = e^y$$

$$45. \frac{\partial r}{\partial x} = \frac{1}{x+y}, \frac{\partial r}{\partial y} = \frac{1}{x+y}, \frac{\partial^2 r}{\partial x^2} = \frac{-1}{(x+y)^2}, \frac{\partial^2 r}{\partial y^2} = \frac{-1}{(x+y)^2}, \frac{\partial^2 r}{\partial y \partial x} = \frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{(x+y)^2}$$

$$46. \frac{\partial s}{\partial x} = \left[\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right] \cdot \frac{\partial}{\partial x} \left(\frac{y}{x} \right) = \left(-\frac{y}{x^2} \right) \left[\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right] = \frac{-y}{x^2 + y^2}, \frac{\partial s}{\partial y} = \left[\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right] \cdot \frac{\partial}{\partial y} \left(\frac{y}{x} \right) = \left(\frac{1}{x} \right) \left[\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right] = \frac{x}{x^2 + y^2},$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{y(2x)}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}, \frac{\partial^2 s}{\partial y^2} = \frac{-x(2y)}{(x^2 + y^2)^2} = -\frac{2xy}{(x^2 + y^2)^2}, \frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} = \frac{(x^2 + y^2)(-1) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$47. \frac{\partial w}{\partial x} = 2x \tan(xy) + x^2 \sec^2(xy) \cdot y = 2x \tan(xy) + x^2 y \sec^2(xy), \frac{\partial w}{\partial y} = x^2 \sec^2(xy) \cdot x = x^3 \sec^2(xy),$$

$$\frac{\partial^2 w}{\partial x^2} = 2 \tan(xy) + 2x \sec^2(xy) \cdot y + 2xy \sec^2(xy) + x^2 y (2 \sec(xy) \sec(xy) \tan(xy) \cdot y)$$

$$= 2 \tan(xy) + 4xy \sec^2(xy) + 2x^2 y^2 \sec^2(xy) \tan(xy), \frac{\partial^2 w}{\partial y^2} = x^3 (2 \sec(xy) \sec(xy) \tan(xy) \cdot x) = 2x^4 \sec^2(xy) \tan(xy)$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2 \sec^2(xy) + x^3 (2 \sec(xy) \sec(xy) \tan(xy) \cdot y) = 3x^2 \sec^2(xy) + x^3 y \sec^2(xy) \tan(xy)$$

$$48. \frac{\partial w}{\partial x} = ye^{x^2-y} \cdot 2x = 2xy e^{x^2-y}, \frac{\partial w}{\partial y} = (1)e^{x^2-y} + ye^{x^2-y} \cdot (-1) = e^{x^2-y}(1-y),$$

$$\frac{\partial^2 w}{\partial x^2} = 2y e^{x^2-y} + 2xy(e^{x^2-y} \cdot 2x) = 2ye^{x^2-y}(1+2x^2), \frac{\partial^2 w}{\partial y^2} = (e^{x^2-y} \cdot (-1))(1-y) + e^{x^2-y}(-1)$$

$$= e^{x^2-y}(y-2), \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = (e^{x^2-y} \cdot 2x)(1-y) = 2x e^{x^2-y}(1-y)$$

$$49. \frac{\partial w}{\partial x} = \sin(x^2 y) + x \cos(x^2 y) \cdot 2xy = \sin(x^2 y) + 2x^2 y \cos(x^2 y), \frac{\partial w}{\partial y} = x \cos(x^2 y) \cdot x^2 = x^3 \cos(x^2 y),$$

$$\frac{\partial^2 w}{\partial x^2} = \cos(x^2 y) \cdot 2xy + 4xy \cos(x^2 y) - 2x^2 y \sin(x^2 y) \cdot 2xy = 6xy \cos(x^2 y) - 4x^3 y^2 \sin(x^2 y),$$

$$\frac{\partial^2 w}{\partial y^2} = -x^3 \sin(x^2 y) \cdot x^2 = -x^5 \sin(x^2 y), \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2 \cos(x^2 y) - x^3 \sin(x^2 y) \cdot 2xy = 3x^2 \cos(x^2 y) - 2x^4 y \sin(x^2 y)$$

50. $\frac{\partial w}{\partial x} = \frac{(x^2+y) - (x-y)(2x)}{(x^2+y)^2} = \frac{-x^2+2xy+y}{(x^2+y)^2}$, $\frac{\partial w}{\partial y} = \frac{(x^2+y)(-1) - (x-y)}{(x^2+y)^2} = \frac{-x^2-x}{(x^2+y)^2}$,
 $\frac{\partial^2 w}{\partial x^2} = \frac{(x^2+y)^2(-2x+2y) - (-x^2+2xy+y)2(x^2+y)(2x)}{[(x^2+y)^2]^2} = \frac{2(x^3-3x^2y-3xy+y^2)}{(x^2+y)^3}$,
 $\frac{\partial^2 w}{\partial y^2} = \frac{(x^2+y)^2 \cdot 0 - (-x^2-x)2(x^2+y) \cdot 1}{[(x^2+y)^2]^2} = \frac{2x^2+2x}{(x^2+y)^3}$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = \frac{(x^2+y)^2(2x+1) - (-x^2+2xy+y)2(x^2+y) \cdot 1}{[(x^2+y)^2]^2}$
 $= \frac{2x^3+3x^2-2xy-y}{(x^2+y)^3}$
51. $\frac{\partial w}{\partial x} = \frac{2}{2x+3y}$, $\frac{\partial w}{\partial y} = \frac{3}{2x+3y}$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$
52. $\frac{\partial w}{\partial x} = e^x + \ln y + \frac{y}{x}$, $\frac{\partial w}{\partial y} = \frac{x}{y} + \ln x$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{1}{y} + \frac{1}{x}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{y} + \frac{1}{x}$
53. $\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4$, $\frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3$, $\frac{\partial^2 w}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3$, and $\frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$
54. $\frac{\partial w}{\partial x} = \sin y + y \cos x + y$, $\frac{\partial w}{\partial y} = x \cos y + \sin x + x$, $\frac{\partial^2 w}{\partial y \partial x} = \cos y + \cos x + 1$, and $\frac{\partial^2 w}{\partial x \partial y} = \cos y + \cos x + 1$
55. (a) x first (b) y first (c) x first (d) x first (e) y first (f) y first
56. (a) y first three times (b) y first three times (c) y first twice (d) x first twice
57. $f_x(1,2) = \lim_{h \rightarrow 0} \frac{f(1+h,2) - f(1,2)}{h} = \lim_{h \rightarrow 0} \frac{[1 - (1+h) + 2 - 6(1+h)^2] - (2-6)}{h} = \lim_{h \rightarrow 0} \frac{-h - 6(1+2h+h^2) + 6}{h}$
 $= \lim_{h \rightarrow 0} \frac{-13h - 6h^2}{h} = \lim_{h \rightarrow 0} (-13 - 6h) = -13$,
 $f_y(1,2) = \lim_{h \rightarrow 0} \frac{f(1,2+h) - f(1,2)}{h} = \lim_{h \rightarrow 0} \frac{[1 - 1 + (2+h) - 3(2+h)] - (2-6)}{h} = \lim_{h \rightarrow 0} \frac{(2-6-2h) - (2-6)}{h}$
 $= \lim_{h \rightarrow 0} (-2) = -2$
58. $f_x(-2,1) = \lim_{h \rightarrow 0} \frac{f(-2+h,1) - f(-2,1)}{h} = \lim_{h \rightarrow 0} \frac{[4 + 2(-2+h) - 3 - (-2+h)] - (-3+2)}{h}$
 $= \lim_{h \rightarrow 0} \frac{(2h-1-h)+1}{h} = \lim_{h \rightarrow 0} 1 = 1$,
 $f_y(-2,1) = \lim_{h \rightarrow 0} \frac{f(-2,1+h) - f(-2,1)}{h} = \lim_{h \rightarrow 0} \frac{[4 - 4 - 3(1+h) + 2(1+h)^2] - (-3+2)}{h}$
 $= \lim_{h \rightarrow 0} \frac{(-3-3h+2+4h+2h^2)+1}{h} = \lim_{h \rightarrow 0} \frac{h+2h^2}{h} = \lim_{h \rightarrow 0} (1+2h) = 1$
59. $f_x(-2,3) = \lim_{h \rightarrow 0} \frac{f(-2+h,3) - f(-2,3)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2(-2+h)+9-1} - \sqrt{-4+9-1}}{h}$
 $= \lim_{h \rightarrow 0} \frac{\sqrt{2h+4}-2}{h} = \lim_{h \rightarrow 0} \left(\frac{\sqrt{2h+4}-2}{h} \cdot \frac{\sqrt{2h+4}+2}{\sqrt{2h+4}+2} \right) = \lim_{h \rightarrow 0} \frac{2}{\sqrt{2h+4}+2} = \frac{1}{2}$,
 $f_y(-2,3) = \lim_{h \rightarrow 0} \frac{f(-2,3+h) - f(-2,3)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{-4+3(3+h)-1} - \sqrt{-4+9-1}}{h}$
 $= \lim_{h \rightarrow 0} \frac{\sqrt{3h+4}-2}{h} = \lim_{h \rightarrow 0} \left(\frac{\sqrt{3h+4}-2}{h} \cdot \frac{\sqrt{3h+4}+2}{\sqrt{3h+4}+2} \right) = \lim_{h \rightarrow 0} \frac{3}{\sqrt{2h+4}+2} = \frac{3}{4}$
60. $f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(h^3+0)}{h^2+0} - 0}{h} = \lim_{h \rightarrow 0} \frac{\sin h^3}{h^3} = 1$
 $f_y(0,0) = \lim_{h \rightarrow 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(0+h^4)}{0+h^2} - 0}{h} = \lim_{h \rightarrow 0} \frac{\sin h^4}{h^3} = \lim_{h \rightarrow 0} \left(h \cdot \frac{\sin h^4}{h^4} \right) = 0 \cdot 1 = 0$
61. (a) In the plane $x = 2 \Rightarrow f_y(x, y) = 3 \Rightarrow f_y(2, -1) = 3 \Rightarrow m = 3$
(b) In the plane $y = -1 \Rightarrow f_x(x, y) = 2 \Rightarrow f_x(2, -1) = 2 \Rightarrow m = 2$

62. (a) In the plane $x = -1 \Rightarrow f_y(x, y) = 3y^2 \Rightarrow f_y(-1, 1) = 3(1)^2 = 3 \Rightarrow m = 3$

(b) In the plane $y = 1 \Rightarrow f_x(x, y) = 2x \Rightarrow f_x(-1, 1) = 2(-1) = -2 \Rightarrow m = -2$

63. $f_z(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0, z_0 + h) - f(x_0, y_0, z_0)}{h}$;

$$f_z(1, 2, 3) = \lim_{h \rightarrow 0} \frac{f(1, 2, 3+h) - f(1, 2, 3)}{h} = \lim_{h \rightarrow 0} \frac{2(3+h)^2 - 2(9)}{h} = \lim_{h \rightarrow 0} \frac{12h + 2h^2}{h} = \lim_{h \rightarrow 0} (12 + 2h) = 12$$

64. $f_y(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h, z_0) - f(x_0, y_0, z_0)}{h}$;

$$f_y(-1, 0, 3) = \lim_{h \rightarrow 0} \frac{f(-1, h, 3) - f(-1, 0, 3)}{h} = \lim_{h \rightarrow 0} \frac{(2h^2 + 9h) - 0}{h} = \lim_{h \rightarrow 0} (2h + 9) = 9$$

65. $y + (3z^2 \frac{\partial z}{\partial x})x + z^3 - 2y \frac{\partial z}{\partial x} = 0 \Rightarrow (3xz^2 - 2y) \frac{\partial z}{\partial x} = -y - z^3 \Rightarrow$ at $(1, 1, 1)$ we have $(3 - 2) \frac{\partial z}{\partial x} = -1 - 1$ or $\frac{\partial z}{\partial x} = -2$

66. $(\frac{\partial x}{\partial z})z + x + (\frac{y}{x}) \frac{\partial x}{\partial z} - 2x \frac{\partial x}{\partial z} = 0 \Rightarrow (z + \frac{y}{x} - 2x) \frac{\partial x}{\partial z} = -x \Rightarrow$ at $(1, -1, -3)$ we have $(-3 - 1 - 2) \frac{\partial x}{\partial z} = -1$ or $\frac{\partial x}{\partial z} = \frac{1}{6}$

67. $a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow 2a = (2bc \sin A) \frac{\partial A}{\partial a} \Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc \sin A}$; also $0 = 2b - 2c \cos A + (2bc \sin A) \frac{\partial A}{\partial b}$
 $\Rightarrow 2c \cos A - 2b = (2bc \sin A) \frac{\partial A}{\partial b} \Rightarrow \frac{\partial A}{\partial b} = \frac{c \cos A - b}{bc \sin A}$

68. $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{(\sin A) \frac{\partial a}{\partial A} - a \cos A}{\sin^2 A} = 0 \Rightarrow (\sin A) \frac{\partial a}{\partial A} - a \cos A = 0 \Rightarrow \frac{\partial a}{\partial A} = \frac{a \cos A}{\sin A}$; also
 $(\frac{1}{\sin A}) \frac{\partial a}{\partial B} = b(-\csc B \cot B) \Rightarrow \frac{\partial a}{\partial B} = -b \csc B \cot B \sin A$

69. Differentiating each equation implicitly gives $1 = v_x \ln u + (\frac{v}{u}) u_x$ and $0 = u_x \ln v + (\frac{u}{v}) v_x$ or

$$\left. \begin{aligned} (\ln u) v_x + (\frac{v}{u}) u_x &= 1 \\ (\frac{u}{v}) v_x + (\ln v) u_x &= 0 \end{aligned} \right\} \Rightarrow v_x = \frac{\begin{vmatrix} 1 & \frac{v}{u} \\ 0 & \ln v \end{vmatrix}}{\begin{vmatrix} \ln u & \frac{v}{u} \\ \frac{u}{v} & \ln v \end{vmatrix}} = \frac{\ln v}{(\ln u)(\ln v) - 1}$$

70. Differentiating each equation implicitly gives $1 = (2x)x_u - (2y)y_u$ and $0 = (2x)x_u - y_u$ or

$$\left. \begin{aligned} (2x)x_u - (2y)y_u &= 1 \\ (2x)x_u - y_u &= 0 \end{aligned} \right\} \Rightarrow x_u = \frac{\begin{vmatrix} 1 & -2y \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2x & -2y \\ 2x & -1 \end{vmatrix}} = \frac{-1}{-2x + 4xy} = \frac{1}{2x - 4xy} \text{ and}$$

$$y_u = \frac{\begin{vmatrix} 2x & 1 \\ 2x & 0 \end{vmatrix}}{-2x + 4xy} = \frac{-2x}{-2x + 4xy} = \frac{2x}{2x - 4xy} = \frac{1}{1 - 2y}; \text{ next } s = x^2 + y^2 \Rightarrow \frac{\partial s}{\partial u} = 2x \frac{\partial x}{\partial u} + 2y \frac{\partial y}{\partial u}$$

$$= 2x \left(\frac{1}{2x - 4xy} \right) + 2y \left(\frac{1}{1 - 2y} \right) = \frac{1}{1 - 2y} + \frac{2y}{1 - 2y} = \frac{1 + 2y}{1 - 2y}$$

71. $f_x(x, y) = \begin{cases} 0 & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases} \Rightarrow f_x(x, y) = 0$ for all points (x, y) ; at $y = 0$, $f_y(x, 0) = \lim_{h \rightarrow 0} \frac{f(x, 0+h) - f(x, 0)}{h} = \lim_{h \rightarrow 0} \frac{f(x, h) - 0}{h}$
 $= \lim_{h \rightarrow 0} \frac{f(x, h)}{h} = 0$ because $\lim_{h \rightarrow 0^-} \frac{f(x, h)}{h} = \lim_{h \rightarrow 0^-} \frac{h^3}{h} = 0$ and $\lim_{h \rightarrow 0^+} \frac{f(x, h)}{h} = \lim_{h \rightarrow 0^+} \frac{h^2}{h} = 0 \Rightarrow f_y(x, y) = \begin{cases} 3y^2 & \text{if } y \geq 0 \\ -2y & \text{if } y < 0 \end{cases}$;
 $f_{yx}(x, y) = f_{xy}(x, y) = 0$ for all points (x, y)

72. At $x = 0$, $f_x(0, y) = \lim_{h \rightarrow 0} \frac{f(0+h, y) - f(0, y)}{h} = \lim_{h \rightarrow 0} \frac{f(h, y) - 0}{h} = \lim_{h \rightarrow 0} \frac{f(h, y)}{h}$ which does not exist because $\lim_{h \rightarrow 0^-} \frac{f(h, y)}{h}$

$$= \lim_{h \rightarrow 0^-} \frac{h^2}{h} = 0 \text{ and } \lim_{h \rightarrow 0^+} \frac{f(h, y)}{h} = \lim_{h \rightarrow 0^+} \frac{\sqrt{h}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = +\infty \Rightarrow f_x(x, y) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x > 0 \\ 2x & \text{if } x < 0 \end{cases};$$

$f_y(x, y) = \begin{cases} 0 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \Rightarrow f_y(x, y) = 0$ for all points (x, y) ; $f_{yx}(x, y) = 0$ for all points (x, y) , while $f_{xy}(x, y) = 0$ for all points (x, y) such that $x \neq 0$.

73. $\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = -4z \Rightarrow \frac{\partial^2 f}{\partial x^2} = 2, \frac{\partial^2 f}{\partial y^2} = 2, \frac{\partial^2 f}{\partial z^2} = -4 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 + (-4) = 0$
74. $\frac{\partial f}{\partial x} = -6xz, \frac{\partial f}{\partial y} = -6yz, \frac{\partial f}{\partial z} = 6z^2 - 3(x^2 + y^2), \frac{\partial^2 f}{\partial x^2} = -6z, \frac{\partial^2 f}{\partial y^2} = -6z, \frac{\partial^2 f}{\partial z^2} = 12z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -6z - 6z + 12z = 0$
75. $\frac{\partial f}{\partial x} = -2e^{-2y} \sin 2x, \frac{\partial f}{\partial y} = -2e^{-2y} \cos 2x, \frac{\partial^2 f}{\partial x^2} = -4e^{-2y} \cos 2x, \frac{\partial^2 f}{\partial y^2} = 4e^{-2y} \cos 2x \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -4e^{-2y} \cos 2x + 4e^{-2y} \cos 2x = 0$
76. $\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2}, \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2}, \frac{\partial^2 f}{\partial x^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0$
77. $\frac{\partial f}{\partial x} = 3, \frac{\partial f}{\partial y} = 2, \frac{\partial^2 f}{\partial x^2} = 0, \frac{\partial^2 f}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 + 0 = 0$
78. $\frac{\partial f}{\partial x} = \frac{1/y}{1 + (\frac{x}{y})^2} = \frac{y}{y^2 + x^2}, \frac{\partial f}{\partial y} = \frac{-x/y^2}{1 + (\frac{x}{y})^2} = \frac{-x}{y^2 + x^2}, \frac{\partial^2 f}{\partial x^2} = \frac{(y^2 + x^2) \cdot 0 - y \cdot 2x}{(y^2 + x^2)^2} = \frac{-2xy}{(y^2 + x^2)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{(y^2 + x^2) \cdot 0 - (-x) \cdot 2y}{(y^2 + x^2)^2} = \frac{2xy}{(y^2 + x^2)^2}$
 $\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(y^2 + x^2)^2} + \frac{2xy}{(y^2 + x^2)^2} = 0$
79. $\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}, \frac{\partial f}{\partial y} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2y) = -y(x^2 + y^2 + z^2)^{-3/2}, \frac{\partial f}{\partial z} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2z) = -z(x^2 + y^2 + z^2)^{-3/2};$
 $\frac{\partial^2 f}{\partial x^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2}, \frac{\partial^2 f}{\partial y^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3y^2(x^2 + y^2 + z^2)^{-5/2},$
 $\frac{\partial^2 f}{\partial z^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3z^2(x^2 + y^2 + z^2)^{-5/2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
 $= \left[-(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2} \right] + \left[-(x^2 + y^2 + z^2)^{-3/2} + 3y^2(x^2 + y^2 + z^2)^{-5/2} \right]$
 $+ \left[-(x^2 + y^2 + z^2)^{-3/2} + 3z^2(x^2 + y^2 + z^2)^{-5/2} \right] = -3(x^2 + y^2 + z^2)^{-3/2} + (3x^2 + 3y^2 + 3z^2)(x^2 + y^2 + z^2)^{-5/2} = 0$
80. $\frac{\partial f}{\partial x} = 3e^{3x+4y} \cos 5z, \frac{\partial f}{\partial y} = 4e^{3x+4y} \cos 5z, \frac{\partial f}{\partial z} = -5e^{3x+4y} \sin 5z; \frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y} \cos 5z, \frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y} \cos 5z,$
 $\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y} \cos 5z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 9e^{3x+4y} \cos 5z + 16e^{3x+4y} \cos 5z - 25e^{3x+4y} \cos 5z = 0$
81. $\frac{\partial w}{\partial x} = \cos(x + ct), \frac{\partial w}{\partial t} = c \cos(x + ct); \frac{\partial^2 w}{\partial x^2} = -\sin(x + ct), \frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x + ct) \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-\sin(x + ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$
82. $\frac{\partial w}{\partial x} = -2 \sin(2x + 2ct), \frac{\partial w}{\partial t} = -2c \sin(2x + 2ct); \frac{\partial^2 w}{\partial x^2} = -4 \cos(2x + 2ct), \frac{\partial^2 w}{\partial t^2} = -4c^2 \cos(2x + 2ct)$
 $\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-4 \cos(2x + 2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$
83. $\frac{\partial w}{\partial x} = \cos(x + ct) - 2 \sin(2x + 2ct), \frac{\partial w}{\partial t} = c \cos(x + ct) - 2c \sin(2x + 2ct);$
 $\frac{\partial^2 w}{\partial x^2} = -\sin(x + ct) - 4 \cos(2x + 2ct), \frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x + ct) - 4c^2 \cos(2x + 2ct)$
 $\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-\sin(x + ct) - 4 \cos(2x + 2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$
84. $\frac{\partial w}{\partial x} = \frac{1}{x + ct}, \frac{\partial w}{\partial t} = \frac{c}{x + ct}; \frac{\partial^2 w}{\partial x^2} = \frac{-1}{(x + ct)^2}, \frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x + ct)^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 \left[\frac{-1}{(x + ct)^2} \right] = c^2 \frac{\partial^2 w}{\partial x^2}$
85. $\frac{\partial w}{\partial x} = 2 \sec^2(2x - 2ct), \frac{\partial w}{\partial t} = -2c \sec^2(2x - 2ct); \frac{\partial^2 w}{\partial x^2} = 8 \sec^2(2x - 2ct) \tan(2x - 2ct),$
 $\frac{\partial^2 w}{\partial t^2} = 8c^2 \sec^2(2x - 2ct) \tan(2x - 2ct) \Rightarrow u_x \frac{\partial^2 w}{\partial t^2} = c^2[8 \sec^2(2x - 2ct) \tan(2x - 2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$

$$86. \frac{\partial w}{\partial x} = -15 \sin(3x + 3ct) + e^{x+ct}, \frac{\partial w}{\partial t} = -15c \sin(3x + 3ct) + ce^{x+ct}; \frac{\partial^2 w}{\partial x^2} = -45 \cos(3x + 3ct) + e^{x+ct}, \\ \frac{\partial^2 w}{\partial t^2} = -45c^2 \cos(3x + 3ct) + c^2 e^{x+ct} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 [-45 \cos(3x + 3ct) + e^{x+ct}] = c^2 \frac{\partial^2 w}{\partial x^2}$$

$$87. \frac{\partial w}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} = \frac{\partial f}{\partial u} (ac) \Rightarrow \frac{\partial^2 w}{\partial t^2} = (ac) \left(\frac{\partial^2 f}{\partial u^2} \right) (ac) = a^2 c^2 \frac{\partial^2 f}{\partial u^2}; \frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \cdot a \Rightarrow \frac{\partial^2 w}{\partial x^2} = \left(a \frac{\partial^2 f}{\partial u^2} \right) \cdot a \\ = a^2 \frac{\partial^2 f}{\partial u^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = a^2 c^2 \frac{\partial^2 f}{\partial u^2} = c^2 \left(a^2 \frac{\partial^2 f}{\partial u^2} \right) = c^2 \frac{\partial^2 w}{\partial x^2}$$

88. If the first partial derivatives are continuous throughout an open region R , then by Theorem 3 in this section of the text, $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$, where $\epsilon_1, \epsilon_2 \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$. Then as $(x, y) \rightarrow (x_0, y_0)$, $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0 \Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0) \Rightarrow f$ is continuous at every point (x_0, y_0) in R .

89. Yes, since f_{xx}, f_{yy}, f_{xy} , and f_{yx} are all continuous on R , use the same reasoning as in Exercise 76 with $f_x(x, y) = f_x(x_0, y_0) + f_{xx}(x_0, y_0) \Delta x + f_{xy}(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ and $f_y(x, y) = f_y(x_0, y_0) + f_{yx}(x_0, y_0) \Delta x + f_{yy}(x_0, y_0) \Delta y + \hat{\epsilon}_1 \Delta x + \hat{\epsilon}_2 \Delta y$. Then $\lim_{(x, y) \rightarrow (x_0, y_0)} f_x(x, y) = f_x(x_0, y_0)$ and $\lim_{(x, y) \rightarrow (x_0, y_0)} f_y(x, y) = f_y(x_0, y_0)$.

90. To find α and β so that $u_t = u_{xx} \Rightarrow u_t = -\beta \sin(\alpha x)e^{-\beta t}$ and $u_x = \alpha \cos(\alpha x)e^{-\beta t} \Rightarrow u_{xx} = -\alpha^2 \sin(\alpha x)e^{-\beta t}$; then $u_t = u_{xx} \Rightarrow -\beta \sin(\alpha x)e^{-\beta t} = -\alpha^2 \sin(\alpha x)e^{-\beta t}$, thus $u_t = u_{xx}$ only if $\beta = \alpha^2$

$$91. f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{h^2}{h^2+0^4} - 0}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0; f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0 \cdot h^2}{0^2+h^4} - 0}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0;$$

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } x = ky^2}} f(x, y) = \lim_{y \rightarrow 0} \frac{(ky^2)y^2}{(ky^2)^2 + y^4} = \lim_{y \rightarrow 0} \frac{ky^4}{k^2y^4 + y^4} = \lim_{y \rightarrow 0} \frac{k}{k^2 + 1} = \frac{k}{k^2 + 1} \Rightarrow \text{different limits for different}$$

values of $k \Rightarrow \lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist $\Rightarrow f(x, y)$ is not continuous at $(0, 0) \Rightarrow$ by Theorem 4, $f(x, y)$ is not differentiable at $(0, 0)$.

$$92. f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{f(h, 0) - 1}{h} = \lim_{h \rightarrow 0} \frac{1-1}{h} = 0; f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{f(0, h) - 1}{h} = \lim_{h \rightarrow 0} \frac{1-1}{h} = 0;$$

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y = x^2}} f(x, y) = \lim_{y \rightarrow 0} 0 = 0 \text{ but } \lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y = 1.5x^2}} f(x, y) = \lim_{y \rightarrow 0} 1 = 1 \Rightarrow \lim_{(x, y) \rightarrow (0, 0)} f(x, y) \text{ does not exist}$$

$\Rightarrow f(x, y)$ is not continuous at $(0, 0) \Rightarrow$ by Theorem 4, $f(x, y)$ is not differentiable at $(0, 0)$.

14.4 THE CHAIN RULE

$$1. (a) \frac{\partial w}{\partial x} = 2x, \frac{\partial w}{\partial y} = 2y, \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t \Rightarrow \frac{dw}{dt} = -2x \sin t + 2y \cos t = -2 \cos t \sin t + 2 \sin t \cos t = 0; w = x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow \frac{dw}{dt} = 0$$

$$(b) \frac{dw}{dt}(\pi) = 0$$

$$2. (a) \frac{\partial w}{\partial x} = 2x, \frac{\partial w}{\partial y} = 2y, \frac{dx}{dt} = -\sin t + \cos t, \frac{dy}{dt} = -\sin t - \cos t \Rightarrow \frac{dw}{dt} = (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t) \\ = 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) = (2 \cos^2 t - 2 \sin^2 t) - (2 \cos^2 t - 2 \sin^2 t) = 0; w = x^2 + y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2 = 2 \cos^2 t + 2 \sin^2 t = 2 \Rightarrow \frac{dw}{dt} = 0$$

$$(b) \frac{dw}{dt}(0) = 0$$

$$3. (a) \frac{\partial w}{\partial x} = \frac{1}{z}, \frac{\partial w}{\partial y} = \frac{1}{z}, \frac{\partial w}{\partial z} = \frac{-(x+y)}{z^2}, \frac{dx}{dt} = -2 \cos t \sin t, \frac{dy}{dt} = 2 \sin t \cos t, \frac{dz}{dt} = -\frac{1}{t^2}$$

$$\Rightarrow \frac{dw}{dt} = -\frac{2}{z} \cos t \sin t + \frac{2}{z} \sin t \cos t + \frac{x+y}{z^2 t^2} = \frac{\cos^2 t + \sin^2 t}{\left(\frac{1}{t^2}\right)(t^2)} = 1; w = \frac{x}{z} + \frac{y}{z} = \frac{\cos^2 t}{\left(\frac{1}{t}\right)} + \frac{\sin^2 t}{\left(\frac{1}{t}\right)} = t \Rightarrow \frac{dw}{dt} = 1$$

$$(b) \frac{dw}{dt}(3) = 1$$

$$4. (a) \frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2}, \frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2}, \frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}, \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t, \frac{dz}{dt} = 2t^{-1/2}$$

$$\Rightarrow \frac{dw}{dt} = \frac{-2x \sin t}{x^2+y^2+z^2} + \frac{2y \cos t}{x^2+y^2+z^2} + \frac{4zt^{-1/2}}{x^2+y^2+z^2} = \frac{-2 \cos t \sin t + 2 \sin t \cos t + 4(4t^{1/2})t^{-1/2}}{\cos^2 t + \sin^2 t + 16t}$$

$$= \frac{16}{1+16t}; w = \ln(x^2+y^2+z^2) = \ln(\cos^2 t + \sin^2 t + 16t) = \ln(1+16t) \Rightarrow \frac{dw}{dt} = \frac{16}{1+16t}$$

$$(b) \frac{dw}{dt}(3) = \frac{16}{49}$$

$$5. (a) \frac{\partial w}{\partial x} = 2ye^x, \frac{\partial w}{\partial y} = 2e^x, \frac{\partial w}{\partial z} = -\frac{1}{z}, \frac{dx}{dt} = \frac{2t}{t^2+1}, \frac{dy}{dt} = \frac{1}{t^2+1}, \frac{dz}{dt} = e^t \Rightarrow \frac{dw}{dt} = \frac{4yte^x}{t^2+1} + \frac{2e^x}{t^2+1} - \frac{e^t}{z}$$

$$= \frac{(4t)(\tan^{-1} t)(t^2+1)}{t^2+1} + \frac{2(t^2+1)}{t^2+1} - \frac{e^t}{e^t} = 4t \tan^{-1} t + 1; w = 2ye^x - \ln z = (2 \tan^{-1} t)(t^2+1) - t$$

$$\Rightarrow \frac{dw}{dt} = \left(\frac{2}{t^2+1}\right)(t^2+1) + (2 \tan^{-1} t)(2t) - 1 = 4t \tan^{-1} t + 1$$

$$(b) \frac{dw}{dt}(1) = (4)(1)\left(\frac{\pi}{4}\right) + 1 = \pi + 1$$

$$6. (a) \frac{\partial w}{\partial x} = -y \cos xy, \frac{\partial w}{\partial y} = -x \cos xy, \frac{\partial w}{\partial z} = 1, \frac{dx}{dt} = 1, \frac{dy}{dt} = \frac{1}{t}, \frac{dz}{dt} = e^{t-1} \Rightarrow \frac{dw}{dt} = -y \cos xy - \frac{x \cos xy}{t} + e^{t-1}$$

$$= -(\ln t)[\cos(t \ln t)] - \frac{t \cos(t \ln t)}{t} + e^{t-1} = -(\ln t)[\cos(t \ln t)] - \cos(t \ln t) + e^{t-1}; w = z - \sin xy$$

$$= e^{t-1} - \sin(t \ln t) \Rightarrow \frac{dw}{dt} = e^{t-1} - [\cos(t \ln t)] \left[\ln t + t\left(\frac{1}{t}\right)\right] = e^{t-1} - (1 + \ln t) \cos(t \ln t)$$

$$(b) \frac{dw}{dt}(1) = 1 - (1+0)(1) = 0$$

$$7. (a) \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = (4e^x \ln y) \left(\frac{\cos v}{u \cos v}\right) + \left(\frac{4e^x}{y}\right)(\sin v) = \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y}$$

$$= \frac{4(u \cos v) \ln(u \sin v)}{u} + \frac{4(u \cos v)(\sin v)}{u \sin v} = (4 \cos v) \ln(u \sin v) + 4 \cos v;$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = (4e^x \ln y) \left(\frac{-u \sin v}{u \cos v}\right) + \left(\frac{4e^x}{y}\right)(u \cos v) = -(4e^x \ln y)(\tan v) + \frac{4e^x u \cos v}{y}$$

$$= [-4(u \cos v) \ln(u \sin v)](\tan v) + \frac{4(u \cos v)(u \cos v)}{u \sin v} = (-4u \sin v) \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v};$$

$$z = 4e^x \ln y = 4(u \cos v) \ln(u \sin v) \Rightarrow \frac{\partial z}{\partial u} = (4 \cos v) \ln(u \sin v) + 4(u \cos v) \left(\frac{\sin v}{u \sin v}\right)$$

$$= (4 \cos v) \ln(u \sin v) + 4 \cos v; \text{ also } \frac{\partial z}{\partial v} = (-4u \sin v) \ln(u \sin v) + 4(u \cos v) \left(\frac{u \cos v}{u \sin v}\right)$$

$$= (-4u \sin v) \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v}$$

$$(b) \text{ At } \left(2, \frac{\pi}{4}\right): \frac{\partial z}{\partial u} = 4 \cos \frac{\pi}{4} \ln \left(2 \sin \frac{\pi}{4}\right) + 4 \cos \frac{\pi}{4} = 2\sqrt{2} \ln \sqrt{2} + 2\sqrt{2} = \sqrt{2}(\ln 2 + 2);$$

$$\frac{\partial z}{\partial v} = (-4)(2) \sin \frac{\pi}{4} \ln \left(2 \sin \frac{\pi}{4}\right) + \frac{(4)(2)(\cos^2 \frac{\pi}{4})}{(\sin \frac{\pi}{4})} = -4\sqrt{2} \ln \sqrt{2} + 4\sqrt{2} = -2\sqrt{2} \ln 2 + 4\sqrt{2}$$

$$8. (a) \frac{\partial z}{\partial u} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right] \cos v + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right] \sin v = \frac{y \cos v}{x^2+y^2} - \frac{x \sin v}{x^2+y^2} = \frac{(u \sin v)(\cos v) - (u \cos v)(\sin v)}{u^2} = 0;$$

$$\frac{\partial z}{\partial v} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right] (-u \sin v) + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right] u \cos v = -\frac{yu \sin v}{x^2+y^2} - \frac{xu \cos v}{x^2+y^2} = \frac{-(u \sin v)(u \sin v) - (u \cos v)(u \cos v)}{u^2}$$

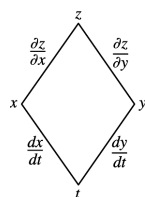
$$= -\sin^2 v - \cos^2 v = -1; z = \tan^{-1} \left(\frac{x}{y}\right) = \tan^{-1}(\cot v) \Rightarrow \frac{\partial z}{\partial u} = 0 \text{ and } \frac{\partial z}{\partial v} = \left(\frac{1}{1+\cot^2 v}\right)(-\csc^2 v)$$

$$= \frac{-1}{\sin^2 v + \cos^2 v} = -1$$

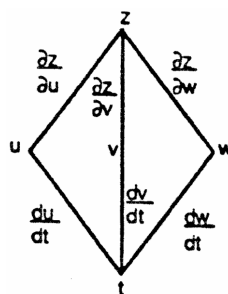
$$(b) \text{ At } \left(1.3, \frac{\pi}{6}\right): \frac{\partial z}{\partial u} = 0 \text{ and } \frac{\partial z}{\partial v} = -1$$

9. (a) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} = (y+z)(1) + (x+z)(1) + (y+x)(v) = x+y+2z+v(y+x)$
 $= (u+v) + (u-v) + 2uv + v(2u) = 2u + 4uv$; $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$
 $= (y+z)(1) + (x+z)(-1) + (y+x)(u) = y-x+(y+x)u = -2v + (2u)u = -2v + 2u^2$;
 $w = xy + yz + xz = (u^2 - v^2) + (u^2v - uv^2) + (u^2v + uv^2) = u^2 - v^2 + 2u^2v \Rightarrow \frac{\partial w}{\partial u} = 2u + 4uv$ and
 $\frac{\partial w}{\partial v} = -2v + 2u^2$
- (b) At $(\frac{1}{2}, 1)$: $\frac{\partial w}{\partial u} = 2(\frac{1}{2}) + 4(\frac{1}{2})(1) = 3$ and $\frac{\partial w}{\partial v} = -2(1) + 2(\frac{1}{2})^2 = -\frac{3}{2}$
10. (a) $\frac{\partial w}{\partial u} = \left(\frac{2x}{x^2+y^2+z^2}\right)(e^v \sin u + ue^v \cos u) + \left(\frac{2y}{x^2+y^2+z^2}\right)(e^v \cos u - ue^v \sin u) + \left(\frac{2z}{x^2+y^2+z^2}\right)(e^v)$
 $= \left(\frac{2ue^v \sin u}{u^2e^{2v} \sin^2 u + u^2e^{2v} \cos^2 u + u^2e^{2v}}\right)(e^v \sin u + ue^v \cos u)$
 $+ \left(\frac{2ue^v \cos u}{u^2e^{2v} \sin^2 u + u^2e^{2v} \cos^2 u + u^2e^{2v}}\right)(e^v \cos u - ue^v \sin u)$
 $+ \left(\frac{2ue^v}{u^2e^{2v} \sin^2 u + u^2e^{2v} \cos^2 u + u^2e^{2v}}\right)(e^v) = \frac{2}{u}$;
 $\frac{\partial w}{\partial v} = \left(\frac{2x}{x^2+y^2+z^2}\right)(ue^v \sin u) + \left(\frac{2y}{x^2+y^2+z^2}\right)(ue^v \cos u) + \left(\frac{2z}{x^2+y^2+z^2}\right)(ue^v)$
 $= \left(\frac{2ue^v \sin u}{u^2e^{2v} \sin^2 u + u^2e^{2v} \cos^2 u + u^2e^{2v}}\right)(ue^v \sin u)$
 $+ \left(\frac{2ue^v \cos u}{u^2e^{2v} \sin^2 u + u^2e^{2v} \cos^2 u + u^2e^{2v}}\right)(ue^v \cos u)$
 $+ \left(\frac{2ue^v}{u^2e^{2v} \sin^2 u + u^2e^{2v} \cos^2 u + u^2e^{2v}}\right)(ue^v) = 2$; $w = \ln(u^2e^{2v} \sin^2 u + u^2e^{2v} \cos^2 u + u^2e^{2v}) = \ln(2u^2e^{2v})$
 $= \ln 2 + 2 \ln u + 2v \Rightarrow \frac{\partial w}{\partial u} = \frac{2}{u}$ and $\frac{\partial w}{\partial v} = 2$
- (b) At $(-2, 0)$: $\frac{\partial w}{\partial u} = \frac{2}{-2} = -1$ and $\frac{\partial w}{\partial v} = 2$
11. (a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} = \frac{1}{q-r} + \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r+r-p+p-q}{(q-r)^2} = 0$;
 $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} = \frac{1}{q-r} - \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r-r+p+p-q}{(q-r)^2} = \frac{2p-2r}{(q-r)^2}$
 $= \frac{(2x+2y+2z)-(2x+2y-2z)}{(2z-2y)^2} = \frac{z}{(z-y)^2}$; $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial z}$
 $= \frac{1}{q-r} + \frac{r-p}{(q-r)^2} - \frac{p-q}{(q-r)^2} = \frac{q-r+r-p-p+q}{(q-r)^2} = \frac{2q-2p}{(q-r)^2} = \frac{-4y}{(2z-2y)^2} = -\frac{y}{(z-y)^2}$;
 $u = \frac{p-q}{q-r} = \frac{2y}{2z-2y} = \frac{y}{z-y} \Rightarrow \frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = \frac{(z-y)-y(-1)}{(z-y)^2} = \frac{z}{(z-y)^2}$, and $\frac{\partial u}{\partial z} = \frac{(z-y)(0)-y(1)}{(z-y)^2}$
 $= -\frac{y}{(z-y)^2}$
- (b) At $(\sqrt{3}, 2, 1)$: $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = \frac{1}{(1-2)^2} = 1$, and $\frac{\partial u}{\partial z} = \frac{-2}{(1-2)^2} = -2$
12. (a) $\frac{\partial u}{\partial x} = \frac{e^{qr}}{\sqrt{1-p^2}}(\cos x) + (re^{qr} \sin^{-1} p)(0) + (qe^{qr} \sin^{-1} p)(0) = \frac{e^{qr} \cos x}{\sqrt{1-p^2}} = \frac{e^{z \ln y} \cos x}{\sqrt{1-\sin^2 x}} = y^z$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$;
 $\frac{\partial u}{\partial y} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr} \sin^{-1} p)\left(\frac{z^2}{y}\right) + (qe^{qr} \sin^{-1} p)(0) = \frac{z^2 re^{qr} \sin^{-1} p}{y} = \frac{z^2 (\frac{1}{y}) y^z x}{y} = xzy^{z-1}$;
 $\frac{\partial u}{\partial z} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr} \sin^{-1} p)(2z \ln y) + (qe^{qr} \sin^{-1} p)\left(-\frac{1}{z^2}\right) = (2zre^{qr} \sin^{-1} p)(\ln y) - \frac{qe^{qr} \sin^{-1} p}{z^2}$
 $= (2z)\left(\frac{1}{z}\right)(y^z x \ln y) - \frac{(z^2 \ln y)(y^z) x}{z^2} = xy^z \ln y$; $u = e^{z \ln y} \sin^{-1}(\sin x) = xy^z$ if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \Rightarrow \frac{\partial u}{\partial x} = y^z$,
 $\frac{\partial u}{\partial y} = xzy^{z-1}$, and $\frac{\partial u}{\partial z} = xy^z \ln y$ from direct calculations
- (b) At $(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2})$: $\frac{\partial u}{\partial x} = (\frac{1}{2})^{-1/2} = \sqrt{2}$, $\frac{\partial u}{\partial y} = (\frac{\pi}{4})(-\frac{1}{2})(\frac{1}{2})^{(-1/2)-1} = -\frac{\pi\sqrt{2}}{4}$, $\frac{\partial u}{\partial z} = (\frac{\pi}{4})(\frac{1}{2})^{-1/2} \ln(\frac{1}{2}) = -\frac{\pi\sqrt{2} \ln 2}{4}$

13. $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

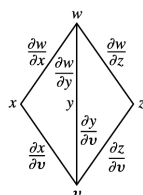
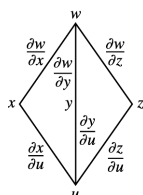


14. $\frac{dz}{dt} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} + \frac{\partial z}{\partial w} \frac{dw}{dt}$



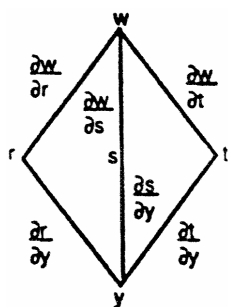
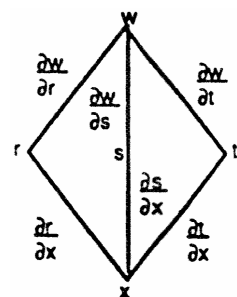
15. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$$



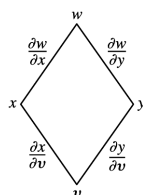
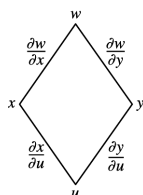
16. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial x}$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial y}$$

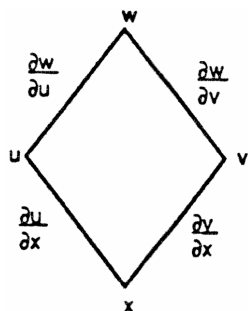


17. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}$

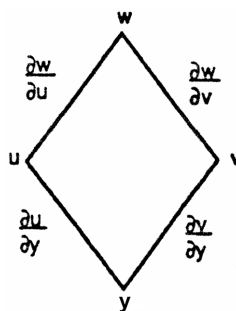
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}$$



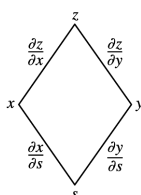
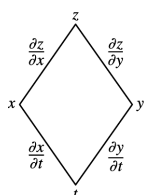
18. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x}$



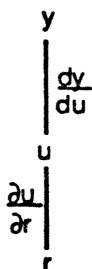
$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y}$$



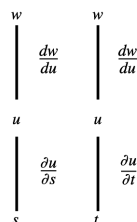
19. $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$



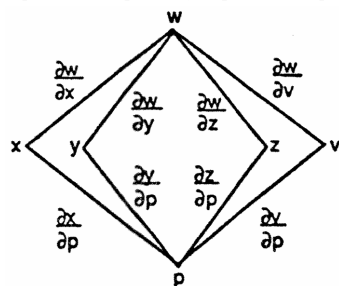
20. $\frac{\partial y}{\partial r} = \frac{dy}{du} \frac{\partial u}{\partial r}$



21. $\frac{\partial w}{\partial s} = \frac{dw}{du} \frac{\partial u}{\partial s} \quad \frac{\partial w}{\partial t} = \frac{dw}{du} \frac{\partial u}{\partial t}$

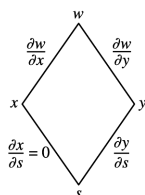
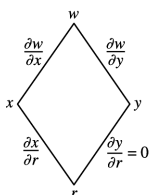


22. $\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial p} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial p} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial p}$

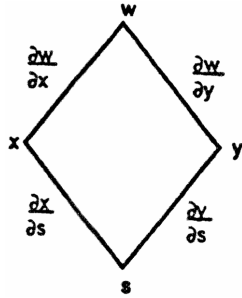


23. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{dx}{dr} + \frac{\partial w}{\partial y} \frac{dy}{dr} = \frac{\partial w}{\partial x} \frac{dx}{dr}$ since $\frac{dy}{dr} = 0$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{dx}{ds} + \frac{\partial w}{\partial y} \frac{dy}{ds} = \frac{\partial w}{\partial y} \frac{dy}{ds}$$
 since $\frac{dx}{ds} = 0$



$$24. \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$



$$25. \text{ Let } F(x, y) = x^3 - 2y^2 + xy = 0 \Rightarrow F_x(x, y) = 3x^2 + y \\ \text{and } F_y(x, y) = -4y + x \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 + y}{-4y + x} \\ \Rightarrow \frac{dy}{dx}(1, 1) = \frac{4}{3}$$

$$26. \text{ Let } F(x, y) = xy + y^2 - 3x - 3 = 0 \Rightarrow F_x(x, y) = y - 3 \text{ and } F_y(x, y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y} \\ \Rightarrow \frac{dy}{dx}(-1, 1) = 2$$

$$27. \text{ Let } F(x, y) = x^2 + xy + y^2 - 7 = 0 \Rightarrow F_x(x, y) = 2x + y \text{ and } F_y(x, y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y} \\ \Rightarrow \frac{dy}{dx}(1, 2) = -\frac{4}{5}$$

$$28. \text{ Let } F(x, y) = xe^y + \sin xy + y - \ln 2 = 0 \Rightarrow F_x(x, y) = e^y + y \cos xy \text{ and } F_y(x, y) = xe^y + x \sin xy + 1 \\ \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y \cos xy}{xe^y + x \sin xy + 1} \Rightarrow \frac{dy}{dx}(0, \ln 2) = -(2 + \ln 2)$$

$$29. \text{ Let } F(x, y, z) = z^3 - xy + yz + y^3 - 2 = 0 \Rightarrow F_x(x, y, z) = -y, F_y(x, y, z) = -x + z + 3y^2, F_z(x, y, z) = 3z^2 + y \\ \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2 + y} = \frac{y}{3z^2 + y} \Rightarrow \frac{\partial z}{\partial x}(1, 1, 1) = \frac{1}{4}; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x + z + 3y^2}{3z^2 + y} = \frac{x - z - 3y^2}{3z^2 + y} \\ \Rightarrow \frac{\partial z}{\partial y}(1, 1, 1) = -\frac{3}{4}$$

$$30. \text{ Let } F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \Rightarrow F_x(x, y, z) = -\frac{1}{x^2}, F_y(x, y, z) = -\frac{1}{y^2}, F_z(x, y, z) = -\frac{1}{z^2} \\ \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{x^2} \Rightarrow \frac{\partial z}{\partial x}(2, 3, 6) = -9; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(-\frac{1}{y^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{y^2} \Rightarrow \frac{\partial z}{\partial y}(2, 3, 6) = -4$$

$$31. \text{ Let } F(x, y, z) = \sin(x + y) + \sin(y + z) + \sin(x + z) = 0 \Rightarrow F_x(x, y, z) = \cos(x + y) + \cos(x + z), \\ F_y(x, y, z) = \cos(x + y) + \cos(y + z), F_z(x, y, z) = \cos(y + z) + \cos(x + z) \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \\ = -\frac{\cos(x + y) + \cos(x + z)}{\cos(y + z) + \cos(x + z)} \Rightarrow \frac{\partial z}{\partial x}(\pi, \pi, \pi) = -1; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x + y) + \cos(y + z)}{\cos(y + z) + \cos(x + z)} \Rightarrow \frac{\partial z}{\partial y}(\pi, \pi, \pi) = -1$$

$$32. \text{ Let } F(x, y, z) = xe^y + ye^z + 2 \ln x - 2 - 3 \ln 2 = 0 \Rightarrow F_x(x, y, z) = e^y + \frac{2}{x}, F_y(x, y, z) = xe^y + e^z, F_z(x, y, z) = ye^z \\ \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(e^y + \frac{2}{x}\right)}{ye^z} \Rightarrow \frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{4}{3 \ln 2}; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xe^y + e^z}{ye^z} \Rightarrow \frac{\partial z}{\partial y}(1, \ln 2, \ln 3) = -\frac{5}{3 \ln 2}$$

$$33. \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} = 2(x + y + z)(1) + 2(x + y + z)[- \sin(r + s)] + 2(x + y + z)[\cos(r + s)] \\ = 2(x + y + z)[1 - \sin(r + s) + \cos(r + s)] = 2[r - s + \cos(r + s) + \sin(r + s)][1 - \sin(r + s) + \cos(r + s)] \\ \Rightarrow \frac{\partial w}{\partial r} \Big|_{r=1, s=-1} = 2(3)(2) = 12$$

$$34. \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} = y \left(\frac{2v}{u} \right) + x(1) + \left(\frac{1}{z} \right) (0) = (u + v) \left(\frac{2v}{u} \right) + \frac{v^2}{u} \Rightarrow \frac{\partial w}{\partial v} \Big|_{u=-1, v=2} = (1) \left(\frac{4}{-1} \right) + \left(\frac{4}{-1} \right) = -8$$

$$35. \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = \left(2x - \frac{y}{x^2} \right) (-2) + \left(\frac{1}{x} \right) (1) = \left[2(u - 2v + 1) - \frac{2u + v - 2}{(u - 2v + 1)^2} \right] (-2) + \frac{1}{u - 2v + 1} \\ \Rightarrow \frac{\partial w}{\partial v} \Big|_{u=0, v=0} = -7$$

36. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = (y \cos xy + \sin y)(2u) + (x \cos xy + x \cos y)(v)$
 $= [uv \cos(u^3v + uv^3) + \sin uv](2u) + [(u^2 + v^2) \cos(u^3v + uv^3) + (u^2 + v^2) \cos uv](v)$
 $\Rightarrow \frac{\partial z}{\partial u} \Big|_{u=0, v=1} = 0 + (\cos 0 + \cos 0)(1) = 2$
37. $\frac{\partial z}{\partial u} = \frac{dz}{dx} \frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right) e^u = \left[\frac{5}{1+(e^u + \ln v)^2}\right] e^u \Rightarrow \frac{\partial z}{\partial u} \Big|_{u=\ln 2, v=1} = \left[\frac{5}{1+(2)^2}\right] (2) = 2;$
 $\frac{\partial z}{\partial v} = \frac{dz}{dx} \frac{\partial x}{\partial v} = \left(\frac{5}{1+x^2}\right) \left(\frac{1}{v}\right) = \left[\frac{5}{1+(e^u + \ln v)^2}\right] \left(\frac{1}{v}\right) \Rightarrow \frac{\partial z}{\partial v} \Big|_{u=\ln 2, v=1} = \left[\frac{5}{1+(2)^2}\right] (1) = 1$
38. $\frac{\partial z}{\partial u} = \frac{dz}{dq} \frac{\partial q}{\partial u} = \left(\frac{1}{q}\right) \left(\frac{\sqrt{v+3}}{1+u^2}\right) = \left(\frac{1}{\sqrt{v+3} \tan^{-1} u}\right) \left(\frac{\sqrt{v+3}}{1+u^2}\right) = \frac{1}{(\tan^{-1} u)(1+u^2)} \Rightarrow \frac{\partial z}{\partial u} \Big|_{u=1, v=-2} = \frac{1}{(\tan^{-1} 1)(1+1^2)} = \frac{2}{\pi};$
 $\frac{\partial z}{\partial v} = \frac{dz}{dq} \frac{\partial q}{\partial v} = \left(\frac{1}{q}\right) \left(\frac{\tan^{-1} u}{2\sqrt{v+3}}\right) = \left(\frac{1}{\sqrt{v+3} \tan^{-1} u}\right) \left(\frac{\tan^{-1} u}{2\sqrt{v+3}}\right) = \frac{1}{2(v+3)} \Rightarrow \frac{\partial z}{\partial v} \Big|_{u=1, v=-2} = \frac{1}{2}$
39. Let $x = s^3 + t^2 \Rightarrow w = f(s^3 + t^2) = f(x) \Rightarrow \frac{\partial w}{\partial s} = \frac{dw}{dx} \frac{\partial x}{\partial s} = f'(x) \cdot 3s^2 = 3s^2 e^{s^3+t^2}, \frac{\partial w}{\partial t} = \frac{dw}{dx} \frac{\partial x}{\partial t} = f'(x) \cdot 2t = 2t e^{s^3+t^2}$
40. Let $x = t s^2$ and $y = \frac{s}{t} \Rightarrow w = f(t s^2, \frac{s}{t}) = f(x, y) \Rightarrow \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} = f_x(x, y) \cdot 2ts + f_y(x, y) \cdot \frac{1}{t}$
 $= (ts^2) \left(\frac{s}{t}\right) \cdot 2ts + \frac{(ts^2)^2}{2} \cdot \frac{1}{t} = 2s^4t + \frac{s^4t}{2} = \frac{5s^4t}{2}; \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} = f_x(x, y) \cdot s^2 + f_y(x, y) \cdot \frac{-s}{t^2}$
 $= (ts^2) \left(\frac{s}{t}\right) \cdot s^2 + \frac{(ts^2)^2}{2} \cdot \left(-\frac{s}{t^2}\right) = s^5 - \frac{s^5}{2} = \frac{s^5}{2}$
41. $V = IR \Rightarrow \frac{\partial V}{\partial I} = R$ and $\frac{\partial V}{\partial R} = I; \frac{dV}{dt} = \frac{\partial V}{\partial I} \frac{dI}{dt} + \frac{\partial V}{\partial R} \frac{dR}{dt} = R \frac{dI}{dt} + I \frac{dR}{dt} \Rightarrow -0.01 \text{ volts/sec}$
 $= (600 \text{ ohms}) \frac{dI}{dt} + (0.04 \text{ amps})(0.5 \text{ ohms/sec}) \Rightarrow \frac{dI}{dt} = -0.00005 \text{ amps/sec}$
42. $V = abc \Rightarrow \frac{dV}{dt} = \frac{\partial V}{\partial a} \frac{da}{dt} + \frac{\partial V}{\partial b} \frac{db}{dt} + \frac{\partial V}{\partial c} \frac{dc}{dt} = (bc) \frac{da}{dt} + (ac) \frac{db}{dt} + (ab) \frac{dc}{dt}$
 $\Rightarrow \frac{dV}{dt} \Big|_{a=1, b=2, c=3} = (2 \text{ m})(3 \text{ m})(1 \text{ m/sec}) + (1 \text{ m})(3 \text{ m})(1 \text{ m/sec}) + (1 \text{ m})(2 \text{ m})(-3 \text{ m/sec}) = 3 \text{ m}^3/\text{sec}$
and the volume is increasing; $S = 2ab + 2ac + 2bc \Rightarrow \frac{dS}{dt} = \frac{\partial S}{\partial a} \frac{da}{dt} + \frac{\partial S}{\partial b} \frac{db}{dt} + \frac{\partial S}{\partial c} \frac{dc}{dt}$
 $= 2(b+c) \frac{da}{dt} + 2(a+c) \frac{db}{dt} + 2(a+b) \frac{dc}{dt} \Rightarrow \frac{dS}{dt} \Big|_{a=1, b=2, c=3}$
 $= 2(5 \text{ m})(1 \text{ m/sec}) + 2(4 \text{ m})(1 \text{ m/sec}) + 2(3 \text{ m})(-3 \text{ m/sec}) = 0 \text{ m}^2/\text{sec}$ and the surface area is not changing;
 $D = \sqrt{a^2 + b^2 + c^2} \Rightarrow \frac{dD}{dt} = \frac{\partial D}{\partial a} \frac{da}{dt} + \frac{\partial D}{\partial b} \frac{db}{dt} + \frac{\partial D}{\partial c} \frac{dc}{dt} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \left(a \frac{da}{dt} + b \frac{db}{dt} + c \frac{dc}{dt}\right) \Rightarrow \frac{dD}{dt} \Big|_{a=1, b=2, c=3}$
 $= \left(\frac{1}{\sqrt{14} \text{ m}}\right) [(1 \text{ m})(1 \text{ m/sec}) + (2 \text{ m})(1 \text{ m/sec}) + (3 \text{ m})(-3 \text{ m/sec})] = -\frac{6}{\sqrt{14}} \text{ m/sec} < 0 \Rightarrow \text{the diagonals are decreasing in length}$
43. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} (1) + \frac{\partial f}{\partial v} (0) + \frac{\partial f}{\partial w} (-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w},$
 $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial y} = \frac{\partial f}{\partial u} (-1) + \frac{\partial f}{\partial v} (1) + \frac{\partial f}{\partial w} (0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v},$ and
 $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial z} = \frac{\partial f}{\partial u} (0) + \frac{\partial f}{\partial v} (-1) + \frac{\partial f}{\partial w} (1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$
44. (a) $\frac{\partial w}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} = f_x \cos \theta + f_y \sin \theta$ and $\frac{\partial w}{\partial \theta} = f_x(-r \sin \theta) + f_y(r \cos \theta) \Rightarrow \frac{1}{r} \frac{\partial w}{\partial \theta} = -f_x \sin \theta + f_y \cos \theta$
(b) $\frac{\partial w}{\partial r} \sin \theta = f_x \sin \theta \cos \theta + f_y \sin^2 \theta$ and $\left(\frac{\cos \theta}{r}\right) \frac{\partial w}{\partial \theta} = -f_x \sin \theta \cos \theta + f_y \cos^2 \theta$
 $\Rightarrow f_y = (\sin \theta) \frac{\partial w}{\partial r} + \left(\frac{\cos \theta}{r}\right) \frac{\partial w}{\partial \theta};$ then $\frac{\partial w}{\partial r} = f_x \cos \theta + [(\sin \theta) \frac{\partial w}{\partial r} + \left(\frac{\cos \theta}{r}\right) \frac{\partial w}{\partial \theta}] (\sin \theta) \Rightarrow f_x \cos \theta$
 $= \frac{\partial w}{\partial r} - (\sin^2 \theta) \frac{\partial w}{\partial r} - \left(\frac{\sin \theta \cos \theta}{r}\right) \frac{\partial w}{\partial \theta} = (1 - \sin^2 \theta) \frac{\partial w}{\partial r} - \left(\frac{\sin \theta \cos \theta}{r}\right) \frac{\partial w}{\partial \theta} \Rightarrow f_x = (\cos \theta) \frac{\partial w}{\partial r} - \left(\frac{\sin \theta}{r}\right) \frac{\partial w}{\partial \theta}$
(c) $(f_x)^2 = (\cos^2 \theta) \left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{2 \sin \theta \cos \theta}{r}\right) \left(\frac{\partial w}{\partial r} \frac{\partial w}{\partial \theta}\right) + \left(\frac{\sin^2 \theta}{r^2}\right) \left(\frac{\partial w}{\partial \theta}\right)^2$ and
 $(f_y)^2 = (\sin^2 \theta) \left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{2 \sin \theta \cos \theta}{r}\right) \left(\frac{\partial w}{\partial r} \frac{\partial w}{\partial \theta}\right) + \left(\frac{\cos^2 \theta}{r^2}\right) \left(\frac{\partial w}{\partial \theta}\right)^2 \Rightarrow (f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta}\right)^2$

$$\begin{aligned}
45. \quad w_x &= \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} = x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v} \Rightarrow w_{xx} = \frac{\partial w}{\partial u} + x \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial u} \right) + y \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial v} \right) \\
&= \frac{\partial w}{\partial u} + x \left(\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial x} \right) + y \left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial x} \right) = \frac{\partial w}{\partial u} + x \left(x \frac{\partial^2 w}{\partial u^2} + y \frac{\partial^2 w}{\partial v \partial u} \right) + y \left(x \frac{\partial^2 w}{\partial u \partial v} + y \frac{\partial^2 w}{\partial v^2} \right) \\
&= \frac{\partial w}{\partial u} + x^2 \frac{\partial^2 w}{\partial u^2} + 2xy \frac{\partial^2 w}{\partial v \partial u} + y^2 \frac{\partial^2 w}{\partial v^2}; \quad w_y = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = -y \frac{\partial w}{\partial u} + x \frac{\partial w}{\partial v} \\
\Rightarrow w_{yy} &= -\frac{\partial w}{\partial u} - y \left(\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial y} \right) + x \left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial y} \right) \\
&= -\frac{\partial w}{\partial u} - y \left(-y \frac{\partial^2 w}{\partial u^2} + x \frac{\partial^2 w}{\partial v \partial u} \right) + x \left(-y \frac{\partial^2 w}{\partial u \partial v} + x \frac{\partial^2 w}{\partial v^2} \right) = -\frac{\partial w}{\partial u} + y^2 \frac{\partial^2 w}{\partial u^2} - 2xy \frac{\partial^2 w}{\partial v \partial u} + x^2 \frac{\partial^2 w}{\partial v^2}; \text{ thus} \\
w_{xx} + w_{yy} &= (x^2 + y^2) \frac{\partial^2 w}{\partial u^2} + (x^2 + y^2) \frac{\partial^2 w}{\partial v^2} = (x^2 + y^2) (w_{uu} + w_{vv}) = 0, \text{ since } w_{uu} + w_{vv} = 0
\end{aligned}$$

$$\begin{aligned}
46. \quad \frac{\partial w}{\partial x} &= f'(u)(1) + g'(v)(1) = f'(u) + g'(v) \Rightarrow w_{xx} = f''(u)(1) + g''(v)(1) = f''(u) + g''(v); \\
\frac{\partial w}{\partial y} &= f'(u)(i) + g'(v)(-i) \Rightarrow w_{yy} = f''(u)(i^2) + g''(v)(i^2) = -f''(u) - g''(v) \Rightarrow w_{xx} + w_{yy} = 0
\end{aligned}$$

$$\begin{aligned}
47. \quad f_x(x, y, z) &= \cos t, f_y(x, y, z) = \sin t, \text{ and } f_z(x, y, z) = t^2 + t - 2 \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \\
&= (\cos t)(-\sin t) + (\sin t)(\cos t) + (t^2 + t - 2)(1) = t^2 + t - 2; \quad \frac{df}{dt} = 0 \Rightarrow t^2 + t - 2 = 0 \Rightarrow t = -2 \\
\text{or } t &= 1; \quad t = -2 \Rightarrow x = \cos(-2), y = \sin(-2), z = -2 \text{ for the point } (\cos(-2), \sin(-2), -2); \quad t = 1 \Rightarrow x = \cos 1, \\
y &= \sin 1, z = 1 \text{ for the point } (\cos 1, \sin 1, 1)
\end{aligned}$$

$$\begin{aligned}
48. \quad \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = (2xe^{2y} \cos 3z)(-\sin t) + (2x^2e^{2y} \cos 3z) \left(\frac{1}{t+2} \right) + (-3x^2e^{2y} \sin 3z)(1) \\
&= -2xe^{2y} \cos 3z \sin t + \frac{2x^2e^{2y} \cos 3z}{t+2} - 3x^2e^{2y} \sin 3z; \text{ at the point on the curve } z = 0 \Rightarrow t = z = 0 \\
\Rightarrow \frac{dw}{dt} \Big|_{(1, \ln 2, 0)} &= 0 + \frac{2(1)^2(4)(1)}{2} - 0 = 4
\end{aligned}$$

$$\begin{aligned}
49. \quad (a) \quad \frac{\partial T}{\partial x} &= 8x - 4y \text{ and } \frac{\partial T}{\partial y} = 8y - 4x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} = (8x - 4y)(-\sin t) + (8y - 4x)(\cos t) \\
&= (8 \cos t - 4 \sin t)(-\sin t) + (8 \sin t - 4 \cos t)(\cos t) = 4 \sin^2 t - 4 \cos^2 t \Rightarrow \frac{d^2T}{dt^2} = 16 \sin t \cos t; \\
\frac{dT}{dt} &= 0 \Rightarrow 4 \sin^2 t - 4 \cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t \Rightarrow \sin t = \cos t \text{ or } \sin t = -\cos t \Rightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4} \text{ on} \\
&\text{the interval } 0 \leq t \leq 2\pi;
\end{aligned}$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{\pi}{4}} = 16 \sin \frac{\pi}{4} \cos \frac{\pi}{4} > 0 \Rightarrow T \text{ has a minimum at } (x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{3\pi}{4}} = 16 \sin \frac{3\pi}{4} \cos \frac{3\pi}{4} < 0 \Rightarrow T \text{ has a maximum at } (x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{5\pi}{4}} = 16 \sin \frac{5\pi}{4} \cos \frac{5\pi}{4} > 0 \Rightarrow T \text{ has a minimum at } (x, y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{7\pi}{4}} = 16 \sin \frac{7\pi}{4} \cos \frac{7\pi}{4} < 0 \Rightarrow T \text{ has a maximum at } (x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$$

$$\begin{aligned}
(b) \quad T &= 4x^2 - 4xy + 4y^2 \Rightarrow \frac{\partial T}{\partial x} = 8x - 4y, \text{ and } \frac{\partial T}{\partial y} = 8y - 4x \text{ so the extreme values occur at the four points} \\
&\text{found in part (a): } T \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) = T \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) = 4 \left(\frac{1}{2} \right) - 4 \left(-\frac{1}{2} \right) + 4 \left(\frac{1}{2} \right) = 6, \text{ the maximum and} \\
T \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) &= T \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) = 4 \left(\frac{1}{2} \right) - 4 \left(\frac{1}{2} \right) + 4 \left(\frac{1}{2} \right) = 2, \text{ the minimum}
\end{aligned}$$

$$\begin{aligned}
50. \quad (a) \quad \frac{\partial T}{\partial x} &= y \text{ and } \frac{\partial T}{\partial y} = x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} = y \left(-2\sqrt{2} \sin t \right) + x \left(\sqrt{2} \cos t \right) \\
&= \left(\sqrt{2} \sin t \right) \left(-2\sqrt{2} \sin t \right) + \left(2\sqrt{2} \cos t \right) \left(\sqrt{2} \cos t \right) = -4 \sin^2 t + 4 \cos^2 t = -4 \sin^2 t + 4(1 - \sin^2 t) \\
&= 4 - 8 \sin^2 t \Rightarrow \frac{d^2T}{dt^2} = -16 \sin t \cos t; \quad \frac{dT}{dt} = 0 \Rightarrow 4 - 8 \sin^2 t = 0 \Rightarrow \sin^2 t = \frac{1}{2} \Rightarrow \sin t = \pm \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}, \\
\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} &\text{ on the interval } 0 \leq t \leq 2\pi; \\
\left. \frac{d^2T}{dt^2} \right|_{t=\frac{\pi}{4}} &= -8 \sin 2 \left(\frac{\pi}{4} \right) = -8 \Rightarrow T \text{ has a maximum at } (x, y) = (2, 1); \\
\left. \frac{d^2T}{dt^2} \right|_{t=\frac{3\pi}{4}} &= -8 \sin 2 \left(\frac{3\pi}{4} \right) = 8 \Rightarrow T \text{ has a minimum at } (x, y) = (-2, 1);
\end{aligned}$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{5\pi}{4}} = -8 \sin 2\left(\frac{5\pi}{4}\right) = -8 \Rightarrow T \text{ has a maximum at } (x, y) = (-2, -1);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{7\pi}{4}} = -8 \sin 2\left(\frac{7\pi}{4}\right) = 8 \Rightarrow T \text{ has a minimum at } (x, y) = (2, -1)$$

(b) $T = xy - 2 \Rightarrow \frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x$ so the extreme values occur at the four points found in part (a):

$$T(2, 1) = T(-2, -1) = 0, \text{ the maximum and } T(-2, 1) = T(2, -1) = -4, \text{ the minimum}$$

51. $G(u, x) = \int_a^u g(t, x) dt$ where $u = f(x) \Rightarrow \frac{dG}{dx} = \frac{\partial G}{\partial u} \frac{du}{dx} + \frac{\partial G}{\partial x} \frac{dx}{dx} = g(u, x)f'(x) + \int_a^u g_x(t, x) dt$; thus

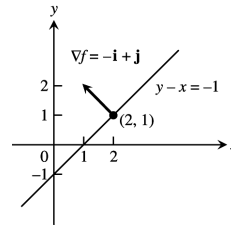
$$F(x) = \int_0^{x^2} \sqrt{t^4 + x^3} dt \Rightarrow F'(x) = \sqrt{(x^2)^4 + x^3} (2x) + \int_0^{x^2} \frac{\partial}{\partial x} \sqrt{t^4 + x^3} dt = 2x\sqrt{x^8 + x^3} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4 + x^3}} dt$$

52. Using the result in Exercise 51, $F(x) = \int_{x^2}^1 \sqrt{t^3 + x^2} dt = - \int_1^{x^2} \sqrt{t^3 + x^2} dt \Rightarrow F'(x)$

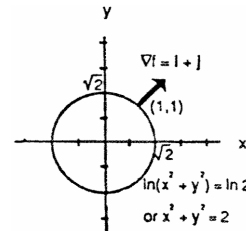
$$= \left[-\sqrt{(x^2)^3 + x^2} x^2 - \int_1^{x^2} \frac{\partial}{\partial x} \sqrt{t^3 + x^2} dt \right] = -x^2 \sqrt{x^6 + x^2} + \int_{x^2}^1 \frac{x}{\sqrt{t^3 + x^2}} dt$$

14.5 DIRECTIONAL DERIVATIVES AND GRADIENT VECTORS

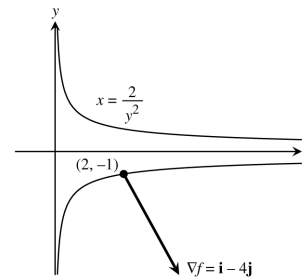
1. $\frac{\partial f}{\partial x} = -1, \frac{\partial f}{\partial y} = 1 \Rightarrow \nabla f = -\mathbf{i} + \mathbf{j}; f(2, 1) = -1$
 $\Rightarrow -1 = y - x$ is the level curve



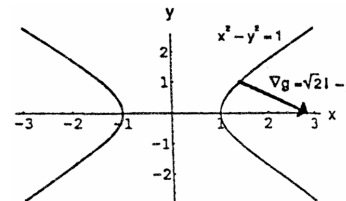
2. $\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2} \Rightarrow \frac{\partial f}{\partial x}(1, 1) = 1; \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$
 $\Rightarrow \frac{\partial f}{\partial y}(1, 1) = 1 \Rightarrow \nabla f = \mathbf{i} + \mathbf{j}; f(1, 1) = \ln 2 \Rightarrow \ln 2$
 $= \ln(x^2 + y^2) \Rightarrow 2 = x^2 + y^2$ is the level curve



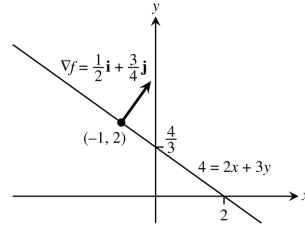
3. $\frac{\partial g}{\partial x} = y^2 \Rightarrow \frac{\partial g}{\partial x}(2, -1) = 1; \frac{\partial g}{\partial y} = 2xy \Rightarrow \frac{\partial g}{\partial y}(2, -1) = -4;$
 $\Rightarrow \nabla g = \mathbf{i} - 4\mathbf{j}; g(2, -1) = 2 \Rightarrow x = \frac{2}{y^2}$ is the level curve



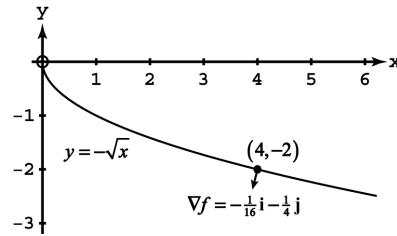
4. $\frac{\partial g}{\partial x} = x \Rightarrow \frac{\partial g}{\partial x}(\sqrt{2}, 1) = \sqrt{2}; \frac{\partial g}{\partial y} = -y$
 $\Rightarrow \frac{\partial g}{\partial y}(\sqrt{2}, 1) = -1 \Rightarrow \nabla g = \sqrt{2}\mathbf{i} - \mathbf{j};$
 $g(\sqrt{2}, 1) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{x^2}{2} - \frac{y^2}{2}$ or $1 = x^2 - y^2$ is the level curve



$$\begin{aligned}
 5. \quad \frac{\partial f}{\partial x} &= \frac{1}{\sqrt{2x+3y}} \Rightarrow \frac{\partial f}{\partial x}(-1, 2) = \frac{1}{2}; \quad \frac{\partial f}{\partial y} = \frac{3}{2\sqrt{2x+3y}} \\
 &\Rightarrow \frac{\partial f}{\partial y}(-1, 2) = \frac{3}{4}; \Rightarrow \nabla f = \frac{1}{2}\mathbf{i} + \frac{3}{4}\mathbf{j}; \quad f(-1, 2) = 2 \\
 &\Rightarrow 4 = 2x + 3y \text{ is the level curve}
 \end{aligned}$$



$$\begin{aligned}
 6. \quad \frac{\partial f}{\partial x} &= \frac{y}{2y^2\sqrt{x}+2x^{3/2}} \Rightarrow \frac{\partial f}{\partial x}(4, -2) = -\frac{1}{16}; \\
 \frac{\partial f}{\partial y} &= -\frac{\sqrt{x}}{2y^2+x} \Rightarrow \frac{\partial f}{\partial y}(4, -2) = -\frac{1}{4} \Rightarrow \nabla f = -\frac{1}{16}\mathbf{i} - \frac{1}{4}\mathbf{j}; \\
 f(4, -2) &= -\frac{\pi}{4} \Rightarrow y = -\sqrt{x} \text{ is the level curve}
 \end{aligned}$$



$$\begin{aligned}
 7. \quad \frac{\partial f}{\partial x} &= 2x + \frac{z}{x} \Rightarrow \frac{\partial f}{\partial x}(1, 1, 1) = 3; \quad \frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(1, 1, 1) = 2; \quad \frac{\partial f}{\partial z} = -4z + \ln x \Rightarrow \frac{\partial f}{\partial z}(1, 1, 1) = -4; \\
 \text{thus } \nabla f &= 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \frac{\partial f}{\partial x} &= -6xz + \frac{z}{x^2z^2+1} \Rightarrow \frac{\partial f}{\partial x}(1, 1, 1) = -\frac{11}{2}; \quad \frac{\partial f}{\partial y} = -6yz \Rightarrow \frac{\partial f}{\partial y}(1, 1, 1) = -6; \quad \frac{\partial f}{\partial z} = 6z^2 - 3(x^2 + y^2) + \frac{x}{x^2z^2+1} \\
 &\Rightarrow \frac{\partial f}{\partial z}(1, 1, 1) = \frac{1}{2}; \text{ thus } \nabla f = -\frac{11}{2}\mathbf{i} - 6\mathbf{j} + \frac{1}{2}\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \frac{\partial f}{\partial x} &= -\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{x} \Rightarrow \frac{\partial f}{\partial x}(-1, 2, -2) = -\frac{26}{27}; \quad \frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{y} \Rightarrow \frac{\partial f}{\partial y}(-1, 2, -2) = \frac{23}{54}; \\
 \frac{\partial f}{\partial z} &= -\frac{z}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{z} \Rightarrow \frac{\partial f}{\partial z}(-1, 2, -2) = -\frac{23}{54}; \text{ thus } \nabla f = -\frac{26}{27}\mathbf{i} + \frac{23}{54}\mathbf{j} - \frac{23}{54}\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \frac{\partial f}{\partial x} &= e^{x+y} \cos z + \frac{y+1}{\sqrt{1-x^2}} \Rightarrow \frac{\partial f}{\partial x}(0, 0, \frac{\pi}{6}) = \frac{\sqrt{3}}{2} + 1; \quad \frac{\partial f}{\partial y} = e^{x+y} \cos z + \sin^{-1} x \Rightarrow \frac{\partial f}{\partial y}(0, 0, \frac{\pi}{6}) = \frac{\sqrt{3}}{2}; \\
 \frac{\partial f}{\partial z} &= -e^{x+y} \sin z \Rightarrow \frac{\partial f}{\partial z}(0, 0, \frac{\pi}{6}) = -\frac{1}{2}; \text{ thus } \nabla f = \left(\frac{\sqrt{3}+2}{2}\right)\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \mathbf{u} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{4\mathbf{i}+3\mathbf{j}}{\sqrt{4^2+3^2}} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}; \quad f_x(x, y) = 2y \Rightarrow f_x(5, 5) = 10; \quad f_y(x, y) = 2x - 6y \Rightarrow f_y(5, 5) = -20 \\
 &\Rightarrow \nabla f = 10\mathbf{i} - 20\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right) = -4
 \end{aligned}$$

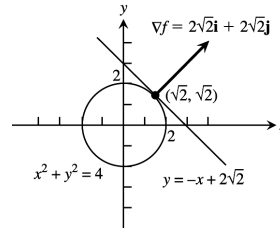
$$\begin{aligned}
 12. \quad \mathbf{u} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}-4\mathbf{j}}{\sqrt{3^2+(-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}; \quad f_x(x, y) = 4x \Rightarrow f_x(-1, 1) = -4; \quad f_y(x, y) = 2y \Rightarrow f_y(-1, 1) = 2 \\
 &\Rightarrow \nabla f = -4\mathbf{i} + 2\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{12}{5} - \frac{8}{5} = -4
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \mathbf{u} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{12\mathbf{i}+5\mathbf{j}}{\sqrt{12^2+5^2}} = \frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}; \quad g_x(x, y) = \frac{y^2+2}{(xy+2)^2} \Rightarrow g_x(1, -1) = 3; \quad g_y(x, y) = -\frac{x^2+2}{(xy+2)^2} \Rightarrow g_y(1, -1) = -3 \\
 &\Rightarrow \nabla g = 3\mathbf{i} - 3\mathbf{j} \Rightarrow (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = \frac{36}{13} - \frac{15}{13} = \frac{21}{13}
 \end{aligned}$$

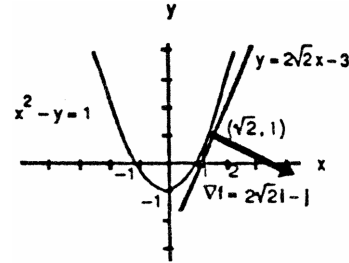
$$\begin{aligned}
 14. \quad \mathbf{u} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}-2\mathbf{j}}{\sqrt{3^2+(-2)^2}} = \frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}; \quad h_x(x, y) = \frac{\left(\frac{-y}{x^2}\right)}{\left(\frac{y}{x}\right)^2+1} + \frac{\left(\frac{y}{x}\right)\sqrt{3}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_x(1, 1) = \frac{1}{2}; \\
 h_y(x, y) &= \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2+1} + \frac{\left(\frac{y}{x}\right)\sqrt{3}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_y(1, 1) = \frac{3}{2} \Rightarrow \nabla h = \frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} \Rightarrow (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{3}{2\sqrt{13}} - \frac{6}{2\sqrt{13}} \\
 &= -\frac{3}{2\sqrt{13}}
 \end{aligned}$$

15. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{3^2+6^2+(-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$; $f_x(x, y, z) = y + z \Rightarrow f_x(1, -1, 2) = 1$; $f_y(x, y, z) = x + z \Rightarrow f_y(1, -1, 2) = 3$; $f_z(x, y, z) = y + x \Rightarrow f_z(1, -1, 2) = 0 \Rightarrow \nabla f = \mathbf{i} + 3\mathbf{j} \Rightarrow (\mathbf{D}_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = \frac{3}{7} + \frac{18}{7} = 3$
16. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$; $f_x(x, y, z) = 2x \Rightarrow f_x(1, 1, 1) = 2$; $f_y(x, y, z) = 4y \Rightarrow f_y(1, 1, 1) = 4$; $f_z(x, y, z) = -6z \Rightarrow f_z(1, 1, 1) = -6 \Rightarrow \nabla f = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} \Rightarrow (\mathbf{D}_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 2\left(\frac{1}{\sqrt{3}}\right) + 4\left(\frac{1}{\sqrt{3}}\right) - 6\left(\frac{1}{\sqrt{3}}\right) = 0$
17. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{2\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+1^2+(-2)^2}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$; $g_x(x, y, z) = 3e^x \cos yz \Rightarrow g_x(0, 0, 0) = 3$; $g_y(x, y, z) = -3ze^x \sin yz \Rightarrow g_y(0, 0, 0) = 0$; $g_z(x, y, z) = -3ye^x \sin yz \Rightarrow g_z(0, 0, 0) = 0 \Rightarrow \nabla g = 3\mathbf{i} \Rightarrow (\mathbf{D}_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = 2$
18. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$; $h_x(x, y, z) = -y \sin xy + \frac{1}{x} \Rightarrow h_x(1, 0, \frac{1}{2}) = 1$; $h_y(x, y, z) = -x \sin xy + ze^{yz} \Rightarrow h_y(1, 0, \frac{1}{2}) = \frac{1}{2}$; $h_z(x, y, z) = ye^{yz} + \frac{1}{z} \Rightarrow h_z(1, 0, \frac{1}{2}) = 2 \Rightarrow \nabla h = \mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k} \Rightarrow (\mathbf{D}_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{1}{3} + \frac{1}{3} + \frac{4}{3} = 2$
19. $\nabla f = (2x + y)\mathbf{i} + (x + 2y)\mathbf{j} \Rightarrow \nabla f(-1, 1) = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{-\mathbf{i}+\mathbf{j}}{\sqrt{(-1)^2+1^2}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$; $(\mathbf{D}_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = \sqrt{2}$ and $(\mathbf{D}_{-\mathbf{u}}f)_{P_0} = -\sqrt{2}$
20. $\nabla f = (2xy + ye^{xy} \sin y)\mathbf{i} + (x^2 + xe^{xy} \sin y + e^{xy} \cos y)\mathbf{j} \Rightarrow \nabla f(1, 0) = 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = \mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\mathbf{j}$; $(\mathbf{D}_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 2$ and $(\mathbf{D}_{-\mathbf{u}}f)_{P_0} = -2$
21. $\nabla f = \frac{1}{y}\mathbf{i} - \left(\frac{x}{y^2} + z\right)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla f(4, 1, 1) = \mathbf{i} - 5\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{i}-5\mathbf{j}-\mathbf{k}}{\sqrt{1^2+(-5)^2+(-1)^2}} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}$; f increases most rapidly in the direction of $\mathbf{u} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{3\sqrt{3}}\mathbf{i} + \frac{5}{3\sqrt{3}}\mathbf{j} + \frac{1}{3\sqrt{3}}\mathbf{k}$; $(\mathbf{D}_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 3\sqrt{3}$ and $(\mathbf{D}_{-\mathbf{u}}f)_{P_0} = -3\sqrt{3}$
22. $\nabla g = e^y\mathbf{i} + xe^y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla g(1, \ln 2, \frac{1}{2}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla g}{|\nabla g|} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$; g increases most rapidly in the direction $\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$; $(\mathbf{D}_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = |\nabla g| = 3$ and $(\mathbf{D}_{-\mathbf{u}}g)_{P_0} = -3$
23. $\nabla f = \left(\frac{1}{x} + \frac{1}{x}\right)\mathbf{i} + \left(\frac{1}{y} + \frac{1}{y}\right)\mathbf{j} + \left(\frac{1}{z} + \frac{1}{z}\right)\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$; f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$; $(\mathbf{D}_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 2\sqrt{3}$ and $(\mathbf{D}_{-\mathbf{u}}f)_{P_0} = -2\sqrt{3}$
24. $\nabla h = \left(\frac{2x}{x^2+y^2-1}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2-1} + 1\right)\mathbf{j} + 6\mathbf{k} \Rightarrow \nabla h(1, 1, 0) = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla h}{|\nabla h|} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$; h increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$; $(\mathbf{D}_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = |\nabla h| = 7$ and $(\mathbf{D}_{-\mathbf{u}}h)_{P_0} = -7$

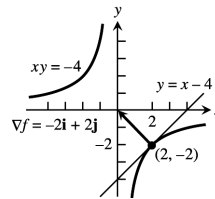
$$\begin{aligned}
 25. \quad \nabla f &= 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f(\sqrt{2}, \sqrt{2}) = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} \\
 &\Rightarrow \text{Tangent line: } 2\sqrt{2}(x - \sqrt{2}) + 2\sqrt{2}(y - \sqrt{2}) = 0 \\
 &\Rightarrow \sqrt{2}x + \sqrt{2}y = 4
 \end{aligned}$$



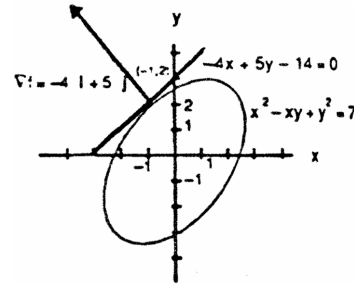
$$\begin{aligned}
 26. \quad \nabla f &= 2x\mathbf{i} - \mathbf{j} \Rightarrow \nabla f(\sqrt{2}, 1) = 2\sqrt{2}\mathbf{i} - \mathbf{j} \\
 &\Rightarrow \text{Tangent line: } 2\sqrt{2}(x - \sqrt{2}) - (y - 1) = 0 \\
 &\Rightarrow y = 2\sqrt{2}x - 3
 \end{aligned}$$



$$\begin{aligned}
 27. \quad \nabla f &= y\mathbf{i} + x\mathbf{j} \Rightarrow \nabla f(2, -2) = -2\mathbf{i} + 2\mathbf{j} \\
 &\Rightarrow \text{Tangent line: } -2(x - 2) + 2(y + 2) = 0 \\
 &\Rightarrow y = x - 4
 \end{aligned}$$



$$\begin{aligned}
 28. \quad \nabla f &= (2x - y)\mathbf{i} + (2y - x)\mathbf{j} \Rightarrow \nabla f(-1, 2) = -4\mathbf{i} + 5\mathbf{j} \\
 &\Rightarrow \text{Tangent line: } -4(x + 1) + 5(y - 2) = 0 \\
 &\Rightarrow -4x + 5y - 14 = 0
 \end{aligned}$$



$$\begin{aligned}
 29. \quad \nabla f &= (2x - y)\mathbf{i} + (-x + 2y - 1)\mathbf{j} \\
 (a) \quad \nabla f(1, -1) &= 3\mathbf{i} - 4\mathbf{j} \Rightarrow |\nabla f(1, -1)| = 5 \Rightarrow D_{\mathbf{u}}f(1, -1) = 5 \text{ in the direction of } \mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} \\
 (b) \quad -\nabla f(1, -1) &= -3\mathbf{i} + 4\mathbf{j} \Rightarrow |\nabla f(1, -1)| = 5 \Rightarrow D_{\mathbf{u}}f(1, -1) = -5 \text{ in the direction of } \mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \\
 (c) \quad D_{\mathbf{u}}f(1, -1) &= 0 \text{ in the direction of } \mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j} \text{ or } \mathbf{u} = -\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j} \\
 (d) \quad \text{Let } \mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1; D_{\mathbf{u}}f(1, -1) = \nabla f(1, -1) \cdot \mathbf{u} = (3\mathbf{i} - 4\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\
 &= 3u_1 - 4u_2 = 4 \Rightarrow u_2 = \frac{3}{4}u_1 - 1 \Rightarrow u_1^2 + \left(\frac{3}{4}u_1 - 1\right)^2 = 1 \Rightarrow \frac{25}{16}u_1^2 - \frac{3}{2}u_1 = 0 \Rightarrow u_1 = 0 \text{ or } u_1 = \frac{24}{25}; \\
 &u_1 = 0 \Rightarrow u_2 = -1 \Rightarrow \mathbf{u} = -\mathbf{j}, \text{ or } u_1 = \frac{24}{25} \Rightarrow u_2 = -\frac{7}{25} \Rightarrow \mathbf{u} = \frac{24}{25}\mathbf{i} - \frac{7}{25}\mathbf{j} \\
 (e) \quad \text{Let } \mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1; D_{\mathbf{u}}f(1, -1) = \nabla f(1, -1) \cdot \mathbf{u} = (3\mathbf{i} - 4\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\
 &= 3u_1 - 4u_2 = -3 \Rightarrow u_1 = \frac{4}{3}u_2 - 1 \Rightarrow \left(\frac{4}{3}u_2 - 1\right)^2 + u_2^2 = 1 \Rightarrow \frac{25}{9}u_2^2 - \frac{8}{3}u_2 = 0 \Rightarrow u_2 = 0 \text{ or } u_2 = \frac{24}{25}; \\
 &u_2 = 0 \Rightarrow u_1 = -1 \Rightarrow \mathbf{u} = -\mathbf{i}, \text{ or } u_2 = \frac{24}{25} \Rightarrow u_1 = \frac{7}{25} \Rightarrow \mathbf{u} = \frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \nabla f &= \frac{2y}{(x+y)^2}\mathbf{i} - \frac{2x}{(x+y)^2}\mathbf{j} \\
 (a) \quad \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) &= 3\mathbf{i} + \mathbf{j} \Rightarrow |\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right)| = \sqrt{10} \Rightarrow D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \sqrt{10} \text{ in the direction of } \mathbf{u} = \frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j} \\
 (b) \quad -\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) &= -3\mathbf{i} - \mathbf{j} \Rightarrow |\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right)| = \sqrt{10} \Rightarrow D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = -\sqrt{10} \text{ in the direction of } \mathbf{u} = -\frac{3}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}
 \end{aligned}$$

(c) $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$ in the direction of $\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}$ or $\mathbf{u} = -\frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

(d) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \mathbf{u} = (3\mathbf{i} + \mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
 $= 3u_1 + u_2 = -2 \Rightarrow u_2 = -3u_1 - 2 \Rightarrow u_1^2 + (-3u_1 - 2)^2 = 1 \Rightarrow 10u_1^2 + 12u_1 + 3 = 0 \Rightarrow u_1 = \frac{-6 \pm \sqrt{6}}{10}$
 $u_1 = \frac{-6 + \sqrt{6}}{10} \Rightarrow u_2 = \frac{-2 - 3\sqrt{6}}{10} \Rightarrow \mathbf{u} = \frac{-6 + \sqrt{6}}{10}\mathbf{i} + \frac{-2 - 3\sqrt{6}}{10}\mathbf{j}$, or $u_1 = \frac{-6 - \sqrt{6}}{10} \Rightarrow u_2 = \frac{-2 + 3\sqrt{6}}{10}$
 $\Rightarrow \mathbf{u} = \frac{-6 - \sqrt{6}}{10}\mathbf{i} + \frac{-2 + 3\sqrt{6}}{10}\mathbf{j}$

(e) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \mathbf{u} = (3\mathbf{i} + \mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
 $= 3u_1 + u_2 = 1 \Rightarrow u_2 = 1 - 3u_1 \Rightarrow u_1^2 + (1 - 3u_1)^2 = 1 \Rightarrow 10u_1^2 - 6u_1 = 0 \Rightarrow u_1 = 0$ or $u_1 = \frac{3}{5}$;
 $u_1 = 0 \Rightarrow u_2 = 1 \Rightarrow \mathbf{u} = \mathbf{j}$, or $u_1 = \frac{3}{5} \Rightarrow u_2 = -\frac{4}{5} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

31. $\nabla f = y\mathbf{i} + (x + 2y)\mathbf{j} \Rightarrow \nabla f(3, 2) = 2\mathbf{i} + 7\mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = 7\mathbf{i} - 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7\mathbf{i} - 2\mathbf{j}}{\sqrt{7^2 + (-2)^2}}$
 $= \frac{7}{\sqrt{53}}\mathbf{i} - \frac{2}{\sqrt{53}}\mathbf{j}$ and $-\mathbf{u} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j}$ are the directions where the derivative is zero

32. $\nabla f = \frac{4xy^2}{(x^2 + y^2)^2}\mathbf{i} - \frac{4x^2y}{(x^2 + y^2)^2}\mathbf{j} \Rightarrow \nabla f(1, 1) = \mathbf{i} - \mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = \mathbf{i} + \mathbf{j}$
 $\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ are the directions where the derivative is zero

33. $\nabla f = (2x - 3y)\mathbf{i} + (-3x + 8y)\mathbf{j} \Rightarrow \nabla f(1, 2) = -4\mathbf{i} + 13\mathbf{j} \Rightarrow |\nabla f(1, 2)| = \sqrt{(-4)^2 + (13)^2} = \sqrt{185}$; no, the maximum rate of change is $\sqrt{185} < 14$

34. $\nabla T = 2y\mathbf{i} + (2x - z)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla T(1, -1, 1) = -2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla T(1, -1, 1)| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$; no, the minimum rate of change is $-\sqrt{6} > -3$

35. $\nabla f = f_x(1, 2)\mathbf{i} + f_y(1, 2)\mathbf{j}$ and $\mathbf{u}_1 = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)(1, 2) = f_x(1, 2)\left(\frac{1}{\sqrt{2}}\right) + f_y(1, 2)\left(\frac{1}{\sqrt{2}}\right)$
 $= 2\sqrt{2} \Rightarrow f_x(1, 2) + f_y(1, 2) = 4$; $\mathbf{u}_2 = -\mathbf{j} \Rightarrow (D_{\mathbf{u}_2}f)(1, 2) = f_x(1, 2)(0) + f_y(1, 2)(-1) = -3 \Rightarrow -f_y(1, 2) = -3$
 $\Rightarrow f_y(1, 2) = 3$; then $f_x(1, 2) + 3 = 4 \Rightarrow f_x(1, 2) = 1$; thus $\nabla f(1, 2) = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{i} - 2\mathbf{j}}{\sqrt{(-1)^2 + (-2)^2}}$
 $= -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$

36. (a) $(D_{\mathbf{u}}f)_P = 2\sqrt{3} \Rightarrow |\nabla f| = 2\sqrt{3}$; $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$; thus $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$
 $\Rightarrow \nabla f = |\nabla f|\mathbf{u} \Rightarrow \nabla f = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

(b) $\mathbf{v} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 2\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) - 2(0) = 2\sqrt{2}$

37. The directional derivative is the scalar component. With ∇f evaluated at P_0 , the scalar component of ∇f in the direction of $\mathbf{u}$ is $\nabla f \cdot \mathbf{u} = (D_{\mathbf{u}}f)_{P_0}$.

38. $D_{\mathbf{i}}f = \nabla f \cdot \mathbf{i} = (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) \cdot \mathbf{i} = f_x$; similarly, $D_{\mathbf{j}}f = \nabla f \cdot \mathbf{j} = f_y$ and $D_{\mathbf{k}}f = \nabla f \cdot \mathbf{k} = f_z$

39. If (x, y) is a point on the line, then $\mathbf{T}(x, y) = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j}$ is a vector parallel to the line $\Rightarrow \mathbf{T} \cdot \mathbf{N} = 0$
 $\Rightarrow A(x - x_0) + B(y - y_0) = 0$, as claimed.

40. (a) $\nabla(kf) = \frac{\partial(kf)}{\partial x}\mathbf{i} + \frac{\partial(kf)}{\partial y}\mathbf{j} + \frac{\partial(kf)}{\partial z}\mathbf{k} = k\left(\frac{\partial f}{\partial x}\right)\mathbf{i} + k\left(\frac{\partial f}{\partial y}\right)\mathbf{j} + k\left(\frac{\partial f}{\partial z}\right)\mathbf{k} = k\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = k\nabla f$

$$\begin{aligned}
\text{(b)} \quad \nabla(f+g) &= \frac{\partial(f+g)}{\partial x} \mathbf{i} + \frac{\partial(f+g)}{\partial y} \mathbf{j} + \frac{\partial(f+g)}{\partial z} \mathbf{k} = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \right) \mathbf{i} + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} \right) \mathbf{j} + \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} \right) \mathbf{k} \\
&= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} + \frac{\partial g}{\partial z} \mathbf{k} = \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) + \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right) = \nabla f + \nabla g \\
\text{(c)} \quad \nabla(f-g) &= \nabla f - \nabla g \text{ (Substitute } -g \text{ for } g \text{ in part (b) above)} \\
\text{(d)} \quad \nabla(fg) &= \frac{\partial(fg)}{\partial x} \mathbf{i} + \frac{\partial(fg)}{\partial y} \mathbf{j} + \frac{\partial(fg)}{\partial z} \mathbf{k} = \left(\frac{\partial f}{\partial x} g + \frac{\partial g}{\partial x} f \right) \mathbf{i} + \left(\frac{\partial f}{\partial y} g + \frac{\partial g}{\partial y} f \right) \mathbf{j} + \left(\frac{\partial f}{\partial z} g + \frac{\partial g}{\partial z} f \right) \mathbf{k} \\
&= \left(\frac{\partial f}{\partial x} g \right) \mathbf{i} + \left(\frac{\partial g}{\partial x} f \right) \mathbf{i} + \left(\frac{\partial f}{\partial y} g \right) \mathbf{j} + \left(\frac{\partial g}{\partial y} f \right) \mathbf{j} + \left(\frac{\partial f}{\partial z} g \right) \mathbf{k} + \left(\frac{\partial g}{\partial z} f \right) \mathbf{k} \\
&= f \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right) + g \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) = f \nabla g + g \nabla f \\
\text{(e)} \quad \nabla \left(\frac{f}{g} \right) &= \frac{\partial \left(\frac{f}{g} \right)}{\partial x} \mathbf{i} + \frac{\partial \left(\frac{f}{g} \right)}{\partial y} \mathbf{j} + \frac{\partial \left(\frac{f}{g} \right)}{\partial z} \mathbf{k} = \left(\frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2} \right) \mathbf{i} + \left(\frac{g \frac{\partial f}{\partial y} - f \frac{\partial g}{\partial y}}{g^2} \right) \mathbf{j} + \left(\frac{g \frac{\partial f}{\partial z} - f \frac{\partial g}{\partial z}}{g^2} \right) \mathbf{k} \\
&= \left(\frac{g \frac{\partial f}{\partial x} \mathbf{i} + g \frac{\partial f}{\partial y} \mathbf{j} + g \frac{\partial f}{\partial z} \mathbf{k}}{g^2} \right) - \left(\frac{f \frac{\partial g}{\partial x} \mathbf{i} + f \frac{\partial g}{\partial y} \mathbf{j} + f \frac{\partial g}{\partial z} \mathbf{k}}{g^2} \right) = \frac{g \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right)}{g^2} - \frac{f \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right)}{g^2} \\
&= \frac{g \nabla f}{g^2} - \frac{f \nabla g}{g^2} = \frac{g \nabla f - f \nabla g}{g^2}
\end{aligned}$$

14.6 TANGENT PLANES AND DIFFERENTIALS

- (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent plane: $2(x-1) + 2(y-1) + 2(z-1) = 0$
 $\Rightarrow x + y + z = 3$;

(b) Normal line: $x = 1 + 2t, y = 1 + 2t, z = 1 + 2t$
- (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(3, 5, -4) = 6\mathbf{i} + 10\mathbf{j} + 8\mathbf{k} \Rightarrow$ Tangent plane: $6(x-3) + 10(y-5) + 8(z+4) = 0$
 $\Rightarrow 3x + 5y + 4z = 18$;

(b) Normal line: $x = 3 + 6t, y = 5 + 10t, z = -4 + 8t$
- (a) $\nabla f = -2x\mathbf{i} + 2z\mathbf{k} \Rightarrow \nabla f(2, 0, 2) = -4\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent plane: $-4(x-2) + 2(z-2) = 0$
 $\Rightarrow -4x + 2z + 4 = 0 \Rightarrow -2x + z + 2 = 0$;

(b) Normal line: $x = 2 - 4t, y = 0, z = 2 + 2t$
- (a) $\nabla f = (2x + 2y)\mathbf{i} + (2x - 2y)\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1, -1, 3) = 4\mathbf{j} + 6\mathbf{k} \Rightarrow$ Tangent plane: $4(y+1) + 6(z-3) = 0$
 $\Rightarrow 2y + 3z = 7$;

(b) Normal line: $x = 1, y = -1 + 4t, z = 3 + 6t$
- (a) $\nabla f = (-\pi \sin \pi x - 2xy + ze^{xz})\mathbf{i} + (-x^2 + z)\mathbf{j} + (xe^{xz} + y)\mathbf{k} \Rightarrow \nabla f(0, 1, 2) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane:
 $2(x-0) + 2(y-1) + 1(z-2) = 0 \Rightarrow 2x + 2y + z - 4 = 0$;

(b) Normal line: $x = 2t, y = 1 + 2t, z = 2 + t$
- (a) $\nabla f = (2x - y)\mathbf{i} - (x + 2y)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1, 1, -1) = \mathbf{i} - 3\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane:
 $1(x-1) - 3(y-1) - 1(z+1) = 0 \Rightarrow x - 3y - z = -1$;

(b) Normal line: $x = 1 + t, y = 1 - 3t, z = -1 - t$
- (a) $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k}$ for all points $\Rightarrow \nabla f(0, 1, 0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $1(x-0) + 1(y-1) + 1(z-0) = 0$
 $\Rightarrow x + y + z - 1 = 0$;

(b) Normal line: $x = t, y = 1 + t, z = t$
- (a) $\nabla f = (2x - 2y - 1)\mathbf{i} + (2y - 2x + 3)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(2, -3, 18) = 9\mathbf{i} - 7\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane:
 $9(x-2) - 7(y+3) - 1(z-18) = 0 \Rightarrow 9x - 7y - z = 21$;

(b) Normal line: $x = 2 + 9t, y = -3 - 7t, z = 18 - t$

9. $z = f(x, y) = \ln(x^2 + y^2) \Rightarrow f_x(x, y) = \frac{2x}{x^2 + y^2}$ and $f_y(x, y) = \frac{2y}{x^2 + y^2} \Rightarrow f_x(1, 0) = 2$ and $f_y(1, 0) = 0 \Rightarrow$ from Eq. (4) the tangent plane at $(1, 0, 0)$ is $2(x - 1) - z = 0$ or $2x - z - 2 = 0$

10. $z = f(x, y) = e^{-(x^2 + y^2)} \Rightarrow f_x(x, y) = -2xe^{-(x^2 + y^2)}$ and $f_y(x, y) = -2ye^{-(x^2 + y^2)} \Rightarrow f_x(0, 0) = 0$ and $f_y(0, 0) = 0 \Rightarrow$ from Eq. (4) the tangent plane at $(0, 0, 1)$ is $z - 1 = 0$ or $z = 1$

11. $z = f(x, y) = \sqrt{y - x} \Rightarrow f_x(x, y) = -\frac{1}{2}(y - x)^{-1/2}$ and $f_y(x, y) = \frac{1}{2}(y - x)^{-1/2} \Rightarrow f_x(1, 2) = -\frac{1}{2}$ and $f_y(1, 2) = \frac{1}{2} \Rightarrow$ from Eq. (4) the tangent plane at $(1, 2, 1)$ is $-\frac{1}{2}(x - 1) + \frac{1}{2}(y - 2) - (z - 1) = 0 \Rightarrow x - y + 2z - 1 = 0$

12. $z = f(x, y) = 4x^2 + y^2 \Rightarrow f_x(x, y) = 8x$ and $f_y(x, y) = 2y \Rightarrow f_x(1, 1) = 8$ and $f_y(1, 1) = 2 \Rightarrow$ from Eq. (4) the tangent plane at $(1, 1, 5)$ is $8(x - 1) + 2(y - 1) - (z - 5) = 0$ or $8x + 2y - z - 5 = 0$

13. $\nabla f = \mathbf{i} + 2y\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{i}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
 $\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1, y = 1 + 2t, z = 1 - 2t$

14. $\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\nabla g = 2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k} \Rightarrow \nabla g(1, 1, 1) = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$;
 $\Rightarrow \mathbf{v} = \nabla f \times \nabla g \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1 + 2t, y = 1 - 4t, z = 1 + 2t$

15. $\nabla f = 2x\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f(1, 1, \frac{1}{2}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
 $\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1 - 2t, y = 1, z = \frac{1}{2} + 2t$

16. $\nabla f = \mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f(\frac{1}{2}, 1, \frac{1}{2}) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
 $\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{k} \Rightarrow$ Tangent line: $x = \frac{1}{2} - t, y = 1, z = \frac{1}{2} + t$

17. $\nabla f = (3x^2 + 6xy^2 + 4y)\mathbf{i} + (6x^2y + 3y^2 + 4x)\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(1, 1, 3) = 13\mathbf{i} + 13\mathbf{j} - 6\mathbf{k}$; $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$
 $\Rightarrow \nabla g(1, 1, 3) = 2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$; $\mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} = 90\mathbf{i} - 90\mathbf{j} \Rightarrow$ Tangent line:
 $x = 1 + 90t, y = 1 - 90t, z = 3$

18. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f(\sqrt{2}, \sqrt{2}, 4) = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j}$; $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla g(\sqrt{2}, \sqrt{2}, 4)$
 $= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \mathbf{k}$; $\mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} = -2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} \Rightarrow$ Tangent line:
 $x = \sqrt{2} - 2\sqrt{2}t, y = \sqrt{2} + 2\sqrt{2}t, z = 4$

19. $\nabla f = \left(\frac{x}{x^2 + y^2 + z^2}\right)\mathbf{i} + \left(\frac{y}{x^2 + y^2 + z^2}\right)\mathbf{j} + \left(\frac{z}{x^2 + y^2 + z^2}\right)\mathbf{k} \Rightarrow \nabla f(3, 4, 12) = \frac{3}{169}\mathbf{i} + \frac{4}{169}\mathbf{j} + \frac{12}{169}\mathbf{k}$;
 $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{3^2 + 6^2 + (-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{9}{1183}$ and $df = (\nabla f \cdot \mathbf{u}) ds = \left(\frac{9}{1183}\right)(0.1) \approx 0.0008$

20. $\nabla f = (e^x \cos yz)\mathbf{i} - (ze^x \sin yz)\mathbf{j} - (ye^x \sin yz)\mathbf{k} \Rightarrow \nabla f(0, 0, 0) = \mathbf{i}; \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}}{\sqrt{2^2 + 2^2 + (-2)^2}}$
 $= \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{3}}$ and $df = (\nabla f \cdot \mathbf{u}) ds = \frac{1}{\sqrt{3}}(0.1) \approx 0.0577$
21. $\nabla g = (1 + \cos z)\mathbf{i} + (1 - \sin z)\mathbf{j} + (-x \sin z - y \cos z)\mathbf{k} \Rightarrow \nabla g(2, -1, 0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}; \mathbf{A} = \vec{P_0 P_1} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
 $\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{\sqrt{(-2)^2 + 2^2 + 2^2}} = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \nabla g \cdot \mathbf{u} = 0$ and $dg = (\nabla g \cdot \mathbf{u}) ds = (0)(0.2) = 0$
22. $\nabla h = [-\pi y \sin(\pi xy) + z^2]\mathbf{i} - [\pi x \sin(\pi xy)]\mathbf{j} + 2xz\mathbf{k} \Rightarrow \nabla h(-1, -1, -1) = (\pi \sin \pi + 1)\mathbf{i} + (\pi \sin \pi)\mathbf{j} + 2\mathbf{k}$
 $= \mathbf{i} + 2\mathbf{k}; \mathbf{v} = \vec{P_0 P_1} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ where $P_1 = (0, 0, 0) \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
 $\Rightarrow \nabla h \cdot \mathbf{u} = \frac{3}{\sqrt{3}} = \sqrt{3}$ and $dh = (\nabla h \cdot \mathbf{u}) ds = \sqrt{3}(0.1) \approx 0.1732$
23. (a) The unit tangent vector at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the direction of motion is $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$;
 $\nabla T = (\sin 2y)\mathbf{i} + (2x \cos 2y)\mathbf{j} \Rightarrow \nabla T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (\sin \sqrt{3})\mathbf{i} + (\cos \sqrt{3})\mathbf{j} \Rightarrow D_u T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \nabla T \cdot \mathbf{u}$
 $= \frac{\sqrt{3}}{2} \sin \sqrt{3} - \frac{1}{2} \cos \sqrt{3} \approx 0.935^\circ \text{ C/ft}$
- (b) $\mathbf{r}(t) = (\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j} \Rightarrow \mathbf{v}(t) = (2 \cos 2t)\mathbf{i} - (2 \sin 2t)\mathbf{j}$ and $|\mathbf{v}| = 2$; $\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt}$
 $= \nabla T \cdot \mathbf{v} = \left(\nabla T \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\right) |\mathbf{v}| = (D_u T) |\mathbf{v}|$, where $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$; at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we have $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ from part (a)
 $\Rightarrow \frac{dT}{dt} = \left(\frac{\sqrt{3}}{2} \sin \sqrt{3} - \frac{1}{2} \cos \sqrt{3}\right) \cdot 2 = \sqrt{3} \sin \sqrt{3} - \cos \sqrt{3} \approx 1.87^\circ \text{ C/sec}$
24. (a) $\nabla T = (4x - yz)\mathbf{i} - xz\mathbf{j} - xy\mathbf{k} \Rightarrow \nabla T(8, 6, -4) = 56\mathbf{i} + 32\mathbf{j} - 48\mathbf{k}; \mathbf{r}(t) = 2t^2\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k} \Rightarrow$ the particle is
at the point $P(8, 6, -4)$ when $t = 2$; $\mathbf{v}(t) = 4t\mathbf{i} + 3\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{v}(2) = 8\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \frac{8}{\sqrt{89}}\mathbf{i} + \frac{3}{\sqrt{89}}\mathbf{j} - \frac{4}{\sqrt{89}}\mathbf{k} \Rightarrow D_u T(8, 6, -4) = \nabla T \cdot \mathbf{u} = \frac{1}{\sqrt{89}} [56 \cdot 8 + 32 \cdot 3 - 48 \cdot (-4)] = \frac{736}{\sqrt{89}}^\circ \text{ C/m}$
- (b) $\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} = \nabla T \cdot \mathbf{v} = (\nabla T \cdot \mathbf{u}) |\mathbf{v}| \Rightarrow$ at $t = 2$, $\frac{dT}{dt} = D_u T|_{t=2} \mathbf{v}(2) = \left(\frac{736}{\sqrt{89}}\right) \sqrt{89} = 736^\circ \text{ C/sec}$
25. (a) $f(0, 0) = 1, f_x(x, y) = 2x \Rightarrow f_x(0, 0) = 0, f_y(x, y) = 2y \Rightarrow f_y(0, 0) = 0 \Rightarrow L(x, y) = 1 + 0(x - 0) + 0(y - 0) = 1$
(b) $f(1, 1) = 3, f_x(1, 1) = 2, f_y(1, 1) = 2 \Rightarrow L(x, y) = 3 + 2(x - 1) + 2(y - 1) = 2x + 2y - 1$
26. (a) $f(0, 0) = 4, f_x(x, y) = 2(x + y + 2) \Rightarrow f_x(0, 0) = 4, f_y(x, y) = 2(x + y + 2) \Rightarrow f_y(0, 0) = 4$
 $\Rightarrow L(x, y) = 4 + 4(x - 0) + 4(y - 0) = 4x + 4y + 4$
- (b) $f(1, 2) = 25, f_x(1, 2) = 10, f_y(1, 2) = 10 \Rightarrow L(x, y) = 25 + 10(x - 1) + 10(y - 2) = 10x + 10y - 5$
27. (a) $f(0, 0) = 5, f_x(x, y) = 3$ for all $(x, y), f_y(x, y) = -4$ for all $(x, y) \Rightarrow L(x, y) = 5 + 3(x - 0) - 4(y - 0) = 3x - 4y + 5$
(b) $f(1, 1) = 4, f_x(1, 1) = 3, f_y(1, 1) = -4 \Rightarrow L(x, y) = 4 + 3(x - 1) - 4(y - 1) = 3x - 4y + 5$
28. (a) $f(1, 1) = 1, f_x(x, y) = 3x^2y^4 \Rightarrow f_x(1, 1) = 3, f_y(x, y) = 4x^3y^3 \Rightarrow f_y(1, 1) = 4$
 $\Rightarrow L(x, y) = 1 + 3(x - 1) + 4(y - 1) = 3x + 4y - 6$
- (b) $f(0, 0) = 0, f_x(0, 0) = 0, f_y(0, 0) = 0 \Rightarrow L(x, y) = 0$
29. (a) $f(0, 0) = 1, f_x(x, y) = e^x \cos y \Rightarrow f_x(0, 0) = 1, f_y(x, y) = -e^x \sin y \Rightarrow f_y(0, 0) = 0$
 $\Rightarrow L(x, y) = 1 + 1(x - 0) + 0(y - 0) = x + 1$
- (b) $f\left(0, \frac{\pi}{2}\right) = 0, f_x\left(0, \frac{\pi}{2}\right) = 0, f_y\left(0, \frac{\pi}{2}\right) = -1 \Rightarrow L(x, y) = 0 + 0(x - 0) - 1\left(y - \frac{\pi}{2}\right) = -y + \frac{\pi}{2}$

30. (a) $f(0, 0) = 1, f_x(x, y) = -e^{2y-x} \Rightarrow f_x(0, 0) = -1, f_y(x, y) = 2e^{2y-x} \Rightarrow f_y(0, 0) = 2$
 $\Rightarrow L(x, y) = 1 - 1(x - 0) + 2(y - 0) = -x + 2y + 1$
 (b) $f(1, 2) = e^3, f_x(1, 2) = -e^3, f_y(1, 2) = 2e^3 \Rightarrow L(x, y) = e^3 - e^3(x - 1) + 2e^3(y - 2) = -e^3x + 2e^3y - 2e^3$
31. (a) $W(20, 25) = 11^\circ\text{F}; W(30, -10) = -39^\circ\text{F}; W(15, 15) = 0^\circ\text{F}$
 (b) $W(10, -40) = -65.5^\circ\text{F}; W(50, -40) = -88^\circ\text{F}; W(60, 30) = 10.2^\circ\text{F};$
 (c) $W(25, 5) = -17.4088^\circ\text{F}; \frac{\partial W}{\partial V} = -\frac{5.72}{\sqrt{0.84}} + \frac{0.0684t}{\sqrt{0.84}} \Rightarrow \frac{\partial W}{\partial V}(25, 5) = -0.36; \frac{\partial W}{\partial T} = 0.6215 + 0.4275v^{0.16}$
 $\Rightarrow \frac{\partial W}{\partial T}(25, 5) = 1.3370 \Rightarrow L(V, T) = -17.4088 - 0.36(V - 25) + 1.337(T - 5) = 1.337T - 0.36V - 15.0938$
 (d) i) $W(24, 6) \approx L(24, 6) = -15.7118 \approx -15.7^\circ\text{F}$
 ii) $W(27, 2) \approx L(27, 2) = -22.1398 \approx -22.1^\circ\text{F}$
 ii) $W(5, -10) \approx L(5, -10) = -30.2638 \approx -30.2^\circ\text{F}$ This value is very different because the point $(5, -10)$ is not close to the point $(25, 5)$.
32. $W(50, -20) = -59.5298^\circ\text{F}; \frac{\partial W}{\partial V} = -\frac{5.72}{\sqrt{0.84}} + \frac{0.0684t}{\sqrt{0.84}} \Rightarrow \frac{\partial W}{\partial V}(50, -20) = -0.2651; \frac{\partial W}{\partial T} = 0.6215 + 0.4275v^{0.16}$
 $\Rightarrow \frac{\partial W}{\partial T}(50, -20) = 1.4209 \Rightarrow L(V, T) = -59.5298 - 0.2651(V - 50) + 1.4209(T + 20)$
 $= 1.4209T - 0.2651V - 17.8568$
 (a) $W(49, -22) \approx L(49, -22) = -62.1065 \approx -62.1^\circ\text{F}$
 (b) $W(53, -19) \approx L(53, -19) = -58.9042 \approx -58.9^\circ\text{F}$
 (c) $W(60, -30) \approx L(60, -30) = -76.3898 \approx -76.4^\circ\text{F}$
33. $f(2, 1) = 3, f_x(x, y) = 2x - 3y \Rightarrow f_x(2, 1) = 1, f_y(x, y) = -3x \Rightarrow f_y(2, 1) = -6 \Rightarrow L(x, y) = 3 + 1(x - 2) - 6(y - 1)$
 $= 7 + x - 6y; f_{xx}(x, y) = 2, f_{yy}(x, y) = 0, f_{xy}(x, y) = -3 \Rightarrow M = 3; \text{ thus } |E(x, y)| \leq \left(\frac{1}{2}\right)(3)(|x - 2| + |y - 1|)^2$
 $\leq \left(\frac{3}{2}\right)(0.1 + 0.1)^2 = 0.06$
34. $f(2, 2) = 11, f_x(x, y) = x + y + 3 \Rightarrow f_x(2, 2) = 7, f_y(x, y) = x + \frac{y}{2} - 3 \Rightarrow f_y(2, 2) = 0$
 $\Rightarrow L(x, y) = 11 + 7(x - 2) + 0(y - 2) = 7x - 3; f_{xx}(x, y) = 1, f_{yy}(x, y) = \frac{1}{2}, f_{xy}(x, y) = 1$
 $\Rightarrow M = 1; \text{ thus } |E(x, y)| \leq \left(\frac{1}{2}\right)(1)(|x - 2| + |y - 2|)^2 \leq \left(\frac{1}{2}\right)(0.1 + 0.1)^2 = 0.02$
35. $f(0, 0) = 1, f_x(x, y) = \cos y \Rightarrow f_x(0, 0) = 1, f_y(x, y) = 1 - x \sin y \Rightarrow f_y(0, 0) = 1$
 $\Rightarrow L(x, y) = 1 + 1(x - 0) + 1(y - 0) = x + y + 1; f_{xx}(x, y) = 0, f_{yy}(x, y) = -x \cos y, f_{xy}(x, y) = -\sin y \Rightarrow M = 1;$
 thus $|E(x, y)| \leq \left(\frac{1}{2}\right)(1)(|x| + |y|)^2 \leq \left(\frac{1}{2}\right)(0.2 + 0.2)^2 = 0.08$
36. $f(1, 2) = 6, f_x(x, y) = y^2 - y \sin(x - 1) \Rightarrow f_x(1, 2) = 4, f_y(x, y) = 2xy + \cos(x - 1) \Rightarrow f_y(1, 2) = 5$
 $\Rightarrow L(x, y) = 6 + 4(x - 1) + 5(y - 2) = 4x + 5y - 8; f_{xx}(x, y) = -y \cos(x - 1), f_{yy}(x, y) = 2x,$
 $f_{xy}(x, y) = 2y - \sin(x - 1); |x - 1| \leq 0.1 \Rightarrow 0.9 \leq x \leq 1.1 \text{ and } |y - 2| \leq 0.1 \Rightarrow 1.9 \leq y \leq 2.1; \text{ thus the max of }$
 $|f_{xx}(x, y)| \text{ on } R \text{ is } 2.1, \text{ the max of } |f_{yy}(x, y)| \text{ on } R \text{ is } 2.2, \text{ and the max of } |f_{xy}(x, y)| \text{ on } R \text{ is } 2(2.1) - \sin(0.9 - 1)$
 $\leq 4.3 \Rightarrow M = 4.3; \text{ thus } |E(x, y)| \leq \left(\frac{1}{2}\right)(4.3)(|x - 1| + |y - 2|)^2 \leq (2.15)(0.1 + 0.1)^2 = 0.086$
37. $f(0, 0) = 1, f_x(x, y) = e^x \cos y \Rightarrow f_x(0, 0) = 1, f_y(x, y) = -e^x \sin y \Rightarrow f_y(0, 0) = 0$
 $\Rightarrow L(x, y) = 1 + 1(x - 0) + 0(y - 0) = 1 + x; f_{xx}(x, y) = e^x \cos y, f_{yy}(x, y) = -e^x \cos y, f_{xy}(x, y) = -e^x \sin y;$
 $|x| \leq 0.1 \Rightarrow -0.1 \leq x \leq 0.1 \text{ and } |y| \leq 0.1 \Rightarrow -0.1 \leq y \leq 0.1; \text{ thus the max of } |f_{xx}(x, y)| \text{ on } R \text{ is } e^{0.1} \cos(0.1)$
 $\leq 1.11, \text{ the max of } |f_{yy}(x, y)| \text{ on } R \text{ is } e^{0.1} \cos(0.1) \leq 1.11, \text{ and the max of } |f_{xy}(x, y)| \text{ on } R \text{ is } e^{0.1} \sin(0.1)$
 $\leq 0.12 \Rightarrow M = 1.11; \text{ thus } |E(x, y)| \leq \left(\frac{1}{2}\right)(1.11)(|x| + |y|)^2 \leq (0.555)(0.1 + 0.1)^2 = 0.0222$

38. $f(1, 1) = 0$, $f_x(x, y) = \frac{1}{x} \Rightarrow f_x(1, 1) = 1$, $f_y(x, y) = \frac{1}{y} \Rightarrow f_y(1, 1) = 1 \Rightarrow L(x, y) = 0 + 1(x - 1) + 1(y - 1)$
 $= x + y - 2$; $f_{xx}(x, y) = -\frac{1}{x^2}$, $f_{yy}(x, y) = -\frac{1}{y^2}$, $f_{xy}(x, y) = 0$; $|x - 1| \leq 0.2 \Rightarrow 0.98 \leq x \leq 1.2$ so the max of
 $|f_{xx}(x, y)|$ on R is $\frac{1}{(0.98)^2} \leq 1.04$; $|y - 1| \leq 0.2 \Rightarrow 0.98 \leq y \leq 1.2$ so the max of $|f_{yy}(x, y)|$ on R is
 $\frac{1}{(0.98)^2} \leq 1.04 \Rightarrow M = 1.04$; thus $|E(x, y)| \leq \left(\frac{1}{2}\right)(1.04)(|x - 1| + |y - 1|)^2 \leq (0.52)(0.2 + 0.2)^2 = 0.0832$
39. (a) $f(1, 1, 1) = 3$, $f_x(1, 1, 1) = y + z|_{(1,1,1)} = 2$, $f_y(1, 1, 1) = x + z|_{(1,1,1)} = 2$, $f_z(1, 1, 1) = y + x|_{(1,1,1)} = 2$
 $\Rightarrow L(x, y, z) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = 2x + 2y + 2z - 3$
 (b) $f(1, 0, 0) = 0$, $f_x(1, 0, 0) = 0$, $f_y(1, 0, 0) = 1$, $f_z(1, 0, 0) = 1 \Rightarrow L(x, y, z) = 0 + 0(x - 1) + (y - 0) + (z - 0) = y + z$
 (c) $f(0, 0, 0) = 0$, $f_x(0, 0, 0) = 0$, $f_y(0, 0, 0) = 0$, $f_z(0, 0, 0) = 0 \Rightarrow L(x, y, z) = 0$
40. (a) $f(1, 1, 1) = 3$, $f_x(1, 1, 1) = 2x|_{(1,1,1)} = 2$, $f_y(1, 1, 1) = 2y|_{(1,1,1)} = 2$, $f_z(1, 1, 1) = 2z|_{(1,1,1)} = 2$
 $\Rightarrow L(x, y, z) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = 2x + 2y + 2z - 3$
 (b) $f(0, 1, 0) = 1$, $f_x(0, 1, 0) = 0$, $f_y(0, 1, 0) = 2$, $f_z(0, 1, 0) = 0 \Rightarrow L(x, y, z) = 1 + 0(x - 0) + 2(y - 1) + 0(z - 0)$
 $= 2y - 1$
 (c) $f(1, 0, 0) = 1$, $f_x(1, 0, 0) = 2$, $f_y(1, 0, 0) = 0$, $f_z(1, 0, 0) = 0 \Rightarrow L(x, y, z) = 1 + 2(x - 1) + 0(y - 0) + 0(z - 0)$
 $= 2x - 1$
41. (a) $f(1, 0, 0) = 1$, $f_x(1, 0, 0) = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \Big|_{(1,0,0)} = 1$, $f_y(1, 0, 0) = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \Big|_{(1,0,0)} = 0$,
 $f_z(1, 0, 0) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \Big|_{(1,0,0)} = 0 \Rightarrow L(x, y, z) = 1 + 1(x - 1) + 0(y - 0) + 0(z - 0) = x$
 (b) $f(1, 1, 0) = \sqrt{2}$, $f_x(1, 1, 0) = \frac{1}{\sqrt{2}}$, $f_y(1, 1, 0) = \frac{1}{\sqrt{2}}$, $f_z(1, 1, 0) = 0$
 $\Rightarrow L(x, y, z) = \sqrt{2} + \frac{1}{\sqrt{2}}(x - 1) + \frac{1}{\sqrt{2}}(y - 1) + 0(z - 0) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$
 (c) $f(1, 2, 2) = 3$, $f_x(1, 2, 2) = \frac{1}{3}$, $f_y(1, 2, 2) = \frac{2}{3}$, $f_z(1, 2, 2) = \frac{2}{3} \Rightarrow L(x, y, z) = 3 + \frac{1}{3}(x - 1) + \frac{2}{3}(y - 2) + \frac{2}{3}(z - 2)$
 $= \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z$
42. (a) $f\left(\frac{\pi}{2}, 1, 1\right) = 1$, $f_x\left(\frac{\pi}{2}, 1, 1\right) = \frac{y \cos xy}{z} \Big|_{(\frac{\pi}{2},1,1)} = 0$, $f_y\left(\frac{\pi}{2}, 1, 1\right) = \frac{x \cos xy}{z} \Big|_{(\frac{\pi}{2},1,1)} = 0$,
 $f_z\left(\frac{\pi}{2}, 1, 1\right) = \frac{-\sin xy}{z^2} \Big|_{(\frac{\pi}{2},1,1)} = -1 \Rightarrow L(x, y, z) = 1 + 0(x - \frac{\pi}{2}) + 0(y - 1) - 1(z - 1) = 2 - z$
 (b) $f(2, 0, 1) = 0$, $f_x(2, 0, 1) = 0$, $f_y(2, 0, 1) = 2$, $f_z(2, 0, 1) = 0 \Rightarrow L(x, y, z) = 0 + 0(x - 2) + 2(y - 0) + 0(z - 1) = 2y$
43. (a) $f(0, 0, 0) = 2$, $f_x(0, 0, 0) = e^x|_{(0,0,0)} = 1$, $f_y(0, 0, 0) = -\sin(y + z)|_{(0,0,0)} = 0$,
 $f_z(0, 0, 0) = -\sin(y + z)|_{(0,0,0)} = 0 \Rightarrow L(x, y, z) = 2 + 1(x - 0) + 0(y - 0) + 0(z - 0) = 2 + x$
 (b) $f\left(0, \frac{\pi}{2}, 0\right) = 1$, $f_x\left(0, \frac{\pi}{2}, 0\right) = 1$, $f_y\left(0, \frac{\pi}{2}, 0\right) = -1$, $f_z\left(0, \frac{\pi}{2}, 0\right) = -1 \Rightarrow L(x, y, z)$
 $= 1 + 1(x - 0) - 1(y - \frac{\pi}{2}) - 1(z - 0) = x - y - z + \frac{\pi}{2} + 1$
 (c) $f\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = 1$, $f_x\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = 1$, $f_y\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = -1$, $f_z\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = -1 \Rightarrow L(x, y, z)$
 $= 1 + 1(x - 0) - 1(y - \frac{\pi}{4}) - 1(z - \frac{\pi}{4}) = x - y - z + \frac{\pi}{2} + 1$
44. (a) $f(1, 0, 0) = 0$, $f_x(1, 0, 0) = \frac{yz}{(xyz)^2 + 1} \Big|_{(1,0,0)} = 0$, $f_y(1, 0, 0) = \frac{xz}{(xyz)^2 + 1} \Big|_{(1,0,0)} = 0$,
 $f_z(1, 0, 0) = \frac{xy}{(xyz)^2 + 1} \Big|_{(1,0,0)} = 0 \Rightarrow L(x, y, z) = 0$
 (b) $f(1, 1, 0) = 0$, $f_x(1, 1, 0) = 0$, $f_y(1, 1, 0) = 0$, $f_z(1, 1, 0) = 1 \Rightarrow L(x, y, z) = 0 + 0(x - 1) + 0(y - 1) + 1(z - 0) = z$
 (c) $f(1, 1, 1) = \frac{\pi}{4}$, $f_x(1, 1, 1) = \frac{1}{2}$, $f_y(1, 1, 1) = \frac{1}{2}$, $f_z(1, 1, 1) = \frac{1}{2} \Rightarrow L(x, y, z) = \frac{\pi}{4} + \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) + \frac{1}{2}(z - 1)$
 $= \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z + \frac{\pi}{4} - \frac{3}{2}$

45. $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2) \Rightarrow f(1, 1, 2) = -2; f_x = z, f_y = -3z, f_z = x - 3y \Rightarrow L(x, y, z) = -2 + 2(x - 1) - 6(y - 1) - 2(z - 2) = 2x - 6y - 2z + 6; f_{xx} = 0, f_{yy} = 0, f_{zz} = 0, f_{xy} = 0, f_{yz} = -3 \Rightarrow M = 3; \text{ thus, } |E(x, y, z)| \leq \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.02)^2 = 0.0024$
46. $f(x, y, z) = x^2 + xy + yz + \frac{1}{4}z^2$ at $P_0(1, 1, 2) \Rightarrow f(1, 1, 2) = 5; f_x = 2x + y, f_y = x + z, f_z = y + \frac{1}{2}z \Rightarrow L(x, y, z) = 5 + 3(x - 1) + 3(y - 1) + 2(z - 2) = 3x + 3y + 2z - 5; f_{xx} = 2, f_{yy} = 0, f_{zz} = \frac{1}{2}, f_{xy} = 1, f_{xz} = 0, f_{yz} = 1 \Rightarrow M = 2; \text{ thus } |E(x, y, z)| \leq \left(\frac{1}{2}\right)(2)(0.01 + 0.01 + 0.08)^2 = 0.01$
47. $f(x, y, z) = xy + 2yz - 3xz$ at $P_0(1, 1, 0) \Rightarrow f(1, 1, 0) = 1; f_x = y - 3z, f_y = x + 2z, f_z = 2y - 3x \Rightarrow L(x, y, z) = 1 + (x - 1) + (y - 1) - (z - 0) = x + y - z - 1; f_{xx} = 0, f_{yy} = 0, f_{zz} = 0, f_{xy} = 1, f_{xz} = -3, f_{yz} = 2 \Rightarrow M = 3; \text{ thus } |E(x, y, z)| \leq \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.01)^2 = 0.00135$
48. $f(x, y, z) = \sqrt{2} \cos x \sin(y + z)$ at $P_0(0, 0, \frac{\pi}{4}) \Rightarrow f(0, 0, \frac{\pi}{4}) = 1; f_x = -\sqrt{2} \sin x \sin(y + z), f_y = \sqrt{2} \cos x \cos(y + z), f_z = \sqrt{2} \cos x \cos(y + z) \Rightarrow L(x, y, z) = 1 - 0(x - 0) + (y - 0) + (z - \frac{\pi}{4}) = y + z - \frac{\pi}{4} + 1; f_{xx} = -\sqrt{2} \cos x \sin(y + z), f_{yy} = -\sqrt{2} \cos x \sin(y + z), f_{zz} = -\sqrt{2} \cos x \sin(y + z), f_{xy} = -\sqrt{2} \sin x \cos(y + z), f_{xz} = -\sqrt{2} \sin x \cos(y + z), f_{yz} = -\sqrt{2} \cos x \sin(y + z). \text{ The absolute value of each of these second partial derivatives is bounded above by } \sqrt{2} \Rightarrow M = \sqrt{2}; \text{ thus } |E(x, y, z)| \leq \left(\frac{1}{2}\right)(\sqrt{2})(0.01 + 0.01 + 0.01)^2 = 0.000636.$
49. $T_x(x, y) = e^y + e^{-y}$ and $T_y(x, y) = x(e^y - e^{-y}) \Rightarrow dT = T_x(x, y) dx + T_y(x, y) dy = (e^y + e^{-y}) dx + x(e^y - e^{-y}) dy \Rightarrow dT|_{(2, \ln 2)} = 2.5 dx + 3.0 dy. \text{ If } |dx| \leq 0.1 \text{ and } |dy| \leq 0.02, \text{ then the maximum possible error in the computed value of } T \text{ is } (2.5)(0.1) + (3.0)(0.02) = 0.31 \text{ in magnitude.}$
50. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r dr + V_h dh \Rightarrow \frac{dV}{V} = \frac{2\pi rh dr + \pi r^2 dh}{\pi r^2 h} = \frac{2}{r} dr + \frac{1}{h} dh; \text{ now } \left|\frac{dr}{r} \cdot 100\right| \leq 1 \text{ and } \left|\frac{dh}{h} \cdot 100\right| \leq 1 \Rightarrow \left|\frac{dV}{V} \cdot 100\right| \leq \left|(2 \frac{dr}{r})(100) + \left(\frac{dh}{h}\right)(100)\right| \leq 2 \left|\frac{dr}{r} \cdot 100\right| + \left|\frac{dh}{h} \cdot 100\right| \leq 2(1) + 1 = 3 \Rightarrow 3\%$
51. $\frac{dx}{x} \leq 0.02, \frac{dy}{y} \leq 0.03$
 (a) $S = 2x^2 + 4xy \Rightarrow dS = (4x + 4y)dx + 4x dy = (4x^2 + 4xy)\frac{dx}{x} + 4xy \frac{dy}{y} \leq (4x^2 + 4xy)(0.02) + (4xy)(0.03) = 0.04(2x^2) + 0.05(4xy) \leq 0.05(2x^2) + 0.05(4xy) = (0.05)(2x^2 + 4xy) = 0.05S$
 (b) $V = x^2y \Rightarrow dV = 2xy dx + x^2 dy = 2x^2y \frac{dx}{x} + x^2y \frac{dy}{y} \leq (2x^2y)(0.02) + (x^2y)(0.03) = 0.07(x^2y) = 0.07V$
52. $V = \frac{4\pi}{3}r^3 + \pi r^2 h \Rightarrow dV = (4\pi r^2 + 2\pi rh)dr + \pi r^2 dh; r = 10, h = 15, dr = \frac{1}{2} \text{ and } dh = 0 \Rightarrow dV = \left(4\pi(10)^2 + 2\pi(10)(15)\right)\left(\frac{1}{2}\right) + \pi(10)^2(0) = 350\pi \text{ cm}^3$
53. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r dr + V_h dh \Rightarrow dV = 2\pi rh dr + \pi r^2 dh \Rightarrow dV|_{(5, 12)} = 120\pi dr + 25\pi dh; |dr| \leq 0.1 \text{ cm and } |dh| \leq 0.1 \text{ cm} \Rightarrow dV \leq (120\pi)(0.1) + (25\pi)(0.1) = 14.5\pi \text{ cm}^3; V(5, 12) = 300\pi \text{ cm}^3 \Rightarrow \text{maximum percentage error is } \pm \frac{14.5\pi}{300\pi} \times 100 = \pm 4.83\%$
54. (a) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow -\frac{1}{R^2} dR = -\frac{1}{R_1^2} dR_1 - \frac{1}{R_2^2} dR_2 \Rightarrow dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$
 (b) $dR = R^2 \left[\left(\frac{1}{R_1^2}\right) dR_1 + \left(\frac{1}{R_2^2}\right) dR_2\right] \Rightarrow dR|_{(100, 400)} = R^2 \left[\frac{1}{(100)^2} dR_1 + \frac{1}{(400)^2} dR_2\right] \Rightarrow R \text{ will be more sensitive to a variation in } R_1 \text{ since } \frac{1}{(100)^2} > \frac{1}{(400)^2}$

(c) From part (a), $dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$ so that R_1 changing from 20 to 20.1 ohms $\Rightarrow dR_1 = 0.1$ ohm and R_2 changing from 25 to 24.9 ohms $\Rightarrow dR_2 = -0.1$ ohms; $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R = \frac{100}{9}$ ohms

$$\Rightarrow dR|_{(20,25)} = \frac{\left(\frac{100}{9}\right)^2}{(20)^2} (0.1) + \frac{\left(\frac{100}{9}\right)^2}{(25)^2} (-0.1) \approx 0.011 \text{ ohms} \Rightarrow \text{percentage change is } \frac{dR}{R}|_{(20,25)} \times 100$$

$$= \frac{0.011}{\left(\frac{100}{9}\right)} \times 100 \approx 0.1\%$$

55. $A = xy \Rightarrow dA = x dy + y dx$; if $x > y$ then a 1-unit change in y gives a greater change in dA than a 1-unit change in x . Thus, pay more attention to y which is the smaller of the two dimensions.

56. (a) $f_x(x, y) = 2x(y + 1) \Rightarrow f_x(1, 0) = 2$ and $f_y(x, y) = x^2 \Rightarrow f_y(1, 0) = 1 \Rightarrow df = 2 dx + 1 dy \Rightarrow df$ is more sensitive to changes in x

(b) $df = 0 \Rightarrow 2 dx + dy = 0 \Rightarrow 2 \frac{dx}{dy} + 1 = 0 \Rightarrow \frac{dx}{dy} = -\frac{1}{2}$

57. (a) $r^2 = x^2 + y^2 \Rightarrow 2r dr = 2x dx + 2y dy \Rightarrow dr = \frac{x}{r} dx + \frac{y}{r} dy \Rightarrow dr|_{(3,4)} = \left(\frac{3}{5}\right) (\pm 0.01) + \left(\frac{4}{5}\right) (\pm 0.01)$

$$= \pm \frac{0.07}{5} = \pm 0.014 \Rightarrow \left|\frac{dr}{r} \times 100\right| = \left|\pm \frac{0.014}{5} \times 100\right| = 0.28\%; d\theta = \frac{\left(-\frac{y}{x^2}\right)}{\left(\frac{y}{x}\right)^2 + 1} dx + \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1} dy$$

$$= \frac{-y}{y^2 + x^2} dx + \frac{x}{y^2 + x^2} dy \Rightarrow d\theta|_{(3,4)} = \left(\frac{-4}{25}\right) (\pm 0.01) + \left(\frac{3}{25}\right) (\pm 0.01) = \frac{\mp 0.04}{25} + \frac{\pm 0.03}{25}$$

$$\Rightarrow \text{maximum change in } d\theta \text{ occurs when } dx \text{ and } dy \text{ have opposite signs (} dx = 0.01 \text{ and } dy = -0.01 \text{ or vice versa)} \Rightarrow d\theta = \frac{\pm 0.07}{25} \approx \pm 0.0028; \theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.927255218 \Rightarrow \left|\frac{d\theta}{\theta} \times 100\right| = \left|\frac{\pm 0.0028}{0.927255218} \times 100\right|$$

$$\approx 0.30\%$$

(b) the radius r is more sensitive to changes in y , and the angle θ is more sensitive to changes in x

58. (a) $V = \pi r^2 h \Rightarrow dV = 2\pi r h dr + \pi r^2 dh \Rightarrow$ at $r = 1$ and $h = 5$ we have $dV = 10\pi dr + \pi dh \Rightarrow$ the volume is about 10 times more sensitive to a change in r

(b) $dV = 0 \Rightarrow 0 = 2\pi r h dr + \pi r^2 dh = 2h dr + r dh = 10 dr + dh \Rightarrow dr = -\frac{1}{10} dh$; choose $dh = 1.5$

$$\Rightarrow dr = -0.15 \Rightarrow h = 6.5 \text{ in. and } r = 0.85 \text{ in. is one solution for } \Delta V \approx dV = 0$$

59. $f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \Rightarrow f_a = d, f_b = -c, f_c = -b, f_d = a \Rightarrow df = d da - c db - b dc + a dd$; since $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, the function f is most sensitive to a change in d .

60. $u_x = e^y, u_y = xe^y + \sin z, u_z = y \cos z \Rightarrow du = e^y dx + (xe^y + \sin z) dy + (y \cos z) dz$

$$\Rightarrow du|_{(2, \ln 3, \frac{\pi}{2})} = 3 dx + 7 dy + 0 dz = 3 dx + 7 dy \Rightarrow \text{magnitude of the maximum possible error}$$

$$\leq 3(0.2) + 7(0.6) = 4.8$$

61. $Q_K = \frac{1}{2} \left(\frac{2KM}{h}\right)^{-1/2} \left(\frac{2M}{h}\right), Q_M = \frac{1}{2} \left(\frac{2KM}{h}\right)^{-1/2} \left(\frac{2K}{h}\right)$, and $Q_h = \frac{1}{2} \left(\frac{2KM}{h}\right)^{-1/2} \left(\frac{-2KM}{h^2}\right)$

$$\Rightarrow dQ = \frac{1}{2} \left(\frac{2KM}{h}\right)^{-1/2} \left(\frac{2M}{h}\right) dK + \frac{1}{2} \left(\frac{2KM}{h}\right)^{-1/2} \left(\frac{2K}{h}\right) dM + \frac{1}{2} \left(\frac{2KM}{h}\right)^{-1/2} \left(\frac{-2KM}{h^2}\right) dh$$

$$= \frac{1}{2} \left(\frac{2KM}{h}\right)^{-1/2} \left[\frac{2M}{h} dK + \frac{2K}{h} dM - \frac{2KM}{h^2} dh\right] \Rightarrow dQ|_{(2,20,0.005)}$$

$$= \frac{1}{2} \left[\frac{(2)(2)(20)}{0.05}\right]^{-1/2} \left[\frac{(2)(20)}{0.05} dK + \frac{(2)(2)}{0.05} dM - \frac{(2)(2)(20)}{(0.05)^2} dh\right] = (0.0125)(800 dK + 80 dM - 32,000 dh)$$

$$\Rightarrow Q \text{ is most sensitive to changes in } h$$

62. $A = \frac{1}{2} ab \sin C \Rightarrow A_a = \frac{1}{2} b \sin C, A_b = \frac{1}{2} a \sin C, A_c = \frac{1}{2} ab \cos C$

$$\Rightarrow dA = \left(\frac{1}{2} b \sin C\right) da + \left(\frac{1}{2} a \sin C\right) db + \left(\frac{1}{2} ab \cos C\right) dC; dC = |2^\circ| = |0.0349| \text{ radians, } da = |0.5| \text{ ft,}$$

$$db = |0.5| \text{ ft; at } a = 150 \text{ ft, } b = 200 \text{ ft, and } C = 60^\circ, \text{ we see that the change is approximately}$$

$$dA = \frac{1}{2} (200)(\sin 60^\circ) |0.5| + \frac{1}{2} (150)(\sin 60^\circ) |0.5| + \frac{1}{2} (200)(150)(\cos 60^\circ) |0.0349| = \pm 338 \text{ ft}^2$$

63. $z = f(x, y) \Rightarrow g(x, y, z) = f(x, y) - z = 0 \Rightarrow g_x(x, y, z) = f_x(x, y), g_y(x, y, z) = f_y(x, y)$ and $g_z(x, y, z) = -1$
 $\Rightarrow g_x(x_0, y_0, f(x_0, y_0)) = f_x(x_0, y_0), g_y(x_0, y_0, f(x_0, y_0)) = f_y(x_0, y_0)$ and $g_z(x_0, y_0, f(x_0, y_0)) = -1 \Rightarrow$ the tangent
 plane at the point P_0 is $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - [z - f(x_0, y_0)] = 0$ or
 $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$

64. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} = 2(\cos t + t \sin t)\mathbf{i} + 2(\sin t - t \cos t)\mathbf{j}$ and $\mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \frac{(t \cos t)\mathbf{i} + (t \sin t)\mathbf{j}}{\sqrt{(t \cos t)^2 + (t \sin t)^2}} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ since $t > 0 \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$
 $= 2(\cos t + t \sin t)(\cos t) + 2(\sin t - t \cos t)(\sin t) = 2$

65. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
 $= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}}{\sqrt{(\sin t)^2 + (\cos t)^2 + 1}} = \left(\frac{-\sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t}{\sqrt{2}}\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$
 $= (2 \cos t) \left(\frac{-\sin t}{\sqrt{2}}\right) + (2 \sin t) \left(\frac{\cos t}{\sqrt{2}}\right) + (2t) \left(\frac{1}{\sqrt{2}}\right) = \frac{2t}{\sqrt{2}} \Rightarrow (D_{\mathbf{u}}f) \left(\frac{-\pi}{4}\right) = \frac{-\pi}{2\sqrt{2}}, (D_{\mathbf{u}}f)(0) = 0$ and
 $(D_{\mathbf{u}}f) \left(\frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{2}}$

66. $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t+3)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} - \frac{1}{4}\mathbf{k}; t = 1 \Rightarrow x = 1, y = 1, z = -1 \Rightarrow P_0 = (1, 1, -1)$
 and $\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{4}\mathbf{k}; f(x, y, z) = x^2 + y^2 - z - 3 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$
 $\Rightarrow \nabla f(1, 1, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k};$ therefore $\mathbf{v} = \frac{1}{4}(\nabla f) \Rightarrow$ the curve is normal to the surface

67. $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t-1)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} + 2\mathbf{k}; t = 1 \Rightarrow x = 1, y = 1, z = 1 \Rightarrow P_0 = (1, 1, 1)$ and
 $\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}; f(x, y, z) = x^2 + y^2 - z - 1 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k};$
 now $\mathbf{v}(1) \cdot \nabla f(1, 1, 1) = 0$, thus the curve is tangent to the surface when $t = 1$

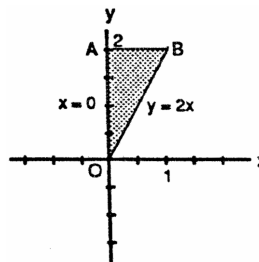
14.7 EXTREME VALUES AND SADDLE POINTS

- $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2y - 3 = 0 \Rightarrow x = -3$ and $y = 3 \Rightarrow$ critical point is $(-3, 3);$
 $f_{xx}(-3, 3) = 2, f_{yy}(-3, 3) = 2, f_{xy}(-3, 3) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of
 $f(-3, 3) = -5$
- $f_x(x, y) = 2y - 10x + 4 = 0$ and $f_y(x, y) = 2x - 4y + 4 = 0 \Rightarrow x = \frac{2}{3}$ and $y = \frac{4}{3} \Rightarrow$ critical point is $(\frac{2}{3}, \frac{4}{3});$
 $f_{xx}(\frac{2}{3}, \frac{4}{3}) = -10, f_{yy}(\frac{2}{3}, \frac{4}{3}) = -4, f_{xy}(\frac{2}{3}, \frac{4}{3}) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of
 $f(\frac{2}{3}, \frac{4}{3}) = 0$
- $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2 = 0 \Rightarrow x = -2$ and $y = 1 \Rightarrow$ critical point is $(-2, 1);$
 $f_{xx}(-2, 1) = 2, f_{yy}(-2, 1) = 0, f_{xy}(-2, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point
- $f_x(x, y) = 5y - 14x + 3 = 0$ and $f_y(x, y) = 5x - 6 = 0 \Rightarrow x = \frac{6}{5}$ and $y = \frac{69}{25} \Rightarrow$ critical point is $(\frac{6}{5}, \frac{69}{25});$
 $f_{xx}(\frac{6}{5}, \frac{69}{25}) = -14, f_{yy}(\frac{6}{5}, \frac{69}{25}) = 0, f_{xy}(\frac{6}{5}, \frac{69}{25}) = 5 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -25 < 0 \Rightarrow$ saddle point
- $f_x(x, y) = 2y - 2x + 3 = 0$ and $f_y(x, y) = 2x - 4y = 0 \Rightarrow x = 3$ and $y = \frac{3}{2} \Rightarrow$ critical point is $(3, \frac{3}{2});$
 $f_{xx}(3, \frac{3}{2}) = -2, f_{yy}(3, \frac{3}{2}) = -4, f_{xy}(3, \frac{3}{2}) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of
 $f(3, \frac{3}{2}) = \frac{17}{2}$
- $f_x(x, y) = 2x - 4y = 0$ and $f_y(x, y) = -4x + 2y + 6 = 0 \Rightarrow x = 2$ and $y = 1 \Rightarrow$ critical point is $(2, 1);$
 $f_{xx}(2, 1) = 2, f_{yy}(2, 1) = 2, f_{xy}(2, 1) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point

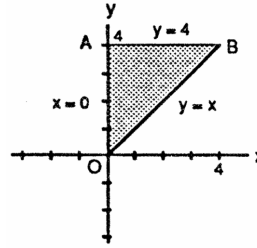
7. $f_x(x, y) = 4x + 3y - 5 = 0$ and $f_y(x, y) = 3x + 8y + 2 = 0 \Rightarrow x = 2$ and $y = -1 \Rightarrow$ critical point is $(2, -1)$;
 $f_{xx}(2, -1) = 4$, $f_{yy}(2, -1) = 8$, $f_{xy}(2, -1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 23 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2, -1) = -6$
8. $f_x(x, y) = 2x - 2y - 2 = 0$ and $f_y(x, y) = -2x + 4y + 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow$ critical point is $(1, 0)$;
 $f_{xx}(1, 0) = 2$, $f_{yy}(1, 0) = 4$, $f_{xy}(1, 0) = -2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1, 0) = 0$
9. $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow$ critical point is $(1, 2)$; $f_{xx}(1, 2) = 2$,
 $f_{yy}(1, 2) = -2$, $f_{xy}(1, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point
10. $f_x(x, y) = 2x + 2y = 0$ and $f_y(x, y) = 2x = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ critical point is $(0, 0)$; $f_{xx}(0, 0) = 2$,
 $f_{yy}(0, 0) = 0$, $f_{xy}(0, 0) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point
11. $f_x(x, y) = \frac{112x - 8x}{\sqrt{56x^2 - 8y^2 - 16x - 31}} - 8 = 0$ and $f_y(x, y) = \frac{-8y}{\sqrt{56x^2 - 8y^2 - 16x - 31}} = 0 \Rightarrow$ critical point is $(\frac{16}{7}, 0)$;
 $f_{xx}(\frac{16}{7}, 0) = -\frac{8}{15}$, $f_{yy}(\frac{16}{7}, 0) = -\frac{8}{15}$, $f_{xy}(\frac{16}{7}, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{64}{225} > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of
 $f(\frac{16}{7}, 0) = -\frac{16}{7}$
12. $f_x(x, y) = \frac{-2x}{3(x^2 + y^2)^{2/3}} = 0$ and $f_y(x, y) = \frac{-2y}{3(x^2 + y^2)^{2/3}} = 0 \Rightarrow$ there are no solutions to the system $f_x(x, y) = 0$ and
 $f_y(x, y) = 0$, however, we must also consider where the partials are undefined, and this occurs when $x = 0$ and $y = 0$
 $\Rightarrow$ critical point is $(0, 0)$. Note that the partial derivatives are defined at every other point other than $(0, 0)$. We cannot use
the second derivative test, but this is the only possible local maximum, local minimum, or saddle point. $f(x, y)$ has a local
maximum of $f(0, 0) = 1$ at $(0, 0)$ since $f(x, y) = 1 - \sqrt[3]{x^2 + y^2} \leq 1$ for all (x, y) other than $(0, 0)$.
13. $f_x(x, y) = 3x^2 - 2y = 0$ and $f_y(x, y) = -3y^2 - 2x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -\frac{2}{3}$ and $y = \frac{2}{3} \Rightarrow$ critical points
are $(0, 0)$ and $(-\frac{2}{3}, \frac{2}{3})$; for $(0, 0)$: $f_{xx}(0, 0) = 6x|_{(0,0)} = 0$, $f_{yy}(0, 0) = -6y|_{(0,0)} = 0$, $f_{xy}(0, 0) = -2$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point; for $(-\frac{2}{3}, \frac{2}{3})$: $f_{xx}(-\frac{2}{3}, \frac{2}{3}) = -4$, $f_{yy}(-\frac{2}{3}, \frac{2}{3}) = -4$, $f_{xy}(-\frac{2}{3}, \frac{2}{3}) = -2$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-\frac{2}{3}, \frac{2}{3}) = \frac{170}{27}$
14. $f_x(x, y) = 3x^2 + 3y = 0$ and $f_y(x, y) = 3x + 3y^2 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -1$ and $y = -1 \Rightarrow$ critical points
are $(0, 0)$ and $(-1, -1)$; for $(0, 0)$: $f_{xx}(0, 0) = 6x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 6y|_{(0,0)} = 0$, $f_{xy}(0, 0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$
 $= -9 < 0 \Rightarrow$ saddle point; for $(-1, -1)$: $f_{xx}(-1, -1) = -6$, $f_{yy}(-1, -1) = -6$, $f_{xy}(-1, -1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$
 $= 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1, -1) = 1$
15. $f_x(x, y) = 12x - 6x^2 + 6y = 0$ and $f_y(x, y) = 6y + 6x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = -1 \Rightarrow$ critical
points are $(0, 0)$ and $(1, -1)$; for $(0, 0)$: $f_{xx}(0, 0) = 12 - 12x|_{(0,0)} = 12$, $f_{yy}(0, 0) = 6$, $f_{xy}(0, 0) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$
 $= 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 0) = 0$; for $(1, -1)$: $f_{xx}(1, -1) = 0$, $f_{yy}(1, -1) = 6$,
 $f_{xy}(1, -1) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point
16. $f_x(x, y) = 3x^2 + 6x = 0 \Rightarrow x = 0$ or $x = -2$; $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ or $y = 2 \Rightarrow$ the critical points are
 $(0, 0)$, $(0, 2)$, $(-2, 0)$, and $(-2, 2)$; for $(0, 0)$: $f_{xx}(0, 0) = 6x + 6|_{(0,0)} = 6$, $f_{yy}(0, 0) = 6y - 6|_{(0,0)} = -6$,
 $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point; for $(0, 2)$: $f_{xx}(0, 2) = 6$, $f_{yy}(0, 2) = 6$, $f_{xy}(0, 2) = 0$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 2) = -12$; for $(-2, 0)$: $f_{xx}(-2, 0) = -6$,
 $f_{yy}(-2, 0) = -6$, $f_{xy}(-2, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2, 0) = -4$;
for $(-2, 2)$: $f_{xx}(-2, 2) = -6$, $f_{yy}(-2, 2) = 6$, $f_{xy}(-2, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

17. $f_x(x, y) = 3x^2 + 3y^2 - 15 = 0$ and $f_y(x, y) = 6xy + 3y^2 - 15 = 0 \Rightarrow$ critical points are $(2, 1)$, $(-2, -1)$, $(0, \sqrt{5})$, and $(0, -\sqrt{5})$; for $(2, 1)$: $f_{xx}(2, 1) = 6x|_{(2,1)} = 12$, $f_{yy}(2, 1) = (6x + 6y)|_{(2,1)} = 18$, $f_{xy}(2, 1) = 6y|_{(2,1)} = 6$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 180 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2, 1) = -30$; for $(-2, -1)$: $f_{xx}(-2, -1) = 6x|_{(-2,-1)} = -12$, $f_{yy}(-2, -1) = (6x + 6y)|_{(-2,-1)} = -18$, $f_{xy}(-2, -1) = 6y|_{(-2,-1)} = -6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 180 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2, -1) = 30$; for $(0, \sqrt{5})$: $f_{xx}(0, \sqrt{5}) = 6x|_{(0,\sqrt{5})} = 0$, $f_{yy}(0, \sqrt{5}) = (6x + 6y)|_{(0,\sqrt{5})} = 6\sqrt{5}$, $f_{xy}(0, \sqrt{5}) = 6y|_{(0,\sqrt{5})} = 6\sqrt{5} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -180 < 0 \Rightarrow$ saddle point;
for $(0, -\sqrt{5})$: $f_{xx}(0, -\sqrt{5}) = 6x|_{(0,-\sqrt{5})} = 0$, $f_{yy}(0, -\sqrt{5}) = (6x + 6y)|_{(0,-\sqrt{5})} = -6\sqrt{5}$, $f_{xy}(0, -\sqrt{5}) = 6y|_{(0,-\sqrt{5})} = -6\sqrt{5} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -180 < 0 \Rightarrow$ saddle point.
18. $f_x(x, y) = 6x^2 - 18x = 0 \Rightarrow 6x(x - 3) = 0 \Rightarrow x = 0$ or $x = 3$; $f_y(x, y) = 6y^2 + 6y - 12 = 0 \Rightarrow 6(y + 2)(y - 1) = 0 \Rightarrow y = -2$ or $y = 1 \Rightarrow$ the critical points are $(0, -2)$, $(0, 1)$, $(3, -2)$, and $(3, 1)$; $f_{xx}(x, y) = 12x - 18$, $f_{yy}(x, y) = 12y + 6$, and $f_{xy}(x, y) = 0$; for $(0, -2)$: $f_{xx}(0, -2) = -18$, $f_{yy}(0, -2) = -18$, $f_{xy}(0, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 324 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0, -2) = 20$; for $(0, 1)$: $f_{xx}(0, 1) = -18$, $f_{yy}(0, 1) = 18$, $f_{xy}(0, 1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -324 < 0 \Rightarrow$ saddle point; for $(3, -2)$: $f_{xx}(3, -2) = 18$, $f_{yy}(3, -2) = -18$, $f_{xy}(3, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -324 < 0 \Rightarrow$ saddle point; for $(3, 1)$: $f_{xx}(3, 1) = 18$, $f_{yy}(3, 1) = 18$, $f_{xy}(3, 1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 324 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(3, 1) = -34$
19. $f_x(x, y) = 4y - 4x^3 = 0$ and $f_y(x, y) = 4x - 4y^3 = 0 \Rightarrow x = y \Rightarrow x(1 - x^2) = 0 \Rightarrow x = 0, 1, -1 \Rightarrow$ the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$; for $(0, 0)$: $f_{xx}(0, 0) = -12x^2|_{(0,0)} = 0$, $f_{yy}(0, 0) = -12y^2|_{(0,0)} = 0$, $f_{xy}(0, 0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1, 1)$: $f_{xx}(1, 1) = -12$, $f_{yy}(1, 1) = -12$, $f_{xy}(1, 1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(1, 1) = 2$; for $(-1, -1)$: $f_{xx}(-1, -1) = -12$, $f_{yy}(-1, -1) = -12$, $f_{xy}(-1, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1, -1) = 2$
20. $f_x(x, y) = 4x^3 + 4y = 0$ and $f_y(x, y) = 4y^3 + 4x = 0 \Rightarrow x = -y \Rightarrow -x^3 + x = 0 \Rightarrow x(1 - x^2) = 0 \Rightarrow x = 0, 1, -1 \Rightarrow$ the critical points are $(0, 0)$, $(1, -1)$, and $(-1, 1)$; $f_{xx}(x, y) = 12x^2$, $f_{yy}(x, y) = 12y^2$, and $f_{xy}(x, y) = 4$; for $(0, 0)$: $f_{xx}(0, 0) = 0$, $f_{yy}(0, 0) = 0$, $f_{xy}(0, 0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1, -1)$: $f_{xx}(1, -1) = 12$, $f_{yy}(1, -1) = 12$, $f_{xy}(1, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1, -1) = -2$; for $(-1, 1)$: $f_{xx}(-1, 1) = 12$, $f_{yy}(-1, 1) = 12$, $f_{xy}(-1, 1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(-1, 1) = -2$
21. $f_x(x, y) = \frac{-2x}{(x^2 + y^2 - 1)^2} = 0$ and $f_y(x, y) = \frac{-2y}{(x^2 + y^2 - 1)^2} = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ the critical point is $(0, 0)$;
 $f_{xx} = \frac{4x^2 - 2y^2 + 2}{(x^2 + y^2 - 1)^3}$, $f_{yy} = \frac{-2x^2 + 4y^2 + 2}{(x^2 + y^2 - 1)^3}$, $f_{xy} = \frac{8xy}{(x^2 + y^2 - 1)^3}$; $f_{xx}(0, 0) = -2$, $f_{yy}(0, 0) = -2$, $f_{xy}(0, 0) = 0$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0, 0) = -1$
22. $f_x(x, y) = -\frac{1}{x^2} + y = 0$ and $f_y(x, y) = x - \frac{1}{y^2} = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow$ the critical point is $(1, 1)$; $f_{xx} = \frac{2}{x^3}$, $f_{yy} = \frac{2}{y^3}$, $f_{xy} = 1$; $f_{xx}(1, 1) = 2$, $f_{yy}(1, 1) = 2$, $f_{xy}(1, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 2 \Rightarrow$ local minimum of $f(1, 1) = 3$
23. $f_x(x, y) = y \cos x = 0$ and $f_y(x, y) = \sin x = 0 \Rightarrow x = n\pi$, n an integer, and $y = 0 \Rightarrow$ the critical points are $(n\pi, 0)$, n an integer (Note: $\cos x$ and $\sin x$ cannot both be 0 for the same x , so $\sin x$ must be 0 and $y = 0$);
 $f_{xx} = -y \sin x$, $f_{yy} = 0$, $f_{xy} = \cos x$; $f_{xx}(n\pi, 0) = 0$, $f_{yy}(n\pi, 0) = 0$, $f_{xy}(n\pi, 0) = 1$ if n is even and $f_{xy}(n\pi, 0) = -1$ if n is odd $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point.

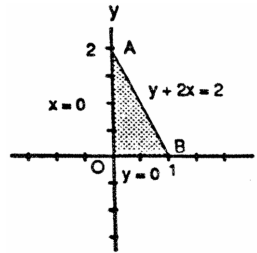
24. $f_x(x, y) = 2e^{2x} \cos y = 0$ and $f_y(x, y) = -e^{2x} \sin y = 0 \Rightarrow$ no solution since $e^{2x} \neq 0$ for any x and the functions $\cos y$ and $\sin y$ cannot equal 0 for the same $y \Rightarrow$ no critical points $\Rightarrow$ no extrema and no saddle points
25. $f_x(x, y) = (2x - 4)e^{x^2 + y^2 - 4x} = 0$ and $f_y(x, y) = 2ye^{x^2 + y^2 - 4x} = 0 \Rightarrow$ critical point is $(2, 0)$; $f_{xx}(2, 0) = \frac{2}{e^4}$, $f_{xy}(2, 0) = 0$, $f_{yy}(2, 0) = \frac{2}{e^4} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{4}{e^8} > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2, 0) = \frac{1}{e^4}$
26. $f_x(x, y) = -ye^x = 0$ and $f_y(x, y) = e^y - e^x = 0 \Rightarrow$ critical point is $(0, 0)$; $f_{xx}(0, 0) = 0$, $f_{xy}(0, 0) = -1$, $f_{yy}(0, 0) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point
27. $f_x(x, y) = 2xe^{-y} = 0$ and $f_y(x, y) = 2ye^{-y} - e^{-y}(x^2 + y^2) = 0 \Rightarrow$ critical points are $(0, 0)$ and $(0, 2)$; for $(0, 0)$: $f_{xx}(0, 0) = 2e^{-y}|_{(0,0)} = 2$, $f_{yy}(0, 0) = (2e^{-y} - 4ye^{-y} + e^{-y}(x^2 + y^2))|_{(0,0)} = 2$, $f_{xy}(0, 0) = -2xe^{-y}|_{(0,0)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 0) = 0$; for $(0, 2)$: $f_{xx}(0, 2) = 2e^{-y}|_{(0,2)} = \frac{2}{e^2}$, $f_{yy}(0, 2) = (2e^{-y} - 4ye^{-y} + e^{-y}(x^2 + y^2))|_{(0,2)} = -\frac{2}{e^2}$, $f_{xy}(0, 2) = -2xe^{-y}|_{(0,2)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -\frac{4}{e^4} < 0 \Rightarrow$ saddle point
28. $f_x(x, y) = e^x(x^2 - 2x + y^2) = 0$ and $f_y(x, y) = -2ye^x = 0 \Rightarrow$ critical points are $(0, 0)$ and $(-2, 0)$; for $(0, 0)$: $f_{xx}(0, 0) = e^x(x^2 + 4x + 2 - y^2)|_{(0,0)} = 2$, $f_{yy}(0, 0) = -2e^x|_{(0,0)} = -2$, $f_{xy}(0, 0) = -2ye^x|_{(0,0)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$ and $f_{xx} > 0 \Rightarrow$ saddle point; for $(-2, 0)$: $f_{xx}(-2, 0) = e^x(x^2 + 4x + 2 - y^2)|_{(-2,0)} = -\frac{2}{e^2}$, $f_{yy}(-2, 0) = -2e^x|_{(-2,0)} = -\frac{2}{e^2}$, $f_{xy}(-2, 0) = -2ye^x|_{(-2,0)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{4}{e^4} > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2, 0) = \frac{4}{e^2}$
29. $f_x(x, y) = -4 + \frac{2}{x} = 0$ and $f_y(x, y) = -1 + \frac{1}{y} = 0 \Rightarrow$ critical point is $(\frac{1}{2}, 1)$; $f_{xx}(\frac{1}{2}, 1) = -8$, $f_{yy}(\frac{1}{2}, 1) = -1$, $f_{xy}(\frac{1}{2}, 1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 8 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(\frac{1}{2}, 1) = -3 - 2\ln 2$
30. $f_x(x, y) = 2x + \frac{1}{x+y} = 0$ and $f_y(x, y) = -1 + \frac{1}{x+y} = 0 \Rightarrow$ critical point is $(-\frac{1}{2}, \frac{3}{2})$; $f_{xx}(-\frac{1}{2}, \frac{3}{2}) = 1$, $f_{yy}(-\frac{1}{2}, \frac{3}{2}) = -1$, $f_{xy}(-\frac{1}{2}, \frac{3}{2}) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -2 < 0 \Rightarrow$ saddle point
31. (i) On OA, $f(x, y) = f(0, y) = y^2 - 4y + 1$ on $0 \leq y \leq 2$;
 $f'(0, y) = 2y - 4 = 0 \Rightarrow y = 2$;
 $f(0, 0) = 1$ and $f(0, 2) = -3$
- (ii) On AB, $f(x, y) = f(x, 2) = 2x^2 - 4x - 3$ on $0 \leq x \leq 1$;
 $f'(x, 2) = 4x - 4 = 0 \Rightarrow x = 1$;
 $f(0, 2) = -3$ and $f(1, 2) = -5$
- (iii) On OB, $f(x, y) = f(x, 2x) = 6x^2 - 12x + 1$ on $0 \leq x \leq 1$; endpoint values have been found above;
 $f'(x, 2x) = 12x - 12 = 0 \Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of OB
- (iv) For interior points of the triangular region, $f_x(x, y) = 4x - 4 = 0$ and $f_y(x, y) = 2y - 4 = 0 \Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of the region. Therefore, the absolute maximum is 1 at $(0, 0)$ and the absolute minimum is -5 at $(1, 2)$.



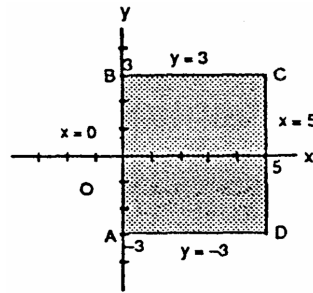
32. (i) On OA, $D(x, y) = D(0, y) = y^2 + 1$ on $0 \leq y \leq 4$;
 $D'(0, y) = 2y = 0 \Rightarrow y = 0$; $D(0, 0) = 1$ and
 $D(0, 4) = 17$
- (ii) On AB, $D(x, y) = D(x, 4) = x^2 - 4x + 17$ on
 $0 \leq x \leq 4$; $D'(x, 4) = 2x - 4 = 0 \Rightarrow x = 2$ and $(2, 4)$
 is an interior point of AB; $D(2, 4) = 13$ and
 $D(4, 4) = D(0, 4) = 17$
- (iii) On OB, $D(x, y) = D(x, x) = x^2 + 1$ on $0 \leq x \leq 4$;
 $D'(x, x) = 2x = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of OB; endpoint values have been found above
- (iv) For interior points of the triangular region, $f_x(x, y) = 2x - y = 0$ and $f_y(x, y) = -x + 2y = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of the region. Therefore, the absolute maximum is 17 at $(0, 4)$ and $(4, 4)$, and the absolute minimum is 1 at $(0, 0)$.



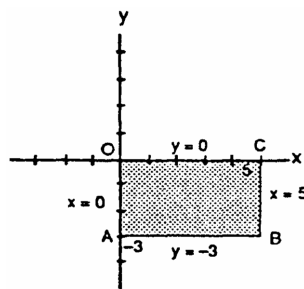
33. (i) On OA, $f(x, y) = f(0, y) = y^2$ on $0 \leq y \leq 2$;
 $f'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $f(0, 0) = 0$ and
 $f(0, 2) = 4$
- (ii) On OB, $f(x, y) = f(x, 0) = x^2$ on $0 \leq x \leq 1$;
 $f'(x, 0) = 2x = 0 \Rightarrow x = 0$ and $y = 0$; $f(0, 0) = 0$ and
 $f(1, 0) = 1$
- (iii) On AB, $f(x, y) = f(x, -2x + 2) = 5x^2 - 8x + 4$ on
 $0 \leq x \leq 1$; $f'(x, -2x + 2) = 10x - 8 = 0 \Rightarrow x = \frac{4}{5}$
 and $y = \frac{2}{5}$; $f(\frac{4}{5}, \frac{2}{5}) = \frac{4}{5}$; endpoint values have been found above.
- (iv) For interior points of the triangular region, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0 \Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of the region. Therefore the absolute maximum is 4 at $(0, 2)$ and the absolute minimum is 0 at $(0, 0)$.



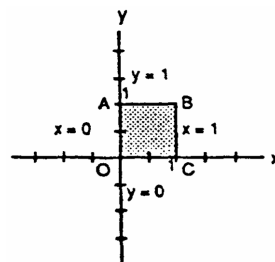
34. (i) On AB, $T(x, y) = T(0, y) = y^2$ on $-3 \leq y \leq 3$;
 $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $T(0, 0) = 0$,
 $T(0, -3) = 9$, and $T(0, 3) = 9$
- (ii) On BC, $T(x, y) = T(x, 3) = x^2 - 3x + 9$ on $0 \leq x \leq 5$;
 $T'(x, 3) = 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$ and $y = 3$;
 $T(\frac{3}{2}, 3) = \frac{27}{4}$ and $T(5, 3) = 19$
- (iii) On CD, $T(x, y) = T(5, y) = y^2 + 5y - 5$ on
 $-3 \leq y \leq 3$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and
 $x = 5$; $T(5, -\frac{5}{2}) = -\frac{45}{4}$, $T(5, -3) = -11$ and $T(5, 3) = 19$
- (iv) On AD, $T(x, y) = T(x, -3) = x^2 - 9x + 9$ on $0 \leq x \leq 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$;
 $T(\frac{9}{2}, -3) = -\frac{45}{4}$, $T(0, -3) = 9$ and $T(5, -3) = -11$
- (v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2 \Rightarrow (4, -2)$ is an interior critical point with $T(4, -2) = -12$. Therefore the absolute maximum is 19 at $(5, 3)$ and the absolute minimum is -12 at $(4, -2)$.



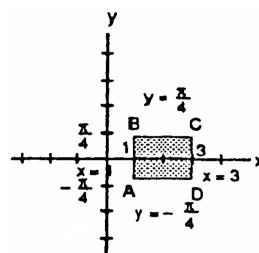
35. (i) On OC, $T(x, y) = T(x, 0) = x^2 - 6x + 2$ on $0 \leq x \leq 5$; $T'(x, 0) = 2x - 6 = 0 \Rightarrow x = 3$ and $y = 0$; $T(3, 0) = -7$, $T(0, 0) = 2$, and $T(5, 0) = -3$
- (ii) On CB, $T(x, y) = T(5, y) = y^2 + 5y - 3$ on $-3 \leq y \leq 0$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T(5, -\frac{5}{2}) = -\frac{37}{4}$ and $T(5, -3) = -9$
- (iii) On AB, $T(x, y) = T(x, -3) = x^2 - 9x + 11$ on $0 \leq x \leq 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T(\frac{9}{2}, -3) = -\frac{37}{4}$ and $T(0, -3) = 11$
- (iv) On AO, $T(x, y) = T(0, y) = y^2 + 2$ on $-3 \leq y \leq 0$; $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of AO
- (v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2$, an interior critical point with $T(4, -2) = -10$. Therefore the absolute maximum is 11 at $(0, -3)$ and the absolute minimum is -10 at $(4, -2)$.



36. (i) On OA, $f(x, y) = f(0, y) = -24y^2$ on $0 \leq y \leq 1$; $f'(0, y) = -48y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of OA; $f(0, 0) = 0$ and $f(0, 1) = -24$
- (ii) On AB, $f(x, y) = f(x, 1) = 48x - 32x^3 - 24$ on $0 \leq x \leq 1$; $f'(x, 1) = 48 - 96x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$ and $y = 1$, or $x = -\frac{1}{\sqrt{2}}$ and $y = 1$, but $(-\frac{1}{\sqrt{2}}, 1)$ is not in the interior of AB; $f(\frac{1}{\sqrt{2}}, 1) = 16\sqrt{2} - 24$ and $f(1, 1) = -8$
- (iii) On BC, $f(x, y) = f(1, y) = 48y - 32 - 24y^2$ on $0 \leq y \leq 1$; $f'(1, y) = 48 - 48y = 0 \Rightarrow y = 1$ and $x = 1$, but $(1, 1)$ is not an interior point of BC; $f(1, 0) = -32$ and $f(1, 1) = -8$
- (iv) On OC, $f(x, y) = f(x, 0) = -32x^3$ on $0 \leq x \leq 1$; $f'(x, 0) = -96x^2 = 0 \Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of OC; $f(0, 0) = 0$ and $f(1, 0) = -32$
- (v) For interior points of the rectangular region, $f_x(x, y) = 48y - 96x^2 = 0$ and $f_y(x, y) = 48x - 48y = 0 \Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{2}$, but $(0, 0)$ is not an interior point of the region; $f(\frac{1}{2}, \frac{1}{2}) = 2$. Therefore the absolute maximum is 2 at $(\frac{1}{2}, \frac{1}{2})$ and the absolute minimum is -32 at $(1, 0)$.



37. (i) On AB, $f(x, y) = f(1, y) = 3 \cos y$ on $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$; $f'(1, y) = -3 \sin y = 0 \Rightarrow y = 0$ and $x = 1$; $f(1, 0) = 3$, $f(1, -\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$, and $f(1, \frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$
- (ii) On CD, $f(x, y) = f(3, y) = 3 \cos y$ on $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$; $f'(3, y) = -3 \sin y = 0 \Rightarrow y = 0$ and $x = 3$; $f(3, 0) = 3$, $f(3, -\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$ and $f(3, \frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$
- (iii) On BC, $f(x, y) = f(x, \frac{\pi}{4}) = \frac{\sqrt{2}}{2} (4x - x^2)$ on $1 \leq x \leq 3$; $f'(x, \frac{\pi}{4}) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = \frac{\pi}{4}$; $f(2, \frac{\pi}{4}) = 2\sqrt{2}$, $f(1, \frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$, and $f(3, \frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$
- (iv) On AD, $f(x, y) = f(x, -\frac{\pi}{4}) = \frac{\sqrt{2}}{2} (4x - x^2)$ on $1 \leq x \leq 3$; $f'(x, -\frac{\pi}{4}) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = -\frac{\pi}{4}$; $f(2, -\frac{\pi}{4}) = 2\sqrt{2}$, $f(1, -\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$, and $f(3, -\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$



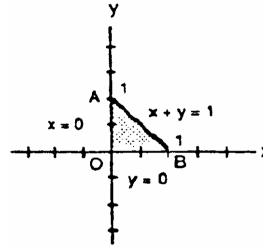
- (v) For interior points of the region, $f_x(x, y) = (4 - 2x) \cos y = 0$ and $f_y(x, y) = -(4x - x^2) \sin y = 0 \Rightarrow x = 2$ and $y = 0$, which is an interior critical point with $f(2, 0) = 4$. Therefore the absolute maximum is 4 at $(2, 0)$ and the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $(3, -\frac{\pi}{4})$, $(3, \frac{\pi}{4})$, $(1, -\frac{\pi}{4})$, and $(1, \frac{\pi}{4})$.

38. (i) On OA, $f(x, y) = f(0, y) = 2y + 1$ on $0 \leq y \leq 1$;
 $f'(0, y) = 2 \Rightarrow$ no interior critical points; $f(0, 0) = 1$
and $f(0, 1) = 3$

- (ii) On OB, $f(x, y) = f(x, 0) = 4x + 1$ on $0 \leq x \leq 1$;
 $f'(x, 0) = 4 \Rightarrow$ no interior critical points; $f(1, 0) = 5$

- (iii) On AB, $f(x, y) = f(x, -x + 1) = 8x^2 - 6x + 3$ on
 $0 \leq x \leq 1$; $f'(x, -x + 1) = 16x - 6 = 0 \Rightarrow x = \frac{3}{8}$
and $y = \frac{5}{8}$; $f(\frac{3}{8}, \frac{5}{8}) = \frac{15}{8}$, $f(0, 1) = 3$, and $f(1, 0) = 5$

- (iv) For interior points of the triangular region, $f_x(x, y) = 4 - 8y = 0$ and $f_y(x, y) = -8x + 2 = 0$
 $\Rightarrow y = \frac{1}{2}$ and $x = \frac{1}{4}$ which is an interior critical point with $f(\frac{1}{4}, \frac{1}{2}) = 2$. Therefore the absolute maximum is 5 at $(1, 0)$ and the absolute minimum is 1 at $(0, 0)$.



39. Let $F(a, b) = \int_a^b (6 - x - x^2) dx$ where $a \leq b$. The boundary of the domain of F is the line $a = b$ in the ab -plane, and $F(a, a) = 0$, so F is identically 0 on the boundary of its domain. For interior critical points we have:
 $\frac{\partial F}{\partial a} = -(6 - a - a^2) = 0 \Rightarrow a = -3, 2$ and $\frac{\partial F}{\partial b} = (6 - b - b^2) = 0 \Rightarrow b = -3, 2$. Since $a \leq b$, there is only one interior critical point $(-3, 2)$ and $F(-3, 2) = \int_{-3}^2 (6 - x - x^2) dx$ gives the area under the parabola $y = 6 - x - x^2$ that is above the x -axis. Therefore, $a = -3$ and $b = 2$.

40. Let $F(a, b) = \int_a^b (24 - 2x - x^2)^{1/3} dx$ where $a \leq b$. The boundary of the domain of F is the line $a = b$ and on this line F is identically 0. For interior critical points we have: $\frac{\partial F}{\partial a} = -(24 - 2a - a^2)^{1/3} = 0 \Rightarrow a = 4, -6$ and
 $\frac{\partial F}{\partial b} = (24 - 2b - b^2)^{1/3} = 0 \Rightarrow b = 4, -6$. Since $a \leq b$, there is only one critical point $(-6, 4)$ and
 $F(-6, 4) = \int_{-6}^4 (24 - 2x - x^2) dx$ gives the area under the curve $y = (24 - 2x - x^2)^{1/3}$ that is above the x -axis. Therefore, $a = -6$ and $b = 4$.

41. $T_x(x, y) = 2x - 1 = 0$ and $T_y(x, y) = 4y = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$ with $T(\frac{1}{2}, 0) = -\frac{1}{4}$; on the boundary
 $x^2 + y^2 = 1$: $T(x, y) = -x^2 - x + 2$ for $-1 \leq x \leq 1 \Rightarrow T'(x, y) = -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = \pm \frac{\sqrt{3}}{2}$;
 $T(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{9}{4}$, $T(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{9}{4}$, $T(-1, 0) = 2$, and $T(1, 0) = 0 \Rightarrow$ the hottest is $2\frac{1}{4}^\circ$ at $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and
 $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$; the coldest is $-\frac{1}{4}^\circ$ at $(\frac{1}{2}, 0)$.

42. $f_x(x, y) = y + 2 - \frac{2}{x} = 0$ and $f_y(x, y) = x - \frac{1}{y} = 0 \Rightarrow x = \frac{1}{2}$ and $y = 2$; $f_{xx}(\frac{1}{2}, 2) = \frac{2}{x^2} \Big|_{(\frac{1}{2}, 2)} = 8$,
 $f_{yy}(\frac{1}{2}, 2) = \frac{1}{y^2} \Big|_{(\frac{1}{2}, 2)} = \frac{1}{4}$, $f_{xy}(\frac{1}{2}, 2) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 1 > 0$ and $f_{xx} > 0 \Rightarrow$ a local minimum of $f(\frac{1}{2}, 2)$
 $= 2 - \ln \frac{1}{2} = 2 + \ln 2$

43. (a) $f_x(x, y) = 2x - 4y = 0$ and $f_y(x, y) = 2y - 4x = 0 \Rightarrow x = 0$ and $y = 0$; $f_{xx}(0, 0) = 2$, $f_{yy}(0, 0) = 2$,
 $f_{xy}(0, 0) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point at $(0, 0)$
(b) $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = 2y - 4 = 0 \Rightarrow x = 1$ and $y = 2$; $f_{xx}(1, 2) = 2$, $f_{yy}(1, 2) = 2$,
 $f_{xy}(1, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1, 2)$

- (c) $f_x(x, y) = 9x^2 - 9 = 0$ and $f_y(x, y) = 2y + 4 = 0 \Rightarrow x = \pm 1$ and $y = -2$; $f_{xx}(1, -2) = 18x|_{(1, -2)} = 18$,
 $f_{yy}(1, -2) = 2$, $f_{xy}(1, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1, -2)$;
 $f_{xx}(-1, -2) = -18$, $f_{yy}(-1, -2) = 2$, $f_{xy}(-1, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point at $(-1, -2)$

44. (a) Minimum at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y)
 (b) Maximum of 1 at $(0, 0)$ since $f(x, y) < 1$ for all other (x, y)
 (c) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$
 (d) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$
 (e) Neither since $f(x, y) < 0$ for $x < 0$ and $y > 0$, but $f(x, y) > 0$ for $x > 0$ and $y > 0$
 (f) Minimum at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y)
45. If $k = 0$, then $f(x, y) = x^2 + y^2 \Rightarrow f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow (0, 0)$ is the only critical point. If $k \neq 0$, $f_x(x, y) = 2x + ky = 0 \Rightarrow y = -\frac{2}{k}x$; $f_y(x, y) = kx + 2y = 0 \Rightarrow kx + 2(-\frac{2}{k}x) = 0 \Rightarrow kx - \frac{4x}{k} = 0 \Rightarrow (k - \frac{4}{k})x = 0 \Rightarrow x = 0$ or $k = \pm 2 \Rightarrow y = (-\frac{2}{k})(0) = 0$ or $y = \pm x$; in any case $(0, 0)$ is a critical point.
46. (See Exercise 45 above): $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 2$, and $f_{xy}(x, y) = k \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 - k^2$; f will have a saddle point at $(0, 0)$ if $4 - k^2 < 0 \Rightarrow k > 2$ or $k < -2$; f will have a local minimum at $(0, 0)$ if $4 - k^2 > 0 \Rightarrow -2 < k < 2$; the test is inconclusive if $4 - k^2 = 0 \Rightarrow k = \pm 2$.
47. No; for example $f(x, y) = xy$ has a saddle point at $(a, b) = (0, 0)$ where $f_x = f_y = 0$.
48. If $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign, then $f_{xx}(a, b)f_{yy}(a, b) < 0$ so $f_{xx}f_{yy} - f_{xy}^2 < 0$. The surface must therefore have a saddle point at (a, b) by the second derivative test.
49. We want the point on $z = 10 - x^2 - y^2$ where the tangent plane is parallel to the plane $x + 2y + 3z = 0$. To find a normal vector to $z = 10 - x^2 - y^2$ let $w = z + x^2 + y^2 - 10$. Then $\nabla w = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ is normal to $z = 10 - x^2 - y^2$ at (x, y) . The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ which is normal to the plane $x + 2y + 3z = 0$ if $6x\mathbf{i} + 6y\mathbf{j} + 3\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ or $x = \frac{1}{6}$ and $y = \frac{1}{3}$. Thus the point is $(\frac{1}{6}, \frac{1}{3}, 10 - \frac{1}{36} - \frac{1}{9})$ or $(\frac{1}{6}, \frac{1}{3}, \frac{355}{36})$.
50. We want the point on $z = x^2 + y^2 + 10$ where the tangent plane is parallel to the plane $x + 2y - z = 0$. Let $w = z - x^2 - y^2 - 10$, then $\nabla w = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ is normal to $z = x^2 + y^2 + 10$ at (x, y) . The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ which is normal to the plane if $x = \frac{1}{2}$ and $y = 1$. Thus the point $(\frac{1}{2}, 1, \frac{1}{4} + 1 + 10)$ or $(\frac{1}{2}, 1, \frac{45}{4})$ is the point on the surface $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.
51. $d(x, y, z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \Rightarrow$ we can minimize $d(x, y, z)$ by minimizing $D(x, y, z) = x^2 + y^2 + z^2$;
 $3x + 2y + z = 6 \Rightarrow z = 6 - 3x - 2y \Rightarrow D(x, y) = x^2 + y^2 + (6 - 3x - 2y)^2 \Rightarrow D_x(x, y) = 2x - 6(6 - 3x - 2y) = 0$
 and $D_y(x, y) = 2y - 4(6 - 3x - 2y) = 0 \Rightarrow$ critical point is $(\frac{9}{7}, \frac{6}{7}) \Rightarrow z = \frac{3}{7}$; $D_{xx}(\frac{9}{7}, \frac{6}{7}) = 20$, $D_{yy}(\frac{9}{7}, \frac{6}{7}) = 10$,
 $D_{xy}(\frac{9}{7}, \frac{6}{7}) = 12 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 56 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}) = \frac{3\sqrt{14}}{7}$
52. $d(x, y, z) = \sqrt{(x-2)^2 + (y+1)^2 + (z-1)^2} \Rightarrow$ we can minimize $d(x, y, z)$ by minimizing
 $D(x, y, z) = (x-2)^2 + (y+1)^2 + (z-1)^2$; $x + y - z = 2 \Rightarrow z = x + y - 2$
 $\Rightarrow D(x, y) = (x-2)^2 + (y+1)^2 + (x+y-3)^2 \Rightarrow D_x(x, y) = 2(x-2) + 2(x+y-3) = 0$
 and $D_y(x, y) = 2(y+1) + 2(x+y-3) = 0 \Rightarrow$ critical point is $(\frac{8}{3}, -\frac{1}{3}) \Rightarrow z = \frac{1}{3}$; $D_{xx}(\frac{8}{3}, -\frac{1}{3}) = 4$, $D_{yy}(\frac{8}{3}, -\frac{1}{3}) = 4$,
 $D_{xy}(\frac{8}{3}, -\frac{1}{3}) = 2 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 12 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d(\frac{8}{3}, -\frac{1}{3}, \frac{1}{3}) = \frac{2}{\sqrt{3}}$

53. $s(x, y, z) = x^2 + y^2 + z^2$; $x + y + z = 9 \Rightarrow z = 9 - x - y \Rightarrow s(x, y) = x^2 + y^2 + (9 - x - y)^2$
 $\Rightarrow s_x(x, y) = 2x - 2(9 - x - y) = 0$ and $s_y(x, y) = 2y - 2(9 - x - y) = 0 \Rightarrow$ critical point is $(3, 3) \Rightarrow z = 3$;
 $s_{xx}(3, 3) = 4$, $s_{yy}(3, 3) = 4$, $s_{xy}(3, 3) = 2 \Rightarrow s_{xx}s_{yy} - s_{xy}^2 = 12 > 0$ and $s_{xx} > 0 \Rightarrow$ local minimum of $s(3, 3, 3) = 27$
54. $p(x, y, z) = xyz$; $x + y + z = 3 \Rightarrow z = 3 - x - y \Rightarrow p(x, y) = xy(3 - x - y) = 3xy - x^2y - xy^2$
 $\Rightarrow p_x(x, y) = 3y - 2xy - y^2 = 0$ and $p_y(x, y) = 3x - x^2 - 2xy = 0 \Rightarrow$ critical points are $(0, 0)$, $(0, 3)$, $(3, 0)$, and $(1, 1)$; for $(0, 0) \Rightarrow z = 3$; $p_{xx}(0, 0) = 0$, $p_{yy}(0, 0) = 0$, $p_{xy}(0, 0) = 3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(0, 3) \Rightarrow z = 0$; $p_{xx}(0, 3) = -6$, $p_{yy}(0, 3) = 0$, $p_{xy}(0, 3) = -3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(3, 0) \Rightarrow z = 0$; $p_{xx}(3, 0) = 0$, $p_{yy}(3, 0) = -6$, $p_{xy}(3, 0) = -3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(1, 1) \Rightarrow z = 1$; $p_{xx}(1, 1) = -2$, $p_{yy}(1, 1) = -2$, $p_{xy}(1, 1) = -1 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = 3 > 0$ and $p_{xx} < 0 \Rightarrow$ local maximum of $p(1, 1, 1) = 1$
55. $s(x, y, z) = xy + yz + xz$; $x + y + z = 6 \Rightarrow z = 6 - x - y \Rightarrow s(x, y) = xy + y(6 - x - y) + x(6 - x - y)$
 $= 6x + 6y - xy - x^2 - y^2 \Rightarrow s_x(x, y) = 6 - 2x - y = 0$ and $s_y(x, y) = 6 - x - 2y = 0 \Rightarrow$ critical point is $(2, 2)$
 $\Rightarrow z = 2$; $s_{xx}(2, 2) = -2$, $s_{yy}(2, 2) = -2$, $s_{xy}(2, 2) = -1 \Rightarrow s_{xx}s_{yy} - s_{xy}^2 = 3 > 0$ and $s_{xx} < 0 \Rightarrow$ local maximum of $s(2, 2, 2) = 12$
56. $d(x, y, z) = \sqrt{(x+6)^2 + (y-4)^2 + (z-0)^2} \Rightarrow$ we can minimize $d(x, y, z)$ by minimizing
 $D(x, y, z) = (x+6)^2 + (y-4)^2 + z^2$; $z = \sqrt{x^2 + y^2} \Rightarrow D(x, y) = (x+6)^2 + (y-4)^2 + x^2 + y^2$
 $= 2x^2 + 2y^2 + 12x - 8y + 52 \Rightarrow D_x(x, y) = 4x + 12 = 0$ and $D_y(x, y) = 4y - 8 = 0 \Rightarrow$ critical point is $(-3, 2)$
 $\Rightarrow z = \sqrt{13}$; $D_{xx}(-3, 2) = 4$, $D_{yy}(-3, 2) = 4$, $D_{xy}(-3, 2) = 0 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 16 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d(-3, 2, \sqrt{13}) = \sqrt{26}$
57. $V(x, y, z) = (2x)(2y)(2z) = 8xyz$; $x^2 + y^2 + z^2 = 4 \Rightarrow z = \sqrt{4 - x^2 - y^2} \Rightarrow V(x, y) = 8xy\sqrt{4 - x^2 - y^2}$,
 $x \geq 0$ and $y \geq 0 \Rightarrow V_x(x, y) = \frac{32y - 16x^2y - 8y^3}{\sqrt{4 - x^2 - y^2}} = 0$ and $V_y(x, y) = \frac{32x - 16xy^2 - 8x^3}{\sqrt{4 - x^2 - y^2}} = 0 \Rightarrow$ critical points are
 $(0, 0)$, $(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$, $(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$, $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$, and $(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$. Only $(0, 0)$ and $(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ satisfy $x \geq 0$ and $y \geq 0$.
 $V(0, 0) = 0$ and $V(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}) = \frac{64}{3\sqrt{3}}$; On $x = 0$, $0 \leq y \leq 2 \Rightarrow V(0, y) = 8(0)y\sqrt{4 - 0^2 - y^2} = 0$, no critical points,
 $V(0, 0) = 0$, $V(0, 2) = 0$; On $y = 0$, $0 \leq x \leq 2 \Rightarrow V(x, 0) = 8x(0)\sqrt{4 - x^2 - 0^2} = 0$, no critical points, $V(0, 0) = 0$,
 $V(0, 2) = 0$; On $y = \sqrt{4 - x^2}$, $0 \leq x \leq 2 \Rightarrow V(x, \sqrt{4 - x^2}) = 8x\sqrt{4 - x^2}\sqrt{4 - x^2 - (\sqrt{4 - x^2})^2} = 0$
no critical points, $V(0, 2) = 0$, $V(2, 0) = 0$. Thus, there is a maximum volume of $\frac{64}{3\sqrt{3}}$ if the box is $\frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}$.
58. $S(x, y, z) = 2xy + 2yz + 2xz$; $xyz = 27 \Rightarrow z = \frac{27}{xy} \Rightarrow S(x, y, z) = 2xy + 2y(\frac{27}{xy}) + 2x(\frac{27}{xy}) = 2xy + \frac{54}{x} + \frac{54}{y}$, $x > 0$,
 $y > 0$; $S_x(x, y) = 2y - \frac{54}{x^2} = 0$ and $S_y(x, y) = 2x - \frac{54}{y^2} = 0 \Rightarrow$ Critical point is $(3, 3) \Rightarrow z = 3$; $S_{xx}(3, 3) = 4$,
 $S_{yy}(3, 3) = 4$, $D_{xy}(3, 3) = 2 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 12 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $S(3, 3, 3) = 54$

59. Let x = height of the box, y = width, and z = length, cut out squares of length x from corner of the material See diagram at right. Fold along the dashed lines to form the box. From the diagram we see that the length of the material is $2x + y$ and the width is $2x + z$. Thus $(2x + y)(2x + z) = 12$

$$\Rightarrow z = \frac{2(6 - 2x^2 + xy)}{2x + y}. \text{ Since } V(x, y, z) = xyz$$

$$\Rightarrow V(x, y) = \frac{2xy(6 - 2x^2 + xy)}{2x + y}, \text{ where } x > 0, y > 0.$$

$$V_x(x, y) = \frac{4(3y^2 - 4x^3y - 4x^2y^2 - xy^3)}{(2x + y)^2} = 0 \text{ and}$$

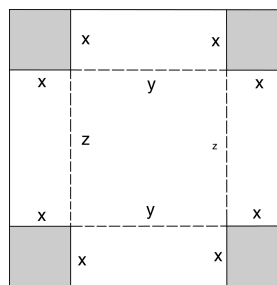
$$V_y(x, y) = \frac{2(12x^2 - 4x^4 - 4x^3y - x^2y^2)}{(2x + y)^2} = 0 \Rightarrow \text{critical points are } (\sqrt{3}, 0), (-\sqrt{3}, 0), \left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right),$$

and $\left(-\frac{1}{\sqrt{3}}, -\frac{4}{\sqrt{3}}\right)$. Only $(\sqrt{3}, 0)$ and $\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$ satisfy $x > 0$ and $y > 0$. For $(\sqrt{3}, 0)$: $z = 0$; $V_{xx}(\sqrt{3}, 0) = 0$,

$$V_{yy}(\sqrt{3}, 0) = -2\sqrt{3}, V_{xy}(\sqrt{3}, 0) = -4\sqrt{3} \Rightarrow V_{xx}V_{yy} - V_{xy}^2 = -48 < 0 \Rightarrow \text{saddle point. For } \left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right): z = \frac{4}{\sqrt{3}};$$

$$V_{xx}\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{80}{3\sqrt{3}}, V_{yy}\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}, V_{xy}\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{4}{3\sqrt{3}} \Rightarrow V_{xx}V_{yy} - V_{xy}^2 = \frac{16}{3} > 0 \text{ and}$$

$$V_{xx} < 0 \Rightarrow \text{local maximum of } V\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = \frac{16}{3\sqrt{3}}$$



60. (a) (i) On $x = 0$, $f(x, y) = f(0, y) = y^2 - y + 1$ for $0 \leq y \leq 1$; $f'(0, y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$ and $x = 0$;
 $f(0, \frac{1}{2}) = \frac{3}{4}$, $f(0, 0) = 1$, and $f(0, 1) = 1$
 (ii) On $y = 1$, $f(x, y) = f(x, 1) = x^2 + x + 1$ for $0 \leq x \leq 1$; $f'(x, 1) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = 1$, but $(-\frac{1}{2}, 1)$ is outside the domain; $f(0, 1) = 1$ and $f(1, 1) = 3$
 (iii) On $x = 1$, $f(x, y) = f(1, y) = y^2 + y + 1$ for $0 \leq y \leq 1$; $f'(1, y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ and $x = 1$, but $(1, -\frac{1}{2})$ is outside the domain; $f(1, 0) = 1$ and $f(1, 1) = 3$
 (iv) On $y = 0$, $f(x, y) = f(x, 0) = x^2 - x + 1$ for $0 \leq x \leq 1$; $f'(x, 0) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$;
 $f(\frac{1}{2}, 0) = \frac{3}{4}$; $f(0, 0) = 1$, and $f(1, 0) = 1$
 (v) On the interior of the square, $f_x(x, y) = 2x + 2y - 1 = 0$ and $f_y(x, y) = 2y + 2x - 1 = 0 \Rightarrow 2x + 2y = 1 \Rightarrow (x + y) = \frac{1}{2}$. Then $f(x, y) = x^2 + y^2 + 2xy - x - y + 1 = (x + y)^2 - (x + y) + 1 = \frac{3}{4}$ is the absolute minimum value when $2x + 2y = 1$.
 (b) The absolute maximum is $f(1, 1) = 3$.

61. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -2 \sin t + 2 \cos t = 0 \Rightarrow \cos t = \sin t \Rightarrow x = y$
 (i) On the semicircle $x^2 + y^2 = 4$, $y \geq 0$, we have $t = \frac{\pi}{4}$ and $x = y = \sqrt{2} \Rightarrow f(\sqrt{2}, \sqrt{2}) = 2\sqrt{2}$. At the endpoints, $f(-2, 0) = -2$ and $f(2, 0) = 2$. Therefore the absolute minimum is $f(-2, 0) = -2$ when $t = \pi$; the absolute maximum is $f(\sqrt{2}, \sqrt{2}) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$.
 (ii) On the quartercircle $x^2 + y^2 = 4$, $x \geq 0$ and $y \geq 0$, the endpoints give $f(0, 2) = 2$ and $f(2, 0) = 2$. Therefore the absolute minimum is $f(2, 0) = 2$ and $f(0, 2) = 2$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f(\sqrt{2}, \sqrt{2}) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$.
 (b) $\frac{dg}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} = y \frac{dx}{dt} + x \frac{dy}{dt} = -4 \sin^2 t + 4 \cos^2 t = 0 \Rightarrow \cos t = \pm \sin t \Rightarrow x = \pm y$.
 (i) On the semicircle $x^2 + y^2 = 4$, $y \geq 0$, we obtain $x = y = \sqrt{2}$ at $t = \frac{\pi}{4}$ and $x = -\sqrt{2}$, $y = \sqrt{2}$ at $t = \frac{3\pi}{4}$. Then $g(\sqrt{2}, \sqrt{2}) = 2$ and $g(-\sqrt{2}, \sqrt{2}) = -2$. At the endpoints, $g(-2, 0) = g(2, 0) = 0$. Therefore the absolute minimum is $g(-\sqrt{2}, \sqrt{2}) = -2$ when $t = \frac{3\pi}{4}$; the absolute maximum is $g(\sqrt{2}, \sqrt{2}) = 2$ when $t = \frac{\pi}{4}$.

- (ii) On the quartercircle $x^2 + y^2 = 4$, $x \geq 0$ and $y \geq 0$, the endpoints give $g(0, 2) = 0$ and $g(2, 0) = 0$. Therefore the absolute minimum is $g(2, 0) = 0$ and $g(0, 2) = 0$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g(\sqrt{2}, \sqrt{2}) = 2$ when $t = \frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x} \frac{dx}{dt} + \frac{\partial h}{\partial y} \frac{dy}{dt} = 4x \frac{dx}{dt} + 2y \frac{dy}{dt} = (8 \cos t)(-2 \sin t) + (4 \sin t)(2 \cos t) = -8 \cos t \sin t = 0$
 $\Rightarrow t = 0, \frac{\pi}{2}, \pi$ yielding the points $(2, 0), (0, 2)$ for $0 \leq t \leq \pi$.

- (i) On the semicircle $x^2 + y^2 = 4$, $y \geq 0$ we have $h(2, 0) = 8$, $h(0, 2) = 4$, and $h(-2, 0) = 8$. Therefore, the absolute minimum is $h(0, 2) = 4$ when $t = \frac{\pi}{2}$; the absolute maximum is $h(2, 0) = 8$ and $h(-2, 0) = 8$ when $t = 0, \pi$ respectively.
- (ii) On the quartercircle $x^2 + y^2 = 4$, $x \geq 0$ and $y \geq 0$ the absolute minimum is $h(0, 2) = 4$ when $t = \frac{\pi}{2}$; the absolute maximum is $h(2, 0) = 8$ when $t = 0$.

62. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 2 \frac{dx}{dt} + 3 \frac{dy}{dt} = -6 \sin t + 6 \cos t = 0 \Rightarrow \sin t = \cos t \Rightarrow t = \frac{\pi}{4}$ for $0 \leq t \leq \pi$.

- (i) On the semi-ellipse, $\frac{x^2}{9} + \frac{y^2}{4} = 1$, $y \geq 0$, $f(x, y) = 2x + 3y = 6 \cos t + 6 \sin t = 6 \left(\frac{3\sqrt{2}}{2} \right) + 6 \left(\frac{\sqrt{2}}{2} \right) = 6\sqrt{2}$ at $t = \frac{\pi}{4}$. At the endpoints, $f(-3, 0) = -6$ and $f(3, 0) = 6$. The absolute minimum is $f(-3, 0) = -6$ when $t = \pi$; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

- (ii) On the quarter ellipse, at the endpoints $f(0, 2) = 6$ and $f(3, 0) = 6$. The absolute minimum is $f(3, 0) = 6$ and $f(0, 2) = 6$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} = y \frac{dx}{dt} + x \frac{dy}{dt} = (2 \sin t)(-3 \sin t) + (3 \cos t)(2 \cos t) = 6(\cos^2 t - \sin^2 t) = 6 \cos 2t = 0$
 $\Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}$ for $0 \leq t \leq \pi$.

- (i) On the semi-ellipse, $g(x, y) = xy = 6 \sin t \cos t$. Then $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$, and $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$. At the endpoints, $g(-3, 0) = g(3, 0) = 0$. The absolute minimum is $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

- (ii) On the quarter ellipse, at the endpoints $g(0, 2) = 0$ and $g(3, 0) = 0$. The absolute minimum is $g(3, 0) = 0$ and $g(0, 2) = 0$ at $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x} \frac{dx}{dt} + \frac{\partial h}{\partial y} \frac{dy}{dt} = 2x \frac{dx}{dt} + 6y \frac{dy}{dt} = (6 \cos t)(-3 \sin t) + (12 \sin t)(2 \cos t) = 6 \sin t \cos t = 0$
 $\Rightarrow t = 0, \frac{\pi}{2}, \pi$ for $0 \leq t \leq \pi$, yielding the points $(3, 0), (0, 2)$, and $(-3, 0)$.

- (i) On the semi-ellipse, $y \geq 0$ so that $h(3, 0) = 9$, $h(0, 2) = 12$, and $h(-3, 0) = 9$. The absolute minimum is $h(3, 0) = 9$ and $h(-3, 0) = 9$ when $t = 0, \pi$ respectively; the absolute maximum is $h(0, 2) = 12$ when $t = \frac{\pi}{2}$.
- (ii) On the quarter ellipse, the absolute minimum is $h(3, 0) = 9$ when $t = 0$; the absolute maximum is $h(0, 2) = 12$ when $t = \frac{\pi}{2}$.

63. $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = y \frac{dx}{dt} + x \frac{dy}{dt}$

- (i) $x = 2t$ and $y = t + 1 \Rightarrow \frac{df}{dt} = (t + 1)(2) + (2t)(1) = 4t + 2 = 0 \Rightarrow t = -\frac{1}{2} \Rightarrow x = -1$ and $y = \frac{1}{2}$ with $f(-1, \frac{1}{2}) = -\frac{1}{2}$. The absolute minimum is $f(-1, \frac{1}{2}) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; there is no absolute maximum.
- (ii) For the endpoints: $t = -1 \Rightarrow x = -2$ and $y = 0$ with $f(-2, 0) = 0$; $t = 0 \Rightarrow x = 0$ and $y = 1$ with $f(0, 1) = 0$. The absolute minimum is $f(-1, \frac{1}{2}) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; the absolute maximum is $f(0, 1) = 0$ and $f(-2, 0) = 0$ when $t = -1, 0$ respectively.
- (iii) There are no interior critical points. For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 1$ with $f(0, 1) = 0$; $t = 1 \Rightarrow x = 2$ and $y = 2$ with $f(2, 2) = 4$. The absolute minimum is $f(0, 1) = 0$ when $t = 0$; the absolute maximum is $f(2, 2) = 4$ when $t = 1$.

64. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$
- (i) $x = t$ and $y = 2 - 2t \Rightarrow \frac{df}{dt} = (2t)(1) + 2(2 - 2t)(-2) = 10t - 8 = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} = \frac{4}{5}$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ when $t = \frac{4}{5}$; there is no absolute maximum along the line.
- (ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $f(0, 2) = 4$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $f(1, 0) = 1$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ at the interior critical point when $t = \frac{4}{5}$; the absolute maximum is $f(0, 2) = 4$ at the endpoint when $t = 0$.
- (b) $\frac{dg}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} = \left[\frac{-2x}{(x^2 + y^2)^2} \right] \frac{dx}{dt} + \left[\frac{-2y}{(x^2 + y^2)^2} \right] \frac{dy}{dt}$
- (i) $x = t$ and $y = 2 - 2t \Rightarrow x^2 + y^2 = 5t^2 - 8t + 4 \Rightarrow \frac{dg}{dt} = -(5t^2 - 8t + 4)^{-2} [(-2t)(1) + (-2)(2 - 2t)(-2)]$
 $= -(5t^2 - 8t + 4)^{-2} (-10t + 8) = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{1}{\left(\frac{4}{5}\right)} = \frac{5}{4}$. The absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$; there is no absolute minimum along the line since x and y can be as large as we please.
- (ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $g(0, 2) = \frac{1}{4}$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $g(1, 0) = 1$. The absolute minimum is $g(0, 2) = \frac{1}{4}$ when $t = 0$; the absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$.

65. $w = (mx_1 + b - y_1)^2 + (mx_2 + b - y_2)^2 + \cdots + (mx_n + b - y_n)^2$

$$\Rightarrow \frac{\partial w}{\partial m} = 2(mx_1 + b - y_1)(x_1) + 2(mx_2 + b - y_2)(x_2) + \cdots + 2(mx_n + b - y_n)(x_n)$$

$$\Rightarrow \frac{\partial w}{\partial b} = 2(mx_1 + b - y_1)(1) + 2(mx_2 + b - y_2)(1) + \cdots + 2(mx_n + b - y_n)(1)$$

$$\frac{\partial w}{\partial m} = 0 \Rightarrow 2[(mx_1 + b - y_1)(x_1) + (mx_2 + b - y_2)(x_2) + \cdots + (mx_n + b - y_n)(x_n)] = 0$$

$$\Rightarrow mx_1^2 + bx_1 - x_1y_1 + mx_2^2 + bx_2 - x_2y_2 + \cdots + mx_n^2 + bx_n - x_ny_n = 0$$

$$\Rightarrow m(x_1^2 + x_2^2 + \cdots + x_n^2) + b(x_1 + x_2 + \cdots + x_n) - (x_1y_1 + x_2y_2 + \cdots + x_ny_n) = 0$$

$$\Rightarrow m \sum_{k=1}^n (x_k^2) + b \sum_{k=1}^n x_k - \sum_{k=1}^n (x_k y_k) = 0$$

$$\frac{\partial w}{\partial b} = 0 \Rightarrow 2[(mx_1 + b - y_1) + (mx_2 + b - y_2) + \cdots + (mx_n + b - y_n)] = 0$$

$$\Rightarrow mx_1 + b - y_1 + mx_2 + b - y_2 + \cdots + mx_n + b - y_n = 0$$

$$\Rightarrow m(x_1 + x_2 + \cdots + x_n) + (b + b + \cdots + b) - (y_1 + y_2 + \cdots + y_n) = 0$$

$$\Rightarrow m \sum_{k=1}^n x_k + b \sum_{k=1}^n 1 - \sum_{k=1}^n y_k = 0 \Rightarrow m \sum_{k=1}^n x_k + bn - \sum_{k=1}^n y_k = 0 \Rightarrow b = \frac{1}{n} \left(\sum_{k=1}^n y_k - m \sum_{k=1}^n x_k \right).$$

Substituting for b in the equation obtained for $\frac{\partial w}{\partial m}$ we get $m \sum_{k=1}^n (x_k^2) + \frac{1}{n} \left(\sum_{k=1}^n y_k - m \sum_{k=1}^n x_k \right) \sum_{k=1}^n x_k - \sum_{k=1}^n (x_k y_k) = 0$.

Multiply both sides by n to obtain $m n \sum_{k=1}^n (x_k^2) + \left(\sum_{k=1}^n y_k - m \sum_{k=1}^n x_k \right) \sum_{k=1}^n x_k - n \sum_{k=1}^n (x_k y_k) = 0$

$$\Rightarrow m n \sum_{k=1}^n (x_k^2) + \left(\sum_{k=1}^n x_k \right) \left(\sum_{k=1}^n y_k \right) - m \left(\sum_{k=1}^n x_k \right)^2 - n \sum_{k=1}^n (x_k y_k) = 0$$

$$\Rightarrow m n \sum_{k=1}^n (x_k^2) - m \left(\sum_{k=1}^n x_k \right)^2 = n \sum_{k=1}^n (x_k y_k) - \left(\sum_{k=1}^n x_k \right) \left(\sum_{k=1}^n y_k \right)$$

$$\Rightarrow m \left[n \sum_{k=1}^n (x_k^2) - \left(\sum_{k=1}^n x_k \right)^2 \right] = n \sum_{k=1}^n (x_k y_k) - \left(\sum_{k=1}^n x_k \right) \left(\sum_{k=1}^n y_k \right)$$

$$\Rightarrow m = \frac{n \sum_{k=1}^n (x_k y_k) - \left(\sum_{k=1}^n x_k \right) \left(\sum_{k=1}^n y_k \right)}{n \sum_{k=1}^n (x_k^2) - \left(\sum_{k=1}^n x_k \right)^2} = \frac{\left(\sum_{k=1}^n x_k \right) \left(\sum_{k=1}^n y_k \right) - n \sum_{k=1}^n (x_k y_k)}{\left(\sum_{k=1}^n x_k \right)^2 - n \sum_{k=1}^n (x_k^2)}$$

To show that these values for m and b minimize the sum of the squares of the distances, use second derivative test.

$$\frac{\partial^2 w}{\partial m^2} = 2x_1^2 + 2x_2^2 + \cdots + 2x_n^2 = 2 \sum_{k=1}^n (x_k^2); \frac{\partial^2 w}{\partial m \partial b} = 2x_1 + 2x_2 + \cdots + 2x_n = 2 \sum_{k=1}^n x_k; \frac{\partial^2 w}{\partial b^2} = 2 + 2 + \cdots + 2 = 2n$$

The discriminant is: $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m \partial b}\right)^2 = \left[2 \sum_{k=1}^n (x_k^2)\right](2n) - \left[2 \sum_{k=1}^n x_k\right]^2 = 4 \left[n \sum_{k=1}^n (x_k^2) - \left(\sum_{k=1}^n x_k\right)^2\right]$.

$$\begin{aligned} \text{Now, } n \sum_{k=1}^n (x_k^2) - \left(\sum_{k=1}^n x_k\right)^2 &= n(x_1^2 + x_2^2 + \cdots + x_n^2) - (x_1 + x_2 + \cdots + x_n)(x_1 + x_2 + \cdots + x_n) \\ &= n x_1^2 + n x_2^2 + \cdots + n x_n^2 - x_1^2 - x_1 x_2 - \cdots - x_1 x_n - x_2 x_1 - x_2^2 - \cdots - x_2 x_n - x_n x_1 - x_n x_2 - \cdots - x_n^2 \\ &= (n-1)x_1^2 + (n-1)x_2^2 + \cdots + (n-1)x_n^2 - 2x_1 x_2 - 2x_1 x_3 - \cdots - 2x_1 x_n - 2x_2 x_3 - \cdots - 2x_2 x_n - \cdots - 2x_{n-1} x_n \\ &= (x_1^2 - 2x_1 x_2 + x_2^2) + (x_1^2 - 2x_1 x_3 + x_3^2) + \cdots + (x_1^2 - 2x_1 x_n + x_n^2) + (x_2^2 - 2x_2 x_3 + x_3^2) + \cdots + (x_2^2 - 2x_2 x_n + x_n^2) \\ &\quad + \cdots + (x_{n-1}^2 - 2x_{n-1} x_n + x_n^2) \\ &= (x_1 - x_2)^2 + (x_1 - x_3)^2 + \cdots + (x_1 - x_n)^2 + (x_2 - x_3)^2 + \cdots + (x_2 - x_n)^2 + \cdots + (x_{n-1} - x_n)^2 \geq 0. \end{aligned}$$

Thus we have: $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m \partial b}\right)^2 = 4 \left[n \sum_{k=1}^n (x_k^2) - \left(\sum_{k=1}^n x_k\right)^2\right] \geq 4(0) = 0$. If $x_1 = x_2 = \cdots = x_n$ then

$$\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m \partial b}\right)^2 = 0. \text{ Also, } \frac{\partial^2 w}{\partial m^2} = 2 \sum_{k=1}^n (x_k^2) \geq 0. \text{ If } x_1 = x_2 = \cdots = x_n = 0, \text{ then } \frac{\partial^2 w}{\partial m^2} = 0.$$

Provided that at least one x_i is nonzero and different from the rest of $x_j, j \neq i$, then $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m \partial b}\right)^2 > 0$ and $\frac{\partial^2 w}{\partial m^2} > 0 \Rightarrow$ the values given above for m and b minimize w .

66. $m = \frac{(0)(5) - 3(6)}{(0)^2 - 3(8)} = \frac{3}{4}$ and

$$b = \frac{1}{3} \left[5 - \frac{3}{4}(0)\right] = \frac{5}{3}$$

$$\Rightarrow y = \frac{3}{4}x + \frac{5}{3}; y|_{x=4} = \frac{14}{3}$$

k	x_k	y_k	x_k^2	$x_k y_k$
1	-2	0	4	0
2	0	2	0	0
3	2	3	4	6
Σ	0	5	8	6

67. $m = \frac{(2)(-1) - 3(-14)}{(2)^2 - 3(10)} = -\frac{20}{13}$ and

$$b = \frac{1}{3} \left[-1 - \left(-\frac{20}{13}\right)(2)\right] = \frac{9}{13}$$

$$\Rightarrow y = -\frac{20}{13}x + \frac{9}{13}; y|_{x=4} = -\frac{71}{13}$$

k	x_k	y_k	x_k^2	$x_k y_k$
1	-1	2	1	-2
2	0	1	0	0
3	3	-4	9	-12
Σ	2	-1	10	-14

68. $m = \frac{(3)(5) - 3(8)}{(3)^2 - 3(5)} = \frac{3}{2}$ and

$$b = \frac{1}{3} \left[5 - \frac{3}{2}(3)\right] = \frac{1}{6}$$

$$\Rightarrow y = \frac{3}{2}x + \frac{1}{6}; y|_{x=4} = \frac{37}{6}$$

k	x_k	y_k	x_k^2	$x_k y_k$
1	0	0	0	0
2	1	2	1	2
3	2	3	4	6
Σ	3	5	5	8

69-74. Example CAS commands:

Maple:

```
f := (x,y) -> x^2+y^3-3*x*y;
```

```
x0,x1 := -5,5;
```

```
y0,y1 := -5,5;
```

```
plot3d( f(x,y), x=x0..x1, y=y0..y1, axes=boxed, shading=zhue, title="#69(a) (Section 14.7)" );
```

```
plot3d( f(x,y), x=x0..x1, y=y0..y1, grid=[40,40], axes=boxed, shading=zhue, style=patchcontour, title="#69(b) (Section 14.7)" );
```

```
fx := D[1](f);
```

(c)

```
fy := D[2](f);
```

```
crit_pts := solve( {fx(x,y)=0,fy(x,y)=0}, {x,y} );
```

```
fxx := D[1](fx);
```

(d)

```
fxy := D[2](fx);
```

```

fyy := D[2](fy);
discr := unapply( fxx(x,y)*fyy(x,y)-fxy(x,y)^2, (x,y) );
for CP in {crit_pts} do                                     # (e)
  eval( [x,y,fxx(x,y),discr(x,y)], CP );
end do;
# (0,0) is a saddle point
# ( 9/4, 3/2) is a local minimum

```

Mathematica: (assigned functions and bounds will vary)

```

Clear[x,y,f]
f[x_,y_]:= x^2 + y^3 - 3x y
xmin=-5; xmax=5; ymin=-5; ymax=5;
Plot3D[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, AxesLabel -> {x, y, z}]
ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, ContourShading -> False, Contours -> 40]
fx= D[f[x,y], x];
fy= D[f[x,y], y];
critical=Solve[{fx==0, fy==0},{x, y}]
fxx= D[fx, x];
fxy= D[fx, y];
fyy= D[fy, y];
discriminant= fxx fyy - fxy^2
{{x, y}, f[x, y], discriminant, fxx} /.critical

```

14.8 LAGRANGE MULTIPLIERS

1. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 4y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 4y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 4y\lambda$
 $\Rightarrow x = 8x\lambda^2 \Rightarrow \lambda = \pm \frac{\sqrt{2}}{4}$ or $x = 0$.

CASE 1: If $x = 0$, then $y = 0$. But $(0, 0)$ is not on the ellipse so $x \neq 0$.

CASE 2: $x \neq 0 \Rightarrow \lambda = \pm \frac{\sqrt{2}}{4} \Rightarrow x = \pm \sqrt{2}y \Rightarrow \left(\pm \sqrt{2}y\right)^2 + 2y^2 = 1 \Rightarrow y = \pm \frac{1}{2}$.

Therefore f takes on its extreme values at $\left(\pm \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ and $\left(\pm \frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$. The extreme values of f on the ellipse are $\pm \frac{\sqrt{2}}{2}$.

2. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 2y\lambda$
 $\Rightarrow x = 4x\lambda^2 \Rightarrow x = 0$ or $\lambda = \pm \frac{1}{2}$.

CASE 1: If $x = 0$, then $y = 0$. But $(0, 0)$ is not on the circle $x^2 + y^2 - 10 = 0$ so $x \neq 0$.

CASE 2: $x \neq 0 \Rightarrow \lambda = \pm \frac{1}{2} \Rightarrow y = 2x \left(\pm \frac{1}{2}\right) = \pm x \Rightarrow x^2 + (\pm x)^2 - 10 = 0 \Rightarrow x = \pm \sqrt{5} \Rightarrow y = \pm \sqrt{5}$.

Therefore f takes on its extreme values at $\left(\pm \sqrt{5}, \sqrt{5}\right)$ and $\left(\pm \sqrt{5}, -\sqrt{5}\right)$. The extreme values of f on the circle are 5 and -5.

3. $\nabla f = -2x\mathbf{i} - 2y\mathbf{j}$ and $\nabla g = \mathbf{i} + 3\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow -2x\mathbf{i} - 2y\mathbf{j} = \lambda(\mathbf{i} + 3\mathbf{j}) \Rightarrow x = -\frac{\lambda}{2}$ and $y = -\frac{3\lambda}{2}$
 $\Rightarrow \left(-\frac{\lambda}{2}\right) + 3\left(-\frac{3\lambda}{2}\right) = 10 \Rightarrow \lambda = -2 \Rightarrow x = 1$ and $y = 3 \Rightarrow f$ takes on its extreme value at $(1, 3)$ on the line.
 The extreme value is $f(1, 3) = 49 - 1 - 9 = 39$.

4. $\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2xy\mathbf{i} + x^2\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow 2xy = \lambda$ and $x^2 = \lambda$
 $\Rightarrow 2xy = x^2 \Rightarrow x = 0$ or $2y = x$.

CASE 1: If $x = 0$, then $x + y = 3 \Rightarrow y = 3$.

CASE 2: If $x \neq 0$, then $2y = x$ so that $x + y = 3 \Rightarrow 2y + y = 3 \Rightarrow y = 1 \Rightarrow x = 2$.

Therefore f takes on its extreme values at $(0, 3)$ and $(2, 1)$. The extreme values of f are $f(0, 3) = 0$ and $f(2, 1) = 4$.

5. We optimize $f(x, y) = x^2 + y^2$, the square of the distance to the origin, subject to the constraint $g(x, y) = xy^2 - 54 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = y^2\mathbf{i} + 2xy\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} = \lambda(y^2\mathbf{i} + 2xy\mathbf{j}) \Rightarrow 2x = \lambda y^2$ and $2y = 2\lambda xy$.

CASE 1: If $y = 0$, then $x = 0$. But $(0, 0)$ does not satisfy the constraint $xy^2 = 54$ so $y \neq 0$.

CASE 2: If $y \neq 0$, then $2 = 2\lambda x \Rightarrow x = \frac{1}{\lambda} \Rightarrow 2\left(\frac{1}{\lambda}\right) = \lambda y^2 \Rightarrow y^2 = \frac{2}{\lambda^2}$. Then $xy^2 = 54 \Rightarrow \left(\frac{1}{\lambda}\right)\left(\frac{2}{\lambda^2}\right) = 54 \Rightarrow \lambda^3 = \frac{1}{27} \Rightarrow \lambda = \frac{1}{3} \Rightarrow x = 3$ and $y^2 = 18 \Rightarrow x = 3$ and $y = \pm 3\sqrt{2}$.

Therefore $\left(3, \pm 3\sqrt{2}\right)$ are the points on the curve $xy^2 = 54$ nearest the origin (since $xy^2 = 54$ has points increasingly far away as y gets close to 0, no points are farthest away).

6. We optimize $f(x, y) = x^2 + y^2$, the square of the distance to the origin subject to the constraint $g(x, y) = x^2y - 2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2xy\mathbf{i} + x^2\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = 2xy\lambda$ and $2y = x^2\lambda \Rightarrow \lambda = \frac{2y}{x^2}$, since $x = 0 \Rightarrow y = 0$ (but $g(0, 0) \neq 0$). Thus $x \neq 0$ and $2x = 2xy\left(\frac{2y}{x^2}\right) \Rightarrow x^2 = 2y^2 \Rightarrow (2y^2)y - 2 = 0 \Rightarrow y = 1$ (since $y > 0$) $\Rightarrow x = \pm\sqrt{2}$. Therefore $\left(\pm\sqrt{2}, 1\right)$ are the points on the curve $x^2y = 2$ nearest the origin (since $x^2y = 2$ has points increasingly far away as x gets close to 0, no points are farthest away).
7. (a) $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = \lambda y$ and $1 = \lambda x \Rightarrow y = \frac{1}{\lambda}$ and $x = \frac{1}{\lambda} \Rightarrow \frac{1}{\lambda^2} = 16 \Rightarrow \lambda = \pm\frac{1}{4}$. Use $\lambda = \frac{1}{4}$ since $x > 0$ and $y > 0$. Then $x = 4$ and $y = 4 \Rightarrow$ the minimum value is 8 at the point $(4, 4)$. Now, $xy = 16$, $x > 0$, $y > 0$ is a branch of a hyperbola in the first quadrant with the x - and y -axes as asymptotes. The equations $x + y = c$ give a family of parallel lines with $m = -1$. As these lines move away from the origin, the number c increases. Thus the minimum value of c occurs where $x + y = c$ is tangent to the hyperbola's branch.
- (b) $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow y = \lambda = x$ and $y = 16 \Rightarrow y = 8 \Rightarrow x = 8 \Rightarrow f(8, 8) = 64$ is the maximum value. The equations $xy = c$ ($x > 0$ and $y > 0$ or $x < 0$ and $y < 0$ to get a maximum value) give a family of hyperbolas in the first and third quadrants with the x - and y -axes as asymptotes. The maximum value of c occurs where the hyperbola $xy = c$ is tangent to the line $x + y = 16$.
8. Let $f(x, y) = x^2 + y^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x + y)\mathbf{i} + (2y + x)\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda(2x + y)$ and $2y = \lambda(2y + x) \Rightarrow \frac{2y}{2y+x} = \lambda \Rightarrow 2x = \left(\frac{2y}{2y+x}\right)(2x + y) \Rightarrow x(2y + x) = y(2x + y) \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$.
- CASE 1: $y = x \Rightarrow x^2 + x(x) + x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ and $y = x$.
- CASE 2: $y = -x \Rightarrow x^2 + x(-x) + (-x)^2 - 1 = 0 \Rightarrow x = \pm 1$ and $y = -x$. Thus $f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3} = f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $f(1, -1) = 2 = f(-1, 1)$.
- Therefore the points $(1, -1)$ and $(-1, 1)$ are the farthest away; $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are the closest points to the origin.
9. $V = \pi r^2 h \Rightarrow 16\pi = \pi r^2 h \Rightarrow 16 = r^2 h \Rightarrow g(r, h) = r^2 h - 16$; $S = 2\pi r h + 2\pi r^2 \Rightarrow \nabla S = (2\pi h + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2r h\mathbf{i} + r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow (2\pi r h + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j} = \lambda(2r h\mathbf{i} + r^2\mathbf{j}) \Rightarrow 2\pi r h + 4\pi r = 2r h\lambda$ and $2\pi r = \lambda r^2 \Rightarrow r = 0$ or $\lambda = \frac{2\pi}{r}$. But $r = 0$ gives no physical can, so $r \neq 0 \Rightarrow \lambda = \frac{2\pi}{r} \Rightarrow 2\pi h + 4\pi r = 2r h\left(\frac{2\pi}{r}\right) \Rightarrow 2r = h \Rightarrow 16 = r^2(2r) \Rightarrow r = 2 \Rightarrow h = 4$; thus $r = 2$ cm and $h = 4$ cm give the only extreme surface area of 24π cm². Since $r = 4$ cm and $h = 1$ cm $\Rightarrow V = 16\pi$ cm³ and $S = 40\pi$ cm², which is a larger surface area, then 24π cm² must be the minimum surface area.

10. For a cylinder of radius r and height h we want to maximize the surface area $S = 2\pi rh$ subject to the constraint $g(r, h) = r^2 + \left(\frac{h}{2}\right)^2 - a^2 = 0$. Thus $\nabla S = 2\pi h\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2r\mathbf{i} + \frac{h}{2}\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow 2\pi h = 2\lambda r$ and $2\pi r = \frac{\lambda h}{2} \Rightarrow \frac{\pi h}{r} = \lambda$ and $2\pi r = \left(\frac{\pi h}{r}\right)\left(\frac{h}{2}\right) \Rightarrow 4r^2 = h^2 \Rightarrow h = 2r \Rightarrow r^2 + \frac{4r^2}{4} = a^2 \Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}} \Rightarrow h = a\sqrt{2} \Rightarrow S = 2\pi\left(\frac{a}{\sqrt{2}}\right)(a\sqrt{2}) = 2\pi a^2$.
11. $A = (2x)(2y) = 4xy$ subject to $g(x, y) = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$; $\nabla A = 4y\mathbf{i} + 4x\mathbf{j}$ and $\nabla g = \frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}$ so that $\nabla A = \lambda \nabla g \Rightarrow 4y\mathbf{i} + 4x\mathbf{j} = \lambda\left(\frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}\right) \Rightarrow 4y = \left(\frac{x}{8}\right)\lambda$ and $4x = \left(\frac{2y}{9}\right)\lambda \Rightarrow \lambda = \frac{32y}{x}$ and $4x = \left(\frac{2y}{9}\right)\left(\frac{32y}{x}\right) \Rightarrow y = \pm \frac{3}{4}x \Rightarrow \frac{x^2}{16} + \frac{\left(\pm\frac{3}{4}x\right)^2}{9} = 1 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$. We use $x = 2\sqrt{2}$ since x represents distance. Then $y = \pm \frac{3}{4}(2\sqrt{2}) = \pm \frac{3\sqrt{2}}{2}$, so the length is $2x = 4\sqrt{2}$ and the width is $2y = 3\sqrt{2}$.
12. $P = 4x + 4y$ subject to $g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; $\nabla P = 4\mathbf{i} + 4\mathbf{j}$ and $\nabla g = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j}$ so that $\nabla P = \lambda \nabla g \Rightarrow 4 = \left(\frac{2x}{a^2}\right)\lambda$ and $4 = \left(\frac{2y}{b^2}\right)\lambda \Rightarrow \lambda = \frac{2a^2}{x}$ and $4 = \left(\frac{2y}{b^2}\right)\left(\frac{2a^2}{x}\right) \Rightarrow y = \left(\frac{b^2}{a^2}\right)x \Rightarrow \frac{x^2}{a^2} + \frac{\left(\frac{b^2}{a^2}\right)^2 x^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{b^2 x^2}{a^4} = 1 \Rightarrow (a^2 + b^2)x^2 = a^4 \Rightarrow x = \frac{a^2}{\sqrt{a^2 + b^2}}$, since $x > 0 \Rightarrow y = \left(\frac{b^2}{a^2}\right)x = \frac{b^2}{\sqrt{a^2 + b^2}} \Rightarrow \text{width} = 2x = \frac{2a^2}{\sqrt{a^2 + b^2}}$ and height $= 2y = \frac{2b^2}{\sqrt{a^2 + b^2}} \Rightarrow \text{perimeter is } P = 4x + 4y = \frac{4a^2 + 4b^2}{\sqrt{a^2 + b^2}} = 4\sqrt{a^2 + b^2}$.
13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x - 2)\mathbf{i} + (2y - 4)\mathbf{j}$ so that $\nabla f = \lambda \nabla g = 2x\mathbf{i} + 2y\mathbf{j} = \lambda[(2x - 2)\mathbf{i} + (2y - 4)\mathbf{j}] \Rightarrow 2x = \lambda(2x - 2)$ and $2y = \lambda(2y - 4) \Rightarrow x = \frac{\lambda}{\lambda - 1}$ and $y = \frac{2\lambda}{\lambda - 1}$, $\lambda \neq 1 \Rightarrow y = 2x \Rightarrow x^2 - 2x + (2x)^2 - 4(2x) = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 2$ and $y = 4$. Therefore $f(0, 0) = 0$ is the minimum value and $f(2, 4) = 20$ is the maximum value. (Note that $\lambda = 1$ gives $2x = 2x - 2$ or $0 = -2$, which is impossible.)
14. $\nabla f = 3\mathbf{i} - \mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3 = 2\lambda x$ and $-1 = 2\lambda y \Rightarrow \lambda = \frac{3}{2x}$ and $-1 = 2\left(\frac{3}{2x}\right)y \Rightarrow y = -\frac{x}{3} \Rightarrow x^2 + \left(-\frac{x}{3}\right)^2 = 4 \Rightarrow 10x^2 = 36 \Rightarrow x = \pm \frac{6}{\sqrt{10}} \Rightarrow x = \frac{6}{\sqrt{10}}$ and $y = -\frac{2}{\sqrt{10}}$, or $x = -\frac{6}{\sqrt{10}}$ and $y = \frac{2}{\sqrt{10}}$. Therefore $f\left(\frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = \frac{20}{\sqrt{10}} + 6 = 2\sqrt{10} + 6 \approx 12.325$ is the maximum value, and $f\left(-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) = -2\sqrt{10} + 6 \approx -0.325$ is the minimum value.
15. $\nabla T = (8x - 4y)\mathbf{i} + (-4x + 2y)\mathbf{j}$ and $g(x, y) = x^2 + y^2 - 25 = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla T = \lambda \nabla g \Rightarrow (8x - 4y)\mathbf{i} + (-4x + 2y)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 8x - 4y = 2\lambda x$ and $-4x + 2y = 2\lambda y \Rightarrow y = \frac{-2x}{\lambda - 1}$, $\lambda \neq 1 \Rightarrow 8x - 4\left(\frac{-2x}{\lambda - 1}\right) = 2\lambda x \Rightarrow x = 0$, or $\lambda = 0$, or $\lambda = 5$.
- CASE 1: $x = 0 \Rightarrow y = 0$; but $(0, 0)$ is not on $x^2 + y^2 = 25$ so $x \neq 0$.
- CASE 2: $\lambda = 0 \Rightarrow y = 2x \Rightarrow x^2 + (2x)^2 = 25 \Rightarrow x = \pm\sqrt{5}$ and $y = 2x$.
- CASE 3: $\lambda = 5 \Rightarrow y = \frac{-2x}{4} = -\frac{x}{2} \Rightarrow x^2 + \left(-\frac{x}{2}\right)^2 = 25 \Rightarrow x = \pm 2\sqrt{5} \Rightarrow x = 2\sqrt{5}$ and $y = -\sqrt{5}$, or $x = -2\sqrt{5}$ and $y = \sqrt{5}$.
- Therefore $T(\sqrt{5}, 2\sqrt{5}) = 0^\circ = T(-\sqrt{5}, -2\sqrt{5})$ is the minimum value and $T(2\sqrt{5}, -\sqrt{5}) = 125^\circ = T(-2\sqrt{5}, \sqrt{5})$ is the maximum value. (Note: $\lambda = 1 \Rightarrow x = 0$ from the equation $-4x + 2y = 2\lambda y$; but we found $x \neq 0$ in CASE 1.)
16. The surface area is given by $S = 4\pi r^2 + 2\pi rh$ subject to the constraint $V(r, h) = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$. Thus $\nabla S = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla V = (4\pi r^2 + 2\pi rh)\mathbf{i} + \pi r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla V = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j} = \lambda[(4\pi r^2 + 2\pi rh)\mathbf{i} + \pi r^2\mathbf{j}] \Rightarrow 8\pi r + 2\pi h = \lambda(4\pi r^2 + 2\pi rh)$ and $2\pi r = \lambda\pi r^2 \Rightarrow r = 0$ or $2 = r\lambda$. But $r \neq 0$

so $2 = r\lambda \Rightarrow \lambda = \frac{2}{r} \Rightarrow 4r + h = \frac{2}{r}(2r^2 + rh) \Rightarrow h = 0 \Rightarrow$ the tank is a sphere (there is no cylindrical part) and $\frac{4}{3}\pi r^3 = 8000 \Rightarrow r = 10\left(\frac{6}{\pi}\right)^{1/3}$.

17. Let $f(x, y, z) = (x - 1)^2 + (y - 1)^2 + (z - 1)^2$ be the square of the distance from $(1, 1, 1)$. Then

$\nabla f = 2(x - 1)\mathbf{i} + 2(y - 1)\mathbf{j} + 2(z - 1)\mathbf{k}$ and $\nabla g = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ so that $\nabla f = \lambda \nabla g$
 $\Rightarrow 2(x - 1)\mathbf{i} + 2(y - 1)\mathbf{j} + 2(z - 1)\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \Rightarrow 2(x - 1) = \lambda, 2(y - 1) = 2\lambda, 2(z - 1) = 3\lambda$
 $\Rightarrow 2(y - 1) = 2[2(x - 1)]$ and $2(z - 1) = 3[2(x - 1)] \Rightarrow x = \frac{y+1}{2} \Rightarrow z + 2 = 3\left(\frac{y+1}{2}\right)$ or $z = \frac{3y-1}{2}$; thus
 $\frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) - 13 = 0 \Rightarrow y = 2 \Rightarrow x = \frac{3}{2}$ and $z = \frac{5}{2}$. Therefore the point $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ is closest (since no point on the plane is farthest from the point $(1, 1, 1)$).

18. Let $f(x, y, z) = (x - 1)^2 + (y + 1)^2 + (z - 1)^2$ be the square of the distance from $(1, -1, 1)$. Then

$\nabla f = 2(x - 1)\mathbf{i} + 2(y + 1)\mathbf{j} + 2(z - 1)\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow x - 1 = \lambda x, y + 1 = \lambda y$ and $z - 1 = \lambda z \Rightarrow x = \frac{1}{1-\lambda}, y = -\frac{1}{1-\lambda},$ and $z = \frac{1}{1-\lambda}$ for $\lambda \neq 1 \Rightarrow \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 4$
 $\Rightarrow \frac{1}{1-\lambda} = \pm \frac{2}{\sqrt{3}} \Rightarrow x = \frac{2}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}, z = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}}, y = \frac{2}{\sqrt{3}}, z = -\frac{2}{\sqrt{3}}$. The largest value of f occurs where $x < 0, y > 0,$ and $z < 0$ or at the point $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ on the sphere.

19. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and

$\nabla g = 2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 2x\lambda, 2y = 2y\lambda,$ and $2z = -2z\lambda \Rightarrow x = 0$ or $\lambda = 1$.

CASE 1: $\lambda = 1 \Rightarrow 2y = -2y \Rightarrow y = 0; 2z = -2z \Rightarrow z = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ and $y = z = 0$.

CASE 2: $x = 0 \Rightarrow y^2 - z^2 = 1$, which has no solution.

Therefore the points on the unit circle $x^2 + y^2 = 1$, are the points on the surface $x^2 + y^2 - z^2 = 1$ closest to the origin. The minimum distance is 1.

20. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and

$\nabla g = y\mathbf{i} + x\mathbf{j} - \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(y\mathbf{i} + x\mathbf{j} - \mathbf{k}) \Rightarrow 2x = \lambda y, 2y = \lambda x,$ and $2z = -\lambda$
 $\Rightarrow x = \frac{\lambda y}{2} \Rightarrow 2y = \lambda\left(\frac{\lambda y}{2}\right) \Rightarrow y = 0$ or $\lambda = \pm 2$.

CASE 1: $y = 0 \Rightarrow x = 0 \Rightarrow -z + 1 = 0 \Rightarrow z = 1$.

CASE 2: $\lambda = 2 \Rightarrow x = y$ and $z = -1 \Rightarrow x^2 - (-1) + 1 = 0 \Rightarrow x^2 + 2 = 0$, so no solution.

CASE 3: $\lambda = -2 \Rightarrow x = -y$ and $z = 1 \Rightarrow (-y)y - 1 + 1 = 0 \Rightarrow y = 0$, again.

Therefore $(0, 0, 1)$ is the point on the surface closest to the origin since this point gives the only extreme value and there is no maximum distance from the surface to the origin.

21. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and

$\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(-y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}) \Rightarrow 2x = -y\lambda, 2y = -x\lambda,$ and $2z = 2z\lambda \Rightarrow \lambda = 1$ or $z = 0$.

CASE 1: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow y = 0$ and $x = 0 \Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ and $x = y = 0$.

CASE 2: $z = 0 \Rightarrow -xy - 4 = 0 \Rightarrow y = -\frac{4}{x}$. Then $2x = \frac{4}{x}\lambda \Rightarrow \lambda = \frac{x^2}{2}$, and $-\frac{8}{x} = -x\lambda \Rightarrow -\frac{8}{x} = -x\left(\frac{x^2}{2}\right)$
 $\Rightarrow x^4 = 16 \Rightarrow x = \pm 2$. Thus, $x = 2$ and $y = -2$, or $x = -2$ and $y = 2$.

Therefore we get four points: $(2, -2, 0), (-2, 2, 0), (0, 0, 2)$ and $(0, 0, -2)$. But the points $(0, 0, 2)$ and $(0, 0, -2)$ are closest to the origin since they are 2 units away and the others are $2\sqrt{2}$ units away.

22. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and

$\nabla g = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda yz, 2y = \lambda xz,$ and $2z = \lambda xy \Rightarrow 2x^2 = \lambda xyz$ and $2y^2 = \lambda yxz$
 $\Rightarrow x^2 = y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x \Rightarrow x(\pm x)(\pm x) = 1 \Rightarrow x = \pm 1 \Rightarrow$ the points are $(1, 1, 1), (1, -1, -1), (-1, -1, 1),$ and $(-1, 1, -1)$.

23. $\nabla f = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-2 = 2y\lambda$, and $5 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}$, $y = -\frac{1}{\lambda} = -2x$, and $z = \frac{5}{2\lambda} = 5x \Rightarrow x^2 + (-2x)^2 + (5x)^2 = 30 \Rightarrow x = \pm 1$.
Thus, $x = 1, y = -2, z = 5$ or $x = -1, y = 2, z = -5$. Therefore $f(1, -2, 5) = 30$ is the maximum value and $f(-1, 2, -5) = -30$ is the minimum value.
24. $\nabla f = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $2 = 2y\lambda$, and $3 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}$, $y = \frac{1}{\lambda} = 2x$, and $z = \frac{3}{2\lambda} = 3x \Rightarrow x^2 + (2x)^2 + (3x)^2 = 25 \Rightarrow x = \pm \frac{5}{\sqrt{14}}$.
Thus, $x = \frac{5}{\sqrt{14}}, y = \frac{10}{\sqrt{14}}, z = \frac{15}{\sqrt{14}}$ or $x = -\frac{5}{\sqrt{14}}, y = -\frac{10}{\sqrt{14}}, z = -\frac{15}{\sqrt{14}}$. Therefore $f\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right) = 5\sqrt{14}$ is the maximum value and $f\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right) = -5\sqrt{14}$ is the minimum value.
25. $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = x + y + z - 9 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow 2x = \lambda, 2y = \lambda$, and $2z = \lambda \Rightarrow x = y = z \Rightarrow x + x + x - 9 = 0 \Rightarrow x = 3, y = 3$, and $z = 3$.
26. $f(x, y, z) = xyz$ and $g(x, y, z) = x + y + z^2 - 16 = 0 \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + 2z\mathbf{k}) \Rightarrow yz = \lambda, xz = \lambda$, and $xy = 2z\lambda \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$.
But $z > 0$ so that $y = x \Rightarrow x^2 = 2z\lambda$ and $xz = \lambda$. Then $x^2 = 2z(xz) \Rightarrow x = 0$ or $x = 2z^2$. But $x > 0$ so that $x = 2z^2 \Rightarrow y = 2z^2 \Rightarrow 2z^2 + 2z^2 + z^2 = 16 \Rightarrow z = \pm \frac{4}{\sqrt{5}}$. We use $z = \frac{4}{\sqrt{5}}$ since $z > 0$. Then $x = \frac{32}{5}$ and $y = \frac{32}{5}$ which yields $f\left(\frac{32}{5}, \frac{32}{5}, \frac{4}{\sqrt{5}}\right) = \frac{4096}{25\sqrt{5}}$.
27. $V = xyz$ and $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \Rightarrow \nabla V = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow yz = \lambda x, xz = \lambda y$, and $xy = \lambda z \Rightarrow xyz = \lambda x^2$ and $xyz = \lambda y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x \Rightarrow x^2 + x^2 + x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$ since $x > 0 \Rightarrow$ the dimensions of the box are $\frac{1}{\sqrt{3}}$ by $\frac{1}{\sqrt{3}}$ by $\frac{1}{\sqrt{3}}$ for maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)
28. $V = xyz$ with x, y, z all positive and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$; thus $V = xyz$ and $g(x, y, z) = bcx + acy + abz - abc = 0 \Rightarrow \nabla V = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = b\mathbf{i} + c\mathbf{j} + a\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow yz = \lambda bc, xz = \lambda ac$, and $xy = \lambda ab \Rightarrow xyz = \lambda bcx, xyz = \lambda acy$, and $xyz = \lambda abz \Rightarrow \lambda \neq 0$. Also, $\lambda bcx = \lambda acy = \lambda abz \Rightarrow bx = ay, cy = bz$, and $cx = az \Rightarrow y = \frac{b}{a}x$ and $z = \frac{c}{a}x$. Then $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \frac{x}{a} + \frac{1}{b}\left(\frac{b}{a}x\right) + \frac{1}{c}\left(\frac{c}{a}x\right) = 1 \Rightarrow \frac{3x}{a} = 1 \Rightarrow x = \frac{a}{3} \Rightarrow y = \left(\frac{b}{a}\right)\left(\frac{a}{3}\right) = \frac{b}{3}$ and $z = \left(\frac{c}{a}\right)\left(\frac{a}{3}\right) = \frac{c}{3} \Rightarrow V = xyz = \left(\frac{a}{3}\right)\left(\frac{b}{3}\right)\left(\frac{c}{3}\right) = \frac{abc}{27}$ is the maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)
29. $\nabla T = 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k}$ and $\nabla g = 8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$ so that $\nabla T = \lambda \nabla g \Rightarrow 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k} = \lambda(8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}) \Rightarrow 16x = 8x\lambda, 4z = 2y\lambda$, and $4y - 16 = 8z\lambda \Rightarrow \lambda = 2$ or $x = 0$.
CASE 1: $\lambda = 2 \Rightarrow 4z = 2y(2) \Rightarrow z = y$. Then $4z - 16 = 16z \Rightarrow z = -\frac{4}{3} \Rightarrow y = -\frac{4}{3}$. Then $4x^2 + \left(-\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)^2 = 16 \Rightarrow x = \pm \frac{4}{3}$.
CASE 2: $x = 0 \Rightarrow \lambda = \frac{2z}{y} \Rightarrow 4y - 16 = 8z\left(\frac{2z}{y}\right) \Rightarrow y^2 - 4y = 4z^2 \Rightarrow 4(0)^2 + y^2 + (y^2 - 4y) - 16 = 0 \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow (y - 4)(y + 2) = 0 \Rightarrow y = 4$ or $y = -2$. Now $y = 4 \Rightarrow 4z^2 = 4^2 - 4(4) \Rightarrow z = 0$ and $y = -2 \Rightarrow 4z^2 = (-2)^2 - 4(-2) \Rightarrow z = \pm \sqrt{3}$.
The temperatures are $T\left(\pm \frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right) = 642\frac{2}{3}^\circ$, $T(0, 4, 0) = 600^\circ$, $T\left(0, -2, \sqrt{3}\right) = \left(600 - 24\sqrt{3}\right)^\circ$, and $T\left(0, -2, -\sqrt{3}\right) = \left(600 + 24\sqrt{3}\right)^\circ \approx 641.6^\circ$. Therefore $\left(\pm \frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ are the hottest points on the space probe.

30. $\nabla T = 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla T = \lambda \nabla g$
 $\Rightarrow 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 400yz^2 = 2x\lambda, 400xz^2 = 2y\lambda, \text{ and } 800xyz = 2z\lambda.$
 Solving this system yields the points $(0, \pm 1, 0), (\pm 1, 0, 0),$ and $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2})$. The corresponding
 temperatures are $T(0, \pm 1, 0) = 0, T(\pm 1, 0, 0) = 0,$ and $T(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}) = \pm 50$. Therefore 50 is the
 maximum temperature at $(\frac{1}{2}, \frac{1}{2}, \pm \frac{\sqrt{2}}{2})$ and $(-\frac{1}{2}, -\frac{1}{2}, \pm \frac{\sqrt{2}}{2})$; -50 is the minimum temperature at
 $(\frac{1}{2}, -\frac{1}{2}, \pm \frac{\sqrt{2}}{2})$ and $(-\frac{1}{2}, \frac{1}{2}, \pm \frac{\sqrt{2}}{2})$.
31. $\nabla U = (y+2)\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2\mathbf{i} + \mathbf{j}$ so that $\nabla U = \lambda \nabla g \Rightarrow (y+2)\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + \mathbf{j}) \Rightarrow y+2 = 2\lambda$ and
 $x = \lambda \Rightarrow y+2 = 2x \Rightarrow y = 2x-2 \Rightarrow 2x + (2x-2) = 30 \Rightarrow x = 8$ and $y = 14$. Therefore $U(8, 14) = \$128$
 is the maximum value of U under the constraint.
32. $\nabla M = (6+z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla M = \lambda \nabla g \Rightarrow (6+z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$
 $= \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 6+z = 2x\lambda, -2y = 2y\lambda, x = 2z\lambda \Rightarrow \lambda = -1$ or $y = 0$.
 CASE 1: $\lambda = -1 \Rightarrow 6+z = -2x$ and $x = -2z \Rightarrow 6+z = -2(-2z) \Rightarrow z = 2$ and $x = -4$. Then
 $(-4)^2 + y^2 + 2^2 - 36 = 0 \Rightarrow y = \pm 4$.
 CASE 2: $y = 0, 6+z = 2x\lambda$, and $x = 2z\lambda \Rightarrow \lambda = \frac{x}{2z} \Rightarrow 6+z = 2x(\frac{x}{2z}) \Rightarrow 6z + z^2 = x^2$
 $\Rightarrow (6z + z^2) + 0^2 + z^2 = 36 \Rightarrow z = -6$ or $z = 3$. Now $z = -6 \Rightarrow x^2 = 0 \Rightarrow x = 0; z = 3$
 $\Rightarrow x^2 = 27 \Rightarrow x = \pm 3\sqrt{3}$.
 Therefore we have the points $(\pm 3\sqrt{3}, 0, 3), (0, 0, -6),$ and $(-4, \pm 4, 2)$. Then $M(3\sqrt{3}, 0, 3) = 27\sqrt{3} + 60$
 $\approx 106.8, M(-3\sqrt{3}, 0, 3) = 60 - 27\sqrt{3} \approx 13.2, M(0, 0, -6) = 60,$ and $M(-4, 4, 2) = 12 = M(-4, -4, 2)$. Therefore,
 the weakest field is at $(-4, \pm 4, 2)$.
33. Let $g_1(x, y, z) = 2x - y = 0$ and $g_2(x, y, z) = y + z = 0 \Rightarrow \nabla g_1 = 2\mathbf{i} - \mathbf{j}, \nabla g_2 = \mathbf{j} + \mathbf{k},$ and $\nabla f = 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k}$
 so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = \lambda(2\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} + \mathbf{k}) \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = 2\lambda\mathbf{i} + (\mu - \lambda)\mathbf{j} + \mu\mathbf{k}$
 $\Rightarrow 2x = 2\lambda, 2 = \mu - \lambda,$ and $-2z = \mu \Rightarrow x = \lambda$. Then $2 = -2z - x \Rightarrow x = -2z - 2$ so that $2x - y = 0$
 $\Rightarrow 2(-2z - 2) - y = 0 \Rightarrow -4z - 4 - y = 0$. This equation coupled with $y + z = 0$ implies $z = -\frac{4}{3}$ and $y = \frac{4}{3}$. Then
 $x = \frac{2}{3}$ so that $(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3})$ is the point that gives the maximum value $f(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}) = (\frac{2}{3})^2 + 2(\frac{4}{3}) - (-\frac{4}{3})^2 = \frac{4}{3}$.
34. Let $g_1(x, y, z) = x + 2y + 3z - 6 = 0$ and $g_2(x, y, z) = x + 3y + 9z - 9 = 0 \Rightarrow \nabla g_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k},$
 $\nabla g_2 = \mathbf{i} + 3\mathbf{j} + 9\mathbf{k},$ and $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$
 $= \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}) \Rightarrow 2x = \lambda + \mu, 2y = 2\lambda + 3\mu,$ and $2z = 3\lambda + 9\mu$. Then $0 = x + 2y + 3z - 6$
 $= \frac{1}{2}(\lambda + \mu) + (2\lambda + 3\mu) + (\frac{3}{2}\lambda + \frac{9}{2}\mu) - 6 \Rightarrow 7\lambda + 17\mu = 6; 0 = x + 3y + 9z - 9$
 $\Rightarrow \frac{1}{2}(\lambda + \mu) + (3\lambda + \frac{9}{2}\mu) + (\frac{27}{2}\lambda + \frac{81}{2}\mu) - 9 \Rightarrow 34\lambda + 91\mu = 18$. Solving these two equations for λ and μ gives
 $\lambda = \frac{240}{59}$ and $\mu = -\frac{78}{59} \Rightarrow x = \frac{\lambda + \mu}{2} = \frac{81}{59}, y = \frac{2\lambda + 3\mu}{2} = \frac{123}{59},$ and $z = \frac{3\lambda + 9\mu}{2} = \frac{9}{59}$. The minimum value is
 $f(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}) = \frac{21,771}{59^2} = \frac{369}{59}$. (Note that there is no maximum value of f subject to the constraints because
 at least one of the variables $x, y,$ or z can be made arbitrary and assume a value as large as we please.)
35. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x, y, z)$ subject to the
 constraints $g_1(x, y, z) = y + 2z - 12 = 0$ and $g_2(x, y, z) = x + y - 6 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \nabla g_1 = \mathbf{j} + 2\mathbf{k},$
 and $\nabla g_2 = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x = \mu, 2y = \lambda + \mu,$ and $2z = 2\lambda$. Then $0 = y + 2z - 12$
 $= (\frac{\lambda}{2} + \frac{\mu}{2}) + 2\lambda - 12 \Rightarrow \frac{5}{2}\lambda + \frac{1}{2}\mu = 12 \Rightarrow 5\lambda + \mu = 24; 0 = x + y - 6 = \frac{\mu}{2} + (\frac{\lambda}{2} + \frac{\mu}{2}) - 6 \Rightarrow \frac{1}{2}\lambda + \mu = 6$
 $\Rightarrow \lambda + 2\mu = 12$. Solving these two equations for λ and μ gives $\lambda = 4$ and $\mu = 4 \Rightarrow x = \frac{\mu}{2} = 2, y = \frac{\lambda + \mu}{2} = 4,$ and
 $z = \lambda = 4$. The point $(2, 4, 4)$ on the line of intersection is closest to the origin. (There is no maximum distance from the
 origin since points on the line can be arbitrarily far away.)

36. The maximum value is $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$ from Exercise 33 above.

37. Let $g_1(x, y, z) = z - 1 = 0$ and $g_2(x, y, z) = x^2 + y^2 + z^2 - 10 = 0 \Rightarrow \nabla g_1 = \mathbf{k}$, $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, and $\nabla f = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k} = \lambda(\mathbf{k}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$
 $\Rightarrow 2xyz = 2x\mu$, $x^2z = 2y\mu$, and $x^2y = 2z\mu + \lambda \Rightarrow xyz = x\mu \Rightarrow x = 0$ or $yz = \mu \Rightarrow \mu = y$ since $z = 1$.

CASE 1: $x = 0$ and $z = 1 \Rightarrow y^2 - 9 = 0$ (from g_2) $\Rightarrow y = \pm 3$ yielding the points $(0, \pm 3, 1)$.

CASE 2: $\mu = y \Rightarrow x^2z = 2y^2 \Rightarrow x^2 = 2y^2$ (since $z = 1$) $\Rightarrow 2y^2 + y^2 + 1 - 10 = 0$ (from g_2) $\Rightarrow 3y^2 - 9 = 0$
 $\Rightarrow y = \pm \sqrt{3} \Rightarrow x^2 = 2(\pm \sqrt{3})^2 \Rightarrow x = \pm \sqrt{6}$ yielding the points $(\pm \sqrt{6}, \pm \sqrt{3}, 1)$.

Now $f(0, \pm 3, 1) = 1$ and $f(\pm \sqrt{6}, \pm \sqrt{3}, 1) = 6(\pm \sqrt{3}) + 1 = 1 \pm 6\sqrt{3}$. Therefore the maximum of f is $1 + 6\sqrt{3}$ at $(\pm \sqrt{6}, \sqrt{3}, 1)$, and the minimum of f is $1 - 6\sqrt{3}$ at $(\pm \sqrt{6}, -\sqrt{3}, 1)$.

38. (a) Let $g_1(x, y, z) = x + y + z - 40 = 0$ and $g_2(x, y, z) = x + y - z = 0 \Rightarrow \nabla g_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\nabla g_2 = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\nabla w = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla w = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$
 $\Rightarrow yz = \lambda + \mu$, $xz = \lambda + \mu$, and $xy = \lambda - \mu \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$.

CASE 1: $z = 0 \Rightarrow x + y = 40$ and $x + y = 0 \Rightarrow$ no solution.

CASE 2: $x = y \Rightarrow 2x + z - 40 = 0$ and $2x - z = 0 \Rightarrow z = 20 \Rightarrow x = 10$ and $y = 10 \Rightarrow w = (10)(10)(20) = 2000$

(b) $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j}$ is parallel to the line of intersection $\Rightarrow$ the line is $x = -2t + 10$,

$y = 2t + 10$, $z = 20$. Since $z = 20$, we see that $w = xyz = (-2t + 10)(2t + 10)(20) = (-4t^2 + 100)(20)$ which has its maximum when $t = 0 \Rightarrow x = 10$, $y = 10$, and $z = 20$.

39. Let $g_1(x, y, z) = y - x = 0$ and $g_2(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$. Then $\nabla f = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = -\mathbf{i} + \mathbf{j}$, and $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k} = \lambda(-\mathbf{i} + \mathbf{j}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$
 $\Rightarrow y = -\lambda + 2x\mu$, $x = \lambda + 2y\mu$, and $2z = 2z\mu \Rightarrow z = 0$ or $\mu = 1$.

CASE 1: $z = 0 \Rightarrow x^2 + y^2 - 4 = 0 \Rightarrow 2x^2 - 4 = 0$ (since $x = y$) $\Rightarrow x = \pm \sqrt{2}$ and $y = \pm \sqrt{2}$ yielding the points $(\pm \sqrt{2}, \pm \sqrt{2}, 0)$.

CASE 2: $\mu = 1 \Rightarrow y = -\lambda + 2x$ and $x = \lambda + 2y \Rightarrow x + y = 2(x + y) \Rightarrow 2x = 2(2x)$ since $x = y \Rightarrow x = 0 \Rightarrow y = 0$
 $\Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ yielding the points $(0, 0, \pm 2)$.

Now, $f(0, 0, \pm 2) = 4$ and $f(\pm \sqrt{2}, \pm \sqrt{2}, 0) = 2$. Therefore the maximum value of f is 4 at $(0, 0, \pm 2)$ and the minimum value of f is 2 at $(\pm \sqrt{2}, \pm \sqrt{2}, 0)$.

40. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x, y, z)$ subject to the constraints $g_1(x, y, z) = 2y + 4z - 5 = 0$ and $g_2(x, y, z) = 4x^2 + 4y^2 - z^2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = 2\mathbf{j} + 4\mathbf{k}$, and $\nabla g_2 = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2\mathbf{j} + 4\mathbf{k}) + \mu(8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 8x\mu$, $2y = 2\lambda + 8y\mu$, and $2z = 4\lambda - 2z\mu \Rightarrow x = 0$ or $\mu = \frac{1}{4}$.

CASE 1: $x = 0 \Rightarrow 4(0)^2 + 4y^2 - z^2 = 0 \Rightarrow z = \pm 2y \Rightarrow 2y + 4(2y) - 5 = 0 \Rightarrow y = \frac{1}{2}$, or $2y + 4(-2y) - 5 = 0$
 $\Rightarrow y = -\frac{5}{6}$ yielding the points $(0, \frac{1}{2}, 1)$ and $(0, -\frac{5}{6}, \frac{5}{3})$.

CASE 2: $\mu = \frac{1}{4} \Rightarrow y = \lambda + y \Rightarrow \lambda = 0 \Rightarrow 2z = 4(0) - 2z(\frac{1}{4}) \Rightarrow z = 0 \Rightarrow 2y + 4(0) = 5 \Rightarrow y = \frac{5}{2}$ and $(0)^2 = 4x^2 + 4(\frac{5}{2})^2 \Rightarrow$ no solution.

Then $f(0, \frac{1}{2}, 1) = \frac{5}{4}$ and $f(0, -\frac{5}{6}, \frac{5}{3}) = 25(\frac{1}{36} + \frac{1}{9}) = \frac{125}{36} \Rightarrow$ the point $(0, \frac{1}{2}, 1)$ is closest to the origin.

41. $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = y\lambda$ and $1 = x\lambda \Rightarrow y = x \Rightarrow y^2 = 16 \Rightarrow y = \pm 4 \Rightarrow (4, 4)$ and $(-4, -4)$ are candidates for the location of extreme values. But as $x \rightarrow \infty$, $y \rightarrow \infty$ and $f(x, y) \rightarrow \infty$; as $x \rightarrow -\infty$, $y \rightarrow 0$ and $f(x, y) \rightarrow -\infty$. Therefore no maximum or minimum value exists subject to the constraint.

42. Let $f(A, B, C) = \sum_{k=1}^4 (Ax_k + By_k + C - z_k)^2 = C^2 + (B + C - 1)^2 + (A + B + C - 1)^2 + (A + C + 1)^2$. We want to minimize f . Then $f_A(A, B, C) = 4A + 2B + 4C$, $f_B(A, B, C) = 2A + 4B + 4C - 4$, and $f_C(A, B, C) = 4A + 4B + 8C - 2$. Set each partial derivative equal to 0 and solve the system to get $A = -\frac{1}{2}$, $B = \frac{3}{2}$, and $C = -\frac{1}{4}$ or the critical point of f is $(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{4})$.

43. (a) Maximize $f(a, b, c) = a^2b^2c^2$ subject to $a^2 + b^2 + c^2 = r^2$. Thus $\nabla f = 2ab^2c^2\mathbf{i} + 2a^2bc^2\mathbf{j} + 2a^2b^2c\mathbf{k}$ and $\nabla g = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2ab^2c^2 = 2a\lambda$, $2a^2bc^2 = 2b\lambda$, and $2a^2b^2c = 2c\lambda \Rightarrow 2a^2b^2c^2 = 2a^2\lambda = 2b^2\lambda = 2c^2\lambda \Rightarrow \lambda = 0$ or $a^2 = b^2 = c^2$.
CASE 1: $\lambda = 0 \Rightarrow a^2b^2c^2 = 0$.

CASE 2: $a^2 = b^2 = c^2 \Rightarrow f(a, b, c) = a^2a^2a^2$ and $3a^2 = r^2 \Rightarrow f(a, b, c) = \left(\frac{r^2}{3}\right)^3$ is the maximum value.

(b) The point $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is on the sphere if $a + b + c = r^2$. Moreover, by part (a), $abc = f(\sqrt{a}, \sqrt{b}, \sqrt{c}) \leq \left(\frac{r^2}{3}\right)^3 \Rightarrow (abc)^{1/3} \leq \frac{r^2}{3} = \frac{a+b+c}{3}$, as claimed.

44. Let $f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n a_i x_i = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ and $g(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2 - 1$. Then we want $\nabla f = \lambda \nabla g \Rightarrow a_1 = \lambda(2x_1)$, $a_2 = \lambda(2x_2)$, $\dots$, $a_n = \lambda(2x_n)$, $\lambda \neq 0 \Rightarrow x_i = \frac{a_i}{2\lambda} \Rightarrow \frac{a_1^2}{4\lambda^2} + \frac{a_2^2}{4\lambda^2} + \dots + \frac{a_n^2}{4\lambda^2} = 1 \Rightarrow 4\lambda^2 = \sum_{i=1}^n a_i^2 \Rightarrow 2\lambda = \left(\sum_{i=1}^n a_i^2\right)^{1/2} \Rightarrow f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i \left(\frac{a_i}{2\lambda}\right) = \frac{1}{2\lambda} \sum_{i=1}^n a_i^2 = \left(\sum_{i=1}^n a_i^2\right)^{1/2}$ is the maximum value.

45-50. Example CAS commands:

Maple:

```
f := (x,y,z) -> x*y+y*z;
g1 := (x,y,z) -> x^2+y^2-2;
g2 := (x,y,z) -> x^2+z^2-2;
h := unapply( f(x,y,z)-lambda[1]*g1(x,y,z)-lambda[2]*g2(x,y,z), (x,y,z,lambda[1],lambda[2]) ); # (a)
hx := diff( h(x,y,z,lambda[1],lambda[2]), x ); # (b)
hy := diff( h(x,y,z,lambda[1],lambda[2]), y );
hz := diff( h(x,y,z,lambda[1],lambda[2]), z );
hl1 := diff( h(x,y,z,lambda[1],lambda[2]), lambda[1] );
hl2 := diff( h(x,y,z,lambda[1],lambda[2]), lambda[2] );
sys := { hx=0, hy=0, hz=0, hl1=0, hl2=0 };
q1 := solve( sys, {x,y,z,lambda[1],lambda[2]} ); # (c)
q2 := map(allvalues,{q1});
for p in q2 do # (d)
    eval( [x,y,z,f(x,y,z)], p );
    ``=evalf(eval( [x,y,z,f(x,y,z)], p ));
end do;
```

Mathematica: (assigned functions will vary)

```
Clear[x, y, z, lambda1, lambda2]
f[x_, y_, z_] := x y + y z
g1[x_, y_, z_] := x^2 + y^2 - 2
g2[x_, y_, z_] := x^2 + z^2 - 2
h = f[x, y, z] - lambda1 g1[x, y, z] - lambda2 g2[x, y, z];
hx = D[h, x]; hy = D[h, y]; hz = D[h, z]; hL1 = D[h, lambda1]; hL2 = D[h, lambda2];
critical = Solve[{hx == 0, hy == 0, hz == 0, hL1 == 0, hL2 == 0, g1[x, y, z] == 0, g2[x, y, z] == 0},
{x, y, z, lambda1, lambda2}]/N
{{x, y, z}, f[x, y, z]}/.critical
```

14.9 TAYLOR'S FORMULA FOR TWO VARIABLES

- $f(x, y) = xe^y \Rightarrow f_x = e^y, f_y = xe^y, f_{xx} = 0, f_{xy} = e^y, f_{yy} = xe^y$
 $\Rightarrow f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}[x^2f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2f_{yy}(0, 0)]$
 $= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0) = x + xy$ quadratic approximation;
 $f_{xxx} = 0, f_{xxy} = 0, f_{xyy} = e^y, f_{yyy} = xe^y$
 $\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}[x^3f_{xxx}(0, 0) + 3x^2yf_{xxy}(0, 0) + 3xy^2f_{xyy}(0, 0) + y^3f_{yyy}(0, 0)]$
 $= x + xy + \frac{1}{6}(x^3 \cdot 0 + 3x^2y \cdot 0 + 3xy^2 \cdot 1 + y^3 \cdot 0) = x + xy + \frac{1}{2}xy^2$, cubic approximation
- $f(x, y) = e^x \cos y \Rightarrow f_x = e^x \cos y, f_y = -e^x \sin y, f_{xx} = e^x \cos y, f_{xy} = -e^x \sin y, f_{yy} = -e^x \cos y$
 $\Rightarrow f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}[x^2f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2f_{yy}(0, 0)]$
 $= 1 + x \cdot 1 + y \cdot 0 + \frac{1}{2}[x^2 \cdot 1 + 2xy \cdot 0 + y^2 \cdot (-1)] = 1 + x + \frac{1}{2}(x^2 - y^2)$, quadratic approximation;
 $f_{xxx} = e^x \cos y, f_{xxy} = -e^x \sin y, f_{xyy} = -e^x \cos y, f_{yyy} = e^x \sin y$
 $\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}[x^3f_{xxx}(0, 0) + 3x^2yf_{xxy}(0, 0) + 3xy^2f_{xyy}(0, 0) + y^3f_{yyy}(0, 0)]$
 $= 1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{6}[x^3 \cdot 1 + 3x^2y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0]$
 $= 1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{6}(x^3 - 3xy^2)$, cubic approximation
- $f(x, y) = y \sin x \Rightarrow f_x = y \cos x, f_y = \sin x, f_{xx} = -y \sin x, f_{xy} = \cos x, f_{yy} = 0$
 $\Rightarrow f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}[x^2f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2f_{yy}(0, 0)]$
 $= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0) = xy$, quadratic approximation;
 $f_{xxx} = -y \cos x, f_{xxy} = -\sin x, f_{xyy} = 0, f_{yyy} = 0$
 $\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}[x^3f_{xxx}(0, 0) + 3x^2yf_{xxy}(0, 0) + 3xy^2f_{xyy}(0, 0) + y^3f_{yyy}(0, 0)]$
 $= xy + \frac{1}{6}(x^3 \cdot 0 + 3x^2y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = xy$, cubic approximation
- $f(x, y) = \sin x \cos y \Rightarrow f_x = \cos x \cos y, f_y = -\sin x \sin y, f_{xx} = -\sin x \cos y, f_{xy} = -\cos x \sin y,$
 $f_{yy} = -\sin x \cos y \Rightarrow f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}[x^2f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2f_{yy}(0, 0)]$
 $= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0) = x$, quadratic approximation;
 $f_{xxx} = -\cos x \cos y, f_{xxy} = \sin x \sin y, f_{xyy} = -\cos x \cos y, f_{yyy} = \sin x \sin y$
 $\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}[x^3f_{xxx}(0, 0) + 3x^2yf_{xxy}(0, 0) + 3xy^2f_{xyy}(0, 0) + y^3f_{yyy}(0, 0)]$
 $= x + \frac{1}{6}[x^3 \cdot (-1) + 3x^2y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0] = x - \frac{1}{6}(x^3 + 3xy^2)$, cubic approximation
- $f(x, y) = e^x \ln(1 + y) \Rightarrow f_x = e^x \ln(1 + y), f_y = \frac{e^x}{1 + y}, f_{xx} = e^x \ln(1 + y), f_{xy} = \frac{e^x}{1 + y}, f_{yy} = -\frac{e^x}{(1 + y)^2}$
 $\Rightarrow f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}[x^2f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2f_{yy}(0, 0)]$
 $= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)] = y + \frac{1}{2}(2xy - y^2)$, quadratic approximation;
 $f_{xxx} = e^x \ln(1 + y), f_{xxy} = \frac{e^x}{1 + y}, f_{xyy} = -\frac{e^x}{(1 + y)^2}, f_{yyy} = \frac{2e^x}{(1 + y)^3}$

$$\begin{aligned} &\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] \\ &= y + \frac{1}{2} (2xy - y^2) + \frac{1}{6} [x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot (-1) + y^3 \cdot 2] \\ &= y + \frac{1}{2} (2xy - y^2) + \frac{1}{6} (3x^2 y - 3xy^2 + 2y^3), \text{ cubic approximation} \end{aligned}$$

$$\begin{aligned} 6. \quad f(x, y) &= \ln(2x + y + 1) \Rightarrow f_x = \frac{2}{2x + y + 1}, f_y = \frac{1}{2x + y + 1}, f_{xx} = \frac{-4}{(2x + y + 1)^2}, f_{xy} = \frac{-2}{(2x + y + 1)^2}, \\ f_{yy} &= \frac{-1}{(2x + y + 1)^2} \Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] \\ &= 0 + x \cdot 2 + y \cdot 1 + \frac{1}{2} [x^2 \cdot (-4) + 2xy \cdot (-2) + y^2 \cdot (-1)] = 2x + y + \frac{1}{2} (-4x^2 - 4xy - y^2) \\ &= (2x + y) - \frac{1}{2} (2x + y)^2, \text{ quadratic approximation;} \\ f_{xxx} &= \frac{16}{(2x + y + 1)^3}, f_{xxy} = \frac{8}{(2x + y + 1)^3}, f_{xyy} = \frac{4}{(2x + y + 1)^3}, f_{yyy} = \frac{2}{(2x + y + 1)^3} \\ &\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] \\ &= (2x + y) - \frac{1}{2} (2x + y)^2 + \frac{1}{6} (x^3 \cdot 16 + 3x^2 y \cdot 8 + 3xy^2 \cdot 4 + y^3 \cdot 2) \\ &= (2x + y) - \frac{1}{2} (2x + y)^2 + \frac{1}{3} (8x^3 + 12x^2 y + 6xy^2 + y^2) \\ &= (2x + y) - \frac{1}{2} (2x + y)^2 + \frac{1}{3} (2x + y)^3, \text{ cubic approximation} \end{aligned}$$

$$\begin{aligned} 7. \quad f(x, y) &= \sin(x^2 + y^2) \Rightarrow f_x = 2x \cos(x^2 + y^2), f_y = 2y \cos(x^2 + y^2), f_{xx} = 2 \cos(x^2 + y^2) - 4x^2 \sin(x^2 + y^2), \\ f_{xy} &= -4xy \sin(x^2 + y^2), f_{yy} = 2 \cos(x^2 + y^2) - 4y^2 \sin(x^2 + y^2) \\ &\Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] \\ &= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2} (x^2 \cdot 2 + 2xy \cdot 0 + y^2 \cdot 2) = x^2 + y^2, \text{ quadratic approximation;} \\ f_{xxx} &= -12x \sin(x^2 + y^2) - 8x^3 \cos(x^2 + y^2), f_{xxy} = -4y \sin(x^2 + y^2) - 8x^2 y \cos(x^2 + y^2), \\ f_{xyy} &= -4x \sin(x^2 + y^2) - 8xy^2 \cos(x^2 + y^2), f_{yyy} = -12y \sin(x^2 + y^2) - 8y^3 \cos(x^2 + y^2) \\ &\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] \\ &= x^2 + y^2 + \frac{1}{6} (x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = x^2 + y^2, \text{ cubic approximation} \end{aligned}$$

$$\begin{aligned} 8. \quad f(x, y) &= \cos(x^2 + y^2) \Rightarrow f_x = -2x \sin(x^2 + y^2), f_y = -2y \sin(x^2 + y^2), \\ f_{xx} &= -2 \sin(x^2 + y^2) - 4x^2 \cos(x^2 + y^2), f_{xy} = -4xy \cos(x^2 + y^2), f_{yy} = -2 \sin(x^2 + y^2) - 4y^2 \cos(x^2 + y^2) \\ &\Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] \\ &= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2} [x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0] = 1, \text{ quadratic approximation;} \\ f_{xxx} &= -12x \cos(x^2 + y^2) + 8x^3 \sin(x^2 + y^2), f_{xxy} = -4y \cos(x^2 + y^2) + 8x^2 y \sin(x^2 + y^2), \\ f_{xyy} &= -4x \cos(x^2 + y^2) + 8xy^2 \sin(x^2 + y^2), f_{yyy} = -12y \cos(x^2 + y^2) + 8y^3 \sin(x^2 + y^2) \\ &\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] \\ &= 1 + \frac{1}{6} (x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = 1, \text{ cubic approximation} \end{aligned}$$

$$\begin{aligned} 9. \quad f(x, y) &= \frac{1}{1 - x - y} \Rightarrow f_x = \frac{1}{(1 - x - y)^2} = f_y, f_{xx} = \frac{2}{(1 - x - y)^3} = f_{xy} = f_{yy} \\ &\Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] \\ &= 1 + x \cdot 1 + y \cdot 1 + \frac{1}{2} (x^2 \cdot 2 + 2xy \cdot 2 + y^2 \cdot 2) = 1 + (x + y) + (x^2 + 2xy + y^2) \\ &= 1 + (x + y) + (x + y)^2, \text{ quadratic approximation;} f_{xxx} = \frac{6}{(1 - x - y)^4} = f_{xxy} = f_{xyy} = f_{yyy} \\ &\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] \\ &= 1 + (x + y) + (x + y)^2 + \frac{1}{6} (x^3 \cdot 6 + 3x^2 y \cdot 6 + 3xy^2 \cdot 6 + y^3 \cdot 6) \\ &= 1 + (x + y) + (x + y)^2 + (x^3 + 3x^2 y + 3xy^2 + y^3) = 1 + (x + y) + (x + y)^2 + (x + y)^3, \text{ cubic approximation} \end{aligned}$$

$$\begin{aligned} 10. \quad f(x, y) &= \frac{1}{1 - x - y + xy} \Rightarrow f_x = \frac{1 - y}{(1 - x - y + xy)^2}, f_y = \frac{1 - x}{(1 - x - y + xy)^2}, f_{xx} = \frac{2(1 - y)^2}{(1 - x - y + xy)^3}, \\ f_{xy} &= \frac{1}{(1 - x - y + xy)^2}, f_{yy} = \frac{2(1 - x)^2}{(1 - x - y + xy)^3} \\ &\Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] \\ &= 1 + x \cdot 1 + y \cdot 1 + \frac{1}{2} (x^2 \cdot 2 + 2xy \cdot 1 + y^2 \cdot 2) = 1 + x + y + x^2 + xy + y^2, \text{ quadratic approximation;} \end{aligned}$$

$$\begin{aligned}
 f_{xxx} &= \frac{6(1-y)^3}{(1-x-y+xy)^4}, f_{xxy} = \frac{[-4(1-x-y+xy) + 6(1-y)(1-x)(1-y)]}{(1-x-y+xy)^4}, \\
 f_{xyy} &= \frac{[-4(1-x-y+xy) + 6(1-x)(1-y)(1-x)]}{(1-x-y+xy)^4}, f_{yyy} = \frac{6(1-x)^3}{(1-x-y+xy)^4} \\
 &\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] \\
 &= 1 + x + y + x^2 + xy + y^2 + \frac{1}{6} (x^3 \cdot 6 + 3x^2 y \cdot 2 + 3xy^2 \cdot 2 + y^3 \cdot 6) \\
 &= 1 + x + y + x^2 + xy + y^2 + x^3 + x^2 y + xy^2 + y^3, \text{cubic approximation}
 \end{aligned}$$

11. $f(x, y) = \cos x \cos y \Rightarrow f_x = -\sin x \cos y, f_y = -\cos x \sin y, f_{xx} = -\cos x \cos y, f_{xy} = \sin x \sin y,$
 $f_{yy} = -\cos x \cos y \Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$
 $= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2} [x^2 \cdot (-1) + 2xy \cdot 0 + y^2 \cdot (-1)] = 1 - \frac{x^2}{2} - \frac{y^2}{2}, \text{quadratic approximation. Since all partial}$
 derivatives of f are products of sines and cosines, the absolute value of these derivatives is less than or equal
 to 1 $\Rightarrow E(x, y) \leq \frac{1}{6} [(0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + 0.1^3] \leq 0.00134.$

12. $f(x, y) = e^x \sin y \Rightarrow f_x = e^x \sin y, f_y = e^x \cos y, f_{xx} = e^x \sin y, f_{xy} = e^x \cos y, f_{yy} = -e^x \sin y$
 $\Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$
 $= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2} (x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0) = y + xy, \text{quadratic approximation. Now, } f_{xxx} = e^x \sin y,$
 $f_{xxy} = e^x \cos y, f_{xyy} = -e^x \sin y, \text{ and } f_{yyy} = -e^x \cos y. \text{ Since } |x| \leq 0.1, |e^x \sin y| \leq |e^{0.1} \sin 0.1| \approx 0.11 \text{ and}$
 $|e^x \cos y| \leq |e^{0.1} \cos 0.1| \approx 1.11. \text{ Therefore,}$
 $E(x, y) \leq \frac{1}{6} [(0.11)(0.1)^3 + 3(1.11)(0.1)^3 + 3(0.11)(0.1)^3 + (1.11)(0.1)^3] \leq 0.000814.$

14.10 PARTIAL DERIVATIVES WITH CONSTRAINED VARIABLES

1. $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$:

(a) $\begin{pmatrix} y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial y} \right)_z = \frac{\partial w}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial y}; \frac{\partial z}{\partial y} = 0 \text{ and } \frac{\partial z}{\partial y} = 2x \frac{\partial x}{\partial y} + 2y \frac{\partial y}{\partial y}$
 $= 2x \frac{\partial x}{\partial y} + 2y \Rightarrow 0 = 2x \frac{\partial x}{\partial y} + 2y \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x} \Rightarrow \left(\frac{\partial w}{\partial y} \right)_z = (2x) \left(-\frac{y}{x} \right) + (2y)(1) + (2z)(0) = -2y + 2y = 0$

(b) $\begin{pmatrix} x \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x = x \\ y = y(x, z) \\ z = z \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial z} \right)_x = \frac{\partial w}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial z}; \frac{\partial x}{\partial z} = 0 \text{ and } \frac{\partial z}{\partial z} = 2x \frac{\partial x}{\partial z} + 2y \frac{\partial y}{\partial z}$
 $\Rightarrow 1 = 2y \frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = \frac{1}{2y} \Rightarrow \left(\frac{\partial w}{\partial z} \right)_x = (2x)(0) + (2y) \left(\frac{1}{2y} \right) + (2z)(1) = 1 + 2z$

(c) $\begin{pmatrix} y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial z} \right)_y = \frac{\partial w}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial z}; \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial z}{\partial z} = 2x \frac{\partial x}{\partial z} + 2y \frac{\partial y}{\partial z}$
 $\Rightarrow 1 = 2x \frac{\partial x}{\partial z} \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x} \Rightarrow \left(\frac{\partial w}{\partial z} \right)_y = (2x) \left(\frac{1}{2x} \right) + (2y)(0) + (2z)(1) = 1 + 2z$

2. $w = x^2 + y - z + \sin t$ and $x + y = t$:

(a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial y} \right)_{x,z} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial y}; \frac{\partial x}{\partial y} = 0, \frac{\partial z}{\partial y} = 0, \text{ and}$
 $\frac{\partial t}{\partial y} = 1 \Rightarrow \left(\frac{\partial w}{\partial y} \right)_{x,t} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t = 1 + \cos(x + y)$

(b) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \rightarrow \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial y} \right)_{z,t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial y}; \frac{\partial z}{\partial y} = 0 \text{ and } \frac{\partial t}{\partial y} = 0$
 $\Rightarrow \frac{\partial x}{\partial y} = \frac{\partial t}{\partial y} - \frac{\partial y}{\partial y} = -1 \Rightarrow \left(\frac{\partial w}{\partial y} \right)_{z,t} = (2x)(-1) + (1)(1) + (-1)(0) + (\cos t)(0) = 1 - 2(t - y) = 1 + 2y - 2t$

$$(c) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial z} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial z} \cdot \frac{\partial x}{\partial z} = 0 \text{ and } \frac{\partial y}{\partial z} = 0$$

$$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$$

$$(d) \begin{pmatrix} y \\ z \\ t \end{pmatrix} \rightarrow \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial z} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial z} \cdot \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial t}{\partial z} = 0$$

$$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$$

$$(e) \begin{pmatrix} x \\ z \\ t \end{pmatrix} \rightarrow \begin{pmatrix} x = x \\ y = t - x \\ z = z \\ t = t \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial t} \cdot \frac{\partial x}{\partial t} = 0 \text{ and } \frac{\partial z}{\partial t} = 0$$

$$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t$$

$$(f) \begin{pmatrix} y \\ z \\ t \end{pmatrix} \rightarrow \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial t} \cdot \frac{\partial y}{\partial t} = 0 \text{ and } \frac{\partial z}{\partial t} = 0$$

$$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = (2x)(1) + (1)(0) + (-1)(0) + (\cos t)(1) = \cos t + 2x = \cos t + 2(t - y)$$

3. $U = f(P, V, T)$ and $PV = nRT$

$$(a) \begin{pmatrix} P \\ V \end{pmatrix} \rightarrow \begin{pmatrix} P = P \\ V = V \\ T = \frac{PV}{nR} \end{pmatrix} \rightarrow U \Rightarrow \left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial U}{\partial P} \frac{\partial P}{\partial P} + \frac{\partial U}{\partial V} \frac{\partial V}{\partial P} + \frac{\partial U}{\partial T} \frac{\partial T}{\partial P} = \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial V}\right)(0) + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$$

$$= \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$$

$$(b) \begin{pmatrix} V \\ T \end{pmatrix} \rightarrow \begin{pmatrix} P = \frac{nRT}{V} \\ V = V \\ T = T \end{pmatrix} \rightarrow U \Rightarrow \left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial U}{\partial P} \frac{\partial P}{\partial T} + \frac{\partial U}{\partial V} \frac{\partial V}{\partial T} + \frac{\partial U}{\partial T} \frac{\partial T}{\partial T} = \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \left(\frac{\partial U}{\partial V}\right)(0) + \frac{\partial U}{\partial T}$$

$$= \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \frac{\partial U}{\partial T}$$

4. $w = x^2 + y^2 + z^2$ and $y \sin z + z \sin x = 0$

$$(a) \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial x}\right)_y = \frac{\partial w}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial x} \cdot \frac{\partial y}{\partial x} = 0 \text{ and}$$

$$(y \cos z) \frac{\partial z}{\partial x} + (\sin x) \frac{\partial y}{\partial x} + z \cos x = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-z \cos x}{y \cos z + \sin x}. \text{ At } (0, 1, \pi), \frac{\partial z}{\partial x} = \frac{-\pi}{-1} = \pi$$

$$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y|_{(0,1,\pi)}} = (2x)(1) + (2y)(0) + (2z)(\pi)|_{(0,1,\pi)} = 2\pi^2$$

$$(b) \begin{pmatrix} y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial z} = (2x) \frac{\partial x}{\partial z} + (2y)(0) + (2z)(1)$$

$$= (2x) \frac{\partial x}{\partial z} + 2z. \text{ Now } (\sin z) \frac{\partial y}{\partial z} + y \cos z + \sin x + (z \cos x) \frac{\partial x}{\partial z} = 0 \text{ and } \frac{\partial y}{\partial z} = 0$$

$$\Rightarrow y \cos z + \sin x + (z \cos x) \frac{\partial x}{\partial z} = 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{-y \cos z - \sin x}{z \cos x}. \text{ At } (0, 1, \pi), \frac{\partial x}{\partial z} = \frac{1-0}{(\pi)(1)} = \frac{1}{\pi}$$

$$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y|_{(0,1,\pi)}} = 2(0) \left(\frac{1}{\pi}\right) + 2\pi = 2\pi$$

5. $w = x^2 y^2 + yz - z^3$ and $x^2 + y^2 + z^2 = 6$

$$(a) \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial y}$$

$$= (2xy^2)(0) + (2x^2y + z)(1) + (y - 3z^2) \frac{\partial z}{\partial y} = 2x^2y + z + (y - 3z^2) \frac{\partial z}{\partial y}. \text{ Now } (2x) \frac{\partial x}{\partial y} + 2y + (2z) \frac{\partial z}{\partial y} = 0 \text{ and } \frac{\partial x}{\partial y} = 0 \Rightarrow 2y + (2z) \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}. \text{ At } (w, x, y, z) = (4, 2, 1, -1), \frac{\partial z}{\partial y} = -\frac{1}{-1} = 1 \Rightarrow \left(\frac{\partial w}{\partial y} \right)_x \Big|_{(4, 2, 1, -1)} = [(2)(2)^2(1) + (-1)] + [1 - 3(-1)^2](1) = 5$$

$$(b) \begin{pmatrix} y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \rightarrow w \Rightarrow \left(\frac{\partial w}{\partial y} \right)_z = \frac{\partial w}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial y} = (2xy^2) \frac{\partial x}{\partial y} + (2x^2y + z)(1) + (y - 3z^2)(0) = (2x^2y) \frac{\partial x}{\partial y} + 2x^2y + z. \text{ Now } (2x) \frac{\partial x}{\partial y} + 2y + (2z) \frac{\partial z}{\partial y} = 0 \text{ and } \frac{\partial z}{\partial y} = 0 \Rightarrow (2x) \frac{\partial x}{\partial y} + 2y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x}. \text{ At } (w, x, y, z) = (4, 2, 1, -1), \frac{\partial x}{\partial y} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial y} \right)_z \Big|_{(4, 2, 1, -1)} = (2)(2)(1)^2 \left(-\frac{1}{2}\right) + (2)(2)^2(1) + (-1) = 5$$

$$6. y = uv \Rightarrow 1 = v \frac{\partial u}{\partial y} + u \frac{\partial v}{\partial y}; x = u^2 + v^2 \text{ and } \frac{\partial x}{\partial y} = 0 \Rightarrow 0 = 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = \left(-\frac{u}{v}\right) \frac{\partial u}{\partial y} \Rightarrow 1 = v \frac{\partial u}{\partial y} + u \left(-\frac{u}{v} \frac{\partial u}{\partial y}\right) = \left(\frac{v^2 - u^2}{v}\right) \frac{\partial u}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = \frac{v}{v^2 - u^2}. \text{ At } (u, v) = (\sqrt{2}, 1), \frac{\partial u}{\partial y} = \frac{1}{1^2 - (\sqrt{2})^2} = -1 \Rightarrow \left(\frac{\partial u}{\partial y} \right)_x = -1$$

$$7. \begin{pmatrix} r \\ \theta \end{pmatrix} \rightarrow \begin{pmatrix} x = r \cos \theta \\ y = r \sin \theta \end{pmatrix} \Rightarrow \left(\frac{\partial x}{\partial r} \right)_\theta = \cos \theta; x^2 + y^2 = r^2 \Rightarrow 2x + 2y \frac{\partial y}{\partial x} = 2r \frac{\partial r}{\partial x} \text{ and } \frac{\partial y}{\partial x} = 0 \Rightarrow 2x = 2r \frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \left(\frac{\partial r}{\partial x} \right)_y = \frac{x}{\sqrt{x^2 + y^2}}$$

$$8. \text{ If } x, y, \text{ and } z \text{ are independent, then } \left(\frac{\partial w}{\partial x} \right)_{y,z} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial x} = (2x)(1) + (-2y)(0) + (4)(0) + (1) \left(\frac{\partial t}{\partial x} \right) = 2x + \frac{\partial t}{\partial x}. \text{ Thus } x + 2z + t = 25 \Rightarrow 1 + 0 + \frac{\partial t}{\partial x} = 0 \Rightarrow \frac{\partial t}{\partial x} = -1 \Rightarrow \left(\frac{\partial w}{\partial x} \right)_{y,z} = 2x - 1. \text{ On the other hand, if } x, y, \text{ and } t \text{ are independent, then } \left(\frac{\partial w}{\partial x} \right)_{y,t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial x} = (2x)(1) + (-2y)(0) + 4 \frac{\partial z}{\partial x} + (1)(0) = 2x + 4 \frac{\partial z}{\partial x}. \text{ Thus, } x + 2z + t = 25 \Rightarrow 1 + 2 \frac{\partial z}{\partial x} + 0 = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial x} \right)_{y,t} = 2x + 4 \left(-\frac{1}{2}\right) = 2x - 2.$$

$$9. \text{ If } x \text{ is a differentiable function of } y \text{ and } z, \text{ then } f(x, y, z) = 0 \Rightarrow \frac{\partial f}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} = 0 \Rightarrow \left(\frac{\partial x}{\partial y} \right)_z = -\frac{\partial f / \partial y}{\partial f / \partial x}. \text{ Similarly, if } y \text{ is a differentiable function of } x \text{ and } z, \left(\frac{\partial y}{\partial z} \right)_x = -\frac{\partial f / \partial z}{\partial f / \partial x} \text{ and if } z \text{ is a differentiable function of } x \text{ and } y, \left(\frac{\partial z}{\partial x} \right)_y = -\frac{\partial f / \partial x}{\partial f / \partial y}. \text{ Then } \left(\frac{\partial x}{\partial y} \right)_z \left(\frac{\partial y}{\partial z} \right)_x \left(\frac{\partial z}{\partial x} \right)_y = \left(-\frac{\partial f / \partial y}{\partial f / \partial x} \right) \left(-\frac{\partial f / \partial z}{\partial f / \partial y} \right) \left(-\frac{\partial f / \partial x}{\partial f / \partial z} \right) = -1.$$

$$10. z = z + f(u) \text{ and } u = xy \Rightarrow \frac{\partial z}{\partial x} = 1 + \frac{df}{du} \frac{\partial u}{\partial x} = 1 + y \frac{df}{du}; \text{ also } \frac{\partial z}{\partial y} = 0 + \frac{df}{du} \frac{\partial u}{\partial y} = x \frac{df}{du} \text{ so that } x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = x \left(1 + y \frac{df}{du} \right) - y \left(x \frac{df}{du} \right) = x$$

$$11. \text{ If } x \text{ and } y \text{ are independent, then } g(x, y, z) = 0 \Rightarrow \frac{\partial g}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial y} = 0 \text{ and } \frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial y} = 0 \Rightarrow \left(\frac{\partial z}{\partial y} \right)_x = -\frac{\partial g / \partial y}{\partial g / \partial z}, \text{ as claimed.}$$

$$12. \text{ Let } x \text{ and } y \text{ be independent. Then } f(x, y, z, w) = 0, g(x, y, z, w) = 0 \text{ and } \frac{\partial y}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = 0 \text{ and } \frac{\partial g}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = 0 \text{ imply}$$

$$\begin{cases} \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = -\frac{\partial g}{\partial x} \end{cases} \Rightarrow \left(\frac{\partial z}{\partial x} \right)_y = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial w} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial x} \frac{\partial g}{\partial w} + \frac{\partial g}{\partial x} \frac{\partial f}{\partial w}}{\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial x} \frac{\partial g}{\partial w} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial w}}{\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial w}}, \text{ as claimed.}$$

Likewise, $f(x, y, z, w) = 0$, $g(x, y, z, w) = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial y} = 0$ and (similarly) $\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial y} = 0$ imply

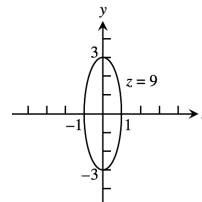
$$\begin{cases} \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial y} = -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial y} = -\frac{\partial g}{\partial y} \end{cases} \Rightarrow \left(\frac{\partial w}{\partial y} \right)_x = \frac{\begin{vmatrix} \frac{\partial f}{\partial z} & -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z} & -\frac{\partial g}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial z} \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial z} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial w}}, \text{ as claimed.}$$

CHAPTER 14 PRACTICE EXERCISES

1. Domain: All points in the xy -plane

Range: $z \geq 0$

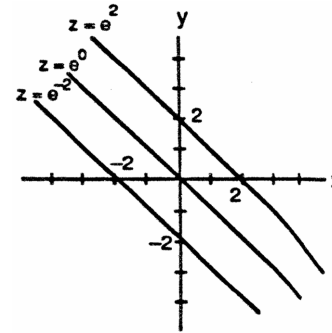
Level curves are ellipses with major axis along the y -axis and minor axis along the x -axis.



2. Domain: All points in the xy -plane

Range: $0 < z < \infty$

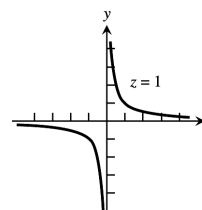
Level curves are the straight lines $x + y = \ln z$ with slope -1 , and $z > 0$.



3. Domain: All (x, y) such that $x \neq 0$ and $y \neq 0$

Range: $z \neq 0$

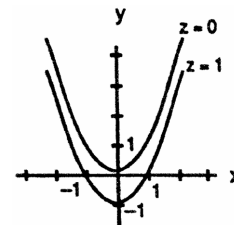
Level curves are hyperbolas with the x - and y -axes as asymptotes.



4. Domain: All (x, y) so that $x^2 - y \geq 0$

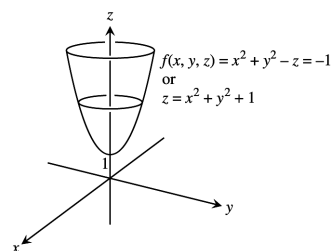
Range: $z \geq 0$

Level curves are the parabolas $y = x^2 - c$, $c \geq 0$.



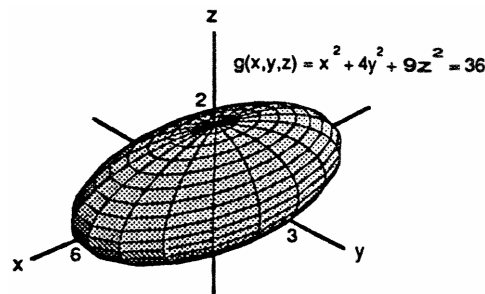
5. Domain: All points (x, y, z) in space
Range: All real numbers

Level surfaces are paraboloids of revolution with the z -axis as axis.



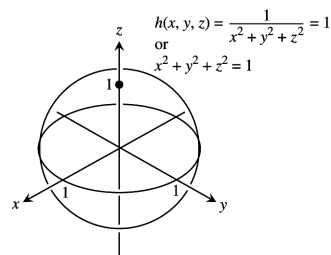
6. Domain: All points (x, y, z) in space
Range: Nonnegative real numbers

Level surfaces are ellipsoids with center $(0, 0, 0)$.



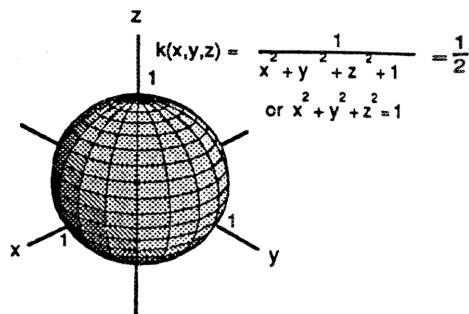
7. Domain: All (x, y, z) such that $(x, y, z) \neq (0, 0, 0)$
Range: Positive real numbers

Level surfaces are spheres with center $(0, 0, 0)$ and radius $r > 0$.



8. Domain: All points (x, y, z) in space
Range: $(0, 1]$

Level surfaces are spheres with center $(0, 0, 0)$ and radius $r > 0$.



9. $\lim_{(x,y) \rightarrow (\pi, \ln 2)} e^y \cos x = e^{\ln 2} \cos \pi = (2)(-1) = -2$

10. $\lim_{(x,y) \rightarrow (0,0)} \frac{2+y}{x+\cos y} = \frac{2+0}{0+\cos 0} = 2$

11. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq \pm y}} \frac{x-y}{x^2-y^2} = \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq \pm y}} \frac{x-y}{(x-y)(x+y)} = \lim_{(x,y) \rightarrow (1,1)} \frac{1}{x+y} = \frac{1}{1+1} = \frac{1}{2}$

12. $\lim_{(x,y) \rightarrow (1,1)} \frac{x^3y^3-1}{xy-1} = \lim_{(x,y) \rightarrow (1,1)} \frac{(xy-1)(x^2y^2+xy+1)}{xy-1} = \lim_{(x,y) \rightarrow (1,1)} (x^2y^2+xy+1) = 1^2 \cdot 1^2 + 1 \cdot 1 + 1 = 3$

13. $\lim_{P \rightarrow (1,-1,e)} \ln |x+y+z| = \ln |1+(-1)+e| = \ln e = 1$

14. $\lim_{P \rightarrow (1,-1,-1)} \tan^{-1}(x+y+z) = \tan^{-1}(1+(-1)+(-1)) = \tan^{-1}(-1) = -\frac{\pi}{4}$

15. Let $y = kx^2$, $k \neq 1$. Then $\lim_{\substack{(x,y) \rightarrow (0,0) \\ y \neq x^2}} \frac{y}{x^2 - y} = \lim_{(x, kx^2) \rightarrow (0,0)} \frac{kx^2}{x^2 - kx^2} = \frac{k}{1 - k^2}$ which gives different limits for different values of $k \Rightarrow$ the limit does not exist.

16. Let $y = kx$, $k \neq 0$. Then $\lim_{\substack{(x,y) \rightarrow (0,0) \\ xy \neq 0}} \frac{x^2 + y^2}{xy} = \lim_{(x, kx) \rightarrow (0,0)} \frac{x^2 + (kx)^2}{x(kx)} = \frac{1 + k^2}{k}$ which gives different limits for different values of $k \Rightarrow$ the limit does not exist.

17. Let $y = kx$. Then $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2 - k^2x^2}{x^2 + k^2x^2} = \frac{1 - k^2}{1 + k^2}$ which gives different limits for different values of $k \Rightarrow$ the limit does not exist so $f(0, 0)$ cannot be defined in a way that makes f continuous at the origin.

18. Along the x -axis, $y = 0$ and $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x-y)}{|x|+|y|} = \lim_{x \rightarrow 0} \frac{\sin x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$, so the limit fails to exist $\Rightarrow f$ is not continuous at $(0, 0)$.

$$19. \frac{\partial g}{\partial r} = \cos \theta + \sin \theta, \frac{\partial g}{\partial \theta} = -r \sin \theta + r \cos \theta$$

$$20. \frac{\partial f}{\partial x} = \frac{1}{2} \left(\frac{2x}{x^2 + y^2} \right) + \frac{\left(-\frac{y}{x^2} \right)}{1 + \left(\frac{y}{x} \right)^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} = \frac{x - y}{x^2 + y^2}, \frac{\partial f}{\partial y} = \frac{1}{2} \left(\frac{2y}{x^2 + y^2} \right) + \frac{\left(\frac{1}{x} \right)}{1 + \left(\frac{y}{x} \right)^2} = \frac{y}{x^2 + y^2} + \frac{x}{x^2 + y^2} = \frac{x + y}{x^2 + y^2}$$

$$21. \frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}, \frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}, \frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$$

$$22. h_x(x, y, z) = 2\pi \cos(2\pi x + y - 3z), h_y(x, y, z) = \cos(2\pi x + y - 3z), h_z(x, y, z) = -3 \cos(2\pi x + y - 3z)$$

$$23. \frac{\partial P}{\partial n} = \frac{RT}{V}, \frac{\partial P}{\partial R} = \frac{nT}{V}, \frac{\partial P}{\partial T} = \frac{nR}{V}, \frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$$

$$24. f_r(r, \ell, T, w) = -\frac{1}{2r\ell} \sqrt{\frac{T}{\pi w}}, f_\ell(r, \ell, T, w) = -\frac{1}{2r\ell^2} \sqrt{\frac{T}{\pi w}}, f_r(r, \ell, T, w) = \left(\frac{1}{2r\ell} \right) \left(\frac{1}{\sqrt{\pi w}} \right) \left(\frac{1}{2\sqrt{T}} \right) \\ = \frac{1}{4r\ell} \sqrt{\frac{1}{T\pi w}} = \frac{1}{4r\ell T} \sqrt{\frac{T}{\pi w}}, f_w(r, \ell, T, w) = \left(\frac{1}{2r\ell} \right) \sqrt{\frac{T}{\pi}} \left(-\frac{1}{2} w^{-3/2} \right) = -\frac{1}{4r\ell w} \sqrt{\frac{T}{\pi w}}$$

$$25. \frac{\partial g}{\partial x} = \frac{1}{y}, \frac{\partial g}{\partial y} = 1 - \frac{x}{y^2} \Rightarrow \frac{\partial^2 g}{\partial x^2} = 0, \frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}, \frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{y^2}$$

$$26. g_x(x, y) = e^x + y \cos x, g_y(x, y) = \sin x \Rightarrow g_{xx}(x, y) = e^x - y \sin x, g_{yy}(x, y) = 0, g_{xy}(x, y) = g_{yx}(x, y) = \cos x$$

$$27. \frac{\partial f}{\partial x} = 1 + y - 15x^2 + \frac{2x}{x^2 + 1}, \frac{\partial f}{\partial y} = x \Rightarrow \frac{\partial^2 f}{\partial x^2} = -30x + \frac{2 - 2x^2}{(x^2 + 1)^2}, \frac{\partial^2 f}{\partial y^2} = 0, \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$$

$$28. f_x(x, y) = -3y, f_y(x, y) = 2y - 3x - \sin y + 7e^y \Rightarrow f_{xx}(x, y) = 0, f_{yy}(x, y) = 2 - \cos y + 7e^y, f_{xy}(x, y) = f_{yx}(x, y) = -3$$

$$29. \frac{\partial w}{\partial x} = y \cos(xy + \pi), \frac{\partial w}{\partial y} = x \cos(xy + \pi), \frac{dx}{dt} = e^t, \frac{dy}{dt} = \frac{1}{t+1} \\ \Rightarrow \frac{dw}{dt} = [y \cos(xy + \pi)]e^t + [x \cos(xy + \pi)] \left(\frac{1}{t+1} \right); t = 0 \Rightarrow x = 1 \text{ and } y = 0 \\ \Rightarrow \frac{dw}{dt} \Big|_{t=0} = 0 \cdot 1 + [1 \cdot (-1)] \left(\frac{1}{0+1} \right) = -1$$

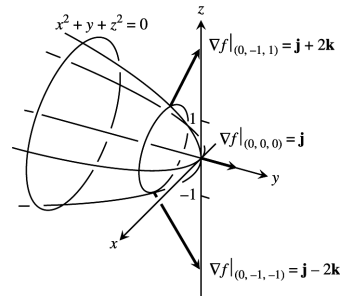
$$30. \frac{\partial w}{\partial x} = e^y, \frac{\partial w}{\partial y} = xe^y + \sin z, \frac{\partial w}{\partial z} = y \cos z + \sin z, \frac{dx}{dt} = t^{-1/2}, \frac{dy}{dt} = 1 + \frac{1}{t}, \frac{dz}{dt} = \pi \\ \Rightarrow \frac{dw}{dt} = e^y t^{-1/2} + (xe^y + \sin z) \left(1 + \frac{1}{t} \right) + (y \cos z + \sin z)\pi; t = 1 \Rightarrow x = 2, y = 0, \text{ and } z = \pi \\ \Rightarrow \frac{dw}{dt} \Big|_{t=1} = 1 \cdot 1 + (2 \cdot 1 + 0)(2) + (0 + 0)\pi = 5$$

31. $\frac{\partial w}{\partial x} = 2 \cos(2x - y)$, $\frac{\partial w}{\partial y} = -\cos(2x - y)$, $\frac{\partial x}{\partial r} = 1$, $\frac{\partial x}{\partial s} = \cos s$, $\frac{\partial y}{\partial r} = s$, $\frac{\partial y}{\partial s} = r$
 $\Rightarrow \frac{\partial w}{\partial r} = [2 \cos(2x - y)](1) + [-\cos(2x - y)](s)$; $r = \pi$ and $s = 0 \Rightarrow x = \pi$ and $y = 0$
 $\Rightarrow \frac{\partial w}{\partial r} \Big|_{(\pi, 0)} = (2 \cos 2\pi) - (\cos 2\pi)(0) = 2$; $\frac{\partial w}{\partial s} = [2 \cos(2x - y)](\cos s) + [-\cos(2x - y)](r)$
 $\Rightarrow \frac{\partial w}{\partial s} \Big|_{(\pi, 0)} = (2 \cos 2\pi)(\cos 0) - (\cos 2\pi)(\pi) = 2 - \pi$
32. $\frac{\partial w}{\partial u} = \frac{dw}{dx} \frac{\partial x}{\partial u} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(2e^u \cos v)$; $u = v = 0 \Rightarrow x = 2 \Rightarrow \frac{\partial w}{\partial u} \Big|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(2) = \frac{2}{5}$;
 $\frac{\partial w}{\partial v} = \frac{dw}{dx} \frac{\partial x}{\partial v} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(-2e^u \sin v) \Rightarrow \frac{\partial w}{\partial v} \Big|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(0) = 0$
33. $\frac{\partial f}{\partial x} = y + z$, $\frac{\partial f}{\partial y} = x + z$, $\frac{\partial f}{\partial z} = y + x$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = -2 \sin 2t$
 $\Rightarrow \frac{df}{dt} = -(y + z)(\sin t) + (x + z)(\cos t) - 2(y + x)(\sin 2t)$; $t = 1 \Rightarrow x = \cos 1$, $y = \sin 1$, and $z = \cos 2$
 $\Rightarrow \frac{df}{dt} \Big|_{t=1} = -(\sin 1 + \cos 2)(\sin 1) + (\cos 1 + \cos 2)(\cos 1) - 2(\sin 1 + \cos 1)(\sin 2)$
34. $\frac{\partial w}{\partial x} = \frac{dw}{ds} \frac{\partial s}{\partial x} = (5) \frac{dw}{ds}$ and $\frac{\partial w}{\partial y} = \frac{dw}{ds} \frac{\partial s}{\partial y} = (1) \frac{dw}{ds} = \frac{dw}{ds} \Rightarrow \frac{\partial w}{\partial x} - 5 \frac{\partial w}{\partial y} = 5 \frac{dw}{ds} - 5 \frac{dw}{ds} = 0$
35. $F(x, y) = 1 - x - y^2 - \sin xy \Rightarrow F_x = -1 - y \cos xy$ and $F_y = -2y - x \cos xy \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-1 - y \cos xy}{-2y - x \cos xy}$
 $= \frac{1 + y \cos xy}{-2y - x \cos xy} \Rightarrow$ at $(x, y) = (0, 1)$ we have $\frac{dy}{dx} \Big|_{(0,1)} = \frac{1+1}{-2} = -1$
36. $F(x, y) = 2xy + e^{x+y} - 2 \Rightarrow F_x = 2y + e^{x+y}$ and $F_y = 2x + e^{x+y} \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$
 $\Rightarrow$ at $(x, y) = (0, \ln 2)$ we have $\frac{dy}{dx} \Big|_{(0, \ln 2)} = -\frac{2 \ln 2 + 2}{0 + 2} = -(\ln 2 + 1)$
37. $\nabla f = (-\sin x \cos y)\mathbf{i} - (\cos x \sin y)\mathbf{j} \Rightarrow \nabla f \Big|_{(\frac{\pi}{4}, \frac{\pi}{4})} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} \Rightarrow |\nabla f| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$;
 $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \Rightarrow f$ increases most rapidly in the direction $\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \frac{\sqrt{2}}{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\frac{\sqrt{2}}{2}$;
 $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 4\mathbf{j}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = \left(-\frac{1}{2}\right)\left(\frac{3}{5}\right) + \left(-\frac{1}{2}\right)\left(\frac{4}{5}\right) = -\frac{7}{10}$
38. $\nabla f = 2xe^{-2y}\mathbf{i} - 2x^2e^{-2y}\mathbf{j} \Rightarrow \nabla f \Big|_{(1,0)} = 2\mathbf{i} - 2\mathbf{j} \Rightarrow |\nabla f| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$
 $\Rightarrow f$ increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 2\sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{2}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$
 $\Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (2)\left(\frac{1}{\sqrt{2}}\right) + (-2)\left(\frac{1}{\sqrt{2}}\right) = 0$
39. $\nabla f = \left(\frac{2}{2x + 3y + 6z}\right)\mathbf{i} + \left(\frac{3}{2x + 3y + 6z}\right)\mathbf{j} + \left(\frac{6}{2x + 3y + 6z}\right)\mathbf{k} \Rightarrow \nabla f \Big|_{(-1, -1, 1)} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$;
 $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow f$ increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 7$, $(D_{-\mathbf{u}}f)_{P_0} = -7$;
 $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = (D_{\mathbf{u}}f)_{P_0} = 7$
40. $\nabla f = (2x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j} + (1 - 2z)\mathbf{k} \Rightarrow \nabla f \Big|_{(0,0,0)} = 2\mathbf{j} + \mathbf{k}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k} \Rightarrow f$ increases most rapidly in the direction $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$;
 $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \sqrt{5}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{5}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
 $\Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (0)\left(\frac{1}{\sqrt{3}}\right) + (2)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$

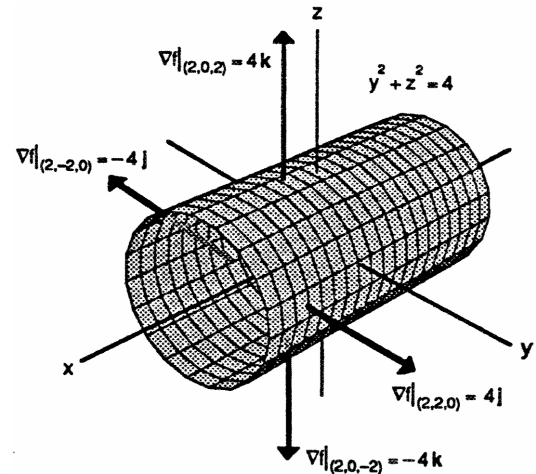
41. $\mathbf{r} = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v}(t) = (-3 \sin 3t)\mathbf{i} + (3 \cos 3t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{3}\right) = -3\mathbf{j} + 3\mathbf{k}$
 $\Rightarrow \mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}; f(x, y, z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}; t = \frac{\pi}{3} \text{ yields the point on the helix } (-1, 0, \pi)$
 $\Rightarrow \nabla f|_{(-1, 0, \pi)} = -\pi\mathbf{j} \Rightarrow \nabla f \cdot \mathbf{u} = (-\pi\mathbf{j}) \cdot \left(-\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) = \frac{\pi}{\sqrt{2}}$
42. $f(x, y, z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}; \text{ at } (1, 1, 1) \text{ we get } \nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \text{the maximum value of}$
 $D_{\mathbf{u}}f|_{(1, 1, 1)} = |\nabla f| = \sqrt{3}$
43. (a) Let $\nabla f = a\mathbf{i} + b\mathbf{j}$ at $(1, 2)$. The direction toward $(2, 2)$ is determined by $\mathbf{v}_1 = (2 - 1)\mathbf{i} + (2 - 2)\mathbf{j} = \mathbf{i} = \mathbf{u}$
 so that $\nabla f \cdot \mathbf{u} = 2 \Rightarrow a = 2$. The direction toward $(1, 1)$ is determined by $\mathbf{v}_2 = (1 - 1)\mathbf{i} + (1 - 2)\mathbf{j} = -\mathbf{j} = \mathbf{u}$
 so that $\nabla f \cdot \mathbf{u} = -2 \Rightarrow -b = -2 \Rightarrow b = 2$. Therefore $\nabla f = 2\mathbf{i} + 2\mathbf{j}; f_x(1, 2) = f_y(1, 2) = 2$.
- (b) The direction toward $(4, 6)$ is determined by $\mathbf{v}_3 = (4 - 1)\mathbf{i} + (6 - 2)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
 $\Rightarrow \nabla f \cdot \mathbf{u} = \frac{14}{5}$.

44. (a) True (b) False (c) True (d) True

45. $\nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow$
 $\nabla f|_{(0, -1, -1)} = \mathbf{j} - 2\mathbf{k},$
 $\nabla f|_{(0, 0, 0)} = \mathbf{j},$
 $\nabla f|_{(0, -1, 1)} = \mathbf{j} + 2\mathbf{k}$



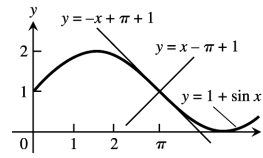
46. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$
 $\nabla f|_{(2, 2, 0)} = 4\mathbf{j},$
 $\nabla f|_{(2, -2, 0)} = -4\mathbf{j},$
 $\nabla f|_{(2, 0, 2)} = 4\mathbf{k},$
 $\nabla f|_{(2, 0, -2)} = -4\mathbf{k}$



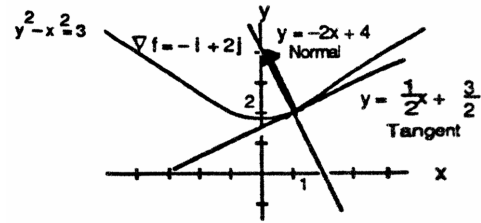
47. $\nabla f = 2x\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow \nabla f|_{(2, -1, 1)} = 4\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow \text{Tangent Plane: } 4(x - 2) - (y + 1) - 5(z - 1) = 0$
 $\Rightarrow 4x - y - 5z = 4; \text{ Normal Line: } x = 2 + 4t, y = -1 - t, z = 1 - 5t$
48. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f|_{(1, 1, 2)} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \text{Tangent Plane: } 2(x - 1) + 2(y - 1) + (z - 2) = 0$
 $\Rightarrow 2x + 2y + z - 6 = 0; \text{ Normal Line: } x = 1 + 2t, y = 1 + 2t, z = 2 + t$
49. $\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2} \Rightarrow \frac{\partial z}{\partial x}|_{(0, 1, 0)} = 0 \text{ and } \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2} \Rightarrow \frac{\partial z}{\partial y}|_{(0, 1, 0)} = 2; \text{ thus the tangent plane is}$
 $2(y - 1) - (z - 0) = 0 \text{ or } 2y - z - 2 = 0$

50. $\frac{\partial z}{\partial x} = -2x(x^2 + y^2)^{-2} \Rightarrow \frac{\partial z}{\partial x}\bigg|_{(1,1,\frac{1}{2})} = -\frac{1}{2}$ and $\frac{\partial z}{\partial y} = -2y(x^2 + y^2)^{-2} \Rightarrow \frac{\partial z}{\partial y}\bigg|_{(1,1,\frac{1}{2})} = -\frac{1}{2}$; thus the tangent plane is $-\frac{1}{2}(x-1) - \frac{1}{2}(y-1) - (z-\frac{1}{2}) = 0$ or $x + y + 2z - 3 = 0$

51. $\nabla f = (-\cos x)\mathbf{i} + \mathbf{j} \Rightarrow \nabla f|_{(\pi,1)} = \mathbf{i} + \mathbf{j} \Rightarrow$ the tangent line is $(x - \pi) + (y - 1) = 0 \Rightarrow x + y = \pi + 1$; the normal line is $y - 1 = 1(x - \pi) \Rightarrow y = x - \pi + 1$



52. $\nabla f = -x\mathbf{i} + y\mathbf{j} \Rightarrow \nabla f|_{(1,2)} = -\mathbf{i} + 2\mathbf{j} \Rightarrow$ the tangent line is $-(x - 1) + 2(y - 2) = 0 \Rightarrow y = \frac{1}{2}x + \frac{3}{2}$; the normal line is $y - 2 = -2(x - 1) \Rightarrow y = -2x + 4$



53. Let $f(x, y, z) = x^2 + 2y + 2z - 4$ and $g(x, y, z) = y - 1$. Then $\nabla f = 2x\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}|_{(1,1,\frac{1}{2})} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$$\text{and } \nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow \text{the line is } x = 1 - 2t, y = 1, z = \frac{1}{2} + 2t$$

54. Let $f(x, y, z) = x + y^2 + z - 2$ and $g(x, y, z) = y - 1$. Then $\nabla f = \mathbf{i} + 2y\mathbf{j} + \mathbf{k}|_{(\frac{1}{2},1,\frac{1}{2})} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{k} \Rightarrow \text{the line is } x = \frac{1}{2} - t, y = 1, z = \frac{1}{2} + t$$

55. $f(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{1}{2}$, $f_x(\frac{\pi}{4}, \frac{\pi}{4}) = \cos x \cos y|_{(\pi/4, \pi/4)} = \frac{1}{2}$, $f_y(\frac{\pi}{4}, \frac{\pi}{4}) = -\sin x \sin y|_{(\pi/4, \pi/4)} = -\frac{1}{2}$
 $\Rightarrow L(x, y) = \frac{1}{2} + \frac{1}{2}(x - \frac{\pi}{4}) - \frac{1}{2}(y - \frac{\pi}{4}) = \frac{1}{2} + \frac{1}{2}x - \frac{1}{2}y$; $f_{xx}(x, y) = -\sin x \cos y$, $f_{yy}(x, y) = -\sin x \cos y$, and $f_{xy}(x, y) = -\cos x \sin y$. Thus an upper bound for E depends on the bound M used for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$.

$$\text{With } M = \frac{\sqrt{2}}{2} \text{ we have } |E(x, y)| \leq \frac{1}{2} \left(\frac{\sqrt{2}}{2} \right) (|x - \frac{\pi}{4}| + |y - \frac{\pi}{4}|)^2 \leq \frac{\sqrt{2}}{4} (0.2)^2 \leq 0.0142;$$

$$\text{with } M = 1, |E(x, y)| \leq \frac{1}{2} (1) (|x - \frac{\pi}{4}| + |y - \frac{\pi}{4}|)^2 = \frac{1}{2} (0.2)^2 = 0.02.$$

56. $f(1, 1) = 0$, $f_x(1, 1) = y|_{(1,1)} = 1$, $f_y(1, 1) = x - 6y|_{(1,1)} = -5 \Rightarrow L(x, y) = (x - 1) - 5(y - 1) = x - 5y + 4$;

$$f_{xx}(x, y) = 0, f_{yy}(x, y) = -6, \text{ and } f_{xy}(x, y) = 1 \Rightarrow \text{maximum of } |f_{xx}|, |f_{yy}|, \text{ and } |f_{xy}| \text{ is } 6 \Rightarrow M = 6$$

$$\Rightarrow |E(x, y)| \leq \frac{1}{2} (6) (|x - 1| + |y - 1|)^2 = \frac{1}{2} (6)(0.1 + 0.2)^2 = 0.27$$

57. $f(1, 0, 0) = 0$, $f_x(1, 0, 0) = y - 3z|_{(1,0,0)} = 0$, $f_y(1, 0, 0) = x + 2z|_{(1,0,0)} = 1$, $f_z(1, 0, 0) = 2y - 3x|_{(1,0,0)} = -3$

$$\Rightarrow L(x, y, z) = 0(x - 1) + (y - 0) - 3(z - 0) = y - 3z; f(1, 1, 0) = 1, f_x(1, 1, 0) = 1, f_y(1, 1, 0) = 1, f_z(1, 1, 0) = -1$$

$$\Rightarrow L(x, y, z) = 1 + (x - 1) + (y - 1) - 1(z - 0) = x + y - z - 1$$

58. $f(0, 0, \frac{\pi}{4}) = 1$, $f_x(0, 0, \frac{\pi}{4}) = -\sqrt{2} \sin x \sin(y + z)|_{(0,0,\pi/4)} = 0$, $f_y(0, 0, \frac{\pi}{4}) = \sqrt{2} \cos x \cos(y + z)|_{(0,0,\pi/4)} = 1$,

$$f_z(0, 0, \frac{\pi}{4}) = \sqrt{2} \cos x \cos(y + z)|_{(0,0,\pi/4)} = 1 \Rightarrow L(x, y, z) = 1 + 1(y - 0) + 1(z - \frac{\pi}{4}) = 1 + y + z - \frac{\pi}{4};$$

$$f(\frac{\pi}{4}, \frac{\pi}{4}, 0) = \frac{\sqrt{2}}{2}, f_x(\frac{\pi}{4}, \frac{\pi}{4}, 0) = -\frac{\sqrt{2}}{2}, f_y(\frac{\pi}{4}, \frac{\pi}{4}, 0) = \frac{\sqrt{2}}{2}, f_z(\frac{\pi}{4}, \frac{\pi}{4}, 0) = \frac{\sqrt{2}}{2}$$

$$\Rightarrow L(x, y, z) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) + \frac{\sqrt{2}}{2}(y - \frac{\pi}{4}) + \frac{\sqrt{2}}{2}(z - 0) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}z$$

59. $V = \pi r^2 h \Rightarrow dV = 2\pi r h dr + \pi r^2 dh \Rightarrow dV|_{(1.5, 5280)} = 2\pi(1.5)(5280) dr + \pi(1.5)^2 dh = 15,840\pi dr + 2.25\pi dh$.

You should be more careful with the diameter since it has a greater effect on dV .

60. $df = (2x - y) dx + (-x + 2y) dy \Rightarrow df|_{(1, 2)} = 3 dy \Rightarrow f$ is more sensitive to changes in y ; in fact, near the point $(1, 2)$ a change in x does not change f .

61. $dI = \frac{1}{R} dV - \frac{V}{R^2} dR \Rightarrow dI|_{(24, 100)} = \frac{1}{100} dV - \frac{24}{100^2} dR \Rightarrow dI|_{dV=-1, dR=-20} = -0.01 + (480)(.0001) = 0.038$,
or increases by 0.038 amps; % change in $V = (100) \left(-\frac{1}{24}\right) \approx -4.17\%$; % change in $R = \left(-\frac{20}{100}\right)(100) = -20\%$;
 $I = \frac{24}{100} = 0.24 \Rightarrow$ estimated % change in $I = \frac{dI}{I} \times 100 = \frac{0.038}{0.24} \times 100 \approx 15.83\% \Rightarrow$ more sensitive to voltage change.

62. $A = \pi ab \Rightarrow dA = \pi b da + \pi a db \Rightarrow dA|_{(10, 16)} = 16\pi da + 10\pi db$; $da = \pm 0.1$ and $db = \pm 0.1$
 $\Rightarrow dA = \pm 26\pi(0.1) = \pm 2.6\pi$ and $A = \pi(10)(16) = 160\pi \Rightarrow \left|\frac{dA}{A} \times 100\right| = \left|\frac{2.6\pi}{160\pi} \times 100\right| \approx 1.625\%$

63. (a) $y = uv \Rightarrow dy = v du + u dv$; percentage change in $u \leq 2\% \Rightarrow |du| \leq 0.02$, and percentage change in $v \leq 3\%$
 $\Rightarrow |dv| \leq 0.03$; $\frac{dy}{y} = \frac{v du + u dv}{uv} = \frac{du}{u} + \frac{dv}{v} \Rightarrow \left|\frac{dy}{y} \times 100\right| = \left|\frac{du}{u} \times 100 + \frac{dv}{v} \times 100\right| \leq \left|\frac{du}{u} \times 100\right| + \left|\frac{dv}{v} \times 100\right|$
 $\leq 2\% + 3\% = 5\%$

(b) $z = u + v \Rightarrow \frac{dz}{z} = \frac{du + dv}{u + v} = \frac{du}{u + v} + \frac{dv}{u + v} \leq \frac{du}{u} + \frac{dv}{v}$ (since $u > 0, v > 0$)
 $\Rightarrow \left|\frac{dz}{z} \times 100\right| \leq \left|\frac{du}{u} \times 100 + \frac{dv}{v} \times 100\right| = \left|\frac{dy}{y} \times 100\right|$

64. $C = \frac{7}{71.84w^{0.425}h^{0.725}} \Rightarrow C_w = \frac{(-0.425)(7)}{71.84w^{1.425}h^{0.725}}$ and $C_h = \frac{(-0.725)(7)}{71.84w^{0.425}h^{1.725}}$
 $\Rightarrow dC = \frac{-2.975}{71.84w^{1.425}h^{0.725}} dw + \frac{-5.075}{71.84w^{0.425}h^{1.725}} dh$; thus when $w = 70$ and $h = 180$ we have
 $dC|_{(70, 180)} \approx -(0.00000225) dw - (0.00000149) dh \Rightarrow 1 \text{ kg error in weight has more effect}$

65. $f_x(x, y) = 2x - y + 2 = 0$ and $f_y(x, y) = -x + 2y + 2 = 0 \Rightarrow x = -2$ and $y = -2 \Rightarrow (-2, -2)$ is the critical point;
 $f_{xx}(-2, -2) = 2, f_{yy}(-2, -2) = 2, f_{xy}(-2, -2) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value
of $f(-2, -2) = -8$

66. $f_x(x, y) = 10x + 4y + 4 = 0$ and $f_y(x, y) = 4x - 4y - 4 = 0 \Rightarrow x = 0$ and $y = -1 \Rightarrow (0, -1)$ is the critical point;
 $f_{xx}(0, -1) = 10, f_{yy}(0, -1) = -4, f_{xy}(0, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -56 < 0 \Rightarrow$ saddle point with $f(0, -1) = 2$

67. $f_x(x, y) = 6x^2 + 3y = 0$ and $f_y(x, y) = 3x + 6y^2 = 0 \Rightarrow y = -2x^2$ and $3x + 6(4x^4) = 0 \Rightarrow x(1 + 8x^3) = 0$
 $\Rightarrow x = 0$ and $y = 0$, or $x = -\frac{1}{2}$ and $y = -\frac{1}{2} \Rightarrow$ the critical points are $(0, 0)$ and $(-\frac{1}{2}, -\frac{1}{2})$. For $(0, 0)$:
 $f_{xx}(0, 0) = 12x|_{(0, 0)} = 0, f_{yy}(0, 0) = 12y|_{(0, 0)} = 0, f_{xy}(0, 0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with
 $f(0, 0) = 0$. For $(-\frac{1}{2}, -\frac{1}{2})$: $f_{xx} = -6, f_{yy} = -6, f_{xy} = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum
value of $f(-\frac{1}{2}, -\frac{1}{2}) = \frac{1}{4}$

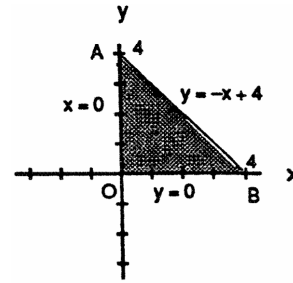
68. $f_x(x, y) = 3x^2 - 3y = 0$ and $f_y(x, y) = 3y^2 - 3x = 0 \Rightarrow y = x^2$ and $x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow$ the critical
points are $(0, 0)$ and $(1, 1)$. For $(0, 0)$: $f_{xx}(0, 0) = 6x|_{(0, 0)} = 0, f_{yy}(0, 0) = 6y|_{(0, 0)} = 0, f_{xy}(0, 0) = -3$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0, 0) = 15$. For $(1, 1)$: $f_{xx}(1, 1) = 6, f_{yy}(1, 1) = 6, f_{xy}(1, 1) = -3$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(1, 1) = 14$

69. $f_x(x, y) = 3x^2 + 6x = 0$ and $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow x(x + 2) = 0$ and $y(y - 2) = 0 \Rightarrow x = 0$ or $x = -2$ and
 $y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0, 0), (0, 2), (-2, 0)$, and $(-2, 2)$. For $(0, 0)$: $f_{xx}(0, 0) = 6x + 6|_{(0, 0)}$
 $= 6, f_{yy}(0, 0) = 6y - 6|_{(0, 0)} = -6, f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(0, 0) = 0$. For
 $(0, 2)$: $f_{xx}(0, 2) = 6, f_{yy}(0, 2) = 6, f_{xy}(0, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of

$f(0, 2) = -4$. For $(-2, 0)$: $f_{xx}(-2, 0) = -6$, $f_{yy}(-2, 0) = -6$, $f_{xy}(-2, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0$
 $\Rightarrow$ local maximum value of $f(-2, 0) = 4$. For $(-2, 2)$: $f_{xx}(-2, 2) = -6$, $f_{yy}(-2, 2) = 6$, $f_{xy}(-2, 2) = 0$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(-2, 2) = 0$.

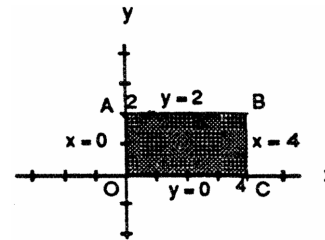
70. $f_x(x, y) = 4x^3 - 16x = 0 \Rightarrow 4x(x^2 - 4) = 0 \Rightarrow x = 0, 2, -2$; $f_y(x, y) = 6y - 6 = 0 \Rightarrow y = 1$. Therefore the critical points are $(0, 1)$, $(2, 1)$, and $(-2, 1)$. For $(0, 1)$: $f_{xx}(0, 1) = 12x^2 - 16|_{(0,1)} = -16$, $f_{yy}(0, 1) = 6$, $f_{xy}(0, 1) = 0$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -96 < 0 \Rightarrow$ saddle point with $f(0, 1) = -3$. For $(2, 1)$: $f_{xx}(2, 1) = 32$, $f_{yy}(2, 1) = 6$, $f_{xy}(2, 1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(2, 1) = -19$. For $(-2, 1)$: $f_{xx}(-2, 1) = 32$, $f_{yy}(-2, 1) = 6$, $f_{xy}(-2, 1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2, 1) = -19$.

71. (i) On OA, $f(x, y) = f(0, y) = y^2 + 3y$ for $0 \leq y \leq 4$
 $\Rightarrow f'(0, y) = 2y + 3 = 0 \Rightarrow y = -\frac{3}{2}$. But $(0, -\frac{3}{2})$ is not in the region.



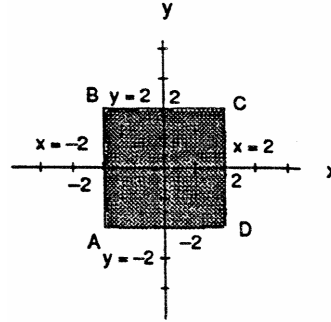
- Endpoints: $f(0, 0) = 0$ and $f(0, 4) = 28$.
(ii) On AB, $f(x, y) = f(x, -x + 4) = x^2 - 10x + 28$
for $0 \leq x \leq 4 \Rightarrow f'(x, -x + 4) = 2x - 10 = 0$
 $\Rightarrow x = 5, y = -1$. But $(5, -1)$ is not in the region.
Endpoints: $f(4, 0) = 4$ and $f(0, 4) = 28$.
(iii) On OB, $f(x, y) = f(x, 0) = x^2 - 3x$ for $0 \leq x \leq 4 \Rightarrow f'(x, 0) = 2x - 3 \Rightarrow x = \frac{3}{2}$ and $y = 0 \Rightarrow (\frac{3}{2}, 0)$ is a critical point with $f(\frac{3}{2}, 0) = -\frac{9}{4}$.
Endpoints: $f(0, 0) = 0$ and $f(4, 0) = 4$.
(iv) For the interior of the triangular region, $f_x(x, y) = 2x + y - 3 = 0$ and $f_y(x, y) = x + 2y + 3 = 0 \Rightarrow x = 3$ and $y = -3$. But $(3, -3)$ is not in the region. Therefore the absolute maximum is 28 at $(0, 4)$ and the absolute minimum is $-\frac{9}{4}$ at $(\frac{3}{2}, 0)$.

72. (i) On OA, $f(x, y) = f(0, y) = -y^2 + 4y + 1$ for $0 \leq y \leq 2 \Rightarrow f'(0, y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 0$. But $(0, 2)$ is not in the interior of OA.
Endpoints: $f(0, 0) = 1$ and $f(0, 2) = 5$.

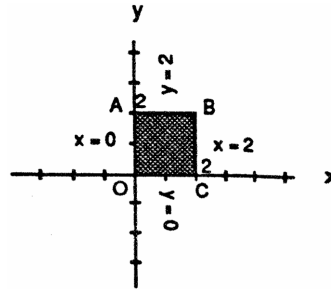


- (ii) On AB, $f(x, y) = f(x, 2) = x^2 - 2x + 5$ for $0 \leq x \leq 4 \Rightarrow f'(x, 2) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 2$
 $\Rightarrow (1, 2)$ is an interior critical point of AB with $f(1, 2) = 4$. Endpoints: $f(4, 2) = 13$ and $f(0, 2) = 5$.
(iii) On BC, $f(x, y) = f(4, y) = -y^2 + 4y + 9$ for $0 \leq y \leq 2 \Rightarrow f'(4, y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 4$. But $(4, 2)$ is not in the interior of BC. Endpoints: $f(4, 0) = 9$ and $f(4, 2) = 13$.
(iv) On OC, $f(x, y) = f(x, 0) = x^2 - 2x + 1$ for $0 \leq x \leq 4 \Rightarrow f'(x, 0) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1, 0)$ is an interior critical point of OC with $f(1, 0) = 0$. Endpoints: $f(0, 0) = 1$ and $f(4, 0) = 9$.
(v) For the interior of the rectangular region, $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2$. But $(1, 2)$ is not in the interior of the region. Therefore the absolute maximum is 13 at $(4, 2)$ and the absolute minimum is 0 at $(1, 0)$.

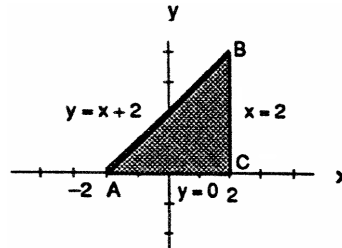
73. (i) On AB, $f(x, y) = f(-2, y) = y^2 - y - 4$ for $-2 \leq y \leq 2 \Rightarrow f'(-2, y) = 2y - 1 \Rightarrow y = \frac{1}{2}$ and $x = -2 \Rightarrow (-2, \frac{1}{2})$ is an interior critical point in AB with $f(-2, \frac{1}{2}) = -\frac{17}{4}$. Endpoints: $f(-2, -2) = 2$ and $f(-2, 2) = -2$.
- (ii) On BC, $f(x, y) = f(x, 2) = -2$ for $-2 \leq x \leq 2 \Rightarrow f'(x, 2) = 0 \Rightarrow$ no critical points in the interior of BC. Endpoints: $f(-2, 2) = -2$ and $f(2, 2) = -2$.
- (iii) On CD, $f(x, y) = f(2, y) = y^2 - 5y + 4$ for $-2 \leq y \leq 2 \Rightarrow f'(2, y) = 2y - 5 = 0 \Rightarrow y = \frac{5}{2}$ and $x = 2$. But $(2, \frac{5}{2})$ is not in the region. Endpoints: $f(2, -2) = 18$ and $f(2, 2) = -2$.
- (iv) On AD, $f(x, y) = f(x, -2) = 4x + 10$ for $-2 \leq x \leq 2 \Rightarrow f'(x, -2) = 4 \Rightarrow$ no critical points in the interior of AD. Endpoints: $f(-2, -2) = 2$ and $f(2, -2) = 18$.
- (v) For the interior of the square, $f_x(x, y) = -y + 2 = 0$ and $f_y(x, y) = 2y - x - 3 = 0 \Rightarrow y = 2$ and $x = 1 \Rightarrow (1, 2)$ is an interior critical point of the square with $f(1, 2) = -2$. Therefore the absolute maximum is 18 at $(2, -2)$ and the absolute minimum is $-\frac{17}{4}$ at $(-2, \frac{1}{2})$.



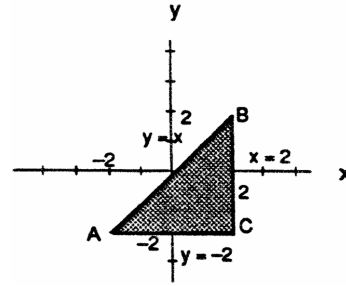
74. (i) On OA, $f(x, y) = f(0, y) = 2y - y^2$ for $0 \leq y \leq 2 \Rightarrow f'(0, y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 0 \Rightarrow (0, 1)$ is an interior critical point of OA with $f(0, 1) = 1$. Endpoints: $f(0, 0) = 0$ and $f(0, 2) = 0$.
- (ii) On AB, $f(x, y) = f(x, 2) = 2x - x^2$ for $0 \leq x \leq 2 \Rightarrow f'(x, 2) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1, 2)$ is an interior critical point of AB with $f(1, 2) = 1$. Endpoints: $f(0, 2) = 0$ and $f(2, 2) = 0$.
- (iii) On BC, $f(x, y) = f(2, y) = 2y - y^2$ for $0 \leq y \leq 2 \Rightarrow f'(2, y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 2 \Rightarrow (2, 1)$ is an interior critical point of BC with $f(2, 1) = 1$. Endpoints: $f(2, 0) = 0$ and $f(2, 2) = 0$.
- (iv) On OC, $f(x, y) = f(x, 0) = 2x - x^2$ for $0 \leq x \leq 2 \Rightarrow f'(x, 0) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1, 0)$ is an interior critical point of OC with $f(1, 0) = 1$. Endpoints: $f(0, 0) = 0$ and $f(2, 0) = 0$.
- (v) For the interior of the rectangular region, $f_x(x, y) = 2 - 2x = 0$ and $f_y(x, y) = 2 - 2y = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow (1, 1)$ is an interior critical point of the square with $f(1, 1) = 2$. Therefore the absolute maximum is 2 at $(1, 1)$ and the absolute minimum is 0 at the four corners $(0, 0)$, $(0, 2)$, $(2, 2)$, and $(2, 0)$.



75. (i) On AB, $f(x, y) = f(x, x + 2) = -2x + 4$ for $-2 \leq x \leq 2 \Rightarrow f'(x, x + 2) = -2 = 0 \Rightarrow$ no critical points in the interior of AB. Endpoints: $f(-2, 0) = 8$ and $f(2, 4) = 0$.
- (ii) On BC, $f(x, y) = f(2, y) = -y^2 + 4y$ for $0 \leq y \leq 4 \Rightarrow f'(2, y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 2 \Rightarrow (2, 2)$ is an interior critical point of BC with $f(2, 2) = 4$. Endpoints: $f(2, 0) = 0$ and $f(2, 4) = 0$.
- (iii) On AC, $f(x, y) = f(x, 0) = x^2 - 2x$ for $-2 \leq x \leq 2 \Rightarrow f'(x, 0) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1, 0)$ is an interior critical point of AC with $f(1, 0) = -1$. Endpoints: $f(-2, 0) = 8$ and $f(2, 0) = 0$.
- (iv) For the interior of the triangular region, $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1, 2)$ is an interior critical point of the region with $f(1, 2) = 3$. Therefore the absolute maximum is 8 at $(-2, 0)$ and the absolute minimum is -1 at $(1, 0)$.

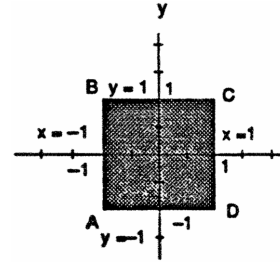


76. (i) On AB, $f(x, y) = f(x, x) = 4x^2 - 2x^4 + 16$ for $-2 \leq x \leq 2 \Rightarrow f'(x, x) = 8x - 8x^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$, or $x = -1$ and $y = -1 \Rightarrow (0, 0), (1, 1), (-1, -1)$ are all interior points of AB with $f(0, 0) = 16$, $f(1, 1) = 18$, and $f(-1, -1) = 18$. Endpoints: $f(-2, -2) = 0$ and $f(2, 2) = 0$.



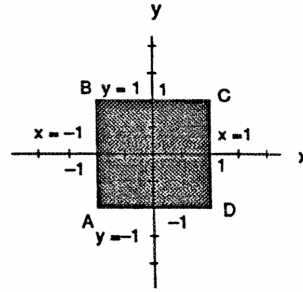
- (ii) On BC, $f(x, y) = f(2, y) = 8y - y^4$ for $-2 \leq y \leq 2 \Rightarrow f'(2, y) = 8 - 4y^3 = 0 \Rightarrow y = \sqrt[3]{2}$ and $x = 2 \Rightarrow (2, \sqrt[3]{2})$ is an interior critical point of BC with $f(2, \sqrt[3]{2}) = 6\sqrt[3]{2}$. Endpoints: $f(2, -2) = -32$ and $f(2, 2) = 0$.
- (iii) On AC, $f(x, y) = f(x, -2) = -8x - x^4$ for $-2 \leq x \leq 2 \Rightarrow f'(x, -2) = -8 - 4x^3 = 0 \Rightarrow x = \sqrt[3]{-2}$ and $y = -2 \Rightarrow (\sqrt[3]{-2}, -2)$ is an interior critical point of AC with $f(\sqrt[3]{-2}, -2) = 6\sqrt[3]{2}$. Endpoints: $f(-2, -2) = 0$ and $f(2, -2) = -32$.
- (iv) For the interior of the triangular region, $f_x(x, y) = 4y - 4x^3 = 0$ and $f_y(x, y) = 4x - 4y^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$ or $x = -1$ and $y = -1$. But neither of the points $(0, 0)$ and $(1, 1)$, or $(-1, -1)$ are interior to the region. Therefore the absolute maximum is 18 at $(1, 1)$ and $(-1, -1)$, and the absolute minimum is -32 at $(2, -2)$.

77. (i) On AB, $f(x, y) = f(-1, y) = y^3 - 3y^2 + 2$ for $-1 \leq y \leq 1 \Rightarrow f'(-1, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = -1$, or $y = 2$ and $x = -1 \Rightarrow (-1, 0)$ is an interior critical point of AB with $f(-1, 0) = 2$; $(-1, 2)$ is outside the boundary. Endpoints: $f(-1, -1) = -2$ and $f(-1, 1) = 0$.



- (ii) On BC, $f(x, y) = f(x, 1) = x^3 + 3x^2 - 2$ for $-1 \leq x \leq 1 \Rightarrow f'(x, 1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = 1$, or $x = -2$ and $y = 1 \Rightarrow (0, 1)$ is an interior critical point of BC with $f(0, 1) = -2$; $(-2, 1)$ is outside the boundary. Endpoints: $f(-1, 1) = 0$ and $f(1, 1) = 2$.
- (iii) On CD, $f(x, y) = f(1, y) = y^3 - 3y^2 + 4$ for $-1 \leq y \leq 1 \Rightarrow f'(1, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = 1$, or $y = 2$ and $x = 1 \Rightarrow (1, 0)$ is an interior critical point of CD with $f(1, 0) = 4$; $(1, 2)$ is outside the boundary. Endpoints: $f(1, 1) = 2$ and $f(1, -1) = 0$.
- (iv) On AD, $f(x, y) = f(x, -1) = x^3 + 3x^2 - 4$ for $-1 \leq x \leq 1 \Rightarrow f'(x, -1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = -1$, or $x = -2$ and $y = -1 \Rightarrow (0, -1)$ is an interior point of AD with $f(0, -1) = -4$; $(-2, -1)$ is outside the boundary. Endpoints: $f(-1, -1) = -2$ and $f(1, -1) = 0$.
- (v) For the interior of the square, $f_x(x, y) = 3x^2 + 6x = 0$ and $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow x = 0$ or $x = -2$, and $y = 0$ or $y = 2 \Rightarrow (0, 0)$ is an interior critical point of the square region with $f(0, 0) = 0$; the points $(0, 2)$, $(-2, 0)$, and $(-2, 2)$ are outside the region. Therefore the absolute maximum is 4 at $(1, 0)$ and the absolute minimum is -4 at $(0, -1)$.

78. (i) On AB, $f(x, y) = f(-1, y) = y^3 - 3y$ for $-1 \leq y \leq 1$
 $\Rightarrow f'(-1, y) = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$ and $x = -1$
yielding the corner points $(-1, -1)$ and $(-1, 1)$ with
 $f(-1, -1) = 2$ and $f(-1, 1) = -2$.
- (ii) On BC, $f(x, y) = f(x, 1) = x^3 + 3x + 2$ for
 $-1 \leq x \leq 1 \Rightarrow f'(x, 1) = 3x^2 + 3 = 0 \Rightarrow$ no
solution. Endpoints: $f(-1, 1) = -2$ and $f(1, 1) = 6$.
- (iii) On CD, $f(x, y) = f(1, y) = y^3 + 3y + 2$ for
 $-1 \leq y \leq 1 \Rightarrow f'(1, y) = 3y^2 + 3 = 0 \Rightarrow$ no
solution. Endpoints: $f(1, 1) = 6$ and $f(1, -1) = -2$.
- (iv) On AD, $f(x, y) = f(x, -1) = x^3 - 3x$ for $-1 \leq x \leq 1 \Rightarrow f'(x, -1) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$ and $y = -1$
yielding the corner points $(-1, -1)$ and $(1, -1)$ with $f(-1, -1) = 2$ and $f(1, -1) = -2$.
- (v) For the interior of the square, $f_x(x, y) = 3x^2 + 3y = 0$ and $f_y(x, y) = 3y^2 + 3x = 0 \Rightarrow y = -x^2$ and
 $x^4 + x = 0 \Rightarrow x = 0$ or $x = -1 \Rightarrow y = 0$ or $y = -1 \Rightarrow (0, 0)$ is an interior critical point of the square
region with $f(0, 0) = 1$; $(-1, -1)$ is on the boundary. Therefore the absolute maximum is 6 at $(1, 1)$ and
the absolute minimum is -2 at $(1, -1)$ and $(-1, 1)$.



79. $\nabla f = 3x^2\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3x^2\mathbf{i} + 2y\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 3x^2 = 2x\lambda$ and
 $2y = 2y\lambda \Rightarrow \lambda = 1$ or $y = 0$.
- CASE 1: $\lambda = 1 \Rightarrow 3x^2 = 2x \Rightarrow x = 0$ or $x = \frac{2}{3}$; $x = 0 \Rightarrow y = \pm 1$ yielding the points $(0, 1)$ and $(0, -1)$; $x = \frac{2}{3}$
 $\Rightarrow y = \pm \frac{\sqrt{5}}{3}$ yielding the points $(\frac{2}{3}, \frac{\sqrt{5}}{3})$ and $(\frac{2}{3}, -\frac{\sqrt{5}}{3})$.
- CASE 2: $y = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ yielding the points $(1, 0)$ and $(-1, 0)$.
- Evaluations give $f(0, \pm 1) = 1$, $f(\frac{2}{3}, \pm \frac{\sqrt{5}}{3}) = \frac{23}{27}$, $f(1, 0) = 1$, and $f(-1, 0) = -1$. Therefore the absolute
maximum is 1 at $(0, \pm 1)$ and $(1, 0)$, and the absolute minimum is -1 at $(-1, 0)$.
80. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2\lambda x$ and
 $xy = 2\lambda y \Rightarrow x = 2\lambda(2\lambda x) = 4\lambda^2 x \Rightarrow x = 0$ or $4\lambda^2 = 1$.
- CASE 1: $x = 0 \Rightarrow y = 0$ but $(0, 0)$ does not lie on the circle, so no solution.
- CASE 2: $4\lambda^2 = 1 \Rightarrow \lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = y = \pm \frac{1}{\sqrt{2}}$ yielding the
points $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. For $\lambda = -\frac{1}{2}$, $y = -x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$ and
 $y = -x$ yielding the points $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.
- Evaluations give the absolute maximum value $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2}$ and the absolute minimum
value $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\frac{1}{2}$.
81. (i) $f(x, y) = x^2 + 3y^2 + 2y$ on $x^2 + y^2 = 1 \Rightarrow \nabla f = 2x\mathbf{i} + (6y + 2)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g$
 $\Rightarrow 2x\mathbf{i} + (6y + 2)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x = 2x\lambda$ and $6y + 2 = 2y\lambda \Rightarrow \lambda = 1$ or $x = 0$.
- CASE 1: $\lambda = 1 \Rightarrow 6y + 2 = 2y \Rightarrow y = -\frac{1}{2}$ and $x = \pm \frac{\sqrt{3}}{2}$ yielding the points $(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2})$.
- CASE 2: $x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ yielding the points $(0, \pm 1)$.
- Evaluations give $f(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}) = \frac{1}{2}$, $f(0, 1) = 5$, and $f(0, -1) = 1$. Therefore $\frac{1}{2}$ and 5 are the extreme
values on the boundary of the disk.
- (ii) For the interior of the disk, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 6y + 2 = 0 \Rightarrow x = 0$ and $y = -\frac{1}{3}$
 $\Rightarrow (0, -\frac{1}{3})$ is an interior critical point with $f(0, -\frac{1}{3}) = -\frac{1}{3}$. Therefore the absolute maximum of f on the
disk is 5 at $(0, 1)$ and the absolute minimum of f on the disk is $-\frac{1}{3}$ at $(0, -\frac{1}{3})$.

82. (i) $f(x, y) = x^2 + y^2 - 3x - xy$ on $x^2 + y^2 = 9 \Rightarrow \nabla f = (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x - 3 - y = 2x\lambda$ and $2y - x = 2y\lambda \Rightarrow 2x(1 - \lambda) - y = 3$ and $-x + 2y(1 - \lambda) = 0 \Rightarrow 1 - \lambda = \frac{x}{2y}$ and $(2x)\left(\frac{x}{2y}\right) - y = 3 \Rightarrow x^2 - y^2 = 3y \Rightarrow x^2 = y^2 + 3y$. Thus, $9 = x^2 + y^2 = y^2 + 3y + y^2 \Rightarrow 2y^2 + 3y - 9 = 0 \Rightarrow (2y - 3)(y + 3) = 0 \Rightarrow y = -3, \frac{3}{2}$. For $y = -3$, $x^2 + y^2 = 9 \Rightarrow x = 0$ yielding the point $(0, -3)$. For $y = \frac{3}{2}$, $x^2 + y^2 = 9 \Rightarrow x^2 + \frac{9}{4} = 9 \Rightarrow x^2 = \frac{27}{4} \Rightarrow x = \pm \frac{3\sqrt{3}}{2}$. Evaluations give $f(0, -3) = 9$, $f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 + \frac{27\sqrt{3}}{4} \approx 20.691$, and $f\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 - \frac{27\sqrt{3}}{4} \approx -2.691$.
- (ii) For the interior of the disk, $f_x(x, y) = 2x - 3 - y = 0$ and $f_y(x, y) = 2y - x = 0 \Rightarrow x = 2$ and $y = 1 \Rightarrow (2, 1)$ is an interior critical point of the disk with $f(2, 1) = -3$. Therefore, the absolute maximum of f on the disk is $9 + \frac{27\sqrt{3}}{4}$ at $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ and the absolute minimum of f on the disk is -3 at $(2, 1)$.
83. $\nabla f = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - \mathbf{j} + \mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-1 = 2y\lambda$, $1 = 2z\lambda \Rightarrow x = -y = z = \frac{1}{\lambda}$. Thus $x^2 + y^2 + z^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$ yielding the points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Evaluations give the absolute maximum value of $f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$ and the absolute minimum value of $f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\sqrt{3}$.
84. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin and $g(x, y, z) = x^2 - zy - 4$. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = 2x\mathbf{i} - z\mathbf{j} - y\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = 2\lambda x$, $2y = -\lambda z$, and $2z = -\lambda y \Rightarrow x = 0$ or $\lambda = 1$.
- CASE 1: $x = 0 \Rightarrow zy = -4 \Rightarrow z = -\frac{4}{y}$ and $y = -\frac{4}{z} \Rightarrow 2\left(-\frac{4}{y}\right) = -\lambda y$ and $2\left(-\frac{4}{z}\right) = -\lambda z \Rightarrow \frac{8}{\lambda} = y^2$ and $\frac{8}{\lambda} = z^2 \Rightarrow y^2 = z^2 \Rightarrow y = \pm z$. But $y = x \Rightarrow z^2 = -4$ leads to no solution, so $y = -z \Rightarrow z^2 = 4 \Rightarrow z = \pm 2$ yielding the points $(0, -2, 2)$ and $(0, 2, -2)$.
- CASE 2: $\lambda = 1 \Rightarrow 2z = -y$ and $2y = -z \Rightarrow 2y = -\left(-\frac{y}{2}\right) \Rightarrow 4y = y \Rightarrow y = 0 \Rightarrow z = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$ yielding the points $(-2, 0, 0)$ and $(2, 0, 0)$.
- Evaluations give $f(0, -2, 2) = f(0, 2, -2) = 8$ and $f(-2, 0, 0) = f(2, 0, 0) = 4$. Thus the points $(-2, 0, 0)$ and $(2, 0, 0)$ on the surface are closest to the origin.
85. The cost is $f(x, y, z) = 2axy + 2bxz + 2cyz$ subject to the constraint $xyz = V$. Then $\nabla f = \lambda \nabla g \Rightarrow 2ay + 2bz = \lambda yz$, $2ax + 2cz = \lambda xz$, and $2bx + 2cy = \lambda xy \Rightarrow 2axy + 2bxz = \lambda xyz$, $2axy + 2cyz = \lambda xyz$, and $2bxz + 2cyz = \lambda xyz \Rightarrow 2axy + 2bxz = 2axy + 2cyz \Rightarrow y = \left(\frac{b}{c}\right)x$. Also $2axy + 2bxz = 2bxz + 2cyz \Rightarrow z = \left(\frac{a}{c}\right)x$. Then $x\left(\frac{b}{c}x\right)\left(\frac{a}{c}x\right) = V \Rightarrow x^3 = \frac{c^2V}{ab} \Rightarrow \text{width} = x = \left(\frac{c^2V}{ab}\right)^{1/3}$, $\text{Depth} = y = \left(\frac{b}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{b^2V}{ac}\right)^{1/3}$, and $\text{Height} = z = \left(\frac{a}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{a^2V}{bc}\right)^{1/3}$.
86. The volume of the pyramid in the first octant formed by the plane is $V(a, b, c) = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{1}{6}abc$. The point $(2, 1, 2)$ on the plane $\Rightarrow \frac{2}{a} + \frac{1}{b} + \frac{2}{c} = 1$. We want to minimize V subject to the constraint $2bc + ac + 2ab = abc$. Thus, $\nabla V = \frac{bc}{6}\mathbf{i} + \frac{ac}{6}\mathbf{j} + \frac{ab}{6}\mathbf{k}$ and $\nabla g = (c + 2b - bc)\mathbf{i} + (2c + 2a - ac)\mathbf{j} + (2b + a - ab)\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow \frac{bc}{6} = \lambda(c + 2b - bc)$, $\frac{ac}{6} = \lambda(2c + 2a - ac)$, and $\frac{ab}{6} = \lambda(2b + a - ab) \Rightarrow \frac{abc}{6} = \lambda(ac + 2ab - abc)$, $\frac{abc}{6} = \lambda(2bc + 2ab - abc)$, and $\frac{abc}{6} = \lambda(2bc + ac - abc) \Rightarrow \lambda ac = 2\lambda bc$ and $2\lambda ab = 2\lambda bc$. Now $\lambda \neq 0$ since $a \neq 0$, $b \neq 0$, and $c \neq 0 \Rightarrow ac = 2bc$ and $ab = bc \Rightarrow a = 2b = c$. Substituting into the constraint equation gives $\frac{2}{a} + \frac{2}{a} + \frac{2}{a} = 1 \Rightarrow a = 6 \Rightarrow b = 3$ and $c = 6$. Therefore the desired plane is $\frac{x}{6} + \frac{y}{3} + \frac{z}{6} = 1$ or $x + 2y + z = 6$.

87. $\nabla f = (y+z)\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$, and $\nabla h = z\mathbf{i} + x\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h$
 $\Rightarrow (y+z)\mathbf{i} + x\mathbf{j} + x\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(z\mathbf{i} + x\mathbf{k}) \Rightarrow y+z = 2\lambda x + \mu z, x = 2\lambda y, x = \mu x \Rightarrow x = 0$ or $\mu = 1$.

CASE 1: $x = 0$ which is impossible since $xz = 1$.

CASE 2: $\mu = 1 \Rightarrow y+z = 2\lambda x + z \Rightarrow y = 2\lambda x$ and $x = 2\lambda y \Rightarrow y = (2\lambda)(2\lambda y) \Rightarrow y = 0$ or $4\lambda^2 = 1$. If $y = 0$, then $x^2 = 1 \Rightarrow x = \pm 1$ so with $xz = 1$ we obtain the points $(1, 0, 1)$ and $(-1, 0, -1)$. If $4\lambda^2 = 1$, then $\lambda = \pm \frac{1}{2}$. For $\lambda = -\frac{1}{2}$, $y = -x$ so $x^2 + y^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm \sqrt{2}$, and we obtain the points $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2})$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm \sqrt{2}$, and we obtain the points $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2})$.

Evaluations give $f(1, 0, 1) = 1$, $f(-1, 0, -1) = 1$, $f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}) = \frac{1}{2}$, $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}) = \frac{1}{2}$, $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}) = \frac{3}{2}$, and $f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}) = \frac{3}{2}$. Therefore the absolute maximum is $\frac{3}{2}$ at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2})$, and the absolute minimum is $\frac{1}{2}$ at $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2})$.

88. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\nabla h = 4x\mathbf{i} + 4y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow 2x = \lambda + 4x\mu, 2y = \lambda + 4y\mu$, and $2z = \lambda - 2z\mu \Rightarrow \lambda = 2x(1 - 2\mu) = 2y(1 - 2\mu) = 2z(1 + 2\mu) \Rightarrow x = y$ or $\mu = \frac{1}{2}$.

CASE 1: $x = y \Rightarrow z^2 = 4x^2 \Rightarrow z = \pm 2x$ so that $x + y + z = 1 \Rightarrow x + x + 2x = 1$ or $x + x - 2x = 1$ (impossible) $\Rightarrow x = \frac{1}{4} \Rightarrow y = \frac{1}{4}$ and $z = \frac{1}{2}$ yielding the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

CASE 2: $\mu = \frac{1}{2} \Rightarrow \lambda = 0 \Rightarrow 0 = 2z(1 + 1) \Rightarrow z = 0$ so that $2x^2 + 2y^2 = 0 \Rightarrow x = y = 0$. But the origin $(0, 0, 0)$ fails to satisfy the first constraint $x + y + z = 1$.

Therefore, the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ on the curve of intersection is closest to the origin.

89. (a) y, z are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial y}$
 $= (2xe^{yz}) \frac{\partial x}{\partial y} + (zx^2 e^{yz})(1) + (yx^2 e^{yz})(0); z = x^2 - y^2 \Rightarrow 0 = 2x \frac{\partial x}{\partial y} - 2y \Rightarrow \frac{\partial x}{\partial y} = \frac{y}{x}$; therefore,
 $(\frac{\partial w}{\partial y})_z = (2xe^{yz}) (\frac{y}{x}) + zx^2 e^{yz} = (2y + zx^2) e^{yz}$
- (b) z, x are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial z}$
 $= (2xe^{yz})(0) + (zx^2 e^{yz}) \frac{\partial y}{\partial z} + (yx^2 e^{yz})(1); z = x^2 - y^2 \Rightarrow 1 = 0 - 2y \frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = -\frac{1}{2y}$; therefore,
 $(\frac{\partial w}{\partial z})_x = (zx^2 e^{yz}) (-\frac{1}{2y}) + yx^2 e^{yz} = x^2 e^{yz} (y - \frac{z}{2y})$
- (c) z, y are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial z}$
 $= (2xe^{yz}) \frac{\partial x}{\partial z} + (zx^2 e^{yz})(0) + (yx^2 e^{yz})(1); z = x^2 - y^2 \Rightarrow 1 = 2x \frac{\partial x}{\partial z} - 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x}$; therefore,
 $(\frac{\partial w}{\partial z})_y = (2xe^{yz}) (\frac{1}{2x}) + yx^2 e^{yz} = (1 + x^2 y) e^{yz}$

90. (a) T, P are independent with $U = f(P, V, T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial T} = \frac{\partial U}{\partial P} \frac{\partial P}{\partial T} + \frac{\partial U}{\partial V} \frac{\partial V}{\partial T} + \frac{\partial U}{\partial T} \frac{\partial T}{\partial T}$
 $= (\frac{\partial U}{\partial P})(0) + (\frac{\partial U}{\partial V}) (\frac{\partial V}{\partial T}) + (\frac{\partial U}{\partial T})(1); PV = nRT \Rightarrow P \frac{\partial V}{\partial T} = nR \Rightarrow \frac{\partial V}{\partial T} = \frac{nR}{P}$; therefore,
 $(\frac{\partial U}{\partial T})_P = (\frac{\partial U}{\partial V}) (\frac{nR}{P}) + \frac{\partial U}{\partial T}$
- (b) V, T are independent with $U = f(P, V, T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial V} = \frac{\partial U}{\partial P} \frac{\partial P}{\partial V} + \frac{\partial U}{\partial V} \frac{\partial V}{\partial V} + \frac{\partial U}{\partial T} \frac{\partial T}{\partial V}$
 $= (\frac{\partial U}{\partial P}) (\frac{\partial P}{\partial V}) + (\frac{\partial U}{\partial V})(1) + (\frac{\partial U}{\partial T})(0); PV = nRT \Rightarrow V \frac{\partial P}{\partial V} + P = (nR) (\frac{\partial T}{\partial V}) = 0 \Rightarrow \frac{\partial P}{\partial V} = -\frac{P}{V}$; therefore,
 $(\frac{\partial U}{\partial V})_T = (\frac{\partial U}{\partial P}) (-\frac{P}{V}) + \frac{\partial U}{\partial V}$

91. Note that $x = r \cos \theta$ and $y = r \sin \theta \Rightarrow r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \left(\frac{y}{x} \right)$. Thus,

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta} \frac{\partial \theta}{\partial x} = \left(\frac{\partial w}{\partial r} \right) \left(\frac{x}{\sqrt{x^2 + y^2}} \right) + \left(\frac{\partial w}{\partial \theta} \right) \left(\frac{-y}{x^2 + y^2} \right) = (\cos \theta) \frac{\partial w}{\partial r} - \left(\frac{\sin \theta}{r} \right) \frac{\partial w}{\partial \theta}; \\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta} \frac{\partial \theta}{\partial y} = \left(\frac{\partial w}{\partial r} \right) \left(\frac{y}{\sqrt{x^2 + y^2}} \right) + \left(\frac{\partial w}{\partial \theta} \right) \left(\frac{x}{x^2 + y^2} \right) = (\sin \theta) \frac{\partial w}{\partial r} + \left(\frac{\cos \theta}{r} \right) \frac{\partial w}{\partial \theta}\end{aligned}$$

92. $z_x = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} = af_u + af_v$, and $z_y = f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} = bf_u - bf_v$

93. $\frac{\partial u}{\partial y} = b$ and $\frac{\partial u}{\partial x} = a \Rightarrow \frac{\partial w}{\partial x} = \frac{dw}{du} \frac{\partial u}{\partial x} = a \frac{dw}{du}$ and $\frac{\partial w}{\partial y} = \frac{dw}{du} \frac{\partial u}{\partial y} = b \frac{dw}{du} \Rightarrow \frac{1}{a} \frac{\partial w}{\partial x} = \frac{dw}{du}$ and $\frac{1}{b} \frac{\partial w}{\partial y} = \frac{dw}{du}$
 $\Rightarrow \frac{1}{a} \frac{\partial w}{\partial x} = \frac{1}{b} \frac{\partial w}{\partial y} \Rightarrow b \frac{\partial w}{\partial x} = a \frac{\partial w}{\partial y}$

94. $\frac{\partial w}{\partial x} = \frac{2x}{x^2 + y^2 + 2z} = \frac{2(r+s)}{(r+s)^2 + (r-s)^2 + 4rs} = \frac{2(r+s)}{2(r^2 + 2rs + s^2)} = \frac{1}{r+s}$, $\frac{\partial w}{\partial y} = \frac{2y}{x^2 + y^2 + 2z} = \frac{2(r-s)}{2(r+s)^2} = \frac{r-s}{(r+s)^2}$,
 and $\frac{\partial w}{\partial z} = \frac{2}{x^2 + y^2 + 2z} = \frac{1}{(r+s)^2} \Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} = \frac{1}{r+s} + \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2} \right] (2s) = \frac{2r+2s}{(r+s)^2}$
 $= \frac{2}{r+s}$ and $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} = \frac{1}{r+s} - \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2} \right] (2r) = \frac{2}{r+s}$

95. $e^u \cos v - x = 0 \Rightarrow (e^u \cos v) \frac{\partial u}{\partial x} - (e^u \sin v) \frac{\partial v}{\partial x} = 1$; $e^u \sin v - y = 0 \Rightarrow (e^u \sin v) \frac{\partial u}{\partial x} + (e^u \cos v) \frac{\partial v}{\partial x} = 0$.

Solving this system yields $\frac{\partial u}{\partial x} = e^{-u} \cos v$ and $\frac{\partial v}{\partial x} = -e^{-u} \sin v$. Similarly, $e^u \cos v - x = 0$

$\Rightarrow (e^u \cos v) \frac{\partial u}{\partial y} - (e^u \sin v) \frac{\partial v}{\partial y} = 0$ and $e^u \sin v - y = 0 \Rightarrow (e^u \sin v) \frac{\partial u}{\partial y} + (e^u \cos v) \frac{\partial v}{\partial y} = 1$. Solving this

second system yields $\frac{\partial u}{\partial y} = e^{-u} \sin v$ and $\frac{\partial v}{\partial y} = e^{-u} \cos v$. Therefore $\left(\frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \right) \cdot \left(\frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j} \right)$

$= [(e^{-u} \cos v) \mathbf{i} + (e^{-u} \sin v) \mathbf{j}] \cdot [(-e^{-u} \sin v) \mathbf{i} + (e^{-u} \cos v) \mathbf{j}] = 0 \Rightarrow$ the vectors are orthogonal $\Rightarrow$ the angle between the vectors is the constant $\frac{\pi}{2}$.

96. $\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = (-r \sin \theta) \frac{\partial f}{\partial x} + (r \cos \theta) \frac{\partial f}{\partial y}$
 $\Rightarrow \frac{\partial^2 g}{\partial \theta^2} = (-r \sin \theta) \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial \theta} \right) - (r \cos \theta) \frac{\partial f}{\partial x} + (r \cos \theta) \left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial \theta} \right) - (r \sin \theta) \frac{\partial f}{\partial y}$
 $= (-r \sin \theta) \left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta} \right) - (r \cos \theta) + (r \cos \theta) \left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta} \right) - (r \sin \theta)$
 $= (-r \sin \theta + r \cos \theta)(-r \sin \theta + r \cos \theta) - (r \cos \theta + r \sin \theta) = (-2)(-2) - (0 + 2) = 4 - 2 = 2$ at $(r, \theta) = \left(2, \frac{\pi}{2} \right)$.

97. $(y+z)^2 + (z-x)^2 = 16 \Rightarrow \nabla f = -2(z-x)\mathbf{i} + 2(y+z)\mathbf{j} + 2(y+2z-x)\mathbf{k}$; if the normal line is parallel to the yz -plane, then x is constant $\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow -2(z-x) = 0 \Rightarrow z = x \Rightarrow (y+z)^2 + (z-z)^2 = 16 \Rightarrow y+z = \pm 4$.
 Let $x = t \Rightarrow z = t \Rightarrow y = -t \pm 4$. Therefore the points are $(t, -t \pm 4, t)$, t a real number.

98. Let $f(x, y, z) = xy + yz + zx - x - z^2 = 0$. If the tangent plane is to be parallel to the xy -plane, then ∇f is perpendicular to the xy -plane $\Rightarrow \nabla f \cdot \mathbf{i} = 0$ and $\nabla f \cdot \mathbf{j} = 0$. Now $\nabla f = (y+z-1)\mathbf{i} + (x+z)\mathbf{j} + (y+x-2z)\mathbf{k}$ so that $\nabla f \cdot \mathbf{i} = y+z-1 = 0 \Rightarrow y+z = 1 \Rightarrow y = 1-z$, and $\nabla f \cdot \mathbf{j} = x+z = 0 \Rightarrow x = -z$. Then $-z(1-z) + (1-z)z + z(-z) - (-z) - z^2 = 0 \Rightarrow z - 2z^2 = 0 \Rightarrow z = \frac{1}{2}$ or $z = 0$. Now $z = \frac{1}{2} \Rightarrow x = -\frac{1}{2}$ and $y = \frac{1}{2} \Rightarrow \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$ is one desired point; $z = 0 \Rightarrow x = 0$ and $y = 1 \Rightarrow (0, 1, 0)$ is a second desired point.

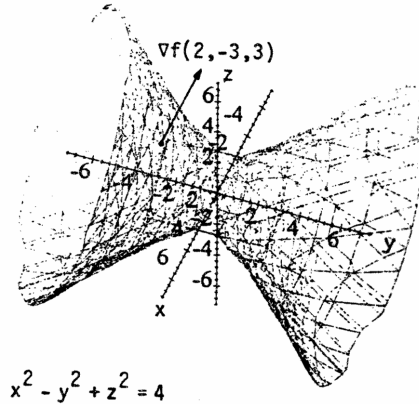
99. $\nabla f = \lambda(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \Rightarrow \frac{\partial f}{\partial x} = \lambda x \Rightarrow f(x, y, z) = \frac{1}{2} \lambda x^2 + g(y, z)$ for some function $g \Rightarrow \lambda y = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$
 $\Rightarrow g(y, z) = \frac{1}{2} \lambda y^2 + h(z)$ for some function $h \Rightarrow \lambda z = \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} = h'(z) \Rightarrow h(z) = \frac{1}{2} \lambda z^2 + C$ for some arbitrary constant $C \Rightarrow g(y, z) = \frac{1}{2} \lambda y^2 + \left(\frac{1}{2} \lambda z^2 + C \right) \Rightarrow f(x, y, z) = \frac{1}{2} \lambda x^2 + \frac{1}{2} \lambda y^2 + \frac{1}{2} \lambda z^2 + C \Rightarrow f(0, 0, a) = \frac{1}{2} \lambda a^2 + C$ and $f(0, 0, -a) = \frac{1}{2} \lambda (-a)^2 + C \Rightarrow f(0, 0, a) = f(0, 0, -a)$ for any constant a , as claimed.

$$\begin{aligned}
 100. \left(\frac{df}{ds} \right)_{\mathbf{u}, (0,0,0)} &= \lim_{s \rightarrow 0} \frac{f(0+su_1, 0+su_2, 0+su_3) - f(0,0,0)}{s}, s > 0 \\
 &= \lim_{s \rightarrow 0} \frac{\sqrt{s^2u_1^2 + s^2u_2^2 + s^2u_3^2} - 0}{s}, s > 0 \\
 &= \lim_{s \rightarrow 0} \frac{s\sqrt{u_1^2 + u_2^2 + u_3^2}}{s} = \lim_{s \rightarrow 0} |\mathbf{u}| = 1;
 \end{aligned}$$

however, $\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}} \mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}} \mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}} \mathbf{k}$ fails to exist at the origin $(0, 0, 0)$

101. Let $f(x, y, z) = xy + z - 2 \Rightarrow \nabla f = y\mathbf{i} + x\mathbf{j} + \mathbf{k}$. At $(1, 1, 1)$, we have $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the normal line is $x = 1 + t, y = 1 + t, z = 1 + t$, so at $t = -1 \Rightarrow x = 0, y = 0, z = 0$ and the normal line passes through the origin.

102. (b) $f(x, y, z) = x^2 - y^2 + z^2 = 4$
 $\Rightarrow \nabla f = 2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$ at $(2, -3, 3)$
the gradient is $\nabla f = 4\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ which is
normal to the surface
(c) Tangent plane: $4x + 6y + 6z = 8$ or
 $2x + 3y + 3z = 4$
Normal line: $x = 2 + 4t, y = -3 + 6t, z = 3 + 6t$



CHAPTER 14 ADDITIONAL AND ADVANCED EXERCISES

- By definition, $f_{xy}(0, 0) = \lim_{h \rightarrow 0} \frac{f_x(0, h) - f_x(0, 0)}{h}$ so we need to calculate the first partial derivatives in the numerator. For $(x, y) \neq (0, 0)$ we calculate $f_x(x, y)$ by applying the differentiation rules to the formula for $f(x, y)$: $f_x(x, y) = \frac{x^2y - y^3}{x^2 + y^2} + (xy) \frac{(x^2 + y^2)(2x) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2} = \frac{x^2y - y^3}{x^2 + y^2} + \frac{4x^2y^3}{(x^2 + y^2)^2} \Rightarrow f_x(0, h) = -\frac{h^3}{h^2} = -h$. For $(x, y) = (0, 0)$ we apply the definition: $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$. Then by definition $f_{xy}(0, 0) = \lim_{h \rightarrow 0} \frac{-h - 0}{h} = -1$. Similarly, $f_{yx}(0, 0) = \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$, so for $(x, y) \neq (0, 0)$ we have $f_y(x, y) = \frac{x^3 - xy^2}{x^2 + y^2} - \frac{4x^3y^2}{(x^2 + y^2)^2} \Rightarrow f_y(h, 0) = \frac{h^3}{h^2} = h$; for $(x, y) = (0, 0)$ we obtain $f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$. Then by definition $f_{yx}(0, 0) = \lim_{h \rightarrow 0} \frac{h - 0}{h} = 1$. Note that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ in this case.
- $\frac{\partial w}{\partial x} = 1 + e^x \cos y \Rightarrow w = x + e^x \cos y + g(y)$; $\frac{\partial w}{\partial y} = -e^x \sin y + g'(y) = 2y - e^x \sin y \Rightarrow g'(y) = 2y$
 $\Rightarrow g(y) = y^2 + C$; $w = \ln 2$ when $x = \ln 2$ and $y = 0 \Rightarrow \ln 2 = \ln 2 + e^{\ln 2} \cos 0 + 0^2 + C \Rightarrow 0 = 2 + C$
 $\Rightarrow C = -2$. Thus, $w = x + e^x \cos y + g(y) = x + e^x \cos y + y^2 - 2$.
- Substitution of $u = u(x)$ and $v = v(x)$ in $g(u, v)$ gives $g(u(x), v(x))$ which is a function of the independent variable x . Then, $g(u, v) = \int_u^v f(t) dt \Rightarrow \frac{dg}{dx} = \frac{\partial g}{\partial u} \frac{du}{dx} + \frac{\partial g}{\partial v} \frac{dv}{dx} = \left(\frac{\partial}{\partial u} \int_u^v f(t) dt \right) \frac{du}{dx} + \left(\frac{\partial}{\partial v} \int_u^v f(t) dt \right) \frac{dv}{dx}$
 $= \left(-\frac{\partial}{\partial u} \int_u^u f(t) dt \right) \frac{du}{dx} + \left(\frac{\partial}{\partial v} \int_u^v f(t) dt \right) \frac{dv}{dx} = -f(u(x)) \frac{du}{dx} + f(v(x)) \frac{dv}{dx} = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}$
- Applying the chain rules, $f_x = \frac{df}{dr} \frac{\partial r}{\partial x} \Rightarrow f_{xx} = \left(\frac{d^2f}{dr^2} \right) \left(\frac{\partial r}{\partial x} \right)^2 + \frac{df}{dr} \frac{\partial^2 r}{\partial x^2}$. Similarly, $f_{yy} = \left(\frac{d^2f}{dr^2} \right) \left(\frac{\partial r}{\partial y} \right)^2 + \frac{df}{dr} \frac{\partial^2 r}{\partial y^2}$ and $f_{zz} = \left(\frac{d^2f}{dr^2} \right) \left(\frac{\partial r}{\partial z} \right)^2 + \frac{df}{dr} \frac{\partial^2 r}{\partial z^2}$. Moreover, $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}$; $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}$; and $\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{(\sqrt{x^2+y^2+z^2})^3}$. Next, $f_{xx} + f_{yy} + f_{zz} = 0$

$$\begin{aligned} &\Rightarrow \left(\frac{d^2f}{dr^2}\right) \left(\frac{x^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right) \left(\frac{y^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}\right) + \left(\frac{d^2f}{dr^2}\right) \left(\frac{y^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right) \left(\frac{x^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}\right) \\ &+ \left(\frac{d^2f}{dr^2}\right) \left(\frac{z^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right) \left(\frac{x^2+y^2}{(\sqrt{x^2+y^2+z^2})^3}\right) = 0 \Rightarrow \frac{d^2f}{dr^2} + \left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right) \frac{df}{dr} = 0 \Rightarrow \frac{d^2f}{dr^2} + \frac{2}{r} \frac{df}{dr} = 0 \\ &\Rightarrow \frac{d}{dr} \left(\frac{f'}{r}\right) = \left(-\frac{1}{r}\right) f', \text{ where } f' = \frac{df}{dr} \Rightarrow \frac{f'}{r} = -\frac{2}{r} \Rightarrow \ln f' = -2 \ln r + \ln C \Rightarrow f' = Cr^{-2}, \text{ or} \\ &\frac{df}{dr} = Cr^{-2} \Rightarrow f(r) = -\frac{C}{r} + b = \frac{a}{r} + b \text{ for some constants } a \text{ and } b \text{ (setting } a = -C) \end{aligned}$$

5. (a) Let $u = tx$, $v = ty$, and $w = f(u, v) = f(u(t, x), v(t, y)) = f(tx, ty) = t^n f(x, y)$, where t , x , and y are independent variables. Then $nt^{n-1}f(x, y) = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial t} = x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v}$. Now,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} = \left(\frac{\partial w}{\partial u}\right)(t) + \left(\frac{\partial w}{\partial v}\right)(0) = t \frac{\partial w}{\partial u} \Rightarrow \frac{\partial w}{\partial u} = \left(\frac{1}{t}\right) \left(\frac{\partial w}{\partial x}\right). \text{ Likewise,}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = \left(\frac{\partial w}{\partial u}\right)(0) + \left(\frac{\partial w}{\partial v}\right)(t) \Rightarrow \frac{\partial w}{\partial v} = \left(\frac{1}{t}\right) \left(\frac{\partial w}{\partial y}\right). \text{ Therefore,}$$

$$nt^{n-1}f(x, y) = x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v} = \left(\frac{x}{t}\right) \left(\frac{\partial w}{\partial x}\right) + \left(\frac{y}{t}\right) \left(\frac{\partial w}{\partial y}\right). \text{ When } t = 1, u = x, v = y, \text{ and } w = f(x, y)$$

$$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} \text{ and } \frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} \Rightarrow nf(x, y) = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}, \text{ as claimed.}$$

- (b) From part (a), $nt^{n-1}f(x, y) = x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v}$. Differentiating with respect to t again we obtain

$$n(n-1)t^{n-2}f(x, y) = x \frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial t} + x \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial t} + y \frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial t} + y \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial t} = x^2 \frac{\partial^2 w}{\partial u^2} + 2xy \frac{\partial^2 w}{\partial u \partial v} + y^2 \frac{\partial^2 w}{\partial v^2}.$$

$$\text{Also from part (a), } \frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x} \left(t \frac{\partial w}{\partial u}\right) = t \frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial x} + t \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial x} = t^2 \frac{\partial^2 w}{\partial u^2}, \frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y}\right)$$

$$= \frac{\partial}{\partial y} \left(t \frac{\partial w}{\partial v}\right) = t \frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial y} + t \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial y} = t^2 \frac{\partial^2 w}{\partial v^2}, \text{ and } \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y} \left(t \frac{\partial w}{\partial u}\right) = t \frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial y} + t \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial y}$$

$$= t^2 \frac{\partial^2 w}{\partial v \partial u} \Rightarrow \left(\frac{1}{t^2}\right) \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial u^2}, \left(\frac{1}{t^2}\right) \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial v^2}, \text{ and } \left(\frac{1}{t^2}\right) \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial v \partial u}$$

$$\Rightarrow n(n-1)t^{n-2}f(x, y) = \left(\frac{x^2}{t^2}\right) \left(\frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{2xy}{t^2}\right) \left(\frac{\partial^2 w}{\partial y \partial x}\right) + \left(\frac{y^2}{t^2}\right) \left(\frac{\partial^2 w}{\partial y^2}\right) \text{ for } t \neq 0. \text{ When } t = 1, w = f(x, y) \text{ and}$$

$$\text{we have } n(n-1)f(x, y) = x^2 \left(\frac{\partial^2 f}{\partial x^2}\right) + 2xy \left(\frac{\partial^2 f}{\partial x \partial y}\right) + y^2 \left(\frac{\partial^2 f}{\partial y^2}\right) \text{ as claimed.}$$

6. (a) $\lim_{r \rightarrow 0} \frac{\sin 6r}{6r} = \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$, where $t = 6r$

$$\begin{aligned} \text{(b) } f_r(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{\sin 6h}{6h}\right) - 1}{h} = \lim_{h \rightarrow 0} \frac{\sin 6h - 6h}{6h^2} = \lim_{h \rightarrow 0} \frac{6 \cos 6h - 6}{12h} \\ &= \lim_{h \rightarrow 0} \frac{-36 \sin 6h}{12} = 0 \quad (\text{applying l'Hôpital's rule twice}) \end{aligned}$$

$$\text{(c) } f_\theta(r, \theta) = \lim_{h \rightarrow 0} \frac{f(r, \theta+h) - f(r, \theta)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{\sin 6r}{6r}\right) - \left(\frac{\sin 6r}{6r}\right)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

7. (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and $\nabla r = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k} = \frac{\mathbf{r}}{r}$

$$\text{(b) } r^n = (\sqrt{x^2 + y^2 + z^2})^n$$

$$\Rightarrow \nabla(r^n) = nx(x^2 + y^2 + z^2)^{(n/2)-1}\mathbf{i} + ny(x^2 + y^2 + z^2)^{(n/2)-1}\mathbf{j} + nz(x^2 + y^2 + z^2)^{(n/2)-1}\mathbf{k} = nr^{n-2}\mathbf{r}$$

$$\text{(c) Let } n = 2 \text{ in part (b). Then } \frac{1}{2} \nabla(r^2) = \mathbf{r} \Rightarrow \nabla\left(\frac{1}{2}r^2\right) = \mathbf{r} \Rightarrow \frac{r^2}{2} = \frac{1}{2}(x^2 + y^2 + z^2) \text{ is the function.}$$

$$\begin{aligned} \text{(d) } d\mathbf{r} &= dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \Rightarrow \mathbf{r} \cdot d\mathbf{r} = x dx + y dy + z dz, \text{ and } d\mathbf{r} = r_x dx + r_y dy + r_z dz = \frac{x}{r} dx + \frac{y}{r} dy + \frac{z}{r} dz \\ &\Rightarrow \mathbf{r} \cdot d\mathbf{r} = x dx + y dy + z dz = \mathbf{r} \cdot d\mathbf{r} \end{aligned}$$

$$\text{(e) } \mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \Rightarrow \mathbf{A} \cdot \mathbf{r} = ax + by + cz \Rightarrow \nabla(\mathbf{A} \cdot \mathbf{r}) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{A}$$

8. $f(g(t), h(t)) = c \Rightarrow 0 = \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right)$, where $\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ is the tangent vector
 $\Rightarrow \nabla f$ is orthogonal to the tangent vector

9. $f(x, y, z) = xz^2 - yz + \cos xy - 1 \Rightarrow \nabla f = (z^2 - y \sin xy)\mathbf{i} + (-z - x \sin xy)\mathbf{j} + (2xz - y)\mathbf{k} \Rightarrow \nabla f(0, 0, 1) = \mathbf{i} - \mathbf{j}$
 $\Rightarrow$ the tangent plane is $x - y = 0$; $\mathbf{r} = (\ln t)\mathbf{i} + (t \ln t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{r}' = \left(\frac{1}{t}\right)\mathbf{i} + (\ln t + 1)\mathbf{j} + \mathbf{k}$; $x = y = 0, z = 1$
 $\Rightarrow t = 1 \Rightarrow \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Since $(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j}) = \mathbf{r}'(1) \cdot \nabla f = 0$, $\mathbf{r}$ is parallel to the plane, and
 $\mathbf{r}(1) = 0\mathbf{i} + 0\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

10. Let $f(x, y, z) = x^3 + y^3 + z^3 - xyz \Rightarrow \nabla f = (3x^2 - yz)\mathbf{i} + (3y^2 - xz)\mathbf{j} + (3z^2 - xy)\mathbf{k} \Rightarrow \nabla f(0, -1, 1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$
 $\Rightarrow$ the tangent plane is $x + 3y + 3z = 0$; $\mathbf{r} = \left(\frac{t^3}{4} - 2\right)\mathbf{i} + \left(\frac{t}{4} - 3\right)\mathbf{j} + (\cos(t - 2))\mathbf{k}$
 $\Rightarrow \mathbf{r}' = \left(\frac{3t^2}{4}\right)\mathbf{i} - \left(\frac{1}{4}\right)\mathbf{j} - (\sin(t - 2))\mathbf{k}$; $x = 0, y = -1, z = 1 \Rightarrow t = 2 \Rightarrow \mathbf{r}'(2) = 3\mathbf{i} - \mathbf{j}$. Since
 $\mathbf{r}'(2) \cdot \nabla f = 0 \Rightarrow \mathbf{r}$ is parallel to the plane, and $\mathbf{r}(2) = -\mathbf{i} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.
11. $\frac{\partial z}{\partial x} = 3x^2 - 9y = 0$ and $\frac{\partial z}{\partial y} = 3y^2 - 9x = 0 \Rightarrow y = \frac{1}{3}x^2$ and $3\left(\frac{1}{3}x^2\right)^2 - 9x = 0 \Rightarrow \frac{1}{3}x^4 - 9x = 0$
 $\Rightarrow x(x^3 - 27) = 0 \Rightarrow x = 0$ or $x = 3$. Now $x = 0 \Rightarrow y = 0$ or $(0, 0)$ and $x = 3 \Rightarrow y = 3$ or $(3, 3)$. Next
 $\frac{\partial^2 z}{\partial x^2} = 6x, \frac{\partial^2 z}{\partial y^2} = 6y$, and $\frac{\partial^2 z}{\partial x \partial y} = -9$. For $(0, 0)$, $\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -81 \Rightarrow$ no extremum (a saddle point),
and for $(3, 3)$, $\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 243 > 0$ and $\frac{\partial^2 z}{\partial x^2} = 18 > 0 \Rightarrow$ a local minimum.
12. $f(x, y) = 6xye^{-(2x+3y)} \Rightarrow f_x(x, y) = 6y(1 - 2x)e^{-(2x+3y)} = 0$ and $f_y(x, y) = 6x(1 - 3y)e^{-(2x+3y)} = 0 \Rightarrow x = 0$ and
 $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{3}$. The value $f(0, 0) = 0$ is on the boundary, and $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{e^2}$. On the positive y -axis,
 $f(0, y) = 0$, and on the positive x -axis, $f(x, 0) = 0$. As $x \rightarrow \infty$ or $y \rightarrow \infty$ we see that $f(x, y) \rightarrow 0$. Thus the absolute
maximum of f in the closed first quadrant is $\frac{1}{e^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\right)$.
13. Let $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \Rightarrow \nabla f = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j} + \frac{2z}{c^2}\mathbf{k} \Rightarrow$ an equation of the plane tangent at the point
 $P_0(x_0, y_0, z_0)$ is $\left(\frac{2x_0}{a^2}\right)x + \left(\frac{2y_0}{b^2}\right)y + \left(\frac{2z_0}{c^2}\right)z = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} + \frac{2z_0^2}{c^2} = 2$ or $\left(\frac{x_0}{a^2}\right)x + \left(\frac{y_0}{b^2}\right)y + \left(\frac{z_0}{c^2}\right)z = 1$.
The intercepts of the plane are $\left(\frac{a^2}{x_0}, 0, 0\right)$, $\left(0, \frac{b^2}{y_0}, 0\right)$ and $\left(0, 0, \frac{c^2}{z_0}\right)$. The volume of the tetrahedron formed by the
plane and the coordinate planes is $V = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{a^2}{x_0}\right)\left(\frac{b^2}{y_0}\right)\left(\frac{c^2}{z_0}\right) \Rightarrow$ we need to maximize $V(x, y, z) = \frac{(abc)^2}{6}(xyz)^{-1}$
subject to the constraint $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Thus, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{x^2yz}\right) = \frac{2x}{a^2}\lambda$, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xy^2z}\right) = \frac{2y}{b^2}\lambda$,
and $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xyz^2}\right) = \frac{2z}{c^2}\lambda$. Multiply the first equation by a^2yz , the second by b^2xz , and the third by c^2xy . Then equate
the first and second $\Rightarrow a^2y^2 = b^2x^2 \Rightarrow y = \frac{b}{a}x, x > 0$; equate the first and third $\Rightarrow a^2z^2 = c^2x^2 \Rightarrow z = \frac{c}{a}x, x > 0$;
substitute into $f(x, y, z) = 0 \Rightarrow x = \frac{a}{\sqrt{3}} \Rightarrow y = \frac{b}{\sqrt{3}} \Rightarrow z = \frac{c}{\sqrt{3}} \Rightarrow V = \frac{\sqrt{3}}{2}abc$.
14. $2(x - u) = -\lambda, 2(y - v) = \lambda, -2(x - u) = \mu$, and $-2(y - v) = -2\mu v \Rightarrow x - u = v - y, x - u = -\frac{\mu}{2}$, and
 $y - v = \mu v \Rightarrow x - u = -\mu v = -\frac{\mu}{2} \Rightarrow v = \frac{1}{2}$ or $\mu = 0$.
CASE 1: $\mu = 0 \Rightarrow x = u, y = v$, and $\lambda = 0$; then $y = x + 1 \Rightarrow v = u + 1$ and $v^2 = u \Rightarrow v = v^2 + 1$
 $\Rightarrow v^2 - v + 1 = 0 \Rightarrow v = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow$ no real solution.
CASE 2: $v = \frac{1}{2}$ and $u = v^2 \Rightarrow u = \frac{1}{4}; x - \frac{1}{4} = \frac{1}{2} - y$ and $y = x + 1 \Rightarrow x - \frac{1}{4} = -x - \frac{1}{2} \Rightarrow 2x = -\frac{1}{4} \Rightarrow x = -\frac{1}{8}$
 $\Rightarrow y = \frac{7}{8}$. Then $f\left(-\frac{1}{8}, \frac{7}{8}, \frac{1}{4}, \frac{1}{2}\right) = \left(-\frac{1}{8} - \frac{1}{4}\right)^2 + \left(\frac{7}{8} - \frac{1}{2}\right)^2 = 2\left(\frac{3}{8}\right)^2 \Rightarrow$ the minimum distance is $\frac{3}{8}\sqrt{2}$.
(Notice that f has no maximum value.)
15. Let (x_0, y_0) be any point in $\mathbb{R}$. We must show $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$ or, equivalently that
 $\lim_{(h, k) \rightarrow (0, 0)} |f(x_0 + h, y_0 + k) - f(x_0, y_0)| = 0$. Consider $f(x_0 + h, y_0 + k) - f(x_0, y_0)$
 $= [f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)] + [f(x_0, y_0 + k) - f(x_0, y_0)]$. Let $F(x) = f(x, y_0 + k)$ and apply the Mean Value
Theorem: there exists ξ with $x_0 < \xi < x_0 + h$ such that $F'(\xi)h = F(x_0 + h) - F(x_0) \Rightarrow hf_x(\xi, y_0 + k)$
 $= f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)$. Similarly, $kf_y(x_0, \eta) = f(x_0, y_0 + k) - f(x_0, y_0)$ for some η with
 $y_0 < \eta < y_0 + k$. Then $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \leq |hf_x(\xi, y_0 + k)| + |kf_y(x_0, \eta)|$. If M, N are positive real
numbers such that $|f_x| \leq M$ and $|f_y| \leq N$ for all (x, y) in the xy -plane, then $|f(x_0 + h, y_0 + k) - f(x_0, y_0)|$
 $\leq M|h| + N|k|$. As $(h, k) \rightarrow 0, |f(x_0 + h, y_0 + k) - f(x_0, y_0)| \rightarrow 0 \Rightarrow \lim_{(h, k) \rightarrow (0, 0)} |f(x_0 + h, y_0 + k) - f(x_0, y_0)|$
 $= 0 \Rightarrow f$ is continuous at (x_0, y_0) .

16. At extreme values, ∇f and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ are orthogonal because $\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = 0$ by the First Derivative Theorem for Local Extreme Values.
17. $\frac{\partial f}{\partial x} = 0 \Rightarrow f(x, y) = h(y)$ is a function of y only. Also, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial x} = 0 \Rightarrow g(x, y) = k(x)$ is a function of x only. Moreover, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow h'(y) = k'(x)$ for all x and y . This can happen only if $h'(y) = k'(x) = c$ is a constant. Integration gives $h(y) = cy + c_1$ and $k(x) = cx + c_2$, where c_1 and c_2 are constants. Therefore $f(x, y) = cy + c_1$ and $g(x, y) = cx + c_2$. Then $f(1, 2) = g(1, 2) = 5 \Rightarrow 5 = 2c + c_1 = c + c_2$, and $f(0, 0) = 4 \Rightarrow c_1 = 4 \Rightarrow c = \frac{1}{2} \Rightarrow c_2 = \frac{9}{2}$. Thus, $f(x, y) = \frac{1}{2}y + 4$ and $g(x, y) = \frac{1}{2}x + \frac{9}{2}$.
18. Let $g(x, y) = D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$. Then $D_u g(x, y) = g_x(x, y)a + g_y(x, y)b = f_{xx}(x, y)a^2 + f_{xy}(x, y)ab + f_{xy}(x, y)ba + f_{yy}(x, y)b^2 = f_{xx}(x, y)a^2 + 2f_{xy}(x, y)ab + f_{yy}(x, y)b^2$.
19. Since the particle is heat-seeking, at each point (x, y) it moves in the direction of maximal temperature increase, that is in the direction of $\nabla T(x, y) = (e^{-2y} \sin x)\mathbf{i} + (2e^{-2y} \cos x)\mathbf{j}$. Since $\nabla T(x, y)$ is parallel to the particle's velocity vector, it is tangent to the path $y = f(x)$ of the particle $\Rightarrow f'(x) = \frac{2e^{-2y} \cos x}{e^{-2y} \sin x} = 2 \cot x$. Integration gives $f(x) = 2 \ln |\sin x| + C$ and $f(\frac{\pi}{4}) = 0 \Rightarrow 0 = 2 \ln |\sin \frac{\pi}{4}| + C \Rightarrow C = -2 \ln \frac{\sqrt{2}}{2} = \ln \left(\frac{2}{\sqrt{2}}\right)^2 = \ln 2$. Therefore, the path of the particle is the graph of $y = 2 \ln |\sin x| + \ln 2$.
20. The line of travel is $x = t, y = t, z = 30 - 5t$, and the bullet hits the surface $z = 2x^2 + 3y^2$ when $30 - 5t = 2t^2 + 3t^2 \Rightarrow t^2 + t - 6 = 0 \Rightarrow (t+3)(t-2) = 0 \Rightarrow t = 2$ (since $t > 0$). Thus the bullet hits the surface at the point $(2, 2, 20)$. Now, the vector $4x\mathbf{i} + 6y\mathbf{j} - \mathbf{k}$ is normal to the surface at any (x, y, z) , so that $\mathbf{n} = 8\mathbf{i} + 12\mathbf{j} - \mathbf{k}$ is normal to the surface at $(2, 2, 20)$. If $\mathbf{v} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$, then the velocity of the particle after the ricochet is $\mathbf{w} = \mathbf{v} - 2 \text{proj}_{\mathbf{n}} \mathbf{v} = \mathbf{v} - \left(\frac{2\mathbf{v} \cdot \mathbf{n}}{|\mathbf{n}|^2}\right) \mathbf{n} = \mathbf{v} - \left(\frac{2 \cdot 25}{209}\right) \mathbf{n} = (\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - \left(\frac{400}{209}\mathbf{i} + \frac{600}{209}\mathbf{j} - \frac{50}{209}\mathbf{k}\right) = -\frac{191}{209}\mathbf{i} - \frac{391}{209}\mathbf{j} - \frac{995}{209}\mathbf{k}$.
21. (a) $\mathbf{k}$ is a vector normal to $z = 10 - x^2 - y^2$ at the point $(0, 0, 10)$. So directions tangential to S at $(0, 0, 10)$ will be unit vectors $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$. Also, $\nabla T(x, y, z) = (2xy + 4)\mathbf{i} + (x^2 + 2yz + 14)\mathbf{j} + (y^2 + 1)\mathbf{k} \Rightarrow \nabla T(0, 0, 10) = 4\mathbf{i} + 14\mathbf{j} + \mathbf{k}$. We seek the unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ such that $D_u T(0, 0, 10) = (4\mathbf{i} + 14\mathbf{j} + \mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j}) = (4\mathbf{i} + 14\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j})$ is a maximum. The maximum will occur when $a\mathbf{i} + b\mathbf{j}$ has the same direction as $4\mathbf{i} + 14\mathbf{j}$, or $\mathbf{u} = \frac{1}{\sqrt{53}}(2\mathbf{i} + 7\mathbf{j})$.
- (b) A vector normal to S at $(1, 1, 8)$ is $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Now, $\nabla T(1, 1, 8) = 6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}$ and we seek the unit vector $\mathbf{u}$ such that $D_u T(1, 1, 8) = \nabla T \cdot \mathbf{u}$ has its largest value. Now write $\nabla T = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is parallel to ∇T and $\mathbf{w}$ is orthogonal to ∇T . Then $D_u T = \nabla T \cdot \mathbf{u} = (\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u}$. Thus $D_u T(1, 1, 8)$ is a maximum when $\mathbf{u}$ has the same direction as $\mathbf{w}$. Now, $\mathbf{w} = \nabla T - \left(\frac{\nabla T \cdot \mathbf{n}}{|\mathbf{n}|^2}\right) \mathbf{n} = (6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}) - \left(\frac{12+62+2}{4+4+1}\right)(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \left(6 - \frac{152}{9}\right)\mathbf{i} + \left(31 - \frac{152}{9}\right)\mathbf{j} + \left(2 - \frac{76}{9}\right)\mathbf{k} = -\frac{98}{9}\mathbf{i} + \frac{127}{9}\mathbf{j} - \frac{58}{9}\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{w}}{|\mathbf{w}|} = -\frac{1}{\sqrt{29,097}}(98\mathbf{i} - 127\mathbf{j} + 58\mathbf{k})$.
22. Suppose the surface (boundary) of the mineral deposit is the graph of $z = f(x, y)$ (where the z -axis points up into the air). Then $-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$ is an outer normal to the mineral deposit at (x, y) and $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ points in the direction of steepest ascent of the mineral deposit. This is in the direction of the vector $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ at $(0, 0)$ (the location of the 1st borehole) that the geologists should drill their fourth borehole. To approximate this vector we use the fact that $(0, 0, -1000)$, $(0, 100, -950)$, and $(100, 0, -1025)$ lie on the graph of $z = f(x, y)$. The plane containing these three points is a good approximation to the tangent plane to $z = f(x, y)$ at the point $(0, 0, 0)$. A normal to this plane is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 100 & 50 \\ 100 & 0 & -25 \end{vmatrix}$

$= -2500\mathbf{i} + 5000\mathbf{j} - 10,000\mathbf{k}$, or $-\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. So at $(0, 0)$ the vector $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ is approximately $-\mathbf{i} + 2\mathbf{j}$. Thus the geologists should drill their fourth borehole in the direction of $\frac{1}{\sqrt{5}}(-\mathbf{i} + 2\mathbf{j})$ from the first borehole.

$$\begin{aligned} 23. \quad w &= e^{rt} \sin \pi x \Rightarrow w_t = re^{rt} \sin \pi x \text{ and } w_x = \pi e^{rt} \cos \pi x \Rightarrow w_{xx} = -\pi^2 e^{rt} \sin \pi x; w_{xx} = \frac{1}{c^2} w_t, \text{ where } c^2 \text{ is the} \\ &\text{positive constant determined by the material of the rod} \Rightarrow -\pi^2 e^{rt} \sin \pi x = \frac{1}{c^2} (re^{rt} \sin \pi x) \\ &\Rightarrow (r + c^2 \pi^2) e^{rt} \sin \pi x = 0 \Rightarrow r = -c^2 \pi^2 \Rightarrow w = e^{-c^2 \pi^2 t} \sin \pi x \end{aligned}$$

$$\begin{aligned} 24. \quad w &= e^{rt} \sin kx \Rightarrow w_t = re^{rt} \sin kx \text{ and } w_x = ke^{rt} \cos kx \Rightarrow w_{xx} = -k^2 e^{rt} \sin kx; w_{xx} = \frac{1}{c^2} w_t \\ &\Rightarrow -k^2 e^{rt} \sin kx = \frac{1}{c^2} (re^{rt} \sin kx) \Rightarrow (r + c^2 k^2) e^{rt} \sin kx = 0 \Rightarrow r = -c^2 k^2 \Rightarrow w = e^{-c^2 k^2 t} \sin kx. \\ \text{Now, } w(L, t) &= 0 \Rightarrow e^{-c^2 k^2 t} \sin kL = 0 \Rightarrow kL = n\pi \text{ for } n \text{ an integer} \Rightarrow k = \frac{n\pi}{L} \Rightarrow w = e^{-c^2 n^2 \pi^2 t / L^2} \sin \left(\frac{n\pi}{L} x \right). \\ \text{As } t &\rightarrow \infty, w \rightarrow 0 \text{ since } \left| \sin \left(\frac{n\pi}{L} x \right) \right| \leq 1 \text{ and } e^{-c^2 n^2 \pi^2 t / L^2} \rightarrow 0. \end{aligned}$$

CHAPTER 15 MULTIPLE INTEGRALS

15.1 DOUBLE AND ITERATED INTEGRALS OVER RECTANGLES

1. $\int_1^2 \int_0^4 2xy \, dy \, dx = \int_1^2 [x y^2]_0^4 \, dx = \int_1^2 16x \, dx = [8x^2]_1^2 = 24$
2. $\int_0^2 \int_{-1}^1 (x - y) \, dy \, dx = \int_0^2 [xy - \frac{1}{2}y^2]_{-1}^1 \, dx = \int_0^2 2x \, dx = [x^2]_0^2 = 4$
3. $\int_{-1}^0 \int_{-1}^1 (x + y + 1) \, dx \, dy = \int_{-1}^0 [\frac{x^2}{2} + yx + x]_{-1}^1 \, dy = \int_{-1}^0 (2y + 2) \, dy = [y^2 + 2y]_{-1}^0 = 1$
4. $\int_0^1 \int_0^1 (1 - \frac{x^2 + y^2}{2}) \, dx \, dy = \int_0^1 [x - \frac{x^3}{6} - \frac{xy^2}{2}]_0^1 \, dy = \int_0^1 (\frac{5}{6} - \frac{y^2}{2}) \, dy = [\frac{5}{6}y - \frac{y^3}{6}]_0^1 = \frac{2}{3}$
5. $\int_0^3 \int_0^2 (4 - y^2) \, dy \, dx = \int_0^3 [4y - \frac{y^3}{3}]_0^2 \, dx = \int_0^3 \frac{16}{3} \, dx = [\frac{16}{3}x]_0^3 = 16$
6. $\int_0^3 \int_{-2}^0 (x^2y - 2xy) \, dy \, dx = \int_0^3 [\frac{x^2y^2}{2} - xy^2]_{-2}^0 \, dx = \int_0^3 (4x - 2x^2) \, dx = [2x^2 - \frac{2x^3}{3}]_0^3 = 0$
7. $\int_0^1 \int_0^1 \frac{y}{1+xy} \, dx \, dy = \int_0^1 [\ln|1 + xy|]_0^1 \, dy = \int_0^1 \ln|1 + y| \, dy = [y \ln|1 + y| - y + \ln|1 + y|]_0^1 = 2 \ln 2 - 1$
8. $\int_1^4 \int_0^4 (\frac{x}{2} + \sqrt{y}) \, dx \, dy = \int_1^4 [\frac{1}{4}x^2 + x\sqrt{y}]_0^4 \, dy = \int_1^4 (4 + 4y^{1/2}) \, dy = [4y + \frac{8}{3}y^{3/2}]_1^4 = \frac{92}{3}$
9. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} \, dy \, dx = \int_0^{\ln 2} [e^{2x+y}]_1^{\ln 5} \, dx = \int_0^{\ln 2} (5e^{2x} - e^{2x+1}) \, dx = [\frac{5}{2}e^{2x} - \frac{1}{2}e^{2x+1}]_0^{\ln 2} = \frac{3}{2}(5 - e)$
10. $\int_0^1 \int_1^2 x y e^x \, dy \, dx = \int_0^1 [\frac{1}{2}x y^2 e^x]_1^2 \, dx = \int_0^1 \frac{3}{2}x e^x \, dx = [\frac{3}{2}x e^x - \frac{3}{2}e^x]_0^1 = \frac{3}{2}$
11. $\int_{-1}^2 \int_0^{\pi/2} y \sin x \, dx \, dy = \int_{-1}^2 [-y \cos x]_0^{\pi/2} \, dy = \int_{-1}^2 y \, dy = [\frac{1}{2}y^2]_{-1}^2 = \frac{3}{2}$
12. $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y) \, dx \, dy = \int_{\pi}^{2\pi} [-\cos x + x \cos y]_0^{\pi} \, dy = \int_{\pi}^{2\pi} (2 + \pi \cos y) \, dy = [2y + \pi \sin y]_{\pi}^{2\pi} = 2\pi$
13. $\iint_R (6y^2 - 2x) \, dA = \int_0^1 \int_0^2 (6y^2 - 2x) \, dy \, dx = \int_0^1 [2y^3 - 2xy]_0^2 \, dx = \int_0^1 (16 - 4x) \, dx = [16x - 2x^2]_0^1 = 14$
14. $\iint_R \frac{\sqrt{x}}{y^2} \, dA = \int_0^4 \int_1^2 \frac{\sqrt{x}}{y^2} \, dy \, dx = \int_0^4 [-\frac{\sqrt{x}}{y}]_1^2 \, dx = \int_0^4 \frac{1}{2}x^{1/2} \, dx = [\frac{1}{3}x^{3/2}]_0^4 = \frac{8}{3}$
15. $\iint_R xy \cos y \, dA = \int_{-1}^1 \int_0^{\pi} xy \cos y \, dy \, dx = \int_{-1}^1 [xy \sin y + x \cos y]_0^{\pi} \, dx = \int_{-1}^1 (-2x) \, dx = [-x^2]_{-1}^1 = 0$
16. $\iint_R y \sin(x + y) \, dA = \int_{-\pi}^0 \int_0^{\pi} y \sin(x + y) \, dy \, dx = \int_{-\pi}^0 [-y \cos(x + y) + \sin(x + y)]_0^{\pi} \, dx$
 $= \int_{-\pi}^0 (\sin(x + \pi) - \pi \cos(x + \pi) - \sin x) \, dx = [-\cos(x + \pi) - \pi \sin(x + \pi) + \cos x]_{-\pi}^0 = 4$

$$17. \iint_R e^{x-y} dA = \int_0^{\ln 2} \int_0^{\ln 2} e^{x-y} dy dx = \int_0^{\ln 2} [-e^{x-y}]_0^{\ln 2} dx = \int_0^{\ln 2} (-e^{x-\ln 2} + e^x) dx = [-e^{x-\ln 2} + e^x]_0^{\ln 2} = \frac{1}{2}$$

$$18. \iint_R xy e^{xy^2} dA = \int_0^2 \int_0^1 xy e^{xy^2} dy dx = \int_0^2 \left[\frac{1}{2} e^{xy^2} \right]_0^1 dx = \int_0^2 \left(\frac{1}{2} e^x - \frac{1}{2} \right) dx = \left[\frac{1}{2} e^x - \frac{1}{2} x \right]_0^2 = \frac{1}{2} (e^2 - 3)$$

$$19. \iint_R \frac{xy^3}{x^2+1} dA = \int_0^1 \int_0^2 \frac{xy^3}{x^2+1} dy dx = \int_0^1 \left[\frac{xy^4}{4(x^2+1)} \right]_0^2 dx = \int_0^1 \frac{4x}{x^2+1} dx = [2 \ln|x^2+1|]_0^1 = 2 \ln 2$$

$$20. \iint_R \frac{y}{x^2y^2+1} dA = \int_0^1 \int_0^1 \frac{y}{(xy)^2+1} dx dy = \int_0^1 [\tan^{-1}(xy)]_0^1 dy = \int_0^1 \tan^{-1} y dy = [y \tan^{-1} y - \frac{1}{2} \ln|1+y^2|]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2$$

$$21. \int_1^2 \int_1^2 \frac{1}{xy} dy dx = \int_1^2 \frac{1}{x} (\ln 2 - \ln 1) dx = (\ln 2) \int_1^2 \frac{1}{x} dx = (\ln 2)^2$$

$$22. \int_0^1 \int_0^\pi y \cos xy dx dy = \int_0^1 [\sin xy]_0^\pi dy = \int_0^1 \sin \pi y dy = [-\frac{1}{\pi} \cos \pi y]_0^1 = -\frac{1}{\pi} (-1 - 1) = \frac{2}{\pi}$$

$$23. V = \iint_R f(x, y) dA = \int_{-1}^1 \int_{-1}^1 (x^2 + y^2) dy dx = \int_{-1}^1 [x^2 y + \frac{1}{3} y^3]_{-1}^1 dx = \int_{-1}^1 (2x^2 + \frac{2}{3}) dx = [\frac{2}{3} x^3 + \frac{2}{3} x]_{-1}^1 = \frac{8}{3}$$

$$24. V = \iint_R f(x, y) dA = \int_0^2 \int_0^2 (16 - x^2 - y^2) dy dx = \int_0^2 [16y - x^2 y - \frac{1}{3} y^3]_0^2 dx = \int_0^2 (\frac{88}{3} - 2x^2) dx = [\frac{88}{3} x - \frac{2}{3} x^3]_0^2 = \frac{160}{3}$$

$$25. V = \iint_R f(x, y) dA = \int_0^1 \int_0^1 (2 - x - y) dy dx = \int_0^1 [2y - xy - \frac{1}{2} y^2]_0^1 dx = \int_0^1 (\frac{3}{2} - x) dx = [\frac{3}{2} x - \frac{1}{2} x^2]_0^1 = 1$$

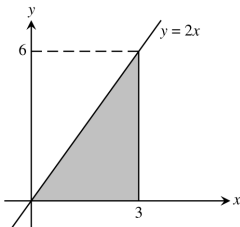
$$26. V = \iint_R f(x, y) dA = \int_0^4 \int_0^2 \frac{y}{2} dy dx = \int_0^4 \left[\frac{y^2}{4} \right]_0^2 dx = \int_0^4 1 dx = [x]_0^4 = 4$$

$$27. V = \iint_R f(x, y) dA = \int_0^{\pi/2} \int_0^{\pi/4} 2 \sin x \cos y dy dx = \int_0^{\pi/2} [2 \sin x \sin y]_0^{\pi/4} dx = \int_0^{\pi/2} (\sqrt{2} \sin x) dx = [-\sqrt{2} \cos x]_0^{\pi/2} = \sqrt{2}$$

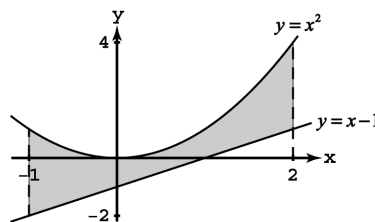
$$28. V = \iint_R f(x, y) dA = \int_0^1 \int_0^2 (4 - y^2) dy dx = \int_0^1 [4y - \frac{1}{3} y^3]_0^2 dx = \int_0^1 (\frac{16}{3}) dx = [\frac{16}{3} x]_0^1 = \frac{16}{3}$$

15.2 DOUBLE INTEGRALS OVER GENERAL REGIONS

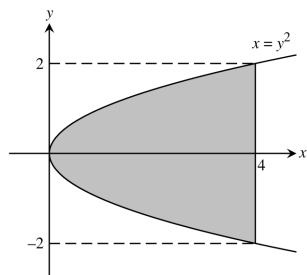
1.



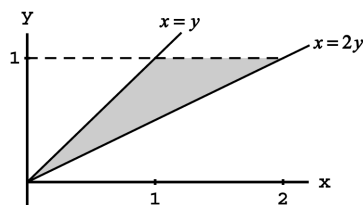
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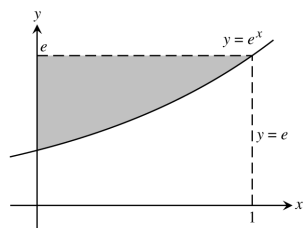
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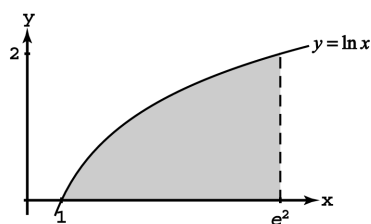
4.



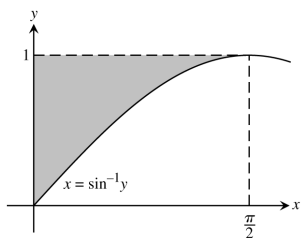
5.



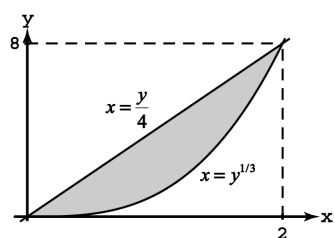
6.



7.



8.



9. (a) $\int_0^2 \int_{x^3}^8 dy dx$

(b) $\int_0^8 \int_0^{y^{1/3}} dx dy$

10. (a) $\int_0^3 \int_0^{2x} dy dx$

(b) $\int_0^6 \int_{y/2}^3 dx dy$

11. (a) $\int_0^3 \int_{x^2}^{3x} dy dx$

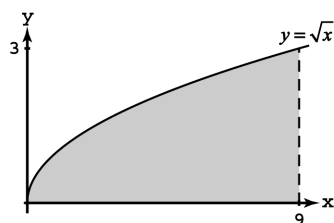
(b) $\int_0^9 \int_{y/3}^{\sqrt{y}} dx dy$

12. (a) $\int_0^2 \int_1^{e^x} dy dx$

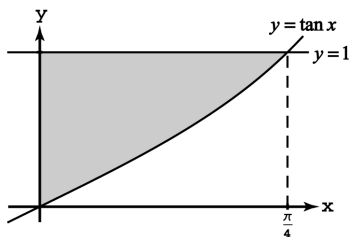
(b) $\int_1^{e^2} \int_{\ln y}^2 dx dy$

13. (a) $\int_0^9 \int_0^{\sqrt{x}} dy dx$

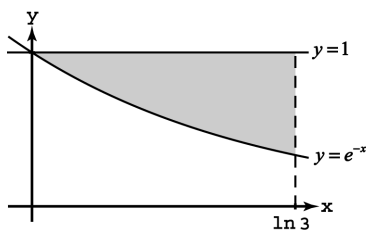
(b) $\int_0^3 \int_{y^2}^9 dx dy$



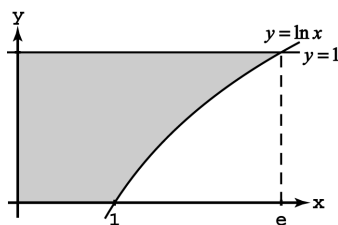
14. (a) $\int_0^{\pi/4} \int_{\tan x}^1 dy \, dx$
 (b) $\int_0^1 \int_0^{\tan^{-1} y} dx \, dy$



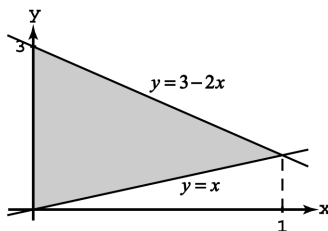
15. (a) $\int_0^{\ln 3} \int_{e^{-x}}^1 dy \, dx$
 (b) $\int_{1/3}^1 \int_{-\ln y}^{\ln 3} dx \, dy$



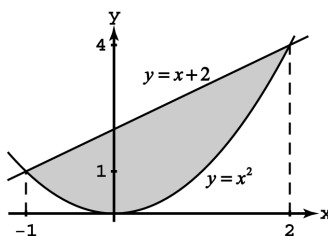
16. (a) $\int_0^1 \int_0^1 dy \, dx + \int_1^e \int_{\ln x}^1 dy \, dx$
 (b) $\int_0^1 \int_0^{e^y} dx \, dy$



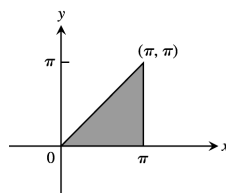
17. (a) $\int_0^1 \int_x^{3-2x} dy \, dx$
 (b) $\int_0^1 \int_0^y dx \, dy + \int_1^3 \int_0^{(3-y)/2} dx \, dy$



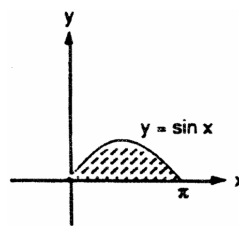
18. (a) $\int_{-1}^2 \int_{x^2}^{x+2} dy \, dx$
 (b) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy + \int_1^3 \int_{y-2}^{\sqrt{y}} dx \, dy$



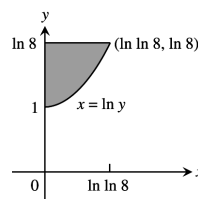
19. $\int_0^\pi \int_0^x (x \sin y) \, dy \, dx = \int_0^\pi [-x \cos y]_0^x \, dx$
 $= \int_0^\pi (x - x \cos x) \, dx = \left[\frac{x^2}{2} - (\cos x + x \sin x) \right]_0^\pi$
 $= \frac{\pi^2}{2} + 2$



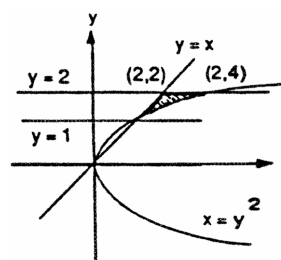
$$\begin{aligned}
 20. \int_0^\pi \int_0^{\sin x} y \, dy \, dx &= \int_0^\pi \left[\frac{y^2}{2} \right]_0^{\sin x} dx = \int_0^\pi \frac{1}{2} \sin^2 x \, dx \\
 &= \frac{1}{4} \int_0^\pi (1 - \cos 2x) \, dx = \frac{1}{4} \left[x - \frac{1}{2} \sin 2x \right]_0^\pi = \frac{\pi}{4}
 \end{aligned}$$



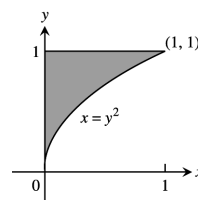
$$\begin{aligned}
 21. \int_1^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx \, dy &= \int_1^{\ln 8} [e^{x+y}]_0^{\ln y} dy = \int_1^{\ln 8} (ye^y - e^y) dy \\
 &= [(y-1)e^y - e^y]_1^{\ln 8} = 8(\ln 8 - 1) - 8 + e \\
 &= 8 \ln 8 - 16 + e
 \end{aligned}$$



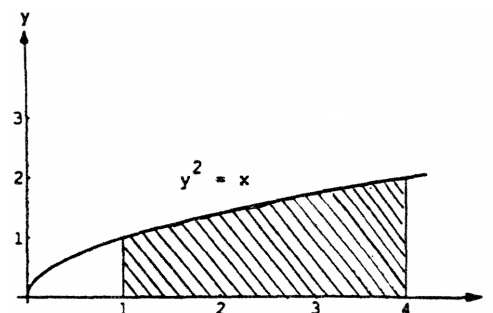
$$\begin{aligned}
 22. \int_1^2 \int_y^{y^2} dx \, dy &= \int_1^2 (y^2 - y) \, dy = \left[\frac{y^3}{3} - \frac{y^2}{2} \right]_1^2 \\
 &= \left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - \frac{1}{2} \right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}
 \end{aligned}$$



$$\begin{aligned}
 23. \int_0^1 \int_0^{y^2} 3y^3 e^{xy} \, dx \, dy &= \int_0^1 [3y^2 e^{xy}]_0^{y^2} dy \\
 &= \int_0^1 (3y^2 e^{y^3} - 3y^2) \, dy = [e^{y^3} - y^3]_0^1 = e - 2
 \end{aligned}$$



$$\begin{aligned}
 24. \int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} \, dy \, dx &= \int_1^4 \left[\frac{3}{2} \sqrt{x} e^{y/\sqrt{x}} \right]_0^{\sqrt{x}} dx \\
 &= \frac{3}{2} (e - 1) \int_1^4 \sqrt{x} \, dx = \left[\frac{3}{2} (e - 1) \left(\frac{2}{3} x^{3/2} \right) \right]_1^4 = 7(e - 1)
 \end{aligned}$$



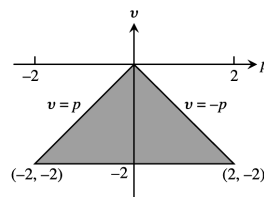
$$25. \int_1^2 \int_x^{2x} \frac{x}{y} \, dy \, dx = \int_1^2 [x \ln y]_x^{2x} dx = (\ln 2) \int_1^2 x \, dx = \frac{3}{2} \ln 2$$

$$\begin{aligned}
 26. \int_0^1 \int_0^{1-x} (x^2 + y^2) \, dy \, dx &= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx = \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} \right] dx \\
 &= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12} \right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0 \right) - \left(0 - 0 - \frac{1}{12} \right) = \frac{1}{6}
 \end{aligned}$$

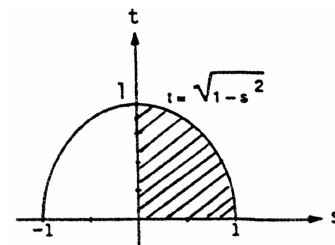
$$\begin{aligned}
 27. \int_0^1 \int_0^{1-u} (v - \sqrt{u}) \, dv \, du &= \int_0^1 \left[\frac{v^2}{2} - v\sqrt{u} \right]_0^{1-u} du = \int_0^1 \left[\frac{(1-u)^2}{2} - \sqrt{u}(1-u) \right] du \\
 &= \int_0^1 \left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2} \right) du = \left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3} u^{3/2} + \frac{2}{5} u^{5/2} \right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{10}
 \end{aligned}$$

$$\begin{aligned}
 28. \int_1^2 \int_0^{\ln t} e^s \ln t \, ds \, dt &= \int_1^2 [e^s \ln t]_0^{\ln t} \, dt = \int_1^2 (t \ln t - \ln t) \, dt = \left[\frac{t^2}{2} \ln t - \frac{t^2}{4} - t \ln t + t \right]_1^2 \\
 &= (2 \ln 2 - 1 - 2 \ln 2 + 2) - \left(-\frac{1}{4} + 1 \right) = \frac{1}{4}
 \end{aligned}$$

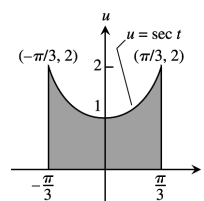
$$\begin{aligned}
 29. \int_{-2}^0 \int_v^{-v} 2 \, dp \, dv &= 2 \int_{-2}^0 [p]_v^{-v} \, dv = 2 \int_{-2}^0 -2v \, dv \\
 &= -2 [v^2]_{-2}^0 = 8
 \end{aligned}$$



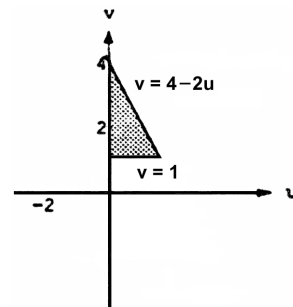
$$\begin{aligned}
 30. \int_0^1 \int_0^{\sqrt{1-s^2}} 8t \, dt \, ds &= \int_0^1 [4t^2]_0^{\sqrt{1-s^2}} \, ds \\
 &= \int_0^1 4(1-s^2) \, ds = 4 \left[s - \frac{s^3}{3} \right]_0^1 = \frac{8}{3}
 \end{aligned}$$



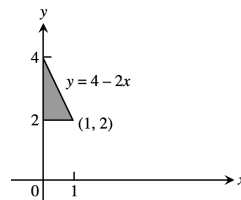
$$\begin{aligned}
 31. \int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t \, du \, dt &= \int_{-\pi/3}^{\pi/3} [(3 \cos t)u]_0^{\sec t} \, dt \\
 &= \int_{-\pi/3}^{\pi/3} 3 \, dt = 2\pi
 \end{aligned}$$



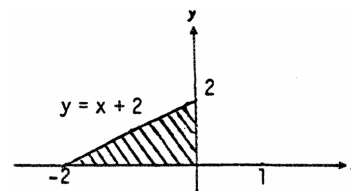
$$\begin{aligned}
 32. \int_0^{3/2} \int_1^{4-2u} \frac{4-2u}{v^2} \, dv \, du &= \int_0^{3/2} \left[\frac{2u-4}{v} \right]_1^{4-2u} \, du \\
 &= \int_0^{3/2} (3-2u) \, du = [3u - u^2]_0^{3/2} = \frac{9}{2}
 \end{aligned}$$



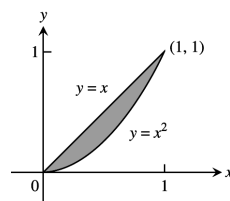
$$33. \int_2^4 \int_0^{(4-y)/2} dx \, dy$$



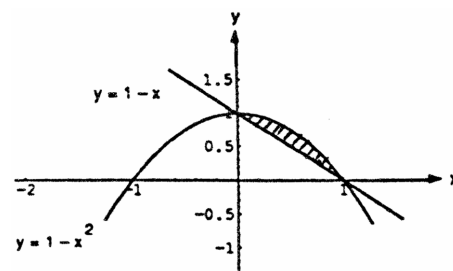
$$34. \int_{-2}^0 \int_0^{x+2} dy \, dx$$



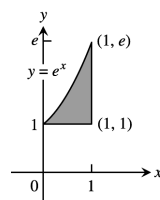
35. $\int_0^1 \int_{x^2}^x dy \, dx$



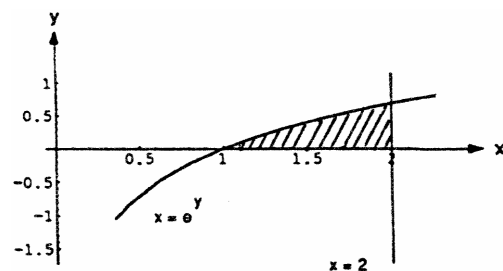
36. $\int_0^1 \int_{1-y}^{\sqrt{1-y}} dx \, dy$



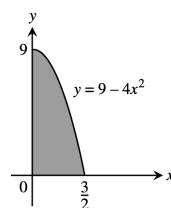
37. $\int_1^e \int_{\ln y}^1 dx \, dy$



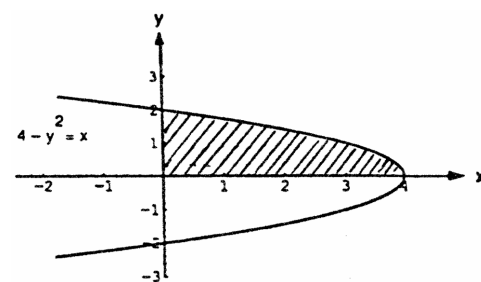
38. $\int_1^2 \int_0^{\ln x} dy \, dx$



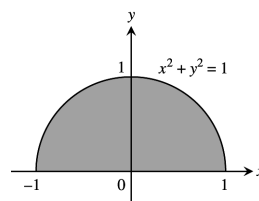
39. $\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x \, dx \, dy$



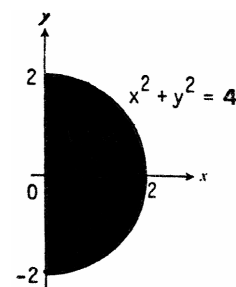
40. $\int_0^4 \int_0^{\sqrt{4-x}} y \, dy \, dx$



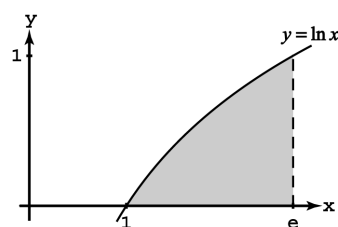
41. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y \, dy \, dx$



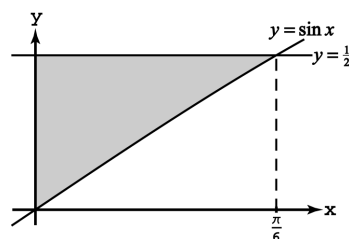
42. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} 6x \, dx \, dy$



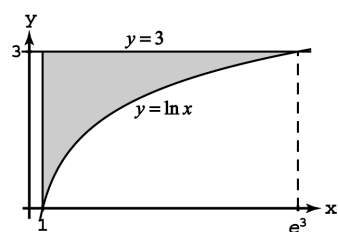
43. $\int_0^1 \int_{e^y}^e x y \, dx \, dy$



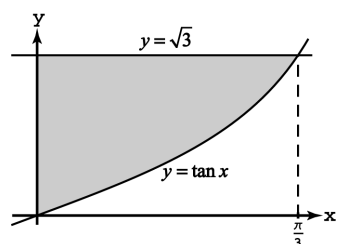
44. $\int_0^{1/2} \int_0^{\sin^{-1} y} x y^2 \, dx \, dy$



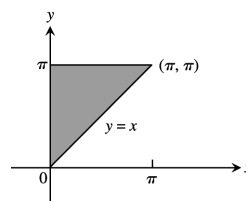
45. $\int_1^{e^3} \int_{\ln x}^3 (x + y) \, dy \, dx$



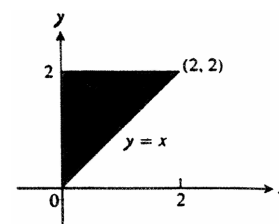
46. $\int_0^{\pi/3} \int_{\tan x}^{\sqrt{3}} \sqrt{xy} \, dy \, dx$



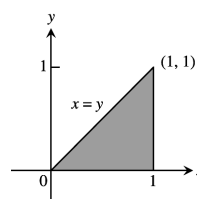
$$47. \int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx = \int_0^\pi \int_0^y \frac{\sin y}{y} dx dy = \int_0^\pi \sin y dy = 2$$



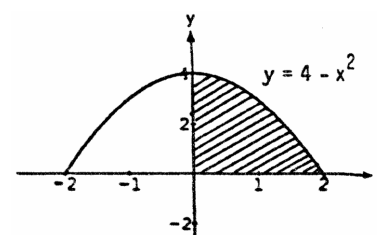
$$\begin{aligned} 48. \int_0^2 \int_x^2 2y^2 \sin xy dy dx &= \int_0^2 \int_0^y 2y^2 \sin xy dx dy \\ &= \int_0^2 [-2y \cos xy]_0^y dy = \int_0^2 (-2y \cos y^2 + 2y) dy \\ &= [-\sin y^2 + y^2]_0^2 = 4 - \sin 4 \end{aligned}$$



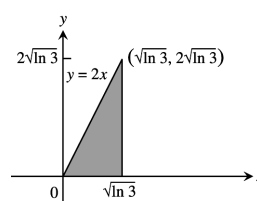
$$\begin{aligned} 49. \int_0^1 \int_y^1 x^2 e^{xy} dx dy &= \int_0^1 \int_0^x x^2 e^{xy} dy dx = \int_0^1 [x e^{xy}]_0^x dx \\ &= \int_0^1 (x e^{x^2} - x) dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2} \right]_0^1 = \frac{e-2}{2} \end{aligned}$$



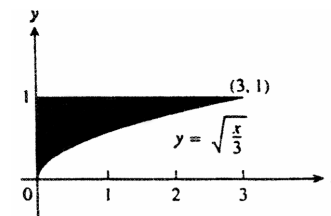
$$\begin{aligned} 50. \int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} dy dx &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} dx dy \\ &= \int_0^4 \left[\frac{x^2 e^{2y}}{2(4-y)} \right]_0^{\sqrt{4-y}} dy = \int_0^4 \frac{e^{2y}}{2} dy = \left[\frac{e^{2y}}{4} \right]_0^4 = \frac{e^8 - 1}{4} \end{aligned}$$



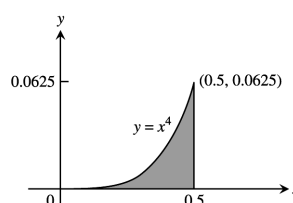
$$\begin{aligned} 51. \int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} dx dy &= \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2} dy dx \\ &= \int_0^{\sqrt{\ln 3}} 2x e^{x^2} dx = [e^{x^2}]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2 \end{aligned}$$



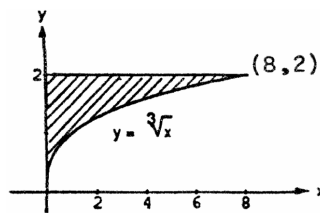
$$\begin{aligned} 52. \int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} dy dx &= \int_0^1 \int_0^{3y^2} e^{y^3} dx dy \\ &= \int_0^1 3y^2 e^{y^3} dy = [e^{y^3}]_0^1 = e - 1 \end{aligned}$$



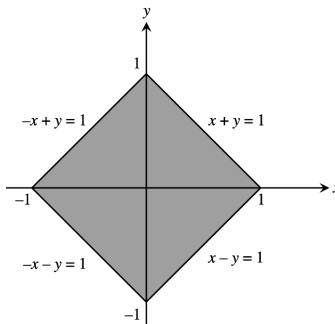
$$\begin{aligned} 53. \int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) dx dy &= \int_0^{1/2} \int_0^{x^4} \cos(16\pi x^5) dy dx \\ &= \int_0^{1/2} x^4 \cos(16\pi x^5) dx = \left[\frac{\sin(16\pi x^5)}{80\pi} \right]_0^{1/2} = \frac{1}{80\pi} \end{aligned}$$



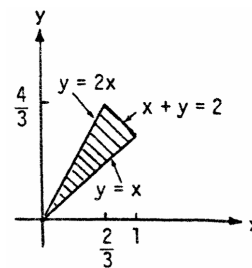
$$\begin{aligned}
 54. \int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1} dy dx &= \int_0^2 \int_0^{y^3} \frac{1}{y^4+1} dx dy \\
 &= \int_0^2 \frac{y^3}{y^4+1} dy = \frac{1}{4} [\ln(y^4+1)]_0^2 = \frac{\ln 17}{4}
 \end{aligned}$$



$$\begin{aligned}
 55. \iint_R (y - 2x^2) dA &= \int_{-1}^0 \int_{-x-1}^{x+1} (y - 2x^2) dy dx + \int_0^1 \int_{x-1}^{1-x} (y - 2x^2) dy dx \\
 &= \int_{-1}^0 \left[\frac{1}{2} y^2 - 2x^2 y \right]_{-x-1}^{x+1} dx + \int_0^1 \left[\frac{1}{2} y^2 - 2x^2 y \right]_{x-1}^{1-x} dx \\
 &= \int_{-1}^0 \left[\frac{1}{2} (x+1)^2 - 2x^2(x+1) - \frac{1}{2} (-x-1)^2 + 2x^2(-x-1) \right] dx \\
 &\quad + \int_0^1 \left[\frac{1}{2} (1-x)^2 - 2x^2(1-x) - \frac{1}{2} (x-1)^2 + 2x^2(x-1) \right] dx \\
 &= -4 \int_{-1}^0 (x^3 + x^2) dx + 4 \int_0^1 (x^3 - x^2) dx \\
 &= -4 \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^0 + 4 \left[\frac{x^4}{4} - \frac{x^3}{3} \right]_0^1 = 4 \left[\frac{(-1)^4}{4} + \frac{(-1)^3}{3} \right] + 4 \left(\frac{1}{4} - \frac{1}{3} \right) = 8 \left(\frac{3}{12} - \frac{4}{12} \right) = -\frac{8}{12} = -\frac{2}{3}
 \end{aligned}$$



$$\begin{aligned}
 56. \iint_R xy dA &= \int_0^{2/3} \int_x^{2x} xy dy dx + \int_{2/3}^1 \int_x^{2-x} xy dy dx \\
 &= \int_0^{2/3} \left[\frac{1}{2} xy^2 \right]_x^{2x} dx + \int_{2/3}^1 \left[\frac{1}{2} xy^2 \right]_x^{2-x} dx \\
 &= \int_0^{2/3} \left(2x^3 - \frac{1}{2} x^3 \right) dx + \int_{2/3}^1 \left[\frac{1}{2} x(2-x)^2 - \frac{1}{2} x^3 \right] dx \\
 &= \int_0^{2/3} \frac{3}{2} x^3 dx + \int_{2/3}^1 (2x - x^2) dx \\
 &= \left[\frac{3}{8} x^4 \right]_0^{2/3} + \left[x^2 - \frac{2}{3} x^3 \right]_{2/3}^1 = \left(\frac{3}{8} \right) \left(\frac{16}{81} \right) + \left(1 - \frac{2}{3} \right) - \left[\frac{4}{9} - \left(\frac{2}{3} \right) \left(\frac{8}{27} \right) \right] = \frac{6}{81} + \frac{27}{81} - \left(\frac{36}{81} - \frac{16}{81} \right) = \frac{13}{81}
 \end{aligned}$$



$$\begin{aligned}
 57. V &= \int_0^1 \int_x^{2-x} (x^2 + y^2) dy dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3} \right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} + \frac{(2-x)^4}{12} \right]_0^1 \\
 &= \left(\frac{2}{3} - \frac{7}{12} + \frac{1}{12} \right) - (0 - 0 - \frac{16}{12}) = \frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 58. V &= \int_{-2}^1 \int_x^{2-x^2} x^2 dy dx = \int_{-2}^1 [x^2 y]_x^{2-x^2} dx = \int_{-2}^1 (2x^2 - x^4 - x^3) dx = \left[\frac{2}{3} x^3 - \frac{1}{5} x^5 - \frac{1}{4} x^4 \right]_{-2}^1 \\
 &= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4} \right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4} \right) = \left(\frac{40}{60} - \frac{12}{60} - \frac{15}{60} \right) - \left(-\frac{320}{60} + \frac{384}{60} - \frac{240}{60} \right) = \frac{189}{60} = \frac{63}{20}
 \end{aligned}$$

$$\begin{aligned}
 59. V &= \int_{-4}^1 \int_{3x}^{4-x^2} (x+4) dy dx = \int_{-4}^1 [xy + 4y]_{3x}^{4-x^2} dx = \int_{-4}^1 [x(4-x^2) + 4(4-x^2) - 3x^2 - 12x] dx \\
 &= \int_{-4}^1 (-x^3 - 7x^2 - 8x + 16) dx = \left[-\frac{1}{4} x^4 - \frac{7}{3} x^3 - 4x^2 + 16x \right]_{-4}^1 = \left(-\frac{1}{4} - \frac{7}{3} + 12 \right) - \left(\frac{64}{3} - 64 \right) = \frac{157}{3} - \frac{1}{4} = \frac{625}{12}
 \end{aligned}$$

$$\begin{aligned}
 60. V &= \int_0^2 \int_0^{\sqrt{4-x^2}} (3-y) dy dx = \int_0^2 \left[3y - \frac{y^2}{2} \right]_0^{\sqrt{4-x^2}} dx = \int_0^2 \left[3\sqrt{4-x^2} - \frac{(4-x^2)}{2} \right] dx \\
 &= \left[\frac{3}{2} x \sqrt{4-x^2} + 6 \sin^{-1} \left(\frac{x}{2} \right) - 2x + \frac{x^3}{6} \right]_0^2 = 6 \left(\frac{\pi}{2} \right) - 4 + \frac{8}{6} = 3\pi - \frac{16}{6} = \frac{9\pi-8}{3}
 \end{aligned}$$

$$61. V = \int_0^2 \int_0^3 (4-y^2) dx dy = \int_0^2 [4x - y^2 x]_0^3 dy = \int_0^2 (12 - 3y^2) dy = [12y - y^3]_0^2 = 24 - 8 = 16$$

$$\begin{aligned}
 62. \quad V &= \int_0^2 \int_0^{4-x^2} (4-x^2-y) \, dy \, dx = \int_0^2 \left[(4-x^2)y - \frac{y^2}{2} \right]_0^{4-x^2} dx = \int_0^2 \frac{1}{2} (4-x^2)^2 dx = \int_0^2 \left(8-4x^2+\frac{x^4}{2} \right) dx \\
 &= \left[8x - \frac{4}{3}x^3 + \frac{1}{10}x^5 \right]_0^2 = 16 - \frac{32}{3} + \frac{32}{10} = \frac{480-320+96}{30} = \frac{128}{15}
 \end{aligned}$$

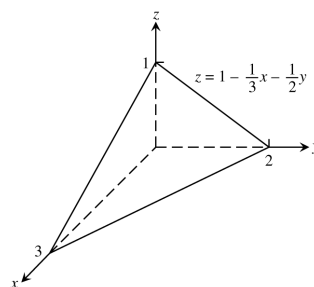
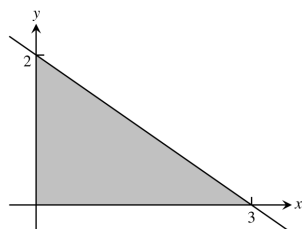
$$63. \quad V = \int_0^2 \int_0^{2-x} (12-3y^2) \, dy \, dx = \int_0^2 [12y - y^3]_0^{2-x} dx = \int_0^2 [24-12x - (2-x)^3] dx = \left[24x - 6x^2 + \frac{(2-x)^4}{4} \right]_0^2 = 20$$

$$64. \quad V = \int_{-1}^0 \int_{-x-1}^{x+1} (3-3x) \, dy \, dx + \int_0^1 \int_{x-1}^{1-x} (3-3x) \, dy \, dx = 6 \int_{-1}^0 (1-x^2) \, dx + 6 \int_0^1 (1-x)^2 \, dx = 4 + 2 = 6$$

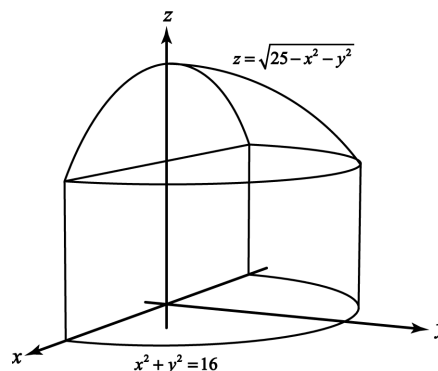
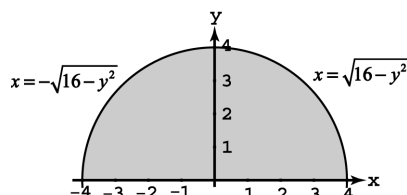
$$\begin{aligned}
 65. \quad V &= \int_1^2 \int_{-1/x}^{1/x} (x+1) \, dy \, dx = \int_1^2 [xy + y]_{-1/x}^{1/x} dx = \int_1^2 \left[1 + \frac{1}{x} - \left(-1 - \frac{1}{x} \right) \right] dx = 2 \int_1^2 \left(1 + \frac{1}{x} \right) dx = 2 [x + \ln x]_1^2 \\
 &= 2(1 + \ln 2)
 \end{aligned}$$

$$\begin{aligned}
 66. \quad V &= 4 \int_0^{\pi/3} \int_0^{\sec x} (1+y^2) \, dy \, dx = 4 \int_0^{\pi/3} \left[y + \frac{y^3}{3} \right]_0^{\sec x} dx = 4 \int_0^{\pi/3} \left(\sec x + \frac{\sec^3 x}{3} \right) dx \\
 &= \frac{2}{3} [7 \ln |\sec x + \tan x| + \sec x \tan x]_0^{\pi/3} = \frac{2}{3} [7 \ln (2 + \sqrt{3}) + 2\sqrt{3}]
 \end{aligned}$$

67.



68.



$$69. \quad \int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y} \, dy \, dx = \int_1^\infty \left[\frac{\ln y}{x^3} \right]_{e^{-x}}^1 dx = \int_1^\infty -\left(\frac{-x}{x^3} \right) dx = -\lim_{b \rightarrow \infty} \left[\frac{1}{x} \right]_1^b = -\lim_{b \rightarrow \infty} \left(\frac{1}{b} - 1 \right) = 1$$

$$\begin{aligned}
 70. \quad \int_{-1}^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y+1) \, dy \, dx &= \int_{-1}^1 [y^2 + y]_{-1/(1-x^2)^{1/2}}^{1/(1-x^2)^{1/2}} dx = \int_{-1}^1 \frac{2}{\sqrt{1-x^2}} dx = 4 \lim_{b \rightarrow 1^-} [\sin^{-1} x]_0^b \\
 &= 4 \lim_{b \rightarrow 1^-} [\sin^{-1} b - 0] = 2\pi
 \end{aligned}$$

$$\begin{aligned}
 71. \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)} \, dx \, dy &= 2 \int_0^\infty \left(\frac{2}{y^2+1} \right) \left(\lim_{b \rightarrow \infty} \tan^{-1} b - \tan^{-1} 0 \right) dy = 2\pi \lim_{b \rightarrow \infty} \int_0^b \frac{1}{y^2+1} dy \\
 &= 2\pi \left(\lim_{b \rightarrow \infty} \tan^{-1} b - \tan^{-1} 0 \right) = (2\pi) \left(\frac{\pi}{2} \right) = \pi^2
 \end{aligned}$$

$$\begin{aligned}
 72. \int_0^\infty \int_0^\infty x e^{-(x+2y)} dx dy &= \int_0^\infty e^{-2y} \lim_{b \rightarrow \infty} [-x e^{-x} - e^{-x}]_0^b dy = \int_0^\infty e^{-2y} \lim_{b \rightarrow \infty} (-b e^{-b} - e^{-b} + 1) dy \\
 &= \int_0^\infty e^{-2y} dy = \frac{1}{2} \lim_{b \rightarrow \infty} (-e^{-2b} + 1) = \frac{1}{2}
 \end{aligned}$$

$$73. \iint_R f(x, y) dA \approx \frac{1}{4} f\left(-\frac{1}{2}, 0\right) + \frac{1}{8} f(0, 0) + \frac{1}{8} f\left(\frac{1}{4}, 0\right) = \frac{1}{4} \left(-\frac{1}{2}\right) + \frac{1}{8} \left(0 + \frac{1}{4}\right) = -\frac{3}{32}$$

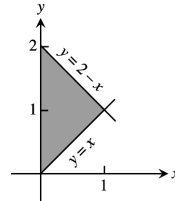
$$74. \iint_R f(x, y) dA \approx \frac{1}{4} \left[f\left(\frac{7}{4}, \frac{11}{4}\right) + f\left(\frac{9}{4}, \frac{11}{4}\right) + f\left(\frac{7}{4}, \frac{13}{4}\right) + f\left(\frac{9}{4}, \frac{13}{4}\right) \right] = \frac{1}{16} (29 + 31 + 33 + 35) = \frac{128}{16} = 8$$

75. The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2 + y^2 = 4$ at the point $(\sqrt{3}, 1) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$. Thus,

$$\iint_R f(x, y) dA = \int_0^{\sqrt{3}} \int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2} dy dx = \int_0^{\sqrt{3}} \left[(4-x^2) - \frac{x}{\sqrt{3}} \sqrt{4-x^2} \right] dx = \left[4x - \frac{x^3}{3} + \frac{(4-x^2)^{3/2}}{3\sqrt{3}} \right]_0^{\sqrt{3}} = \frac{20\sqrt{3}}{9}$$

$$\begin{aligned}
 76. \int_2^\infty \int_0^{2-x} \frac{1}{(x^2-x)(y-1)^{2/3}} dy dx &= \int_2^\infty \left[\frac{3(y-1)^{1/3}}{(x^2-x)} \right]_0^{2-x} dx = \int_2^\infty \left(\frac{3}{x^2-x} + \frac{3}{x^2-x} \right) dx = 6 \int_2^\infty \frac{dx}{x(x-1)} \\
 &= 6 \lim_{b \rightarrow \infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x} \right) dx = 6 \lim_{b \rightarrow \infty} [\ln(x-1) - \ln x]_2^b = 6 \lim_{b \rightarrow \infty} [\ln(b-1) - \ln b - \ln 1 + \ln 2] \\
 &= 6 \left[\lim_{b \rightarrow \infty} \ln\left(1 - \frac{1}{b}\right) + \ln 2 \right] = 6 \ln 2
 \end{aligned}$$

$$\begin{aligned}
 77. V &= \int_0^1 \int_x^{2-x} (x^2 + y^2) dy dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx \\
 &= \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3} \right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} + \frac{(2-x)^4}{12} \right]_0^1 \\
 &= \left(\frac{2}{3} - \frac{7}{12} + \frac{1}{12} \right) - \left(0 - 0 + \frac{16}{12} \right) = \frac{4}{3}
 \end{aligned}$$



$$\begin{aligned}
 78. \int_0^2 (\tan^{-1} \pi x - \tan^{-1} x) dx &= \int_0^2 \int_x^{\pi x} \frac{1}{1+y^2} dy dx = \int_0^2 \int_{y/\pi}^y \frac{1}{1+y^2} dx dy + \int_2^{2\pi} \int_{y/\pi}^2 \frac{1}{1+y^2} dx dy \\
 &= \int_0^2 \frac{(1-\frac{1}{\pi})y}{1+y^2} dy + \int_2^{2\pi} \frac{(2-\frac{y}{\pi})}{1+y^2} dy = \left(\frac{\pi-1}{2\pi} \right) [\ln(1+y^2)]_0^2 + \left[2 \tan^{-1} y + \frac{1}{2\pi} \ln(1+y^2) \right]_2^{2\pi} \\
 &= \left(\frac{\pi-1}{2\pi} \right) \ln 5 + 2 \tan^{-1} 2\pi - \frac{1}{2\pi} \ln(1+4\pi^2) - 2 \tan^{-1} 2 + \frac{1}{2\pi} \ln 5 \\
 &= 2 \tan^{-1} 2\pi - 2 \tan^{-1} 2 - \frac{1}{2\pi} \ln(1+4\pi^2) + \frac{\ln 5}{2}
 \end{aligned}$$

79. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points (x, y) such that $4 - x^2 - 2y^2 \geq 0$ or $x^2 + 2y^2 \leq 4$, which is the ellipse $x^2 + 2y^2 = 4$ together with its interior.

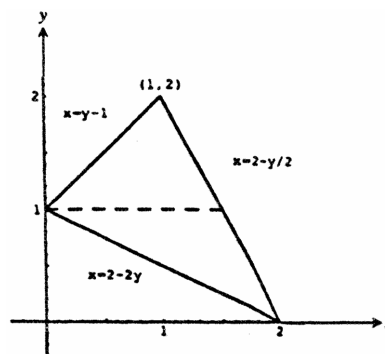
80. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points (x, y) such that $x^2 + y^2 - 9 \leq 0$ or $x^2 + y^2 \leq 9$, which is the closed disk of radius 3 centered at the origin.

81. No, it is not possible. By Fubini's theorem, the two orders of integration must give the same result.

82. One way would be to partition R into two triangles with the line $y = 1$. The integral of f over R could then be written as a sum of integrals that could be evaluated by integrating first with respect to x and then with respect to y :

$$\begin{aligned} \iint_R f(x, y) \, dA \\ = \int_0^1 \int_{2-2y}^{2-(y/2)} f(x, y) \, dx \, dy + \int_1^2 \int_{y-1}^{2-(y/2)} f(x, y) \, dx \, dy. \end{aligned}$$

Partitioning R with the line $x = 1$ would let us write the integral of f over R as a sum of iterated integrals with order $dy \, dx$.



$$\begin{aligned} 83. \int_{-b}^b \int_{-b}^b e^{-x^2-y^2} \, dx \, dy &= \int_{-b}^b \int_{-b}^b e^{-y^2} e^{-x^2} \, dx \, dy = \int_{-b}^b e^{-y^2} \left(\int_{-b}^b e^{-x^2} \, dx \right) dy = \left(\int_{-b}^b e^{-x^2} \, dx \right) \left(\int_{-b}^b e^{-y^2} \, dy \right) \\ &= \left(\int_{-b}^b e^{-x^2} \, dx \right)^2 = \left(2 \int_0^b e^{-x^2} \, dx \right)^2 = 4 \left(\int_0^b e^{-x^2} \, dx \right)^2; \text{ taking limits as } b \rightarrow \infty \text{ gives the stated result.} \end{aligned}$$

$$\begin{aligned} 84. \int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}} \, dy \, dx &= \int_0^3 \int_0^1 \frac{x^2}{(y-1)^{2/3}} \, dx \, dy = \int_0^3 \frac{1}{(y-1)^{2/3}} \left[\frac{x^3}{3} \right]_0^1 dy = \frac{1}{3} \int_0^3 \frac{dy}{(y-1)^{2/3}} \\ &= \frac{1}{3} \lim_{b \rightarrow 1^-} \int_0^b \frac{dy}{(y-1)^{2/3}} + \frac{1}{3} \lim_{b \rightarrow 1^+} \int_b^3 \frac{dy}{(y-1)^{2/3}} = \lim_{b \rightarrow 1^-} \left[(y-1)^{1/3} \right]_0^b + \lim_{b \rightarrow 1^+} \left[(y-1)^{1/3} \right]_b^3 \\ &= \left[\lim_{b \rightarrow 1^-} (b-1)^{1/3} - (-1)^{1/3} \right] - \left[\lim_{b \rightarrow 1^+} (b-1)^{1/3} - (2)^{1/3} \right] = (0+1) - (0-\sqrt[3]{2}) = 1 + \sqrt[3]{2} \end{aligned}$$

- 85-88. Example CAS commands:

Maple:

```
f := (x,y) -> 1/x/y;
q1 := Int( Int( f(x,y), y=1..x ), x=1..3 );
evalf( q1 );
value( q1 );
evalf( value(q1) );
```

- 89-94. Example CAS commands:

Maple:

```
f := (x,y) -> exp(x^2);
c,d := 0,1;
g1 := y -> 2*y;
g2 := y -> 4;
q5 := Int( Int( f(x,y), x=g1(y)..g2(y) ), y=c..d );
value( q5 );
plot3d( 0, x=g1(y)..g2(y), y=c..d, color=pink, style=patchnogrid, axes=boxed, orientation=[-90,0],
        scaling=constrained, title="#89 (Section 15.2)" );
r5 := Int( Int( f(x,y), y=0..x/2 ), x=0..2 ) + Int( Int( f(x,y), y=0..1 ), x=2..4 );
value( r5 );
value( q5-r5 );
```

- 85-94. Example CAS commands:

Mathematica: (functions and bounds will vary)

You can integrate using the built-in integral signs or with the command **Integrate**. In the **Integrate** command, the integration begins with the variable on the right. (In this case, y going from 1 to x).

```
Clear[x, y, f]
f[x_, y_] := 1 / (x y)
Integrate[f[x, y], {x, 1, 3}, {y, 1, x}]
```

To reverse the order of integration, it is best to first plot the region over which the integration extends. This can be done with `ImplicitPlot` and all bounds involving both x and y can be plotted. A graphics package must be loaded. Remember to use the double equal sign for the equations of the bounding curves.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==2y, x==4, y==0, y==1}, {x, 0, 4.1}, {y, 0, 1.1}];
f[x_, y_] := Exp[x^2]
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + Integrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
```

To get a numerical value for the result, use the numerical integrator, **NIntegrate**. Verify that this equals the original.

```
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + NIntegrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
NIntegrate[f[x, y], {y, 0, 1}, {x, 2y, 4}]
```

Another way to show a region is with the `FilledPlot` command. This assumes that functions are given as $y = f(x)$.

```
Clear[x, y, f]
<<Graphics`FilledPlot`
FilledPlot[{x^2, 9}, {x, 0, 3}, AxesLabels -> {x, y}];
f[x_, y_] := x Cos[y^2]
Integrate[f[x, y], {y, 0, 9}, {x, 0, Sqrt[y]}]
```

$$85. \int_1^3 \int_1^x \frac{1}{xy} dy dx \approx 0.603$$

$$86. \int_0^1 \int_0^1 e^{-(x^2+y^2)} dy dx \approx 0.558$$

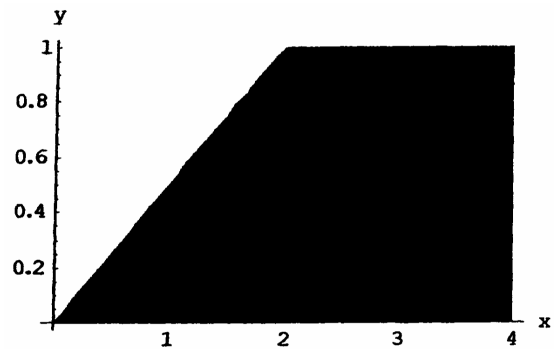
$$87. \int_0^1 \int_0^1 \tan^{-1} xy dy dx \approx 0.233$$

$$88. \int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2} dy dx \approx 3.142$$

89. Evaluate the integrals:

$$\begin{aligned} & \int_0^1 \int_{2y}^4 e^{x^2} dx dy \\ &= \int_0^2 \int_0^{x/2} e^{x^2} dy dx + \int_2^4 \int_0^1 e^{x^2} dy dx \\ &= -\frac{1}{4} + \frac{1}{4}(e^4 - 2\sqrt{\pi} \operatorname{erfi}(2) + 2\sqrt{\pi} \operatorname{erfi}(4)) \\ &\approx 1.1494 \times 10^6 \end{aligned}$$

The following graph was generated using Mathematica.

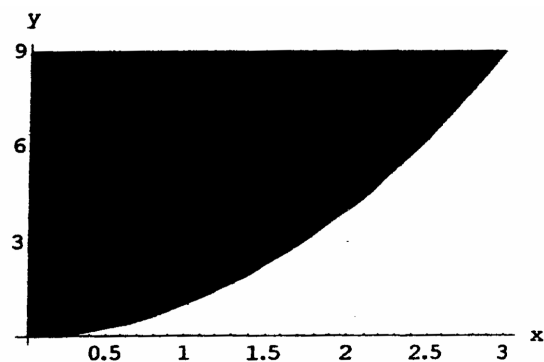


90. Evaluate the integrals:

$$\int_0^3 \int_{x^2}^9 x \cos(y^2) dy dx = \int_0^9 \int_0^{\sqrt{y}} x \cos(y^2) dx dy$$

$$= \frac{\sin(81)}{4} \approx -0.157472$$

The following graph was generated using Mathematica.



91. Evaluate the integrals:

$$\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2) dx dy = \int_0^8 \int_{x^2/32}^{\sqrt[3]{x}} (x^2y - xy^2) dy dx$$

$$= \frac{67,520}{693} \approx 97.4315$$

The following graph was generated using Mathematica.

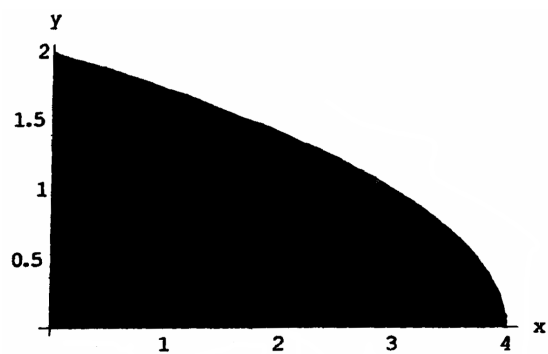


92. Evaluate the integrals:

$$\int_0^2 \int_0^{4-y^2} e^{xy} dx dy = \int_0^4 \int_0^{\sqrt{4-x}} e^{xy} dy dx$$

$$\approx 20.5648$$

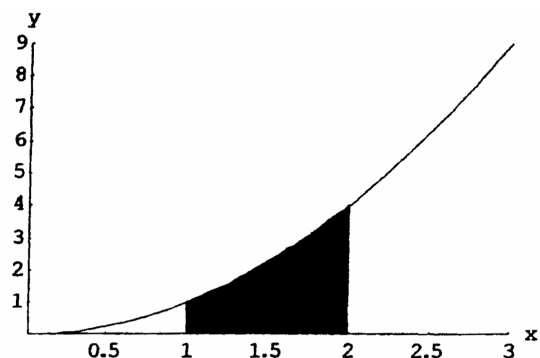
The following graph was generated using Mathematica.



93. Evaluate the integrals:

$$\begin{aligned}
 & \int_1^2 \int_0^{x^2} \frac{1}{x+y} dy dx \\
 &= \int_0^1 \int_1^2 \frac{1}{x+y} dx dy + \int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y} dx dy \\
 &= -1 + \ln\left(\frac{27}{4}\right) \approx 0.909543
 \end{aligned}$$

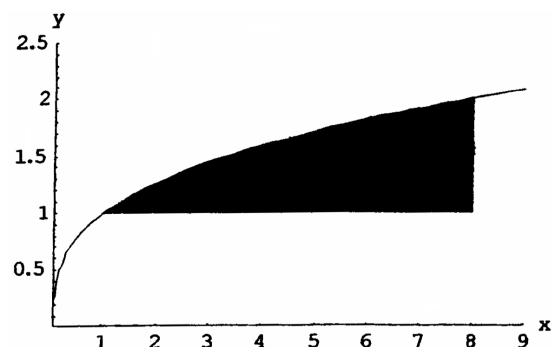
The following graph was generated using Mathematica.



94. Evaluate the integrals:

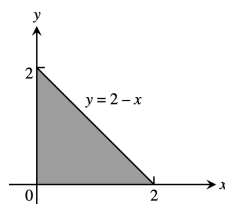
$$\begin{aligned}
 & \int_1^8 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}} dx dy = \int_1^8 \int_1^{\sqrt[3]{x}} \frac{1}{\sqrt{x^2+y^2}} dy dx \\
 & \approx 0.866649
 \end{aligned}$$

The following graph was generated using Mathematica.

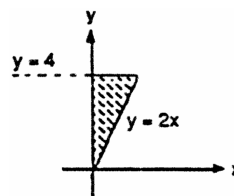


15.3 AREA BY DOUBLE INTEGRATION

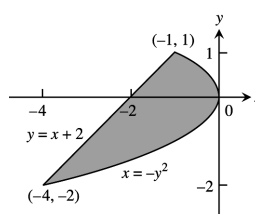
$$\begin{aligned}
 1. \quad & \int_0^2 \int_0^{2-x} dy dx = \int_0^2 (2-x) dx = \left[2x - \frac{x^2}{2} \right]_0^2 = 2, \\
 & \text{or } \int_0^2 \int_0^{2-y} dx dy = \int_0^2 (2-y) dy = 2
 \end{aligned}$$



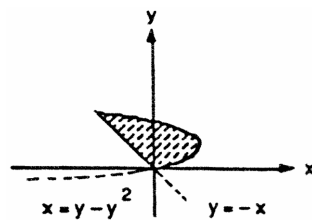
$$\begin{aligned}
 2. \quad & \int_0^2 \int_{2x}^4 dy dx = \int_0^2 (4-2x) dx = [4x - x^2]_0^2 = 4, \\
 & \text{or } \int_0^4 \int_0^{y/2} dx dy = \int_0^4 \frac{y}{2} dy = 4
 \end{aligned}$$



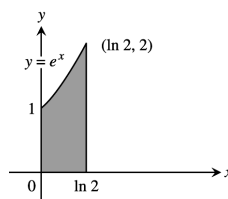
$$\begin{aligned}
 3. \quad & \int_{-2}^1 \int_{y-2}^{-y^2} dx dy = \int_{-2}^1 (-y^2 - y + 2) dy \\
 &= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y \right]_{-2}^1 \\
 &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) = \frac{9}{2}
 \end{aligned}$$



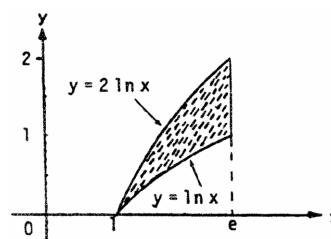
$$\begin{aligned}
 4. \quad \int_0^2 \int_{-y}^{y-y^2} dx \, dy &= \int_0^2 (2y - y^2) \, dy = \left[y^2 - \frac{y^3}{3} \right]_0^2 \\
 &= 4 - \frac{8}{3} = \frac{4}{3}
 \end{aligned}$$



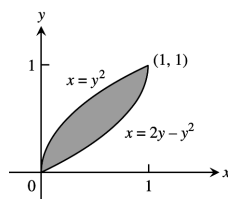
$$5. \quad \int_0^{\ln 2} \int_0^{e^x} dy \, dx = \int_0^{\ln 2} e^x \, dx = [e^x]_0^{\ln 2} = 2 - 1 = 1$$



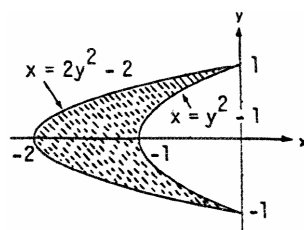
$$\begin{aligned}
 6. \quad \int_1^e \int_{\ln x}^{2 \ln x} dy \, dx &= \int_1^e \ln x \, dx = [x \ln x - x]_1^e \\
 &= (e - e) - (0 - 1) = 1
 \end{aligned}$$



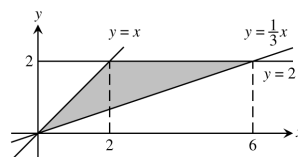
$$\begin{aligned}
 7. \quad \int_0^1 \int_{y^2}^{2y-y^2} dx \, dy &= \int_0^1 (2y - 2y^2) \, dy = \left[y^2 - \frac{2}{3} y^3 \right]_0^1 \\
 &= \frac{1}{3}
 \end{aligned}$$



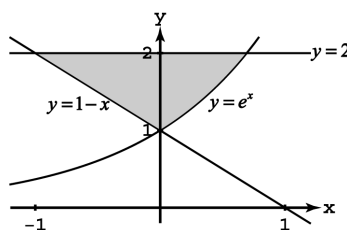
$$\begin{aligned}
 8. \quad \int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx \, dy &= \int_{-1}^1 (y^2 - 1 - 2y^2 + 2) \, dy \\
 &= \int_{-1}^1 (1 - y^2) \, dy = \left[y - \frac{y^3}{3} \right]_{-1}^1 = \frac{4}{3}
 \end{aligned}$$



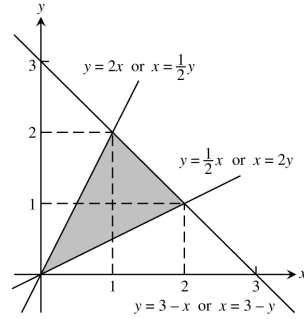
$$\begin{aligned}
 9. \quad \int_0^2 \int_y^{3y} 1 \, dx \, dy &= \int_0^2 [x]_y^{3y} \, dy \\
 &= \int_0^2 (2y) \, dy = [y^2]_0^2 = 4
 \end{aligned}$$



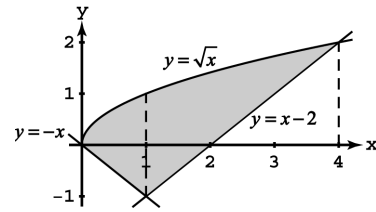
$$\begin{aligned}
 10. \quad \int_1^2 \int_{1-y}^{\ln y} 1 \, dx \, dy &= \int_1^2 [x]_{1-y}^{\ln y} \, dy \\
 &= \int_1^2 (\ln y - 1 + y) \, dy = \left[y \ln y - 2y + \frac{y^2}{2} \right]_1^2 \\
 &= 2 \ln 2 - \frac{1}{2}
 \end{aligned}$$



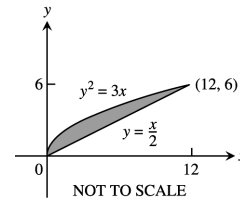
$$\begin{aligned}
 11. \quad & \int_0^1 \int_{x/2}^{2x} 1 \, dy \, dx + \int_1^2 \int_{x/2}^{3-x} 1 \, dy \, dx \\
 &= \int_0^1 [y]_{x/2}^{2x} dx + \int_1^2 [y]_{x/2}^{3-x} dx \\
 &= \int_0^1 \left(\frac{3}{2}x\right) dx + \int_1^2 \left(3 - \frac{3}{2}x\right) dx \\
 &= \left[\frac{3}{4}x^2\right]_0^1 + \left[3x - \frac{3}{4}x^2\right]_1^2 = \frac{3}{2}
 \end{aligned}$$



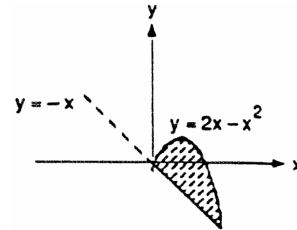
$$\begin{aligned}
 12. \quad & \int_0^1 \int_{-x}^{\sqrt{x}} 1 \, dy \, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} 1 \, dy \, dx \\
 &= \int_0^1 [y]_{-x}^{\sqrt{x}} dx + \int_1^4 [y]_{x-2}^{\sqrt{x}} dx \\
 &= \int_0^1 (\sqrt{x} + x) dx + \int_1^4 (\sqrt{x} - x + 2) dx \\
 &= \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \frac{13}{3}
 \end{aligned}$$



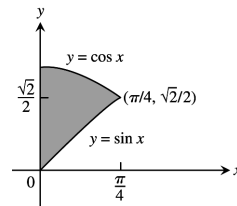
$$\begin{aligned}
 13. \quad & \int_0^6 \int_{y^2/3}^{2y} dx \, dy = \int_0^6 \left(2y - \frac{y^2}{3}\right) dy = \left[y^2 - \frac{y^3}{9}\right]_0^6 \\
 &= 36 - \frac{216}{9} = 12
 \end{aligned}$$



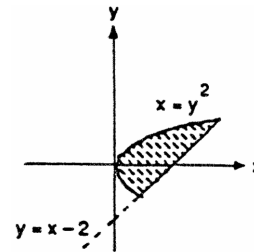
$$\begin{aligned}
 14. \quad & \int_0^3 \int_{-x}^{2x-x^2} dy \, dx = \int_0^3 (3x - x^2) dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 \\
 &= \frac{27}{2} - 9 = \frac{9}{2}
 \end{aligned}$$



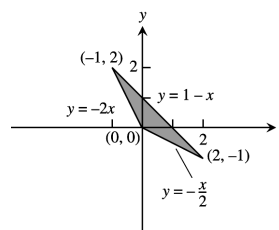
$$\begin{aligned}
 15. \quad & \int_0^{\pi/4} \int_{\sin x}^{\cos x} dy \, dx \\
 &= \int_0^{\pi/4} (\cos x - \sin x) dx = [\sin x + \cos x]_0^{\pi/4} \\
 &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1
 \end{aligned}$$



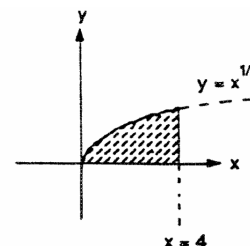
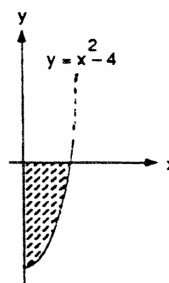
$$\begin{aligned}
 16. \quad & \int_{-1}^2 \int_{y^2}^{y+2} dx \, dy = \int_{-1}^2 (y + 2 - y^2) dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^2 \\
 &= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 5 - \frac{1}{2} = \frac{9}{2}
 \end{aligned}$$



$$\begin{aligned}
 17. \quad & \int_{-1}^0 \int_{-2x}^{1-x} dy \, dx + \int_0^2 \int_{-x/2}^{1-x} dy \, dx \\
 &= \int_{-1}^0 (1+x) \, dx + \int_0^2 (1-\frac{x}{2}) \, dx \\
 &= \left[x + \frac{x^2}{2} \right]_{-1}^0 + \left[x - \frac{x^2}{4} \right]_0^2 = -(-1 + \frac{1}{2}) + (2-1) = \frac{3}{2}
 \end{aligned}$$



$$\begin{aligned}
 18. \quad & \int_0^2 \int_{x^2-4}^0 dy \, dx + \int_0^4 \int_0^{\sqrt{x}} dy \, dx \\
 &= \int_0^2 (4-x^2) \, dx + \int_0^4 x^{1/2} \, dx \\
 &= \left[4x - \frac{x^3}{3} \right]_0^2 + \left[\frac{2}{3} x^{3/2} \right]_0^4 = (8 - \frac{8}{3}) + \frac{16}{3} = \frac{32}{3}
 \end{aligned}$$



$$\begin{aligned}
 19. \quad (a) \quad & \text{average} = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \sin(x+y) \, dy \, dx = \frac{1}{\pi^2} \int_0^\pi [-\cos(x+y)]_0^\pi \, dx = \frac{1}{\pi^2} \int_0^\pi [-\cos(x+\pi) + \cos x] \, dx \\
 &= \frac{1}{\pi^2} [-\sin(x+\pi) + \sin x]_0^\pi = \frac{1}{\pi^2} [(-\sin 2\pi + \sin \pi) - (-\sin \pi + \sin 0)] = 0 \\
 (b) \quad & \text{average} = \frac{1}{(\frac{\pi}{2})^2} \int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y) \, dy \, dx = \frac{2}{\pi^2} \int_0^{\pi/2} [-\cos(x+y)]_0^{\pi/2} \, dx = \frac{2}{\pi^2} \int_0^{\pi/2} [-\cos(x+\frac{\pi}{2}) + \cos x] \, dx \\
 &= \frac{2}{\pi^2} [-\sin(x+\frac{\pi}{2}) + \sin x]_0^{\pi/2} = \frac{2}{\pi^2} [(-\sin \frac{3\pi}{2} + \sin \pi) - (-\sin \frac{\pi}{2} + \sin 0)] = \frac{4}{\pi^2}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & \text{average value over the square} = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 \left[\frac{xy^2}{2} \right]_0^1 \, dx = \int_0^1 \frac{x}{2} \, dx = \frac{1}{4} = 0.25; \\
 & \text{average value over the quarter circle} = \frac{1}{(\frac{\pi}{4})} \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2} \right]_0^{\sqrt{1-x^2}} \, dx \\
 &= \frac{2}{\pi} \int_0^1 (x - x^3) \, dx = \frac{2}{\pi} \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2\pi} \approx 0.159. \text{ The average value over the square is larger.}
 \end{aligned}$$

$$21. \quad \text{average height} = \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2) \, dy \, dx = \frac{1}{4} \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_0^2 \, dx = \frac{1}{4} \int_0^2 (2x^2 + \frac{8}{3}) \, dx = \frac{1}{2} \left[\frac{x^3}{3} + \frac{4x}{3} \right]_0^2 = \frac{8}{3}$$

$$\begin{aligned}
 22. \quad & \text{average} = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2 \ln 2} \int_{\ln 2}^{2 \ln 2} \frac{1}{xy} \, dy \, dx = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2 \ln 2} \left[\frac{\ln y}{x} \right]_{\ln 2}^{2 \ln 2} \, dx \\
 &= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2 \ln 2} \frac{1}{x} (\ln 2 + \ln \ln 2 - \ln \ln 2) \, dx = \left(\frac{1}{\ln 2} \right) \int_{\ln 2}^{2 \ln 2} \frac{dx}{x} = \left(\frac{1}{\ln 2} \right) [\ln x]_{\ln 2}^{2 \ln 2} \\
 &= \left(\frac{1}{\ln 2} \right) (\ln 2 + \ln \ln 2 - \ln \ln 2) = 1
 \end{aligned}$$

$$\begin{aligned}
 23. \quad & \int_{-5}^5 \int_{-2}^0 \frac{10,000e^y}{1+\frac{|x|}{2}} \, dy \, dx = 10,000 (1 - e^{-2}) \int_{-5}^5 \frac{dx}{1+\frac{|x|}{2}} = 10,000 (1 - e^{-2}) \left[\int_{-5}^0 \frac{dx}{1-\frac{x}{2}} + \int_0^5 \frac{dx}{1+\frac{x}{2}} \right] \\
 &= 10,000 (1 - e^{-2}) \left[-2 \ln \left(1 - \frac{x}{2} \right) \right]_{-5}^0 + 10,000 (1 - e^{-2}) \left[2 \ln \left(1 + \frac{x}{2} \right) \right]_0^5 \\
 &= 10,000 (1 - e^{-2}) \left[2 \ln \left(1 + \frac{5}{2} \right) \right] + 10,000 (1 - e^{-2}) \left[2 \ln \left(1 + \frac{5}{2} \right) \right] = 40,000 (1 - e^{-2}) \ln \left(\frac{7}{2} \right) \approx 43,329
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & \int_0^1 \int_{y^2}^{2y-y^2} 100(y+1) \, dx \, dy = \int_0^1 [100(y+1)x]_{y^2}^{2y-y^2} \, dy = \int_0^1 100(y+1)(2y-2y^2) \, dy = 200 \int_0^1 (y-y^3) \, dy \\
 &= 200 \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = (200) \left(\frac{1}{4} \right) = 50
 \end{aligned}$$

25. Let (x_i, y_i) be the location of the weather station in county i for $i = 1, \dots, 254$. The average temperature

in Texas at time t_0 is approximately $\frac{\sum_{i=1}^{254} T(x_i, y_i) \Delta_i A}{A}$, where $T(x_i, y_i)$ is the temperature at time t_0 at the weather station in county i , $\Delta_i A$ is the area of county i , and A is the area of Texas.

26. Let $y = f(x)$ be a nonnegative, continuous function on $[a, b]$, then $A = \int_R dA = \int_a^b \int_0^{f(x)} dy dx = \int_a^b [y]_0^{f(x)} dx = \int_a^b f(x) dx$

15.4 DOUBLE INTEGRALS IN POLAR FORM

1. $x^2 + y^2 = 9^2 \Rightarrow r = 9 \Rightarrow \frac{\pi}{2} \leq \theta \leq 2\pi, 0 \leq r \leq 9$
2. $x^2 + y^2 = 1^2 \Rightarrow r = 1, x^2 + y^2 = 4^2 \Rightarrow r = 4 \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 4$
3. $y = x \Rightarrow \theta = \frac{\pi}{4}, y = -x \Rightarrow \theta = \frac{3\pi}{4}, y = 1 \Rightarrow r = \csc \theta \Rightarrow \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, 0 \leq r \leq \csc \theta$
4. $x = 1 \Rightarrow r = \sec \theta, y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3} \Rightarrow 0 \leq \theta \leq \frac{\pi}{3}, 0 \leq r \leq \sec \theta$
5. $x^2 + y^2 = 1^2 \Rightarrow r = 1, x = 2\sqrt{3} \Rightarrow r = 2\sqrt{3} \sec \theta, y = 2 \Rightarrow r = 2 \csc \theta; 2\sqrt{3} \sec \theta = 2 \csc \theta \Rightarrow \theta = \frac{\pi}{6} \Rightarrow 0 \leq \theta \leq \frac{\pi}{6}, 1 \leq r \leq 2\sqrt{3} \sec \theta; \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2\sqrt{3} \csc \theta$
6. $x^2 + y^2 = 2^2 \Rightarrow r = 2, x = 1 \Rightarrow r = \sec \theta; 2 = \sec \theta \Rightarrow \theta = \frac{\pi}{3} \text{ or } \theta = -\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, \sec \theta \leq r \leq 2$
7. $x^2 + y^2 = 2x \Rightarrow r = 2 \cos \theta \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2 \cos \theta$
8. $x^2 + y^2 = 2y \Rightarrow r = 2 \sin \theta \Rightarrow 0 \leq \theta \leq \pi, 0 \leq r \leq 2 \sin \theta$
9. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy dx = \int_0^\pi \int_0^1 r dr d\theta = \frac{1}{2} \int_0^\pi d\theta = \frac{\pi}{2}$
10. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy = \int_0^{\pi/2} \int_0^1 r^3 dr d\theta = \frac{1}{4} \int_0^{\pi/2} d\theta = \frac{\pi}{8}$
11. $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2) dx dy = \int_0^{\pi/2} \int_0^2 r^3 dr d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$
12. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy dx = \int_0^{2\pi} \int_0^a r dr d\theta = \frac{a^2}{2} \int_0^{2\pi} d\theta = \pi a^2$
13. $\int_0^6 \int_0^y x dx dy = \int_{\pi/4}^{\pi/2} \int_0^{6 \csc \theta} r^2 \cos \theta dr d\theta = 72 \int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta d\theta = -36 [\cot^2 \theta]_{\pi/4}^{\pi/2} = 36$
14. $\int_0^2 \int_0^x y dy dx = \int_0^{\pi/4} \int_0^{2 \sec \theta} r^2 \sin \theta dr d\theta = \frac{8}{3} \int_0^{\pi/4} \tan \theta \sec^2 \theta d\theta = \frac{4}{3}$
15. $\int_1^{\sqrt{3}} \int_1^x dy dx = \int_{\pi/6}^{\pi/4} \int_{\csc \theta}^{\sqrt{3} \sec \theta} r dr d\theta = \int_{\pi/6}^{\pi/4} (\frac{3}{2} \sec^2 \theta - \frac{1}{2} \csc^2 \theta) d\theta = [\frac{3}{2} \tan \theta + \frac{1}{2} \cot \theta]_{\pi/6}^{\pi/4} = 2 - \sqrt{3}$
16. $\int_{\sqrt{2}}^2 \int_{\sqrt{4-y^2}}^y dy dx = \int_{\pi/4}^{\pi/2} \int_2^{2 \csc \theta} r dr d\theta = \int_{\pi/6}^{\pi/4} (2 \csc^2 \theta - 2) d\theta = [-2 \cot \theta - \frac{1}{2} \theta]_{\pi/4}^{\pi/2} = 2 - \frac{\pi}{2}$

$$17. \int_{-1}^0 \int_{-\sqrt{1-x^2}}^{\frac{2}{1+\sqrt{x^2+y^2}}} dy dx = \int_{\pi}^{3\pi/2} \int_0^1 \frac{2r}{1+r} dr d\theta = 2 \int_{\pi}^{3\pi/2} \int_0^1 \left(1 - \frac{1}{1+r}\right) dr d\theta = 2 \int_{\pi}^{3\pi/2} (1 - \ln 2) d\theta = (1 - \ln 2)\pi$$

$$18. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy dx = 4 \int_0^{\pi/2} \int_0^1 \frac{2r}{(1+r^2)^2} dr d\theta = 4 \int_0^{\pi/2} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = 2 \int_0^{\pi/2} d\theta = \pi$$

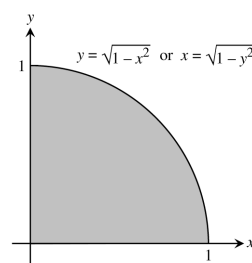
$$19. \int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} dx dy = \int_0^{\pi/2} \int_0^{\ln 2} re^r dr d\theta = \int_0^{\pi/2} (2 \ln 2 - 1) d\theta = \frac{\pi}{2} (2 \ln 2 - 1)$$

$$20. \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy = 4 \int_0^{\pi/2} \int_0^1 \ln(r^2 + 1) r dr d\theta = 2 \int_0^{\pi/2} (\ln 4 - 1) d\theta = \pi(\ln 4 - 1)$$

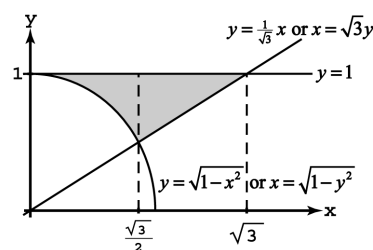
$$21. \int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y) dy dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} (r \cos \theta + 2r \sin \theta) r dr d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^3}{3} \cos \theta + \frac{2r^3}{3} \sin \theta \right]_0^{\sqrt{2}} d\theta = \int_{\pi/4}^{\pi/2} \left(\frac{2\sqrt{2}}{3} \cos \theta + \frac{4\sqrt{2}}{3} \sin \theta \right) d\theta = \left[\frac{2\sqrt{2}}{3} \sin \theta - \frac{4\sqrt{2}}{3} \cos \theta \right]_{\pi/4}^{\pi/2} = \frac{2(1+\sqrt{2})}{3}$$

$$22. \int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2} dy dx = \int_0^{\pi/4} \int_{\sec \theta}^{2 \cos \theta} \frac{1}{r^4} r dr d\theta = \int_0^{\pi/4} \left[-\frac{1}{2r^2} \right]_{\sec \theta}^{2 \cos \theta} d\theta = \int_0^{\pi/4} \left(\frac{1}{2} \cos^2 \theta - \frac{1}{8} \sec^2 \theta \right) d\theta = \left[\frac{1}{4} \theta + \frac{1}{8} \sin 2\theta - \frac{1}{8} \tan \theta \right]_0^{\pi/4} = \frac{\pi}{16}$$

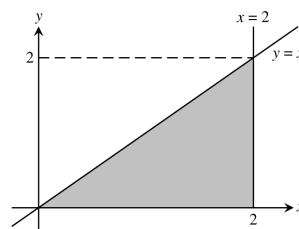
$$23. \int_0^1 \int_0^{\sqrt{1-x^2}} xy dy dx \text{ or } \int_0^1 \int_0^{\sqrt{1-y^2}} xy dx dy$$



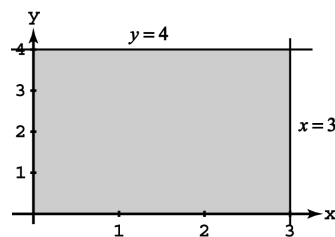
$$24. \int_{1/2}^1 \int_{\sqrt{1-y^2}}^{\sqrt{3}y} x dx dy \text{ or } \int_0^{\sqrt{3}/2} \int_{\sqrt{1-x^2}}^1 x dy dx + \int_{\sqrt{3}/2}^1 \int_{x/\sqrt{3}}^1 x dy dx$$



$$25. \int_0^2 \int_0^x y^2(x^2+y^2) dy dx \text{ or } \int_0^2 \int_y^2 y^2(x^2+y^2) dx dy$$



$$26. \int_0^3 \int_0^4 (x^2 + y^2)^3 dy dx \text{ or } \int_0^4 \int_0^3 (x^2 + y^2)^3 dx dy$$



$$27. \int_0^{\pi/2} \int_0^{2\sqrt{2-\sin 2\theta}} r dr d\theta = 2 \int_0^{\pi/2} (2 - \sin 2\theta) d\theta = 2(\pi - 1)$$

$$28. A = 2 \int_0^{\pi/2} \int_1^{1+\cos \theta} r dr d\theta = \int_0^{\pi/2} (2 \cos \theta + \cos^2 \theta) d\theta = \frac{8+\pi}{4}$$

$$29. A = 2 \int_0^{\pi/6} \int_0^{12 \cos 3\theta} r dr d\theta = 144 \int_0^{\pi/6} \cos^2 3\theta d\theta = 12\pi$$

$$30. A = \int_0^{2\pi} \int_0^{4\theta/3} r dr d\theta = \frac{8}{9} \int_0^{2\pi} \theta^2 d\theta = \frac{64\pi^3}{27}$$

$$31. A = \int_0^{\pi/2} \int_0^{1+\sin \theta} r dr d\theta = \frac{1}{2} \int_0^{\pi/2} \left(\frac{3}{2} + 2 \sin \theta - \frac{\cos 2\theta}{2} \right) d\theta = \frac{3\pi}{8} + 1$$

$$32. A = 4 \int_0^{\pi/2} \int_0^{1-\cos \theta} r dr d\theta = 2 \int_0^{\pi/2} \left(\frac{3}{2} - 2 \cos \theta + \frac{\cos 2\theta}{2} \right) d\theta = \frac{3\pi}{2} - 4$$

$$33. \text{average} = \frac{4}{\pi a^2} \int_0^{\pi/2} \int_0^a r \sqrt{a^2 - r^2} dr d\theta = \frac{4}{3\pi a^2} \int_0^{\pi/2} a^3 d\theta = \frac{2a}{3}$$

$$34. \text{average} = \frac{4}{\pi a^2} \int_0^{\pi/2} \int_0^a r^2 dr d\theta = \frac{4}{3\pi a^2} \int_0^{\pi/2} a^3 d\theta = \frac{2a}{3}$$

$$35. \text{average} = \frac{1}{\pi a^2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} dy dx = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a r^2 dr d\theta = \frac{a}{3\pi} \int_0^{2\pi} d\theta = \frac{2a}{3}$$

$$36. \text{average} = \frac{1}{\pi} \iint_R [(1-x)^2 + y^2] dy dx = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 [(1-r \cos \theta)^2 + r^2 \sin^2 \theta] r dr d\theta \\ = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (r^3 - 2r^2 \cos \theta + r) dr d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{3}{4} - \frac{2 \cos \theta}{3} \right) d\theta = \frac{1}{\pi} \left[\frac{3}{4} \theta - \frac{2 \sin \theta}{3} \right]_0^{2\pi} = \frac{3}{2}$$

$$37. \int_0^{2\pi} \int_1^{\sqrt{e}} \left(\frac{\ln r^2}{r} \right) r dr d\theta = \int_0^{2\pi} \int_1^{\sqrt{e}} 2 \ln r dr d\theta = 2 \int_0^{2\pi} [r \ln r - r]_1^{e^{1/2}} d\theta = 2 \int_0^{2\pi} \sqrt{e} \left[\left(\frac{1}{2} - 1 \right) + 1 \right] d\theta = 2\pi(2 - \sqrt{e})$$

$$38. \int_0^{2\pi} \int_1^e \left(\frac{\ln r^2}{r} \right) dr d\theta = \int_0^{2\pi} \int_1^e \left(\frac{2 \ln r}{r} \right) dr d\theta = \int_0^{2\pi} [(\ln r)^2]_1^e d\theta = \int_0^{2\pi} d\theta = 2\pi$$

$$39. V = 2 \int_0^{\pi/2} \int_1^{1+\cos \theta} r^2 \cos \theta dr d\theta = \frac{2}{3} \int_0^{\pi/2} (3 \cos^2 \theta + 3 \cos^3 \theta + \cos^4 \theta) d\theta \\ = \frac{2}{3} \left[\frac{15\theta}{8} + \sin 2\theta + 3 \sin \theta - \sin^3 \theta + \frac{\sin 4\theta}{32} \right]_0^{\pi/2} = \frac{4}{3} + \frac{5\pi}{8}$$

$$40. V = 4 \int_0^{\pi/4} \int_0^{\sqrt{2 \cos 2\theta}} r \sqrt{2-r^2} dr d\theta = -\frac{4}{3} \int_0^{\pi/4} [(2-2 \cos 2\theta)^{3/2} - 2^{3/2}] d\theta \\ = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3} \int_0^{\pi/4} (1 - \cos^2 \theta) \sin \theta d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3} \left[\frac{\cos^3 \theta}{3} - \cos \theta \right]_0^{\pi/4} = \frac{6\pi\sqrt{2} + 40\sqrt{2} - 64}{9}$$

$$41. (a) \quad I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\pi/2} \int_0^\infty (e^{-r^2}) r dr d\theta = \int_0^{\pi/2} \left[\lim_{b \rightarrow \infty} \int_0^b re^{-r^2} dr \right] d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \lim_{b \rightarrow \infty} (e^{-b^2} - 1) d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$$

$$(b) \quad \lim_{x \rightarrow \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \left(\frac{2}{\sqrt{\pi}} \right) \left(\frac{\sqrt{\pi}}{2} \right) = 1, \text{ from part (a)}$$

$$42. \quad \int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx dy = \int_0^{\pi/2} \int_0^\infty \frac{r}{(1+r^2)^2} dr d\theta = \frac{\pi}{2} \lim_{b \rightarrow \infty} \int_0^b \frac{r}{(1+r^2)^2} dr = \frac{\pi}{4} \lim_{b \rightarrow \infty} \left[-\frac{1}{1+r^2} \right]_0^b$$

$$= \frac{\pi}{4} \lim_{b \rightarrow \infty} \left(1 - \frac{1}{1+b^2} \right) = \frac{\pi}{4}$$

$$43. \quad \text{Over the disk } x^2 + y^2 \leq \frac{3}{4}: \int_R \frac{1}{1-x^2-y^2} dA = \int_0^{2\pi} \int_0^{\sqrt{3}/2} \frac{r}{1-r^2} dr d\theta = \int_0^{2\pi} \left[-\frac{1}{2} \ln(1-r^2) \right]_0^{\sqrt{3}/2} d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{2} \ln \frac{1}{4} \right) d\theta = (\ln 2) \int_0^{2\pi} d\theta = \pi \ln 4$$

Over the disk $x^2 + y^2 \leq 1$: $\int_R \frac{1}{1-x^2-y^2} dA = \int_0^{2\pi} \int_0^1 \frac{r}{1-r^2} dr d\theta = \int_0^{2\pi} \left[\lim_{a \rightarrow 1^-} \int_0^a \frac{r}{1-r^2} dr \right] d\theta$

$$= \int_0^{2\pi} \lim_{a \rightarrow 1^-} \left[-\frac{1}{2} \ln(1-a^2) \right] d\theta = 2\pi \cdot \lim_{a \rightarrow 1^-} \left[-\frac{1}{2} \ln(1-a^2) \right] = 2\pi \cdot \infty, \text{ so the integral does not exist over } x^2 + y^2 \leq 1$$

$$44. \quad \text{The area in polar coordinates is given by } A = \int_\alpha^\beta \int_0^{f(\theta)} r dr d\theta = \int_\alpha^\beta \left[\frac{r^2}{2} \right]_0^{f(\theta)} d\theta = \frac{1}{2} \int_\alpha^\beta f^2(\theta) d\theta = \int_\alpha^\beta \frac{1}{2} r^2 d\theta,$$

where $r = f(\theta)$

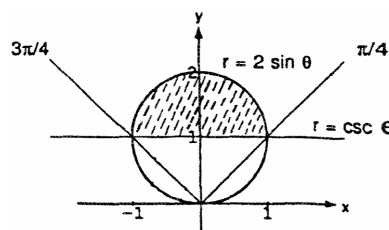
$$45. \quad \text{average} = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a [(r \cos \theta - h)^2 + r^2 \sin^2 \theta] r dr d\theta = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a (r^3 - 2r^2 h \cos \theta + rh^2) dr d\theta$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} \left(\frac{a^4}{4} - \frac{2a^3 h \cos \theta}{3} + \frac{a^2 h^2}{2} \right) d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{a^2}{4} - \frac{2ah \cos \theta}{3} + \frac{h^2}{2} \right) d\theta = \frac{1}{\pi} \left[\frac{a^2 \theta}{4} - \frac{2ah \sin \theta}{3} + \frac{h^2 \theta}{2} \right]_0^{2\pi}$$

$$= \frac{1}{2} (a^2 + 2h^2)$$

$$46. \quad A = \int_{\pi/4}^{3\pi/4} \int_{\csc \theta}^{2 \sin \theta} r dr d\theta = \frac{1}{2} \int_{\pi/4}^{3\pi/4} (4 \sin^2 \theta - \csc^2 \theta) d\theta$$

$$= \frac{1}{2} [2\theta - \sin 2\theta + \cot \theta]_{\pi/4}^{3\pi/4} = \frac{\pi}{2}$$



47-50. Example CAS commands:

Maple:

`f := (x,y) -> y/(x^2+y^2);`

`a,b := 0,1;`

`f1 := x -> x;`

`f2 := x -> 1;`

`plot3d( f(x,y), y=f1(x)..f2(x), x=a..b, axes=boxed, style=patchnogrid, shading=zhue, orientation=[0,180], title="#47(a)`
(a)

`q1 := eval( x=a, [x=r*cos(theta),y=r*sin(theta)] );`

(b)

`q2 := eval( x=b, [x=r*cos(theta),y=r*sin(theta)] );`

`q3 := eval( y=f1(x), [x=r*cos(theta),y=r*sin(theta)] );`

`q4 := eval( y=f2(x), [x=r*cos(theta),y=r*sin(theta)] );`

`theta1 := solve( q3, theta );`

```

theta2 := solve( q1, theta );
r1 := 0;
r2 := solve( q4, r );
plot3d(0,r=r1..r2, theta=theta1..theta2, axes=boxed, style=patchnogrid, shading=zhue, orientation=[-90,0],
        title="#47(c) (Section 15.4)" );
fP := simplify(eval( f(x,y), [x=r*cos(theta),y=r*sin(theta)] ));      # (d)
q5 := Int( Int( fP*r, r=r1..r2 ), theta=theta1..theta2 );
value( q5 );

```

Mathematica: (functions and bounds will vary)

For 47 and 48, begin by drawing the region of integration with the **FilledPlot** command.

```

Clear[x, y, r, t]
<<Graphics`FilledPlot`
FilledPlot[{x, 1}, {x, 0, 1}, AspectRatio -> 1, AxesLabel -> {x,y}];

```

The picture demonstrates that r goes from 0 to the line $y=1$ or $r = 1/\sin[t]$, while t goes from $\pi/4$ to $\pi/2$.

```

f:= y / (x^2 + y^2)
topolar={x -> r Cos[t], y -> r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, pi/4, pi/2}, {r, 0, 1/Sin[t]}]

```

For 49 and 50, drawing the region of integration with the **ImplicitPlot** command.

```

Clear[x, y]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==y, x==2 - y, y==0, y==1}, {x, 0, 2.1}, {y, 0, 1.1}];

```

The picture shows that as t goes from 0 to $\pi/4$, r goes from 0 to the line $x=2-y$. **Solve** will find the bound for r .

```

bdr=Solve[r Cos[t]==2 - r Sin[t], r]//Simplify
f:=Sqrt[x + y]
topolar={x -> r Cos[t], y -> r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, pi/4}, {r, 0, bdr[[1, 1, 2]]}]

```

15.5 TRIPLE INTEGRALS IN RECTANGULAR COORDINATES

- $$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z) \, dy \, dz \, dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy \, dz \, dx = \int_0^1 \int_0^{1-x} (1 - x - z) \, dz \, dx$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx = \int_0^1 \frac{(1-x)^2}{2} dx = \left[-\frac{(1-x)^3}{6} \right]_0^1 = \frac{1}{6}$$
- $$\int_0^1 \int_0^2 \int_0^3 dz \, dy \, dx = \int_0^1 \int_0^2 3 \, dy \, dx = \int_0^1 6 \, dx = 6, \int_0^2 \int_0^1 \int_0^3 dz \, dx \, dy, \int_0^3 \int_0^2 \int_0^1 dx \, dy \, dz, \int_0^2 \int_0^3 \int_0^1 dx \, dz \, dy,$$

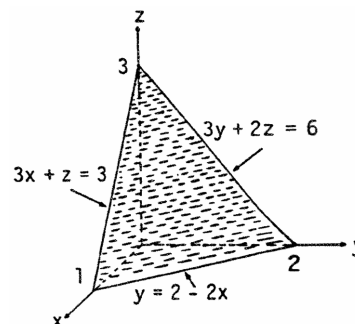
$$\int_0^3 \int_0^1 \int_0^2 dy \, dx \, dz, \int_0^1 \int_0^3 \int_0^2 dy \, dz \, dx$$
- $$\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz \, dy \, dx$$

$$= \int_0^1 \int_0^{2-2x} \left(3 - 3x - \frac{3}{2}y \right) dy \, dx$$

$$= \int_0^1 \left[3(1-x) \cdot 2(1-x) - \frac{3}{4} \cdot 4(1-x)^2 \right] dx$$

$$= 3 \int_0^1 (1-x)^2 dx = \left[-(1-x)^3 \right]_0^1 = 1,$$

$$\int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dz \, dx \, dy, \int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3} dy \, dz \, dx,$$



$$\int_0^3 \int_0^{1-z/3} \int_0^{2-2x-2z/3} dy \, dx \, dz, \int_0^2 \int_0^{3-3y/2} \int_0^{1-y/2-z/3} dx \, dz \, dy, \int_0^3 \int_0^{2-2z/3} \int_0^{1-y/2-z/3} dx \, dy \, dz$$

$$4. \int_0^2 \int_0^3 \int_0^{\sqrt{4-x^2}} dz \, dy \, dx = \int_0^2 \int_0^3 \sqrt{4-x^2} \, dy \, dx = \int_0^2 3\sqrt{4-x^2} \, dx = \frac{3}{2} \left[x\sqrt{4-x^2} + 4 \sin^{-1} \frac{x}{2} \right]_0^2 = 6 \sin^{-1} 1 = 3\pi,$$

$$\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} dz \, dx \, dy, \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^3 dy \, dz \, dx, \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^3 dy \, dx \, dz, \int_0^2 \int_0^3 \int_0^{\sqrt{4-z^2}} dx \, dy \, dz, \int_0^3 \int_0^2 \int_0^{\sqrt{4-z^2}} dx \, dz \, dy$$

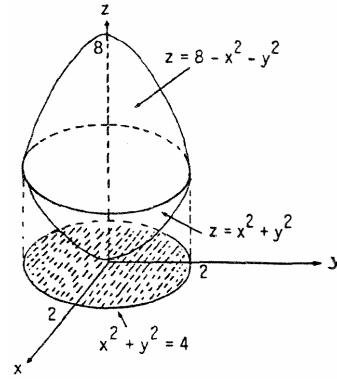
$$5. \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz \, dy \, dx = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz \, dy \, dx$$

$$= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} [8 - 2(x^2 + y^2)] \, dy \, dx$$

$$= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2) \, dy \, dx$$

$$= 8 \int_0^{\pi/2} \int_0^2 (4 - r^2) r \, dr \, d\theta = 8 \int_0^{\pi/2} \left[2r^2 - \frac{r^4}{4} \right]_0^2 d\theta$$

$$= 32 \int_0^{\pi/2} d\theta = 32 \left(\frac{\pi}{2} \right) = 16\pi,$$



$$\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz \, dx \, dy,$$

$$\int_{-2}^2 \int_{y^2}^4 \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx \, dz \, dy + \int_{-2}^2 \int_4^{8-y^2} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx \, dz \, dy,$$

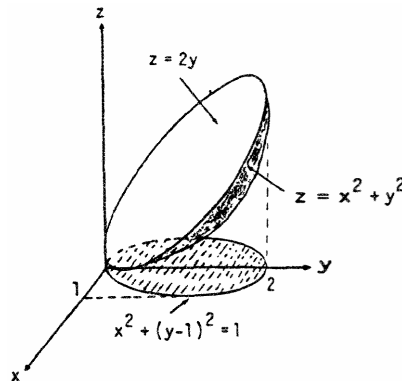
$$\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx \, dy \, dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx \, dy \, dz, \int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy \, dz \, dx + \int_{-2}^2 \int_4^{8-x^2} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy \, dz \, dx,$$

$$\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy \, dx \, dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy \, dx \, dz$$

6. The projection of D onto the xy-plane has the boundary $x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$, which is a circle.

Therefore the two integrals are:

$$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz \, dx \, dy \text{ and } \int_{-1}^1 \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} dz \, dy \, dx$$



$$7. \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) \, dz \, dy \, dx = \int_0^1 \int_0^1 \left(x^2 + y^2 + \frac{1}{3} \right) \, dy \, dx = \int_0^1 \left(x^2 + \frac{2}{3} \right) \, dx = 1$$

$$8. \int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz \, dx \, dy = \int_0^{\sqrt{2}} \int_0^{3y} (8 - 2x^2 - 4y^2) \, dx \, dy = \int_0^{\sqrt{2}} \left[8x - \frac{2}{3} x^3 - 4xy^2 \right]_0^{3y} dy$$

$$= \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3) \, dy = \left[12y^2 - \frac{15}{2} y^4 \right]_0^{\sqrt{2}} = 24 - 30 = -6$$

$$9. \int_1^e \int_1^e \int_1^e \frac{1}{xyz} \, dx \, dy \, dz = \int_1^e \int_1^e \left[\frac{\ln x}{yz} \right]_1^e dy \, dz = \int_1^e \int_1^e \frac{3}{yz} \, dy \, dz = \int_1^e \left[\frac{\ln y}{z} \right]_1^e dz = \int_1^e \frac{6}{z} \, dz = 6$$

$$10. \int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz \, dy \, dx = \int_0^1 \int_0^{3-3x} (3 - 3x - y) \, dy \, dx = \int_0^1 \left[(3 - 3x)y - \frac{1}{2} y^2 \right]_0^{3-3x} dx = \frac{9}{2} \int_0^1 (1 - x)^2 \, dx$$

$$= -\frac{3}{2} [(1 - x)^3]_0^1 = \frac{3}{2}$$

$$11. \int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \, dx \, dy \, dz = \int_0^{\pi/6} \int_0^1 5y \sin z \, dy \, dz = \frac{5}{2} \int_0^{\pi/6} \sin z \, dz = \frac{5(2-\sqrt{3})}{4}$$

$$12. \int_{-1}^1 \int_0^1 \int_0^2 (x+y+z) \, dy \, dx \, dz = \int_{-1}^1 \int_0^1 [xy + \frac{1}{2}y^2 + zy]_0^2 \, dx \, dz = \int_{-1}^1 \int_0^1 (2x+2+2z) \, dx \, dz \\ = \int_{-1}^1 [x^2 + 2x + 2zx]_0^1 \, dz = \int_{-1}^1 (3+2z) \, dz = [3z + z^2]_{-1}^1 = 6$$

$$13. \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz \, dy \, dx = \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-x^2} \, dy \, dx = \int_0^3 (9-x^2) \, dx = \left[9x - \frac{x^3}{3}\right]_0^3 = 18$$

$$14. \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz \, dx \, dy = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x+y) \, dx \, dy = \int_0^2 [x^2 + xy]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \, dy = \int_0^2 (4-y^2)^{1/2} (2y) \, dy \\ = \left[-\frac{2}{3}(4-y^2)^{3/2}\right]_0^2 = \frac{2}{3}(4)^{3/2} = \frac{16}{3}$$

$$15. \int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz \, dy \, dx = \int_0^1 \int_0^{2-x} (2-x-y) \, dy \, dx = \int_0^1 [(2-x)^2 - \frac{1}{2}(2-x)^2] \, dx = \frac{1}{2} \int_0^1 (2-x)^2 \, dx \\ = \left[-\frac{1}{6}(2-x)^3\right]_0^1 = -\frac{1}{6} + \frac{8}{6} = \frac{7}{6}$$

$$16. \int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \, dz \, dy \, dx = \int_0^1 \int_0^{1-x^2} x(1-x^2-y) \, dy \, dx = \int_0^1 x \left[(1-x^2)^2 - \frac{1}{2}(1-x^2)^2\right] \, dx = \int_0^1 \frac{1}{2} x (1-x^2)^2 \, dx \\ = \left[-\frac{1}{12}(1-x^2)^3\right]_0^1 = \frac{1}{12}$$

$$17. \int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w) \, du \, dv \, dw = \int_0^\pi \int_0^\pi [\sin(w+v+\pi) - \sin(w+v)] \, dv \, dw \\ = \int_0^\pi [(-\cos(w+2\pi) + \cos(w+\pi)) + (\cos(w+\pi) - \cos w)] \, dw \\ = [-\sin(w+2\pi) + \sin(w+\pi) - \sin w + \sin(w+\pi)]_0^\pi = 0$$

$$18. \int_0^1 \int_1^{\sqrt{e}} \int_1^e s e^s \ln r \frac{(\ln t)^2}{t} \, dt \, dr \, ds = \int_0^1 \int_1^{\sqrt{e}} (s e^s \ln r) \left[\frac{1}{3}(\ln t)^3\right]_1^e \, dr \, ds = \int_0^1 \int_1^{\sqrt{e}} \frac{s e^s}{3} \ln r \, dr \, ds = \int_0^1 \frac{s e^s}{3} [r \ln r - r]_1^{\sqrt{e}} \, ds \\ = \frac{2-\sqrt{e}}{6} \int_0^1 s e^s \, ds = \frac{2-\sqrt{e}}{6} [s e^s - e^s]_0^1 = \frac{2-\sqrt{e}}{6}$$

$$19. \int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x \, dx \, dt \, dv = \int_0^{\pi/4} \int_0^{\ln \sec v} \lim_{b \rightarrow -\infty} (e^{2t} - e^b) \, dt \, dv = \int_0^{\pi/4} \int_0^{\ln \sec v} e^{2t} \, dt \, dv = \int_0^{\pi/4} \left(\frac{1}{2} e^{2 \ln \sec v} - \frac{1}{2}\right) \, dv \\ = \int_0^{\pi/4} \left(\frac{\sec^2 v}{2} - \frac{1}{2}\right) \, dv = \left[\frac{\tan v}{2} - \frac{v}{2}\right]_0^{\pi/4} = \frac{1}{2} - \frac{\pi}{8}$$

$$20. \int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} \, dp \, dq \, dr = \int_0^7 \int_0^2 \frac{q\sqrt{4-q^2}}{r+1} \, dq \, dr = \int_0^7 \frac{1}{3(r+1)} \left[-(4-q^2)^{3/2}\right]_0^2 \, dr = \frac{8}{3} \int_0^7 \frac{1}{r+1} \, dr = \frac{8 \ln 8}{3} = 8 \ln 2$$

$$21. \begin{array}{lll} \text{(a)} \int_{-1}^1 \int_0^{1-x^2} \int_x^{1-z} dy \, dz \, dx & \text{(b)} \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} dy \, dx \, dz & \text{(c)} \int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy \, dz \\ \text{(d)} \int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dz \, dy & \text{(e)} \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz \, dx \, dy & \end{array}$$

$$22. \begin{array}{lll} \text{(a)} \int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy \, dz \, dx & \text{(b)} \int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy \, dx \, dz & \text{(c)} \int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx \, dy \, dz \\ \text{(d)} \int_{-1}^0 \int_0^{y^2} \int_0^1 dx \, dz \, dy & \text{(e)} \int_{-1}^0 \int_0^1 \int_0^{y^2} dz \, dx \, dy & \end{array}$$

$$23. V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz \, dy \, dx = \int_0^1 \int_{-1}^1 y^2 \, dy \, dx = \frac{2}{3} \int_0^1 dx = \frac{2}{3}$$

$$24. V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy \, dz \, dx = \int_0^1 \int_0^{1-x} (2 - 2z) \, dz \, dx = \int_0^1 [2z - z^2]_0^{1-x} \, dx = \int_0^1 (1 - x^2) \, dx = \left[x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}$$

$$25. V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz \, dy \, dx = \int_0^4 \int_0^{\sqrt{4-x}} (2 - y) \, dy \, dx = \int_0^4 \left[2\sqrt{4-x} - \left(\frac{4-x}{2} \right) \right] \, dx \\ = \left[-\frac{4}{3} (4-x)^{3/2} + \frac{1}{4} (4-x)^2 \right]_0^4 = \frac{4}{3} (4)^{3/2} - \frac{1}{4} (16) = \frac{32}{3} - 4 = \frac{20}{3}$$

$$26. V = 2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz \, dy \, dx = -2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 y \, dy \, dx = \int_0^1 (1 - x^2) \, dx = \frac{2}{3}$$

$$27. V = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz \, dy \, dx = \int_0^1 \int_0^{2-2x} (3 - 3x - \frac{3}{2}y) \, dy \, dx = \int_0^1 \left[6(1-x)^2 - \frac{3}{4} \cdot 4(1-x)^2 \right] \, dx \\ = \int_0^1 3(1-x)^2 \, dx = \left[-(1-x)^3 \right]_0^1 = 1$$

$$28. V = \int_0^1 \int_0^{1-x} \int_0^{\cos(\pi x/2)} dz \, dy \, dx = \int_0^1 \int_0^{1-x} \cos\left(\frac{\pi x}{2}\right) \, dy \, dx = \int_0^1 \left(\cos \frac{\pi x}{2}\right) (1-x) \, dx \\ = \int_0^1 \cos\left(\frac{\pi x}{2}\right) \, dx - \int_0^1 x \cos\left(\frac{\pi x}{2}\right) \, dx = \left[\frac{2}{\pi} \sin \frac{\pi x}{2} \right]_0^1 - \frac{4}{\pi^2} \int_0^{\pi/2} u \cos u \, du = \frac{2}{\pi} - \frac{4}{\pi^2} [\cos u + u \sin u]_0^{\pi/2} \\ = \frac{2}{\pi} - \frac{4}{\pi^2} \left(\frac{\pi}{2} - 1 \right) = \frac{4}{\pi^2}$$

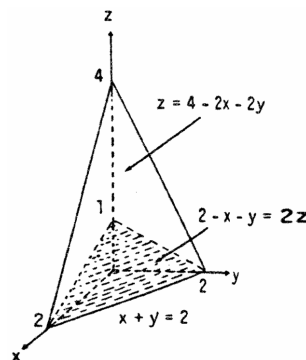
$$29. V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz \, dy \, dx = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} \, dy \, dx = 8 \int_0^1 (1-x^2) \, dx = \frac{16}{3}$$

$$30. V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz \, dy \, dx = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y) \, dy \, dx = \int_0^2 \left[(4-x^2)^2 - \frac{1}{2} (4-x^2)^2 \right] \, dx \\ = \frac{1}{2} \int_0^2 (4-x^2)^2 \, dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2} \right) \, dx = \frac{128}{15}$$

$$31. V = \int_0^4 \int_0^{(\sqrt{16-y^2})/2} \int_0^{4-y} dx \, dz \, dy = \int_0^4 \int_0^{(\sqrt{16-y^2})/2} (4-y) \, dz \, dy = \int_0^4 \frac{\sqrt{16-y^2}}{2} (4-y) \, dy \\ = \int_0^4 2\sqrt{16-y^2} \, dy - \frac{1}{2} \int_0^4 y \sqrt{16-y^2} \, dy = \left[y\sqrt{16-y^2} + 16 \sin^{-1} \frac{y}{4} \right]_0^4 + \left[\frac{1}{6} (16-y^2)^{3/2} \right]_0^4 \\ = 16 \left(\frac{\pi}{2} \right) - \frac{1}{6} (16)^{3/2} = 8\pi - \frac{32}{3}$$

$$32. V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dz \, dy \, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x) \, dy \, dx = 2 \int_{-2}^2 (3-x) \sqrt{4-x^2} \, dx \\ = 3 \int_{-2}^2 2\sqrt{4-x^2} \, dx - 2 \int_{-2}^2 x \sqrt{4-x^2} \, dx = 3 \left[x\sqrt{4-x^2} + 4 \sin^{-1} \frac{x}{2} \right]_{-2}^2 + \left[\frac{2}{3} (4-x^2)^{3/2} \right]_{-2}^2 \\ = 12 \sin^{-1} 1 - 12 \sin^{-1} (-1) = 12 \left(\frac{\pi}{2} \right) - 12 \left(-\frac{\pi}{2} \right) = 12\pi$$

$$33. \int_0^2 \int_0^{2-x} \int_{(2-x-y)/2}^{4-2x-2y} dz \, dy \, dx = \int_0^2 \int_0^{2-x} \left(3 - \frac{3x}{2} - \frac{3y}{2} \right) \, dy \, dx \\ = \int_0^2 \left[3 \left(1 - \frac{x}{2} \right) (2-x) - \frac{3}{4} (2-x)^2 \right] \, dx \\ = \int_0^2 \left[6 - 6x + \frac{3x^2}{2} - \frac{3(2-x)^2}{4} \right] \, dx \\ = \left[6x - 3x^2 + \frac{x^3}{2} + \frac{(2-x)^3}{4} \right]_0^2 = (12 - 12 + 4 + 0) - \frac{2^3}{4} = 2$$



$$34. V = \int_0^4 \int_z^8 \int_z^{8-z} dx dy dz = \int_0^4 \int_z^8 (8-2z) dy dz = \int_0^4 (8-2z)(8-z) dz = \int_0^4 (64-24z+2z^2) dz \\ = [64z - 12z^2 + \frac{2}{3}z^3]_0^4 = \frac{320}{3}$$

$$35. V = 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} \int_0^{x+2} dz dy dx = 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} (x+2) dy dx = \int_{-2}^2 (x+2)\sqrt{4-x^2} dx \\ = \int_{-2}^2 2\sqrt{4-x^2} dx + \int_{-2}^2 x\sqrt{4-x^2} dx = \left[x\sqrt{4-x^2} + 4 \sin^{-1} \frac{x}{2} \right]_{-2}^2 + \left[-\frac{1}{3}(4-x^2)^{3/2} \right]_{-2}^2 \\ = 4\left(\frac{\pi}{2}\right) - 4\left(-\frac{\pi}{2}\right) = 4\pi$$

$$36. V = 2 \int_0^1 \int_0^{1-y^2} \int_0^{x^2+y^2} dz dx dy = 2 \int_0^1 \int_0^{1-y^2} (x^2+y^2) dx dy = 2 \int_0^1 \left[\frac{x^3}{3} + xy^2 \right]_0^{1-y^2} dy \\ = 2 \int_0^1 (1-y^2) \left[\frac{1}{3}(1-y^2)^2 + y^2 \right] dy = 2 \int_0^1 (1-y^2) \left(\frac{1}{3} + \frac{1}{3}y^2 + \frac{1}{3}y^4 \right) dy = \frac{2}{3} \int_0^1 (1-y^6) dy \\ = \frac{2}{3} \left[y - \frac{y^7}{7} \right]_0^1 = \left(\frac{2}{3} \right) \left(\frac{6}{7} \right) = \frac{4}{7}$$

$$37. \text{average} = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 (x^2+9) dz dy dx = \frac{1}{8} \int_0^2 \int_0^2 (2x^2+18) dy dx = \frac{1}{8} \int_0^2 (4x^2+36) dx = \frac{31}{8}$$

$$38. \text{average} = \frac{1}{2} \int_0^1 \int_0^1 \int_0^2 (x+y-z) dz dy dx = \frac{1}{2} \int_0^1 \int_0^1 (2x+2y-2) dy dx = \frac{1}{2} \int_0^1 (2x-1) dx = 0$$

$$39. \text{average} = \int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2) dz dy dx = \int_0^1 \int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx = 1$$

$$40. \text{average} = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyz dz dy dx = \frac{1}{4} \int_0^2 \int_0^2 xy dy dx = \frac{1}{2} \int_0^2 x dx = 1$$

$$41. \int_0^4 \int_0^1 \int_{2y}^2 \frac{4 \cos(x^2)}{2\sqrt{z}} dx dy dz = \int_0^4 \int_0^2 \int_0^{x/2} \frac{4 \cos(x^2)}{2\sqrt{z}} dy dx dz = \int_0^4 \int_0^2 \frac{x \cos(x^2)}{\sqrt{z}} dx dz = \int_0^4 \left(\frac{\sin 4}{2} \right) z^{-1/2} dz \\ = [(\sin 4)z^{1/2}]_0^4 = 2 \sin 4$$

$$42. \int_0^1 \int_0^1 \int_{x^2}^1 12xz e^{zy^2} dy dx dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xz e^{zy^2} dx dy dz = \int_0^1 \int_0^1 6yz e^{zy^2} dy dz = \int_0^1 [3e^{zy^2}]_0^1 dz \\ = 3 \int_0^1 (e^z - z) dz = 3[e^z - \frac{1}{2}z^2]_0^1 = 3e - \frac{3}{2}$$

$$43. \int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} dx dy dz = \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{4\pi \sin(\pi y^2)}{y^2} dy dz = \int_0^1 \int_0^{y^3} \frac{4\pi \sin(\pi y^2)}{y^2} dz dy \\ = \int_0^1 4\pi y \sin(\pi y^2) dy = [-2 \cos(\pi y^2)]_0^1 = -2(-1) + 2(1) = 4$$

$$44. \int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} dy dz dx = \int_0^2 \int_0^{4-x^2} \frac{x \sin 2z}{4-z} dz dx = \int_0^2 \int_0^{\sqrt{4-z}} \left(\frac{\sin 2z}{4-z} \right) x dx dz = \int_0^2 \left(\frac{\sin 2z}{4-z} \right) \frac{1}{2} (4-z) dz \\ = \left[-\frac{1}{4} \cos 2z \right]_0^4 = \left[-\frac{1}{4} + \frac{1}{2} \sin^2 z \right]_0^4 = \frac{\sin^2 4}{2}$$

$$45. \int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz dy dx = \frac{4}{15} \Rightarrow \int_0^1 \int_0^{4-a-x^2} (4-x^2-y-a) dy dx = \frac{4}{15} \\ \Rightarrow \int_0^1 \left[(4-a-x^2)^2 - \frac{1}{2}(4-a-x^2)^2 \right] dx = \frac{4}{15} \Rightarrow \frac{1}{2} \int_0^1 (4-a-x^2)^2 dx = \frac{4}{15} \Rightarrow \int_0^1 [(4-a)^2 - 2x^2(4-a) + x^4] dx \\ = \frac{8}{15} \Rightarrow \left[(4-a)^2 x - \frac{2}{3} x^3 (4-a) + \frac{x^5}{5} \right]_0^1 = \frac{8}{15} \Rightarrow (4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5} = \frac{8}{15} \Rightarrow 15(4-a)^2 - 10(4-a) - 5 = 0 \\ \Rightarrow 3(4-a)^2 - 2(4-a) - 1 = 0 \Rightarrow [3(4-a)+1][(4-a)-1] = 0 \Rightarrow 4-a = -\frac{1}{3} \text{ or } 4-a = 1 \Rightarrow a = \frac{13}{3} \text{ or } a = 3$$

46. The volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4abc\pi}{3}$ so that $\frac{4(1)(2)(c)\pi}{3} = 8\pi \Rightarrow c = 3$.
47. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where it is positive. These criteria are met by the points (x, y, z) such that $4x^2 + 4y^2 + z^2 - 4 \leq 0$ or $4x^2 + 4y^2 + z^2 \leq 4$, which is a solid ellipsoid centered at the origin.
48. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where it is negative. These criteria are met by the points (x, y, z) such that $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, which is a solid sphere of radius 1 centered at the origin.

49-52. Example CAS commands:

Maple:

```
F := (x,y,z) -> x^2*y^2*z;
q1 := Int( Int( Int( F(x,y,z), y=-sqrt(1-x^2)..sqrt(1-x^2) ), x=-1..1 ), z=0..1 );
value( q1 );
```

Mathematica: (functions and bounds will vary)

```
Clear[f, x, y, z];
f:= x^2 y^2 z
Integrate[f, {x,-1,1}, {y,-Sqrt[1-x^2], Sqrt[1-x^2]}, {z, 0, 1}]
N[%]
topolar={x -> r Cos[t], y -> r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, 2π}, {r, 0, 1},{z, 0, 1}]
N[%]
```

15.6 MOMENTS AND CENTERS OF MASS

- $$M = \int_0^1 \int_x^{2-x^2} 3 \, dy \, dx = 3 \int_0^1 (2 - x^2 - x) \, dx = \frac{7}{2}; M_y = \int_0^1 \int_x^{2-x^2} 3x \, dy \, dx = 3 \int_0^1 [xy]_x^{2-x^2} \, dx$$

$$= 3 \int_0^1 (2x - x^3 - x^2) \, dx = \frac{5}{4}; M_x = \int_0^1 \int_x^{2-x^2} 3y \, dy \, dx = \frac{3}{2} \int_0^1 [y^2]_x^{2-x^2} \, dx = \frac{3}{2} \int_0^1 (4 - 5x^2 + x^4) \, dx = \frac{19}{5}$$

$$\Rightarrow \bar{x} = \frac{5}{14} \text{ and } \bar{y} = \frac{38}{35}$$
- $$M = \delta \int_0^3 \int_0^3 dy \, dx = \delta \int_0^3 3 \, dx = 9\delta; I_x = \delta \int_0^3 \int_0^3 y^2 \, dy \, dx = \delta \int_0^3 \left[\frac{y^3}{3} \right]_0^3 \, dx = 27\delta;$$

$$I_y = \delta \int_0^3 \int_0^3 x^2 \, dy \, dx = \delta \int_0^3 [x^2 y]_0^3 \, dx = \delta \int_0^3 3x^2 \, dx = 27\delta$$
- $$M = \int_0^2 \int_{y^2/2}^{4-y} dx \, dy = \int_0^2 \left(4 - y - \frac{y^2}{2} \right) dy = \frac{14}{3}; M_y = \int_0^2 \int_{y^2/2}^{4-y} x \, dx \, dy = \frac{1}{2} \int_0^2 [x^2]_{y^2/2}^{4-y} \, dy$$

$$= \frac{1}{2} \int_0^2 \left(16 - 8y + y^2 - \frac{y^4}{4} \right) dy = \frac{128}{15}; M_x = \int_0^2 \int_{y^2/2}^{4-y} y \, dx \, dy = \int_0^2 \left(4y - y^2 - \frac{y^3}{2} \right) dy = \frac{10}{3}$$

$$\Rightarrow \bar{x} = \frac{64}{35} \text{ and } \bar{y} = \frac{5}{7}$$
- $$M = \int_0^3 \int_0^{3-x} dy \, dx = \int_0^3 (3 - x) \, dx = \frac{9}{2}; M_y = \int_0^3 \int_0^{3-x} x \, dy \, dx = \int_0^3 [xy]_0^{3-x} \, dx = \int_0^3 (3x - x^2) \, dx = \frac{9}{2}$$

$$\Rightarrow \bar{x} = 1 \text{ and } \bar{y} = 1, \text{ by symmetry}$$
- $$M = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy \, dx = \frac{\pi a^2}{4}; M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} x \, dy \, dx = \int_0^a [xy]_0^{\sqrt{a^2-x^2}} \, dx = \int_0^a x \sqrt{a^2 - x^2} \, dx = \frac{a^3}{3}$$

$$\Rightarrow \bar{x} = \bar{y} = \frac{4a}{3\pi}, \text{ by symmetry}$$

6. $M = \int_0^\pi \int_0^{\sin x} dy dx = \int_0^\pi \sin x dx = 2$; $M_x = \int_0^\pi \int_0^{\sin x} y dy dx = \frac{1}{2} \int_0^\pi [y^2]_0^{\sin x} dx = \frac{1}{2} \int_0^\pi \sin^2 x dx$
 $= \frac{1}{4} \int_0^\pi (1 - \cos 2x) dx = \frac{\pi}{4} \Rightarrow \bar{x} = \frac{\pi}{2} \text{ and } \bar{y} = \frac{\pi}{8}$
7. $I_x = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2 dy dx = \int_{-2}^2 \left[\frac{y^3}{3} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \frac{2}{3} \int_{-2}^2 (4-x^2)^{3/2} dx = 4\pi$; $I_y = 4\pi$, by symmetry;
 $I_0 = I_x + I_y = 8\pi$
8. $I_y = \int_\pi^{2\pi} \int_0^{(\sin^2 x)/x^2} x^2 dy dx = \int_\pi^{2\pi} (\sin^2 x - 0) dx = \frac{1}{2} \int_\pi^{2\pi} (1 - \cos 2x) dx = \frac{\pi}{2}$
9. $M = \int_{-\infty}^0 \int_0^{e^x} dy dx = \int_{-\infty}^0 e^x dx = \lim_{b \rightarrow -\infty} \int_b^0 e^x dx = 1 - \lim_{b \rightarrow -\infty} e^b = 1$; $M_y = \int_{-\infty}^0 \int_0^{e^x} x dy dx = \int_{-\infty}^0 x e^x dx$
 $= \lim_{b \rightarrow -\infty} \int_b^0 x e^x dx = \lim_{b \rightarrow -\infty} [x e^x - e^x]_b^0 = -1 - \lim_{b \rightarrow -\infty} (b e^b - e^b) = -1$; $M_x = \int_{-\infty}^0 \int_0^{e^x} y dy dx$
 $= \frac{1}{2} \int_{-\infty}^0 e^{2x} dx = \frac{1}{2} \lim_{b \rightarrow -\infty} \int_b^0 e^{2x} dx = \frac{1}{4} \Rightarrow \bar{x} = -1 \text{ and } \bar{y} = \frac{1}{4}$
10. $M_y = \int_0^\infty \int_0^{e^{-x^2/2}} x dy dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-x^2/2} dx = - \lim_{b \rightarrow \infty} \left[\frac{1}{e^{x^2/2}} - 1 \right]_0^b = 1$
11. $M = \int_0^2 \int_{-y}^{y-y^2} (x+y) dx dy = \int_0^2 \left[\frac{x^2}{2} + xy \right]_{-y}^{y-y^2} dy = \int_0^2 \left(\frac{y^4}{2} - 2y^3 + 2y^2 \right) dy = \left[\frac{y^5}{10} - \frac{y^4}{2} + \frac{2y^3}{3} \right]_0^2 = \frac{8}{15}$;
 $I_x = \int_0^2 \int_{-y}^{y-y^2} y^2(x+y) dx dy = \int_0^2 \left[\frac{x^2 y^2}{2} + xy^3 \right]_{-y}^{y-y^2} dy = \int_0^2 \left(\frac{y^6}{2} - 2y^5 + 2y^4 \right) dy = \frac{64}{105}$;
12. $M = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{4y^2}^{\sqrt{12-4y^2}} 5x dx dy = 5 \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \left[\frac{x^2}{2} \right]_{4y^2}^{\sqrt{12-4y^2}} dy = \frac{5}{2} \int_{-\sqrt{3}/2}^{\sqrt{3}/2} (12 - 4y^2 - 16y^4) dy = 23\sqrt{3}$
13. $M = \int_0^1 \int_x^{2-x} (6x + 3y + 3) dy dx = \int_0^1 \left[6xy + \frac{3}{2} y^2 + 3y \right]_x^{2-x} dx = \int_0^1 (12 - 12x^2) dx = 8$;
 $M_y = \int_0^1 \int_x^{2-x} x(6x + 3y + 3) dy dx = \int_0^1 (12x - 12x^3) dx = 3$; $M_x = \int_0^1 \int_x^{2-x} y(6x + 3y + 3) dy dx$
 $= \int_0^1 (14 - 6x - 6x^2 - 2x^3) dx = \frac{17}{2} \Rightarrow \bar{x} = \frac{3}{8} \text{ and } \bar{y} = \frac{17}{16}$
14. $M = \int_0^1 \int_{y^2}^{2y-y^2} (y+1) dx dy = \int_0^1 (2y - 2y^3) dy = \frac{1}{2}$; $M_x = \int_0^1 \int_{y^2}^{2y-y^2} y(y+1) dx dy = \int_0^1 (2y^2 - 2y^4) dy = \frac{4}{15}$;
 $M_y = \int_0^1 \int_{y^2}^{2y-y^2} x(y+1) dx dy = \int_0^1 (2y^2 - 2y^4) dy = \frac{4}{15} \Rightarrow \bar{x} = \frac{8}{15} \text{ and } \bar{y} = \frac{8}{15}$; $I_x = \int_0^1 \int_{y^2}^{2y-y^2} y^2(y+1) dx dy$
 $= 2 \int_0^1 (y^3 - y^5) dy = \frac{1}{6}$
15. $M = \int_0^1 \int_0^6 (x+y+1) dx dy = \int_0^1 (6y + 24) dy = 27$; $M_x = \int_0^1 \int_0^6 y(x+y+1) dx dy = \int_0^1 y(6y + 24) dy = 14$;
 $M_y = \int_0^1 \int_0^6 x(x+y+1) dx dy = \int_0^1 (18y + 90) dy = 99 \Rightarrow \bar{x} = \frac{11}{3} \text{ and } \bar{y} = \frac{14}{27}$; $I_y = \int_0^1 \int_0^6 x^2(x+y+1) dx dy$
 $= 216 \int_0^1 \left(\frac{y}{3} + \frac{11}{6} \right) dy = 432$
16. $M = \int_{-1}^1 \int_{x^2}^1 (y+1) dy dx = - \int_{-1}^1 \left(\frac{x^4}{2} + x^2 - \frac{3}{2} \right) dx = \frac{32}{15}$; $M_x = \int_{-1}^1 \int_{x^2}^1 y(y+1) dy dx = \int_{-1}^1 \left(\frac{5}{6} - \frac{x^6}{3} - \frac{x^4}{2} \right) dx$
 $= \frac{48}{35}$; $M_y = \int_{-1}^1 \int_{x^2}^1 x(y+1) dy dx = \int_{-1}^1 \left(\frac{3x}{2} - \frac{x^5}{2} - x^3 \right) dx = 0 \Rightarrow \bar{x} = 0 \text{ and } \bar{y} = \frac{9}{14}$; $I_y = \int_{-1}^1 \int_{x^2}^1 x^2(y+1) dy dx$
 $= \int_{-1}^1 \left(\frac{3x^2}{2} - \frac{x^6}{2} - x^4 \right) dx = \frac{16}{35}$

$$\begin{aligned}
 17. \quad M &= \int_{-1}^1 \int_0^{x^2} (7y + 1) \, dy \, dx = \int_{-1}^1 \left(\frac{7x^4}{2} + x^2 \right) \, dx = \frac{31}{15}; \quad M_x = \int_{-1}^1 \int_0^{x^2} y(7y + 1) \, dy \, dx = \int_{-1}^1 \left(\frac{7x^6}{3} + \frac{x^4}{2} \right) \, dx = \frac{13}{15}; \\
 M_y &= \int_{-1}^1 \int_0^{x^2} x(7y + 1) \, dy \, dx = \int_{-1}^1 \left(\frac{7x^5}{2} + x^3 \right) \, dx = 0 \Rightarrow \bar{x} = 0 \text{ and } \bar{y} = \frac{13}{31}; \quad I_y = \int_{-1}^1 \int_0^{x^2} x^2(7y + 1) \, dy \, dx \\
 &= \int_{-1}^1 \left(\frac{7x^6}{2} + x^4 \right) \, dx = \frac{7}{5}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad M &= \int_0^{20} \int_{-1}^1 \left(1 + \frac{x}{20} \right) \, dy \, dx = \int_0^{20} \left(2 + \frac{x}{10} \right) \, dx = 60; \quad M_x = \int_0^{20} \int_{-1}^1 y \left(1 + \frac{x}{20} \right) \, dy \, dx = \int_0^{20} \left[\left(1 + \frac{x}{20} \right) \left(\frac{y^2}{2} \right) \right]_{-1}^1 \, dx = 0; \\
 M_y &= \int_0^{20} \int_{-1}^1 x \left(1 + \frac{x}{20} \right) \, dy \, dx = \int_0^{20} \left(2x + \frac{x^2}{10} \right) \, dx = \frac{2000}{3} \Rightarrow \bar{x} = \frac{100}{9} \text{ and } \bar{y} = 0; \quad I_x = \int_0^{20} \int_{-1}^1 y^2 \left(1 + \frac{x}{20} \right) \, dy \, dx \\
 &= \frac{2}{3} \int_0^{20} \left(1 + \frac{x}{20} \right) \, dx = 20
 \end{aligned}$$

$$\begin{aligned}
 19. \quad M &= \int_0^1 \int_{-y}^y (y + 1) \, dx \, dy = \int_0^1 (2y^2 + 2y) \, dy = \frac{5}{3}; \quad M_x = \int_0^1 \int_{-y}^y y(y + 1) \, dx \, dy = 2 \int_0^1 (y^3 + y^2) \, dy = \frac{7}{6}; \\
 M_y &= \int_0^1 \int_{-y}^y x(y + 1) \, dx \, dy = \int_0^1 0 \, dy = 0 \Rightarrow \bar{x} = 0 \text{ and } \bar{y} = \frac{7}{10}; \quad I_x = \int_0^1 \int_{-y}^y y^2(y + 1) \, dx \, dy = \int_0^1 (2y^4 + 2y^3) \, dy \\
 &= \frac{9}{10}; \quad I_y = \int_0^1 \int_{-y}^y x^2(y + 1) \, dx \, dy = \frac{1}{3} \int_0^1 (2y^4 + 2y^3) \, dy = \frac{3}{10} \Rightarrow I_o = I_x + I_y = \frac{6}{5}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad M &= \int_0^1 \int_{-y}^y (3x^2 + 1) \, dx \, dy = \int_0^1 (2y^3 + 2y) \, dy = \frac{3}{2}; \quad M_x = \int_0^1 \int_{-y}^y y(3x^2 + 1) \, dx \, dy = \int_0^1 (2y^4 + 2y^2) \, dy = \frac{16}{15}; \\
 M_y &= \int_0^1 \int_{-y}^y x(3x^2 + 1) \, dx \, dy = 0 \Rightarrow \bar{x} = 0 \text{ and } \bar{y} = \frac{32}{45}; \quad I_x = \int_0^1 \int_{-y}^y y^2(3x^2 + 1) \, dx \, dy = \int_0^1 (2y^5 + 2y^3) \, dy = \frac{5}{6}; \\
 I_y &= \int_0^1 \int_{-y}^y x^2(3x^2 + 1) \, dx \, dy = 2 \int_0^1 \left(\frac{3}{5}y^5 + \frac{1}{3}y^3 \right) \, dy = \frac{11}{30} \Rightarrow I_o = I_x + I_y = \frac{6}{5}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad I_x &= \int_0^a \int_0^b \int_0^c (y^2 + z^2) \, dz \, dy \, dx = \int_0^a \int_0^b \left(cy^2 + \frac{c^3}{3} \right) \, dy \, dx = \int_0^a \left(\frac{cb^3}{3} + \frac{c^3b}{3} \right) \, dx = \frac{abc(b^2 + c^2)}{3} \\
 &= \frac{M}{3} (b^2 + c^2) \text{ where } M = abc; \quad I_y = \frac{M}{3} (a^2 + c^2) \text{ and } I_z = \frac{M}{3} (a^2 + b^2), \text{ by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \text{The plane } z &= \frac{4-2y}{3} \text{ is the top of the wedge} \Rightarrow I_x = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} (y^2 + z^2) \, dz \, dy \, dx \\
 &= \int_{-3}^3 \int_{-2}^4 \left[\frac{8y^2}{3} - \frac{2y^3}{3} + \frac{8(2-y)^3}{81} + \frac{64}{81} \right] \, dy \, dx = \int_{-3}^3 \frac{104}{3} \, dx = 208; \quad I_y = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} (x^2 + z^2) \, dz \, dy \, dx \\
 &= \int_{-3}^3 \int_{-2}^4 \left[\frac{(4-2y)^3}{81} + \frac{x^2(4-2y)}{3} + \frac{4x^2}{3} + \frac{64}{81} \right] \, dy \, dx = \int_{-3}^3 \left(12x^2 + \frac{32}{3} \right) \, dx = 280; \\
 I_z &= \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} (x^2 + y^2) \, dz \, dy \, dx = \int_{-3}^3 \int_{-2}^4 (x^2 + y^2) \left(\frac{8}{3} - \frac{2y}{3} \right) \, dy \, dx = 12 \int_{-3}^3 (x^2 + 2) \, dx = 360
 \end{aligned}$$

$$\begin{aligned}
 23. \quad M &= 4 \int_0^1 \int_0^1 \int_{4y^2}^4 dz \, dy \, dx = 4 \int_0^1 \int_0^1 (4 - 4y^2) \, dy \, dx = 16 \int_0^1 \frac{2}{3} \, dx = \frac{32}{3}; \quad M_{xy} = 4 \int_0^1 \int_0^1 \int_{4y^2}^4 z \, dz \, dy \, dx \\
 &= 2 \int_0^1 \int_0^1 (16 - 16y^4) \, dy \, dx = \frac{128}{5} \int_0^1 dx = \frac{128}{5} \Rightarrow \bar{z} = \frac{12}{5}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}; \\
 I_x &= 4 \int_0^1 \int_0^1 \int_{4y^2}^4 (y^2 + z^2) \, dz \, dy \, dx = 4 \int_0^1 \int_0^1 \left[(4y^2 + \frac{64}{3}) - (4y^4 + \frac{64y^6}{3}) \right] \, dy \, dx = 4 \int_0^1 \frac{1976}{105} \, dx = \frac{7904}{105}; \\
 I_y &= 4 \int_0^1 \int_0^1 \int_{4y^2}^4 (x^2 + z^2) \, dz \, dy \, dx = 4 \int_0^1 \int_0^1 \left[(4x^2 + \frac{64}{3}) - (4x^2y^2 + \frac{64y^6}{3}) \right] \, dy \, dx = 4 \int_0^1 \left(\frac{8}{3}x^2 + \frac{128}{7} \right) \, dx \\
 &= \frac{4832}{63}; \quad I_z = 4 \int_0^1 \int_0^1 \int_{4y^2}^4 (x^2 + y^2) \, dz \, dy \, dx = 16 \int_0^1 \int_0^1 (x^2 - x^2y^2 + y^2 - y^4) \, dy \, dx \\
 &= 16 \int_0^1 \left(\frac{2x^2}{3} + \frac{2}{15} \right) \, dx = \frac{256}{45}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad (a) \quad M &= \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} \int_0^{2-x} dz \, dy \, dx = \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} (2-x) \, dy \, dx = \int_{-2}^2 (2-x) \left(\sqrt{4-x^2} \right) \, dx = 4\pi; \\
 M_{yz} &= \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} \int_0^{2-x} x \, dz \, dy \, dx = \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} x(2-x) \, dy \, dx = \int_{-2}^2 x(2-x) \left(\sqrt{4-x^2} \right) \, dx = -2\pi;
 \end{aligned}$$

$$\begin{aligned} M_{xz} &= \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} \int_0^{2-x} y \, dz \, dy \, dx = \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} y(2-x) \, dy \, dx \\ &= \frac{1}{2} \int_{-2}^2 (2-x) \left[\frac{4-x^2}{4} - \frac{4-x^2}{4} \right] dx = 0 \Rightarrow \bar{x} = -\frac{1}{2} \text{ and } \bar{y} = 0 \end{aligned}$$

$$\begin{aligned} \text{(b) } M_{xy} &= \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} \int_0^{2-x} z \, dz \, dy \, dx = \frac{1}{2} \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} (2-x)^2 \, dy \, dx = \frac{1}{2} \int_{-2}^2 (2-x)^2 (\sqrt{4-x^2}) \, dx \\ &= 5\pi \Rightarrow \bar{z} = \frac{5}{4} \end{aligned}$$

$$\begin{aligned} 25. \text{ (a) } M &= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 dz \, dy \, dx = 4 \int_0^{\pi/2} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^2 (4r - r^3) \, dr \, d\theta = 4 \int_0^{\pi/2} 4 \, d\theta = 8\pi; \\ M_{xy} &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 zr \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \frac{r}{2} (16 - r^4) \, dr \, d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{8}{3}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry} \end{aligned}$$

$$\begin{aligned} \text{(b) } M &= 8\pi \Rightarrow 4\pi = \int_0^{2\pi} \int_0^{\sqrt{c}} \int_{r^2}^c r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{c}} (cr - r^3) \, dr \, d\theta = \int_0^{2\pi} \frac{c^2}{4} d\theta = \frac{c^2\pi}{2} \Rightarrow c^2 = 8 \Rightarrow c = 2\sqrt{2}, \\ &\text{since } c > 0 \end{aligned}$$

$$\begin{aligned} 26. M &= 8; M_{xy} = \int_{-1}^1 \int_3^5 \int_{-1}^1 z \, dz \, dy \, dx = \int_{-1}^1 \int_3^5 \left[\frac{z^2}{2} \right]_{-1}^1 dy \, dx = 0; M_{yz} = \int_{-1}^1 \int_3^5 \int_{-1}^1 x \, dz \, dy \, dx \\ &= 2 \int_{-1}^1 \int_3^5 x \, dy \, dx = 4 \int_{-1}^1 x \, dx = 0; M_{xz} = \int_{-1}^1 \int_3^5 \int_{-1}^1 y \, dz \, dy \, dx = 2 \int_{-1}^1 \int_3^5 y \, dy \, dx = 16 \int_{-1}^1 dx = 32 \\ &\Rightarrow \bar{x} = 0, \bar{y} = 4, \bar{z} = 0; I_x = \int_{-1}^1 \int_3^5 \int_{-1}^1 (y^2 + z^2) \, dz \, dy \, dx = \int_{-1}^1 \int_3^5 (2y^2 + \frac{2}{3}) \, dy \, dx = \frac{2}{3} \int_{-1}^1 100 \, dx = \frac{400}{3}; \\ I_y &= \int_{-1}^1 \int_3^5 \int_{-1}^1 (x^2 + z^2) \, dz \, dy \, dx = \int_{-1}^1 \int_3^5 (2x^2 + \frac{2}{3}) \, dy \, dx = \frac{4}{3} \int_{-1}^1 (3x^2 + 1) \, dx = \frac{16}{3}; \\ I_z &= \int_{-1}^1 \int_3^5 \int_{-1}^1 (x^2 + y^2) \, dz \, dy \, dx = 2 \int_{-1}^1 \int_3^5 (x^2 + y^2) \, dy \, dx = 2 \int_{-1}^1 (2x^2 + \frac{98}{3}) \, dx = \frac{400}{3} \end{aligned}$$

$$\begin{aligned} 27. \text{ The plane } y + 2z = 2 \text{ is the top of the wedge } &\Rightarrow I_L = \int_{-2}^2 \int_{-2}^4 \int_{-1}^{(2-y)/2} [(y-6)^2 + z^2] \, dz \, dy \, dx \\ &= \int_{-2}^2 \int_{-2}^4 \left[\frac{(y-6)^2(4-y)}{2} + \frac{(2-y)^3}{24} + \frac{1}{3} \right] dy \, dx; \text{ let } t = 2 - y \Rightarrow I_L = 4 \int_{-2}^4 \left(\frac{13t^3}{24} + 5t^2 + 16t + \frac{49}{3} \right) dt = 1386; \\ M &= \frac{1}{2} (3)(6)(4) = 36 \end{aligned}$$

$$\begin{aligned} 28. \text{ The plane } y + 2z = 2 \text{ is the top of the wedge } &\Rightarrow I_L = \int_{-2}^2 \int_{-2}^4 \int_{-1}^{(2-y)/2} [(x-4)^2 + y^2] \, dz \, dy \, dx \\ &= \frac{1}{2} \int_{-2}^2 \int_{-2}^4 (x^2 - 8x + 16 + y^2) (4-y) \, dy \, dx = \int_{-2}^2 (9x^2 - 72x + 162) \, dx = 696; M = \frac{1}{2} (3)(6)(4) = 36 \end{aligned}$$

$$\begin{aligned} 29. \text{ (a) } M &= \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x \, dz \, dy \, dx = \int_0^2 \int_0^{2-x} (4x - 2x^2 - 2xy) \, dy \, dx = \int_0^2 (x^3 - 4x^2 + 4x) \, dx = \frac{4}{3} \\ \text{(b) } M_{xy} &= \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2xz \, dz \, dy \, dx = \int_0^2 \int_0^{2-x} x(2-x-y)^2 \, dy \, dx = \int_0^2 \frac{x(2-x)^3}{3} \, dx = \frac{8}{15}; M_{xz} = \frac{8}{15} \text{ by} \\ &\text{symmetry; } M_{yz} = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x^2 \, dz \, dy \, dx = \int_0^2 \int_0^{2-x} 2x^2(2-x-y) \, dy \, dx = \int_0^2 (2x - x^2)^2 \, dx = \frac{16}{15} \\ &\Rightarrow \bar{x} = \frac{4}{5}, \text{ and } \bar{y} = \bar{z} = \frac{2}{5} \end{aligned}$$

$$\begin{aligned} 30. \text{ (a) } M &= \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxy \, dz \, dy \, dx = k \int_0^2 \int_0^{\sqrt{x}} xy(4-x^2) \, dy \, dx = \frac{k}{2} \int_0^2 (4x^2 - x^4) \, dx = \frac{32k}{15} \\ \text{(b) } M_{yz} &= \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kx^2y \, dz \, dy \, dx = k \int_0^2 \int_0^{\sqrt{x}} x^2y(4-x^2) \, dy \, dx = \frac{k}{2} \int_0^2 (4x^3 - x^5) \, dx = \frac{8k}{3} \\ &\Rightarrow \bar{x} = \frac{5}{4}; M_{xz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxy^2 \, dz \, dy \, dx = k \int_0^2 \int_0^{\sqrt{x}} xy^2(4-x^2) \, dy \, dx = \frac{k}{3} \int_0^2 (4x^{5/2} - x^{9/2}) \, dx \\ &= \frac{256\sqrt{2}k}{231} \Rightarrow \bar{y} = \frac{40\sqrt{2}}{77}; M_{xy} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxyz \, dz \, dy \, dx = \int_0^2 \int_0^{\sqrt{x}} xy(4-x^2)^2 \, dy \, dx \\ &= \frac{k}{4} \int_0^2 (16x^2 - 8x^4 + x^6) \, dx = \frac{256k}{105} \Rightarrow \bar{z} = \frac{8}{7} \end{aligned}$$

$$31. \text{ (a) } M = \int_0^1 \int_0^1 \int_0^1 (x+y+z+1) \, dz \, dy \, dx = \int_0^1 \int_0^1 (x+y+\frac{3}{2}) \, dy \, dx = \int_0^1 (x+2) \, dx = \frac{5}{2}$$

$$(b) \quad M_{xy} = \int_0^1 \int_0^1 \int_0^1 z(x+y+z+1) \, dz \, dy \, dx = \frac{1}{2} \int_0^1 \int_0^1 (x+y+\frac{5}{3}) \, dy \, dx = \frac{1}{2} \int_0^1 (x+\frac{13}{6}) \, dx = \frac{4}{3}$$

$$\Rightarrow M_{xy} = M_{yz} = M_{xz} = \frac{4}{3}, \text{ by symmetry } \Rightarrow \bar{x} = \bar{y} = \bar{z} = \frac{8}{15}$$

$$(c) \quad I_z = \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2)(x+y+z+1) \, dz \, dy \, dx = \int_0^1 \int_0^1 (x^2 + y^2)(x+y+\frac{3}{2}) \, dy \, dx$$

$$= \int_0^1 (x^3 + 2x^2 + \frac{1}{3}x + \frac{3}{4}) \, dx = \frac{11}{6} \Rightarrow I_x = I_y = I_z = \frac{11}{6}, \text{ by symmetry}$$

32. The plane $y + 2z = 2$ is the top of the wedge.

$$(a) \quad M = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 (x+1)(2-\frac{y}{2}) \, dy \, dx = 18$$

$$(b) \quad M_{yz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} x(x+1) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 x(x+1)(2-\frac{y}{2}) \, dy \, dx = 6;$$

$$M_{xz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} y(x+1) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 y(x+1)(2-\frac{y}{2}) \, dy \, dx = 0;$$

$$M_{xy} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} z(x+1) \, dz \, dy \, dx = \frac{1}{2} \int_{-1}^1 \int_{-2}^4 (x+1)(\frac{y^2}{4} - y) \, dy \, dx = 0 \Rightarrow \bar{x} = \frac{1}{3}, \text{ and } \bar{y} = \bar{z} = 0$$

$$(c) \quad I_x = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(y^2 + z^2) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 (x+1) \left[2y^2 + \frac{1}{3} - \frac{y^3}{2} + \frac{1}{3}(1-\frac{y}{2})^3 \right] \, dy \, dx = 45;$$

$$I_y = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(x^2 + z^2) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 (x+1) \left[2x^2 + \frac{1}{3} - \frac{x^2 y}{2} + \frac{1}{3}(1-\frac{y}{2})^3 \right] \, dy \, dx = 15;$$

$$I_z = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(x^2 + y^2) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 (x+1)(2-\frac{y}{2})(x^2 + y^2) \, dy \, dx = 42$$

$$33. \quad M = \int_0^1 \int_{z-1}^1 \int_0^{\sqrt{z}} (2y+5) \, dy \, dx \, dz = \int_0^1 \int_{z-1}^1 (z+5\sqrt{z}) \, dx \, dz = \int_0^1 2(z+5\sqrt{z})(1-z) \, dz$$

$$= 2 \int_0^1 (5z^{1/2} + z - 5z^{3/2} - z^2) \, dz = 2 \left[\frac{10}{3} z^{3/2} + \frac{1}{2} z^2 - 2z^{5/2} - \frac{1}{3} z^3 \right]_0^1 = 2 \left(\frac{9}{3} - \frac{3}{2} \right) = 3$$

$$34. \quad M = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2(x^2+y^2)}^{16-2(x^2+y^2)} \sqrt{x^2+y^2} \, dz \, dy \, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2} [16-4(x^2+y^2)] \, dy \, dx$$

$$= 4 \int_0^{2\pi} \int_0^2 r(4-r^2)r \, dr \, d\theta = 4 \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 d\theta = 4 \int_0^{2\pi} \frac{64}{15} d\theta = \frac{512\pi}{15}$$

$$35. (a) \quad \bar{x} = \frac{M_{yz}}{M} = 0 \Rightarrow \iiint_R x \delta(x, y, z) \, dx \, dy \, dz = 0 \Rightarrow M_{yz} = 0$$

$$(b) \quad I_L = \iiint_D |\mathbf{v} - h\mathbf{i}|^2 \, dm = \iiint_D |(x-h)\mathbf{i} + y\mathbf{j}|^2 \, dm = \iiint_D (x^2 - 2xh + h^2 + y^2) \, dm$$

$$= \iiint_D (x^2 + y^2) \, dm - 2h \iiint_D x \, dm + h^2 \iiint_D dm = I_x - 0 + h^2 m = I_{c.m.} + h^2 m$$

$$36. \quad I_L = I_{c.m.} + mh^2 = \frac{2}{5} ma^2 + ma^2 = \frac{7}{5} ma^2$$

$$37. (a) \quad (\bar{x}, \bar{y}, \bar{z}) = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \Rightarrow I_z = I_{c.m.} + abc \left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}} \right)^2 \Rightarrow I_{c.m.} = I_z - \frac{abc(a^2+b^2)}{4}$$

$$= \frac{abc(a^2+b^2)}{3} - \frac{abc(a^2+b^2)}{4} = \frac{abc(a^2+b^2)}{12}; R_{c.m.} = \sqrt{\frac{I_{c.m.}}{M}} = \sqrt{\frac{a^2+b^2}{12}}$$

$$(b) \quad I_L = I_{c.m.} + abc \left(\sqrt{\frac{a^2}{4} + \left(\frac{b}{2} - 2b\right)^2} \right)^2 = \frac{abc(a^2+b^2)}{12} + \frac{abc(a^2+9b^2)}{4} = \frac{abc(4a^2+28b^2)}{12}$$

$$= \frac{abc(a^2+7b^2)}{3}; R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{a^2+7b^2}{3}}$$

$$38. \quad M = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} dz \, dy \, dx = \int_{-3}^3 \int_{-2}^4 \frac{2}{3} (4-y) \, dy \, dx = \int_{-3}^3 \frac{2}{3} \left[4y - \frac{y^2}{2} \right]_{-2}^4 dx = 12 \int_{-3}^3 dx = 72;$$

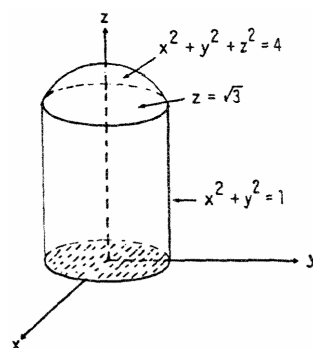
$$\bar{x} = \bar{y} = \bar{z} = 0 \text{ from Exercise 22 } \Rightarrow I_x = I_{c.m.} + 72 \left(\sqrt{0^2 + 0^2} \right)^2 = I_{c.m.} \Rightarrow I_L = I_{c.m.} + 72 \left(\sqrt{16 + \frac{16}{9}} \right)^2$$

$$= 208 + 72 \left(\frac{160}{9} \right) = 1488$$

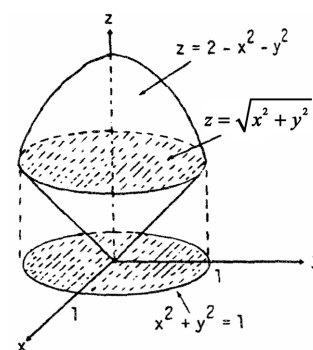
15.7 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

1.
$$\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left[r(2-r^2)^{1/2} - r^2 \right] dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^3}{3} \right]_0^1 d\theta$$
$$= \int_0^{2\pi} \left(\frac{2^{3/2}}{3} - \frac{2}{3} \right) d\theta = \frac{4\pi(\sqrt{2}-1)}{3}$$
2.
$$\int_0^{2\pi} \int_0^3 \int_{r^2/3}^{\sqrt{18-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^3 \left[r(18-r^2)^{1/2} - \frac{r^3}{3} \right] dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(18-r^2)^{3/2} - \frac{r^4}{12} \right]_0^3 d\theta$$
$$= \frac{9\pi(8\sqrt{2}-7)}{2}$$
3.
$$\int_0^{2\pi} \int_0^{\theta/2\pi} \int_0^{3+24r^2} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^{\theta/2\pi} (3r + 24r^3) dr \, d\theta = \int_0^{2\pi} \left[\frac{3}{2}r^2 + 6r^4 \right]_0^{\theta/2\pi} d\theta = \frac{3}{2} \int_0^{2\pi} \left(\frac{\theta^2}{4\pi^2} + \frac{4\theta^4}{16\pi^4} \right) d\theta$$
$$= \frac{3}{2} \left[\frac{\theta^3}{12\pi^2} + \frac{\theta^5}{20\pi^4} \right]_0^{2\pi} = \frac{17\pi}{5}$$
4.
$$\int_0^\pi \int_0^{\theta/\pi} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z \, dz \, r \, dr \, d\theta = \int_0^\pi \int_0^{\theta/\pi} \frac{1}{2} [9(4-r^2) - (4-r^2)] r \, dr \, d\theta = 4 \int_0^\pi \int_0^{\theta/\pi} (4r - r^3) dr \, d\theta$$
$$= 4 \int_0^\pi \left[2r^2 - \frac{r^4}{4} \right]_0^{\theta/\pi} d\theta = 4 \int_0^\pi \left(\frac{2\theta^2}{\pi^2} - \frac{\theta^4}{4\pi^4} \right) d\theta = \frac{37\pi}{15}$$
5.
$$\int_0^{2\pi} \int_0^1 \int_r^{(2-r^2)^{-1/2}} 3 \, dz \, r \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^1 \left[r(2-r^2)^{-1/2} - r^2 \right] dr \, d\theta = 3 \int_0^{2\pi} \left[-(2-r^2)^{1/2} - \frac{r^3}{3} \right]_0^1 d\theta$$
$$= 3 \int_0^{2\pi} \left(\sqrt{2} - \frac{4}{3} \right) d\theta = \pi(6\sqrt{2} - 8)$$
6.
$$\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} (r^2 \sin^2 \theta + z^2) \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left(r^3 \sin^2 \theta + \frac{r}{12} \right) dr \, d\theta = \int_0^{2\pi} \left(\frac{\sin^2 \theta}{4} + \frac{1}{24} \right) d\theta = \frac{\pi}{3}$$
7.
$$\int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 \, dr \, dz \, d\theta = \int_0^{2\pi} \int_0^3 \frac{z^4}{324} \, dz \, d\theta = \int_0^{2\pi} \frac{3}{20} \, d\theta = \frac{3\pi}{10}$$
8.
$$\int_{-1}^1 \int_0^{2\pi} \int_0^{1+\cos \theta} 4r \, dr \, d\theta \, dz = \int_{-1}^1 \int_0^{2\pi} 2(1+\cos \theta)^2 \, d\theta \, dz = \int_{-1}^1 6\pi \, dz = 12\pi$$
9.
$$\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r \, d\theta \, dr \, dz = \int_0^1 \int_0^{\sqrt{z}} \left[\frac{r^2 \theta}{2} + \frac{r^2 \sin 2\theta}{4} + z^2 \theta \right]_0^{2\pi} r \, dr \, dz = \int_0^1 \int_0^{\sqrt{z}} (\pi r^3 + 2\pi r z^2) \, dr \, dz$$
$$= \int_0^1 \left[\frac{\pi r^4}{4} + \pi r^2 z^2 \right]_0^{\sqrt{z}} dz = \int_0^1 \left(\frac{\pi z^2}{4} + \pi z^3 \right) dz = \left[\frac{\pi z^3}{12} + \frac{\pi z^4}{4} \right]_0^1 = \frac{\pi}{3}$$
10.
$$\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r \sin \theta + 1) r \, d\theta \, dz \, dr = \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} 2\pi r \, dz \, dr = 2\pi \int_0^2 \left[r(4-r^2)^{1/2} - r^2 + 2r \right] dr$$
$$= 2\pi \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} + r^2 \right]_0^2 = 2\pi \left[-\frac{8}{3} + 4 + \frac{1}{3}(4)^{3/2} \right] = 8\pi$$

11. (a) $\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$
 (b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta$
 (c) $\int_0^1 \int_0^{\sqrt{4-r^2}} \int_0^{2\pi} r \, d\theta \, dz \, dr$



12. (a) $\int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz \, r \, dr \, d\theta$
 (b) $\int_0^{2\pi} \int_0^1 \int_0^z r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r \, dr \, dz \, d\theta$
 (c) $\int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r \, d\theta \, dz \, dr$



13. $\int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} \int_0^{3r^2} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

14. $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_0^{\cos \theta} r^3 \, dz \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 r^4 \cos \theta \, dr \, d\theta = \frac{1}{5} \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta = \frac{2}{5}$

15. $\int_0^{\pi} \int_0^{2 \sin \theta} \int_0^{4-r \sin \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

16. $\int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} \int_0^{5-r \cos \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

17. $\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} \int_0^4 f(r, \theta, z) \, dz \, r \, dr \, d\theta$

18. $\int_{-\pi/2}^{\pi/2} \int_{\cos \theta}^{\cos \theta} \int_0^{3-r \sin \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

19. $\int_0^{\pi/4} \int_0^{\sec \theta} \int_0^{2-r \sin \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

20. $\int_{\pi/4}^{\pi/2} \int_0^{\csc \theta} \int_0^{2-r \sin \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

21. $\int_0^{\pi} \int_0^{\pi} \int_0^{2 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{\pi} \int_0^{\pi} \sin^4 \phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{\pi} \left(\left[-\frac{\sin^3 \phi \cos \phi}{4} \right]_0^{\pi} + \frac{3}{4} \int_0^{\pi} \sin^2 \phi \, d\phi \right) d\theta$
 $= 2 \int_0^{\pi} \int_0^{\pi} \sin^2 \phi \, d\phi \, d\theta = \int_0^{\pi} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} d\theta = \int_0^{\pi} \pi \, d\theta = \pi^2$

22. $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} 4 \cos \phi \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} [2 \sin^2 \phi]_0^{\pi/4} d\theta = \int_0^{2\pi} d\theta = 2\pi$

23. $\int_0^{2\pi} \int_0^{\pi} \int_0^{(1-\cos \phi)/2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{24} \int_0^{2\pi} \int_0^{\pi} (1-\cos \phi)^3 \sin \phi \, d\phi \, d\theta = \frac{1}{96} \int_0^{2\pi} [(1-\cos \phi)^4]_0^{\pi} d\theta$
 $= \frac{1}{96} \int_0^{2\pi} (2^4 - 0) \, d\theta = \frac{16}{96} \int_0^{2\pi} d\theta = \frac{1}{6} (2\pi) = \frac{\pi}{3}$

24. $\int_0^{3\pi/2} \int_0^{\pi} \int_0^1 5\rho^3 \sin^3 \phi \, d\rho \, d\phi \, d\theta = \frac{5}{4} \int_0^{3\pi/2} \int_0^{\pi} \sin^3 \phi \, d\phi \, d\theta = \frac{5}{4} \int_0^{3\pi/2} \left(\left[-\frac{\sin^2 \phi \cos \phi}{3} \right]_0^{\pi} + \frac{2}{3} \int_0^{\pi} \sin \phi \, d\phi \right) d\theta$
 $= \frac{5}{6} \int_0^{3\pi/2} [-\cos \phi]_0^{\pi} d\theta = \frac{5}{3} \int_0^{3\pi/2} d\theta = \frac{5\pi}{2}$

$$25. \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 3\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} (8 - \sec^3 \phi) \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \left[-8 \cos \phi - \frac{1}{2} \sec^2 \phi \right]_0^{\pi/3} d\theta \\ = \int_0^{2\pi} \left[(-4 - 2) - \left(-8 - \frac{1}{2} \right) \right] d\theta = \frac{5}{2} \int_0^{2\pi} d\theta = 5\pi$$

$$26. \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^3 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^{\pi/4} \tan \phi \sec^2 \phi \, d\phi \, d\theta = \frac{1}{4} \int_0^{2\pi} \left[\frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4}$$

$$27. \int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho = \int_0^2 \int_{-\pi}^0 \rho^3 \left[-\frac{\cos 2\phi}{2} \right]_{\pi/4}^{\pi/2} d\theta \, d\rho = \int_0^2 \int_{-\pi}^0 \frac{\rho^3}{2} d\theta \, d\rho = \int_0^2 \frac{\rho^3 \pi}{2} d\rho = \left[\frac{\pi \rho^4}{8} \right]_0^2 = 2\pi$$

$$28. \int_{\pi/6}^{\pi/3} \int_{\csc \phi}^{2 \csc \phi} \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi = 2\pi \int_{\pi/6}^{\pi/3} \int_{\csc \phi}^{2 \csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi = \frac{2\pi}{3} \int_{\pi/6}^{\pi/3} [\rho^3 \sin \phi]_{\csc \phi}^{2 \csc \phi} d\phi = \frac{14\pi}{3} \int_{\pi/6}^{\pi/3} \csc^2 \phi \, d\phi = \frac{28\pi}{3\sqrt{3}}$$

$$29. \int_0^1 \int_0^{\pi} \int_0^{\pi/4} 12\rho \sin^3 \phi \, d\phi \, d\theta \, d\rho = \int_0^1 \int_0^{\pi} \left(12\rho \left[\frac{-\sin^2 \phi \cos \phi}{3} \right]_0^{\pi/4} + 8\rho \int_0^{\pi/4} \sin \phi \, d\phi \right) d\theta \, d\rho \\ = \int_0^1 \int_0^{\pi} \left(-\frac{2\rho}{\sqrt{2}} - 8\rho [\cos \phi]_0^{\pi/4} \right) d\theta \, d\rho = \int_0^1 \int_0^{\pi} \left(8\rho - \frac{10\rho}{\sqrt{2}} \right) d\theta \, d\rho = \pi \int_0^1 \left(8\rho - \frac{10\rho}{\sqrt{2}} \right) d\rho = \pi \left[4\rho^2 - \frac{5\rho^2}{\sqrt{2}} \right]_0^1 \\ = \frac{(4\sqrt{2} - 5)\pi}{\sqrt{2}}$$

$$30. \int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{\csc \phi}^2 5\rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi = \int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} (32 - \csc^5 \phi) \sin^3 \phi \, d\theta \, d\phi = \int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} (32 \sin^3 \phi - \csc^2 \phi) \, d\theta \, d\phi \\ = \pi \int_{\pi/6}^{\pi/2} (32 \sin^3 \phi - \csc^2 \phi) \, d\phi = \pi \left[-\frac{32 \sin^2 \phi \cos \phi}{3} \right]_{\pi/6}^{\pi/2} + \frac{64\pi}{3} \int_{\pi/6}^{\pi/2} \sin \phi \, d\phi + \pi [\cot \phi]_{\pi/6}^{\pi/2} \\ = \pi \left(\frac{32\sqrt{3}}{24} \right) - \frac{64\pi}{3} [\cos \phi]_{\pi/6}^{\pi/2} - \pi (\sqrt{3}) = \frac{\sqrt{3}}{3} \pi + \left(\frac{64\pi}{3} \right) \left(\frac{\sqrt{3}}{2} \right) = \frac{33\pi\sqrt{3}}{3} = 11\pi\sqrt{3}$$

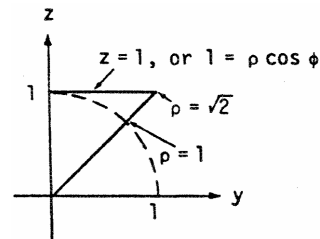
$$31. (a) \, x^2 + y^2 = 1 \Rightarrow \rho^2 \sin^2 \phi = 1, \text{ and } \rho \sin \phi = 1 \Rightarrow \rho = \csc \phi; \text{ thus}$$

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$(b) \, \int_0^{2\pi} \int_1^2 \int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_0^2 \int_0^{\pi/6} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

$$32. (a) \, \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$(b) \, \int_0^{2\pi} \int_0^1 \int_0^{\pi/4} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta \\ + \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{\cos^{-1}(1/\rho)}^{\pi/4} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$



$$33. \, V = \int_0^{2\pi} \int_0^{\pi/2} \int_{\cos \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (8 - \cos^3 \phi) \sin \phi \, d\phi \, d\theta \\ = \frac{1}{3} \int_0^{2\pi} \left[-8 \cos \phi + \frac{\cos^4 \phi}{4} \right]_0^{\pi/2} d\theta = \frac{1}{3} \int_0^{2\pi} \left(8 - \frac{1}{4} \right) d\theta = \left(\frac{31}{12} \right) (2\pi) = \frac{31\pi}{6}$$

$$34. \, V = \int_0^{2\pi} \int_0^{\pi/2} \int_1^{1+\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (3 \cos \phi + 3 \cos^2 \phi + \cos^3 \phi) \sin \phi \, d\phi \, d\theta \\ = \frac{1}{3} \int_0^{2\pi} \left[-\frac{3}{2} \cos^2 \phi - \cos^3 \phi - \frac{1}{4} \cos^4 \phi \right]_0^{\pi/2} d\theta = \frac{1}{3} \int_0^{2\pi} \left(\frac{3}{2} + 1 + \frac{1}{4} \right) d\theta = \frac{11}{12} \int_0^{2\pi} d\theta = \left(\frac{11}{12} \right) (2\pi) = \frac{11\pi}{6}$$

$$35. \, V = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} (1 - \cos \phi)^3 \sin \phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{(1 - \cos \phi)^4}{4} \right]_0^{\pi} d\theta \\ = \frac{1}{12} (2)^4 \int_0^{2\pi} d\theta = \frac{4}{3} (2\pi) = \frac{8\pi}{3}$$

$$\begin{aligned}
 36. \quad V &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{1-\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{(1-\cos\phi)^4}{4} \right]_0^{\pi/2} d\theta \\
 &= \frac{1}{12} \int_0^{2\pi} d\theta = \frac{1}{12} (2\pi) = \frac{\pi}{6}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad V &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \cos^3\phi \sin\phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \left[-\frac{\cos^4\phi}{4} \right]_{\pi/4}^{\pi/2} d\theta \\
 &= \left(\frac{8}{3}\right) \left(\frac{1}{16}\right) \int_0^{2\pi} d\theta = \frac{1}{6} (2\pi) = \frac{\pi}{3}
 \end{aligned}$$

$$38. \quad V = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \sin\phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} [-\cos\phi]_{\pi/3}^{\pi/2} d\theta = \frac{4}{3} \int_0^{2\pi} d\theta = \frac{8\pi}{3}$$

$$\begin{aligned}
 39. \quad (a) \quad & 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta & (b) \quad & 8 \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta \\
 (c) \quad & 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} dz \, dy \, dx
 \end{aligned}$$

$$\begin{aligned}
 40. \quad (a) \quad & \int_0^{\pi/2} \int_0^{3/\sqrt{2}} \int_r^{\sqrt{9-r^2}} dz \, r \, dr \, d\theta & (b) \quad & \int_0^{\pi/2} \int_0^{\pi/4} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
 (c) \quad & \int_0^{\pi/2} \int_0^{\pi/4} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 9 \int_0^{\pi/2} \int_0^{\pi/4} \sin\phi \, d\phi \, d\theta = -9 \int_0^{\pi/2} \left(\frac{1}{\sqrt{2}} - 1 \right) d\theta = \frac{9\pi(2-\sqrt{2})}{4}
 \end{aligned}$$

$$\begin{aligned}
 41. \quad (a) \quad & V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta & (b) \quad & V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta \\
 (c) \quad & V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx \\
 (d) \quad & V = \int_0^{2\pi} \int_0^{\sqrt{3}} \left[r(4-r^2)^{1/2} - r \right] dr \, d\theta = \int_0^{2\pi} \left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^2}{2} \right]_0^{\sqrt{3}} d\theta = \int_0^{2\pi} \left(-\frac{1}{3} - \frac{3}{2} + \frac{4^{3/2}}{3} \right) d\theta \\
 & = \frac{5}{6} \int_0^{2\pi} d\theta = \frac{5\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 42. \quad (a) \quad & I_z = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} r^2 \, dz \, r \, dr \, d\theta \\
 (b) \quad & I_z = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho^2 \sin^2\phi) (\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta, \text{ since } r^2 = x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi \\
 (c) \quad & I_z = \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{5} \sin^3\phi \, d\phi \, d\theta = \frac{1}{5} \int_0^{2\pi} \left(\left[-\frac{\sin^2\phi \cos\phi}{3} \right]_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \sin\phi \, d\phi \right) d\theta = \frac{2}{15} \int_0^{2\pi} [-\cos\phi]_0^{\pi/2} d\theta \\
 & = \frac{2}{15} (2\pi) = \frac{4\pi}{15}
 \end{aligned}$$

$$43. \quad V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^4-1}^{4-4r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 (5r - 4r^3 - r^5) \, dr \, d\theta = 4 \int_0^{\pi/2} \left(\frac{5}{2} - 1 - \frac{1}{6} \right) d\theta = 4 \int_0^{\pi/2} d\theta = \frac{8\pi}{3}$$

$$\begin{aligned}
 44. \quad V &= 4 \int_0^{\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{1-r} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(r - r^2 + r\sqrt{1-r^2} \right) dr \, d\theta = 4 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3} (1-r^2)^{3/2} \right]_0^1 d\theta \\
 &= 4 \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} \right) d\theta = 2 \int_0^{\pi/2} d\theta = 2 \left(\frac{\pi}{2} \right) = \pi
 \end{aligned}$$

$$\begin{aligned}
 45. \quad V &= \int_{3\pi/2}^{2\pi} \int_0^{3\cos\theta} \int_0^{-r\sin\theta} dz \, r \, dr \, d\theta = \int_{3\pi/2}^{2\pi} \int_0^{3\cos\theta} -r^2 \sin\theta \, dr \, d\theta = \int_{3\pi/2}^{2\pi} (-9\cos^3\theta) (\sin\theta) \, d\theta = \left[\frac{9}{4} \cos^4\theta \right]_{3\pi/2}^{2\pi} \\
 &= \frac{9}{4} - 0 = \frac{9}{4}
 \end{aligned}$$

$$\begin{aligned}
 46. \quad V &= 2 \int_{\pi/2}^{\pi} \int_0^{-3\cos\theta} \int_0^r dz \, r \, dr \, d\theta = 2 \int_{\pi/2}^{\pi} \int_0^{-3\cos\theta} r^2 \, dr \, d\theta = \frac{2}{3} \int_{\pi/2}^{\pi} -27\cos^3\theta \, d\theta \\
 &= -18 \left(\left[\frac{\cos^2\theta \sin\theta}{3} \right]_{\pi/2}^{\pi} + \frac{2}{3} \int_{\pi/2}^{\pi} \cos\theta \, d\theta \right) = -12 [\sin\theta]_{\pi/2}^{\pi} = 12
 \end{aligned}$$

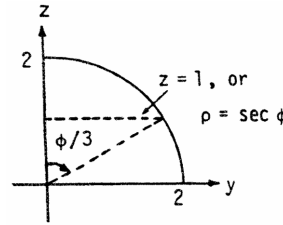
$$\begin{aligned}
 47. \quad V &= \int_0^{\pi/2} \int_0^{\sin \theta} \int_0^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^{\sin \theta} r\sqrt{1-r^2} \, dr \, d\theta = \int_0^{\pi/2} \left[-\frac{1}{3}(1-r^2)^{3/2} \right]_0^{\sin \theta} d\theta \\
 &= -\frac{1}{3} \int_0^{\pi/2} \left[(1-\sin^2 \theta)^{3/2} - 1 \right] d\theta = -\frac{1}{3} \int_0^{\pi/2} (\cos^3 \theta - 1) \, d\theta = -\frac{1}{3} \left(\left[\frac{\cos^2 \theta \sin \theta}{3} \right]_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \cos \theta \, d\theta \right) + \left[\frac{\theta}{3} \right]_0^{\pi/2} \\
 &= -\frac{2}{9} [\sin \theta]_0^{\pi/2} + \frac{\pi}{6} = \frac{-4+3\pi}{18}
 \end{aligned}$$

$$\begin{aligned}
 48. \quad V &= \int_0^{\pi/2} \int_0^{\cos \theta} \int_0^{3\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^{\cos \theta} 3r\sqrt{1-r^2} \, dr \, d\theta = \int_0^{\pi/2} \left[-(1-r^2)^{3/2} \right]_0^{\cos \theta} d\theta \\
 &= \int_0^{\pi/2} \left[-(1-\cos^2 \theta)^{3/2} + 1 \right] d\theta = \int_0^{\pi/2} (1-\sin^3 \theta) \, d\theta = \left[\theta + \frac{\sin^2 \theta \cos \theta}{3} \right]_0^{\pi/2} - \frac{2}{3} \int_0^{\pi/2} \sin \theta \, d\theta \\
 &= \frac{\pi}{2} + \frac{2}{3} [\cos \theta]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3} = \frac{3\pi-4}{6}
 \end{aligned}$$

$$49. \quad V = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \frac{a^3}{3} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} [-\cos \phi]_{\pi/3}^{2\pi/3} d\theta = \frac{a^3}{3} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \right) d\theta = \frac{2\pi a^3}{3}$$

$$50. \quad V = \int_0^{\pi/6} \int_0^{\pi/2} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{\pi/6} \int_0^{\pi/2} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{\pi/6} d\theta = \frac{a^3 \pi}{18}$$

$$\begin{aligned}
 51. \quad V &= \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} (8 \sin \phi - \tan \phi \sec^2 \phi) \, d\phi \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \left[-8 \cos \phi - \frac{1}{2} \tan^2 \phi \right]_0^{\pi/3} d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \left[-4 - \frac{1}{2}(3) + 8 \right] d\theta = \frac{1}{3} \int_0^{2\pi} \frac{5}{2} d\theta = \frac{5}{6} (2\pi) = \frac{5\pi}{3}
 \end{aligned}$$



$$\begin{aligned}
 52. \quad V &= 4 \int_0^{\pi/2} \int_0^{\pi/4} \int_{\sec \phi}^{2 \sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{4}{3} \int_0^{\pi/2} \int_0^{\pi/4} (8 \sec^3 \phi - \sec^3 \phi) \sin \phi \, d\phi \, d\theta \\
 &= \frac{28}{3} \int_0^{\pi/2} \int_0^{\pi/4} \sec^3 \phi \sin \phi \, d\phi \, d\theta = \frac{28}{3} \int_0^{\pi/2} \int_0^{\pi/4} \tan \phi \sec^2 \phi \, d\phi \, d\theta = \frac{28}{3} \int_0^{\pi/2} \left[\frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} d\theta = \frac{14}{3} \int_0^{\pi/2} d\theta = \frac{7\pi}{3}
 \end{aligned}$$

$$53. \quad V = 4 \int_0^{\pi/2} \int_0^1 \int_0^{r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r^3 \, dr \, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

$$54. \quad V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r^2+1} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r \, dr \, d\theta = 2 \int_0^{\pi/2} d\theta = \pi$$

$$55. \quad V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^r dz \, r \, dr \, d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2 \, dr \, d\theta = 8 \left(\frac{2\sqrt{2}-1}{3} \right) \int_0^{\pi/2} d\theta = \frac{4\pi(2\sqrt{2}-1)}{3}$$

$$56. \quad V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r\sqrt{2-r^2} \, dr \, d\theta = 8 \int_0^{\pi/2} \left[-\frac{1}{3}(2-r^2)^{3/2} \right]_1^{\sqrt{2}} d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}$$

$$57. \quad V = \int_0^{2\pi} \int_0^2 \int_0^{4-r \sin \theta} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (4r - r^2 \sin \theta) \, dr \, d\theta = 8 \int_0^{2\pi} \left(1 - \frac{\sin \theta}{3} \right) d\theta = 16\pi$$

$$58. \quad V = \int_0^{2\pi} \int_0^2 \int_0^{4-r \cos \theta - r \sin \theta} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 [4r - r^2 (\cos \theta + \sin \theta)] \, dr \, d\theta = \frac{8}{3} \int_0^{2\pi} (3 - \cos \theta - \sin \theta) \, d\theta = 16\pi$$

$$\begin{aligned}
 59. \quad &\text{The paraboloids intersect when } 4x^2 + 4y^2 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1 \text{ and } z = 4 \\
 \Rightarrow \quad &V = 4 \int_0^{\pi/2} \int_0^1 \int_{4r^2}^{5-r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 (5r - 5r^3) \, dr \, d\theta = 20 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta = 5 \int_0^{\pi/2} d\theta = \frac{5\pi}{2}
 \end{aligned}$$

$$60. \text{ The paraboloid intersects the } xy\text{-plane when } 9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9 \Rightarrow V = 4 \int_0^{\pi/2} \int_1^3 \int_0^{9-r^2} dz \, r \, dr \, d\theta \\ = 4 \int_0^{\pi/2} \int_1^3 (9r - r^3) \, dr \, d\theta = 4 \int_0^{\pi/2} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_1^3 d\theta = 4 \int_0^{\pi/2} \left(\frac{81}{4} - \frac{17}{4} \right) d\theta = 64 \int_0^{\pi/2} d\theta = 32\pi$$

$$61. V = 8 \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta = 8 \int_0^{2\pi} \int_0^1 r(4-r^2)^{1/2} \, dr \, d\theta = 8 \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{3/2} \right]_0^1 d\theta \\ = -\frac{8}{3} \int_0^{2\pi} (3^{3/2} - 8) \, d\theta = \frac{4\pi(8-3\sqrt{3})}{3}$$

$$62. \text{ The sphere and paraboloid intersect when } x^2 + y^2 + z^2 = 2 \text{ and } z = x^2 + y^2 \Rightarrow z^2 + z - 2 = 0 \\ \Rightarrow (z+2)(z-1) = 0 \Rightarrow z = 1 \text{ or } z = -2 \Rightarrow z = 1 \text{ since } z \geq 0. \text{ Thus, } x^2 + y^2 = 1 \text{ and the volume is} \\ \text{given by the triple integral } V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 [r(2-r^2)^{1/2} - r^3] \, dr \, d\theta \\ = 4 \int_0^{\pi/2} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^4}{4} \right]_0^1 d\theta = 4 \int_0^{\pi/2} \left(\frac{2\sqrt{2}}{3} - \frac{7}{12} \right) d\theta = \frac{\pi(8\sqrt{2}-7)}{6}$$

$$63. \text{ average} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2 \, dz \, dr \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 2r^2 \, dr \, d\theta = \frac{1}{3\pi} \int_0^{2\pi} d\theta = \frac{2}{3}$$

$$64. \text{ average} = \frac{1}{(\frac{4\pi}{3})} \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 \, dz \, dr \, d\theta = \frac{3}{4\pi} \int_0^{2\pi} \int_0^1 2r^2 \sqrt{1-r^2} \, dr \, d\theta \\ = \frac{3}{2\pi} \int_0^{2\pi} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r \sqrt{1-r^2} (1-2r^2) \right]_0^1 d\theta = \frac{3}{16\pi} \int_0^{2\pi} \left(\frac{\pi}{2} + 0 \right) d\theta = \frac{3}{32} \int_0^{2\pi} d\theta = \left(\frac{3}{32} \right) (2\pi) = \frac{3\pi}{16}$$

$$65. \text{ average} = \frac{1}{(\frac{4\pi}{3})} \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{3}{16\pi} \int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta = \frac{3}{8\pi} \int_0^{2\pi} d\theta = \frac{3}{4}$$

$$66. \text{ average} = \frac{1}{(\frac{2\pi}{3})} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} d\theta \\ = \frac{3}{16\pi} \int_0^{2\pi} d\theta = \left(\frac{3}{16\pi} \right) (2\pi) = \frac{3}{8}$$

$$67. M = 4 \int_0^{\pi/2} \int_0^1 \int_0^r dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r^2 \, dr \, d\theta = \frac{4}{3} \int_0^{\pi/2} d\theta = \frac{2\pi}{3}; M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^r z \, dz \, r \, dr \, d\theta \\ = \frac{1}{2} \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4} \right) \left(\frac{3}{2\pi} \right) = \frac{3}{8}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}$$

$$68. M = \int_0^{\pi/2} \int_0^2 \int_0^r dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 r^2 \, dr \, d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}; M_{yz} = \int_0^{\pi/2} \int_0^2 \int_0^r x \, dz \, r \, dr \, d\theta \\ = \int_0^{\pi/2} \int_0^2 r^3 \cos \theta \, dr \, d\theta = 4 \int_0^{\pi/2} \cos \theta \, d\theta = 4; M_{xz} = \int_0^{\pi/2} \int_0^2 \int_0^r y \, dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 r^3 \sin \theta \, dr \, d\theta \\ = 4 \int_0^{\pi/2} \sin \theta \, d\theta = 4; M_{xy} = \int_0^{\pi/2} \int_0^2 \int_0^r z \, dz \, r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/2} \int_0^2 r^3 \, dr \, d\theta = 2 \int_0^{\pi/2} d\theta = \pi \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}, \\ \bar{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{3}{4}$$

$$69. M = \frac{8\pi}{3}; M_{xy} = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 z \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \rho^3 \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta = 4 \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta \\ = 4 \int_0^{2\pi} \left[\frac{\sin^2 \phi}{2} \right]_{\pi/3}^{\pi/2} d\theta = 4 \int_0^{2\pi} \left(\frac{1}{2} - \frac{3}{8} \right) d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi \Rightarrow \bar{z} = \frac{M_{xy}}{M} = (\pi) \left(\frac{3}{8\pi} \right) = \frac{3}{8}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}$$

$$70. M = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \frac{2-\sqrt{2}}{2} d\theta = \frac{\pi a^3 (2-\sqrt{2})}{3}; \\ M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^3 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta = \frac{a^4}{4} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \cos \phi \, d\phi \, d\theta = \frac{a^4}{16} \int_0^{2\pi} d\theta = \frac{\pi a^4}{8}$$

$$\Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi a^4}{8}\right) \left[\frac{3}{\pi a^3 (2-\sqrt{2})}\right] = \left(\frac{3a}{8}\right) \left(\frac{2+\sqrt{2}}{2}\right) = \frac{3(2+\sqrt{2})a}{16}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}$$

$$\begin{aligned} 71. M &= \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^4 r^{3/2} \, dr \, d\theta = \frac{64}{5} \int_0^{2\pi} d\theta = \frac{128\pi}{5}; M_{xy} = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} z \, dz \, r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^4 r^2 \, dr \, d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{5}{6}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry} \end{aligned}$$

$$\begin{aligned} 72. M &= \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \int_0^1 2r\sqrt{1-r^2} \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \left[-\frac{2}{3}(1-r^2)^{3/2}\right]_0^1 d\theta \\ &= \frac{2}{3} \int_{-\pi/3}^{\pi/3} d\theta = \left(\frac{2}{3}\right) \left(\frac{2\pi}{3}\right) = \frac{4\pi}{9}; M_{yz} = \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 \cos \theta \, dz \, r \, dr \, d\theta = 2 \int_{-\pi/3}^{\pi/3} \int_0^1 r^2 \sqrt{1-r^2} \cos \theta \, dr \, d\theta \\ &= 2 \int_{-\pi/3}^{\pi/3} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r \sqrt{1-r^2} (1-2r^2)\right]_0^1 \cos \theta \, d\theta = \frac{\pi}{8} \int_{-\pi/3}^{\pi/3} \cos \theta \, d\theta = \frac{\pi}{8} [\sin \theta]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{8}\right) \left(2 \cdot \frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{8} \\ &\Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{9\sqrt{3}}{32}, \text{ and } \bar{y} = \bar{z} = 0, \text{ by symmetry} \end{aligned}$$

73. We orient the cone with its vertex at the origin and axis along the z-axis $\Rightarrow \phi = \frac{\pi}{4}$. We use the the x-axis

$$\begin{aligned} \text{which is through the vertex and parallel to the base of the cone} &\Rightarrow I_x = \int_0^{2\pi} \int_0^1 \int_r^1 (r^2 \sin^2 \theta + z^2) \, dz \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r^3 \sin^2 \theta - r^4 \sin^2 \theta + \frac{r}{3} - \frac{r^4}{3}\right) \, dr \, d\theta = \int_0^{2\pi} \left(\frac{\sin^2 \theta}{20} + \frac{1}{10}\right) \, d\theta = \left[\frac{\theta}{40} - \frac{\sin 2\theta}{80} + \frac{\theta}{10}\right]_0^{2\pi} = \frac{\pi}{20} + \frac{\pi}{5} = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} 74. I_z &= \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a 2r^3 \sqrt{a^2-r^2} \, dr \, d\theta = 2 \int_0^{2\pi} \left[\left(-\frac{r^2}{5} - \frac{2a^2}{15}\right)(a^2-r^2)^{3/2}\right]_0^a d\theta = 2 \int_0^{2\pi} \frac{2}{15} a^5 \, d\theta \\ &= \frac{8\pi a^5}{15} \end{aligned}$$

$$\begin{aligned} 75. I_z &= \int_0^{2\pi} \int_0^a \int_{\frac{h}{a}r}^h (x^2 + y^2) \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \int_{\frac{h}{a}r}^h r^3 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \left(hr^3 - \frac{hr^4}{a}\right) \, dr \, d\theta = \int_0^{2\pi} h \left[\frac{r^4}{4} - \frac{r^5}{5a}\right]_0^a d\theta \\ &= \int_0^{2\pi} h \left(\frac{a^4}{4} - \frac{a^5}{5a}\right) \, d\theta = \frac{ha^4}{20} \int_0^{2\pi} d\theta = \frac{\pi ha^4}{10} \end{aligned}$$

$$\begin{aligned} 76. (a) M &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} z \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{2} r^5 \, dr \, d\theta = \frac{1}{12} \int_0^{2\pi} d\theta = \frac{\pi}{6}; M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^{r^2} z^2 \, dz \, r \, dr \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^1 r^7 \, dr \, d\theta = \frac{1}{24} \int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow \bar{z} = \frac{1}{2}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}; I_z = \int_0^{2\pi} \int_0^1 \int_0^{r^2} zr^3 \, dz \, r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^7 \, dr \, d\theta = \frac{1}{16} \int_0^{2\pi} d\theta = \frac{\pi}{8} \end{aligned}$$

$$\begin{aligned} (b) M &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} r^2 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r^4 \, dr \, d\theta = \frac{1}{5} \int_0^{2\pi} d\theta = \frac{2\pi}{5}; M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^{r^2} zr^2 \, dz \, r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^6 \, dr \, d\theta = \frac{1}{14} \int_0^{2\pi} d\theta = \frac{\pi}{7} \Rightarrow \bar{z} = \frac{5}{14}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}; I_z = \int_0^{2\pi} \int_0^1 \int_0^{r^2} r^4 \, dz \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 r^6 \, dr \, d\theta = \frac{1}{7} \int_0^{2\pi} d\theta = \frac{2\pi}{7} \end{aligned}$$

$$\begin{aligned} 77. (a) M &= \int_0^{2\pi} \int_0^1 \int_r^1 z \, dz \, r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 (r-r^3) \, dr \, d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4}; M_{xy} = \int_0^{2\pi} \int_0^1 \int_r^1 z^2 \, dz \, r \, dr \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^1 (r-r^4) \, dr \, d\theta = \frac{1}{10} \int_0^{2\pi} d\theta = \frac{\pi}{5} \Rightarrow \bar{z} = \frac{4}{5}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}; I_z = \int_0^{2\pi} \int_0^1 \int_r^1 zr^3 \, dz \, r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^1 (r^3-r^5) \, dr \, d\theta = \frac{1}{24} \int_0^{2\pi} d\theta = \frac{\pi}{12} \end{aligned}$$

$$\begin{aligned} (b) M &= \int_0^{2\pi} \int_0^1 \int_r^1 z^2 \, dz \, r \, dr \, d\theta = \frac{\pi}{5} \text{ from part (a)}; M_{xy} = \int_0^{2\pi} \int_0^1 \int_r^1 z^3 \, dz \, r \, dr \, d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^1 (r-r^5) \, dr \, d\theta \\ &= \frac{1}{12} \int_0^{2\pi} d\theta = \frac{\pi}{6} \Rightarrow \bar{z} = \frac{5}{6}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}; I_z = \int_0^{2\pi} \int_0^1 \int_r^1 z^2 r^3 \, dz \, r \, dr \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^1 (r^3-r^6) \, dr \, d\theta \\ &= \frac{1}{28} \int_0^{2\pi} d\theta = \frac{\pi}{14} \end{aligned}$$

$$78. (a) \quad M = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^4 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{a^5}{5} \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta = \frac{2a^5}{5} \int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}; \quad I_z = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^6 \sin^3 \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{a^7}{7} \int_0^{2\pi} \int_0^\pi (1 - \cos^2 \phi) \sin \phi \, d\phi \, d\theta = \frac{a^7}{7} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^\pi d\theta = \frac{4a^7}{21} \int_0^{2\pi} d\theta = \frac{8a^7\pi}{21}$$

$$(b) \quad M = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^3 \sin^2 \phi \, d\rho \, d\phi \, d\theta = \frac{a^4}{4} \int_0^{2\pi} \int_0^\pi \frac{(1 - \cos 2\phi)}{2} \, d\phi \, d\theta = \frac{\pi a^4}{8} \int_0^{2\pi} d\theta = \frac{\pi^2 a^4}{4};$$

$$I_z = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^5 \sin^4 \phi \, d\rho \, d\phi \, d\theta = \frac{a^6}{6} \int_0^{2\pi} \int_0^\pi \sin^4 \phi \, d\phi \, d\theta$$

$$= \frac{a^6}{6} \int_0^{2\pi} \left(\left[-\frac{\sin^3 \phi \cos \phi}{4} \right]_0^\pi + \frac{3}{4} \int_0^\pi \sin^2 \phi \, d\phi \right) d\theta = \frac{a^6}{8} \int_0^{2\pi} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right]_0^\pi d\theta = \frac{\pi a^6}{16} \int_0^{2\pi} d\theta = \frac{a^6 \pi^2}{8}$$

$$79. \quad M = \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2 - r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \frac{h}{a} r \sqrt{a^2 - r^2} \, dr \, d\theta = \frac{h}{a} \int_0^{2\pi} \left[-\frac{1}{3} (a^2 - r^2)^{3/2} \right]_0^a d\theta$$

$$= \frac{h}{a} \int_0^{2\pi} \frac{a^3}{3} d\theta = \frac{2\pi a^2 h}{3}; \quad M_{xy} = \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2 - r^2}} z \, dz \, r \, dr \, d\theta = \frac{h^2}{2a^2} \int_0^{2\pi} \int_0^a (a^2 r - r^3) \, dr \, d\theta$$

$$= \frac{h^2}{2a^2} \int_0^{2\pi} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) d\theta = \frac{a^2 h^2 \pi}{4} \Rightarrow \bar{z} = \left(\frac{\pi a^2 h^2}{4} \right) \left(\frac{3}{2\pi a^2 \pi} \right) = \frac{3}{8} h, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}$$

80. Let the base radius of the cone be a and the height h , and place the cone's axis of symmetry along the z -axis with the vertex at the origin. Then $M = \frac{\pi a^2 h}{3}$ and $M_{xy} = \int_0^{2\pi} \int_0^a \int_{\frac{h}{a}r}^h z \, dz \, r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^a \left(h^2 r - \frac{h^2}{a^2} r^3 \right) dr \, d\theta$

$$= \frac{h^2}{2} \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4a^2} \right]_0^a d\theta = \frac{h^2}{2} \int_0^{2\pi} \left(\frac{a^2}{2} - \frac{a^2}{4} \right) d\theta = \frac{h^2 a^2}{8} \int_0^{2\pi} d\theta = \frac{h^2 a^2 \pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{h^2 a^2 \pi}{4} \right) \left(\frac{3}{\pi a^2 h} \right) = \frac{3}{4} h, \text{ and}$$

$\bar{x} = \bar{y} = 0$, by symmetry $\Rightarrow$ the centroid is one fourth of the way from the base to the vertex

81. The density distribution function is linear so it has the form $\delta(\rho) = k\rho + C$, where ρ is the distance from the center of the planet. Now, $\delta(R) = 0 \Rightarrow kR + C = 0$, and $\delta(\rho) = k\rho - kR$. It remains to determine the constant k :

$$M = \int_0^{2\pi} \int_0^\pi \int_0^R (k\rho - kR) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \left[k \frac{\rho^4}{4} - kR \frac{\rho^3}{3} \right]_0^R \sin \phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^\pi k \left(\frac{R^4}{4} - \frac{R^4}{3} \right) \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} -\frac{k}{12} R^4 [-\cos \phi]_0^\pi d\theta = \int_0^{2\pi} -\frac{k}{6} R^4 d\theta = -\frac{k\pi R^4}{3} \Rightarrow k = -\frac{3M}{\pi R^4}$$

$$\Rightarrow \delta(\rho) = -\frac{3M}{\pi R^4} \rho + \frac{3M}{\pi R^4} R. \text{ At the center of the planet } \rho = 0 \Rightarrow \delta(0) = \left(\frac{3M}{\pi R^4} \right) R = \frac{3M}{\pi R^3}.$$

82. The mass of the planet's atmosphere to an altitude h above the surface of the planet is the triple integral

$$M(h) = \int_0^{2\pi} \int_0^\pi \int_R^h \mu_0 e^{-c(\rho-R)} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_R^h \int_0^{2\pi} \int_0^\pi \mu_0 e^{-c(\rho-R)} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho$$

$$= \int_R^h \int_0^{2\pi} [\mu_0 e^{-c(\rho-R)} \rho^2 (-\cos \phi)]_0^\pi d\theta \, d\rho = 2 \int_R^h \int_0^{2\pi} \mu_0 e^{cR} e^{-c\rho} \rho^2 \, d\theta \, d\rho = 4\pi \mu_0 e^{cR} \int_R^h e^{-c\rho} \rho^2 \, d\rho$$

$$= 4\pi \mu_0 e^{cR} \left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3} \right]_R^h \quad (\text{by parts})$$

$$= 4\pi \mu_0 e^{cR} \left(-\frac{h^2 e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2 e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3} \right).$$

The mass of the planet's atmosphere is therefore $M = \lim_{h \rightarrow \infty} M(h) = 4\pi \mu_0 \left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3} \right).$

83. (a) A plane perpendicular to the x -axis has the form $x = a$ in rectangular coordinates $\Rightarrow r \cos \theta = a \Rightarrow r = \frac{a}{\cos \theta}$

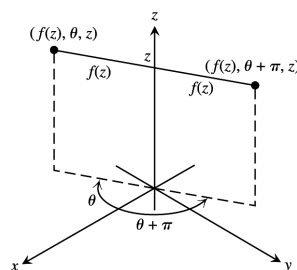
$\Rightarrow r = a \sec \theta$, in cylindrical coordinates.

(b) A plane perpendicular to the y -axis has the form $y = b$ in rectangular coordinates $\Rightarrow r \sin \theta = b \Rightarrow r = \frac{b}{\sin \theta}$

$\Rightarrow r = b \csc \theta$, in cylindrical coordinates.

84. $ax + by = c \Rightarrow a(r \cos \theta) + b(r \sin \theta) = c \Rightarrow r(a \cos \theta + b \sin \theta) = c \Rightarrow r = \frac{c}{a \cos \theta + b \sin \theta}.$

85. The equation $r = f(z)$ implies that the point (r, θ, z)
 $= (f(z), \theta, z)$ will lie on the surface for all θ . In particular
 $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ does
 $\Rightarrow$ the surface is symmetric with respect to the z -axis.



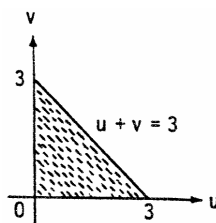
86. The equation $\rho = f(\phi)$ implies that the point $(\rho, \phi, \theta) = (f(\phi), \phi, \theta)$ lies on the surface for all θ . In particular, if
 $(f(\phi), \phi, \theta)$ lies on the surface, then $(f(\phi), \phi, \theta + \pi)$ lies on the surface, so the surface is symmetric with respect to the
 z -axis.

15.8 SUBSTITUTIONS IN MULTIPLE INTEGRALS

1. (a) $x - y = u$ and $2x + y = v \Rightarrow 3x = u + v$ and $y = x - u \Rightarrow x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(-2u + v)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

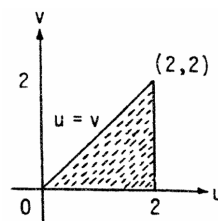
- (b) The line segment $y = x$ from $(0, 0)$ to $(1, 1)$ is $x - y = 0 \Rightarrow u = 0$; the line segment $y = -2x$ from $(0, 0)$ to $(1, -2)$ is $2x + y = 0 \Rightarrow v = 0$; the line segment $x = 1$ from $(1, 1)$ to $(1, -2)$ is $(x - y) + (2x + y) = 3 \Rightarrow u + v = 3$. The transformed region is sketched at the right.



2. (a) $x + 2y = u$ and $x - y = v \Rightarrow 3y = u - v$ and $x = v + y \Rightarrow y = \frac{1}{3}(u - v)$ and $x = \frac{1}{3}(u + 2v)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

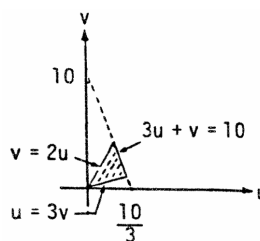
- (b) The triangular region in the xy -plane has vertices $(0, 0)$, $(2, 0)$, and $(\frac{2}{3}, \frac{2}{3})$. The line segment $y = x$ from $(0, 0)$ to $(\frac{2}{3}, \frac{2}{3})$ is $x - y = 0 \Rightarrow v = 0$; the line segment $y = 0$ from $(0, 0)$ to $(2, 0) \Rightarrow u = v$; the line segment $x + 2y = 2$ from $(\frac{2}{3}, \frac{2}{3})$ to $(2, 0) \Rightarrow u = 2$. The transformed region is sketched at the right.



3. (a) $3x + 2y = u$ and $x + 4y = v \Rightarrow -5x = -2u + v$ and $y = \frac{1}{2}(u - 3x) \Rightarrow x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(3v - u)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$$

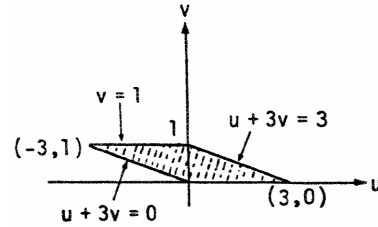
- (b) The x -axis $y = 0 \Rightarrow u = 3v$; the y -axis $x = 0 \Rightarrow v = 2u$; the line $x + y = 1 \Rightarrow \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1 \Rightarrow 2(2u - v) + (3v - u) = 10 \Rightarrow 3u + v = 10$. The transformed region is sketched at the right.



4. (a) $2x - 3y = u$ and $-x + y = v \Rightarrow -x = u + 3v$ and $y = v + x \Rightarrow x = -u - 3v$ and $y = -u - 2v$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1$$

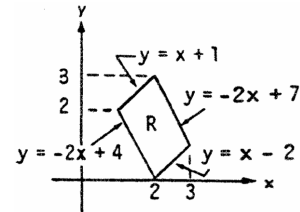
- (b) The line $x = -3 \Rightarrow -u - 3v = -3$ or $u + 3v = 3$;
 $x = 0 \Rightarrow u + 3v = 0$; $y = x \Rightarrow v = 0$; $y = x + 1 \Rightarrow v = 1$. The transformed region is the parallelogram sketched at the right.



$$\begin{aligned} 5. \int_0^4 \int_{y/2}^{(y/2)+1} (x - \frac{y}{2}) dx dy &= \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} \right]_{y/2}^{(y/2)+1} dy = \frac{1}{2} \int_0^4 \left[\left(\frac{y}{2} + 1 \right)^2 - \left(\frac{y}{2} \right)^2 - \left(\frac{y}{2} + 1 \right)y + \left(\frac{y}{2} \right)y \right] dy \\ &= \frac{1}{2} \int_0^4 (y + 1 - y) dy = \frac{1}{2} \int_0^4 dy = \frac{1}{2} (4) = 2 \end{aligned}$$

$$\begin{aligned} 6. \iint_R (2x^2 - xy - y^2) dx dy &= \iint_R (x - y)(2x + y) dx dy \\ &= \iint_G uv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv = \frac{1}{3} \iint_G uv du dv; \end{aligned}$$

We find the boundaries of G from the boundaries of R , shown in the accompanying figure:

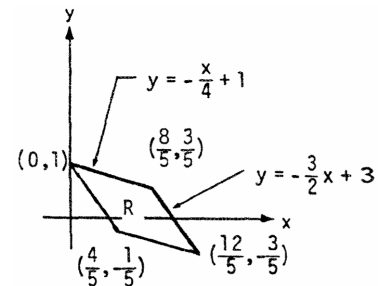


xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = -2x + 4$	$\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 4$	$v = 4$
$y = -2x + 7$	$\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 7$	$v = 7$
$y = x - 2$	$\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) - 2$	$u = 2$
$y = x + 1$	$\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) + 1$	$u = -1$

$$\Rightarrow \frac{1}{3} \iint_G uv du dv = \frac{1}{3} \int_{-1}^2 \int_4^7 uv dv du = \frac{1}{3} \int_{-1}^2 u \left[\frac{v^2}{2} \right]_4^7 du = \frac{11}{2} \int_{-1}^2 u du = \left(\frac{11}{2} \right) \left[\frac{u^2}{2} \right]_{-1}^2 = \left(\frac{11}{4} \right) (4 - 1) = \frac{33}{4}$$

$$\begin{aligned} 7. \iint_R (3x^2 + 14xy + 8y^2) dx dy &= \iint_R (3x + 2y)(x + 4y) dx dy \\ &= \iint_G uv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv = \frac{1}{10} \iint_G uv du dv; \end{aligned}$$

We find the boundaries of G from the boundaries of R , shown in the accompanying figure:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = -\frac{3}{2}x + 1$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 1$	$u = 2$
$y = -\frac{3}{2}x + 3$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 3$	$u = 6$
$y = -\frac{1}{4}x$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v)$	$v = 0$
$y = -\frac{1}{4}x + 1$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v) + 1$	$v = 4$

$$\Rightarrow \frac{1}{10} \iint_G uv \, du \, dv = \frac{1}{10} \int_2^6 \int_0^4 uv \, dv \, du = \frac{1}{10} \int_2^6 u \left[\frac{v^2}{2} \right]_0^4 du = \frac{4}{5} \int_2^6 u \, du = \left(\frac{4}{5} \right) \left[\frac{u^2}{2} \right]_2^6 = \left(\frac{4}{5} \right) (18 - 2) = \frac{64}{5}$$

8. $\iint_R 2(x - y) \, dx \, dy = \iint_G -2v \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \, dv = \iint_G -2v \, du \, dv$; the region G is sketched in Exercise 4

$$\Rightarrow \iint_G -2v \, du \, dv = \int_0^1 \int_{-3v}^{3-3v} -2v \, du \, dv = \int_0^1 -2v(3 - 3v + 3v) \, dv = \int_0^1 -6v \, dv = [-3v^2]_0^1 = -3$$

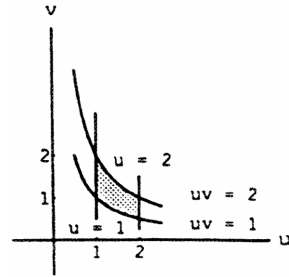
9. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 9 \Rightarrow u = 3$; thus

$$\begin{aligned} \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx \, dy &= \int_1^3 \int_1^2 (v + u) \left(\frac{2u}{v} \right) dv \, du = \int_1^3 \int_1^2 \left(2u + \frac{2u^2}{v} \right) dv \, du = \int_1^3 [2uv + 2u^2 \ln v]_1^2 du \\ &= \int_1^3 (2u + 2u^2 \ln 2) \, du = \left[u^2 + \frac{2}{3} u^2 \ln 2 \right]_1^3 = 8 + \frac{2}{3} (26)(\ln 2) = 8 + \frac{52}{3} (\ln 2) \end{aligned}$$

10. (a) $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$, and

the region G is sketched at the right



(b) $x = 1 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 2$; $y = 1 \Rightarrow uv = 1 \Rightarrow v = \frac{1}{u}$, and $y = 2 \Rightarrow uv = 2 \Rightarrow v = \frac{2}{u}$; thus,

$$\begin{aligned} \int_1^2 \int_{1/u}^{2/u} \frac{y}{x} \, dy \, dx &= \int_1^2 \int_{1/u}^{2/u} \left(\frac{uv}{u} \right) u \, dv \, du = \int_1^2 \int_{1/u}^{2/u} uv \, dv \, du = \int_1^2 u \left[\frac{v^2}{2} \right]_{1/u}^{2/u} du = \int_1^2 u \left(\frac{2}{u^2} - \frac{1}{2u^2} \right) du \\ &= \frac{3}{2} \int_1^2 u \left(\frac{1}{u^2} \right) du = \frac{3}{2} [\ln u]_1^2 = \frac{3}{2} \ln 2; \quad \int_1^2 \int_{1/u}^{2/u} \frac{y}{x} \, dy \, dx = \int_1^2 \left[\frac{1}{x} \cdot \frac{y^2}{2} \right]_{1/u}^{2/u} dx = \frac{3}{2} \int_1^2 \frac{dx}{x} = \frac{3}{2} [\ln x]_1^2 = \frac{3}{2} \ln 2 \end{aligned}$$

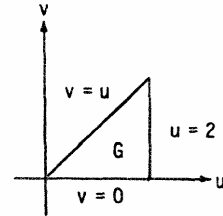
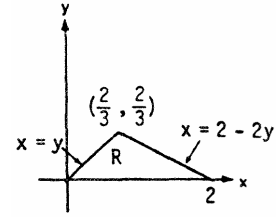
11. $x = a \cos \theta$ and $y = a \sin \theta \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = J(r,\theta) = \begin{vmatrix} a \cos \theta & -a \sin \theta \\ b \sin \theta & b \cos \theta \end{vmatrix} = abr \cos^2 \theta + abr \sin^2 \theta = abr$;

$$\begin{aligned} I_0 &= \iint_R (x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^1 r^2 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) |J(r,\theta)| \, dr \, d\theta = \int_0^{2\pi} \int_0^1 abr^3 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) \, dr \, d\theta \\ &= \frac{ab}{4} \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) \, d\theta = \frac{ab}{4} \left[\frac{a^2 \theta}{2} + \frac{a^2 \sin 2\theta}{4} + \frac{b^2 \theta}{2} - \frac{b^2 \sin 2\theta}{4} \right]_0^{2\pi} = \frac{ab\pi(a^2 + b^2)}{4} \end{aligned}$$

12. $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$; $A = \iint_R dy \, dx = \iint_G ab \, du \, dv = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab \, dv \, du$

$$= 2ab \int_{-1}^1 \sqrt{1-u^2} \, du = 2ab \left[\frac{u}{2} \sqrt{1-u^2} + \frac{1}{2} \sin^{-1} u \right]_{-1}^1 = ab [\sin^{-1} 1 - \sin^{-1} (-1)] = ab \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = ab\pi$$

13. The region of integration R in the xy -plane is sketched in the figure at the right. The boundaries of the image G are obtained as follows, with G sketched at the right:



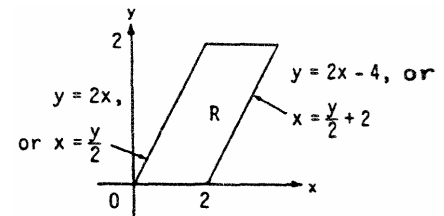
xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y$	$\frac{1}{3}(u + 2v) = \frac{1}{3}(u - v)$	$v = 0$
$x = 2 - 2y$	$\frac{1}{3}(u + 2v) = 2 - \frac{2}{3}(u - v)$	$u = 2$
$y = 0$	$0 = \frac{1}{3}(u - v)$	$v = u$

Also, from Exercise 2, $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = -\frac{1}{3} \Rightarrow \int_0^{2/3} \int_y^{2-2y} (x+2y)e^{(y-x)} dx dy = \int_0^2 \int_0^u ue^{-v} \left| -\frac{1}{3} \right| dv du$
 $= \frac{1}{3} \int_0^2 u [-e^{-v}]_0^u du = \frac{1}{3} \int_0^2 u (1 - e^{-u}) du = \frac{1}{3} \left[u(u + e^{-u}) - \frac{u^2}{2} + e^{-u} \right]_0^2 = \frac{1}{3} [2(2 + e^{-2}) - 2 + e^{-2} - 1]$
 $= \frac{1}{3} (3e^{-2} + 1) \approx 0.4687$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u + v) - v = 2u$ and

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1; \text{ next, } u = x - \frac{v}{2}$$

$= x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy -plane are transformed to the boundaries of G :



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3(2x-y)e^{(2x-y)^2} dx dy = \int_0^2 \int_0^2 v^3(2u)e^{4u^2} du dv = \int_0^2 v^3 \left[\frac{1}{4} e^{4u^2} \right]_0^2 dv = \frac{1}{4} \int_0^2 v^3 (e^{16} - 1) dv$$

$$= \frac{1}{4} (e^{16} - 1) \left[\frac{v^4}{4} \right]_0^2 = e^{16} - 1$$

15. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 4 \Rightarrow u = 2$; thus

$$\int_1^2 \int_{1/y}^y (x^2 + y^2) dx dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2) dx dy = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2 v^2 \right) \left(\frac{2u}{v} \right) du dv = \int_1^2 \int_1^2 \left(\frac{2u^3}{v^3} + 2u^3 v \right) du dv$$

$$= \int_1^2 \left[\frac{u^4}{2v^3} + \frac{1}{2}u^4v \right]_1^2 dv = \int_1^2 \left(\frac{15}{2v^3} + \frac{15v}{2} \right) dv = \left[-\frac{15}{4v^2} + \frac{15v^2}{4} \right]_1^2 = \frac{225}{16}$$

$$16. \quad x = u^2 - v^2 \text{ and } y = 2uv; \quad \frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = 4(u^2 + v^2);$$

$$y = 2\sqrt{1-x} \Rightarrow y^2 = 4(1-x) \Rightarrow (2uv)^2 = 4(1-(u^2-v^2)) \Rightarrow u = \pm 1; y = 0 \Rightarrow 2uv = 0 \Rightarrow u = 0 \text{ or } v = 0;$$

$x = 0 \Rightarrow u^2 - v^2 = 0 \Rightarrow u = v \text{ or } u = -v$; This gives us four triangular regions, but only the one in the quadrant where both u, v are positive maps into the region R in the xy -plane.

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x}} \sqrt{x^2 + y^2} \, dx \, dy &= \int_0^1 \int_0^u \sqrt{(u^2 - v^2)^2 + (2uv)^2} \cdot 4(u^2 + v^2) \, dv \, du = 4 \int_0^1 \int_0^u (u^2 + v^2)^2 \, dv \, du \\ &= 4 \int_1^2 \left[u^4v + \frac{2}{3}u^2v^3 + \frac{1}{5}v^5 \right]_0^u du = \frac{112}{15} \int_1^2 u^5 \, du = \frac{112}{15} \left[\frac{1}{6}u^6 \right]_1^2 = \frac{56}{45} \end{aligned}$$

$$17. \quad (a) \quad x = u \cos v \text{ and } y = u \sin v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} = u \cos^2 v + u \sin^2 v = u$$

$$(b) \quad x = u \sin v \text{ and } y = u \cos v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} = -u \sin^2 v - u \cos^2 v = -u$$

$$18. \quad (a) \quad x = u \cos v, y = u \sin v, z = w \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u \sin v & 0 \\ \sin v & u \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \cos^2 v + u \sin^2 v = u$$

$$(b) \quad x = 2u - 1, y = 3v - 4, z = \frac{1}{2}(w - 4) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = (2)(3)\left(\frac{1}{2}\right) = 3$$

$$\begin{aligned} 19. \quad & \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} \\ &= (\cos \phi) \begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} + (\rho \sin \phi) \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} \\ &= (\rho^2 \cos \phi) (\sin \phi \cos \phi \cos^2 \theta + \sin \phi \cos \phi \sin^2 \theta) + (\rho^2 \sin \phi) (\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta) \\ &= \rho^2 \sin \phi \cos^2 \phi + \rho^2 \sin^3 \phi = (\rho^2 \sin \phi) (\cos^2 \phi + \sin^2 \phi) = \rho^2 \sin \phi \end{aligned}$$

$$20. \quad \text{Let } u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x) \Rightarrow \int_a^b f(u) \, du = \int_{g(a)}^{g(b)} f(g(x))g'(x) \, dx \text{ in accordance with Theorem 7 in Section 5.6. Note that } g'(x) \text{ represents the Jacobian of the transformation } u = g(x) \text{ or } x = g^{-1}(u).$$

$$\begin{aligned} 21. \quad & \int_0^3 \int_0^4 \int_{y/2}^{1+(y/2)} \left(\frac{2x-y}{2} + \frac{z}{3} \right) \, dx \, dy \, dz = \int_0^3 \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} + \frac{xz}{3} \right]_{y/2}^{1+(y/2)} dy \, dz = \int_0^3 \int_0^4 \left[\frac{1}{2}(y+1) - \frac{y}{2} + \frac{z}{3} \right] dy \, dz \\ &= \int_0^3 \left[\frac{(y+1)^2}{4} - \frac{y^2}{4} + \frac{yz}{3} \right]_0^4 dz = \int_0^3 \left(\frac{9}{4} + \frac{4z}{3} - \frac{1}{4} \right) dz = \int_0^3 \left(2 + \frac{4z}{3} \right) dz = \left[2z + \frac{2z^2}{3} \right]_0^3 = 12 \end{aligned}$$

$$22. \quad J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc; \text{ the transformation takes the ellipsoid region } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \text{ in } xyz\text{-space}$$

into the spherical region $u^2 + v^2 + w^2 \leq 1$ in uvw -space (which has volume $V = \frac{4}{3}\pi$)

$$\Rightarrow V = \iiint_R dx \, dy \, dz = \iiint_G abc \, du \, dv \, dw = \frac{4\pi abc}{3}$$

$$\begin{aligned}
 23. J(u, v, w) &= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc; \text{ for } R \text{ and } G \text{ as in Exercise 22, } \int_R \int \int |xyz| \, dx \, dy \, dz \\
 &= \int_G \int \int a^2 b^2 c^2 uvw \, dw \, dv \, du = 8a^2 b^2 c^2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi) (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta \\
 &= \frac{4a^2 b^2 c^2}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin \theta \cos \theta \sin^3 \phi \cos \phi \, d\phi \, d\theta = \frac{a^2 b^2 c^2}{3} \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{a^2 b^2 c^2}{6}
 \end{aligned}$$

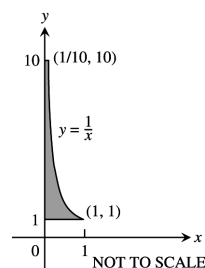
$$\begin{aligned}
 24. u = x, v = xy, \text{ and } w = 3z \Rightarrow x = u, y = \frac{v}{u}, \text{ and } z = \frac{1}{3} w \Rightarrow J(u, v, w) &= \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}; \\
 \int_D \int \int (x^2 y + 3xyz) \, dx \, dy \, dz &= \int_G \int \int [u^2 (\frac{v}{u}) + 3u (\frac{v}{u}) (\frac{w}{3})] |J(u, v, w)| \, du \, dv \, dw = \frac{1}{3} \int_0^3 \int_0^2 \int_1^2 (v + \frac{vw}{u}) \, du \, dv \, dw \\
 &= \frac{1}{3} \int_0^3 \int_0^2 (v + vw \ln 2) \, dv \, dw = \frac{1}{3} \int_0^3 (1 + w \ln 2) \left[\frac{v^2}{2} \right]_0^2 \, dw = \frac{2}{3} \int_0^3 (1 + w \ln 2) \, dw = \frac{2}{3} \left[w + \frac{w^2}{2} \ln 2 \right]_0^3 \\
 &= \frac{2}{3} \left(3 + \frac{9}{2} \ln 2 \right) = 2 + 3 \ln 2 = 2 + \ln 8
 \end{aligned}$$

$$\begin{aligned}
 25. \text{ The first moment about the } xy\text{-coordinate plane for the semi-ellipsoid, } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ using the} \\
 \text{transformation in Exercise 23 is, } M_{xy} &= \int_D \int \int z \, dz \, dy \, dx = \int_G \int \int cw |J(u, v, w)| \, du \, dv \, dw \\
 &= abc^2 \int_G \int \int w \, du \, dv \, dw = (abc^2) \cdot (M_{xy} \text{ of the hemisphere } x^2 + y^2 + z^2 = 1, z \geq 0) = \frac{abc^2 \pi}{4}; \\
 \text{the mass of the semi-ellipsoid is } \frac{2abc\pi}{3} \Rightarrow \bar{z} &= \left(\frac{abc^2 \pi}{4} \right) \left(\frac{3}{2abc\pi} \right) = \frac{3}{8} c
 \end{aligned}$$

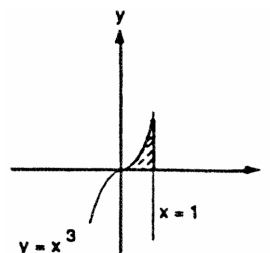
$$\begin{aligned}
 26. \text{ A solid of revolution is symmetric about the axis of revolution, therefore, the height of the solid is solely a function of } r. \\
 \text{That is, } y = f(x) = f(r). \text{ Using cylindrical coordinates with } x = r \cos \theta, y = y \text{ and } z = r \sin \theta, \text{ we have} \\
 V = \int_G \int \int r \, dy \, d\theta \, dr = \int_a^b \int_0^{2\pi} \int_0^{f(r)} r \, dy \, d\theta \, dr = \int_a^b \int_0^{2\pi} [r y]_0^{f(r)} \, d\theta \, dr = \int_a^b \int_0^{2\pi} r f(r) \, d\theta \, dr = \int_a^b [r\theta f(r)]_0^{2\pi} \, dr \\
 = \int_a^b 2\pi r f(r) \, dr. \text{ In the last integral, } r \text{ is a dummy or stand-in variable and as such it can be replaced by any variable name.} \\
 \text{Choosing } x \text{ instead of } r \text{ we have } V = \int_a^b 2\pi x f(x) \, dx, \text{ which is the same result obtained using the shell method.}
 \end{aligned}$$

CHAPTER 15 PRACTICE EXERCISES

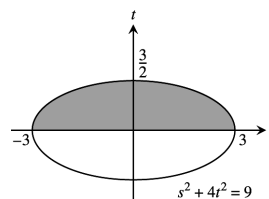
$$\begin{aligned}
 1. \int_1^{10} \int_0^{1/y} y e^{xy} \, dx \, dy &= \int_1^{10} [e^{xy}]_0^{1/y} \, dy \\
 &= \int_1^{10} (e - 1) \, dy = 9e - 9
 \end{aligned}$$



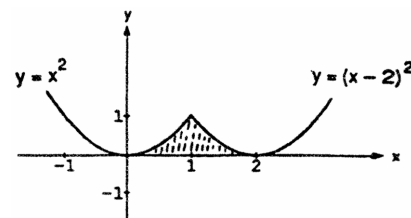
$$\begin{aligned}
 2. \int_0^1 \int_0^{x^3} e^{y/x} \, dy \, dx &= \int_0^1 x [e^{y/x}]_0^{x^3} \, dx \\
 &= \int_0^1 (x e^{x^2} - x) \, dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2} \right]_0^1 = \frac{e-2}{2}
 \end{aligned}$$



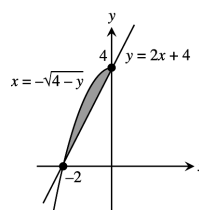
$$\begin{aligned}
 3. \quad \int_0^{3/2} \int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t \, ds \, dt &= \int_0^{3/2} [ts]_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} dt \\
 &= \int_0^{3/2} 2t\sqrt{9-4t^2} \, dt = \left[-\frac{1}{6}(9-4t^2)^{3/2} \right]_0^{3/2} \\
 &= -\frac{1}{6}(0^{3/2} - 9^{3/2}) = \frac{27}{6} = \frac{9}{2}
 \end{aligned}$$



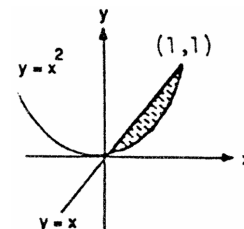
$$\begin{aligned}
 4. \quad \int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy \, dx \, dy &= \int_0^1 y \left[\frac{x^2}{2} \right]_{\sqrt{y}}^{2-\sqrt{y}} dy \\
 &= \frac{1}{2} \int_0^1 y(4 - 4\sqrt{y} + y - y) dy \\
 &= \int_0^1 (2y - 2y^{3/2}) dy = \left[y^2 - \frac{4y^{5/2}}{5} \right]_0^1 = \frac{1}{5}
 \end{aligned}$$



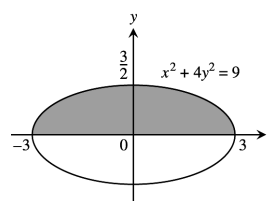
$$\begin{aligned}
 5. \quad \int_{-2}^0 \int_{2x+4}^{4-x^2} dy \, dx &= \int_{-2}^0 (-x^2 - 2x) \, dx \\
 &= \left[-\frac{x^3}{3} - x^2 \right]_{-2}^0 = -\left(\frac{8}{3} - 4 \right) = \frac{4}{3} \\
 \int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx \, dy &= \int_0^4 \left(\frac{y-4}{2} + \sqrt{4-y} \right) dy \\
 &= \left[\frac{y^2}{2} - 2y - \frac{2}{3}(4-y)^{3/2} \right]_0^4 = 4 - 8 + \frac{2}{3} \cdot 4^{3/2} \\
 &= -4 + \frac{16}{3} = \frac{4}{3}
 \end{aligned}$$



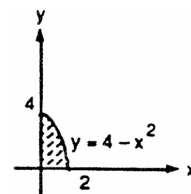
$$\begin{aligned}
 6. \quad \int_0^1 \int_y^{\sqrt{y}} \sqrt{x} \, dx \, dy &= \int_0^1 \left[\frac{2}{3} x^{3/2} \right]_y^{\sqrt{y}} dy \\
 &= \frac{2}{3} \int_0^1 (y^{3/4} - y^{3/2}) dy = \frac{2}{3} \left[\frac{4}{7} y^{7/4} - \frac{2}{5} y^{5/2} \right]_0^1 \\
 &= \frac{2}{3} \left(\frac{4}{7} - \frac{2}{5} \right) = \frac{4}{35} \\
 \int_0^1 \int_{x^2}^x \sqrt{x} \, dy \, dx &= \int_0^1 x^{1/2}(x - x^2) \, dx = \int_0^1 (x^{3/2} - x^{5/2}) \, dx \\
 &= \left[\frac{2}{5} x^{5/2} - \frac{2}{7} x^{7/2} \right]_0^1 = \frac{2}{5} - \frac{2}{7} = \frac{4}{35}
 \end{aligned}$$



$$\begin{aligned}
 7. \quad \int_{-3}^3 \int_0^{(1/2)\sqrt{9-x^2}} y \, dy \, dx &= \int_{-3}^3 \left[\frac{y^2}{2} \right]_0^{(1/2)\sqrt{9-x^2}} dx \\
 &= \int_{-3}^3 \frac{1}{8}(9-x^2) \, dx = \left[\frac{9x}{8} - \frac{x^3}{24} \right]_{-3}^3 \\
 &= \left(\frac{27}{8} - \frac{27}{24} \right) - \left(-\frac{27}{8} + \frac{27}{24} \right) = \frac{27}{6} = \frac{9}{2} \\
 \int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y \, dx \, dy &= \int_0^{3/2} 2y\sqrt{9-4y^2} \, dy \\
 &= -\frac{1}{4} \cdot \frac{2}{3}(9-4y^2)^{3/2} \Big|_0^{3/2} = \frac{1}{6} \cdot 9^{3/2} = \frac{27}{6} = \frac{9}{2}
 \end{aligned}$$



$$\begin{aligned}
 8. \quad \int_0^2 \int_0^{4-x^2} 2x \, dy \, dx &= \int_0^2 [2xy]_0^{4-x^2} dx \\
 &= \int_0^2 (2x(4-x^2)) \, dx = \int_0^2 (8x - 2x^3) \, dx \\
 &= \left[4x^2 - \frac{x^4}{2} \right]_0^2 = 16 - \frac{16}{2} = 8 \\
 \int_0^4 \int_0^{\sqrt{4-y}} 2x \, dx \, dy &= \int_0^4 [x^2]_0^{\sqrt{4-y}} dy \\
 &= \int_0^4 (4-y) \, dy = \left[4y - \frac{y^2}{2} \right]_0^4 = 16 - \frac{16}{2} = 8
 \end{aligned}$$



9. $\int_0^1 \int_{2y}^2 4 \cos(x^2) dx dy = \int_0^2 \int_0^{x/2} 4 \cos(x^2) dy dx = \int_0^2 2x \cos(x^2) dx = [\sin(x^2)]_0^2 = \sin 4$
10. $\int_0^2 \int_{y/2}^1 e^{x^2} dx dy = \int_0^1 \int_0^{2x} e^{x^2} dy dx = \int_0^1 2xe^{x^2} dx = [e^{x^2}]_0^1 = e - 1$
11. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4 + 1} dy dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4 + 1} dx dy = \frac{1}{4} \int_0^2 \frac{4y^3}{y^4 + 1} dy = \frac{\ln 17}{4}$
12. $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin(\pi x^2)}{x^2} dx dy = \int_0^1 \int_0^{x^3} \frac{2\pi \sin(\pi x^2)}{x^2} dy dx = \int_0^1 2\pi x \sin(\pi x^2) dx = [-\cos(\pi x^2)]_0^1 = -(-1) - (-1) = 2$
13. $A = \int_{-2}^0 \int_{2x+4}^{4-x^2} dy dx = \int_{-2}^0 (-x^2 - 2x) dx = \frac{4}{3}$
14. $A = \int_1^4 \int_{2-y}^{\sqrt{y}} dx dy = \int_1^4 (\sqrt{y} - 2 + y) dy = \frac{37}{6}$
15. $V = \int_0^1 \int_x^{2-x} (x^2 + y^2) dy dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx = \int_0^1 \left[2x^2 + \frac{(2-x)^3}{3} - \frac{7x^3}{3} \right] dx = \left[\frac{2x^3}{3} - \frac{(2-x)^4}{12} - \frac{7x^4}{12} \right]_0^1 = \left(\frac{2}{3} - \frac{1}{12} - \frac{7}{12} \right) + \frac{2^4}{12} = \frac{4}{3}$
16. $V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx = \int_{-3}^2 [x^2 y]_x^{6-x^2} dx = \int_{-3}^2 (6x^2 - x^4 - x^3) dx = \frac{125}{4}$
17. average value $= \int_0^1 \int_0^1 xy dy dx = \int_0^1 \left[\frac{xy^2}{2} \right]_0^1 dx = \int_0^1 \frac{x}{2} dx = \frac{1}{4}$
18. average value $= \frac{1}{(\frac{\pi}{4})} \int_0^1 \int_0^{\sqrt{1-x^2}} xy dy dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2} \right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_0^1 (x - x^3) dx = \frac{1}{2\pi}$
19. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy dx = \int_0^{2\pi} \int_0^1 \frac{2r}{(1+r^2)^2} dr d\theta = \int_0^{2\pi} \left[-\frac{1}{1+r^2} \right]_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$
20. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy = \int_0^{2\pi} \int_0^1 r \ln(r^2 + 1) dr d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{2} \ln u du d\theta = \frac{1}{2} \int_0^{2\pi} [u \ln u - u]_1^2 d\theta = \frac{1}{2} \int_0^{2\pi} (2 \ln 2 - 1) d\theta = [\ln(4) - 1] \pi$
21. $(x^2 + y^2)^2 - (x^2 - y^2) = 0 \Rightarrow r^4 - r^2 \cos 2\theta = 0 \Rightarrow r^2 = \cos 2\theta$ so the integral is $\int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} dr d\theta = \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2(1+r^2)} \right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta} \right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2 \cos^2 \theta} \right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{\sec^2 \theta}{2} \right) d\theta = \frac{1}{2} \left[\theta - \frac{\tan \theta}{2} \right]_{-\pi/4}^{\pi/4} = \frac{\pi-2}{4}$
22. (a) $\iint_R \frac{1}{(1+x^2+y^2)^2} dx dy = \int_0^{\pi/3} \int_0^{\sec \theta} \frac{r}{(1+r^2)^2} dr d\theta = \int_0^{\pi/3} \left[-\frac{1}{2(1+r^2)} \right]_0^{\sec \theta} d\theta = \int_0^{\pi/3} \left[\frac{1}{2} - \frac{1}{2(1+\sec^2 \theta)} \right] d\theta = \frac{1}{2} \int_0^{\pi/3} \frac{\sec^2 \theta}{1+\sec^2 \theta} d\theta; \left[\frac{u = \tan \theta}{du = \sec^2 \theta d\theta} \right] \rightarrow \frac{1}{2} \int_0^{\sqrt{3}} \frac{du}{2+u^2} = \frac{1}{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} \right]_0^{\sqrt{3}} = \frac{\sqrt{2}}{4} \tan^{-1} \sqrt{\frac{3}{2}}$
- (b) $\iint_R \frac{1}{(1+x^2+y^2)^2} dx dy = \int_0^{\pi/2} \int_0^{\infty} \frac{r}{(1+r^2)^2} dr d\theta = \int_0^{\pi/2} \lim_{b \rightarrow \infty} \left[-\frac{1}{2(1+r^2)} \right]_0^b d\theta = \int_0^{\pi/2} \lim_{b \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{2(1+b^2)} \right] d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}$

$$23. \int_0^\pi \int_0^\pi \int_0^\pi \cos(x+y+z) \, dx \, dy \, dz = \int_0^\pi \int_0^\pi [\sin(z+y+\pi) - \sin(z+y)] \, dy \, dz \\ = \int_0^\pi [-\cos(z+2\pi) + \cos(z+\pi) - \cos z + \cos(z+\pi)] \, dz = 0$$

$$24. \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)} \, dz \, dy \, dx = \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} e^{(x+y)} \, dy \, dx = \int_{\ln 6}^{\ln 7} e^x \, dx = 1$$

$$25. \int_0^1 \int_0^{x^2} \int_0^{x+y} (2x-y-z) \, dz \, dy \, dx = \int_0^1 \int_0^{x^2} \left(\frac{3x^2}{2} - \frac{3y^2}{2} \right) \, dy \, dx = \int_0^1 \left(\frac{3x^4}{2} - \frac{x^6}{2} \right) \, dx = \frac{8}{35}$$

$$26. \int_1^e \int_1^x \int_0^{\frac{2y}{z^3}} dy \, dz \, dx = \int_1^e \int_1^x \frac{1}{z} \, dz \, dx = \int_1^e \ln x \, dx = [x \ln x - x]_1^e = 1$$

$$27. V = 2 \int_0^{\pi/2} \int_{-\cos y}^0 \int_0^{-2x} dz \, dx \, dy = 2 \int_0^{\pi/2} \int_{-\cos y}^0 -2x \, dx \, dy = 2 \int_0^{\pi/2} \cos^2 y \, dy = 2 \left[\frac{y}{2} + \frac{\sin 2y}{4} \right]_0^{\pi/2} = \frac{\pi}{2}$$

$$28. V = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2} dz \, dy \, dx = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2) \, dy \, dx = 4 \int_0^2 (4-x^2)^{3/2} \, dx \\ = \left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24 \sin^{-1} \frac{x}{2} \right]_0^2 = 24 \sin^{-1} 1 = 12\pi$$

$$29. \text{average} = \frac{1}{3} \int_0^1 \int_0^3 \int_0^1 30xz\sqrt{x^2+y} \, dz \, dy \, dx = \frac{1}{3} \int_0^1 \int_0^3 15x\sqrt{x^2+y} \, dy \, dx = \frac{1}{3} \int_0^1 \int_0^1 15x\sqrt{x^2+y} \, dx \, dy \\ = \frac{1}{3} \int_0^1 \left[5(x^2+y)^{3/2} \right]_0^1 dy = \frac{1}{3} \int_0^1 [5(1+y)^{3/2} - 5y^{3/2}] \, dy = \frac{1}{3} [2(1+y)^{5/2} - 2y^{5/2}]_0^1 = \frac{1}{3} [2(4)^{5/2} - 2(3)^{5/2} - 2] \\ = \frac{1}{3} [2(31 - 3^{5/2})]$$

$$30. \text{average} = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^\pi \int_0^a \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{3a}{16\pi} \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta = \frac{3a}{8\pi} \int_0^{2\pi} d\theta = \frac{3a}{4}$$

$$31. (a) \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3 \, dz \, dx \, dy$$

$$(b) \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 3\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$(c) \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3 \, dz \, r \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^{\sqrt{2}} [r(4-r^2)^{1/2} - r^2] \, dr \, d\theta = 3 \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} \right]_0^{\sqrt{2}} d\theta \\ = \int_0^{2\pi} (-2^{3/2} - 2^{3/2} + 4^{3/2}) \, d\theta = (8 - 4\sqrt{2}) \int_0^{2\pi} d\theta = 2\pi(8 - 4\sqrt{2})$$

$$32. (a) \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^{r^2} 21r \cos \theta (r \sin \theta)^2 \, dz \, r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^{r^2} 21r^3 \cos \theta \sin^2 \theta \, dz \, r \, dr \, d\theta$$

$$(b) \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^{r^2} 21r^3 \cos \theta \sin^2 \theta \, dz \, r \, dr \, d\theta = 84 \int_0^{\pi/2} \int_0^1 r^6 \sin^2 \theta \cos \theta \, dr \, d\theta = 12 \int_0^{\pi/2} \sin^2 \theta \cos \theta \, d\theta = 4$$

$$33. (a) \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$(b) \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} (\sec \phi)(\sec \phi \tan \phi) \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} d\theta = \frac{1}{6} \int_0^{2\pi} d\theta = \frac{\pi}{3}$$

$$34. (a) \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} (6+4y) \, dz \, dy \, dx \quad (b) \int_0^{\pi/2} \int_0^1 \int_0^r (6+4r \sin \theta) \, dz \, r \, dr \, d\theta$$

$$(c) \int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{\csc \phi} (6+4\rho \sin \phi \sin \theta) (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta$$

$$(d) \int_0^{\pi/2} \int_0^1 \int_0^r (6+4r \sin \theta) \, dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3 \sin \theta) \, dr \, d\theta = \int_0^{\pi/2} [2r^3 + r^4 \sin \theta]_0^1 d\theta \\ = \int_0^{\pi/2} (2 + \sin \theta) \, d\theta = [2\theta - \cos \theta]_0^{\pi/2} = \pi + 1$$

$$35. \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 y x \, dz \, dy \, dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 y x \, dz \, dy \, dx$$

36. (a) Bounded on the top and bottom by the sphere $x^2 + y^2 + z^2 = 4$, on the right by the right circular cylinder $(x-1)^2 + y^2 = 1$, on the left by the plane $y = 0$

$$(b) \int_0^{\pi/2} \int_0^{2 \cos \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$$

$$37. (a) V = \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (r\sqrt{8-r^2} - 2r) \, dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(8-r^2)^{3/2} - r^2 \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}(4)^{3/2} - 4 + \frac{1}{3}(8)^{3/2} \right] d\theta = \int_0^{2\pi} \frac{4}{3} (-2 - 3 + 2\sqrt{8}) \, d\theta = \frac{4}{3} (4\sqrt{2} - 5) \int_0^{2\pi} d\theta = \frac{8\pi(4\sqrt{2}-5)}{3}$$

$$(b) V = \int_0^{2\pi} \int_0^{\pi/4} \int_{2 \sec \phi}^{\sqrt{8}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} (2\sqrt{2} \sin \phi - \sec^3 \phi \sin \phi) \, d\phi \, d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} (2\sqrt{2} \sin \phi - \tan \phi \sec^2 \phi) \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \left[-2\sqrt{2} \cos \phi - \frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} \left(-2 - \frac{1}{2} + 2\sqrt{2} \right) d\theta = \frac{8}{3} \int_0^{2\pi} \left(\frac{-5+4\sqrt{2}}{2} \right) d\theta = \frac{8\pi(4\sqrt{2}-5)}{3}$$

$$38. I_z = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 (\rho \sin \phi)^2 (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{32}{5} \int_0^{2\pi} \int_0^{\pi/3} (\sin \phi - \cos^2 \phi \sin \phi) \, d\phi \, d\theta = \frac{32}{5} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^{\pi/3} d\theta = \frac{8\pi}{3}$$

$$39. \text{ With the centers of the spheres at the origin, } I_z = \int_0^{2\pi} \int_0^{\pi} \int_a^b \delta (\rho \sin \phi)^2 (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta$$

$$= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi} \sin^3 \phi \, d\phi \, d\theta = \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi} (\sin \phi - \cos^2 \phi \sin \phi) \, d\phi \, d\theta$$

$$= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^{\pi} d\theta = \frac{4\delta(b^5 - a^5)}{15} \int_0^{2\pi} d\theta = \frac{8\pi\delta(b^5 - a^5)}{15}$$

$$40. I_z = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos \theta} (\rho \sin \phi)^2 (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos \theta} \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi} (1 - \cos \phi)^5 \sin^3 \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} (1 - \cos \phi)^6 (1 + \cos \phi) \sin \phi \, d\phi \, d\theta;$$

$$\left[\begin{array}{l} u = 1 - \cos \phi \\ du = \sin \phi \, d\phi \end{array} \right] \rightarrow \frac{1}{5} \int_0^{2\pi} \int_0^2 u^6 (2 - u) \, du \, d\theta = \frac{1}{5} \int_0^{2\pi} \left[\frac{2u^7}{7} - \frac{u^8}{8} \right]_0^2 d\theta = \frac{1}{5} \int_0^{2\pi} \left(\frac{1}{7} - \frac{1}{8} \right) 2^8 d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \frac{2^3 \cdot 2^5}{56} d\theta = \frac{32}{35} \int_0^{2\pi} d\theta = \frac{64\pi}{35}$$

$$41. M = \int_1^2 \int_{2/x}^2 dy \, dx = \int_1^2 \left(2 - \frac{2}{x} \right) dx = 2 - \ln 4; M_y = \int_1^2 \int_{2/x}^2 x \, dy \, dx = \int_1^2 x \left(2 - \frac{2}{x} \right) dx = 1;$$

$$M_x = \int_1^2 \int_{2/x}^2 y \, dy \, dx = \int_1^2 \left(2 - \frac{2}{x^2} \right) dx = 1 \Rightarrow \bar{x} = \bar{y} = \frac{1}{2 - \ln 4}$$

$$42. M = \int_0^4 \int_{-2y}^{2y-y^2} dx \, dy = \int_0^4 (4y - y^2) \, dy = \frac{32}{3}; M_x = \int_0^4 \int_{-2y}^{2y-y^2} y \, dx \, dy = \int_0^4 (4y^2 - y^3) \, dy = \left[\frac{4y^3}{3} - \frac{y^4}{4} \right]_0^4 = \frac{64}{3};$$

$$M_y = \int_0^4 \int_{-2y}^{2y-y^2} x \, dx \, dy = \int_0^4 \left[\frac{(2y-y^2)^2}{2} - 2y^2 \right] dy = \left[\frac{y^5}{10} - \frac{y^4}{2} \right]_0^4 = -\frac{128}{5} \Rightarrow \bar{x} = \frac{M_y}{M} = -\frac{12}{5} \text{ and } \bar{y} = \frac{M_x}{M} = 2$$

$$43. I_0 = \int_0^2 \int_{2x}^4 (x^2 + y^2) (3) \, dy \, dx = 3 \int_0^2 \left(4x^2 + \frac{64}{3} - \frac{14x^3}{3} \right) dx = 104$$

$$44. (a) I_0 = \int_{-2}^2 \int_{-1}^1 (x^2 + y^2) \, dy \, dx = \int_{-2}^2 \left(2x^2 + \frac{2}{3} \right) dx = \frac{40}{3}$$

$$(b) \quad I_x = \int_{-a}^a \int_{-b}^b y^2 \, dy \, dx = \int_{-a}^a \frac{2b^3}{3} \, dx = \frac{4ab^3}{3}; \quad I_y = \int_{-b}^b \int_{-a}^a x^2 \, dx \, dy = \int_{-b}^b \frac{2a^3}{3} \, dy = \frac{4a^3b}{3} \Rightarrow I_o = I_x + I_y \\ = \frac{4ab^3}{3} + \frac{4a^3b}{3} = \frac{4ab(b^2 + a^2)}{3}$$

$$45. \quad M = \delta \int_0^3 \int_0^{2x/3} dy \, dx = \delta \int_0^3 \frac{2x}{3} \, dx = 3\delta; \quad I_x = \delta \int_0^3 \int_0^{2x/3} y^2 \, dy \, dx = \frac{8\delta}{81} \int_0^3 x^3 \, dx = \left(\frac{8\delta}{81}\right) \left(\frac{3^4}{4}\right) = 2\delta$$

$$46. \quad M = \int_0^1 \int_{x^2}^x (x+1) \, dy \, dx = \int_0^1 (x - x^3) \, dx = \frac{1}{4}; \quad M_x = \int_0^1 \int_{x^2}^x y(x+1) \, dy \, dx = \frac{1}{2} \int_0^1 (x^3 - x^5 + x^2 - x^4) \, dx = \frac{13}{120}; \\ M_y = \int_0^1 \int_{x^2}^x x(x+1) \, dy \, dx = \int_0^1 (x^2 - x^4) \, dx = \frac{2}{15} \Rightarrow \bar{x} = \frac{8}{15} \text{ and } \bar{y} = \frac{13}{30}; \quad I_x = \int_0^1 \int_{x^2}^x y^2(x+1) \, dy \, dx \\ = \frac{1}{3} \int_0^1 (x^4 - x^7 + x^3 - x^6) \, dx = \frac{17}{280} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{17}{70}}; \quad I_y = \int_0^1 \int_{x^2}^x x^2(x+1) \, dy \, dx = \int_0^1 (x^3 - x^5) \, dx = \frac{1}{12}$$

$$47. \quad M = \int_{-1}^1 \int_{-1}^1 (x^2 + y^2 + \frac{1}{3}) \, dy \, dx = \int_{-1}^1 (2x^2 + \frac{4}{3}) \, dx = 4; \quad M_x = \int_{-1}^1 \int_{-1}^1 y(x^2 + y^2 + \frac{1}{3}) \, dy \, dx = \int_{-1}^1 0 \, dx = 0; \\ M_y = \int_{-1}^1 \int_{-1}^1 x(x^2 + y^2 + \frac{1}{3}) \, dy \, dx = \int_{-1}^1 (2x^3 + \frac{4}{3}x) \, dx = 0$$

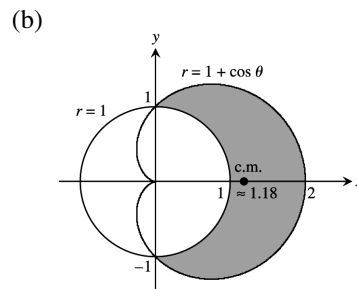
48. Place the $\triangle ABC$ with its vertices at $A(0, 0)$, $B(b, 0)$ and $C(a, h)$. The line through the points A and C is

$$y = \frac{h}{a}x; \text{ the line through the points } C \text{ and } B \text{ is } y = \frac{h}{a-b}(x-b). \text{ Thus, } M = \int_0^h \int_{ay/h}^{(a-b)y/h+b} \delta \, dx \, dy \\ = b\delta \int_0^h (1 - \frac{y}{h}) \, dy = \frac{\delta bh}{2}; \quad I_x = \int_0^h \int_{ay/h}^{(a-b)y/h+b} y^2 \delta \, dx \, dy = b\delta \int_0^h (y^2 - \frac{y^3}{h}) \, dy = \frac{\delta bh^3}{12}$$

$$49. \quad M = \int_{-\pi/3}^{\pi/3} \int_0^3 r \, dr \, d\theta = \frac{9}{2} \int_{-\pi/3}^{\pi/3} d\theta = 3\pi; \quad M_y = \int_{-\pi/3}^{\pi/3} \int_0^3 r^2 \cos \theta \, dr \, d\theta = 9 \int_{-\pi/3}^{\pi/3} \cos \theta \, d\theta = 9\sqrt{3} \Rightarrow \bar{x} = \frac{3\sqrt{3}}{\pi}, \\ \text{and } \bar{y} = 0 \text{ by symmetry}$$

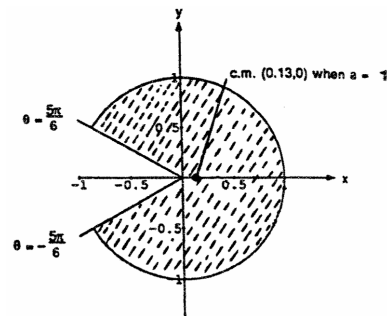
$$50. \quad M = \int_0^{\pi/2} \int_1^3 r \, dr \, d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi; \quad M_y = \int_0^{\pi/2} \int_1^3 r^2 \cos \theta \, dr \, d\theta = \frac{26}{3} \int_0^{\pi/2} \cos \theta \, d\theta = \frac{26}{3} \Rightarrow \bar{x} = \frac{13}{3\pi}, \text{ and } \\ \bar{y} = \frac{13}{3\pi} \text{ by symmetry}$$

$$51. (a) \quad M = 2 \int_0^{\pi/2} \int_1^{1+\cos \theta} r \, dr \, d\theta \\ = \int_0^{\pi/2} (2 \cos \theta + \frac{1+\cos 2\theta}{2}) \, d\theta = \frac{8+\pi}{4}; \\ M_y = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} (r \cos \theta) r \, dr \, d\theta \\ = \int_{-\pi/2}^{\pi/2} (\cos^2 \theta + \cos^3 \theta + \frac{\cos^4 \theta}{3}) \, d\theta \\ = \frac{32+15\pi}{24} \Rightarrow \bar{x} = \frac{15\pi+32}{6\pi+48}, \text{ and } \\ \bar{y} = 0 \text{ by symmetry}$$



$$52. (a) \quad M = \int_{-\alpha}^{\alpha} \int_0^a r \, dr \, d\theta = \int_{-\alpha}^{\alpha} \frac{a^2}{2} \, d\theta = a^2\alpha; \quad M_y = \int_{-\alpha}^{\alpha} \int_0^a (r \cos \theta) r \, dr \, d\theta = \int_{-\alpha}^{\alpha} \frac{a^3 \cos \theta}{3} \, d\theta = \frac{2a^3 \sin \alpha}{3} \\ \Rightarrow \bar{x} = \frac{2a \sin \alpha}{3\alpha}, \text{ and } \bar{y} = 0 \text{ by symmetry; } \lim_{\alpha \rightarrow \pi^-} \bar{x} = \lim_{\alpha \rightarrow \pi^-} \frac{2a \sin \alpha}{3\alpha} = 0$$

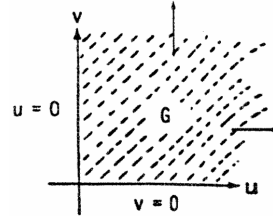
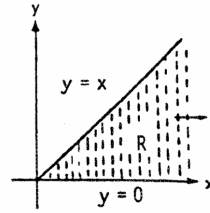
$$(b) \quad \bar{x} = \frac{2a}{5\pi} \text{ and } \bar{y} = 0$$



53. $x = u + y$ and $y = v \Rightarrow x = u + v$ and $y = v$

$$\Rightarrow J(u, v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1; \text{ the boundary of the}$$

image G is obtained from the boundary of R as follows:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = x$	$v = u + v$	$u = 0$
$y = 0$	$v = 0$	$v = 0$

$$\Rightarrow \int_0^\infty \int_0^x e^{-sx} f(x - y, y) dy dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u, v) du dv$$

54. If $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$ where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$, then $x = \frac{\delta s - \beta t}{\alpha\delta - \beta\gamma}$, $y = \frac{-\gamma s + \alpha t}{\alpha\delta - \beta\gamma}$,

$$\text{and } J(s, t) = \frac{1}{(\alpha\delta - \beta\gamma)^2} \begin{vmatrix} \delta & -\beta \\ -\gamma & \alpha \end{vmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \Rightarrow \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(s^2+t^2)} \frac{1}{\sqrt{ac-b^2}} ds dt$$

$$= \frac{1}{\sqrt{ac-b^2}} \int_0^{2\pi} \int_0^\infty r e^{-r^2} dr d\theta = \frac{1}{2\sqrt{ac-b^2}} \int_0^{2\pi} d\theta = \frac{\pi}{\sqrt{ac-b^2}}. \text{ Therefore, } \frac{\pi}{\sqrt{ac-b^2}} = 1 \Rightarrow ac - b^2 = \pi^2.$$

CHAPTER 15 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx$ (b) $V = \int_{-3}^2 \int_x^{6-x^2} \int_0^{x^2} dz dy dx$

(c) $V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx = \int_{-3}^2 \int_x^{6-x^2} (6x^2 - x^4 - x^3) dx = \left[2x^3 - \frac{x^5}{5} - \frac{x^4}{4} \right]_{-3}^2 = \frac{125}{4}$

2. Place the sphere's center at the origin with the surface of the water at $z = -3$. Then

$$9 = 25 - x^2 - y^2 \Rightarrow x^2 + y^2 = 16 \text{ is the projection of the volume of water onto the } xy\text{-plane}$$

$$\Rightarrow V = \int_0^{2\pi} \int_0^4 \int_{-\sqrt{25-r^2}}^{-3} dz r dr d\theta = \int_0^{2\pi} \int_0^4 (r\sqrt{25-r^2} - 3r) dr d\theta = \int_0^{2\pi} \left[-\frac{1}{3} (25-r^2)^{3/2} - \frac{3}{2} r^2 \right]_0^4 d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} (9)^{3/2} - 24 + \frac{1}{3} (25)^{3/2} \right] d\theta = \int_0^{2\pi} \frac{26}{3} d\theta = \frac{52\pi}{3}$$

3. Using cylindrical coordinates, $V = \int_0^{2\pi} \int_0^1 \int_0^{2-r(\cos \theta + \sin \theta)} dz r dr d\theta = \int_0^{2\pi} \int_0^1 (2r - r^2 \cos \theta - r^2 \sin \theta) dr d\theta$

$$= \int_0^{2\pi} \left(1 - \frac{1}{3} \cos \theta - \frac{1}{3} \sin \theta \right) d\theta = \left[\theta - \frac{1}{3} \sin \theta + \frac{1}{3} \cos \theta \right]_0^{2\pi} = 2\pi$$

4. $V = 4 \int_0^{\pi/2} \int_0^1 \int_r^{\sqrt{2-r^2}} dz r dr d\theta = 4 \int_0^{\pi/2} \int_0^1 (r\sqrt{2-r^2} - r^3) dr d\theta = 4 \int_0^{\pi/2} \left[-\frac{1}{3} (2-r^2)^{3/2} - \frac{r^4}{4} \right]_0^1 d\theta$

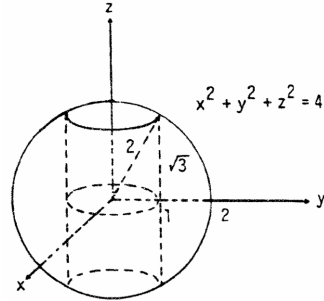
$$= 4 \int_0^{\pi/2} \left(-\frac{1}{3} - \frac{1}{4} + \frac{2\sqrt{2}}{3} \right) d\theta = \left(\frac{8\sqrt{2}-7}{3} \right) \int_0^{\pi/2} d\theta = \frac{\pi(8\sqrt{2}-7)}{6}$$

5. The surfaces intersect when $3 - x^2 - y^2 = 2x^2 + 2y^2 \Rightarrow x^2 + y^2 = 1$. Thus the volume is

$$V = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{2x^2+2y^2}^{3-x^2-y^2} dz dy dx = 4 \int_0^{\pi/2} \int_0^1 \int_{2r^2}^{3-r^2} dz r dr d\theta = 4 \int_0^{\pi/2} \int_0^1 (3r - 3r^3) dr d\theta = 3 \int_0^{\pi/2} d\theta = \frac{3\pi}{2}$$

$$\begin{aligned} 6. V &= 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2\sin\phi} \rho^2 \sin\phi d\rho d\phi d\theta = \frac{64}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin^4\phi d\phi d\theta \\ &= \frac{64}{3} \int_0^{\pi/2} \left[-\frac{\sin^3\phi \cos\phi}{4} \Big|_0^{\pi/2} + \frac{3}{4} \int_0^{\pi/2} \sin^2\phi d\phi \right] d\theta = 16 \int_0^{\pi/2} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right]_0^{\pi/2} d\theta = 4\pi \int_0^{\pi/2} d\theta = 2\pi^2 \end{aligned}$$

7. (a) The radius of the hole is 1, and the radius of the sphere is 2.



$$(b) V = 2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} r dr dz d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} (3 - z^2) dz d\theta = 2\sqrt{3} \int_0^{2\pi} d\theta = 4\sqrt{3}\pi$$

$$\begin{aligned} 8. V &= \int_0^{\pi} \int_0^{3\sin\theta} \int_0^{\sqrt{9-r^2}} dz r dr d\theta = \int_0^{\pi} \int_0^{3\sin\theta} r \sqrt{9-r^2} dr d\theta = \int_0^{\pi} \left[-\frac{1}{3} (9-r^2)^{3/2} \right]_0^{3\sin\theta} d\theta \\ &= \int_0^{\pi} \left[-\frac{1}{3} (9-9\sin^2\theta)^{3/2} + \frac{1}{3} (9)^{3/2} \right] d\theta = 9 \int_0^{\pi} \left[1 - (1-\sin^2\theta)^{3/2} \right] d\theta = 9 \int_0^{\pi} (1 - \cos^3\theta) d\theta \\ &= \int_0^{\pi} (1 - \cos\theta + \sin^2\theta \cos\theta) d\theta = 9 \left[\theta - \sin\theta + \frac{\sin^3\theta}{3} \right]_0^{\pi} = 9\pi \end{aligned}$$

9. The surfaces intersect when $x^2 + y^2 = \frac{x^2 + y^2 + 1}{2} \Rightarrow x^2 + y^2 = 1$. Thus the volume in cylindrical coordinates is

$$V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{(r^2+1)/2} dz r dr d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(\frac{r}{2} - \frac{r^3}{2} \right) dr d\theta = 4 \int_0^{\pi/2} \left[\frac{r^2}{4} - \frac{r^4}{8} \right]_0^1 d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

$$\begin{aligned} 10. V &= \int_0^{\pi/2} \int_1^2 \int_0^{r^2 \sin\theta \cos\theta} dz r dr d\theta = \int_0^{\pi/2} \int_1^2 r^3 \sin\theta \cos\theta dr d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_1^2 \sin\theta \cos\theta d\theta \\ &= \frac{15}{4} \int_0^{\pi/2} \sin\theta \cos\theta d\theta = \frac{15}{4} \left[\frac{\sin^2\theta}{2} \right]_0^{\pi/2} = \frac{15}{8} \end{aligned}$$

$$\begin{aligned} 11. \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx &= \int_0^{\infty} \int_a^b e^{-xy} dy dx = \int_a^b \int_0^{\infty} e^{-xy} dx dy = \int_a^b \left(\lim_{t \rightarrow \infty} \int_0^t e^{-xy} dx \right) dy \\ &= \int_a^b \lim_{t \rightarrow \infty} \left[-\frac{e^{-xy}}{y} \right]_0^t dy = \int_a^b \lim_{t \rightarrow \infty} \left(\frac{1}{y} - \frac{e^{-yt}}{y} \right) dy = \int_a^b \frac{1}{y} dy = [\ln y]_a^b = \ln \left(\frac{b}{a} \right) \end{aligned}$$

12. (a) The region of integration is sketched at the right

$$\Rightarrow \int_0^{a \sin \beta} \int_{y \cot \beta}^{\sqrt{a^2 - y^2}} \ln(x^2 + y^2) dx dy$$

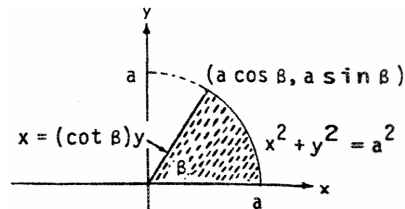
$$= \int_0^{\beta} \int_0^a r \ln(r^2) dr d\theta;$$

$$\left[\begin{array}{l} u = r^2 \\ du = 2r dr \end{array} \right] \rightarrow \frac{1}{2} \int_0^{\beta} \int_0^{a^2} \ln u du d\theta$$

$$= \frac{1}{2} \int_0^{\beta} [u \ln u - u]_0^{a^2} d\theta$$

$$= \frac{1}{2} \int_0^{\beta} \left[2a^2 \ln a - a^2 - \lim_{t \rightarrow 0} t \ln t \right] d\theta = \frac{a^2}{2} \int_0^{\beta} (2 \ln a - 1) d\theta = a^2 \beta \left(\ln a - \frac{1}{2} \right)$$

$$(b) \int_0^{a \cos \beta} \int_0^{(\tan \beta)x} \ln(x^2 + y^2) dy dx + \int_{a \cos \beta}^a \int_0^{\sqrt{a^2 - x^2}} \ln(x^2 + y^2) dy dx$$



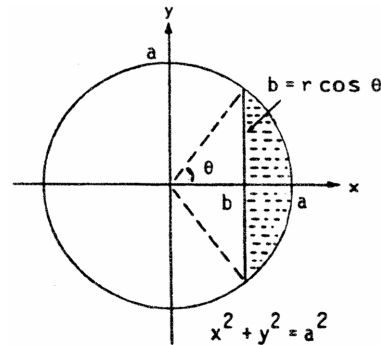
$$\begin{aligned}
13. \int_0^x \int_0^u e^{m(x-t)} f(t) dt du &= \int_0^x \int_t^x e^{m(x-t)} f(t) du dt = \int_0^x (x-t) e^{m(x-t)} f(t) dt; \text{ also} \\
\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t) dt du dv &= \int_0^x \int_t^x \int_t^v e^{m(x-t)} f(t) du dv dt = \int_0^x \int_t^x (v-t) e^{m(x-t)} f(t) dv dt \\
&= \int_0^x \left[\frac{1}{2} (v-t)^2 e^{m(x-t)} f(t) \right]_t^x dt = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t) dt
\end{aligned}$$

$$\begin{aligned}
14. \int_0^1 f(x) \left(\int_0^x g(x-y) f(y) dy \right) dx &= \int_0^1 \int_0^x g(x-y) f(x) f(y) dy dx \\
&= \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy = \int_0^1 f(y) \left(\int_y^1 g(x-y) f(x) dx \right) dy; \\
\int_0^1 \int_0^1 g(|x-y|) f(x) f(y) dx dy &= \int_0^1 \int_0^x g(x-y) f(x) f(y) dy dx + \int_0^1 \int_x^1 g(y-x) f(x) f(y) dy dx \\
&= \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy + \int_0^1 \int_x^1 g(y-x) f(x) f(y) dy dx \\
&= \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy + \underbrace{\int_0^1 \int_y^1 g(x-y) f(y) f(x) dx dy}_{\text{simply interchange } x \text{ and } y \\ \text{variable names}} \\
&= 2 \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy, \text{ and the statement now follows.}
\end{aligned}$$

$$\begin{aligned}
15. I_0(a) &= \int_0^a \int_0^{x/a^2} (x^2 + y^2) dy dx = \int_0^a \left[x^2 y + \frac{y^3}{3} \right]_0^{x/a^2} dx = \int_0^a \left(\frac{x^3}{a^2} + \frac{x^3}{3a^6} \right) dx = \left[\frac{x^4}{4a^2} + \frac{x^4}{12a^6} \right]_0^a \\
&= \frac{a^2}{4} + \frac{1}{12} a^{-2}; I_0'(a) = \frac{1}{2} a - \frac{1}{6} a^{-3} = 0 \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \sqrt[4]{\frac{1}{3}} = \frac{1}{\sqrt[4]{3}}. \text{ Since } I_0''(a) = \frac{1}{2} + \frac{1}{2} a^{-4} > 0, \text{ the} \\
&\text{value of } a \text{ does provide a } \underline{\text{minimum}} \text{ for the polar moment of inertia } I_0(a).
\end{aligned}$$

$$16. I_0 = \int_0^2 \int_{2x}^4 (x^2 + y^2) (3) dy dx = 3 \int_0^2 \left(4x^2 - \frac{14x^3}{3} + \frac{64}{3} \right) dx = 104$$

$$\begin{aligned}
17. M &= \int_{-\theta}^{\theta} \int_{b \sec \theta}^a r dr d\theta = \int_{-\theta}^{\theta} \left(\frac{a^2}{2} - \frac{b^2}{2} \sec^2 \theta \right) d\theta \\
&= a^2 \theta - b^2 \tan \theta = a^2 \cos^{-1} \left(\frac{b}{a} \right) - b^2 \left(\frac{\sqrt{a^2 - b^2}}{b} \right) \\
&= a^2 \cos^{-1} \left(\frac{b}{a} \right) - b \sqrt{a^2 - b^2}; I_0 = \int_{-\theta}^{\theta} \int_{b \sec \theta}^a r^3 dr d\theta \\
&= \frac{1}{4} \int_{-\theta}^{\theta} (a^4 + b^4 \sec^4 \theta) d\theta \\
&= \frac{1}{4} \int_{-\theta}^{\theta} [a^4 + b^4 (1 + \tan^2 \theta) (\sec^2 \theta)] d\theta \\
&= \frac{1}{4} \left[a^4 \theta - b^4 \tan \theta - \frac{b^4 \tan^3 \theta}{3} \right]_{-\theta}^{\theta} \\
&= \frac{a^4 \theta}{2} - \frac{b^4 \tan \theta}{2} - \frac{b^4 \tan^3 \theta}{6} \\
&= \frac{1}{2} a^4 \cos^{-1} \left(\frac{b}{a} \right) - \frac{1}{2} b^3 \sqrt{a^2 - b^2} - \frac{1}{6} b^3 (a^2 - b^2)^{3/2}
\end{aligned}$$



$$\begin{aligned}
18. M &= \int_{-2}^2 \int_{1-(y^2/4)}^{2-(y^2/2)} dx dy = \int_{-2}^2 \left(1 - \frac{y^2}{4} \right) dy = \left[y - \frac{y^3}{12} \right]_{-2}^2 = \frac{8}{3}; M_y = \int_{-2}^2 \int_{1-(y^2/4)}^{2-(y^2/2)} x dx dy \\
&= \int_{-2}^2 \left[\frac{x^2}{2} \right]_{1-(y^2/4)}^{2-(y^2/2)} dy = \int_{-2}^2 \frac{3}{32} (4 - y^2) dy = \frac{3}{32} \int_{-2}^2 (16 - 8y^2 + y^4) dy = \frac{3}{16} \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\
&= \frac{3}{16} \left(32 - \frac{64}{3} + \frac{32}{5} \right) = \left(\frac{3}{16} \right) \left(\frac{32 \cdot 8}{15} \right) = \frac{48}{15} \Rightarrow \bar{x} = \frac{M_y}{M} = \left(\frac{48}{15} \right) \left(\frac{3}{8} \right) = \frac{6}{5}, \text{ and } \bar{y} = 0 \text{ by symmetry}
\end{aligned}$$

$$19. \int_0^a \int_0^b e^{\max(b^2 x^2, a^2 y^2)} dy dx = \int_0^a \int_0^{bx/a} e^{b^2 x^2} dy dx + \int_0^b \int_0^{ay/b} e^{a^2 y^2} dx dy$$

$$\begin{aligned}
&= \int_0^a \left(\frac{b}{a}x\right) e^{b^2x^2} dx + \int_0^b \left(\frac{a}{b}y\right) e^{a^2y^2} dy = \left[\frac{1}{2ab} e^{b^2x^2}\right]_0^a + \left[\frac{1}{2ba} e^{a^2y^2}\right]_0^b = \frac{1}{2ab} (e^{b^2a^2} - 1) + \frac{1}{2ab} (e^{a^2b^2} - 1) \\
&= \frac{1}{ab} (e^{a^2b^2} - 1)
\end{aligned}$$

$$\begin{aligned}
20. \int_{y_0}^{y_1} \int_{x_0}^{x_1} \frac{\partial^2 F(x, y)}{\partial x \partial y} dx dy &= \int_{y_0}^{y_1} \left[\frac{\partial F(x, y)}{\partial y} \right]_{x_0}^{x_1} dy = \int_{y_0}^{y_1} \left[\frac{\partial F(x_1, y)}{\partial y} - \frac{\partial F(x_0, y)}{\partial y} \right] dy = [F(x_1, y) - F(x_0, y)]_{y_0}^{y_1} \\
&= F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0)
\end{aligned}$$

21. (a) (i) Fubini's Theorem

(ii) Treating $G(y)$ as a constant

(iii) Algebraic rearrangement

(iv) The definite integral is a constant number

$$(b) \int_0^{\ln 2} \int_0^{\pi/2} e^x \cos y dy dx = \left(\int_0^{\ln 2} e^x dx \right) \left(\int_0^{\pi/2} \cos y dy \right) = (e^{\ln 2} - e^0) (\sin \frac{\pi}{2} - \sin 0) = (1)(1) = 1$$

$$(c) \int_1^2 \int_{-1}^1 \frac{x}{y^2} dx dy = \left(\int_1^2 \frac{1}{y^2} dy \right) \left(\int_{-1}^1 x dx \right) = \left[-\frac{1}{y} \right]_1^2 \left[\frac{x^2}{2} \right]_{-1}^1 = \left(-\frac{1}{2} + 1 \right) \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

22. (a) $\nabla f = x\mathbf{i} + y\mathbf{j} \Rightarrow D_u f = u_1 x + u_2 y$; the area of the region of integration is $\frac{1}{2}$

$$\begin{aligned}
&\Rightarrow \text{average} = 2 \int_0^1 \int_0^{1-x} (u_1 x + u_2 y) dy dx = 2 \int_0^1 \left[u_1 x(1-x) + \frac{1}{2} u_2 (1-x)^2 \right] dx \\
&= 2 \left[u_1 \left(\frac{x^2}{2} - \frac{x^3}{3} \right) - \left(\frac{1}{2} u_2 \right) \left(\frac{(1-x)^3}{3} \right) \right]_0^1 = 2 \left(\frac{1}{6} u_1 + \frac{1}{6} u_2 \right) = \frac{1}{3} (u_1 + u_2)
\end{aligned}$$

$$(b) \text{average} = \frac{1}{\text{area}} \iint_R (u_1 x + u_2 y) dA = \frac{u_1}{\text{area}} \iint_R x dA + \frac{u_2}{\text{area}} \iint_R y dA = u_1 \left(\frac{M_y}{M} \right) + u_2 \left(\frac{M_x}{M} \right) = u_1 \bar{x} + u_2 \bar{y}$$

$$\begin{aligned}
23. (a) I^2 &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\pi/2} \int_0^\infty (e^{-r^2}) r dr d\theta = \int_0^{\pi/2} \left[\lim_{b \rightarrow \infty} \int_0^b r e^{-r^2} dr \right] d\theta \\
&= -\frac{1}{2} \int_0^{\pi/2} \lim_{b \rightarrow \infty} (e^{-b^2} - 1) d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}
\end{aligned}$$

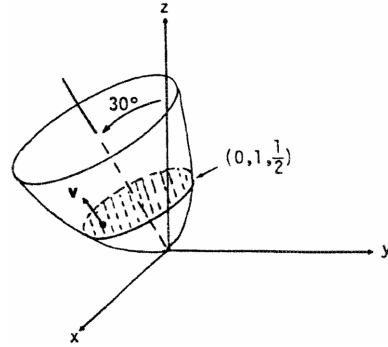
$$(b) \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} dt = \int_0^\infty (y^2)^{-1/2} e^{-y^2} (2y) dy = 2 \int_0^\infty e^{-y^2} dy = 2 \left(\frac{\sqrt{\pi}}{2} \right) = \sqrt{\pi}, \text{ where } y = \sqrt{t}$$

$$24. Q = \int_0^{2\pi} \int_0^R kr^2(1 - \sin \theta) dr d\theta = \frac{kR^3}{3} \int_0^{2\pi} (1 - \sin \theta) d\theta = \frac{kR^3}{3} [\theta + \cos \theta]_0^{2\pi} = \frac{2\pi kR^3}{3}$$

$$\begin{aligned}
25. \text{For a height } h \text{ in the bowl the volume of water is } V &= \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \int_{x^2+y^2}^h dz dy dx \\
&= \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (h - x^2 - y^2) dy dx = \int_0^{2\pi} \int_0^{\sqrt{h}} (h - r^2) r dr d\theta = \int_0^{2\pi} \left[\frac{hr^2}{2} - \frac{r^4}{4} \right]_0^{\sqrt{h}} d\theta = \int_0^{2\pi} \frac{h^2}{4} d\theta = \frac{h^2\pi}{2}.
\end{aligned}$$

Since the top of the bowl has area 10π , then we calibrate the bowl by comparing it to a right circular cylinder whose cross sectional area is 10π from $z = 0$ to $z = 10$. If such a cylinder contains $\frac{h^2\pi}{2}$ cubic inches of water to a depth w then we have $10\pi w = \frac{h^2\pi}{2} \Rightarrow w = \frac{h^2}{20}$. So for 1 inch of rain, $w = 1$ and $h = \sqrt{20}$; for 3 inches of rain, $w = 3$ and $h = \sqrt{60}$.

26. (a) An equation for the satellite dish in standard position is $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since the axis is tilted 30° , a unit vector $\mathbf{v} = 0\mathbf{i} + \mathbf{a}\mathbf{j} + \mathbf{b}\mathbf{k}$ normal to the plane of the



$$\begin{aligned} \text{water level satisfies } \mathbf{b} &= \mathbf{v} \cdot \mathbf{k} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ \Rightarrow \mathbf{a} &= -\sqrt{1 - \mathbf{b}^2} = -\frac{1}{2} \Rightarrow \mathbf{v} = -\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k} \\ \Rightarrow -\frac{1}{2}(y - 1) + \frac{\sqrt{3}}{2}\left(z - \frac{1}{2}\right) &= 0 \\ \Rightarrow z &= \frac{1}{\sqrt{3}}y + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right) \end{aligned}$$

is an equation of the plane of the water level. Therefore

the volume of water is $V = \iint_R \int_{\frac{1}{\sqrt{3}}y + \frac{1}{2} - \frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}y + \frac{1}{2} - \frac{1}{\sqrt{3}}} dz dy dx$, where R is the interior of the ellipse

$$x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 + \frac{2}{\sqrt{3}} = 0. \text{ When } x = 0, \text{ then } y = \alpha \text{ or } y = \beta, \text{ where } \alpha = \frac{\frac{2}{\sqrt{3}} + \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$$

$$\text{and } \beta = \frac{\frac{2}{\sqrt{3}} - \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2} \Rightarrow V = \int_{\alpha}^{\beta} \int_{-\left(\frac{2}{3}y + 1 - \frac{2}{\sqrt{3}}y^2\right)^{1/2}}^{\left(\frac{2}{3}y + 1 - \frac{2}{\sqrt{3}}y^2\right)^{1/2}} \int_{\frac{1}{\sqrt{3}}y + \frac{1}{2} - \frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}y + \frac{1}{2} - \frac{1}{\sqrt{3}}} 1 dz dx dy$$

- (b) $x = 0 \Rightarrow z = \frac{1}{2}y^2$ and $\frac{dz}{dy} = y$; $y = 1 \Rightarrow \frac{dz}{dy} = 1 \Rightarrow$ the tangent line has slope 1 or a 45° slant
 $\Rightarrow$ at 45° and thereafter, the dish will not hold water.

27. The cylinder is given by $x^2 + y^2 = 1$ from $z = 1$ to $\infty \Rightarrow \iiint_D z(r^2 + z^2)^{-5/2} dV$

$$\begin{aligned} &= \int_0^{2\pi} \int_0^1 \int_1^\infty \frac{z}{(r^2 + z^2)^{5/2}} dz r dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \int_1^a \frac{rz}{(r^2 + z^2)^{5/2}} dz dr d\theta \\ &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \left[\left(-\frac{1}{3}\right) \frac{r}{(r^2 + z^2)^{3/2}} \right]_1^a dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \left[\left(-\frac{1}{3}\right) \frac{r}{(r^2 + a^2)^{3/2}} + \left(\frac{1}{3}\right) \frac{r}{(r^2 + 1)^{3/2}} \right] dr d\theta \\ &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \left[\frac{1}{3} (r^2 + a^2)^{-1/2} - \frac{1}{3} (r^2 + 1)^{-1/2} \right]_0^1 d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \left[\frac{1}{3} (1 + a^2)^{-1/2} - \frac{1}{3} (2^{-1/2}) - \frac{1}{3} (a^2)^{-1/2} + \frac{1}{3} \right] d\theta \\ &= \lim_{a \rightarrow \infty} 2\pi \left[\frac{1}{3} (1 + a^2)^{-1/2} - \frac{1}{3} \left(\frac{\sqrt{2}}{2}\right) - \frac{1}{3} \left(\frac{1}{a}\right) + \frac{1}{3} \right] = 2\pi \left[\frac{1}{3} - \left(\frac{1}{3}\right) \frac{\sqrt{2}}{2} \right]. \end{aligned}$$

28. Let's see?

The length of the "unit" line segment is: $L = 2 \int_0^1 dx = 2$.

The area of the unit circle is: $A = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} dy dx = \pi$.

The volume of the unit sphere is: $V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx = \frac{4}{3}\pi$.

Therefore, the hypervolume of the unit 4-sphere should be:

$$V_{\text{hyper}} = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2-z^2}} dw dz dy dx.$$

Mathematica is able to handle this integral, but we'll use the brute force approach.

$$\begin{aligned} V_{\text{hyper}} &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2-z^2}} dw dz dy dx = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2-z^2} dz dy dx \\ &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2} \sqrt{1 - \frac{z^2}{1-x^2-y^2}} dz dy dx = \left[\frac{z}{\sqrt{1-x^2-y^2}} = \cos \theta \right] \\ &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2) \int_{\pi/2}^0 -\sqrt{1-\cos^2 \theta} \sin \theta d\theta dy dx = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2) \int_{\pi/2}^0 -\sin^2 \theta d\theta dy dx \\ &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{4} (1-x^2-y^2) dy dx = 4\pi \int_0^1 \left(\sqrt{1-x^2} - x^2 \sqrt{1-x^2} - \frac{1}{3} (1-x^2)^{3/2} \right) dx \\ &= 4\pi \int_0^1 \sqrt{1-x^2} \left[(1-x^2) - \frac{1-x^3}{3} \right] dx = \frac{8}{3}\pi \int_0^1 (1-x^2)^{3/2} dx = \left[\frac{x = \cos \theta}{dx = -\sin \theta d\theta} \right] = -\frac{8}{3}\pi \int_{\pi/2}^0 \sin^4 \theta d\theta \\ &= -\frac{8}{3}\pi \int_{\pi/2}^0 \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta = -\frac{2}{3}\pi \int_{\pi/2}^0 (1-2\cos 2\theta + \cos^2 2\theta) d\theta = -\frac{2}{3}\pi \int_{\pi/2}^0 \left(\frac{3}{2} - 2\cos 2\theta + \frac{\cos 4\theta}{2} \right) d\theta = \frac{\pi^2}{2} \end{aligned}$$

NOTES:

CHAPTER 16 INTEGRATION IN VECTOR FIELDS

16.1 LINE INTEGRALS

1. $\mathbf{r} = t\mathbf{i} + (1 - t)\mathbf{j} \Rightarrow x = t \text{ and } y = 1 - t \Rightarrow y = 1 - x \Rightarrow (c)$
2. $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, \text{ and } z = t \Rightarrow (e)$
3. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} \Rightarrow x = 2 \cos t \text{ and } y = 2 \sin t \Rightarrow x^2 + y^2 = 4 \Rightarrow (g)$
4. $\mathbf{r} = t\mathbf{i} \Rightarrow x = t, y = 0, \text{ and } z = 0 \Rightarrow (a)$
5. $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t, y = t, \text{ and } z = t \Rightarrow (d)$
6. $\mathbf{r} = t\mathbf{j} + (2 - 2t)\mathbf{k} \Rightarrow y = t \text{ and } z = 2 - 2t \Rightarrow z = 2 - 2y \Rightarrow (b)$
7. $\mathbf{r} = (t^2 - 1)\mathbf{j} + 2t\mathbf{k} \Rightarrow y = t^2 - 1 \text{ and } z = 2t \Rightarrow y = \frac{z^2}{4} - 1 \Rightarrow (f)$
8. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{k} \Rightarrow x = 2 \cos t \text{ and } z = 2 \sin t \Rightarrow x^2 + z^2 = 4 \Rightarrow (h)$
9. $\mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2}; x = t \text{ and } y = 1 - t \Rightarrow x + y = t + (1 - t) = 1$
 $\Rightarrow \int_C f(x, y, z) \, ds = \int_0^1 f(t, 1 - t, 0) \left| \frac{d\mathbf{r}}{dt} \right| \, dt = \int_0^1 (1) (\sqrt{2}) \, dt = \left[\sqrt{2}t \right]_0^1 = \sqrt{2}$
10. $\mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2}; x = t, y = 1 - t, \text{ and } z = 1 \Rightarrow x - y + z - 2$
 $= t - (1 - t) + 1 - 2 = 2t - 2 \Rightarrow \int_C f(x, y, z) \, ds = \int_0^1 (2t - 2) \sqrt{2} \, dt = \sqrt{2} [t^2 - 2t]_0^1 = -\sqrt{2}$
11. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{4 + 1 + 4} = 3; xy + y + z$
 $= (2t)t + t + (2 - 2t) \Rightarrow \int_C f(x, y, z) \, ds = \int_0^1 (2t^2 - t + 2) 3 \, dt = 3 \left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t \right]_0^1 = 3 \left(\frac{2}{3} - \frac{1}{2} + 2 \right) = \frac{13}{2}$
12. $\mathbf{r}(t) = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k}, -2\pi \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-4 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j} + 3\mathbf{k}$
 $\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} = 5; \sqrt{x^2 + y^2} = \sqrt{16 \cos^2 t + 16 \sin^2 t} = 4 \Rightarrow \int_C f(x, y, z) \, ds = \int_{-2\pi}^{2\pi} (4)(5) \, dt$
 $= [20t]_{-2\pi}^{2\pi} = 80\pi$
13. $\mathbf{r}(t) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (1 - t)\mathbf{i} + (2 - 3t)\mathbf{j} + (3 - 2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$
 $\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 9 + 4} = \sqrt{14}; x + y + z = (1 - t) + (2 - 3t) + (3 - 2t) = 6 - 6t \Rightarrow \int_C f(x, y, z) \, ds$
 $= \int_0^1 (6 - 6t) \sqrt{14} \, dt = 6\sqrt{14} \left[t - \frac{t^2}{2} \right]_0^1 = \left(6\sqrt{14} \right) \left(\frac{1}{2} \right) = 3\sqrt{14}$
14. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 1 \leq t \leq \infty \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{3}; \frac{\sqrt{3}}{x^2 + y^2 + z^2} = \frac{\sqrt{3}}{t^2 + t^2 + t^2} = \frac{\sqrt{3}}{3t^2}$
 $\Rightarrow \int_C f(x, y, z) \, ds = \int_1^\infty \left(\frac{\sqrt{3}}{3t^2} \right) \sqrt{3} \, dt = \left[-\frac{1}{t} \right]_1^\infty = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1$

$$15. C_1: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 4t^2}; x + \sqrt{y} - z^2 = t + \sqrt{t^2} - 0 = t + |t| = 2t$$

since $t \geq 0 \Rightarrow \int_{C_1} f(x, y, z) \, ds = \int_0^1 2t\sqrt{1 + 4t^2} \, dt = \left[\frac{1}{6} (1 + 4t^2)^{3/2} \right]_0^1 = \frac{1}{6} (5)^{3/2} - \frac{1}{6} = \frac{1}{6} (5\sqrt{5} - 1);$

$$C_2: \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y} - z^2 = 1 + \sqrt{1} - t^2 = 2 - t^2$$

$$\Rightarrow \int_{C_2} f(x, y, z) \, ds = \int_0^1 (2 - t^2)(1) \, dt = \left[2t - \frac{1}{3}t^3 \right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}; \text{ therefore } \int_C f(x, y, z) \, ds$$

$$= \int_{C_1} f(x, y, z) \, ds + \int_{C_2} f(x, y, z) \, ds = \frac{5}{6}\sqrt{5} + \frac{5}{3}$$

$$16. C_1: \mathbf{r}(t) = t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y} - z^2 = 0 + \sqrt{0} - t^2 = -t^2$$

$$\Rightarrow \int_{C_1} f(x, y, z) \, ds = \int_0^1 (-t^2)(1) \, dt = \left[-\frac{t^3}{3} \right]_0^1 = -\frac{1}{3};$$

$$C_2: \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y} - z^2 = 0 + \sqrt{t} - 1 = \sqrt{t} - 1$$

$$\Rightarrow \int_{C_2} f(x, y, z) \, ds = \int_0^1 (\sqrt{t} - 1)(1) \, dt = \left[\frac{2}{3}t^{3/2} - t \right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3};$$

$$C_3: \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y} - z^2 = t + \sqrt{1} - 1 = t$$

$$\Rightarrow \int_{C_3} f(x, y, z) \, ds = \int_0^1 (t)(1) \, dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2} \Rightarrow \int_C f(x, y, z) \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds + \int_{C_3} f \, ds = -\frac{1}{3} + \left(-\frac{1}{3}\right) + \frac{1}{2}$$

$$= -\frac{1}{6}$$

$$17. \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 < a \leq t \leq b \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{3}; \frac{x+y+z}{x^2+y^2+z^2} = \frac{t+t+t}{t^2+t^2+t^2} = \frac{1}{t}$$

$$\Rightarrow \int_C f(x, y, z) \, ds = \int_a^b \left(\frac{1}{t}\right) \sqrt{3} \, dt = \left[\sqrt{3} \ln |t| \right]_a^b = \sqrt{3} \ln \left(\frac{b}{a}\right), \text{ since } 0 < a \leq b$$

$$18. \mathbf{r}(t) = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k}, 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{j} + (a \cos t)\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = |a|;$$

$$-\sqrt{x^2 + z^2} = -\sqrt{0 + a^2 \sin^2 t} = \begin{cases} -|a| \sin t, & 0 \leq t \leq \pi \\ |a| \sin t, & \pi \leq t \leq 2\pi \end{cases} \Rightarrow \int_C f(x, y, z) \, ds = \int_0^\pi -|a|^2 \sin t \, dt + \int_\pi^{2\pi} |a|^2 \sin t \, dt$$

$$= [a^2 \cos t]_0^\pi - [a^2 \cos t]_\pi^{2\pi} = [a^2(-1) - a^2] - [a^2 - a^2(-1)] = -4a^2$$

$$19. (a) \mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t\mathbf{j}, 0 \leq t \leq 4 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \frac{\sqrt{5}}{2} \Rightarrow \int_C x \, ds = \int_0^4 t \frac{\sqrt{5}}{2} dt = \frac{\sqrt{5}}{2} \int_0^4 t \, dt = \left[\frac{\sqrt{5}}{4} t^2 \right]_0^4 = 4\sqrt{5}$$

$$(b) \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 4t^2} \Rightarrow \int_C x \, ds = \int_0^2 t \sqrt{1 + 4t^2} dt$$

$$= \left[\frac{1}{12} (1 + 4t^2)^{3/2} \right]_0^2 = \frac{17\sqrt{17}-1}{12}$$

$$20. (a) \mathbf{r}(t) = t\mathbf{i} + 4t\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 4\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{17} \Rightarrow \int_C \sqrt{x+2y} \, ds = \int_0^1 \sqrt{t+2(4t)} \sqrt{17} \, dt$$

$$= \sqrt{17} \int_0^1 \sqrt{9t} \, dt = 3\sqrt{17} \int_0^1 \sqrt{t} \, dt = \left[2\sqrt{17} t^{3/2} \right]_0^1 = 2\sqrt{17}$$

$$(b) C_1: \mathbf{r}(t) = t\mathbf{i}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; C_2: \mathbf{r}(t) = \mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$$

$$\int_C \sqrt{x+2y} \, ds = \int_{C_1} \sqrt{x+2y} \, ds + \int_{C_2} \sqrt{x+2y} \, ds = \int_0^1 \sqrt{t+2(0)} \, dt + \int_0^2 \sqrt{1+2(t)} \, dt$$

$$= \int_0^1 \sqrt{t} \, dt + \int_0^2 \sqrt{1+2t} \, dt = \left[\frac{2}{3} t^{3/2} \right]_0^1 + \left[\frac{1}{3} (1+2t)^{3/2} \right]_0^2 = \frac{2}{3} + \left(\frac{5\sqrt{5}}{3} - \frac{1}{3} \right) = \frac{5\sqrt{5}+1}{3}$$

$$21. \mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j}, -1 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = 4\mathbf{i} - 3\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 5 \Rightarrow \int_C y e^{x^2} \, ds = \int_{-1}^2 (-3t) e^{(4t)^2} \cdot 5 \, dt$$

$$= -15 \int_{-1}^2 t e^{16t^2} \, dt = \left[-\frac{15}{32} e^{16t^2} \right]_{-1}^2 = -\frac{15}{32} e^{64} + \frac{15}{32} e^{16} = \frac{15}{32} (e^{16} - e^{64})$$

22. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t} = 1 \Rightarrow \int_C (x - y + 3) \, ds$
 $= \int_0^{2\pi} (\cos t - \sin t + 3) \cdot 1 \, dt = [\sin t + \cos t + 3t]_0^{2\pi} = 6\pi$
23. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, $1 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + 3t^2\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(2t)^2 + (3t^2)^2} = t\sqrt{4 + 9t^2} \Rightarrow \int_C \frac{x^2}{y^{4/3}} \, ds$
 $= \int_1^2 \frac{(t^2)^2}{(t^3)^{4/3}} \cdot t\sqrt{4 + 9t^2} \, dt = \int_1^2 t\sqrt{4 + 9t^2} \, dt = \left[\frac{1}{27}(4 + 9t^2)^{3/2} \right]_1^2 = \frac{80\sqrt{10} - 13\sqrt{13}}{27}$
24. $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}$, $\frac{1}{2} \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 4t^3\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(3t^2)^2 + (4t^3)^2} = t^2\sqrt{9 + 16t^2} \Rightarrow \int_C \frac{\sqrt{y}}{x} \, ds$
 $= \int_{1/2}^1 \frac{\sqrt{t^4}}{t^3} \cdot t^2\sqrt{9 + 16t^2} \, dt = \int_{1/2}^1 t\sqrt{9 + 16t^2} \, dt = \left[\frac{1}{48}(9 + 16t^2)^{3/2} \right]_{1/2}^1 = \frac{125 - 13\sqrt{13}}{48}$
25. $C_1: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 4t^2}$; $C_2: \mathbf{r}(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j}$, $0 \leq t \leq 1$
 $\Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2} \Rightarrow \int_C (x + \sqrt{y}) \, ds = \int_{C_1} (x + \sqrt{y}) \, ds + \int_{C_2} (x + \sqrt{y}) \, ds$
 $= \int_0^1 (t + \sqrt{t^2}) \sqrt{1 + 4t^2} \, dt + \int_0^1 ((1 - t) + \sqrt{1 - t}) \sqrt{2} \, dt = \int_0^1 2t\sqrt{1 + 4t^2} \, dt + \int_0^1 (1 - t + \sqrt{1 - t}) \sqrt{2} \, dt$
 $= \left[\frac{1}{6}(1 + 4t^2)^{3/2} \right]_0^1 + \sqrt{2} \left[t - \frac{1}{2}t^2 - \frac{2}{3}(1 - t)^{3/2} \right]_0^1 = \frac{5\sqrt{5} - 1}{6} + \frac{7\sqrt{2}}{6} = \frac{5\sqrt{5} + 7\sqrt{2} - 1}{6}$
26. $C_1: \mathbf{r}(t) = t\mathbf{i}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$; $C_2: \mathbf{r}(t) = \mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$;
 $C_3: \mathbf{r}(t) = (1 - t)\mathbf{i} + \mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$; $C_4: \mathbf{r}(t) = (1 - t)\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$;
 $\Rightarrow \int_C \frac{1}{x^2 + y^2 + 1} \, ds = \int_{C_1} \frac{1}{x^2 + y^2 + 1} \, ds + \int_{C_2} \frac{1}{x^2 + y^2 + 1} \, ds + \int_{C_3} \frac{1}{x^2 + y^2 + 1} \, ds + \int_{C_4} \frac{1}{x^2 + y^2 + 1} \, ds$
 $= \int_0^1 \frac{dt}{t^2 + 1} + \int_0^1 \frac{dt}{t^2 + 2} + \int_0^1 \frac{dt}{(1 - t)^2 + 2} + \int_0^1 \frac{dt}{(1 - t)^2 + 1}$
 $= [\tan^{-1}t]_0^1 + \frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{t}{\sqrt{2}} \right) \right]_0^1 + \frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{t - 1}{\sqrt{2}} \right) \right]_0^1 + [-\tan^{-1}(1 - t)]_0^1 = \frac{\pi}{2} + \frac{2}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \right)$
27. $\mathbf{r}(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + \frac{x^2}{2}\mathbf{j}$, $0 \leq x \leq 2 \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dx} \right| = \sqrt{1 + x^2}$; $f(x, y) = f \left(x, \frac{x^2}{2} \right) = \frac{x^3}{\left(\frac{x^2}{2} \right)} = 2x \Rightarrow \int_C f \, ds$
 $= \int_0^2 (2x) \sqrt{1 + x^2} \, dx = \left[\frac{2}{3} (1 + x^2)^{3/2} \right]_0^2 = \frac{2}{3} (5^{3/2} - 1) = \frac{10\sqrt{5} - 2}{3}$
28. $\mathbf{r}(t) = (1 - t)\mathbf{i} + \frac{1}{2}(1 - t)^2\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + (1 - t)^2}$; $f(x, y) = f \left((1 - t), \frac{1}{2}(1 - t)^2 \right) = \frac{(1 - t) + \frac{1}{4}(1 - t)^4}{\sqrt{1 + (1 - t)^2}}$
 $\Rightarrow \int_C f \, ds = \int_0^1 \frac{(1 - t) + \frac{1}{4}(1 - t)^4}{\sqrt{1 + (1 - t)^2}} \sqrt{1 + (1 - t)^2} \, dt = \int_0^1 \left((1 - t) + \frac{1}{4}(1 - t)^4 \right) \, dt = \left[-\frac{1}{2}(1 - t)^2 - \frac{1}{20}(1 - t)^5 \right]_0^1$
 $= 0 - \left(-\frac{1}{2} - \frac{1}{20} \right) = \frac{11}{20}$
29. $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j}$, $0 \leq t \leq \frac{\pi}{2} \Rightarrow \frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2$; $f(x, y) = f(2 \cos t, 2 \sin t)$
 $= 2 \cos t + 2 \sin t \Rightarrow \int_C f \, ds = \int_0^{\pi/2} (2 \cos t + 2 \sin t)(2) \, dt = [4 \sin t - 4 \cos t]_0^{\pi/2} = 4 - (-4) = 8$
30. $\mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j}$, $0 \leq t \leq \frac{\pi}{4} \Rightarrow \frac{d\mathbf{r}}{dt} = (2 \cos t)\mathbf{i} + (-2 \sin t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2$; $f(x, y) = f(2 \sin t, 2 \cos t)$
 $= 4 \sin^2 t - 2 \cos t \Rightarrow \int_C f \, ds = \int_0^{\pi/4} (4 \sin^2 t - 2 \cos t)(2) \, dt = [4t - 2 \sin 2t - 4 \sin t]_0^{\pi/4} = \pi - 2(1 + \sqrt{2})$
31. $y = x^2$, $0 \leq x \leq 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 4t^2} \Rightarrow \int_C f(x, y) \, ds$
 $= \int_C (x + \sqrt{y}) \, ds = \int_0^2 (t + \sqrt{t^2}) \sqrt{1 + 4t^2} \, dt = \int_0^2 2t\sqrt{1 + 4t^2} \, dt = \left[\frac{1}{6}(1 + 4t^2)^{3/2} \right]_0^2 = \frac{17\sqrt{17} - 1}{6}$

$$32. 2x + 3y = 6, 0 \leq x \leq 6 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + (2 - \frac{2}{3}t)\mathbf{j}, 0 \leq t \leq 6 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \frac{2}{3}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \frac{\sqrt{13}}{3} \Rightarrow A = \int_C f(x, y) ds$$

$$= \int_C (4 + 3x + 2y) ds = \int_0^6 (4 + 3t + 2(2 - \frac{2}{3}t)) \frac{\sqrt{13}}{3} dt = \frac{\sqrt{13}}{3} \int_0^6 (8 + \frac{5}{3}t) dt = \frac{\sqrt{13}}{3} [8t + \frac{5}{6}t^2]_0^6 = 26\sqrt{13}$$

$$33. \mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2\sqrt{t^2 + 1}; M = \int_C \delta(x, y, z) ds = \int_0^1 \delta(t) (2\sqrt{t^2 + 1}) dt$$

$$= \int_0^1 (\frac{3}{2}t) (2\sqrt{t^2 + 1}) dt = \left[(t^2 + 1)^{3/2} \right]_0^1 = 2^{3/2} - 1 = 2\sqrt{2} - 1$$

$$34. \mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, -1 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k}$$

$$\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2\sqrt{t^2 + 1}; M = \int_C \delta(x, y, z) ds$$

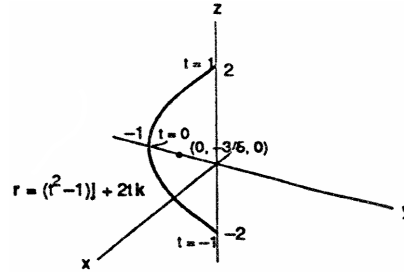
$$= \int_{-1}^1 (15\sqrt{t^2 - 1} + 2) (2\sqrt{t^2 + 1}) dt$$

$$= \int_{-1}^1 30(t^2 + 1) dt = \left[30 \left(\frac{t^3}{3} + t \right) \right]_{-1}^1 = 60 \left(\frac{1}{3} + 1 \right) = 80;$$

$$M_{xz} = \int_C y \delta(x, y, z) ds = \int_{-1}^1 (t^2 - 1) [30(t^2 + 1)] dt$$

$$= \int_{-1}^1 30(t^4 - 1) dt = \left[30 \left(\frac{t^5}{5} - t \right) \right]_{-1}^1 = 60 \left(\frac{1}{5} - 1 \right)$$

$$= -48 \Rightarrow \bar{y} = \frac{M_{xz}}{M} = -\frac{48}{80} = -\frac{3}{5}; M_{yz} = \int_C x \delta(x, y, z) ds = \int_C 0 \delta ds = 0 \Rightarrow \bar{x} = 0; \bar{z} = 0 \text{ by symmetry (since } \delta \text{ is independent of } z) \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = (0, -\frac{3}{5}, 0)$$



$$35. \mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2 + 2 + 4t^2} = 2\sqrt{1 + t^2};$$

$$(a) M = \int_C \delta ds = \int_0^1 (3t) (2\sqrt{1 + t^2}) dt = \left[2(1 + t^2)^{3/2} \right]_0^1 = 2(2^{3/2} - 1) = 4\sqrt{2} - 2$$

$$(b) M = \int_C \delta ds = \int_0^1 (1) (2\sqrt{1 + t^2}) dt = \left[t\sqrt{1 + t^2} + \ln(t + \sqrt{1 + t^2}) \right]_0^1 = \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right] - (0 + \ln 1)$$

$$= \sqrt{2} + \ln(1 + \sqrt{2})$$

$$36. \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} + t^{1/2}\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 4 + t} = \sqrt{5 + t};$$

$$M = \int_C \delta ds = \int_0^2 (3\sqrt{5 + t}) (\sqrt{5 + t}) dt = \int_0^2 3(5 + t) dt = \left[\frac{3}{2}(5 + t)^2 \right]_0^2 = \frac{3}{2}(7^2 - 5^2) = \frac{3}{2}(24) = 36;$$

$$M_{yz} = \int_C x \delta ds = \int_0^2 t[3(5 + t)] dt = \int_0^2 (15t + 3t^2) dt = \left[\frac{15}{2}t^2 + t^3 \right]_0^2 = 30 + 8 = 38;$$

$$M_{xz} = \int_C y \delta ds = \int_0^2 2t[3(5 + t)] dt = 2 \int_0^2 (15t + 3t^2) dt = 76; M_{xy} = \int_C z \delta ds = \int_0^2 \frac{2}{3}t^{3/2}[3(5 + t)] dt$$

$$= \int_0^2 (10t^{3/2} + 2t^{5/2}) dt = \left[4t^{5/2} + \frac{4}{7}t^{7/2} \right]_0^2 = 4(2)^{5/2} + \frac{4}{7}(2)^{7/2} = 16\sqrt{2} + \frac{32}{7}\sqrt{2} = \frac{144}{7}\sqrt{2} \Rightarrow \bar{x} = \frac{M_{yz}}{M}$$

$$= \frac{38}{36} = \frac{19}{18}, \bar{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{144\sqrt{2}}{7 \cdot 36} = \frac{4}{7}\sqrt{2}$$

$$37. \text{ Let } x = a \cos t \text{ and } y = a \sin t, 0 \leq t \leq 2\pi. \text{ Then } \frac{dx}{dt} = -a \sin t, \frac{dy}{dt} = a \cos t, \frac{dz}{dt} = 0$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2} dt = a dt; I_z = \int_C (x^2 + y^2) \delta ds = \int_0^{2\pi} (a^2 \sin^2 t + a^2 \cos^2 t) a \delta dt$$

$$= \int_0^{2\pi} a^3 \delta dt = 2\pi a^3.$$

$$38. \mathbf{r}(t) = t\mathbf{j} + (2 - 2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} - 2\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{5}; M = \int_C \delta ds = \int_0^1 \delta \sqrt{5} dt = \delta \sqrt{5};$$

$$I_x = \int_C (y^2 + z^2) \delta ds = \int_0^1 [t^2 + (2 - 2t)^2] \delta \sqrt{5} dt = \int_0^1 (5t^2 - 8t + 4) \delta \sqrt{5} dt = \delta \sqrt{5} \left[\frac{5}{3}t^3 - 4t^2 + 4t \right]_0^1 = \frac{5}{3} \delta \sqrt{5};$$

$$I_y = \int_C (x^2 + z^2) \delta \, ds = \int_0^1 [0^2 + (2 - 2t)^2] \delta \sqrt{5} \, dt = \int_0^1 (4t^2 - 8t + 4) \delta \sqrt{5} \, dt = \delta \sqrt{5} \left[\frac{4}{3} t^3 - 4t^2 + 4t \right]_0^1 = \frac{4}{3} \delta \sqrt{5};$$

$$I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^1 (0^2 + t^2) \delta \sqrt{5} \, dt = \delta \sqrt{5} \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3} \delta \sqrt{5}$$

39. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2};$

(a) $I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t) \delta \sqrt{2} \, dt = 2\pi \delta \sqrt{2}$

(b) $I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^{4\pi} \delta \sqrt{2} \, dt = 4\pi \delta \sqrt{2}$

40. $\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + \frac{2\sqrt{2}}{3} t^{3/2} \mathbf{k}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} + \sqrt{2} t \mathbf{k}$

$\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(t+1)^2} = t+1$ for $0 \leq t \leq 1$; $M = \int_C \delta \, ds = \int_0^1 (t+1) \, dt = \left[\frac{1}{2} (t+1)^2 \right]_0^1 = \frac{1}{2} (2^2 - 1^2) = \frac{3}{2};$

$M_{xy} = \int_C z \delta \, ds = \int_0^1 \left(\frac{2\sqrt{2}}{3} t^{3/2} \right) (t+1) \, dt = \frac{2\sqrt{2}}{3} \int_0^1 (t^{5/2} + t^{3/2}) \, dt = \frac{2\sqrt{2}}{3} \left[\frac{2}{7} t^{7/2} + \frac{2}{5} t^{5/2} \right]_0^1$
 $= \frac{2\sqrt{2}}{3} \left(\frac{2}{7} + \frac{2}{5} \right) = \frac{2\sqrt{2}}{3} \left(\frac{24}{35} \right) = \frac{16\sqrt{2}}{35} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{16\sqrt{2}}{35} \right) \left(\frac{2}{3} \right) = \frac{32\sqrt{2}}{105}; I_z = \int_C (x^2 + y^2) \delta \, ds$
 $= \int_0^1 (t^2 \cos^2 t + t^2 \sin^2 t) (t+1) \, dt = \int_0^1 (t^3 + t^2) \, dt = \left[\frac{t^4}{4} + \frac{t^3}{3} \right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$

41. $\delta(x, y, z) = 2 - z$ and $\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, $0 \leq t \leq \pi \Rightarrow M = 2\pi - 2$ as found in Example 3 of the text;

also $\left| \frac{d\mathbf{r}}{dt} \right| = 1$; $I_x = \int_C (y^2 + z^2) \delta \, ds = \int_0^\pi (\cos^2 t + \sin^2 t) (2 - \sin t) \, dt = \int_0^\pi (2 - \sin t) \, dt = 2\pi - 2$

42. $\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3} t^{3/2} \mathbf{j} + \frac{t^2}{2} \mathbf{k}$, $0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \sqrt{2} t^{1/2} \mathbf{j} + t \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 2t + t^2} = \sqrt{(1+t)^2} = 1+t$ for

$0 \leq t \leq 2$; $M = \int_C \delta \, ds = \int_0^2 \left(\frac{1}{t+1} \right) (1+t) \, dt = \int_0^2 dt = 2$; $M_{yz} = \int_C x \delta \, ds = \int_0^2 t \left(\frac{1}{t+1} \right) (1+t) \, dt = \left[\frac{t^2}{2} \right]_0^2 = 2$;

$M_{xz} = \int_C y \delta \, ds = \int_0^2 \frac{2\sqrt{2}}{3} t^{3/2} \, dt = \left[\frac{4\sqrt{2}}{15} t^{5/2} \right]_0^2 = \frac{32}{15}$; $M_{xy} = \int_C z \delta \, ds = \int_0^2 \frac{t^2}{2} \, dt = \left[\frac{t^3}{6} \right]_0^2 = \frac{4}{3} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = 1$,

$\bar{y} = \frac{M_{xz}}{M} = \frac{16}{15}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{2}{3}$; $I_x = \int_C (y^2 + z^2) \delta \, ds = \int_0^2 \left(\frac{8}{9} t^3 + \frac{1}{4} t^4 \right) \, dt = \left[\frac{2}{9} t^4 + \frac{t^5}{20} \right]_0^2 = \frac{32}{9} + \frac{32}{20} = \frac{232}{45}$;

$I_y = \int_C (x^2 + z^2) \delta \, ds = \int_0^2 \left(t^2 + \frac{1}{4} t^4 \right) \, dt = \left[\frac{t^3}{3} + \frac{t^5}{20} \right]_0^2 = \frac{8}{3} + \frac{32}{20} = \frac{64}{15}$; $I_z = \int_C (x^2 + y^2) \delta \, ds$

$= \int_0^2 \left(t^2 + \frac{8}{9} t^3 \right) \, dt = \left[\frac{t^3}{3} + \frac{2}{9} t^4 \right]_0^2 = \frac{8}{3} + \frac{32}{9} = \frac{56}{9}$

43-46. Example CAS commands:

Maple:

`f := (x,y,z) -> sqrt(1+30*x^2+10*y);`

`g := t -> t;`

`h := t -> t^2;`

`k := t -> 3*t^2;`

`a,b := 0,2;`

`ds := ( D(g)^2 + D(h)^2 + D(k)^2 )^(1/2);` # (a)

`'ds' = ds(t)*dt;`

`F := f(g,h,k);` # (b)

`F(t) = F(t);`

`Int( f, s=C..NULL ) = Int( simplify(F(t)*ds(t)), t=a..b );` # (c)

``` = value(rhs(%));`

Mathematica: (functions and domains may vary)

```
Clear[x, y, z, r, t, f]
f[x_, y_, z_] := Sqrt[1 + 30x^2 + 10y]
{a, b} = {0, 2};
x[t_] := t
y[t_] := t^2
z[t_] := 3t^2
r[t_] := {x[t], y[t], z[t]}
v[t_] := D[r[t], t]
mag[vector_] := Sqrt[vector.vector]
Integrate[f[x[t], y[t], z[t]] mag[v[t]], {t, a, b}]
N[%]
```

## 16.2 VECTOR FIELDS, WORK, CIRCULATION, AND FLUX

1.  $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}$ ; similarly,  
 $\frac{\partial f}{\partial y} = -y(x^2 + y^2 + z^2)^{-3/2}$  and  $\frac{\partial f}{\partial z} = -z(x^2 + y^2 + z^2)^{-3/2} \Rightarrow \nabla f = \frac{-xi - yj - zk}{(x^2 + y^2 + z^2)^{3/2}}$
2.  $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \ln(x^2 + y^2 + z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{2} \left( \frac{1}{x^2 + y^2 + z^2} \right) (2x) = \frac{x}{x^2 + y^2 + z^2}$ ;  
 similarly,  $\frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2}$  and  $\frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2} \Rightarrow \nabla f = \frac{xi + yj + zk}{x^2 + y^2 + z^2}$
3.  $g(x, y, z) = e^z - \ln(x^2 + y^2) \Rightarrow \frac{\partial g}{\partial x} = -\frac{2x}{x^2 + y^2}, \frac{\partial g}{\partial y} = -\frac{2y}{x^2 + y^2}$  and  $\frac{\partial g}{\partial z} = e^z$   
 $\Rightarrow \nabla g = \left( \frac{-2x}{x^2 + y^2} \right) \mathbf{i} - \left( \frac{2y}{x^2 + y^2} \right) \mathbf{j} + e^z \mathbf{k}$
4.  $g(x, y, z) = xy + yz + xz \Rightarrow \frac{\partial g}{\partial x} = y + z, \frac{\partial g}{\partial y} = x + z, \text{ and } \frac{\partial g}{\partial z} = y + x \Rightarrow \nabla g = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$
5.  $|\mathbf{F}|$  inversely proportional to the square of the distance from  $(x, y)$  to the origin  $\Rightarrow \sqrt{(M(x, y))^2 + (N(x, y))^2} = \frac{k}{x^2 + y^2}, k > 0$ ;  $\mathbf{F}$  points toward the origin  $\Rightarrow \mathbf{F}$  is in the direction of  $\mathbf{n} = \frac{-x}{\sqrt{x^2 + y^2}} \mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j}$   
 $\Rightarrow \mathbf{F} = a\mathbf{n}$ , for some constant  $a > 0$ . Then  $M(x, y) = \frac{-ax}{\sqrt{x^2 + y^2}}$  and  $N(x, y) = \frac{-ay}{\sqrt{x^2 + y^2}}$   
 $\Rightarrow \sqrt{(M(x, y))^2 + (N(x, y))^2} = a \Rightarrow a = \frac{k}{x^2 + y^2} \Rightarrow \mathbf{F} = \frac{-kx}{(x^2 + y^2)^{3/2}} \mathbf{i} - \frac{ky}{(x^2 + y^2)^{3/2}} \mathbf{j}$ , for any constant  $k > 0$
6. Given  $x^2 + y^2 = a^2 + b^2$ , let  $x = \sqrt{a^2 + b^2} \cos t$  and  $y = -\sqrt{a^2 + b^2} \sin t$ . Then  
 $\mathbf{r} = \left( \sqrt{a^2 + b^2} \cos t \right) \mathbf{i} - \left( \sqrt{a^2 + b^2} \sin t \right) \mathbf{j}$  traces the circle in a clockwise direction as  $t$  goes from 0 to  $2\pi$   
 $\Rightarrow \mathbf{v} = \left( -\sqrt{a^2 + b^2} \sin t \right) \mathbf{i} - \left( \sqrt{a^2 + b^2} \cos t \right) \mathbf{j}$  is tangent to the circle in a clockwise direction. Thus, let  
 $\mathbf{F} = \mathbf{v} \Rightarrow \mathbf{F} = y\mathbf{i} - x\mathbf{j}$  and  $\mathbf{F}(0, 0) = \mathbf{0}$ .
7. Substitute the parametric representations for  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  representing each path into the vector field  $\mathbf{F}$ , and calculate  $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$ .  
 (a)  $\mathbf{F} = 3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 9t \Rightarrow \int_0^1 9t dt = \frac{9}{2}$   
 (b)  $\mathbf{F} = 3t^2\mathbf{i} + 2t\mathbf{j} + 4t^4\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 7t^2 + 16t^7 \Rightarrow \int_0^1 (7t^2 + 16t^7) dt = \left[ \frac{7}{3}t^3 + 2t^8 \right]_0^1 = \frac{7}{3} + 2 = \frac{13}{3}$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = 3t\mathbf{i} + 2t\mathbf{j} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 5t \Rightarrow \int_0^1 5t \, dt = \frac{5}{2};$$

$$\mathbf{F}_2 = 3\mathbf{i} + 2\mathbf{j} + 4t\mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 4t \Rightarrow \int_0^1 4t \, dt = 2 \Rightarrow \frac{5}{2} + 2 = \frac{9}{2}$$

8. Substitute the parametric representation for  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  representing each path into the vector field  $\mathbf{F}$ , and calculate  $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$ .

$$(a) \mathbf{F} = \left(\frac{1}{t^2+1}\right)\mathbf{j} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{t^2+1} \Rightarrow \int_0^1 \frac{1}{t^2+1} \, dt = [\tan^{-1} t]_0^1 = \frac{\pi}{4}$$

$$(b) \mathbf{F} = \left(\frac{1}{t^2+1}\right)\mathbf{j} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{2t}{t^2+1} \Rightarrow \int_0^1 \frac{2t}{t^2+1} \, dt = [\ln(t^2+1)]_0^1 = \ln 2$$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = \left(\frac{1}{t^2+1}\right)\mathbf{j} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = \frac{1}{t^2+1}; \mathbf{F}_2 = \frac{1}{2}\mathbf{j} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k}$$

$$\Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0 \Rightarrow \int_0^1 \frac{1}{t^2+1} \, dt = \frac{\pi}{4}$$

9. Substitute the parametric representation for  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  representing each path into the vector field  $\mathbf{F}$ , and calculate  $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$ .

$$(a) \mathbf{F} = \sqrt{t}\mathbf{i} - 2t\mathbf{j} + \sqrt{t}\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2\sqrt{t} - 2t \Rightarrow \int_0^1 (2\sqrt{t} - 2t) \, dt = \left[\frac{4}{3}t^{3/2} - t^2\right]_0^1 = \frac{1}{3}$$

$$(b) \mathbf{F} = t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t^4 - 3t^2 \Rightarrow \int_0^1 (4t^4 - 3t^2) \, dt = \left[\frac{4}{5}t^5 - t^3\right]_0^1 = -\frac{1}{5}$$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = -2t\mathbf{j} + \sqrt{t}\mathbf{k} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -2t \Rightarrow \int_0^1 -2t \, dt = -1;$$

$$\mathbf{F}_2 = \sqrt{t}\mathbf{i} - 2\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow \int_0^1 1 \, dt = 1 \Rightarrow -1 + 1 = 0$$

10. Substitute the parametric representation for  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  representing each path into the vector field  $\mathbf{F}$ , and calculate  $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$ .

$$(a) \mathbf{F} = t^2\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 \Rightarrow \int_0^1 3t^2 \, dt = 1$$

$$(b) \mathbf{F} = t^3\mathbf{i} - t^6\mathbf{j} + t^5\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^7 + 4t^8 \Rightarrow \int_0^1 (t^3 + 2t^7 + 4t^8) \, dt$$

$$= \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4}{9}t^9\right]_0^1 = \frac{17}{18}$$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = t^2\mathbf{i} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = t^2 \Rightarrow \int_0^1 t^2 \, dt = \frac{1}{3};$$

$$\mathbf{F}_2 = \mathbf{i} + t\mathbf{j} + t\mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \int_0^1 t \, dt = \frac{1}{2} \Rightarrow \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

11. Substitute the parametric representation for  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  representing each path into the vector field  $\mathbf{F}$ , and calculate  $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$ .

$$(a) \mathbf{F} = (3t^2 - 3t)\mathbf{i} + 3t\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 + 1 \Rightarrow \int_0^1 (3t^2 + 1) \, dt = [t^3 + t]_0^1 = 2$$

$$(b) \mathbf{F} = (3t^2 - 3t)\mathbf{i} + 3t^4\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 4t^3 + 3t^2 - 3t$$

$$\Rightarrow \int_0^1 (6t^5 + 4t^3 + 3t^2 - 3t) \, dt = \left[t^6 + t^4 + t^3 - \frac{3}{2}t^2\right]_0^1 = \frac{3}{2}$$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = (3t^2 - 3t)\mathbf{i} + \mathbf{k} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 3t^2 - 3t$$

$$\Rightarrow \int_0^1 (3t^2 - 3t) \, dt = \left[t^3 - \frac{3}{2}t^2\right]_0^1 = -\frac{1}{2}; \mathbf{F}_2 = 3t\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow \int_0^1 1 \, dt = 1$$

$$\Rightarrow -\frac{1}{2} + 1 = \frac{1}{2}$$

12. Substitute the parametric representation for  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  representing each path into the vector field  $\mathbf{F}$ , and calculate  $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$ .

$$(a) \mathbf{F} = 2t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t \Rightarrow \int_0^1 6t \, dt = [3t^2]_0^1 = 3$$

$$(b) \mathbf{F} = (t^2 + t^4)\mathbf{i} + (t^4 + t)\mathbf{j} + (t + t^2)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 5t^4 + 3t^2 \\ \Rightarrow \int_0^1 (6t^5 + 5t^4 + 3t^2) dt = [t^6 + t^5 + t^3]_0^1 = 3$$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 2t \Rightarrow \int_0^1 2t dt = 1; \\ \mathbf{F}_2 = (1 + t)\mathbf{i} + (t + 1)\mathbf{j} + 2\mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 2 \Rightarrow \int_0^1 2 dt = 2 \Rightarrow 1 + 2 = 3$$

$$13. x = t, y = 2t + 1, 0 \leq t \leq 3 \Rightarrow dx = dt \Rightarrow \int_C (x - y) dx = \int_0^3 (t - (2t + 1)) dt = \int_0^3 (-t - 1) dt = [-\frac{1}{2}t^2 - t]_0^3 = -\frac{15}{2}$$

$$14. x = t, y = t^2, 1 \leq t \leq 2 \Rightarrow dy = 2t dt \Rightarrow \int_C \frac{x}{y} dy = \int_1^2 \frac{1}{t^2} (2t) dt = \int_1^2 2 dt = [2t]_1^2 = 2$$

$$15. C_1: x = t, y = 0, 0 \leq t \leq 3 \Rightarrow dy = 0; C_2: x = 3, y = t, 0 \leq t \leq 3 \Rightarrow dy = dt \Rightarrow \int_C (x^2 + y^2) dy \\ = \int_{C_1} (x^2 + y^2) dx + \int_{C_2} (x^2 + y^2) dx = \int_0^3 (t^2 + 0^2) \cdot 0 + \int_0^3 (3^2 + t^2) dt = \int_0^3 (9 + t^2) dt = [9t + \frac{1}{3}t^3]_0^3 = 36$$

$$16. C_1: x = t, y = 3t, 0 \leq t \leq 1 \Rightarrow dx = dt; C_2: x = 1 - t, y = 3, 0 \leq t \leq 1 \Rightarrow dx = -dt; C_3: x = 0, y = 3 - t, 0 \leq t \leq 3 \\ \Rightarrow dx = 0 \Rightarrow \int_C \sqrt{x + y} dx = \int_{C_1} \sqrt{x + y} dx + \int_{C_2} \sqrt{x + y} dx + \int_{C_3} \sqrt{x + y} dx \\ = \int_0^1 \sqrt{t + 3t} dt + \int_0^1 \sqrt{(1 - t) + 3} (-1) dt + \int_0^3 \sqrt{0 + (3 - t)} \cdot 0 = \int_0^1 2\sqrt{t} dt - \int_0^1 \sqrt{4 - t} dt \\ = [\frac{4}{3}t^{3/2}]_0^1 + [\frac{2}{3}(4 - t)^{3/2}]_0^1 = \frac{4}{3} + (2\sqrt{3} - \frac{16}{3}) = 2\sqrt{3} - 4$$

$$17. \mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}, 0 \leq t \leq 1 \Rightarrow dx = dt, dy = 0, dz = 2t dt$$

$$(a) \int_C (x + y - z) dx = \int_0^1 (t - 1 - t^2) dt = [\frac{1}{2}t^2 - t - \frac{1}{3}t^3]_0^1 = -\frac{5}{6}$$

$$(b) \int_C (x + y - z) dy = \int_0^1 (t - 1 - t^2) \cdot 0 = 0$$

$$(c) \int_C (x + y - z) dz = \int_0^1 (t - 1 - t^2) 2t dt = \int_0^1 (2t^2 - 2t - 2t^3) dt = [\frac{2}{3}t^3 - t^2 - \frac{1}{2}t^4]_0^1 = -\frac{5}{6}$$

$$18. \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}, 0 \leq t \leq \pi \Rightarrow dx = -\sin t dt, dy = \cos t dt, dz = \sin t dt$$

$$(a) \int_C xz dx = \int_0^\pi (\cos t) (-\cos t) (-\sin t) dt = \int_0^\pi \cos^2 t \sin t dt = [-\frac{1}{3}(\cos t)^3]_0^\pi = \frac{2}{3}$$

$$(b) \int_C xz dy = \int_0^\pi (\cos t) (-\cos t) (\cos t) dt = -\int_0^\pi \cos^3 t dt = -\int_0^\pi (1 - \sin^2 t) \cos t dt = [\frac{1}{3}(\sin t)^3 - \sin t]_0^\pi = 0$$

$$(c) \int_C xy dz = \int_0^\pi (\cos t) (\sin t) (-\cos t) (\sin t) dt = -\int_0^\pi \cos^2 t \sin^2 t dt = -\frac{1}{4} \int_0^\pi \sin^2 2t dt = -\frac{1}{4} \int_0^\pi \frac{1 - \cos 4t}{2} dt \\ = -\frac{1}{8} \int_0^\pi (1 - \cos 4t) dt = [-\frac{1}{8}t + \frac{1}{32}\sin 4t]_0^\pi = -\frac{\pi}{8}$$

$$19. \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1, \text{ and } \mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} \Rightarrow \mathbf{F} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$$

$$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t^3 \Rightarrow \text{work} = \int_0^1 2t^3 dt = \frac{1}{2}$$

$$20. \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k}, 0 \leq t \leq 2\pi, \text{ and } \mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$$

$$\Rightarrow \mathbf{F} = (2 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + (\cos t + \sin t)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \\ = 3 \cos^2 t - 2 \sin^2 t + \frac{1}{6} \cos t + \frac{1}{6} \sin t \Rightarrow \text{work} = \int_0^{2\pi} (3 \cos^2 t - 2 \sin^2 t + \frac{1}{6} \cos t + \frac{1}{6} \sin t) dt \\ = [\frac{3}{2}t + \frac{3}{4}\sin 2t - t + \frac{\sin 2t}{2} + \frac{1}{6}\sin t - \frac{1}{6}\cos t]_0^{2\pi} = \pi$$

21.  $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq 2\pi$ , and  $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$  and  
 $\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin^2 t + \cos t \Rightarrow \text{work} = \int_0^{2\pi} (t \cos t - \sin^2 t + \cos t) dt$   
 $= \left[ \cos t + t \sin t - \frac{t}{2} + \frac{\sin 2t}{4} + \sin t \right]_0^{2\pi} = -\pi$
22.  $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{1}{6}\mathbf{k}$ ,  $0 \leq t \leq 2\pi$ , and  $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\cos^2 t)\mathbf{j} + (12 \sin t)\mathbf{k}$  and  
 $\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin t \cos^2 t + 2 \sin t$   
 $\Rightarrow \text{work} = \int_0^{2\pi} (t \cos t - \sin t \cos^2 t + 2 \sin t) dt = \left[ \cos t + t \sin t + \frac{1}{3} \cos^3 t - 2 \cos t \right]_0^{2\pi} = 0$
23.  $x = t$  and  $y = x^2 = t^2 \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j}$ ,  $-1 \leq t \leq 2$ , and  $\mathbf{F} = xy\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t^3\mathbf{i} + (t+t^2)\mathbf{j}$  and  
 $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + (2t^2 + 2t^3) = 3t^3 + 2t^2 \Rightarrow \int_C xy dx + (x+y) dy = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_{-1}^2 (3t^3 + 2t^2) dt$   
 $= \left[ \frac{3}{4}t^4 + \frac{2}{3}t^3 \right]_{-1}^2 = \left( 12 + \frac{16}{3} \right) - \left( \frac{3}{4} - \frac{2}{3} \right) = \frac{45}{4} + \frac{18}{3} = \frac{69}{4}$
24. Along  $(0, 0)$  to  $(1, 0)$ :  $\mathbf{r} = t\mathbf{i}$ ,  $0 \leq t \leq 1$ , and  $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t\mathbf{i} + t\mathbf{j}$  and  $\frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t$ ;  
 Along  $(1, 0)$  to  $(0, 1)$ :  $\mathbf{r} = (1-t)\mathbf{i} + t\mathbf{j}$ ,  $0 \leq t \leq 1$ , and  $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (1-2t)\mathbf{i} + \mathbf{j}$  and  
 $\frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$ ;  
 Along  $(0, 1)$  to  $(0, 0)$ :  $\mathbf{r} = (1-t)\mathbf{j}$ ,  $0 \leq t \leq 1$ , and  $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (t-1)\mathbf{i} + (1-t)\mathbf{j}$  and  
 $\frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t-1 \Rightarrow \int_C (x-y) dx + (x+y) dy = \int_0^1 t dt + \int_0^1 2t dt + \int_0^1 (t-1) dt = \int_0^1 (4t-1) dt$   
 $= [2t^2 - t]_0^1 = 2 - 1 = 1$
25.  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = y^2\mathbf{i} + y\mathbf{j}$ ,  $2 \geq y \geq -1$ , and  $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} = y^4\mathbf{i} - y\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dy} = 2y\mathbf{i} + \mathbf{j}$  and  $\mathbf{F} \cdot \frac{d\mathbf{r}}{dy} = 2y^5 - y$   
 $\Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_2^{-1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dy} dy = \int_2^{-1} (2y^5 - y) dy = \left[ \frac{1}{3}y^6 - \frac{1}{2}y^2 \right]_2^{-1} = \left( \frac{1}{3} - \frac{1}{2} \right) - \left( \frac{64}{3} - \frac{4}{2} \right) = \frac{3}{2} - \frac{63}{3} = -\frac{39}{2}$
26.  $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ ,  $0 \leq t \leq \frac{\pi}{2}$ , and  $\mathbf{F} = y\mathbf{i} - x\mathbf{j} \Rightarrow \mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j}$  and  $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$   
 $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin^2 t - \cos^2 t = -1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-1) dt = -\frac{\pi}{2}$
27.  $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j}$ ,  $0 \leq t \leq 1$ , and  $\mathbf{F} = xy\mathbf{i} + (y-x)\mathbf{j} \Rightarrow \mathbf{F} = (1+3t+2t^2)\mathbf{i} + t\mathbf{j}$  and  
 $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 1+5t+2t^2 \Rightarrow \text{work} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^1 (1+5t+2t^2) dt = \left[ t + \frac{5}{2}t^2 + \frac{2}{3}t^3 \right]_0^1 = \frac{25}{6}$
28.  $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j}$ ,  $0 \leq t \leq 2\pi$ , and  $\mathbf{F} = \nabla f = 2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}$   
 $\Rightarrow \mathbf{F} = 4(\cos t + \sin t)\mathbf{i} + 4(\cos t + \sin t)\mathbf{j}$  and  $\frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$   
 $= -8(\sin t \cos t + \sin^2 t) + 8(\cos^2 t + \cos t \sin t) = 8(\cos^2 t - \sin^2 t) = 8 \cos 2t \Rightarrow \text{work} = \int_C \nabla f \cdot d\mathbf{r}$   
 $= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{2\pi} 8 \cos 2t dt = [4 \sin 2t]_0^{2\pi} = 0$
29. (a)  $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ ,  $0 \leq t \leq 2\pi$ ,  $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$ , and  $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ ,  
 $\mathbf{F}_1 = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ , and  $\mathbf{F}_2 = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 0$  and  $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1$   
 $\Rightarrow \text{Circ}_1 = \int_0^{2\pi} 0 dt = 0$  and  $\text{Circ}_2 = \int_0^{2\pi} 1 dt = 2\pi$ ;  $\mathbf{n} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n} = \cos^2 t + \sin^2 t = 1$  and  
 $\mathbf{F}_2 \cdot \mathbf{n} = 0 \Rightarrow \text{Flux}_1 = \int_0^{2\pi} 1 dt = 2\pi$  and  $\text{Flux}_2 = \int_0^{2\pi} 0 dt = 0$
- (b)  $\mathbf{r} = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$ ,  $0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (4 \cos t)\mathbf{j}$ ,  $\mathbf{F}_1 = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$ , and  
 $\mathbf{F}_2 = (-4 \sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 15 \sin t \cos t$  and  $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = 4 \Rightarrow \text{Circ}_1 = \int_0^{2\pi} 15 \sin t \cos t dt$   
 $= \left[ \frac{15}{2} \sin^2 t \right]_0^{2\pi} = 0$  and  $\text{Circ}_2 = \int_0^{2\pi} 4 dt = 8\pi$ ;  $\mathbf{n} = \left( \frac{4}{\sqrt{17}} \cos t \right)\mathbf{i} + \left( \frac{1}{\sqrt{17}} \sin t \right)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n}$



$$= \frac{4}{\sqrt{17}} \cos^2 t + \frac{4}{\sqrt{17}} \sin^2 t \text{ and } \mathbf{F}_2 \cdot \mathbf{n} = -\frac{15}{\sqrt{17}} \sin t \cos t \Rightarrow \text{Flux}_1 = \int_0^{2\pi} (\mathbf{F}_1 \cdot \mathbf{n}) |\mathbf{v}| dt = \int_0^{2\pi} \left( \frac{4}{\sqrt{17}} \right) \sqrt{17} dt$$

$$= 8\pi \text{ and } \text{Flux}_2 = \int_0^{2\pi} (\mathbf{F}_2 \cdot \mathbf{n}) |\mathbf{v}| dt = \int_0^{2\pi} \left( -\frac{15}{\sqrt{17}} \sin t \cos t \right) \sqrt{17} dt = \left[ -\frac{15}{2} \sin^2 t \right]_0^{2\pi} = 0$$

30.  $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$ ,  $0 \leq t \leq 2\pi$ ,  $\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j}$ , and  $\mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$ ,  
 $\mathbf{F}_1 = (2a \cos t)\mathbf{i} - (3a \sin t)\mathbf{j}$ , and  $\mathbf{F}_2 = (2a \cos t)\mathbf{i} + (a \cos t - a \sin t)\mathbf{j} \Rightarrow \mathbf{n} |\mathbf{v}| = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$ ,  
 $\mathbf{F}_1 \cdot \mathbf{n} |\mathbf{v}| = 2a^2 \cos^2 t - 3a^2 \sin^2 t$ , and  $\mathbf{F}_2 \cdot \mathbf{n} |\mathbf{v}| = 2a^2 \cos^2 t + a^2 \sin t \cos t - a^2 \sin^2 t$   
 $\Rightarrow \text{Flux}_1 = \int_0^{2\pi} (2a^2 \cos^2 t - 3a^2 \sin^2 t) dt = 2a^2 \left[ \frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} - 3a^2 \left[ \frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = -\pi a^2$ , and  
 $\text{Flux}_2 = \int_0^{2\pi} (2a^2 \cos^2 t - a^2 \sin t \cos t - a^2 \sin^2 t) dt = 2a^2 \left[ \frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} + \frac{a^2}{2} [\sin^2 t]_0^{2\pi} - a^2 \left[ \frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = \pi a^2$

31.  $\mathbf{F}_1 = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$ ,  $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 0 \Rightarrow \text{Circ}_1 = 0$ ;  $M_1 = a \cos t$ ,  
 $N_1 = a \sin t$ ,  $dx = -a \sin t dt$ ,  $dy = a \cos t dt \Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (a^2 \cos^2 t + a^2 \sin^2 t) dt$   
 $= \int_0^\pi a^2 dt = a^2 \pi$ ;  
 $\mathbf{F}_2 = t\mathbf{i}$ ,  $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \text{Circ}_2 = \int_{-a}^a t dt = 0$ ;  $M_2 = t$ ,  $N_2 = 0$ ,  $dx = dt$ ,  $dy = 0 \Rightarrow \text{Flux}_2$   
 $= \int_C M_2 dy - N_2 dx = \int_{-a}^a 0 dt = 0$ ; therefore,  $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$  and  $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = a^2 \pi$

32.  $\mathbf{F}_1 = (a^2 \cos^2 t)\mathbf{i} + (a^2 \sin^2 t)\mathbf{j}$ ,  $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t$   
 $\Rightarrow \text{Circ}_1 = \int_0^\pi (-a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t) dt = -\frac{2a^3}{3}$ ;  $M_1 = a^2 \cos^2 t$ ,  $N_1 = a^2 \sin^2 t$ ,  $dy = a \cos t dt$ ,  
 $dx = -a \sin t dt \Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (a^3 \cos^3 t + a^3 \sin^3 t) dt = \frac{4}{3} a^3$ ;  
 $\mathbf{F}_2 = t^2 \mathbf{i}$ ,  $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t^2 \Rightarrow \text{Circ}_2 = \int_{-a}^a t^2 dt = \frac{2a^3}{3}$ ;  $M_2 = t^2$ ,  $N_2 = 0$ ,  $dy = 0$ ,  $dx = dt$   
 $\Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2 dx = 0$ ; therefore,  $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$  and  $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = \frac{4}{3} a^3$

33.  $\mathbf{F}_1 = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$ ,  $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^2 \sin^2 t + a^2 \cos^2 t = a^2$   
 $\Rightarrow \text{Circ}_1 = \int_0^\pi a^2 dt = a^2 \pi$ ;  $M_1 = -a \sin t$ ,  $N_1 = a \cos t$ ,  $dx = -a \sin t dt$ ,  $dy = a \cos t dt$   
 $\Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (-a^2 \sin t \cos t + a^2 \sin t \cos t) dt = 0$ ;  $\mathbf{F}_2 = t\mathbf{j}$ ,  $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$   
 $\Rightarrow \text{Circ}_2 = 0$ ;  $M_2 = 0$ ,  $N_2 = t$ ,  $dx = dt$ ,  $dy = 0 \Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2 dx = \int_{-a}^a -t dt = 0$ ; therefore,  
 $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = a^2 \pi$  and  $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

34.  $\mathbf{F}_1 = (-a^2 \sin^2 t)\mathbf{i} + (a^2 \cos^2 t)\mathbf{j}$ ,  $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^3 \sin^3 t + a^3 \cos^3 t$   
 $\Rightarrow \text{Circ}_1 = \int_0^\pi (a^3 \sin^3 t + a^3 \cos^3 t) dt = \frac{4}{3} a^3$ ;  $M_1 = -a^2 \sin^2 t$ ,  $N_1 = a^2 \cos^2 t$ ,  $dy = a \cos t dt$ ,  $dx = -a \sin t dt$   
 $\Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (-a^3 \cos t \sin^2 t + a^3 \sin t \cos^2 t) dt = \frac{2}{3} a^3$ ;  $\mathbf{F}_2 = t^2 \mathbf{j}$ ,  $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$   
 $\Rightarrow \text{Circ}_2 = 0$ ;  $M_2 = 0$ ,  $N_2 = t^2$ ,  $dy = 0$ ,  $dx = dt \Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2 dx = \int_{-a}^a -t^2 dt = -\frac{2}{3} a^3$ ; therefore,  
 $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = \frac{4}{3} a^3$  and  $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

35. (a)  $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ ,  $0 \leq t \leq \pi$ , and  $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$  and  
 $\mathbf{F} = (\cos t + \sin t)\mathbf{i} - (\cos^2 t + \sin^2 t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t - \sin^2 t - \cos t \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} ds$   
 $= \int_0^\pi (-\sin t \cos t - \sin^2 t - \cos t) dt = \left[ -\frac{1}{2} \sin^2 t - \frac{t}{2} + \frac{\sin 2t}{4} - \sin t \right]_0^\pi = -\frac{\pi}{2}$   
 (b)  $\mathbf{r} = (1 - 2t)\mathbf{i}$ ,  $0 \leq t \leq 1$ , and  $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = -2\mathbf{i}$  and  $\mathbf{F} = (1 - 2t)\mathbf{i} - (1 - 2t)^2 \mathbf{j} \Rightarrow$   
 $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t - 2 \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^1 (4t - 2) dt = [2t^2 - 2t]_0^1 = 0$

$$\begin{aligned}
(c) \quad \mathbf{r}_1 &= (1-t)\mathbf{i} - t\mathbf{j}, 0 \leq t \leq 1, \text{ and } \mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j} \text{ and } \mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j} \\
&\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = (2t-1) + (1-2t+2t^2) = 2t^2 \Rightarrow \text{Flow}_1 = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = \int_0^1 2t^2 dt = \frac{2}{3}; \mathbf{r}_2 = -t\mathbf{i} + (t-1)\mathbf{j}, \\
0 \leq t \leq 1, \text{ and } \mathbf{F} &= (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j} \text{ and } \mathbf{F} = -\mathbf{i} - (t^2+t^2-2t+1)\mathbf{j} \\
&= -\mathbf{i} - (2t^2-2t+1)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = 1 - (2t^2-2t+1) = 2t-2t^2 \Rightarrow \text{Flow}_2 = \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = \int_0^1 (2t-2t^2) dt \\
&= \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1
\end{aligned}$$

$$\begin{aligned}
36. \text{ From } (1, 0) \text{ to } (0, 1): \quad \mathbf{r}_1 &= (1-t)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1, \text{ and } \mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} + \mathbf{j}, \\
\mathbf{F} &= \mathbf{i} - (1-2t+2t^2)\mathbf{j}, \text{ and } \mathbf{n}_1 |\mathbf{v}_1| = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_1 |\mathbf{v}_1| = 2t-2t^2 \Rightarrow \text{Flux}_1 = \int_0^1 (2t-2t^2) dt \\
&= \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3}; \\
\text{From } (0, 1) \text{ to } (-1, 0): \quad \mathbf{r}_2 &= -t\mathbf{i} + (1-t)\mathbf{j}, 0 \leq t \leq 1, \text{ and } \mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} - \mathbf{j}, \\
\mathbf{F} &= (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}, \text{ and } \mathbf{n}_2 |\mathbf{v}_2| = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_2 |\mathbf{v}_2| = (2t-1) + (-1+2t-2t^2) = -2+4t-2t^2 \\
&\Rightarrow \text{Flux}_2 = \int_0^1 (-2+4t-2t^2) dt = \left[-2t+2t^2-\frac{2}{3}t^3\right]_0^1 = -\frac{2}{3}; \\
\text{From } (-1, 0) \text{ to } (1, 0): \quad \mathbf{r}_3 &= (-1+t)\mathbf{i}, 0 \leq t \leq 1, \text{ and } \mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_3}{dt} = 2\mathbf{i}, \\
\mathbf{F} &= (-1+2t)\mathbf{i} - (1-4t+4t^2)\mathbf{j}, \text{ and } \mathbf{n}_3 |\mathbf{v}_3| = -2\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_3 |\mathbf{v}_3| = 2(1-4t+4t^2) \\
&\Rightarrow \text{Flux}_3 = 2 \int_0^1 (1-4t+4t^2) dt = 2 \left[t-2t^2+\frac{4}{3}t^3\right]_0^1 = \frac{2}{3} \Rightarrow \text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}
\end{aligned}$$

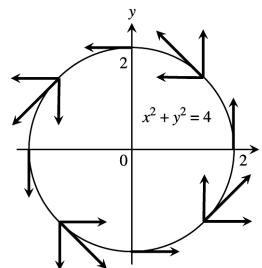
$$\begin{aligned}
37. (a) \quad y &= 2x, 0 \leq x \leq 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((2t)^2\mathbf{i} + 2(t)(2t)\mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) \\
&= 4t^2 + 8t^2 = 12t^2 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^2 12t^2 dt = [4t^3]_0^2 = 32 \\
(b) \quad y &= x^2, 0 \leq x \leq 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((t^2)^2\mathbf{i} + 2(t)(t^2)\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) \\
&= t^4 + 4t^4 = 5t^4 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^2 5t^4 dt = [t^5]_0^2 = 32 \\
(c) \quad \text{answers will vary, one possible path is } y &= \frac{1}{2}x^3, 0 \leq x \leq 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^3\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 3t^2\mathbf{j} \\
&\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \left(\left(\frac{1}{2}t^3\right)^2\mathbf{i} + 2(t)\left(\frac{1}{2}t^3\right)\mathbf{j}\right) \cdot (\mathbf{i} + 3t^2\mathbf{j}) = \frac{1}{4}t^6 + \frac{3}{2}t^6 = \frac{7}{4}t^6 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^2 \frac{7}{4}t^6 dt = \left[\frac{1}{4}t^7\right]_0^2 \\
&= 32
\end{aligned}$$

$$\begin{aligned}
38. (a) \quad C_1: \mathbf{r}(t) &= (1-t)\mathbf{i} + \mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1)\mathbf{i} + ((1-t) + 2(1))\mathbf{j}) \cdot (-\mathbf{i}) = -1; \\
C_2: \mathbf{r}(t) &= -\mathbf{i} + (1-t)\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1-t)\mathbf{i} + ((-1) + 2(1-t))\mathbf{j}) \cdot (-\mathbf{j}) = 2t-1; \\
C_3: \mathbf{r}(t) &= (t-1)\mathbf{i} - \mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((-1)\mathbf{i} + ((t-1) + 2(-1))\mathbf{j}) \cdot (\mathbf{i}) = -1; \\
C_4: \mathbf{r}(t) &= \mathbf{i} + (t-1)\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((t-1)\mathbf{i} + ((1) + 2(t-1))\mathbf{j}) \cdot (\mathbf{j}) = 2t-1; \\
&\Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_3} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_4} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\
&= \int_0^2 (-1) dt + \int_0^2 (2t-1) dt + \int_0^2 (-1) dt + \int_0^2 (2t-1) dt = [-t]_0^2 + [t^2-t]_0^2 + [-t]_0^2 + [t^2-t]_0^2 \\
&= -2+2-2+2=0 \\
(b) \quad x^2 + y^2 &= 4 \Rightarrow \mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}, 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \\
&\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((2\sin t)\mathbf{i} + (2\cos t + 2(2\sin t))\mathbf{j}) \cdot ((-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}) = -4\sin^2 t + 4\cos^2 t + 8\sin t \cos t \\
&= 4\cos 2t + 4\sin 2t \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{2\pi} (4\cos 2t + 4\sin 2t) dt = [2\sin 2t - 2\cos 2t]_0^{2\pi} = 0 \\
(c) \quad \text{answers will vary, one possible path is:} \\
C_1: \mathbf{r}(t) &= t\mathbf{i}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((0)\mathbf{i} + (t+2(1))\mathbf{j}) \cdot (\mathbf{i}) = 0; \\
C_2: \mathbf{r}(t) &= (1-t)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = (t\mathbf{i} + ((1-t) + 2t)\mathbf{j}) \cdot (-\mathbf{i} + \mathbf{j}) = 1; \\
C_3: \mathbf{r}(t) &= (1-t)\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1-t)\mathbf{i} + (0+2(1-t))\mathbf{j}) \cdot (-\mathbf{j}) = 2t-1;
\end{aligned}$$

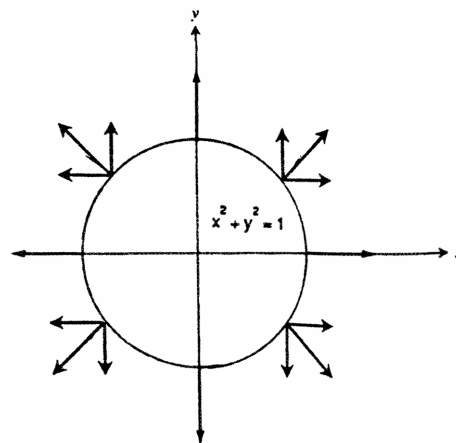
$$\Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_3} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^1 (0) dt + \int_0^1 (1) dt + \int_0^1 (2t - 1) dt$$

$$= 0 + [t]_0^1 + [t^2 - t]_0^1 = 1 + (-1) = 0$$

39.  $\mathbf{F} = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$  on  $x^2 + y^2 = 4$ ;  
 at  $(2, 0)$ ,  $\mathbf{F} = \mathbf{j}$ ; at  $(0, 2)$ ,  $\mathbf{F} = -\mathbf{i}$ ; at  $(-2, 0)$ ,  
 $\mathbf{F} = -\mathbf{j}$ ; at  $(0, -2)$ ,  $\mathbf{F} = \mathbf{i}$ ; at  $(\sqrt{2}, \sqrt{2})$ ,  $\mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$ ;  
 at  $(\sqrt{2}, -\sqrt{2})$ ,  $\mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$ ; at  $(-\sqrt{2}, \sqrt{2})$ ,  
 $\mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ ; at  $(-\sqrt{2}, -\sqrt{2})$ ,  $\mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$



40.  $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$  on  $x^2 + y^2 = 1$ ; at  $(1, 0)$ ,  $\mathbf{F} = \mathbf{i}$ ;  
 at  $(-1, 0)$ ,  $\mathbf{F} = -\mathbf{i}$ ; at  $(0, 1)$ ,  $\mathbf{F} = \mathbf{j}$ ; at  $(0, -1)$ ,  
 $\mathbf{F} = -\mathbf{j}$ ; at  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ ,  $\mathbf{F} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ ;  
 at  $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ ,  $\mathbf{F} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ ;  
 at  $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ ,  $\mathbf{F} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$ ;  
 at  $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ ,  $\mathbf{F} = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$ .



41. (a)  $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$  is to have a magnitude  $\sqrt{a^2 + b^2}$  and to be tangent to  $x^2 + y^2 = a^2 + b^2$  in a counterclockwise direction. Thus  $x^2 + y^2 = a^2 + b^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$  is the slope of the tangent line at any point on the circle  $\Rightarrow y' = -\frac{a}{b}$  at  $(a, b)$ . Let  $\mathbf{v} = -b\mathbf{i} + a\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{a^2 + b^2}$ , with  $\mathbf{v}$  in a counterclockwise direction and tangent to the circle. Then let  $P(x, y) = -y$  and  $Q(x, y) = x$   
 $\Rightarrow \mathbf{G} = -y\mathbf{i} + x\mathbf{j} \Rightarrow$  for  $(a, b)$  on  $x^2 + y^2 = a^2 + b^2$  we have  $\mathbf{G} = -b\mathbf{i} + a\mathbf{j}$  and  $|\mathbf{G}| = \sqrt{a^2 + b^2}$ .  
 (b)  $\mathbf{G} = (\sqrt{x^2 + y^2})\mathbf{F} = (\sqrt{a^2 + b^2})\mathbf{F}$ .

42. (a) From Exercise 41, part a,  $-y\mathbf{i} + x\mathbf{j}$  is a vector tangent to the circle and pointing in a counterclockwise direction  $\Rightarrow y\mathbf{i} - x\mathbf{j}$  is a vector tangent to the circle pointing in a clockwise direction  $\Rightarrow \mathbf{G} = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$  is a unit vector tangent to the circle and pointing in a clockwise direction.  
 (b)  $\mathbf{G} = -\mathbf{F}$

43. The slope of the line through  $(x, y)$  and the origin is  $\frac{y}{x} \Rightarrow \mathbf{v} = x\mathbf{i} + y\mathbf{j}$  is a vector parallel to that line and pointing away from the origin  $\Rightarrow \mathbf{F} = -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$  is the unit vector pointing toward the origin.

44. (a) From Exercise 43,  $-\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$  is a unit vector through  $(x, y)$  pointing toward the origin and we want  $|\mathbf{F}|$  to have magnitude  $\sqrt{x^2 + y^2} \Rightarrow \mathbf{F} = \sqrt{x^2 + y^2} \left( -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} \right) = -x\mathbf{i} - y\mathbf{j}$ .  
 (b) We want  $|\mathbf{F}| = \frac{C}{\sqrt{x^2 + y^2}}$  where  $C \neq 0$  is a constant  $\Rightarrow \mathbf{F} = \frac{C}{\sqrt{x^2 + y^2}} \left( -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} \right) = -C \left( \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2} \right)$ .

45. Yes. The work and area have the same numerical value because  $\text{work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y \mathbf{i} \cdot d\mathbf{r}$   
 $= \int_b^a [f(t)\mathbf{i}] \cdot \left[\mathbf{i} + \frac{df}{dt}\mathbf{j}\right] dt$  [On the path,  $y$  equals  $f(t)$ ]  
 $= \int_a^b f(t) dt = \text{Area under the curve}$  [because  $f(t) > 0$ ]

46.  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + f'(x)\mathbf{j}$ ;  $\mathbf{F} = \frac{k}{\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j})$  has constant magnitude  $k$  and points away from the origin  $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dx} = \frac{kx}{\sqrt{x^2 + y^2}} + \frac{k \cdot y \cdot f'(x)}{\sqrt{x^2 + y^2}} = \frac{kx + k \cdot f(x) \cdot f'(x)}{\sqrt{x^2 + [f(x)]^2}} = k \frac{d}{dx} \sqrt{x^2 + [f(x)]^2}$ , by the chain rule  
 $\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dx} dx = \int_a^b k \frac{d}{dx} \sqrt{x^2 + [f(x)]^2} dx = k [\sqrt{x^2 + [f(x)]^2}]_a^b$   
 $= k (\sqrt{b^2 + [f(b)]^2} - \sqrt{a^2 + [f(a)]^2})$ , as claimed.

47.  $\mathbf{F} = -4t^3\mathbf{i} + 8t^2\mathbf{j} + 2\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 12t^3 \Rightarrow \text{Flow} = \int_0^2 12t^3 dt = [3t^4]_0^2 = 48$

48.  $\mathbf{F} = 12t^2\mathbf{j} + 9t^2\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = 3\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 72t^2 \Rightarrow \text{Flow} = \int_0^1 72t^2 dt = [24t^3]_0^1 = 24$

49.  $\mathbf{F} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + 1$   
 $\Rightarrow \text{Flow} = \int_0^\pi (-\sin t \cos t + 1) dt = \left[\frac{1}{2} \cos^2 t + t\right]_0^\pi = \left(\frac{1}{2} + \pi\right) - \left(\frac{1}{2} + 0\right) = \pi$

50.  $\mathbf{F} = (-2 \sin t)\mathbf{i} - (2 \cos t)\mathbf{j} + 2\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4 \sin^2 t - 4 \cos^2 t + 4 = 0$   
 $\Rightarrow \text{Flow} = 0$

51.  $C_1$ :  $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq \frac{\pi}{2} \Rightarrow \mathbf{F} = (2 \cos t)\mathbf{i} + 2t\mathbf{j} + (2 \sin t)\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$   
 $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -2 \cos t \sin t + 2t \cos t + 2 \sin t = -\sin 2t + 2t \cos t + 2 \sin t$   
 $\Rightarrow \text{Flow}_1 = \int_0^{\pi/2} (-\sin 2t + 2t \cos t + 2 \sin t) dt = \left[\frac{1}{2} \cos 2t + 2t \sin t + 2 \cos t - 2 \cos t\right]_0^{\pi/2} = -1 + \pi$ ;  
 $C_2$ :  $\mathbf{r} = \mathbf{j} + \frac{\pi}{2}(1-t)\mathbf{k}$ ,  $0 \leq t \leq 1 \Rightarrow \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = -\frac{\pi}{2}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\pi$   
 $\Rightarrow \text{Flow}_2 = \int_0^1 -\pi dt = [-\pi t]_0^1 = -\pi$ ;  
 $C_3$ :  $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j}$ ,  $0 \leq t \leq 1 \Rightarrow \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{k}$  and  $\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$   
 $\Rightarrow \text{Flow}_3 = \int_0^1 2t dt = [t^2]_0^1 = 1 \Rightarrow \text{Circulation} = (-1 + \pi) - \pi + 1 = 0$

52.  $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$ , where  $f(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(f(\mathbf{r}(t)))$   
by the chain rule  $\Rightarrow \text{Circulation} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t))) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ . Since  $C$  is an entire ellipse,  $\mathbf{r}(b) = \mathbf{r}(a)$ , thus the Circulation = 0.

53. Let  $x = t$  be the parameter  $\Rightarrow y = x^2 = t^2$  and  $z = x = t \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq 1$  from  $(0, 0, 0)$  to  $(1, 1, 1)$   
 $\Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$  and  $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^3 - t^3 = 2t^3 \Rightarrow \text{Flow} = \int_0^1 2t^3 dt = \frac{1}{2}$

54. (a)  $\mathbf{F} = \nabla(xy^2z^3) \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = \frac{df}{dt}$ , where  $f(x, y, z) = xy^2z^3 \Rightarrow \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t))) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0$  since  $C$  is an entire ellipse.  
(b)  $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_{(1,1,1)}^{(2,1,-1)} \frac{d}{dt}(xy^2z^3) dt = [xy^2z^3]_{(1,1,1)}^{(2,1,-1)} = (2)(1)^2(-1)^3 - (1)(1)^2(1)^3 = -2 - 1 = -3$

55-60. Example CAS commands:

Maple:

```
with(LinearAlgebra);#55
F := r -> < r[1]*r[2]^6 | 3*r[1]*(r[1]*r[2]^5+2) >;
r := t -> < 2*cos(t) | sin(t) >;
a,b := 0,2*Pi;
dr := map(diff,r(t),t); # (a)
F(r(t)); # (b)
q1 := simplify(F(r(t)) . dr) assuming t::real; # (c)
q2 := Int(q1, t=a..b);
value(q2);
```

Mathematica: (functions and bounds will vary):

Exercises 55 and 56 use vectors in 2 dimensions

```
Clear[x, y, t, f, r, v]
f[x_, y_]:= {x y^6, 3x (x y^5 + 2)}
{a, b}={0, 2Pi};
x[t_]:= 2 Cos[t]
y[t_]:= Sin[t]
r[t_]:= {x[t], y[t]}
v[t_]:= r'[t]
integrand= f[x[t], y[t]] . v[t] //Simplify
Integrate[integrand,{t, a, b}]
N[%]
```

If the integration takes too long or cannot be done, use NIntegrate to integrate numerically. This is suggested for exercises 57 - 60 that use vectors in 3 dimensions. Be certain to leave spaces between variables to be multiplied.

```
Clear[x, y, z, t, f, r, v]
f[x_, y_, z_]:= {y + y z Cos[x y z], x^2 + x z Cos[x y z], z + x y Cos[x y z]}
{a, b}={0, 2Pi};
x[t_]:= 2 Cos[t]
y[t_]:= 3 Sin[t]
z[t_]:= 1
r[t_]:= {x[t], y[t], z[t]}
v[t_]:= r'[t]
integrand= f[x[t], y[t], z[t]] . v[t] //Simplify
NIntegrate[integrand,{t, a, b}]
```

### 16.3 PATH INDEPENDENCE, POTENTIAL FUNCTIONS, AND CONSERVATIVE FIELDS

1.  $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow$  Conservative
2.  $\frac{\partial P}{\partial y} = x \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y \cos z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \sin z = \frac{\partial M}{\partial y} \Rightarrow$  Conservative
3.  $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial z} \Rightarrow$  Not Conservative
4.  $\frac{\partial N}{\partial x} = 1 \neq -1 = \frac{\partial M}{\partial y} \Rightarrow$  Not Conservative
5.  $\frac{\partial N}{\partial x} = 0 \neq 1 = \frac{\partial M}{\partial y} \Rightarrow$  Not Conservative
6.  $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y} \Rightarrow$  Conservative

7.  $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow g(y, z) = \frac{3y^2}{2} + h(z) \Rightarrow f(x, y, z) = x^2 + \frac{3y^2}{2} + h(z)$   
 $\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C \Rightarrow f(x, y, z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$
8.  $\frac{\partial f}{\partial x} = y + z \Rightarrow f(x, y, z) = (y + z)x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \Rightarrow \frac{\partial g}{\partial y} = z \Rightarrow g(y, z) = zy + h(z)$   
 $\Rightarrow f(x, y, z) = (y + z)x + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = x + y + h'(z) = x + y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = (y + z)x + zy + C$
9.  $\frac{\partial f}{\partial x} = e^{y+2z} \Rightarrow f(x, y, z) = xe^{y+2z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xe^{y+2z} + \frac{\partial g}{\partial y} = xe^{y+2z} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow f(x, y, z) = xe^{y+2z} + h(z)$   
 $\Rightarrow \frac{\partial f}{\partial z} = 2xe^{y+2z} + h'(z) = 2xe^{y+2z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{y+2z} + C$
10.  $\frac{\partial f}{\partial x} = y \sin z \Rightarrow f(x, y, z) = xy \sin z + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$   
 $\Rightarrow f(x, y, z) = xy \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy \cos z + h'(z) = xy \cos z \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xy \sin z + C$
11.  $\frac{\partial f}{\partial z} = \frac{z}{y^2 + z^2} \Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2 + z^2) + g(x, y) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \ln x + \sec^2(x + y) \Rightarrow g(x, y) = (x \ln x - x) + \tan(x + y) + h(y)$   
 $\Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2 + z^2) + (x \ln x - x) + \tan(x + y) + h(y)$   
 $\Rightarrow \frac{\partial f}{\partial y} = \frac{y}{y^2 + z^2} + \sec^2(x + y) + h'(y) = \sec^2(x + y) + \frac{y}{y^2 + z^2} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C \Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2 + z^2) + (x \ln x - x) + \tan(x + y) + C$
12.  $\frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2} \Rightarrow f(x, y, z) = \tan^{-1}(xy) + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2} + \frac{\partial g}{\partial y} = \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}$   
 $\Rightarrow \frac{\partial g}{\partial y} = \frac{z}{\sqrt{1-y^2z^2}} \Rightarrow g(y, z) = \sin^{-1}(yz) + h(z) \Rightarrow f(x, y, z) = \tan^{-1}(xy) + \sin^{-1}(yz) + h(z)$   
 $\Rightarrow \frac{\partial f}{\partial z} = \frac{y}{\sqrt{1-y^2z^2}} + h'(z) = \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z} \Rightarrow h'(z) = \frac{1}{z} \Rightarrow h(z) = \ln|z| + C$   
 $\Rightarrow f(x, y, z) = \tan^{-1}(xy) + \sin^{-1}(yz) + \ln|z| + C$
13. Let  $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz$  is exact;  $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y \Rightarrow g(y, z) = y^2 + h(z) \Rightarrow f(x, y, z) = x^2 + y^2 + h(z)$   
 $\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 2z \Rightarrow h(z) = z^2 + C \Rightarrow f(x, y, z) = x^2 + y^2 + z^2 + C \Rightarrow \int_{(0,0,0)}^{(2,3,-6)} 2x dx + 2y dy + 2z dz$   
 $= f(2, 3, -6) - f(0, 0, 0) = 2^2 + 3^2 + (-6)^2 = 49$
14. Let  $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz$  is exact;  $\frac{\partial f}{\partial x} = yz \Rightarrow f(x, y, z) = xyz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = xyz + h(z)$   
 $\Rightarrow \frac{\partial f}{\partial z} = xy + h'(z) = xy \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xyz + C$   
 $\Rightarrow \int_{(1,1,2)}^{(3,5,0)} yz dx + xz dy + xy dz = f(3, 5, 0) - f(1, 1, 2) = 0 - 2 = -2$
15. Let  $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 - z^2)\mathbf{j} - 2yz\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -2z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}$   
 $\Rightarrow M dx + N dy + P dz$  is exact;  $\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2 \Rightarrow \frac{\partial g}{\partial y} = -z^2$   
 $\Rightarrow g(y, z) = -yz^2 + h(z) \Rightarrow f(x, y, z) = x^2y - yz^2 + h(z) \Rightarrow \frac{\partial f}{\partial z} = -2yz + h'(z) = -2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$   
 $\Rightarrow f(x, y, z) = x^2y - yz^2 + C \Rightarrow \int_{(0,0,0)}^{(1,2,3)} 2xy dx + (x^2 - z^2) dy - 2yz dz = f(1, 2, 3) - f(0, 0, 0) = 2 - 2(3)^2 = -16$
16. Let  $\mathbf{F}(x, y, z) = 2x\mathbf{i} - y^2\mathbf{j} - \left(\frac{4}{1+z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$   
 $\Rightarrow M dx + N dy + P dz$  is exact;  $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -y^2 \Rightarrow g(y, z) = -\frac{y^3}{3} + h(z)$

$$\begin{aligned} \Rightarrow f(x, y, z) &= x^2 - \frac{y^3}{3} + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = -\frac{4}{1+z^2} \Rightarrow h(z) = -4 \tan^{-1} z + C \Rightarrow f(x, y, z) \\ &= x^2 - \frac{y^3}{3} - 4 \tan^{-1} z + C \Rightarrow \int_{(0,0,0)}^{(3,3,1)} 2x \, dx - y^2 \, dy - \frac{4}{1+z^2} \, dz = f(3, 3, 1) - f(0, 0, 0) \\ &= \left(9 - \frac{27}{3} - 4 \cdot \frac{\pi}{4}\right) - (0 - 0 - 0) = -\pi \end{aligned}$$

$$\begin{aligned} 17. \text{ Let } \mathbf{F}(x, y, z) &= (\sin y \cos x)\mathbf{i} + (\cos y \sin x)\mathbf{j} + \mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \cos y \cos x = \frac{\partial M}{\partial y} \\ &\Rightarrow M \, dx + N \, dy + P \, dz \text{ is exact; } \frac{\partial f}{\partial x} = \sin y \cos x \Rightarrow f(x, y, z) = \sin y \sin x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \cos y \sin x + \frac{\partial g}{\partial y} \\ &= \cos y \sin x \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = \sin y \sin x + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 1 \Rightarrow h(z) = z + C \\ &\Rightarrow f(x, y, z) = \sin y \sin x + z + C \Rightarrow \int_{(1,0,0)}^{(0,1,1)} \sin y \cos x \, dx + \cos y \sin x \, dy + dz = f(0, 1, 1) - f(1, 0, 0) \\ &= (0 + 1) - (0 + 0) = 1 \end{aligned}$$

$$\begin{aligned} 18. \text{ Let } \mathbf{F}(x, y, z) &= (2 \cos y)\mathbf{i} + \left(\frac{1}{y} - 2x \sin y\right)\mathbf{j} + \left(\frac{1}{z}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2 \sin y = \frac{\partial M}{\partial y} \\ &\Rightarrow M \, dx + N \, dy + P \, dz \text{ is exact; } \frac{\partial f}{\partial x} = 2 \cos y \Rightarrow f(x, y, z) = 2x \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -2x \sin y + \frac{\partial g}{\partial y} \\ &= \frac{1}{y} - 2x \sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y, z) = \ln |y| + h(z) \Rightarrow f(x, y, z) = 2x \cos y + \ln |y| + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = \frac{1}{z} \\ &\Rightarrow h(z) = \ln |z| + C \Rightarrow f(x, y, z) = 2x \cos y + \ln |y| + \ln |z| + C \\ &\Rightarrow \int_{(0,2,1)}^{(1,\pi/2,2)} 2 \cos y \, dx + \left(\frac{1}{y} - 2x \sin y\right) \, dy + \frac{1}{z} \, dz = f\left(1, \frac{\pi}{2}, 2\right) - f(0, 2, 1) \\ &= \left(2 \cdot 0 + \ln \frac{\pi}{2} + \ln 2\right) - (0 \cdot \cos 2 + \ln 2 + \ln 1) = \ln \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} 19. \text{ Let } \mathbf{F}(x, y, z) &= 3x^2\mathbf{i} + \left(\frac{z^2}{y}\right)\mathbf{j} + (2z \ln y)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = \frac{2z}{y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \\ &\Rightarrow M \, dx + N \, dy + P \, dz \text{ is exact; } \frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x, y, z) = x^3 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = \frac{z^2}{y} \Rightarrow g(y, z) = z^2 \ln y + h(z) \\ &\Rightarrow f(x, y, z) = x^3 + z^2 \ln y + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2z \ln y + h'(z) = 2z \ln y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) \\ &= x^3 + z^2 \ln y + C \Rightarrow \int_{(1,1,1)}^{(1,2,3)} 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln y \, dz = f(1, 2, 3) - f(1, 1, 1) \\ &= (1 + 9 \ln 2 + C) - (1 + 0 + C) = 9 \ln 2 \end{aligned}$$

$$\begin{aligned} 20. \text{ Let } \mathbf{F}(x, y, z) &= (2x \ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - (xy)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y} \\ &\Rightarrow M \, dx + N \, dy + P \, dz \text{ is exact; } \frac{\partial f}{\partial x} = 2x \ln y - yz \Rightarrow f(x, y, z) = x^2 \ln y - xyz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y} \\ &= \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = x^2 \ln y - xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = -xy + h'(z) = -xy \Rightarrow h'(z) = 0 \\ &\Rightarrow h(z) = C \Rightarrow f(x, y, z) = x^2 \ln y - xyz + C \Rightarrow \int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz) \, dx + \left(\frac{x^2}{y} - xz\right) \, dy - xy \, dz \\ &= f(2, 1, 1) - f(1, 2, 1) = (4 \ln 1 - 2 + C) - (\ln 2 - 2 + C) = -\ln 2 \end{aligned}$$

$$\begin{aligned} 21. \text{ Let } \mathbf{F}(x, y, z) &= \left(\frac{1}{y}\right)\mathbf{i} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} - \left(\frac{y}{z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{y^2} = \frac{\partial M}{\partial y} \\ &\Rightarrow M \, dx + N \, dy + P \, dz \text{ is exact; } \frac{\partial f}{\partial x} = \frac{1}{y} \Rightarrow f(x, y, z) = \frac{x}{y} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{\partial g}{\partial y} = \frac{1}{z} - \frac{x}{y^2} \\ &\Rightarrow \frac{\partial g}{\partial y} = \frac{1}{z} \Rightarrow g(y, z) = \frac{y}{z} + h(z) \Rightarrow f(x, y, z) = \frac{x}{y} + \frac{y}{z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = -\frac{y}{z^2} + h'(z) = -\frac{y}{z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \\ &\Rightarrow f(x, y, z) = \frac{x}{y} + \frac{y}{z} + C \Rightarrow \int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} \, dx + \left(\frac{1}{z} - \frac{x}{y^2}\right) \, dy - \frac{y}{z^2} \, dz = f(2, 2, 2) - f(1, 1, 1) = \left(\frac{2}{2} + \frac{2}{2} + C\right) - \left(\frac{1}{1} + \frac{1}{1} + C\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} 22. \text{ Let } \mathbf{F}(x, y, z) &= \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{x^2 + y^2 + z^2} \left( \text{and let } \rho^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \frac{\partial \rho}{\partial z} = \frac{z}{\rho} \right) \\ &\Rightarrow \frac{\partial P}{\partial y} = -\frac{4yz}{\rho^4} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{4xz}{\rho^4} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{4xy}{\rho^4} = \frac{\partial M}{\partial y} \Rightarrow M \, dx + N \, dy + P \, dz \text{ is exact; } \\ &\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + z^2} \Rightarrow f(x, y, z) = \ln(x^2 + y^2 + z^2) + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + z^2} + \frac{\partial g}{\partial y} = \frac{2y}{x^2 + y^2 + z^2} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial g}{\partial y} = 0 &\Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = \ln(x^2 + y^2 + z^2) + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{2z}{x^2 + y^2 + z^2} + h'(z) \\ &= \frac{2z}{x^2 + y^2 + z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = \ln(x^2 + y^2 + z^2) + C \\ &\Rightarrow \int_{(-1, -1, -1)}^{(2, 2, 2)} \frac{2x dx + 2y dy + 2z dz}{x^2 + y^2 + z^2} = f(2, 2, 2) - f(-1, -1, -1) = \ln 12 - \ln 3 = \ln 4 \end{aligned}$$

$$\begin{aligned} 23. \mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) &= (1 + t)\mathbf{i} + (1 + 2t)\mathbf{j} + (1 - 2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow dx = dt, dy = 2 dt, dz = -2 dt \\ &\Rightarrow \int_{(1, 1, 1)}^{(2, 3, -1)} y dx + x dy + 4 dz = \int_0^1 (2t + 1) dt + (t + 1)(2 dt) + 4(-2) dt = \int_0^1 (4t - 5) dt = [2t^2 - 5t]_0^1 = -3 \end{aligned}$$

$$\begin{aligned} 24. \mathbf{r} = t(3\mathbf{j} + 4\mathbf{k}), 0 \leq t \leq 1 &\Rightarrow dx = 0, dy = 3 dt, dz = 4 dt \Rightarrow \int_{(0, 0, 0)}^{(0, 3, 4)} x^2 dx + yz dy + \left(\frac{y^2}{2}\right) dz \\ &= \int_0^1 (12t^2)(3 dt) + \left(\frac{9t^2}{2}\right)(4 dt) = \int_0^1 54t^2 dt = [18t^3]_0^1 = 18 \end{aligned}$$

$$\begin{aligned} 25. \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} &\Rightarrow M dx + N dy + P dz \text{ is exact} \Rightarrow \mathbf{F} \text{ is conservative} \\ &\Rightarrow \text{path independence} \end{aligned}$$

$$\begin{aligned} 26. \frac{\partial P}{\partial y} = -\frac{yz}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{xz}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{xy}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial M}{\partial y} \\ &\Rightarrow M dx + N dy + P dz \text{ is exact} \Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{path independence} \end{aligned}$$

$$\begin{aligned} 27. \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y} &\Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{there exists an } f \text{ so that } \mathbf{F} = \nabla f; \\ \frac{\partial f}{\partial x} = \frac{2x}{y} &\Rightarrow f(x, y) = \frac{x^2}{y} + g(y) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} \Rightarrow g'(y) = \frac{1}{y^2} \Rightarrow g(y) = -\frac{1}{y} + C \\ &\Rightarrow f(x, y) = \frac{x^2}{y} - \frac{1}{y} + C \Rightarrow \mathbf{F} = \nabla \left( \frac{x^2 - 1}{y} \right) \end{aligned}$$

$$\begin{aligned} 28. \frac{\partial P}{\partial y} = \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y} &\Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{there exists an } f \text{ so that } \mathbf{F} = \nabla f; \\ \frac{\partial f}{\partial x} = e^x \ln y &\Rightarrow f(x, y, z) = e^x \ln y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin z \Rightarrow \frac{\partial g}{\partial y} = \sin z \Rightarrow g(y, z) \\ &= y \sin z + h(z) \Rightarrow f(x, y, z) = e^x \ln y + y \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = y \cos z + h'(z) = y \cos z \Rightarrow h'(z) = 0 \\ &\Rightarrow h(z) = C \Rightarrow f(x, y, z) = e^x \ln y + y \sin z + C \Rightarrow \mathbf{F} = \nabla (e^x \ln y + y \sin z) \end{aligned}$$

$$\begin{aligned} 29. \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} &\Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{there exists an } f \text{ so that } \mathbf{F} = \nabla f; \\ \frac{\partial f}{\partial x} = x^2 + y &\Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y, z) = \frac{1}{3}y^3 + h(z) \\ &\Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z - e^z + C \Rightarrow f(x, y, z) \\ &= \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla \left( \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z \right) \\ \text{(a) work} &= \int_A^B \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[ \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z \right]_{(1, 0, 0)}^{(1, 0, 1)} = \left( \frac{1}{3} + 0 + 0 + e - e \right) - \left( \frac{1}{3} + 0 + 0 - 1 \right) \\ &= 1 \\ \text{(b) work} &= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[ \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z \right]_{(1, 0, 0)}^{(1, 0, 1)} = 1 \\ \text{(c) work} &= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[ \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z \right]_{(1, 0, 0)}^{(1, 0, 1)} = 1 \end{aligned}$$

Note: Since  $\mathbf{F}$  is conservative,  $\int_A^B \mathbf{F} \cdot d\mathbf{r}$  is independent of the path from  $(1, 0, 0)$  to  $(1, 0, 1)$ .

$$\begin{aligned} 30. \frac{\partial P}{\partial y} = xe^{yz} + xye^{yz} + \cos y = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = ye^{yz} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = ze^{yz} = \frac{\partial M}{\partial y} &\Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{there exists an } f \text{ so} \\ \text{that } \mathbf{F} = \nabla f; \frac{\partial f}{\partial x} = e^{yz} &\Rightarrow f(x, y, z) = xe^{yz} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z \cos y \Rightarrow \frac{\partial g}{\partial y} = z \cos y \\ &\Rightarrow g(y, z) = z \sin y + h(z) \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + h(z) \Rightarrow \frac{\partial f}{\partial z} = xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y \\ &\Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + C \Rightarrow \mathbf{F} = \nabla (xe^{yz} + z \sin y) \end{aligned}$$



$$(a) \text{ work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z \sin y]_{(1,0,1)}^{(1,\pi/2,0)} = (1+0) - (1+0) = 0$$

$$(b) \text{ work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z \sin y]_{(1,0,1)}^{(1,\pi/2,0)} = 0$$

$$(c) \text{ work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z \sin y]_{(1,0,1)}^{(1,\pi/2,0)} = 0$$

Note: Since  $\mathbf{F}$  is conservative,  $\int_A^B \mathbf{F} \cdot d\mathbf{r}$  is independent of the path from  $(1, 0, 1)$  to  $(1, \frac{\pi}{2}, 0)$ .

31. (a)  $\mathbf{F} = \nabla(x^3y^2) \Rightarrow \mathbf{F} = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$ ; let  $C_1$  be the path from  $(-1, 1)$  to  $(0, 0) \Rightarrow x = t - 1$  and  $y = -t + 1, 0 \leq t \leq 1 \Rightarrow \mathbf{F} = 3(t-1)^2(-t+1)^2\mathbf{i} + 2(t-1)^3(-t+1)\mathbf{j} = 3(t-1)^4\mathbf{i} - 2(t-1)^4\mathbf{j}$  and  $\mathbf{r}_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j} \Rightarrow d\mathbf{r}_1 = dt\mathbf{i} - dt\mathbf{j} \Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_0^1 [3(t-1)^4 - 2(t-1)^4] dt = \int_0^1 5(t-1)^4 dt = [(t-1)^5]_0^1 = 1$ ; let  $C_2$  be the path from  $(0, 0)$  to  $(1, 1) \Rightarrow x = t$  and  $y = t, 0 \leq t \leq 1 \Rightarrow \mathbf{F} = 3t^4\mathbf{i} + 2t^4\mathbf{j}$  and  $\mathbf{r}_2 = t\mathbf{i} + t\mathbf{j} \Rightarrow d\mathbf{r}_2 = dt\mathbf{i} + dt\mathbf{j} \Rightarrow \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_0^1 (3t^4 + 2t^4) dt = \int_0^1 5t^4 dt = 1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = 2$

(b) Since  $f(x, y) = x^3y^2$  is a potential function for  $\mathbf{F}$ ,  $\int_{(-1,1)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(-1, 1) = 2$

32.  $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2x \sin y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$  is conservative  $\Rightarrow$  there exists an  $f$  so that  $\mathbf{F} = \nabla f$ ;  
 $\frac{\partial f}{\partial x} = 2x \cos y \Rightarrow f(x, y, z) = x^2 \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin y + \frac{\partial g}{\partial y} = -x^2 \sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$   
 $\Rightarrow f(x, y, z) = x^2 \cos y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x^2 \cos y + C \Rightarrow \mathbf{F} = \nabla(x^2 \cos y)$

(a)  $\int_C 2x \cos y dx - x^2 \sin y dy = [x^2 \cos y]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b)  $\int_C 2x \cos y dx - x^2 \sin y dy = [x^2 \cos y]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c)  $\int_C 2x \cos y dx - x^2 \sin y dy = [x^2 \cos y]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$

(d)  $\int_C 2x \cos y dx - x^2 \sin y dy = [x^2 \cos y]_{(1,0)}^{(1,0)} = 1 - 1 = 0$

33. (a) If the differential form is exact, then  $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy$  for all  $y \Rightarrow 2a = c, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$  for all  $x$ , and  $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2ay$  for all  $y \Rightarrow b = 2a$  and  $c = 2a$

(b)  $\mathbf{F} = \nabla f \Rightarrow$  the differential form with  $a = 1$  in part (a) is exact  $\Rightarrow b = 2$  and  $c = 2$

34.  $\mathbf{F} = \nabla f \Rightarrow g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f \cdot d\mathbf{r} = f(x, y, z) - f(0, 0, 0) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0, \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$ , and  $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 0 \Rightarrow \nabla g = \nabla f = \mathbf{F}$ , as claimed

35. The path will not matter; the work along any path will be the same because the field is conservative.

36. The field is not conservative, for otherwise the work would be the same along  $C_1$  and  $C_2$ .

37. Let the coordinates of points A and B be  $(x_A, y_A, z_A)$  and  $(x_B, y_B, z_B)$ , respectively. The force  $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  is conservative because all the partial derivatives of M, N, and P are zero. Therefore, the potential function is  $f(x, y, z) = ax + by + cz + C$ , and the work done by the force in moving a particle along any path from A to B is  $f(B) - f(A) = f(x_B, y_B, z_B) - f(x_A, y_A, z_A) = (ax_B + by_B + cz_B + C) - (ax_A + by_A + cz_A + C) = a(x_B - x_A) + b(y_B - y_A) + c(z_B - z_A) = \mathbf{F} \cdot \vec{BA}$

38. (a) Let  $-GmM = C \Rightarrow \mathbf{F} = C \left[ \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} \right]$   
 $\Rightarrow \frac{\partial P}{\partial y} = \frac{-3yzC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{-3xyC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} = \nabla f$  for  
 some  $f$ ;  $\frac{\partial f}{\partial x} = \frac{x C}{(x^2 + y^2 + z^2)^{3/2}} \Rightarrow f(x, y, z) = -\frac{C}{(x^2 + y^2 + z^2)^{1/2}} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{y C}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial g}{\partial y}$   
 $= \frac{y C}{(x^2 + y^2 + z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{z C}{(x^2 + y^2 + z^2)^{3/2}} + h'(z) = \frac{z C}{(x^2 + y^2 + z^2)^{3/2}}$   
 $\Rightarrow h(z) = C_1 \Rightarrow f(x, y, z) = -\frac{C}{(x^2 + y^2 + z^2)^{1/2}} + C_1$ . Let  $C_1 = 0 \Rightarrow f(x, y, z) = \frac{GmM}{(x^2 + y^2 + z^2)^{1/2}}$  is a potential  
 function for  $\mathbf{F}$ .  
 (b) If  $s$  is the distance of  $(x, y, z)$  from the origin, then  $s = \sqrt{x^2 + y^2 + z^2}$ . The work done by the gravitational field  
 $\mathbf{F}$  is work  $= \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left[ \frac{GmM}{\sqrt{x^2 + y^2 + z^2}} \right]_{P_1}^{P_2} = \frac{GmM}{s_2} - \frac{GmM}{s_1} = GmM \left( \frac{1}{s_2} - \frac{1}{s_1} \right)$ , as claimed.

## 16.4 GREEN'S THEOREM IN THE PLANE

1.  $M = -y = -a \sin t$ ,  $N = x = a \cos t$ ,  $dx = -a \sin t dt$ ,  $dy = a \cos t dt \Rightarrow \frac{\partial M}{\partial x} = 0$ ,  $\frac{\partial M}{\partial y} = -1$ ,  $\frac{\partial N}{\partial x} = 1$ , and  
 $\frac{\partial N}{\partial y} = 0$ ;  
 Equation (3):  $\oint_C M dy - N dx = \int_0^{2\pi} [(-a \sin t)(a \cos t) - (a \cos t)(-a \sin t)] dt = \int_0^{2\pi} 0 dt = 0$ ;  
 $\iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R 0 dx dy = 0$ , Flux  
 Equation (4):  $\oint_C M dx + N dy = \int_0^{2\pi} [(-a \sin t)(-a \sin t) - (a \cos t)(a \cos t)] dt = \int_0^{2\pi} a^2 dt = 2\pi a^2$ ;  
 $\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \int_{-a}^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} 2 dy dx = \int_{-a}^a 4\sqrt{a^2 - x^2} dx = 4 \left[ \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_{-a}^a$   
 $= 2a^2 \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = 2a^2\pi$ , Circulation
2.  $M = y = a \sin t$ ,  $N = 0$ ,  $dx = -a \sin t dt$ ,  $dy = a \cos t dt \Rightarrow \frac{\partial M}{\partial x} = 0$ ,  $\frac{\partial M}{\partial y} = 1$ ,  $\frac{\partial N}{\partial x} = 0$ , and  $\frac{\partial N}{\partial y} = 0$ ;  
 Equation (3):  $\oint_C M dy - N dx = \int_0^{2\pi} a^2 \sin t \cos t dt = a^2 \left[ \frac{1}{2} \sin^2 t \right]_0^{2\pi} = 0$ ;  $\iint_R 0 dx dy = 0$ , Flux  
 Equation (4):  $\oint_C M dx + N dy = \int_0^{2\pi} (-a^2 \sin^2 t) dt = -a^2 \left[ \frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = -\pi a^2$ ;  $\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$   
 $= \iint_R -1 dx dy = \int_0^{2\pi} \int_0^a -r dr d\theta = \int_0^{2\pi} -\frac{a^2}{2} d\theta = -\pi a^2$ , Circulation
3.  $M = 2x = 2a \cos t$ ,  $N = -3y = -3a \sin t$ ,  $dx = -a \sin t dt$ ,  $dy = a \cos t dt \Rightarrow \frac{\partial M}{\partial x} = 2$ ,  $\frac{\partial M}{\partial y} = 0$ ,  $\frac{\partial N}{\partial x} = 0$ , and  
 $\frac{\partial N}{\partial y} = -3$ ;  
 Equation (3):  $\oint_C M dy - N dx = \int_0^{2\pi} [(2a \cos t)(a \cos t) + (3a \sin t)(-a \sin t)] dt$   
 $= \int_0^{2\pi} (2a^2 \cos^2 t - 3a^2 \sin^2 t) dt = 2a^2 \left[ \frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} - 3a^2 \left[ \frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = 2\pi a^2 - 3\pi a^2 = -\pi a^2$ ;  
 $\iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R -1 dx dy = \int_0^{2\pi} \int_0^a -r dr d\theta = \int_0^{2\pi} -\frac{a^2}{2} d\theta = -\pi a^2$ , Flux  
 Equation (4):  $\oint_C M dx + N dy = \int_0^{2\pi} [(2a \cos t)(-a \sin t) + (-3a \sin t)(a \cos t)] dt$   
 $= \int_0^{2\pi} (-2a^2 \sin t \cos t - 3a^2 \sin t \cos t) dt = -5a^2 \left[ \frac{1}{2} \sin^2 t \right]_0^{2\pi} = 0$ ;  $\iint_R 0 dx dy = 0$ , Circulation
4.  $M = -x^2 y = -a^3 \cos^2 t$ ,  $N = xy^2 = a^3 \cos t \sin^2 t$ ,  $dx = -a \sin t dt$ ,  $dy = a \cos t dt$   
 $\Rightarrow \frac{\partial M}{\partial x} = -2xy$ ,  $\frac{\partial M}{\partial y} = -x^2$ ,  $\frac{\partial N}{\partial x} = y^2$ , and  $\frac{\partial N}{\partial y} = 2xy$ ;  
 Equation (3):  $\oint_C M dy - N dx = \int_0^{2\pi} (-a^4 \cos^3 t \sin t + a^4 \cos t \sin^3 t) dt = \left[ \frac{a^4}{4} \cos^4 t + \frac{a^4}{4} \sin^4 t \right]_0^{2\pi} = 0$ ;

$$\iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R (-2xy + 2xy) dx dy = 0, \text{ Flux}$$

$$\begin{aligned} \text{Equation (4): } \oint_C M dx + N dy &= \int_0^{2\pi} (a^4 \cos^2 t \sin^2 t + a^4 \cos^2 t \sin^2 t) dt = \int_0^{2\pi} (2a^4 \cos^2 t \sin^2 t) dt \\ &= \int_0^{2\pi} \frac{1}{2} a^4 \sin^2 2t dt = \frac{a^4}{4} \int_0^{4\pi} \sin^2 u du = \frac{a^4}{4} \left[ \frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{4\pi} = \frac{\pi a^4}{2}; \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R (y^2 + x^2) dx dy \\ &= \int_0^{2\pi} \int_0^a r^2 \cdot r dr d\theta = \int_0^{2\pi} \frac{a^4}{4} d\theta = \frac{\pi a^4}{2}, \text{ Circulation} \end{aligned}$$

$$\begin{aligned} 5. \quad M = x - y, N = y - x &\Rightarrow \frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = -1, \frac{\partial N}{\partial y} = 1 \Rightarrow \text{Flux} = \iint_R 2 dx dy = \int_0^1 \int_0^1 2 dx dy = 2; \\ \text{Circ} &= \iint_R [-1 - (-1)] dx dy = 0 \end{aligned}$$

$$\begin{aligned} 6. \quad M = x^2 + 4y, N = x + y^2 &\Rightarrow \frac{\partial M}{\partial x} = 2x, \frac{\partial M}{\partial y} = 4, \frac{\partial N}{\partial x} = 1, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (2x + 2y) dx dy \\ &= \int_0^1 \int_0^1 (2x + 2y) dx dy = \int_0^1 [x^2 + 2xy]_0^1 dy = \int_0^1 (1 + 2y) dy = [y + y^2]_0^1 = 2; \text{Circ} = \iint_R (1 - 4) dx dy \\ &= \int_0^1 \int_0^1 -3 dx dy = -3 \end{aligned}$$

$$\begin{aligned} 7. \quad M = y^2 - x^2, N = x^2 + y^2 &\Rightarrow \frac{\partial M}{\partial x} = -2x, \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2x, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (-2x + 2y) dx dy \\ &= \int_0^3 \int_0^x (-2x + 2y) dy dx = \int_0^3 (-2x^2 + x^2) dx = \left[ -\frac{1}{3} x^3 \right]_0^3 = -9; \text{Circ} = \iint_R (2x - 2y) dx dy \\ &= \int_0^3 \int_0^x (2x - 2y) dy dx = \int_0^3 x^2 dx = 9 \end{aligned}$$

$$\begin{aligned} 8. \quad M = x + y, N = -(x^2 + y^2) &\Rightarrow \frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = -2x, \frac{\partial N}{\partial y} = -2y \Rightarrow \text{Flux} = \iint_R (1 - 2y) dx dy \\ &= \int_0^1 \int_0^x (1 - 2y) dy dx = \int_0^1 (x - x^2) dx = \frac{1}{6}; \text{Circ} = \iint_R (-2x - 1) dx dy = \int_0^1 \int_0^x (-2x - 1) dy dx \\ &= \int_0^1 (-2x^2 - x) dx = -\frac{7}{6} \end{aligned}$$

$$\begin{aligned} 9. \quad M = xy + y^2, N = x - y &\Rightarrow \frac{\partial M}{\partial x} = y, \frac{\partial M}{\partial y} = x + 2y, \frac{\partial N}{\partial x} = 1, \frac{\partial N}{\partial y} = -1 \Rightarrow \text{Flux} = \iint_R (y + (-1)) dy dx \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} (y - 1) dy dx = \int_0^1 \left( \frac{1}{2}x - \sqrt{x} - \frac{1}{2}x^4 + x^2 \right) dx = -\frac{11}{60}; \text{Circ} = \iint_R (1 - (x + 2y)) dy dx \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y) dy dx = \int_0^1 (\sqrt{x} - x^{3/2} - x - x^2 + x^3 + x^4) dx = -\frac{7}{60} \end{aligned}$$

$$\begin{aligned} 10. \quad M = x + 3y, N = 2x - y &\Rightarrow \frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = 3, \frac{\partial N}{\partial x} = 2, \frac{\partial N}{\partial y} = -1 \Rightarrow \text{Flux} = \iint_R (1 + (-1)) dy dx = 0 \\ \text{Circ} &= \iint_R (2 - 3) dy dx = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-x^2)/2}}^{\sqrt{(2-x^2)/2}} (-1) dy dx = -\frac{2}{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2} dx = -\pi\sqrt{2} \end{aligned}$$

$$\begin{aligned} 11. \quad M = x^3 y^2, N = \frac{1}{2} x^4 y &\Rightarrow \frac{\partial M}{\partial x} = 3x^2 y^2, \frac{\partial M}{\partial y} = 2x^3 y, \frac{\partial N}{\partial x} = 2x^3 y, \frac{\partial N}{\partial y} = \frac{1}{2} x^4 \Rightarrow \text{Flux} = \iint_R (3x^2 y^2 + \frac{1}{2} x^4) dy dx \\ &= \int_0^2 \int_{x^2-x}^x (3x^2 y^2 + \frac{1}{2} x^4) dy dx = \int_0^2 (3x^5 - \frac{7}{2} x^6 + 3x^7 - x^8) dx = \frac{64}{9}; \text{Circ} = \iint_R (2x^3 y - 2x^3 y) dy dx = 0 \end{aligned}$$

$$\begin{aligned}
 12. \quad M &= \frac{x}{1+y^2}, N = \tan^{-1}y \Rightarrow \frac{\partial M}{\partial x} = \frac{1}{1+y^2}, \frac{\partial M}{\partial y} = \frac{-2xy}{(1+y^2)^2}, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = \frac{1}{1+y^2} \Rightarrow \text{Flux} = \iint_R \left( \frac{1}{1+y^2} + \frac{1}{1+y^2} \right) dx dy \\
 &= \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{1+y^2} dx dy = \int_{-1}^1 \frac{4\sqrt{1-y^2}}{1+y^2} dy = 4\pi\sqrt{2} - 4\pi; \text{Circ} = \iint_R \left( 0 - \left( \frac{-2xy}{(1+y^2)^2} \right) \right) dy dx \\
 &= \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left( \frac{2xy}{(1+y^2)^2} \right) dy dx = \int_{-1}^1 (0) dx = 0
 \end{aligned}$$

$$\begin{aligned}
 13. \quad M &= x + e^x \sin y, N = x + e^x \cos y \Rightarrow \frac{\partial M}{\partial x} = 1 + e^x \sin y, \frac{\partial M}{\partial y} = e^x \cos y, \frac{\partial N}{\partial x} = 1 + e^x \cos y, \frac{\partial N}{\partial y} = -e^x \sin y \\
 &\Rightarrow \text{Flux} = \iint_R dx dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r dr d\theta = \int_{-\pi/4}^{\pi/4} \left( \frac{1}{2} \cos 2\theta \right) d\theta = \left[ \frac{1}{4} \sin 2\theta \right]_{-\pi/4}^{\pi/4} = \frac{1}{2}; \\
 \text{Circ} &= \iint_R (1 + e^x \cos y - e^x \cos y) dx dy = \iint_R dx dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r dr d\theta = \int_{-\pi/4}^{\pi/4} \left( \frac{1}{2} \cos 2\theta \right) d\theta = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad M &= \tan^{-1} \frac{y}{x}, N = \ln(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2 + y^2}, \frac{\partial M}{\partial y} = \frac{x}{x^2 + y^2}, \frac{\partial N}{\partial x} = \frac{2x}{x^2 + y^2}, \frac{\partial N}{\partial y} = \frac{2y}{x^2 + y^2} \\
 &\Rightarrow \text{Flux} = \iint_R \left( \frac{-y}{x^2 + y^2} + \frac{2y}{x^2 + y^2} \right) dx dy = \int_0^\pi \int_1^2 \left( \frac{r \sin \theta}{r^2} \right) r dr d\theta = \int_0^\pi \sin \theta d\theta = 2; \\
 \text{Circ} &= \iint_R \left( \frac{2x}{x^2 + y^2} - \frac{x}{x^2 + y^2} \right) dx dy = \int_0^\pi \int_1^2 \left( \frac{r \cos \theta}{r^2} \right) r dr d\theta = \int_0^\pi \cos \theta d\theta = 0
 \end{aligned}$$

$$\begin{aligned}
 15. \quad M &= xy, N = y^2 \Rightarrow \frac{\partial M}{\partial x} = y, \frac{\partial M}{\partial y} = x, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (y + 2y) dy dx = \int_0^1 \int_x^x 3y dy dx \\
 &= \int_0^1 \left( \frac{3x^2}{2} - \frac{3x^4}{2} \right) dx = \frac{1}{5}; \text{Circ} = \iint_R -x dy dx = \int_0^1 \int_{x^2}^x -x dy dx = \int_0^1 (-x^2 + x^3) dx = -\frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad M &= -\sin y, N = x \cos y \Rightarrow \frac{\partial M}{\partial x} = 0, \frac{\partial M}{\partial y} = -\cos y, \frac{\partial N}{\partial x} = \cos y, \frac{\partial N}{\partial y} = -x \sin y \\
 &\Rightarrow \text{Flux} = \iint_R (-x \sin y) dx dy = \int_0^{\pi/2} \int_0^{\pi/2} (-x \sin y) dx dy = \int_0^{\pi/2} \left( -\frac{\pi^2}{8} \sin y \right) dy = -\frac{\pi^2}{8}; \\
 \text{Circ} &= \iint_R [\cos y - (-\cos y)] dx dy = \int_0^{\pi/2} \int_0^{\pi/2} 2 \cos y dx dy = \int_0^{\pi/2} \pi \cos y dy = [\pi \sin y]_0^{\pi/2} = \pi
 \end{aligned}$$

$$\begin{aligned}
 17. \quad M &= 3xy - \frac{x}{1+y^2}, N = e^x + \tan^{-1}y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}, \frac{\partial N}{\partial y} = \frac{1}{1+y^2} \\
 &\Rightarrow \text{Flux} = \iint_R \left( 3y - \frac{1}{1+y^2} + \frac{1}{1+y^2} \right) dx dy = \iint_R 3y dx dy = \int_0^{2\pi} \int_0^{a(1+\cos \theta)} (3r \sin \theta) r dr d\theta \\
 &= \int_0^{2\pi} a^3 (1 + \cos \theta)^3 (\sin \theta) d\theta = \left[ -\frac{a^3}{4} (1 + \cos \theta)^4 \right]_0^{2\pi} = -4a^3 - (-4a^3) = 0
 \end{aligned}$$

$$\begin{aligned}
 18. \quad M &= y + e^x \ln y, N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = 1 + \frac{e^x}{y}, \frac{\partial N}{\partial x} = \frac{e^x}{y} \Rightarrow \text{Circ} = \iint_R \left[ \frac{e^x}{y} - \left( 1 + \frac{e^x}{y} \right) \right] dx dy = \iint_R (-1) dx dy \\
 &= \int_{-1}^1 \int_{x^4+1}^{3-x^2} -1 dy dx = - \int_{-1}^1 [(3-x^2) - (x^4+1)] dx = \int_{-1}^1 (x^4 + x^2 - 2) dx = -\frac{44}{15}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad M &= 2xy^3, N = 4x^2y^2 \Rightarrow \frac{\partial M}{\partial y} = 6xy^2, \frac{\partial N}{\partial x} = 8xy^2 \Rightarrow \text{work} = \oint_C 2xy^3 dx + 4x^2y^2 dy = \iint_R (8xy^2 - 6xy^2) dx dy \\
 &= \int_0^1 \int_0^{x^3} 2xy^2 dy dx = \int_0^1 \frac{2}{3} x^{10} dx = \frac{2}{33}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad M &= 4x - 2y, N = 2x - 4y \Rightarrow \frac{\partial M}{\partial y} = -2, \frac{\partial N}{\partial x} = 2 \Rightarrow \text{work} = \oint_C (4x - 2y) dx + (2x - 4y) dy \\
 &= \iint_R [2 - (-2)] dx dy = 4 \iint_R dx dy = 4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi
 \end{aligned}$$

$$\begin{aligned}
 21. \quad M = y^2, N = x^2 &\Rightarrow \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2x \Rightarrow \oint_C y^2 dx + x^2 dy = \iint_R (2x - 2y) dy dx \\
 &= \int_0^1 \int_0^{1-x} (2x - 2y) dy dx = \int_0^1 (-3x^2 + 4x - 1) dx = [-x^3 + 2x^2 - x]_0^1 = -1 + 2 - 1 = 0
 \end{aligned}$$

$$\begin{aligned}
 22. \quad M = 3y, N = 2x &\Rightarrow \frac{\partial M}{\partial y} = 3, \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C 3y dx + 2x dy = \iint_R (2 - 3) dx dy = \int_0^\pi \int_0^{\sin x} (-1) dy dx \\
 &= -\int_0^\pi \sin x dx = -2
 \end{aligned}$$

$$\begin{aligned}
 23. \quad M = 6y + x, N = y + 2x &\Rightarrow \frac{\partial M}{\partial y} = 6, \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C (6y + x) dx + (y + 2x) dy = \iint_R (2 - 6) dy dx \\
 &= -4(\text{Area of the circle}) = -16\pi
 \end{aligned}$$

$$24. \quad M = 2x + y^2, N = 2xy + 3y \Rightarrow \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2y \Rightarrow \oint_C (2x + y^2) dx + (2xy + 3y) dy = \iint_R (2y - 2y) dx dy = 0$$

$$\begin{aligned}
 25. \quad M = x = a \cos t, N = y = a \sin t &\Rightarrow dx = -a \sin t dt, dy = a \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx \\
 &= \frac{1}{2} \int_0^{2\pi} (a^2 \cos^2 t + a^2 \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} a^2 dt = \pi a^2
 \end{aligned}$$

$$\begin{aligned}
 26. \quad M = x = a \cos t, N = y = b \sin t &\Rightarrow dx = -a \sin t dt, dy = b \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx \\
 &= \frac{1}{2} \int_0^{2\pi} (ab \cos^2 t + ab \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} ab dt = \pi ab
 \end{aligned}$$

$$\begin{aligned}
 27. \quad M = x = \cos^3 t, N = y = \sin^3 t &\Rightarrow dx = -3 \cos^2 t \sin t dt, dy = 3 \sin^2 t \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx \\
 &= \frac{1}{2} \int_0^{2\pi} (3 \sin^2 t \cos^2 t) (\cos^2 t + \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} (3 \sin^2 t \cos^2 t) dt = \frac{3}{8} \int_0^{2\pi} \sin^2 2t dt = \frac{3}{16} \int_0^{4\pi} \sin^2 u du \\
 &= \frac{3}{16} \left[ \frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{4\pi} = \frac{3}{8} \pi
 \end{aligned}$$

$$\begin{aligned}
 28. \quad C_1: M = x = t, N = y = 0 &\Rightarrow dx = dt, dy = 0; C_2: M = x = (2\pi - t) - \sin(2\pi - t) = 2\pi - t + \sin t, N = y \\
 &= 1 - \cos(2\pi - t) = 1 - \cos t \Rightarrow dx = (\cos t - 1) dt, dy = \sin t dt \\
 &\Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \oint_{C_1} x dy - y dx + \frac{1}{2} \oint_{C_2} x dy - y dx \\
 &= \frac{1}{2} \int_0^{2\pi} (0) dt + \frac{1}{2} \int_0^{2\pi} [(2\pi - t + \sin t)(\sin t) - (1 - \cos t)(\cos t - 1)] dt = -\frac{1}{2} \int_0^{2\pi} (2 \cos t + t \sin t - 2 - 2\pi \sin t) dt \\
 &= -\frac{1}{2} [3 \sin t - t \cos t - 2t - 2\pi \cos t]_0^{2\pi} = 3\pi
 \end{aligned}$$

$$29. \quad (a) \quad M = f(x), N = g(y) \Rightarrow \frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C f(x) dx + g(y) dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R 0 dx dy = 0$$

$$\begin{aligned}
 (b) \quad M = ky, N = hx &\Rightarrow \frac{\partial M}{\partial y} = k, \frac{\partial N}{\partial x} = h \Rightarrow \oint_C ky dx + hx dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\
 &= \iint_R (h - k) dx dy = (h - k)(\text{Area of the region})
 \end{aligned}$$

$$\begin{aligned}
 30. \quad M = xy^2, N = x^2y + 2x &\Rightarrow \frac{\partial M}{\partial y} = 2xy, \frac{\partial N}{\partial x} = 2xy + 2 \Rightarrow \oint_C xy^2 dx + (x^2y + 2x) dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\
 &= \iint_R (2xy + 2 - 2xy) dx dy = 2 \iint_R dx dy = 2 \text{ times the area of the square}
 \end{aligned}$$

31. The integral is 0 for any simple closed plane curve  $C$ . The reasoning: By the tangential form of Green's

$$\begin{aligned} \text{Theorem, with } M = 4x^3y \text{ and } N = x^4, \oint_C 4x^3y \, dx + x^4 \, dy &= \iint_R \left[ \frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial y}(4x^3y) \right] dx \, dy \\ &= \iint_R \underbrace{(4x^3 - 4x^3)}_0 dx \, dy = 0. \end{aligned}$$

32. The integral is 0 for any simple closed curve  $C$ . The reasoning: By the normal form of Green's theorem, with

$$M = x^3 \text{ and } N = -y^3, \oint_C -y^3 \, dy + x^3 \, dx = \iint_R \left[ \underbrace{\frac{\partial}{\partial x}(-y^3)}_0 - \underbrace{\frac{\partial}{\partial y}(x^3)}_0 \right] dx \, dy = 0.$$

$$\begin{aligned} 33. \text{ Let } M = x \text{ and } N = 0 &\Rightarrow \frac{\partial M}{\partial x} = 1 \text{ and } \frac{\partial N}{\partial y} = 0 \Rightarrow \oint_C M \, dy - N \, dx = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy \Rightarrow \oint_C x \, dy \\ &= \iint_R (1 + 0) dx \, dy \Rightarrow \text{Area of } R = \iint_R dx \, dy = \oint_C x \, dy; \text{ similarly, } M = y \text{ and } N = 0 \Rightarrow \frac{\partial M}{\partial y} = 1 \text{ and } \\ \frac{\partial N}{\partial x} = 0 &\Rightarrow \oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} \right) dy \, dx \Rightarrow \oint_C y \, dx = \iint_R (0 + 1) dy \, dx \Rightarrow - \oint_C y \, dx \\ &= \iint_R dx \, dy = \text{Area of } R \end{aligned}$$

$$34. \int_a^b f(x) \, dx = \text{Area of } R = - \oint_C y \, dx, \text{ from Exercise 33}$$

$$\begin{aligned} 35. \text{ Let } \delta(x, y) = 1 &\Rightarrow \bar{x} = \frac{M_x}{M} = \frac{\iint_R x \delta(x, y) \, dA}{\iint_R \delta(x, y) \, dA} = \frac{\iint_R x \, dA}{\iint_R dA} = \frac{\iint_R x \, dA}{A} \Rightarrow A\bar{x} = \iint_R x \, dA = \iint_R (x + 0) dx \, dy \\ &= \oint_C \frac{x^2}{2} dy, A\bar{x} = \iint_R x \, dA = \iint_R (0 + x) dx \, dy = - \oint_C xy \, dx, \text{ and } A\bar{x} = \iint_R x \, dA = \iint_R \left( \frac{2}{3}x + \frac{1}{3}x \right) dx \, dy \\ &= \oint_C \frac{1}{3}x^2 dy - \frac{1}{3}xy \, dx \Rightarrow \frac{1}{2} \oint_C x^2 dy = - \oint_C xy \, dx = \frac{1}{3} \oint_C x^2 dy - xy \, dx = A\bar{x} \end{aligned}$$

$$\begin{aligned} 36. \text{ If } \delta(x, y) = 1, \text{ then } I_y &= \iint_R x^2 \delta(x, y) \, dA = \iint_R x^2 \, dA = \iint_R (x^2 + 0) dy \, dx = \frac{1}{3} \oint_C x^3 dy, \\ \iint_R x^2 \, dA &= \iint_R (0 + x^2) dy \, dx = - \oint_C x^2 y \, dx, \text{ and } \iint_R x^2 \, dA = \iint_R \left( \frac{3}{4}x^2 + \frac{1}{4}x^2 \right) dy \, dx \\ &= \oint_C \frac{1}{4}x^3 dy - \frac{1}{4}x^2 y \, dx = \frac{1}{4} \oint_C x^3 dy - x^2 y \, dx \Rightarrow \frac{1}{3} \oint_C x^3 dy = - \oint_C x^2 y \, dx = \frac{1}{4} \oint_C x^3 dy - x^2 y \, dx = I_y \end{aligned}$$

$$37. M = \frac{\partial f}{\partial y}, N = -\frac{\partial f}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y^2}, \frac{\partial N}{\partial x} = -\frac{\partial^2 f}{\partial x^2} \Rightarrow \oint_C \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = \iint_R \left( -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} \right) dx \, dy = 0 \text{ for such curves } C$$

$$\begin{aligned} 38. M = \frac{1}{4}x^2y + \frac{1}{3}y^3, N = x &\Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2, \frac{\partial N}{\partial x} = 1 \Rightarrow \text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left( \frac{1}{4}x^2 + y^2 \right) > 0 \text{ in the interior of the} \\ \text{ellipse } \frac{1}{4}x^2 + y^2 = 1 &\Rightarrow \text{work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( 1 - \frac{1}{4}x^2 - y^2 \right) dx \, dy \text{ will be maximized on the region} \\ R = \{(x, y) \mid \text{curl } \mathbf{F} \geq 0\} &\geq 0 \text{ or over the region enclosed by } 1 = \frac{1}{4}x^2 + y^2 \end{aligned}$$

$$\begin{aligned} 39. (a) \nabla f &= \left( \frac{2x}{x^2+y^2} \right) \mathbf{i} + \left( \frac{2y}{x^2+y^2} \right) \mathbf{j} \Rightarrow M = \frac{2x}{x^2+y^2}, N = \frac{2y}{x^2+y^2}; \text{ since } M, N \text{ are discontinuous at } (0, 0), \text{ we} \\ \text{compute } \int_C \nabla f \cdot \mathbf{n} \, ds &\text{ directly since Green's Theorem does not apply. Let } x = a \cos t, y = a \sin t \Rightarrow dx = -a \sin t \, dt, \\ dy &= a \cos t \, dt, M = \frac{2}{a} \cos t, N = \frac{2}{a} \sin t, 0 \leq t \leq 2\pi, \text{ so } \int_C \nabla f \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx \\ &= \int_0^{2\pi} \left[ \left( \frac{2}{a} \cos t \right) (a \cos t) - \left( \frac{2}{a} \sin t \right) (-a \sin t) \right] dt = \int_0^{2\pi} 2(\cos^2 t + \sin^2 t) dt = 4\pi. \text{ Note that this holds for any} \end{aligned}$$

$a > 0$ , so  $\int_C \nabla f \cdot \mathbf{n} \, ds = 4\pi$  for any circle  $C$  centered at  $(0, 0)$  traversed counterclockwise and  $\int_C \nabla f \cdot \mathbf{n} \, ds = -4\pi$  if  $C$  is traversed clockwise.

- (b) If  $K$  does not enclose the point  $(0, 0)$  we may apply Green's Theorem:  $\int_C \nabla f \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx$   
 $= \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy = \iint_R \left( \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \right) dx \, dy = \iint_R 0 \, dx \, dy = 0$ . If  $K$  does enclose the point  $(0, 0)$  we proceed as follows:

Choose  $a$  small enough so that the circle  $C$  centered at  $(0, 0)$  of radius  $a$  lies entirely within  $K$ . Green's Theorem applies to the region  $R$  that lies between  $K$  and  $C$ . Thus, as before,  $0 = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy$

$$= \int_K M \, dy - N \, dx + \int_C M \, dy - N \, dx \text{ where } K \text{ is traversed counterclockwise and } C \text{ is traversed clockwise.}$$

Hence by part (a)  $0 = \left[ \int_K M \, dy - N \, dx \right] - 4\pi \Rightarrow 4\pi = \int_K M \, dy - N \, dx = \int_K \nabla f \cdot \mathbf{n} \, ds$ . We have shown:

$$\int_K \nabla f \cdot \mathbf{n} \, ds = \begin{cases} 0 & \text{if } (0, 0) \text{ lies inside } K \\ 4\pi & \text{if } (0, 0) \text{ lies outside } K \end{cases}$$

40. Assume a particle has a closed trajectory in  $R$  and let  $C_1$  be the path  $\Rightarrow C_1$  encloses a simply connected region  $R_1 \Rightarrow C_1$  is a simple closed curve. Then the flux over  $R_1$  is  $\oint_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds = 0$ , since the velocity vectors  $\mathbf{F}$  are tangent to  $C_1$ . But  $0 = \oint_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds = \oint_{C_1} M \, dy - N \, dx = \iint_{R_1} \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy \Rightarrow M_x + N_y = 0$ , which is a contradiction. Therefore,  $C_1$  cannot be a closed trajectory.

$$\begin{aligned} 41. \int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x} dx \, dy &= N(g_2(y), y) - N(g_1(y), y) \Rightarrow \int_c^d \int_{g_1(y)}^{g_2(y)} \left( \frac{\partial N}{\partial x} \right) dx \, dy = \int_c^d [N(g_2(y), y) - N(g_1(y), y)] dy \\ &= \int_c^d N(g_2(y), y) dy - \int_c^d N(g_1(y), y) dy = \int_c^d N(g_2(y), y) dy + \int_d^c N(g_1(y), y) dy = \int_{C_2} N \, dy + \int_{C_1} N \, dy \\ &= \oint_C N \, dy \Rightarrow \oint_C N \, dy = \iint_R \frac{\partial N}{\partial x} dx \, dy \end{aligned}$$

42. The curl of a conservative two-dimensional field is zero. The reasoning: A two-dimensional field  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$  can be considered to be the restriction to the  $xy$ -plane of a three-dimensional field whose  $k$  component is zero, and whose  $\mathbf{i}$  and  $\mathbf{j}$  components are independent of  $z$ . For such a field to be conservative, we must have  $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$  by the component test in Section 16.3  $\Rightarrow \text{curl } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ .

43-46. Example CAS commands:

Maple:

```
with(plots);#43
M := (x,y) -> 2*x-y;
N := (x,y) -> x+3*y;
C := x^2 + 4*y^2 = 4;
implicitplot(C, x=-2..2, y=-2..2, scaling=constrained, title="#43(a) (Section 16.4)");
curlF_k := D[1](N) - D[2](M): # (b)
'curlF_k' = curlF_k(x,y);
top,bot := solve(C, y); # (c)
left,right := -2, 2;
q1 := Int(Int(curlF_k(x,y), y=bot..top), x=left..right);
value(q1);
```

Mathematica: (functions and bounds will vary)

The **ImplicitPlot** command will be useful for 43 and 44, but is not needed for 43 and 44. In 44, the equation of the line from  $(0, 4)$  to  $(2, 0)$  must be determined first.

```

Clear[x, y, f]
<<Graphics`ImplicitPlot`
f[x_, y_]:= {2x - y, x + 3y}
curve= x^2 + 4y^2 ==4
ImplicitPlot[curve, {x, -3, 3}, {y, -2, 2}, AspectRatio -> Automatic, AxesLabel -> {x, y}];
ybounds= Solve[curve, y]
{y1, y2}=y/.ybounds;
integrand:=D[f[x,y][[2]], x] - D[f[x,y][[1]], y]//Simplify
Integrate[integrand, {x, -2, 2}, {y, y1, y2}]
N[%]

```

Bounds for  $y$  are determined differently in 45 and 46. In 46, note equation of the line from  $(0, 4)$  to  $(2, 0)$ .

```

Clear[x, y, f]
f[x_, y_]:= {x Exp[y], 4x^2 Log[y]}
ybound = 4 - 2x
Plot[{0, ybound}, {x, 0, 2}, AspectRatio -> Automatic, AxesLabel -> {x, y}];
integrand:=D[f[x, y][[2]], x] - D[f[x, y][[1]], y]//Simplify
Integrate[integrand, {x, 0, 2}, {y, 0, ybound}]
N[%]

```

## 16.5 SURFACES AND AREA

- In cylindrical coordinates, let  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = (\sqrt{x^2 + y^2})^2 = r^2$ . Then  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r^2\mathbf{k}$ ,  $0 \leq r \leq 2$ ,  $0 \leq \theta \leq 2\pi$ .
- In cylindrical coordinates, let  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = 9 - x^2 - y^2 = 9 - r^2$ . Then  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (9 - r^2)\mathbf{k}$ ;  $z \geq 0 \Rightarrow 9 - r^2 \geq 0 \Rightarrow r^2 \leq 9 \Rightarrow -3 \leq r \leq 3$ ,  $0 \leq \theta \leq 2\pi$ . But  $-3 \leq r \leq 0$  gives the same points as  $0 \leq r \leq 3$ , so let  $0 \leq r \leq 3$ .
- In cylindrical coordinates, let  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = \frac{\sqrt{x^2 + y^2}}{2} \Rightarrow z = \frac{r}{2}$ . Then  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (\frac{r}{2})\mathbf{k}$ . For  $0 \leq z \leq 3$ ,  $0 \leq \frac{r}{2} \leq 3 \Rightarrow 0 \leq r \leq 6$ ; to get only the first octant, let  $0 \leq \theta \leq \frac{\pi}{2}$ .
- In cylindrical coordinates, let  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = 2\sqrt{x^2 + y^2} \Rightarrow z = 2r$ . Then  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + 2r\mathbf{k}$ . For  $2 \leq z \leq 4$ ,  $2 \leq 2r \leq 4 \Rightarrow 1 \leq r \leq 2$ , and let  $0 \leq \theta \leq 2\pi$ .
- In cylindrical coordinates, let  $x = r \cos \theta$ ,  $y = r \sin \theta$ ; since  $x^2 + y^2 = r^2 \Rightarrow z^2 = 9 - (x^2 + y^2) = 9 - r^2 \Rightarrow z = \sqrt{9 - r^2}$ ,  $z \geq 0$ . Then  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \sqrt{9 - r^2}\mathbf{k}$ . Let  $0 \leq \theta \leq 2\pi$ . For the domain of  $r$ :  $z = \sqrt{x^2 + y^2}$  and  $x^2 + y^2 + z^2 = 9 \Rightarrow x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 9 \Rightarrow 2(x^2 + y^2) = 9 \Rightarrow 2r^2 = 9 \Rightarrow r = \frac{3}{\sqrt{2}} \Rightarrow 0 \leq r \leq \frac{3}{\sqrt{2}}$ .
- In cylindrical coordinates,  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \sqrt{4 - r^2}\mathbf{k}$  (see Exercise 5 above with  $x^2 + y^2 + z^2 = 4$ , instead of  $x^2 + y^2 + z^2 = 9$ ). For the first octant, let  $0 \leq \theta \leq \frac{\pi}{2}$ . For the domain of  $r$ :  $z = \sqrt{x^2 + y^2}$  and  $x^2 + y^2 + z^2 = 4 \Rightarrow x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 4 \Rightarrow 2(x^2 + y^2) = 4 \Rightarrow 2r^2 = 4 \Rightarrow r = \sqrt{2}$ . Thus, let  $\sqrt{2} \leq r \leq 2$  (to get the portion of the sphere between the cone and the  $xy$ -plane).



7. In spherical coordinates,  $x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ ,  $\rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow \rho^2 = 3 \Rightarrow \rho = \sqrt{3}$   
 $\Rightarrow z = \sqrt{3} \cos \phi$  for the sphere;  $z = \frac{\sqrt{3}}{2} = \sqrt{3} \cos \phi \Rightarrow \cos \phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$ ;  $z = -\frac{\sqrt{3}}{2} \Rightarrow -\frac{\sqrt{3}}{2} = \sqrt{3} \cos \phi$   
 $\Rightarrow \cos \phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$ . Then  $\mathbf{r}(\phi, \theta) = (\sqrt{3} \sin \phi \cos \theta) \mathbf{i} + (\sqrt{3} \sin \phi \sin \theta) \mathbf{j} + (\sqrt{3} \cos \phi) \mathbf{k}$ ,  
 $\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$  and  $0 \leq \theta \leq 2\pi$ .
8. In spherical coordinates,  $x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ ,  $\rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow \rho^2 = 8 \Rightarrow \rho = \sqrt{8} = 2\sqrt{2}$   
 $\Rightarrow x = 2\sqrt{2} \sin \phi \cos \theta$ ,  $y = 2\sqrt{2} \sin \phi \sin \theta$ , and  $z = 2\sqrt{2} \cos \phi$ . Thus let  
 $\mathbf{r}(\phi, \theta) = (2\sqrt{2} \sin \phi \cos \theta) \mathbf{i} + (2\sqrt{2} \sin \phi \sin \theta) \mathbf{j} + (2\sqrt{2} \cos \phi) \mathbf{k}$ ;  $z = -2 \Rightarrow -2 = 2\sqrt{2} \cos \phi$   
 $\Rightarrow \cos \phi = -\frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4}$ ;  $z = 2\sqrt{2} \Rightarrow 2\sqrt{2} = 2\sqrt{2} \cos \phi \Rightarrow \cos \phi = 1 \Rightarrow \phi = 0$ . Thus  $0 \leq \phi \leq \frac{3\pi}{4}$  and  
 $0 \leq \theta \leq 2\pi$ .
9. Since  $z = 4 - y^2$ , we can let  $\mathbf{r}$  be a function of  $x$  and  $y \Rightarrow \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}$ . Then  $z = 0$   
 $\Rightarrow 0 = 4 - y^2 \Rightarrow y = \pm 2$ . Thus, let  $-2 \leq y \leq 2$  and  $0 \leq x \leq 2$ .
10. Since  $y = x^2$ , we can let  $\mathbf{r}$  be a function of  $x$  and  $z \Rightarrow \mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$ . Then  $y = 2$   
 $\Rightarrow x^2 = 2 \Rightarrow x = \pm \sqrt{2}$ . Thus, let  $-\sqrt{2} \leq x \leq \sqrt{2}$  and  $0 \leq z \leq 3$ .
11. When  $x = 0$ , let  $y^2 + z^2 = 9$  be the circular section in the  $yz$ -plane. Use polar coordinates in the  $yz$ -plane  
 $\Rightarrow y = 3 \cos \theta$  and  $z = 3 \sin \theta$ . Thus let  $x = u$  and  $\theta = v \Rightarrow \mathbf{r}(u, v) = u\mathbf{i} + (3 \cos v)\mathbf{j} + (3 \sin v)\mathbf{k}$  where  
 $0 \leq u \leq 3$ , and  $0 \leq v \leq 2\pi$ .
12. When  $y = 0$ , let  $x^2 + z^2 = 4$  be the circular section in the  $xz$ -plane. Use polar coordinates in the  $xz$ -plane  
 $\Rightarrow x = 2 \cos \theta$  and  $z = 2 \sin \theta$ . Thus let  $y = u$  and  $\theta = v \Rightarrow \mathbf{r}(u, v) = (2 \cos v)\mathbf{i} + u\mathbf{j} + (2 \sin v)\mathbf{k}$  where  
 $-2 \leq u \leq 2$ , and  $0 \leq v \leq \pi$  (since we want the portion above the  $xy$ -plane).
13. (a)  $x + y + z = 1 \Rightarrow z = 1 - x - y$ . In cylindrical coordinates, let  $x = r \cos \theta$  and  $y = r \sin \theta$   
 $\Rightarrow z = 1 - r \cos \theta - r \sin \theta \Rightarrow \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1 - r \cos \theta - r \sin \theta)\mathbf{k}$ ,  $0 \leq \theta \leq 2\pi$  and  
 $0 \leq r \leq 3$ .
- (b) In a fashion similar to cylindrical coordinates, but working in the  $yz$ -plane instead of the  $xy$ -plane, let  
 $y = u \cos v$ ,  $z = u \sin v$  where  $u = \sqrt{y^2 + z^2}$  and  $v$  is the angle formed by  $(x, y, z)$ ,  $(x, 0, 0)$ , and  $(x, y, 0)$   
with  $(x, 0, 0)$  as vertex. Since  $x + y + z = 1 \Rightarrow x = 1 - y - z \Rightarrow x = 1 - u \cos v - u \sin v$ , then  $\mathbf{r}$  is a  
function of  $u$  and  $v \Rightarrow \mathbf{r}(u, v) = (1 - u \cos v - u \sin v)\mathbf{i} + (u \cos v)\mathbf{j} + (u \sin v)\mathbf{k}$ ,  $0 \leq u \leq 3$  and  $0 \leq v \leq 2\pi$ .
14. (a) In a fashion similar to cylindrical coordinates, but working in the  $xz$ -plane instead of the  $xy$ -plane, let  
 $x = u \cos v$ ,  $z = u \sin v$  where  $u = \sqrt{x^2 + z^2}$  and  $v$  is the angle formed by  $(x, y, z)$ ,  $(y, 0, 0)$ , and  $(x, y, 0)$   
with vertex  $(y, 0, 0)$ . Since  $x - y + 2z = 2 \Rightarrow y = x + 2z - 2$ , then  $\mathbf{r}(u, v)$   
 $= (u \cos v)\mathbf{i} + (u \cos v + 2u \sin v - 2)\mathbf{j} + (u \sin v)\mathbf{k}$ ,  $0 \leq u \leq \sqrt{3}$  and  $0 \leq v \leq 2\pi$ .
- (b) In a fashion similar to cylindrical coordinates, but working in the  $yz$ -plane instead of the  $xy$ -plane, let  
 $y = u \cos v$ ,  $z = u \sin v$  where  $u = \sqrt{y^2 + z^2}$  and  $v$  is the angle formed by  $(x, y, z)$ ,  $(x, 0, 0)$ , and  $(x, y, 0)$   
with vertex  $(x, 0, 0)$ . Since  $x - y + 2z = 2 \Rightarrow x = y - 2z + 2$ , then  $\mathbf{r}(u, v)$   
 $= (u \cos v - 2u \sin v + 2)\mathbf{i} + (u \cos v)\mathbf{j} + (u \sin v)\mathbf{k}$ ,  $0 \leq u \leq \sqrt{2}$  and  $0 \leq v \leq 2\pi$ .
15. Let  $x = w \cos v$  and  $z = w \sin v$ . Then  $(x - 2)^2 + z^2 = 4 \Rightarrow x^2 - 4x + z^2 = 0 \Rightarrow w^2 \cos^2 v - 4w \cos v + w^2 \sin^2 v$   
 $= 0 \Rightarrow w^2 - 4w \cos v = 0 \Rightarrow w = 0$  or  $w - 4 \cos v = 0 \Rightarrow w = 0$  or  $w = 4 \cos v$ . Now  $w = 0 \Rightarrow x = 0$  and  $y = 0$ ,  
which is a line not a cylinder. Therefore, let  $w = 4 \cos v \Rightarrow x = (4 \cos v)(\cos v) = 4 \cos^2 v$  and  $z = 4 \cos v \sin v$ .  
Finally, let  $y = u$ . Then  $\mathbf{r}(u, v) = (4 \cos^2 v)\mathbf{i} + u\mathbf{j} + (4 \cos v \sin v)\mathbf{k}$ ,  $-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$  and  $0 \leq u \leq 3$ .

16. Let  $y = w \cos v$  and  $z = w \sin v$ . Then  $y^2 + (z - 5)^2 = 25 \Rightarrow y^2 + z^2 - 10z = 0$   
 $\Rightarrow w^2 \cos^2 v + w^2 \sin^2 v - 10w \sin v = 0 \Rightarrow w^2 - 10w \sin v = 0 \Rightarrow w(w - 10 \sin v) = 0 \Rightarrow w = 0$  or  
 $w = 10 \sin v$ . Now  $w = 0 \Rightarrow y = 0$  and  $z = 0$ , which is a line not a cylinder. Therefore, let  $w = 10 \sin v$   
 $\Rightarrow y = 10 \sin v \cos v$  and  $z = 10 \sin^2 v$ . Finally, let  $x = u$ . Then  $\mathbf{r}(u, v) = u\mathbf{i} + (10 \sin v \cos v)\mathbf{j} + (10 \sin^2 v)\mathbf{k}$ ,  
 $0 \leq u \leq 10$  and  $0 \leq v \leq \pi$ .

17. Let  $x = r \cos \theta$  and  $y = r \sin \theta$ . Then  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \left(\frac{2-r \sin \theta}{2}\right)\mathbf{k}$ ,  $0 \leq r \leq 1$  and  $0 \leq \theta \leq 2\pi$   
 $\Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - \left(\frac{\sin \theta}{2}\right)\mathbf{k}$  and  $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} - \left(\frac{r \cos \theta}{2}\right)\mathbf{k}$   
 $\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -\frac{\sin \theta}{2} \\ -r \sin \theta & r \cos \theta & -\frac{r \cos \theta}{2} \end{vmatrix}$   
 $= \left(\frac{-r \sin \theta \cos \theta}{2} + \frac{(\sin \theta)(r \cos \theta)}{2}\right)\mathbf{i} + \left(\frac{r \sin^2 \theta}{2} + \frac{r \cos^2 \theta}{2}\right)\mathbf{j} + (r \cos^2 \theta + r \sin^2 \theta)\mathbf{k} = \frac{r}{2}\mathbf{j} + r\mathbf{k}$   
 $\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{r^2}{4} + r^2} = \frac{\sqrt{5}r}{2} \Rightarrow A = \int_0^{2\pi} \int_0^1 \frac{\sqrt{5}r}{2} dr d\theta = \int_0^{2\pi} \left[\frac{\sqrt{5}r^2}{4}\right]_0^1 d\theta = \int_0^{2\pi} d\theta = \frac{\pi\sqrt{5}}{2}$

18. Let  $x = r \cos \theta$  and  $y = r \sin \theta \Rightarrow z = -x = -r \cos \theta$ ,  $0 \leq r \leq 2$  and  $0 \leq \theta \leq 2\pi$ . Then  
 $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - (r \cos \theta)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - (\cos \theta)\mathbf{k}$  and  
 $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} + (r \sin \theta)\mathbf{k}$   
 $\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -\cos \theta \\ -r \sin \theta & r \cos \theta & r \sin \theta \end{vmatrix}$   
 $= (r \sin^2 \theta + r \cos^2 \theta)\mathbf{i} + (r \sin \theta \cos \theta - r \sin \theta \cos \theta)\mathbf{j} + (r \cos^2 \theta + r \sin^2 \theta)\mathbf{k} = r\mathbf{i} + r\mathbf{k}$   
 $\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 + r^2} = r\sqrt{2} \Rightarrow A = \int_0^{2\pi} \int_0^2 r\sqrt{2} dr d\theta = \int_0^{2\pi} \left[\frac{r^2\sqrt{2}}{2}\right]_0^2 d\theta = \int_0^{2\pi} 2\sqrt{2} d\theta = 4\pi\sqrt{2}$

19. Let  $x = r \cos \theta$  and  $y = r \sin \theta \Rightarrow z = 2\sqrt{x^2 + y^2} = 2r$ ,  $1 \leq r \leq 3$  and  $0 \leq \theta \leq 2\pi$ . Then  
 $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + 2r\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + 2\mathbf{k}$  and  $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j}$   
 $\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (-2r \cos \theta)\mathbf{i} - (2r \sin \theta)\mathbf{j} + (r \cos^2 \theta + r \sin^2 \theta)\mathbf{k}$   
 $= (-2r \cos \theta)\mathbf{i} - (2r \sin \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta + r^2} = \sqrt{5r^2} = r\sqrt{5}$   
 $\Rightarrow A = \int_0^{2\pi} \int_1^3 r\sqrt{5} dr d\theta = \int_0^{2\pi} \left[\frac{r^2\sqrt{5}}{2}\right]_1^3 d\theta = \int_0^{2\pi} 4\sqrt{5} d\theta = 8\pi\sqrt{5}$

20. Let  $x = r \cos \theta$  and  $y = r \sin \theta \Rightarrow z = \frac{\sqrt{x^2 + y^2}}{3} = \frac{r}{3}$ ,  $3 \leq r \leq 4$  and  $0 \leq \theta \leq 2\pi$ . Then  
 $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \left(\frac{r}{3}\right)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \left(\frac{1}{3}\right)\mathbf{k}$  and  $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j}$   
 $\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & \frac{1}{3} \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \left(-\frac{1}{3}r \cos \theta\right)\mathbf{i} - \left(\frac{1}{3}r \sin \theta\right)\mathbf{j} + (r \cos^2 \theta + r \sin^2 \theta)\mathbf{k}$   
 $= \left(-\frac{1}{3}r \cos \theta\right)\mathbf{i} - \left(\frac{1}{3}r \sin \theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{1}{9}r^2 \cos^2 \theta + \frac{1}{9}r^2 \sin^2 \theta + r^2} = \sqrt{\frac{10r^2}{9}} = \frac{r\sqrt{10}}{3}$   
 $\Rightarrow A = \int_0^{2\pi} \int_3^4 \frac{r\sqrt{10}}{3} dr d\theta = \int_0^{2\pi} \left[\frac{r^2\sqrt{10}}{6}\right]_3^4 d\theta = \int_0^{2\pi} \frac{7\sqrt{10}}{6} d\theta = \frac{7\pi\sqrt{10}}{3}$

21. Let  $x = r \cos \theta$  and  $y = r \sin \theta \Rightarrow r^2 = x^2 + y^2 = 1$ ,  $1 \leq z \leq 4$  and  $0 \leq \theta \leq 2\pi$ . Then  
 $\mathbf{r}(z, \theta) = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_z = \mathbf{k}$  and  $\mathbf{r}_\theta = (-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} \Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

$$\Rightarrow A = \int_0^{2\pi} \int_1^4 1 \, dr \, d\theta = \int_0^{2\pi} 3 \, d\theta = 6\pi$$

22. Let  $x = u \cos v$  and  $z = u \sin v \Rightarrow u^2 = x^2 + z^2 = 10, -1 \leq y \leq 1, 0 \leq v \leq 2\pi$ . Then

$$\mathbf{r}(y, v) = (u \cos v)\mathbf{i} + y\mathbf{j} + (u \sin v)\mathbf{k} = (\sqrt{10} \cos v)\mathbf{i} + y\mathbf{j} + (\sqrt{10} \sin v)\mathbf{k}$$

$$\Rightarrow \mathbf{r}_v = (-\sqrt{10} \sin v)\mathbf{i} + (\sqrt{10} \cos v)\mathbf{k} \text{ and } \mathbf{r}_y = \mathbf{j} \Rightarrow \mathbf{r}_v \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{10} \sin v & 0 & \sqrt{10} \cos v \\ 0 & 1 & 0 \end{vmatrix}$$

$$= (-\sqrt{10} \cos v)\mathbf{i} - (\sqrt{10} \sin v)\mathbf{k} \Rightarrow |\mathbf{r}_v \times \mathbf{r}_y| = \sqrt{10} \Rightarrow A = \int_0^{2\pi} \int_{-1}^1 \sqrt{10} \, du \, dv = \int_0^{2\pi} [\sqrt{10}u]_{-1}^1 \, dv$$

$$= \int_0^{2\pi} 2\sqrt{10} \, dv = 4\pi\sqrt{10}$$

23.  $z = 2 - x^2 - y^2$  and  $z = \sqrt{x^2 + y^2} \Rightarrow z = 2 - z^2 \Rightarrow z^2 + z - 2 = 0 \Rightarrow z = -2$  or  $z = 1$ . Since  $z = \sqrt{x^2 + y^2} \geq 0$ , we get  $z = 1$  where the cone intersects the paraboloid. When  $x = 0$  and  $y = 0, z = 2 \Rightarrow$  the vertex of the paraboloid is  $(0, 0, 2)$ . Therefore,  $z$  ranges from 1 to 2 on the “cap”  $\Rightarrow r$  ranges from 1 (when  $x^2 + y^2 = 1$ ) to 0 (when  $x = 0$  and  $y = 0$  at the vertex). Let  $x = r \cos \theta, y = r \sin \theta$ , and  $z = 2 - r^2$ . Then

$$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (2 - r^2)\mathbf{k}, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - 2r\mathbf{k} \text{ and}$$

$$\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix}$$

$$= (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4 \cos^2 \theta + 4r^4 \sin^2 \theta + r^2} = r\sqrt{4r^2 + 1}$$

$$\Rightarrow A = \int_0^{2\pi} \int_0^1 r\sqrt{4r^2 + 1} \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{1}{12} (4r^2 + 1)^{3/2} \right]_0^1 \, d\theta = \int_0^{2\pi} \left( \frac{5\sqrt{5}-1}{12} \right) \, d\theta = \frac{\pi}{6} (5\sqrt{5} - 1)$$

24. Let  $x = r \cos \theta, y = r \sin \theta$  and  $z = x^2 + y^2 = r^2$ . Then  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r^2\mathbf{k}, 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + 2r\mathbf{k}$  and  $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (-2r^2 \cos \theta)\mathbf{i} - (2r^2 \sin \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$$

$$= \sqrt{4r^4 \cos^2 \theta + 4r^4 \sin^2 \theta + r^2} = r\sqrt{4r^2 + 1} \Rightarrow A = \int_0^{2\pi} \int_1^2 r\sqrt{4r^2 + 1} \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{1}{12} (4r^2 + 1)^{3/2} \right]_1^2 \, d\theta$$

$$= \int_0^{2\pi} \left( \frac{17\sqrt{17}-5\sqrt{5}}{12} \right) \, d\theta = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5})$$

25. Let  $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta$ , and  $z = \rho \cos \phi \Rightarrow \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2}$  on the sphere. Next,  $x^2 + y^2 + z^2 = 2$  and  $z = \sqrt{x^2 + y^2} \Rightarrow z^2 + z^2 = 2 \Rightarrow z^2 = 1 \Rightarrow z = 1$  since  $z \geq 0 \Rightarrow \phi = \frac{\pi}{4}$ . For the lower portion of the sphere cut by the cone, we get  $\phi = \pi$ . Then

$$\mathbf{r}(\phi, \theta) = (\sqrt{2} \sin \phi \cos \theta)\mathbf{i} + (\sqrt{2} \sin \phi \sin \theta)\mathbf{j} + (\sqrt{2} \cos \phi)\mathbf{k}, \frac{\pi}{4} \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \mathbf{r}_\phi = (\sqrt{2} \cos \phi \cos \theta)\mathbf{i} + (\sqrt{2} \cos \phi \sin \theta)\mathbf{j} - (\sqrt{2} \sin \phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-\sqrt{2} \sin \phi \sin \theta)\mathbf{i} + (\sqrt{2} \sin \phi \cos \theta)\mathbf{j}$$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2} \cos \phi \cos \theta & \sqrt{2} \cos \phi \sin \theta & -\sqrt{2} \sin \phi \\ -\sqrt{2} \sin \phi \sin \theta & \sqrt{2} \sin \phi \cos \theta & 0 \end{vmatrix}$$

$$= (2 \sin^2 \phi \cos \theta)\mathbf{i} + (2 \sin^2 \phi \sin \theta)\mathbf{j} + (2 \sin \phi \cos \phi)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{4 \sin^4 \phi \cos^2 \theta + 4 \sin^4 \phi \sin^2 \theta + 4 \sin^2 \phi \cos^2 \phi} = \sqrt{4 \sin^2 \phi} = 2 |\sin \phi| = 2 \sin \phi$$

$$\Rightarrow A = \int_0^{2\pi} \int_{\pi/4}^{\pi} 2 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} (2 + \sqrt{2}) \, d\theta = (4 + 2\sqrt{2}) \pi$$

26. Let  $x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ , and  $z = \rho \cos \phi \Rightarrow \rho = \sqrt{x^2 + y^2 + z^2} = 2$  on the sphere. Next,  
 $z = -1 \Rightarrow -1 = 2 \cos \phi \Rightarrow \cos \phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$ ;  $z = \sqrt{3} \Rightarrow \sqrt{3} = 2 \cos \phi \Rightarrow \cos \phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$ . Then  
 $\mathbf{r}(\phi, \theta) = (2 \sin \phi \cos \theta)\mathbf{i} + (2 \sin \phi \sin \theta)\mathbf{j} + (2 \cos \phi)\mathbf{k}$ ,  $\frac{\pi}{6} \leq \phi \leq \frac{2\pi}{3}$ ,  $0 \leq \theta \leq 2\pi$

$$\Rightarrow \mathbf{r}_\phi = (2 \cos \phi \cos \theta)\mathbf{i} + (2 \cos \phi \sin \theta)\mathbf{j} - (2 \sin \phi)\mathbf{k} \text{ and}$$

$$\mathbf{r}_\theta = (-2 \sin \phi \sin \theta)\mathbf{i} + (2 \sin \phi \cos \theta)\mathbf{j}$$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos \phi \cos \theta & 2 \cos \phi \sin \theta & -2 \sin \phi \\ -2 \sin \phi \sin \theta & 2 \sin \phi \cos \theta & 0 \end{vmatrix}$$

$$= (4 \sin^2 \phi \cos \theta)\mathbf{i} + (4 \sin^2 \phi \sin \theta)\mathbf{j} + (4 \sin \phi \cos \phi)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16 \sin^4 \phi \cos^2 \theta + 16 \sin^4 \phi \sin^2 \theta + 16 \sin^2 \phi \cos^2 \phi} = \sqrt{16 \sin^2 \phi} = 4 |\sin \phi| = 4 \sin \phi$$

$$\Rightarrow A = \int_0^{2\pi} \int_{\pi/6}^{2\pi/3} 4 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} (2 + 2\sqrt{3}) \, d\theta = (4 + 4\sqrt{3}) \pi$$

27. The parametrization  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$

$$\text{at } P_0 = (\sqrt{2}, \sqrt{2}, 2) \Rightarrow \theta = \frac{\pi}{4}, r = 2,$$

$$\mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k} \text{ and}$$

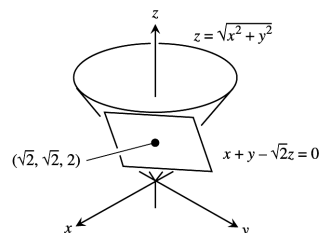
$$\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix}$$

$$= -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k} \Rightarrow \text{the tangent plane is}$$

$$0 = (-\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}) \cdot [(x - \sqrt{2})\mathbf{i} + (y - \sqrt{2})\mathbf{j} + (z - 2)\mathbf{k}] \Rightarrow \sqrt{2}x + \sqrt{2}y - 2z = 0, \text{ or } x + y - \sqrt{2}z = 0.$$

$$\text{The parametrization } \mathbf{r}(r, \theta) \Rightarrow x = r \cos \theta, y = r \sin \theta \text{ and } z = r \Rightarrow x^2 + y^2 = r^2 = z^2 \Rightarrow \text{the surface is } z = \sqrt{x^2 + y^2}.$$



28. The parametrization  $\mathbf{r}(\phi, \theta)$

$$= (4 \sin \phi \cos \theta)\mathbf{i} + (4 \sin \phi \sin \theta)\mathbf{j} + (4 \cos \phi)\mathbf{k}$$

$$\text{at } P_0 = (\sqrt{2}, \sqrt{2}, 2\sqrt{3}) \Rightarrow \rho = 4 \text{ and } z = 2\sqrt{3}$$

$$= 4 \cos \phi \Rightarrow \phi = \frac{\pi}{6}; \text{ also } x = \sqrt{2} \text{ and } y = \sqrt{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}. \text{ Then } \mathbf{r}_\phi$$

$$= (4 \cos \phi \cos \theta)\mathbf{i} + (4 \cos \phi \sin \theta)\mathbf{j} - (4 \sin \phi)\mathbf{k}$$

$$= \sqrt{6}\mathbf{i} + \sqrt{6}\mathbf{j} - 2\mathbf{k} \text{ and}$$

$$\mathbf{r}_\theta = (-4 \sin \phi \sin \theta)\mathbf{i} + (4 \sin \phi \cos \theta)\mathbf{j}$$

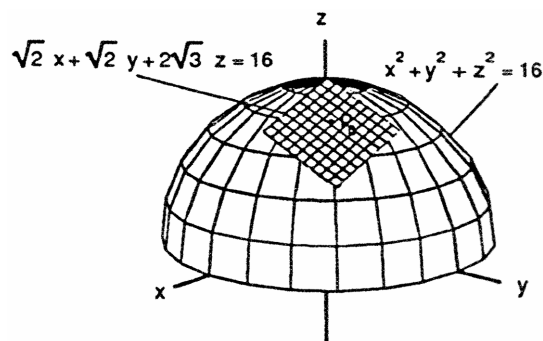
$$= -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} \text{ at } P_0 \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{6} & \sqrt{6} & -2 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix}$$

$$= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k} \Rightarrow \text{the tangent plane is}$$

$$(2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}) \cdot [(x - \sqrt{2})\mathbf{i} + (y - \sqrt{2})\mathbf{j} + (z - 2\sqrt{3})\mathbf{k}] = 0 \Rightarrow \sqrt{2}x + \sqrt{2}y + 2\sqrt{3}z = 16,$$

$$\text{or } x + y + \sqrt{6}z = 8\sqrt{2}. \text{ The parametrization } \Rightarrow x = 4 \sin \phi \cos \theta, y = 4 \sin \phi \sin \theta, z = 4 \cos \phi$$

$$\Rightarrow \text{the surface is } x^2 + y^2 + z^2 = 16, z \geq 0.$$



29. The parametrization
- $\mathbf{r}(\theta, z) = (3 \sin 2\theta)\mathbf{i} + (6 \sin^2 \theta)\mathbf{j} + z\mathbf{k}$

at  $P_0 = \left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right) \Rightarrow \theta = \frac{\pi}{3}$  and  $z = 0$ . Then

$$\mathbf{r}_\theta = (6 \cos 2\theta)\mathbf{i} + (12 \sin \theta \cos \theta)\mathbf{j}$$

$$= -3\mathbf{i} + 3\sqrt{3}\mathbf{j} \text{ and } \mathbf{r}_z = \mathbf{k} \text{ at } P_0$$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3\sqrt{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\sqrt{3}\mathbf{i} + 3\mathbf{j}$$

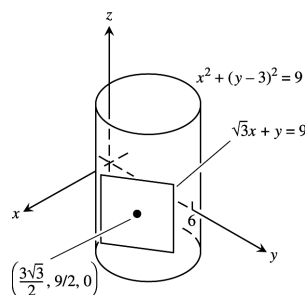
 $\Rightarrow$  the tangent plane is

$$(3\sqrt{3}\mathbf{i} + 3\mathbf{j}) \cdot \left[\left(x - \frac{3\sqrt{3}}{2}\right)\mathbf{i} + \left(y - \frac{9}{2}\right)\mathbf{j} + (z - 0)\mathbf{k}\right] = 0$$

$$\Rightarrow \sqrt{3}x + y = 9. \text{ The parametrization } \Rightarrow x = 3 \sin 2\theta$$

$$\text{and } y = 6 \sin^2 \theta \Rightarrow x^2 + y^2 = 9 \sin^2 2\theta + (6 \sin^2 \theta)^2$$

$$= 9(4 \sin^2 \theta \cos^2 \theta) + 36 \sin^4 \theta = 6(6 \sin^2 \theta) = 6y \Rightarrow x^2 + y^2 - 6y + 9 = 9 \Rightarrow x^2 + (y - 3)^2 = 9$$



30. The parametrization
- $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$
- at

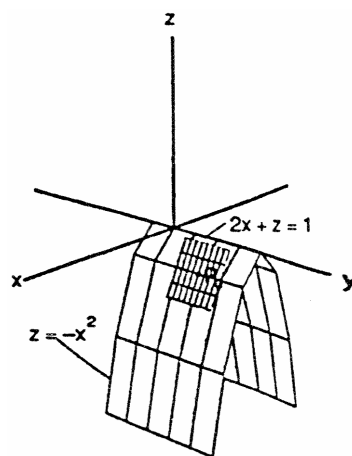
 $P_0 = (1, 2, -1) \Rightarrow \mathbf{r}_x = \mathbf{i} - 2x\mathbf{k} = \mathbf{i} - 2\mathbf{k}$  and  $\mathbf{r}_y = \mathbf{j}$  at  $P_0$ 

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + \mathbf{k} \Rightarrow \text{the tangent plane}$$

$$\text{is } (2\mathbf{i} + \mathbf{k}) \cdot [(x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z + 1)\mathbf{k}] = 0$$

$$\Rightarrow 2x + z = 1. \text{ The parametrization } \Rightarrow x = x, y = y \text{ and}$$

$$z = -x^2 \Rightarrow \text{the surface is } z = -x^2$$



31. (a) An arbitrary point on the circle
- $C$
- is
- $(x, z) = (R + r \cos u, r \sin u) \Rightarrow (x, y, z)$
- is on the torus with

$$x = (R + r \cos u) \cos v, y = (R + r \cos u) \sin v, \text{ and } z = r \sin u, 0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$$

- (b)
- $\mathbf{r}_u = (-r \sin u \cos v)\mathbf{i} - (r \sin u \sin v)\mathbf{j} + (r \cos u)\mathbf{k}$
- and
- $\mathbf{r}_v = (-(R + r \cos u) \sin v)\mathbf{i} + ((R + r \cos u) \cos v)\mathbf{j}$

$$\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin u \cos v & -r \sin u \sin v & r \cos u \\ -(R + r \cos u) \sin v & (R + r \cos u) \cos v & 0 \end{vmatrix}$$

$$= -(R + r \cos u)(r \cos v \cos u)\mathbf{i} - (R + r \cos u)(r \sin v \cos u)\mathbf{j} + (-r \sin u)(R + r \cos u)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = (R + r \cos u)^2 (r^2 \cos^2 v \cos^2 u + r^2 \sin^2 v \cos^2 u + r^2 \sin^2 u) \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = r(R + r \cos u)$$

$$\Rightarrow A = \int_0^{2\pi} \int_0^{2\pi} (rR + r^2 \cos u) du dv = \int_0^{2\pi} 2\pi rR dv = 4\pi^2 rR$$

32. (a) The point
- $(x, y, z)$
- is on the surface for fixed
- $x = f(u)$
- when
- $y = g(u) \sin\left(\frac{\pi}{2} - v\right)$
- and
- $z = g(u) \cos\left(\frac{\pi}{2} - v\right)$

$$\Rightarrow x = f(u), y = g(u) \cos v, \text{ and } z = g(u) \sin v \Rightarrow \mathbf{r}(u, v) = f(u)\mathbf{i} + (g(u) \cos v)\mathbf{j} + (g(u) \sin v)\mathbf{k}, 0 \leq v \leq 2\pi,$$

$$a \leq u \leq b$$

- (b) Let
- $u = y$
- and
- $x = u^2 \Rightarrow f(u) = u^2$
- and
- $g(u) = u \Rightarrow \mathbf{r}(u, v) = u^2\mathbf{i} + (u \cos v)\mathbf{j} + (u \sin v)\mathbf{k}, 0 \leq v \leq 2\pi, 0 \leq u$

33. (a) Let
- $w^2 + \frac{z^2}{c^2} = 1$
- where
- $w = \cos \phi$
- and
- $\frac{z}{c} = \sin \phi \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \phi \Rightarrow \frac{x}{a} = \cos \phi \cos \theta$
- and
- $\frac{y}{b} = \cos \phi \sin \theta$

$$\Rightarrow x = a \cos \theta \cos \phi, y = b \sin \theta \cos \phi, \text{ and } z = c \sin \phi$$

$$\Rightarrow \mathbf{r}(\theta, \phi) = (a \cos \theta \cos \phi)\mathbf{i} + (b \sin \theta \cos \phi)\mathbf{j} + (c \sin \phi)\mathbf{k}$$

$$(b) \mathbf{r}_\theta = (-a \sin \theta \cos \phi) \mathbf{i} + (b \cos \theta \cos \phi) \mathbf{j} \text{ and } \mathbf{r}_\phi = (-a \cos \theta \sin \phi) \mathbf{i} - (b \sin \theta \sin \phi) \mathbf{j} + (c \cos \phi) \mathbf{k}$$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin \theta \cos \phi & b \cos \theta \cos \phi & 0 \\ -a \cos \theta \sin \phi & -b \sin \theta \sin \phi & c \cos \phi \end{vmatrix}$$

$$= (bc \cos \theta \cos^2 \phi) \mathbf{i} + (ac \sin \theta \cos^2 \phi) \mathbf{j} + (ab \sin \phi \cos \phi) \mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_\phi|^2 = b^2 c^2 \cos^2 \theta \cos^4 \phi + a^2 c^2 \sin^2 \theta \cos^4 \phi + a^2 b^2 \sin^2 \phi \cos^2 \phi, \text{ and the result follows.}$$

$$A \Rightarrow \int_0^{2\pi} \int_0^\pi |\mathbf{r}_\theta \times \mathbf{r}_\phi| d\phi d\theta = \int_0^{2\pi} \int_0^\pi [a^2 b^2 \sin^2 \phi \cos^2 \phi + b^2 c^2 \cos^2 \theta \cos^4 \phi + a^2 c^2 \sin^2 \theta \cos^4 \phi]^{1/2} d\phi d\theta$$

$$34. (a) \mathbf{r}(\theta, u) = (\cosh u \cos \theta) \mathbf{i} + (\cosh u \sin \theta) \mathbf{j} + (\sinh u) \mathbf{k}$$

$$(b) \mathbf{r}(\theta, u) = (a \cosh u \cos \theta) \mathbf{i} + (b \cosh u \sin \theta) \mathbf{j} + (c \sinh u) \mathbf{k}$$

$$35. \mathbf{r}(\theta, u) = (5 \cosh u \cos \theta) \mathbf{i} + (5 \cosh u \sin \theta) \mathbf{j} + (5 \sinh u) \mathbf{k} \Rightarrow \mathbf{r}_\theta = (-5 \cosh u \sin \theta) \mathbf{i} + (5 \cosh u \cos \theta) \mathbf{j} \text{ and } \mathbf{r}_u = (5 \sinh u \cos \theta) \mathbf{i} + (5 \sinh u \sin \theta) \mathbf{j} + (5 \cosh u) \mathbf{k}$$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 \cosh u \sin \theta & 5 \cosh u \cos \theta & 0 \\ 5 \sinh u \cos \theta & 5 \sinh u \sin \theta & 5 \cosh u \end{vmatrix}$$

$$= (25 \cosh^2 u \cos \theta) \mathbf{i} + (25 \cosh^2 u \sin \theta) \mathbf{j} - (25 \cosh u \sinh u) \mathbf{k}. \text{ At the point } (x_0, y_0, 0), \text{ where } x_0^2 + y_0^2 = 25$$

$$\text{we have } 5 \sinh u = 0 \Rightarrow u = 0 \text{ and } x_0 = 25 \cos \theta, y_0 = 25 \sin \theta \Rightarrow \text{the tangent plane is}$$

$$5(x_0 \mathbf{i} + y_0 \mathbf{j}) \cdot [(x - x_0) \mathbf{i} + (y - y_0) \mathbf{j} + z \mathbf{k}] = 0 \Rightarrow x_0 x - x_0^2 + y_0 y - y_0^2 = 0 \Rightarrow x_0 x + y_0 y = 25$$

$$36. \text{ Let } \frac{z^2}{c^2} - w^2 = 1 \text{ where } \frac{z}{c} = \cosh u \text{ and } w = \sinh u \Rightarrow w^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \frac{x}{a} = w \cosh u \text{ and } \frac{y}{b} = w \sinh u$$

$$\Rightarrow x = a \sinh u \cosh u, y = b \sinh u \sinh u, \text{ and } z = c \cosh u$$

$$\Rightarrow \mathbf{r}(\theta, u) = (a \sinh u \cosh u) \mathbf{i} + (b \sinh u \sinh u) \mathbf{j} + (c \cosh u) \mathbf{k}, 0 \leq \theta \leq 2\pi, -\infty < u < \infty$$

$$37. \mathbf{p} = \mathbf{k}, \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2} = \sqrt{4x^2 + 4y^2 + 1} \text{ and } |\nabla f \cdot \mathbf{p}| = 1;$$

$$z = 2 \Rightarrow x^2 + y^2 = 2; \text{ thus } S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{4x^2 + 4y^2 + 1} dx dy$$

$$= \iint_R \sqrt{4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta + 1} r dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \left[ \frac{1}{12} (4r^2 + 1)^{3/2} \right]_0^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \frac{13}{6} d\theta = \frac{13}{3} \pi$$

$$38. \mathbf{p} = \mathbf{k}, \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 1} \text{ and } |\nabla f \cdot \mathbf{p}| = 1; 2 \leq x^2 + y^2 \leq 6$$

$$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{4x^2 + 4y^2 + 1} dx dy = \iint_R \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \int_{\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2 + 1} r dr d\theta$$

$$= \int_0^{2\pi} \left[ \frac{1}{12} (4r^2 + 1)^{3/2} \right]_{\sqrt{2}}^{\sqrt{6}} d\theta = \int_0^{2\pi} \frac{49}{6} d\theta = \frac{49}{3} \pi$$

$$39. \mathbf{p} = \mathbf{k}, \nabla f = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\nabla f| = 3 \text{ and } |\nabla f \cdot \mathbf{p}| = 2; x = y^2 \text{ and } x = 2 - y^2 \text{ intersect at } (1, 1) \text{ and } (1, -1)$$

$$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{3}{2} dx dy = \int_{-1}^1 \int_{y^2}^{2-y^2} \frac{3}{2} dx dy = \int_{-1}^1 (3 - 3y^2) dy = 4$$

$$40. \mathbf{p} = \mathbf{k}, \nabla f = 2x\mathbf{i} - 2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1} \text{ and } |\nabla f \cdot \mathbf{p}| = 2 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA$$

$$= \iint_R \frac{2\sqrt{x^2 + 1}}{2} dx dy = \int_0^{\sqrt{3}} \int_0^x \sqrt{x^2 + 1} dy dx = \int_0^{\sqrt{3}} x \sqrt{x^2 + 1} dx = \left[ \frac{1}{3} (x^2 + 1)^{3/2} \right]_0^{\sqrt{3}} = \frac{1}{3} (4)^{3/2} - \frac{1}{3} = \frac{7}{3}$$

$$\begin{aligned}
 41. \quad \mathbf{p} = \mathbf{k}, \quad \nabla f = 2x\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} &\Rightarrow |\nabla f| = \sqrt{(2x)^2 + (-2)^2 + (-2)^2} = \sqrt{4x^2 + 8} = 2\sqrt{x^2 + 2} \text{ and } |\nabla f \cdot \mathbf{p}| = 2 \\
 &\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2\sqrt{x^2+2}}{2} dx dy = \int_0^2 \int_0^{3x} \sqrt{x^2+2} dy dx = \int_0^2 3x\sqrt{x^2+2} dx = \left[ (x^2+2)^{3/2} \right]_0^2 \\
 &= 6\sqrt{6} - 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 42. \quad \mathbf{p} = \mathbf{k}, \quad \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} &\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{8} = 2\sqrt{2} \text{ and } |\nabla f \cdot \mathbf{p}| = 2z; x^2 + y^2 + z^2 = 2 \text{ and } \\
 z = \sqrt{x^2 + y^2} &\Rightarrow x^2 + y^2 = 1; \text{ thus, } S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2\sqrt{2}}{2z} dA = \sqrt{2} \iint_R \frac{1}{z} dA \\
 &= \sqrt{2} \iint_R \frac{1}{\sqrt{2-(x^2+y^2)}} dA = \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r dr d\theta}{\sqrt{2-r^2}} = \sqrt{2} \int_0^{2\pi} (-1 + \sqrt{2}) d\theta = 2\pi(2 - \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \mathbf{p} = \mathbf{k}, \quad \nabla f = c\mathbf{i} - \mathbf{k} &\Rightarrow |\nabla f| = \sqrt{c^2 + 1} \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{c^2 + 1} dx dy \\
 &= \int_0^{2\pi} \int_0^1 \sqrt{c^2 + 1} r dr d\theta = \int_0^{2\pi} \frac{\sqrt{c^2+1}}{2} d\theta = \pi\sqrt{c^2 + 1}
 \end{aligned}$$

$$\begin{aligned}
 44. \quad \mathbf{p} = \mathbf{k}, \quad \nabla f = 2x\mathbf{i} + 2z\mathbf{j} &\Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2z)^2} = 2 \text{ and } |\nabla f \cdot \mathbf{p}| = 2z \text{ for the upper surface, } z \geq 0 \\
 &\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2}{2z} dA = \iint_R \frac{1}{\sqrt{1-x^2}} dy dx = 2 \int_{-1/2}^{1/2} \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dy dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} dx \\
 &= [\sin^{-1} x]_{-1/2}^{1/2} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \mathbf{p} = \mathbf{i}, \quad \nabla f = \mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} &\Rightarrow |\nabla f| = \sqrt{1^2 + (2y)^2 + (2z)^2} = \sqrt{1 + 4y^2 + 4z^2} \text{ and } |\nabla f \cdot \mathbf{p}| = 1; 1 \leq y^2 + z^2 \leq 4 \\
 &\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{1 + 4y^2 + 4z^2} dy dz = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta} r dr d\theta \\
 &= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} r dr d\theta = \int_0^{2\pi} \left[ \frac{1}{12} (1 + 4r^2)^{3/2} \right]_1^2 d\theta = \int_0^{2\pi} \frac{1}{12} (17\sqrt{17} - 5\sqrt{5}) d\theta = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5})
 \end{aligned}$$

$$\begin{aligned}
 46. \quad \mathbf{p} = \mathbf{j}, \quad \nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} &\Rightarrow |\nabla f| = \sqrt{4x^2 + 4z^2 + 1} \text{ and } |\nabla f \cdot \mathbf{p}| = 1; y = 0 \text{ and } x^2 + y + z^2 = 2 \Rightarrow x^2 + z^2 = 2; \\
 \text{thus, } S &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{4x^2 + 4z^2 + 1} dx dz = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \frac{13}{6} d\theta = \frac{13}{3} \pi
 \end{aligned}$$

$$\begin{aligned}
 47. \quad \mathbf{p} = \mathbf{k}, \quad \nabla f = (2x - \frac{2}{x})\mathbf{i} + \sqrt{15}\mathbf{j} - \mathbf{k} &\Rightarrow |\nabla f| = \sqrt{(2x - \frac{2}{x})^2 + (\sqrt{15})^2 + (-1)^2} = \sqrt{4x^2 + 8 + \frac{4}{x^2}} = \sqrt{(2x + \frac{2}{x})^2} \\
 &= 2x + \frac{2}{x}, \text{ on } 1 \leq x \leq 2 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R (2x + 2x^{-1}) dx dy \\
 &= \int_0^1 \int_1^2 (2x + 2x^{-1}) dx dy = \int_0^1 [x^2 + 2 \ln x]_1^2 dy = \int_0^1 (3 + 2 \ln 2) dy = 3 + 2 \ln 2
 \end{aligned}$$

$$\begin{aligned}
 48. \quad \mathbf{p} = \mathbf{k}, \quad \nabla f = 3\sqrt{x}\mathbf{i} + 3\sqrt{y}\mathbf{j} - 3\mathbf{k} &\Rightarrow |\nabla f| = \sqrt{9x + 9y + 9} = 3\sqrt{x + y + 1} \text{ and } |\nabla f \cdot \mathbf{p}| = 3 \\
 &\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{x + y + 1} dx dy = \int_0^1 \int_0^1 \sqrt{x + y + 1} dx dy = \int_0^1 \left[ \frac{2}{3} (x + y + 1)^{3/2} \right]_0^1 dy \\
 &= \int_0^1 \left[ \frac{2}{3} (y + 2)^{3/2} - \frac{2}{3} (y + 1)^{3/2} \right] dy = \left[ \frac{4}{15} (y + 2)^{5/2} - \frac{4}{15} (y + 1)^{5/2} \right]_0^1 = \frac{4}{15} [(3)^{5/2} - (2)^{5/2} - (2)^{5/2} + 1] \\
 &= \frac{4}{15} (9\sqrt{3} - 8\sqrt{2} + 1)
 \end{aligned}$$

$$\begin{aligned}
 49. \quad f_x(x, y) = 2x, f_y(x, y) = 2y &\Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} \Rightarrow \text{Area} = \iint_R \sqrt{4x^2 + 4y^2 + 1} dx dy \\
 &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{4r^2 + 1} r dr d\theta = \frac{\pi}{6} (13\sqrt{13} - 1)
 \end{aligned}$$

$$50. f_y(y, z) = -2y, f_z(y, z) = -2z \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{4y^2 + 4z^2 + 1} \Rightarrow \text{Area} = \iint_R \sqrt{4y^2 + 4z^2 + 1} \, dy \, dz \\ = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} \, r \, dr \, d\theta = \frac{\pi}{6} (5\sqrt{5} - 1)$$

$$51. f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} = \sqrt{2} \Rightarrow \text{Area} = \iint_{R_{xy}} \sqrt{2} \, dx \, dy \\ = \sqrt{2}(\text{Area between the ellipse and the circle}) = \sqrt{2}(6\pi - \pi) = 5\pi\sqrt{2}$$

$$52. \text{Over } R_{xy}: z = 2 - \frac{2}{3}x - 2y \Rightarrow f_x(x, y) = -\frac{2}{3}, f_y(x, y) = -2 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{4}{9} + 4 + 1} = \frac{7}{3} \\ \Rightarrow \text{Area} = \iint_{R_{xy}} \frac{7}{3} \, dA = \frac{7}{3} (\text{Area of the shadow triangle in the } xy\text{-plane}) = \left(\frac{7}{3}\right) \left(\frac{3}{2}\right) = \frac{7}{2}.$$

$$\text{Over } R_{xz}: y = 1 - \frac{1}{3}x - \frac{1}{2}z \Rightarrow f_x(x, z) = -\frac{1}{3}, f_z(x, z) = -\frac{1}{2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{\frac{1}{9} + \frac{1}{4} + 1} = \frac{7}{6} \\ \Rightarrow \text{Area} = \iint_{R_{xz}} \frac{7}{6} \, dA = \frac{7}{6} (\text{Area of the shadow triangle in the } xz\text{-plane}) = \left(\frac{7}{6}\right) (3) = \frac{7}{2}.$$

$$\text{Over } R_{yz}: x = 3 - 3y - \frac{3}{2}z \Rightarrow f_y(y, z) = -3, f_z(y, z) = -\frac{3}{2} \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{9 + \frac{9}{4} + 1} = \frac{7}{2} \\ \Rightarrow \text{Area} = \iint_{R_{yz}} \frac{7}{2} \, dA = \frac{7}{2} (\text{Area of the shadow triangle in the } yz\text{-plane}) = \left(\frac{7}{2}\right) (1) = \frac{7}{2}.$$

$$53. y = \frac{2}{3}z^{3/2} \Rightarrow f_x(x, z) = 0, f_z(x, z) = z^{1/2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{z + 1}; y = \frac{16}{3} \Rightarrow \frac{16}{3} = \frac{2}{3}z^{3/2} \Rightarrow z = 4 \\ \Rightarrow \text{Area} = \int_0^4 \int_0^1 \sqrt{z + 1} \, dx \, dz = \int_0^4 \sqrt{z + 1} \, dz = \frac{2}{3} (5\sqrt{5} - 1)$$

$$54. y = 4 - z \Rightarrow f_x(x, z) = 0, f_z(x, z) = -1 \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{2} \Rightarrow \text{Area} = \iint_{R_{xz}} \sqrt{2} \, dA = \int_0^2 \int_0^{4-z^2} \sqrt{2} \, dx \, dz \\ = \sqrt{2} \int_0^2 (4 - z^2) \, dz = \frac{16\sqrt{2}}{3}$$

$$55. \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k} \Rightarrow \mathbf{r}_x(x, y) = \mathbf{i} + f_x(x, y)\mathbf{k}, \mathbf{r}_y(x, y) = \mathbf{j} + f_y(x, y)\mathbf{k}$$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x, y) \\ 0 & 1 & f_y(x, y) \end{vmatrix} = -f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}$$

$$\Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(-f_x(x, y))^2 + (-f_y(x, y))^2 + 1^2} = \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1}$$

$$\Rightarrow d\sigma = \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1} \, dA$$

$$56. S \text{ is obtained by rotating } y = f(x), a \leq x \leq b \text{ about the } x\text{-axis where } f(x) \geq 0$$

(a) Let  $(x, y, z)$  be a point on  $S$ . Consider the cross section when  $x = x^*$ , the cross section is a circle with radius  $r = f(x^*)$ .

The set of parametric equations for this circle are given by  $y(\theta) = r \cos \theta = f(x^*) \cos \theta$  and  $z(\theta) = r \sin \theta$

$= f(x^*) \sin \theta$  where  $0 \leq \theta \leq 2\pi$ . Since  $x$  can take on any value between  $a$  and  $b$  we have  $x(x, \theta) = x$ ,  $y(x, \theta)$

$= f(x) \cos \theta$ ,  $z(x, \theta) = f(x) \sin \theta$  where  $a \leq x \leq b$  and  $0 \leq \theta \leq 2\pi$ . Thus  $\mathbf{r}(x, \theta) = x\mathbf{i} + f(x) \cos \theta \mathbf{j} + f(x) \sin \theta \mathbf{k}$

(b)  $\mathbf{r}_x(x, \theta) = \mathbf{i} + f'(x) \cos \theta \mathbf{j} + f'(x) \sin \theta \mathbf{k}$  and  $\mathbf{r}_\theta(x, \theta) = -f(x) \sin \theta \mathbf{j} + f(x) \cos \theta \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) \cos \theta & f'(x) \sin \theta \\ 0 & -f(x) \sin \theta & f(x) \cos \theta \end{vmatrix} = f(x) \cdot f'(x) \mathbf{i} - f(x) \cos \theta \mathbf{j} - f(x) \sin \theta \mathbf{k}$$

$$\Rightarrow |\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{(f(x) \cdot f'(x))^2 + (-f(x) \cos \theta)^2 + (-f(x) \sin \theta)^2} = f(x) \sqrt{1 + (f'(x))^2}$$

$$A = \int_a^b \int_0^{2\pi} f(x) \sqrt{1 + (f'(x))^2} \, d\theta \, dx = \int_a^b \left[ \left( f(x) \sqrt{1 + (f'(x))^2} \right) \theta \right]_0^{2\pi} \, dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx$$



## 16.6 SURFACE INTEGRALS

$$\begin{aligned}
 1. \text{ Let the parametrization be } \mathbf{r}(x, z) &= x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j} \text{ and } \mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= 2x\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2 + 1} \Rightarrow \iint_S G(x, y, z) \, d\sigma = \int_0^3 \int_0^2 x\sqrt{4x^2 + 1} \, dx \, dz = \int_0^3 \left[ \frac{1}{12} (4x^2 + 1)^{3/2} \right]_0^2 dz \\
 &= \int_0^3 \frac{1}{12} (17\sqrt{17} - 1) \, dz = \frac{17\sqrt{17} - 1}{4}
 \end{aligned}$$

$$\begin{aligned}
 2. \text{ Let the parametrization be } \mathbf{r}(x, y) &= x\mathbf{i} + y\mathbf{j} + \sqrt{4 - y^2}\mathbf{k}, -2 \leq y \leq 2 \Rightarrow \mathbf{r}_x = \mathbf{i} \text{ and } \mathbf{r}_y = \mathbf{j} - \frac{y}{\sqrt{4 - y^2}}\mathbf{k} \\
 \Rightarrow \mathbf{r}_x \times \mathbf{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{\sqrt{4 - y^2}} \end{vmatrix} = \frac{y}{\sqrt{4 - y^2}}\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\frac{y^2}{4 - y^2} + 1} = \frac{2}{\sqrt{4 - y^2}} \\
 \Rightarrow \iint_S G(x, y, z) \, d\sigma &= \int_1^4 \int_{-2}^2 \sqrt{4 - y^2} \left( \frac{2}{\sqrt{4 - y^2}} \right) dy \, dx = 24
 \end{aligned}$$

$$3. \text{ Let the parametrization be } \mathbf{r}(\phi, \theta) = (\sin \phi \cos \theta)\mathbf{i} + (\sin \phi \sin \theta)\mathbf{j} + (\cos \phi)\mathbf{k} \text{ (spherical coordinates with } \rho = 1 \text{ on the sphere), } 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_\phi = (\cos \phi \cos \theta)\mathbf{i} + (\cos \phi \sin \theta)\mathbf{j} - (\sin \phi)\mathbf{k} \text{ and}$$

$$\begin{aligned}
 \mathbf{r}_\theta &= (-\sin \phi \sin \theta)\mathbf{i} + (\sin \phi \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{vmatrix} \\
 &= (\sin^2 \phi \cos \theta)\mathbf{i} + (\sin^2 \phi \sin \theta)\mathbf{j} + (\sin \phi \cos \phi)\mathbf{k} \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{\sin^4 \phi \cos^2 \theta + \sin^4 \phi \sin^2 \theta + \sin^2 \phi \cos^2 \phi} \\
 &= \sin \phi; x = \sin \phi \cos \theta \Rightarrow G(x, y, z) = \cos^2 \theta \sin^2 \phi \Rightarrow \iint_S G(x, y, z) \, d\sigma = \int_0^{2\pi} \int_0^\pi (\cos^2 \theta \sin^2 \phi) (\sin \phi) \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^\pi (\cos^2 \theta) (1 - \cos^2 \phi) (\sin \phi) \, d\phi \, d\theta; \left[ \begin{matrix} u = \cos \phi \\ du = -\sin \phi \, d\phi \end{matrix} \right] \rightarrow \int_0^{2\pi} \int_1^{-1} (\cos^2 \theta) (u^2 - 1) \, du \, d\theta \\
 &= \int_0^{2\pi} (\cos^2 \theta) \left[ \frac{u^3}{3} - u \right]_1^{-1} d\theta = \frac{4}{3} \int_0^{2\pi} \cos^2 \theta \, d\theta = \frac{4}{3} \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi} = \frac{4\pi}{3}
 \end{aligned}$$

$$4. \text{ Let the parametrization be } \mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k} \text{ (spherical coordinates with } \rho = a, a \geq 0, \text{ on the sphere), } 0 \leq \phi \leq \frac{\pi}{2} \text{ (since } z \geq 0), 0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_\phi = (a \cos \phi \cos \theta)\mathbf{i} + (a \cos \phi \sin \theta)\mathbf{j} - (a \sin \phi)\mathbf{k} \text{ and}$$

$$\begin{aligned}
 \mathbf{r}_\theta &= (-a \sin \phi \sin \theta)\mathbf{i} + (a \sin \phi \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\
 &= (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k} \\
 \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi} = a^2 \sin \phi; z = a \cos \phi \\
 \Rightarrow G(x, y, z) &= a^2 \cos^2 \phi \Rightarrow \iint_S G(x, y, z) \, d\sigma = \int_0^{2\pi} \int_0^{\pi/2} (a^2 \cos^2 \phi) (a^2 \sin \phi) \, d\phi \, d\theta = \frac{2}{3} \pi a^4
 \end{aligned}$$

$$5. \text{ Let the parametrization be } \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - x - y)\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k} \text{ and } \mathbf{r}_y = \mathbf{j} - \mathbf{k}$$

$$\begin{aligned}
 \Rightarrow \mathbf{r}_x \times \mathbf{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3} \Rightarrow \iint_S F(x, y, z) \, d\sigma = \int_0^1 \int_0^1 (4 - x - y) \sqrt{3} \, dy \, dx \\
 &= \int_0^1 \sqrt{3} \left[ 4y - xy - \frac{y^2}{2} \right]_0^1 dx = \int_0^1 \sqrt{3} \left( \frac{7}{2} - x \right) dx = \sqrt{3} \left[ \frac{7}{2}x - \frac{x^2}{2} \right]_0^1 = 3\sqrt{3}
 \end{aligned}$$

6. Let the parametrization be  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$ ,  $0 \leq r \leq 1$  (since  $0 \leq z \leq 1$ ) and  $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\ &= (-r \cos \theta)\mathbf{i} - (r \sin \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r \cos \theta)^2 + (-r \sin \theta)^2 + r^2} = r\sqrt{2}; z = r \text{ and } x = r \cos \theta \\ \Rightarrow F(x, y, z) &= r - r \cos \theta \Rightarrow \iint_S F(x, y, z) \, d\sigma = \int_0^{2\pi} \int_0^1 (r - r \cos \theta) (r\sqrt{2}) \, dr \, d\theta = \sqrt{2} \int_0^{2\pi} \int_0^1 (1 - \cos \theta) r^2 \, dr \, d\theta \\ &= \frac{2\pi\sqrt{2}}{3} \end{aligned}$$

7. Let the parametrization be  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1 - r^2)\mathbf{k}$ ,  $0 \leq r \leq 1$  (since  $0 \leq z \leq 1$ ) and  $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - 2r\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\ &= (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(2r^2 \cos \theta)^2 + (2r^2 \sin \theta)^2 + r^2} = r\sqrt{1 + 4r^2}; z = 1 - r^2 \text{ and } \\ x &= r \cos \theta \Rightarrow H(x, y, z) = (r^2 \cos^2 \theta) \sqrt{1 + 4r^2} \Rightarrow \iint_S H(x, y, z) \, d\sigma \\ &= \int_0^{2\pi} \int_0^1 (r^2 \cos^2 \theta) (\sqrt{1 + 4r^2}) (r\sqrt{1 + 4r^2}) \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r^3 (1 + 4r^2) \cos^2 \theta \, dr \, d\theta = \frac{11\pi}{12} \end{aligned}$$

8. Let the parametrization be  $\mathbf{r}(\phi, \theta) = (2 \sin \phi \cos \theta)\mathbf{i} + (2 \sin \phi \sin \theta)\mathbf{j} + (2 \cos \phi)\mathbf{k}$  (spherical coordinates with  $\rho = 2$  on the sphere),  $0 \leq \phi \leq \frac{\pi}{4}$ ;  $x^2 + y^2 + z^2 = 4$  and  $z = \sqrt{x^2 + y^2} \Rightarrow z^2 + z^2 = 4 \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$  (since  $z \geq 0$ )  $\Rightarrow 2 \cos \phi = \sqrt{2} \Rightarrow \cos \phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$ ,  $0 \leq \theta \leq 2\pi$ ;  $\mathbf{r}_\phi = (2 \cos \phi \cos \theta)\mathbf{i} + (2 \cos \phi \sin \theta)\mathbf{j} - (2 \sin \phi)\mathbf{k}$

$$\begin{aligned} \text{and } \mathbf{r}_\theta &= (-2 \sin \phi \sin \theta)\mathbf{i} + (2 \sin \phi \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos \phi \cos \theta & 2 \cos \phi \sin \theta & -2 \sin \phi \\ -2 \sin \phi \sin \theta & 2 \sin \phi \cos \theta & 0 \end{vmatrix} \\ &= (4 \sin^2 \phi \cos \theta)\mathbf{i} + (4 \sin^2 \phi \sin \theta)\mathbf{j} + (4 \sin \phi \cos \phi)\mathbf{k} \\ \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{16 \sin^4 \phi \cos^2 \theta + 16 \sin^4 \phi \sin^2 \theta + 16 \sin^2 \phi \cos^2 \phi} = 4 \sin \phi; y = 2 \sin \phi \sin \theta \text{ and } \\ z &= 2 \cos \phi \Rightarrow H(x, y, z) = 4 \cos \phi \sin \phi \sin \theta \Rightarrow \iint_S H(x, y, z) \, d\sigma = \int_0^{2\pi} \int_0^{\pi/4} (4 \cos \phi \sin \phi \sin \theta) (4 \sin \phi) \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} 16 \sin^2 \phi \cos \phi \sin \theta \, d\phi \, d\theta = 0 \end{aligned}$$

9. The bottom face  $S$  of the cube is in the  $xy$ -plane  $\Rightarrow z = 0 \Rightarrow G(x, y, 0) = x + y$  and  $f(x, y, z) = z = 0 \Rightarrow \mathbf{p} = \mathbf{k}$

$$\text{and } \nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx \, dy \Rightarrow \iint_S G \, d\sigma = \iint_R (x + y) \, dx \, dy$$

$$= \int_0^a \int_0^a (x + y) \, dx \, dy = \int_0^a \left( \frac{x^2}{2} + ay \right) dy = a^3. \text{ Because of symmetry, we also get } a^3 \text{ over the face of the cube}$$

in the  $xz$ -plane and  $a^3$  over the face of the cube in the  $yz$ -plane. Next, on the top of the cube,  $G(x, y, z)$

$$= G(x, y, a) = x + y + a \text{ and } f(x, y, z) = z = a \Rightarrow \mathbf{p} = \mathbf{k} \text{ and } \nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx \, dy$$

$$\iint_S G \, d\sigma = \iint_R (x + y + a) \, dx \, dy = \int_0^a \int_0^a (x + y + a) \, dx \, dy = \int_0^a \int_0^a (x + y) \, dx \, dy + \int_0^a \int_0^a a \, dx \, dy = 2a^3.$$

Because of symmetry, the integral is also  $2a^3$  over each of the other two faces. Therefore,

$$\iint_{\text{cube}} (x + y + z) \, d\sigma = 3(a^3 + 2a^3) = 9a^3.$$

10. On the face  $S$  in the  $xz$ -plane, we have  $y = 0 \Rightarrow f(x, y, z) = y = 0$  and  $G(x, y, z) = G(x, 0, z) = z \Rightarrow \mathbf{p} = \mathbf{j}$  and  $\nabla f = \mathbf{j}$

$$\Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx \, dz \Rightarrow \iint_S G \, d\sigma = \iint_S (y + z) \, d\sigma = \int_0^1 \int_0^2 z \, dx \, dz = \int_0^1 2z \, dz = 1.$$

On the face in the  $xy$ -plane, we have  $z = 0 \Rightarrow f(x, y, z) = z = 0$  and  $G(x, y, z) = G(x, y, 0) = y \Rightarrow \mathbf{p} = \mathbf{k}$  and  $\nabla f = \mathbf{k}$

$$\Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dy \Rightarrow \iint_S G d\sigma = \iint_S y d\sigma = \int_0^1 \int_0^2 y dx dy = 1.$$

On the triangular face in the plane  $x = 2$  we have  $f(x, y, z) = x = 2$  and  $G(x, y, z) = G(2, y, z) = y + z \Rightarrow \mathbf{p} = \mathbf{i}$  and  $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$  and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz dy \Rightarrow \iint_S G d\sigma = \iint_S (y + z) d\sigma = \int_0^1 \int_0^{1-y} (y + z) dz dy$   
 $= \int_0^1 \frac{1}{2} (1 - y^2) dy = \frac{1}{3}.$

On the triangular face in the  $yz$ -plane, we have  $x = 0 \Rightarrow f(x, y, z) = x = 0$  and  $G(x, y, z) = G(0, y, z) = y + z$   
 $\Rightarrow \mathbf{p} = \mathbf{i}$  and  $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$  and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz dy \Rightarrow \iint_S G d\sigma = \iint_S (y + z) d\sigma$   
 $= \int_0^1 \int_0^{1-y} (y + z) dz dy = \frac{1}{3}.$

Finally, on the sloped face, we have  $y + z = 1 \Rightarrow f(x, y, z) = y + z = 1$  and  $G(x, y, z) = y + z = 1 \Rightarrow \mathbf{p} = \mathbf{k}$  and  $\nabla f = \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{2}$  and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{2} dx dy \Rightarrow \iint_S G d\sigma = \iint_S (y + z) d\sigma$   
 $= \int_0^1 \int_0^2 \sqrt{2} dx dy = 2\sqrt{2}.$  Therefore,  $\iint_{\text{wedge}} G(x, y, z) d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}$

11. On the faces in the coordinate planes,  $G(x, y, z) = 0 \Rightarrow$  the integral over these faces is 0.

On the face  $x = a$ , we have  $f(x, y, z) = x = a$  and  $G(x, y, z) = G(a, y, z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$  and  $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$   
and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy dz \Rightarrow \iint_S G d\sigma = \iint_S ayz d\sigma = \int_0^c \int_0^b ayz dy dz = \frac{ab^2c^2}{4}.$

On the face  $y = b$ , we have  $f(x, y, z) = y = b$  and  $G(x, y, z) = G(x, b, z) = bxz \Rightarrow \mathbf{p} = \mathbf{j}$  and  $\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1$   
and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dz \Rightarrow \iint_S G d\sigma = \iint_S bxz d\sigma = \int_0^c \int_0^a bxz dx dz = \frac{a^2bc^2}{4}.$

On the face  $z = c$ , we have  $f(x, y, z) = z = c$  and  $G(x, y, z) = G(x, y, c) = cxy \Rightarrow \mathbf{p} = \mathbf{k}$  and  $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$   
and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy dx \Rightarrow \iint_S G d\sigma = \iint_S cxy d\sigma = \int_0^b \int_0^a cxy dx dy = \frac{a^2b^2c}{4}.$  Therefore,  
 $\iint_S G(x, y, z) d\sigma = \frac{abc(ab + ac + bc)}{4}.$

12. On the face  $x = a$ , we have  $f(x, y, z) = x = a$  and  $G(x, y, z) = G(a, y, z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$  and  $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$   
and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz dy \Rightarrow \iint_S G d\sigma = \iint_S ayz d\sigma = \int_{-b}^b \int_{-c}^c ayz dz dy = 0.$  Because of the symmetry  
of  $G$  on all the other faces, all the integrals are 0, and  $\iint_S G(x, y, z) d\sigma = 0.$

13.  $f(x, y, z) = 2x + 2y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$  and  $G(x, y, z) = x + y + (2 - 2x - 2y) = 2 - x - y \Rightarrow \mathbf{p} = \mathbf{k},$   
 $|\nabla f| = 3$  and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = 3 dy dx; z = 0 \Rightarrow 2x + 2y = 2 \Rightarrow y = 1 - x \Rightarrow \iint_S G d\sigma = \iint_S (2 - x - y) d\sigma$   
 $= 3 \int_0^1 \int_0^{1-x} (2 - x - y) dy dx = 3 \int_0^1 [(2 - x)(1 - x) - \frac{1}{2}(1 - x)^2] dx = 3 \int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2}\right) dx = 2$

14.  $f(x, y, z) = y^2 + 4z = 16 \Rightarrow \nabla f = 2y\mathbf{j} + 4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 16} = 2\sqrt{y^2 + 4}$  and  $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$   
 $\Rightarrow d\sigma = \frac{2\sqrt{y^2 + 4}}{4} dx dy \Rightarrow \iint_S G d\sigma = \int_{-4}^4 \int_0^1 (x\sqrt{y^2 + 4}) \left(\frac{\sqrt{y^2 + 4}}{2}\right) dx dy = \int_{-4}^4 \int_0^1 \frac{x(y^2 + 4)}{2} dx dy$   
 $= \int_{-4}^4 \frac{1}{4} (y^2 + 4) dy = \frac{1}{2} \left[\frac{y^3}{3} + 4y\right]_0^4 = \frac{1}{2} \left(\frac{64}{3} + 16\right) = \frac{56}{3}$

$$\begin{aligned}
15. \quad f(x, y, z) = x + y^2 - z = 0 &\Rightarrow \nabla f = \mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 2} = \sqrt{2}\sqrt{2y^2 + 1} \text{ and } \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \\
&\Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2y^2+1}}{1} dx dy \Rightarrow \iint_S G d\sigma = \int_0^1 \int_0^y (x + y^2 - x) \sqrt{2}\sqrt{2y^2+1} dx dy = \sqrt{2} \int_0^1 \int_0^y y^2 \sqrt{2y^2+1} dx dy \\
&= \sqrt{2} \int_0^1 y^3 \sqrt{2y^2+1} dy = \frac{6\sqrt{6}+\sqrt{2}}{30}
\end{aligned}$$

$$\begin{aligned}
16. \quad f(x, y, z) = x^2 + y - z = 0 &\Rightarrow \nabla f = 2x\mathbf{i} + \mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 2} = \sqrt{2}\sqrt{2x^2 + 1} \text{ and } \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \\
&\Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2x^2+1}}{1} dx dy \Rightarrow \iint_S G d\sigma = \int_{-1}^1 \int_0^1 x \sqrt{2}\sqrt{2x^2+1} dx dy = \sqrt{2} \int_{-1}^1 \int_0^1 x \sqrt{2x^2+1} dx dy \\
&= \frac{3\sqrt{6}-\sqrt{2}}{6} \int_0^1 dy = \frac{3\sqrt{6}-\sqrt{2}}{3}
\end{aligned}$$

$$\begin{aligned}
17. \quad f(x, y, z) = 2x + y + z = 2 &\Rightarrow \nabla f = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{6} \text{ and } \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{6}}{1} dy dx \\
&\Rightarrow \iint_S G d\sigma = \int_0^1 \int_{1-x}^{2-2x} x y (2 - 2x - y) \sqrt{6} dy dx = \sqrt{6} \int_0^1 \int_{1-x}^{2-2x} (2x y - 2x^2 y - x y^2) dy dx \\
&= \sqrt{6} \int_0^1 \left( \frac{2}{3} x - 2x^2 + 2x^3 - \frac{2}{3} x^4 \right) dx = \frac{\sqrt{6}}{30}
\end{aligned}$$

$$\begin{aligned}
18. \quad f(x, y, z) = x + y = 1 &\Rightarrow \nabla f = \mathbf{i} + \mathbf{j} \Rightarrow |\nabla f| = \sqrt{2} \text{ and } \mathbf{p} = \mathbf{j} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \\
&\Rightarrow d\sigma = \frac{\sqrt{2}}{1} dz dx \Rightarrow \iint_S G d\sigma = \int_0^1 \int_0^1 (x - (1-x) - z) \sqrt{2} dz dx = \sqrt{2} \int_0^1 \int_0^1 (2x - z - 1) dz dx \\
&= \sqrt{2} \int_0^1 \left( 2x - \frac{3}{2} \right) dx = -\frac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
19. \quad \text{Let the parametrization be } \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}, 0 \leq x \leq 1, -2 \leq y \leq 2; z = 0 &\Rightarrow 0 = 4 - y^2 \\
&\Rightarrow y = \pm 2; \mathbf{r}_x = \mathbf{i} \text{ and } \mathbf{r}_y = \mathbf{j} - 2y\mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} d\sigma \\
&= \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y| dy dx = (2xy - 3z) dy dx = [2xy - 3(4 - y^2)] dy dx \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma \\
&= \int_0^1 \int_{-2}^2 (2xy + 3y^2 - 12) dy dx = \int_0^1 [xy^2 + y^3 - 12y]_{-2}^2 dx = \int_0^1 -32 dx = -32
\end{aligned}$$

$$\begin{aligned}
20. \quad \text{Let the parametrization be } \mathbf{r}(x, y) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}, -1 \leq x \leq 1, 0 \leq z \leq 2 &\Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j} \text{ and } \mathbf{r}_z = \mathbf{k} \\
&\Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} |\mathbf{r}_x \times \mathbf{r}_z| dz dx = -x^2 dz dx \\
&\Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \int_{-1}^1 \int_0^2 -x^2 dz dx = -\frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
21. \quad \text{Let the parametrization be } \mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k} &\text{ (spherical coordinates with } \rho = a, a \geq 0, \text{ on the sphere), } 0 \leq \phi \leq \frac{\pi}{2} \text{ (for the first octant), } 0 \leq \theta \leq \frac{\pi}{2} \text{ (for the first octant)} \\
&\Rightarrow \mathbf{r}_\phi = (a \cos \phi \cos \theta)\mathbf{i} + (a \cos \phi \sin \theta)\mathbf{j} - (a \sin \phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-a \sin \phi \sin \theta)\mathbf{i} + (a \sin \phi \cos \theta)\mathbf{j} \\
&\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\
&= (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} |\mathbf{r}_\phi \times \mathbf{r}_\theta| d\theta d\phi \\
&= a^3 \cos^2 \phi \sin \phi d\theta d\phi \text{ since } \mathbf{F} = z\mathbf{k} = (a \cos \phi)\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} a^3 \cos^2 \phi \sin \phi d\phi d\theta = \frac{\pi a^3}{6}
\end{aligned}$$

22. Let the parametrization be  $\mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k}$  (spherical coordinates with  $\rho = a$ ,  $a \geq 0$ , on the sphere),  $0 \leq \phi \leq \pi$ ,  $0 \leq \theta \leq 2\pi$

$$\Rightarrow \mathbf{r}_\phi = (a \cos \phi \cos \theta)\mathbf{i} + (a \cos \phi \sin \theta)\mathbf{j} - (a \sin \phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-a \sin \phi \sin \theta)\mathbf{i} + (a \sin \phi \cos \theta)\mathbf{j}$$

$$\begin{aligned} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\ &= (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\phi \, d\theta \\ &= (a^3 \sin^3 \phi \cos^2 \theta + a^3 \sin^3 \phi \sin^2 \theta + a^3 \sin \phi \cos^2 \phi) \, d\theta \, d\phi = a^3 \sin \phi \, d\theta \, d\phi \text{ since } \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &= (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k} \Rightarrow \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^\pi a^3 \sin \phi \, d\phi \, d\theta = 4\pi a^3 \end{aligned}$$

23. Let the parametrization be  $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (2a - x - y)\mathbf{k}$ ,  $0 \leq x \leq a$ ,  $0 \leq y \leq a \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$  and  $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$$\begin{aligned} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y| \, dy \, dx \\ &= [2xy + 2y(2a - x - y) + 2x(2a - x - y)] \, dy \, dx \text{ since } \mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k} \\ &= 2xy\mathbf{i} + 2y(2a - x - y)\mathbf{j} + 2x(2a - x - y)\mathbf{k} \Rightarrow \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma \\ &= \int_0^a \int_0^a [2xy + 2y(2a - x - y) + 2x(2a - x - y)] \, dy \, dx = \int_0^a \int_0^a (4ay - 2y^2 + 4ax - 2x^2 - 2xy) \, dy \, dx \\ &= \int_0^a \left( \frac{4}{3}a^3 + 3a^2x - 2ax^2 \right) \, dx = \left( \frac{4}{3} + \frac{3}{2} - \frac{2}{3} \right) a^4 = \frac{13a^4}{6} \end{aligned}$$

24. Let the parametrization be  $\mathbf{r}(\theta, z) = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + z\mathbf{k}$ ,  $0 \leq z \leq a$ ,  $0 \leq \theta \leq 2\pi$  (where  $r = \sqrt{x^2 + y^2} = 1$  on the cylinder)  $\Rightarrow \mathbf{r}_\theta = (-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}$  and  $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$

$$\begin{aligned} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_z}{|\mathbf{r}_\theta \times \mathbf{r}_z|} |\mathbf{r}_\theta \times \mathbf{r}_z| \, dz \, d\theta = (\cos^2 \theta + \sin^2 \theta) \, dz \, d\theta = dz \, d\theta, \text{ since } \mathbf{F} = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + z\mathbf{k} \\ \Rightarrow \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int_0^{2\pi} \int_0^a 1 \, dz \, d\theta = 2\pi a \end{aligned}$$

25. Let the parametrization be  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$ ,  $0 \leq r \leq 1$  (since  $0 \leq z \leq 1$ ) and  $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 1 \end{vmatrix} \\ &= (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r| \, d\theta \, dr = (r^3 \sin \theta \cos^2 \theta + r^2) \, d\theta \, dr \text{ since } \\ \mathbf{F} &= (r^2 \sin \theta \cos \theta)\mathbf{i} - r\mathbf{k} \Rightarrow \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^1 (r^3 \sin \theta \cos^2 \theta + r^2) \, dr \, d\theta = \int_0^{2\pi} \left( \frac{1}{4} \sin \theta \cos^2 \theta + \frac{1}{3} \right) \, d\theta \\ &= \left[ -\frac{1}{12} \cos^3 \theta + \frac{\theta}{3} \right]_0^{2\pi} = \frac{2\pi}{3} \end{aligned}$$

26. Let the parametrization be  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + 2r\mathbf{k}$ ,  $0 \leq r \leq 1$  (since  $0 \leq z \leq 2$ ) and  $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 2 \end{vmatrix} \\ &= (2r \cos \theta)\mathbf{i} + (2r \sin \theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r| \, d\theta \, dr \\ &= (2r^3 \sin^2 \theta \cos \theta + 4r^3 \cos \theta \sin \theta + r) \, d\theta \, dr \text{ since } \\ \mathbf{F} &= (r^2 \sin^2 \theta)\mathbf{i} + (2r^2 \cos \theta)\mathbf{j} - \mathbf{k} \Rightarrow \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^1 (2r^3 \sin^2 \theta \cos \theta + 4r^3 \cos \theta \sin \theta + r) \, dr \, d\theta \\ &= \int_0^{2\pi} \left( \frac{1}{2} \sin^2 \theta \cos \theta + \cos \theta \sin \theta + \frac{1}{2} \right) \, d\theta = \left[ \frac{1}{6} \sin^3 \theta + \frac{1}{2} \sin^2 \theta + \frac{1}{2} \theta \right]_0^{2\pi} = \pi \end{aligned}$$

27. Let the parametrization be  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$ ,  $1 \leq r \leq 2$  (since  $1 \leq z \leq 2$ ) and  $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 1 \end{vmatrix} \\ &= (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r| \, d\theta \, dr = (-r^2 \cos^2 \theta - r^2 \sin^2 \theta - r^3) \, d\theta \, dr \\ &= (-r^2 - r^3) \, d\theta \, dr \text{ since } \mathbf{F} = (-r \cos \theta)\mathbf{i} - (r \sin \theta)\mathbf{j} + r^2\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_1^2 (-r^2 - r^3) \, dr \, d\theta = -\frac{73\pi}{6} \end{aligned}$$

28. Let the parametrization be  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r^2\mathbf{k}$ ,  $0 \leq r \leq 1$  (since  $0 \leq z \leq 1$ ) and  $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + 2r\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 2r \end{vmatrix} \\ &= (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r| \, d\theta \, dr = (8r^3 \cos^2 \theta + 8r^3 \sin^2 \theta - 2r) \, d\theta \, dr \\ &= (8r^3 - 2r) \, d\theta \, dr \text{ since } \mathbf{F} = (4r \cos \theta)\mathbf{i} + (4r \sin \theta)\mathbf{j} + 2\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^1 (8r^3 - 2r) \, dr \, d\theta = 2\pi \end{aligned}$$

29.  $g(x, y, z) = z$ ,  $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$  and  $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (\mathbf{F} \cdot \mathbf{k}) \, dA$   
 $= \int_0^2 \int_0^3 3 \, dy \, dx = 18$

30.  $g(x, y, z) = y$ ,  $\mathbf{p} = -\mathbf{j} \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$  and  $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (\mathbf{F} \cdot -\mathbf{j}) \, dA$   
 $= \int_{-1}^2 \int_2^7 2 \, dz \, dx = \int_{-1}^2 2(7-2) \, dx = 10(2+1) = 30$

31.  $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$ ;  $\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{z^2}{a}$ ;  
 $|\nabla g \cdot \mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z} \, dA \Rightarrow \text{Flux} = \iint_R \left(\frac{z^2}{a}\right) \left(\frac{a}{z}\right) \, dA = \iint_R z \, dA = \iint_R \sqrt{a^2 - (x^2 + y^2)} \, dx \, dy$   
 $= \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2} \, r \, dr \, d\theta = \frac{\pi a^3}{6}$

32.  $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$ ;  $\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{-xy}{a} + \frac{xy}{a}$   
 $= 0$ ;  $|\nabla g \cdot \mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z} \, dA \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S 0 \, d\sigma = 0$

33. From Exercise 31,  $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$  and  $d\sigma = \frac{a}{z} \, dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{xy}{a} - \frac{xy}{a} + \frac{z}{a} = \frac{z}{a} \Rightarrow \text{Flux} = \iint_R \left(\frac{z}{a}\right) \left(\frac{a}{z}\right) \, dA$   
 $= \iint_R 1 \, dA = \frac{\pi a^2}{4}$

34. From Exercise 31,  $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$  and  $d\sigma = \frac{a}{z} \, dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{zx^2}{a} + \frac{zy^2}{a} + \frac{z^3}{a} = z \left(\frac{x^2 + y^2 + z^2}{a}\right) = az$   
 $\Rightarrow \text{Flux} = \iint_R (za) \left(\frac{a}{z}\right) \, dx \, dy = \iint_R a^2 \, dx \, dy = a^2(\text{Area of } R) = \frac{1}{4} \pi a^4$

35. From Exercise 31,  $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$  and  $d\sigma = \frac{a}{z} \, dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a} = a \Rightarrow \text{Flux}$   
 $= \iint_R a \left(\frac{a}{z}\right) \, dA = \iint_R \frac{a^2}{z} \, dA = \iint_R \frac{a^2}{\sqrt{a^2 - (x^2 + y^2)}} \, dA = \int_0^{\pi/2} \int_0^a \frac{a^2}{\sqrt{a^2 - r^2}} \, r \, dr \, d\theta$   
 $= \int_0^{\pi/2} a^2 \left[-\sqrt{a^2 - r^2}\right]_0^a \, d\theta = \frac{\pi a^3}{2}$

$$36. \text{ From Exercise 31, } \mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \text{ and } d\sigma = \frac{a}{z} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{\left(\frac{x^2}{a}\right) + \left(\frac{y^2}{a}\right) + \left(\frac{z^2}{a}\right)}{\sqrt{x^2 + y^2 + z^2}} = \frac{\left(\frac{a^2}{a}\right)}{a} = 1$$

$$\Rightarrow \text{Flux} = \iint_R \frac{a}{z} dx dy = \iint_R \frac{a}{\sqrt{a^2 - (x^2 + y^2)}} dx dy = \int_0^{\pi/2} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} r dr d\theta = \frac{\pi a^2}{2}$$

$$37. g(x, y, z) = y^2 + z = 4 \Rightarrow \nabla g = 2y\mathbf{j} + \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4y^2 + 1} \Rightarrow \mathbf{n} = \frac{2y\mathbf{j} + \mathbf{k}}{\sqrt{4y^2 + 1}}$$

$$\Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{2xy - 3z}{\sqrt{4y^2 + 1}}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2 + 1} dA \Rightarrow \text{Flux}$$

$$= \iint_R \left( \frac{2xy - 3z}{\sqrt{4y^2 + 1}} \right) \sqrt{4y^2 + 1} dA = \iint_R (2xy - 3z) dA; z = 0 \text{ and } z = 4 - y^2 \Rightarrow y^2 = 4$$

$$\Rightarrow \text{Flux} = \iint_R [2xy - 3(4 - y^2)] dA = \int_0^1 \int_{-2}^2 (2xy - 12 + 3y^2) dy dx = \int_0^1 [xy^2 - 12y + y^3]_{-2}^2 dx$$

$$= \int_0^1 -32 dx = -32$$

$$38. g(x, y, z) = x^2 + y^2 - z = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4(x^2 + y^2) + 1}$$

$$\Rightarrow \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}}{\sqrt{4(x^2 + y^2) + 1}} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{8x^2 + 8y^2 - 2}{\sqrt{4(x^2 + y^2) + 1}}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4(x^2 + y^2) + 1} dA$$

$$\Rightarrow \text{Flux} = \iint_R \left( \frac{8x^2 + 8y^2 - 2}{\sqrt{4(x^2 + y^2) + 1}} \right) \sqrt{4(x^2 + y^2) + 1} dA = \iint_R (8x^2 + 8y^2 - 2) dA; z = 1 \text{ and } x^2 + y^2 = z$$

$$\Rightarrow x^2 + y^2 = 1 \Rightarrow \text{Flux} = \int_0^{2\pi} \int_0^1 (8r^2 - 2) r dr d\theta = 2\pi$$

$$39. g(x, y, z) = y - e^x = 0 \Rightarrow \nabla g = -e^x\mathbf{i} + \mathbf{j} \Rightarrow |\nabla g| = \sqrt{e^{2x} + 1} \Rightarrow \mathbf{n} = \frac{e^x\mathbf{i} - \mathbf{j}}{\sqrt{e^{2x} + 1}} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{-2e^x - 2y}{\sqrt{e^{2x} + 1}}; \mathbf{p} = \mathbf{i}$$

$$\Rightarrow |\nabla g \cdot \mathbf{p}| = e^x \Rightarrow d\sigma = \frac{\sqrt{e^{2x} + 1}}{e^x} dA \Rightarrow \text{Flux} = \iint_R \left( \frac{-2e^x - 2y}{\sqrt{e^{2x} + 1}} \right) \left( \frac{\sqrt{e^{2x} + 1}}{e^x} \right) dA = \iint_R \frac{-2e^x - 2e^x}{e^x} dA$$

$$= \iint_R -4 dA = \int_0^1 \int_1^2 -4 dy dz = -4$$

$$40. g(x, y, z) = y - \ln x = 0 \Rightarrow \nabla g = -\frac{1}{x}\mathbf{i} + \mathbf{j} \Rightarrow |\nabla g| = \sqrt{\frac{1}{x^2} + 1} = \frac{\sqrt{1 + x^2}}{x} \text{ since } 1 \leq x \leq e$$

$$\Rightarrow \mathbf{n} = \frac{\left(-\frac{1}{x}\mathbf{i} + \mathbf{j}\right)}{\left(\frac{\sqrt{1+x^2}}{x}\right)} = \frac{-\mathbf{i} + x\mathbf{j}}{\sqrt{1+x^2}} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{2xy}{\sqrt{1+x^2}}; \mathbf{p} = \mathbf{j} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{1+x^2}}{x} dA$$

$$\Rightarrow \text{Flux} = \iint_R \left( \frac{2xy}{\sqrt{1+x^2}} \right) \left( \frac{\sqrt{1+x^2}}{x} \right) dA = \int_0^1 \int_1^e 2y dx dz = \int_1^e \int_0^1 2 \ln x dz dx = \int_1^e 2 \ln x dx$$

$$= 2[x \ln x - x]_1^e = 2(e - e) - 2(0 - 1) = 2$$

$$41. \text{ On the face } z = a: g(x, y, z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2xz = 2ax \text{ since } z = a;$$

$$d\sigma = dx dy \Rightarrow \text{Flux} = \iint_R 2ax dx dy = \int_0^a \int_0^a 2ax dx dy = a^4.$$

$$\text{On the face } z = 0: g(x, y, z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2xz = 0 \text{ since } z = 0;$$

$$d\sigma = dx dy \Rightarrow \text{Flux} = \iint_R 0 dx dy = 0.$$

$$\text{On the face } x = a: g(x, y, z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2xy = 2ay \text{ since } x = a;$$

$$d\sigma = dy dz \Rightarrow \text{Flux} = \int_0^a \int_0^a 2ay dy dz = a^4.$$

$$\text{On the face } x = 0: g(x, y, z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{i} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2xy = 0 \text{ since } x = 0$$

$$\Rightarrow \text{Flux} = 0.$$

$$\text{On the face } y = a: g(x, y, z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2yz = 2az \text{ since } y = a;$$

$$d\sigma = dz dx \Rightarrow \text{Flux} = \int_0^a \int_0^a 2az dz dx = a^4.$$

$$\text{On the face } y = 0: g(x, y, z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2yz = 0 \text{ since } y = 0$$

$$\Rightarrow \text{Flux} = 0. \text{ Therefore, Total Flux} = 3a^4.$$

$$\begin{aligned}
42. \text{ Across the cap: } g(x, y, z) = x^2 + y^2 + z^2 = 25 &\Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10 \\
&\Rightarrow \mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{5} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{x^2z}{5} + \frac{y^2z}{5} + \frac{z}{5}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 2z \text{ since } z \geq 0 \Rightarrow d\sigma = \frac{10}{2z} dA \\
&\Rightarrow \text{Flux}_{\text{cap}} = \iint_{\text{cap}} \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{\text{R}} \left( \frac{x^2z}{5} + \frac{y^2z}{5} + \frac{z}{5} \right) \left( \frac{5}{z} \right) dA = \iint_{\text{R}} (x^2 + y^2 + 1) dx dy = \int_0^{2\pi} \int_0^4 (r^2 + 1) r dr d\theta \\
&= \int_0^{2\pi} 72 d\theta = 144\pi.
\end{aligned}$$

$$\begin{aligned}
\text{Across the bottom: } g(x, y, z) = z = 3 &\Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1 \Rightarrow \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -1; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \\
&\Rightarrow d\sigma = dA \Rightarrow \text{Flux}_{\text{bottom}} = \iint_{\text{bottom}} \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{\text{R}} -1 dA = -1(\text{Area of the circular region}) = -16\pi. \text{ Therefore,} \\
\text{Flux} = \text{Flux}_{\text{cap}} + \text{Flux}_{\text{bottom}} &= 128\pi
\end{aligned}$$

$$\begin{aligned}
43. \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} &\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \text{ since } z \geq 0 \Rightarrow d\sigma = \frac{2a}{2z} dA \\
&= \frac{a}{z} dA; M = \iint_S \delta d\sigma = \frac{\delta}{8} (\text{surface area of sphere}) = \frac{\delta\pi a^2}{2}; M_{xy} = \iint_S z\delta d\sigma = \delta \iint_R z \left( \frac{a}{z} \right) dA = a\delta \iint_R dA \\
&= a\delta \int_0^{\pi/2} \int_0^a r dr d\theta = \frac{\delta\pi a^3}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left( \frac{\delta\pi a^3}{4} \right) \left( \frac{2}{\delta\pi a^2} \right) = \frac{a}{2}. \text{ Because of symmetry, } \bar{x} = \bar{y} = \frac{a}{2} \Rightarrow \text{the centroid is} \\
&\left( \frac{a}{2}, \frac{a}{2}, \frac{a}{2} \right).
\end{aligned}$$

$$\begin{aligned}
44. \nabla f = 2y\mathbf{j} + 2z\mathbf{k} &\Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = \sqrt{4(y^2 + z^2)} = 6; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{k}| = 2z \text{ since } z \geq 0 \Rightarrow d\sigma = \frac{6}{2z} dA \\
&= \frac{3}{z} dA; M = \iint_S 1 d\sigma = \int_{-3}^3 \int_0^3 \frac{3}{z} dx dy = \int_{-3}^3 \int_0^3 \frac{3}{\sqrt{9-y^2}} dx dy = 9\pi; M_{xy} = \iint_S z d\sigma = \int_{-3}^3 \int_0^3 z \left( \frac{3}{z} \right) dx dy = 54; \\
M_{xz} &= \iint_S y d\sigma = \int_{-3}^3 \int_0^3 y \left( \frac{3}{z} \right) dx dy = \int_{-3}^3 \int_0^3 \frac{3y}{\sqrt{9-y^2}} dx dy = 0; M_{yz} = \iint_S x d\sigma = \int_{-3}^3 \int_0^3 \frac{3x}{\sqrt{9-y^2}} dx dy = \frac{27}{2}\pi. \\
\text{Therefore, } \bar{x} &= \frac{\left( \frac{27}{2}\pi \right)}{9\pi} = \frac{3}{2}, \bar{y} = 0, \text{ and } \bar{z} = \frac{54}{9\pi} = \frac{6}{\pi}
\end{aligned}$$

$$\begin{aligned}
45. \text{ Because of symmetry, } \bar{x} = \bar{y} = 0; M &= \iint_S \delta d\sigma = \delta \iint_S d\sigma = (\text{Area of } S)\delta = 3\pi\sqrt{2}\delta; \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \\
&\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{x^2 + y^2 + z^2}}{2z} dA \\
&= \frac{\sqrt{x^2 + y^2 + (x^2 + y^2)}}{z} dA = \frac{\sqrt{2}\sqrt{x^2 + y^2}}{z} dA \Rightarrow M_{xy} = \delta \iint_R z \left( \frac{\sqrt{2}\sqrt{x^2 + y^2}}{z} \right) dA = \delta \iint_R \sqrt{2}\sqrt{x^2 + y^2} dA \\
&= \delta \int_0^{2\pi} \int_1^2 \sqrt{2} r^2 dr d\theta = \frac{14\pi\sqrt{2}}{3} \delta \Rightarrow \bar{z} = \frac{\left( \frac{14\pi\sqrt{2}}{3} \delta \right)}{3\pi\sqrt{2}\delta} = \frac{14}{9} \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left( 0, 0, \frac{14}{9} \right). \text{ Next, } I_z = \iint_S (x^2 + y^2) \delta d\sigma \\
&= \iint_R (x^2 + y^2) \left( \frac{\sqrt{2}\sqrt{x^2 + y^2}}{z} \right) \delta dA = \delta\sqrt{2} \iint_R (x^2 + y^2) dA = \delta\sqrt{2} \int_0^{2\pi} \int_1^2 r^3 dr d\theta = \frac{15\pi\sqrt{2}}{2} \delta \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{10}}{2}
\end{aligned}$$

$$\begin{aligned}
46. f(x, y, z) = 4x^2 + 4y^2 - z^2 = 0 &\Rightarrow \nabla f = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{64x^2 + 64y^2 + 4z^2} \\
&= 2\sqrt{16x^2 + 16y^2 + z^2} = 2\sqrt{4z^2 + z^2} = 2\sqrt{5}z \text{ since } z \geq 0; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{5}z}{2z} dA = \sqrt{5} dA \\
&\Rightarrow I_z = \iint_S (x^2 + y^2) \delta d\sigma = \delta\sqrt{5} \iint_R (x^2 + y^2) dx dy = \delta\sqrt{5} \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^3 dr d\theta = \frac{3\sqrt{5}\pi\delta}{2}
\end{aligned}$$

$$\begin{aligned}
47. (a) \text{ Let the diameter lie on the } z\text{-axis and let } f(x, y, z) = x^2 + y^2 + z^2 = a^2, z \geq 0 &\text{ be the upper hemisphere} \\
&\Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a, a > 0; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \text{ since } z \geq 0 \\
&\Rightarrow d\sigma = \frac{a}{z} dA \Rightarrow I_z = \iint_S \delta (x^2 + y^2) \left( \frac{a}{z} \right) d\sigma = a\delta \iint_R \frac{x^2 + y^2}{\sqrt{a^2 - (x^2 + y^2)}} dA = a\delta \int_0^{2\pi} \int_0^a \frac{r^2}{\sqrt{a^2 - r^2}} r dr d\theta \\
&= a\delta \int_0^{2\pi} \left[ -r^2 \sqrt{a^2 - r^2} - \frac{2}{3} (a^2 - r^2)^{3/2} \right]_0^a d\theta = a\delta \int_0^{2\pi} \frac{2}{3} a^3 d\theta = \frac{4\pi}{3} a^4 \delta \Rightarrow \text{the moment of inertia is } \frac{8\pi}{3} a^4 \delta \text{ for} \\
&\text{the whole sphere}
\end{aligned}$$



(b)  $I_L = I_{c.m.} + mh^2$ , where  $m$  is the mass of the body and  $h$  is the distance between the parallel lines; now,

$$\begin{aligned} I_{c.m.} &= \frac{8\pi}{3} a^4 \delta \text{ (from part a) and } \frac{m}{2} = \iint_S \delta \, d\sigma = \delta \iint_R \left(\frac{a}{z}\right) dA = a\delta \iint_R \frac{1}{\sqrt{a^2 - (x^2 + y^2)}} dy \, dx \\ &= a\delta \int_0^{2\pi} \int_0^a \frac{1}{\sqrt{a^2 - r^2}} r \, dr \, d\theta = a\delta \int_0^{2\pi} \left[-\sqrt{a^2 - r^2}\right]_0^a d\theta = a\delta \int_0^{2\pi} a \, d\theta = 2\pi a^2 \delta \text{ and } h = a \\ \Rightarrow I_L &= \frac{8\pi}{3} a^4 \delta + 4\pi a^2 \delta a^2 = \frac{20\pi}{3} a^4 \delta \end{aligned}$$

48. Let  $z = \frac{h}{a} \sqrt{x^2 + y^2}$  be the cone from  $z = 0$  to  $z = h$ ,  $h > 0$ . Because of symmetry,  $\bar{x} = 0$  and  $\bar{y} = 0$ ;

$$\begin{aligned} z &= \frac{h}{a} \sqrt{x^2 + y^2} \Rightarrow f(x, y, z) = \frac{h^2}{a^2} (x^2 + y^2) - z^2 = 0 \Rightarrow \nabla f = \frac{2xh^2}{a^2} \mathbf{i} + \frac{2yh^2}{a^2} \mathbf{j} - 2z\mathbf{k} \\ \Rightarrow |\nabla f| &= \sqrt{\frac{4x^2h^4}{a^4} + \frac{4y^2h^4}{a^4} + 4z^2} = 2\sqrt{\frac{h^4}{a^4} (x^2 + y^2) + \frac{h^2}{a^2} (x^2 + y^2)} = 2\sqrt{\left(\frac{h^2}{a^2}\right) (x^2 + y^2) \left(\frac{h^2}{a^2} + 1\right)} \\ &= 2\sqrt{z^2 \left(\frac{h^2 + a^2}{a^2}\right)} = \left(\frac{2z}{a}\right) \sqrt{h^2 + a^2} \text{ since } z \geq 0; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{\left(\frac{2z}{a}\right) \sqrt{h^2 + a^2}}{2z} dA \\ &= \frac{\sqrt{h^2 + a^2}}{a} dA; M = \iint_S d\sigma = \iint_R \frac{\sqrt{h^2 + a^2}}{a} dA = \frac{\sqrt{h^2 + a^2}}{a} (\pi a^2) = \pi a \sqrt{h^2 + a^2}; \\ M_{xy} &= \iint_S z \, d\sigma = \iint_R z \left(\frac{\sqrt{h^2 + a^2}}{a}\right) dA = \frac{\sqrt{h^2 + a^2}}{a} \iint_R \frac{h}{a} \sqrt{x^2 + y^2} \, dx \, dy = \frac{h\sqrt{h^2 + a^2}}{a^2} \int_0^{2\pi} \int_0^a r^2 \, dr \, d\theta \\ &= \frac{2\pi ah\sqrt{h^2 + a^2}}{3} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{2h}{3} \Rightarrow \text{the centroid is } (0, 0, \frac{2h}{3}) \end{aligned}$$

## 16.7 STOKES' THEOREM

$$\begin{aligned} 1. \quad \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (2 - 0)\mathbf{k} = 2\mathbf{k} \text{ and } \mathbf{n} = \mathbf{k} \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} = 2 \Rightarrow d\sigma = dx \, dy \\ \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R 2 \, dA = 2(\text{Area of the ellipse}) = 4\pi \end{aligned}$$

$$\begin{aligned} 2. \quad \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (3 - 2)\mathbf{k} = \mathbf{k} \text{ and } \mathbf{n} = \mathbf{k} \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} = 1 \Rightarrow d\sigma = dx \, dy \\ \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R dx \, dy = \text{Area of circle} = 9\pi \end{aligned}$$

$$\begin{aligned} 3. \quad \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\mathbf{i} - 2x\mathbf{j} + (z - 1)\mathbf{k} \text{ and } \mathbf{n} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} \\ &= \frac{1}{\sqrt{3}} (-x - 2x + z - 1) \Rightarrow d\sigma = \frac{\sqrt{3}}{1} dA \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{1}{\sqrt{3}} (-3x + z - 1) \sqrt{3} \, dA \\ &= \int_0^1 \int_0^{1-x} [-3x + (1 - x - y) - 1] \, dy \, dx = \int_0^1 \int_0^{1-x} (-4x - y) \, dy \, dx = \int_0^1 \left[4x(1 - x) + \frac{1}{2}(1 - x)^2\right] \, dx \\ &= -\int_0^1 \left(\frac{1}{2} + 3x - \frac{7}{2}x^2\right) \, dx = -\frac{5}{6} \end{aligned}$$

$$\begin{aligned} 4. \quad \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 & x^2 + z^2 & x^2 + y^2 \end{vmatrix} = (2y - 2z)\mathbf{i} + (2z - 2x)\mathbf{j} + (2x - 2y)\mathbf{k} \text{ and } \mathbf{n} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} \\ \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} &= \frac{1}{\sqrt{3}} (2y - 2z + 2z - 2x + 2x - 2y) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S 0 \, d\sigma = 0 \end{aligned}$$

$$5. \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 & x^2 + y^2 & x^2 + y^2 \end{vmatrix} = 2y\mathbf{i} + (2z - 2x)\mathbf{j} + (2x - 2y)\mathbf{k} \text{ and } \mathbf{n} = \mathbf{k}$$

$$\Rightarrow \operatorname{curl} \mathbf{F} \cdot \mathbf{n} = 2x - 2y \Rightarrow d\sigma = dx dy \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^1 \int_{-1}^1 (2x - 2y) dx dy = \int_{-1}^1 [x^2 - 2xy]_{-1}^1 dy \\ = \int_{-1}^1 -4y dy = 0$$

$$6. \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y^3 & 1 & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 3x^2 y^2 \mathbf{k} \text{ and } \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{4}$$

$$\Rightarrow \operatorname{curl} \mathbf{F} \cdot \mathbf{n} = -\frac{3}{4} x^2 y^2 z; d\sigma = \frac{4}{z} dA \text{ (Section 16.6, Example 6, with } a = 4) \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(-\frac{3}{4} x^2 y^2 z\right) \left(\frac{4}{z}\right) dA \\ = -3 \int_0^{2\pi} \int_0^2 (r^2 \cos^2 \theta) (r^2 \sin^2 \theta) r dr d\theta = -3 \int_0^{2\pi} \left[\frac{r^6}{6}\right]_0^2 (\cos \theta \sin \theta)^2 d\theta = -32 \int_0^{2\pi} \frac{1}{4} \sin^2 2\theta d\theta = -4 \int_0^{4\pi} \sin^2 u du \\ = -4 \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = -8\pi$$

$$7. x = 3 \cos t \text{ and } y = 2 \sin t \Rightarrow \mathbf{F} = (2 \sin t)\mathbf{i} + (9 \cos^2 t)\mathbf{j} + (9 \cos^2 t + 16 \sin^4 t) \sin e^{\sqrt{(6 \sin t \cos t)(0)}} \mathbf{k} \text{ at the} \\ \text{base of the shell; } \mathbf{r} = (3 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} \Rightarrow d\mathbf{r} = (-3 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -6 \sin^2 t + 18 \cos^3 t \\ \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \int_0^{2\pi} (-6 \sin^2 t + 18 \cos^3 t) dt = \left[-3t + \frac{3}{2} \sin 2t + 6(\sin t)(\cos^2 t + 2)\right]_0^{2\pi} = -6\pi$$

$$8. \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z + \frac{1}{2+x} & \tan^{-1} y & x + \frac{1}{4+z} \end{vmatrix} = -2\mathbf{j}; f(x, y, z) = 4x^2 + y + z^2 \Rightarrow \nabla f = 8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k}$$

$$\Rightarrow \mathbf{n} = \frac{\nabla f}{|\nabla f|} \text{ and } \mathbf{p} = \mathbf{j} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = |\nabla f| dA; \nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{1}{|\nabla f|} (-2\mathbf{j} \cdot \nabla f) = \frac{-2}{|\nabla f|} \\ \Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = -2 dA \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_R -2 dA = -2(\text{Area of } R) = -2(\pi \cdot 1 \cdot 2) = -4\pi, \text{ where } R \\ \text{is the elliptic region in the } xz\text{-plane enclosed by } 4x^2 + z^2 = 4.$$

$$9. \text{Flux of } \nabla \times \mathbf{F} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}, \text{ so let } C \text{ be parametrized by } \mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, \\ 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ay \sin t + ax \cos t = a^2 \sin^2 t + a^2 \cos^2 t = a^2 \\ \Rightarrow \text{Flux of } \nabla \times \mathbf{F} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} a^2 dt = 2\pi a^2$$

$$10. \nabla \times (y\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\mathbf{k}; \mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\Rightarrow \nabla \times (y\mathbf{i}) \cdot \mathbf{n} = -z; d\sigma = \frac{1}{z} dA \text{ (Section 16.6, Example 6, with } a = 1) \Rightarrow \iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n} d\sigma \\ = \iint_R (-z) \left(\frac{1}{z} dA\right) = -\iint_R dA = -\pi, \text{ where } R \text{ is the disk } x^2 + y^2 \leq 1 \text{ in the } xy\text{-plane.}$$

$$11. \text{Let } S_1 \text{ and } S_2 \text{ be oriented surfaces that span } C \text{ and that induce the same positive direction on } C. \text{ Then} \\ \iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}_1 d\sigma_1 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}_2 d\sigma_2$$

12.  $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma + \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$ , and since  $S_1$  and  $S_2$  are joined by the simple closed curve  $C$ , each of the above integrals will be equal to a circulation integral on  $C$ . But for one surface the circulation will be counterclockwise, and for the other surface the circulation will be clockwise. Since the integrands are the same, the sum will be 0  $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0$ .

$$\begin{aligned}
 13. \quad \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}; \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - 2r\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \\
 &\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} + r\mathbf{k}; \mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} \text{ and } d\sigma = |\mathbf{r}_r \times \mathbf{r}_\theta| \, dr \, d\theta \\
 &\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta = (10r^2 \cos \theta + 4r^2 \sin \theta + 3r) \, dr \, d\theta \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma \\
 &= \int_0^{2\pi} \int_0^2 (10r^2 \cos \theta + 4r^2 \sin \theta + 3r) \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{10}{3} r^3 \cos \theta + \frac{4}{3} r^3 \sin \theta + \frac{3}{2} r^2 \right]_0^2 \, d\theta \\
 &= \int_0^{2\pi} \left( \frac{80}{3} \cos \theta + \frac{32}{3} \sin \theta + 6 \right) \, d\theta = 6(2\pi) = 12\pi
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x+z \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}; \mathbf{r}_r \times \mathbf{r}_\theta = (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} + r\mathbf{k} \text{ and} \\
 \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta \text{ (see Exercise 13 above)} \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma \\
 &= \int_0^{2\pi} \int_0^3 (-2r^2 \cos \theta - 4r^2 \sin \theta - 2r) \, dr \, d\theta = \int_0^{2\pi} \left[ -\frac{2}{3} r^3 \cos \theta - \frac{4}{3} r^3 \sin \theta - r^2 \right]_0^3 \, d\theta \\
 &= \int_0^{2\pi} (-18 \cos \theta - 36 \sin \theta - 9) \, d\theta = -9(2\pi) = -18\pi
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix} = -2y^3\mathbf{i} + 0\mathbf{j} - x^2\mathbf{k}; \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\
 &= (-r \cos \theta)\mathbf{i} - (r \sin \theta)\mathbf{j} + r\mathbf{k} \text{ and } \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta \text{ (see Exercise 13 above)} \\
 &\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (2ry^3 \cos \theta - rx^2) \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (2r^4 \sin^3 \theta \cos \theta - r^3 \cos^2 \theta) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left( \frac{2}{5} \sin^3 \theta \cos \theta - \frac{1}{4} \cos^2 \theta \right) \, d\theta = \left[ \frac{1}{10} \sin^4 \theta - \frac{1}{4} \left( \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \right]_0^{2\pi} = -\frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}; \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\
 &= (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k} \text{ and } \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta \text{ (see Exercise 13 above)} \\
 &\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^5 (r \cos \theta + r \sin \theta + r) \, dr \, d\theta = \int_0^{2\pi} \left[ (\cos \theta + \sin \theta + 1) \frac{r^2}{2} \right]_0^5 \, d\theta = \left( \frac{25}{2} \right) (2\pi) = 25\pi
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 5\mathbf{k}; \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3} \cos \phi \cos \theta & \sqrt{3} \cos \phi \sin \theta & -\sqrt{3} \sin \phi \\ -\sqrt{3} \sin \phi \sin \theta & \sqrt{3} \sin \phi \cos \theta & 0 \end{vmatrix} \\
 &= (3 \sin^2 \phi \cos \theta)\mathbf{i} + (3 \sin^2 \phi \sin \theta)\mathbf{j} + (3 \sin \phi \cos \phi)\mathbf{k}; \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \, d\phi \, d\theta \text{ (see Exercise} \\
 &13 \text{ above)} \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^{\pi/2} -15 \cos \phi \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \left[ \frac{15}{2} \cos^2 \phi \right]_0^{\pi/2} \, d\theta = \int_0^{2\pi} -\frac{15}{2} \, d\theta = -15\pi
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x \end{vmatrix} = -2z\mathbf{i} - \mathbf{j} - 2y\mathbf{k}; \quad \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos \phi \cos \theta & 2 \cos \phi \sin \theta & -2 \sin \phi \\ -2 \sin \phi \sin \theta & 2 \sin \phi \cos \theta & 0 \end{vmatrix} \\
 &= (4 \sin^2 \phi \cos \theta)\mathbf{i} + (4 \sin^2 \phi \sin \theta)\mathbf{j} + (4 \sin \phi \cos \phi)\mathbf{k}; \quad \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \, d\phi \, d\theta \text{ (see Exercise 13 above)} \\
 &\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (-8z \sin^2 \phi \cos \theta - 4 \sin^2 \phi \sin \theta - 8y \sin \phi \cos \theta) \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^{\pi/2} (-16 \sin^2 \phi \cos \phi \cos \theta - 4 \sin^2 \phi \sin \theta - 16 \sin^2 \phi \sin \theta \cos \theta) \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \left[ -\frac{16}{3} \sin^3 \phi \cos \theta - 4 \left( \frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) (\sin \theta) - 16 \left( \frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) (\sin \theta \cos \theta) \right]_0^{\pi/2} d\theta \\
 &= \int_0^{2\pi} \left( -\frac{16}{3} \cos \theta - \pi \sin \theta - 4\pi \sin \theta \cos \theta \right) d\theta = \left[ -\frac{16}{3} \sin \theta + \pi \cos \theta - 2\pi \sin^2 \theta \right]_0^{2\pi} = 0
 \end{aligned}$$

$$19. \text{ (a) } \mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \operatorname{curl} \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S 0 \, d\sigma = 0$$

$$\text{(b) Let } f(x, y, z) = x^2 y^2 z^3 \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \operatorname{curl} \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S 0 \, d\sigma = 0$$

$$\text{(c) } \mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{0} \Rightarrow \nabla \times \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S 0 \, d\sigma = 0$$

$$\text{(d) } \mathbf{F} = \nabla f \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S 0 \, d\sigma = 0$$

$$\begin{aligned}
 20. \quad \mathbf{F} &= \nabla f = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x)\mathbf{i} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2y)\mathbf{j} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2z)\mathbf{k} \\
 &= -x(x^2 + y^2 + z^2)^{-3/2}\mathbf{i} - y(x^2 + y^2 + z^2)^{-3/2}\mathbf{j} - z(x^2 + y^2 + z^2)^{-3/2}\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{(a) } \mathbf{r} &= (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \\
 &\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -x(x^2 + y^2 + z^2)^{-3/2}(-a \sin t) - y(x^2 + y^2 + z^2)^{-3/2}(a \cos t) \\
 &= \left(-\frac{a \cos t}{a^3}\right)(-a \sin t) - \left(\frac{a \sin t}{a^3}\right)(a \cos t) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0
 \end{aligned}$$

$$\text{(b) } \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S \nabla \times \nabla f \cdot \mathbf{n} \, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n} \, d\sigma = \iint_S 0 \, d\sigma = 0$$

$$\begin{aligned}
 21. \quad \text{Let } \mathbf{F} &= 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k} \Rightarrow \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}; \quad \mathbf{n} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3} \\
 &\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = -2 \Rightarrow \oint_C 2y \, dx + 3z \, dy - x \, dz = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S -2 \, d\sigma \\
 &= -2 \iint_S d\sigma, \text{ where } \iint_S d\sigma \text{ is the area of the region enclosed by } C \text{ on the plane } S: 2x + 2y + z = 2
 \end{aligned}$$

$$22. \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

$$\begin{aligned}
 23. \quad \text{Suppose } \mathbf{F} &= M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \text{ exists such that } \nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \\
 &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \text{ Then } \frac{\partial}{\partial x} \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) = \frac{\partial}{\partial x}(x) \Rightarrow \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} = 1. \text{ Likewise, } \frac{\partial}{\partial y} \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial y}(y) \\
 &\Rightarrow \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} = 1 \text{ and } \frac{\partial}{\partial z} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{\partial}{\partial z}(z) \Rightarrow \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 1. \text{ Summing the calculated equations} \\
 &\Rightarrow \left(\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 P}{\partial y \partial x}\right) + \left(\frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 N}{\partial x \partial z}\right) + \left(\frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 M}{\partial z \partial y}\right) = 3 \text{ or } 0 = 3 \text{ (assuming the second mixed partials are equal).} \\
 &\text{This result is a contradiction, so there is no field } \mathbf{F} \text{ such that } \operatorname{curl} \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.
 \end{aligned}$$

24. Yes: If  $\nabla \times \mathbf{F} = \mathbf{0}$ , then the circulation of  $\mathbf{F}$  around the boundary  $C$  of any oriented surface  $S$  in the domain of

$\mathbf{F}$  is zero. The reason is this: By Stokes's theorem, circulation  $= \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n} d\sigma = 0$ .

$$\begin{aligned} 25. \mathbf{r} &= \sqrt{x^2 + y^2} \Rightarrow \mathbf{r}^4 = (x^2 + y^2)^2 \Rightarrow \mathbf{F} = \nabla(\mathbf{r}^4) = 4x(x^2 + y^2)\mathbf{i} + 4y(x^2 + y^2)\mathbf{j} = M\mathbf{i} + N\mathbf{j} \\ &\Rightarrow \oint_C \nabla(\mathbf{r}^4) \cdot \mathbf{n} ds = \oint_C \mathbf{F} \cdot \mathbf{n} ds = \oint_C M dy - N dx = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \\ &= \iint_R [4(x^2 + y^2) + 8x^2 + 4(x^2 + y^2) + 8y^2] dA = \iint_R 16(x^2 + y^2) dA = 16 \iint_R x^2 dA + 16 \iint_R y^2 dA \\ &= 16I_y + 16I_x. \end{aligned}$$

$$26. \frac{\partial P}{\partial y} = 0, \frac{\partial N}{\partial z} = 0, \frac{\partial M}{\partial z} = 0, \frac{\partial P}{\partial x} = 0, \frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \Rightarrow \text{curl } \mathbf{F} = \left[ \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] \mathbf{k} = \mathbf{0}.$$

However,  $x^2 + y^2 = 1 \Rightarrow \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$$\Rightarrow \mathbf{F} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_0^{2\pi} 1 dt = 2\pi \text{ which is not zero.}$$

## 16.8 THE DIVERGENCE THEOREM AND A UNIFIED THEORY

$$1. \mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}} \Rightarrow \text{div } \mathbf{F} = \frac{xy - xy}{(x^2 + y^2)^{3/2}} = 0 \qquad 2. \mathbf{F} = x\mathbf{i} + y\mathbf{j} \Rightarrow \text{div } \mathbf{F} = 1 + 1 = 2$$

$$\begin{aligned} 3. \mathbf{F} &= -\frac{GM(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}} \Rightarrow \text{div } \mathbf{F} = -GM \left[ \frac{(x^2 + y^2 + z^2)^{3/2} - 3x^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \right] \\ &\quad - GM \left[ \frac{(x^2 + y^2 + z^2)^{3/2} - 3y^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \right] - GM \left[ \frac{(x^2 + y^2 + z^2)^{3/2} - 3z^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \right] \\ &= -GM \left[ \frac{3(x^2 + y^2 + z^2)^2 - 3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{7/2}} \right] = 0 \end{aligned}$$

$$4. z = a^2 - r^2 \text{ in cylindrical coordinates} \Rightarrow z = a^2 - (x^2 + y^2) \Rightarrow \mathbf{v} = (a^2 - x^2 - y^2)\mathbf{k} \Rightarrow \text{div } \mathbf{v} = 0$$

$$5. \frac{\partial}{\partial x}(y - x) = -1, \frac{\partial}{\partial y}(z - y) = -1, \frac{\partial}{\partial z}(y - x) = 0 \Rightarrow \nabla \cdot \mathbf{F} = -2 \Rightarrow \text{Flux} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 -2 dx dy dz = -2(2^3) = -16$$

$$6. \frac{\partial}{\partial x}(x^2) = 2x, \frac{\partial}{\partial y}(y^2) = 2y, \frac{\partial}{\partial z}(z^2) = 2z \Rightarrow \nabla \cdot \mathbf{F} = 2x + 2y + 2z$$

$$\begin{aligned} \text{(a) Flux} &= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) dx dy dz = \int_0^1 \int_0^1 [x^2 + 2x(y + z)]_0^1 dy dz = \int_0^1 \int_0^1 (1 + 2y + 2z) dy dz \\ &= \int_0^1 [y(1 + 2z) + y^2]_0^1 dz = \int_0^1 (2 + 2z) dz = [2z + z^2]_0^1 = 3 \end{aligned}$$

$$\begin{aligned} \text{(b) Flux} &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + 2y + 2z) dx dy dz = \int_{-1}^1 \int_{-1}^1 [x^2 + 2x(y + z)]_{-1}^1 dy dz = \int_{-1}^1 \int_{-1}^1 (4y + 4z) dy dz \\ &= \int_{-1}^1 [2y^2 + 4yz]_{-1}^1 dz = \int_{-1}^1 8z dz = [4z^2]_{-1}^1 = 0 \end{aligned}$$

$$\begin{aligned} \text{(c) In cylindrical coordinates, Flux} &= \iiint_D (2x + 2y + 2z) dx dy dz \\ &= \int_0^1 \int_0^{2\pi} \int_0^2 (2r \cos \theta + 2r \sin \theta + 2z) r dr d\theta dz = \int_0^1 \int_0^{2\pi} \left[ \frac{2}{3} r^3 \cos \theta + \frac{2}{3} r^3 \sin \theta + zr^2 \right]_0^2 d\theta dz \\ &= \int_0^1 \int_0^{2\pi} \left( \frac{16}{3} \cos \theta + \frac{16}{3} \sin \theta + 4z \right) d\theta dz = \int_0^1 \left[ \frac{16}{3} \sin \theta - \frac{16}{3} \cos \theta + 4z\theta \right]_0^{2\pi} dz = \int_0^1 8\pi z dz = [4\pi z^2]_0^1 = 4\pi \end{aligned}$$

$$7. \frac{\partial}{\partial x}(y) = 0, \frac{\partial}{\partial y}(xy) = x, \frac{\partial}{\partial z}(-z) = -1 \Rightarrow \nabla \cdot \mathbf{F} = x - 1; z = x^2 + y^2 \Rightarrow z = r^2 \text{ in cylindrical coordinates}$$

$$\begin{aligned} \Rightarrow \text{Flux} &= \iiint_D (x - 1) dz dy dx = \int_0^{2\pi} \int_0^2 \int_0^{r^2} (r \cos \theta - 1) dz r dr d\theta = \int_0^{2\pi} \int_0^2 (r^3 \cos \theta - r^2) r dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{r^5}{5} \cos \theta - \frac{r^4}{4} \right]_0^2 d\theta = \int_0^{2\pi} \left( \frac{32}{5} \cos \theta - 4 \right) d\theta = \left[ \frac{32}{5} \sin \theta - 4\theta \right]_0^{2\pi} = -8\pi \end{aligned}$$

8.  $\frac{\partial}{\partial x}(x^2) = 2x$ ,  $\frac{\partial}{\partial y}(xz) = 0$ ,  $\frac{\partial}{\partial z}(3z) = 3 \Rightarrow \nabla \cdot \mathbf{F} = 2x + 3 \Rightarrow \text{Flux} = \iiint_D (2x + 3) dV$   
 $= \int_0^{2\pi} \int_0^\pi \int_0^2 (2\rho \sin \phi \cos \theta + 3)(\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\pi \left[ \frac{\rho^4}{2} \sin \phi \cos \theta + \rho^3 \right]_0^2 \sin \phi d\phi d\theta$   
 $= \int_0^{2\pi} \int_0^\pi (8 \sin \phi \cos \theta + 8) \sin \phi d\phi d\theta = \int_0^{2\pi} \left[ 8 \left( \frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) \cos \theta - 8 \cos \phi \right]_0^\pi d\theta = \int_0^{2\pi} (4\pi \cos \theta + 16) d\theta = 32\pi$
9.  $\frac{\partial}{\partial x}(x^2) = 2x$ ,  $\frac{\partial}{\partial y}(-2xy) = -2x$ ,  $\frac{\partial}{\partial z}(3xz) = 3x \Rightarrow \text{Flux} = \iiint_D 3x dx dy dz$   
 $= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 (3\rho \sin \phi \cos \theta)(\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} 12 \sin^2 \phi \cos \theta d\phi d\theta = \int_0^{\pi/2} 3\pi \cos \theta d\theta = 3\pi$
10.  $\frac{\partial}{\partial x}(6x^2 + 2xy) = 12x + 2y$ ,  $\frac{\partial}{\partial y}(2y + x^2z) = 2$ ,  $\frac{\partial}{\partial z}(4x^2y^3) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 12x + 2y + 2$   
 $\Rightarrow \text{Flux} = \iiint_D (12x + 2y + 2) dV = \int_0^3 \int_0^{\pi/2} \int_0^2 (12r \cos \theta + 2r \sin \theta + 2)r dr d\theta dz$   
 $= \int_0^3 \int_0^{\pi/2} (32 \cos \theta + \frac{16}{3} \sin \theta + 4) d\theta dz = \int_0^3 (32 + 2\pi + \frac{16}{3}) dz = 112 + 6\pi$
11.  $\frac{\partial}{\partial x}(2xz) = 2z$ ,  $\frac{\partial}{\partial y}(-xy) = -x$ ,  $\frac{\partial}{\partial z}(-z^2) = -2z \Rightarrow \nabla \cdot \mathbf{F} = -x \Rightarrow \text{Flux} = \iiint_D -x dV$   
 $= \int_0^2 \int_0^{\sqrt{16-4x^2}} \int_0^{4-y} -x dz dy dx = \int_0^2 \int_0^{\sqrt{16-4x^2}} (xy - 4x) dy dx = \int_0^2 \left[ \frac{1}{2} x (16 - 4x^2) - 4x \sqrt{16 - 4x^2} \right] dx$   
 $= \left[ 4x^2 - \frac{1}{2} x^4 + \frac{1}{3} (16 - 4x^2)^{3/2} \right]_0^2 = -\frac{40}{3}$
12.  $\frac{\partial}{\partial x}(x^3) = 3x^2$ ,  $\frac{\partial}{\partial y}(y^3) = 3y^2$ ,  $\frac{\partial}{\partial z}(z^3) = 3z^2 \Rightarrow \nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 3z^2 \Rightarrow \text{Flux} = \iiint_D 3(x^2 + y^2 + z^2) dV$   
 $= 3 \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 (\rho^2 \sin \phi) d\rho d\phi d\theta = 3 \int_0^{2\pi} \int_0^\pi \frac{a^5}{5} \sin \phi d\phi d\theta = 3 \int_0^{2\pi} \frac{2a^5}{5} d\theta = \frac{12\pi a^5}{5}$
13. Let  $\rho = \sqrt{x^2 + y^2 + z^2}$ . Then  $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$ ,  $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$ ,  $\frac{\partial \rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}(\rho x) = \left( \frac{\partial \rho}{\partial x} \right) x + \rho = \frac{x^2}{\rho} + \rho$ ,  $\frac{\partial}{\partial y}(\rho y) = \left( \frac{\partial \rho}{\partial y} \right) y + \rho = \frac{y^2}{\rho} + \rho$ ,  $\frac{\partial}{\partial z}(\rho z) = \left( \frac{\partial \rho}{\partial z} \right) z + \rho = \frac{z^2}{\rho} + \rho \Rightarrow \nabla \cdot \mathbf{F} = \frac{x^2 + y^2 + z^2}{\rho} + 3\rho = 4\rho$ , since  $\rho = \sqrt{x^2 + y^2 + z^2}$   
 $\Rightarrow \text{Flux} = \iiint_D 4\rho dV = \int_0^{2\pi} \int_0^\pi \int_1^{\sqrt{2}} (4\rho)(\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\pi 3 \sin \phi d\phi d\theta = \int_0^{2\pi} 6 d\theta = 12\pi$
14. Let  $\rho = \sqrt{x^2 + y^2 + z^2}$ . Then  $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$ ,  $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$ ,  $\frac{\partial \rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x} \left( \frac{x}{\rho} \right) = \frac{1}{\rho} - \left( \frac{x}{\rho^2} \right) \frac{\partial \rho}{\partial x} = \frac{1}{\rho} - \frac{x^2}{\rho^3}$ . Similarly,  
 $\frac{\partial}{\partial y} \left( \frac{y}{\rho} \right) = \frac{1}{\rho} - \frac{y^2}{\rho^3}$  and  $\frac{\partial}{\partial z} \left( \frac{z}{\rho} \right) = \frac{1}{\rho} - \frac{z^2}{\rho^3} \Rightarrow \nabla \cdot \mathbf{F} = \frac{3}{\rho} - \frac{x^2 + y^2 + z^2}{\rho^3} = \frac{2}{\rho}$   
 $\Rightarrow \text{Flux} = \iiint_D \frac{2}{\rho} dV = \int_0^{2\pi} \int_0^\pi \int_1^2 \left( \frac{2}{\rho} \right) (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\pi 3 \sin \phi d\phi d\theta = \int_0^{2\pi} 6 d\theta = 12\pi$
15.  $\frac{\partial}{\partial x}(5x^3 + 12xy^2) = 15x^2 + 12y^2$ ,  $\frac{\partial}{\partial y}(y^3 + e^y \sin z) = 3y^2 + e^y \sin z$ ,  $\frac{\partial}{\partial z}(5z^3 + e^y \cos z) = 15z^2 - e^y \sin z$   
 $\Rightarrow \nabla \cdot \mathbf{F} = 15x^2 + 15y^2 + 15z^2 = 15\rho^2 \Rightarrow \text{Flux} = \iiint_D 15\rho^2 dV = \int_0^{2\pi} \int_0^\pi \int_1^{\sqrt{2}} (15\rho^2)(\rho^2 \sin \phi) d\rho d\phi d\theta$   
 $= \int_0^{2\pi} \int_0^\pi (12\sqrt{2} - 3) \sin \phi d\phi d\theta = \int_0^{2\pi} (24\sqrt{2} - 6) d\theta = (48\sqrt{2} - 12)\pi$
16.  $\frac{\partial}{\partial x}[\ln(x^2 + y^2)] = \frac{2x}{x^2 + y^2}$ ,  $\frac{\partial}{\partial y} \left( -\frac{2z}{x} \tan^{-1} \frac{y}{x} \right) = \left( -\frac{2z}{x} \right) \left[ \frac{\left( \frac{1}{x} \right)}{1 + \left( \frac{y}{x} \right)^2} \right] = -\frac{2z}{x^2 + y^2}$ ,  $\frac{\partial}{\partial z}(z\sqrt{x^2 + y^2}) = \sqrt{x^2 + y^2}$   
 $\Rightarrow \nabla \cdot \mathbf{F} = \frac{2x}{x^2 + y^2} - \frac{2z}{x^2 + y^2} + \sqrt{x^2 + y^2} \Rightarrow \text{Flux} = \iiint_D \left( \frac{2x}{x^2 + y^2} - \frac{2z}{x^2 + y^2} + \sqrt{x^2 + y^2} \right) dz dy dx$

$$\begin{aligned}
&= \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{-1}^2 \left( \frac{2r \cos \theta}{r^2} - \frac{2z}{r^2} + r \right) dz r dr d\theta = \int_0^{2\pi} \int_1^{\sqrt{2}} \left( 6 \cos \theta - \frac{3}{r} + 3r^2 \right) dr d\theta \\
&= \int_0^{2\pi} \left[ 6 \left( \sqrt{2} - 1 \right) \cos \theta - 3 \ln \sqrt{2} + 2\sqrt{2} - 1 \right] d\theta = 2\pi \left( -\frac{3}{2} \ln 2 + 2\sqrt{2} - 1 \right)
\end{aligned}$$

17. (a)  $\mathbf{G} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \Rightarrow \nabla \times \mathbf{G} = \text{curl } \mathbf{G} = \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \Rightarrow \nabla \cdot \nabla \times \mathbf{G}$

$$\begin{aligned}
&= \text{div}(\text{curl } \mathbf{G}) = \frac{\partial}{\partial x} \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\
&= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0 \text{ if all first and second partial derivatives are continuous}
\end{aligned}$$

(b) By the Divergence Theorem, the outward flux of  $\nabla \times \mathbf{G}$  across a closed surface is zero because

$$\begin{aligned}
&\text{outward flux of } \nabla \times \mathbf{G} = \int_S (\nabla \times \mathbf{G}) \cdot \mathbf{n} d\sigma \\
&= \int_D \int \int \nabla \cdot \nabla \times \mathbf{G} dV \quad [\text{Divergence Theorem with } \mathbf{F} = \nabla \times \mathbf{G}] \\
&= \int_D \int \int (0) dV = 0 \quad [\text{by part (a)}]
\end{aligned}$$

18. (a) Let  $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$  and  $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k} \Rightarrow a\mathbf{F}_1 + b\mathbf{F}_2$

$$\begin{aligned}
&= (aM_1 + bM_2)\mathbf{i} + (aN_1 + bN_2)\mathbf{j} + (aP_1 + bP_2)\mathbf{k} \Rightarrow \nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) \\
&= \left( a \frac{\partial M_1}{\partial x} + b \frac{\partial M_2}{\partial x} \right) + \left( a \frac{\partial N_1}{\partial y} + b \frac{\partial N_2}{\partial y} \right) + \left( a \frac{\partial P_1}{\partial z} + b \frac{\partial P_2}{\partial z} \right) \\
&= a \left( \frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z} \right) + b \left( \frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} \right) = a(\nabla \cdot \mathbf{F}_1) + b(\nabla \cdot \mathbf{F}_2)
\end{aligned}$$

(b) Define  $\mathbf{F}_1$  and  $\mathbf{F}_2$  as in part a  $\Rightarrow \nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2)$

$$\begin{aligned}
&= \left[ \left( a \frac{\partial P_1}{\partial y} + b \frac{\partial P_2}{\partial y} \right) - \left( a \frac{\partial N_1}{\partial z} + b \frac{\partial N_2}{\partial z} \right) \right] \mathbf{i} + \left[ \left( a \frac{\partial M_1}{\partial z} + b \frac{\partial M_2}{\partial z} \right) - \left( a \frac{\partial P_1}{\partial x} + b \frac{\partial P_2}{\partial x} \right) \right] \mathbf{j} \\
&+ \left[ \left( a \frac{\partial N_1}{\partial x} + b \frac{\partial N_2}{\partial x} \right) - \left( a \frac{\partial M_1}{\partial y} + b \frac{\partial M_2}{\partial y} \right) \right] \mathbf{k} = a \left[ \left( \frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z} \right) \mathbf{i} + \left( \frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x} \right) \mathbf{j} + \left( \frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y} \right) \mathbf{k} \right] \\
&+ b \left[ \left( \frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z} \right) \mathbf{i} + \left( \frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x} \right) \mathbf{j} + \left( \frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y} \right) \mathbf{k} \right] = a \nabla \times \mathbf{F}_1 + b \nabla \times \mathbf{F}_2
\end{aligned}$$

(c)  $\mathbf{F}_1 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M_1 & N_1 & P_1 \\ M_2 & N_2 & P_2 \end{vmatrix} = (N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k} \Rightarrow \nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2)$

$$\begin{aligned}
&= \nabla \cdot [(N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}] \\
&= \frac{\partial}{\partial x} (N_1P_2 - P_1N_2) - \frac{\partial}{\partial y} (M_1P_2 - P_1M_2) + \frac{\partial}{\partial z} (M_1N_2 - N_1M_2) = \left( P_2 \frac{\partial N_1}{\partial x} + N_1 \frac{\partial P_2}{\partial x} - N_2 \frac{\partial P_1}{\partial x} - P_1 \frac{\partial N_2}{\partial x} \right) \\
&- \left( M_1 \frac{\partial P_2}{\partial y} + P_2 \frac{\partial M_1}{\partial y} - P_1 \frac{\partial M_2}{\partial y} - M_2 \frac{\partial P_1}{\partial y} \right) + \left( M_1 \frac{\partial N_2}{\partial z} + N_2 \frac{\partial M_1}{\partial z} - N_1 \frac{\partial M_2}{\partial z} - M_2 \frac{\partial N_1}{\partial z} \right) \\
&= M_2 \left( \frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z} \right) + N_2 \left( \frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x} \right) + P_2 \left( \frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y} \right) + M_1 \left( \frac{\partial N_2}{\partial z} - \frac{\partial P_2}{\partial y} \right) + N_1 \left( \frac{\partial P_2}{\partial x} - \frac{\partial M_2}{\partial z} \right) \\
&+ P_1 \left( \frac{\partial M_2}{\partial y} - \frac{\partial N_2}{\partial x} \right) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2
\end{aligned}$$

19. (a)  $\text{div}(\mathbf{gF}) = \nabla \cdot \mathbf{gF} = \frac{\partial}{\partial x} (\mathbf{gM}) + \frac{\partial}{\partial y} (\mathbf{gN}) + \frac{\partial}{\partial z} (\mathbf{gP}) = \left( \mathbf{g} \frac{\partial M}{\partial x} + M \frac{\partial \mathbf{g}}{\partial x} \right) + \left( \mathbf{g} \frac{\partial N}{\partial y} + N \frac{\partial \mathbf{g}}{\partial y} \right) + \left( \mathbf{g} \frac{\partial P}{\partial z} + P \frac{\partial \mathbf{g}}{\partial z} \right)$

$$= \left( M \frac{\partial \mathbf{g}}{\partial x} + N \frac{\partial \mathbf{g}}{\partial y} + P \frac{\partial \mathbf{g}}{\partial z} \right) + \mathbf{g} \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) = \mathbf{g} \nabla \cdot \mathbf{F} + \nabla \mathbf{g} \cdot \mathbf{F}$$

(b)  $\nabla \times (\mathbf{gF}) = \left[ \frac{\partial}{\partial y} (\mathbf{gP}) - \frac{\partial}{\partial z} (\mathbf{gN}) \right] \mathbf{i} + \left[ \frac{\partial}{\partial z} (\mathbf{gM}) - \frac{\partial}{\partial x} (\mathbf{gP}) \right] \mathbf{j} + \left[ \frac{\partial}{\partial x} (\mathbf{gN}) - \frac{\partial}{\partial y} (\mathbf{gM}) \right] \mathbf{k}$

$$\begin{aligned}
&= \left( P \frac{\partial \mathbf{g}}{\partial y} + \mathbf{g} \frac{\partial P}{\partial y} - N \frac{\partial \mathbf{g}}{\partial z} - \mathbf{g} \frac{\partial N}{\partial z} \right) \mathbf{i} + \left( M \frac{\partial \mathbf{g}}{\partial z} + \mathbf{g} \frac{\partial M}{\partial z} - P \frac{\partial \mathbf{g}}{\partial x} - \mathbf{g} \frac{\partial P}{\partial x} \right) \mathbf{j} + \left( N \frac{\partial \mathbf{g}}{\partial x} + \mathbf{g} \frac{\partial N}{\partial x} - M \frac{\partial \mathbf{g}}{\partial y} - \mathbf{g} \frac{\partial M}{\partial y} \right) \mathbf{k} \\
&= \left( P \frac{\partial \mathbf{g}}{\partial y} - N \frac{\partial \mathbf{g}}{\partial z} \right) \mathbf{i} + \left( \mathbf{g} \frac{\partial P}{\partial y} - \mathbf{g} \frac{\partial N}{\partial z} \right) \mathbf{i} + \left( M \frac{\partial \mathbf{g}}{\partial z} - P \frac{\partial \mathbf{g}}{\partial x} \right) \mathbf{j} + \left( \mathbf{g} \frac{\partial M}{\partial z} - \mathbf{g} \frac{\partial P}{\partial x} \right) \mathbf{j} + \left( N \frac{\partial \mathbf{g}}{\partial x} - M \frac{\partial \mathbf{g}}{\partial y} \right) \mathbf{k} \\
&+ \left( \mathbf{g} \frac{\partial N}{\partial x} - \mathbf{g} \frac{\partial M}{\partial y} \right) \mathbf{k} = \mathbf{g} \nabla \times \mathbf{F} + \nabla \mathbf{g} \times \mathbf{F}
\end{aligned}$$

20. Let  $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$  and  $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$ .

$$\begin{aligned} \text{(a)} \quad \mathbf{F}_1 \times \mathbf{F}_2 &= (N_1P_2 - P_1N_2)\mathbf{i} + (P_1M_2 - M_1P_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k} \Rightarrow \nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) \\ &= \left[ \frac{\partial}{\partial y} (M_1N_2 - N_1M_2) - \frac{\partial}{\partial z} (P_1M_2 - M_1P_2) \right] \mathbf{i} + \left[ \frac{\partial}{\partial z} (N_1P_2 - P_1N_2) - \frac{\partial}{\partial x} (M_1N_2 - N_1M_2) \right] \mathbf{j} \\ &\quad + \left[ \frac{\partial}{\partial x} (P_1M_2 - M_1P_2) - \frac{\partial}{\partial y} (N_1P_2 - P_1N_2) \right] \mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{and consider the } \mathbf{i}\text{-component only: } &\frac{\partial}{\partial y} (M_1N_2 - N_1M_2) - \frac{\partial}{\partial z} (P_1M_2 - M_1P_2) \\ &= N_2 \frac{\partial M_1}{\partial y} + M_1 \frac{\partial N_2}{\partial y} - M_2 \frac{\partial N_1}{\partial y} - N_1 \frac{\partial M_2}{\partial y} - M_2 \frac{\partial P_1}{\partial z} - P_1 \frac{\partial M_2}{\partial z} + P_2 \frac{\partial M_1}{\partial z} + M_1 \frac{\partial P_2}{\partial z} \\ &= \left( N_2 \frac{\partial M_1}{\partial y} + P_2 \frac{\partial M_1}{\partial z} \right) - \left( N_1 \frac{\partial M_2}{\partial y} + P_1 \frac{\partial M_2}{\partial z} \right) + \left( \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} \right) M_1 - \left( \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z} \right) M_2 \\ &= \left( M_2 \frac{\partial M_1}{\partial x} + N_2 \frac{\partial M_1}{\partial y} + P_2 \frac{\partial M_1}{\partial z} \right) - \left( M_1 \frac{\partial M_2}{\partial x} + N_1 \frac{\partial M_2}{\partial y} + P_1 \frac{\partial M_2}{\partial z} \right) + \left( \frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} \right) M_1 \\ &\quad - \left( \frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z} \right) M_2. \text{ Now, } \mathbf{i}\text{-comp of } (\mathbf{F}_2 \cdot \nabla) \mathbf{F}_1 = \left( M_2 \frac{\partial}{\partial x} + N_2 \frac{\partial}{\partial y} + P_2 \frac{\partial}{\partial z} \right) M_1 \\ &= \left( M_2 \frac{\partial M_1}{\partial x} + N_2 \frac{\partial M_1}{\partial y} + P_2 \frac{\partial M_1}{\partial z} \right); \text{ likewise, } \mathbf{i}\text{-comp of } (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 = \left( M_1 \frac{\partial M_2}{\partial x} + N_1 \frac{\partial M_2}{\partial y} + P_1 \frac{\partial M_2}{\partial z} \right); \\ \mathbf{i}\text{-comp of } (\nabla \cdot \mathbf{F}_2) \mathbf{F}_1 &= \left( \frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} \right) M_1 \text{ and } \mathbf{i}\text{-comp of } (\nabla \cdot \mathbf{F}_1) \mathbf{F}_2 = \left( \frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z} \right) M_2. \end{aligned}$$

Similar results hold for the  $\mathbf{j}$  and  $\mathbf{k}$  components of  $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2)$ . In summary, since the corresponding components are equal, we have the result

$$\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla) \mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2) \mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1) \mathbf{F}_2$$

(b) Here again we consider only the  $\mathbf{i}$ -component of each expression. Thus, the  $\mathbf{i}$ -comp of  $\nabla (\mathbf{F}_1 \cdot \mathbf{F}_2)$

$$= \frac{\partial}{\partial x} (M_1M_2 + N_1N_2 + P_1P_2) = \left( M_1 \frac{\partial M_2}{\partial x} + M_2 \frac{\partial M_1}{\partial x} + N_1 \frac{\partial N_2}{\partial x} + N_2 \frac{\partial N_1}{\partial x} + P_1 \frac{\partial P_2}{\partial x} + P_2 \frac{\partial P_1}{\partial x} \right)$$

$$\mathbf{i}\text{-comp of } (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 = \left( M_1 \frac{\partial M_2}{\partial x} + N_1 \frac{\partial M_2}{\partial y} + P_1 \frac{\partial M_2}{\partial z} \right),$$

$$\mathbf{i}\text{-comp of } (\mathbf{F}_2 \cdot \nabla) \mathbf{F}_1 = \left( M_2 \frac{\partial M_1}{\partial x} + N_2 \frac{\partial M_1}{\partial y} + P_2 \frac{\partial M_1}{\partial z} \right),$$

$$\mathbf{i}\text{-comp of } \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) = N_1 \left( \frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y} \right) - P_1 \left( \frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x} \right), \text{ and}$$

$$\mathbf{i}\text{-comp of } \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1) = N_2 \left( \frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y} \right) - P_2 \left( \frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x} \right).$$

Since corresponding components are equal, we see that

$$\nabla (\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla) \mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1), \text{ as claimed.}$$

21. The integral's value never exceeds the surface area of  $S$ . Since  $|\mathbf{F}| \leq 1$ , we have  $|\mathbf{F} \cdot \mathbf{n}| = |\mathbf{F}| |\mathbf{n}| \leq (1)(1) = 1$  and

$$\begin{aligned} \iint_D \nabla \cdot \mathbf{F} \, d\sigma &= \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma && [\text{Divergence Theorem}] \\ &\leq \iint_S |\mathbf{F} \cdot \mathbf{n}| \, d\sigma && [\text{A property of integrals}] \\ &\leq \iint_S (1) \, d\sigma && [|\mathbf{F} \cdot \mathbf{n}| \leq 1] \\ &= \text{Area of } S. \end{aligned}$$

22. Yes, the outward flux through the top is 5. The reason is this: Since  $\nabla \cdot \mathbf{F} = \nabla \cdot (x\mathbf{i} - 2y\mathbf{j} + (z+3)\mathbf{k}) = 1 - 2 + 1 = 0$ , the outward flux across the closed cubelike surface is 0 by the Divergence Theorem. The flux across the top is therefore the negative of the flux across the sides and base. Routine calculations show that the sum of these latter fluxes is  $-5$ . (The flux across the sides that lie in the  $xz$ -plane and the  $yz$ -plane are 0, while the flux across the  $xy$ -plane is  $-3$ .) Therefore the flux across the top is 5.

23. (a)  $\frac{\partial}{\partial x}(x) = 1, \frac{\partial}{\partial y}(y) = 1, \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow \text{Flux} = \iiint_D 3 \, dV = 3 \iiint_D dV = 3(\text{Volume of the solid})$

(b) If  $\mathbf{F}$  is orthogonal to  $\mathbf{n}$  at every point of  $S$ , then  $\mathbf{F} \cdot \mathbf{n} = 0$  everywhere  $\Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0$ . But the flux is  $3(\text{Volume of the solid}) \neq 0$ , so  $\mathbf{F}$  is not orthogonal to  $\mathbf{n}$  at every point.



24.  $\nabla \cdot \mathbf{F} = -2x - 4y - 6z + 12 \Rightarrow \text{Flux} = \int_0^a \int_0^b \int_0^1 (-2x - 4y - 6z + 12) dz dy dx = \int_0^a \int_0^b (-2x - 4y + 9) dy dx$   
 $= \int_0^a (-2xb - 2b^2 + 9b) dx = -a^2b - 2ab^2 + 9ab = ab(-a - 2b + 9) = f(a, b)$ ;  $\frac{\partial f}{\partial a} = -2ab - 2b^2 + 9b$  and  
 $\frac{\partial f}{\partial b} = -a^2 - 4ab + 9a$  so that  $\frac{\partial f}{\partial a} = 0$  and  $\frac{\partial f}{\partial b} = 0 \Rightarrow b(-2a - 2b + 9) = 0$  and  $a(-a - 4b + 9) = 0 \Rightarrow b = 0$  or  
 $-2a - 2b + 9 = 0$ , and  $a = 0$  or  $-a - 4b + 9 = 0$ . Now  $b = 0$  or  $a = 0 \Rightarrow \text{Flux} = 0$ ;  $-2a - 2b + 9 = 0$  and  
 $-a - 4b + 9 = 0 \Rightarrow 3a - 9 = 0 \Rightarrow a = 3 \Rightarrow b = \frac{3}{2}$  so that  $f(3, \frac{3}{2}) = \frac{27}{2}$  is the maximum flux.

25.  $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV = \iiint_D 3 dV \Rightarrow \frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D dV = \text{Volume of } D$

26.  $\mathbf{F} = \mathbf{C} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV = \iiint_D 0 dV = 0$

27. (a) From the Divergence Theorem,  $\iint_S \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \nabla f dV = \iiint_D \nabla^2 f dV = \iiint_D 0 dV = 0$

(b) From the Divergence Theorem,  $\iint_S f \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot f \nabla f dV$ . Now,

$$f \nabla f = \left(f \frac{\partial f}{\partial x}\right) \mathbf{i} + \left(f \frac{\partial f}{\partial y}\right) \mathbf{j} + \left(f \frac{\partial f}{\partial z}\right) \mathbf{k} \Rightarrow \nabla \cdot f \nabla f = \left[f \frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2\right] + \left[f \frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)^2\right] + \left[f \frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial f}{\partial z}\right)^2\right]$$

$$= f \nabla^2 f + |\nabla f|^2 = 0 + |\nabla f|^2 \text{ since } f \text{ is harmonic} \Rightarrow \iint_S f \nabla f \cdot \mathbf{n} d\sigma = \iiint_D |\nabla f|^2 dV, \text{ as claimed.}$$

28. From the Divergence Theorem,  $\iint_S \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \nabla f dV = \iiint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dV$ . Now,

$$f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \ln(x^2 + y^2 + z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2 + z^2}, \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2}, \frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2 + z^2}{(x^2 + y^2 + z^2)^2}, \frac{\partial^2 f}{\partial z^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2}, \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2} \Rightarrow \iint_S \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \frac{dV}{x^2 + y^2 + z^2} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \frac{\rho^2 \sin \phi}{\rho^2} d\rho d\phi d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} a \sin \phi d\phi d\theta = \int_0^{\pi/2} [-a \cos \phi]_0^{\pi/2} d\theta = \int_0^{\pi/2} a d\theta = \frac{\pi a}{2}$$

29.  $\iint_S f \nabla g \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot f \nabla g dV = \iiint_D \nabla \cdot \left(f \frac{\partial g}{\partial x} \mathbf{i} + f \frac{\partial g}{\partial y} \mathbf{j} + f \frac{\partial g}{\partial z} \mathbf{k}\right) dV$   
 $= \iiint_D \left(f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z} \frac{\partial g}{\partial z}\right) dV$   
 $= \iiint_D \left[f \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial g}{\partial z}\right)\right] dV = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g) dV$

30. By Exercise 29,  $\iint_S f \nabla g \cdot \mathbf{n} d\sigma = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g) dV$  and by interchanging the roles of  $f$  and  $g$ ,

$$\iint_S g \nabla f \cdot \mathbf{n} d\sigma = \iiint_D (g \nabla^2 f + \nabla g \cdot \nabla f) dV. \text{ Subtracting the second equation from the first yields:}$$

$$\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} d\sigma = \iiint_D (f \nabla^2 g - g \nabla^2 f) dV \text{ since } \nabla f \cdot \nabla g = \nabla g \cdot \nabla f.$$

31. (a) The integral  $\iiint_D p(t, x, y, z) dV$  represents the mass of the fluid at any time  $t$ . The equation says that

the instantaneous rate of change of mass is flux of the fluid through the surface  $S$  enclosing the region  $D$ :  
the mass decreases if the flux is outward (so the fluid flows out of  $D$ ), and increases if the flow is inward  
(interpreting  $\mathbf{n}$  as the outward pointing unit normal to the surface).

(b)  $\iiint_D \frac{\partial p}{\partial t} dV = \frac{d}{dt} \iiint_D p dV = - \iint_S p \mathbf{v} \cdot \mathbf{n} d\sigma = - \iiint_D \nabla \cdot p \mathbf{v} dV \Rightarrow \frac{\partial p}{\partial t} = - \nabla \cdot p \mathbf{v}$

Since the law is to hold for all regions  $D$ ,  $\nabla \cdot p \mathbf{v} + \frac{\partial p}{\partial t} = 0$ , as claimed

32. (a)  $\nabla T$  points in the direction of maximum change of the temperature, so if the solid is heating up at the point the temperature is greater in a region surrounding the point  $\Rightarrow \nabla T$  points away from the point  $\Rightarrow -\nabla T$  points toward the point  $\Rightarrow -\nabla T$  points in the direction the heat flows.
- (b) Assuming the Law of Conservation of Mass (Exercise 31) with  $-k \nabla T = \mathbf{p}\mathbf{v}$  and  $c\rho T = p$ , we have
- $$\frac{d}{dt} \int_D \int \int c\rho T \, dV = - \int_S \int -k \nabla T \cdot \mathbf{n} \, d\sigma \Rightarrow \text{the continuity equation, } \nabla \cdot (-k \nabla T) + \frac{\partial}{\partial t} (c\rho T) = 0$$
- $$\Rightarrow c\rho \frac{\partial T}{\partial t} = -\nabla \cdot (-k \nabla T) = k \nabla^2 T \Rightarrow \frac{\partial T}{\partial t} = \frac{k}{c\rho} \nabla^2 T = K \nabla^2 T, \text{ as claimed}$$

## CHAPTER 16 PRACTICE EXERCISES

1. Path 1:  $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t, y = t, z = t, 0 \leq t \leq 1 \Rightarrow f(g(t), h(t), k(t)) = 3 - 3t^2$  and  $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{3} dt \Rightarrow \int_C f(x, y, z) \, ds = \int_0^1 \sqrt{3} (3 - 3t^2) \, dt = 2\sqrt{3}$
- Path 2:  $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1 \Rightarrow x = t, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = 2t - 3t^2 + 3$  and  $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{2} dt \Rightarrow \int_{C_1} f(x, y, z) \, ds = \int_0^1 \sqrt{2} (2t - 3t^2 + 3) \, dt = 3\sqrt{2}$ ;
- $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, z = t \Rightarrow f(g(t), h(t), k(t)) = 2 - 2t$  and  $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$
- $$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_2} f(x, y, z) \, ds = \int_0^1 (2 - 2t) \, dt = 1$$
- $$\Rightarrow \int_C f(x, y, z) \, ds = \int_{C_1} f(x, y, z) \, ds + \int_{C_2} f(x, y, z) \, ds = 3\sqrt{2} + 1$$
2. Path 1:  $\mathbf{r}_1 = t\mathbf{i} \Rightarrow x = t, y = 0, z = 0 \Rightarrow f(g(t), h(t), k(t)) = t^2$  and  $\frac{dx}{dt} = 1, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$
- $$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_1} f(x, y, z) \, ds = \int_0^1 t^2 \, dt = \frac{1}{3}$$
- $\mathbf{r}_2 = \mathbf{i} + t\mathbf{j} \Rightarrow x = 1, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = 1 + t$  and  $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$
- $$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_2} f(x, y, z) \, ds = \int_0^1 (1 + t) \, dt = \frac{3}{2}$$
- $\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, z = t \Rightarrow f(g(t), h(t), k(t)) = 2 - t$  and  $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$
- $$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_3} f(x, y, z) \, ds = \int_0^1 (2 - t) \, dt = \frac{3}{2}$$
- $$\Rightarrow \int_{\text{Path 1}} f(x, y, z) \, ds = \int_{C_1} f(x, y, z) \, ds + \int_{C_2} f(x, y, z) \, ds + \int_{C_3} f(x, y, z) \, ds = \frac{10}{3}$$
- Path 2:  $\mathbf{r}_4 = t\mathbf{i} + t\mathbf{j} \Rightarrow x = t, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = t^2 + t$  and  $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$
- $$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{2} dt \Rightarrow \int_{C_4} f(x, y, z) \, ds = \int_0^1 \sqrt{2} (t^2 + t) \, dt = \frac{5}{6} \sqrt{2}$$
- $\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$  (see above)  $\Rightarrow \int_{C_3} f(x, y, z) \, ds = \frac{3}{2}$
- $$\Rightarrow \int_{\text{Path 2}} f(x, y, z) \, ds = \int_{C_4} f(x, y, z) \, ds + \int_{C_3} f(x, y, z) \, ds = \frac{5}{6} \sqrt{2} + \frac{3}{2} = \frac{5\sqrt{2} + 9}{6}$$
- Path 3:  $\mathbf{r}_5 = t\mathbf{k} \Rightarrow x = 0, y = 0, z = t, 0 \leq t \leq 1 \Rightarrow f(g(t), h(t), k(t)) = -t$  and  $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$
- $$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_5} f(x, y, z) \, ds = \int_0^1 -t \, dt = -\frac{1}{2}$$
- $\mathbf{r}_6 = t\mathbf{j} + \mathbf{k} \Rightarrow x = 0, y = t, z = 1, 0 \leq t \leq 1 \Rightarrow f(g(t), h(t), k(t)) = t - 1$  and  $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$
- $$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_6} f(x, y, z) \, ds = \int_0^1 (t - 1) \, dt = -\frac{1}{2}$$
- $\mathbf{r}_7 = t\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow x = t, y = 1, z = 1, 0 \leq t \leq 1 \Rightarrow f(g(t), h(t), k(t)) = t^2$  and  $\frac{dx}{dt} = 1, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_7} f(x, y, z) ds = \int_0^1 t^2 dt = \frac{1}{3}$$

$$\Rightarrow \int_{\text{Path } 3} f(x, y, z) ds = \int_{C_5} f(x, y, z) ds + \int_{C_6} f(x, y, z) ds + \int_{C_7} f(x, y, z) ds = -\frac{1}{2} - \frac{1}{2} + \frac{1}{3} = -\frac{2}{3}$$

3.  $\mathbf{r} = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k} \Rightarrow x = 0, y = a \cos t, z = a \sin t \Rightarrow f(g(t), h(t), k(t)) = \sqrt{a^2 \sin^2 t} = a |\sin t|$  and

$$\frac{dx}{dt} = 0, \frac{dy}{dt} = -a \sin t, \frac{dz}{dt} = a \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = a dt$$

$$\Rightarrow \int_C f(x, y, z) ds = \int_0^{2\pi} a^2 |\sin t| dt = \int_0^\pi a^2 \sin t dt + \int_\pi^{2\pi} -a^2 \sin t dt = 4a^2$$

4.  $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow x = \cos t + t \sin t, y = \sin t - t \cos t, z = 0$

$$\Rightarrow f(g(t), h(t), k(t)) = \sqrt{(\cos t + t \sin t)^2 + (\sin t - t \cos t)^2} = \sqrt{1 + t^2} \text{ and } \frac{dx}{dt} = -\sin t + \sin t + t \cos t$$

$$= t \cos t, \frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t, \frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

$$= \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} dt = |t| dt = t dt \text{ since } 0 \leq t \leq \sqrt{3} \Rightarrow \int_C f(x, y, z) ds = \int_0^{\sqrt{3}} t \sqrt{1 + t^2} dt = \frac{7}{3}$$

5.  $\frac{\partial P}{\partial y} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial M}{\partial y}$

$$\Rightarrow M dx + N dy + P dz \text{ is exact; } \frac{\partial f}{\partial x} = \frac{1}{\sqrt{x+y+z}} \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{\sqrt{x+y+z}} + \frac{\partial g}{\partial y}$$

$$= \frac{1}{\sqrt{x+y+z}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{1}{\sqrt{x+y+z}} + h'(z)$$

$$= \frac{1}{\sqrt{x+y+z}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + C \Rightarrow \int_{(-1,1,1)}^{(4,-3,0)} \frac{dx+dy+dz}{\sqrt{x+y+z}}$$

$$= f(4, -3, 0) - f(-1, 1, 1) = 2\sqrt{1} - 2\sqrt{1} = 0$$

6.  $\frac{\partial P}{\partial y} = -\frac{1}{2\sqrt{yz}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz \text{ is exact; } \frac{\partial f}{\partial x} = 1 \Rightarrow f(x, y, z)$

$$= x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -\sqrt{\frac{z}{y}} \Rightarrow g(y, z) = -2\sqrt{yz} + h(z) \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + h(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = -\sqrt{\frac{y}{z}} + h'(z) = -\sqrt{\frac{y}{z}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + C$$

$$\Rightarrow \int_{(1,1,1)}^{(10,3,3)} dx - \sqrt{\frac{z}{y}} dy - \sqrt{\frac{y}{z}} dz = f(10, 3, 3) - f(1, 1, 1) = (10 - 2 \cdot 3) - (1 - 2 \cdot 1) = 4 + 1 = 5$$

7.  $\frac{\partial M}{\partial z} = -y \cos z \neq y \cos z = \frac{\partial P}{\partial x} \Rightarrow \mathbf{F}$  is not conservative;  $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} - \mathbf{k}, 0 \leq t \leq 2\pi$

$$\Rightarrow d\mathbf{r} = (-2 \sin t)\mathbf{i} - (2 \cos t)\mathbf{j} \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} [ -(-2 \sin t)(\sin(-1))(-2 \sin t) + (2 \cos t)(\sin(-1))(-2 \cos t) ] dt$$

$$= 4 \sin(1) \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 8\pi \sin(1)$$

8.  $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 3x^2 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$  is conservative  $\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0$

9. Let  $M = 8x \sin y$  and  $N = -8y \cos x \Rightarrow \frac{\partial M}{\partial y} = 8x \cos y$  and  $\frac{\partial N}{\partial x} = 8y \sin x \Rightarrow \int_C 8x \sin y dx - 8y \cos x dy$

$$= \int_R \int (8y \sin x - 8x \cos y) dy dx = \int_0^{\pi/2} \int_0^{\pi/2} (8y \sin x - 8x \cos y) dy dx = \int_0^{\pi/2} (\pi^2 \sin x - 8x) dx = -\pi^2 + \pi^2 = 0$$

10. Let  $M = y^2$  and  $N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y$  and  $\frac{\partial N}{\partial x} = 2x \Rightarrow \int_C y^2 dx + x^2 dy = \int_R \int (2x - 2y) dx dy$

$$= \int_0^{2\pi} \int_0^2 (2r \cos \theta - 2r \sin \theta) r dr d\theta = \int_0^{2\pi} \frac{16}{3} (\cos \theta - \sin \theta) d\theta = 0$$

11. Let  $z = 1 - x - y \Rightarrow f_x(x, y) = -1$  and  $f_y(x, y) = -1 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3} \Rightarrow \text{Surface Area} = \iint_R \sqrt{3} \, dx \, dy$   
 $= \sqrt{3}(\text{Area of the circular region in the } xy\text{-plane}) = \pi\sqrt{3}$
12.  $\nabla f = -3\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{i} \Rightarrow |\nabla f| = \sqrt{9 + 4y^2 + 4z^2}$  and  $|\nabla f \cdot \mathbf{p}| = 3$   
 $\Rightarrow \text{Surface Area} = \iint_R \frac{\sqrt{9 + 4y^2 + 4z^2}}{3} \, dy \, dz = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{\sqrt{9 + 4r^2}}{3} r \, dr \, d\theta = \frac{1}{3} \int_0^{2\pi} \left( \frac{7}{4} \sqrt{21} - \frac{9}{4} \right) d\theta = \frac{\pi}{6} (7\sqrt{21} - 9)$
13.  $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2$  and  $|\nabla f \cdot \mathbf{p}| = |2z| = 2z$  since  $z \geq 0 \Rightarrow \text{Surface Area} = \iint_R \frac{2}{2z} \, dA = \iint_R \frac{1}{z} \, dA = \iint_R \frac{1}{\sqrt{1-x^2-y^2}} \, dx \, dy = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-r^2}} r \, dr \, d\theta$   
 $= \int_0^{2\pi} \left[ -\sqrt{1-r^2} \right]_0^{1/\sqrt{2}} d\theta = \int_0^{2\pi} \left( 1 - \frac{1}{\sqrt{2}} \right) d\theta = 2\pi \left( 1 - \frac{1}{\sqrt{2}} \right)$
14. (a)  $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 4$  and  $|\nabla f \cdot \mathbf{p}| = 2z$  since  $z \geq 0 \Rightarrow \text{Surface Area} = \iint_R \frac{4}{2z} \, dA = \iint_R \frac{2}{z} \, dA = 2 \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{2}{\sqrt{4-r^2}} r \, dr \, d\theta = 4\pi - 8$   
 (b)  $\mathbf{r} = 2 \cos \theta \Rightarrow d\mathbf{r} = -2 \sin \theta \, d\theta; ds^2 = r^2 \, d\theta^2 + dr^2$  (Arc length in polar coordinates)  
 $\Rightarrow ds^2 = (2 \cos \theta)^2 \, d\theta^2 + dr^2 = 4 \cos^2 \theta \, d\theta^2 + 4 \sin^2 \theta \, d\theta^2 = 4 \, d\theta^2 \Rightarrow ds = 2 \, d\theta$ ; the height of the cylinder is  $z = \sqrt{4-r^2} = \sqrt{4-4\cos^2\theta} = 2|\sin\theta| = 2\sin\theta$  if  $0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \text{Surface Area} = \int_{-\pi/2}^{\pi/2} h \, ds$   
 $= 2 \int_0^{\pi/2} (2 \sin \theta)(2 \, d\theta) = 8$
15.  $f(x, y, z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \nabla f = \left(\frac{1}{a}\right)\mathbf{i} + \left(\frac{1}{b}\right)\mathbf{j} + \left(\frac{1}{c}\right)\mathbf{k} \Rightarrow |\nabla f| = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$  and  $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = \frac{1}{c}$   
 since  $c > 0 \Rightarrow \text{Surface Area} = \iint_R \frac{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}{\left(\frac{1}{c}\right)} \, dA = c \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \iint_R \, dA = \frac{1}{2} abc \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$ ,  
 since the area of the triangular region  $R$  is  $\frac{1}{2} ab$ . To check this result, let  $\mathbf{v} = a\mathbf{i} + c\mathbf{k}$  and  $\mathbf{w} = -a\mathbf{i} + b\mathbf{j}$ ; the area can be found by computing  $\frac{1}{2} |\mathbf{v} \times \mathbf{w}|$ .
16. (a)  $\nabla f = 2y\mathbf{j} - \mathbf{k}, \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 1}$  and  $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2 + 1} \, dx \, dy$   
 $\Rightarrow \iint_S g(x, y, z) \, d\sigma = \iint_R \frac{yz}{\sqrt{4y^2 + 1}} \sqrt{4y^2 + 1} \, dx \, dy = \iint_R y(y^2 - 1) \, dx \, dy = \int_{-1}^1 \int_0^3 (y^3 - y) \, dx \, dy$   
 $= \int_{-1}^1 3(y^3 - y) \, dy = 3 \left[ \frac{y^4}{4} - \frac{y^2}{2} \right]_{-1}^1 = 0$   
 (b)  $\iint_S g(x, y, z) \, d\sigma = \iint_R \frac{z}{\sqrt{4y^2 + 1}} \sqrt{4y^2 + 1} \, dx \, dy = \int_{-1}^1 \int_0^3 (y^2 - 1) \, dx \, dy = \int_{-1}^1 3(y^2 - 1) \, dy$   
 $= 3 \left[ \frac{y^3}{3} - y \right]_{-1}^1 = -4$
17.  $\nabla f = 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = 2\sqrt{y^2 + z^2} = 10$  and  $|\nabla f \cdot \mathbf{p}| = 2z$  since  $z \geq 0$   
 $\Rightarrow d\sigma = \frac{10}{2z} \, dx \, dy = \frac{5}{z} \, dx \, dy = \iint_S g(x, y, z) \, d\sigma = \iint_R (x^4 y) (y^2 + z^2) \left(\frac{5}{z}\right) \, dx \, dy$   
 $= \iint_R (x^4 y) (25) \left(\frac{5}{\sqrt{25-y^2}}\right) \, dx \, dy = \int_0^4 \int_0^1 \frac{125y}{\sqrt{25-y^2}} x^4 \, dx \, dy = \int_0^4 \frac{25y}{\sqrt{25-y^2}} \, dy = 50$
18. Define the coordinate system so that the origin is at the center of the earth, the  $z$ -axis is the earth's axis (north is the positive  $z$  direction), and the  $xz$ -plane contains the earth's prime meridian. Let  $S$  denote the surface which is Wyoming so then  $S$  is part of the surface  $z = (R^2 - x^2 - y^2)^{1/2}$ . Let  $R_{xy}$  be the projection of  $S$  onto the  $xy$ -plane. The surface area of

$$\begin{aligned}
\text{Wyoming is } \iint_S 1 \, d\sigma &= \iint_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA = \iint_{R_{xy}} \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1} \, dA = \iint_{R_{xy}} \frac{R}{(R^2 - x^2 - y^2)^{1/2}} \, dA \\
&= \int_{\theta_1}^{\theta_2} \int_{R \sin 45^\circ}^{R \sin 49^\circ} R (R^2 - r^2)^{-1/2} r \, dr \, d\theta \text{ (where } \theta_1 \text{ and } \theta_2 \text{ are the radian equivalent to } 104^\circ 3' \text{ and } 111^\circ 3', \text{ respectively)} \\
&= \int_{\theta_1}^{\theta_2} -R (R^2 - r^2)^{1/2} \Big|_{R \sin 45^\circ}^{R \sin 49^\circ} = \int_{\theta_1}^{\theta_2} R (R^2 - R^2 \sin^2 45^\circ)^{1/2} - R (R^2 - R^2 \sin^2 49^\circ)^{1/2} \, d\theta \\
&= (\theta_2 - \theta_1) R^2 (\cos 45^\circ - \cos 49^\circ) = \frac{7\pi}{180} R^2 (\cos 45^\circ - \cos 49^\circ) = \frac{7\pi}{180} (3959)^2 (\cos 45^\circ - \cos 49^\circ) \approx 97,751 \text{ sq. mi.}
\end{aligned}$$

19. A possible parametrization is  $\mathbf{r}(\phi, \theta) = (6 \sin \phi \cos \theta)\mathbf{i} + (6 \sin \phi \sin \theta)\mathbf{j} + (6 \cos \phi)\mathbf{k}$  (spherical coordinates);  
 now  $\rho = 6$  and  $z = -3 \Rightarrow -3 = 6 \cos \phi \Rightarrow \cos \phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$  and  $z = 3\sqrt{3} \Rightarrow 3\sqrt{3} = 6 \cos \phi$   
 $\Rightarrow \cos \phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} \leq \phi \leq \frac{2\pi}{3}$ ; also  $0 \leq \theta \leq 2\pi$

20. A possible parametrization is  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - \left(\frac{r^2}{2}\right)\mathbf{k}$  (cylindrical coordinates);  
 now  $r = \sqrt{x^2 + y^2} \Rightarrow z = -\frac{r^2}{2}$  and  $-2 \leq z \leq 0 \Rightarrow -2 \leq -\frac{r^2}{2} \leq 0 \Rightarrow 4 \geq r^2 \geq 0 \Rightarrow 0 \leq r \leq 2$  since  $r \geq 0$ ;  
 also  $0 \leq \theta \leq 2\pi$

21. A possible parametrization is  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1 + r)\mathbf{k}$  (cylindrical coordinates);  
 now  $r = \sqrt{x^2 + y^2} \Rightarrow z = 1 + r$  and  $1 \leq z \leq 3 \Rightarrow 1 \leq 1 + r \leq 3 \Rightarrow 0 \leq r \leq 2$ ; also  $0 \leq \theta \leq 2\pi$

22. A possible parametrization is  $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \left(3 - x - \frac{y}{2}\right)\mathbf{k}$  for  $0 \leq x \leq 2$  and  $0 \leq y \leq 2$

23. Let  $x = u \cos v$  and  $z = u \sin v$ , where  $u = \sqrt{x^2 + z^2}$  and  $v$  is the angle in the  $xz$ -plane with the  $x$ -axis  
 $\Rightarrow \mathbf{r}(u, v) = (u \cos v)\mathbf{i} + 2u^2\mathbf{j} + (u \sin v)\mathbf{k}$  is a possible parametrization;  $0 \leq y \leq 2 \Rightarrow 2u^2 \leq 2 \Rightarrow u^2 \leq 1$   
 $\Rightarrow 0 \leq u \leq 1$  since  $u \geq 0$ ; also, for just the upper half of the paraboloid,  $0 \leq v \leq \pi$

24. A possible parametrization is  $\left(\sqrt{10} \sin \phi \cos \theta\right)\mathbf{i} + \left(\sqrt{10} \sin \phi \sin \theta\right)\mathbf{j} + \left(\sqrt{10} \cos \phi\right)\mathbf{k}$ ,  $0 \leq \phi \leq \frac{\pi}{2}$  and  $0 \leq \theta \leq \frac{\pi}{2}$

25.  $\mathbf{r}_u = \mathbf{i} + \mathbf{j}$ ,  $\mathbf{r}_v = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{6}$   
 $\Rightarrow \text{Surface Area} = \iint_{R_{uv}} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \int_0^1 \int_0^1 \sqrt{6} \, du \, dv = \sqrt{6}$

26.  $\iint_S (xy - z^2) \, d\sigma = \int_0^1 \int_0^1 [(u + v)(u - v) - v^2] \sqrt{6} \, du \, dv = \sqrt{6} \int_0^1 \int_0^1 (u^2 - 2v^2) \, du \, dv$   
 $= \sqrt{6} \int_0^1 \left[ \frac{u^3}{3} - 2uv^2 \right]_0^1 dv = \sqrt{6} \int_0^1 \left( \frac{1}{3} - 2v^2 \right) dv = \sqrt{6} \left[ \frac{1}{3}v - \frac{2}{3}v^3 \right]_0^1 = -\frac{\sqrt{6}}{3} = -\sqrt{\frac{2}{3}}$

27.  $\mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$ ,  $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 1 \end{vmatrix}$   
 $= (\sin \theta)\mathbf{i} - (\cos \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\sin^2 \theta + \cos^2 \theta + r^2} = \sqrt{1 + r^2} \Rightarrow \text{Surface Area} = \int_{R_{r\theta}} |\mathbf{r}_r \times \mathbf{r}_\theta| \, dr \, d\theta$   
 $= \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{r}{2} \sqrt{1 + r^2} + \frac{1}{2} \ln \left( r + \sqrt{1 + r^2} \right) \right]_0^1 d\theta = \int_0^{2\pi} \left[ \frac{1}{2} \sqrt{2} + \frac{1}{2} \ln \left( 1 + \sqrt{2} \right) \right] d\theta$   
 $= \pi \left[ \sqrt{2} + \ln \left( 1 + \sqrt{2} \right) \right]$

$$28. \iint_S \sqrt{x^2 + y^2 + 1} \, d\sigma = \int_0^{2\pi} \int_0^1 \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + 1} \sqrt{1 + r^2} \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (1 + r^2) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[ r + \frac{r^3}{3} \right]_0^1 d\theta = \int_0^{2\pi} \frac{4}{3} \, d\theta = \frac{8}{3} \pi$$

$$29. \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

$$30. \frac{\partial P}{\partial y} = \frac{-3zy}{(x^2 + y^2 + z^2)^{-5/2}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{-3xz}{(x^2 + y^2 + z^2)^{-5/2}} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{-3xy}{(x^2 + y^2 + z^2)^{-5/2}} = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

$$31. \frac{\partial P}{\partial y} = 0 \neq ye^z = \frac{\partial N}{\partial z} \Rightarrow \text{Not Conservative}$$

$$32. \frac{\partial P}{\partial y} = \frac{x}{(x+yz)^2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{-y}{(x+yz)^2} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{-z}{(x+yz)^2} = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

$$33. \frac{\partial f}{\partial x} = 2 \Rightarrow f(x, y, z) = 2x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y + z \Rightarrow g(y, z) = y^2 + zy + h(z)$$

$$\Rightarrow f(x, y, z) = 2x + y^2 + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = y + h'(z) = y + 1 \Rightarrow h'(z) = 1 \Rightarrow h(z) = z + C$$

$$\Rightarrow f(x, y, z) = 2x + y^2 + zy + z + C$$

$$34. \frac{\partial f}{\partial x} = z \cos xz \Rightarrow f(x, y, z) = \sin xz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = e^y \Rightarrow g(y, z) = e^y + h(z)$$

$$\Rightarrow f(x, y, z) = \sin xz + e^y + h(z) \Rightarrow \frac{\partial f}{\partial z} = x \cos xz + h'(z) = x \cos xz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$$

$$\Rightarrow f(x, y, z) = \sin xz + e^y + C$$

$$35. \text{Over Path 1: } \mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow x = t, y = t, z = t \text{ and } d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$$

$$\Rightarrow \mathbf{F} \cdot d\mathbf{r} = (3t^2 + 1) dt \Rightarrow \text{Work} = \int_0^1 (3t^2 + 1) dt = 2;$$

$$\text{Over Path 2: } \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1 \Rightarrow x = t, y = t, z = 0 \text{ and } d\mathbf{r}_1 = (\mathbf{i} + \mathbf{j}) dt \Rightarrow \mathbf{F}_1 = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$$

$$\Rightarrow \mathbf{F}_1 \cdot d\mathbf{r}_1 = (2t^2 + 1) dt \Rightarrow \text{Work}_1 = \int_0^1 (2t^2 + 1) dt = \frac{5}{3}; \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow x = 1, y = 1, z = t \text{ and}$$

$$d\mathbf{r}_2 = \mathbf{k} dt \Rightarrow \mathbf{F}_2 = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot d\mathbf{r}_2 = dt \Rightarrow \text{Work}_2 = \int_0^1 dt = 1 \Rightarrow \text{Work} = \text{Work}_1 + \text{Work}_2 = \frac{5}{3} + 1 = \frac{8}{3}$$

$$36. \text{Over Path 1: } \mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow x = t, y = t, z = t \text{ and } d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$$

$$\Rightarrow \mathbf{F} \cdot d\mathbf{r} = (3t^2 + 1) dt \Rightarrow \text{Work} = \int_0^1 (3t^2 + 1) dt = 2;$$

Over Path 2: Since  $f$  is conservative,  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  around any simple closed curve  $C$ . Thus consider

$$\int_{\text{curve}} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}, \text{ where } C_1 \text{ is the path from } (0, 0, 0) \text{ to } (1, 1, 0) \text{ to } (1, 1, 1) \text{ and } C_2 \text{ is the path}$$

$$\text{from } (1, 1, 1) \text{ to } (0, 0, 0). \text{ Now, from Path 1 above, } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -2 \Rightarrow 0 = \int_{\text{curve}} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + (-2)$$

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2$$

$$37. (a) \mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} \Rightarrow x = e^t \cos t, y = e^t \sin t \text{ from } (1, 0) \text{ to } (e^{2\pi}, 0) \Rightarrow 0 \leq t \leq 2\pi$$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} \text{ and } \mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}} = \frac{(e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}}{(e^{2t} \cos^2 t + e^{2t} \sin^2 t)^{3/2}}$$

$$= \left( \frac{\cos t}{e^{2t}} \right)\mathbf{i} + \left( \frac{\sin t}{e^{2t}} \right)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \left( \frac{\cos^2 t}{e^t} - \frac{\sin t \cos t}{e^t} + \frac{\sin^2 t}{e^t} + \frac{\sin t \cos t}{e^t} \right) = e^{-t}$$

$$\Rightarrow \text{Work} = \int_0^{2\pi} e^{-t} dt = 1 - e^{-2\pi}$$

$$(b) \mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}} \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{(x^2 + y^2)^{3/2}} \Rightarrow f(x, y, z) = -(x^2 + y^2)^{-1/2} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{y}{(x^2 + y^2)^{3/2}} + \frac{\partial g}{\partial y}$$

$$= \frac{y}{(x^2 + y^2)^{3/2}} \Rightarrow g(y, z) = C \Rightarrow f(x, y, z) = -(x^2 + y^2)^{-1/2} \text{ is a potential function for } \mathbf{F} \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= f(e^{2\pi}, 0) - f(1, 0) = 1 - e^{-2\pi}$$

38. (a)  $\mathbf{F} = \nabla (x^2ze^y) \Rightarrow \mathbf{F}$  is conservative  $\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  for any closed path  $C$

(b)  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(1,0,0)}^{(1,0,2\pi)} \nabla (x^2ze^y) \cdot d\mathbf{r} = (x^2ze^y)|_{(1,0,2\pi)} - (x^2ze^y)|_{(1,0,0)} = 2\pi - 0 = 2\pi$

39.  $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -y & 3z^2 \end{vmatrix} = -2y\mathbf{k}$ ; unit normal to the plane is  $\mathbf{n} = \frac{2\mathbf{i}+6\mathbf{j}-3\mathbf{k}}{\sqrt{4+36+9}} = \frac{2}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$

$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{6}{7}y$ ;  $\mathbf{p} = \mathbf{k}$  and  $f(x, y, z) = 2x + 6y - 3z \Rightarrow |\nabla f \cdot \mathbf{p}| = 3 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{7}{3} dA$

$\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_R \int \frac{6}{7}y d\sigma = \int_R \int \left(\frac{6}{7}y\right) \left(\frac{7}{3} dA\right) = \int_R \int 2y dA = \int_0^{2\pi} \int_0^1 2r \sin \theta r dr d\theta = \int_0^{2\pi} \frac{2}{3} \sin \theta d\theta = 0$

40.  $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y & x + y & 4y^2 - z \end{vmatrix} = 8y\mathbf{i}$ ; the circle lies in the plane  $f(x, y, z) = y + z = 0$  with unit normal

$\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_R \int \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \int_R \int 0 d\sigma = 0$

41. (a)  $\mathbf{r} = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow x = \sqrt{2}t, y = \sqrt{2}t, z = 4 - t^2 \Rightarrow \frac{dx}{dt} = \sqrt{2}, \frac{dy}{dt} = \sqrt{2}, \frac{dz}{dt} = -2t$   
 $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{4 + 4t^2} dt \Rightarrow M = \int_C \delta(x, y, z) ds = \int_0^1 3t\sqrt{4 + 4t^2} dt = \left[\frac{1}{4}(4 + 4t)^{3/2}\right]_0^1$   
 $= 4\sqrt{2} - 2$

(b)  $M = \int_C \delta(x, y, z) ds = \int_0^1 \sqrt{4 + 4t^2} dt = \left[t\sqrt{1 + t^2} + \ln(t + \sqrt{1 + t^2})\right]_0^1 = \sqrt{2} + \ln(1 + \sqrt{2})$

42.  $\mathbf{r} = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}, 0 \leq t \leq 2 \Rightarrow x = t, y = 2t, z = \frac{2}{3}t^{3/2} \Rightarrow \frac{dx}{dt} = 1, \frac{dy}{dt} = 2, \frac{dz}{dt} = t^{1/2}$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{t + 5} dt \Rightarrow M = \int_C \delta(x, y, z) ds = \int_0^2 3\sqrt{5 + t} \sqrt{t + 5} dt$

$= \int_0^2 3(t + 5) dt = 36$ ;  $M_{yz} = \int_C x\delta ds = \int_0^2 3t(t + 5) dt = 38$ ;  $M_{xz} = \int_C y\delta ds = \int_0^2 6t(t + 5) dt = 76$ ;

$M_{xy} = \int_C z\delta ds = \int_0^2 2t^{3/2}(t + 5) dt = \frac{144}{7}\sqrt{2} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{38}{36} = \frac{19}{18}, \bar{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}, \bar{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{144}{7}\sqrt{2}\right)}{36}$   
 $= \frac{4}{7}\sqrt{2}$

43.  $\mathbf{r} = t\mathbf{i} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\mathbf{j} + \left(\frac{t^2}{2}\right)\mathbf{k}, 0 \leq t \leq 2 \Rightarrow x = t, y = \frac{2\sqrt{2}}{3}t^{3/2}, z = \frac{t^2}{2} \Rightarrow \frac{dx}{dt} = 1, \frac{dy}{dt} = \sqrt{2}t^{1/2}, \frac{dz}{dt} = t$   
 $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{1 + 2t + t^2} dt = \sqrt{(t + 1)^2} dt = |t + 1| dt = (t + 1) dt$  on the domain given.

Then  $M = \int_C \delta ds = \int_0^2 \left(\frac{1}{t+1}\right)(t + 1) dt = \int_0^2 dt = 2$ ;  $M_{yz} = \int_C x\delta ds = \int_0^2 t\left(\frac{1}{t+1}\right)(t + 1) dt = \int_0^2 t dt = 2$ ;

$M_{xz} = \int_C y\delta ds = \int_0^2 \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\left(\frac{1}{t+1}\right)(t + 1) dt = \int_0^2 \frac{2\sqrt{2}}{3}t^{3/2} dt = \frac{32}{15}$ ;  $M_{xy} = \int_C z\delta ds$

$= \int_0^2 \left(\frac{t^2}{2}\right)\left(\frac{1}{t+1}\right)(t + 1) dt = \int_0^2 \frac{t^2}{2} dt = \frac{4}{3} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{2}{2} = 1; \bar{y} = \frac{M_{xz}}{M} = \frac{\left(\frac{32}{15}\right)}{2} = \frac{16}{15}; \bar{z} = \frac{M_{xy}}{M}$

$= \frac{\left(\frac{4}{3}\right)}{2} = \frac{2}{3}$ ;  $I_x = \int_C (y^2 + z^2)\delta ds = \int_0^2 \left(\frac{8}{9}t^3 + \frac{t^4}{4}\right) dt = \frac{232}{45}$ ;  $I_y = \int_C (x^2 + z^2)\delta ds = \int_0^2 \left(t^2 + \frac{t^4}{4}\right) dt = \frac{64}{15}$ ;

$I_z = \int_C (y^2 + x^2)\delta ds = \int_0^2 \left(t^2 + \frac{8}{9}t^3\right) dt = \frac{56}{9}$

44.  $\bar{z} = 0$  because the arch is in the  $xy$ -plane, and  $\bar{x} = 0$  because the mass is distributed symmetrically with respect

to the  $y$ -axis;  $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, 0 \leq t \leq \pi \Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

$= \sqrt{(-a \sin t)^2 + (a \cos t)^2} dt = a dt$ , since  $a \geq 0$ ;  $M = \int_C \delta ds = \int_C (2a - y) ds = \int_0^\pi (2a - a \sin t) a dt$

$$= 2a^2\pi - 2a^2; M_{xz} = \int_C y \delta \, dt = \int_C y(2a - y) \, ds = \int_0^\pi (a \sin t)(2a - a \sin t) \, dt = \int_0^\pi (2a^2 \sin t - a^2 \sin^2 t) \, dt$$

$$= \left[ -2a^2 \cos t - a^2 \left( \frac{t}{2} - \frac{\sin 2t}{4} \right) \right]_0^\pi = 4a^2 - \frac{a^2\pi}{2} \Rightarrow \bar{y} = \frac{(4a^2 - \frac{a^2\pi}{2})}{2a^2\pi - 2a^2} = \frac{8 - \pi}{4\pi - 4} \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = (0, \frac{8 - \pi}{4\pi - 4}, 0)$$

45.  $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}, 0 \leq t \leq \ln 2 \Rightarrow x = e^t \cos t, y = e^t \sin t, z = e^t \Rightarrow \frac{dx}{dt} = (e^t \cos t - e^t \sin t),$

$$\frac{dy}{dt} = (e^t \sin t + e^t \cos t), \frac{dz}{dt} = e^t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2} \, dt = \sqrt{3e^{2t}} \, dt = \sqrt{3} e^t \, dt; M = \int_C \delta \, ds = \int_0^{\ln 2} \sqrt{3} e^t \, dt$$

$$= \sqrt{3}; M_{xy} = \int_C z \delta \, ds = \int_0^{\ln 2} (\sqrt{3} e^t) (e^t) \, dt = \int_0^{\ln 2} \sqrt{3} e^{2t} \, dt = \frac{3\sqrt{3}}{2} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{3\sqrt{3}}{2}\right)}{\sqrt{3}} = \frac{3}{2};$$

$$I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^{\ln 2} (e^{2t} \cos^2 t + e^{2t} \sin^2 t) (\sqrt{3} e^t) \, dt = \int_0^{\ln 2} \sqrt{3} e^{3t} \, dt = \frac{7\sqrt{3}}{3}$$

46.  $\mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 3t\mathbf{k}, 0 \leq t \leq 2\pi \Rightarrow x = 2 \sin t, y = 2 \cos t, z = 3t \Rightarrow \frac{dx}{dt} = 2 \cos t, \frac{dy}{dt} = -2 \sin t,$

$$\frac{dz}{dt} = 3 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt = \sqrt{4 + 9} \, dt = \sqrt{13} \, dt; M = \int_C \delta \, ds = \int_0^{2\pi} \delta \sqrt{13} \, dt = 2\pi\delta\sqrt{13};$$

$$M_{xy} = \int_C z \delta \, ds = \int_0^{2\pi} (3t) (\delta \sqrt{13}) \, dt = 6\delta\pi^2\sqrt{13}; M_{yz} = \int_C x \delta \, ds = \int_0^{2\pi} (2 \sin t) (\delta \sqrt{13}) \, dt = 0;$$

$$M_{xz} = \int_C y \delta \, ds = \int_0^{2\pi} (2 \cos t) (\delta \sqrt{13}) \, dt = 0 \Rightarrow \bar{x} = \bar{y} = 0 \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{6\delta\pi^2\sqrt{13}}{2\delta\pi\sqrt{13}} = 3\pi \Rightarrow (0, 0, 3\pi) \text{ is the center of mass}$$

47. Because of symmetry  $\bar{x} = \bar{y} = 0$ . Let  $f(x, y, z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$$\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10 \text{ and } \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z, \text{ since } z \geq 0 \Rightarrow M = \int_R \delta(x, y, z) \, d\sigma$$

$$= \int_R z \left(\frac{10}{2z}\right) \, dA = \int_R 5 \, dA = 5(\text{Area of the circular region}) = 80\pi; M_{xy} = \int_R z \delta \, d\sigma = \int_R 5z \, dA$$

$$= \int_R 5\sqrt{25 - x^2 - y^2} \, dx \, dy = \int_0^{2\pi} \int_0^4 (5\sqrt{25 - r^2}) \, r \, dr \, d\theta = \int_0^{2\pi} \frac{490}{3} \, d\theta = \frac{980}{3} \pi \Rightarrow \bar{z} = \frac{\left(\frac{980}{3}\pi\right)}{80\pi} = \frac{49}{12}$$

$$\Rightarrow (\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{49}{12}); I_z = \int_R (x^2 + y^2) \delta \, d\sigma = \int_R 5(x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \int_0^4 5r^3 \, dr \, d\theta = \int_0^{2\pi} 320 \, d\theta = 640\pi$$

48. On the face  $z = 1$ :  $g(x, y, z) = z = 1$  and  $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$  and  $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$$\Rightarrow I = \int_R (x^2 + y^2) \, dA = 2 \int_0^{\pi/4} \int_0^{\sec \theta} r^3 \, dr \, d\theta = \frac{2}{3}; \text{ On the face } z = 0: g(x, y, z) = z = 0 \Rightarrow \nabla g = \mathbf{k} \text{ and } \mathbf{p} = \mathbf{k}$$

$$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \int_R (x^2 + y^2) \, dA = \frac{2}{3}; \text{ On the face } y = 0: g(x, y, z) = y = 0$$

$$\Rightarrow \nabla g = \mathbf{j} \text{ and } \mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \int_R (x^2 + 0) \, dA = \int_0^1 \int_0^1 x^2 \, dx \, dz = \frac{1}{3};$$

On the face  $y = 1$ :  $g(x, y, z) = y = 1 \Rightarrow \nabla g = \mathbf{j}$  and  $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$$\Rightarrow I = \int_R (x^2 + 1^2) \, dA = \int_0^1 \int_0^1 (x^2 + 1) \, dx \, dz = \frac{4}{3}; \text{ On the face } x = 1: g(x, y, z) = x = 1 \Rightarrow \nabla g = \mathbf{i} \text{ and } \mathbf{p} = \mathbf{i}$$

$$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \int_R (1^2 + y^2) \, dA = \int_0^1 \int_0^1 (1 + y^2) \, dy \, dz = \frac{4}{3}; \text{ On the face}$$

$x = 0: g(x, y, z) = x = 0 \Rightarrow \nabla g = \mathbf{i}$  and  $\mathbf{p} = \mathbf{i} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$$\Rightarrow I = \int_R (0^2 + y^2) \, dA = \int_0^1 \int_0^1 y^2 \, dy \, dz = \frac{1}{3} \Rightarrow I_z = \frac{2}{3} + \frac{2}{3} + \frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{1}{3} = \frac{14}{3}$$



$$\begin{aligned}
49. \quad M &= 2xy + x \text{ and } N = xy - y \Rightarrow \frac{\partial M}{\partial x} = 2y + 1, \frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = y, \frac{\partial N}{\partial y} = x - 1 \Rightarrow \text{Flux} = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \\
&= \iint_R (2y + 1 + x - 1) dy dx = \int_0^1 \int_0^1 (2y + x) dy dx = \frac{3}{2}; \text{Circ} = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\
&= \iint_R (y - 2x) dy dx = \int_0^1 \int_0^1 (y - 2x) dy dx = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
50. \quad M &= y - 6x^2 \text{ and } N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = -12x, \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 1, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \\
&= \iint_R (-12x + 2y) dx dy = \int_0^1 \int_y^1 (-12x + 2y) dx dy = \int_0^1 (4y^2 + 2y - 6) dy = -\frac{11}{3}; \\
\text{Circ} &= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R (1 - 1) dx dy = 0
\end{aligned}$$

$$\begin{aligned}
51. \quad M &= -\frac{\cos y}{x} \text{ and } N = \ln x \sin y \Rightarrow \frac{\partial M}{\partial y} = \frac{\sin y}{x} \text{ and } \frac{\partial N}{\partial x} = \frac{\sin y}{x} \Rightarrow \oint_C \ln x \sin y dy - \frac{\cos y}{x} dx \\
&= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R \left( \frac{\sin y}{x} - \frac{\sin y}{x} \right) dx dy = 0
\end{aligned}$$

$$\begin{aligned}
52. \quad (a) \quad \text{Let } M &= x \text{ and } N = y \Rightarrow \frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = 1 \Rightarrow \text{Flux} = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \\
&= \iint_R (1 + 1) dx dy = 2 \iint_R dx dy = 2(\text{Area of the region})
\end{aligned}$$

(b) Let  $C$  be a closed curve to which Green's Theorem applies and let  $\mathbf{n}$  be the unit normal vector to  $C$ . Let  $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$  and assume  $\mathbf{F}$  is orthogonal to  $\mathbf{n}$  at every point of  $C$ . Then the flux density of  $\mathbf{F}$  at every point of  $C$  is 0 since  $\mathbf{F} \cdot \mathbf{n} = 0$  at every point of  $C \Rightarrow \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$  at every point of  $C$

$$\Rightarrow \text{Flux} = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R 0 dx dy = 0.$$

But part (a) above states that the flux is  $2(\text{Area of the region}) \Rightarrow$  the area of the region would be 0  $\Rightarrow$  contradiction. Therefore,  $\mathbf{F}$  cannot be orthogonal to  $\mathbf{n}$  at every point of  $C$ .

$$\begin{aligned}
53. \quad \frac{\partial}{\partial x} (2xy) &= 2y, \frac{\partial}{\partial y} (2yz) = 2z, \frac{\partial}{\partial z} (2xz) = 2x \Rightarrow \nabla \cdot \mathbf{F} = 2y + 2z + 2x \Rightarrow \text{Flux} = \iiint_D (2x + 2y + 2z) dV \\
&= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) dx dy dz = \int_0^1 \int_0^1 (1 + 2y + 2z) dy dz = \int_0^1 (2 + 2z) dz = 3
\end{aligned}$$

$$\begin{aligned}
54. \quad \frac{\partial}{\partial x} (xz) &= z, \frac{\partial}{\partial y} (yz) = z, \frac{\partial}{\partial z} (1) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 2z \Rightarrow \text{Flux} = \iiint_D 2z r dr d\theta dz \\
&= \int_0^{2\pi} \int_0^4 \int_3^{\sqrt{25-r^2}} 2z dz r dr d\theta = \int_0^{2\pi} \int_0^4 r(16 - r^2) dr d\theta = \int_0^{2\pi} 64 d\theta = 128\pi
\end{aligned}$$

$$\begin{aligned}
55. \quad \frac{\partial}{\partial x} (-2x) &= -2, \frac{\partial}{\partial y} (-3y) = -3, \frac{\partial}{\partial z} (z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = -4; x^2 + y^2 + z^2 = 2 \text{ and } x^2 + y^2 = z \Rightarrow z = 1 \\
\Rightarrow x^2 + y^2 &= 1 \Rightarrow \text{Flux} = \iiint_D -4 dV = -4 \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz r dr d\theta = -4 \int_0^{2\pi} \int_0^1 (r\sqrt{2-r^2} - r^3) dr d\theta \\
&= -4 \int_0^{2\pi} \left( -\frac{7}{12} + \frac{2}{3} \sqrt{2} \right) d\theta = \frac{2}{3} \pi (7 - 8\sqrt{2})
\end{aligned}$$

$$\begin{aligned}
56. \quad \frac{\partial}{\partial x} (6x + y) &= 6, \frac{\partial}{\partial y} (-x - z) = 0, \frac{\partial}{\partial z} (4yz) = 4y \Rightarrow \nabla \cdot \mathbf{F} = 6 + 4y; z = \sqrt{x^2 + y^2} = r \\
\Rightarrow \text{Flux} &= \iiint_D (6 + 4y) dV = \int_0^{\pi/2} \int_0^1 \int_0^r (6 + 4r \sin \theta) dz r dr d\theta = \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3 \sin \theta) dr d\theta \\
&= \int_0^{\pi/2} (2 + \sin \theta) d\theta = \pi + 1
\end{aligned}$$

$$57. \mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow \text{Flux} = \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_D \nabla \cdot \mathbf{F} \, dV = 0$$

$$58. \mathbf{F} = 3xz^2\mathbf{i} + y\mathbf{j} - z^3\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3z^2 + 1 - 3z^2 = 1 \Rightarrow \text{Flux} = \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_D \nabla \cdot \mathbf{F} \, dV \\ = \int_0^4 \int_0^{\sqrt{16-x^2}/2} \int_0^{y/2} 1 \, dz \, dy \, dx = \int_0^4 \left( \frac{16-x^2}{16} \right) dx = \left[ x - \frac{x^3}{48} \right]_0^4 = \frac{8}{3}$$

$$59. \mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = y^2 + x^2 + 0 \Rightarrow \text{Flux} = \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_D \nabla \cdot \mathbf{F} \, dV \\ = \int \int_D (x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 2r^3 \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2} \, d\theta = \pi$$

$$60. (a) \mathbf{F} = (3z + 1)\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow \text{Flux across the hemisphere} = \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_D \nabla \cdot \mathbf{F} \, dV = \int_D 3 \, dV \\ = 3 \left( \frac{1}{2} \right) \left( \frac{4}{3} \pi a^3 \right) = 2\pi a^3$$

$$(b) f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4a^2} = 2a \text{ since} \\ a \geq 0 \Rightarrow \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2a} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = (3z + 1) \left( \frac{z}{a} \right); \mathbf{p} = \mathbf{k} \Rightarrow \nabla f \cdot \mathbf{p} = \nabla f \cdot \mathbf{k} = 2z \\ \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \text{ since } z \geq 0 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{2a}{2z} dA = \frac{a}{z} dA \Rightarrow \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_{R_{xy}} (3z + 1) \left( \frac{z}{a} \right) \left( \frac{a}{z} \right) dA \\ = \int_{R_{xy}} (3z + 1) \, dx \, dy = \int_{R_{xy}} (3\sqrt{a^2 - x^2 - y^2} + 1) \, dx \, dy = \int_0^{2\pi} \int_0^a (3\sqrt{a^2 - r^2} + 1) \, r \, dr \, d\theta \\ = \int_0^{2\pi} \left( \frac{a^2}{2} + a^3 \right) d\theta = \pi a^2 + 2\pi a^3, \text{ which is the flux across the hemisphere. Across the base we find} \\ \mathbf{F} = [3(0) + 1]\mathbf{k} = \mathbf{k} \text{ since } z = 0 \text{ in the } xy\text{-plane} \Rightarrow \mathbf{n} = -\mathbf{k} \text{ (outward normal)} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -1 \Rightarrow \text{Flux across the} \\ \text{base} = \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_{R_{xy}} -1 \, dx \, dy = -\pi a^2. \text{ Therefore, the total flux across the closed surface is} \\ (\pi a^2 + 2\pi a^3) - \pi a^2 = 2\pi a^3.$$

## CHAPTER 16 ADDITIONAL AND ADVANCED EXERCISES

- $dx = (-2 \sin t + 2 \sin 2t) \, dt$  and  $dy = (2 \cos t - 2 \cos 2t) \, dt$ ; Area  $= \frac{1}{2} \oint_C x \, dy - y \, dx$   
 $= \frac{1}{2} \int_0^{2\pi} [(2 \cos t - \cos 2t)(2 \cos t - 2 \cos 2t) - (2 \sin t - \sin 2t)(-2 \sin t + 2 \sin 2t)] \, dt$   
 $= \frac{1}{2} \int_0^{2\pi} [6 - (6 \cos t \cos 2t + 6 \sin t \sin 2t)] \, dt = \frac{1}{2} \int_0^{2\pi} (6 - 6 \cos t) \, dt = 6\pi$
- $dx = (-2 \sin t - 2 \sin 2t) \, dt$  and  $dy = (2 \cos t - 2 \cos 2t) \, dt$ ; Area  $= \frac{1}{2} \oint_C x \, dy - y \, dx$   
 $= \frac{1}{2} \int_0^{2\pi} [(2 \cos t + \cos 2t)(2 \cos t - 2 \cos 2t) - (2 \sin t - \sin 2t)(-2 \sin t - 2 \sin 2t)] \, dt$   
 $= \frac{1}{2} \int_0^{2\pi} [2 - 2(\cos t \cos 2t - \sin t \sin 2t)] \, dt = \frac{1}{2} \int_0^{2\pi} (2 - 2 \cos 3t) \, dt = \frac{1}{2} [2t - \frac{2}{3} \sin 3t]_0^{2\pi} = 2\pi$
- $dx = \cos 2t \, dt$  and  $dy = \cos t \, dt$ ; Area  $= \frac{1}{2} \oint_C x \, dy - y \, dx = \frac{1}{2} \int_0^\pi (\frac{1}{2} \sin 2t \cos t - \sin t \cos 2t) \, dt$   
 $= \frac{1}{2} \int_0^\pi [\sin t \cos^2 t - (\sin t)(2 \cos^2 t - 1)] \, dt = \frac{1}{2} \int_0^\pi (-\sin t \cos^2 t + \sin t) \, dt = \frac{1}{2} [\frac{1}{3} \cos^3 t - \cos t]_0^\pi = -\frac{1}{3} + 1 = \frac{2}{3}$
- $dx = (-2a \sin t - 2a \cos 2t) \, dt$  and  $dy = (b \cos t) \, dt$ ; Area  $= \frac{1}{2} \oint_C x \, dy - y \, dx$   
 $= \frac{1}{2} \int_0^{2\pi} [(2ab \cos^2 t - ab \cos t \sin 2t) - (-2ab \sin^2 t - 2ab \sin t \cos 2t)] \, dt$

$$= \frac{1}{2} \int_0^{2\pi} [2ab - 2ab \cos^2 t \sin t + 2ab(\sin t)(2 \cos^2 t - 1)] dt = \frac{1}{2} \int_0^{2\pi} (2ab + 2ab \cos^2 t \sin t - 2ab \sin t) dt$$

$$= \frac{1}{2} [2abt - \frac{2}{3} ab \cos^3 t + 2ab \cos t]_0^{2\pi} = 2\pi ab$$

5. (a)  $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$  is  $\mathbf{0}$  only at the point  $(0, 0, 0)$ , and  $\text{curl } \mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$  is never  $\mathbf{0}$ .  
 (b)  $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{k}$  is  $\mathbf{0}$  only on the line  $x = t, y = 0, z = 0$  and  $\text{curl } \mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j}$  is never  $\mathbf{0}$ .  
 (c)  $\mathbf{F}(x, y, z) = z\mathbf{i}$  is  $\mathbf{0}$  only when  $z = 0$  (the  $xy$ -plane) and  $\text{curl } \mathbf{F}(x, y, z) = \mathbf{j}$  is never  $\mathbf{0}$ .

6.  $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$  and  $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{R}$ , so  $\mathbf{F}$  is parallel to  $\mathbf{n}$  when  $yz^2 = \frac{cx}{R}$ ,  $xz^2 = \frac{cy}{R}$ , and  $2xyz = \frac{cz}{R} \Rightarrow \frac{yz^2}{x} = \frac{xz^2}{y} = 2xy \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$  and  $z^2 = \pm \frac{c}{R} = 2x^2 \Rightarrow z = \pm \sqrt{2}x$ . Also,  $x^2 + y^2 + z^2 = R^2 \Rightarrow x^2 + x^2 + 2x^2 = R^2 \Rightarrow 4x^2 = R^2 \Rightarrow x = \pm \frac{R}{2}$ . Thus the points are:  $(\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2})$ ,  $(\frac{R}{2}, \frac{R}{2}, -\frac{\sqrt{2}R}{2})$ ,  $(-\frac{R}{2}, -\frac{R}{2}, \frac{\sqrt{2}R}{2})$ ,  $(-\frac{R}{2}, -\frac{R}{2}, -\frac{\sqrt{2}R}{2})$ ,  $(\frac{R}{2}, -\frac{R}{2}, \frac{\sqrt{2}R}{2})$ ,  $(\frac{R}{2}, -\frac{R}{2}, -\frac{\sqrt{2}R}{2})$ ,  $(-\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2})$ ,  $(-\frac{R}{2}, \frac{R}{2}, -\frac{\sqrt{2}R}{2})$ .

7. Set up the coordinate system so that  $(a, b, c) = (0, R, 0) \Rightarrow \delta(x, y, z) = \sqrt{x^2 + (y - R)^2 + z^2}$   
 $= \sqrt{x^2 + y^2 + z^2 - 2Ry + R^2} = \sqrt{2R^2 - 2Ry}$ ; let  $f(x, y, z) = x^2 + y^2 + z^2 - R^2$  and  $\mathbf{p} = \mathbf{i}$   
 $\Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2R \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{i}|} dz dy = \frac{2R}{2x} dz dy$   
 $\Rightarrow \text{Mass} = \iint_S \delta(x, y, z) d\sigma = \iint_{R_{yz}} \sqrt{2R^2 - 2Ry} \left(\frac{R}{x}\right) dz dy = R \iint_{R_{yz}} \frac{\sqrt{2R^2 - 2Ry}}{\sqrt{R^2 - y^2 - z^2}} dz dy$   
 $= 4R \int_{-R}^R \int_0^{\sqrt{R^2 - y^2}} \frac{\sqrt{2R^2 - 2Ry}}{\sqrt{R^2 - y^2 - z^2}} dz dy = 4R \int_{-R}^R \sqrt{2R^2 - 2Ry} \sin^{-1} \left( \frac{z}{\sqrt{R^2 - y^2}} \right) \Big|_0^{\sqrt{R^2 - y^2}} dy$   
 $= 2\pi R \int_{-R}^R \sqrt{2R^2 - 2Ry} dy = 2\pi R \left( \frac{-1}{3R} \right) (2R^2 - 2Ry)^{3/2} \Big|_{-R}^R = \frac{16\pi R^3}{3}$

8.  $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \theta\mathbf{k}$ ,  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 1 \end{vmatrix}$   
 $= (\sin \theta)\mathbf{i} - (\cos \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{1 + r^2}$ ;  $\delta = 2\sqrt{x^2 + y^2} = 2\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = 2r$   
 $\Rightarrow \text{Mass} = \iint_S \delta(x, y, z) d\sigma = \int_0^{2\pi} \int_0^1 2r\sqrt{1 + r^2} dr d\theta = \int_0^{2\pi} \left[ \frac{2}{3} (1 + r^2)^{3/2} \right]_0^1 d\theta = \int_0^{2\pi} \frac{2}{3} (2\sqrt{2} - 1) d\theta$   
 $= \frac{4\pi}{3} (2\sqrt{2} - 1)$

9.  $M = x^2 + 4xy$  and  $N = -6y \Rightarrow \frac{\partial M}{\partial x} = 2x + 4y$  and  $\frac{\partial N}{\partial x} = -6 \Rightarrow \text{Flux} = \int_0^b \int_0^a (2x + 4y - 6) dx dy$   
 $= \int_0^b (a^2 + 4ay - 6a) dy = a^2b + 2ab^2 - 6ab$ . We want to minimize  $f(a, b) = a^2b + 2ab^2 - 6ab = ab(a + 2b - 6)$ .  
 Thus,  $f_a(a, b) = 2ab + 2b^2 - 6b = 0$  and  $f_b(a, b) = a^2 + 4ab - 6a = 0 \Rightarrow b(2a + 2b - 6) = 0 \Rightarrow b = 0$  or  $b = -a + 3$ .  
 Now  $b = 0 \Rightarrow a^2 - 6a = 0 \Rightarrow a = 0$  or  $a = 6 \Rightarrow (0, 0)$  and  $(6, 0)$  are critical points. On the other hand,  $b = -a + 3$   
 $\Rightarrow a^2 + 4a(-a + 3) - 6a = 0 \Rightarrow -3a^2 + 6a = 0 \Rightarrow a = 0$  or  $a = 2 \Rightarrow (0, 3)$  and  $(2, 1)$  are also critical points. The flux at  $(0, 0) = 0$ , the flux at  $(6, 0) = 0$ , the flux at  $(0, 3) = 0$  and the flux at  $(2, 1) = -4$ . Therefore, the flux is minimized at  $(2, 1)$  with value  $-4$ .

10. A plane through the origin has equation  $ax + by + cz = 0$ . Consider first the case when  $c \neq 0$ . Assume the plane is given by  $z = ax + by$  and let  $f(x, y, z) = x^2 + y^2 + z^2 = 4$ . Let  $C$  denote the circle of intersection of the plane with the sphere. By Stokes's Theorem,  $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$ , where  $\mathbf{n}$  is a unit normal to the plane. Let

$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (ax + by)\mathbf{k}$  be a parametrization of the surface. Then  $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & a \\ 0 & 1 & b \end{vmatrix} = -a\mathbf{i} - b\mathbf{j} + \mathbf{k}$

$$\Rightarrow d\sigma = |\mathbf{r}_x \times \mathbf{r}_y| dx dy = \sqrt{a^2 + b^2 + 1} dx dy. \text{ Also, } \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ and } \mathbf{n} = \frac{a\mathbf{i} + b\mathbf{j} - \mathbf{k}}{\sqrt{a^2 + b^2 + 1}}$$

$$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{R_{xy}} \frac{a+b-1}{\sqrt{a^2+b^2+1}} \sqrt{a^2+b^2+1} dx dy = \iint_{R_{xy}} (a+b-1) dx dy = (a+b-1) \iint_{R_{xy}} dx dy. \text{ Now}$$

$$x^2 + y^2 + (ax + by)^2 = 4 \Rightarrow \left(\frac{a^2+1}{4}\right)x^2 + \left(\frac{b^2+1}{4}\right)y^2 + \left(\frac{ab}{2}\right)xy = 1 \Rightarrow \text{the region } R_{xy} \text{ is the interior of the ellipse}$$

$Ax^2 + Bxy + Cy^2 = 1$  in the  $xy$ -plane, where  $A = \frac{a^2+1}{4}$ ,  $B = \frac{ab}{2}$ , and  $C = \frac{b^2+1}{4}$ . The area of the ellipse is

$$\frac{2\pi}{\sqrt{4AC - B^2}} = \frac{4\pi}{\sqrt{a^2 + b^2 + 1}} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = h(a, b) = \frac{4\pi(a+b-1)}{\sqrt{a^2 + b^2 + 1}}. \text{ Thus we optimize } H(a, b) = \frac{(a+b-1)^2}{a^2 + b^2 + 1}:$$

$$\frac{\partial H}{\partial a} = \frac{2(a+b-1)(b^2+1+a-ab)}{(a^2+b^2+1)^2} = 0 \text{ and } \frac{\partial H}{\partial b} = \frac{2(a+b-1)(a^2+1+b-ab)}{(a^2+b^2+1)^2} = 0 \Rightarrow a+b-1=0, \text{ or } b^2+1+a-ab=0$$

$$\text{and } a^2+1+b-ab=0 \Rightarrow a+b-1=0, \text{ or } a^2-b^2+(b-a)=0 \Rightarrow a+b-1=0, \text{ or } (a-b)(a+b-1)=0$$

$$\Rightarrow a+b-1=0 \text{ or } a=b. \text{ The critical values } a+b-1=0 \text{ give a saddle. If } a=b, \text{ then } 0=b^2+1+a-ab$$

$$\Rightarrow a^2+1+a-a^2=0 \Rightarrow a=-1 \Rightarrow b=-1. \text{ Thus, the point } (a, b) = (-1, -1) \text{ gives a local extremum for } \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$\Rightarrow z = -x - y \Rightarrow x + y + z = 0 \text{ is the desired plane, if } c \neq 0.$$

**Note:** Since  $h(-1, -1)$  is negative, the circulation about  $\mathbf{n}$  is clockwise, so  $-\mathbf{n}$  is the correct pointing normal for the counterclockwise circulation. Thus  $\iint_S \nabla \times \mathbf{F} \cdot (-\mathbf{n}) d\sigma$  actually gives the maximum circulation.

If  $c = 0$ , one can see that the corresponding problem is equivalent to the calculation above when  $b = 0$ , which does not lead to a local extreme.

11. (a) Partition the string into small pieces. Let  $\Delta_i s$  be the length of the  $i^{\text{th}}$  piece. Let  $(x_i, y_i)$  be a point in the  $i^{\text{th}}$  piece. The work done by gravity in moving the  $i^{\text{th}}$  piece to the  $x$ -axis is approximately

$$W_i = (gx_i y_i \Delta_i s) y_i \text{ where } x_i y_i \Delta_i s \text{ is approximately the mass of the } i^{\text{th}} \text{ piece. The total work done by}$$

$$\text{gravity in moving the string to the } x\text{-axis is } \sum_i W_i = \sum_i gx_i y_i^2 \Delta_i s \Rightarrow \text{Work} = \int_C gxy^2 ds$$

$$(b) \text{Work} = \int_C gxy^2 ds = \int_0^{\pi/2} g(2 \cos t)(4 \sin^2 t) \sqrt{4 \sin^2 t + 4 \cos^2 t} dt = 16g \int_0^{\pi/2} \cos t \sin^2 t dt \\ = \left[ 16g \left( \frac{\sin^3 t}{3} \right) \right]_0^{\pi/2} = \frac{16}{3} g$$

$$(c) \bar{x} = \frac{\int_C x(xy) ds}{\int_C xy ds} \text{ and } \bar{y} = \frac{\int_C y(xy) ds}{\int_C xy ds}; \text{ the mass of the string is } \int_C xy ds \text{ and the weight of the string is}$$

$$g \int_C xy ds. \text{ Therefore, the work done in moving the point mass at } (\bar{x}, \bar{y}) \text{ to the } x\text{-axis is}$$

$$W = \left( g \int_C xy ds \right) \bar{y} = g \int_C xy^2 ds = \frac{16}{3} g.$$

12. (a) Partition the sheet into small pieces. Let  $\Delta_i \sigma$  be the area of the  $i^{\text{th}}$  piece and select a point  $(x_i, y_i, z_i)$  in the  $i^{\text{th}}$  piece. The mass of the  $i^{\text{th}}$  piece is approximately  $x_i y_i \Delta_i \sigma$ . The work done by gravity in moving the  $i^{\text{th}}$  piece to the  $xy$ -plane is approximately  $(gx_i y_i \Delta_i \sigma) z_i = gx_i y_i z_i \Delta_i \sigma \Rightarrow \text{Work} = \iiint_S gxyz d\sigma.$

$$(b) \iint_S gxyz d\sigma = g \iint_{R_{xy}} xy(1-x-y) \sqrt{1+(-1)^2+(-1)^2} dA = \sqrt{3}g \int_0^1 \int_0^{1-x} (xy - x^2y - xy^2) dy dx \\ = \sqrt{3}g \int_0^1 \left[ \frac{1}{2} xy^2 - \frac{1}{2} x^2 y^2 - \frac{1}{3} xy^3 \right]_0^{1-x} dx = \sqrt{3}g \int_0^1 \left[ \frac{1}{6} x - \frac{1}{2} x^2 + \frac{1}{2} x^3 - \frac{1}{6} x^4 \right] dx \\ = \sqrt{3}g \left[ \frac{1}{12} x^2 - \frac{1}{6} x^3 + \frac{1}{6} x^4 - \frac{1}{30} x^5 \right]_0^1 = \sqrt{3}g \left( \frac{1}{12} - \frac{1}{30} \right) = \frac{\sqrt{3}g}{20}$$

- (c) The center of mass of the sheet is the point  $(\bar{x}, \bar{y}, \bar{z})$  where  $\bar{z} = \frac{M_{xy}}{M}$  with  $M_{xy} = \int_S xyz \, d\sigma$  and  $M = \int_S xy \, d\sigma$ . The work done by gravity in moving the point mass at  $(\bar{x}, \bar{y}, \bar{z})$  to the  $xy$ -plane is  $gM\bar{z} = gM \left( \frac{M_{xy}}{M} \right) = gM_{xy} = \int_S gxyz \, d\sigma = \frac{\sqrt{3}g}{20}$ .

13. (a) Partition the sphere  $x^2 + y^2 + (z - 2)^2 = 1$  into small pieces. Let  $\Delta_i\sigma$  be the surface area of the  $i^{\text{th}}$  piece and let  $(x_i, y_i, z_i)$  be a point on the  $i^{\text{th}}$  piece. The force due to pressure on the  $i^{\text{th}}$  piece is approximately  $w(4 - z_i)\Delta_i\sigma$ . The total force on  $S$  is approximately  $\sum_i w(4 - z_i)\Delta_i\sigma$ . This gives the actual force to be  $\int_S w(4 - z) \, d\sigma$ .
- (b) The upward buoyant force is a result of the  $\mathbf{k}$ -component of the force on the ball due to liquid pressure. The force on the ball at  $(x, y, z)$  is  $w(4 - z)(-\mathbf{n}) = w(z - 4)\mathbf{n}$ , where  $\mathbf{n}$  is the outer unit normal at  $(x, y, z)$ . Hence the  $\mathbf{k}$ -component of this force is  $w(z - 4)\mathbf{n} \cdot \mathbf{k} = w(z - 4)\mathbf{k} \cdot \mathbf{n}$ . The (magnitude of the) buoyant force on the ball is obtained by adding up all these  $\mathbf{k}$ -components to obtain  $\int_S w(z - 4)\mathbf{k} \cdot \mathbf{n} \, d\sigma$ .
- (c) The Divergence Theorem says  $\int_S w(z - 4)\mathbf{k} \cdot \mathbf{n} \, d\sigma = \int_D \operatorname{div}(w(z - 4)\mathbf{k}) \, dV = \int_D w \, dV$ , where  $D$  is  $x^2 + y^2 + (z - 2)^2 \leq 1 \Rightarrow \int_S w(z - 4)\mathbf{k} \cdot \mathbf{n} \, d\sigma = w \int_D 1 \, dV = \frac{4}{3}\pi w$ , the weight of the fluid if it were to occupy the region  $D$ .
14. The surface  $S$  is  $z = \sqrt{x^2 + y^2}$  from  $z = 1$  to  $z = 2$ . Partition  $S$  into small pieces and let  $\Delta_i\sigma$  be the area of the  $i^{\text{th}}$  piece. Let  $(x_i, y_i, z_i)$  be a point on the  $i^{\text{th}}$  piece. Then the magnitude of the force on the  $i^{\text{th}}$  piece due to liquid pressure is approximately  $F_i = w(2 - z_i)\Delta_i\sigma \Rightarrow$  the total force on  $S$  is approximately  $\sum_i F_i = \sum w(2 - z_i)\Delta_i\sigma \Rightarrow$  the actual force is  $\int_S w(2 - z) \, d\sigma = \int_{R_{xy}} \int w(2 - \sqrt{x^2 + y^2}) \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} \, dA$   
 $= \int_{R_{xy}} \int \sqrt{2} w(2 - \sqrt{x^2 + y^2}) \, dA = \int_0^{2\pi} \int_1^2 \sqrt{2} w(2 - r) r \, dr \, d\theta = \int_0^{2\pi} \sqrt{2} w \left[ r^2 - \frac{1}{3} r^3 \right]_1^2 \, d\theta = \int_0^{2\pi} \frac{2\sqrt{2}w}{3} \, d\theta = \frac{4\sqrt{2}\pi w}{3}$
15. Assume that  $S$  is a surface to which Stokes's Theorem applies. Then  $\oint_C \mathbf{E} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n} \, d\sigma$   
 $= \int_S \left( -\frac{\partial \mathbf{B}}{\partial t} \right) \cdot \mathbf{n} \, d\sigma = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n} \, d\sigma$ . Thus the voltage around a loop equals the negative of the rate of change of magnetic flux through the loop.
16. According to Gauss's Law,  $\int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi GmM$  for any surface enclosing the origin. But if  $\mathbf{F} = \nabla \times \mathbf{H}$  then the integral over such a closed surface would have to be 0 by the Divergence Theorem since  $\operatorname{div} \mathbf{F} = 0$ .
17.  $\oint_C f \nabla g \cdot d\mathbf{r} = \int_S \nabla \times (f \nabla g) \cdot \mathbf{n} \, d\sigma$  (Stokes's Theorem)  
 $= \int_S (f \nabla \times \nabla g + \nabla f \times \nabla g) \cdot \mathbf{n} \, d\sigma$  (Section 16.8, Exercise 19b)  
 $= \int_S [(f)(\mathbf{0}) + \nabla f \times \nabla g] \cdot \mathbf{n} \, d\sigma$  (Section 16.7, Equation 8)  
 $= \int_S (\nabla f \times \nabla g) \cdot \mathbf{n} \, d\sigma$
18.  $\nabla \times \mathbf{F}_1 = \nabla \times \mathbf{F}_2 \Rightarrow \nabla \times (\mathbf{F}_2 - \mathbf{F}_1) = \mathbf{0} \Rightarrow \mathbf{F}_2 - \mathbf{F}_1$  is conservative  $\Rightarrow \mathbf{F}_2 - \mathbf{F}_1 = \nabla f$ ; also,  $\nabla \cdot \mathbf{F}_1 = \nabla \cdot \mathbf{F}_2 \Rightarrow \nabla \cdot (\mathbf{F}_2 - \mathbf{F}_1) = 0 \Rightarrow \nabla^2 f = 0$  (so  $f$  is harmonic). Finally, on the surface  $S$ ,  $\nabla f \cdot \mathbf{n} = (\mathbf{F}_2 - \mathbf{F}_1) \cdot \mathbf{n} = \mathbf{F}_2 \cdot \mathbf{n} - \mathbf{F}_1 \cdot \mathbf{n} = 0$ . Now,  $\nabla \cdot (f \nabla f) = \nabla f \cdot \nabla f + f \nabla^2 f$  so the Divergence Theorem gives

$$\int_D \int \int |\nabla f|^2 dV + \int_D \int \int f \nabla^2 f dV = \int_D \int \int \nabla \cdot (f \nabla f) dV = \int_S \int f \nabla f \cdot \mathbf{n} d\sigma = 0, \text{ and since } \nabla^2 f = 0 \text{ we have}$$

$$\int_D \int \int |\nabla f|^2 dV + 0 = 0 \Rightarrow \int_D \int \int |\mathbf{F}_2 - \mathbf{F}_1|^2 dV = 0 \Rightarrow \mathbf{F}_2 - \mathbf{F}_1 = \mathbf{0} \Rightarrow \mathbf{F}_2 = \mathbf{F}_1, \text{ as claimed.}$$

19. False; let  $\mathbf{F} = y\mathbf{i} + x\mathbf{j} \neq \mathbf{0} \Rightarrow \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$  and  $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

20.  $|\mathbf{r}_u \times \mathbf{r}_v|^2 = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 \sin^2 \theta = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 (1 - \cos^2 \theta) = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 \cos^2 \theta = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2$   
 $\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = EG - F^2 \Rightarrow d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| du dv = \sqrt{EG - F^2} du dv$

21.  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3 \Rightarrow \int_D \int \int \nabla \cdot \mathbf{r} dV = 3 \int_D \int \int dV = 3V \Rightarrow V = \frac{1}{3} \int_D \int \int \nabla \cdot \mathbf{r} dV$   
 $= \frac{1}{3} \int_S \int \mathbf{r} \cdot \mathbf{n} d\sigma, \text{ by the Divergence Theorem}$